Unsteady Free Convection from Elliptic Tubes at Large Grashof Numbers

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

This study solves the problem of unsteady free convection from an inclined heated tube both numerically and analytically. The tube is taken to have an elliptic cross-section having a constant heat flux applied to its surface. The surrounding fluid is viscous and incompressible and infinite in extent. The Boussinesq approximation is used to describe the buoyancy force driving the flow. The underlying assumptions made in this work are that the flow remains laminar and two-dimensional for all time. This enables the Navier-Stokes and energy equations to be formulated in terms of the streamfunction, and vorticity.

We assume that initially an impulsive heat flux is applied to the surface and that both the tube and surrounding fluid have the same initial temperature. The problem is solved subject to the no-slip and constant heat flux conditions on the surface together with quiescent far-field and initial conditions.

An approximate analytical-numerical solution was derived for small times, t and large Grashof numbers, Gr. This was done by expanding the flow variables in a double series in terms of two small parameters and reduces to solving a set of differential equations. The first few terms were solved exactly while the higher-order terms were determined numerically.

Flow characteristics presented include average surface temperature plots as well as surface vorticity and surface temperature distributions. The results demonstrate that the approximate analytical-numerical solution is in good agreement with the fully numerical solution for small t and large Gr.

Acknowledgements

I would like to give a great big thanks for my supervisor Dr. Serge D'Alessio for his enormous support, inspirations and encouragement; without him this wouldn't have been possible. Also, I would like to thank my wife Kushlani, my parents Madubala and Sunil for their love and support throughout my life. Thanks as well to my examining committee: Dr. Francis Poulin and Dr. Marek Stastna. To my family and friends, thank you. Finally, I would also like to thank all the other members of the applied mathematics department.

Dedication

This is dedicated to the ones I love.

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Chapter 1

Introduction

1.1 Setting Up The Problem

Unsteady free, or natural, convection from a horizontal two-dimensional body is a fundamental thermal-fluid problem. Numerous theoretical, experimental and numerical studies related to the this problem have been done over the years. This thesis deals with the unsteady behavior of laminar, two-dimensional flow caused by free convection from an inclined elliptic cylinder in a fluid which is otherwise at rest. This problem is of interest for both theoretical and practical reasons since it has important applications in engineering such as hot wire anemometry, flow past heated tubes or wires, thermal pollution, dispersion of pollutants, and even in the design of heat exchangers.

This thesis investigates the initial development of the unsteady free convection problem in an unbounded domain. The surface of the cylinder has a constant heat flux and the fluid is assumed to be viscous, incompressible and Boussinesq. Also, the governing equations are the Navier-Stokes and energy equations.

The physical configuration is illustrated in Figure 1.1. The Cartesian axes x and y are rotated to coincide with the major and minor axes of the ellipse having lengths $2a$ and $2b$, respectively. Gravity acts in the vertical direction, and the ellipse is inclined at an angle η measured relative to the horizontal. The cylinder surface has a constant heat flux Q, while the far-field temperature of the fluid is T_∞, with $T > T_\infty$. Here, T denotes the temperature of the cylinder surface which varies with time and with position on the surface. The buoyancy-induced flow is considered to be unsteady and laminar with negligible viscous dissipation.

An approximate analytical-numerical solution valid for small times, t, and large Grashof numbers, Gr, is obtained by applying double series expansions for the flow variables. The expansions generate a hierarchy of simplified problems to the full governing equations. Boundary and initial conditions are defined for each problem, and integral conditions are derived for the vorticity using Green's Theorem. The leading-order terms as well as several higher-order terms were determined analytically, while the higher-order terms were determined numerically. This approximate

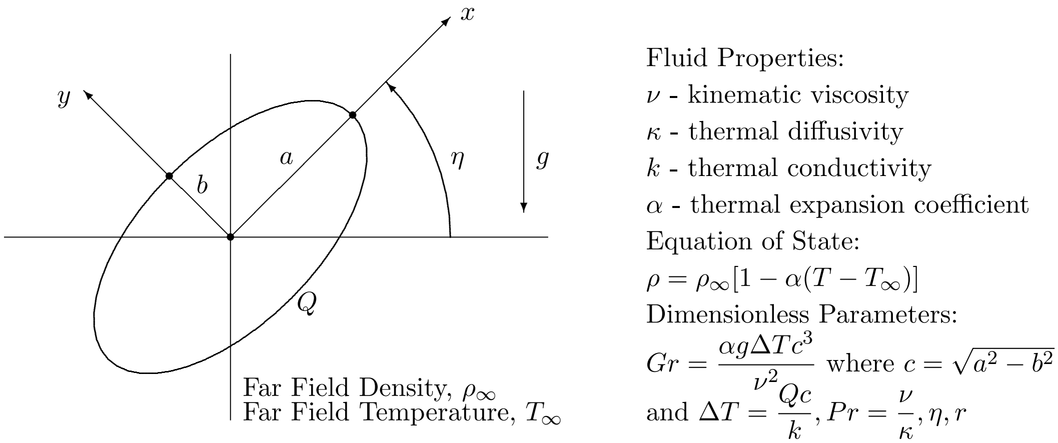

Figure 1.1: The flow configuration

solution serves to provide a reasonably accurate description of the initial flow and heat transfer process.

The thesis is organized as follows. In the next section a literature review is given. In chapter 2, derivation of the governing equations,boundary and initial conditions are explained. Chapter 3 describes the analytical method used to solve the problem while chapter 4 outlines the numerical method used in solving the higher order problems. In chapter 5, results and comparisons are presented. Finally, a summary is given in chapter 6.

1.2 Literature Review

The problem considered in this thesis is similar to that studied by Williams [35]. He also examined the unsteady problem of laminar two-dimensional free convective heat transfer from an inclined elliptic cylinder and also focused on the initial development of heat transfer and flow. However, to solve the problem analytically, he used perturbation theory and expanded the flow variables in powers of $\sqrt{t}$ and considered the isothermal case whereby the surface was maintained at constant temperature. The expansion used is valid for small t and small Gr, and analytic solutions to the problem were found for the case when the Prandtl number, Pr, is equal to unity. The scaling adopted in that study is identical to that outlined in section 2.2.3. The main difference between the work by Williams [35] and this study are as follows. First, we solve the unsteady free convection problem resulting from a cylinder emitting a constant heat flux. Second, a double series expansion is applied in the parameters $\lambda = \sqrt{\frac{4t}{\sqrt{Gr}}}$ and t. These parameters are better suited for investigating the early unsteady flow for large Gr. Third, analytical solutions have been found for the cases when $Pr = 1$ and $Pr \neq 1$. Lastly, a different scaling given in section 2.2.2, has been utilized. The main difference in the scaling of the equations lies in the velocity scale U. Williams [35] uses $U = \frac{\nu}{c}$ whereas the scale used here is $U = \sqrt{\alpha g \Delta T c}$.

In this present study an expansion procedure involving the parameter $\lambda = \sqrt{\frac{4t}{\sqrt{Gr}}}$ is carried out. This parameter will be small if t is small and Gr is large. We will see that the resulting differential equations in the series expansion in λ are still too complicated to solve analytically. If we take t to be a second small parameter, we can expand each of the equations in another series in terms of t. This amounts to doing a double expansion and enables us to further simplify the resulting equations and to solve them analytically. For small times we expect the gradients in the radial-direction (normal to the cylinder surface) to be much larger than those in the angular-direction. Also, for small times velocities will be very small. Thus, for small times the main mechanism of heat transfer will be by conduction. From this and the similarity solution to the heat conduction equation, the form of λ can be obtained.

A similar double expansion has been previously used in the study of two-dimensional laminar viscous flows at constant temperature. In these problems the parameter λ is defined as $\lambda = \sqrt{\frac{8t}{R}}$ where R is the Reynolds number. It turns out that our equation for the streamfunction and vorticity are identical if we replace $\frac{1}{\sqrt{Gr}}$ by $\frac{2}{R}$. Some previous studies that have used this approach include Rohlf and D'Alessio [28], Badr and Dennis [1], Collins and Dennis [5] for the case of a circular geometry, while Staniforth [32] was one of the first studies that successfully applied this approach to the elliptic geometry. In this thesis the double expansion procedure has been extended to solve the problem of free convection.

Another problem worth mentioning is the laminar plume formed above a line source explained in Leal [21]. This refers to a well known analytical solution for steady free convection from a line source of constant heat flux. This corresponds to the solution at large distances since far away the cylinder can be treated as a line source. For large Gr the equations can be simplified by making boundary layer approximations and rescaling the horizontal coordinate and velocity by a factor of $Gr^{\frac{1}{4}}$. It turns out that an exact similarity solution can be found for Prandtl numbers $Pr = 2, \frac{5}{9}$ (Yih [36]). The problem studied here can be thought of the other extreme case in that the laminar plume case is valid for large t and at large distances while our problem is valid for small t and close to the cylinder. It is interesting to point out that both the laminar plume problem and this problem have the factor of $Gr^{\frac{1}{4}}$ appearing in the solutions.

Extensive research has been conducted for circular cylinders experimentally, analytically, and numerically. Among the pioneers, Langmuir [20] concluded that the heat transfer rate could be evaluated as the amount of heat conveyed by pure conduction through a film of stationary fluid surrounding the wire, which is called the film theory. McAdams [24] correlated experimental data of flows over circular cylinders by other workers for Rayleigh numbers in the range between 10^{-4} and 10^{9}. The problem of laminar free convection from a horizontal circular cylinder with constant surface temperature or constant surface heat flux has been dealt with by Koh [17]. Analytical studies based on the boundary-layer approximation were conducted by Muntasser and Mulligan [26]. Numerical studies of the circular cylinder case with an isothermal cylinder surface have been reported in Saitoh, Sajiki and Maruhara [29]. They adopted a high-accuracy fourth-order finite difference method and a coordinate transformation technique to solve the two-dimensional natural convection problem for $Pr = 0.7$. Numerical solutions of the governing Navier-Stokes and energy equations were obtained by Kuehn and Goldstein [18], Farouk and Guceri [12], and Wang, Kahawita and Nguyen [34] to name a few.

Although problems related to heated circular cylinders are well studied, relatively little work has been done for the more general geometry of an elliptic cylinder which is considered here. Lin and Chao [22] utilized a suitable coordinate transformation to solve the boundary-layer equations for two-dimensional and axisymmetric bodies of arbitrary contour in terms of series solutions. Results were obtained for Prandtl numbers in the range between 0.72 and infinity, aspect ratio of the elliptic cross-section from 0.25 to 1, and the cases when the major axis either vertical or horizontal,respectively. Raithby and Hollands [27], following the film theory of Langmuir, proposed an approximate procedure for the prediction of the amount of heat exchanged by free convection at the surface of slender elliptic cylinders over a wide range of the Rayleigh numbers. Later Hassani [14] simplified the Raithby-Hollands method and extended its application to asymmetric horizontal cylinders having an arbitrary convex cross-section.

Merkin [25] numerically solved the boundary-layer equations for horizontal cylinders with the major axis oriented vertically to the direction of gravity. For a slender elliptic cylinder suspended in air, numerical solutions of the full conservation equations of mass, momentum and energy were obtained by Badr and Shamsher [2]. The problem has been solved for Rayleigh numbers ranging from 10 to 1000 and for a constant value of Prandtl number (Pr = 0.7). The cylinder axis ratio varies from 0.1 to 0.964 approaching a flat plate at one end and a circular cylinder at the other. Results were presented for the local and average Nusselt numbers along with details of the thermal and velocity fields given in the form of isotherm and streamline patterns.

Finlay [13] solved the steady problem of free convective heat transfer from an isothermal inclined elliptic cylinder numerically and derived asymptotic boundary conditions that could be applied at large distances. She also studied the stability of the flow to a specific type of disturbance. D'Alessio, Finlay and Pascal [8] considered the steady and unsteady problems of free convection from elliptic cylinders for small Gr numbers. The current study is an unsteady extension of [8] for large Gr numbers.

Work on the case of uniform heat flux has been done by Mahfouz and Kocabiyik [23]. They performed a numerical study of the transient buoyancy driven flow adjacent to a cylinder of elliptic cross-section with major axis horizontal, whose surface is subjected to a sudden uniform heat flux. This was executed for different values of the modified Rayleigh number in the range between 10^3 and 10^7, the Prandtl number in the range between 0.1 and 10, and the axis ratio in the range between 0.05 and 0.998.

Experiments have been performed by Elsayed, Ibrahim and Elsayed [11] for free convection of air around the outer surface of a horizontal elliptic tube with constant heat flux. The local and average Nusselt number distributions were presented at different values of Rayleigh numbers ranging from 1.1×10^7 to 8×10^7 and different tube inclination angles. Average Nusselt numbers were evaluated and correlated with the Rayleigh number corresponding to the elliptic tube with vertical major axis. Comparison between the convection characteristics of isothermal and constant heat flux elliptic tubes has been presented. The experiment showed that the maximum average Nusselt number is achieved when the major axis of the tube is vertical. Huang and Mayinger [15] also performed experiments in which laminar natural convection from elliptic tubes was investigated for different orientations and for different axis ratios.

Works on the extreme geometry of a flat plate were also conducted. Badr and Shamsher [2] performed a numerical study of the free convection from an elliptic cylinder with major axis vertical, for different values of the axis ratio in the range between 0.05 and 0.998 and validated the method by comparing results with the available experimental and numerical data for the circular cylinder and the flat plate as limiting cases. Experiments for the flat plate case was done by Saunders [31] while numerical solutions were obtained by Suriano and Yang [33].

The combined problems of forced and mixed convection from inclined elliptic cylinders undergoing uniform acceleration was studied by D'Alessio, Saunders and Harmsworth [9]. An approximate analytical solution valid for small times and large Reynolds numbers was presented. Also, a numerical solution was obtained by solving the full Navier-Stokes and energy equations using a spectral-finite difference method. More recently, the problem of unsteady heat transfer from an elliptic cylinder was solved numerically by Juncu [16]. In his paper, the influence of the volume heat capacity ratio and axis ratio on the heat transfer rate was investigated for various Reynolds and Prandtl numbers for the case of elliptic cylinders with spatially uniform, but changing with time, temperature.

In summary, this work offers an approximate analytical-numerical solution procedure to solve the problem of unsteady free convection from an inclined elliptical cylinder with constant heat flux which is limited to small times following the impulsive startup. Comparisons between this approximate solution and the fully numerical solution [7] have also been carried out.

Chapter 2

The Governing Equations

2.1 Deriving The Equations

Let us start with the continuity equation, taken from Kundu[19]:

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \vec{u} = 0 \; , \tag{2.1}$$

where $\vec{u} = (u, v)$ is the velocity of the flow, with u the velocity component in the x-direction, v in the y-direction and $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{u} \cdot \nabla)$ is the material derivative. We assume that the fluid is incompressible, and simplify the continuity equation to

$$\nabla \cdot \vec{u} = 0 \; . \tag{2.2}$$

We also assume that the fluid is viscous, and has a constant dynamic viscosity μ. Then the Navier-Stokes equations are given by

$$\rho\frac{D\vec{u}}{Dt} = -\nabla p + \rho\vec{g} + \mu\nabla^2\vec{u} \; . \tag{2.3}$$

Here, $\vec{g} = -g(\sin\eta, \cos\eta)$ is the gravitational vector with g denoting the gravitational constant; η is the angle of inclination of the elliptical cylinder with the horizontal plane. Using the Boussinesq approximation (replace ρ by ρ_0 everywhere except in the gravitational term), we obtain the following form for the Navier-Stokes equations

$$\rho_0\frac{D\vec{u}}{Dt} = -\nabla p + \rho\vec{g} + \mu\nabla^2\vec{u} \; . \tag{2.4}$$

Introducing the kinematic viscosity $\nu = \frac{\mu}{\rho_0}$, the Navier-Stokes equations can be written as

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho_0}\nabla p + \frac{\rho}{\rho_0}\vec{g} + \nu\nabla^2\vec{u} \; . \tag{2.5}$$

We use a linear equation of state as in Kundu[19]:

$$\frac{\rho}{\rho_0} = 1 - \alpha(T - T_\infty) \; , \tag{2.6}$$

where α is the thermal expansion coefficient, T is the temperature and T_∞ is the far-field temperature. Then the Navier-Stokes equations become

$$\frac{D\vec{u}}{Dt} = -\nabla\left(\frac{p}{\rho_0}\right) + (1-\alpha(T-T_\infty))\vec{g} + \nu\nabla^2\vec{u} \; . \tag{2.7}$$

Expanding the material derivative gives

$$\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} = -\nabla\left(\frac{p}{\rho_0}\right) + \vec{g} - \alpha\vec{g}(T-T_\infty)) + \nu\nabla^2\vec{u} \; . \tag{2.8}$$

We know that vorticity $\vec{\omega}$ is defined as $\vec{\omega} = \nabla\times\vec{u}$. To obtain an equation which involves vorticity, we use the following vector identity

$$\nabla\times(\nabla\times\vec{f}) = \nabla(\nabla\cdot\vec{f}) - \nabla^2\vec{f} \; . \tag{2.9}$$

Plugging $\vec{f} = \vec{u}$ and the continuity equation (2.2) into (2.9), we obtain the following equation for vorticity in terms of $\vec{u}$:

$$\nabla\times\vec{\omega} = -\nabla^2\vec{u} \; .$$

Substituting this into (2.8) we obtain

$$\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} = -\nabla\left(\frac{p}{\rho_0}\right) + \vec{g} - \alpha\vec{g}(T-T_\infty)) - \nu(\nabla\times\vec{\omega}) \; . \tag{2.10}$$

Using a second vector identity

$$(\vec{u}\cdot\nabla)\vec{u} = \frac{1}{2}\nabla(\vec{u}\cdot\vec{u}) - \vec{u}\times\vec{\omega} \; , \tag{2.11}$$

(2.10) becomes

$$\frac{\partial\vec{u}}{\partial t} - \vec{u}\times\vec{\omega} = -\nabla\left(\frac{p}{\rho_0} + \frac{1}{2}\vec{u}\cdot\vec{u}\right) + \vec{g} - \alpha\vec{g}(T-T_\infty)) - \nu(\nabla\times\vec{\omega}) \; . \tag{2.12}$$

We write the gravitational vector $\vec{g}$, as a gradient of a scalar $\vec{g} = \nabla\Lambda$, where

$$\Lambda = -g\left(x\sin\eta + y\cos\eta\right) \; .$$

Next, we introduce the quantity Γ as

$$\Gamma = \frac{p}{\rho_0} + \frac{1}{2}\vec{u}\cdot\vec{u} - \Lambda - \alpha T_\infty\Lambda \; .$$

Then, (2.12) becomes

$$\frac{\partial\vec{u}}{\partial t} - \vec{u}\times\vec{\omega} = -\nabla\Gamma - \alpha\vec{g}T - \nu\left(\nabla\times\vec{\omega}\right) \; . \tag{2.13}$$

Now we take the curl of the above to obtain

$$\frac{\partial}{\partial t}(\nabla \times \vec{u}) - \nabla \times (\vec{u} \times \vec{\omega}) = -\nabla \times (\nabla \Gamma) - \alpha(\nabla \times \vec{g}T) - \nu(\nabla \times (\nabla \times \vec{\omega})) \ . \quad (2.14)$$

Since $\nabla \times (\nabla \Gamma) = \vec{0}$, and by the definition of vorticity, we can reduce (2.14) to the following form

$$\frac{\partial \vec{\omega}}{\partial t} - \nabla \times (\vec{u} \times \vec{\omega}) = -\alpha(\nabla \times \vec{g}T) - \nu(\nabla \times (\nabla \times \vec{\omega})) \ . \quad (2.15)$$

Using vector identity (2.9), and setting $\vec{f} = \vec{\omega}$, we can further simplify (2.15) to

$$\frac{\partial \vec{\omega}}{\partial t} - \nabla \times (\vec{u} \times \vec{\omega}) = -\alpha(\nabla \times \vec{g}T) + \nu \nabla^2 \vec{\omega} \ . \quad (2.16)$$

Using the vector identity

$$\nabla \times (\vec{u} \times \vec{\omega}) = \vec{u}(\nabla \cdot \vec{\omega}) - \vec{\omega}(\nabla \cdot \vec{u}) + (\vec{\omega} \cdot \nabla)\vec{u} - (\vec{u} \cdot \nabla)\vec{\omega} \ , \quad (2.17)$$

along with the continuity equation and noting that the flow is 2-D, the above reduces to

$$\nabla \times (\vec{u} \times \vec{\omega}) = -(\vec{u} \cdot \nabla)\vec{\omega} \ . \quad (2.18)$$

Finally, we obtain the following for the vorticity equation in vector form

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{u} \cdot \nabla)\vec{\omega} = -\alpha(\nabla \times \vec{g}T) + \nu \nabla^2 \vec{\omega} \ . \quad (2.19)$$

The term $-\alpha(\nabla \times \vec{g}T)$, corresponds to the generation of vorticity due to the baroclinicity of the flow, or generation of vorticity when surfaces of constant pressure and density are not parallel. The term $\nu \nabla^2 \vec{\omega}$, corresponds to the change in vorticity due to molecular diffusion.

Since we assumed that the flow is 2-D, we can introduce the streamfunction ψ, where

$$u = -\frac{\partial \psi}{\partial y}, \ v = \frac{\partial \psi}{\partial x} \ . \quad (2.20)$$

If $\vec{\omega} = \nabla \times \vec{u} = (0, 0, \zeta)$, then

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \ . \quad (2.21)$$

Also,

$$\nabla \times T\vec{g} = \nabla \times (-g(T \sin \eta, T \cos \eta, 0)) = -g\left(0, 0, \frac{\partial T}{\partial x}\cos \eta - \frac{\partial T}{\partial y}\sin \eta\right) \ . \quad (2.22)$$

Using equations (2.21) and (2.22), and plugging them to equation (2.19), we obtain the governing equation for the scalar vorticity ζ,

$$\frac{\partial \zeta}{\partial t} + (\vec{u} \cdot \nabla)\zeta = \alpha g\left(\frac{\partial T}{\partial x}\cos \eta - \frac{\partial T}{\partial y}\sin \eta\right) + \nu \nabla^2 \zeta \ . \quad (2.23)$$

Using

$$\vec{u}\cdot\nabla\zeta = (u,v)\cdot\left(\frac{\partial\zeta}{\partial x},\frac{\partial\zeta}{\partial y}\right) = \left(-\frac{\partial\psi}{\partial y},\frac{\partial\psi}{\partial x}\right)\cdot\left(\frac{\partial\zeta}{\partial x},\frac{\partial\zeta}{\partial y}\right) = -\frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x}+\frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y}\ , \quad (2.24)$$

and substituting this into equation (2.23), we can further reduce the scalar vorticity equation to

$$\frac{\partial\zeta}{\partial t} = \frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y} + \nu\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^\zeta}{\partial y^2}\right) + \alpha g\left(\frac{\partial T}{\partial x}\cos\eta - \frac{\partial T}{\partial y}\sin\eta\right)\ . \quad (2.25)$$

To derive an equation governing the fluid temperature,T, we begin with the thermal energy equation

$$\rho\frac{De}{Dt} = -\nabla\cdot\vec{q} - p(\nabla\cdot\vec{u}) + \epsilon\ , \quad (2.26)$$

where e is the internal energy of the fluid, $\vec{q}$ is the heat flux per unit area, and ϵ is viscous dissipation. We will neglect viscous dissipation and thus set $\epsilon = 0$.

We can write e and $\vec{q}$ in terms of the fluid temperature T, the specific heat at constant pressure C_p, and the thermal conductivity of the fluid k, as

$$e = C_pT,\ \vec{q} = -k\nabla T\ . \quad (2.27)$$

Using the continuity equation and (2.27), the thermal energy equation simplifies to

$$\rho C_p\frac{DT}{Dt} = k\nabla^2 T\ . \quad (2.28)$$

Expanding for constant k and C_p, the material derivative gives

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \vec{u}\cdot\nabla T = \frac{\partial T}{\partial t} - \frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y}\ . \quad (2.29)$$

If we define the thermal diffusivity κ, with $\kappa = \frac{k}{\rho C_p}$, then the thermal energy equation reduces to

$$\frac{\partial T}{\partial t} = \frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y} + \kappa\nabla^2 T\ . \quad (2.30)$$

Thus, the three main governing equations in terms of streamfunction ψ, vorticity ζ and temperature T are

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \zeta\ , \quad (2.31)$$

$$\frac{\partial\zeta}{\partial t} = \frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y} + \nu\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^\zeta}{\partial y^2}\right) + \alpha g\left(\frac{\partial T}{\partial x}\cos\eta - \frac{\partial T}{\partial y}\sin\eta\right)\ , \quad (2.32)$$

$$\frac{\partial T}{\partial t} = \frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y} + \kappa\nabla^2 T\ . \quad (2.33)$$

2.2 Non-dimensionalization Methods

2.2.1 Non-dimensionalization Method 1

At large distances, we expect a balance to exist between the viscous force and the buoyancy force. For a sphere of radius R we have that

$$6\pi\mu R u_\infty = \frac{4}{3}\pi R^3 \Delta\rho g \ , \tag{2.34}$$

according to creep flow.

From the linear equation of state (2.6), we obtain

$$\Delta\rho = \rho_0 - \rho = \alpha\rho_0\Delta T \ ,$$

and (2.34) gives

$$u_\infty = \frac{2}{9}\frac{R^2}{\nu}\alpha g\Delta T \ . \tag{2.35}$$

If we set the length scale to $c = \sqrt{a^2 - b^2}$, the semi-focal length of the ellipse (see figure 1), then (2.35) suggests the following non-dimensionalization:

$$(\tilde{x}, \tilde{y}) = \left(\frac{x}{c}, \frac{y}{c}\right) \ , U = \frac{c^2\alpha g\Delta T}{\nu},$$

$$\phi = \frac{T - T_\infty}{\Delta T} \ , \Delta T = c\frac{Q}{\kappa},$$

$$\tilde{t} = \frac{U}{c}t \ , \tilde{\psi} = \frac{1}{Uc}\psi \ , \tilde{\zeta} = \frac{c}{U}\zeta \ , \tag{2.36}$$

where Q is the constant heat flux and the tildes denote dimensionless quantities.

Applying the above scaling to the streamfunction equation (2.31) leads to

$$\frac{\partial^2\tilde{\psi}}{\partial\tilde{x}^2} + \frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2} = \tilde{\zeta} \ . \tag{2.37}$$

Further applying this scaling to the vorticity-transport equation (2.32), we obtain

$$\frac{\partial\tilde{\zeta}}{\partial\tilde{t}} = \left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial\tilde{\zeta}}{\partial\tilde{x}} - \frac{\partial\tilde{\psi}}{\partial\tilde{x}}\frac{\partial\tilde{\zeta}}{\partial\tilde{y}}\right) + \frac{\nu}{cU}\left(\frac{\partial^2\tilde{\zeta}}{\partial\tilde{x}^2} + \frac{\partial^2\tilde{\zeta}}{\partial\tilde{y}^2}\right)$$

$$+ \frac{\alpha g c\Delta T}{U^2}\left(\frac{\partial\phi}{\partial\tilde{x}}\cos\eta - \frac{\partial\phi}{\partial\tilde{y}}\sin\eta\right) \ . \tag{2.38}$$

Finally, the temperature equation (2.33) under this scaling becomes

$$\frac{\partial\phi}{\partial\tilde{t}} = \left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial\phi}{\partial\tilde{x}} - \frac{\partial\tilde{\psi}}{\partial\tilde{x}}\frac{\partial\phi}{\partial\tilde{y}}\right) + \frac{\kappa}{cU}\left(\frac{\partial^2\phi}{\partial\tilde{x}^2} + \frac{\partial^2\phi}{\partial\tilde{y}^2}\right) \ . \tag{2.39}$$

We define the following dimensionless parameters: Prandtl number Pr, where $Pr = \frac{\nu}{\kappa}$ and Grashof number Gr, where $Gr = \frac{c^3 \alpha g \Delta T}{\nu^2}$. The Prandtl number represents the ratio of momentum diffusivity to thermal diffusivity of the fluid, while the Grashof number represents the relative strength of the buoyancy force to the viscous force.

In terms of these parameters our equations in dimensionless form become

$$\frac{\partial^2 \tilde{\psi}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{\psi}}{\partial \tilde{y}^2} = \tilde{\zeta} \ , \tag{2.40}$$

$$\begin{aligned} \frac{\partial \tilde{\zeta}}{\partial \tilde{t}} = \left(\frac{\partial \tilde{\psi}}{\partial \tilde{y}} \frac{\partial \tilde{\zeta}}{\partial \tilde{x}} - \frac{\partial \tilde{\psi}}{\partial \tilde{x}} \frac{\partial \tilde{\zeta}}{\partial \tilde{y}} \right) + \frac{1}{Gr} \left(\frac{\partial^2 \tilde{\zeta}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{\zeta}}{\partial \tilde{y}^2} \right) \\ + \frac{1}{Gr} \left(\frac{\partial \phi}{\partial \tilde{x}} \cos \eta - \frac{\partial \phi}{\partial \tilde{y}} \sin \eta \right) \ , \end{aligned} \tag{2.41}$$

$$\frac{\partial \phi}{\partial \tilde{t}} = \left(\frac{\partial \tilde{\psi}}{\partial \tilde{y}} \frac{\partial \phi}{\partial \tilde{x}} - \frac{\partial \tilde{\psi}}{\partial \tilde{x}} \frac{\partial \phi}{\partial \tilde{y}} \right) + \frac{1}{Gr Pr} \left(\frac{\partial^2 \phi}{\partial \tilde{x}^2} + \frac{\partial^2 \phi}{\partial \tilde{y}^2} \right) \ . \tag{2.42}$$

2.2.2 Non-dimensionalization Method 2

Near the surface we expect a balance to occur between inertia and buoyancy. If u is the speed of a fluid parcel rising and h the distance from the surface we can write

$$\frac{1}{2} \rho_0 u^2 = (\rho - \rho_0) g h \ , \tag{2.43}$$

from which we obtain

$$u^2 = 2 \frac{(\rho - \rho_0)}{\rho_0} g h \ .$$

From the linear equation of state (2.6), it follows that

$$\frac{\rho - \rho_0}{\rho_0} = \alpha \Delta T \ .$$

Hence,

$$u = \sqrt{2 \alpha \Delta T g h} \ . \tag{2.44}$$

If we set the length scale to $c = \sqrt{a^2 - b^2}$, then (2.44) suggests the following non-dimensionalization:

$$\begin{aligned} (\tilde{x}, \tilde{y}) = \left(\frac{x}{c}, \frac{y}{c} \right) \ , \ U = \sqrt{\alpha g \Delta T c}, \\ \phi = \frac{T - T_\infty}{\Delta T} \ , \ \Delta T = c \frac{Q}{\kappa}, \\ \tilde{t} = \frac{U}{c} t \ , \ \tilde{\psi} = \frac{1}{Uc} \psi \ , \ \tilde{\zeta} = \frac{c}{U} \zeta \ . \end{aligned} \tag{2.45}$$

Applying the above scaling to the streamfunction equation (2.31), we obtain

$$\frac{\partial^2\tilde{\psi}}{\partial\tilde{x}^2} + \frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2} = \tilde{\zeta} \ , \tag{2.46}$$

while the vorticity-transport equation (2.32) becomes

$$\begin{aligned} \frac{\partial\tilde{\zeta}}{\partial\tilde{t}} = \left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial\tilde{\zeta}}{\partial\tilde{x}} - \frac{\partial\tilde{\psi}}{\partial\tilde{x}}\frac{\partial\tilde{\zeta}}{\partial\tilde{y}}\right) + \frac{\nu}{cU}\left(\frac{\partial^2\tilde{\zeta}}{\partial\tilde{x}^2} + \frac{\partial^2\tilde{\zeta}}{\partial\tilde{y}^2}\right) \\ + \frac{\alpha gc\Delta T}{U^2}\left(\frac{\partial\phi}{\partial\tilde{x}}\cos\eta - \frac{\partial\phi}{\partial\tilde{y}}\sin\eta\right) \ , \end{aligned} \tag{2.47}$$

and the temperature equation (2.33) yields to

$$\frac{\partial\phi}{\partial\tilde{t}} = \left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial\phi}{\partial\tilde{x}} - \frac{\partial\tilde{\psi}}{\partial\tilde{x}}\frac{\partial\phi}{\partial\tilde{y}}\right) + \frac{\kappa}{cU}\left(\frac{\partial^2\phi}{\partial\tilde{x}^2} + \frac{\partial^2\phi}{\partial\tilde{y}^2}\right) \ . \tag{2.48}$$

In terms of Gr and Pr the dimensionless equations read

$$\frac{\partial^2\tilde{\psi}}{\partial\tilde{x}^2} + \frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2} = \tilde{\zeta} \ , \tag{2.49}$$

$$\frac{\partial\tilde{\zeta}}{\partial\tilde{t}} = \left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial\tilde{\zeta}}{\partial\tilde{x}} - \frac{\partial\tilde{\psi}}{\partial\tilde{x}}\frac{\partial\tilde{\zeta}}{\partial\tilde{y}}\right) + \frac{1}{\sqrt{Gr}}\left(\frac{\partial^2\tilde{\zeta}}{\partial\tilde{x}^2} + \frac{\partial^2\tilde{\zeta}}{\partial\tilde{y}^2}\right) + \left(\frac{\partial\phi}{\partial\tilde{x}}\cos\eta - \frac{\partial\phi}{\partial\tilde{y}}\sin\eta\right) \ , \tag{2.50}$$

$$\frac{\partial\phi}{\partial\tilde{t}} = \left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial\phi}{\partial\tilde{x}} - \frac{\partial\tilde{\psi}}{\partial\tilde{x}}\frac{\partial\phi}{\partial\tilde{y}}\right) + \frac{1}{\sqrt{Gr}Pr}\left(\frac{\partial^2\phi}{\partial\tilde{x}^2} + \frac{\partial^2\phi}{\partial\tilde{y}^2}\right) \ . \tag{2.51}$$

2.2.3 Non-dimensionalization Method 3

Another possible scaling is given by

$$\begin{aligned} (\tilde{x}, \tilde{y}) = \left(\frac{x}{c}, \frac{y}{c}\right) \ , U = \frac{\nu}{c}, \\ \phi = \frac{T - T_\infty}{\Delta T} \ , \tilde{t} = \frac{\nu}{c^2}t, \\ \tilde{\psi} = \frac{1}{\nu}\psi \ , \tilde{\zeta} = \frac{c^2}{\nu}\zeta \ , \end{aligned} \tag{2.52}$$

from which we obtain:

$$\frac{\partial^2\tilde{\psi}}{\partial\tilde{x}^2} + \frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2} = \tilde{\zeta} \ , \tag{2.53}$$

$$\frac{\partial\tilde{\zeta}}{\partial\tilde{t}} = \left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial\tilde{\zeta}}{\partial\tilde{x}} - \frac{\partial\tilde{\psi}}{\partial\tilde{x}}\frac{\partial\tilde{\zeta}}{\partial\tilde{y}}\right) + \left(\frac{\partial^2\tilde{\zeta}}{\partial\tilde{x}^2} + \frac{\partial^2\tilde{\zeta}}{\partial\tilde{y}^2}\right) + Gr\left(\frac{\partial\phi}{\partial\tilde{x}}\cos\eta - \frac{\partial\phi}{\partial\tilde{y}}\sin\eta\right) \ , \quad (2.54)$$

$$\frac{\partial\phi}{\partial\tilde{t}} = \left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial\phi}{\partial\tilde{x}} - \frac{\partial\tilde{\psi}}{\partial\tilde{x}}\frac{\partial\phi}{\partial\tilde{y}}\right) + \frac{1}{Pr}\left(\frac{\partial^2\phi}{\partial\tilde{x}^2} + \frac{\partial^2\phi}{\partial\tilde{y}^2}\right) \ . \quad (2.55)$$

2.2.4 Non-dimensionalization Method 4

This last scaling is similar to the previous one and is given by

$$(\tilde{x},\tilde{y}) = \left(\frac{x}{c},\frac{y}{c}\right) \ , U = \frac{\kappa}{c},$$
$$\phi = \frac{T - T_\infty}{\Delta T} \ , \tilde{t} = \frac{\kappa}{c^2}t,$$
$$\tilde{\psi} = \frac{1}{\kappa}\psi \ , \tilde{\zeta} = \frac{c^2}{\kappa}\zeta \ . \quad (2.56)$$

This scaling introduces another dimensionless parameter known as the Rayleigh number, Ra, given by $Ra = PrGr = \frac{c^3\alpha g\Delta T}{\nu\kappa}$. The Rayleigh number represents the relative strength of the viscous force to the buoyancy force within the fluid which can be expressed as

$$\frac{\alpha g c^3\Delta T}{\kappa^2} = \frac{\alpha g c^3\Delta T}{\kappa\nu}\frac{\nu}{\kappa} = RaPr \ .$$

This leads to the dimensionless set:

$$\frac{\partial^2\tilde{\psi}}{\partial\tilde{x}^2} + \frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2} = \tilde{\zeta} \ , \quad (2.57)$$

$$\frac{\partial\tilde{\zeta}}{\partial\tilde{t}} = \left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial\tilde{\zeta}}{\partial\tilde{x}} - \frac{\partial\tilde{\psi}}{\partial\tilde{x}}\frac{\partial\tilde{\zeta}}{\partial\tilde{y}}\right) + \left(\frac{\partial^2\tilde{\zeta}}{\partial\tilde{x}^2} + \frac{\partial^2\tilde{\zeta}}{\partial\tilde{y}^2}\right) + RaPr\left(\frac{\partial\phi}{\partial\tilde{x}}\cos\eta - \frac{\partial\phi}{\partial\tilde{y}}\sin\eta\right) \ , \quad (2.58)$$

$$\frac{\partial\phi}{\partial\tilde{t}} = \left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial\phi}{\partial\tilde{x}} - \frac{\partial\tilde{\psi}}{\partial\tilde{x}}\frac{\partial\phi}{\partial\tilde{y}}\right) + \left(\frac{\partial^2\phi}{\partial\tilde{x}^2} + \frac{\partial^2\phi}{\partial\tilde{y}^2}\right) \ . \quad (2.59)$$

2.2.5 Non-dimensionalization Method Used

In the previous subsections, we have presented four possible non-dimensionalization methods. Although any one of these can be adopted, we have selected method 2. Henceforth, our governing equations will be

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \zeta \ , \tag{2.60}$$

$$\frac{\partial\zeta}{\partial t} = \left(\frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y}\right) + \frac{1}{\sqrt{Gr}}\left(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2}\right) + \left(\frac{\partial\phi}{\partial x}\cos\eta - \frac{\partial\phi}{\partial y}\sin\eta\right) \ , \tag{2.61}$$

$$\frac{\partial\phi}{\partial t} = \left(\frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial y}\right) + \frac{1}{\sqrt{Gr}Pr}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) \ , \tag{2.62}$$

where the tildes have been dropped for the ease of notation.

2.3 Transformation to Elliptical Coordinates

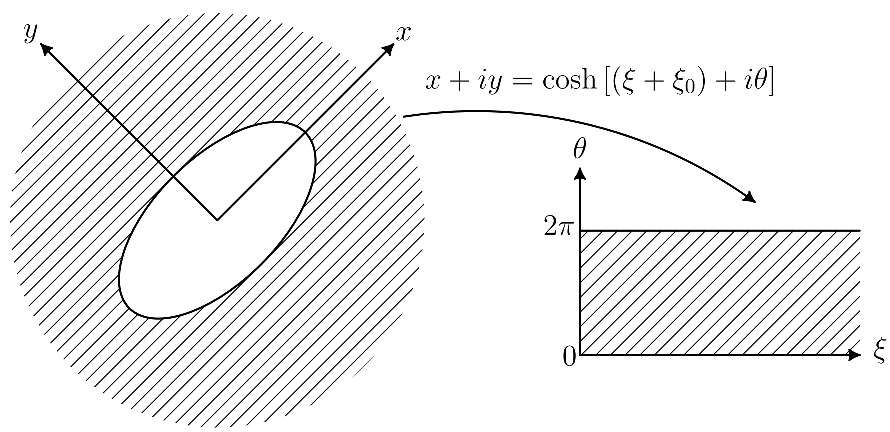

Figure 2.1: The mapping from Cartesian to elliptical co-ordinates

To make computations easier, we use a conformal mapping shown in figure 2.1. The transformation is given by

$$x + iy = \cosh[(\xi + \xi_0) + i\theta] \ , \tag{2.63}$$

where $\tanh\xi_0 = r$, and $r = \frac{b}{a}$ is the ratio of the semi-minor and semi-major axis lengths of the ellipse. This choice of the constant ξ_0 is such that the contour

$\xi = 0$ will coincide with the surface of the cylinder. Applying the identity $\cosh z = \frac{1}{2}(e^z + e^{-z})$ to (2.63), and comparing the real and imaginary parts we obtain

$$x = \cosh(\xi + \xi_0)\cos\theta \ , \ y = \sinh(\xi + \xi_0)\sin\theta \ . \tag{2.64}$$

D'Alessio [6] and Saunders [30] also used this transformation in their work.

The scale factors for the elliptical co-ordinate system (ξ, θ), are defined by

$$h_\xi = \sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2} \ , h_\theta = \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2} \ .$$

Using equation (2.64), we obtain

$$h_\xi = \sqrt{\cosh^2(\xi + \xi_0) - \cos^2\theta} \ , h_\theta = \sqrt{\cosh^2(\xi + \xi_0) - \cos^2\theta} \ . \tag{2.65}$$

We define the metric M of the transformation (2.63) as $M(\xi, \theta) = h_\xi = h_\theta$. Then,

$$M^2(\xi, \theta) = \cosh^2(\xi + \xi_0) - \cos^2\theta = \frac{1}{2}\left[\cosh 2(\xi + \xi_0) - \cos 2\theta\right] \ . \tag{2.66}$$

For the remainder of the thesis, the dependence of M on ξ and θ will not be explicitly shown, but will assumed to exist.

We differentiate equations (2.64) with respect to x and y to obtain the following set of equations

$$\frac{\partial \theta}{\partial x} = -\frac{\cosh(\xi + \xi_0)\sin\theta}{M^2} \ , \tag{2.67}$$

$$\frac{\partial \theta}{\partial y} = \frac{\sinh(\xi + \xi_0)\cos\theta}{M^2} \ , \tag{2.68}$$

$$\frac{\partial \xi}{\partial x} = \frac{\sinh(\xi + \xi_0)\cos\theta}{M^2} \ , \tag{2.69}$$

$$\frac{\partial \xi}{\partial y} = \frac{\cosh(\xi + \xi_0)\sin\theta}{M^2} \ . \tag{2.70}$$

Differentiating (2.67) – (2.70) with respect to x and y gives

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\sin\theta\cos\theta}{M^4}\left[1 + \frac{2\cosh^2(\xi + \xi_0)(\sinh^2(\xi + \xi_0) - \sin^2\theta)}{M^2}\right] \ , \tag{2.71}$$

$$\frac{\partial^2 \theta}{\partial y^2} = \frac{\sin\theta\cos\theta}{M^4}\left[1 - \frac{2\sinh^2(\xi + \xi_0)(\cosh^2(\xi + \xi_0) - \cos^2\theta)}{M^2}\right] \ , \tag{2.72}$$

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{\sinh(\xi + \xi_0)\cosh(\xi + \xi_0)}{M^4}\left[1 - \frac{2\cos^2(\theta)(\sinh^2(\xi + \xi_0) - \sin^2\theta)}{M^2}\right] \ , \tag{2.73}$$

$$\frac{\partial^2 \xi}{\partial y^2} = \frac{\sinh(\xi + \xi_0)\cosh(\xi + \xi_0)}{M^4}\left[1 - \frac{2\sin^2(\theta)(\cosh^2(\xi + \xi_0) - \cos^2\theta)}{M^2}\right] \ . \tag{2.74}$$

We apply the chain rule for differentiation:

$$\frac{\partial \Omega}{\partial x_i} = \frac{\partial \Omega}{\partial \xi}\frac{\partial \xi}{\partial x_i} + \frac{\partial \Omega}{\partial \theta}\frac{\partial \theta}{\partial x_i} \ ,$$

$$\frac{\partial^2 \Omega}{\partial {x_i}^2} = \frac{\partial^2 \Omega}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x_i}\right)^2 + \frac{\partial \Omega}{\partial \xi}\frac{\partial^2 \xi}{\partial {x_i}^2} + \frac{\partial^2 \Omega}{\partial \theta^2}\left(\frac{\partial \theta}{\partial x_i}\right)^2 + \frac{\partial \Omega}{\partial \theta}\frac{\partial^2 \theta}{\partial {x_i}^2} \ ,$$

where Ω is one of ϕ,ψ, or ζ and x_i is either x or y. Using the above derivatives and equations (2.67) – (2.74), the equation for the streamfunction (2.60) in elliptical co-ordinate system becomes

$$\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \theta^2} = M^2 \zeta \ . \tag{2.75}$$

The vorticity equation (2.61) becomes

$$\frac{\partial \zeta}{\partial t} = \frac{1}{M^2}\left[\frac{\partial \psi}{\partial \theta}\frac{\partial \zeta}{\partial \xi} - \frac{\partial \psi}{\partial \xi}\frac{\partial \zeta}{\partial \theta} + \frac{1}{\sqrt{Gr}}\left(\frac{\partial^2 \zeta}{\partial \xi^2} + \frac{\partial^2 \zeta}{\partial \theta^2}\right)\right] + \frac{1}{M^2}\left[A(\xi,\theta)\frac{\partial \phi}{\partial \xi} - B(\xi,\theta)\frac{\partial \phi}{\partial \theta}\right] \ , \tag{2.76}$$

where functions $A(\xi,\theta)$ and $B(\xi,\theta)$ are introduced for brevity and are defined as

$$A = \sinh(\xi+\xi_0)\cos(\eta)\cos(\theta) - \cosh(\xi+\xi_0)\sin(\eta)\sin(\theta) \ , \tag{2.77}$$

$$B = \cosh(\xi+\xi_0)\cos(\eta)\sin(\theta) + \sinh(\xi+\xi_0)\sin(\eta)\cos(\theta) \ . \tag{2.78}$$

Finally, the temperature equation (2.62) becomes

$$\frac{\partial \phi}{\partial t} = \frac{1}{M^2}\left[\frac{\partial \psi}{\partial \theta}\frac{\partial \phi}{\partial \xi} - \frac{\partial \psi}{\partial \xi}\frac{\partial \phi}{\partial \theta} + \frac{1}{\sqrt{Gr}Pr}\left(\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \theta^2}\right)\right] \ . \tag{2.79}$$

2.4 Boundary Conditions

To solve the governing equations, we need to impose some boundary conditions. On the cylinder surface ($\xi = 0$) we require the no-slip condition, no flow through the surface, and constant heat flux at the surface. To carry out the no-slip and impermeable conditions, we need to express the flow velocity in terms of the streamfunction, ψ. We express the flow velocity as: $\vec{u} = (u_\xi, u_\theta)$ where u_ξ and u_θ are the ξ and θ-components of the velocity, respectively. Then the no-slip condition becomes

$$u_\theta = 0 \text{ on } \xi = 0 \ , \tag{2.80}$$

and the impermeable condition becomes

$$u_\xi = 0 \text{ on } \xi = 0 \ . \tag{2.81}$$

To write the velocity components in terms of the streamfunction, we use the divergence of a curvilinear co-ordinate system:

$$\nabla \cdot \vec{u} = \frac{1}{h_\xi h_\theta}\left[\frac{\partial(h_\theta u_\xi)}{\partial \xi} + \frac{\partial(h_\xi u_\theta)}{\partial \theta}\right] ,$$

where $h_\xi = h_\theta = M$ as previously found. Using the continuity equation (2.2), we simplify the above equation to

$$\frac{1}{M^2}\left[\frac{\partial(M u_\xi)}{\partial \xi} + \frac{\partial(M u_\theta)}{\partial \theta}\right] = 0 .$$

Thus, we can define the streamfunction ψ in the (ξ, θ)-plane such that

$$M u_\xi = -\frac{\partial \psi}{\partial \theta} \text{ and } M u_\theta = \frac{\partial \psi}{\partial \xi}$$

so that the velocity is given by

$$(u_\xi, u_\theta) = \left(-\frac{1}{M}\frac{\partial \psi}{\partial \theta}, \frac{1}{M}\frac{\partial \psi}{\partial \xi}\right) . \tag{2.82}$$

The impermeable condition (2.81) in terms of the streamfunction becomes

$$\frac{\partial \psi}{\partial \theta} = 0 \text{ on } \xi = 0 .$$

This implies that ψ is constant on $\xi = 0$. Because there is only one solid boundary, without loss of generality we set this constant to zero. Thus, the no-slip and the impermeable conditions in terms of the streamfunction yield

$$\psi = \frac{\partial \psi}{\partial \xi} = 0 \text{ on } \xi = 0 . \tag{2.83}$$

The condition of constant heat flux gives

$$\frac{\partial \phi}{\partial \xi} = -M \text{ on } \xi = 0 . \tag{2.84}$$

Considering the stagnation streamline that extends from the surface of the cylinder ($\xi = 0$) to infinity, we develop far-field conditions for the streamfunction and vorticity. On the cylinder surface a stagnation point is a point at which the fluid comes to rest inviscidly. The stagnation streamline is the streamline that goes through the stagnation point. Since we have a quiescent flow far away from the cylinder, it follows that

$$u_\xi \, , \, u_\theta \to 0 \text{ as } \xi \to \infty ,$$

or in terms of the streamfunction this becomes

$$\frac{1}{M}\frac{\partial \psi}{\partial \xi} \, , \, \frac{1}{M}\frac{\partial \psi}{\partial \theta} \to 0 \text{ as } \xi \to \infty .$$

Since ψ is constant everywhere on the stagnation streamline, the value must be same as the constant on the surface $\xi = 0$. However, we showed that $\psi = 0$ on $\xi = 0$. The far-field condition for the streamfunction then becomes

$$\psi \to 0 \text{ as } \xi \to \infty \ .$$

For a quiescent flow, the far-field condition for vorticity is

$$\zeta \to 0 \text{ as } \xi \to \infty \ .$$

Since $T \to T_\infty$ as $\xi \to \infty$, the far-field condition for temperature is simply

$$\phi \to 0 \text{ as } \xi \to \infty \ .$$

Therefore, the far-field conditions for streamfunction, vorticity, and temperature are

$$\psi \ , \ \zeta \ , \ \phi \to 0 \text{ as } \xi \to \infty \ . \tag{2.85}$$

2.5 Integral Conditions

We now show that the vorticity satisfies integral conditions. We start with the Green's second identity as demonstrated in [10]

$$\int\int_D (g\nabla^2 h - h\nabla^2 g)dA = \oint_C (g\frac{\partial h}{\partial \hat{n}} - h\frac{\partial g}{\partial \hat{n}})dS \ , \tag{2.86}$$

where D is the fluid domain, C is the closed curve surrounding the domain D, and $\hat{n}$ is the direction normal to the curve C and g,h are twice differentiable functions. If we let

$$g = \psi \text{ and } h_n = e^{-n\xi}\begin{pmatrix} \sin(n\theta) \\ \cos(n\theta) \end{pmatrix} \text{ for } n = 0, 1, 2, \cdots \ ,$$

then

$$\nabla^2 g = \nabla^2 \psi = M^2 \zeta \ .$$

Since the functions h_n are harmonic, it follows that

$$\nabla^2 h_n = 0 \ .$$

From the boundary conditions, we obtain the following along the solid boundary

$$g = \psi = 0 \ ,$$
$$\frac{\partial g}{\partial \hat{n}} = \frac{\partial \psi}{\partial \xi} = 0 \ .$$

As $\xi \to \infty$,

$$h_n \to 0 \text{ for } n \neq 0 \ ,$$
$$\frac{\partial h_n}{\partial \hat{n}} = \frac{\partial h_n}{\partial \xi} \to 0 \ .$$

Plugging the above conditions into equation (2.86), we obtain the integral conditions for the vorticity

$$\int_0^\infty \int_{-\pi}^{\pi} e^{-n\xi} M^2 \zeta \begin{pmatrix} (\sin(n\theta) \\ \cos(n\theta) \end{pmatrix} d\theta d\xi = 0 \text{ for } n = 0, 1, 2, \cdots \ . \tag{2.87}$$

2.6 Initial Conditions

Since the fluid is at rest at $t = 0$, we have that

$$\psi(\xi, \theta, t = 0) = 0 \ , \tag{2.88}$$

$$\zeta(\xi, \theta, t = 0) = 0 \ . \tag{2.89}$$

We assume that initially both the fluid and cylinder are maintained at temperature T_∞. Thus,

$$\phi(\xi, \theta, t = 0) = 0 \ . \tag{2.90}$$

2.7 Boundary-layer Coordinate

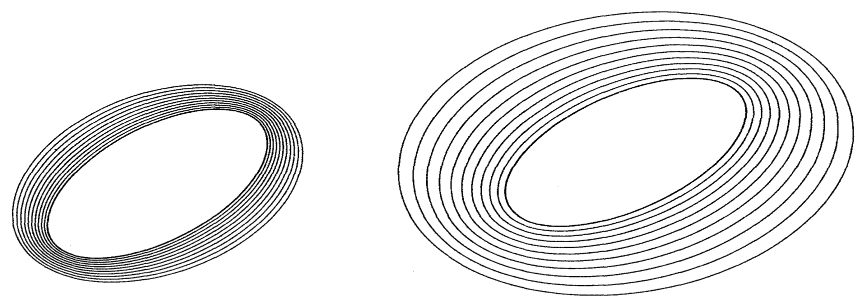

Figure 2.2: Illustration of the grid expansion with time

At $t = 0$ we impulsively apply a constant heat flux on the cylinder surface. To better resolve the early stages of the flow following this impulsive startup at $t = 0$, we define a boundary-layer coordinate z by

$$\xi = \lambda z \quad \text{where} \quad \lambda = \sqrt{\frac{4t}{\sqrt{Gr}}} \ . \tag{2.91}$$

The left and right sets of ellipses of figure 2.2 correspond to lines of constant ξ at times $t = t_1$ and $t = t_2$, respectively, where $t_2 > t_1$. Figure 2.2 illustrates how lines of constant ξ evolve in time in the boundary-layer coordinate system. We have that,

$$\frac{\partial \Phi}{\partial \xi} = \frac{1}{\lambda}\frac{\partial \Phi}{\partial z} \ ,$$

$$\frac{\partial^2 \Phi}{\partial \xi^2} = \frac{1}{\lambda^2}\frac{\partial^2 \Phi}{\partial z^2} \ ,$$

$$\frac{\partial \Phi}{\partial t} = \frac{\partial \Phi}{\partial t} - \frac{z}{2t}\frac{\partial \Phi}{\partial z} \ ,$$

where Φ is one of ψ, ζ, or ϕ.

Substituting the above into equations (2.75),(2.76) and (2.79), yields the following transformed equations for the streamfunction, vorticity and temperature.

$$\frac{\partial^2 \psi}{\partial z^2} + \lambda^2 \frac{\partial^2 \psi}{\partial \theta^2} = \lambda^2 M^2 \zeta \ , \tag{2.92}$$

$$\begin{aligned} \frac{1}{M^2}\frac{\partial^2 \zeta}{\partial z^2} + 2z\frac{\partial \zeta}{\partial z} = 4t\frac{\partial \zeta}{\partial t} - \frac{\lambda^2}{M^2}\frac{\partial^2 \zeta}{\partial \theta^2} + \frac{4t}{\lambda M^2}\left(\frac{\partial \psi}{\partial z}\frac{\partial \zeta}{\partial \theta} - \frac{\partial \psi}{\partial \theta}\frac{\partial \zeta}{\partial z}\right) \\ - \frac{4tA}{\lambda M^2}\frac{\partial \phi}{\partial z} + \frac{4tB}{M^2}\frac{\partial \phi}{\partial \theta} \ , \end{aligned} \tag{2.93}$$

$$\frac{1}{PrM^2}\frac{\partial^2 \phi}{\partial z^2} + 2z\frac{\partial \phi}{\partial z} = 4t\frac{\partial \phi}{\partial t} - \frac{\lambda^2}{PrM^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{4t}{\lambda M^2}\left(\frac{\partial \psi}{\partial z}\frac{\partial \phi}{\partial \theta} - \frac{\partial \psi}{\partial \theta}\frac{\partial \phi}{\partial z}\right) \ . \tag{2.94}$$

Since we will be concerned with the early stages of the flow, we will work with this set of equations for the remainder of the thesis.

The boundary conditions become

$$\psi = \frac{\partial \psi}{\partial z} = 0 \text{ on } z = 0 \ . \tag{2.95}$$

The constant heat flux condition is

$$\frac{\partial \phi}{\partial z} = -M\lambda \text{ on } z = 0 \ . \tag{2.96}$$

The far-field and the integral conditions yield

$$\psi \ , \ \zeta \ , \ \phi \to 0 \text{ as } z \to \infty \ , \tag{2.97}$$

$$\int_0^\infty \int_{-\pi}^{\pi} \lambda e^{-n\lambda z} M^2 \zeta \begin{pmatrix} (\sin(n\theta) \\ \cos(n\theta) \end{pmatrix} d\theta dz = 0 \text{ for } n = 0, 1, 2, \cdots \ . \tag{2.98}$$

Finally, the initial conditions become

$$\psi(z, \theta, t = 0) = \zeta(z, \theta, t = 0) = \phi(z, \theta, t = 0) = 0 \ . \tag{2.99}$$

Chapter 3

Analytical Solution Procedure

3.1 Series Expansion Procedure

The governing equations represent three coupled non-linear second-order partial differential equations (PDEs). Because of the complexity of the equations, it is very difficult to find an exact analytical solution. Instead we attempt to find an approximate analytical solution using perturbation theory. If Gr is large and t is small, then λ is also small, and it is possible to expand the flow variables in a double series in terms of λ and t. First, we expand the flow variables ϕ , ζ and ψ in a series of the form

$$\phi(z,\theta,t) = \phi_0(z,\theta,t) + \lambda\phi_1(z,\theta,t) + \lambda^2\phi_2(z,\theta,t) + \cdots \ , \tag{3.1}$$

$$\zeta(z,\theta,t) = \zeta_0(z,\theta,t) + \lambda\zeta_1(z,\theta,t) + \lambda^2\zeta_2(z,\theta,t) + \cdots \ , \tag{3.2}$$

$$\psi(z,\theta,t) = \psi_0(z,\theta,t) + \lambda\psi_1(z,\theta,t) + \lambda^2\psi_2(z,\theta,t) + \cdots \ . \tag{3.3}$$

Then each $\phi_n, \zeta_n, \psi_n, n = 0, 1, 2, \cdots$, can be further expanded in a series of the form

$$\phi_n(z,\theta,t) = \phi_{n0}(z,\theta) + t\phi_{n1}(z,\theta) + \cdots \ , \tag{3.4}$$

$$\zeta_n(z,\theta,t) = \zeta_{n0}(z,\theta) + t\zeta_{n1}(z,\theta) + \cdots \ , \tag{3.5}$$

$$\psi_n(z,\theta,t) = \psi_{n0}(z,\theta) + t\psi_{n1}(z,\theta) + \cdots \ . \tag{3.6}$$

We also need to expand the functions M^2,A,B and $\mathrm{e}^{-n\lambda t}$:

$$M^2 = M_0^2(\theta) + \sinh(2\xi_0)\lambda z + \cosh(2\xi_0)\lambda^2 z^2 + \cdots \ ,$$

$$A(z,\theta) = A_0(\theta) + A_1(\theta)\lambda z + \frac{A_0(\theta)}{2}\lambda^2 z^2 + \cdots \ ,$$

$$B(z,\theta) = B_0(\theta) + B_1(\theta)\lambda z + \frac{B_0(\theta)}{2}\lambda^2 z^2 + \cdots \ .$$

$$\mathrm{e}^{-n\lambda z} = 1 - n\lambda z + \frac{n^2\lambda^2 z^2}{2} - \frac{n^3\lambda^3 z^3}{6} + \cdots \ .$$

where

$$M_0^2 = \frac{1}{2}\left[\cosh(2\xi_0) - \cos(2\theta)\right] \ ,$$

$$A_0(\theta) = \sinh(\xi_0)\cos(\eta)\cos(\theta) - \cosh(\xi_0)\sin(\eta)\sin(\theta) \ ,$$

$$A_1(\theta) = \cosh(\xi_0)\cos(\eta)\cos(\theta) - \sinh(\xi_0)\sin(\eta)\sin(\theta) \ ,$$

$$B_0(\theta) = \cosh(\xi_0)\cos(\eta)\sin(\theta) + \sinh(\xi_0)\sin(\eta)\cos(\theta) \ ,$$

$$B_1(\theta) = \sinh(\xi_0)\cos(\eta)\sin(\theta) + \cosh(\xi_0)\sin(\eta)\cos(\theta) \ .$$

Substituting the above series into equations (2.92)-(2.99) produces a hierarchy of problems at various orders.

We see that there are two small parameters appearing in this problem t and λ. These parameters will be the same when $t = \lambda$ or when $t = \frac{4}{\sqrt{Gr}}$. For a fixed value of Gr our procedure is expected to be valid for $t << \frac{4}{\sqrt{Gr}}$. This will be important later when we make comparisons with the fully numerical solution.

An alternate expansion procedure is presented in Appendix B.

3.2 Solving for the Temperature

3.2.1 The O(1) problem

The first term in the series (3.1), ϕ_0, satisfies the following equation

$$\frac{1}{PrM_0^2}\frac{\partial^2\phi_0}{\partial z^2} + 2z\frac{\partial\phi_0}{\partial z} = 4t\frac{\partial\phi_0}{\partial t} \ . \tag{3.7}$$

The constant heat flux and the far-field conditions are

$$\frac{\partial\phi_0}{\partial z} = 0 \text{ on } z = 0 \ , \ \phi_0 \to 0 \text{ as } z \to \infty \ . \tag{3.8}$$

Since equation (3.7) is too complicated to solve analytically, an expansion in t is also necessary to make analytical progress. We now continue the procedure of determining some of the other terms in the double series (3.4). As we will see the procedure gets more and more complicated as more terms are sought.

If we introduce the transformation $w = \sqrt{Pr}M_0 z$ we can eliminate the explicit θ-dependence, and the problem reduces to essentially solving the following differential equations

$$\frac{\partial^2\phi_{0N}}{\partial w^2} + 2w\frac{\partial\phi_{0N}}{\partial w} - 4N\phi_{0N} = 0 \ , \tag{3.9}$$

subject to the constant heat flux conditions

$$\frac{\partial\phi_{0N}}{\partial w} = 0 \text{ when } w = 0 \ , N = 0, 1, 2, \cdots \ , \tag{3.10}$$

and the far-field conditions

$$\phi_{0N} \to 0 \text{ as } w \to \infty \ , N = 0, 1, 2, \cdots \ . \tag{3.11}$$

In order to solve equation (3.9), we first make the transformation $\phi_{0N} = F(\theta)f(w)\mathrm{e}^{-\frac{w^2}{2}}$, to obtain

$$\frac{d^2 f}{dw^2} + (-w^2 - 1 - 4N)f = 0 \ . \tag{3.12}$$

Using the transformation $w = \frac{u}{\sqrt{2}}$ then yields

$$\frac{d^2 f}{du^2} + ((-2N - 1) + \frac{1}{2} - \frac{u^2}{4})f = 0 \ . \tag{3.13}$$

In this form the solution can be expressed in terms of the Parabolic Cylinder Functions (see [3]). If we let $\nu = -2N - 1$, a negative integer, the solution to (3.13) is

$$f = AD_{-2N-1}(u) + BD_{2N}(iu) \ , \tag{3.14}$$

where A and B are arbitrary constants. If we substitute $u = \sqrt{2}w$ into (3.14) we obtain

$$\phi_{0N} = F(\theta)A\mathrm{e}^{-\frac{w^2}{2}} D_{-2N-1}(\sqrt{2}w) + F(\theta)B\mathrm{e}^{-\frac{w^2}{2}} D_{2N}(i\sqrt{2}w) \ , \tag{3.15}$$

and by properties of Parabolic Cylinder Functions, it follows that

$$D_{2N}(i\sqrt{2}w) = \frac{\mathrm{e}^{\frac{w^2}{2}}}{2^N} H_{2N}(iw) \ , \tag{3.16}$$

where H_{2N} represents a Hermite polynomial in degree 2N. Further,

$$D_{-2N-1}(\sqrt{2}w) =$$

$$(\sqrt{2})^{-2N-1}\mathrm{e}^{\frac{w^2}{2}} \left[\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{1}{2}(-2N-1)\right)} \Phi\left(-\frac{1}{2}(-2N-1), \frac{1}{2}, w^2 \right) \right]$$

$$+ (\sqrt{2})^{-2N-1}\mathrm{e}^{\frac{w^2}{2}} \left[w \frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2}(-2N-1)\right)} \Phi\left(\frac{1}{2} - \frac{1}{2}(-2N-1), \frac{3}{2}, w^2 \right) \right] \ , \tag{3.17}$$

where

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \ ,$$

$$\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi} \ ,$$

$$\Gamma\left(\frac{1}{2} - \frac{1}{2}(-2N-1)\right) = N! \ ,$$

$$\Gamma\left(-\frac{1}{2}(-2N-1)\right) = \frac{(2N-1)!!}{2^N}\sqrt{\pi} \ ,$$

$$\Phi\left(-\frac{1}{2}\left(-2N-1\right),\frac{1}{2},w^2\right) = 1 + \frac{(N+\frac{1}{2})}{\frac{1}{2}!!}w^2 + \frac{(N+\frac{1}{2})(N+\frac{3}{2})}{\frac{1}{2}\frac{3}{2}}w^4 + \cdots \ ,$$

$$\Phi\left(\frac{1}{2}-\frac{1}{2}\left(-2N-1\right),\frac{3}{2},w^2\right) = 1 + \frac{(N+\frac{3}{2})}{\frac{1}{2}!!}w^2 + \frac{(N+1)(N+2)}{\frac{3}{2}\frac{5}{2}}w^4 + \cdots \ ,$$

taken from [3].

Since H_{2N} is a polynomial of degree $2N$ (>1), $H_{2N}(iw)$ is unbounded as $w \to \infty$. Therefore, to satisfy the far-field condition (3.11), we must have $B = 0$.

Then (3.15) becomes

$$\phi_{0N} = F(\theta)A\mathrm{e}^{-\frac{w^2}{2}}D_{-2N-1}(\sqrt{2}w) \ . \tag{3.18}$$

Plugging (3.18) into constant heat flux condition (3.10) we obtain that $A = 0$. Thus, ϕ_{0N} becomes

$$\phi_{0N}(w) = 0 \ , N = 0, 1, 2, \cdots \ , \tag{3.19}$$

from which it immediately follows that

$$\phi_0(w, \theta, t) = 0 \ . \tag{3.20}$$

3.2.2 The O(λ) problem

The non-zero leading-order term for the temperature corresponds to ϕ_1 and satisfies the equation

$$\frac{1}{PrM_0^2}\frac{\partial^2\phi_1}{\partial z^2} + 2z\frac{\partial\phi_1}{\partial z} - 2\phi_1 = 4t\frac{\partial\phi_1}{\partial t} \ . \tag{3.21}$$

The constant heat flux and the far-field conditions are

$$\frac{\partial\phi_1}{\partial z} = -M_0 \text{ on } z = 0 \ , \ \phi_1 \to 0 \text{ as } z \to \infty \ . \tag{3.22}$$

Since equation (3.21) is too complicated to solve analytically, an expansion in t is also necessary to make analytical progress. If we introduce the transformation $w = \sqrt{Pr}M_0z$ the problem reduces to essentially solving the following set of differential equations

$$\frac{\partial^2\phi_{10}}{\partial w^2} + 2w\frac{\partial\phi_{10}}{\partial w} - 2\phi_{10} = 0 \ , \tag{3.23}$$

$$\frac{\partial^2\phi_{1N}}{\partial w^2} + 2w\frac{\partial\phi_{1N}}{\partial w} - (2+4N)\phi_{1N} = 0 \ , \tag{3.24}$$

for $N = 1, 2, 3, \cdots$, subject to the constant heat flux condition

$$\frac{\partial\phi_{10}}{\partial w} = -\frac{1}{\sqrt{Pr}} \ , \ \frac{\partial\phi_{1N}}{\partial w} = 0 \quad \text{ on } \ w = 0 \tag{3.25}$$

along with the far-field conditions

$$\phi_{10} \to 0 \ , \ \phi_{1N} \to 0 \ \text{ as } \ w \to \infty \ . \tag{3.26}$$

For the $N = 0$ case, the homogeneous differential equation has two linearly independent solutions of the following form

$$\phi_{10_1} = w \ ,$$

and

$$\phi_{10_2} = \mathrm{e}^{-w^2} + \sqrt{\pi} w \mathrm{erf}(w) \ ,$$

where $\mathrm{erf}(w) = \frac{2}{\sqrt{\pi}} \int_0^w \mathrm{e}^{-u^2} du$. So, the solution for ϕ_{10} can be written as

$$\phi_{10}(w) = Cw + D(\mathrm{e}^{-w^2} + \sqrt{\pi} w \mathrm{erf}(w)) \ ,$$

where $C(\theta)$ and $D(\theta)$ are arbitrary functions. Rearranging the above equation gives

$$\phi_{10}(w) = w \left(C + D\sqrt{\pi}\mathrm{erf}(w) \right) + D\mathrm{e}^{-w^2} \ , \tag{3.27}$$

and applying the heat flux condition we find that $C = -\frac{1}{\sqrt{Pr}}$.

Then equation (3.27) becomes

$$\phi_{10}(w) = w \left(-\frac{1}{\sqrt{Pr}} + D\sqrt{\pi}\mathrm{erf}(w) \right) + D\mathrm{e}^{-w^2} \ . \tag{3.28}$$

To satisfy the far-field condition (3.26), we must have

$$\left(-\frac{1}{\sqrt{Pr}} + D\sqrt{\pi}\mathrm{erf}(w) \right) \to 0 \ \text{ as } \ w \to \infty \ .$$

Thus, we can conclude that

$$D = \frac{1}{\sqrt{\pi}\sqrt{Pr}} \ .$$

Thus, the solution for ϕ_{10} is

$$\phi_{10}(w) = \frac{1}{\sqrt{\pi}\sqrt{Pr}}\mathrm{e}^{-w^2} - \frac{w}{\sqrt{Pr}}\mathrm{erfc}(w) \ , \tag{3.29}$$

where $\mathrm{erfc}(w) = 1 - \mathrm{erf}(w)$.

For the $N = 1, 2, 3, \cdots$ cases, the differential equations can be expressed in terms of the Parabolic Cylinder Functions (discussed in section 3.2.1). Using a similar argument we can show that

$$\phi_{1N}(w) = 0 \ \text{ for } \ N = 1, 2, 3, \cdots \ . \tag{3.30}$$

Thus, we arrive at

$$\phi_1(t, w) = \phi_{10}(w) = \frac{1}{\sqrt{\pi}\sqrt{Pr}}\mathrm{e}^{-w^2} - \frac{w}{\sqrt{Pr}}\mathrm{erfc}(w) \ . \tag{3.31}$$

3.2.3 The O(λ^2) problem

The equation for ϕ_2 is given by

$$\frac{1}{Pr}\frac{\partial^2\phi_2}{\partial z^2}+2M_0^2 z\frac{\partial\phi_2}{\partial z}-4tM_0^2\frac{\partial\phi_2}{\partial t}-4M_0^2\phi_2=4t\frac{\partial\phi_1}{\partial\theta}\frac{\partial\psi_2}{\partial z}-4t\frac{\partial\phi_1}{\partial z}\frac{\partial\psi_2}{\partial\theta}$$

$$-2\sinh(2\xi_0)\frac{\partial\phi_1}{\partial z}z^2+4t\sinh(2\xi_0)\left(\frac{\partial\phi_1}{\partial t}+\frac{\phi_1}{2t}\right)z\ . \qquad (3.32)$$

The constant heat flux and the far-field conditions are

$$\frac{\partial\phi_2}{\partial z}=0 \text{ on } z=0\ ,\phi_2\to 0 \text{ as } z\to\infty\ . \qquad (3.33)$$

Using the double expansion, the problem reduces to solving the differential equations

$$\frac{1}{PrM_0^2}\frac{\partial^2\phi_{20}}{\partial z^2}+2z\frac{\partial\phi_{20}}{\partial z}-4\phi_{20}=\frac{2\sinh(2\xi_0)}{M_0^2}z\left(\phi_{10}-z\frac{\partial\phi_{10}}{\partial z}\right)\ , \qquad (3.34)$$

$$\frac{1}{PrM_0^2}\frac{\partial^2\phi_{2N}}{\partial z^2}+2z\frac{\partial\phi_{2N}}{\partial z}-4(1+N)\phi_{2N}=0\ , N=1,3,4,\cdots\ , \qquad (3.35)$$

$$\frac{1}{PrM_0^2}\frac{\partial^2\phi_{22}}{\partial z^2}+2z\frac{\partial\phi_{22}}{\partial z}-12\phi_{22}=\frac{4}{M_0^2}z\left(\frac{\partial\phi_{10}}{\partial\theta}\frac{\partial\psi_{21}}{\partial z}-\frac{\partial\phi_{10}}{\partial z}\frac{\partial\psi_{21}}{\partial\theta}\right)\ , \qquad (3.36)$$

subject to the constant heat flux condition

$$\frac{\partial\phi_{2N}}{\partial z}=0\ , \text{ when } z=0\ , N=0,1,2,\cdots\ , \qquad (3.37)$$

and the far-field conditions

$$\phi_{2N}\to 0\ , \text{ as } z\to\infty\ , N=0,1,2,\cdots\ . \qquad (3.38)$$

For the $N=0$ case, we introduce the transformation $w=\sqrt{Pr}M_0 z$, then the equation and boundary conditions become

$$\frac{\partial^2\phi_{20}}{\partial w^2}+2w\frac{\partial\phi_{20}}{\partial w}-4\phi_{20}=\frac{2\sinh(2\xi_0)}{PrM_0^3\sqrt{\pi}}w\mathrm{e}^{-w^2}\ ,$$

$$\frac{\partial\phi_{20}}{\partial w}=0\ , \text{ when } w=0\ ,$$

$$\phi_{20}\to 0\ , \text{ as } w\to\infty\ . \qquad (3.39)$$

If we introduce a new variable $\Phi(w)=M_0^3(\theta)\phi_{20}(w)$, then (3.39) transforms to

$$\frac{d^2\Phi}{dw^2}+2w\frac{d\Phi}{dw}-4\Phi=\frac{2\sinh(2\xi_0)}{Pr\sqrt{\pi}}w\mathrm{e}^{-w^2}\ ,$$

$$\frac{d\Phi}{dw} = 0 \text{ , when } w = 0 \text{ ,}$$
$$\Phi \to 0 \text{ , as } w \to \infty \text{ .} \tag{3.40}$$

We next solve the above problem using the method of variation of parameters.

The homogeneous differential equation has two linearly independent solutions having the following form

$$\Phi_1 = w^2 + \frac{1}{2} \text{ ,}$$

and

$$\Phi_2 = w\mathrm{e}^{-w^2} + \sqrt{\pi}\left(w^2 + \frac{1}{2}\right)\mathrm{erf}(w) \text{ .}$$

Thus, a particular solution to (3.40) can be written as

$$\Phi_p(w) = U\Phi_1 + V\Phi_2 \tag{3.41}$$

where

$$U' = -\frac{\Phi_2 F}{R} \text{ and } V' = \frac{\Phi_1 F}{R} \text{ ,}$$

and R is the Wronskian given by

$$R = \Phi_1\Phi_2' - \Phi_2\Phi_1' = \mathrm{e}^{-w^2} \text{ ,}$$

and

$$F = \frac{2\sinh(2\xi_0)}{Pr\sqrt{\pi}} w\mathrm{e}^{-w^2} \text{ .}$$

By integrating U' and V' with respect to w, and setting the arbitrary constants to zero, we obtain

$$U = \frac{\sinh(2\xi_0)}{Pr}\left[\frac{1}{8}\mathrm{erf}(w) - \frac{1}{4\sqrt{\pi}}w\mathrm{e}^{-w^2} - \frac{1}{2}w^4\mathrm{erf}(w) - \frac{1}{2\sqrt{\pi}}w^3\mathrm{e}^{-w^2} - \frac{1}{2}w^2\mathrm{erf}(w)\right] \text{ ,}$$

$$V = \frac{\sinh(2\xi_0)}{2\sqrt{\pi}Pr}\left(w^4 + w^2\right) \text{ .}$$

Then, the full solution of (3.40) can be written as

$$\Phi(w) = K_1\left(w^2 + \frac{1}{2}\right) + K_2\left(w\mathrm{e}^{-w^2} + \sqrt{\pi}\left(w^2 + \frac{1}{2}\right)\mathrm{erf}(w)\right)$$
$$+ \frac{\sinh(2\xi_0)}{Pr}\left[\frac{1}{8}\mathrm{erf}(w)\left(w^2 + \frac{1}{2}\right) - \frac{1}{8}\frac{w\mathrm{e}^{-w^2}}{\sqrt{\pi}}\right] \text{ .} \tag{3.42}$$

Applying the constant heat flux condition, we obtain

$$K_2 = 0 \text{ ,}$$

and applying the far-field condition, we obtain

$$K_1 = 0 \ .$$

Therefore, (3.42) simplifies to

$$\Phi(w) = \frac{\sinh(2\xi_0)}{Pr}\left[\frac{1}{8}\mathrm{erf}(w)\left(w^2+\frac{1}{2}\right) - \frac{1}{8}\frac{w\mathrm{e}^{-w^2}}{\sqrt{\pi}}\right] \ . \tag{3.43}$$

Since $\phi_{20}(w) = \frac{\Phi(w)}{M_0^3}$, it follows that

$$\phi_{20}(w,\theta) = \frac{\sinh(2\xi_0)}{PrM_0^3}\left[\frac{1}{8}\mathrm{erf}(w)\left(w^2+\frac{1}{2}\right) - \frac{w\mathrm{e}^{-w^2}}{8\sqrt{\pi}}\right] \ . \tag{3.44}$$

For the $N = 1, 3, 4, \cdots$ cases, we introduce the transformation $w = \sqrt{Pr}M_0 z$, then the equations and boundary conditions become

$$\begin{aligned} \frac{\partial^2\phi_{2N}}{\partial w^2} + 2w\frac{\partial\phi_{2N}}{\partial w} - 4(1+N)\phi_{2N} = 0 \ , N = 1,3,4,\cdots \ , \\ \frac{\partial\phi_{2N}}{\partial w} = 0 \ , \text{when } w = 0 \ , N = 1,3,4,\cdots \ , \\ \phi_{2N} \to 0 \ , \text{as } w \to \infty \ , N = 1,3,4,\cdots \ . \end{aligned} \tag{3.45}$$

To solve the above equations we use a procedure similar to the $N = 0$ case. Since the differential equations are homogeneous, and from the constant heat flux and far-field conditions we can derive that

$$\phi_{2N}(w,\theta) = 0 \quad \text{for} \quad N = 1,3,4,\cdots \ . \tag{3.46}$$

$\phi_{22}(w,\theta)$ is numerically solved and is explained in the next chapter.

Thus, we can conclude that

$$\phi_2(t,w,\theta) = \phi_{20} + t^2\phi_{22} \ . \tag{3.47}$$

In the next chapter a numerical method to solve for ϕ_{30} will be outlined. Summarizing, we have that the approximate analytical solution for temperature ϕ is given by

$$\phi(t,w,\theta) = \lambda\phi_{10} + \lambda^2\phi_{20} + \lambda^3\phi_{30} + O(\lambda^3 t + \lambda^4) \ . \tag{3.48}$$

3.3 Solving for the Vorticity

3.3.1 The O(1) problem

The first term in the series (3.2), ζ_0, satisfies the following equation

$$\frac{1}{M_0^2}\frac{\partial^2\zeta_0}{\partial z^2}+2z\frac{\partial\zeta_0}{\partial z}=4t\frac{\partial\zeta_0}{\partial t}-\frac{4tA_0}{M_0^2}\frac{\partial\phi_1}{\partial z}\ . \tag{3.49}$$

The far-field and the integral conditions are

$$\begin{aligned}
&\zeta_0\to 0 \text{ as } z\to\infty\ ,\\
&\int_0^\infty\int_{-\pi}^{\pi} M_0^2\zeta_0\sin(n\theta)d\theta dz=0 \ \text{ for } n=0,1,2,\cdots\ ,\\
&\int_0^\infty\int_{-\pi}^{\pi} M_0^2\zeta_0\cos(n\theta)d\theta dz=0 \ \text{ for } n=0,1,2,\cdots\ .
\end{aligned} \tag{3.50}$$

To solve for the leading-order problem for the vorticity expansion, we introduce the transformation $s=M_0z$. Then, ζ_0, satisfies the equation and conditions given by

$$\begin{aligned}
&\frac{\partial^2\zeta_0}{\partial s^2}+2s\frac{\partial\zeta_0}{\partial s}=4t\frac{\partial\zeta_0}{\partial t}-4tA_0\frac{\partial\phi_1}{\partial s}\ ,\\
&\zeta_0\to 0 \text{ as } s\to\infty\ ,\\
&\int_0^\infty\int_{-\pi}^{\pi} M_0\zeta_0\sin(n\theta)d\theta ds=0 \ \text{ for } n=0,1,2,\cdots\ ,\\
&\int_0^\infty\int_{-\pi}^{\pi} M_0\zeta_0\cos(n\theta)d\theta dz=0 \ \text{ for } n=0,1,2,\cdots\ .
\end{aligned} \tag{3.51}$$

Further expanding in t, the problem reduces to solving the following differential equations

$$\frac{\partial^2\zeta_{00}}{\partial s^2}+2s\frac{\partial\zeta_{00}}{\partial s}=0\ , \tag{3.52}$$

$$\frac{\partial^2\zeta_{01}}{\partial s^2}+2s\frac{\partial\zeta_{01}}{\partial s}-4\zeta_{01}=-4A_0\frac{\partial\phi_{10}}{\partial s}\ , \tag{3.53}$$

$$\frac{\partial^2\zeta_{0N}}{\partial s^2}+2s\frac{\partial\zeta_{0N}}{\partial s}-4\zeta_{0N}=0 \text{ for} N=2,3,4\cdots\ , \tag{3.54}$$

subject to the far-field conditions

$$\zeta_{0N}\to 0 \text{ as } s\to\infty \text{ for } \ N=0,1,2,\cdots\ ,$$

and the integral conditions

$$\int_0^\infty\int_{-\pi}^{\pi} M_0\zeta_{0N}\sin(n\theta)d\theta ds=0\ ,$$

$$\int_0^\infty \int_{-\pi}^{\pi} M_0 \zeta_{0N} \cos(n\theta) d\theta ds = 0 \; ,$$

for $n, N = 0, 1, 2, \cdots$.

It immediately follows from the initial condition (2.99) that

$$\zeta_{00}(s, \theta) = 0 \; . \tag{3.55}$$

Since

$$\frac{\partial \phi_{10}}{\partial s} = \sqrt{Pr}\frac{\partial \phi_{10}}{\partial w} = -\mathrm{erfc}(w) = -\mathrm{erfc}(\sqrt{Pr}s) \; ,$$

we can write equation (3.53) as

$$\frac{\partial^2 \zeta_{01}}{\partial s^2} + 2s\frac{\partial \zeta_{01}}{\partial s} - 4\zeta_{01} = 4A_0 \left(1 - \mathrm{erf}(\sqrt{Pr}s)\right) \; . \tag{3.56}$$

Now we break up the solution $\zeta_{01} = \zeta_{01}^{(1)} + \zeta_{01}^{(2)}$ where $\zeta_{01}^{(1)}$, $\zeta_{01}^{(2)}$ satisfy

$$\frac{\partial^2 \zeta_{01}^{(1)}}{\partial s^2} + 2s\frac{\partial \zeta_{01}^{(1)}}{\partial s} - 4\zeta_{01}^{(1)} = 4A_0 \; , \tag{3.57}$$

and

$$\frac{\partial^2 \zeta_{01}^{(2)}}{\partial s^2} + 2s\frac{\partial \zeta_{01}^{(2)}}{\partial s} - 4\zeta_{01}^{(2)} = -4A_0 \mathrm{erf}(\sqrt{Pr}s) \; . \tag{3.58}$$

A particular solution for (3.57) is

$$\zeta_{01}^{(1)} = -A_0 \; . \tag{3.59}$$

Next, we solve equation (3.58) for the case when $Pr = 1$ using the method of variation of parameters. The homogeneous differential equation has two linearly independent solutions of the following form

$$\zeta_{01,1}^{(2)} = s^2 + \frac{1}{2} \; ,$$

and

$$\zeta_{01,2}^{(2)} = s\mathrm{e}^{-s^2} + \sqrt{\pi}\left(s^2 + \frac{1}{2}\right)\mathrm{erf}(s) \; .$$

Thus, a particular solution to (3.58) is

$$\zeta_{01}^{(2)} = u\zeta_{01,1}^{(2)} + v\zeta_{01,2}^{(2)} \tag{3.60}$$

where

$$u' = -\frac{\zeta_{01,2} f}{R} \; , \; v' = \frac{\zeta_{01,1} f}{R} \; , \; R = \zeta_{01,1}\zeta_{01,2}' - \zeta_{01,2}\zeta_{01,1}' = \mathrm{e}^{-s^2} \; , \; f = -4A_0 \mathrm{erf}(s) \; .$$

By integrating u' and v' with respect to s, and setting the arbitrary constants to zero, we obtain that

$$\frac{u}{-4A_0} = \frac{1}{2}s^2\mathrm{erf}(s) - \frac{1}{4}\mathrm{erf}(s) + \frac{1}{2\sqrt{\pi}}s\mathrm{e}^{-s^2} - \frac{\sqrt{\pi}}{2}s\mathrm{e}^{-s^2}\mathrm{erf}(s)^2 \; ,$$

$$\frac{v}{-4A_0} = \frac{1}{2} s e^{s^2} \mathrm{erf}(s) - \frac{1}{2\sqrt{\pi}} s^2 \ .$$

Then, the full solution of (3.56) can be assembled as

$$\zeta_{01} = D_1(\theta)\zeta_{01,1}^{(2)} + D_2(\theta)\zeta_{01,2}^{(2)} + \zeta_{01}^{(1)} + \zeta_{01}^{(2)} \ .$$

Thus,

$$\zeta_{01} = D_1(\theta)\left(s^2 + \frac{1}{2}\right) + D_2(\theta)\left(s e^{-s^2} + \sqrt{\pi}\left(s^2 + \frac{1}{2}\right)\mathrm{erf}(s)\right)$$

$$- A_0 - 4A_0 \left[\frac{1}{4}\left(s^2 - \frac{1}{2}\right)\mathrm{erf}(s) + \frac{s e^{s^2}}{4\sqrt{\pi}}\right] \ . \qquad (3.61)$$

Applying the far-field condition, we obtain

$$D_1(\theta) = A_0 - \sqrt{\pi} D_2(\theta) \ .$$

Now we can write

$$\zeta_{01} = D_2(\theta)K(s) + A_0(\theta)E(s) \ , \qquad (3.62)$$

where

$$K(s) = s e^{-s^2} - \sqrt{\pi}\left(s^2 + \frac{1}{2}\right)\mathrm{erfc}(s) \ ,$$

$$E(s) = \left(s^2 - \frac{1}{2}\right)\mathrm{erfc}(s) - \frac{s e^{-s^2}}{\sqrt{\pi}} \ .$$

Plugging (3.61) into the integral conditions, and noting that $\int_0^\infty K(s)ds = -\frac{1}{3}$ and $\int_0^\infty E(s)ds = -\frac{2}{3\sqrt{\pi}}$ we obtain the following

$$\int_{-\pi}^{\pi} M_0(\theta)D_2(\theta)\sin(n\theta)d\theta = -\frac{2}{\sqrt{\pi}}\int_{-\pi}^{\pi} M_0(\theta)A_0(\theta)\sin(n\theta)d\theta \ ,$$

$$\int_{-\pi}^{\pi} M_0(\theta)D_2(\theta)\cos(n\theta)d\theta = -\frac{2}{\sqrt{\pi}}\int_{-\pi}^{\pi} M_0(\theta)A_0(\theta)\cos(n\theta)d\theta \ . \qquad (3.63)$$

Now we write $D_2(\theta)M_0(\theta)$ as a Fourier series

$$D_2(\theta)M_0(\theta) = \frac{\alpha_0}{2} + \sum_{l=1}^{\infty}\left(\alpha_l\cos(l\theta) + \beta_l\sin(l\theta)\right) \ . \qquad (3.64)$$

Then using (3.63) and (3.64), we can write

$$\alpha_0 = -\frac{2}{\pi\sqrt{\pi}}\int_{-\pi}^{\pi} M_0(\theta)A_0(\theta)d\theta \ ,$$

$$\alpha_l = -\frac{2}{\pi\sqrt{\pi}}\int_{-\pi}^{\pi} M_0(\theta)A_0(\theta)\cos(n\theta)d\theta \ ,$$

$$\beta_l = -\frac{2}{\pi\sqrt{\pi}} \int_{-\pi}^{\pi} M_0(\theta) A_0(\theta) \sin(n\theta) d\theta \ .$$

Using the MAPLE computer algebra system, the values of $\alpha_0, \alpha_1, \cdots$ and $\beta_1, \beta_2, \cdots$ can easily be computed and are listed in Appendix A. Then $D_2(\theta)$ becomes

$$D_2(\theta) = \frac{1}{M_0} \left(\frac{\alpha_0}{2} + \sum_{l=1}^{\infty} (\alpha_l \cos(l\theta) + \beta_l \sin(l\theta)) \right) \ . \tag{3.65}$$

For $Pr = 1$ the general solution for ζ_{01} becomes

$$\zeta_{01}(s, \theta) = D_2(\theta) \left[s\mathrm{e}^{-s^2} - \sqrt{\pi} \left(s^2 + \frac{1}{2} \right) \mathrm{erfc}(s) \right]$$

$$+ A_0(\theta) \left[\left(s^2 - \frac{1}{2} \right) \mathrm{erfc}(s) - \frac{s\mathrm{e}^{-s^2}}{\sqrt{\pi}} \right] \tag{3.66}$$

For $Pr \neq 1$, we solve equation (3.56) using the method of variation of parameters. The homogeneous differential equation has two linearly independent solutions of the following form

$$\zeta_{01,1}^* = s^2 + \frac{1}{2} \ ,$$

and

$$\zeta_{01,2}^* = s\mathrm{e}^{-s^2} + \sqrt{\pi} \left(s^2 + \frac{1}{2} \right) \mathrm{erf}(s) \ .$$

Thus, a particular solution to (3.56) is

$$\zeta_{01,p}^* = u^* \zeta_{01,1}^* + v^* \zeta_{01,2}^* \tag{3.67}$$

where

$$u^{*\prime} = -\frac{\zeta_{01,2}^* f^*}{R} \ , \ v^{*\prime} = \frac{\zeta_{01,1}^* f^*}{R} \ ,$$

$$R = \zeta_{01,1}^* {\zeta_{01,2}^*}' - \zeta_{01,2}^* {\zeta_{01,1}^*}' = \mathrm{e}^{-s^2} \ , \ f^* = 4A_0 \mathrm{erfc}(\sqrt{Pr} s) \ .$$

By integrating $u^{*\prime}$ and $v^{*\prime}$ with respect to s, and setting the arbitrary constants to zero, we obtain

$$u^* = -\frac{4A_0\sqrt{\pi}}{2} s\mathrm{e}^{s^2} \mathrm{erf}(s) \mathrm{erfc}(\sqrt{Pr} s)$$

$$-4A_0\sqrt{Pr} \frac{\mathrm{e}^{(1-Pr)s^2}}{2(1-Pr)} \mathrm{erf}(s) + \frac{4A_0}{2(1-Pr)} \mathrm{erf}\left(\sqrt{Pr} s\right) \ ,$$

$$v^* = \frac{4A_0}{2} s\mathrm{e}^{s^2} - \frac{4A_0}{2} s\mathrm{e}^{s^2} \mathrm{erf}(\sqrt{Pr} s) + \frac{4A_0\sqrt{Pr}}{\sqrt{\pi}} \frac{\mathrm{e}^{(1-Pr)s^2}}{2(1-Pr)} \ .$$

Then, the full solution of (3.56) for $Pr \neq 1$ can be written as

$$\zeta_{01} = \hat{D}_1(\theta) \zeta_{01,1}^* + \hat{D}_2(\theta) \zeta_{01,2}^* + {\zeta_{01,p}}^*$$

Applying the far-field condition, we obtain

$$\hat{D}_1(\theta) = -\frac{4A_0}{2\,(1-Pr)} - \sqrt{\pi}\hat{D}_2(\theta)\ .$$

Now we can write

$$\zeta_{01} = \hat{D}_2(\theta)K^*(s) + A_0(\theta)E^*(s)\ , \tag{3.68}$$

where

$$K^*(s) = s\mathrm{e}^{-s^2} - \sqrt{\pi}\left(s^2+\frac{1}{2}\right)\mathrm{erfc}(s)\ ,$$

$$E^*(s) = 4\left[-\frac{\left(s^2+\frac{1}{2}\right)}{2\,(1-Pr)}\mathrm{erfc}(\sqrt{Pr}s) + \frac{s^2}{2}\mathrm{erfc}(\sqrt{Pr}s) + \frac{\sqrt{Pr}s}{2\sqrt{\pi}}\frac{\mathrm{e}^{-Prs^2}}{(1-Pr)}\right]\ .$$

Plugging (3.68) into the integral conditions, and noting that $\int_0^\infty K^*(s)ds = -\frac{1}{3}$ and $\int_0^\infty E^*(s)ds = -\frac{2}{3\sqrt{\pi}(1-Pr)\sqrt{Pr}}$, we obtain the following

$$\int_{-\pi}^{\pi} M_0(\theta)\hat{D}_2(\theta)\sin(n\theta)d\theta = -\frac{2}{\sqrt{\pi}\,(1-Pr)\,\sqrt{Pr}}\int_{-\pi}^{\pi} M_0(\theta)A_0(\theta)\sin(n\theta)d\theta\ ,$$
$$\int_{-\pi}^{\pi} M_0(\theta)\hat{D}_2(\theta)\cos(n\theta)d\theta = -\frac{2}{\sqrt{\pi}\,(1-Pr)\,\sqrt{Pr}}\int_{-\pi}^{\pi} M_0(\theta)A_0(\theta)\cos(n\theta)d\theta\ . \tag{3.69}$$

Similar to previous work, we write $\hat{D}_2(\theta)M_0(\theta)$ as a Fourier series

$$\hat{D}_2(\theta)M_0(\theta) = \frac{\hat{\alpha}_0}{2} + \sum_{l=1}^{\infty}\left(\hat{\alpha}_l\cos(l\theta) + \hat{\beta}_l\sin(l\theta)\right)\ . \tag{3.70}$$

Then using (3.69) and (3.70), we can write

$$\hat{\alpha}_0 = -\frac{2}{\pi\sqrt{\pi}\,(1-Pr)\,\sqrt{Pr}}\int_{-\pi}^{\pi} M_0(\theta)A_0(\theta)d\theta\ ,$$

$$\hat{\alpha}_l = -\frac{2}{\pi\sqrt{\pi}\,(1-Pr)\,\sqrt{Pr}}\int_{-\pi}^{\pi} M_0(\theta)A_0(\theta)\cos(n\theta)d\theta\ ,$$
$$\hat{\beta}_l = -\frac{2}{\pi\sqrt{\pi}\,(1-Pr)\,\sqrt{Pr}}\int_{-\pi}^{\pi} M_0(\theta)A_0(\theta)\sin(n\theta)d\theta\ .$$

Again using the MAPLE computer algebra system, the values of $\hat{\alpha}_0, \hat{\alpha}_1, \cdots$ and $\hat{\beta}_1, \hat{\beta}_2, \cdots$ can be easily computed and are listed in Appendix A. Then $\hat{D}_2(\theta)$ becomes

$$\hat{D}_2(\theta) = \frac{1}{M_0}\left(\frac{\hat{\alpha}_0}{2} + \sum_{l=1}^{\infty}\left(\hat{\alpha}_l\cos(l\theta) + \hat{\beta}_l\sin(l\theta)\right)\right)\ . \tag{3.71}$$

For $Pr \neq 1$ the general solution for ζ_{01} becomes

$$\zeta_{01}(s,\theta) = \hat{D}_2(\theta)\left[s\mathrm{e}^{-s^2} - \sqrt{\pi}\left(s^2+\frac{1}{2}\right)\mathrm{erfc}(s)\right]$$

$$+4A_0(\theta)\left[\frac{-1}{2(1-Pr)}(s^2+\frac{1}{2})\mathrm{erfc}(\sqrt{Pr}s)\right]$$

$$+4A_0(\theta)\left[\frac{s^2}{2}\mathrm{erfc}(\sqrt{Pr}s)+\frac{\sqrt{Pr}}{2\sqrt{\pi}(1-Pr)}s\mathrm{e}^{-Prs^2}\right]. \quad (3.72)$$

In the above $D_2(\theta)$ and $\hat{D}_2(\theta)$ have been solved explicitly using the Maple computer algebra system.

To solve for the $N = 2, 3, \cdots$ cases, we write equation (3.54) in terms of the Parabolic Cylinder functions. Using a similar method used in previous sections, we can show that the solution of (3.54) has the following form

$$\zeta_{ON} = H(\theta)\mathrm{e}^{-\frac{1}{2}s^2}D_{-2N-1}(\sqrt{2}s) + J(\theta)\mathrm{e}^{-\frac{1}{2}s^2}D_{2N}(i\sqrt{2}s)\ . \quad (3.73)$$

Using the far-field condition, we can show that

$$J(\theta) = 0$$

then (3.73) will be

$$\zeta_{0N} = H(\theta)\mathrm{e}^{-\frac{1}{2}s^2}D_{-2N-1}(\sqrt{2}s)\ . \quad (3.74)$$

Plugging (3.74) into the integral conditions, we obtain

$$\int_0^\infty\int_{-\pi}^{\pi} M_0H(\theta)\mathrm{e}^{-\frac{1}{2}s^2}D_{-2N-1}(\sqrt{2}s)\sin(n\theta)d\theta ds = 0 \ \text{ for } n = 0,1,2,\cdots,$$

$$\int_0^\infty\int_{-\pi}^{\pi} M_0H(\theta)\mathrm{e}^{-\frac{1}{2}s^2}D_{-2N-1}(\sqrt{2}s)\cos(n\theta)d\theta ds = 0 \ \text{ for } n = 0,1,2,\cdots.$$

Rearranging the above equations

$$\int_0^\infty \mathrm{e}^{-\frac{1}{2}s^2}D_{-2N-1}(\sqrt{2}s)ds\int_{-\pi}^{\pi} M_0H(\theta)\sin(n\theta)d\theta = 0 \ \text{ for } n = 0,1,2,\cdots$$

$$\int_0^\infty \mathrm{e}^{-\frac{1}{2}s^2}D_{-2N-1}(\sqrt{2}s)ds\int_{-\pi}^{\pi} M_0H(\theta)\cos(n\theta)d\theta = 0 \ \text{ for } n = 0,1,2,\cdots$$

Since $\int_0^\infty \mathrm{e}^{-\frac{1}{2}s^2}D_{-2N-1}(\sqrt{2}s)ds \neq 0$, we must have

$$\int_{-\pi}^{\pi} M_0H(\theta)\sin(n\theta)d\theta = 0 \ \text{ for } n = 0,1,2,\cdots, \quad (3.75)$$

$$\int_{-\pi}^{\pi} M_0H(\theta)\cos(n\theta)d\theta = 0 \ \text{ for } n = 0,1,2,\cdots. \quad (3.76)$$

Now we write $H(\theta)$ as a Fourier series

$$H(\theta) = \frac{a_0}{2} + \sum_{l=1}^{\infty}\left(a_l\cos(l\theta) + b_l\sin(l\theta)\right)\ .$$

Applying the above Fourier series into (3.75) and (3.76) we can show that

$$a_0 = a_1 = \cdots = b_1 = b_2 = \cdots = 0 \; .$$

So we conclude that

$$\zeta_{0N}(s, \theta) = 0 \;\; \text{for} \; N = 2, 3, 4, \cdots \; . \tag{3.77}$$

Thus, we obtain

$$\zeta_0(t, s, \theta) = t\zeta_{01} \; . \tag{3.78}$$

In the next chapter a numerical method for determining ζ_{11} will be given. We can now conclude that

$$\zeta_1(t, s, \theta) = t\zeta_{11} + O(t^2) \; . \tag{3.79}$$

Summarizing, we have that the approximate analytical solution for the vorticity ζ is

$$\zeta(t, s, \theta) = t\zeta_{01} + \lambda t\zeta_{11} + O(\lambda t^2 + \lambda^2) \; . \tag{3.80}$$

3.4 Solving for the Streamfunction

3.4.1 The O(1) and O(λ) problem

The first two terms in the series (3.3), ψ_0 and ψ_1, satisfy the following equations

$$\frac{\partial^2 \psi_i}{\partial z^2} = 0 \;\; \text{for} \; i = 0, 1 \; . \tag{3.81}$$

The no-slip and impermeable boundary conditions are

$$\psi_i = 0 \text{ on } z = 0 \; , \;\; \text{for } i = 0, 1 \; ,$$

$$\frac{\partial \psi_i}{\partial z} = 0 \text{ on } z = 0 \; , \;\; \text{for } i = 0, 1 \; . \tag{3.82}$$

Solving these boundary value problems yield

$$\psi_0(z, \theta) = \psi_1(z, \theta) = 0 \; . \tag{3.83}$$

3.4.2 The O(λ^2) problem

Then the non-zero leading-order problem corresponds to solving

$$\frac{\partial^2 \psi_2}{\partial z^2} = M_0^2 t\zeta_0 \; , \tag{3.84}$$

and the no-slip and impermeable boundary conditions are

$$\psi_2 = \frac{\partial \psi_2}{\partial z} = 0 \text{ on } z = 0 \; . \tag{3.85}$$

Making use of the double expansion and the variable $s = M_0 z$, the problem reduces to solving the following differential equations

$$\frac{\partial^2 \psi_{2N}}{\partial s^2} = 0 \;\text{ for }\; N = 0, 2, 3, \cdots \;, \tag{3.86}$$

$$\frac{\partial^2 \psi_{21}}{\partial s^2} = \zeta_{01} \;, \tag{3.87}$$

subject to the no-slip and impermeable conditions

$$\psi_{2N} = \psi_{21} = \frac{\partial \psi_{2N}}{\partial s} = \frac{\partial \psi_{21}}{\partial s} = 0 \;\text{ on }\; s = 0 \; .$$

Solving these equations, we find that

$$\psi_{2N}(s,\theta) = 0 \;\text{ for } N = 0, 2, 3, \cdots \;, \tag{3.88}$$

while for $Pr = 1$

$$\begin{aligned}
\psi_{21}(s,\theta) &= \frac{\sqrt{\pi}}{48}\left(3D_2(\theta) + \frac{9A_0(\theta)}{\sqrt{\pi}}\right)\mathrm{erf}(s) \\
&- \frac{\sqrt{\pi}}{12}\left[D_2(\theta)\mathrm{erfc}(s) - \frac{A_0(\theta)}{\sqrt{\pi}}\mathrm{erfc}(s)\right]s^4 \\
&+ \frac{1}{12}\left(D_2(\theta) - \frac{A_0(\theta)}{\sqrt{\pi}}\right)s^3\mathrm{e}^{-s^2} + \frac{1}{24}\left(5D_2(\theta) + \frac{7A_0(\theta)}{\sqrt{\pi}}\right)s\mathrm{e}^{-s^2} \\
&- \frac{\sqrt{\pi}}{4}\left(D_2(\theta) + \frac{A_0(\theta)}{\sqrt{\pi}}\right)s^2\mathrm{erfc}(s) \; - \frac{1}{3}\left(D_2(\theta) + \frac{2A_0(\theta)}{\sqrt{\pi}}\right)s \;,
\end{aligned} \tag{3.89}$$

and when $Pr \neq 1$

$$\begin{aligned}
\psi_{21}(s,\theta) &= \frac{1}{16}\hat{D}_2(\theta)\sqrt{\pi}erf(s) \\
&+ \left(\frac{3A_0(\theta)}{8\,(1-Pr)} + \frac{A_0(\theta)}{4Pr\,(1-Pr)}\right)erf(\sqrt{Pr}s) \\
&+ \left(\frac{-\hat{D}(\theta)\sqrt{\pi}}{12} - \frac{PrA_0(\theta)}{6\,(1-Pr)}\right)s^4 + \frac{A_0(\theta)}{6\,(1-Pr)}s^4 erf(\sqrt{Pr}s) \\
&+ \frac{\hat{D}(\theta)\sqrt{\pi}}{12}s^4 erf(s) + \frac{\hat{D}_2(\theta)}{12}s^3\mathrm{e}^{-s^2} + \frac{\sqrt{Pr}A_0(\theta)}{6\,(1-Pr)\sqrt{\pi}}s^3\mathrm{e}^{-Prs^2} \\
&+ \frac{5\hat{D}_2(\theta)}{24}s\mathrm{e}^{-s^2} + \left(\frac{-\hat{D}_2(\theta)\sqrt{\pi}}{2} - \frac{A_0(\theta)}{(1-Pr)}\right)\frac{s^2}{2} + \frac{\hat{D}_2(\theta)\sqrt{\pi}}{4}s^2 erf(s)
\end{aligned}$$

$$+\left(\frac{A_0(\theta)}{2\,(1-Pr)\,\sqrt{Pr\pi}}-\frac{\sqrt{Pr}A_0(\theta)}{12\,(1-Pr)\,\sqrt{Pr\pi}}\right)s\mathrm{e}^{-Prs^2}$$

$$+\frac{A_0(\theta)}{2\,(1-Pr)}s^2 erf(\sqrt{Pr}s)+\left(\frac{-\hat{D}_2(\theta)}{3}-\frac{2\sqrt{Pr}A_0(\theta)}{3\,(1-Pr)\,\sqrt{\pi}}\right)s\ . \tag{3.90}$$

Thus, we obtain

$$\psi_2(t,s,\theta)=t\psi_{21}\ . \tag{3.91}$$

Note that ψ_{21} diverges as $s\to\infty$. This is because we chose to apply the surface conditions instead of far-field conditions when solving for ψ_{21}.

Finally, the approximate analytical solution for the streamfunction ψ becomes

$$\psi(t,s,\theta)=\lambda^2 t\psi_{21}+O(\lambda^3)\ . \tag{3.92}$$

Chapter 4

Numerical Solution Procedure

4.1 Solving the O(λ^3) problem for Temperature

Using the series (3.1), we obtain the following equation for ϕ_3

$$\frac{1}{Pr}\frac{\partial^2\phi_3}{\partial z^2} + 2z\left[M_0^2\frac{\partial\phi_3}{\partial z} + \sinh(2\xi_0)z\frac{\partial\phi_2}{\partial z} + \cosh(2\xi_0)z^2\frac{\partial\phi_1}{\partial z}\right] = -\frac{1}{Pr}\frac{\partial^2\phi_1}{\partial\theta^2}$$

$$+4t\left[M_0^2\left(\frac{\partial\phi_3}{\partial t} + \frac{3\phi_3}{2t}\right) + \sinh(2\xi_0)z\left(\frac{\partial\phi_2}{\partial t} + \frac{\phi_2}{t}\right) + \cosh(2\xi_0)z^2\left(\frac{\partial\phi_1}{\partial t} + \frac{\phi_1}{2t}\right)\right]$$

$$+ 4t\left[\frac{\partial\psi_2}{\partial z}\frac{\partial\phi_2}{\partial\theta} + \frac{\partial\psi_3}{\partial z}\frac{\partial\phi_1}{\partial\theta} - \frac{\partial\psi_2}{\partial\theta}\frac{\partial\phi_2}{\partial z} - \frac{\partial\psi_3}{\partial\theta}\frac{\partial\phi_1}{\partial z}\right] . \tag{4.1}$$

The constant heat flux and the far-field conditions are

$$\frac{\partial\phi_3}{\partial z} = 0 \text{ on } z = 0 \ , \ \phi_3 \to 0 \text{ as } z \to \infty \ . \tag{4.2}$$

Further expanding in t, and introducing the transformation $w = \sqrt{Pr}M_0 z$, the problem reduces to solving the following differential equation for ϕ_{30}

$$\frac{\partial^2\phi_{30}}{\partial w^2} + 2w\frac{\partial\phi_{30}}{\partial w} - 6\phi_{30} = -\frac{1}{PrM_0^2}\frac{\partial^2\phi_{10}}{\partial\theta^2}$$

$$+ \frac{2\cosh(2\xi_0)w^2}{PrM_0^4}\left(\phi_{10} - w\frac{\partial\phi_{10}}{\partial w}\right) + \frac{2\sinh(2\xi_0)w}{\sqrt{Pr}M_0^3}\left(\phi_{20} - w\frac{\partial\phi_{20}}{\partial w}\right) , \tag{4.3}$$

subject to the constant heat flux and the far-field conditions

$$\frac{\partial\phi_{30}}{\partial w} = 0 \text{ on } w = 0 \ , \ \phi_{30} \to 0 \text{ as } w \to \infty \ . \tag{4.4}$$

Using the following differentials

$$\frac{dM_0}{d\theta} = \frac{\sin(2\theta)}{2M_0} ,$$

$$\frac{d^2M_0}{d\theta^2} = \frac{\cos(2\theta)}{M_0} - \frac{\sin(2\theta)^2}{4M_0^3} \ ,$$

and the known solutions for ϕ_{10} and ϕ_{20}, we can simplify (4.3) to

$$M_0^6\frac{\partial^2\phi_{30}}{\partial w^2} + 2wM_0^6\frac{\partial\phi_{30}}{\partial w} - 6M_0^6\phi_{30} = f_1(w) + f_2(w)\cos(2\theta) + f_3(w)\cos(4\theta) \ . \quad (4.5)$$

In the above

$$f_1(w) = -w^4 e^{-w^2}\frac{\sinh(2\xi_0)^2}{Pr^{1/2}\pi^{1/2}} - \frac{3}{8Pr^{3/2}} + w\text{erf}(w)\left[\frac{\sinh(2\xi_0)^2}{8Pr^{1/2}} + \frac{3}{8Pr^{3/2}}\right]$$

$$+w^2 e^{-w^2}\left[\frac{\cosh(2\xi_0)^2}{Pr^{3/2}\pi^{1/2}} - \frac{\sinh(2\xi_0)^2}{4Pr^{1/2}\pi^{1/2}} - \frac{1}{4Pr^{3/2}\pi^{1/2}}\right] - w^3\text{erf}(w)\frac{\sinh(2\xi_0)^2}{4Pr^{1/2}} \ ,$$

$$f_2(w) = \cosh(2\xi_0)\left[\frac{1}{2Pr^{3/2}} - \frac{w\text{erf}(w)}{2Pr^{3/2}} - \frac{w^2e^{-w^2}}{Pr^{3/2}\pi^{1/2}}\right] \ ,$$

$$f_3(w) = -\frac{1}{8Pr^{3/2}} + \frac{w\text{erf}(w)}{8Pr^{3/2}} + \frac{w^2e^{-w^2}}{4Pr^{3/2}\pi^{1/2}} \ .$$

If we set $\chi = M_0^6\phi_{30}$, (4.5) becomes

$$\frac{\partial^2\chi}{\partial w^2} + 2w\frac{\partial\chi}{\partial w} - 6\chi = f_1(w) + f_2(w)\cos(2\theta) + f_3(w)\cos(4\theta) \ . \quad (4.6)$$

Letting

$$\chi = \chi_1(w) + \chi_2(w)\cos(2\theta) + \chi_3(w)\cos(4\theta) \ , \quad (4.7)$$

equation (4.6) reduces to solving the following differential equations

$$\frac{d^2\chi_k}{dw^2} + 2w\frac{d\chi_k}{dw} - 6\chi_k = f_k(w) \ \text{ for } \ k = 1, 2, 3 \ , \quad (4.8)$$

subject to the constant heat flux conditions

$$\frac{d\chi_k}{dw} = 0 \ \text{ on } \ w = 0 \quad (4.9)$$

and the far-field conditions

$$\chi_k \to 0 \ \text{ as } \ w \to \infty \ . \quad (4.10)$$

Adopting the notation $\chi_k(w_i) \equiv \chi_{k,i}$, approximating infinity by w_∞, and discretizing the equation for χ_k using central differences with a uniform grid having a spacing of $h = w_\infty/d$, leads to the algebraic system of equations given by

$$\frac{(\chi_{k,i+1} + \chi_{k,i-1} - 2\chi_{k,i})}{h^2} + 2w_i\frac{(\chi_{k,i+1} - \chi_{k,i-1})}{2h} - 6\chi_{k,i} = f_{k,i} \ , \quad (4.11)$$

for $i = 1, 2, \cdots, d-1$.

Simplifying further,we obtain

$$(1-hw_i)\,\chi_{k,i-1}-2\left(1+3h^2\right)\chi_{k,i}+(1+hw_i)\,\chi_{k,i+1}=h^2 f_{k,i}\ . \qquad (4.12)$$

Using the Taylor series, we write

$$\chi_{k,i+1}=\chi_{k,i}+h\chi'_{k,i}+\frac{h^2}{2}\chi''_{k,i}+O(h^3)\ , \qquad (4.13)$$

$$\chi_{k,i+2}=\chi_{k,i}+2h\chi'_{k,i}+2h^2\chi''_{k,i}+O(h^3)\ . \qquad (4.14)$$

Here, the prime denotes differentiation with respect to w.Then, subtracting 4 times equation (4.13) from equation (4.14) gives

$$\chi_{k,i+2}-4\chi_{k,i+1}=-3\chi_{k,i}-2h\chi'_{k,i}\ . \qquad (4.15)$$

Using (4.15) and setting $i=0$, the constant heat flux condition in discretized form becomes

$$\chi_{k,0}=\frac{4}{3}\chi_{k,1}-\frac{1}{3}\chi_{k,2}\ \text{ for }\ k=1,2,3\ . \qquad (4.16)$$

The far-field conditions are simply

$$\chi_{k,d}=0\ \text{ for }\ k=1,2,3\ . \qquad (4.17)$$

The boundary value problem (BVP) in matrix form becomes

$$\begin{pmatrix} 1 & -\frac{4}{3} & \frac{1}{3} & 0 & . \\ (1-hw_1) & -2(1+3h^2) & (1+hw_1) & 0 & . \\ . & . & . & . & . \\ . & 0 & (1-hw_{d-1}) & -2(1+3h^2) & (1+hw_{d-1}) \\ 0 & . & . & 0 & 1 \end{pmatrix}\begin{pmatrix} \chi_{k,0} \\ \chi_{k,1} \\ . \\ \chi_{k,d-1} \\ \chi_{k,d} \end{pmatrix}$$

$$=h^2\begin{pmatrix} 0 \\ f_{k,1} \\ . \\ f_{k,d-1} \\ 0 \end{pmatrix}\ .$$

If we let

$$A^h=\begin{pmatrix} 1 & -\frac{4}{3} & \frac{1}{3} & 0 & . \\ (1-hw_1) & -2(1+3h^2) & (1+hw_1) & 0 & . \\ . & . & . & . & . \\ . & 0 & (1-hw_{d-1}) & -2(1+3h^2) & (1+hw_{d-1}) \\ 0 & . & . & 0 & 1 \end{pmatrix}\ ,$$

$$\chi^h=\begin{pmatrix} \chi_{k,0} \\ \chi_{k,1} \\ . \\ \chi_{k,d-1} \\ \chi_{k,d} \end{pmatrix}\ \text{ and }\ F^h=\begin{pmatrix} 0 \\ f_{k,1} \\ . \\ f_{k,d-1} \\ 0 \end{pmatrix}\ ,$$

then, we can write the BVP as

$$A^h \chi^h = F^h \ . \tag{4.18}$$

Thus, we can easily solve for χ_k for $i = 1, 2, 3$ to obtain

$$\chi^h = {A^h}^{-1} F^h \ . \tag{4.19}$$

We have computed χ_k for $k = 1, 2, 3$ using the MATLAB software package, and then constructed ϕ_{30} using

$$\phi_{30}(w, \theta) = \frac{\chi_1 + \chi_2 \cos(2\theta) + \chi_3 \cos(4\theta)}{M_0^6} \ . \tag{4.20}$$

The solution for ϕ_{22} is solved using a similar procedure. The only difference is that ϕ_{22} has to be expanded in a full Fourier series. We set

$$\phi_{22}(w, \theta) = \frac{\hat{R}_0}{2} + \sum_{l=1}^{\infty} \left(\hat{R}_l \cos(l\theta) + \hat{S}_l \sin(l\theta) \right) \ .$$

Using MATLAB, the values of $\hat{R}_0, \hat{R}_1, \cdots$ and $\hat{S}_1, \hat{S}_2, \cdots$ can be easily computed and are listed in Appendix A.

4.2 Solving the O(λ) problem for Vorticity

Using the series (3.2), we obtain the following equation for ζ_1

$$\begin{aligned} \frac{1}{M_0^2} \frac{\partial^2 \zeta_1}{\partial z^2} + 2z \frac{\partial \zeta_1}{\partial z} + \frac{2}{M_0^2} z^2 \sinh(2\xi_0) \frac{\partial \zeta_0}{\partial z} &= 4t \left(\frac{\partial \zeta_1}{\partial t} + \frac{\zeta_1}{2t} \right) \\ + \frac{4t}{M_0^2} z \sinh(2\xi_0) \frac{\partial \zeta_0}{\partial t} &+ \frac{4t}{M_0^2} \frac{\partial \psi_2}{\partial z} \frac{\partial \zeta_0}{\partial \theta} - \frac{4t}{M_0^2} \frac{\partial \psi_2}{\partial \theta} \frac{\partial \zeta_0}{\partial z} \\ - \frac{4t}{M_0^2} A_0 \frac{\partial \phi_2}{\partial z} &- \frac{4t}{M_0^2} z A_1 \frac{\partial \phi_1}{\partial z} + \frac{4t}{M_0^2} B_0 \frac{\partial \phi_1}{\partial \theta} \ , \end{aligned} \tag{4.21}$$

subject to far-field condition

$$\zeta_1 \to 0 \ \text{ as } \ z \to \infty \ , \tag{4.22}$$

and the integral conditions

$$\int_0^\infty \int_{-\pi}^{\pi} M_0^2 \zeta_1 \sin(n\theta) d\theta dz = \int_0^\infty \int_{-\pi}^{\pi} \left(n M_0^2 - \sinh(2\xi_0) \right) z \zeta_0 \sin(n\theta) d\theta dz \ , \tag{4.23}$$

$$\int_0^\infty \int_{-\pi}^{\pi} M_0^2 \zeta_1 \cos(n\theta) d\theta dz = \int_0^\infty \int_{-\pi}^{\pi} \left(n M_0^2 - \sinh(2\xi_0) \right) z \zeta_0 \cos(n\theta) d\theta dz \ , \tag{4.24}$$

for $n = 0, 1, 2, \cdots$.

Introducing the transformation $s = M_0 z$, expanding in t, and using (3.78), the problem reduces to solving the following differential equations

$$\frac{\partial^2 \zeta_{10}}{\partial s^2} + 2s\frac{\partial \zeta_{10}}{\partial s} - 2\zeta_{10} = 0 \ , \tag{4.25}$$

$$\frac{\partial^2 \zeta_{11}}{\partial s^2} + 2s\frac{\partial \zeta_{11}}{\partial s} - 6\zeta_{11} = -\frac{2}{M_0^3}s^2 \sinh(2\xi_0)\frac{\partial \zeta_{01}}{\partial s} - \frac{4}{M_0}A_0\frac{\partial \phi_{20}}{\partial s}$$
$$+ \frac{4}{M_0^3}s\sinh(2\xi_0)\zeta_{01} - \frac{4}{M_0^2}sA_1\frac{\partial \phi_{10}}{\partial s} + \frac{4}{M_0^2}B_0\frac{\partial \phi_{10}}{\partial \theta} \ , \tag{4.26}$$

$$\frac{\partial^2 \zeta_{13}}{\partial s^2} + 2s\frac{\partial \zeta_{13}}{\partial s} - 14\zeta_{13} = \frac{4}{M_0^3}\frac{\partial \psi_{21}}{\partial s}\frac{\partial \zeta_{01}}{\partial \theta} - \frac{4}{M_0^3}\frac{\partial \psi_{21}}{\partial \theta}\frac{\partial \zeta_{01}}{\partial s} \ , \tag{4.27}$$

$$\frac{\partial^2 \zeta_{1N}}{\partial s^2} + 2s\frac{\partial \zeta_{1N}}{\partial s} - (4N+2)\zeta_{1N} = 0 \ , \ N = 2, 4, 5, 6, \cdots \ , \tag{4.28}$$

subject to the far-field conditions

$$\zeta_{1N} \to 0 \ \text{ as } \ s \to \infty \ , \tag{4.29}$$

and the integral conditions

$$\int_0^\infty \int_{-\pi}^{\pi} M_0^2 \zeta_{1N} \sin(n\theta) d\theta ds =$$
$$\int_0^\infty \int_{-\pi}^{\pi} \left(nM_0 - \frac{\sinh(2\xi_0)}{M_0} \right) s\zeta_{01} \sin(n\theta) d\theta ds \ , \tag{4.30}$$
$$\int_0^\infty \int_{-\pi}^{\pi} M_0^2 \zeta_{1N} \cos(n\theta) d\theta ds =$$
$$\int_0^\infty \int_{-\pi}^{\pi} \left(nM_0 - \frac{\sinh(2\xi_0)}{M_0} \right) s\zeta_{01} \cos(n\theta) d\theta ds \ , \tag{4.31}$$

for fixed N and $n = 0, 1, 2, \cdots$.

For the $N = 0$ case, the homogeneous differential equation has two linearly independent solutions of the following form: s and $(\mathrm{e}^{-s^2} + \sqrt{\pi} s \mathrm{erf}(s))$. Thus, we can write the solution to (4.25) as

$$\zeta_{10} = A(\theta)s + B(\theta)(\mathrm{e}^{-s^2} + \sqrt{\pi} s \mathrm{erf}(s)) \ , \tag{4.32}$$

where $A(\theta)$ and $B(\theta)$ are arbitrary functions.

Using (4.29) we obtain

$$A(\theta) = -\sqrt{\pi}B(\theta) \ ,$$

so, (4.32) becomes

$$\zeta_{10} = B(\theta)\left((\mathrm{e}^{-s^2} - \sqrt{\pi} s \mathrm{erfc}(s) \right) \ .$$

Substituting the above into (4.30),(4.31), and noting that $\int_0^\infty e^{-s^2} ds = \frac{1}{2}\sqrt{\pi}$, and $\int_0^\infty \mathrm{serfc}(s) ds = \frac{1}{4}$ we find that

$$B(\theta) = 0 \; .$$

Thus, we conclude that

$$\zeta_{10} = 0 \; . \tag{4.33}$$

For the $N = 1$ case, using the transformation $\Omega = M_0^2 \zeta_{11}$, the problem reduces to solving

$$\begin{aligned}\frac{\partial^2 \Omega}{\partial s^2} + 2s\frac{\partial \Omega}{\partial s} - 6\Omega = -\frac{4}{M_0} A_0 \frac{\partial \phi_{20}}{\partial s} + 4B_0 \frac{\partial \phi_{10}}{\partial \theta} \\ - 4A_1 s \frac{\partial \phi_{10}}{\partial s} - \frac{2s^2 \sinh(2\xi_0)}{M_0} \frac{\partial \zeta_{01}}{\partial s} + \frac{4s^2}{M_0} \sinh(2\xi_0)\zeta_{01}\end{aligned} \tag{4.34}$$

subject to far-field condition

$$\Omega \to 0 \;\; \text{as} \;\; s \to \infty \; , \tag{4.35}$$

and the integral conditions

$$\int_0^\infty \int_{-\pi}^{\pi} \Omega \sin(n\theta) d\theta ds = \int_0^\infty \int_{-\pi}^{\pi} \left(nM_0 - \frac{\sinh(2\xi_0)}{M_0} \right) s\zeta_{01} \sin(n\theta) d\theta ds \; , \tag{4.36}$$

$$\int_0^\infty \int_{-\pi}^{\pi} \Omega \cos(n\theta) d\theta ds = \int_0^\infty \int_{-\pi}^{\pi} \left(nM_0 - \frac{\sinh(2\xi_0)}{M_0} \right) s\zeta_{01} \cos(n\theta) d\theta ds \; , \tag{4.37}$$

for $n = 0, 1, 2, \cdots$.

We use a similar method developed by Staniforth [32] to solve (4.34). First, we expand $\Omega(s, \theta)$ in the truncated Fourier series

$$\Omega(s, \theta) = \frac{G_0(s)}{2} + \sum_{l=1}^{L} \left(G_l(s) \cos(l\theta) + F_l(s) \sin(l\theta) \right) \; . \tag{4.38}$$

When this series is substituted into (4.34) the Fourier coefficients G_0,F_l and G_l will satisfy differential equations having the form

$$\frac{d^2 P}{ds^2} + 2s\frac{dP}{ds} - 6P = Q(s) \; , \tag{4.39}$$

where $Q(s)$ is a known function. Also, (4.39) must obey conditions of the type

$$P \to 0 \text{ as } s \to \infty \; , \;\; \text{and} \;\; \int_0^\infty P ds = \Gamma \; . \tag{4.40}$$

In order to solve the above problem, we restrict our domain to a finite region. We choose an outer boundary s_∞ to approximate infinity. To find a value for s_∞, we first write the integral as

$$\int_0^\infty P ds = \int_0^{s_\infty} P ds + \int_{s_\infty}^\infty P ds$$

and choose s_∞ such that $\int_{s_\infty}^{\infty} P ds$ is negligible. Thus, the following integral and far-field conditions used to solve (4.39) numerically become

$$P = 0 \text{ when } s = s_\infty \text{ and } \int_0^{s_\infty} P ds = \Gamma \; . \tag{4.41}$$

Since (4.39) is a linear second-order differential equation, we can write the general solution as

$$P = \gamma P_h + P_p \; , \tag{4.42}$$

where P_h, P_p denote the homogeneous and particular solutions, respectively, and γ is a yet to be determined constant.

In discretized form the differential equation (4.39) becomes

$$-2\left(1+3h^2\right) P_1 + \left(1+hs_1\right) P_2 = h^2\left(Q_1+1\right) - 1 \text{ for } i = 1 \; ,$$

$$\left(1-hs_i\right) P_{i-1} - 2\left(1+3h^2\right) P_i + \left(1+hs_i\right) P_{i+1} = h^2 Q_i \text{ for } i = 2, \cdots, d-2 \; ,$$

$$\left(1-hs_{M-1}\right) P_{M-2} - 2\left(1+3h^2\right) P_{M-1} = h^2 Q_{M-1} \text{ for } i = d-1 \; ,$$

where as before h is the uniform grid spacing and s_∞ is used to approximate infinity. We can also write the above set of equations in matrix form as $A^h P^h = Q^h$ where A^h is an $(d-1)\times(d-1)$ tridiagonal matrix. This can be numerically solved by using an efficient routine such as that found in Burden & Faires [4]. A unique solution is guaranteed if the following is satisfied

$$h < \frac{2}{D} \text{ where } D = \max_{0\le s\le s_\infty}(2s) = 2s_\infty .$$

If we construct our solution so that

$$P_h(0) = P_p(0) = 1 \text{ and } P_h(s_\infty) = P_p(s_\infty) = 0 \; ,$$

then it follows from the integral condition that

$$\gamma = \frac{\Gamma - \int_0^{s_\infty} P_p ds}{\int_0^{s_\infty} P_h ds} \; . \tag{4.43}$$

Numerically solving the systems of algebraic equations for the Fourier coefficients together with Simpson's rule to compute the integrals, we are then able to assemble the solution for ζ_{11} through the relation

$$\zeta_{11}(s,\theta) = \Omega / M_0^2 \; . \tag{4.44}$$

Using the MAPLE computer algebra system, the values of $G_0, G_1, \cdots$ and $F_1, F_2, \cdots$ can easily be computed and are listed in Appendix A.

Note that ζ_{1N} for $N = 2, 3, 4, \cdots$ are not yet determined. Lastly, higher-order terms for the streamfunction can also be obtained numerically. However, since these higher-order terms do not enter at the level of approximation considered, we did not pursue this.

Chapter 5

Results and Comparisons

Presented in this chapter are analytical results obtained by the methods described in the previous chapters. The problem is controlled by the following dimensionless parameters: Gr,Pr,η and r. The Prandtl number was fixed at $Pr = 0.7$ which corresponds to air while the remaining parameters were varied. After running various trial cases, the following values for the computational parameters were selected: $w_\infty, s_\infty = 4$, and $L = 20$. As mentioned in chapter 4, a unique solution is guaranteed if the uniform grid spacing, h, satisfies $h < \frac{1}{s_\infty}$. Thus, h must be less than 0.25, so we set $h = 0.05$ as our grid spacing. Also, the analytical results will be compared to the fully-numerical results obtained using the procedure described in D'Alsessio [7]. In the comparisons to be presented the analytical solution corresponds to that which includes all the terms found either numerically or analytically as explained in the previous chapters. Lastly, our comparisons will focus on the temperature and vorticity.

We begin by comparing the time variation of the average surface temperature for the case when $Gr = 10^6$, $r = 0.5$ and $\eta = \frac{\pi}{4}$. Figure 5.1 contrasts the analytical and numerical results over the period $0 \leq t \leq 1$. It is clear from this diagram that the two curves appear to be shifted from one another. The discrepancy seems to originate at very small times following the impulsive startup. To investigate this further the numerical solution procedure was started shortly after $t = 0$ using the analytical solution as an initial condition. Plotted in Figure 5.2 is another comparison between the numerical and analytical solutions. In this plot we have started the numerical solution procedure at $t = 0.01$ instead of $t = 0$ using the leading order terms in the expansions for ϕ, ζ and ψ as initial conditions (including more terms was not necessary as the contributions were small at $t = 0.01$). As shown in Figure 5.2, the agreement improves significantly. This suggests that the source of the problem is associated with the impulsive startup and how the numerical solution procedure responds to this abrupt change. In the plots that follow, the numerical solutions were started at $t = 0.01$. It is interesting to point out that $Gr = 10^6$ for the case illustrated in Figure 5.2 and accordingly the analytical solution is expected to be valid for $t << \frac{4}{\sqrt{Gr}} = 0.004$. However, the agreement is good for much larger times as revealed in Figure 5.3. As expected the agreement worsens as time increases. This is clearly evident if we plot the difference between the analytical and numerical solutions as shown in Figure 5.4. Note that the error is not exactly zero initially because the numerical solution procedure used only the first term in the analytical solution, not the entire solution.

We now compare the surface temperature distribution at various times for the same case as in Figures 5.1 - 5.4 which is $Gr = 10^6$, $r = 0.5$ and $\eta = \frac{\pi}{4}$. As shown in Figure 5.5, the agreement is good over the flatter sections and poor near the tips of the ellipse (located at $\theta = 0$ and $\theta = \pi$). We see that the temperature drops near the tips due to curvature. Since the curvature is larger near the tips, the area to heat up is greater. Since large areas need more energy to heat up than small areas, and since a constant heat flux is applied to the cylinder surface, the energy applied to each section is the same. Thus, the temperature is reduced at sections having larger curvature than those at smaller curvature, and this is clearly evident in Figure 5.5. With the good agreement shown in Figure 5.2 in the average surface temperature, we would also expect the same for the surface temperature distributions. However, the agreement in the surface temperature distribution is not as good as the average surface temperature. The averaging process has the effect of hiding the larger errors occurring at the tips. We must also remember that the comparisons are made well outside the expected region of validity $t << \frac{4}{\sqrt{Gr}} = 0.004$. Since the numerical solution procedure starts at $t = 0.01$, we cannot make comparisons at very small times. Note also that the agreement does not change appreciably for the times $t = 0.05, 0.1$ and 0.2 shown in Figure 5.5. The agreement is also reasonable at the larger times $t = 0.5$ and 1 plotted in Figure 5.6. Shown in Figure 5.7 are the surface vorticity distributions at times $t = 0.05$ and 0.1 for the same parameter values as mentioned above. We see that the surface vorticity distributions resemble negative sine curves. The surface vorticity becomes zero roughly at $\theta = \frac{\pi}{8}$ and $\theta = \frac{9\pi}{8}$. Also the surface vorticity is positive (or rotates counterclockwise)in the ranges $0 \leq \theta \leq \frac{\pi}{8}$ and $\frac{9\pi}{8} \leq \theta \leq 2\pi$ and negative (or rotates clockwise) between $\frac{\pi}{8} \leq \theta \leq \frac{9\pi}{8}$. This makes physical sense since the directions of rotation indicate the paths that buoyant fluid elements near the surface will follow before ultimately rising vertically. Note that the agreement is not as good for the vorticity because fewer terms were retained in that series.

We now change only the Grashof number and keep the other parameter values the same. With $Gr = 10^4$ the region of validity is extended to $t << 0.04$. Figure 5.8 illustrates the surface temperature distributions for the time interval $0.05 \leq t \leq 0.2$. Although we are still not in the appropriate time domain, the agreement in surface temperature does improve over the entire surface, especially at $t = 0.05$. Thus, we can expect the agreement to get even better for smaller t. The agreement in surface vorticity, shown in Figure 5.9, does not change appreciably because the leading and dominant term in the expansion for ζ does not depend on Gr. Decreasing the value of Gr further to $Gr = 100$, increases the region of validity to $t << 0.4$. Here, we would expect the agreement to get even better. However, poor agreement was obtained since Gr is not sufficiently large. Recall that the expansion assumes λ to be small and for a fixed time t it behaves like $Gr^{-\frac{1}{4}}$. Figure 5.10 shows the poor agreement in the average surface temperature distribution. Comparing Figures 5.5, 5.8 and 5.10 for the time interval $0.05 \leq t \leq 0.2$, $Gr = 10^4$ provides the best agreement. For the remaining plots we set $Gr = 10^4$.

The level of agreement will also depend on the parameters r and η. We expect the agreement to improve as r increases (approaching a circular geometry), and when $\eta = 0$ or $\frac{\pi}{2}$ (minor or major axis is aligned with gravity, respectively). Figure 5.11, confirms this prediction for the case $\eta = \frac{\pi}{2}$ and $r = 0.8$. Similar agreement was found when $\eta = 0$. Notice that the temperature distribution is almost uniform as one would expect to get for a circular cylinder, since the curvature is uniform everywhere. On the other hand, we expect the agreement to worsen when r decreases and when $\eta \neq 0, \frac{\pi}{2}$ (that is, the cylinder configuration is not aligned with gravity). This is also confirmed in Figure 5.12 for the case $r = 0.3$ and $\eta = \frac{\pi}{4}$. Note that the temperature distribution gets flatter as r decreases and also the temperature changes near the tips become larger. Recall that the mapping given by equation (2.63) is valid for $0 < r < 1$. We have observed that the analytical solution gives physical results for the range $0.25 < r < 0.9$. Outside this interval the analytical solution breaks down; that is, there is a temperature increment near the tips instead of a decrement.

As a final note, it is worth making some connections between the results obtained here with those obtained by Williams [35]. Both problems involve an abrupt startup resulting from an initial discontinuity. In the problem studied by Williams the initial discontinuity is in temperature while in the problem investigated here the discontinuity is in the heat flux. The difference in the nature of the discontinuities arise from the different boundary conditions applied. Both discontinuities lead to the same results, that is, to heat the fluid adjacent to the surface by the mechanism of conduction thus causing the heated fluid to rise. While quantitative comparisons cannot be made, qualitative comparisons can be drawn. Clearly, comparisons in the surface temperature distributions are not possible since in his case they will be uniform due to the isothermal surface condition used. However, comparisons in surface vorticity distributions are possible. In fact, the surface vorticity distributions obtained here are very similar to those obtained by Williams, apart from a scaling. Both distributions resemble negative sine curves. This is interesting given that the equations used in the two studies are significantly different due to the different scalings adopted. The difference in the scaling of the vorticity distributions is a result of the different ranges in Grashof numbers considered; Williams focused on small Gr while this study focused on large Gr.

As for comparisons with experimental studies, the work that is closest to this study is the recent investigation by Elsayed, Ibrahim and Elsayed [11] which dealt with free convection of air around the outer surface of a horizontal elliptic tube maintained at constant heat flux for large Rayleigh numbers. A stainless steel elliptic tube which had a 90 mm major axis, a 50 mm minor axis, and a 1000 mm length was chosen. A cylindrical 1 kW electrical heater with a 10 mm diameter placed inside the tube's center was used as the heating element. The gap between the heater and the inner surface of the tube was filled with fine sand. The supplied electrical power was measured via a digital watt meter. Eight thermocouples were fixed on the tube surface to measure the local surface temperature. The surface heat flux of the tube was calculated from the electrical power imparted to the heater and the total surface area of the tube. However, in the experiments the measurements were recorded after 3-4 hours of heating when steady-state conditions were achieved. Since this study is only concerned with the small time behaviour, comparisons with those experiments are not possible.

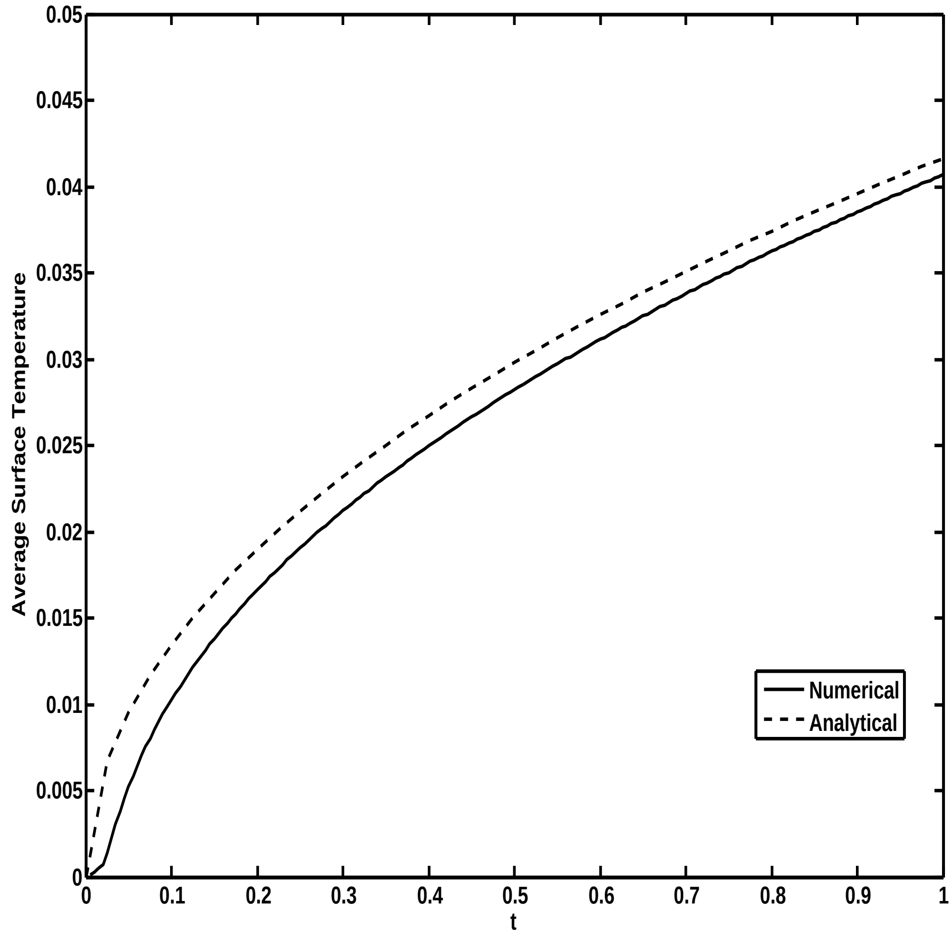

Figure 5.1: The time variation of the average surface temperature for the case $Gr = 10^6, Pr = 0.7, r = 0.5, \eta = \frac{\pi}{4}$

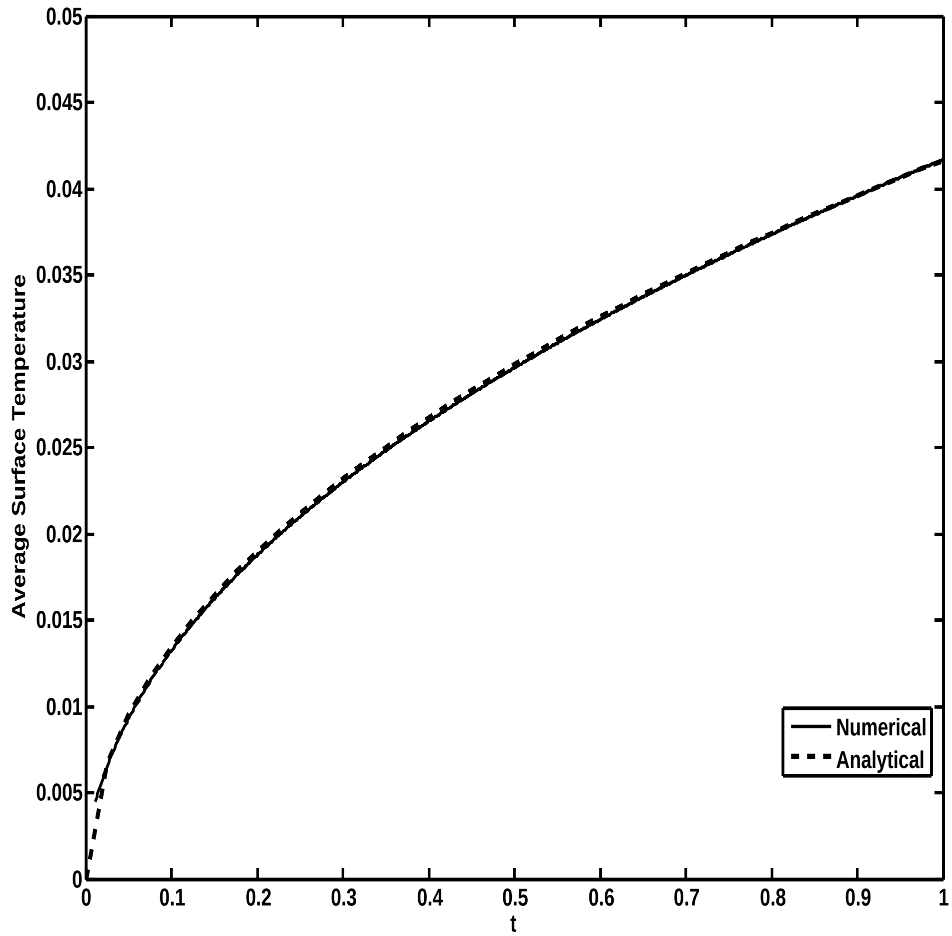

Figure 5.2: The time variation of the average surface temperature for the case $Gr = 10^6, Pr = 0.7, r = 0.5, \eta = \frac{\pi}{4}$ with modified numerical solution

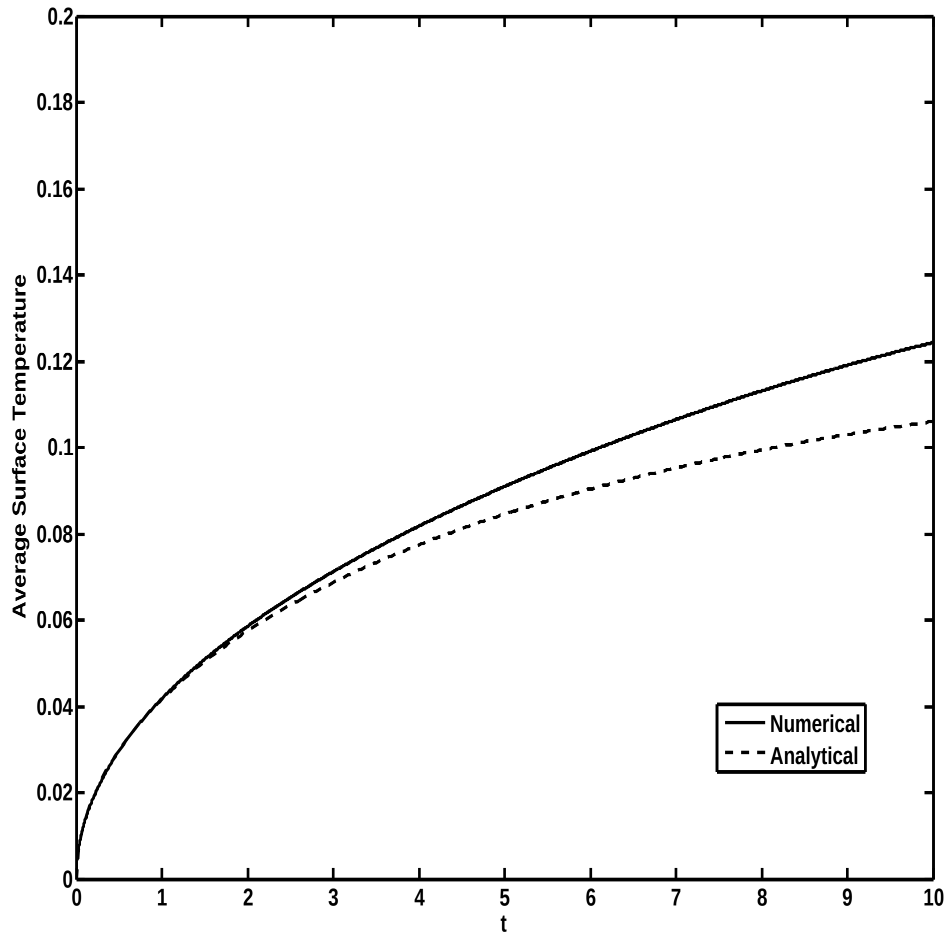

Figure 5.3: The time variation of the average surface temperature for the case $Gr = 10^6, Pr = 0.7, r = 0.5, \eta = \frac{\pi}{4}$ at $0 \leq t \leq 10$

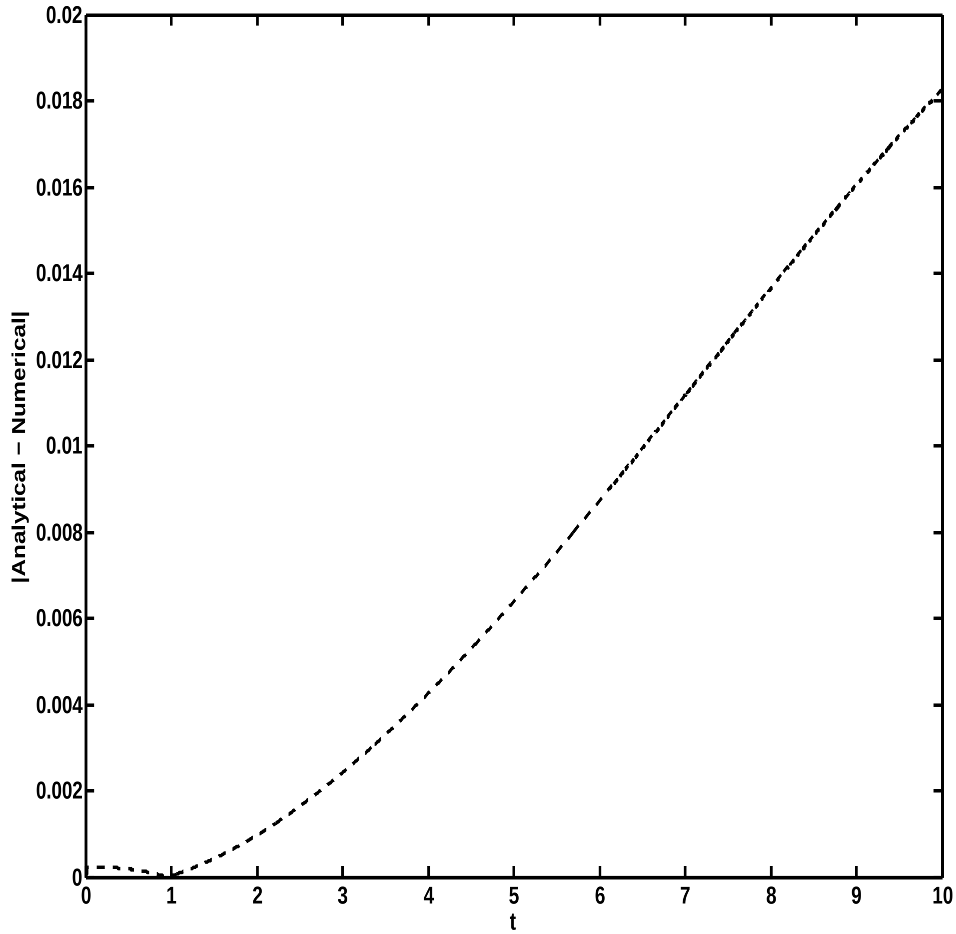

Figure 5.4: The absolute difference between the analytical and numerical solutions of the average surface temperature for the case $Gr = 10^6, Pr = 0.7, r = 0.5, \eta = \frac{\pi}{4}$ at $0 \leq t \leq 10$

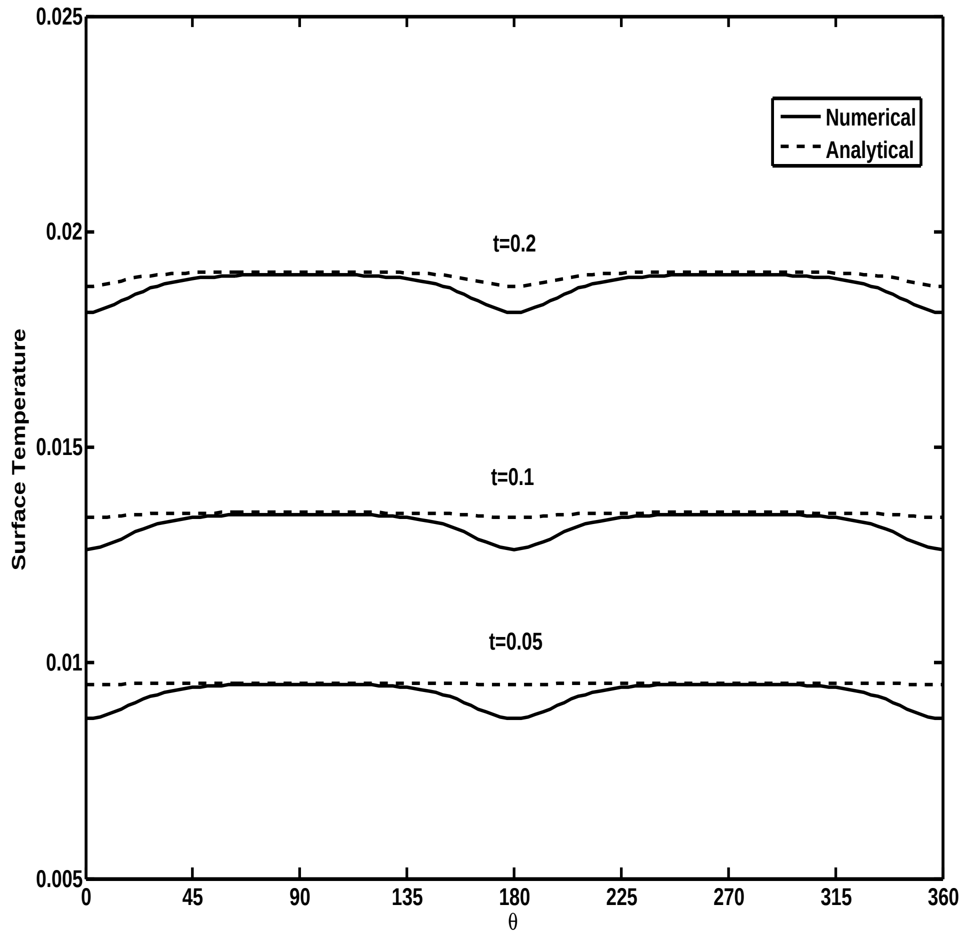

Figure 5.5: The surface temperature distribution for the case case $Gr = 10^6, Pr = 0.7, r = 0.5, \eta = \frac{\pi}{4}$, at times $t = 0.05, 0.1, 0.2$

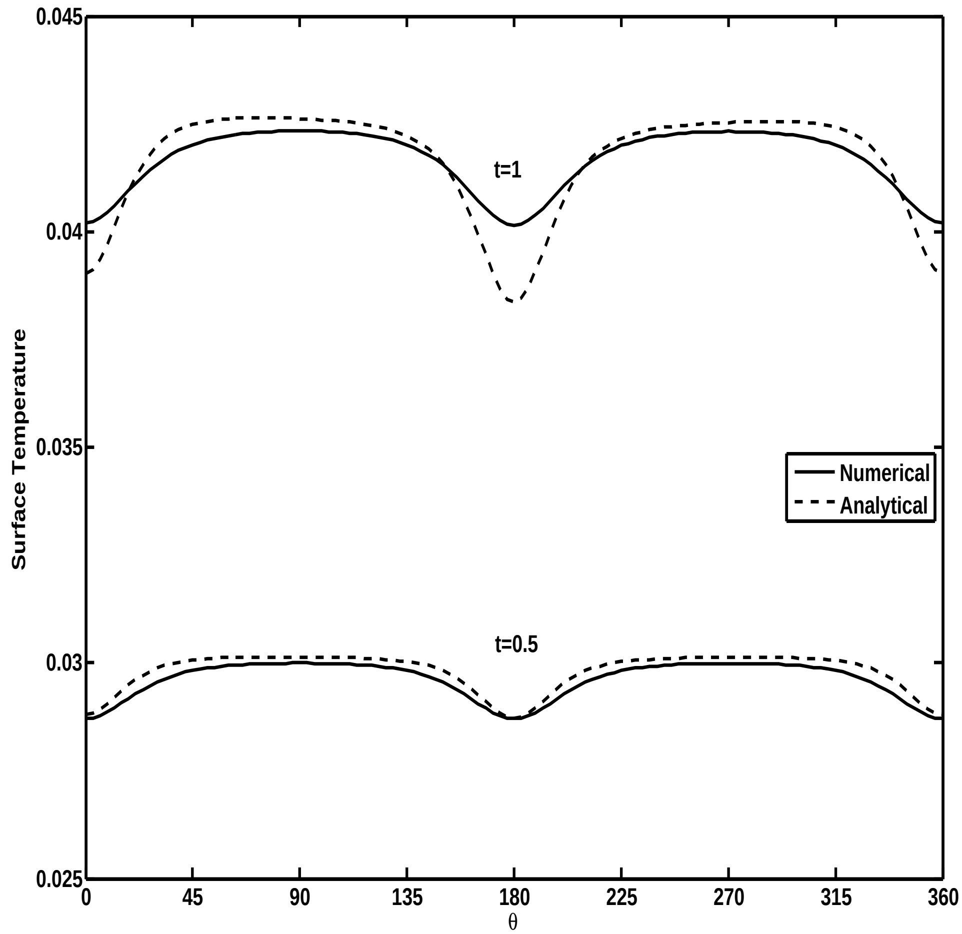

Figure 5.6: The surface temperature distribution for the case case $Gr = 10^6, Pr = 0.7, r = 0.5, \eta = \frac{\pi}{4}$, at times $t = 0.5, 1$

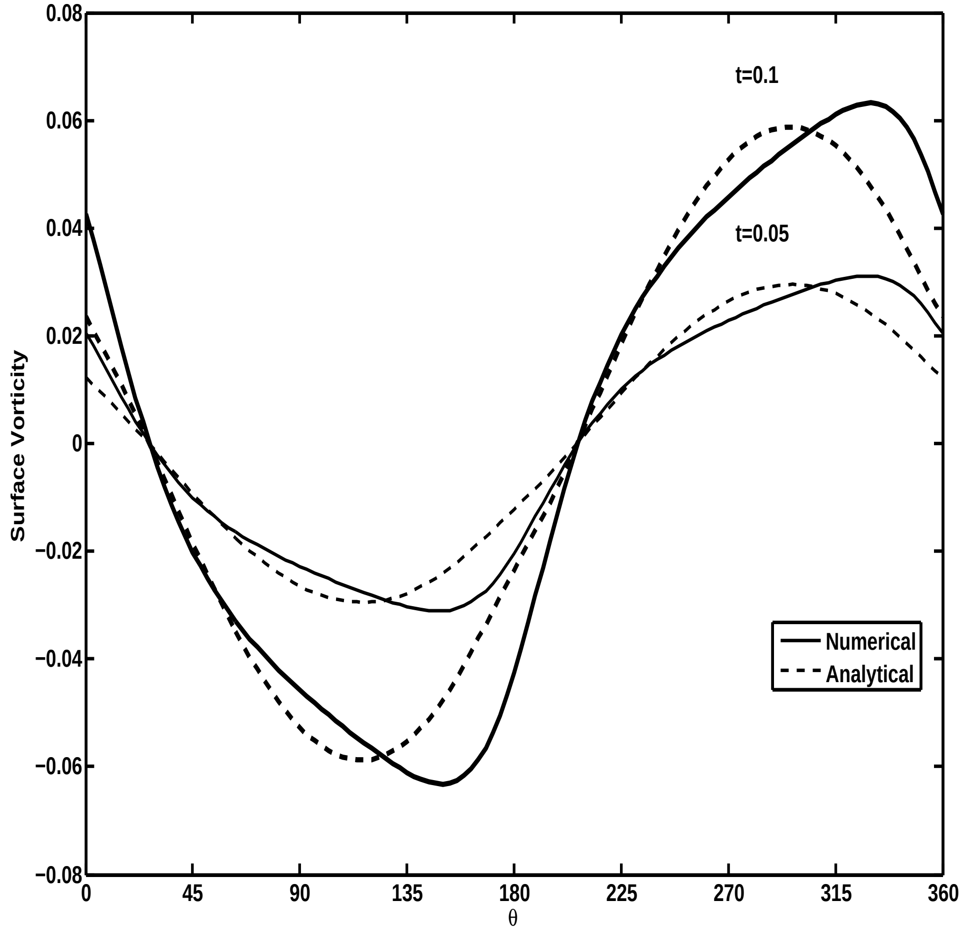

Figure 5.7: The surface vorticity distribution for the case $Gr = 10^6, Pr = 0.7, r = 0.5, \eta = \frac{\pi}{4}$, at times $t = 0.05, 0.1$

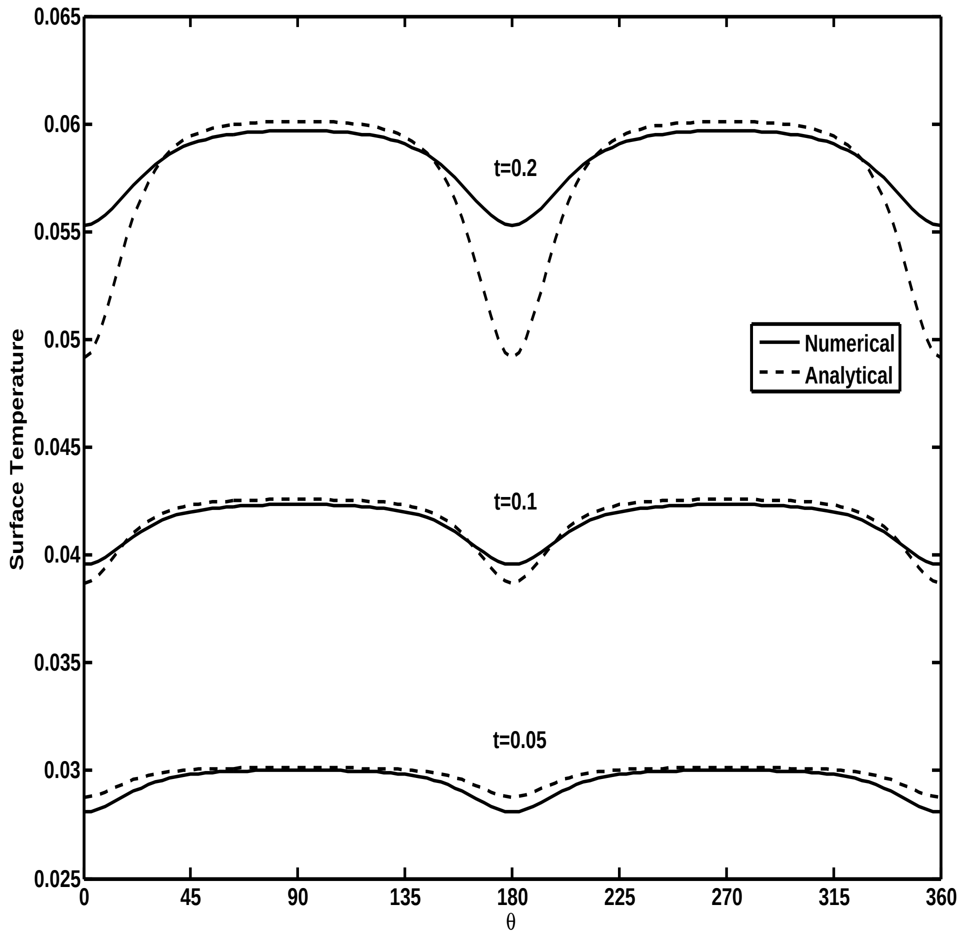

Figure 5.8: The surface temperature distribution for the case case $Gr = 10^4, Pr = 0.7, r = 0.5, \eta = \frac{\pi}{4}$, at times $t = 0.05, 0.1, 0.2$

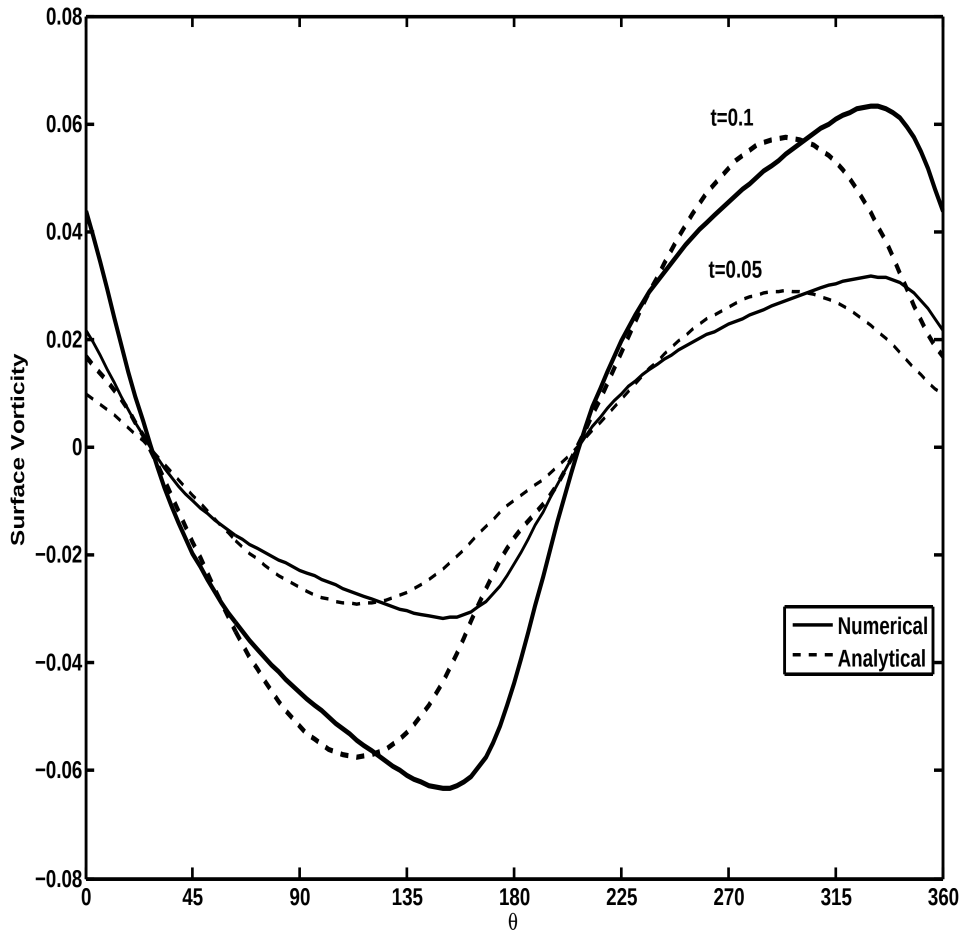

Figure 5.9: The surface vorticity distribution for the case $Gr = 10^4, Pr = 0.7, r = 0.5, \eta = \frac{\pi}{4}$, at times $t = 0.05, 0.1$

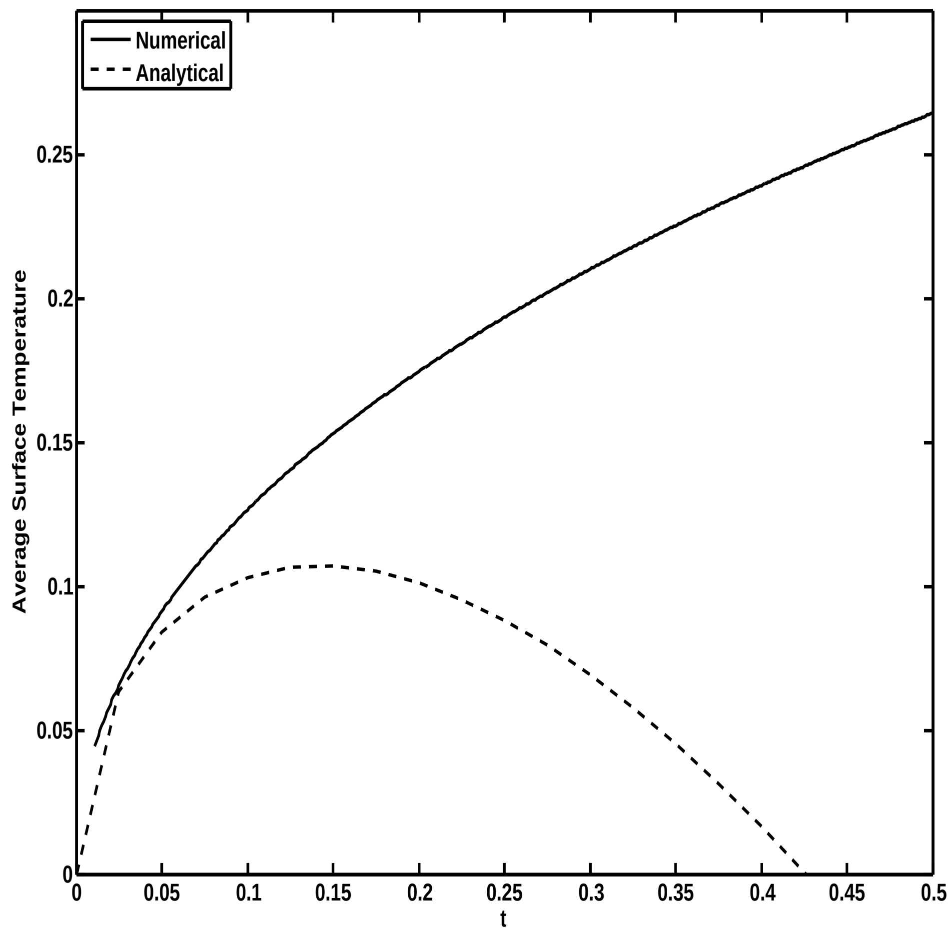

Figure 5.10: The time variation of the average surface temperature for the case $Gr = 10^2, Pr = 0.7, r = 0.5, \eta = \frac{\pi}{4}$ at $0 \leq t \leq 0.5$

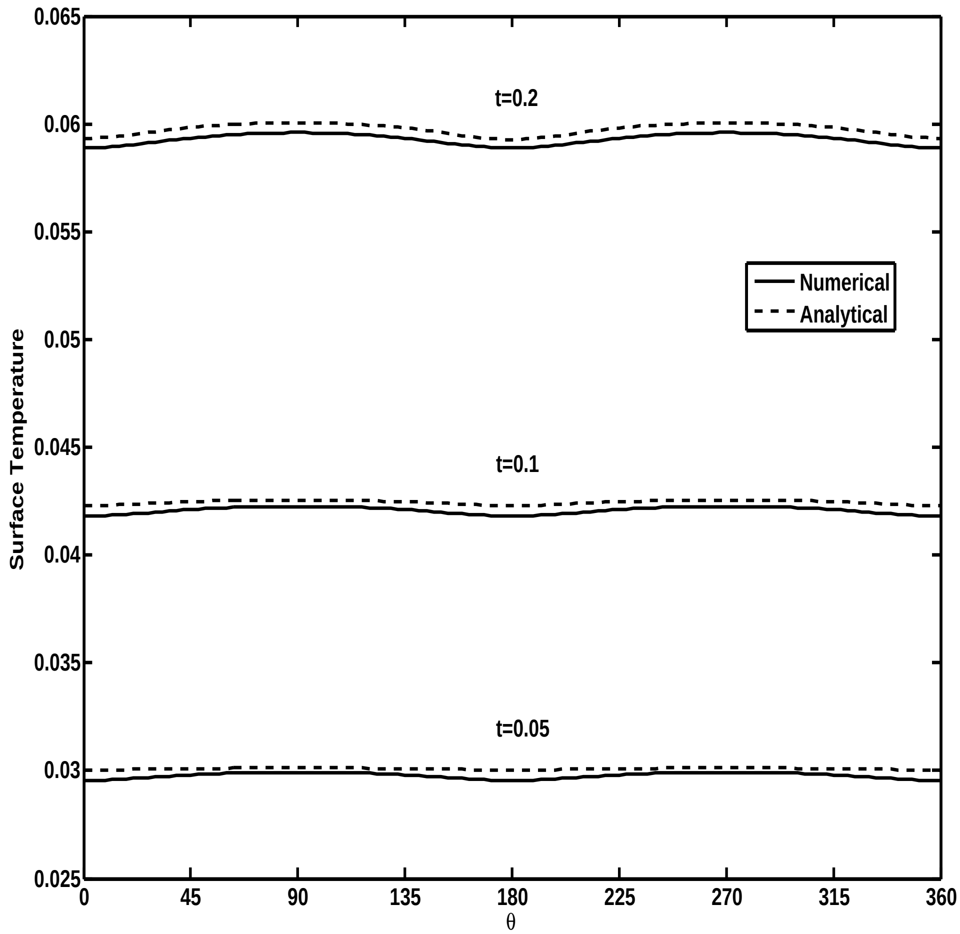

Figure 5.11: The surface temperature distribution for the case case $Gr = 10^4, Pr = 0.7, r = 0.8, \eta = \frac{\pi}{2}$, at times $t = 0.05, 0.1, 0.2$

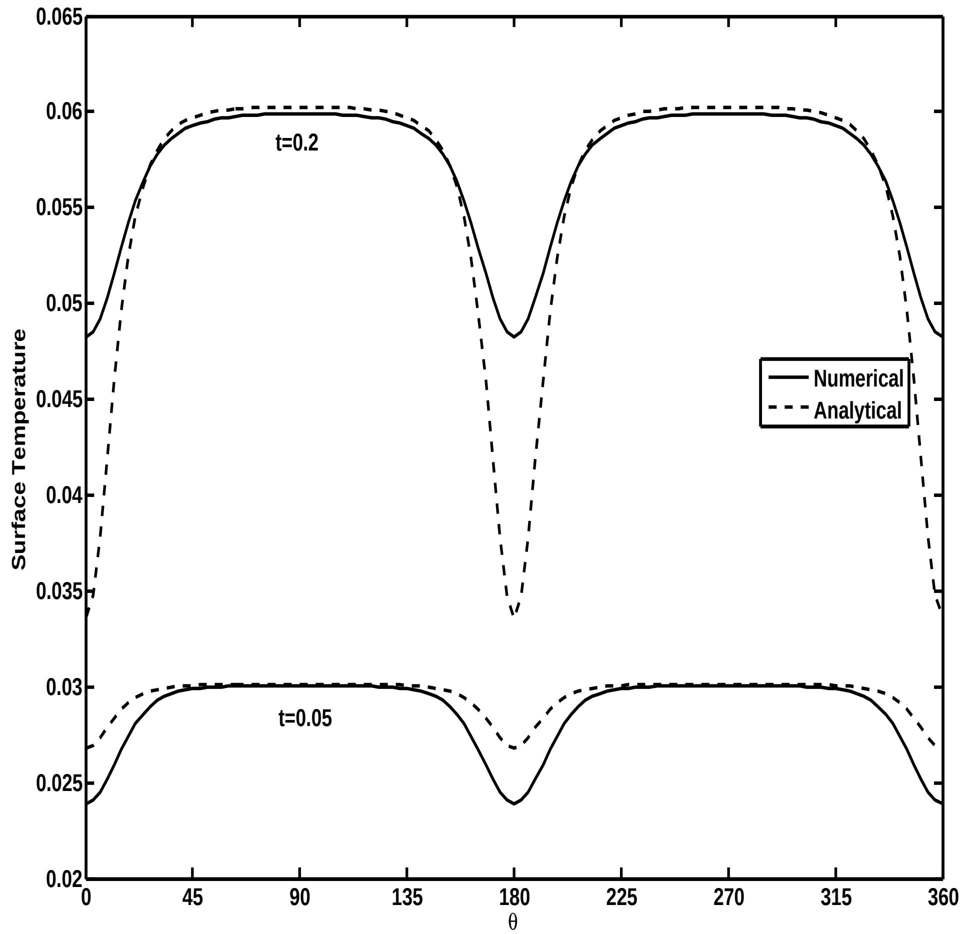

Figure 5.12: The surface temperature distribution for the case $Gr = 10^4, Pr = 0.7, r = 0.3, \eta = \frac{\pi}{4}$, at times $t = 0.05, 0.2$

Chapter 6

Summary

In this study the unsteady problem of laminar two-dimensional free convection from an inclined elliptic tube emitting a constant heat flux at the surface was investigated. The tube is inclined at an arbitrary angle η with the horizontal, and is surrounded by an incompressible viscous Boussinesq fluid which is infinite in extent. Initially, a constant heat flux Q is applied to the surface of the tube, and both the tube and surrounding fluid have the same initial temperature T_∞, where T_∞ is the far-field temperature of the fluid. We were interested specifically in the initial development of the flow and heat transfer process. To solve the Navier-Stokes and energy equations analytically, a double series expansion was applied to the flow variables. This enabled us to obtain a solution valid for small times, t, and large Grashof numbers, Gr. The higher-order terms in the expansion were increasingly difficult to obtain analytically, so they were determined numerically.

The average surface temperature as well as surface vorticity and surface temperature distributions were determined from the derived approximated analytical solutions. Agreement between the analytical and fully numerical results [7] was reasonable provided that $t << 4/\sqrt{Gr}$ and Gr is sufficiently large. Agreement improved as r was increased and when η was set to either 0 or $\frac{\pi}{2}$. The analytical solution accurately predicted the surface temperature behaviour immediately following the abrupt startup while the numerical solution could not respond quickly enough. While the analytical solution may have limited usefulness, it is still important given that the exact solution is still unknown. At the very least, it serves to furnish an initial condition to help get past the impulsive startup, as it was used in this study.

An obvious extension of this work is to examine the limiting case of large t and large Gr. This will likely involve a fully numerical investigation.

Appendix A

Fourier Coefficients

A.1 $\alpha_0, \alpha_1, \cdots$ and $\beta_0, \beta_1, \cdots$ values

Listed in table A.1 are the first 20 Fourier coefficients of (3.65), and they are defined as follows

$$\alpha_0 = -\frac{2}{\pi\sqrt{\pi}} \int_{-\pi}^{\pi} M_0(\theta) A_0(\theta) d\theta \ ,$$
$$\alpha_l = -\frac{2}{\pi\sqrt{\pi}} \int_{-\pi}^{\pi} M_0(\theta) A_0(\theta) \cos(n\theta) d\theta \ ,$$
$$\beta_l = -\frac{2}{\pi\sqrt{\pi}} \int_{-\pi}^{\pi} M_0(\theta) A_0(\theta) \sin(n\theta) d\theta \ .$$

$\alpha_0 = 0$	$\alpha_1 = -0.3445496$	$\alpha_2 = 0$	$\alpha_3 = 0.0709378$
$\alpha_4 = 0$	$\alpha_5 = 0.0062749$	$\alpha_6 = 0$	$\alpha_7 = 0.0010755$
$\alpha_8 = 0$	$\alpha_9 = 0.0002277$	$\alpha_{10} = 0$	$\alpha_{11} = 0.0000537$
$\alpha_{12} = 0$	$\alpha_{13} = 0.0000135$	$\alpha_{14} = 0$	$\alpha_{15} = 0.0000036$
$\alpha_{16} = 0$	$\alpha_{17} = 0.0000009$	$\alpha_{18} = 0$	$\alpha_{19} = 0.0000002$
	$\beta_1 = 0.9513136$	$\beta_2 = 0$	$\beta_3 = -0.1203389$
$\beta_4 = 0$	$\beta_5 = -0.0089868$	$\beta_6 = 0$	$\beta_7 = -0.0014121$
$\beta_8 = 0$	$\beta_9 = -0.0002835$	$\beta_{10} = 0$	$\beta_{11} = -0.0000645$
$\beta_{12} = 0$	$\beta_{13} = -0.0000158$	$\beta_{14} = 0$	$\beta_{15} = -0.0000041$
$\beta_{16} = 0$	$\beta_{17} = -0.0000011$	$\beta_{18} = 0$	$\beta_{19} = -0.0000002$

Table A.1: The first 20 values of $\alpha_0, \alpha_1, \cdots$ and $\beta_1, \beta_2, \cdots$ when $Pr = 1$

We use the following analytical argument to show that α_{even} and β_{even} are zero. Since

$$M_0^2(\theta) = \frac{1}{2}\left[\cosh(2\xi_0) - \cos(2\theta)\right] = \cosh^2(\xi_0) - \cos^2(\theta) \ ,$$

we can write M_0 as

$$M_0(\theta) = \cosh(\xi_0)\left(1 - \frac{\cos^2(\theta)}{\cosh^2(\xi_0)}\right)^{\frac{1}{2}} .$$

Since $-1 \leq \frac{\cos(\theta)}{\cosh(\xi_0)} \leq 1$, we can apply the Binomial series to obtain

$$M_0(\theta) = \cosh(\xi_0)\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}\frac{(-1)^n\cos^{2n}(\theta)}{\cosh^{2n}(\xi_0)} ,$$

and the identity

$$\cos^{2n}(\theta) = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1}\binom{2n}{k}\cos(2(n-k)\theta) ,$$

we can write M_0 as

$$M_0(\theta) = \cosh(\xi_0)\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}\frac{(-1)^n}{\cosh(\xi_0)^{2n}}\left[\frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1}\binom{2n}{k}\cos(2(n-k)\theta)\right] .$$

Thus, $M_0(\theta)$ is a series involves $\cos(2l\theta)$. Since

$$A_0(\theta) = \sinh(\xi_0)\cos(\eta)\cos(\theta) - \cosh(\xi_0)\sin(\eta)\sin(\theta) ,$$

we see that $A_0(\theta)\cos(n\theta)$ contains terms $\cos(\theta)\cos(n\theta)$ and $\sin(\theta)\cos(n\theta)$. Since

$$\cos(\theta)\cos(n\theta) = \frac{1}{2}\left[\cos((n+1)\theta) + \cos((n-1)\theta)\right] ,$$

$$\sin(\theta)\cos(n\theta) = \frac{1}{2}\left[\sin((n+1)\theta) - \sin((n-1)\theta)\right] ,$$

for n even $A_0(\theta)\cos(n\theta)$ contains terms involving odd harmonics. Using orthogonality, it follows that $\alpha_n = 0$ for $n = 0, 2, 4, \cdots$.

Similarly, $A_0(\theta)\sin(n\theta)$ contains terms $\cos(\theta)\sin(n\theta)$ and $\sin(\theta)\sin(n\theta)$. Since

$$\cos(\theta)\sin(n\theta) = \frac{1}{2}\left[\sin((n-1)\theta) + \sin((n+1)\theta)\right] ,$$

$$\sin(\theta)\sin(n\theta) = \frac{1}{2}\left[\cos((n-1)\theta) - \cos((n+1)\theta)\right] ,$$

and using the same argument as above we conclude that $\beta_n = 0$ for $n = 2, 4, 6\cdots$.

A.2 $\hat{\alpha_0}, \hat{\alpha_1}, \cdots$ and $\hat{\beta_0}, \hat{\beta_1}, \cdots$ values

The first 20 Fourier coefficients of (3.71) are defined and listed in table A.2 as follows

$$\hat{\alpha}_0 = -\frac{2}{\pi\sqrt{\pi}\,(1-Pr)\,\sqrt{Pr}}\int_{-\pi}^{\pi} M_0(\theta)A_0(\theta)d\theta\ ,$$

$$\hat{\alpha}_l = -\frac{2}{\pi\sqrt{\pi}\,(1-Pr)\,\sqrt{Pr}}\int_{-\pi}^{\pi} M_0(\theta)A_0(\theta)\cos(n\theta)d\theta\ ,$$

$$\hat{\beta}_l = -\frac{2}{\pi\sqrt{\pi}\,(1-Pr)\,\sqrt{Pr}}\int_{-\pi}^{\pi} M_0(\theta)A_0(\theta)\sin(n\theta)d\theta\ .$$

$\hat{\alpha_0} = 0$	$\hat{\alpha_1} = -1.3727184$	$\hat{\alpha_2} = 0$	$\hat{\alpha_3} = 0.2826229$
$\hat{\alpha_4} = 0$	$\hat{\alpha_5} = 0.02499992$	$\hat{\alpha_6} = 0$	$\hat{\alpha_7} = 0.0042849$
$\hat{\alpha_8} = 0$	$\hat{\alpha_9} = 0.0009073$	$\hat{\alpha_{10}} = 0$	$\hat{\alpha_{11}} = 0.0002139$
$\hat{\alpha_{12}} = 0$	$\hat{\alpha_{13}} = 0.0000539$	$\hat{\alpha_{14}} = 0$	$\hat{\alpha_{15}} = 0.0000142$
$\hat{\alpha_{16}} = 0$	$\hat{\alpha_{17}} = 0.0000038$	$\hat{\alpha_{18}} = 0$	$\hat{\alpha_{19}} = 0.0000008$
	$\hat{\beta_1} = 3.7901242$	$\hat{\beta_2} = 0$	$\hat{\beta_3} = -0.4794415$
$\hat{\beta_4} = 0$	$\hat{\beta_5} = -0.0358043$	$\hat{\beta_6} = 0$	$\hat{\beta_7} = -0.0056257$
$\hat{\beta_8} = 0$	$\hat{\beta_9} = -0.0011296$	$\hat{\beta_{10}} = 0$	$\hat{\beta_{11}} = -0.0002571$
$\hat{\beta_{12}} = 0$	$\hat{\beta_{13}} = -0.0000631$	$\hat{\beta_{14}} = 0$	$\hat{\beta_{15}} = -0.0000163$
$\hat{\beta_{16}} = 0$	$\hat{\beta_{17}} = -0.0000044$	$\hat{\beta_{18}} = 0$	$\hat{\beta_{19}} = -0.0000001$

Table A.2: The first 20 values of $\hat{\alpha_0}, \hat{\alpha_1}, \cdots$ and $\hat{\beta_1}, \hat{\beta_2}, \cdots$ when $Pr = 0.7$

The previous argument also shows that $\hat{\alpha}_{even}$ and $\hat{\beta}_{even}$ are zero.

A.3 $G_0, G_1, \cdots$ and $F_0, F_1, \cdots$ values

Listed in table A.3 are the first 20 Fourier coefficients of (4.38).

$G_0 = 0$	$G_1 = -0.3489500$	$G_2 = 0$	$G_3 = -0.1109000$
$G_4 = 0$	$G_5 = -0.0364720$	$G_6 = 0$	$G_7 = -0.0120720$
$G_8 = 0$	$G_9 = -0.0040055$	$G_{10} = 0$	$G_{11} = -0.0013308$
$G_{12} = 0$	$G_{13} = -0.0004425$	$G_{14} = 0$	$G_{15} = -0.0001472$
$G_{16} = 0$	$G_{17} = -0.0000490$	$G_{18} = 0$	$G_{19} = -0.0000163$
	$F_1 = 0.3639500$	$F_2 = 0$	$F_3 = 0.1121700$
$F_4 = 0$	$F_5 = 0.0366900$	$F_6 = 0$	$F_7 = 0.0121180$
$F_8 = 0$	$F_9 = 0.0040165$	$F_{10} = 0$	$F_{11} = 0.0013336$
$F_{12} = 0$	$F_{13} = 0.0004432$	$F_{14} = 0$	$F_{15} = 0.0001474$
$F_{16} = 0$	$F_{17} = 0.0000490$	$F_{18} = 0$	$F_{19} = 0.0000017$

Table A.3: The first 20 values of $G_0, G_1, \cdots$ and $F_1, F_2, \cdots$ when $Pr = 0.7$

A.4 $\hat{R}_0, \hat{R}_1, \cdots$ and $\hat{S}_0, \hat{S}_1, \cdots$ values

Listed in table A.4 are the first 12 Fourier coefficients of ϕ_{22}.

$\hat{R}_0 = 0$	$\hat{R}_1 = 0.065306$	$\hat{R}_2 = 0$	$\hat{R}_3 = 0.016005$
$\hat{R}_4 = 0$	$\hat{R}_5 = 0.003413$	$\hat{R}_6 = 0$	$\hat{R}_7 = 0.000498$
$\hat{R}_8 = 0$	$\hat{R}_9 = 0.000046$	$\hat{R_{10}} = 0$	$\hat{R_{11}} = 0.000001$
	$\hat{S}_1 = -0.009842$	$\hat{S}_2 = 0$	$\hat{S}_3 = -0.002483$
$\hat{S}_4 = 0$	$\hat{S}_5 = -0.002749$	$\hat{S}_6 = 0$	$\hat{S}_7 = -0.001556$
$\hat{S}_8 = 0$	$\hat{S}_9 = -0.000731$	$\hat{S_{10}} = 0$	$\hat{S_{11}} = -0.000002$

Table A.4: The first 12 values of $\hat{R}_0, \hat{R}_1, \cdots$ and $\hat{S}_0, \hat{S}_1, \cdots$ when $Pr = 0.7$

Appendix B

Alternate Expansion Procedure

In this appendix an alternate expansion procedure is outlined. Recall the temperature, vorticity and streamfunction equations in the elliptical coordinate system are given by

$$M^2\frac{\partial\phi}{\partial t} = \frac{\partial\psi}{\partial\theta}\frac{\partial\phi}{\partial\xi} - \frac{\partial\psi}{\partial\xi}\frac{\partial\phi}{\partial\theta} + \frac{1}{\sqrt{Gr}Pr}\left(\frac{\partial^2\phi}{\partial\xi^2} + \frac{\partial^2\phi}{\partial\theta^2}\right) \ , \qquad (B1)$$

$$M^2\frac{\partial\zeta}{\partial t} = \frac{\partial\psi}{\partial\theta}\frac{\partial\zeta}{\partial\xi} - \frac{\partial\psi}{\partial\xi}\frac{\partial\zeta}{\partial\theta} + \frac{1}{\sqrt{Gr}}\left(\frac{\partial^2\zeta}{\partial\xi^2} + \frac{\partial^2\zeta}{\partial\theta^2}\right) + A(\xi,\theta)\frac{\partial\phi}{\partial\xi} - B(\xi,\theta)\frac{\partial\phi}{\partial\theta} \ , \qquad (B2)$$

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\theta^2} = M^2\zeta \ . \qquad (B3)$$

We define a boundary-layer coordinate z given by

$$\xi = \lambda z \ \text{ where } \ \lambda = \sqrt{\frac{4t}{\sqrt{Gr}}} \ .$$

If we set $t = \epsilon T$ where ϵ denotes a small parameter, then λ becomes

$$\lambda = \sqrt{\frac{4\epsilon T}{\sqrt{Gr}}} \ .$$

Various partial derivatives will have the following form when expressed in terms of ϵ and Gr

$$\frac{\partial\Phi}{\partial\xi} = \frac{1}{\lambda}\frac{\partial\Phi}{\partial z} = \frac{Gr^{\frac{1}{4}}}{2\sqrt{\epsilon T}}\frac{\partial\Phi}{\partial z} \ ,$$

$$\frac{\partial^2\Phi}{\partial\xi^2} = \frac{1}{\lambda^2}\frac{\partial^2\Phi}{\partial z^2} = \frac{\sqrt{Gr}}{4\epsilon T}\frac{\partial^2\Phi}{\partial z^2} \ ,$$

$$\frac{\partial\Phi}{\partial t} = \frac{\partial\Phi}{\partial T}\frac{dT}{dt} + \frac{\partial\Phi}{\partial z}\frac{dz}{dT}\frac{dT}{dt} = \frac{1}{\epsilon}\frac{\partial\Phi}{\partial T} - \frac{z}{2\epsilon T}\frac{\partial\Phi}{\partial z} \ ,$$

where Φ denotes ϕ, ζ, or ψ.

Equations $(B1)$,$(B2)$ and $(B3)$ now become

$$4M^2T\frac{\partial\phi}{\partial T} - 2zM^2\frac{\partial\phi}{\partial z} = 2\sqrt{\epsilon T}Gr^{\frac{1}{4}}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial\phi}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial\phi}{\partial\theta}\right)$$

$$+\frac{1}{Pr}\frac{\partial^2\phi}{\partial z^2} + \frac{4\epsilon T}{\sqrt{Gr}Pr}\frac{\partial^2\phi}{\partial\theta^2}\ , \qquad (B4)$$

$$4M^2T\frac{\partial\zeta}{\partial T} - 2zM^2\frac{\partial\zeta}{\partial z} = 2\sqrt{\epsilon T}Gr^{\frac{1}{4}}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial\zeta}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial\zeta}{\partial\theta}\right)$$

$$+\frac{\partial^2\zeta}{\partial z^2} + \frac{4\epsilon T}{\sqrt{Gr}}\frac{\partial^2\zeta}{\partial\theta^2} + 2\sqrt{\epsilon T}Gr^{\frac{1}{4}}A\frac{\partial\phi}{\partial z} - 4\epsilon TB\frac{\partial\phi}{\partial\theta}\ , \qquad (B5)$$

$$\frac{\partial^2\psi}{\partial z^2} + \frac{4\epsilon T}{\sqrt{Gr}}\frac{\partial^2\psi}{\partial\theta^2} = \frac{4M^2\epsilon T}{\sqrt{Gr}}\zeta\ . \qquad (B6)$$

If $\frac{1}{\sqrt{Gr}} = O(\epsilon)$, then $\frac{1}{\sqrt{Gr}} = \mu\epsilon$ where μ is an $O(1)$ constant. Equations $(B4)$ - $(B6)$ written in terms of the small parameter ϵ now become

$$4M^2T\frac{\partial\phi}{\partial T} - 2zM^2\frac{\partial\phi}{\partial z} = 2\frac{\sqrt{T}}{\sqrt{\mu}}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial\phi}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial\phi}{\partial\theta}\right)$$

$$+\frac{1}{Pr}\frac{\partial^2\phi}{\partial z^2} + \frac{4\mu T\epsilon^2}{Pr}\frac{\partial^2\phi}{\partial\theta^2}\ , \qquad (B7)$$

$$4M^2T\frac{\partial\zeta}{\partial T} - 2zM^2\frac{\partial\zeta}{\partial z} = 2\frac{\sqrt{T}}{\sqrt{\mu}}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial\zeta}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial\zeta}{\partial\theta}\right)$$

$$+\frac{\partial^2\zeta}{\partial z^2} + 4\mu T\epsilon^2\frac{\partial^2\zeta}{\partial\theta^2} + 2\frac{\sqrt{T}}{\sqrt{\mu}}A\frac{\partial\phi}{\partial z} - 4T\epsilon B\frac{\partial\phi}{\partial\theta}\ , \qquad (B8)$$

$$\frac{\partial^2\psi}{\partial z^2} + 4\mu T\epsilon^2\frac{\partial^2\psi}{\partial\theta^2} = 4\mu T\epsilon^2M^2\zeta\ . \qquad (B9)$$

We next expand the flow variables in powers of ϵ as follows

$$\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots\ , \qquad (B10)$$

$$\zeta = \zeta_0 + \epsilon\zeta_1 + \epsilon^2\zeta_2 + \cdots\ , \qquad (B11)$$

$$\psi = \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \cdots\ . \qquad (B12)$$

Various partial derivatives will have the following form

$$\frac{\partial\Phi}{\partial T} = \frac{\partial\Phi_0}{\partial T} + \epsilon\frac{\partial\Phi_1}{\partial T} + \epsilon^2\frac{\partial\Phi_2}{\partial T} + \cdots\ , \qquad (B13)$$

$$\frac{\partial\Phi}{\partial z} = \frac{\partial\Phi_0}{\partial z} + \epsilon\frac{\partial\Phi_1}{\partial z} + \epsilon^2\frac{\partial\Phi_2}{\partial z} + \cdots\ , \qquad (B14)$$

$$\frac{\partial \Phi}{\partial \theta} = \frac{\partial \Phi_0}{\partial \theta} + \epsilon \frac{\partial \Phi_1}{\partial \theta} + \epsilon^2 \frac{\partial \Phi_2}{\partial \theta} + \cdots \; . \tag{B15}$$

where Φ denotes ϕ, ζ, or ψ.

According to Chapter 3 (page 21), the functions M^2, A and B will have the following form when expanded for small ϵ

$$M^2(z,\theta,T) = M_0^2(\theta) + 2\sqrt{\mu}\sqrt{T}\epsilon \sinh(2\xi_0)z + 4\mu T \epsilon^2 \cosh(2\xi_0)z^2 + \cdots \; , \tag{B16}$$

$$A(z,\theta,T) = A_0(\theta) + 2\sqrt{\mu}\sqrt{T}\epsilon A_1(\theta)z + 2\mu T \epsilon^2 A_0(\theta)z^2 + \cdots \; , \tag{B17}$$

$$B(z,\theta,T) = B_0(\theta) + 2\sqrt{\mu}\sqrt{T}\epsilon B_1(\theta)z + 2\mu T \epsilon^2 B_0(\theta)z^2 + \cdots \; , \tag{B18}$$

where the functions $A_0(\theta)$, $B_0(\theta)$, $A_1(\theta)$ and $B_1(\theta)$ are defined on page 22.

Substituting the series ($B10$) - ($B12$) and ($B16$) - ($B18$) into equations ($B7$) - ($B9$) we obtain the following equations at various orders of ϵ.

The $O(1)$ and $O(\epsilon)$ problems for the streamfunction reduce to solving the following equations for ψ_0 and ψ_1, respectively,

$$\frac{\partial^2 \psi_0}{\partial z^2} = 0 \; , \tag{B19}$$

$$\frac{\partial^2 \psi_1}{\partial z^2} = 0 \; . \tag{B20}$$

In order to satisfy the no-slip and impermeable boundary conditions (2.95), both ψ_0 and ψ_1 must be zero. Thus, series ($B12$) becomes

$$\psi = \epsilon^2 \psi_2 + \epsilon^3 \psi_3 + \cdots \; . \tag{B21}$$

The $O(\epsilon^2)$ problem for the streamfunction will have the following form

$$\frac{\partial^2 \psi_2}{\partial z^2} = 4\mu {M_0}^2 T \zeta_0 \; , \tag{B22}$$

and is in agreement with equation (3.84), apart from a factor of 4μ.

Next we proceed to the $O(1)$ problem for the temperature. This corresponds to solving the following equation for ϕ_0

$$4{M_0}^2 T \frac{\partial \phi_0}{\partial T} - 2z{M_0}^2 \frac{\partial \phi_0}{\partial z} = \frac{1}{Pr} \frac{\partial^2 \phi_0}{\partial z^2} \; . \tag{B23}$$

This equation is in exact agreement with equation (3.7). Applying the constant heat flux condition (2.96), it follows that $\phi_0 = 0$. Thus, series ($B10$) becomes

$$\phi = \epsilon \phi_1 + \epsilon^2 \phi_2 + \cdots \; . \tag{B24}$$

Since $\psi_0 = \psi_1 = \phi_0 = 0$, the nonlinear term will not appear in the $O(\epsilon)$ and $O(\epsilon^2)$ problems for the temperature. Thus, the equations for ϕ_1 and ϕ_2 are

$$4{M_0}^2 T \frac{\partial \phi_1}{\partial T} - 2z{M_0}^2 \frac{\partial \phi_1}{\partial z} = \frac{1}{Pr} \frac{\partial^2 \phi_1}{\partial z^2} \; , \tag{B25}$$

$$4{M_0}^2T\frac{\partial\phi_2}{\partial T} - 2z{M_0}^2\frac{\partial\phi_2}{\partial z} + 4\sqrt{\mu}\sqrt{T}z\sinh(2\xi_0)\left(2T\frac{\partial\phi_1}{\partial T} - z\frac{\partial\phi_1}{\partial z}\right)$$
$$= \frac{1}{Pr}\frac{\partial^2\phi_2}{\partial z^2}\ . \qquad (B26)$$

Note that equation ($B25$) is similar to (3.21); the only difference is that (3.21) has the extra term $-2\phi_1$. Equation ($B26$), on the other hand, is quite different from (3.32). The nonlinear term makes an appearance in (3.32) but not in ($B26$). In addition, (3.32) contains two extra terms.

Finally, the $O(1)$, $O(\epsilon)$ and $O(\epsilon^2)$ problems for the vorticity yield the following equations for ζ_0, ζ_1 and ζ_2, respectively,

$$4{M_0}^2T\frac{\partial\zeta_0}{\partial T} - 2z{M_0}^2\frac{\partial\zeta_0}{\partial z} = \frac{\partial^2\zeta_0}{\partial z^2}\ , \qquad (B27)$$

$$4{M_0}^2T\frac{\partial\zeta_1}{\partial T} - 2z{M_0}^2\frac{\partial\zeta_1}{\partial z} + 4\sqrt{\mu}\sqrt{T}z\sinh(2\xi_0)\left(2T\frac{\partial\zeta_0}{\partial T} - z\frac{\partial\zeta_0}{\partial z}\right) = \frac{\partial^2\zeta_1}{\partial z^2}$$
$$+2A_0\frac{\sqrt{T}}{\sqrt{\mu}}\frac{\partial\phi_1}{\partial z}\ , \qquad (B28)$$

$$4{M_0}^2T\frac{\partial\zeta_2}{\partial T} - 2z{M_0}^2\frac{\partial\zeta_2}{\partial z} + 4\sqrt{\mu}\sqrt{T}z\sinh(2\xi_0)\left(2T\frac{\partial\zeta_1}{\partial T} - z\frac{\partial\zeta_1}{\partial z}\right)$$
$$+8\mu Tz^2\cosh(2\xi_0)\left(2T\frac{\partial\zeta_0}{\partial T} - z\frac{\partial\zeta_0}{\partial z}\right) = \frac{\partial^2\zeta_2}{\partial z^2}$$
$$+\frac{2\sqrt{T}}{\sqrt{\mu}}\left(\frac{\partial\psi_2}{\partial\theta}\frac{\partial\zeta_0}{\partial z} - \frac{\partial\psi_2}{\partial z}\frac{\partial\zeta_0}{\partial\theta}\right) + 4\mu T\frac{\partial^2\zeta_0}{\partial\theta^2}$$
$$+\frac{2\sqrt{T}}{\sqrt{\mu}}\left(A_0\frac{\partial\phi_2}{\partial z} + 2A_1\sqrt{\mu}\sqrt{T}z\frac{\partial\phi_1}{\partial z}\right) + 4TB_0\sqrt{\mu}\frac{\partial\phi_1}{\partial\theta}\ . \qquad (B29)$$

When the leading order equation ($B27$) is compared with equation (3.49), we see that the source term in (3.49) is not present in ($B27$). Equation ($B27$) can be solved by separation of variables as follows. In terms of the variable $s = M_0z$, equation ($B27$) transforms to

$$\frac{\partial^2\zeta_0}{\partial s^2} + 2s\frac{\partial\zeta_0}{\partial s} = 4T\frac{\partial\zeta_0}{\partial T}\ . \qquad (B30)$$

Setting $\zeta_0(s,T) = X(s)Y(T)$ and substituting into ($B30$) it follows that

$$Y(T) = aT^{\frac{k}{4}}\ , \qquad (B31)$$

where k is the separation constant and a is an arbitrary constant. Imposing the initial condition (2.99), we must have that $k \geq 0$. Then $X(s)$ satisfies the differential equation

$$\frac{d^2X}{ds^2} + 2s\frac{dX}{ds} - kX = 0\ . \qquad (B32)$$

Applying the integral conditions (3.51), the far-field condition (2.97) and the properties of the Parabolic Cylinder Functions, it can be shown that $\zeta_0(s,T)=0$. Thus, series ($B11$) now becomes

$$\zeta = \epsilon\zeta_1 + \epsilon^2\zeta_2 + \epsilon^3\zeta_3 + \cdots \ . \tag{$B33$}$$

It then follows that $\psi_2 = 0$ from equation ($B22$). Thus, equation ($B28$) simplifies to

$$4{M_0}^2 T\frac{\partial\zeta_1}{\partial T} - 2z{M_0}^2\frac{\partial\zeta_1}{\partial z} = \frac{\partial^2\zeta_1}{\partial z^2} + 2A_0\frac{\sqrt{T}}{\sqrt{\mu}}\frac{\partial\phi_1}{\partial z}\ . \tag{$B34$}$$

Equation ($B34$) is now in close agreement with equation (3.49), apart from a factor of $\frac{2\sqrt{T}}{\sqrt{\mu}}$. Equation ($B29$) also simplifies to

$$4{M_0}^2 T\frac{\partial\zeta_2}{\partial T} - 2z{M_0}^2\frac{\partial\zeta_2}{\partial z} + 4\sqrt{\mu}\sqrt{T}z\sinh(2\xi_0)\left(2T\frac{\partial\zeta_1}{\partial T} - z\frac{\partial\zeta_1}{\partial z}\right)$$

$$= \frac{\partial^2\zeta_2}{\partial z^2} + \frac{2\sqrt{T}}{\sqrt{\mu}}\left(A_0\frac{\partial\phi_2}{\partial z} + 2A_1\sqrt{\mu}\sqrt{T}z\frac{\partial\phi_1}{\partial z}\right) + 4TB_0\sqrt{\mu}\frac{\partial\phi_1}{\partial\theta}\ . \tag{$B36$}$$

The first nonzero term in the streamfunction expansion, ψ_3, satisfies

$$\frac{\partial^2\psi_3}{\partial z^2} = 4\mu{M_0}^2 T\zeta_1\ , \tag{$B36$}$$

which also agrees with equation (3.84), apart from a factor of 4μ.

In summary, the first nonzero terms in the expansions for ψ, ϕ and ζ are ψ_3, ϕ_1 and ζ_1, respectively, and satisfy very similar equations as those obtained using the expansion procedure in Chapter 3. This leads one to ask whether one procedure is advantageous over the other. The answer to this depends on the goal of the exercise. Certainly the approach outlined here is more traditional and less complicated than that of Chapter 3. However, the method of Chapter 3 has at least two advantages over this one. First, the method of Chapter 3 provides a systematic procedure by which analytical solutions can be obtained, whereas the approach here leads to equations (such as ($B26$) and ($B34$)) that are too complicated to solve analytically. Second, the small parameter λ used in Chapter 3 is one that is suggested by the physics. To see this, we examine the equation (2.79). Retaining the dominant terms for small times leads to the conduction equation given by

$$\frac{\partial\phi}{\partial t} = \frac{1}{{M_0}^2 Pr\sqrt{Gr}}\frac{\partial^2\phi}{\partial\xi^2}\ , \tag{$B37$}$$

where

$${M_0}^2(\theta) = \frac{1}{2}\left(\cosh(2\xi_0) - \cos(2\theta)\right)\ . \tag{$B38$}$$

The solution to ($B37$) satisfying the boundary conditions

$$\frac{1}{M_0}\left(\frac{\partial\phi}{\partial\xi}\right) = -1 \text{ on } \xi = 0 \text{ and } \phi\to 0 \text{ as } \xi\to\infty$$

is found to be

$$\phi(\xi,\theta,t) = \frac{t\sqrt{Pr}}{\sqrt{\frac{4t}{\sqrt{Gr}}}}\left(\frac{1}{\sqrt{\pi}M_0}\mathrm{e}^{-\frac{{M_0}^2 Pr\xi^2}{\frac{4t}{\sqrt{Gr}}}} - \frac{\xi\sqrt{Pr}}{\sqrt{\frac{4t}{\sqrt{Gr}}}}\mathrm{erfc}\left(\frac{M_0\sqrt{Pr}\xi}{\sqrt{\frac{4t}{\sqrt{Gr}}}}\right)\right)\frac{4M_0}{\sqrt{Gr}}\,. \quad (B39)$$

Expressed this way we see that the solution naturally involves the similarity variable z^* defined by

$$z^* = \frac{\sqrt{Pr}M_0\xi}{\lambda} \text{ where } \lambda = \sqrt{\frac{4t}{\sqrt{Gr}}}\,.$$

For these reasons together with the fact that this was primarily an analytical study, the procedure of Chapter 3 is more appropriate.

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