Vacuum Energy in Expanding Spacetime and Superoscillation - Induced Resonance

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

This thesis is divided into two parts. The first part is a study of the general problem of vacuum energy or so-called 'zero point fluctuations' of a quantum field on expanding spacetimes and the interplay between the dilution of energy as a mode expands and the generation of energy as new modes enter from below the ultraviolet cutoff. The second deals with the phenomenon of superoscillations and some of the consequences in quantum theory and cosmology.

Modern theoretical cosmology sits upon the theory of **inflation** which assumes that the universe underwent a period of accelerated expansion sometime in the past. Indeed, a whole new scientific discipline was born known as high precision observational cosmology when ground breaking detailed measurements were made in the late 1990's that confirmed some predictions of the theory of inflation. However, inflation is essentially a classical phenomenon with the inclusion of the quantum theory relegated to the provision of initial perturbations of an otherwise homogeneous and isotropic spacetime usually interpreted as the inexorable quantum fluctuations of a (classical) scalar field coupled to the metric.

Quantum field theory on curved spacetime, on the other hand, has some novel features in comparison to it's flat spacetime cousin. The inclusion of some of these effects into a discussion of the novel inflationary picture should provide some very interesting and non-trivial insights to the very early universe and perhaps might shed some light on the fundamental nature of the gravitational interaction itself. Usually when studying quantum fields in curved spacetime and the energetic interaction between gravity and the quanta one works in the semi-classical picture where gravity remains a classical field. This is not only because a fully consistent quantum theory of gravity has not been constructed yet. Indeed, there should exist a presumably quite extensive regime where the picture of quantum fields propagating on a classical background remains a valid approximation. In the first part of this thesis we study this regime and some of the interesting physics that arises. Eventually, however, we go one step further than a semi-classical treatment and investigate the hypothesis that the dynamics of cosmologically significant spacetimes is provided by the spacetime dependence of the **quantum vacuum energy** of a scalar field on that spacetime. Put more simply, we discuss the possibility that the tendency for a spacetime to expand and accelerate it's expansion reduces to the statement that it is vacuum-energetically favorable to do so. The idea that the gravitational degrees of freedom are induced in this way is an old one due to Sakharov and is one represented in this thesis in simplified form and with specific calculations and examples. We find that such an interpretation is at least not excluded and, in fact, sits satisfactorily with the ideas of inflation.

Along the way to our conclusive discussion of the 'induced cosmology' we discuss, after briefly reviewing inflation and quantum field theory in curved spacetime, the general problem of vacuum energy in curved spacetime and some simplified models of the quantum mechanical ground state energy of a collection of harmonic oscillators on expanding spaces including some discrete models. Our philosophy throughout will be one of pragmatism; we assume a cutoff on momenta (or length scales) at an unspecified energy scale and assume our conclusions hold, if not all the way from the Planck scale (which would presumably be subject to beyond the standard model 'quantum gravitational' effects), then at least in some meso-scale between the Hubble scale and the Planck scale. It is certainly true that a quantum scalar field really is a collection of independent harmonic oscillators one for each different comoving length scale (wavelength). The question that we seek to address in this thesis is “what of the vacuum energy of those modes associated with neither the extreme ultraviolet (where ordinary field theory breaks down) nor the extreme infrared (where ordinary general relativity is assumed to break down)?”. After discussing an infrared divergence which we find to be present in a larger class of powerlaw spacetimes than has been previously found, we also implement an infrared cutoff on energies. An interpretation of the infrared cutoff as the realization of a local expansion of a patch of an otherwise flat and very large ‘ambient’ spacetime is attempted and the corresponding picture of an energetic initiation of inflation is provided.

The phenomenon of superoscillations for bandlimited functions is the observation that a function may approximate to arbitrary precision a plane wave not contained in it’s Fourier decomposition. In particular, it is possible for a function with compact support in the frequency domain to approximate with arbitrary precision a high frequency waveform outside of this support on an arbitrary long interval. This phenomenon has only recently begun to be studied in the literature and as yet very few quantitative results have been obtained. In the second part of this thesis we study the energetic response of a classical and quantum harmonic oscillator driven by a superoscillating driving force. We find that the oscillator indeed responds to the ‘imposter’ driving force as if it were real and, dubbing the response ‘ghost resonance’, we investigate some consequences in quantum field theory and cosmology.

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Conventions and Notation

Although this thesis assumes a basic understanding of both general relativity to the level of [1] for the study of cosmology and scalar quantum field theory to the level somewhere between [2] and [3] some introductory remarks are made in the first chapter of Part I in order to make the presentation complete. Throughout we try to stick with the $(-+++)$ signature convention and use the Heisenberg picture of quantum evolution unless explicitly stated. Also, we use geometric notation for derivatives so that $f_{,\mu} := \partial_\mu f$ for functions and $v^\alpha_{;\mu} := \nabla_\mu v^\alpha$ for covariant derivatives. In general ϕ will represent a classical (number valued) field whereas φ stands for a scalar quantum field, $\eta_{\mu\nu}$ will represent the Minkowski metric $\mathrm{diag}(-1, 1, 1, 1)$ and $g_{\mu\nu}$ a general spacetime metric. We work in natural units where $\hbar = c = 1$ and $8\pi G = 1$. For cosmological time derivatives we use 'dot' $\dot{}$ to represent the cosmological time derivative ∂_t and 'prime' $'$ to represent the comoving time derivative ∂_η.

Part I

Quantum Field Theory in Cosmology

Introduction

The reality of the vacuum energy of quantum fields is an experimentally verified fact of nature as exhibited by the Casimir force [4] known to act between two parallel conducting plates. The force is understood to arise from the suppression due to the presence of the conducting plates of the ground state energies of a subset of the independent harmonic oscillators that collectively represent the electromagnetic field. For a scalar field in Minkowski spacetime the vacuum energy density is formally infinite even for a field defined only on a finite region due to the infinite number of harmonic oscillators making up the field. This is a well known and much discussed divergence known as the ultraviolet divergence. The Casimir energy is understood as the *difference* between the formally infinite vacuum energy of the field as it acts with the conducting plates and energy of the field without the plates. The problem of vacuum energy on non-trivial backgrounds is an obvious generalization of the problem of the Casimir energy. Such studies have been carried out using some simplified versions of quantum field theory (conformally coupled theories) and various techniques of regularization, for example [5, 6, 7, 8, 9]. However, these early studies used approximate methods such as the adiabatic approximation (where the spacetime is assumed to be varying slowly) and were conducted at a time when the foundations of quantum field theory in curved spacetime were being first understood. The study of vacuum energy in curved spacetime waned in the years after the initial enthusiasm once the quantum field theory was well understood. However modern theoretical cosmology has motivated a revisiting of these old ideas with the introduction of quantum field theory in curved spacetime into testable science. In particular the theory of inflation utilizes quantum field theory in an essential way in providing a mechanism for the formation of large scale structure in our unverse. In this, the first part of this thesis, we make a systematic study of the role of quantum vacuum energy in expanding spacetime and in particular on cosmologically realistic backgrounds. We emphasize exact solutions and physical justifications for the simplifications we make. Despite the fact that in the standard presentation inflation is formally generated by the dynamics of a *classical* scalar field there is an intuitive understanding in the community that the inflationary dynamics ought to be ultimately a result of the non-trivial evolution of the vacuum energy of the quantum fields of nature. Such an idea has only recently been made more precise in the literature [10, 11, 12, 13] but the idea that spacetime dynamics is nothing but the reaction of spacetime to the dynamical vacuum energy in the presence of a Planck scale cutoff is an old one due to Sakharov [14] proposed well before the theory of inflation or quantum field theory on curved spacetime were developed.

The layout of this part is as follows. We begin with a concise review of the separate ingredients, cosmology and quantum field theory that are necessary for a study of vacuum energy in cosmology. This review is not intended to be exhaustive or complete but serves only to make the presentation complete. However we do highlight some interesting and original points in these sections. We then discuss vacuum energy in general and particularly the idea of a dynamical (time dependent) vacuum energy in the full cosmological scenario. We also study the dynamical vacuum energy in simplified frameworks in order to provide a more complete picture of the physics involved. We end with the main yet rather speculative chapter concerned with realizing the idea that the cosmological dynamics are wholly due to non-trivial dynamical vacuum energy.

Chapter 1

Quantum Field Theory and Cosmology

In this chapter we review the inflationary scenario in cosmology and also aspects of quantum field theory (QFT) on a curved manifold essential for an understanding of the material presented in the following chapters of this thesis.

1.1 Inflationary Cosmology

During the last century, our picture of the large scale structure and history of the universe underwent a drastic revolution. Starting with Edwin Hubble's observation of the recession of distant galaxies interpreted at the time within the new theory of general relativity as an isotropic and homogeneous expansion of space and ending with Penzias and Wilson's observation of the primordial cosmic microwave background radiation (CMB) in 1965 [15], a very detailed picture of the universe as a whole emerged. This so-called standard big bang model (SBBM) in cosmology painted a picture of the universe starting in a state of extremely high energy density and cooling through expansion, the forces of nature and the constituent matter emerging through phase transitions providing the structure and laws of the astrophysics we observe today. The SBBM was, however, always faced with serious conceptual and theoretical obstacles to a position of a rigorous and complete model of the very early universe. These days one usually quotes the 'horizon' and 'flatness' problems of the SBBM when discussing how economically the inflationary model solves them [16]. The horizon problem of SBBM is the observation that two photons arriving at earth telescopes from opposite directions in the sky should have originated from causally disconnected regions and so should not be correlated in any way yet we observe an almost perfectly isotropic CMB. On the other hand, the flatness problem is the observation of almost exact spatial flatness[1] of the observable universe today. Under the standard assumptions of SBBM it is known that spatial flatness is dynamically unstable, and hence in order for space to be so flat today the initial parameters (such as energy density, pressure and radiation content) need to be extremely fine tuned. Such fine tuning is unsatisfactory for a fundamental scientific theory.

[1] These terms will be defined and discussed below.

The simple and far reaching inflationary hypothesis that turns these anomalies into natural features is that *the universe underwent a sufficiently long period of accelerated expansion*. Once the accelerated period was completed the universe is assumed to have settled down to a standard decelerated expansion in accordance with our usual intuitive (and to a degree rigorous due to the singularity theorems[2]) understanding of the attractive nature of the gravitational force, precisely as in SBBM. Under this seemingly innocent single additional assumption, regions previously causally disconnected in spacetime become connected, avoiding the horizon problem, and spatial flatness becomes a stable fixed point of the evolution. On top of the success of the inflationary hypothesis in solving the above mentioned problems and additionally a possible explanation of the origin of large scale structure in the universe (to be discussed below), the theory of inflation made a detailed and consequently confirmed prediction about the fine structure of the CMB and hence entered the realm of real and testable scientific theory. For this reason the picture of an early period of inflation followed by standard decelerated expansion is widely accepted today by cosmologists as an accurate description of the early evolution of our universe. However, as yet there is no completely satisfying mechanism for the onset or even the end of an inflationary period and there is much contention as to the detailed interpretation of the model. It is part of the goal of this thesis to shed at least a little light on this problem based on a more thorough investigation of the role of quantum fluctuations in an accelerating spacetime.

1.1.1 Friedman Lemaître Spacetime

Under the observationally justified assumptions of large scale homogeneity and isotropy (HI), the metric that describes our universe, known as the Friedmann Lemaître (FL) metric, is written in so-called co-moving coordinates as

$$ds^2 = dt^2 + a(t)^2 \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right). \tag{1.1.1}$$

Here a hypersurface of constant coordinate time t is a maximally symmetric three manifold here labeled by the parameter κ. The three types of maximally symmetric three manifold are represented by

$$\begin{aligned} \kappa > 0 \quad &\Rightarrow \quad \text{three sphere} \\ \kappa = 0 \quad &\Rightarrow \quad \text{flat space} \\ \kappa < 0 \quad &\Rightarrow \quad \text{space of constant negative curvature.} \end{aligned}$$

These three cases are known respectively as **closed**, **flat** and **open** universes. In this normalization the scale factor $a(t)$ is dimensionless, the radial coordinate has a dimension of length and κ has the dimension of $(\text{length})^{-2}$ and parameterizes ‘how curved’ the spacial three-slices are. Experimentally [19] it is found that on large scales our universe

[2]The singularity theorems of classical general relativity [17] are essentially rigorous mathematical statements that ‘reasonable’ matter is gravitationally attracted to other reasonable matter resulting in gravitational collapse and singular gravitational fields where ‘reasonable’ is defined in terms of certain intuitively motivated energy conditions. It is interesting to note however that certain quantum fields have been found to violate the energy conditions that go into the singularity theorems [18].

is described by a FL metric with *flat* spatial sections[3]. In what follows we shall set $\kappa = 0$ unless otherwise stated.

The FL model thus encodes the large scale dynamics of the universe into the single function $a(t)$. The Einstein field equation (EE) $G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}$ for this restricted class of spacetimes reduces to the non-linear differential equations for a

$$3\left(\frac{\dot{a}}{a}\right)^2 = \rho + \Lambda \quad - \text{ The "Friedmann equation"} \tag{1.1.2}$$

$$2\left(\frac{\ddot{a}}{a}\right) + \frac{\dot{a}}{a^2} = \Lambda - P \tag{1.1.3}$$

where we model the classical matter content in such an idealized cosmology by an isotropic and homogeneous perfect fluid possessing energy momentum described by the well known tensor (in a co-moving[4] frame)

$$T^\mu_\nu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}. \tag{1.1.4}$$

Here ρ is referred to as the **energy density** and P the **pressure** of the fluid. The dynamics of the fluid are prescribed by specifying an **equation of state** $P(\rho)$ relating the pressure and energy density; a period of time in which $w := P/\rho$, known as the **equation of state parameter**, is approximately constant is referred to as an **epoch**. In general however, the energy momentum tensor (EMT) $T_{\mu\nu}$ that acts as a source for the EE is defined to be proportional to the functional derivative of the matter action $\mathcal{S}_m$ with respect to the metric:

$$T_{\mu\nu} = \frac{-2}{\sqrt{\mathbf{g}}}\frac{\delta \mathcal{S}_m}{\delta g^{\mu\nu}}$$

where $\sqrt{\mathbf{g}}$ is the (absolute value of the) determinant of the metric, and should be subject to its own (coupled) dynamics arising from an action principle for $\mathcal{S}_m$. Matter whose EMT can be put into the form 1.1.4 for some choice of coordinates is also referred to as a perfect fluid.

It will be convenient to introduce a new time coordinate, the conformal time η defined as

$$\eta(t) = \int^t \frac{dt'}{a(t')}, \tag{1.1.5}$$

with respect to which the FL metric is manifestly conformally flat:

$$ds^2 = a(\eta)^2\left(-d\eta^2 + dr^2 + r^2 d\Omega^2\right).$$

These coordinates greatly simplify the d'Alembertian differential operator $\Box = \nabla_\mu \nabla^\mu$ (the contraction of the covariant derivative) and which will be useful in the quantum theory.

[3]Recall that this was the essential observation leading to the horizon problem of SBBM.

[4]The co-moving frame is that frame in which the four velocity $\mathbf{U}$ of the fluid takes the form $\mathbf{U} = (1, 0, 0, 0)$.

De Sitter spacetime

De Sitter spacetime corresponds to a vacuum solution to 1.1.2 in the presence of a positive cosmological constant Λ. In the absence of matter the Friedmann equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\Lambda$$

and hence the de Sitter scale factor is $a(t) = a_0 \mathrm{e}^{Ht}$ with $H = \sqrt{\Lambda/3}$. The other Einstein equation 1.1.3 is automatically satisfied by this solution. For physical reasons we choose when discussing de Sitter spacetime in the context of cosmology the expanding branch of the square root since we observe our universe to be expanding. Technically [20], the de Sitter manifold is the full branched solution where there exists a causally disconnected sector represented by an exponentially contracting space.

Note that, instead of being a gravitational parameter, Λ is interpretable as a diagonal (and rather non-standard) contribution to the EMT. Examining 1.1.2 and 1.1.3 we see that one might view Λ as a perfect fluid with $\rho = \Lambda$ and $P = -\Lambda$ and hence $w = -1$. The cosmological constant thus contributes as if it were a type of matter with *constant positive energy density but negative pressure*. Hence we conclude that matter with $w \approx -1$ gives rise to approximate exponential expansion. We will see in the next section that a scalar field with a particularly slowly varying potential function possesses precisely this type of behaviour. We will almost exclusively work with $\Lambda = 0$ in this thesis since we most certainly do not live in a de Sitter universe and moreover, dynamical evolution corresponding to near exponential expansion is still achievable even with $\Lambda = 0$.

The parameter H of de Sitter spacetime not only controls the rate of expansion (which might be thought to be merely a coordinate choice) but also defines what is known as the **cosmological horizon** present in all expanding spacetimes. Consider an observer at point $A = x(0) = 0$ at time $t = 0$ sending a signal at the speed of light to an observer at point B. In a small time δt the signal travels a proper distance $a(t)\delta x$ and we have

$$\frac{dx}{dt} = a^{-1}(t) = \mathrm{e}^{-Ht}$$

possessing solution $x(t) = \frac{-1}{H}\mathrm{e}^{-Ht} + \frac{1}{H}$. Hence we see that the co-moving distance traversable by the signal is bounded with $x \to \frac{1}{H}$ as $t \to \infty$. We are led to the conclusion that observers separated by a co-moving distance of $2/H$, known as (twice) the horizon distance, or more cannot observe one another. We shall go into more detail on horizons below and also in the next chapter in the quantum theory.

The reason why de Sitter spacetime is of interest to cosmologists is two fold. One the one hand, de Sitter spacetime provides a simple example where the quantum wave equation may be solved exactly and implications of the existence of a cosmological horizon may be (and are below) studied, while on the other hand our own universe while undergoing inflation is expected to have been evolving approximately as a de Sitter spacetime. This is the essence of the slow roll approximation in inflation, discussed later in this chapter.

Powerlaw spacetime

A large class of non-vacuum exact solutions to 1.1.2, 1.1.3 are given by the so called **powerlaw** line element, a FL solution with scale factor $a(t) = (t/t_0)^p$[5]. The matter is assumed to be a perfect fluid where different equations of state correspond to different powers of p as discussed below.

Specifically we realize the perfect fluid by a coupled homogeneous and isotropic scalar field $\phi(t)$ which evolves as [21]

$$\phi = \mu\sqrt{2p}\ln\left(\sqrt{\frac{V_0}{p(3p-1)}}\frac{t}{\mu}\right)$$

with potential

$$V = V_0 \exp\left(-\sqrt{\frac{2}{p}}\frac{\phi}{\mu}\right). \tag{1.1.6}$$

In these expressions μ is an arbitrary mass scale. Here ϕ satisfies its own field equation

$$\Box\phi + \frac{dV}{d\phi} = \frac{1}{\sqrt{\mathbf{g}}}\left(\sqrt{\mathbf{g}}g^{\mu\nu}\phi_{,\nu}\right)_{,\mu} + \frac{dV}{d\phi} = 0,$$

arising from the action

$$\mathcal{S}_m = -\frac{1}{2}\int \sqrt{\mathbf{g}}d^4x \ \left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + 2V(\phi)\right)$$

under the HI assumptions. The EMT associated with such a field is given by 1.1.4 with fluid parameters

$$\begin{aligned} P &= \frac{1}{2}\dot{\phi}^2 - V \\ \rho &= \frac{1}{2}\dot{\phi}^2 + V \end{aligned} \tag{1.1.7}$$

which for powerlaw become

$$\begin{aligned} P &= p(2-3p)\frac{\mu^2}{t^2} \\ \rho &= 3p^2\frac{\mu^2}{t^2}. \end{aligned} \tag{1.1.8}$$

Hence the equation of state parameter w for powerlaw is

$$w = \frac{2}{3p} - 1.$$

We read off some physically relevant special cases found in the literature [21]:

$$\begin{aligned} &\text{Radiation domination}: \quad a(t) = t^{\frac{1}{2}}, \quad w = \frac{1}{3} \\ &\text{Matter (dust) domination}: \quad a(t) = t^{\frac{1}{3}}, \quad w = \frac{2}{3} \\ &\text{Vacuum energy domination}: \quad p \to \infty, \quad w = -1. \end{aligned}$$

[5] Recall that we choose units and coordinates such that the scale factor is dimensionless.

We notice that as long as $p > 1$ or in other words $w < -1/3$, we have $\ddot{a}(t) > 0$ so that $p > 1$ powerlaw spacetimes satisfy the condition for inflation.

The limit $p \to \infty$ is of special interest and we will study this limit more below in the quantum theory. We see that in this limit the equation of state behaves as $w \to -1$ converging to the equation of state for a pure cosmological constant and de Sitter spacetime. This is not surprising since the exponential function is that function whose rate of change is greater than any polynomial power so that one would expect de Sitter spacetime to emerge in the high p limit in this sense.

It should also be noted that, in consistency with its interpretation as an approximation to de Sitter spacetime, the powerlaw solution too possesses a cosmological horizon. Repeating the argument presented above for de Sitter we see that for an observer A at $x = 0$ at $t = t_0$ a signal sent out radially cannot be received at arbitrary comoving separation from A as

$$x(t) = \frac{t^{1-p}}{1-p} - \frac{t_0^{1-p}}{1-p} \to \frac{t_0^{1-p}}{p-1} \quad \text{as } t \to \infty.$$

This is known as the **co-moving particle horizon** distance and is defined [22] at arbitrary times t by[6]

$$d_{CP} = \int_{t_0}^{t} \frac{dt'}{a(t')}.$$

On the other hand the co-moving **event horizon** distance is defined [22] by

$$d_{CE} = \int_{t}^{\infty} \frac{dt'}{a(t')}$$

and is the physically meaningful distance for the dynamics of quantum fields representing (co-moving) distances beyond which events are causally disconnected. We see that the proper event horizon distance for de sitter is special in that it is constant in time

$$d_{PE}^{\text{dS}} = a(t) d_{CE}^{\text{dS}} = \mathrm{e}^{Ht} \frac{\mathrm{e}^{-Ht}}{H} = \frac{1}{H}$$

whereas the proper powerlaw event horizon distance is time dependent

$$d_{PE}^{\text{PL}} = \frac{t}{1-p}, \qquad (t_0 = 0)$$

and expands as time progresses. The result derived in the previous section about the de Sitter horizon displays the fact that the co-moving event horizon converges to the constant physical event horizon in numerical value in the infinite future. Of course, in the conformal coordinates, the worldline $x = |\eta|$ corresponds to the boundary of a light cone. In proper coordinates this world line is described by $\tilde{x} = a(\eta)x = 1/H$ (since $a(\eta) \propto 1/\eta$) which is constant in time. In powerlaw this result is perturbed to read $\tilde{x} \propto |\eta|^{1/(1-p)}$ which is approximately constant for high p. Below we will also see that for sufficiently high p (the so-called slow roll spacetimes) the generalization of the Hubble parameter $H(t) = \dot{a}/a$ for these spacetime behaves like $H \sim \sqrt{V(\phi)}$. Hence for high

[6] The sub and super scripts on the various horizon scales refer to 'physical', 'comoving', 'event', 'particle' and of course 'de Sitter' and 'powerlaw'.

p the proper horizon distance is expressible also as $d_{PE}^{\mathrm{PL}} \sim 1/H(t)$ in analogy to the de Sitter result.

As will be discussed below the presence of causal discontinuity in this physically realistic spacetime is of great significance for inflation where the dynamics of perturbations on scales greater than the horizon during inflation are magnified to macroscopic scales so as to become seeds for structure formation. It is the signatures of these magnified perturbations observed in the CMB that allow cosmologists to study the detailed dynamics of the very early universe and to possibly test candidate theories of Planck scale physics.

For the powerlaw scale factor there exists a singularity in the conformal time variable 1.1.5 at $\eta(0)$ so that we choose the lower limit of integration as some arbitrarily small parameter ϵ and consider the domain of the variable t to be (ϵ, ∞). We will wish eventually to refer to times when a co-moving wavelength possesses an arbitrarily small proper wavelength which occurs for vanishingly small t. Hence ϵ should be as small a quantity as necessary to allow η to represent as much of the limit $t \to 0^+$ as necessary in any limit argument. This procedure will be a modification of the **Bunch Davies** vacuum identification criterion discussed below and will allow us to choose particular solutions to the wave equation by imposing a boundary condition in the limit $t \to 0^+$. To simplify the equations we also choose the reference time t_0 to be $t_0 = 1$. We have

$$\eta(t) = \frac{1}{1-p}\left(t^{1-p} - \epsilon^{1-p}\right)$$

so that the range of η is

$$\eta \in \begin{cases} (0, \infty) & \text{for } p < 1 \\ \left(0, \frac{\epsilon^{1-p}}{(p-1)}\right) & \text{for } p > 1 \end{cases}$$

The special case of $p = 1$ is unique since in that case

$$\eta(t) = \ln(t) - \ln(\epsilon)$$

so that

$$\eta \in (0, \infty) \quad \text{for } p = 1.$$

Further, the scale factor in conformal time is found by a simple calculation to be

$$a(\eta) = \left((1-p)\eta + \epsilon^{1-p}\right)^{\frac{p}{1-p}}, \;\; p \neq 1; \quad a(\eta) = \epsilon e^{\eta}, \;\; p = 1.$$

We shift the coordinate η by

$$\eta \to \overline{\eta} = \eta + \frac{\epsilon^{1-p}}{1-p} \tag{1.1.9}$$

so that the scale factor becomes $a(\eta) = ((1-p)\eta)^{p/(1-p)}$ and the range of η (dropping the overline) is now

$$\text{Range } \eta = \begin{cases} \left(\frac{\epsilon^{1-p}}{1-p}, \infty\right) & \text{for } p < 1 \\ \left(\frac{-\epsilon^{1-p}}{p-1}, 0\right) & \text{for } p > 1 \end{cases}.$$

Note that this shift leaves the metric invariant. We see again one of the many continuity features between high p power law spacetime and de Sitter spacetime. For high p the powerlaw conformal time is entirely negative and the infinite future is represented by the limit $\eta \to 0^+$ as is also the case for the de Sitter conformal time given by $\eta = -\mathrm{e}^{-Ht}/H$. Again, the special case $p = 1$ is unique in that the shift 1.1.9 is undefined in that case.

It should be noted that without implementing the ϵ regulator we would have missed the negative range of the variable η for $p > 1$ and not been able to refer to the limit $\eta \to -\infty$ necessary for identifying the vacuum as we will do below in the quantum field theory. This discussion is not highlighted in the literature.

1.2 Quantum Field Theory in Curved Spacetime

Quantum field theory is the generalization of the principles of quantum mechanics to the description of continuous fields. In curved spacetime the theory of quantum fields is far less developed than the flat spacetime counterpart. The subject received great attention in the late 1960's and early 1970's with the pioneering work of Parker on the creation of particles in expanding universes [23, 24, 25] and the appearance of the first review articles [26]. Other authors such as Ford, Fulling, Unruh, Davies and DeWitt made significant contributions to the subject over the following decade culminating in a satisfactory theory of the renormalization of the energy momentum tensor and description of quantum fields on FL backgrounds (see Ref. [2] and references therein). Modern presentations start with the observation that the objects φ are better understood not as a field of 'operators at a point' but as an operator valued distribution, mapping functions into operators on a Hilbert space or, more generally, an abstract operator algebra [27]. Although we present a brief discussion on the effective action and the so-called induced gravity proposal in a later section we will not be concerned with the development of these and other more sophisticated techniques here in this thesis since the problems we wish to address here may be studied on a more intuitive and conceptual level. In this thesis we will be attempt to take some essential features of what QFT is and study them in a context in which some of the essential features of inflation are important. For this purpose a rigorous formulation is not necessary.

In this section we develop some of the techniques that will be used in the next chapter concerned with the dynamics of vacuum energy in expanding spacetimes. Much of the material in this section is spread throughout the books [28, 3, 2, 29] and various review articles in the literature such as [30, 31] but here we add a unique perspective on these issues. Unless otherwise stated in this chapter we will use the notation $\sum_k$ to stand for either $\frac{1}{\mathrm{Vol}\, X} \sum_k$ or $(2\pi)^{-3/2} \int d^3k$ so that our results apply to both compact and non-compact spaces X. Similarly the notation $\delta_{k,k'}$ is read as the Kronecker and Dirac delta functions respectively in the two cases.

1.2.1 Quantum fields and the choice of vacuum state

We take the philosophy that a quantum field φ is an entity which satisfies the Klein Gordon (KG) operator equation $(\Box - m^2 - \xi R)\varphi = V'(\varphi)$ arising out of a canonical quantization of a classical field satisfying the classical (number valued function) equivalent of the wave equation. The classical wave equation is the Euler Lagrange equation for the action

$$\mathcal{S}[\varphi] = -\frac{1}{2}\int d^4x\sqrt{\mathbf{g}}\ \left(g^{\mu\nu}\varphi_{,\mu}\,\varphi_{,\nu} + (m^2 + \xi R)\varphi^2 + 2V(\varphi)\right), \tag{1.2.1}$$

where R is the Ricci scalar, here written in curved spacetime. That is, to quantize the classical field we postulate the equal time canonical commutation relations

$$[\varphi(t,x), \pi(t,x')] = i\delta^3(x - x')$$
$$[\varphi(t,x), \varphi(t,x')] = 0$$
$$[\pi(t,x), \pi(t,x')] = 0$$

where

$$\pi(t,x) = \frac{\delta\mathcal{S}[\varphi]}{\delta\partial_{x^0}\varphi},$$

and the first order Hamilton equations of motion $\partial_t\phi = [H,\phi]$, $\partial_t\pi = [H,\pi]$ (equivalent to the second order KG equation).

A classical field is said to be **free** if $V(\varphi) = 0$, interacting if it is not (for example the commonly used $V(\varphi) = 1/4!\ \varphi^4$ potential), **minimally coupled** if $\xi = 0$, conformally coupled when $\xi = 1/6$ and **massless** when $m = 0$. Each of these different theories have significantly different structures, particularly at the quantum level and we shall have time to mention at least a few concrete examples of these differences in later chapters. The free, massless, minimally coupled version of the KG equation is thus written as

$$\Box\varphi = 0. \tag{1.2.2}$$

We restrict ourselves to this wave equation in this thesis because this is mathematically the simplest field with which to study the essential quantum phenomena in the context of a cosmologically interesting background spacetime and further, these are the fields which are utilized in inflation to provide the initial perturbations on which the large scale structures of spacetime are seeded.

Fields in Minkowski space

The dynamics of a classical field ϕ specified by 1.2.2 in flat spacetime is equivalent to the dynamics of a collection of independent undamped harmonic oscillators. In flat spacetime we have $\Box\phi = -\partial_t^2\phi + \nabla^2\phi$. Expanding the field ϕ at fixed a time into eigenfunctions to the spatial Laplacian ∇^2

$$f_k(x,t) = \phi_k(t)\mathrm{e}^{ik\cdot x}$$

we see that the coefficient ϕ_k satisfies the oscillator equation of motion $(\partial_t^2 + \omega_k^2)\phi_k(t) = 0$ where $\omega_k^2 = k\cdot k$. Thus a complete description of the dynamics of each of the independent oscillators ϕ_k is equivalent to a complete description of the dynamics of ϕ itself. This is useful since the quantum theory of a harmonic oscillator is very well understood. In this way one understands a field as simply a collection of harmonic oscillators.

It is important to note that due to the generality of the mathematical description each oscillator equation possesses two independent solutions given by the complex exponentials with positive and negative frequency $\pm\omega_k = \pm\sqrt{k\cdot k}$. The full solution space of the classical field (isomorphic to the tensor product space of the oscillator Fourier components

$\mathcal{H} = \otimes_k \mathcal{H}_k$) may therefore be decomposed $\mathcal{H} \simeq \mathcal{S}_p \oplus \overline{\mathcal{S}_p}$ into positive and negative frequency parts $\mathcal{S}_p$ and $\overline{\mathcal{S}_p}$ respectively where $\mathcal{S}_p$ is that space spanned by those oscillators $\phi_k(t)$ for each k with positive frequency ω_k only. Quantum field theory on flat spacetime utilizes this decomposition into positive and negative frequency solutions in an essential way as described below.

By writing a *classical* field as a sum of independent classical harmonic oscillators $\phi_k(t)$ one can, by direct quantization of the oscillators, write a *quantum* field as a sum of independent quantum harmonic oscillators. This is done in the usual way by introducing the auxiliary variables a_k and $a_k^\dagger$ for each k in terms of which the field is written

$$\varphi(t,x) = \sum_k \varphi_k \mathrm{e}^{ik\cdot x}, \quad \varphi_k(t) = \frac{1}{\sqrt{2}}\left(a_k(t) + a_{-k}^\dagger(t)\right).$$

Just as in the quantum mechanics of a harmonic oscillator each degree of freedom $\varphi_k(t)$ is described by the **creation** and **annihilation** operators $a^\dagger(t)$ and $a(t)$ respectively whose time dependence is given by solving the time independent harmonic oscillator equation:

$$a_k(t) = a_k \frac{\mathrm{e}^{-i\omega_k t}}{\sqrt{2\omega_k}}, \quad a^\dagger(t) = a^\dagger \frac{\mathrm{e}^{i\omega_k t}}{\sqrt{2\omega_k}}.$$

with $[a_k, a_{k'}^\dagger] = \mathbb{1}\delta_{k,k'}$ We write this as

$$\varphi_k(t) = \frac{1}{\sqrt{2}}\left(a_k v_k(t) + a_{-k}^\dagger v_k^*\right). \tag{1.2.3}$$

or even more concisely

$$\varphi = \sum_k a_k u_k + a_k^\dagger u_k^*$$

where

$$u_k(x,t) = \frac{1}{\sqrt{2\omega_k}} \mathrm{e}^{-i\omega_k t + k\cdot x}.$$

The **vacuum state** is that state $|0\rangle$ for which $a_k|0\rangle = 0$ for all k. The states of the theory are described by the n-particle states constructed out of the vacuum state as in quantum mechanics. For example the state $|1_k\rangle = a_k^\dagger|0\rangle$ is interpreted as a state containing one particle of momentum k. This basis is known as the **Fock** basis.

It is important to note that specifying a splitting of the basis functions v_k into positive and negative frequencies is the same as specifying a splitting of φ_k into a_k and $a_k^\dagger$ and hence into different notions of what the particle content of a given state is. In flat spacetime as noted above however there is a natural choice for the functions v_k given by the positive and negative frequency exponentials defined with respect to the global inertial coordinate system of Minkowski spacetime[7]. In the second part of this thesis we will discuss a phenomenon known as the **Unruh effect** whereby the vacuum state $|0\rangle$ is seen not to be empty (as in $a_k|0\rangle = 0$) but to contain a thermal spectrum of particles. This is achieved by choosing not the inertial Lorentzian time coordinate but the timelike direction defined by a congruence of constantly accelerating observers. This surprising result, that to an accelerated observer the inertial vacuum state is highly excited, is at the heart of some of the most modern applications of QFT in curved spacetime including Hawking radiation and the holographic principle which we discuss in the final chapter.

[7]The inertial Minkowski coordinates are precisely those coordinate systems in which the line element and hence the d'Alembertian is of the standard form as given above. Thus in all 'natural' coordinate systems in flat space the idea of the 'positive frequency solutions to the KG equation' is a unique one.

Curved spacetime

In the transition from quantum field theory (QFT) on Minkowski spacetime to the curved spacetime theory one is forced to abandon many notions that might have at first seemed necessary for a consistent interpretation. In particular there exists in general no preferred timelike vector field with which to define a time evolution and, more importantly for our purposes, no preferred definition of positive frequency mode function. Furthermore, in general the wave equation

$$\frac{1}{\sqrt{\mathbf{g}}}\left(\sqrt{\mathbf{g}}g^{\mu\nu}\phi_{,\mu}\right)_{,\nu}=0$$

is not separable and even if it is, the freedom to make arbitrary coordinate transformations including time dependent ones makes the decomposition of a field at a fixed time into Fourier modes arbitrary. In this way, the picture of a quantum field as a collection of harmonic oscillators is more obscure in curved spacetime.

We saw that in Minkowski spacetime a split of the solution space into positive and negative frequency subspaces was tantamount to a decomposition of the field into creation and annihilation operators a and $a^\dagger$ respectively. Therefore in curved spacetime we expect the notion of 'particle' to lose meaning being dependent on a choice of coordinates. Another concept, closely related to the definition of particle, made ambiguous in curved spacetime is that of the **vacuum state** for the quantum field. One might expect the vacuum state to simply be given by that state in which all the harmonic oscillators are in their ground state at one time. This prescription is ambiguous in curved spacetime due to the ambiguity of the decomposition of a Fourier mode of the field into creation and annihilation operators.

We will not need to address all the complexities of the general theory here in this thesis since we will be applying the formalism only to FL backgrounds for which a preferred set of observers exists given by the so-called co-moving observers that observe an isotropic expansion of spacetime. Still, however, it is worth keeping in mind the subtle dependence of the content of quantum states on the symmetries of spacetime. We elucidate these comments in the following section with some concrete examples.

1.2.2 Quantum fields in FL spacetime

Let ϕ be a classical scalar field on a FL spacetime. Then the KG field equation is

$$\ddot{\phi}-3\left(\frac{\dot{a}}{a}\right)^2\dot{\phi}-\frac{\ddot{a}}{a}\phi-\nabla^2\phi=0.$$

To quantize this field system we reduce the field to a collection of independent degrees of freedom for which the quantization is known. To this end we choose the conformal time coordinate η and define the new field variable (which we call the **co-moving field**) $\chi=a(\eta)\phi$. With respect to this new time coordinate and field variable we have

$$\chi''-\frac{a''}{a}\chi-\nabla^2\chi=0$$

so that the instantaneous Fourier coefficient $\chi_k(\eta)$ satisfies

$$\chi_k''+\left(k^2-\frac{a''}{a}\right)\chi_k=0. \tag{1.2.4}$$

It is thus the variables χ_k which have a consistent interpretation as the independent harmonic oscillators but now with time dependent frequencies $\omega_k^2 = k^2 - a''/a$. We write

$$\chi_k = \frac{1}{\sqrt{2}}\left(v_k a_k + v_k^* a_{-k}^\dagger\right)$$

where, in addition to solving the classical (number valued) version of 1.2.4, the functions v_k must also satisfy the so-called **Wronskian condition**

$$v_k \dot{v}_k^* - v_k^* \dot{v}_k = -2i$$

in order for the commutation relations $[a_k, a_{k'}^\dagger] = \delta_{k,k'}$ to be consistent with the spacetime commutation $[\phi(x), \pi(x')] = i\delta(x - x')$. The unique exponential mode functions associated with the inertial time coordinate in Minkowski spacetime trivially satisfy this extra condition as can be easily checked. However, in general the Wronskian condition is an extra condition on the v_k fixing a choice of normalization. Notice that due to the cosmological assumption of isotropy the mode equation 1.2.4 only depends on the magnitude of 3-vector k. This allows us to make the simplifying isotropy assumption on the functions v_k allowing us later to integrate out the spherical symmetry in integral expressions such as $\int d^3kF(v_k) = \int dk\ 2\pi k^2 F(v_k)$.

It is the time dependence of the frequencies of the constituent harmonic oscillators that is solely responsible for all the non-trivial features of QFT on FL backgrounds. From 1.2.4 it is clear that what at one time is a positive frequency, at another may not be. For example let us specify a scale factor $a(\eta)$ which is constant in the distant past (or past boundary) and distant future (or future boundary) but these constant values are different from one another. Then we construct particular mode function solutions by providing the initial condition that v_k converge to the usual Minkowski mode functions v_k^0 in past boundary. These mode functions are used to define the operators a_k and $a_k^\dagger$ and hence the vacuum state $a_k|0\rangle = 0$. Since the mode equation 1.2.4 is not the Minkowski one, these particular mode functions will not evolve so as to be equal to the Minkowski ones for all time and indeed will be distinct from them after passing through the non-trivial region where the mode equations differ. That is, at the future boundary one would expect the 'state with no particles in it' or vacuum state to be that state defined by the Minkowski mode functions there. But the 'real' state and 'real' particle content is defined by the 'real' mode functions which are distinct from v_k^0 at the future boundary and hence the vacuum state $|0\rangle$ contains particles with respect to an asymptotic observer who is ignorant to the prior period of non-trivial gravitational field. We can easily obtain a quantitative measure of the number of particles present in the state $|0\rangle$ after the expansion has taken place by expanding v_k in terms of the mode function solutions which converge to the Minkowskian modes in the asymptotic future $\tilde{v}_k$ as

$$v_k = \sum_j \alpha_{kj}\tilde{v}_j + \beta_{kj}\tilde{v}_j^*$$

implying a expansion of the creation and annihilation operators

$$a_k = \sum_j \alpha_{jk}^*\tilde{a}_j - \beta_{jk}^*\tilde{a}_j^\dagger.$$

The two choices of mode functions define two different vacuum states $|0\rangle$ and $|\tilde{0}\rangle$ respectively. Unaccelerated observers in the remote past will of course observe no particles since we assume the (Heisenberg) state to be $|0\rangle$. In the remote future however unaccelerated observers will not detect "no particles" since that would correspond to the quantum state being $|\tilde{0}\rangle$. The number of $\tilde{a}_k$ particles present in the state $|0\rangle$ is given by the expectation of the number operator

$$\langle 0|\tilde{a}_k^\dagger \tilde{a}_k|0\rangle = \sum_j |\beta_{jk}|^2.$$

The coefficients α_{ij}, β_{ij} are known as **Bogoliubov coefficients**. One might say that the anomalous particles were 'produced' by the time dependent gravitational field. This effect is a general feature of QFT in curved spacetime where we interpret the (minimal) coupling between gravity and the field φ as producing field excitation when the spacetime is not static in an analogous way to the production of electromagnetic waves by time dependent electromagnetic fields (for an excellent article on exactly this analogy see [32]).

Even more striking than the ambiguousness of the definition of positive frequency, it seems possible that the frequency $\omega_k(t)^2 = k^2 - a''/a$ might become imaginary if a''/a were to dominate k^2 in 1.2.4. It turns out that this is exactly the feature of the theory that allows inflation to explain the large scale structure of the observable universe. As will be discussed below, the dynamics of perturbations in a FL background are described by an equation of motion essentially identical to 1.2.4. Each mode in an inflationary epoch possesses a time beyond which the effective frequency of its harmonic oscillator description is imaginary and hence branches into a decaying and exponentially growing solution. It is this amplification of possibly initially microscopic fluctuations to observable galactic scales that places inflation in a position to explain the large scale structure of the observable universe.

Bunch Davies vacuum in de Sitter and powerlaw backgrounds

In conformal time, both the de Sitter and $p > 1$ powerlaw spacetime scale factors behave as $a(\eta) \to 0$ as $\eta \to -\infty$. This feature allows one to argue for a particular choice of mode function solutions to define the vacuum state which we will call the **Bunch-Davies** vacuum[8]. The argument proceeds as follows: Since the decoupled degrees of freedom of the quantum field are the co-moving modes which expand with the spacetime, in the limit where $a(\eta) \to 0$ the proper wavelength of the decoupled modes becomes negligible with respect to any curvature scale. Thus in this limit the modes behave as though the curvature was negligible and hence as though they were in Minkowski spacetime. We choose the Bunch Davies mode functions to be those mode function solutions to KG which converge to the Minkowski functions in the remote past. Note that this is a different physical criterion than arguing for Minkowski modes based on an asymptotically flat remote past and is applicable to the realistic cosmological models of spacetime evolution based on SBBM and inflation. In de Sitter spacetime 1.2.4 becomes

$$\chi_k'' + \left(k^2 - \frac{2}{\eta^2}\right)\chi_k = 0$$

possessing the general mode function solution

$$v_k(\eta) = A\sqrt{|\eta|}J_{3/2}(k|\eta|) + B\sqrt{|\eta|}Y_{3/2}(k|\eta|)$$

[8] In the literature this is usually the name reserved for the de Sitter spacetime construction only but here we extend the definition to $p > 1$ powerlaw.

where $J_{3/2}$ and $Y_{3/2}$ are the Bessel J and Bessel Y functions of parameter 3/2. The Bunch Davies solution is obtained by imposing the asymptotic behaviour

$$v_k(\eta) \to \frac{1}{\sqrt{\omega_k}} \mathrm{e}^{i\omega_k \eta}, \quad \frac{v_k'(\eta)}{v_k(\eta)} \to i\omega_k \quad as \quad \eta \to -\infty$$

where

$$\omega_k^2 = k^2 - \frac{2}{\eta^2}.$$

In this limit the Bessel functions behave as [3]

$$J_n(\lambda) \to \sqrt{\frac{2}{\pi\lambda}} \cos\big(\lambda - \frac{n\pi}{2} - \frac{\pi}{4}\big), \quad Y_n(\lambda) \to \sqrt{\frac{2}{\pi\lambda}} \sin\big(\lambda - \frac{n\pi}{2} - \frac{\pi}{4}\big).$$

Noting that in this same limit the frequency behaves as $\omega_k^2 \to k$ we find the particular solution

$$v_k = \sqrt{\frac{\pi|\eta|}{2}} \left(J_{3/2}(k|\eta|) - iY_{3/2}(k|\eta|) \right). \tag{1.2.5}$$

For powerlaw, using the mode equation

$$v_k'' + \left(k^2 - \frac{p(2p-1)}{(1-p)^2\eta^2} \right) v_k = 0$$

the procedure proceeds identically, with the result that

$$v_k(\eta) = \sqrt{\frac{\pi|\eta|}{2}} \left(J_n(k|\eta|) - iY_n(k|\eta|) \right), \quad n = \sqrt{\frac{1}{4} + \frac{p(2p-1)}{(1-p)^2}}. \tag{1.2.6}$$

Hence the spacetime dynamics is entirely encoded into the solutions to the KG through the order parameter n of the Bessel function solutions when we assume Bunch Davies boundary conditions. In Figure 1.1 we plot the Bessel function order parameter as a function of the power p of the associated scale factor. We note that this particular solution converges to the de Sitter particular solution (with order parameter $n = 3/2$) in the $p \to \infty$ and $p \to -\infty$ limits. Curiously there is also another value of p for which this solution is exactly the de Sitter solution, namely for $p = 2/3$. This power law spacetime is of physical relevance as it describes a dust dominated solution to the classical EE. Another feature of note is that to each value n of the Bessel parameter there corresponds exactly two powers p (except for the pathological $p = 1$ case) for which the Bessel functions of order n are particular Bunch Davies mode functions. It should be pointed out however that this does not mean that the QFT of the two spacetimes corresponding to a given n are identical since, as we shall see in the next chapter, quantities of physical interest such as the energy momentum tensor of the quantum field involve the scale factor a so that different powers p give rise to physically distinct theories.

A note on zero modes and quantization

The quantization of a field by a direct quantization of its constituent harmonic oscillators has some technical difficulties. Firstly, the effective frequency $\omega_k(\eta)$ can be imaginary indicating a complete breakdown of the particle picture as provided by the operators

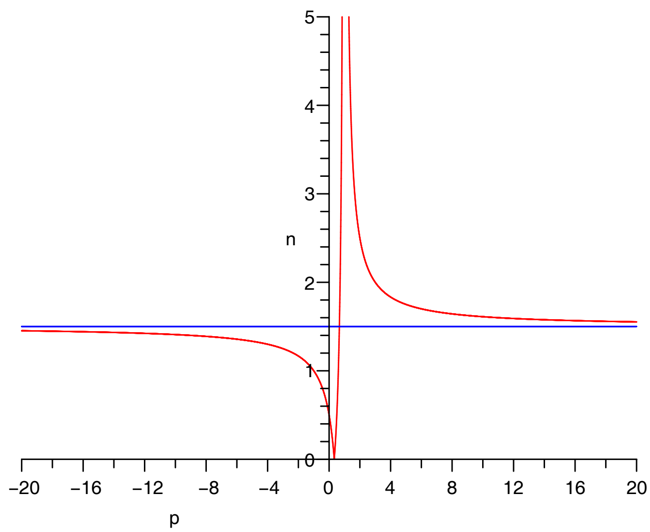

Figure 1.1: The index of the Bessel mode function particular solution specifying the Bunch Davies vacuum in powerlaw spacetime (red) versus the power p. The de Sitter Bessel parameter is displayed as the horizontal asymptote (blue).

a_k and $a^\dagger$ usually associated with the addition of one energy quanta to the state of a harmonic oscillator. It would seem that those oscillators with imaginary frequencies should be quantized by alternative means. However, we take the position adopted in the literature that the mathematical methods of the oscillator quantization - the introduction of the operators a_k taking care of the commutation properties of φ and the functions v_k taking care of the field equation - remain valid even in the case where an oscillator interpretation is unavailable. Secondly (and less problematically) there exists modes for which the frequency vanishes $\omega_k(\eta) = 0$ and hence for which the mode function solutions are linear functions:

$$v_{k^2=a''/a} = A\eta + B$$

since they satisfy $v_k'' = 0$. The reason the zero mode is irrelevant is that zero modes constitute a set of measure zero (at a fixed time) in the set of all modes so that their behaviour does not effect quantities (which we will calculate and utilize later on) such as $\int d^3k\ |v_k|^2$.

A note on $p = 1$ powerlaw

The $p = 1$ powerlaw spacetime seems arise as a special case is almost all of the analysis thus far. It is therefore worth making a study of this particular spacetime and the QFT on it. We shall have more to say about $p = 1$ in a later section on horizons in powerlaw spacetime.

The regularized scale factor for $p = 1$ powerlaw is $a(\eta) = \epsilon e^{\eta}$ where due to a particularity the conformal time in this case $\eta(t) = \ln(t) - \ln(\epsilon)$ is dimensionless. Shifting η

by $\eta \to \tilde{\eta} = \eta + \ln \epsilon$ and dropping the tilde we have $\eta \in (\ln \epsilon, \infty)$ and the mode equation 1.2.4 equation reads

$$\chi_k'' + \left(k^2 - 1\right) \chi_k = 0.$$

Hence the constituent harmonic oscillators possess time independent frequencies just as in Minkowski spacetime. The exact mode function solution satisfying the Wronskian condition is

$$v_k = \frac{1}{(k^2-1)^{1/4}} \mathrm{e}^{-i(k^2-1)\eta}$$

which along with the complex conjugate form a complete basis. It would seem that in this spacetime the issues of the ambiguity of the vacuum state are avoided; what is positive frequency at one time is always positive frequency.

The equation of motion 1.2.2 now has a two parameter family of zero modes given by the boundary of the unit ball in three dimensional k-space. However this is still a set of measure zero for the full k-space.

One interpretation of the absence of particle production is that since the event horizon of powerlaw spacetime expands linearly with t and the co-moving modes with $a(t) = t$ the quantum modes never cross the horizon as they do in de Sitter or $p > 1$ spacetime where the modes expand 'faster' than linearly in t. One might imagine that the co-moving quantum field does not 'know' it lives in an expanding spacetime when expressed in co-moving coordinates since none of the modes ever cross the horizon. Again we regard this case as pathological since it is on the boundary of those spacetimes classified as inflationary and those not. In a realistic transition from inflation to SBBM presumably the scale factor will pass through $p = 1$ power law behaviour but only locally in time avoiding the issues associated with the global $p = 1$ solution.

A note on conformal coupling

Above and elsewhere we have considered a massless minimally coupled scalar field in an external classical gravitational field. It is of interest to study alternative couplings to isolate generic features of such coupled fields. Of particular simplicity is the conformally coupled field possessing action functional 1.2.1 with $\xi = 1/6$ and $m = 0$. This coupling is the one almost exclusively studied in the early papers (see the references quoted in the introduction) in this subject and hence is worth mentioning here.

The Euler Lagrange equation reads

$$(\Box - \frac{1}{6}R)\varphi = 0$$

which is invariant under conformal transformations of the classical metric $\mathbf{g}(x) \mapsto \Omega^2(x)\mathbf{g}(x)$ and simultaneous transformation of the field variable $\varphi \mapsto \Omega\varphi$.

As already discussed, all flat FL models are manifestly conformally flat upon the redefinition of the cosmological time

$$\eta(t) = \int^t \frac{1}{a(t')} dt'$$

where the arbitrary constant is left unspecified in the general case. In particular models the geometry can become singular at finite times and singularities at infinite t may be mapped to finite η times.

By the above discussion we are led to the conclusion that mode function solutions to the conformally coupled wave equation in FL backgrounds are identical to the Minkowski space solutions. One has the time dependent Ricci scalar

$$R(t) = 6\Big[\frac{\ddot{a}}{a} + \Big(\frac{\dot{a}}{a}\Big)^2\Big], \qquad R(\eta) = 6\frac{a''}{a^3}$$

where $\dot{a}$ stands for $\partial_t a$ and a' stands for $\partial_\eta a$. Hence the massless conformally coupled wave equation reads

$$\Big(\Box - \frac{a''}{a^3}\Big)\varphi = 0. \tag{1.2.7}$$

Now,

$$\Box\varphi = -\frac{\varphi''}{a^2} - \frac{2a'}{a^3}\varphi' + \frac{\nabla^2\varphi}{a^2}$$

so that, re-expressing (1.2.7) in terms of the co-moving field $\chi = a\varphi$ we have

$$\Box\varphi = -\frac{\chi''}{a^3} - \frac{a''}{a^3}\chi + \frac{\nabla^2\chi}{a^3}$$

and hence

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\chi = 0 \tag{1.2.8}$$

where $\eta^{\mu\nu}$ is the Minkowski metric. This shows explicitly how the wave equation reduces to the Minkowskian wave equation when the field is conformally coupled to the conformally flat FL spacetime. In this case, like the $p = 1$ powerlaw case there is no trouble with identifying the positive frequency solutions and hence there should be no field excitation for this theory. It is for this reason that we do not study the conformally (or any alternatively) coupled theory in this thesis since, in essence, this choice is merely a mathematical trick which sweeps the issues of the interaction between QFT and non-trivial spacetime evolution 'under the rug' so to speak. Furthermore there have been several studies indicating that non-minimal coupling is an un-physical choice by considering violations of energy conditions in general relativity [18, 33].

1.3 Slow Roll Inflation and cosmological perturbations

In this section we bring together the material of the previous two sections in a discussion of the theory of cosmological perturbations in inflation (for a concise and modern review see [34]). To this end we also elucidate the nature of the so-called **slow-roll** approximation in inflation whereby a classical scalar field is used to provide an epoch of $w \simeq -1$ and hence an approximate de Sitter evolution of the background spacetime.

Perturbations

In order to account for the inhomogeneous structure of the universe one needs to generalize the idealized cosmologies described above to include inhomogeneous matter distributions.

The beauty of the inflationary picture is that these inhomogeneities are not imposed in an ad-hoc way as initial conditions but instead are shown to be a result of taking into account the quantum mechanical behaviour of the relevant perturbative degrees of freedom describing the matter/spacetime system.

The issue of which degrees of freedom one should choose to describe quantum mechanically was a point of debate in the early history of this subject but was solved by Bardeen [35] in 1980. The problem is subtle due to the general covariance of the EE. Let us assume a classical scalar field ϕ^0 coupled to the gravitational field $g^0_{\mu\nu}$. If one wishes to describe small perturbations from homogeneity of the form

$$\phi(t,\mathbf{x}) = \phi^0(t) + \delta\phi(t,\mathbf{x}), \quad g_{\mu\nu}(t,\mathbf{x}) = g^0_{\mu\nu}(t) + \delta g_{\mu\nu}(t,\mathbf{x})$$

and treat them quantum mechanically then one should be sure that all the perturbations that are quantized are 'real' in the sense that they cannot be removed by a suitable choice of coordinates. The following example will help to highlight this issue. Imagine that we add a small perturbation $\delta\phi$ to the field ϕ^0 only, leaving the metric unperturbed. The field ϕ^0 is assumed to be evolving monotonically (say decreasing) with the coordinate t (as is the case in the example of the scalar field supporting the powerlaw background). Then in our co-moving coordinates (see 1.1.1) the previously spatially homogeneous scalar field has a value which varies slightly from place to place on a co-moving spatial section. Let us define a new time variable τ by saying that sections of constant τ are those spatial sub-manifolds (which we will call homogeneous sections) in which the value of the scalar field is constant. In other words we choose to slice the four dimensional spacetime into three dimensional spatial sections such that at a point on the co-moving section where the value of the field is slightly *higher* than the homogeneous value, we choose to slice '*after*' the co-moving section to a time when the field has a lower value, restoring the constant field value. On the other hand we choose to slice '*before*' the co-moving section for points where the perturbed field has a lower value than the homogeneous value. This is all done in a continuous way since we assume the perturbations to be sufficiently smooth. Such a perturbation is clearly un-physical since we were able to completely remove it by a change of coordinates under which, by general covariance, the physics should be invariant. Therefore one requires perturbation variables which are coordinate or **gauge invariant** functions of the field and metric for the classical and especially the quantum theory. Such variables exist and were first described by Bardeen [35] and describe separately scalar, vector and tensor perturbations of the metric/field system. The splitting of perturbations into scalar, vector tensor characters is analogous to the splitting of the electromagnetic field strength tensor into components derivable from the scalar potential, parts derivable from the vector potential and purely tensorial components representing a homogeneous self sustained field configurations (traveling waves). For gravity coupled to one scalar field we write the perturbed metric in full generality as

$$\begin{aligned} ds^2 = a^2(\eta)[&-(1+2\Phi)d\eta^2 + 2(B_{,i} - S_i)dx^i d\eta \\ &+ \left((1-2\Psi)\delta_{ij} + 2E_{,ij} + 2F_{(i,j)} + h_{ij}\right) dx^i dx^j] \end{aligned}$$

where $S_i^{;i} = F_i^{;i} = \delta^{ij}h_{ij} = h_{ij}^{;i} = 0$. Therefore the metric perturbations are described by the scalar functions Φ, Ψ, E and B which contribute (along with the scalar field perturbations discussed shortly) to the scalar perturbations, the vectors $\mathbf{S}$ and $\mathbf{F}$ and the $(0,2)$ tensor $\mathbf{h}$ which describe the vector and tensor perturbations respectively. The gauge invariant variable alluded to above describing the scalar perturbations including the perturbations of the scalar field ϕ itself is given by [34]

$$\zeta(\eta, x) = \left[\delta\phi + \phi_0'(B - E')\left(1 + \left(\frac{a'}{a}\right)^2\right) + \frac{a'}{a}\phi_0'\Phi\right].$$

In the literature [34] it is stated that upon expanding the action to second order in perturbations we obtain the action for perturbations

$$S = -\frac{1}{2}\int d^4x\; z(\eta)^2\eta^{\mu\nu}\zeta_{,\mu}\,\zeta_{,\nu} \tag{1.3.1}$$

followed by the statement that the field equations are

$$(z\zeta)'' - \frac{z''}{z}(z\zeta) - \nabla^2(z\zeta) = 0 \tag{1.3.2}$$

where

$$z(\eta) = \frac{a^2\phi_0'}{\sqrt{2}a'}. \tag{1.3.3}$$

(This is analogous to 1.2.4 where we must scale the physical field $\phi \mapsto \chi = a\phi$ to obtain decoupled harmonic oscillator mode equations.) This action and field equation look very similar to the action for a scalar field on a background FL spacetime with scale factor $z(\eta)$. However, since integration on manifolds only makes sense when one uses the proper integration measure $d^4x\sqrt{\mathbf{g}}$ and that the real background spacetime *is* a FL background with scale factor not $z(\eta)$ but $a(\eta)$ we should really write the action as

$$S = -\frac{1}{2}\int d^4x\;\sqrt{g}\frac{z(\eta)}{a(\eta)}g^{\mu\nu}\zeta_{,\mu}\,\zeta_{,\nu}$$

from which follows the field equation

$$\Box\zeta - \left(\frac{z}{a}\right)_{,\eta}\left(\frac{a}{z}\right)\zeta_{,\eta} = 0$$

by a now legitimate application of the Euler Lagrange equations

$$\nabla_\mu\frac{\delta S}{\delta\nabla_\mu\zeta} = 0.$$

To the author it is unclear what it would mean to quantize such a field equation, and to the authors knowledge all accounts of the quantized scalar perturbations use the 'illegitimate' field equation 1.3.2. We will see in fact that this issue can be partially resolved by assuming a particular form for the background dynamics of the classical field ϕ and hence that of a. This will be the slow roll approximation with *constant* slow roll parameter. We will see that this corresponds to a powerlaw evolution of the unperturbed spacetime.

One important thing to note about the so-called Mukhanov variable $z\zeta$ is that in the limit of de Sitter evolution represented by $\phi_0' \to 0$ or in other words $w \to -1$ for the background homogeneous field, the function $z(\eta)$ vanishes. This means that in de Sitter spacetime the scalar fluctuations vanish (to second order in the action) since their action 1.3.1 is multiplied by the pre-factor z.

Slow roll

The essential assumption of slow roll inflation is an assumption on the dynamics of the spacetime supporting scalar field and reads $V(\phi) >> \frac{1}{2}\dot{\phi}^2$. The terminology comes from the analogy of a ball rolling slowly down a slope of very small gradient - the kinetic energy is small yet the potential is high. If such a condition is satisfied we say that the spacetime is undergoing slow roll inflation. The reason why we add the word inflation to the definition is that during such regimes we have by 1.1.7, $w \approx -1$ and hence approximate de Sitter evolution of the spacetime. Following [34] we define the **slow-roll parameters** ϵ, δ and ξ characterizing the slowness of the roll by

$$\begin{aligned}\epsilon(t) &:= \frac{3}{2}\dot{\phi}^2\left(\frac{1}{2}\dot{\phi}^2 + V\right), \qquad (1.3.4)\\ \delta(t) &:= -\frac{\ddot{\phi}}{H\dot{\phi}}\\ \xi(t) &:= \frac{\dot{\epsilon} - \dot{\delta}}{H}\end{aligned}$$

where in analogy to de Sitter spacetime we make the general definition of the time dependent Hubble parameter

$$H(t) := \frac{\dot{a}}{a}.$$

Essentially, ϵ characterizes the slowness of the roll and δ the slowness of the slow roll. Inflation occurs if $\epsilon \sim \eta << 1$. Thus during slow roll we have by 1.1.7 and 1.1.2 without cosmological constant that

$$H(t) \approx \sqrt{V/3}$$

which might be interpreted as a time dependent Hubble constant; during slow roll the function V varies only very slightly so that H is approximately constant.

The EE for the unperturbed fields 1.1.2 without cosmological constant read

$$3\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{2}\dot{\phi}^2 + V$$

which may be rearranged and written in conformal time[9] to read

$$\phi'^2 = 2\left(\frac{a'}{a}\right)^2\epsilon$$

where $\epsilon(\eta)$ is the first slow roll parameter 1.3.4. That is, we have

$$z(\eta) = a\sqrt{\epsilon}$$

furnishing a consistent interpretation of ζ as a legitimate quantum field defined on the curved background gravitational field with (slightly non-standard) action

$$S = -\frac{1}{2}\int d^4x\sqrt{\mathbf{g}}\epsilon(\eta)g^{\mu\nu}\partial_\mu\zeta\partial_\nu\zeta.$$

[9] Note that $\partial_t = a^{-1}\partial_\eta$.

We are therefore justified in importing the quantization formalism and interpretive results developed in the previous section 1.2.2. The results certainly go through when we assume $a/z = \epsilon \approx$ constant in which case the field equation 1.3is exactly that of a scalar field on the unperturbed background spacetime $\Box\zeta = 0$.

As promised above we examine the case when $\epsilon = c =$ constant. Then it can be shown that

$$H(t) = \mathrm{e}^{\sqrt{\frac{c}{2}}\phi(t)}$$

and consequently that $a(t) = t^{1/c}$ [36] so that we identify the slow roll parameter as the inverse of the powerlaw power $c = 1/p$ when the parameter is constant. We can see that for powerlaw potential and field 1.1.6 the first slow roll parameter becomes

$$\epsilon = \frac{2}{3p}$$

which is small for large p. Recall that for powerlaw $w = \frac{2}{3p} - 1$ so that we conclude that the powerlaw background undergoes slow roll inflation for large p. In this constant ϵ case we have $\frac{z''}{z} = \frac{a''}{a}$ and hence the scalar perturbations obey exactly the equations of motion for a scalar field on the homogeneous FL background with scale factor a.

It is also worth noting that the tensor perturbation automatically satisfy a standard equation of motion without recourse to a slow roll regime [34]

$$h'' - \frac{a''}{a}h - \nabla^2 h = 0$$

where $h := h^{\pm}_{ij}$ are the independent polarizations and tensor components of the tensor perturbations. For these perturbations we therefore also have a straight forward application of the quantization procedure in 1.2.2 describing quantized gravitational waves and that this interpretation is independent on the background dynamics of the gravitational and scalar fields.

Slow roll inflation as a mechanism of structure formation

As we have seen, quantized scalar and tensor perturbations of a FL background coupled to a (classical) scalar field satisfy the quantum KG equation and that the perturbations are only defined for *approximate* (slow roll) de Sitter expansion. Therefore we are led by Heisenberg uncertainty for the quantum modes which, crucially, contain metric degrees of freedom, to the conclusion that exactly FL geometry is impossible if we are to take seriously the quantum behaviour of the fields into account. Further, one can make definite predictions on the character of perturbations based on the quantum theory. One experimental window on the inhomogeneities and anisotropies of the physical universe is through our measurement of the primordial CMB radiation. It is observed that the CMB is subtly anisotropic across the sky and that the correlation of the temperature fluctuations as a function of subtended angle follows the characteristic (and by now iconic) shape 1.2. This correlation is directly calculable from the quantum two point function $G(x, x') := \langle 0|\varphi(x)\varphi(x')|0\rangle$, where $|0\rangle$ is the assumed Bunch Davies vacuum, and the knowledge of the detailed relativistic hydrodynamics of the matter during the subsequent evolution. The n-point functions of a quantum field theory are directly interpreted as

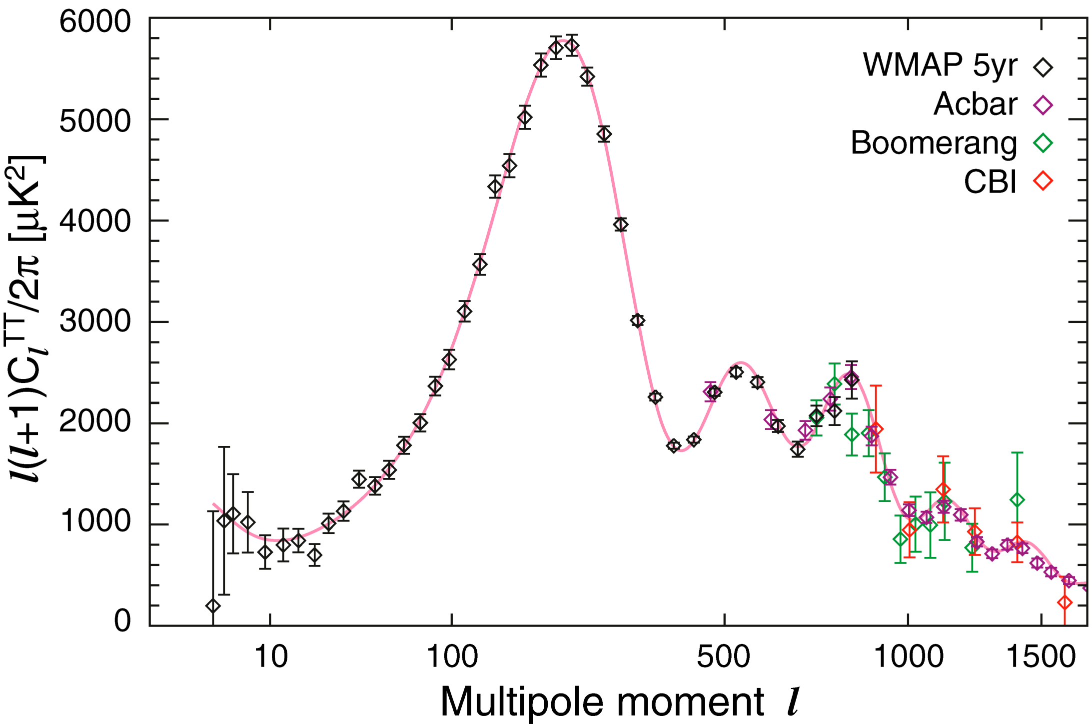

Figure 1.2: The theoretical curve predicted for the angular dependence of the correlations of temperature fluctuations with some experimental data superimposed. (Taken from the NASA web site LAMBDA (Legacy Archive for Microwave Background Data Analysis) and due to the WMAP science team [37]).

correlations between the different realizations of the probabilistic field amplitudes among the ensemble of all possible fields. This interpretation is most transparent in the path integral formulation of QFT where the n-point functions are literally weighted sums (integrals) over all the possible fields of the product of the field amplitude at each of the n points where the n-point function is being evaluated:

$$\langle \varphi(x)\varphi(x')\rangle := \int_{\{\varphi\}} \varphi(x)\varphi(x')\mathrm{e}^{-iS[\varphi]}.$$

Here the weight given to each history $\varphi(x)$ is given by the exponential of the evaluation of the action functional on that trajectory (times i).

However, since quantum fluctuations are naturally microscopic in origin, how is it that they are responsible for such macroscopic effects as CMB fluctuations? The answer is in the KG equation. Assume that sometime during its evolution a quantum mode φ_k, associated with the scalar perturbations for example, becomes classical $\varphi_k \to \phi_k$ by some process such as decoherence. Then the perturbation evolves according to the classical KG equation (since that is what is assumed in the quantization process to begin with!)

$$\phi_k'' + \left(k^2 - \frac{a''}{a}\right)\phi_k = 0, \qquad a''/a \propto 1/\eta^2 \text{ for powerlaw.}$$

For late times in powerlaw, the k^2 term becomes negligible and the 'harmonic oscillators' evolve into the region where their frequencies become imaginary. In this limit the KG equation becomes

$$\phi_k'' \propto \frac{\phi_k}{\eta^2}$$

possessing the general solution

$$\phi_k = A_k|\eta|^{n_+} + B_k|\eta|^{n_-}, \quad n_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{p(2p-1)}{(1-p)^2}} = \frac{1}{2} \pm n.$$

where n was defined previously as the order parameter of powerlaw Bessel exact mode function solutions (using the Bunch Davies vacuum). Indeed, as will also be discussed in the next chapter, the exact Bessel function solutions possess the asymptotics

$$v_k = \sqrt{\frac{\pi|\eta|}{2}}\left(J_n(k|\eta|) - iY_n(k|\eta|)\right) \sim \sqrt{\pi|\eta|}\left((k|\eta|)^n - i(k|\eta|)^{-n}\right) \quad \text{as } \eta \to 0^-. \quad (1.3.5)$$

In this limit the smaller exponent dominates giving rise to the runaway solution. Defining the averaged field on the region $V = L^3$ as $\varphi_V := \int_V d^3x\ \varphi(x)$ and fluctuation spectrum as

$$\delta_L := \langle 0|\varphi_V^2|0\rangle = \int_V d^3x \int_V d^3x' \langle 0|\varphi(x)\varphi(x')|0\rangle$$

it is easy to show that

$$\delta_L(k,\eta) \approx a^{-1}(\eta)k^{3/2}|v_k(\eta)|.$$

It is the function $\delta_L(k\eta)$ that characterizes the magnitude of perturbation of the mode v_k on scales of magnitude L and should be evaluated at some time η_f at the end of inflation when the universe settles down to a regular matter dominated FL spacetime. Hence using 1.3.5 and the expression for the scale factor $a \propto |\eta|^{p/(1-p)}$ we see that

$$\delta_L \propto |\eta|^{\frac{1}{2}-\frac{p}{1-p}-\sqrt{\frac{1}{4}+\frac{p(2p-1)}{(1-p)^2}}} k^{\frac{3}{2}-\sqrt{\frac{1}{4}+\frac{p(2p-1)}{(1-p)^2}}} \approx 1 \quad \text{for large } p.$$

This is one of the most important results of inflation - that *the spectrum of fluctuations is independent of both the co-moving wavenumber and time.* This is known as the **scale invariant spectrum**. This prediction rests strongly on the assumption that the background is evolving as a high p powerlaw.

An intuitive understanding of why the fluctuation spectrum becomes constant on large proper scales is found in the presence of the cosmological horizon. In fact for co-moving modes $k = 2\pi/\lambda$, where λ is the co-moving wavelength, of size approximately the co-moving event horizon scale

$$\lambda \simeq \frac{t^{1-p}}{p-1}$$

we have

$$|\eta|k = \frac{2\pi}{\lambda}|\eta| = \frac{a^{1/p-1}}{(p-1)\lambda} = \frac{2\pi t^{1-p}}{(p-1)\lambda} \simeq 1$$

so that precisely where the frequency $\omega_k^2 = k^2 - A_p/\eta^2$ becomes imaginary the mode crosses the event horizon. It is then clear that such modes should not oscillate as usual and should 'freeze out' since the wave is not in causal contact with parts of itself!

1.4 The state of the art

As an epilogue to the picture we have painted of the physical universe above we mention here a few problems with the standard model of inflation (involving a classical scalar field with potential) as well as some recent developments stemming from some rather exciting observations made in the late 1990's.

The inflationary hypothesis is a very economic one as described above. Assuming the most general case of slow roll inflation the only input to the theory is the classical potential function $V(\phi)$. However this function needs to be of a rather specific form in order to produce inflation, namely it is required to be slowly rolling and of large amplitude, and it is unclear why nature would choose such particular dynamics. Furthermore it is observed today that the universe is entering into a second inflationary era [38]. This is essentially unexplainable by slow roll inflation where the scalar potential is usually assumed to 'settle down' into a ground state at the end of inflation. As we will discuss in Chapter 3 a much more satisfactory interpretation of inflation is as a dynamical response of the spacetime to the presence of vacuum energy. Another problematic feature of standard inflation is that it makes any reference to classical matter at all in a regime of energies that are supposed to exceed even the highest achievable in modern particle accelerators. Again a purely quantum mechanism for inflation would be more satisfactory. Also, throughout the analysis of the generation of structure by quantum perturbations no reference is made of the energetic backreaction of the quantum field upon the background spacetime. In effect the quantum field fluctuations (when the field is in the ground state) are assumed to be 'test fields' providing perturbations of negligible energetic content. In the next chapter we will study precisely this energetic content and show that it is far from behaving like a test field, possessing non-trivial dynamics (time dependence) and large competitive effects between the ultraviolet and infrared which should be taken into account in any complete inflationary theory.

Chapter 2

The Dynamics of Vacuum Energy

The specific problem we wish to discuss in this chapter is the generation or dilution of the vacuum energy[1] in an expanding spacetime as the interplay between the incorporation of new degrees of freedom at a shortest length scale and the dilution of energy as degrees of freedom expand. As exemplified by the theory of inflation, the physics of an expanding spacetime is highly non-trivial while on the other hand QFT, in particular the existence of inexorable vacuum fluctuations, is undeniably novel in its own right. One thus expects some very interesting physics to arise in the interplay between the physics of an expanding spacetime and the physics of vacuum energy. The physical picture that we study is simple: assuming a smallest meaningful length in nature renders the vacuum energy ultraviolet finite and allows one to study the evolution (that is production or dilution) of the total vacuum energy as spacetime expands. In this way one may study the energetic favourability or otherwise of the very expansion of space itself.

Such studies of the dynamics of vacuum energy in curved spacetime are rare in the literature since in order to even begin the discussion authors must be willing to break (local) Lorentz invariance by imposing an ultraviolet (UV) cutoff to render the energy finite or else develop sophisticated (and perhaps obscuring) renormalization techniques. For this reason we motivate carefully our starting assumption of a shortest length cutoff. We argue that, regardless of the expectation that non-trivial quantum gravity effects are expected to become pertinent before we reach the scale where traditional QFT breaks down, a picture of at least how the character of the vacuum energy is modified on a time dependent expanding background space for intermediate energies is useful. Further, we establish and discuss at the end of this chapter a new class of infrared (IR) divergences in the vacuum energy in a class of powerlaw expanding spacetimes. These results essentially justify our 'sweeping under the rug' of the physics in the UV by displaying the significance of the IR physics of vacuum energy. After discussing and justifying a further cutoff, an infrared cutoff, the stage will then be set for a study and interpretation of the results which will be the subject of the following chapter where we interpret the vacuum energy dynamics as that vessel which supports the background spacetime solution on which the field whose vacuum energy we calculate is defined. In this way we begin to address the problem of the quantum **backreaction** on the classical spacetime

[1] At this point the term 'vacuum energy' has not been defined. Below we will adopt various positions on such a definition which all essentially relate to the zero point fluctuations inherent in particle quantum mechanics.

usually embedded within an effective action framework and, in a more speculative vein the proposal of induced gravity where the dynamics of gravity itself is interpreted as a geometrical dependence and evolution of quantum vacuum energy alone (discussed in the following chapter).

After carrying out an exact calculation on the full four dimensional cosmological spacetimes we investigate some simplified models of quantum field theory on an expanding background that possess to increasing degrees of sophistication essentially the same calculation to elucidate the dependence of the results on the various complexities involved in the full cosmological calculation. Initially we write down a discrete model of an expanding piece of string and then move on to low dimensional examples of continuum field theory. In all cases we are concerned with the dynamical evolution of the vacuum energy as an interplay between the incorporation of new degrees of freedom at the shortest length scale and the dilution of energy as degrees of freedom expand.

2.1 Vacuum energy and minimal lengths

It is one of the more interesting predictions of quantum mechanics (QM) that all quantum systems undergo inexorable energetic fluctuations on all scales. The fact that $H|0\rangle = \frac{1}{2}\hbar\omega$ has far reaching consequences in one particle QM not the least that of the stability of atoms. Above we saw how a quantum field may be interpreted as a collection of ordinary quantum harmonic oscillators. The property of a quantum field that all of its constituent harmonic oscillators possess a minimal energy is as necessary a consequence of the principles of QM as it is a paradox and is known as the **ultraviolet divergence** of the **vacuum energy** of the quantum field. For an excellent review article on the mysterious and counterintuitive properties of the quantum vacuum see [39].

To get a handle on the nature of these fluctuations we present the following argument found in [3]. In the schrodinger picture in the ground state each harmonic oscillator mode φ_k of φ will be described by the wave function of the number valued eigenvalues ϕ_k of the operators φ_k

$$\psi_k = \left(\frac{k}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{k\phi_k^2}{2\hbar}\right)$$

possessing an uncertainty in the value ϕ_k of the field

$$\delta\phi_k := \sqrt{\langle\phi_k^2\rangle} \sim k^{-1/2}.$$

Hence for the averaged field in a region $V = L \times L \times L$,

$$\varphi_L = \frac{1}{L^3}\int_V \phi d^3x$$

we have

$$\delta\phi_L^2 = \langle\varphi_L^2\rangle = \int \frac{d^3k}{(2\pi)^3}\delta\phi_k^2 F_k^2$$

where $F_k = \prod_i 2/(k^i L)\,\sin(k^i L/2)$ is negligible for $|kL| \gg 1$. Approximating $\delta\varphi_k$ to be constant on the box we obtain

$$\langle\varphi_L^2\rangle \sim L^{-3}\delta\varphi_k^2|_{k=L^{-1}}$$

and finally

$$\delta\varphi_L \sim \sqrt{\frac{1}{kL^3}} = \frac{1}{L}.$$

The moral of this story is that the fluctuations become arbitrarily large on short length scales and are arbitrarily suppressed on large length scales. The picture that the severity of these quantum fluctuations depends on the length scale on which they act is something that does not generalize to expanding, and in particular inflationary spacetimes as described in the first chapter. However on very short length scales where the non-triviality of the background geometry should be negligible this spectrum should be accurate.

The point here, however, is that since quantum theory itself does not contain an intrinsic length scale below which the picture of fluctuating quantum modes breaks down, we are led by the theory to accommodate for arbitrarily large fluctuations of energy on extremely small length scales. It is clear that the total ground state energy which is the sum of all the constituent harmonic oscillator ground state energies will be badly divergent. This is why we call this the ultraviolet divergence.

A simple example here illustrates the essential issues of the ultraviolet problem. Consider a piece of string of arbitrary length and subject to Dirichlet boundary conditions. In the classical description of a string in arbitrary motion, one finds that the dynamics reduces to a countable family of harmonic oscillators due to the property of superposition for waves. The exact excitement of each independent mode solution is found directly from the initial condition and the conservation of energy. Thus, in classical mechanics as long as one chooses a state of finite energy initially, the energy is rendered finite for all time. In contrast, given that each mode is an independent degree of freedom, the Heisenberg uncertainty principle tells us that each and every mode comes with a finite ground state energy in *any quantum state*. That the total zero point energy is infinite is merely a problem of the continuous nature of the mathematical model of the string since in practice one expects a smallest possible wavelength mode allowable by, for example the atomic nature of the string or the finite thickness of the string. Hence one expects the ground state energy of the piece of string to be the sum of only those modes up to the minimal wavelength and moreover be finite. It is for this reason that pieces of string (which are presumably subject to the laws of quantum mechanics even at macroscopic scales) do not possess an infinite energy!

As with the piece of string above, clearly the problem with the derivation of the result that a quantum field possesses an infinite energy due to arbitrarily large quantum fluctuations on arbitrarily short length scales is in the expectation that the principles of QM and our description of fields holds unmodified all the way to infinitesimal length scales (the high frequency modes in the sum). This is a more fundamental obstruction then the finite thickness of a physical piece of string since the quantum fields of nature are supposed to be fundamental in every sense of the word. In fact, there is good reason to believe that this expectation is naive. It is well known that by applying the principles of QM to the gravitational field one reaches the conclusion that there should be a smallest measurable distance in nature [40] and, further, the quantum ground state fluctuations of the gravitational field will render meaningless a continuum picture of geometry on the smallest length scales [41]. There seems to be two ways out of this conundrum: Either ignore the vacuum energy and argue that only energy *differences* are relevant for physics or else seek to describe the modifications required of either continuum QFT or GR at very short length scales. The first of these two options is the course taken by high energy physicists who do their calculations on flat spacetime. One might argue that in GR, *all* energy gravitates so that this tactic will not be applicable when gravitational effects need to be accounted for. The second option leads one to speculate about the nature of the unfinished theory of quantum gravity and how might QFT be modified when quantum gravity cannot be ignored.

In [42] a modification to the first quantized canonical commutation relations is made which when second quantized yields a field equation that incorporates the notion of a minimal length into its formulation. The essential idea of [42] is to modify spacetime commutation relations such that the modified Heisenberg uncertainty relation states that there exists a minimal uncertainty in spatial position. Brought to the second quantized theory this modification implies a smallest possible proper wavelength for composite modes of a quantum field and hence a cutoff on the mode sum of ground state energies. When the background spacetime is expanding this implies that each mode comes with a creation time when its proper wavelength exceeds the cutoff length and hence a continual generation of vacuum energy is associated with each newly generated mode. In [43] it is found that the vacuum energy contribution of such a modified field is divergent at the generation time and hence the question of the net generation of vacuum energy is unanswerable within that framework. Although the model of [42] is the one of the very few consistent and complete mechanisms proposed in the literature that incorporates an ultraviolet modification to QFT it is probably does not represent accurately the real physics in the ultraviolet on which the full dynamics of the vacuum energy rests strongly.

In [44] a model of expanding spacetime as a growing lattice is presented. The authors use the model to study the problem of the generation of vacuum energy in the presence of a minimal length scale in nature in much the same spirit (but using very different methods) of this thesis. In the next section we present a similar (but significantly simplified) discrete model of dynamical vacuum energy. It would be interesting to compare these two models in higher dimensions.

A very different proposed mechanism for avoiding the UV problem is the freedom in principle to modify the high energy behaviour of the dispersion relation usually assumed, and experimentally confirmed at low energies to be $w_k = k$. Such modifications come from a desire to cure the ultraviolet divergences of QFT without introducing discrete structures or minimal lengths. Further, such modified dispersion relations are a phenomenological way to model expected modifications to QFT arising from the quantum behaviour of spacetime itself in the extreme ultraviolet where we expect the modes of a quantum field to be interacting strongly with the gravitational quanta and be dispersive there. Several modified dispersion relations have been proposed in the literature usually cast in the framework of the trans-Planckian problems of Hawking evaporation and inflation (see [45] and [46] for example). We will not go into these details here and defer such considerations to future study.

In the rest of this chapter we take the position already presented in the introduction that although ultraviolet (or quantum gravitational) physics might have significant dynamical effect on the intricate details of the energetics of inflation, still there exists a meso-scale (from sub-Planckian to galactic scales) of energies associated with a quantum field that could have significant cosmological effects. This is an extremely large energy range encompassing all of the energies tested experimentally by terrestrial experiments, within which we can be sure to trust QFT and GR and hence the conclusions drawn from their (at times awkward) union.

2.2 A Discrete Model of Dynamical Vacuum Energy

Firstly, in this section we consider a simple model of a finite collection of harmonic oscillators as a stripped down model of a real quantum field in curved spacetime. We study the energetics of the ground state fluctuations when we let the number of oscillators be variable.

Consider a collection of independent quantum mechanical harmonic oscillators with fundamental frequencies ω_k and the corresponding total zero point energy

$$E(t) = \sum_k \frac{1}{2}\hbar\omega_k.$$

This sum is clearly dependent on the number of oscillators which we allow to depend on time as

$$E = \frac{1}{2}\hbar \sum_{k=k_0}^{a(t)\Lambda} \omega_k$$

where $a(t)$ is a (discrete[2]) time dependent factor. The sum is to be interpreted as the sum of the zero point fluctuations of the independent modes on a piece of string with a time dependent upper bound on the contributing wave numbers. Thus we form the energy density by dividing by the length of the string which we take to be expanding as $L(t) = a(t)L_0$;

$$\rho(t) = \frac{1}{L(t)}E(t) = \frac{1}{a(t)L_0}\frac{1}{2}\hbar \sum_{k=k_0}^{a(t)\Lambda} \omega_k.$$

where here and below, the sum is to be taken over the discrete modes associated with the finite string length $k = n\pi/L(t)$, $n \in \mathbb{N}$. Then we have

$$\begin{aligned}
\rho(t) &= \frac{\hbar}{2L(t)} \sum_{n=0}^{n=\Lambda L(t)/\pi} \frac{n\pi}{L(t)} \\
&= \frac{\hbar}{2L(t)^2} \sum_{n=0}^{\Lambda L(t)/\pi} n \\
&= \frac{\hbar}{2L(t)^2} \times \frac{1}{2}(\Lambda L(t)/\pi)\left((\Lambda L(t)/\pi) + 1\right) \\
&= \frac{\hbar\Lambda^2}{\pi^2} + \frac{\hbar\Lambda}{\pi L(t)} \qquad (2.2.1)
\end{aligned}$$

Hence the energy density is time dependent but converges to a constant at late times. The decaying term is an interesting modification to the expected quadratic[3] (in Λ) result due to the presence of the time dependent frequencies.

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[2]We remark here that in practice the upper limit of the sum is taken to be very large as to render the 'scale factor' $a(t)$ effectively continuous.

[3]This term is the analogue of the familiar quartic ultraviolet divergence in Minkowski space QFT.

Let us derive this result in a different way which highlights the mechanisms at work. This time we use the expanding piece of string and a discrete time evolution whereby the time steps are such that during one time step the string expands by one unit of the fundamental length Δ.

Consider when the string length L is one cutoff length Δ, $L = \Delta$. Then there is only one contribution to the energy and it is given by

$$E_0 = \frac{1}{2}\hbar\frac{2\pi}{\Delta} = \frac{\hbar\pi}{\Delta}.$$

After the first time step there will be two contributions from the two allowed normal modes that fit on the string

$$E_1 = \frac{\hbar\pi}{\Delta}\left(1 + \frac{1}{2}\right) = \frac{3}{2}\frac{\hbar\pi}{\Delta}.$$

After each step, the sum should include one extra term which will be the new term arising from the incoming smallest mode. In addition, the previous terms all get multiplied by a factor of $(N-1)/N$ since the existing modes wavelengths all stretch by the factor $N/(N-1)$. Notice that, by tracing its history, a mode that came in (as a minimal wavelength / maximal momentum mode) at the nth step would contribute an energy at the $N > n$th step proportional to the stretched momentum

$$k_n^N = \frac{N-1}{N}k_n^{N-1} = \frac{N-1}{N}\frac{N-2}{N-1}k_n^{N-2} = \cdots = \frac{N-1}{N}\frac{N-2}{N-1}\cdots\frac{n}{n+1}k_{\max} = \frac{n}{N}k_{\max}.$$

Hence we express the total energy at the Nth step as

$$E_N = \frac{1}{2}\hbar\sum_n^N k_n^N = \frac{1}{2}\hbar\sum_{n=1}^N \frac{n}{N}k_{\max} = \frac{\pi\hbar}{N\Delta}\sum_{n=1}^N n = \frac{\pi\hbar}{N\Delta}\times\frac{1}{2}N(N+1) = \frac{\pi\hbar N}{2\Delta} + \frac{\pi\hbar}{2\Delta}.$$

We see that the total energy diverges for large time steps N. However, the energy density ρ, given by the total energy divided by the length of the string length $L = N\Delta$ is

$$\rho = \frac{\pi\hbar}{2\Delta^2} + \frac{\pi\hbar}{2\Delta L}$$

which agrees with the first calculated result 2.2.1.

We interpret the result as implying the almost exact cancelation of the ground state energy production due to the new modes entering the sum and the dilution of ground state energy of each mode as its frequency decays with the expanding string.

2.3 Vacuum energy in QFT

2.3.1 Measures of Field Excitation

As we saw in the first chapter, in curved spacetime the choice of basis functions that we choose to expand a quantum field in terms of is arbitrary. This was equivalent to arbitrariness of the 'particle content' of a quantum state as made explicit by the Bogoliubov transformation coefficient β. In certain idealized spacetimes there exists natural definitions of what the particle content should be in asymptotic regions such as FL models with $a(\eta) \rightarrow$ constant. However neither does the physical universe possess such a region nor do we even trust the measure of field excitation based on Bogoliubov transformations as in 1.2.2. This is particularly pertinent in the light of the fact that there exist non-trivial Bogoliubov coefficients between two sets of mode functions that are solutions to the same KG equation written in different coordinates on the same background manifold. After all, these coefficients merely represent a change of basis for the space of functions assumed for the physical model. We therefore look for more objective measures of field excitation.

One such measure is the expectation value of the energy momentum tensor. Certainly this object depends on the choice of coordinates of the background spacetime but the tensorial structure of the energy momentum tensor is related directly to the geometry via the EE giving it an objective significance. More importantly, however, is the expectation that is should be $\langle \hat{T}_{\mu\nu} \rangle$ that describes the classical energy momentum associated with a quantum field[4]. This expectation is motivated by the Ehrenfest theorem for quantum mechanics [47] which states that the expectation values of quantum observables follow the same equations of motion as the classical counterparts. Thus if $G_{\mu\nu} = T_{\mu\nu}$ for classical scalar fields then $G_{\mu\nu}$ should be equal to $\langle \hat{T}_{\mu\nu} \rangle$ for quantum fields. At this point we will not be concerned with solving the equation $G_{\mu\nu} = \langle \hat{T}_{\mu\nu} \rangle$ for the metric but use it only as a justification for the validity of the calculations of $\langle \hat{T}_{\mu\nu} \rangle$ below.

2.3.2 Vacuum energy on realistic expanding spaces

In the following we calculate the vacuum energy density in de Sitter and powerlaw backgrounds in 3+1 dimensions by computing the expectation value of $\hat{T}^0_0$ and discuss several issues that arise in the analysis including a new class of infrared divergences not found in the literature.

De Sitter spacetime - a consistent and exact result

We showed in 1.2.2 how to solve the KG equation in de Sitter space-time for the operator field $\chi_k = a(\eta)\varphi_k$, where $\varphi(x) = \int d^3k(2\pi)^{-3/2}\mathrm{e}^{ikx}\varphi_k$ is the physical scalar field where we use conformal time η and co-moving modes k. We used the Bunch Davies boundary conditions in order to select a (physically well motivated) vacuum state. The mode functions v_k for each k are expressed as a linear combination of Bessel functions of the variable $\lambda = k|\eta|$,

$$v_k = \sqrt{\frac{\pi\lambda}{2k}}\Big(J_{3/2}(\lambda) - iY_{3/2}(\lambda)\Big) \tag{2.3.1}$$

where $\chi_k = 2^{-1/2}(a_k v_k + a^\dagger_{-k} v^*_k)$. These half integer Bessel functions are expressed in terms of elementary functions as

$$\begin{aligned} J_{3/2}(\lambda) &= \sqrt{\frac{2}{\pi\lambda}}\Big(\frac{\sin\lambda}{\lambda} - \cos\lambda\Big) \\ Y_{3/2}(\lambda) &= \sqrt{\frac{2}{\pi\lambda}}\Big(-\sin\lambda - \frac{\cos\lambda}{\lambda}\Big). \end{aligned}$$

[4]By which we mean the gravitational influence of such energy momentum.

We list here some expressions involving these mode functions that will be needed in later calculations

$$|v_k|^2 = \frac{1+k^2\eta^2}{k^3\eta^2}$$
$$|v'_k|^2 = \frac{1-k^2\eta^2+k^4\eta^4}{k^3\eta^4}$$
$$v'^*_k v_k = \frac{1-ik^3|\eta|^3}{k^3|\eta|^3}$$

and of course we have that these mode functions satisfy the standard Wronskian condition

$$v'_k v^*_k - v_k v'^*_k = 2i.$$

Using the form of the expectation value of $\hat{T}^0_0$ B.0.3 derived in the appendix valid for general FL spacetimes

$$\rho := \langle \hat{T}^0_0 \rangle = \frac{1}{4}\frac{a^{-4}}{(2\pi)^3}\int d^3k\ |v'_k|^2 + \left(k^2 + \frac{a''}{a}\right)|v_k|^2 - \partial_\eta\left(\frac{a'}{a}|v_k|^2\right),$$

the de Sitter scale factor $a(\eta) = -1/H\eta$ and the mode functions 2.3.1 one finds

$$\rho = \frac{H^4\eta^4}{8\pi^2}\int_0^\infty dk\ 2k^3 + \frac{k}{\eta^2}.$$

Here we have moved to polar coordinates to impose the isotropy of mode functions that until now has merely been utilized in simplifying calculations.

It remains to impose a proper momentum cutoff to tame the ultraviolet divergence[5] at $k_{max} = \Lambda a(\eta)$ where Λ is a fixed proper momentum value. That is, we cut off the integral for those co-moving modes satisfying

$$k > a(\eta)\Lambda$$

which will depend on time through the scale factor since co-moving modes expand with the spacetime.

The energy density then evolves as

$$\rho(\eta) = \frac{\Lambda^4}{16\pi^2} + \frac{\Lambda^2}{16\pi^2 a^2\eta^2} \tag{2.3.2}$$
$$= \frac{\Lambda^4}{16\pi^2} + \frac{\Lambda^2 H^2}{16\pi^2} \tag{2.3.3}$$

which is independent of time. This result is somewhat expected since de Sitter space-time is time translation invariant. The result 2.3.3 is an finer characterization of the vacuum energy density over the Minkowski quartic Λ^4 divergence usually quoted in the literature. This result is also robust against a modified dispersion as we show in Appendix C

Here we have seen that by interpreting the ultraviolet cutoff as a real constraint arising, as discussed above, from more fundamental quantum gravity considerations, we have

[5]Or alternatively this cutoff might be interpreted as only taking into account a 'meso-scale' of ground state fluctuation of the co-moving constituent harmonic oscillators.

found a quadratic divergence also that depends on the Hubble parameter. This result is also contrary to the results of [48] where infrared divergences are found using Greens function and renormalization techniques. A possible explanation is that the renormalization, usually utilized in taming ultraviolet divergences, also modifies the infrared limit of the theory. We will see below that the situation is drastically different in powerlaw spacetime.

Powerlaw spacetime - a class of infrared divergences

Apart from ultraviolet divergence, quantum fields, usually only massless fields, are sometimes subject to infrared divergences. These divergences are associated with long wavelengths and are present even after the ultraviolet divergences are regulated. A small class of such infrared divergences have been studied in [49] where, using different methods to here, the authors find an infrared divergence in the energy momentum tensor generically for powers $2/3 \leq p \leq 2$. To this end we wish to study the behaviour of a scalar field in a powerlaw background spacetime with metric given as above by

$$ds^2 = -dt^2 + t^p d\mathbf{x}^2.$$

We interpret this spacetime as a justifiable approximation to full exponential expansion (de Sitter spacetime) during slow roll expansion and the field to be either quantized gravitational waves or quantized scalar perturbations. Therefore we will mainly be interested in results that apply for $p > 1$ but for completeness we discuss the powerlaw spacetime for all positive powers which includes some special cases that highlight some interesting points.

Using again the result from the appendix B.0.3 and the exact Bessel function solutions 1.2.6 it is found that ρ is infrared divergent numerically (using Maple) for powers $p \geq 2/3$. The cases of $p = 2/3$, 2, 3/2 and 4/3 are analytically calculable (all powers of the form $(s+1)/s$ give rise to half integer Bessel function solutions for which the results are analytically calculable) where the mode functions take the form of half-integer Bessel functions expressible in terms of elementary functions. For example we have for $p = 2/3$

$$v_k(\eta) = \sqrt{\frac{\pi|\eta|}{2}} \left(\frac{\sin k|\eta|}{k|\eta|} - \cos k|\eta| + i \left(\sin k|\eta| + \frac{\cos k|\eta|}{k|\eta|} \right) \right) \tag{2.3.4}$$

as in 2.3.1 which is integrable (in the variable k) in closed form. For this solution one calculates

$$\rho^{(2/3)} = \frac{1}{4} \frac{6561}{16\pi^2\eta^8} \int dk \; 2k^3 + 4\frac{k}{\eta^2} + \frac{9}{k\eta^4} \tag{2.3.5}$$

which clearly diverges in the infrared due to the inverse power of k in the third term. Somewhat miraculously the $p = 2$ result has the same integrand,

$$\rho^{(2)} = \frac{1}{4} \frac{1}{16\pi^2\eta^4} \int dk \; 2k^3 + 4\frac{k}{\eta^2} + \frac{9}{k\eta^4} \tag{2.3.6}$$

also possessing the described infrared divergence. It would be interesting to investigate this seeming coincidence further using some more powerful computational techniques.

For arbitrary powers one must resort to an asymptotic analysis of the Bessel functions. For small arguments we have [49]

$$J_n(x) \sim x^n \quad \text{for } x \to 0$$
$$Y_n(x) \sim \frac{1}{x^n} \quad \text{for } x \to 0$$

so that the well known and only point $x = 0$ of non-analyticity of the Bessel functions is displayed through the Bessel Y function. This behaviour is confirmed in the explicit solution given above 2.3.4. We see that in terms of $|v_k|^2$ alone

$$k^2|v_k|^2 \sim k^{-2n+2} \tag{2.3.7}$$

so that for $-2n + 2 < -1$, or in other words $n > 3/2$, the infrared divergence of the integrated quantity is explicit. Similarly since the divergent part of $|v'_k|$ behaves as

$$|v'_k| \sim -\frac{1}{2}\frac{Y_n}{\sqrt{2\pi|\eta|}} - \sqrt{\frac{\pi|\eta|}{2}}\left(-kY_{n+1} + \frac{nY_n}{|\eta|}\right)$$

we see the same infrared behaviour in this derivative term

$$|v'_k|^2 \sim k^{-2n}$$

signifying a divergence for $n > 3/2$. This range corresponds to powers $p > 2/3$ as already mentioned. Only the high p part of this result will be of interest in what follows since we will be focusing on the realistic inflationary scenario.

It is important to keep in mind that this divergence as well as the ultraviolet divergence is a divergence of a *local density*; the infrared divergence of the energy *density* ρ cannot be attributed to the infinite volume of space. However, certain separate issues arise in the spatially compact case (the $\kappa = 1$ version of the FL geometry 1.1.1). In that case the spectrum of the spatial Laplacian is discrete and the mode integrals become mode sums possessing a finite minimal value for the offending momentum variable k corresponding to the first harmonic of the three sphere. All the analysis carries through identically apart from the addition of this discreteness. In this way the energy density of a spacetime with compact spatial sections is never infrared divergent and has been studied in the conformally coupled case in [50]. For example the corresponding result to 2.3.5 for $k = 1$ would be

$$\rho \propto \sum_n 2k_n^3 + 4\frac{k_n}{\eta^2} + \frac{9}{k_n\eta^4}$$

where n is a index that runs over the discrete spectrum of ∇ on S^3. Again we cite the experimental observation of almost exact spatial flatness as motivation for not considering closed spatial geometry in this thesis. Furthermore since mode sums are not in general expressible in closed form in contrast to mode integrals we stick to the spatially infinite case where the spectrum of the Laplacian is continuous.

It is interesting to note that not only does the limit $k|\eta| \to 0$ say something about the infrared behaviour of the theory but also about the late time behaviour since for $p > 1$, the infinite future is represented by $\eta \to 0^-$. The quantum field theory seems to tie up the combination $k|\eta|$ in interesting ways; the physics of late times is related to the physics of large scales when spacetime is expanding. That is, we may conclude that there exists a late time divergence of the vacuum energy density even after taking into account the infrared divergence. However, this conclusion is not as robust due to the non-trivial time dependent factors that we neglected in the expansion 2.3.7 and indeed, one would not necessarily place much physical significance on a divergence that occurs at a coordinate singularity such as $\eta = 0$ where objects such as a'/a are singular.

One might inquire as to the avoidance of this infrared divergence in the de Sitter spacetime since that calculation also utilized the Bessel functions 2.3.4. On closer inspection, there is a subtle cancelation of divergent terms in the integrand yielding the convergent result. Presumably, above a certain power p one would expect a similar cancelation due to the analyticity of Bessel functions with respect to the order parameter [51] as $n \to 3/2$ ($p \to \infty$). However since we are interested in the limit as the argument approaches a point of non-analyticity for Bessel functions we should not expect the limit to be uniform and hence not expect necessarily that powerlaw spacetime becomes infrared finite for sufficiently high powers p.

The two-point function in powerlaw spacetime

Before moving on to more simple examples we will point out an observation on the two-point function and the relation to the infrared divergence. For $p = 2$ the equal time two point function is written as

$$\langle 0|\chi(x)\chi(x')|0\rangle = \frac{1}{8\pi^2}\int dk\ |v_k|^2 \mathrm{e}^{ik(x-x')} = \frac{1}{8\pi^2}\int dk\ \frac{3k^2\eta^2 + k^4\eta^4 + 9}{k^3\eta^4}\mathrm{e}^{ik(x-x')}.$$

It is clear that for large spatial separation the correlation converges to zero by a Lebesgue type argument and yet for small separation this result is infrared divergent due to the inverse powers of k in the integrand. Pausing for a minute on this result we realize that it is natural for the two-point correlator to be divergent since the field at a point should clearly be influenced strongly by the field at a very nearby point. However, the divergence here is of a different sort, namely it is an *infrared* divergence and is due to arbitrarily large wavelength modes 'piling up' upon themselves. The point is that the infrared divergence of the EMT is related to this analogous divergence in the two point correlator signifying some sort of causal singularity for this massless theory in the infrared. It has been suggested by Professor Robert Brout[6] that such a divergence might be interpreted as a geometric property of a self avoiding random walk problem in a spacetime with horizon. It would be interesting to investigate this possibility further in the future.

2.3.3 Isolating the source of the divergence

In the following few sections we investigate sequentially more sophisticated models of field theory on expanding spaces in order to isolate the infrared divergence found above in the case of $3 + 1$ powerlaw spacetime.

[6] In private conversation.

Field theory on an expanding circle

Let φ be governed by the action

$$S = \frac{1}{2}\int \sqrt{\mathbf{g}} d^2x \; g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} \tag{2.3.8}$$

defined on the time dependent version of $\mathcal{S}^1$, $ds^2 = -dt^2 + a^2(t)d\theta^2$ which may be written in the manifestly conformally flat form $ds^2 = a(\eta)^2\left(-d\eta^2 + d\theta^2\right)$. In general in d-dimensions the Klein Gordon equation reads

$$\frac{1}{a^d}\partial_\mu\left(a^{d-2}\eta^{\mu\nu}\partial_\nu\phi\right),$$

so that for $d = 2$ we obtain

$$\left(-\partial_\eta^2 + \partial_\theta^2\right)\phi = 0.$$

We see here that there is no friction term present and conclude that the decoupling modes are the proper modes (where by 'proper' modes we mean the 'co-moving modes of the proper field φ). Indeed the wave equation is identical to that of a non-expanding circle except for the irrelevant time dependent pre-factor. However, in 1+1 dimensions the Ricci curvature scalar is $R = 2\ddot{a}/a$ so that only in the case of constant expansion rate (a cone) would the curvature vanish. Thus it would seem that the spacetime curvature alone is not responsible for field excitation.

The quantum field theory on a non-constantly expanding circle is given again by the Minkowski mode functions

$$u_k(\eta,\theta) = \frac{1}{\sqrt{2\omega_k}}\mathrm{e}^{-i\omega_k\eta + ik\theta}$$

where the time independent frequency is given by $\omega_k^2 = k^2$. We have for these functions

$$|u_k'|^2 = \omega_k^2|u_k|^2 = \frac{1}{2}\omega_k.$$

The quantum field is given by the sum over the discrete harmonic (co-moving) modes associated with the circle

$$\varphi(\eta,\theta) = \sum_k \; a_k u_k + a_k^\dagger u_k^*$$

so that the energy density is

$$\begin{aligned}\langle 0|T_0^0|0\rangle &= \langle 0|\frac{1}{2}\varphi'^2 + \frac{1}{2}\delta^{ij}\varphi_{,i}\varphi_{,j}|0\rangle \\ &= \frac{1}{2}\sum_k \; |u_k'|^2 + k^2|u_k|^2 = \frac{1}{2}\sum_k \; \omega_k.\end{aligned}$$

Hence we retrieve the asymptotically finite and time independent result derived in the previous section 2.2.1 justifying our intuitive picture there. In this case the final step to make the identification is to place a proper ultraviolet cutoff on the mode sum and input the time dependent *proper* momentum $k_p = k_c/a$.

Two dimensional expanding space

As above we compute the Euler Lagrange equations associated with the action but now in 2+1 dimensions. The result is

$$\phi'' + \frac{a'}{a}\phi' - \nabla^2\phi = 0$$

where we at last see the appearance of the friction term responsible for gravitational particle creation. Recall that in 3+1 dimensions we make the field redefinition $\chi = a\phi$ in order to eliminate the friction term. This procedure is modified here where we define $\chi = \sqrt{a}\phi$. This results in the decoupled equation

$$\chi'' + \left(\frac{1}{4}\left(\frac{a'}{a}\right)^2 - \frac{1}{2}\frac{a''}{a} - \nabla^2\right)\chi = 0.$$

For the power law scale factor $a(\eta) = [(1-p)\eta]^{\frac{p}{1-p}}$[7] we have

$$\chi_k'' + \left(k^2 + \frac{p^2}{4(1-p)^2\eta^2} - \frac{p(2p-1)}{2(1-p)^2\eta^2}\right)\chi_k = 0$$

possessing Bunch Davies mode function solutions

$$v_k(\eta) = \sqrt{\frac{\pi|\eta|}{2}}\left(J_n(k|\eta|) - iY_n(k|\eta|)\right), \quad n = \sqrt{\frac{1}{4} + \frac{p(3p-2)}{4(1-p)^2}}.$$

A plot of the Bessel function solution order parameter n against power p is given in Fig. 2.1

Not that, in $2+1$ dimensions the de Sitter scale factor $a(\eta) = -1/(H\eta)$ gives rise to the mode equation

$$\chi_k'' + \left(k^2 - \frac{3}{4}\frac{1}{\eta^2}\right)\chi_k = 0$$

possessing Bessel function solutions

$$v_k = \sqrt{\frac{\pi|\eta|}{2}}\left(J_1(k|\eta) - iY_1(k|\eta|)\right).$$

Hence, just as in $3+1$ dimensions, the powerlaw mode functions converge to the de Sitter mode functions as $p \to \infty$ as is expected from the interpretation of powerlaw spacetime as an approximation to de Sitter spacetime.

In $2+1$ the EMT is modified to read

$$\langle \hat{T}_0^0 \rangle = \frac{1}{4}\frac{a^{-3}}{(2\pi)^2}\int d^2k\, |v_k'|^2 + \left(\frac{1}{4}\left(\frac{a'}{a}\right)^2 + k^2\right)|v_k|^2 - \frac{1}{2}\left(\frac{a'}{a}\right)\left(\partial_\eta |v_k|^2\right).$$

[7] Recall that for a proper behaviour the scale factor requires a regularization term. The stated expressions are accurate and refer to the results obtained by regularizing and consequently shifting the definition of η, where the shift required to absorb the regulatory factor in a'^2/a^2 is the same as that required to absorb the factor in a''/a.

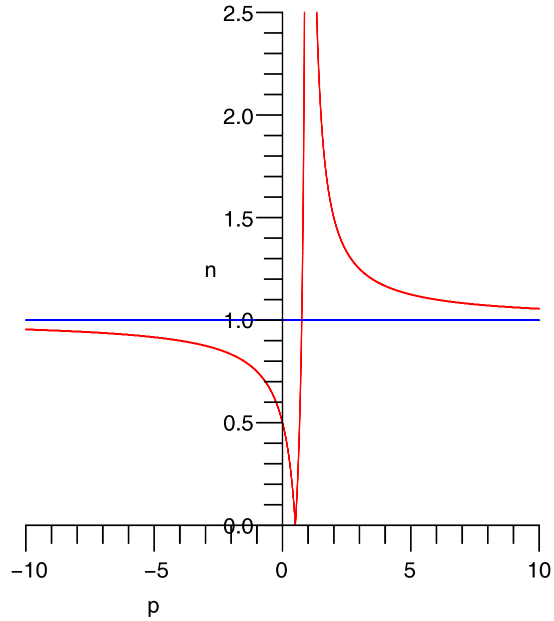

Figure 2.1: The Bessel function order parameter of the mode functions in 2+1 powerlaw spacetime versus power p of the powerlaw scale factor. Also plotted is the asymptotic $n = 1$ order parameter corresponding to the mode functions in $2 + 1$ de Sitter spacetime.

which displays an even more severe infrared divergence than in $3 + 1$ dimensions due to the fact that momentum space is two dimensional contributing only one power of k in the isotropic measure $d^2k \rightarrow 2\pi k\, dk$. Using the asymptotic expansion of the Bessel functions we have

$$k|v_k|^2 \sim k^{-2n+1}$$

so that the integral will be divergent at the lower bound whenever $-2n+1 < -1$ or $n > 1$, once again the precise Bessel parameter of the de Sitter solutions. This corresponds to powerlaw powers $p > 3/4$. Somehow the de Sitter mode functions are playing a limiting role of an infrared divergence. For $2 + 1$ de Sitter the solution is not calculable in closed form due to the fact that the Bessel functions are of non-half integer order and the integrals of their squares are expressible only in terms of hypergeometric series. A numerical calculation reveals that, at least around the origin, the integrand in the de Sitter case actually converges to zero.

Higher dimensions

In general on N dimensional FL spacetime we have

$$\phi'' + (N-2)\frac{a'}{a}\phi' - \nabla^2\phi = 0$$

which decouples for the scaled field $\chi = a^{(N-2)/2}\phi$ as

$$\chi'' + \left(A_N \frac{a'^2}{a^2} + B_N \frac{a''}{a} - \nabla^2 \right) \chi = 0$$

where the functions A_N and B_N are given by

$$A_N = \frac{1}{4}N(N-2) - \frac{1}{2}(N-2)^2$$
$$B_N = -\frac{1}{2}(N-2).$$

It is clear that such a mode equation possesses Bessel function solutions with order parameter

$$n^2 = \frac{1}{4} + A_N \frac{p^2}{(1-p)^2} + B_N \frac{p(2p-1)}{(1-p)^2}$$

which becomes imaginary for high dimension indicating a breakdown of the arguments leading to the isolation of the infrared divergence. In this way one could expect the infrared divergence to be cured for high dimensions as might be intuitively expected by the appearance of higher positive powers of k in the isotropic measure.

A note on massive fields

Throughout we have restricted our attention to the massless scalar field. Massless fields are usually assumed to be more prone to infrared divergences than massive ones. Intuitively, massless quanta travel at the maximum velocity $1 = c$ and so 'spread' their influence maximally to large distances. Infrared divergences arise as an overwhelming contribution from large wavelength (small momentum) modes. So it is of interest to see whether the infrared divergence discussed here is a result of using the massless field or whether it is a genuine feature of the expansion of space. We note that the mode equation is modified in the presence of a mass to be

$$v_k'' + \left(k^2 + a^2 m^2 - \frac{a''}{a} \right) v_k = 0.$$

For powerlaw this is written as

$$v_k'' + \left(k^2 + m^2 \left((1-p)\eta \right)^{\frac{2p}{1-p}} - \frac{p(2p-1)}{(1-p)^2 \eta^2} \right) v_k = 0.$$

which is not soluble in terms of Bessel functions. In fact Maple was unable to solve this differential equation at all in general for arbitrary p. Asymptotically for late times, for any $p > 1$ the k^2 and a''/a (being of inverse power 2 in conformal time) are negligible compared to the m^2 term since $p/(1-p)$ approaches -2 from below as $p \to \infty$. Hence the mode equation reduces for late times to

$$v_k'' + m^2 \left((1-p)\eta \right)^{\frac{2p}{1-p}} v_k = 0.$$

For $p = 2$ this becomes

$$v_k'' + \frac{m^2}{\eta^4} v_k = 0$$

possessing general solution

$$v_k = A\eta \cos\frac{m}{\eta} + B\eta \sin\frac{m}{\eta}.$$

For this asymptotic solution we have that $|v_k|^2$ is badly divergent for $\eta \to 0^+$ suggesting a divergence of the vacuum energy in the infrared[8].

This is a very rough approximation to a full calculation but at least provides circumstantial evidence for an infrared divergence despite the fact that we work with a massive field. Based on this very rough estimate it would seem that the cosmological infrared divergence is not a relic of the masslessness of the fields under consideration.

2.3.4 Interpretation of an infrared cutoff as a model of a local expanding patch of flat spacetime

Above we saw that $\langle \hat{T}^0_0 \rangle$ was not only ultraviolet but also infrared divergent for powerlaw spacetime. In order to proceed further one must regulate such a divergence but also keep in mind any physical interpretation of the regularization. In this case we have the following reasonable physical interpretation of cutting off the co-moving integral at a lower bound. Traditionally, the infrared cutoff energy is interpreted as the vacuum energy of a co-moving box (with proper ultraviolet cutoff) or alternatively as the manifestation of a closed universe (compact spatial sections). That is, one calculates the energy contribution from only those modes less than a maximal co-moving wavelength (or in the closed universe sums over the discrete eigenvalues of the spatial Laplacian) corresponding or proper maximal wavelength that expands with the spacetime. Such a 'box regularization' is sometimes motivated to make the quantum theory more tenable by reducing the number of degrees of freedom to a countable number given by the normal modes (under suitable boundary conditions) of the compact box region. More satisfactorily and importantly for our purposes, however, one might interpret it as the implementation of the idea that *the expansion of spacetime is a process driven by the dynamics of vacuum energy and initiated by a quantum fluctuation in its value on a finite region.* This is the motivation we use here.

[8] Recall that an expanding spacetime couples the $\eta \to 0^-$ limit with the $k \to 0^+$ so that our conclusions here should apply by continuity to the small k infrared limit of the field theory.

Chapter 3

Semiclassical Self Sustained Vacuum Energy Solutions

In this chapter we present some slightly more speculative ideas associated with a particular interpretation of the above calculated vacuum energies. This chapter attempts to address the back-reaction problem of semiclassical gravity but in a novel way. Specifically we attempt below to interpret the time dependence of vacuum energy as the isotropic and homogeneous classical scalar potential supporting the very spacetime solution where the quantum field lives. In this way, one might argue on energetic grounds for the quantum fields alone that a dynamical evolution of the spacetime manifold towards accelerated expansion is provided by the dynamics of the vacuum energy or at least that it is a plausible interpretation.

Up until now we have regarded the quantum field φ as somewhat of a test field with respect to the background geometry of spacetime. We have assumed that the spacetime on which the QFT is defined is supported by the classical matter. In the case of inflation the classical matter and the quantum field are more intimately related, the quantum field being composed of quantized perturbations of the classical field (and metric) but still the assumption that the quantum field is a test field is made in general and throughout the literature. Studies of the **backreaction** of the quantum field on the spacetime dynamics periodically appear in the literature (for example [52, 53]) in a specific context but the results appear to be mostly inconclusive due to the ultraviolet divergence and ambiguity of the vacuum state.

Above we performed a calculation of the vacuum energy density on a fixed background manifold which was assumed to be a solution to the Einstein field equations sustained by a classical (homogeneous and isotropic) scalar field. One might ask however why one would expect the low energy behaviour of quantum fields (and in particular the ubiquitous vacuum energy) to not have significant energetic effects that ruin the finely balanced scalar and gravitational field coupled solutions. At present there is no fully satisfactory way of representing quantum matter in a classical gravitational theory. However, at the end of the previous chapter we investigated the effect of (and provided an interpretation for) the existence of an infrared divergence in the vacuum energy. We then imposed an infrared cutoff and provided an interpretation in terms of an expanding patch of spacetime. This allowed us to get a handle on the time dependence of the vacuum energy (since an infinite quantity is always infinite, infinite quantities have very boring dynamics!) and in particular begin to address the question of the production or not of vacuum energy as spacetime expands.

As mentioned in the introduction, intuitively at least, it is expected that the inflationary dynamics are a result of an interplay with a dynamical vacuum energy of the quantum fields of nature. The idea that spacetime dynamics, and in particular inflation, is a phenomenon associated with the vacuum energy of quantum fields is only recently becoming popular (for example see [10, 11, 12, 13]) with the current interest in 'high energy cosmology'. This research brings together the techniques and results of high energy physics into the context of cosmology in order to describe the very counterintuitive physics of inflation and the current accelerating universe.

As far as our approach is concerned, one could argue that a non-covariantly regularized divergent quantity is as meaningless as a divergent one but we take the position here that it is an interesting and worthwhile endeavor to see where one can get by taking seriously the vacuum energy and in particular the time dependence of the vacuum energy in non-trivial spacetime under a non-covariant cutoff scheme. Is it possible for example for there to exist self consistent vacuum energy supported classical solutions to EE? Is it possible that the entire gravitational dynamics is 'induced' by the spacetime dependence of the vacuum energy? This chapter is devoted to a study of such questions.

The structure of this chapter is as follows. In the first section we review a proposal of exactly the nature alluded to above of trying to make sense of the so-called **semi-classical Einstein equation** where geometry is sourced by quantum matter in the EE. We also discuss the more radical proposal that the gravitational dynamics is *nothing but* the spacetime dependence of dynamical vacuum energy. We then use the results of the previous chapter to apply these ides in the cosmological context. This is an entirely new construction which is not found in the literature.

3.1 Semiclassical and Induced Gravity

In this section we briefly review and discuss some work that attempts to describe the backreaction of quantum fields defined on a spacetime on the dynamics of the spacetime metric itself. Traditionally one starts by writing down the semi-classical Einstein equations

$$G_{\mu\nu} = \langle \hat{T}_{\mu\nu} \rangle$$

where $\langle \hat{T}_{\mu\nu} \rangle$ is the expectation value of the quantum version of the classical energy momentum tensor which for a massless, minimally coupled scalar field φ is again written as

$$\hat{T}_{\mu\nu} = \varphi,_{\mu} \varphi,_{\nu} - \frac{1}{2} g_{\mu\nu} \left(g^{\alpha\beta} \varphi,_{\alpha} \varphi,_{\beta} + 2V(\varphi) \right). \tag{3.1.1}$$

As mentioned previously this equation is somewhat justified by the Ehrenfest theorem of particle quantum mechanics. Serious difficulties present themselves when trying to make sense of 3.1.1 as a field equation for gravity coupled to quantum matter but nonetheless 3.1.1 has been extensively studied in the literature and is the main subject of the standard reference monograph in the field [2]. One of the main difficulties in the early days was making sense of the object $\langle \hat{T}_{\mu\nu} \rangle$ due to the appearance of UV divergences. The issue was settled in the 1970's with the understanding that $\langle\hat{T}_{\mu\nu}\rangle(x)$ is a well defined object at all points where the two point function $G(x,x') = \langle\varphi(x)\varphi(x')\rangle$ is of 'Hadamard form' by a renormalization procedure. This is a technical assumption concerning the singularity structure of the limit $x \to x'$ in G which we will not discuss here (for the details see chapter 9 of [28]) except to say that in any globally hyperbolic[1] spacetime dense in the set of quantum states is a set of states for which $\langle\hat{T}_{\mu\nu}\rangle$ is of Hardamard form. The resulting object is known as the renormalized energy momentum tensor and Wald has provided an axiomatic description of these objects [54]. Constructively, the procedure for defining $\langle\hat{T}_{\mu\nu}\rangle$ involves the usual steps of a renormalization: (i) Regularize the object, (ii) absorb functionally independent divergent (if the regulator were removed) parts into dimensionally equivalent constants of the theory, (iii) remove the regulator leaving only the finite result with the assumption that an infinite shift in the parameters of a theory is irrelevant since we only measure the shifted values for these constants, the 'bare' constants being unobservable due to the omnipresence of the quantum fields. In this case the regularization is done by splitting the points in the quadratic $T_{\mu\nu}$. That is, one calculates

$$\hat{T}_{\mu\nu}(x,x') = \varphi,_{\mu}(x)\varphi,_{\nu}(x') - \frac{1}{2}g_{\mu\nu}\left(g^{\alpha\beta}\varphi,_{\alpha}(x)\varphi,_{\beta}(x')\right)$$

and, knowing the singularity structure of $\langle\varphi(x)\varphi(x')\rangle$ subtracts off the singular pieces and places them inside the gravitational constants as in

$$G_{\text{Renorm}} = G_{\text{Bare}} + \langle\hat{T}_{\mu\nu}\rangle^{G}_{\text{Sing}}$$

such that the dimensions of G_{Bare} and $\langle\hat{T}_{\mu\nu}\rangle^{G}_{\text{Sing}}$ agree. The mathematical details of such an isolation of singularities is beyond the scope of this thesis but suffice it to say that the mathematical theory is only rigorously developed in the Riemannian signature (++++) case and it is an open mathematical problem to extend the results to Lorentzian (-+++) signature.

Up to this point we have taken a rather different approach to $\langle T_{\mu\nu}\rangle$ than the renormalization theory which is the focus of much of the literature in this subject. Here we regard a minimal length in nature as fundamental and not something to be regulated away. This is the same attitude taken in the early work of Sakharov [14] on the quantum backreaction problem. Regardless, we assume that the sophisticated renormalization techniques mentioned here should have no effect on the meso and infrared scale of vacuum energies we study in this thesis. Regardless of which (regularized and with justification) energy momentum tensor one decides to use, the notion of semiclassical gravity is clear: find self consistent solutions to the semiclassical EE.

It is often said that de Sitter spacetime with a scalar field is the only known solution to semiclassical gravity. To show this we need to assume that Minkowski spacetime is also a self consistent solution to the semiclassical EE. That is, we must assume that the quartic divergent vacuum energy is not effected by gravity. Indeed from 2.3.3 it is clear that with the understanding of a natural ultraviolet cutoff at the (for example) Planck momentum Λ the vacuum energy density is constant. The $i \neq j$ component of $\langle T_{\mu\nu}\rangle$ is calculated as

$$\langle T_{i\neq j}\rangle \propto -\frac{1}{a^2}\int d^3k k_i k_j |v_k|^2 = 0$$

[1] Recall that a spacetime is said to be globally hyperbolic essentially if the Cauchy problem for the spacetime in a $3+1$ decomposition is well posed.

since v_k are assumed to be isotropic so that it is an even function of each component k_i. Similarly the components $\langle T_{0i} \rangle$ vanish by the same reasoning. Hence energy momentum tensor is diagonal. However we have (no sum implied on the indices)

$$T^i_i = \frac{1}{2}\varphi_{,0}^2 - \frac{1}{2}(\nabla\varphi)^2 + \varphi_{,i}^2$$

which is evaluated under expectation values (after some more miraculous cancelations) to be

$$P := \langle T^i_i \rangle = -\frac{\Lambda^2 H^2}{16\pi^2} + \langle \varphi_{,i}^2 \rangle.$$

Hence if it is justifiable to ignore the quartic divergence by appealing to the notion that flat spacetime QFT should be also a self consistent solution to the semiclassical EE and also that the anisotropic term $\langle \varphi_{,i} \rangle$ is ignorable then indeed the quantum scalar field is a self consistent solution to EE with equation of spate parameter $w = -1$. The second assumption about the anisotropic term is slightly shaky in the authors opinion. However, this term is difficult to calculate due to the inability to collapse the three dimensional k-space integral down to the one dimensional integral over the magnitude $\sqrt{k \cdot k}$. Nevertheless it would be possible in principle to remove such a term by choosing an appropriately anisotropic set of mode function solutions to KG; the offending term would vanish and the pertinent terms would remain in the same proportionality.

Is powerlaw also subject to an interpretation as a self consistent solution to the semiclassical EE? Once again we argue that the quartic Λ^4 term of the energy density is an artifact of Minkowski space and set this term to zero. We also argue that the anisotropic term can be set to zero by a judicious choice of mode function. Then for $p = 2$ we have

$$\rho = \frac{a^{-4}}{8\pi^2} \int dk \, \left(\frac{4k}{\eta^2} + \frac{9}{k\eta^4} \right)$$

whereas

$$P = \frac{a^{-4}}{8\pi^2} \int dk \, \left(\frac{-2k}{\eta^2} - \frac{9}{k\eta^4} \right).$$

Placing a comoving infrared cutoff and a proper ultraviolet cutoff and moving back to cosmic time t we have, for example

$$\begin{aligned} \rho &= \frac{1}{8\pi^2} \int_{\mu}^{\Lambda a} dk \, \left(\frac{4k}{t^6} + \frac{9}{kt^4} \right) \\ &= \frac{1}{8\pi^2} \left(\frac{\Lambda^2}{t^2} + \frac{9}{t^4} \ln \Lambda t^2 - \frac{\mu^2}{t^6} - \frac{9}{t^4} \ln \mu \right). \end{aligned}$$

Here we see the trade off of ultraviolet modes beginning to contribute to the energy as they enter from below the cutoff scale and the dilution of energy at the infrared cutoff. We see that the energy coming in at a fixed proper momentum is being compensated less and less by the fixed comoving infrared energy being diluted but that the total energy density is decreasing over time. More importantly however (keeping in mind that we have neglected the quartic term) we see that for late times the energy density and also the pressure are dominated by a term proportional to $1/t^2$

$$\rho = \frac{\Lambda^2}{8\pi^2 t^2}, \quad P = -\frac{\Lambda^2}{16\pi^2 t^2}.$$

This is a very interesting and surprising result since we saw in earlier chapters 1.1.8 that *this is the precise time dependence of the energy density* ρ *of the classical scalar field supposed to sustain the slowly rolling inflationary spacetime*. Furthermore the effective equation of state for this vacuum energy is $w = -1/2$. Recall that the equation of state parameter for the classical field sustaining powerlaw spacetime of power p is given by $w = \frac{2}{3p} - 1$. We see that for $p = 2$ the asymptotic vacuum energy equation of state $w = -1/2$ agrees rather well with the one necessary to sustain the spacetime $w = -2/3$. Therefore we interpret this result as reason to believe that 'pure' powerlaw (without classical matter) is a semiclassical self consistent solution. It is interesting to note that the vacuum energy does not possess an equation of state parameter $w = -1$ - the very thing allowing for the possibility that finite p powerlaw might be self consistent. In the literature $w = -1$ is generally referred to as *the* equation of state parameter of vacuum energy. This suggests a change in terminology is necessary.

As far as the generality of this result is concerned, we expect this result to be more robust with higher powers p since high p powerlaw spacetime approximates the semiclassical de Sitter solution more and more closely. However $p > 2$ solutions are not exactly soluble and one would need to resort to more powerful numerical techniques (particularly with regards to the singularity at the origin of the mode functions). The important point is that the vacuum energy does not only consist of the often quoted quartic divergence (which we regard here as unphysical) but takes on a different character precisely of the correct form that lends to it a consistent interpretation as the source of the EE.

In general, divergences other than quartic in the cutoff are expected for the vacuum energy in curved spacetime (see Chapter. 6 of [2]) and this is usually interpreted as invalidating the 'subtract the Minkowski spacetime vacuum energy' tactic to obtain the true vacuum energy. That position assumes that the cutoff is a purely regulatory device not representing a feature of the real world and to be removed at the end of a calculation. Here we interpret the appearance of new divergences as an indication of the ability of vacuum energy to become part of the dynamical equations. What is quite robust is that there should not exist divergences of higher order than quartic and that logarithmic and quadratic divergences are expected.

Induced Gravity

The material presented in the following is of a more mathematically caviler nature due to the indefiniteness of the path integral formulation of Lorentzian QFT.

Recall that the path integral is an attempt at making precise the intuitive notion that to any quantum process every possible evolution from the initial to the final state contributes in an additive way to the total probability amplitude with a phase given by e^{iS} where S is the action functional evaluated for the particular evolution. This is represented by the **path integral** and is written for the case of a quantum scalar field as

$$\int \mathcal{D}\varphi \mathrm{e}^{iS_\varphi}$$

where S_φ is given by the expression 1.2.1. In semiclassical gravity one seeks a framework within which both classical and quantum matter can coexist. The path integral framework provides just this. In principle the full theory of nature would quantize both gravity and matter in the path integral by integrating over both metric and field degrees of freedom. To 'quantize' only the scalar field φ and not gravity $\mathbf{g}$ one only integrates over the φ and leaves $\mathbf{g}$ free:

$$\mathrm{e}^{i\Gamma_{\mathbf{g}}} = \int \mathcal{D}\varphi \mathrm{e}^{i(S_\varphi + S_{\mathbf{g}})}$$

where S_g is the action for gravity (for example the Einstein Hilbert action). Here Γ_g is known as the **effective action** for gravity and is developed in [3] for example. The essential observation of the **induced gravity** proposal is that even without postulating an independent classical gravitational action $S_{\mathbf{g}}$ the effective action $\Gamma_{\mathbf{g}}$ contains purely gravitational terms such as the Ricci scalar R comprising the Einstein Hilbert action. Recall, crucially, that the matter action depends on the metric through the d'Alembertian and density factor in the measure. Such a result involves some sophisticated mathematics in abstract differential geometry (see for example the text [55] for a simple presentation of the relevant material) and is not completely rigorous in the Lorentzian signature case but was anticipated as early as 1967 by Sakharov [14].

In a nutshell, the effective action is expressible as the 'determinant' (the product of the eigenvalues) of the d'Alembertian operator $\Box$, which is meaningless in the Lorentzian case when $\Box$ is hyperbolic with negative and zero eigenvalues. Under the Wick rotation $t \mapsto it$ the signature of the metric becomes Riemannian and $\Box$ an elliptic operator. Then the determinant makes sense and is evaluated using an asymptotic (in the limit $s \to 0$) expansion of the solution kernel of the differential equation $(\partial_s + \Box)\psi = 0$ known as the heat kernel. It is a remarkable fact that the (constant) coefficients in the asymptotic expansion of the heat kernel are geometric invariants depending on the local geometry of the manifold. This observation expresses the intuitive idea that the heat flow on a surface on very short time scales should depend only on the local geometry only and not on the global topology. Heat flow on large time scales should of course depend on the global topology and some very deep and abstract results linking global topology and local geometry have been obtained by studying the flow of information from one to the other using the heat kernel and heat equation [55]. Of course, the expansion is merely an asymptotic one and does not converge in any classical sense - an expression of the ultraviolet divergences of QFT. Regular renormalization theory on curved spacetime merely isolates these divergences in the heat kernel and places them inside the constants of the geometric gravitational action. The key observation of induced gravity (and the philosophy adhered to in this thesis) is that the ultraviolet cutoff in nature is *real* (by some as yet unknown mechanism) so that the divergent pieces of the heat kernel do not even arise. This leaves one with a heat kernel expansion consisting of geometric terms multiplied by finite constants. What need do we have of postulating a bare gravitational action if those geometrical objects arise naturally in the quantum theory of a scalar field? Sakharov answered this question with a resounding 'none' when he showed that with a cutoff at the Planck scale the numerical constants[2] come out to be exactly right for the resulting geometric terms in the heat kernel[3] to be the gravitational action. Put more simply, the quantum amplitude for a particular field evolution depends on the geometry of the manifold and that dependence is precisely given by the Einstein Hilbert action functional (plus higher order corrections[4]).

[2]With the notable caveat of the cosmological constant term interpreted as the vacuum energy.

[3]Although he did not use this modern language. For a modern review of induced gravity see [56].

[4]Which, it should be added have not been ruled out by any of the low energy experimental tests of

In our context of trying to explain the particular cosmological evolution we have seen that the energy momentum tensor is of the correct form (and, of course, dimension) to be equated with the gravitational side of EE in order to argue that de Sitter spacetime was a self consistent solution to the semiclassical equations. With some hand waving we also argued that it was plausible that powerlaw spacetime could be also a semiclassical solution. However in the light of induced gravity we see that an exact matching of the dynamics of the vacuum energy and that of the classical energy momentum necessary to sustain the very spacetime on which the quantum field is defined is not necessary if there is no 'bare' geometrical side of the EE. Without a 'locked' dynamics for the spacetime, the manifold is free do what is chosen for it by the scalar field. If the vacuum energy evolves as if it were on a powerlaw spacetime but that this power is changing over time this would amount to an evolution of the spacetime through the spectrum of powers. Such a scheme amounts to a precise realization of slow roll inflation. It would be very interesting to expand on this idea by studying the evolution of vacuum energy more closely using more powerful computational techniques and under more general conditions.

The essential point of this small section is that a self consistent semiclassical dynamics is not necessary to 'fine tune' in the induced gravity picture where the vacuum energy dynamics is interpreted as geometric dynamics. Such a picture is motivated by the full induced gravity proposal [56] but which finds a simplified and concrete example in the considerations presented in this thesis.

3.2 A Mechanism for Inflation?

As briefly mentioned above, the infrared cutoff can be interpreted as an implementation of the idea that the expansion of spacetime be initiated by a quantum fluctuation of the vacuum energy in flat spacetime giving rise to a dynamical energy source and hence spacetime dynamics as described in the sections above. We have used throughout the vacuum energy density at a spatial point x $\langle \hat{T}_0^0 \rangle$ which of course is independent of x. In practice we would expect inhomogeneous fluctuations of energy on all scales that *average* to the homogeneous value calculated in the quantities $\langle \hat{T}_0^0 \rangle$. From the analysis presented here this possibility seems feasible and we have provided some concrete calculations supporting this intuitively held possibility. This mechanism of course seems to assume that, indeed, Minkowski spacetime is a semiclassical solution to the EE but that it is an unstable one able to decay into an inflating universe. This picture of an unstable field theoretic fixed point decaying into an inflating universe is one commonly held in the field [57].

GR.

Chapter 4

Summary and Outlook

Based on the preliminary studies presented here one could proceed in one of many directions. There are many unsolved problems at the interface of gravitation and QFT. As mentioned at the end of Chapter 1, the glaring dark energy problem in cosmology is widely regarded as one of the greatest unsolved problems in physics. Even, as we have attempted here in a concrete manner, if one interprets dark energy as quantum vacuum energy, somehow the peculiar dynamics including the apparent second inflationary era beginning today needs to be explained. This all requires a study of *dynamical vacuum energy* which presumably can be studied within the context of QFT in curved spacetime.

In this part of the thesis I have introduced and discussed QFT on cosmological backgrounds and the theory of inflation as a phenomenon driven by the dynamics of vacuum energy. The theory of inflation was motivated as an experimentally confirmed hypothesis and some of its short comings were pointed out. I have provided an in-depth and exact study of the powerlaw solution and the quantum field theory on it and also embedded this spacetime in the larger picture of a slow roll inflation scenario. A large class ($p > 2/3$) of infrared divergences were found in powerlaw spacetime superseding results previously obtained in the literature for a limited range of powers p. I discussed an interpretation of the infrared cutoff as a realization of the idea that spacetime is induced by a fluctuation in the vacuum energy density on an otherwise flat ambient spacetime. The idea there was to interpret the inflating spacetime as a self sustaining process whereby a bubble of flat spacetime initially possesses a raised energy density, expands in a backreaction and continues to do so through the non-trivial dynamics of the vacuum energy on the expanding patch. In this thesis I have modeled this process as regular, unbounded (flat) powerlaw spacetime with a co-moving infrared cutoff. I also discussed more radical proposal that not only is the spacetime sourced by the vacuum energy but that the spacetime dynamics itself is nothing but the geometrical dependence of the vacuum energy. It was only in this scenario that a slow roll picture was possible where the spacetime solution is not fixed. It is interesting that a number of desirable properties of the quantum theory rested on the assumption of a slow roll inflating spacetime. Namely, the existence of scalar perturbations at all on the background spacetime, the consistent interpretation of the scalar Mukhanov fluctuations as a scalar field propagating on a background spacetime and the interpretation of the time dependence of the vacuum energy as an ‘induced cosmology’.

Throughout I pointed out a number of opportunities for future work and also made speculative connections with some of the previously existing literature. These included a more detailed comparison of the study of Jacobson's discrete quantum field theory and other discrete dynamical ground state energy calculations, a more thorough numerical investigation of higher p powerlaw spacetime quantum field theory and the generalization to more general types of fields including massive fields to perhaps tame the infrared divergence. I also alluded to a possible connection between the infrared divergence in expanding spacetime with a self avoiding random walk problem.

In conclusion, quantum field theory and cosmology are two very different models of the world and yet the union of some of the lessons provided separately by these theories seems to be a natural one providing a rather novel picture of the early evolution and large scale structure of our universe.

Part II

Superoscillation - Induced Resonance

Introduction

There exists a growing body of research that studies the seemingly paradoxical phenomenon of superoscillation wherein a signal (or function) may locally oscillate faster than the highest Fourier component. Superoscillations have been first studied by Aharonov Berry and others [58] in quantum theory and have seen application in information theory [59], the trans-Planckian problem in black hole evaporation [60], very recently to sub wavelength resolution in optical experiments [61] and shown to imply various other unusual quantum mechanical phenomena such as self acceleration [62, 63]. The relationship between superoscillations, quantum field theory and information theory is presented in the excellent article Ref [64]. To the best of the authors knowledge however there has been to date no attempt at a quantitative study of what might be called the 'reality' of the superoscillations. What we mean by 'reality' will be revealed in the rest of this part of this thesis but as an introductory remark we will make the following mysterious comment: superoscillations are low frequency waveforms in the disguise of high frequency waveforms. The 'reality of the superoscillations' would be the extent to which a superoscillating function can be distinguished from a bona fide high frequency waveform and in particular, the extent to which a mechanical system coupled to a superoscillating source behaves as if it really were driven by a high frequency waveform. In this second part of this thesis we study the reality of superoscillations in the above sense.

The outline is as follows. In the first chapter we explore the methods developed in [62, 63] for constructing superoscillating functions of one variable. This chapter is furnished in the second section by a discussion on harmonic oscillators in order for the material in the following chapter containing the results of a graphical and analytical study to be self contained. We end with a chapter discussing some possible applications in quantum field theory and cosmology.

Chapter 5

Superoscillations and Harmonic Oscillators

5.1 Constructing superoscillations

The general theory of the classes of bandlimited[1] functions provides us with the following result[2] implying the existence of superoscillations:

Theorem 5.1.1. *Let $\mathcal{F}$ be the Fourier operator on $\mathcal{H} = L^2(\mathbb{R})$, $X \subset \mathbb{R}$ be any compact interval and define the projection $P_\Omega \in \mathcal{B}(\mathcal{H})$ by*

$$P_\Omega = \mathcal{F}^{-1}\chi_{[-\Omega,\Omega]}\mathcal{F}$$

and the projection $P_X \in \mathcal{B}(\mathcal{H})$ by $(P_X f)(x) = \chi_X(x)f(x)$ for all $f \in L^2(\mathbb{R})$ where χ is the characteristic function. Then $P_X P_\Omega \mathcal{H}$ is dense in $L^2(X)$ for any non-zero Ω.

That is, the projection of the space of bandlimited functions onto a compact interval X is dense in $L^2(X)$. Put even more simply, the theorem states that one may approximate to arbitrary precision an arbitrary square integrable function on an interval by bandlimited functions. Thus we are not surprised to find bandlimited functions that locally oscillate faster than their highest Fourier mode!

We turn now to the explicit construction of custom made superoscillating functions which will be used in the subsequent section.

5.1.1 Constructive proof

As discussed above we consider functions possessing compactly supported Fourier transforms

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\Omega} dk\ \tilde{f}(k)\mathrm{e}^{ikx}$$

[1] Recall that a square integrable function is said to be 'bandlimited' if its Fourier transform possesses compact support. Since we will be considering real valued functions here we restrict to the case when the Fourier transform is an even function.

[2] For the proof see [63].

and ask whether we can constrain such a function to have the N prescribed amplitudes

$$f(x_i) = a_i, \quad \text{for} \quad i = 1, \ldots, N.$$

That is, we require

$$a_i = \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\Omega} dk\ \tilde{f}(k) \mathrm{e}^{ix_i k}.$$

Of course, there will be many (indeed an infinite number[3]) of functions satisfying the bandlimit and constraints. Therefore, to obtain a unique expression for our superoscillating source we ask in addition for the function to have minimal L^2 norm. Writing this as an optimization problem, we seek to minimize the functional $F[f]$ under the constraints $G_i[f] = 0$ where[4]

$$F[f] = \int dk\ \tilde{f}(k)\tilde{f}^*(k), \quad G_i[f] = a_i - \frac{1}{\sqrt{2\pi}} \int dk\ \tilde{f}(k) \mathrm{e}^{ix_i k}.$$

The standard Euler-Lagrange solution involves the Lagrange multipliers μ_i:

$$\frac{\delta F}{\delta \tilde{f}(k)} = \mu_i \frac{\delta G_i}{\delta \tilde{f}(k)}$$

where the sum over i is implied. Specifically we have

$$\tilde{f}(k) = -\mu_i^* \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-ix_i k}.$$

Integrating both sides of this expression against the constraining plane waves gives

$$\begin{aligned} a_j &= -\mu_i^* \frac{1}{2\pi} \int_{-\Omega}^{\Omega} dk\ \mathrm{e}^{i(x_j - x_i)k} \\ &= -\mu_i^* \frac{1}{2\pi} \frac{\sin(x_j - x_i)\Omega}{\pi(x_j - x_i)} =: -\mu_i^* S_{ji} \end{aligned}$$

so that we may solve for the multipliers as

$$\mu_i^* = -S_{ji}^{-1} a_j.$$

Therefore the unique solution is written as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\Omega} dk\ \tilde{f}(k) \mathrm{e}^{-ikx} = S_{ji}^{-1} a_j \int_{-\Omega}^{\Omega} dk\ \mathrm{e}^{i(x - x_i)k} = S_{ji}^{-1} a_j \frac{\sin \Omega(x - x_i)}{\pi(x - x_i)}.$$

The unique superoscillating function is seen to be a subtle linear combination of shifted sinc functions. By choosing the amplitudes a_i judiciously to approximate any function we

[3]The existence of at least one such function is guaranteed by the theorem. That there exists an infinity of functions is demonstrated by the fact that we may add to our first candidate function any function which vanishes at the prescribed points and which we may choose to take an arbitrary prescribed value at a point distinct from the prescribed points. There are clearly an infinity of such additional functions.

[4]Both G_i and F are functionals of the infinitely many variables $\tilde{f}(k)$ and $\tilde{f}^*(k)$ characterizing a given function f. Formally, the variables $\tilde{f}(k)$ are independent of $\tilde{f}^*(k)$ however this is a redundancy since the two sets of Lagrange equations carry the same information. We will thus only vary with respect to the $f(k)$ variables.

like we are able to custom make a superoscillating signal. For example, given a bandlimit of say $\Omega = 4$ one can prescribe the amplitudes $f(n) = (-1)^n$ for integer n say between -5 and 5 effectively forcing the function to oscillate with wavelength $\lambda = 2$ on this stretch.

Although a rigorous definition of 'superoscillating function' does not exist we will take it as an intuitive notion that has meaning in at least those cases when a bandlimited function 'looks' sinusoidal with frequency higher than the bandlimit on some interval. It is these such superoscillating functions that we will utilize in the next chapter. Firstly however, we provide a short discussion on harmonic oscillators in both the quantum and classical cases. These sections are not entirely necessary for the discussion since the material is standard (for example see Ch. 3 of [3]) but they provide continuity to the results in the rest of this chapter.

5.2 Oscillator mechanics

The basic mathematical problem we are concerned with in this part is the local behaviour of the solutions to the driven oscillator equation

$$\ddot{q} + \omega^2 q = J(t)$$

near a superoscillating region of a bandlimited signal J. We proceed now through a short derivation of the relevant equations used in the numerical studies later on.

The Green's function for the driven harmonic oscillator is written as the kernel of the solution:

$$q(t) = \int_{-\infty}^{\infty} dt' \; J(t') \frac{\sin \omega(t - t')}{\omega} \Theta(t - t') \tag{5.2.1}$$

where $\Theta(t)$ is the Heaviside step function. This is the physically relevant retarded Green's function implementing the initial conditions $q(t_0) = \dot{q}(t_0) = 0$.

The well known classical Hamiltonian functional for the oscillator is given by

$$\mathcal{H} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 - Jq$$

from which follow the equations of motion via Hamilton's equations. We might characterize the energy imparted on the oscillator by tracking the numerical value of the Hamiltonian across the superoscillating region. That is, using 5.2.1, we seek to study the local behaviour of

$$\begin{aligned}
\mathcal{H}(t) \;=\; & \frac{1}{2}\int_{-\infty}^{t} dt' \; dt'' \; J(t')J(t'')\cos \omega(t - t')\cos \omega(t - t'') \\
& + \frac{1}{2}\int_{-\infty}^{t} dt' \; dt'' J(t')J(t'')\sin \omega(t - t')\sin \omega(t - t'') \\
& + J(t)\int_{-\infty}^{t} dt' \; J(t')\frac{\sin \omega(t - t')}{\omega}.
\end{aligned} \tag{5.2.2}$$

An alternative and convenient choice of variables is the choice

$$a(t) = \sqrt{\frac{\omega}{2}} q(t) + i \frac{1}{\sqrt{2\omega}} p(t)$$

and its complex conjugate $a^\dagger := a^*$. It will be necessary to introduce these alternative variables in the quantum case where a direct integral kernel solution analogous to 5.2.1 is unavailable. In terms of these variables the Hamiltonian reduces to the form

$$H(t) = \omega a^\dagger a - \frac{1}{\sqrt{2\omega}} J(a^\dagger + a)$$

providing the equations of motion

$$\dot{a} = -i\omega a + \frac{i}{\sqrt{2\omega}} J, \quad \dot{(a^\dagger)} = (\dot{a})^*. \tag{5.2.3}$$

It should be noted that, despite the fact that we merely take complex linear combinations, the change of variables $(q, p) \to (a, a^\dagger)$ does not correspond to a 'canonical' (in some literature 'contact') transformation - one that preserves the canonical structure. That is, the Poisson bracket for these new variables is $\{a, a^\dagger\} = i$ in contrast to $\{p, q\} = 1$. For this reason the equations of motion are not the Hamiltonian equations associated with the new variables i.e: $\dot{a} \neq \partial_{a^\dagger} H$ but instead are obtained directly from those of (p, q).

It is clear the relations 5.2.3 are exactly soluble by an integrating factor:

$$a(t) = a_0 \mathrm{e}^{-i\omega t} + \frac{i}{\sqrt{2\omega}} \int_{t_0}^{t} dt' \; \mathrm{e}^{i\omega(t'-t)} J(t')$$

where the dependence on the initial conditions has been made explicit with the introduction of the initial time $t = t_0$ and constant a_0. Of course for $a_0 = 0$ the Hamiltonian in terms of this exact solution is written as in 5.2.2.

5.2.1 Quantum oscillator

There are a number of noteworthy differences between the quantum analysis of the harmonic oscillator and the classical one which we highlight here. To make the transition from the classical oscillator the canonical Hamiltonian method is most transparent. Thus we seek to solve the coupled operator equations

$$\dot{q} := \frac{\partial H}{\partial p} = p, \quad \dot{p} := -\frac{\partial H}{\partial q} = -\omega^2 q + J(t)$$

where q and p are the position and momentum operators representing these observables constrained to satisfy the commutation $[q, p] = i$, J is the driving force and

$$H(p, q, t) = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 q^2 - J(t) q$$

is the time dependent Hamiltonian as above. For convenience as usual we introduce the auxiliary operator $a(t)$ and its adjoint $a^\dagger(t)$ by

$$a(t) = \sqrt{\frac{\omega}{2}} \left(q(t) + \frac{i}{\omega} p(t) \right), \quad a^\dagger(t) = (a)^\dagger(t)$$

in terms of which the Hamiltonian reads

$$H(t) = \omega \left(a^\dagger(t) a(t) + \frac{1}{2} \right) - \frac{1}{\sqrt{2\omega}} \left(a^\dagger(t) + a(t) \right) J(t). \tag{5.2.4}$$

Once again we have the equations of motion

$$i\dot{a}(t) = \omega a(t) - \frac{1}{\sqrt{2\omega}}J(t)$$

and commutation $[a(t), a(t)^\dagger] = \mathbb{1}$ soluble again by an integrating factor

$$a(t) = \mathbf{a}\mathrm{e}^{-i\omega t} + \mathbb{1}\frac{i}{\sqrt{2\omega}}\int_{t_0}^{t} dt'\ J(t')\mathrm{e}^{i\omega(t'-t)} \tag{5.2.5}$$

just in case

$$[\mathbf{a}, \mathbf{a}^\dagger] = \mathbb{1}.$$

Here $\mathbf{a}$ is a fixed operator understood as the initial condition $a(t_0)$[5]. The physical meaning of choosing an initial operator $\mathbf{a}$ is that of choosing what might be called the kinematical Hilbert space $\mathcal{H}$. To $\mathcal{H}$ there corresponds a unique vacuum vector $|\Omega\rangle$ satisfying $\mathbf{a}|\Omega\rangle = 0$ and orthonormal basis vectors $\sqrt{n!}|n\rangle = (\mathbf{a}^\dagger)^n|\Omega\rangle$ which are eigenvectors to the kinematical Hamiltonian $\mathbf{H} = H(a, a^\dagger, J = 0) = \omega(\mathbf{a}^\dagger\mathbf{a}) + \omega/2$. This is to be regarded as a reference space (and state); $\mathcal{H}$ is the Hilbert space associated with the un-driven oscillator to which will be compared the family of vacuum states $|\Omega_t\rangle$ associated with the family of operators $a(t)$ and Hamiltonians $H(t)$ as time progresses through the superoscillating region. Of course, our superoscillating driving forces extend non-trivially all the way to infinity where they decay to zero. Hence $\mathcal{H}$ is to be thought of as the asymptotic Hilbert space; the initial time t_0 will be $-\infty$ so that the kinematical Hamiltonian $\mathbf{H}$ is recognized as the limit $\lim_{t\to-\infty}H(t)$.

Here, again, we wish to estimate the excitation of the oscillator during the action of a superoscillating source. One measure of this is the numerical value of the Hamiltonian vacuum expectation interpreted as the statistical average energy imparted to the ground state of the oscillator by the driving source. This quantity is written explicitly as

$$\langle\Omega|H(t)|\Omega\rangle = \omega|\mathcal{J}_t|^2 + \frac{1}{2}\omega + J(t)\int^{t} dt'\ J(t')\frac{\sin\omega(t'-t)}{\omega} \tag{5.2.6}$$

where we define the partial Fourier transform $\mathcal{J}_t$ of J as

$$\mathcal{J}_t = \frac{i}{\sqrt{2\omega}}\int^{t} dt\ J(t')\mathrm{e}^{i\omega t'}.$$

To get an explicit and complete picture of the excitation process we could also calculate the expectation of $q(t)$ in the initial vacuum.

[5]There is an alternative way of obtaining this solution if the use of an intergrating factor for operators makes one a little uneasy: The most general family $\{a(t)\}_{t\in\mathbb{R}}$ of operators satisfying the commutation $[a(t), a^\dagger(t)] = \mathbb{1}$ for all t is parameterized by $a(t) = \mathbf{a}\,v(t) + \mathbb{1}u(t)$ where $[\mathbf{a}, \mathbf{a}^\dagger] = \mathbb{1}$. Reinserting this into the equation of motion we obtain the relation $\mathbf{a}\,(i\dot{v} - \omega v) = \mathbb{1}\left(i\dot{u} - \omega u + J/\sqrt{2\omega}\right)$ implying that both numerical factors vanish. An ordinary integrating factor may then be utilized for the right factor to solve for u which, combined with the requirement of consistency with the commutation relations, gives the solution 5.2.5.

5.2.2 Interpretation of the quantum solution

Examining the derivation of 5.2.6 we find the following result,

$$\begin{aligned} q(t) &= \frac{1}{\sqrt{2\omega}}\left(\left(a^\dagger + \mathcal{J}_t^*\right)\mathrm{e}^{i\omega t} + (a + \mathcal{J}_t)\,\mathrm{e}^{-i\omega t}\right) \\ &= \frac{1}{\sqrt{2\omega}}\left(\mathbf{a}\mathrm{e}^{-i\omega t} + \mathbf{a}^\dagger \mathrm{e}^{i\omega t} + 2\mathfrak{Re}\{\mathcal{J}_t\}\mathbb{1}\right) \\ &= \frac{1}{\sqrt{2\omega}}\left(\mathbf{a}\mathrm{e}^{-i\omega t} + \mathbf{a}^\dagger \mathrm{e}^{i\omega t}\right) + \mathbb{1}\int_{t_0}^{t} dt'\, \frac{\sin(\omega(t-t'))}{\omega} J(t') \end{aligned}$$

from which we obtain

$$\langle\psi|q(t)|\psi\rangle = \int_{t_0}^{t} dt'\, \frac{\sin(\omega(t-t'))}{\omega} J(t')$$

where $|\psi\rangle$ is normalized and arbitrary. This expression is precisely the expression for the classical solution $\overline{q}(t)$ with initial conditions $\dot{\overline{q}}(t_0) = \overline{q}(t_0) = 0$ under the influence a driving force. That is, we have

$$q(t) = \mathbf{q}(t) + \overline{q}(t)\mathbb{1} \tag{5.2.7}$$

where we have introduced the kinematic position operator $\mathbf{q}$ associated with the un-driven oscillator. The expression 5.2.7 provides us with the interpretation of the driven quantum oscillator solution as 'quantum fluctuations' about the classical solution.

5.2.3 Relationship between the classical and quantum Hamiltonians

In an analogous way we may also derive the relation

$$p(t) = \mathbf{p}(t) + \overline{p}(t)$$

from which we obtain the expression for the Hamiltonian expectation

$$\langle H(p,q,t)\rangle = \langle \mathbf{H}(\mathbf{p},\mathbf{q})\rangle + \overline{H}(\overline{p},\overline{q},t)$$

where we use the shorthand $\langle p\rangle := \langle\psi|p|\psi\rangle$. For example when $|\psi\rangle$ is a pure Hamiltonian eigenstate $|\psi\rangle = |n\rangle$ (for example when $n=0$ we consider $|\psi\rangle = |\Omega\rangle$) then the quantum energy expectation is given by the classical expression plus the constant value $\omega(n+1/2)$.

5.2.4 Convergence of quantum states

As noted above, for any finite time the Hamiltonian possesses the non-trivial time dependence given by

$$H(t) = \omega\left(a^\dagger(t)a(t) + \frac{1}{2}\right) - \frac{1}{\sqrt{2\omega}}\left(a^\dagger(t) + a(t)\right)J(t)$$

the ground state $|\Omega_t\rangle$ of which will differ from $|\Omega\rangle$. Here $a(t)$ is given by 5.2.5. We can estimate this difference by expressing $|\Omega\rangle$ in terms of the basis eigenstates $|n\rangle_t$ of $H(t)$. One finds the normalized squeezed state

$$|\Omega\rangle = \exp\left(-\frac{1}{2}|S(t)|^2\right)\sum_{n=0}^{\infty}\frac{S(t)^n}{\sqrt{n!}}|n\rangle_t.$$

where $S(t)$ is the scaled partial Fourier transform

$$S(t) = \frac{i}{\sqrt{2\omega}}\int_{-\infty}^{t} dt'\, \mathrm{e}^{i\omega t'} J(t').$$

This expression shows us that

$$\langle\Omega_t|\Omega\rangle = \exp\left(-\frac{1}{2}|S(t)|^2\right) \tag{5.2.8}$$

implying the convergence of the vacuum states $|\Omega_t\rangle \to |\Omega\rangle$ as $t \to \infty$ for any source whose Fourier decomposition does not contain the resonant frequency ω.

Chapter 6

Ghost Resonance by Bandlimited Signals

We move on now to the main subject of this part of the thesis. The question we wish to address is the following:

> *To what extent does a forced harmonic oscillator driven by a bandlimited signal that does* not *contain the resonant frequency but which nevertheless is constructed to superoscillate at the resonant frequency behave as if it were truly driven by a resonant source?*

The question is non-trivial since the source is precisely constructed to **not** possess the resonant frequency and so should not excite the oscillator at all. This physical fact is displayed by the result 5.2.8 above which shows that the oscillator returns to the (quantum) ground state asymptotically if the source does not contain the resonant frequency and hence is not excited by the driving force. The question arose in a discussion about the physical problem of bandlimited electromagnetic wave propagation which may illuminate the issue we seek to address. In this realistic scenario the above question is restated as

> *Do water molecules in a material 'see' localy (in time) resonant electromagnetic waves in a bandlimited superoscillating source and if so, why does the coupled wave/molecule system behave globally (in time) as if they did not?*

Here, the water molecules act as a filter for the signal, filtering out the resonant frequency as the electromagnetic wave passes through the material. However, we are left with the problem of explaining how the appropriately bandlimited superoscillating wave manages to locally excite the oscillator (and hence is locally filtered) only to have the resonant modes replaced later on, reconstructing the full, unfiltered signal. Some conspiracy between the molecules and wave seems to be taking place! It is in this mood that we present the results of an analytic and numerical study of a harmonic oscillator coupled to a superoscillating driving source.

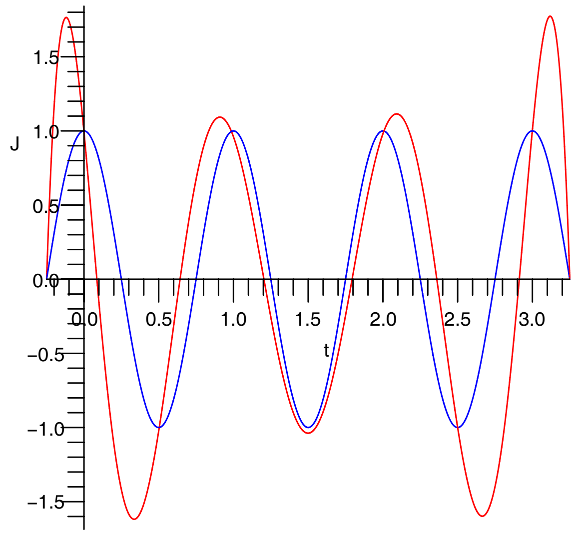

Figure 6.1: Amplitude versus time for the superoscillating section of the signal $J(t)$ (red) and the exactly resonant plane wave (blue).

6.1 Results

Based on the methods of Chapter 5 the signal shown in Figure 6.1 was constructed by specifying nine amplitudes in the constraint procedure. The superoscillating signal is bandlimited to a minimum wavelength of two units and approximates the resonant plane wave of wavelength one unit. The global nature of the superoscillating signal is displayed in Figure 6.2 where the relative magnitude of the superoscillating section to the remainder is displayed. In addition, signals with 5, 7 and 8 prescribed points were constructed all with the same general structure of a small amplitude superoscillating section bounded by large amplitude 'lobes' that decay to zero far from the superoscillations. Indeed it is clear that the signals asymptotically approach zero as $t \to \pm\infty$ by the method of construction since the signal is a linear combination of shifted sinc functions which decay to zero at infinity.

The expectation of the Hamiltonian operator 5.2.6 in the quantum case is plotted in Figure 6.3. As a comparison, we have also included here in Figure 6.4 the shape of the classical total energy function associated with an oscillator driven by an exactly out of phase exactly resonant force. The characteristic parabolic shape indicates the most efficient linear (with time) damping and consequent linear increase in the amplitude of oscillations of the oscillator and it is clear that this behaviour is seen in the energy function of the superoscillation driven oscillator. Recall that the Hamiltonian function is quadratic in the canonical variables. We see that, to become in phase with the driving force, the oscillator's energy actually reduces to zero before being driven at resonance in phase.

By comparison of the actual result 6.3 with the

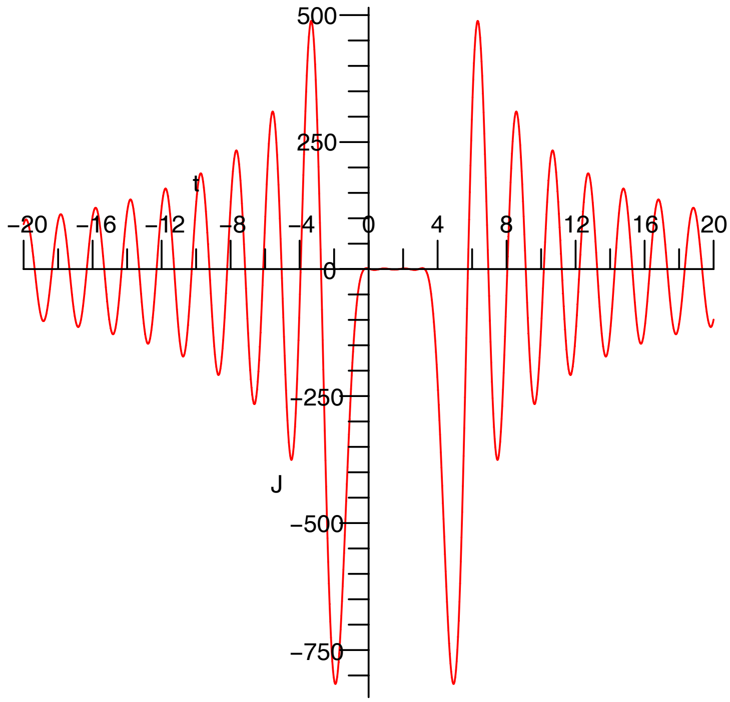

Figure 6.2: The global picture of $J(t)$ (red). Note the large amplitude of the oscillations outside the superoscillating region.

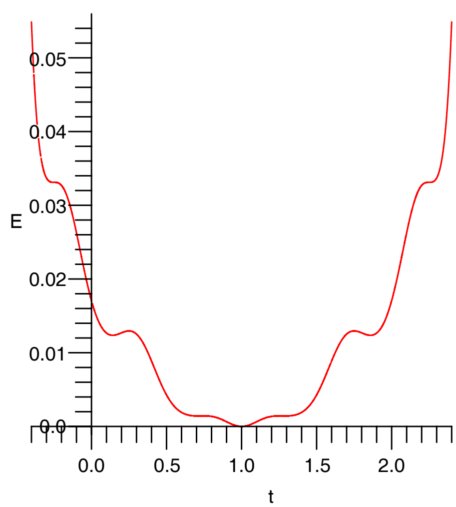

Figure 6.3: The classical Hamiltonian function 5.2.2 (equal to the quantum expectation 5.2.6 value minus the constant $1/2\hbar$) against time for the driven harmonic oscillator in the superoscillating region.

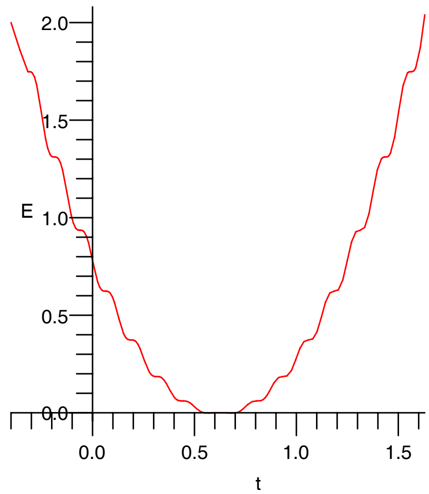

Figure 6.4: The classical Hamiltonian function of an exactly resonant driving force initially completely out of phase with the oscillations near the turning point.

6.2 Interpretation

The behaviour of the quantum driven oscillator above substantiates the (perhaps obvious) claim that 'physics is local'. What we mean by this is that the harmonic oscillator responds only to the local conditions, in this case the local (in time) resonant oscillations, independently of global structures. However this interpretation is as expected as it is mysterious for the reasons stated in the introduction, namely that the resonance experienced by the oscillator somehow gets 'taken back' by the source after the fact so as to render the source globally invisible to the oscillator which is only supposed to respond to one particular frequency - the resonant frequency. This perhaps requires more explanation. During the period before the resonant superoscillations nothing very special seems to be occurring (we will come back to this) since the oscillator is being driven by a function that is varying on characteristic time scale well above that of the resonant frequency ω. During the superoscillations on the other hand, the oscillator behaves exactly as if it were being driven by a resonant driving force. However this 'resonance' has a strange character since the oscillator emerges from the superoscillations with *exactly* the same energy as it had when it entered. This is somewhat expected by a consideration of the symmetry of the driving force J. This indicates that the first (in time) lobe (non-superoscillating) section of the driving force has the effect of placing the oscillator in such a state that the superoscillating section of the driving force exactly removes and subsequently replaces the oscillations before the driving force enters the second (in time) lobe phase. That is, the lobes together 'contain' the resonant frequency even though this is not manifest in the shape of the driving force curve. The sequence of events is thus summarized as follows: The first lobe section induces oscillations which are firstly removed and then replaced by the superoscillations. These residual oscillations a subsequently removed by the second lobe of the driving force.

One might argue against the reality of superoscillations by pointing out that oscillators behave in an approximately resonant manner when subject to approximately resonant driving sources. However, this interpretation is inapplicable in this case since we used in the construction a source that was bandlimited to twice the frequency of the superoscillating section and oscillators driven at twice the resonant frequency do not possess the characteristic quadratic amplification of the energy normally associated with a resonant driving force.

These results open up an interesting possibility for a different interpretation of superoscillations. The time period up until the superoscillating stretch imparts an energy into the oscillator exactly equal to the energy initially removed and subsequently replaced in a resonating fashion by the source during the superoscillations. This seems to suggest that *the high amplitude slowly varying lobes also contain the (relatively) high frequency resonant mode*. The key point here is that the lobes are not of one single frequency as can be seen upon close inspection in Fig. 6.5. Perhaps the requirement of a slowly varying frequency and amplitude will manifest itself in a relatively small amplitude stretch where the signal becomes close to zero. Another interesting possibility is that superoscillating functions are those bandlimited functions that come 'closest' to vanishing on an interval. By a remark due to Feynman in the context of antiparticles [65] a bandlimited function cannot vanish on any finite interval. It would be interesting to make a study of these constructed superoscillating functions under these alternative hypotheses. Certainly it would be possible to use the same methods as above in that analysis.

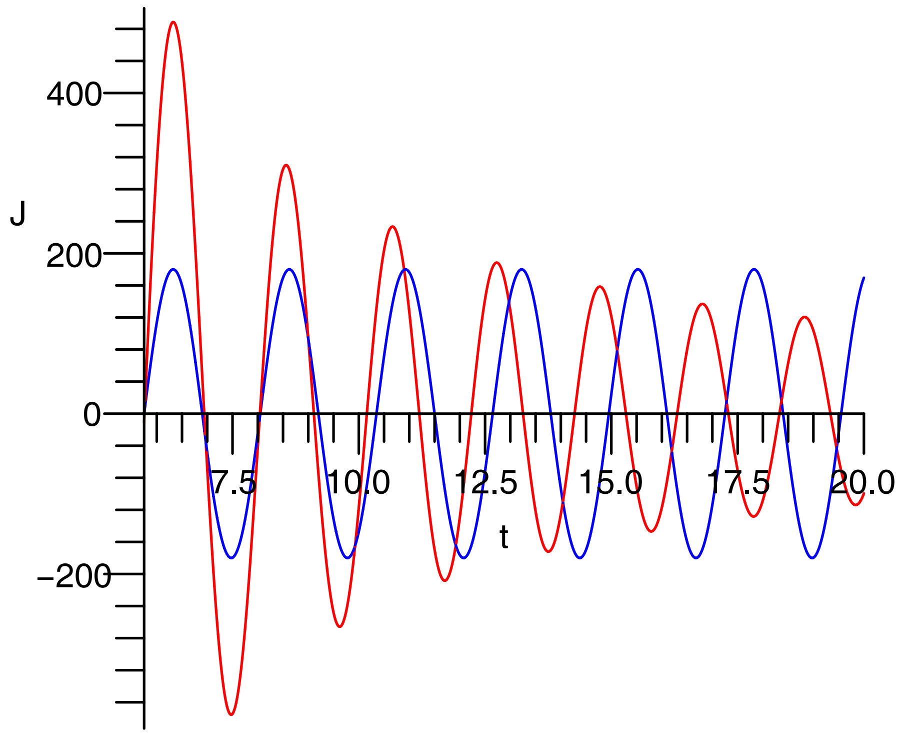

Figure 6.5: The exterior 'lobe' section of the constructed superoscillating signal. For comparison we also display an exact sine wave (which has no special significance other than it matches the initial part of the 'lobe')) and tried to match it to the first period of oscillation of the lobe. Note that the two signals become out of phase after a time indicating that the lobe has a 'slowly varying frequency'.

Chapter 7

Applications of Superoscillations in Gravity and Field Theory

In this chapter we try to make connections between the two seemingly disparate sections presented in this thesis. In fact some connections have already been made in the literature between superoscillations and QFT in the context of Hawking radiation. We here speculate on some further implications for the existence of superoscillations in physics. It is the authors opinion that a true understanding of QFT cannot be achieved without an appreciation of the existence of superoscillations, particularly in the light of the ultraviolet problem and its resolution in terms of a quantum gravity motivated minimal meaningful length.

7.1 The Unruh Effect

In 1.2.2 we briefly mentioned a phenomenon known as the Unruh effect in Minkowski spacetime whereby the 'no particle' Fock vacuum state as defined in inertial coordinates is seen to contain a thermal spectrum of particles in accelerated coordinates. There we drew parallels with the non-uniqueness of the vacuum state in curved spacetime where there exists no preferred class of coordinate systems analogous to the inertial Minkowskian coordinates. The radiation associated with accelerated observers in Minkowski spacetime was first studied in the context of an investigation of Hawking radiation (discussed below) by Davies in 1974 [66] based on earlier work on QFT in Rindler spacetime by Fulling [67]. The use of localized particle detectors was pioneered by Unruh in 1976 [68] from whence the phenomenon was known as the Unruh effect. For a modern review of the subject see [69].

The physical model is simple. Consider a quantum field φ and a point-like detector possessing internal quantum states $|E_n\rangle$ labeled by the energies E_n following the spacetime trajectory $x^\mu(\tau)$ parameterized by the proper time of the detector. To first order in perturbation theory the probability amplitude for the quantum field to make the transition from the vacuum state $|0\rangle$ to $|1_k\rangle$ while simultaneously the coupled detector makes the transition from the ground state $|E_0\rangle$ to the excited $|E\rangle$ at time τ is given at by[1]

$$\frac{ic\langle E|\hat{Q}_0|E_0\rangle}{(2\pi)^{3/2}\sqrt{\omega_k}}\int_{-\infty}^{\tau} \mathrm{e}^{i(E-E_0)\tau'}\mathrm{e}^{-ikx(\tau')}v_k^*(\tau')\epsilon(\tau')\,d\tau' \tag{7.1.1}$$

where $\epsilon(\tau)$ is a function determining the detector efficiency and v_k is the standard mode function evaluated along an arbitrary trajectory. It is usually assumed that this expression gives an objective measure of the content of the Minkowski vacuum state from the perspective of an observer in general (possibly non-inertial) motion at arbitrary times.

Although we will be mainly interested in the single transition amplitude, one can go further than 7.1.1 by summing over all possible final momenta k and all possible excited energy eigenstates $|E_n\rangle$ and evaluating the amplitude at $\tau = \infty$. One obtains the result [2] that the total probability of transition to arbitrary final states is given by

$$P = c^2 \sum_E |\langle E|\hat{Q}_0|E_0\rangle|^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{i(E-E_0)(\tau-\tau')}G^+(x(\tau), x(\tau'))\,d\tau d\tau'$$

where G^+ is the Wightman function associated with the quantum field given in terms of a two point function as $G^+(x, x') = \langle 0|\phi(x)\phi(x')|0\rangle$. Here, $|0\rangle$ is the unique Minkowski vacuum state minimizing the energy of the field.

It is a celebrated result [2] that when the trajectory $x(\tau)$ is one of uniform acceleration, the function P takes on the thermal character

$$P \propto \sum_E \frac{(E-E_0)|\langle E|\eta(0)|E_0\rangle|^2}{\exp(2\pi(E-E_0)\alpha)-1}$$

where α is the magnitude of acceleration. We see the appearance of the Planck factor in the denominator displaying the thermality.

7.1.1 Introduction of superoscillations

The phenomenon of superoscillations has some very counterintuitive implications for the particle concept in QFT. Examining 7.1.1 we see that the transition probability amplitude takes the form of a partial (as defined in Chepter 5) Fourier transform of a product of functions one of which is entirely phase another entirely real and the mode function. Let us assume that one is able to construct a world line such that the integrand in the formula 7.1.1 did not contain the frequency $E - E_0$. Then for $\tau \to \infty$ the transition probability would be zero since the integral would be the Fourier transform of the integrand evaluated at the missing $E - E_0$ frequency. However it would be possible to have a non-zero transition probability in the intermediate region by having the integrand superoscillate at the missing frequency. This cuts right to the heart of the seeming paradox of the superoscillation induced 'ghost resonance' described in the previous chapter. Since in Minkowski spacetime the mode function is entirely a complex phase, the lobes of the superoscillating form could be interpreted as the effect of a peculiar switching function whereas the resonance would be interpreted as an effect due to the path and mode function alone. In any case the picture of the vacuum state and in particular its 'content' is very much modified in the light of superoscillating functions that can 'pretend' to be local particles through an Unruh detector.

[1] See Appendix A for a derivation in the case of a general spacetime.

7.2 Hawking Effect

The use of superoscillations in addressing the trans-Planckian problem in the Hawking effect has been studied in the papers [70, 60]. The Hawking effect suffers from the problem that the derivation seems to make reference to arbitrarily large momentum field modes. In [71] the author argues against the existence of Hawking radiation based on the backreaction of the trans-Planckian modes on the surrounding spacetime via EE. The content of [70] is the observation that the process of Hawking radiation which usually suffers from a reference to ultra Planckian frequencies (a regime where we expect QFT to be somehow replaced by a more fundamental theory and where the energies of the quanta individually approach those able to effect the spacetime geometry itself) is able to be mimicked by a superoscillating field. Recall that a superoscillating function necessarily possesses a relatively much larger amplitude phase. This part of the field is constructed explicitly to lie inside the event horizon and out of causal contact with the rest of the universe. The main point of these studies is that the process of black hole formation and evaporation can still be robust against a Planck scale cutoff on momenta. Such a cutoff would be interpreted as a bandlimit for the fields of nature. Superoscillations allow for the possibility of trans-Planckian waveforms without trans-Planckian frequencies.

7.3 Cosmology

Recall that on proper scales larger than the proper horizon scale comoving mode functions freeze out and increase as an exponent of η or decay with the same exponent whereas on scales smaller than the horizon the mode functions oscillate as regular waves should. In the light of superoscillations, the question would be "would a collection of super-horizon modes behave as if it were sub-horizon if it were constrained to superoscillate on sub-horizon scales?" The question is non-trivial since, based on the expectation that quantum fields make use of *all possible* configurations, the dynamics of super-horizon scale modes would need to be considered in inflationary predictions as a possible source of non-scale invariance. Usually, since super-horizon modes never behave as harmonic oscillators in a theory with a finite duration of inflation, the initial conditions of super horizon modes are left un-assumed but here the existence of superoscillations makes way for the possibility of 'frozen modes at arbitrary scales'.

Chapter 8

Summary and Conclusion

In this second part of this thesis I have introduced superoscillations and conducted a numerical and analytical study of a quantum and a classical harmonic oscillator coupled to a superoscillating source. Specifically we used the methods developed in [62, 63] to construct custom made signals that superoscillate precisely at the resonant frequency of a harmonic oscillator. I found the interesting result that the driving force put the oscillator in such a state that it was exactly out of phase with the superoscillations at the beginning of the superoscillating stretch, was maximally damped down to zero amplitude precisely half way through the superoscillating stretch and consequently was driven at resonance back to the energy it possessed at the beginning of the superoscillations. At that point the signal took the form of a slowly changing low frequency wave again. This opened up a new perspective on superoscillations not found in the literature, namely that superoscillations might alternatively be regarded as the consequence of requiring a bandlimited function to possess a 'slowly varying local frequency'. We ended the analysis with some applications in gravity and QFT.

Possible future research opportunities arising from this work include a more thorough investigation of the alternative definition of superoscillations stated above as well as an investigation of the third possibility that superoscillations are those functions constrained to 'almost vanish' on an interval. Also it would be interesting to investigate the applications in the Unruh effect further to see for example whether it is possible to actually construct time like trajectories or switching functions $\epsilon(\tau)$ that implement the described 'false detector clicks'. Finally it would be interesting to study the possibility of higher dimensional superoscillations or superoscillations in Lorentzian signature spaces with an eye towards physical applications.

Appendix A

Derivation of the Unruh transition probability amplitude in arbitrary spacetime

The following is a new result not published in the literature although it is a trivial extension of the known result for Minkowski spacetime [69]. Let φ be a quantum scalar field in an arbitrary background spacetime manifold, $\hat{Q}$ be a detector observable for a detector possessing internal states $\{|E_n\rangle\}$ following the timelike worldline $x^\mu(\tau)$ and $|0\rangle$ be the vacuum state of the quantum field associated with the mode functions $\{v_k(\eta)\}$ as defined in section 1.2. Let the combined detector/ field system be initially in the state $|\psi\rangle = |E_0\rangle \otimes |0\rangle$ and let the interaction be facilitated by a term in the Hamiltonian

$$\hat{H}_{\text{Int}}(\tau) = \epsilon(\tau)\hat{Q}(\tau)\varphi(x^\mu(\tau))$$

with ϵ a small (with respect to the eigenvalues E_n and characteristic energies of interest) function. We work in the interaction picture where operators evolve with respect to the free hamiltonian and states with respect to the interacting part of the Hamiltonian. Then the time evolution operator for states may be approximated in first order perturbation theory as

$$\hat{U}(\tau) = \mathbb{1} + i\int^\tau \epsilon(\tau')\hat{Q}(\tau')\varphi(x^\mu(\tau')).$$

Hence at a time τ the transition probability for the transition to the state $|E_n\rangle \otimes |\Omega$ is given by

$$(\langle E_n| \otimes \langle\Omega|)\,\hat{U}(\tau)|\psi\rangle = i\int^\tau d\tau'\; \epsilon(\tau')\langle E_n|\hat{Q}(\tau')|E_0\rangle\langle\Omega|\varphi(\tau')|0\rangle.$$

We notice that since $\varphi = \frac{1}{\sqrt{2}}\sum_k v_k a_k + v_k^* a_{-k}$ the only non-trivial transitions allowed for the field are $|0\rangle \to |1_k\rangle$ in which case the probability amplitude is

$$\frac{i\langle E_n|\hat{Q}_0|E_0\rangle}{\sqrt{2}}\int^\tau d\tau'\; \mathrm{e}^{i\Delta E\tau'}\mathrm{e}^{ik\cdot x^i(\tau')}\epsilon(\tau')v_k^*(\eta(\tau')).$$

In the terminology of 5.2.1 this result states that the transition probability for a particle detector to be excited at time τ is given by the partial Fourier transform of the product of the mode function with an exponential and the efficiency function.

Appendix B

Derivation of the expectation of the physical energy momentum tensor

Here we compute the energy momentum tensor for the physical field φ in terms of the mode function solutions associated with the co-moving field χ. Here we perform the calculation in $3+1$ dimensions but the general methods are applicable to the calculation in $2+1$ dimensions done in chapter 2.

The energy momentum tensor in co-moving coordinates and conformal time is expressed as

$$\hat{T}_{\mu\nu} = \phi_{,\mu}\,\phi_{,\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \phi_{,\alpha}\,\phi_{,\beta}$$

so that

$$\begin{aligned}
\hat{T}_{00} &= \frac{1}{2}\left(\partial_\eta \chi a^{-1}\right)^2 + \frac{1}{2} a^{-2} (\nabla\chi)^2 && \text{(B.0.1)} \\
&= \frac{1}{2} a^{-2} \left(\chi'^2 + \chi^2 \left(\frac{a'}{a}\right)^2 - \frac{a'}{a}\left(\chi\chi' + \chi'\chi\right) \right) + \frac{1}{2} a^{-2} (\nabla\chi)^2. && \text{(B.0.2)}
\end{aligned}$$

Where here and throughout, prime will denote differentiation with respect to η.

Recall that the Lagrangian for the physical (massless, free and minimally coupled) field φ is given by

$$L = -\frac{1}{2} \int d^3x \; a^2 \varphi'^2 - a^2 (\nabla\varphi)^2$$

where we have used $\sqrt{\mathbf{g}} = a^4(\eta)$, so that

$$\pi^{(\varphi)} = \frac{\delta L}{\delta \varphi'} = a^2 \varphi'$$

and we obtain the Hamiltonian

$$\hat{H}^{(\varphi)} = \frac{1}{2} \int d^3\, x \frac{\pi^2}{a^2} + a^2 (\nabla\varphi)^2.$$

We wish to express this physical Hamiltonian in terms of the χ variables which we know explicitly as solutions to the appropriate KG equation. We have the relations

$$\chi = a\varphi'$$
$$\pi^{(\chi)} = \frac{\delta L^{(\chi)}}{\delta\chi'} = \chi' = \frac{\pi^{(\varphi)}}{a} + a'\varphi$$

so that the Hamiltonian becomes

$$\hat{H}^{(\phi)} = \frac{1}{2}\int d^3x\ (\pi^{(\chi)})^2 + \left(\frac{a'}{a}\right)^2\chi^2 - \frac{a'}{a}(\pi^{(\chi)}\chi + \chi\pi^{(\chi)}) + (\nabla\chi)^2.$$

We see that the above expression is identical in form to $\hat{T}_{00}$ which of course is expected since we know that in general

$$\hat{H} = \int d^3x\ \sqrt{-g}\hat{T}^0_0.$$

Here we have the same scale factor pre-factor due to $\hat{T}^0_0 = g^{00}\hat{T}_{00} = a^{-2}\hat{T}_{00}$ and $\sqrt{-g} = a^4$.

Now, there are two ways we can think of this quantity. One the one hand we may think of this as the Hamiltonian density existing under a spatial integral whereas on the other we write it explicitly as a function of x. In the first case we may perform such operations as

$$\begin{aligned}\int d^3x\ \pi^2(x) &= \int d^3x\ \left(\int d^3k\ (2\pi)^{-3/2}\pi_k\right)\pi(x)\\ &= \int d^3k\pi_k\Big(\int d^3x(2\pi)^{-3/2}\pi(x)\Big)\\ &= \int d^3k\pi_k\pi_{-k}\end{aligned}$$

where upon taking expectation values we obtain

$$\begin{aligned}\langle\int d^3x\ \pi^2(x)\rangle &= \frac{1}{2}\int d^3k\ \langle(a_k v'^*_k + a^\dagger_{-k}v'_{-k})(a_{-k}v'^*_{-k} + a^\dagger_k v'_k)\rangle\\ &= \frac{1}{2}\int d^3k|v'_k|^2\delta(0)\end{aligned}$$

if we follow the prescription whereby one commutes all creation operators to the left. In the second case we simply expand the operators outright to obtain

$$\begin{aligned}\langle\chi'^2(x)\rangle &= \frac{1}{2}(2\pi)^{-3}\int d^3k\ d^3k'\ \langle\big(a_{k'}v'^*_{k'} + a^\dagger_{-k'}v'_{k'}\big)\big(a_k v'^*_k + a^\dagger_{-k}v'_k\big)\rangle e^{i(k+k')x}\\ &= \frac{1}{2}\int d^3k d^3k' v'^*_{k'}v'_k e^{i(k+k')x}\delta(k+k')\\ &= \frac{1}{2}(2\pi)^{-3}\int d^3k|v'_k|^2.\end{aligned}$$

We see that the two expressions differ by a replacement by $(2\pi)^{-3}$ the factor of $\delta(0)$ in the Hamiltonian.

Inserting the mode expansion into the expression B.0.2 and taking the vacuum expectation value as is done above one obtains the general expression valid in all space-times and for any choice of vacuum

$$T_0^0 \equiv \langle \hat{T}_0^0 \rangle = \frac{1}{4} \frac{a^{-4}}{(2\pi)^3} \int d^3k \; |v_k'|^2 + \left(k^2 + \frac{a''}{a} \right) |v_k|^2 - \partial_\eta \left(\frac{a'}{a} |v_k|^2 \right). \tag{B.0.3}$$

Appendix C

On the vacuum energy in de Sitter spacetime with modified dispersion

Non-trivial dispersion relations such as

$$F^{(1)}(k) = \Lambda \tanh^{1/p}\big[\big(\frac{k}{\Lambda}\big)\big]$$
$$F^{(2)}(k) = \big(k^2 + k^2 b_m \big(\frac{k}{k_c}\big)^{2m}\big)^{1/2}$$

having the asymptotics $F^i(k) \approx k$ for small k are considered in the literature for various reasons including the trans-Planckian problems of inflation [46] and Hawking evaporation [45]. In this appendix we discuss these possibilities briefly and in the context of de Sitter spacetime.

C.0.1 Modified Bunch Davies criterion

One possibility is to introduce the dispersion function into the action for a massless scalar field written in Fourier space

$$S = \frac{1}{2}\int d\eta d^3k \big(a^2 \phi_k'^2 - a^2(\eta) k^2 \phi_k^2\big)$$
$$\longrightarrow \frac{1}{2}\int d\eta d^3k \big(a^2 \phi_k'^2 - a^4 F^2(k/a)\phi_k^2\big).$$

This modification manifests itself in the equation of motion for the scaled field $\chi = a(\eta)\phi$, usually introduced for the purpose of removing first time derivatives in the corresponding Klein Gordon equation, as

$$\hat{\chi}_k'' + \big(a^2 F^2(k/a) - \frac{a''}{a}\big)\hat{\chi}_k = 0.$$

In our test case $F(\lambda) = \Lambda \tanh(\lambda/\Lambda)$ we take the third order Taylor approximation and obtain the field equation

$$\chi_k'' + \big(k^2 + \frac{4}{9}\frac{k^6}{a^4\Lambda^4} - \frac{4}{3}\frac{k^4}{a^2\Lambda^2} - \frac{2}{\eta^2}\big)\hat{\chi}_k.$$

Since the modified dispersion relation differs from the usual dispersion only in the high (proper) k regime, the Bunch Davies vacuum identification criteria must be modified in order to identify the correct vacuum state. In other words, stipulating a modified dispersion relation in the ultraviolet conflicts with the identification of the vacuum mode-functions in the following sense: Usually, in identifying the vacuum state mode-functions, we assume that on very short length scales (equivalently for high k) the vacuum should be that of a Minkowski vacuum. We impose asymptotic conditions on the mode functions that amount to equating these functions with those of the Minkowski vacuum in the $k \to \infty$ limit. However, it is precisely in this regime that the dispersion relation is constructed to diverge from the usual dispersion. It is clear that to be consistent with the modified dispersion, we must equate in the limit the given de Sitter mode functions with, not the usual Minkowski mode functions

$$v_k = \frac{1}{\omega_k^{1/4}} \mathrm{e}^{-i\omega_k},$$

but the modified generalization of the Minkowski mode functions (sometimes called adiabatic mode functions)

$$v_k = \frac{1}{(aF(k/a))^{1/4}} \mathrm{e}^{-i\int^{\eta} d\eta' aF(k/a)}.$$

corresponding to the vacuum state in flat space-time of the field with modified dispersion. This procedure is obviously quite involved since explicit solutions to the initial value problem would seem to be difficult to obtain since the field equation is not of the form such that it can be solved by Bessel functions. We defer such a study to a future project.

C.0.2 Modified dispersion without modified wave equation

We now turn to a simpler realization of modified dispersion. In the following we will show that no suitably regular modified dispersion in de Sitter space time introduces time dependence into the Bunch Davies vacuum expectation value of the energy momentum tensor for a massless scalar field. (One may easily generalize this to a massive scalar field and also to a Power Law spacetime which is proposed to be a more accurate description of our universe during an inflationary epoch).

Let F_Λ be a real function of one variable (called the dispersion function) with the following properties:

$$\begin{aligned}
&F_\Lambda(k) \to k, \quad \text{for} \quad k \to 0, \quad F_\Lambda(0) = 0 \\
&F'_\Lambda(k) \to 1 \quad \text{for} \quad k \to 0, \quad F'_\Lambda(0) = 1 \\
&F_\Lambda(k) \to k \quad \text{for} \quad \Lambda \to \infty \\
&F_\Lambda \quad \text{possesses a Taylor expansion at } k = 0.
\end{aligned}$$

These properties characterize what we mean by a modified dispersion relation that reduces to the usual dispersion at large scales. We explicitly introduce a characteristic scale Λ, a constant in proper coordinates and which is used to generate an absolute scale to which we compare proper lengths. In practice, we interpret Λ as a Planck scale beyond which ordinary quantum field theory (even with a modified dispersion or cutoff) is assumed to break down due to non-trivial gravitational back-reaction. A modified dispersion relation is obtained by making the replacement in the energy momentum tensor

$$k^2 \mapsto a^2 F(k/a)^2. \tag{C.0.1}$$

An Explicit Example

Firstly we make a Taylor approximation to the modified dispersion relations (C.0.1). Specifically we consider the first such relation $F^{(1)}(k)$. Letting $p = 1$ for simplicity we have,

$$F^{(1)}(k) = k - \frac{2}{3}\frac{k^3}{\Lambda^2}.$$

Now, it is the purpose of a modified dispersion relation to somehow take into account the granular or quantum nature of space-time on very short proper scales. To this end we must introduce the modified behavior for very high *proper* momenta $\tilde{k}$ and make the replacement

$$k^2 \longrightarrow a^2 F^2(\tilde{k}) = a^2 F^2(k/a) = k^2 + \frac{4}{9}\frac{k^6}{a^4\Lambda^2} - \frac{4}{3}\frac{k^5}{a\Lambda^2}.$$

The vacuum expectation value of the energy density $\hat{T}^0_0$ is given in general by

$$\begin{aligned} T^0_0 \equiv \langle \hat{T}^o_0 \rangle &= \frac{a^{-4}}{8\pi^2}\int_0^\infty dk\Big[k^2|v'_k|^2 + k^2\Big(\Big(\frac{a'}{a}\Big)^2 + k^2\Big)|v_k|^2 \\ &\qquad -2k^2\frac{a'}{a}(v'^*_k + i)\Big] \end{aligned}$$

so that the non-trivial dispersion mapping gives rise to the new terms in the modified energy density expectation $\tilde{T}^0_0$,

$$\tilde{T}^0_0 = T^0_0 + \frac{a^{-4}}{8\pi^2}\int_0^\infty dk\Big[\frac{4}{9\Lambda^2}\frac{k^8}{a^4} - \frac{4}{3\Lambda^2}\frac{k^3}{a}\Big]|v_k|^2$$

Inserting the Bunch Davies mode functions

$$v_k = \sqrt{\frac{\lambda}{2}}\Big(J_{3/2}(\lambda) - iY_{3/2}(\lambda)\Big), \quad |v_k|^2 = \frac{1+\eta^2 k^2}{\eta^2 k^3},$$

where $J_{3/2}$ and $Y_{3/2}$ are bessel functions of the first and second kind respectively, $\lambda = k|\eta|$, and the de Sitter scale factor is $a(\eta) = -1/(H\eta)$, we obtain after imposing the proper wavenumber cutoff at $k_{proper} = \Lambda$ (in other words we impose the upper limit of integration $k_{max} = a(\eta)\Lambda$),

$$\delta\rho \equiv \tilde{T}^0_0 - T^0_0 = \frac{\Lambda H^2}{18\pi^2}\Big(\frac{\Lambda^3}{6} - 1\Big) + \frac{\Lambda^3}{6\pi^2}\Big(\frac{\Lambda^3}{24} - \frac{1}{5}\Big)\Big]_0^\infty$$

which again is independent of time.

Therefore we have shown that this particular modification to the dispersion relation does not produce a net surplus or net deficit of vacuum energy production in an expanding spacetime. The argument is easily generalized to arbitrary dispersion functions and arbitrarily high order Taylor approximations.

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