On the Polyhedral Lift-and-Project Rank Conjecture for the Fractional Stable Set Polytope

by

Yu Hin Au

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In this thesis, we study the behaviour of Lovász and Schrijver's lift-and-project operators N and N_0 while being applied recursively to the fractional stable set polytope of a graph. We focus on two related conjectures proposed by Lipták and Tunçel: the N-N_0 Conjecture and Rank Conjecture. First, we look at the algebraic derivation of new valid inequalities by the operators N and N_0. We then present algebraic characterizations of these valid inequalities. Tightly based on our algebraic characterizations, we give an alternate proof of a result of Lovász and Schrijver, establishing the equivalence of N and N_0 operators on the fractional stable set polytope. Since the above mentioned conjectures involve also the recursive applications of N and N_0 operators, we also study the valid inequalities obtained by these lift-and-project operators after two applications. We show that the N-N_0 Conjecture is false, while the Rank Conjecture is true for all graphs with no more than 8 nodes.

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Contents

List of Figures

Chapter 1

Introduction

The linear programming problem (LP) is the problem of optimizing a linear function subject to linear constraints. The integer programming problem (IP) is an LP with the additional requirement that all of its variables can only take on integral values.

Integer programming is a very powerful tool of modeling problems in practice, because it captures the discreteness that arises in many decision making processes, in which a choice has to be made within a finite set of alternatives. Most notably, a lot of problems involve binary variables that are used to capture the state of yes-no, build-do not build, true-false of particular objects. This is also why 0-1 programming problems make up an important sub-class of IP's.

While there are polynomial time algorithms for solving LP's (for instance, the ellipsoid method and interior-point methods), it is well known that solving IP's is an $\mathcal{NP}$-hard problem (i.e. there does not exist a polynomial time algorithm for it, unless $\mathcal{P} = \mathcal{NP}$).

Given an IP, usually the first step we take to solve it is to find a (preferably simple) description of a polyhedron P, such that the integral points in P are exactly the feasible solutions to our IP. Then, the problem of optimizing our objective function over P is called the LP*-relaxation* of our IP, and we can find an approximation of the optimal value of our original IP by solving our LP-relaxation.

However, P can be substantially larger than its integer hull (i.e. the convex hull of its integral points), and therefore our approximation can be considerably off. Therefore, it is natural to look for algorithms that, given an LP relaxation, generate inequalities that, together with valid inequalities of P, produce a smaller polyhedron that still contains all the feasible solutions to IP.

One of the classical approaches is to use Gomory-Chvátal cuts. Given an inequality $a^T x \leq \alpha$ valid for a polytope P such that a is an integral vector, we replace it by the inequality $a^T x \leq \lfloor \alpha \rfloor$. We let P' to be the polytope defined by all inequalities that can be obtained from P in this manner, and see that P' contains all the integral points in P, but could be smaller than P. Then we can apply the same process on P' and obtain a yet smaller polytope and so on. Chvátal [6] showed that this process converges to the integer hull of P in finitely many steps. However, under this approach, the number of inequalities generated at each step can be exponential and it may take a very large number of steps before the algorithm arrives at the integer hull. Moreover, the general problem of optimizing a linear function over the first Gomory-Chvátal closure is $\mathcal{NP}$-hard.

The use of lift-and-project operators to generate cuts is another approach that has recently received much attention. More specified in solving 0-1 optimization problems, the lift-and-project operators utilize the idea that a polytope's projection may have more facets than itself, and hence a polytope P that has exponentially many facets can possibly have a simple description if being represented as the projection of another polytope P' in a higher dimension that only has a polynomial number of facets.

Several different lift-and-project operators have been devised, most notably by Sherali and Adams [17], Lovász and Schrijver [16], Balas, Ceria and Cornuéjols [4], Lasserre [11], [12], and most recently by Bienstock and Zuckerberg [5]. These operators possess different properties and are of various strengths and computationally complexity. The reader is encouraged to refer to [3], [8] and [13] for comparisons of the performances of some of the above operators on several well known problems.

Given a convex polytope $S \subseteq [0,1]^n$, all of these operators can shrink S down to its integer hull in n steps. Moreover, if the number of facets of S is polynomial in n, we can optimize a linear function over the polytope obtained by applying a constant number of times any of the lift-and-project operators above to S.

In the negative direction, approximations obtained by applying the operators a constant number times can be quite limited. Take one of the Lovász-Schrijver operators N_+ (a fairly strong operator) as an example. Goemans and Tunçel [10] showed that some simple inequalities take N_+ exactly n rounds to derive. Feige and Krauthgamer [9] showed that in solving the stable set problem on a random graph, the approximate value given by applying $k = o(\log(n))$ rounds of N_+ to the fractional stable set polytope is $\sqrt{n2^{-k}}$, while the optimal value is roughly $2\log_2 n$. Morever, Alekhnovich et al. [1] proved the non-existence of subexponential approximation algorithms for MAX-3SAT, Hypergraph Vertex Cover and Minimum Set Cover using the N_+ approach.

In the thesis, we focus on two of the Lovász-Schrijver operators N_0 and N, whose precise definitions are given in Chapter 2. We want to understand how they behave in the context of approximating the stable set polytope of a graph G (denoted $STAB(G)$) from its fractional stable set polytope (denoted $FRAC(G)$).

Given a graph G, let $N_0^k(G)$ (resp. $N^k(G)$) denote the polytope obtained after recursively applying N_0 (resp. N) k times to $FRAC(G)$. Then we define the N_0-*rank* of a graph G to be the smallest integer k such that $N_0^k(G) = STAB(G)$, and denote it by $r_0(G)$. The N-*rank* of a graph and $r(G)$ are analogously defined.

While in general N is a stronger operator than N_0, Lovász and Schrijver [16] showed that they have the same performance when applied to $FRAC(G)$ for any graph G. Later, Lipták and Tunçel [15] found more results to suggest that the two operators are homogenous in this context, and came to propose the following two conjectures: the "N-N_0 Conjecture"

Conjecture 1.

$$N_0^k(G) = N^k(G) \quad \forall \textit{ graphs } G,\ \forall k \in \mathbb{N},$$

and the "Rank Conjecture"

Conjecture 2.

$$r_0(G) = r(G) \quad \forall \textit{ graphs } G.$$

While the Rank Conjecture suggests that it takes the same number of steps for N_0 and N to trim the fractional stable set polytope to the stable set polytope for any graph, the N-N_0 Conjecture requires that the two intermediate polytopes have to coincide during every step of the trimming process, and thus is stronger than the Rank Conjecture.

In Chapter 2 we give different (yet equivalent) definitions for the operators N_0 and N, study them from several different perspectives, and discuss some of their general properties.

In Chapter 3, we concentrate on their behaviour when applied recursively to the fractional stable set polytope of graphs, and the known results that support Lipták and Tunçel's conjectures. We give an alternate proof to the Lovász-Schrijver result that the polytopes obtained by applying N_0 and N to the fractional stable set polytope of any graph coincide, and are equal to the odd cycle polytope of the graph. Next we give a partial characterization of inequalities that are of N_0-rank 2. After that we give an example in which $N_0^2(G)$ is not equal to $N^2(G)$, disproving the N-N_0 Conjecture. We also slightly generalize Lipták and Tunçel's result on decomposing a graph that contains a clique cut.

In Chapter 4 we show that the Rank Conjecture holds for all graphs with no more than 7 nodes. In Chapter 5, we extend this result to all 8-node graphs and some 9-node graphs. We conclude the thesis by investigating in Chapter 6 the properties of the possible counterexamples to the Rank Conjecture.

Chapter 2

Definitions and preliminaries of N_0 and N operators

In this chapter, we give definitions and some basic properties of Lovász and Schrijver's N_0 and N operator in three different perspectives: Real Algebraic, Lifted Geometric and Geometric.

The Lifted Geometric definition of the operators involves lifting a polytope in $[0,1]^n$ to a space of dimension $O(n^2)$ and projecting it back down to another polytope in $[0,1]^n$, and is the "original" definition of the operators. However, we will rely more on the tools developed by looking into the operators from the Real Algebraic perspective when we give alternate proofs to known results and attempt to characterize inequalities of N_0-rank 2 in Chapter 3. We also give the Geometric characterization of N_0, which is elegant and more intuitive than the Real Algebraic and Lifted Geometric characterizations of it. However, there is currently no known Geometric characterization for the N operator.

2.1 Real Algebraic

2.1.1 Definitions

Given a convex polytope $P \subseteq [0,1]^n$ and $a^T x \leq b$ a facet for P, we can derive from this inequality a series of valid inequalities for $N_0(P)$. First, we consider the (nonlinear) inequalities $x_j a^T x \leq x_j b$ and $(1-x_j)a^T x \leq (1-x_j)b$ for all j between 1 and n (we treat $x_i x_j$ and $x_j x_i$ as different entities). Next, we linearize these inequalities by replacing x_i^2 by x_i, and $x_i x_j$ by y_{ij}.

We repeat the above process with every facet of P. Now for any x, we define that $x \in N_0(P)$ if and only if there exists y, such that the pair (x, y) satisfies all the derived inequalities.

More precisely, let $P := \{x : Ax \leq b\}$ for some $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$. For what follows we let $[k]$ denote the set $\{1, 2, \ldots, k\}$. Also, given a matrix $A \in \mathbb{R}^{m\times n}$ and $S \in [n]$, we let A_S denote the $m \times |T|$ matrix that is A restricted to columns whose indices are in T. In particular, we let A_i denote the i-th column of A. Then

$$
\begin{aligned}
N_0(P) := \Big\{x \ : \ \exists y \in \mathbb{R}^{n(n-1)}\ , \quad \text{s.t.} \quad & (A_j - b)x_j + \sum_{i:i\neq j} A_i y_{ij} \leq 0, \\
& b x_j + \sum_{i:i\neq j} A_i x_i - \sum_{i:i\neq j} A_i y_{ij} \leq b, \\
& \forall j \in [n]\Big\}\,.
\end{aligned} \tag{2.1}
$$

The variables in x, y are ordered as

$$x = (x_1, x_2, \ldots, x_n)^T$$

and

$$y = (y_{21}, y_{31}, \ldots, y_{n1}, y_{12}, y_{32}, \ldots, y_{n2}, \ldots, y_{(n-1)n})^T.$$

Note that now we can express $N_0(P)$ as $\{x : A'x + B'y \leq b'\}$, where $A' \in \mathbb{R}^{2mn\times n}, B' \in \mathbb{R}^{2mn\times(n-1)n}, b' \in \mathbb{R}^{2mn}$,

$$
A' = \begin{pmatrix}
A_1 - b & 0 & \ldots & 0 \\
0 & A_2 - b & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & A_n - b \\
b & A_2 & \ldots & A_n \\
A_1 & b & \ldots & A_n \\
\vdots & \vdots & \ddots & \vdots \\
A_1 & A_2 & \ldots & b
\end{pmatrix},
$$

$$B' = \begin{pmatrix} A_{[n]\setminus\{1\}} & 0 & \dots & 0 \\ 0 & A_{[n]\setminus\{2\}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{[n]\setminus\{n\}} \\ -A_{[n]\setminus\{1\}} & 0 & \dots & 0 \\ 0 & -A_{[n]\setminus\{2\}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -A_{[n]\setminus\{n\}} \end{pmatrix} \quad \text{and} \quad b' = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \\ b \\ \vdots \\ b \end{pmatrix}.$$

We can define $N(P)$ analogously by replicating the derivation of the inequalities, but this time replacing both $x_i x_j$ and $x_j x_i$ by y_{ij} (as x_i, x_j commute and x_i, x_j are 0,1 variables).

In the matrix representation, we have

$$\begin{aligned} N(P) := \Big\{ x : \exists y \in \mathbb{R}^{\frac{n(n-1)}{2}}, \quad \text{s.t.} \quad & (A_j - b)x_j + \sum_{i:i<j} A_i y_{ji} + \sum_{i:i>j} A_i y_{ij} \leq 0, \\ & b x_j + \sum_{i:i\neq j} A_i x_i - \sum_{i:i<j} A_i y_{ji} - \sum_{i:i>j} A_i y_{ij} \leq b, \\ & \forall j \in [n] \Big\}. \end{aligned} \tag{2.2}$$

In this case the variables in x, y are ordered as

$$x = (x_1, x_2, \dots, x_n)^T$$

and

$$y = (y_{21}, y_{31}, \dots, y_{n1}, y_{32}, y_{42}, \dots, y_{n2}, \dots, y_{n(n-1)})^T.$$

Similar to the case in N_0, we can find A'', B'', b'' such that $N(S) = \{x : A''x + B''y \leq b''\}$.

We observe from the derivation process that $A'' = A', b'' = b'$ and $B'' = \begin{pmatrix} \bar{B}'' \\ -\bar{B}'' \end{pmatrix}$, where

$$\bar{B}'' := \begin{pmatrix} A_{[n]\setminus[1]} & 0 & 0 & \dots & 0 \\ A_1 \otimes e_1^T & A_{[n]\setminus[2]} & 0 & \dots & 0 \\ A_1 \otimes e_2^T & A_2 \otimes e_1^T & A_{[n]\setminus[3]} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_1 \otimes e_{n-2}^T & A_2 \otimes e_{n-3}^T & A_3 \otimes e_{n-4}^T & \dots & A_{[n]\setminus[n-1]} \\ A_1 \otimes e_{n-1}^T & A_2 \otimes e_{n-2}^T & A_3 \otimes e_{n-3}^T & \dots & A_{n-1} \otimes e_1^T \end{pmatrix}$$

where $\otimes$ denotes the Kronecker product operation and e_i denotes the i-th unit vector. We will also use e_i's to denote edges in graphs in the subsequent chapters, but it will be clear from the context whether a particular e_i denotes a vector or an edge.

Note that in the bottom of the last column of $\bar{B}''$, $A_{[n]\setminus[n-1]}$ is simply A_n, and $A_{n-1} \otimes e_1^T$ is just A_{n-1}. These expressions are stated in a somewhat clumsy way to make the structure of the matrix more visible. Also, the e_i's above have various sizes, with the ones associated with A_i having size $(n - i)$, for every $i \in [n - 1]$.

Now in both descriptions above, we can "project away" the variable y to give a description of $N_0(P)$ and $N(P)$ that only involves the variable x. Namely, we use nonnegative linear combinations of the inequalities to eliminate the y variable. First, for $N_0(P)$, define a cone $U' := \left\{u : u \geq 0, u^T B' = 0\right\}$. Then, it follows from LP duality that $N_0(P)$ can be rewritten as $\bigcap_{u \in U'} \left\{x : u^T A' x \leq u^T b'\right\}$. In particular, since U' is a cone, we only have to take a u from each of the extreme rays of U' (because other inequalities are implied by those induced by them). We define for any cone K that

$$\text{ext}\,(K) := \{u : u \text{ is an extreme ray of } K, ||u||_1 = 1\}\,.$$

Then we have

$$N_0(P) = \left\{x : u^T A' x \leq u^T b',\ \forall u \in \text{ext}\,(U')\right\}.$$

Similarly for $N(P)$, we can define $U'' := \left\{u : u \geq 0, u^T B'' = 0\right\}$, and we have

$$N(P) = \left\{x : u^T A'' x \leq u^T b'',\ \forall u \in \text{ext}\,(U'')\right\}.$$

Note that since U', U'' are both polyhedral cones, $\text{ext}\,(U')$ and $\text{ext}\,(U'')$ are finite, and hence both $N_0(P)$ and $N(P)$ are polyhedral as well.

It should be noted that the N_0 operator has some resemblance to the Balas-Ceria-Cornuéjols operator. This will be clear when we give the Geometric definition of N_0 in Section 2.3. Also, it is well known that the Sherali-Adams operator coincides with the N operator for the first step [16], but is slightly stronger than N in the subsequent steps [13].

2.1.2 Analysis on N_0

Next, we look into the N_0 operator more closely. Suppose $u \in \mathbb{R}^{mn}$. For every $i \in [n]$, we define the vector $u^{(i)}$ such that $u_j^{(i)} = u_{(i-1)m+j}$ $\forall j \in [m]$ (i.e. u is the concatenation of $u^{(1)}, \ldots, u^{(n)}$). Also, given a matrix $V \in \mathbb{R}^{m \times n}$, we let $\text{vec}\,(V)$ denote the vector in $\mathbb{R}^{mn}$ formed by stacking up the columns of V. Conversely, given a vector $v \in \mathbb{R}^n$ and an integer i that divides n, we define $\text{Mat}_i\,(v)$ to be the $i \times \frac{n}{i}$ matrix such that $\text{vec}\,(\text{Mat}_i\,(v)) = v$. Finally, we let $\bar{B}'$ denote the upper half of B' (so $B' = \begin{pmatrix} \bar{B}' \\ -\bar{B}' \end{pmatrix}$), $\text{Null}\,(A)$ be the null space of A, $\mathbb{D}^n$ denote the set of $n \times n$ diagonal matrices, and I_n denote the $n \times n$ identity matrix. The dimension of I may not be specified in the contexts in which it is clear.

Also, given a vector $v \in \mathbb{R}^n$, we define $v^+, v^- \in \mathbb{R}^n$ such that $v_j^+ := \max\{v_j, 0\}$ $\forall j \in [n]$ and $v_j^- := \max\{-v_j, 0\}$ $\forall j \in [n]$. Notice that both $v^+, v^- \geq 0$ and $v = v^+ - v^-$.

With the above notations, we can give a few alternative characterizations for U'.

Proposition 3. *Suppose $u \in \mathbb{R}^{mn}$. The following are equivalent.*

1. $\begin{pmatrix} u^+ \\ u^- \end{pmatrix} \in U'$;
2. $u \in \text{Null}\left(\bar{B}'^T\right)$;
3. $A^T \text{Mat}_m\,(u) \in \mathbb{D}^n$;
4. $\text{Mat}_n\left((I_n \otimes A^T) u\right) \in \mathbb{D}^n$.

Proof. ((1) $\iff$ (2)) Immediate from the definition of U' and the construction of u^+ and u^-.

((2) $\iff$ (3)) We observe that

$$\begin{aligned}
& \begin{pmatrix} u^+ \\ u^- \end{pmatrix} \in \text{Null}\left(\bar{B}'^T\right) \\
\iff \quad & u^{(j)} \in \text{Null}\left((A_{[n]\setminus\{j\}})^T\right), \ \forall j \in [n] \\
\iff \quad & (u^{(j)})^T A_i = 0 \ \forall i, j \in [n], i \neq j \\
\iff \quad & A^T \text{Mat}_m(u) \in \mathbb{D}^n.
\end{aligned}$$

((3) $\iff$ (4)) This holds because

$$A^T \text{Mat}_m(u) = A^T \text{Mat}_m(u) I_n = \text{Mat}_n\left((I_n \otimes A^T)\text{vec}(\text{Mat}_m(u))\right) = \text{Mat}_n\left((I_n \otimes A^T)u\right).$$

Note that the second equality above follows readily from the fact that, for any matrices P, Q, R such that PQR is well-defined, the identity $\text{vec}(PQR) = (R^T \otimes P)\text{vec}(Q)$ holds. $\square$

Now we give a few lemmas that help characterize $\text{ext}(U')$ and $\text{ext}(U'')$. First, given any $x \in \mathbb{R}^n$, we let $\text{supp}(x) = \{i \in [n] : x_i \neq 0\}$ to denote the *support* of x. Then we define that, given a set $S \subseteq \mathbb{R}^n$,

$$S_{min} := \{s \in S \setminus \{0\} : \nexists s' \in S \setminus \{0\} \text{ s.t. } \text{supp}(s') \subset \text{supp}(s)\}.$$

I.e. S_{min} is the set of non-zero elements in S which are minimal (containment-wise) with respect to their supports.

Since both U' and U'' are cones that are an intersection of the nonnegative orthant with a linear subspace (namely $\text{Null}\left(B'^T\right)$ and $\text{Null}\left(B''^T\right)$), the following lemma is useful.

Lemma 4. *Suppose $K = \mathbb{R}^n_+ \cap \mathcal{L}$ where $\mathcal{L}$ is a linear subspace. Let $u \in K$ such that $||u||_1 = 1$, then*

$$u \in \text{ext}(K) \iff u \in K_{min}.$$

Proof. ($\Rightarrow$) Suppose $u \in \text{ext}(K)$ but $u \notin K_{min}$. Then we have $u' \in K \setminus \{0\}$ such that $\text{supp}(u') \subset \text{supp}(u)$. We take $\lambda := \min\left\{\frac{u_i}{u'_i} \ : \ i \in \text{supp}(u')\right\}$. Now both $(u - \lambda u'), \lambda u'$ belong to $K \setminus \{0\}$, are not multiples of u, and sum up to u, contradicting $u \in \text{ext}(K)$.

($\Leftarrow$) Suppose we are given $u \in K_{min}$ such that $||u||_1 = 1$, but $u \notin \text{ext}\,(K)$. Then there exist vectors $v, w \in K$ such that neither v nor w is a multiple of u, and $v + w = u$. Also, since $v, w \geq 0$, our assumption on u implies that $\text{supp}\,(u) = \text{supp}\,(v) = \text{supp}\,(w)$. Now let $\lambda := \min \left\{ \frac{u_i}{v_i} \ : \ i \in \text{supp}\,(u) \right\}$. Then $(u - \lambda v) \in K \setminus \{0\}$ but $\text{supp}\,(u - \lambda v) \subset \text{supp}\,(u)$, a contradiction. $\square$

Since both B' and B'' possess some special structures, the following two lemmas are telling for U' and U''.

Lemma 5. *Let $A \in \mathbb{R}^{m \times n}$. Then*

$$\begin{aligned} & \text{ext}\left(\left\{ x : x^T \begin{pmatrix} A \\ -A \end{pmatrix} = 0, x \geq 0 \right\} \right) \\ = \ & \left\{ \begin{pmatrix} v^+ \\ v^- \end{pmatrix} : v \in \text{Null}\left(A^T\right)_{min}, ||v||_1 = 1 \right\} \\ \cup \ & \left\{ \frac{1}{2} \begin{pmatrix} e_i \\ e_i \end{pmatrix} \ : \ \exists k \ s.t. \ A_{ik} \neq 0 \right\}. \end{aligned}$$

Proof. Suppose $x := \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix}$, where $x^{(1)}, x^{(2)} \in \mathbb{R}^m$. We show that

$$\begin{aligned} & \text{ext}\left(\left\{ x : x^T \begin{pmatrix} A \\ -A \end{pmatrix} = 0, x \geq 0 \right\} \right) \cap \left\{ x : x^{(1)} = x^{(2)} \right\} \\ = \ & \left\{ \frac{1}{2} \begin{pmatrix} e_i \\ e_i \end{pmatrix} \ : \ \exists k \text{ s.t. } A_{ik} \neq 0 \right\} \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} & \text{ext}\left(\left\{ x : x^T \begin{pmatrix} A \\ -A \end{pmatrix} = 0, x \geq 0 \right\} \right) \cap \left\{ x : x^{(1)} \neq x^{(2)} \right\} \\ = \ & \left\{ \begin{pmatrix} v^+ \\ v^- \end{pmatrix} : v \in \text{Null}\left(A^T\right)_{min}, ||v||_1 = 1 \right\}. \end{aligned} \tag{2.4}$$

($\subseteq$ for (2.3)) Suppose $||x||_1 = 1, x \geq 0, x^{(1)} = x^{(2)}$ but $x \notin \left\{ \frac{1}{2} \begin{pmatrix} e_i \\ e_i \end{pmatrix} : \ \exists k \text{ s.t. } A_{ik} \neq 0 \right\}$. If $x^{(1)} = x^{(2)} = \lambda e_i$ for some i then λ has to equal $\frac{1}{2}$ (since $||x||_1 = 1$), which implies that $A_{ik} = 0 \ \forall k \in [n]$. We let $x' := \begin{pmatrix} e_i \\ 0 \end{pmatrix}$. Then $x' \neq 0, \operatorname{supp}(x') \subset \operatorname{supp}(x)$ and $x'^T \begin{pmatrix} A \\ -A \end{pmatrix} = 0$ (because in this case the i-th row of A is all zeros), hence by Lemma 4 $x \notin \operatorname{ext}\left(\left\{ x : x^T \begin{pmatrix} A \\ -A \end{pmatrix} = 0, x \geq 0 \right\}\right)$.

Otherwise, there exist j, k such that both $x_j^{(1)}, x_k^{(1)} > 0$. We construct x' such that $x'^{(1)} = x'^{(2)} = e_j$. Obviously $x'^T \begin{pmatrix} A \\ -A \end{pmatrix} = 0$ and $\operatorname{supp}(x') \subset \operatorname{supp}(x)$ (because $x_k^{(1)} \neq 0$), so again $x \notin \operatorname{ext}\left(\left\{ x : x^T \begin{pmatrix} A \\ -A \end{pmatrix} = 0, x \geq 0 \right\}\right)$.

($\supseteq$ for (2.3)) Suppose we have an x such that $x^{(1)} = x^{(2)} = \frac{1}{2} e_i$ for some i, but $x \notin \operatorname{ext}\left(\left\{ x : x^T \begin{pmatrix} A \\ -A \end{pmatrix} = 0, x \geq 0 \right\}\right)$. Then we consider $x' := \begin{pmatrix} e_i \\ 0 \end{pmatrix}$ and $x'' := \begin{pmatrix} 0 \\ e_i \end{pmatrix}$. Since $|\operatorname{supp}(x)| = 2$, it follows from Lemma 4 that either x' or x'' is in $\left\{ x : x^T \begin{pmatrix} A \\ -A \end{pmatrix} = 0, x \geq 0 \right\}$, and each of them implies that $A_{ik} = 0 \ \forall k \in [n]$.

($\subseteq$ for (2.4)) Suppose $x \in \operatorname{ext}\left(\left\{ x : x^T \begin{pmatrix} A \\ -A \end{pmatrix} = 0, x \geq 0 \right\}\right)$, , $||x||_1 = 1, x \geq 0$, $x^{(1)} \neq x^{(2)}$ but $x \notin \left\{ \begin{pmatrix} v^+ \\ v^- \end{pmatrix} : v \in \operatorname{Null}\left(A^T\right)_{min}, ||v||_1 = 1 \right\}$.

If $\nexists v \in \operatorname{Null}\left(A^T\right)$ such that $x^{(1)} = v^+, x^{(2)} = v^-$, then $\exists i$ such that $x_i^{(1)}, x_i^{(2)} > 0$. Then $x' := \begin{pmatrix} e_i \\ e_i \end{pmatrix}$ is a certificate that x is not minimal in $\left\{ x : x^T \begin{pmatrix} A \\ -A \end{pmatrix} = 0, x \geq 0 \right\}$.

If $\operatorname{supp}\left(x^{(1)}\right) \cap \operatorname{supp}\left(x^{(2)}\right) = \emptyset$, then $v := x^{(1)} - x^{(2)}$ satisfies $v \in \operatorname{Null}\left(A^T\right), x^{(1)} = v^+$ and $x^{(2)} = v^-$. If v is not minimal in $\operatorname{Null}\left(A^T\right)$, then we have v' such that $v'^T A = 0$ and $\operatorname{supp}(v') \subset \operatorname{supp}(v)$. Define $\lambda := \min\left\{ \frac{|v_j|}{|v'_j|} : j \in \operatorname{supp}(v') \right\}$ and let $v'' := v - \lambda v'$. Define

$x' := \begin{pmatrix} v''^{+} \\ v''^{-} \end{pmatrix}$ and we see that $x \geq 0$, $x'^T \begin{pmatrix} A \\ -A \end{pmatrix} = 0$ and $\operatorname{supp}(x') \subset \operatorname{supp}(x)$. Therefore $x \notin \operatorname{ext}\left(\left\{x : x^T \begin{pmatrix} A \\ -A \end{pmatrix} = 0, x \geq 0\right\}\right)$.

($\supseteq$ for (2.4)) If $x^{(1)} \neq x^{(2)}$ and $\exists v \in \operatorname{Null}\left(A^T\right)_{min}$ such that $x^{(1)} = v^+, x^{(2)} = v^-$ but $x \notin \operatorname{ext}\left(\left\{x : x^T \begin{pmatrix} A \\ -A \end{pmatrix} = 0, x \geq 0\right\}\right)$, then again by Lemma 4 we have a $x' \in \left\{x : x^T \begin{pmatrix} A \\ -A \end{pmatrix} = 0, x \geq 0\right\} \setminus \{0\}$ such that $\operatorname{supp}(x') \subset \operatorname{supp}(x)$. But now we have $x'^{(1)} - x'^{(2)} \in \operatorname{Null}\left(A^T\right)$, contradicting the minimality of v. $\square$

Note that given a polytope $P := \{x : Ax \leq b\}$, while we may assume that A does not have a row of zeros, we will need to apply the above lemma on $A_{[n]\setminus\{i\}}$, which may have a row of zeros (say, when $-x_i \leq 0$ is a facet of P).

Lemma 6. *Suppose we have $A^{(1)}, A^{(2)}, \ldots, A^{(k)}$, with $A^{(i)} \in \mathbb{R}^{m_i \times n_i}$ and*

$$K_i = \left\{x : x^T A^{(i)} = 0, x \geq 0\right\} \ \forall i \in [k].$$

Define

$$K := \left\{ x : x^T \begin{pmatrix} A^{(1)} & 0 & \ldots & 0 \\ 0 & A^{(2)} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A^{(k)} \end{pmatrix} = 0, x \geq 0 \right\}.$$

Suppose $x = (x^{(1)} \oplus x^{(2)} \oplus \cdots \oplus x^{(k)})$, where $x^{(i)} \in \mathbb{R}^{m_i}$ for every $i \in [k]$. Then $x \in \operatorname{ext}(K)$ if and only if $\exists j \in [k]$ such that $x^{(j)} \in \operatorname{ext}(K_j)$, and $x^{(l)} = 0\ \forall l \neq j$.

Proof. ($\Rightarrow$) Suppose we have $x \in K$, $||x||_1 = 1$, and there does not exist $j \in [k]$ such that $x^{(j)} \in \operatorname{ext}(K_j)$ and $x^{(l)} = 0\ \forall l \neq j$.

If $\exists p, q$ such that both $x^{(p)}, x^{(q)} \neq 0$, then the fact $x \in K$ implies that $x^{(p)} \in K_p$ and $x^{(q)} \in K_q$. Now we consider x' such that $x'^{(p)} = x^{(p)}, x'^{(l)} = 0\ \forall l \neq p$. Now $x' \neq 0$ (because $x^{(p)} \neq 0$) and $\operatorname{supp}(x') \subset \operatorname{supp}(x)$ (because $x^{(q)} \neq 0$). However, now we have $x' \in K$, hence $x \notin \operatorname{ext}(K)$ by Lemma 4.

If $\exists p$ such that $x^{(p)} \neq 0, x^{(l)} = 0\ \forall l \neq p$ but $x^{(p)} \notin \mathrm{ext}\,(K_p)$, then we just take any $y \in \mathrm{ext}\,(K_p)$, construct x' such that $x'^{(p)} = y, x'^{(l)} = 0\ \forall l \neq p$. Again by Lemma 4, $x \notin \mathrm{ext}\,(K)$.

($\Leftarrow$) Suppose we have $x \in K$ such that $x^{(p)} \in \mathrm{ext}\,(K_p)$ and $x^{(l)} = 0\ \forall l \neq p$. Assume for a contradiction that $x \notin \mathrm{ext}\,(K)$. Then by Lemma 4 we have a $x' \in K \setminus \{0\}$ such that $\mathrm{supp}\,(x') \subset \mathrm{supp}\,(x)$. This condition implies that $x'^{(p)} \in K_p$ and $\mathrm{supp}\left(x'^{(p)}\right) \subset \mathrm{supp}\left(x^{(p)}\right)$, which contradicts the assumption that $x^{(p)} \in \mathrm{ext}\,(K_p)$. $\square$

With the above lemmas, we are ready to give a complete characterization for $\mathrm{ext}\,(U')$.

Proposition 7. *Suppose $u \in U'$ and $||u||_1 = 1$. Then $u \in \mathrm{ext}\,(U')$ if and only if there exists a special index $i \in [n]$ such that*

1. *either $u^{(i)} = u^{(n+i)} = \frac{1}{2}e_j$ for some j and the j-th row of $A_{[n]\setminus\{i\}}$ is not all zeros, or $\exists v \in \mathrm{Null}\left(A^T_{[n]\setminus\{i\}}\right)_{min}$ such that $u^{(i)} = v^+, u^{(n+i)} = v^-$.*
2. *$u^{(j)} = 0\ \forall j \notin \{i, n+i\}$;*

Now we can have yet another description of $N_0(S)$.

Proposition 8. *Let $S := \{x : Ax \leq b\}$. Then $N_0(S)$ equals the intersection of S with*

$$\bigcap_{i\in[n]} \left\{x : v^T(A_i - b)x_i + (v^-)^T Ax \leq (v^-)^T b, \forall v \in \mathrm{Null}\left((A_{[n]\setminus\{i\}})^T\right)_{min}\right\}.$$

Proof. We know that $N_0(S) = \left\{x : u^T A' x \leq u^T b', \forall u \in \mathrm{ext}\,(U')\right\}$. From Proposition 7, for every $u \in \mathrm{ext}\,(U')$, there is a special index. Let R_i be the set of extreme rays in $\mathrm{ext}\,(U')$ that have special index i (so $\bigcup_{i\in[n]} R_i = \mathrm{ext}\,(U')$). Then we have

$$\begin{aligned}
N_0(S) &= \left\{x : u^T A' x \leq u^T b', \forall u \in \mathrm{ext}\,(U')\right\} \\
&= \left(\bigcap_{i\in[n]} \left\{x : u^T A' x \leq u^T b', \forall u \in R_i\right\}\right) \\
&= \left(\bigcap_{i\in[n]} \left\{x : u^T A' x \leq u^T b', \forall u \in R_i, u^{(i)} = u^{(n+i)}\right\}\right) \cap \\
&\quad \left(\bigcap_{i\in[n]} \left\{x : u^T A' x \leq u^T b', \forall u \in R_i, u^{(i)} \neq u^{(n+i)}\right\}\right).
\end{aligned}$$

For $u \in R_i$, we have

$$\begin{aligned}
& (u^T A')x \leq u^T b' \\
\iff & \left(u^{(i)^T}(A_i - b)\right) x_i + (u^{(n+i)^T} b)x_i + \sum_{j \in [n] \setminus i} (u^{(n+i)^T} A_j)x_j \leq u^{(n+i)^T} b \\
\iff & (u^{(i)} - u^{(n+i)})^T (A_i - b)x_i + u^{(n+i)^T} Ax \leq u^{(n+i)^T} b.
\end{aligned}$$

Also from Proposition 7 we know that $u \in R_i, u^{(i)} \neq u^{(n+i)} \iff \exists v \in \text{Null}\left((A_{[n]\setminus\{i\}})^T\right)_{min}$ such that $u^{(i)} = v^+, u^{(n+i)} = v^-$. So, it is apparent that

$$\begin{aligned}
& \left\{x : u^T A' x \leq u^T b', \forall u \in R_i, u^{(i)} \neq u^{(n+i)}\right\} \\
= & \left\{x : v^T (A_i - b)x_i + (v^-)^T Ax \leq (v^-)^T b, \forall v \in \text{Null}\left((A_{[n]\setminus\{i\}})^T\right)_{min}\right\}
\end{aligned}$$

for every i.

Now, to complete the proof, it suffices to show that

$$S = \bigcap_{i \in [n]} \left\{x : u^T A' x \leq u^T b', \forall u \in R_i, u^{(i)} = u^{(n+i)}\right\}.$$

To show $\subseteq$ we observe that

$$\begin{aligned}
& \left\{u \in R_i, u^{(i)} = u^{(n+i)}\right\} \\
\subseteq & \left\{u \in \mathbb{R}^{2mn} : u^{(i)} = u^{(n+i)} = \frac{1}{2} e_j, u^{(k)} = 0,\ \forall k \notin \{i, n+i\}, j \in [m]\right\}.
\end{aligned}$$

Then we have

$$\begin{aligned}
& \bigcap_{i \in [n]} \left\{x : u^T A' x \leq u^T b', \forall u \in R_i, u^{(i)} = u^{(n+i)}\right\} \\
\supseteq & \bigcap_{\substack{i \in [n] \\ j \in [n]}} \left\{x : u^T A' x \leq u^T b', \forall u \in \mathbb{R}^{2mn} : u^{(i)} = u^{(n+i)} = \frac{1}{2} e_j, u^{(k)} = 0,\ \forall k \notin \{i, n+i\}\right\} \\
= & \bigcap_{\substack{i \in [n] \\ j \in [m]}} \left\{x : \sum_{k=1}^{n} \frac{A_{jk}}{2} x_k \leq \frac{1}{2} b_j\right\} \\
= & \bigcap_{i \in [n]} \{x : Ax \leq b\} \\
= & \{x : Ax \leq b\} \\
= & S.
\end{aligned}$$

For the reverse containment, since A does not contain a row of all zeros, we see that for every $j \in [m], \exists j' \in [n]$ such that $A_{jj'} \neq 0$. We then pick $i \in [n] \setminus \{j'\}$. By Proposition 7, if we have $u^{(i)} = u^{(n+i)} = \frac{1}{2}e_j$ and $u^{(l)} = 0 \ \forall l \notin \{i, n+i\}$, then this $u \in \operatorname{ext}(U')$. In particular, $u \in R_i$ with $u^{(i)} = u^{(n+i)}$. So we obtain that $\sum_{k=1}^{n} \frac{A_{jk}}{2} x_k \leq \frac{1}{2} b_j$ is a valid inequality of the set on the right hand side for every $j \in [m]$, hence it is contained in S. □

We call a set S *lower comprehensive* if $\forall x \in S, \ \forall y \leq x, y \in S$, and a *convex corner* a compact, convex set contained in $\mathbb{R}^n_+$ that is lower comprehensive. Since the objects of our main focus are all convex corners, it is worthwhile to look into the specialization of Proposition 8 on convex corners. In particular, the following result is needed in our proof of $N_0(G) = OC(G)$ in Chapter 3.

Corollary 9. *Let $S := \{x : Ax \leq b, x \geq 0\}$, such that the matrix A only has nonnegative entires. Suppose we have sets $T_1, \ldots, T_n$ such that*

$$T_i \supseteq \left\{ v : \quad \nexists v' \neq 0, \operatorname{supp} \begin{pmatrix} v' \\ A^T v' - (A^T v')_i e_i \end{pmatrix} \subset \operatorname{supp} \begin{pmatrix} v \\ A^T v - (A^T v)_i e_i \end{pmatrix} \right\} \tag{2.5}$$

for every $i \in [n]$. Then $N_0(S)$ equals the intersection of S and

$$\bigcap_{i \in [n]} \left\{ x : v^T (A_i - b) x_i + \left((v^-)^T A - (v^T A - (v^T A)_i e_i)^{-^T} \right) x \leq (v^-)^T b, v \in T_i \right\}.$$

Proof. Let $\bar{A} = \begin{pmatrix} A \\ -I \end{pmatrix}$ and $\bar{b} = \begin{pmatrix} b \\ 0 \end{pmatrix}$, then $S := \left\{ x : \bar{A} x \leq \bar{b} \right\}$. And by Proposition 8 we know that

$$N_0(S) = \bigcap_{i \in [n]} \Bigg\{ x : \left(v^T (A_i - b) - d_i \right) x_i + \left((v^-)^T A - (d^-)^T \right) x \leq (v^-)^T b \, , \\ \begin{pmatrix} v \\ d \end{pmatrix} \in \operatorname{Null} \left(\begin{pmatrix} A \\ -I \end{pmatrix}^T_{[n] \setminus \{i\}} \right)_{min} \Bigg\}.$$

We see that, for any fixed i,

$$\begin{pmatrix} v \\ d \end{pmatrix} \in \operatorname{Null} \left(\begin{pmatrix} A \\ -I \end{pmatrix}^T_{[n] \setminus \{i\}} \right) \iff (v^T A)_j = d_j \ \forall j \neq i.$$

Also, by the minimality assumption and the fact that A only has nonnegative entries, if $d_i \neq 0$ for some i, then we may assume that $d = e_i$ and $v = 0$. In this case, the pair (v, d) induces the constraint $0 \leq 0$. Therefore, we can assume that $d_i = 0$, and hence $d = (v^T A) - (v^T A)_i e_i$.

Observe that for every $v \in \mathbb{R}^m$, we know that $\begin{pmatrix} v \\ A^T - (A^T v)_i e_i \end{pmatrix} \in \text{Null}\left(\begin{pmatrix} A \\ -I \end{pmatrix}^T_{[n]\setminus\{i\}}\right)$.
So the statement is true when $T_i = \mathbb{R}^m \ \forall i \in [n]$. Also by minimality, the statement is also true when

$$T_i = \left\{ v : \quad \nexists v' \neq 0, \text{supp}\begin{pmatrix} v' \\ A^T v' - (A^T v')_i e_i \end{pmatrix} \subset \text{supp}\begin{pmatrix} v \\ A^T v - (A^T v)_i e_i \end{pmatrix} \right\} \ \forall i \in [n].$$

Therefore, the statement is true when all T_i's are in between. $\square$

We saw from above that when S is contained in the nonnegative orthant, every $v \in \mathbb{R}^m$ produces a valid inequality for $N_0(S)$. We say that the constraint is "induced" by v.

We want to characterize the v's that induce constraints that are facets of $N_0(S)$. Before we can do that, we first state a few weaker results.

Proposition 10. *Suppose $S := \{x : Ax \leq b\}$ and $v \in \text{Null}\left((A_{[n]\setminus\{i\}})^T\right)$. If the inequality induced by v is not valid for S, then $0 < v^T b < v^T A_i$.*

Proof. Since $v^T(A_{[n]\setminus\{i\}}) = 0$, we know that $v^+ A_k = v^- A_k \ \forall k \in [n] \setminus \{i\}$. Therefore, when we consider the inequality induced by v, we have

$$v^T(A_i - b)x_i + (v^-)^T Ax \leq (v^-)^T b \tag{2.6}$$
$$\iff \quad (-v^T b)x_i + (v^+)^T Ax \leq -v^T b + (v^+)^T b. \tag{2.7}$$

If $v^T b \leq 0$, then (2.7) is a positive linear combination of valid inequalities that define S, hence the new inequality is valid for S. Also, if in (2.6) we had $v^T b \geq v^T A_i$, then this inequality is again implied by the inequalities that define S. $\square$

Corollary 11. *Suppose $S := \{x : Ax \leq b, x \geq 0\}$ and $v \in \mathbb{R}^m$. If the inequality induced by v is not valid for S, then $0 < v^T b < v^T A_i$.*

Proof. Since $\begin{pmatrix} v \\ A^Tv - (A^Tv)_i e_i \end{pmatrix}^T \begin{pmatrix} b \\ 0 \end{pmatrix} = v^T b$ and $\begin{pmatrix} v \\ A^Tv - (A^Tv)_i e_i \end{pmatrix}^T \begin{pmatrix} A \\ -I \end{pmatrix}_i = v^T A_i$, the claim follows from Proposition 10. ∎

2.1.3 Analysis on N

We now turn our attention to U'' and $\operatorname{ext}(U'')$. First, given $A \in \mathbb{R}^{m\times n}$, we define $\tilde{A} \in \mathbb{R}^{n^2\times mn}$ such that

$$\tilde{A} := \begin{pmatrix} I_n \otimes A_1^T \\ I_n \otimes A_2^T \\ \vdots \\ I_n \otimes A_n^T \end{pmatrix}.$$

We also let $\tilde{\mathbb{S}}^n$ denote the set of $n\times n$ skew-symmetric matrices, and $\operatorname{tril} : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{\frac{n(n-1)}{2}}$ be the operator that maps a $n\times n$ matrix to its lower diagonal part (without the diagonal). Then like Proposition 3, we can have the following for U'':

Proposition 12. *Suppose $u \in \mathbb{R}^{mn}, u \geq 0$. Then the following are equivalent.*

1. $\begin{pmatrix} u^+ \\ u^- \end{pmatrix} \in U''$;
2. $u \in \operatorname{Null}\left(\bar{B}''^T\right)$;
3. $A^T \operatorname{Mat}_m(u) \in \tilde{\mathbb{S}}^n + \mathbb{D}^n$;
4. $\operatorname{tril}\left(\operatorname{Mat}_n\left(((I_n \otimes A^T) + \tilde{A})u\right)\right) = 0.$

Proof. ((1) $\iff$ (2)) Immediate from the definition of U'' and the construction of u^+ and u^-.

((2) $\iff$ (3)) We observe that

$$\begin{aligned} & u \in \operatorname{Null}\left(\bar{B}''^T\right) \\ \iff & (u^{(j)})^T A_i = -(u^{(i)})^T A_j,\ \forall i,j \in [n], i \neq j \\ \iff & A^T \operatorname{Mat}_m(u) \in \tilde{\mathbb{S}}^n + \mathbb{D}^n. \end{aligned}$$

((3) $\iff$ (4)) We have

$$\begin{aligned} & A^T \mathrm{Mat}_m(u) \in \tilde{\mathbb{S}}^n + \mathbb{D}^n \\ \iff\ & \mathrm{tril}\left((A^T \mathrm{Mat}_m(u) + (A^T \mathrm{Mat}_m(u))^T\right) = 0 \\ \iff\ & \mathrm{tril}\left(\mathrm{Mat}_n\left((I_n \otimes A^T)u\right) + (A^T \mathrm{Mat}_m(u))^T\right) = 0. \end{aligned}$$

Concentrating on $(A^T \mathrm{Mat}_m(u))^T$, we see that

$$\begin{aligned} & (A^T \mathrm{Mat}_m(u))^T \\ = \ & \begin{pmatrix} \mathrm{Mat}_m(u))^T A_1 & \mathrm{Mat}_m(u))^T A_2 & \ldots & \mathrm{Mat}_m(u))^T A_n \end{pmatrix} \\ = \ & \begin{pmatrix} (u^T(I_n \otimes A_1))^T & (u^T(I_n \otimes A_2))^T & \ldots & (u^T(I_n \otimes A_n))^T \end{pmatrix} \\ = \ & \begin{pmatrix} (I_n \otimes A_1^T)u & (I_n \otimes A_2^T)u & \ldots & (I_n \otimes A_n^T)u \end{pmatrix} \\ = \ & \mathrm{Mat}_n\left(\begin{pmatrix} I_n \otimes A_1^T \\ I_n \otimes A_2^T \\ \vdots \\ I_n \otimes A_n^T \end{pmatrix} u\right) \\ = \ & \mathrm{Mat}_n\left(\tilde{A}u\right), \end{aligned}$$

and the claim follows. ☐

Here we make a few more observations about U'' and $\mathrm{ext}(U'')$. First we see that $U'' \supseteq U'$, because every column of B'' is the sum of two columns in B'. It turns out that the containment also holds for their extreme rays, as in the following lemma:

Proposition 13.

$$\mathrm{ext}(U') \subseteq \mathrm{ext}(U'').$$

Proof. Suppose $u \in \mathrm{ext}(U')$. Then we have a special index i and $u^{(j)} = 0\ \forall j \notin \{i, n+i\}$. If $u \notin \mathrm{ext}(U'')$, then by Lemma 4 there exists $v \in U'' \setminus \{0\}$ and $\mathrm{supp}(v) \subset \mathrm{supp}(u)$, which implies that $v^{(j)} = 0,\ \forall j \neq \{i, n+i\}$. But now we have $(v^{(j)} - v^{(n+j)})^T A_{[n]\setminus\{j\}} = 0,\ \forall j$, hence $v \in U'$, contradicting $u \in \mathrm{ext}(U')$. ☐

Recall that

$$N(S) = \left\{x : u^T A'' x \leq u^T b'',\ \forall u \in \operatorname{ext}(U'')\right\},$$

where A'', b'' and U'' are as defined in Section 2.1.1. If we define $\operatorname{diag}(\cdot) : \mathbb{R}^{n\times n} \to \mathbb{R}^n$ such that for an $n \times n$ matrix M, $\operatorname{diag}(M)_i := M_{ii}\ \forall i \in [n]$, we can re-write $N(S)$ as

$$\begin{aligned} N(S) &= \Bigg\{ \left(\operatorname{diag}\left(V^T A\right)^T - b^T V + \left(\sum_{i=1}^n V_i^-\right)^T A \right) x \leq \left(\sum_{i=1}^n V_i^-\right)^T b, \\ &\quad (V^T A)_{ij} = -(V^T A)_{ji},\ \forall j \neq i \Big\}. \end{aligned}$$

We again can specialize the above in the case when S is a convex corner. The following result is helpful when we study $N(G)$ in Chapter 3.

Proposition 14. *Suppose $S = \{x : Ax \leq b, x \geq 0\}$ such that every entry in A is nonnegative. Then*

$$\begin{aligned} N(S) &= \Bigg\{ \left(\operatorname{diag}\left(V^T A\right)^T - b^T V + \left(\sum_{i=1}^n V_i^{-T} A - D_i^{-T}\right) \right) x \leq \left(\sum_{i=1}^n V_i^-\right)^T b, \\ &\quad (V^T A - D^T)_{ij} = -(V^T A - D^T)_{ji},\ \forall j \neq i \Big\}. \end{aligned}$$

Furthermore, if $i \neq j$, we may assume that at least one of D_{ij}, D_{ji} is zero.

Proof. Let $\bar{A} = \begin{pmatrix} A \\ -I \end{pmatrix}$ and $\bar{b} = \begin{pmatrix} b \\ 0 \end{pmatrix}$, then we know that $S = \left\{x : \bar{A}x \leq \bar{b}\right\}$. If we let $V \in \mathbb{R}^{n\times m}$ and $D \in \mathbb{R}^{n\times n}$, then

$$\begin{aligned} N(S) &= \Bigg\{ \left(\operatorname{diag}\left(V^T A - D^T\right)^T - b^T V + \left(\sum_{i=1}^n V_i^{-T} A - D_i^{-T}\right) \right) x \leq \left(\sum_{i=1}^n V_i^-\right)^T b, \\ &\quad (V^T A - D^T)_{ij} = -(V^T A - D^T)_{ji},\ \forall j \neq i \Big\}. \end{aligned}$$

If $V = 0$, then the constraint induced by such V, D is trivial $(0 \leq 0)$. Therefore, we can assume by minimality that $\operatorname{diag}(D) = 0$.

The last assertion also follows from minimality, for if there exist i, j such that D_{ij}, D_{ji} are both non-zero, we can set D_{ij} to 0 and D_{ji} to $D_{ji} + D_{ij}$. Now $(V^T A - D^T)_{ij} = -(V^T A - D^T)_{ji}$ is preserved, but D has a smaller support. $\square$

2.2 Lifted Geometric

Recall (2.1), our very first algebraic definition of $N_0(S)$. If we introduce the (redundant) variables $y_{ii}, i \in [n]$ and let $y^{(i)}$ denote the vector $(y_{1i}, y_{2i}, \ldots, y_{ni})^T$, we can slightly re-arrange the inequalities in (2.1) and arrive at the following:

$$\begin{aligned} N_0(S) \quad := \quad & \{x \ : \ \exists y \in \mathbb{R}^{n\times n} \ , \\ \text{s.t.} \quad & Ay^{(i)} \le x_i b, \\ & A(x - y^{(i)}) \le (1 - x_i)b, \\ & y_i^{(i)} = x_i, \ \forall i \in [n]\} \ . \end{aligned} \tag{2.8}$$

Then, we let K be the cone in $\mathbb{R}^{n+1}$,

$$K := \text{cone}\left(\begin{pmatrix} 1 \\ x \end{pmatrix} : x \in S\right).$$

and

$$\begin{aligned} M_0(S) \quad := \quad & \{Y \in \mathbb{R}^{(n+1)\times(n+1)} : \\ & Y_{0i} = Y_{i0} = Y_{ii}, \\ & Y_i, Y_0 - Y_i \in K, \\ & \forall i \in [n]\} \ , \end{aligned}$$

where we have denoted the extra coordinate the 0^{th} coordinate. Now, we can give an alternative definition:

$$N_0(S) := \left\{x : \ \exists Y \in M_0(S), Y_0 = \begin{pmatrix} 1 \\ x \end{pmatrix}\right\}.$$

We can similarly re-arrange (2.2), and conclude that $N(S)$ is (2.8) with the additional condition $y_{ij} = y_{ji} \ \forall i, j \in [n]$. Hence if we define

$$M(S) := \{Y : Y \in M_0(S), Y = Y^T\} \ ,$$

then

$$N(S) := \left\{x : \ \exists Y \in M(S), Y_0 = \begin{pmatrix} 1 \\ x \end{pmatrix}\right\}.$$

2.3 Geometric

Finally, we give the Geometric definition for N_0. Suppose $x \in N_0(S)$. Notice that we may assume without loss of generality that $x \in (0,1)^n$. Otherwise, for example if $x_n = \alpha$ where $\alpha \in \{0,1\}$, then

$$\begin{aligned} x \in N_0(S) &\iff x \in N_0(S) \cap \{x : x_n = \alpha\} \\ &\iff x \in N_0(S \cap \{x : x_n = \alpha\}) \\ &\iff (x_1, x_2, \ldots, x_{n-1})^T \in N_0\left(\left\{ y \in \mathbb{R}^{n-1} \ : \ \begin{pmatrix} y \\ \alpha \end{pmatrix} \in S \right\}\right). \end{aligned}$$

Note that the second " $\iff$ " above follows from the fact that $N_0(S \cap F) = N_0(S) \cap F$ for any F that is a facet of the unit hypercube (a proof of this fact can be found in [10]).

Now we observe from the Lifted Geometric definition that

$$\begin{aligned} & x \in N_0(S) \\ \iff \quad & \exists Y, \begin{pmatrix} 1 & x^T \\ x & Y \end{pmatrix} \in M_0(S) \\ \iff \quad & \exists Y, \frac{1}{x_i} Y_i, \frac{1}{1-x_i}(x - Y_i) \in S, \ \forall i \in [n]. \end{aligned} \tag{2.9}$$

Notice that $\frac{1}{x_i}(Y_i)_i = 1$ and $\frac{1}{1-x_i}(x - Y_i)_i = 0$ for every $i \in [n]$. Therefore,

$$\begin{aligned} (2.9) \iff \quad & \exists v^{(1)}, v^{(2)}, \ldots, v^{(n)}, w^{(1)}, w^{(2)}, \ldots, w^{(n)} \in S, \lambda \in \mathbb{R}^n \\ \text{s.t.} \quad & v_i^{(i)} = 1, w_i^{(i)} = 0 \\ \text{and} \quad & x = \lambda_i v^{(i)} + (1 - \lambda_i) w^{(i)}, \ \forall i \in [n]. \end{aligned} \tag{2.10}$$

$\Rightarrow$ is clear, since we can just let $v^{(i)} = \frac{1}{x_i} Y_i, w^{(i)} = \frac{1}{1-x_i}(x - Y_i)$ $\forall i \in [n]$ and $\lambda = x$, and (2.10) is satisfied. Conversely, given $v^{(i)}$'s, $w^{(i)}$'s and λ that satisfy (2.10), we can solve from the three given conditions that $\lambda = x$, and construct Y such that $Y_i = v^{(i)}$ $\forall i \in [n]$, and such a Y satisfies (2.9).

Moreover, from (2.10), a geometric definition of N_0 naturally arises:

$$N_0(S) := \bigcap_{i \in [n]} \operatorname{conv}\left(x \in S : x_i \in \{0,1\}\right).$$

For comparison, for any given $S \subseteq [0,1]^n$, the Balas-Ceria-Cornuéjols operator yields the set

$$\operatorname{conv}\left(x \in S : x_j \in \{0,1\}\right),$$

for some particular (chosen) $j \in [n]$.

For N, it is not known if an analogous geometric characterization exists. The best result that is currently known is from Lipták and Tunçel [15], which gives a geometric description for $N(S)$ when $S \subseteq [0,1]^2$.

Theorem 15. *(Theorem 27 of Lipták and Tunçel [15]) When $S \subset [0,1]^2$, the polytope $N(S)$ is defined by the following inequalities:*

1. *The valid inequalities of $N_0(S)$;*

2. *Pick any vertex v of the unit square and a direction (clockwise or counterclockwise). Let (a, α) and (β, b) be the first points of S in the chosen direction on the two sides of the unit square not containing v, where $\alpha, \beta \in \{0,1\}$ and a, b are the non-trivial coordinates. Then the inequality defined by the line that passes through v and (a, b) and containing the vertex before v in the chosen direction is valid for $N(S)$.*

It would be nice if similar characterizations of $N(S)$ can be established for sets in higher dimensions.

Chapter 3

N-N_0 Conjecture, Rank Conjecture and relevant results

In this chapter, we study the behaviour of N_0 and N when being applied iteratively to the fractional stable set polytope of graphs. We first give the preliminaries and known results that motivate Lipták and Tunçel's N-N_0 Conjecture and Rank Conjecture. In Section 3.2 we give an alternate proof to Lovász and Schrijver's result ($N_0(G) = N(G) = OC(G)$) based on our algebraic characterizations of N_0 and N given in Chapter 2. In Section 3.3 we look into $N_0^2(G)$, and show some structure of the weight vectors that induce inequalities that are potentially facets of $N_0^2(G)$.

In Section 3.4, we present an example in which $N_0^2(G) \neq N^2(G)$, settling the N-N_0 Conjecture. Finally, we build on Lipták and Tunçel's results of decomposing a graph via clique cuts in Section 3.5, and show some other instances where we can decompose a graph similarly.

3.1 Background

Let G be a finite, simple undirected graph, and let $V(G), E(G)$ denote its node and edge set respectively. Sometimes we use (V, E) instead of $(V(G), E(G))$ when the graph in question is clear. For simplicity, we will also let $V(G) = [n]$.

We let $STAB(G)$ denote the *stable set polytope* of G, which is the convex hull of the incidence vectors of the stable sets of G. In general, $STAB(G)$ can have exponentially many facets and cannot be efficiently computed. A simple approximation to $STAB(G)$ is $FRAC(G)$, the *fractional stable set polytope* of a graph G:

$$FRAC(G) := \left\{ x \in [0,1]^{V(G)} : x_i + x_j \le 1,\ \forall \{i,j\} \in E(G) \right\}.$$

For any graph, $STAB(G)$ is precisely the integer hull of $FRAC(G)$. In general, $FRAC(G) \supset STAB(G)$ unless G is bipartite.

As seen in Chapter 2, we can apply the lift-and-project operators iteratively to a linear relaxation to obtain tighter approximations to its integer hull. We study in this chapter how the operators N and N_0 behave while being applied recursively to $FRAC(G)$.

Recall that, $N_0^k(G)$ denotes the set we obtain from applying N_0 successively to $FRAC(G)$ for k times, and that the N_0-rank of a graph is smallest k such that $N_0^k(G) = STAB(G)$, and is denoted by $r_0(G)$, and $N^k(G)$, N-rank and $r(G)$ are the parallel counterparts for the operator N. These ranks are well-defined as Lovász and Schrijver [16] showed that $N_0^n(P)$ equals the integer hull of P for all $P \subseteq [0,1]^n$. For convenience, we will also use $M_0^k(G), M^k(G)$ instead of $M_0^k(FRAC(G)), M^k(FRAC(G))$.

Given any fixed graph G, an inequality $a^T x \le \alpha$, we can also define the N_0-rank (resp. N-rank) of the inequality relative to G to be the smallest integer k such that $a^T x \le \alpha$ is valid for $N_0^k(G)$ (resp. $N^k(G)$). Then $r_0(G)$ (resp. $r(G)$) can be alternatively defined as the highest N_0-rank (resp. N-rank) among the facets of $STAB(G)$.

We now introduce some of the known results that support Lipták and Tunçel's N-N_0 Conjecture and Rank Conjecture. Recall that the conjectures are:

The N-N_0 Conjecture

$$N_0^k(G) = N^k(G) \quad \forall \text{ graphs } G,\ \forall k \in \mathbb{N}.$$

The Rank Conjecture

$$r_0(G) = r(G) \quad \forall \text{ graphs } G.$$

First, given a graph G and C is a cycle or a walk in G, we let $|C|$ denote the number of edges on C. Then the *odd cycle polytope* of G can be defined as follows:

$$OC(G) := \left\{ x : \sum_{i \in C} x_i \leq \frac{|C|-1}{2},\ \forall \text{ odd cycles } C \text{ in } G \right\} \cap FRAC(G).$$

Then we have

Proposition 16. *(Lovász and Schrijver, 1991) For any graph G, $N_0(G) = N(G) = OC(G)$.*

Here are some other known similarities between the two operators. These results are fundamental in our subsequent analysis on the N- and N_0-ranks of graphs.

Proposition 17. *For all graphs G, we have*

$$r_0(G) \leq r_0(G - v) + 1 \quad \forall v \in V(G).$$

Analogous inequality holds for $r(G)$.

Proposition 18. *(Lemma 5 of Lipták and Tunçel [15]) If $G = G_1 \cup G_2$ such that $G_1 \cap G_2$ is a complete graph, then*

$$r_0(G) = \max\left\{ r_0(G_1), r_0(G_2) \right\}.$$

Analogous identity holds for $r(G)$.

Proposition 19. *For any graph G, $r(G) = r_0(G) = 0 \iff G$ is bipartite.*

Proposition 20. *For all graphs G that are series-parallel (i.e. do not contain a K_4 minor), we have $r_0(G) = r(G) \leq 1$.*

Proposition 21. *If G is a perfect graph and its largest clique has size k, then*

$$r_0(G) = r(G) = k - 2.$$

We now introduce two graph operations. First, the *subdivision of a star* operation takes a node in a graph and introduces a new node on every edge it is incident with, as shown in Figure 3.1.

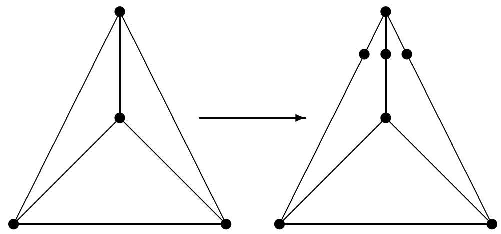

Figure 3.1: Subdivision of a star

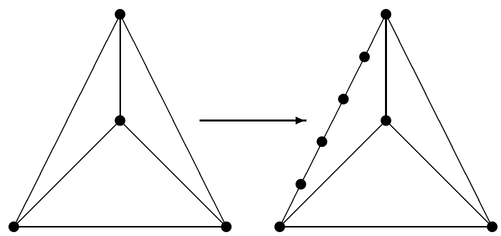

Figure 3.2: Odd subdivision of an edge

The second operation is *odd subdivision of an edge*, which takes an edge and replaces it by a path of odd length, as shown in Figure 3.2.

We call a graph H an *odd-star-subdivision* of G if H can be obtained from G by finitely many subdivision of a star and odd subdivision of an edge operations.

Also, Proposition 17 motivates the examinations of graphs whose N- and/or N_0-rank decreases upon deletion of some node. We let $\mathcal{B}_0$ be the set of graphs G that contain a subset of nodes S of size $r_0(G)$ such that the deleting S from G results in a bipartite graph. We also define $\mathcal{C}_0$ to be the set of graphs whose N_0-rank decrease upon deletion of any node. Note that $\mathcal{C}_0 \not\subseteq \mathcal{B}_0$ (e.g. the 7-antihole). We also define $\mathcal{B}, \mathcal{C}$ analogously, with N-rank instead of N_0-rank.

Then we have the following:

Proposition 22. *(Lipták and Tunçel [15]) If H is an odd-star-subdivision of G, then we have*

$$r_0(H) \geq r_0(G) \quad \textit{and} \quad r(H) \geq r(G),$$

equality holds if $G \in \mathcal{B}_0 \cup \mathcal{C}_0$. Moreover, if $G \in \mathcal{B}$, then $r_0(G) = r(G)$.

To summarize, the N-N_0 Conjecture is true for $k = 1$ for all graphs by Proposition 16. Also since $r_0(G) \geq r(G)$ in general, it is also true for $k = 2$ for graphs which have N_0-rank 2. Another family of graphs for which this conjecture is known to hold is the cliques, since in this case the stronger condition $M_0^k(G) = M^k(G)$ holds for every k (see [10]).

On the other hand, the Rank Conjecture is true for bipartite graphs, series-parallel graphs, perfect graphs and odd-star-subdivisions of graphs in $\mathcal{B}$ (which contains cliques and wheels, among many other graphs). It is also true for antiholes and graphs that have N_0-rank ≤ 2.

We will see in Section 3.4 that the N-N_0 Conjecture is false. However, to date the Rank Conjecture is still open.

3.2 An alternate proof to $N_0(G) = N(G) = OC(G)$

Now we utilize the tools developed in Chapter 2 to give alternate proofs to some known results. First, we give a proof of an elementary result by Lovász and Schrijver about N_0, based on our algebraic characterization of N_0. Before we do that we need some notation. Given a graph G and $i \in V(G)$, we define $(G \ominus i)$ to be the graph obtained from removing node i and all of its neighbours from G, and call $\ominus$ the *destruction* operator. Also, given a vector $z \in \mathbb{R}^V$, we let $\Phi_i(z)$ denote the vector obtained from z by removing the coordinate that corresponds to node i. In other words, $\Phi_i(z)$ is z restricted to the subgraph $(G - i)$. Similarly, we define $\Psi_i(z)$ to be z restricted to the subgraph $(G \ominus i)$.

Given a node $i \in V(G)$ and an inequality $a^T x \leq \alpha$, where $a \in \mathbb{R}^n, \alpha \in \mathbb{R}$, we define $\Phi_i(a)^T \Phi_i(x) \leq \alpha$ and $\Psi_i(a)^T \Psi_i(x) \leq \alpha - a_i$ to be the inequalities obtained from $a^T x \leq \alpha$ by *deleting* and *destroying* i, respectively. Let P be a convex set such that $STAB(G) \subseteq P \subseteq FRAC(G)$. Then we have the following:

Proposition 23. *(Lovász and Schrijver, 1991) If $a^T x \leq \alpha$ is an inequality such that for some $i \in V$, both the deletion and destruction of i give an inequality that is valid for P, then $a^T x \leq \alpha$ is valid for $N_0(P)$.*

Proof. Let $P := \{x : Ax \leq b\}$. We require that the first $|\mathcal{N}(i)|$ rows of $Ax \leq b$ to be the edge constraints $x_i + x_j \leq 1, j \in \mathcal{N}(i)$. Note that these inequalities may not be facets of

P, but we can still use them to derive valid inequalities for $N_0(P)$. We also let the last but one row be the destruction inequality, and the last row be the deletion inequality. We know that all these inequalities are valid for P by hypothesis and the assumption that $P \subseteq FRAC(G)$.

We order the coordinates so that the first coordinate represents i and the next $|\mathcal{N}(i)|$ coordinates represent the neighbours of i, and define the vector $z \in \mathbb{R}^{\mathcal{N}(i)}$ such that $\begin{pmatrix} z \\ \Psi_i(a) \end{pmatrix} = \Phi_i(a)$. Then A, b are in this form:

$$A = \begin{pmatrix} \bar{e} & I & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & \Psi_i(a)^T \\ 0 & z^T & \Psi_i(a)^T \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} \bar{e} \\ \vdots \\ \alpha - a_i \\ \alpha \end{pmatrix},$$

where $\bar{e}$ denotes the vector of all ones.

Now we let $v \in \mathbb{R}^m, v := (z^T, 0, \ldots, 0, 1, -1)$. It is apparent that $v \in \text{Null}\left((A_{[n]\setminus\{1\}})^T\right)$, and the inequality (of $N_0(P)$) induced by v is

$$\begin{aligned} & v^T(A_1 - b)x_1 + \sum_{j=1}^{n}((v^-)^T A_j)x_j \leq (v^-)^T b \\ \iff \quad & (a_i x_i) + \sum_{j \in \mathcal{N}(w)} a_j x_j + \sum_{j \in V(G \ominus i)} a_j x_j \leq \alpha \\ \iff \quad & a^T x \leq \alpha, \end{aligned}$$

which shows that $a^T x \leq \alpha$ is valid for $N_0(P)$. □

We see that in the construction we used in the proof of Proposition 23, the assignment of weights to valid inequalities of P satisfies the following property:

Property 24.

1. *There exists a node i such that the weights on the edge inequalities of edges that are incident with i are all non-negative;*

2. *All other inequalities that has non-zero weight has coefficient 0 at node i.*

In fact, given $N_0^{k-1}(G)$ for some graph G and some integer k, and a pair (v,d) that induces an inequality that is a facet for $N_0^k(G)$, we may assume that (v,d) satisfies Property 24.

Proposition 25. *Let G be a graph, k be an integer and $N_0^{k-1}(G) = \{x : Ax \leq b, x \geq 0\}$, where A, b are chosen such that all edge inequalities of G are present in the system. Then $N_0^k(G)$ is the intersection of $N_0^{k-1}(G)$ and*

$$\bigcap_{i\in[n]} \quad \begin{aligned}&\{x : v^T(A_i - b)x_i + \left((v^-)^T A - (d^-)^T\right) x \leq (v^-)^T b,\\ &(v^T A - d)_j = 0 \qquad \forall j \neq i,\\ &(v,d) \text{ satisfies Property 24.}\}\end{aligned}$$

Proof. The first part of the result follows from Corollary 9. The fact that we may assume (v,d) satisfies Property 24 follows from Lipták's result in [14], which states that if $a^T x \leq \alpha$ is a facet of $N_0(P)$, then there exists a node i whose deletion and destruction from $a^T x \leq \alpha$ both yield valid inequalities for P. However, given $\Phi_i(a)^T \Phi_i(x) \leq \alpha$ and $\Psi_i(a)^T \Psi_i(x) \leq \alpha - a_i$ and the knowledge that they are valid for $N_0^{k-1}(G)$, we have seen in the construction we used in the proof of Proposition 23 an assignment of weights to the valid inequalities of $N_0^{k-1}(G)$ that satisfies Property 24, and induces the inequality $a^T x \leq b$. Therefore, our claim follows. □

Now we focus on the case when $k = 1$, and prove that $N_0(G) = OC(G)$. First, we observe that we may assume all the weights on the non-negativity constraints to be zero.

Lemma 26. *Let A be the incidence matrix of a graph G and b be the all-ones vector. Then $N_0(G)$ is the intersection of $FRAC(G)$ and*

$$\bigcap_{i\in[n]} \quad \begin{aligned}&\{x : v^T(A_i - b)x_i + (v^-)^T A x \leq (v^-)^T b, v \in \operatorname{Null}\left((A_{[n]\setminus\{i\}})^T\right),\\ &v \text{ satisfies Property 24.}\}\end{aligned}$$

Proof. Suppose we have $i \in [n]$, $v \in \mathbb{R}^m$ and $d \in \mathbb{R}^n$ such that $(A^T v - d)_j = 0\ \forall j \neq i$ and $d \neq 0$. Let a_1 be a node such that $d_{a_1} \neq 0$. If $a_1 = i$, then we define $v' = v$ and

$$d'_j := \begin{cases} 0 & \text{if } j = i; \\ d_j & \text{otherwise.} \end{cases}$$

Then the inequality induced by v, d is either the same as that induced by v', d' (if $d_i < 0$) or the inequality induced by v', d' plus $d_i x_i \leq d_i$.

Now if $a_1 \neq i$, then we know there exists an edge e_1 that is incident with a_1 such that $v_{e_1} d_{a_1} > 0$. Let a_2 be the other end of e_1. If $d_{a_2} v_{e_1} > 0$ or $a_2 = i$, then we define $\alpha := \text{sign}\,(d_{a_1}) \min\{|d_{a_1}|, |v_{e_1}|, |d_{a_2}|\}$, where $\text{sign}\,(\cdot)$ is a univariate function such that

$$\text{sign}\,(x) := \begin{cases} 1 & \text{if } x > 0; \\ -1 & \text{if } x < 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Also define v', d' such that

$$v'_j := \begin{cases} v_j - \alpha & \text{if } j = e_1; \\ v_j & \text{otherwise,} \end{cases} \quad \text{and} \quad d'_j := \begin{cases} d_j - \alpha & \text{if } j \in \{a_1, a_2\}; \\ d_j & \text{if } x < 0. \end{cases}$$

The constraint induced by v, d is that induced by v', d' plus $\alpha x_1 \leq \alpha$ if $\alpha < 0$ and $-\alpha x_1 \leq 0$ if $\alpha > 0$.

If $d_{a_2} v_{e_1} \leq 0$, then there exists another edge e_2 that is incident with a_2 such that $v_{e_1} v_{e_2} < 0$. Let a_3 be the other end-node of e_2. Define $\alpha := \text{sign}\,(d_{a_1}) \min\{|d_{a_1}|, |v_{e_1}|, |v_{e_2}|\}$, and v', d' such that

$$v'_j := \begin{cases} v_j - \alpha & \text{if } j = e_1; \\ v_j + \alpha & \text{if } j = e_2; \\ v_j & \text{otherwise,} \end{cases} \quad \text{and} \quad d'_j := \begin{cases} d_j - \alpha & \text{if } j = a_1; \\ d_j + \alpha & \text{if } j = a_3; \\ d_j & \text{otherwise.} \end{cases}$$

Then the constraint induced by v, d is that induced by v', d' plus α times the edge constraint of e_2.

We see that in any of the 3 cases, we have a new pair v', d' whose constraint together with the inequalities of $FRAC(G)$ implies the inequality induced by v, d. If $d' \neq 0$, then we can apply the above process to v', d' to further simplify them.

In all three cases, we have $|\text{supp}\,(v')| + |\text{supp}\,(d')| \leq |\text{supp}\,(v)| + |\text{supp}\,(d)|$. In particular, the inequality is strict for the first two cases, and it holds tight in the third case only when $|\text{supp}\,(v')| = |\text{supp}\,(v)| - 1$ and $|\text{supp}\,(d')| = |\text{supp}\,(d)| + 1$. Since $|\text{supp}\,(v)|$ is finite, we cannot encounter this subcase infinitely many times. Therefore, the algorithm eventually outputs v', d' such that $d' = 0$.

Finally, we see that if (v, d) satisfies Property 24, then so does our output v', and we are finished. $\square$

Now we take a closer look at the incidence matrix of a graph. Let $W := a_1 e_1 a_2 e_2 \ldots e_{k-1} a_k$ be a directed walk (unless otherwise stated, all walks defined subsequently are directed). We construct $\pi(W) \in \mathbb{R}^E$ such that

$$\pi(W)_e := |\{e_i : i \text{ odd}, e_i = e\}| - |\{e_i : i \text{ even}, e_i = e\}|.$$

for every $e \in E$. We call $\pi(W)$ the *alternating incidence vector* of the walk W. Notice that if W is a closed walk, then $(\pi(W)^T A)_v = 0 \; \forall v \in V \setminus \{a_1\}$, and

$$(\pi(W)^T A)_{a_1} = \begin{cases} 0 & \text{if } |W| \text{ is even;} \\ 2 & \text{if } |W| \text{ is odd.} \end{cases}$$

Then we have the following:

Lemma 27. *Suppose A is the incidence matrix of a graph and*

$$\begin{aligned} S \;\; = \;\; & \{t\pi(W) : t \in \mathbb{R} \setminus \{0\}, \\ & W \textit{ an even closed walk, or} \\ & W \textit{ an odd closed walk that starts at } i\}. \end{aligned}$$

Then

$$\mathrm{Null}\left((A_{[n]\setminus\{i\}})^T\right)_{min} \subseteq S \subseteq \mathrm{Null}\left((A_{[n]\setminus\{i\}})^T\right).$$

Proof. (First $\subseteq$) Suppose $v \in \mathrm{Null}\left((A_{[n]\setminus\{i\}})^T\right)_{min}$. If there is an edge e_1 that is incident with i such that $v_{e_1} \neq 0$, then we let $a_1 = i$ and a_2 be the other end-node of e_1. Otherwise, we let e_1 be any edge that is in $\mathrm{supp}(v)$ and a_1, a_2 be the two end-nodes.

The assumption $v^T (A_{[n]\setminus\{i\}}) = 0$ implies that

$$\sum_{j:e \; j} v_e = 0 \quad \forall j \in V \setminus \{i\}. \tag{3.1}$$

If $a_1 \neq i$, then we start constructing an even closed walk. We know that $a_2 \neq i$, and by (3.1) there is an edge e_2 incident with a_2 such that $v_{e_1} v_{e_2} < 0$. We let a_3 denote the other endpoint of e_2, and by assumption that there is no edge in $\mathrm{supp}\,(v)$ that is incident with i, we know that $a_3 \neq i$. So again, we can apply (3.1) and find e_3 such that $v_{e_2} v_{e_3} < 0$, and so on. We stop when we have an even closed walk. I.e. we have a sub-walk $a_j e_j \ldots e_{k-1} a_k$ such that $a_j = a_k$ and $k - j$ is even.

Since there are finitely many nodes, at some point the walk must visit some node more than once. To show that in this case the algorithm must terminate with an even closed walk, we show that if any node is visited 3 times, then we must have an even closed walk.

Suppose we have a sub-walk $a_j e_j \ldots e_{k-1} a_k e_k \ldots e_{l-1} a_l$, where $a_j = a_k = a_l$ and this walk does not contain an even closed sub-walk. This implies that both $k - j$ and $l - k$ are odd. However, that means that $l - j$ is even, and we do have an even closed walk, contradicting the assumption.

We let this even closed walk be W. By the minimality assumption, we know that v has to be a multiple of $\pi(W)$.

Now suppose $a_1 = i$ and construct a walk that starts at i. By (3.1) there exists e_2 that is incident with a_2 such that $v_{e_1} v_{e_2} < 0$. Let a_3 denote the other endpoint of e_2. We keep proceeding in the same manner. Eventually, either we find an even closed walk as above, or the walk visits i again and we cannot apply (3.1). Let W be this closed walk. We know that either W is even or it is odd and starts at i, and we again know that v has to be a multiple of $\pi(W)$, and the claim follows.

(Second $\subseteq$) It is clear that if W is an even closed walk, then $\pi(W)^T A = 0$. Also, if W is an odd closed walk starting at i, we have $\pi(W)^T A_i = 2$ and $\pi(W)^T A_j = 0\ \forall j \neq i$, so $S \subseteq \mathrm{Null}\left((A_{[n]\setminus\{i\}})^T\right)$. $\square$

We now show a simple result that is useful in proving $N_0(G) = OC(G)$ (and later $N(G) = OC(G)$). Suppose $W := a_1 e_1 \ldots a_k e_k a_1$ is an odd closed walk. Then the we call the inequality

$$\sum_{i=1}^{k} x_{a_i} \leq \frac{k-1}{2}$$

the *odd closed walk inequality* of W, and define $OCW(G)$ to be the set of nonnegative vectors that satisfy all odd closed walk inequalities and edge inequalities of a given graph G. Then we have the following:

Lemma 28. *For any graph G, $OC(G) = OCW(G)$.*

Proof. First, since the set of odd cycle constraints is a subset of the odd closed walk constraints, it is clear that $OCW(G) \subseteq OC(G)$.

Now we prove the reverse containment by showing that all odd closed walk constraints are valid inequalities of $OC(G)$, and we do so by induction on the number of edges on the odd closed walk.

When there are 3 edges, the implication is obvious. Now we assume that the statement is true for all odd closed walks with fewer than k edges. Let $W := a_1 e_1 \dots a_k e_k a_1$ be an odd closed walk, and G' the subgraph of G that contains exactly the edges on W. Notice that every node has even degree in G'. Also, since W has odd length, G' must contain an odd cycle. Let this cycle be C_0.

Now we let G'' denote the subgraph obtained by deleting the edges on C_0 from G'. Notice that every node in G'' also has even degree. Hence, the edges in each component of G'' induce a closed walk. Let these closed walks be $C_1, \dots, C_p$.

We know that the odd cycle inequality $\sum_{j:a_j \in C_0} x_{a_j} \leq \frac{|C_0|-1}{2}$ is valid for $OC(G)$. For any fixed $i \in [p]$, if $|C_i|$ is even, then the inequality $\sum_{j:a_j \in C_i} x_{a_j} \leq \frac{|C_i|}{2}$ is exactly half of the sum of the edge constraints of the edges on C_i, and hence is valid for $OC(G)$. If $|C_i|$ is odd, then we know by the inductive hypothesis that $\sum_{j:a_j \in C_i} x_{a_j} \leq \frac{|C_i|-1}{2}$ is valid for $OC(G)$.

And when we sum up the above $p+1$ inequalities, we get

$$\begin{aligned}
& \sum_{i:a_i \in C_0} x_{a_i} + \sum_{j=1}^{p} \sum_{i:a_i \in C_j} x_{a_i} \leq \frac{|C_0|-1}{2} + \sum_{i:|C_i| \text{ odd}} \frac{|C_i|-1}{2} + \sum_{i:|C_i| \text{ even}} \frac{|C_i|}{2} \\
\Rightarrow \quad & \sum_{i=1}^{k} x_{a_i} \leq \frac{|C_0|-1}{2} + \sum_{i:|C_i| \text{ odd}} \frac{|C_i|-1}{2} + \sum_{i:|C_i| \text{ even}} \frac{|C_i|}{2} \\
\Rightarrow \quad & \sum_{i=1}^{k} x_{a_i} \leq \frac{|C_0|-1}{2} + \sum_{i=1}^{p} \frac{|C_i|}{2} \\
\Rightarrow \quad & \sum_{i=1}^{k} x_{a_i} \leq \frac{k-1}{2},
\end{aligned}$$

which is precisely the odd closed walk constraint of W, and the claim follows. $\square$

Now we are ready to prove the result.

Proposition 29. *(Lovász and Schrijver, 1991) For any graph G, $N_0(G) = OC(G)$.*

Proof. Below we give a proof based on our characterization of U'. First, by Lemma 28, it suffices to show that $N_0(G) = OCW(G)$.

By Lemma 26 and 27, we know that $N_0(G)$ is equal to the intersection of $FRAC(G)$ and

$$\bigcap_{i\in[n]} \Big\{x : (t\pi(W))^T(A_i - b)x_i + (t\pi(W))^{-T}Ax \le (t\pi(W))^{-T}b, \\ t \neq 0,\, W \text{ an even closed walk or an odd closed walk that starts at } i\}$$

By Corollary 11, the only case when the induced constraint may not be implied by edge constraints is when $v^T A_i > 0$. So we may assume that $t = 1$ and W is odd and starts at i. But then the constraint induced by $t\pi(W)$ is exactly the odd closed walk constraint of W, hence our claim follows. $\square$

After showing that $N_0(G) = OC(G)$, we also show the other half of Proposition 16.

Proposition 30. *(Lovász and Schrijver, 1991) For any graph G, $N(G) = OC(G)$.*

Proof. Let A be the incidence matrix of a graph G, and $b = \bar{e}$. We know from Proposition 14 that

$$\begin{aligned} N(G) \;=\; & \Bigg\{\Bigg(\big(\operatorname{diag}\big(V^TA\big)\big)^T - b^TV + \left(\sum_{i=1}^n V_i^{-T}A - D_i^{-T}\right)\Bigg)x \le \left(\sum_{i=1}^n V_i^{-}\right)^T b, \\ & (V^TA - D^T)_{ij} = -(V^TA - D^T)_{ji}\ \forall j \neq i\Big\}. \end{aligned} \tag{3.2}$$

Before showing the result, we first introduce an intermediate object. Let H_i the subgraph of G induced by the edges that are in the support of V_i. Define

$$S_{ij} := \left\{k : \exists \text{ an } jk\text{-walk } W \text{ in } H_i, (-1)^{|W|+1}(V^TA)_{ij}(V^TA)_{ik} > 0\right\}.$$

Then we have the following:

Lemma 31. *If $(V^TA)_{ij} \neq 0$, then S_{ij} is non-empty.*

Proof. Let $V_i = v$ and suppose that $(V^T A)_{ij} = v^T A_j \neq 0$, and assume without loss of generality that it is positive. Then we know that j has some neighbour j_1 in G such that $v_{jj_1} > 0$, so the edge $\{j, j_1\}$ is in H_i. If $v^T A_{j_1} > 0$, then $j_1 \in S_{ij}$. Otherwise, $v^T A_{j_1} \leq 0$ and $v_{jj_1} > 0$ together imply that j_1 has a neighbour j_2 such that $v_{j_1 j_2} < 0$. If $v^T A_{ij_2} < 0$, then $j_2 \in S_{ij}$. Otherwise, we proceed and extend our walk. Since there are only finitely many nodes, our sequence of nodes must repeat.

Suppose the node k repeats in the sequence. If the closed walk between the two occurrences of k is odd, then we know that $j \in S_{ij}$ (because there is an odd closed walk that contains j). If the closed walk is even, then there exists a node l that has yet to appear in the sequence that we can extend our walk with. Since the graph is finite, we cannot stay in this even case indefinitely. Therefore, our algorithm must terminate and we conclude that $S_{ij} \neq \emptyset$. $\square$

In the rest of the proof, we restrict our discussions to (V, D) that possess the following properties:

Property 32.

1. (V, D) *satisfies* (3.2)*;*
2. $\nexists (V', D')$ *that satisfies* (3.2) *such that*
   - $\operatorname{supp}(V') \cup \operatorname{supp}(D') \subset \operatorname{supp}(V) \cup \operatorname{supp}(D)$*, or*
   - *the inequality induced by* (V', D')*, together with valid inequalities of* $FRAC(G)$*, implies that induced by* (V, D)*.*

It is clear that we do not lose any meaningful constraints by considering only (V, D)'s that satisfy these properties, since we have excluded only the ones that we know do not induce inequalities that are facets of $N(G)$.

With that, we have the following:

Lemma 33. *Suppose* (V, D) *satisfies Property 32. Then* $D = 0$*.*

Proof. Suppose we have i, j such that $D_{ij} \neq 0$. By minimality we may assume that $i \neq j$. If $(V^T A)_{ij} \neq 0$, then we let $p_1 = i, q_1 = j$. Otherwise, we know by (3.2) and Proposition 14 that $(V^T A)_{ji} \neq 0$, and in this case we let $p_1 = j, q_1 = i$.

Now we find $q_2 \in S_{p_1q_1}$, and let W be the witnessing walk. If $q_2 = p_1$, then by minimality we know that D_{ij} is the only non-zero entry in D and $V = D_{ij}e_{p_1}\pi(W)^T$. If $q_2 \neq p_1$, but one of $D_{p_1q_2}, D_{q_2p_1}$ is non-zero, then we know that D has exactly those two non-zero entries, and V is again $D_{ij}e_{p_1}\pi(W)^T$. In both cases, the constraint induced by (V, D) is a sum of edge constraints.

Otherwise, we know by (3.2) that $(V^TA)_{q_2p_1} \neq 0$. We find $p_2 \in S_{q_2p_1}$, and let W' be the witnessing walk. We define V', D' such that

$$V'_k := \begin{cases} V_k - V_{p_1q_1}\pi(W) & \text{if } k = p_1; \\ V_k - V_{q_2p_1}\pi(W') & \text{if } k = q_1; \\ V_k & \text{otherwise,} \end{cases}$$

and

$$D'_{kl} := \begin{cases} 0 & \text{if } (k,l) = (i,j); \\ D_{kl} + (-1)^{|W|+|W'|}D_{ij} & \text{if } (k,l) = (q_2,p_2), q_2 \neq p_2; \\ D_{kl} & \text{otherwise.} \end{cases}$$

By construction, (V', D') satisfies (3.2), and we see that the constraint induced by (V, D) is that induced by (V', D') plus edge constraints.

Also, since $\sum_{i\in[m],j\in[n]} |V'_{ij}| < \sum_{i\in[m],j\in[n]} |V_{ij}|$, and we can iteratively process (V, D) to arrive at a pair such that $D = 0$, the claim follows. $\square$

Now we know we may assume that $D = 0$, $N(G)$ can be written as

$$\begin{aligned} N(G) \;=\; & \Bigg\{ \left(\left(\operatorname{diag}\left(V^TA\right)\right)^T - b^TV + \left(\sum_{i=1}^n V_i^{-T}A\right)\right) x \leq \left(\sum_{i=1}^n V_i^-\right)^T b, \\ & (V^TA)_{ij} = -(V^TA)_{ji}\ \forall j \neq i \Big\}\,. \end{aligned} \tag{3.3}$$

Focusing on the first term of the left side of the inequality, we see that

$$\begin{aligned}\left(\left(\operatorname{diag}\left(V^TA\right)\right)^T - b^TV\right)_i &= V_i^T(A_i - b)\\ &= V_i^T\left(A_i - \frac{\sum_{j=1}^n A_j}{2}\right)\\ &= V_i^TA_i - \frac{V_i^TA_i}{2} - \sum_{j\neq i}\frac{V_i^TA_j}{2}\\ &= \frac{V_i^TA_i}{2} + \sum_{j\neq i}\frac{V_j^TA_i}{2}\\ &= \sum_{j=1}^n \frac{V_j^TA_i}{2}.\end{aligned}$$

Therefore, the inequality induced by V is

$$\begin{aligned}&\sum_{i=1}^n\left(\left(\operatorname{diag}\left(V^TA\right)\right)^T - b^TV + \left(\sum_{i=1}^n V_i^{-^T}A\right)\right)x \leq \left(\sum_{i=1}^n V_i^-\right)^T b\\ \iff\ &\sum_{i=1}^n\left(\sum_{j=1}^n\frac{V_j^TA_i}{2} + V_j^{-^T}A_i\right)x_i \leq \left(\sum_{i=1}^n V_i^-\right)^T b\\ \iff\ &\left(\sum_{i=1}^n V_i^+ + V_i^-\right)^T Ax \leq 2\left(\sum_{i=1}^n V_i^-\right)^T b.\end{aligned} \tag{3.4}$$

We observe that

$$\begin{aligned}\left(\sum_{i=1}^n V_i^+\right)^T b - \left(\sum_{i=1}^n V_i^-\right)^T b &= \left(\sum_{i=1}^n V_i\right)^T b\\ &= \frac{1}{2}\left(\sum_{i=1}^n V_i\right)^T\left(\sum_{j=1}^n A_j\right)\\ &= \frac{1}{2}\sum_{i=1}^n (V^TA)_{ii}.\end{aligned}$$

The last equality follows from the fact that $(V^TA)_{ij} = -(V^TA)_{ji}\ \forall i,j, i\neq j$.

Therefore, we may assume that $(V^TA)_{ii} > 0$ for some $i \in [n]$, otherwise (3.4) is valid for $FRAC(G)$.

Now we suppose that $(V^TA)_{ii} \neq 0$ for some i. We construct a sequence of ordered pairs $((p_1, q_1), (p_2, q_2), \ldots)$ and set $p_1 = q_1 = i$. Then we find p_2 such that $p_2 \in S_{p_1q_1}$ and let W_1' be the witnessing walk. If $p_2 = p_1$, we terminate. Otherwise, $(V^TA)_{p_2p_1} = -(V^TA)_{p_1p_2} \neq 0$, and we let $q_2 := p_1$.

In general, for every $i \geq 1$, we require $p_{i+1} \in S_{p_iq_i}$ with W_i being the witnessing walk, and let $q_{i+1} = p_i$. We terminate the sequence upon two conditions:

1. We reach some i such that $p_i = q_i$.
2. $\exists i, j, |i - j| \geq 2$ such that $(p_i, q_i) = (p_j, q_j)$.

Since there are finitely many ordered pairs, the algorithm must terminate. If the algorithm terminated by the second condition, we cut off the beginning of each sequence and set p_1, q_1 to be the repeated entry, and terminate the sequence immediately after the second occurrence of this pair.

Let (p_k, q_k) be the last ordered pair in the sequence. We know either of the following is true

- $p_1 = q_1$ and $p_k = q_k$;
- $p_1 \neq q_1, p_1 = p_k$ and $q_1 = q_k$.

We now define that $s_i := \operatorname{sign}\left((V^TA)_{p_iq_i}\right)$. Notice that $s_{i+1} = s_i(-1)^{|W_i|}\ \forall i \in [k-2]$. Also define $V' \in \mathbb{R}^{m\times n}$ such that

$$V' := \sum_{i=1}^{k-1} s_i e_{p_i} \pi(W_i)^T.$$

Since V' satisfies (3.3) and $\operatorname{supp}(V') \subseteq \operatorname{supp}(V)$, we may assume that V is a scalar multiple of V'.

If our sequence is in the $p_1 \neq q_1$ case, then $(V^TA)_{ii} = 0$ for every i, which contradicts our assumption. Therefore, we may assume that $p_1 = q_1$ and $p_k = q_k$.

In this case, we know that $\sum_{i=1}^n (V^TA)_{ii}$ is either $-2, 0$ or 2. We may assume that it is 2 because otherwise $\left(\sum_{i=1}^n V_i^+\right)^T b \geq \left(\sum_{i=1}^n V_i^-\right)^T b$, which implies that (3.4) is implied by edge constraints.

Now we observe that the endpoint of W_i is the starting point for W_{i+2} for every $i \in [k-3]$. This is because W_i is a $q_i p_{i+1}$ -walk, and $q_{i+1} = p_i$ for every i. Therefore, we define two "super" walks $\mathcal{W}_1$ and $\mathcal{W}_2$, such that $\mathcal{W}_1 := W_1 W_3 W_5 \ldots W_{2\lfloor \frac{k-1}{2} \rfloor + 1}$ and $\mathcal{W}_2 := W_2 W_4 W_6 \ldots W_{2\lfloor \frac{k}{2} \rfloor}$. Let $w_1, w_2 \ldots, w_\alpha$ be the sequence of nodes in the (directed) walk $\mathcal{W}_1$. Similarly, let $w'_1, \ldots, w'_\beta$ be the nodes in $\mathcal{W}_2$.

Here is the final piece that completes the proof.

Lemma 34. *If $\sum_{i=1}^{n} (V^T A)_{ii} = 2$, then the union of the edges of $\mathcal{W}_1$ and $\mathcal{W}_2$ induce an odd closed walk in G*

Proof. Since we know that $p_1 = q_1$, it is obvious that the starting points of $\mathcal{W}_1$ and $\mathcal{W}_2$ coincide, same with the ending points (since $p_k = q_k$). So we have a closed walk.

Now we show that our closed walk is odd. $\sum_{i=1}^{n} (V^T A)_{ii} = 2 \Rightarrow (V^T A)_{p_1 q_1} > 0$, so we know that $s_1 = 1$. If $s_{k-1} = -1$, then we know that $\sum_{i=1}^{k-2} |W_i|$ is odd (since $s_{i+1} = s_i(-1)^{|W_i|}\ \forall i$), and $|W_{k-1}|$ is even (by the definition of $S_{p_{k-1} q_{k-1}}$), so the walk has an odd number of edges. Similarly, if $s_{k-1} = 1$, then we know that $\sum_{i=1}^{k-2} |W_i|$ is even and $|W_{k-1}|$ is odd, and our claim follows. $\square$

Now we look at (3.4) for this V. $\sum_{i=1}^{n} (V^T A)_{ii} = 2$ implies that $\left(\sum_{i=1}^{n} V_i^+\right)^T b$ exceeds $\left(\sum_{i=1}^{n} V_i^-\right)^T b$ by exactly 1, and (3.4) is exactly twice the odd closed walk constraint for the walk we constructed by joining $\mathcal{W}_1$ and $\mathcal{W}_2$. Therefore, the inequality induced by V is valid for $OC(G)$, and hence $OC(G) \subseteq N(G)$.

Since it is clear that $N(G) \subseteq N_0(G) = OC(G)$, we are finished. $\square$

3.3 A look at inequalities of N_0-rank 2

We now turn our attention to inequalities that are of N_0-rank 2. Unlike for N_0-rank 1, we do not have a complete characterization for inequalities of N_0-rank 2. However, we give a few (elementary) results in this section, in an attempt to find some structure in the vectors that induce these inequalities.

For the rest of this section, we let $\mathcal{C}$ denote the set of chordless odd cycles in G. We define the $(|E|+|\mathcal{C}|) \times |V|$ matrix A and the vector b of size $(|E|+|\mathcal{C}|)$ such that, in $Ax \leq b$, the first $|E|$ rows are the edge constraints of G and the remaining $|\mathcal{C}|$ rows are the odd cycle inequalities of G. Then we know that $N_0(G) = \{x : Ax \leq b, x \geq 0\}$. By using this A, b, we may have some redundant constraints. Precisely, they are the edge constraints of the edges that are in a triangle. However, it is convenient and unifying to not isolate these edges that are in triangles, as we will see in the analysis below.

Suppose we have vectors $v \in \mathbb{R}^{E \cup \mathcal{C}}$ and $d \in \mathbb{R}^V$. We can look at each coordinate of v as a weight on an edge or a chordless odd cycle in G, and $\operatorname{supp}(v)$ as a set that contains edges and odd cycles. Similarly, we can view d as a weight vector on the nodes of G. From Proposition 25, we know that $N_0^2(G)$ is the intersection of $N_0(G)$ and

$$\bigcap_{i \in [n]} \quad \begin{aligned} &\{x : v^T(A_i - b)x_i + \left((v^-)^T A - (d^-)^T\right) x \leq (v^-)^T b, \\ &(v^T A - d)_j = 0 \quad \forall j \neq i, \\ &(v, d) \text{ satisfies Property 24.}\} \end{aligned}$$

As in the case of $N_0(G)$, we may assume that $d = 0$.

Lemma 35. *Let A, b be defined as above. Then $N_0^2(G)$ is the intersection of $N_0(G)$ and*

$$\bigcap_{i \in [n]} \quad \begin{aligned} &\{x : v^T(A_i - b)x_i + (v^-)^T A x \leq (v^-)^T b, \\ &v^T A_j = 0 \quad \forall j \neq i, \\ &v \textit{ satisfies Property 24.}\} \end{aligned}$$

Proof. Suppose given v, d and a special index i such that $(A^T v - d)_j = 0 \ \forall j \neq i$ and $d \neq 0$. Let a_1 be a node such that $d_{a_1} \neq 0$. If $a_1 = i$, refer to the proof of Lemma 26. Otherwise, if there are no edges $e_1 \in \operatorname{supp}(v)$ such that $v_{e_1} d_{a_1} > 0$, then we know there is a cycle $C_1 \in \mathcal{C}$ such that a_1 is a node on C_1 and $v_{C_1} d_{a_1} > 0$. In that case, we let S be the unique set of $\frac{|C_1|-1}{2}$ edges that cover every node on C_1 except a_1, and let $\alpha := \operatorname{sign}(d_{a_1}) \min\{|d_{a_1}|, |v_{C_1}|\}$. Define v', d' such that

$$v'_j := \begin{cases} v_j + \alpha & \text{if } j \in S; \\ v_j - \alpha & \text{if } j = C_1; \\ v_j & \text{otherwise,} \end{cases} \quad \text{and} \quad d'_j := \begin{cases} d_j - \alpha & \text{if } j = a_1; \\ d_j & \text{otherwise.} \end{cases}$$

Then the inequality induced by v, d is that induced by v', d' plus possibly some edge constraints of the edges on C_1.

Now suppose there does exist an edge $e_1 \in \operatorname{supp}(v)$ that is incident with a_1 and satisfies $v_{e_1} d_{a_1} > 0$. Let a_2 denote the other end of e_1. If $d_{a_2} \neq 0$ or there exists another edge $e_2 \in \operatorname{supp}(v)$ that is incident with a_2, refer to the proof of Lemma 26. Otherwise, we know there exists a cycle C_1 such that $v_{C_1} v_{e_1} < 0$ and a_2 is on C_1. In that case, we let S be the unique set of $\frac{|C_1|-1}{2}$ edges that cover every node on C_1 except a_2, and let $\alpha := \operatorname{sign}(|d_{a_1}) \min\{|d_{a_1}|, |v_{e_1}|, |v_{C_1}|\}$. Define

$$v'_j := \begin{cases} v_j - \alpha & \text{if } j \in S \cup \{e_1\}; \\ v_j + \alpha & \text{if } j = C_1; \\ v_j & \text{otherwise,} \end{cases} \qquad \text{and} \quad d'_j := \begin{cases} w_j - \alpha & \text{if } j = a_1; \\ d_j & \text{otherwise.} \end{cases}$$

Then again the inequality induced by v, d is the one induced by v', d' plus perhaps some edge constraints.

We can replace v, d by v', d' and run the above process again, until we get $d = 0$. Also, none of the edges that are incident with i in our output have negative weight because by our assumption on (v, d), no cycles with non-zero weight passes through i. Therefore our algorithm preserves Property 24 and our claim follows. $\square$

Given $v \in \operatorname{Null}\left((A_{[n]\setminus\{i\}})^T\right)$, we can write v as $\begin{pmatrix} v_E \\ v_{\mathcal{C}} \end{pmatrix}$ such that $v_E \in \mathbb{R}^E$ corresponds to the weights on the edges and $v_{\mathcal{C}} \in \mathbb{R}^{\mathcal{C}}$ corresponds to the weights on the cycles. Since we are only interested in v's that potentially induces an inequality that is a facet for $N_0^2(G)$ and is not valid for $N_0(G)$, we are going to assume that v possesses the following properties throughout the remainder of this section.

Property 36.

1. *v satisfies Property 24*
2. $v \in \operatorname{Null}\left(A_{[n]\setminus\{i\}}\right)_{min}$.
3. $v_{\mathcal{C}} \neq 0$.

We may assume (1) and (2) by obvious reasons. For (3) we see that if $v_{\mathcal{C}} = 0$ then the inequality induced by v is valid for $N_0(G)$. Note that an implication of assuming (2) and (3) is the following:

Lemma 37. *Suppose v satisfies Property 36. Then we know that none of the following exists:*

- *an even closed walk in G such that every edge on the walk is in* $\operatorname{supp}(v)$*;*
- *an odd closed walk that passes through i such that every edge in the walk is in* $\operatorname{supp}(v)$*.*

Proof. We have seen in the proof of $N_0(G) = OC(G)$ that the above are exactly the elements in $\operatorname{Null}\left((A')^T_{[n]\setminus\{i\}}\right)$ where A' is the incidence matrix of G. Since we know that there exists some cycle C such that $v_C \neq 0$, we may assume that the support of v does not contain any of the above type of walks. $\square$

Next, we try to find some structures in such v's. We first show that v_E can be decomposed into "i-paths" (paths that start at i and end at a node on some cycle in $\operatorname{supp}(v)$) and "connecting walks" (that run between two nodes that are both on some, perhaps different, cycles in $\operatorname{supp}(v)$), as in the following lemma:

Proposition 38. *Suppose v satisfies Property 36. Then v_E can be written as*

$$\sum_{j\in[|\mathcal{P}|]} p_j \pi(P_j) + \sum_{j\in[|\mathcal{Q}|]} q_j \pi(Q_j)$$

where

- $\mathcal{P} = \left\{P_1, \ldots, P_{|\mathcal{P}|}\right\}$ *is a set of i-paths, and the union of all edges on the paths induce a tree in G;*
- $\mathcal{Q} = \left\{Q_1, \ldots, Q_{|\mathcal{P}|}\right\}$ *is a set of connecting walks that run between nodes that are in* $\operatorname{supp}\left((v^{\mathcal{C}})^T A^{\mathcal{C}}\right)$*;*
- $p \in \mathbb{R}^{\mathcal{P}}_{++}, q \in \mathbb{R}^{\mathcal{Q}}$.

Proof. We first find the paths that start at i. We know that the weights of the edges that are incident with i are all nonnegative. Let e_1 be an edge in $\operatorname{supp}(v)$ that is incident with i. Let $a_1 := i$ and a_2 be the other end of e_1. In the general step, after finding e_k, we find another edge e_{k+1} that is incident with a_k, such that $v_{e_k} v_{e_{k+1}} < 0$. We let a_{k+1} denote the other endpoint of e_{k+1}.

If at any step we could not find e_{k+1}, then we know that a_k is on a cycle C such that $v_C v_{e_k} < 0$. In that case, we record the walk $a_1 e_1 a_2 e_2 \ldots a_k$ as P_j and set $p_j := \min\left\{|v_{e_j}| : j \in [k]\right\}$. By Lemma 37 we know that P_j is in fact a path.

Now if we let $w := v_E - p_j \pi(P_j)$, then we know that $\operatorname{supp}(w) \subset \operatorname{supp}(v_E)$. If there are still edges in $\operatorname{supp}(w)$ that is incident with i, we repeat the above process and find all i-paths.

Since the paths all have a common node in i and by Lemma 37 the union of the edges on the i-paths cannot induce a cycle, if follows that they induce a tree.

Now suppose we have exhausted all the paths that start with i, and let $w = v_E - \sum_{j \in [|\mathcal{P}|]} p_j \pi(P_j)$. If $w \neq 0$, then we start finding other paths (that do not involve i) similarly. First we show that there exists some node j such that the weights of the edges that are incident with j are all of the same sign.

Suppose for a contradiction that every node is either incident with no edges with $\operatorname{supp}(w)$, or there are two edges e_1, e_2 incident with it that have weights of opposite signs. We let $a_1 e_1 a_2 e_2 \ldots a_k$ be the longest path we can find that satisfies $w_{e_j} w_{e_{j+1}} < 0 \; \forall j \in [k-2]$. By assumption, there exists an edge e_k that is incident with a_k such that $w_{e_k} w_{e_{k-1}} < 0$. Since the path was assumed to be the longest, the other end of e_k must lie on the path. Also by Lemma 37, edges in $\operatorname{supp}(w)$ cannot induce an even closed walk (since $\operatorname{supp}(w) \subseteq \operatorname{supp}(v_E)$), so the other end of e_k must be a_m such that $k - m$ is even. Similarly, there is an edge e_0 incident with a_1 such that $w_{e_0} w_{e_1} < 0$. We let a_n be the other end of e_0, and we know that $n - 1$ must be even.

Now we define the walk

$$W := \begin{cases} a_m e_m a_{m+1} e_{m+1} \ldots a_n & \text{if } m < n; \\ a_m e_{m-1} a_{m-1} \ldots a_n & \text{otherwise.} \end{cases}$$

Then $a_1 e_1 a_2 e_2 \ldots a_k e_k W e_0 a_1$ is an even closed walk, contradicting our assumption on v.

Therefore, there must exist a node l such that the weights of all its incident edges have the same sign. We start constructing a walk with such a l, in the same way we constructed the i-paths. We extend the path by taking edges with weights of alternating signs, and stop when we could not extend the walk further. We let $Q_j := a_1 e_1 \ldots a_{k+1}$ be the walk, and define

$$q_j := \operatorname{sign}(w_{e_1}) \min\left\{|v_{e_j}| : j \in [k]\right\}.$$

We can repeat the above process, find all the connecting walks, and completely decompose v^E. ☐

Remark 39. *Note that a connecting walk could be closed.*

We end this section by showing that, for the v's that induce inequalities that are facets of $N_0^2(G)$, there is a certain level of connectivity between the connecting walks and the cycles in support of $v_{\mathcal{C}}$.

Proposition 40. *Suppose v satisfies Property 36, and $S \subset \operatorname{supp}(v_{\mathcal{C}})$. Define $w \in \mathbb{R}^{E \cup \mathcal{C}}$ such that*

$$w_j = \begin{cases} v_j & \text{if } j \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists a connecting walk that has exactly one end in $\operatorname{supp}\left(w^T A\right)$.

Proof. This follows directly from the above proposition and the minimality of v. If $\exists S$ such that no connecting walk "escapes" the set of nodes involved in S, then we can take the cycles in S and the i-paths that run to nodes involved in S off v and obtain a new vector that satisfies Property 36 with a smaller support. ☐

A way to look at Proposition 40 is that, given v that induces a facet of $N_0^2(G)$, if we construct the auxiliary graph H such that $V(H) = \operatorname{supp}(v_{\mathcal{C}})$ and cycle i is adjacent to cycle j in H if and only if there is a connecting walk that has one end on cycle i and the other one cycle j, then H has to be connected.

With the characterization above and some creativity, one can construct many inequalities that are of N_0-rank 2 for any graph G. For example, the wheel inequality (which has N_0-rank 2) can be induced by using the hub node as i, assigning a weight of 1 on every edge that is incident with the hub, and a weight of -1 on the rim. More examples will be given in Chapter 5, when we show that certain family of inequalities are of N_0-rank 2 by giving an appropriate weight assignment on the edges and odd cycles of the graph.

However, more must be done before we have a complete characterization for $N_0^2(G)$.

3.4 A counterexample to the N-N_0 Conjecture

Here we give an example for which $N^2(G) \subset N_0^2(G)$, hence disproving the N-N_0 Conjecture.

Claim 41. *Let G be the graph in Figure 3.3. Then*

$$\frac{1}{5}(2,1,2,1,2,1,1)^T \in N_0^2(G) \setminus N^2(G)$$

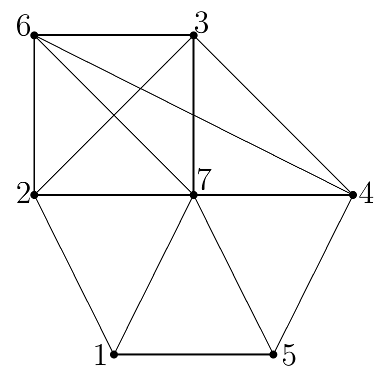

Figure 3.3: A graph G satisfying $N^2(G) \subset N_0^2(G)$

Proof. Let x denote the point $\frac{1}{5}(2,1,2,1,2,1,1)^T$. To show that $x \in N_0^2(G)$, we consider the following matrix

$$\frac{1}{5}\begin{pmatrix} 5 & 2 & 1 & 2 & 1 & 2 & 1 & 1 \\ 2 & 2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that every column and the difference of every column with the first column belongs to $OC(G)$. Thus, the matrix above is in $M_0^2(G)$, and consequently $x \in N_0^2(G)$.

Now suppose for a contradiction that $x \in N^2(G)$. Then there exists Y such that $Y' := \begin{pmatrix} 1 & x^T \\ x & Y \end{pmatrix} \in M^2(G)$. We know that $Y_{ii} = x_i \ \forall i \in [7]$. Also, if $i \sim j$ in G, then the edge inequality $Y_{ij} + Y_{ii} \leq x_i$ applies, which implies that $Y_{ij} = 0$. Therefore, Y' must take the following form;

$$Y' = \frac{1}{5}\begin{pmatrix} 5 & 2 & 1 & 2 & 1 & 2 & 1 & 1 \\ 2 & 2 & 0 & 5Y_{13} & 5Y_{14} & 0 & 5Y_{16} & 0 \\ 1 & 0 & 1 & 0 & 5Y_{24} & 5Y_{25} & 0 & 0 \\ 2 & 5Y_{13} & 0 & 2 & 0 & 5Y_{35} & 0 & 0 \\ 1 & 5Y_{14} & 5Y_{24} & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 5Y_{25} & 5Y_{35} & 0 & 2 & 5Y_{56} & 0 \\ 1 & 5Y_{16} & 0 & 0 & 0 & 5Y_{56} & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now since $Y' \in M^2(G)$, all inequalities in the following table have to hold.

Inequality	Remark
$Y_{13} + Y_{14} + Y_{16} \leq \frac{2}{5}$	Odd cycle inequality of 6-3-4-6 on Y_1'
$Y_{25} + Y_{35} + Y_{56} \leq \frac{2}{5}$	Odd cycle inequality of 6-2-3-6 on Y_5'
$-Y_{13} \leq -\frac{1}{5}$	Odd cycle inequality of 7-1-2-7 on $x - Y_3'$
$-Y_{14} \leq -\frac{1}{5}$	Odd cycle inequality of 7-1-5-7 on $x - Y_4'$
$-Y_{25} \leq -\frac{1}{5}$	Odd cycle inequality of 7-1-5-7 on $x - Y_2'$
$-Y_{35} \leq -\frac{1}{5}$	Odd cycle inequality of 7-4-5-7 on $x - Y_3'$
$-Y_{16} - Y_{56} \leq -\frac{1}{5}$	Odd cycle inequality of 7-1-5-7 on $x - Y_6'$

However, if we sum up all the above inequalities, we get $0 \leq -\frac{1}{5}$, which is a contradiction. $\square$

Therefore, the N-N_0 Conjecture is false. In fact, Claim 41 still holds if we add an additional edge $\{2, 4\}$ to the above graph. Hence, N-N_0 Conjecture does not hold for even perfect graphs, for which we already knew the Rank Conjecture holds.

We will see in Chapter 4 that the N- and N_0-rank of the graph in Figure 3.3 are both 3, hence it is not a counterexample to the Rank Conjecture. However, it is very intriguing that the N-N_0 Conjecture can be disproven by such a small graph. It gives a lot of motivation to verify the Rank Conjecture on other similarly small graphs, and we will do so in Chapters 4 and 5.

3.5 Decomposition of graphs

Before moving on to investigating the ranks of small graphs, we first turn our focus to finding conditions under which the rank of a graph G can be obtained by knowing the ranks of certain proper subgraphs of G. For an example, we saw in Proposition 18 that if G is a union of two subgraphs that intersect at a clique, then the rank of G is equal to the maximum of the ranks of the two subgraphs. In this section, we will slightly generalize that result, and give several other conditions that allow us to "decompose" a graph while studying its N- and N_0-rank.

First, given a graph G, $x \in \mathbb{R}^{V(G)}$ and H a subgraph of G, we let x_H denote the vector x being restricted to H. Then we have the following:

Proposition 42. *Let G be a graph such that $v, w \in V(G), \mathcal{N}(i) = \mathcal{N}(w)$ and $v \not\sim w$. Then $STAB(G)$ is defined by the facets of $STAB(G-v)$ and $STAB(G-w)$.*

Proof. It suffices to show that for any $x \in \mathbb{R}^V, x_{G-v} \in STAB(G-v), x_{G-w} \in STAB(G-w) \iff x \in STAB(G)$. First, "$\Leftarrow$" is clear. For "$\Rightarrow$", suppose we are given x such that $x_{G-v} \in STAB(G-v)$ and $x_{G-w} \in STAB(G-w)$. We assume without loss of generality that $x_v \geq x_w$.

Since $x_{G-v} \in STAB(G-v)$, it can be expressed as a convex combination of incidence vectors of stable sets in $(G-v)$. If S is one of those stable sets and $v \in S$, then by assumption on v, w, $S \cup \{w\}$ is a stable set in G. Therefore, if we define x' such that

$$x'_i := \begin{cases} x_v & \text{if } i = w; \\ x_i & \text{otherwise,} \end{cases}$$

we know that $x' \in STAB(G)$. Since $STAB(G)$ is lower-comprehensive and $x_v \geq x_w$, it follows that $x \in STAB(G)$. $\square$

Let $S \subseteq V(G)$. We let G_S denote the subgraph of G induced by nodes in S. Since the rank of G equals the maximum among the ranks of the facets of $STAB(G)$, the following fact is clear.

Proposition 43. *Let G be a graph and $STAB(G) = \{x : Ax \leq b, x \geq 0\}$, where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$. Then*

$$r_0(G) = \max \left\{ r_0(G_{\mathrm{supp}(A^T)_i}) : i \in [m] \right\}.$$

Analogous identity holds for $r(G)$.

Moreover, Proposition 42 and 43 immediately imply the following:

Corollary 44. *Let G be a graph such that $i, j \in V(G), \mathcal{N}(i) = \mathcal{N}(j)$ and $i \not\sim j$. Then $r_0(G) = r_0(G - i)$ and $r(G) = r(G - i)$.*

Remark 45. *In general, $G = G_1 \cup G_2$ and $G_1 \cong G_2$ do not imply $r_0(G) = r_0(G_1)$.*

With this, Proposition 18 can be slightly generalized.

Proposition 46. *Suppose $G = G_1 \cup G_2$, and $G_1 \cap G_2$ is a complete k-partite graph, with $S_1, \ldots, S_k$ being the partitions. If $\mathcal{N}(v) \cap S_i \in \{\emptyset, S_i\}$ for all $v \in V(G_1) \Delta V(G_2)$ and for all $i \in [k]$, then*

$$r_0(G) = \max \{r_0(G_1), r_0(G_2)\}.$$

Analogous identity holds for $r(G)$.

Proof. We first show that for any fixed $i \in [k]$, $\mathcal{N}(v) = \mathcal{N}(w)\ \forall v, w \in S_i$. First of all, for $u \in V(G_1) \Delta V(G_2)$, $\mathcal{N}(u) \cap S_i \in \{\emptyset, S_i\}$ is equivalent to $u \sim v \iff u \sim w$. This is also true when $u \in V(G_1 \cap G_2)$, since it is a complete k-partite graph.

Now for each $i \in [k]$, we remove all but one node from S_i from G , and call this subgraph G'. Let G_1' be the subgraph in G' that is induced by nodes $V(G_1) \cap V(G')$, and similarly define G_2'. Then $G' = G_1' \cup G_2'$ while $V(G_1' \cap G_2') = \{s_i, i \in [k]\}$, where s_i is the lone representative of S_i in G'. Since $G_1 \cap G_2$ is a complete k-partite graph, the nodes $\{s_i, i \in [k]\}$ have to induce a clique in G'. Therefore, by Proposition 18, $r_0(G') = \max \{r_0(G_1'), r_0(G_2')\}$. Also, by Proposition 44, we have $r_0(G) = r_0(G'), r_0(G_1) = r_0(G_1')$ and $r_0(G_2) = r_0(G_2')$, and we have the desired result by combining the equalities. The proof for the N-rank follows exactly the same steps. ◻

Now we introduce a graph operation called *cloning a node.* Given a graph G and $v \in V(G)$, by cloning v we add a new node that is joined to v and all nodes in G that are adjacent to v. Then we have the following:

Proposition 47. *Suppose G is a graph and $STAB(G) = \{x : Ax \leq b, x \geq 0\}$. Let G' be the graph obtained by cloning i, and let 0 be the new node. Then*

$$STAB(G') = \left\{ \begin{pmatrix} x_0 \\ x \end{pmatrix} : A_i x_0 + Ax \leq b \right\}$$

Proof. It follows directly from the fact that, for any $S \subseteq V(G')$, S is a stable set in G' if and only if $S \cup \{0\} \setminus \{i\}$ is a stable set. □

Proposition 42, 43 and 47 together imply the following:

Proposition 48. *Let G a graph and $S \subseteq V(G)$. If $\mathcal{N}(v) \setminus S = \mathcal{N}(w) \setminus S$ $\forall v, w \in S$ and every component in G_S is a clique, then*

$$r_0(G) = r_0(G_{(V(G)\setminus S)\cup T}),$$

where T is the set of nodes of any largest component in G_S. Analogous identity holds for $r(G)$.

We now show two other cases under which the graph can be decomposed. First, we call a stable set in G *maximal* if there does not exist another stable set in G that properly contains it. Also, for a set $S \subseteq V(G)$, we let χ_S denote the incidence vector of S. Then we have the following:

Proposition 49. *Suppose G has k distinct maximal stable sets and $k < |V(G)|$. Then there exists a node $v \in V(G)$ such that $r_0(G-v) = r_0(G)$ and $r(G-v) = r(G)$. Moreover, if $\exists v \in V(G)$ such that $(G - v) \in \mathcal{C}_0$ (resp. $(G-v) \in \mathcal{C}$), then $r_0(G-v) = r_0(G)$ (resp. $r(G-v) = r(G)$).*

Proof. Suppose $a^T x \leq \alpha$ is a facet of $STAB(G)$. First we show that if the number of distinct maximal stable sets is less than the number of nodes, then $|\mathrm{supp}\,(a)\,| \subset |V(G)|$.

We see that, since $a^T x \leq \alpha$ is a facet of $STAB(G)$, there exist $|V(G)|$ distinct incidence vectors of stable sets that lie on the facet. By assumption, since there are less than $|V(G)|$ distinct maximal stable sets, we know there exists a stable set S such that S is not maximal, and $a^T \chi_S = \alpha$. Let S' be a stable set that properly contains S. Then we know that $a^T \chi_{S'} \geq a^T \chi_S$, which implies that $a^T \chi_{S'} = \alpha$, because $a^T x \leq \alpha$ is valid for $STAB(G)$. Then we take any $i \in S' \setminus S$, and see that a_i has to be 0.

Therefore none of the facets of $STAB(G)$ have full support, and it follows from Proposition 43 that there exists a node $v \in V(G)$ such that $r_0(G - v) = r_0(G)$.

Moreover, if there exists $v \in V(G)$ such that $(G - v) \in \mathcal{C}_I$, for any $w \in V(G) \setminus \{v\}$, we know that $r_0(G - w) \leq r_0((G - w) - v) + 1 = r_0((G - v) - w) + 1 = r_0(G - v)$. If $r_0(G - v) < r_0(G)$, then $r_0(G - w) < r_0(G)$ $\forall w \in V(G)$, which is a contradiction. Therefore, we have $r_0(G - v) = r_0(G)$. The argument for the N-rank is analogous. $\square$

Proposition 50. *Suppose $G = G_1 \cup G_2$. If*

$$\mathcal{N}(v) \cap V(G_1) \cap V(G_2) \in \{\emptyset, V(G_1) \cap V(G_2)\} \qquad \forall v \in V(G_1) \Delta V(G_2),$$

then

$$r_0(G) = \max\{r_0(G_1), r_0(G_2)\}.$$

Analogous identity holds for $r(G)$.

Proof. Again, it suffices to show that given $x \in \mathbb{R}^V, x_{G_1} \in STAB(G_1), x_{G_2} \in STAB(G_2)$ if and only if $x \in STAB(G)$. "$\Leftarrow$" is again trivial. For "$\Rightarrow$", suppose we are given x such that $x_{G_i} \in STAB(G_i)$, $\forall i \in [2]$. First, $x_1 \in STAB(G_1)$ implies that there exist $\lambda \in \mathbb{R}^k_+, ||\lambda||_1 = 1$ and stable sets $P_1, \ldots, P_k$ such that

$$x_{G_1} = \sum_{i=1}^{k} \lambda_i \chi_{P_i}.$$

Notice that for each P_i, we can write it as $P_i' \cup P_i''$ where P_i' is a stable set in $G - G_2$ and P_i'' is a stable set in $G_1 \cap G_2$. Now we can rewrite the above as

$$x_{G_1} = \sum_{i=1}^{k} \lambda_i \chi_{P_i'} + \sum_{i=1}^{k} \lambda_i \chi_{P_i''}.$$

Similarly, for x_{G_2}, we find $\alpha \in \mathbb{R}^l_+$, $||\alpha||_1 = 1$, $Q'_1, \ldots, Q'_l$ stable sets of $G - G_1$ and $Q''_1, \ldots, Q''_l$ stable sets of $G_1 \cap G_2$ such that

$$x_{G_2} = \sum_{i=1}^{l} \alpha_i \chi_{Q'_i} + \sum_{i=1}^{l} \alpha_i \chi_{Q''_i}.$$

Now we define $d_1 := \sum_{i \in [k], P''_i \neq \emptyset} \lambda_i$ and $d_2 := \sum_{i \in [l], Q''_i \neq \emptyset} \alpha_i$, and assume without loss of generality that $d_1 \geq d_2$.

Also, for any $i \in [k]$, if $P''_i \neq \emptyset$, then there exists a node in $V(G_1 \cap G_2)$ that is not adjacent to any node in P'_i. Therefore, we know that $P'_i \cup Q''_j$ is a stable set in G_1 for any $j \in [l]$.

Since we know that

$$x_{G_1 \cap G_2} = \sum_{i=1}^{k} \lambda_i \chi_{P''_i} = \sum_{i=1}^{l} \alpha_i \chi_{Q''_i},$$

we have

$$\begin{aligned} x_{G_1} &= \sum_{i=1}^{k} \lambda_i \chi_{P'_i} + \sum_{i=1}^{l} \alpha_i \chi_{Q''_i} \\ &= \sum_{i \in [k], P''_i = \emptyset} \lambda_i \chi_{P'_i} + \sum_{i \in [k], P''_i \neq \emptyset} \lambda_i \chi_{P'_i} + \sum_{i \in [l], Q''_i \neq \emptyset} \alpha_i \chi_{Q''_i} \\ &= \sum_{i \in [k], P''_i = \emptyset} \lambda_i \chi_{P'_i} + d_2 \left(\sum_{\substack{i \in [k], P''_i \neq \emptyset \\ j \in [l], Q''_j \neq \emptyset}} \frac{\lambda_i}{d_1} \frac{\alpha_j}{d_2} \chi_{P'_i \cup Q''_j} \right). \end{aligned}$$

Now we see that we can express x as

$$(1 - d_2) \left(\sum_{\substack{i \in [k], P''_i = \emptyset \\ j \in [l], Q''_j = \emptyset}} \frac{\lambda_i}{1 - d_1} \frac{\alpha_j}{1 - d_2} \chi_{P'_i \cup Q'_j} \right) + d_2 \left(\sum_{\substack{i \in [k], P''_i \neq \emptyset \\ j \in [l], Q''_j \neq \emptyset}} \frac{\lambda_i}{d_1} \frac{\alpha_j}{d_2} \chi_{P'_i \cup Q''_j \cup Q'_j} \right).$$

Notice that $P'_i \cup Q'_j$ is a stable set in G for every i, j. Also, when $P''_i \neq \emptyset$, $(P'_i \cup Q''_j \cup Q'_j)$

is a stable set in G for any $j \in [l]$. Also we see that

$$\begin{aligned}
& (1-d_2)\left(\sum_{\substack{i\in[k],P_i''=\emptyset \\ j\in[l],Q_j''=\emptyset}} \frac{\lambda_i}{1-d_1}\frac{\alpha_j}{1-d_2}\right) + d_2\left(\sum_{\substack{i\in[k],P_i''\neq\emptyset \\ j\in[l],Q_j''\neq\emptyset}} \frac{\lambda_i}{d_1}\frac{\alpha_j}{d_2}\right) \\
= \; & \frac{1}{1-d_1}\left(\sum_{i\in[k],P_i''=\emptyset}\lambda_i\right)\left(\sum_{j\in[l],Q_j''=\emptyset}\alpha_j\right) + \frac{1}{d_1}\left(\sum_{i\in[k],P_i''\neq\emptyset}\lambda_i\right)\left(\sum_{j\in[l],Q_j''\neq\emptyset}\alpha_j\right) \\
= \; & \frac{1}{1-d_1}(1-d_1)(1-d_2) + \frac{1}{d_1}(d_1)(d_2) \\
= \; & 1.
\end{aligned}$$

Therefore, the above is indeed a convex combination of incidence vectors of stable sets in G, and hence $x \in STAB(G)$. $\square$

Chapter 4

Verifying the Rank Conjecture on graphs with no more than 7 nodes

The fact that the N-N_0 Conjecture can be disproven by a graph of as few as 7 nodes gives us some hope that there also exists a counterexample to the Rank Conjecture that is relatively small. Here we show that the Rank Conjecture holds for all graphs on 7 or fewer nodes. We start at the number 7 because it is the smallest non-trivial case.

Although the proof to the 8-node result in Chapter 5 is self-contained, and thus contains another proof to the 7-node result, the proof we give in this chapter is more elementary, and contains many examples of how we apply the basic tools we saw in the previous chapters to find the N- and N_0-rank of any specific graph. Thus, it serves well as a warm-up for the reader to the more sophisticated proof in Chapter 5.

Now we state a few facts that we will need in the proof of the main result of this chapter. The following two lemmas follow directly from Proposition 17, Proposition 19 and Proposition 20, and will be applied extensively throughout the proof to obtain upper bounds on $r_0(G)$ and $r(G)$.

Lemma 51. *If there exists $S \subseteq V(G)$, such that $G - S$ is bipartite, then $r_0(G) \leq |S|$.*

Lemma 52. *If there exists $S \subseteq V(G)$, such that $G - S$ is series-parallel, then $r_0(G) \leq |S| + 1$.*

On the other hand, the next result is useful in proving lower bounds on $r_0(G)$ and $r(G)$.

Proposition 53. *(Lemma 22 of Lipták and Tunçel [15]) Let $S \subset V(G)$ be a stable set in the graph G. For $k \geq 3$ define the vector $x^{(S,k)} \in \mathbb{R}^{V(G)}$ as follows:*

$$x_i^{(S,k)} = \begin{cases} \frac{k-1}{k} & \text{if } i \in S, \\ \frac{1}{k} & \text{if } i \notin S. \end{cases}$$

If $x^{(S,3)} \in N(G)$, then $x^{(S,k)} \in N^m(G)$ for all $k \geq m+2$ for any $m \geq 1$.

Proposition 53 is a generalization of the following fact that is first shown by Lovász and Schrijver in [16].

Corollary 54. *For any graph G,*

$$\frac{1}{k+2}\bar{e} \in N^k(G) \quad \forall k \geq 0.$$

Here we show another generalization of Corollary 54. Note that the proof of this result relies on Lemma 84 and 85, whose (self-contained) proofs are presented in Chapter 6.

Recall that for any graph G, $z \in \mathbb{R}^{V(G)}$ and $i \in V(G)$, $\Phi_i(z)$ and $\Psi_i(z)$ are z restricted to the subgraphs $(G - i)$ and $(G \ominus i)$ respectively. Then we have the following:

Proposition 55. *For any graph G and any integer $k \geq 0$, we have*

$$\frac{k+2}{k+3} N_0^k(G) \subseteq N_0^{k+1}(G).$$

Analogous containment holds for N.

Proof. We prove the claim by induction on k.

When $k = 0$, given $x \in FRAC(G)$, then $\frac{2}{3}x \in OC(G)$ because for any odd cycle C, $\sum_{i \in C} \frac{2}{3} x_i \leq \frac{|C|}{3} \leq \frac{|C|-1}{2}$.

For the inductive step, given $x \in N_0^k(G)$, we show that $\frac{k+2}{k+3}x \in N_0^{k+1}(G)$. First, $x \in N_0^k(G)$ implies that there exists Y such that $\begin{pmatrix} 1 & x^T \\ x & Y \end{pmatrix} \in M_0^k(G)$. We define Y' such that

$$Y'_{ij} = \begin{cases} \frac{k+2}{k+3} Y_{ij} & \text{if } i = j; \\ \frac{k+1}{k+3} Y_{ij} & \text{if } i \neq j. \end{cases}$$

Now we show that $\begin{pmatrix} 1 & \frac{k+2}{k+3}x^T \\ \frac{k+2}{k+3}x & Y' \end{pmatrix} \in M^{k+1}(G)$.

For any $i \in [n]$, we know that $Y_i \in x_i N^{k-1}(G)$. Then by Lemma 84, we have $\Psi_i(Y_i) \in x_i N^{k-1}(G \ominus i)$. By inductive hypothesis, $\frac{k+1}{k+2}\Psi_i(Y_i) \in x_i N^k(G \ominus i)$. Since $\Psi_i(Y_i) = \frac{k+3}{k+1}\Psi_i(Y_i')$ by the construction of Y', this implies that $\Psi_i(Y_i') \in \frac{k+2}{k+3}x_i N^k(G \ominus i)$, and by Lemma 84 again, we know that $Y_i' \in \frac{k+2}{k+3}x_i N^k(G)$.

To show $(\frac{k+2}{k+3}x - Y_i') \in (1 - \frac{k+2}{k+3}x_i)N^k(G)$ for every $i \in [n]$, we see that

$$
\begin{aligned}
& \frac{k+1}{k+3}(x - Y_i) \in \frac{k+1}{k+3}(1-x_i)N_0^{k-1}(G) \\
\Rightarrow \quad & \frac{k+1}{k+3}\Phi_i(x) - \frac{k+1}{k+3}\Phi_i(Y_i) \in \frac{k+1}{k+3}(1-x_i)N_0^{k-1}(G-i) \quad \text{by Lemma 84} \\
\Rightarrow \quad & \frac{k+1}{k+3}\Phi_i(x) - \Phi_i(Y_i') \in \frac{k+1}{k+3}(1-x_i)N_0^{k-1}(G-i) \quad \text{by construction of } Y' \\
\Rightarrow \quad & \Phi\left(\frac{k+1}{k+3}x - Y_i'\right) \in \frac{k+1}{k+3}(1-x_i)N_0^{k-1}(G-i) \\
\Rightarrow \quad & \frac{k+1}{k+3}x - Y_i' \in \frac{k+1}{k+3}(1-x_i)N^{k-1}(G) \quad \text{by Lemma 84} \\
\Rightarrow \quad & \frac{k+1}{k+3}x - Y_i' \in \frac{k+1}{k+3}(1-x_i)\left(\frac{k+2}{k+1}N_0^k(G)\right) \quad \text{by inductive hypothesis} \\
\Rightarrow \quad & \frac{k+1}{k+3}x - Y_i' \in \left(\frac{k+2}{k+3} - \frac{k+2}{k+3}x_i\right)N_0^k(G) \\
\Rightarrow \quad & \frac{k+1}{k+3}x - Y_i' + \frac{1}{k+3}x \in \left(\frac{k+2}{k+3} - \frac{k+2}{k+3}x_i + \frac{1}{k+3}\right)N_0^k(G) \quad \text{since } x \in N_0^k(G) \\
\Rightarrow \quad & \frac{k+2}{k+3}x - Y_i' \in \left(1 - \frac{k+2}{k+3}x_i\right)N_0^k(G),
\end{aligned}
$$

hence the claim follows.

The same argument also applies to N, as Y' is symmetric if Y is. $\square$

Now we return to proving the main result of this chapter. Given a graph H on n nodes and $S_1, S_2, \ldots, S_k, S_i \subseteq [n],\ \forall i \in [k]$, we define $G = (H, S_1, \ldots, S_k)$ to be the graph with $n+k$ nodes, such that nodes $1, \ldots, n$ induce the graph H, and $\mathcal{N}(n+i) = S_i,\ \forall i \in [k]$. Similarly, we define the graph $[H, S_1, \ldots, S_k]$ to be the graph $(H, S_1, \ldots, S_k)$, except nodes $n+1, n+2, \ldots n+k$ induce a clique.

Then we are ready show the following:

Proposition 56. *Suppose G is a graph and $|V(G)| \leq 7$. Then $r_0(G) = r(G)$.*

Proof. We know that $r_0(G) \geq r(G)$. First, assume there is a graph on no more than 7 nodes that satisfies $r_0(G) > r(G)$. By Lemma 21 G has to be imperfect, and hence has to contain an odd-hole (an induced subgraph that is a chordless odd cycle of length at least 5), or an odd-antihole (the complement of an odd-hole). For graphs with 7 or fewer nodes, that means that G has to contain a 5-hole, a 7-hole or a 7-antihole (the 5-antihole case can be ignored because the 5-antihole is isomorphic to the 5-hole).

If $|V(G)| = 5$ or 6, then either G is the 5-hole (in which case $r_0(G) = r(G) = 1$), or it is a 5-hole plus a node. Let v denote the node that is not on the 5-hole. Then we have $r_0(G - v) = 1$, so $r_0(G) \leq 2$, which implies that $r_0(G) = r(G)$. Also, if $V(G) = 7$ and G contains an induced subgraph of a 7-hole or a 7-antihole, then the graph is the 7-hole or 7-antihole, both of which satisfy $r_0(G) = r(G)$ (for the ranks of odd-antiholes, please refer to Proposition 57 in Chapter 5).

So if we let C_5 denote the 5-cycle, we may assume that $G = (C_5, S_1, S_2)$ or $[C_5, S_1, S_2]$ for some $S_1, S_2 \subseteq [5]$. We assume without loss of generality that $|S_1| \leq |S_2|$.

Notice that no matter what S_1, S_2 are and whether $6 \sim 7$, $G - \{1, 6, 7\}$ is a path (and hence bipartite). Therefore by Lemma 51 we have $r_0(G) \leq 3$ for all imperfect graphs on 7 nodes. Since the Rank Conjecture holds for graphs of N_0-rank ≤ 2, if G is a 7-node counterexample to the Rank Conjecture, G has to satisfy $r_0(G) = 3$ and $r(G) = 2$. Therefore, it suffices to show that every graph in our consideration satisfies either $r_0(G) \leq 2$ or $r(G) \geq 3$.

Now, if $|S_1| \leq 2$, then $G - \{k, 7\}$ is bipartite for any $k \in S_1$, so $r_0(G) \leq 2$ by Lemma 51. So we can assume that $|S_1| \geq 3$. Now we split our discussion into two cases.

Case 1: $6 \not\sim 7$

If $|S_1| = 3$, Then there are only two non-isomorphic cases for $(G - 7)$, either $S_1 = \{1, 2, 3\}$ or $S_1 = \{1, 2, 4\}$. In the latter case, $G - \{1, 7\}$ is bipartite, hence $r_0(G) \leq 2$ by Lemma 51. So we only have to concentrate on the first case.

When $6 \not\sim 7$ and $S_1 = \{1, 2, 3\}$, G must be a subgraph of the following graph H_1.

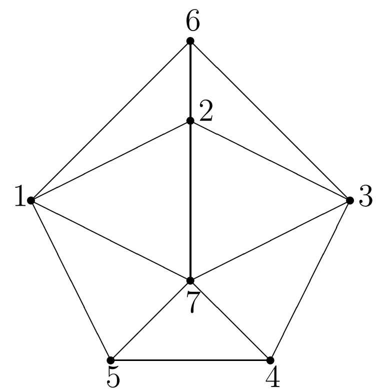

Figure 4.1: The graph H_1

Observe that $(H_1 - 1)$ is series-parallel. Since $(G - 1)$ is a subgraph of $(H_1 - 1)$, and subgraphs of a series-parallel graph are also series-parallel, we have $r_0(G) \leq 2$ by Lemma 52.

If $|S_1| = 4$ and $6 \not\sim 7$, G has to be contained in the following graph H_2.

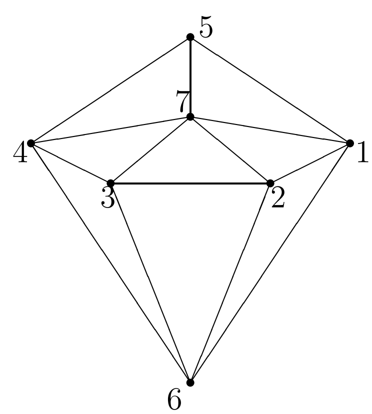

Figure 4.2: The graph H_2

Consider $(H_2 - 1)$. It can be expressed as a union of a 4-wheel (induced by $\{2, 3, 4, 6, 7\}$) and a 3-cycle (induced by $\{4, 5, 7\}$). The 4-wheel and 3-cycle are both of N_0-rank 1, and their intersection is a 2-clique (the edge $\{4, 7\}$). Therefore by Lemma 18, $r_0(H_2 - 1)$ equals the maximum of rank of the 4-wheel and the 3-cycle, which are both 1. Hence

$r_0(H_2 - 1) = 1$.

Now $(G-1)$ can also be decomposed a similar way. As long as $4 \sim 7$, $(G-1)$ can be expressed at a union of two subgraphs, one being a subgraph of a 4-wheel and another being a subgraph of a 3-cycle, that intersect at the edge $\{4, 7\}$. Every subgraph of the 4-wheel, as well as every subgraph of the 3-cycle has N_0-rank at most one. So, we have $r_0(G-1) \leq 1$, which implies that $r_0(G) \leq 2$.

If $4 \not\sim 7$, since $|S_2| \geq |S_1| = 4$ by assumption, we know that 7 is adjacent to all of $1, 2, 3$ and 5. Now we see that $(G-4)$ is isomorphic to $(H_2 - 1)$. So we know that $r_0(G-4) \leq 1$, and hence $r_0(G) \leq 2$.

If $|S_1| = 5$, then by assumption $|S_2| = 5$. So, $S_1 = S_2 = [5]$. By Corollary 44, we have $r_0(G) = r_0(G-6)$. Since $(G-6)$ is the 5-wheel which has N_0-rank 2, $r_0(G) = 2$.

So the case in which $6 \not\sim 7$ is complete.

Case 2: $6 \sim 7$

If G is a counterexample to the Rank Conjecture and satisfies $r_0(G) = 3, r(G) = 2$, we know that $r_0(G-6) = r_0(G-7) = 2$. Since $|S_2| \geq |S_1| \geq 3$, both $(G-6)$ and $(G-7)$ have to be isomorphic to one of the three graphs in Figure 4.3.

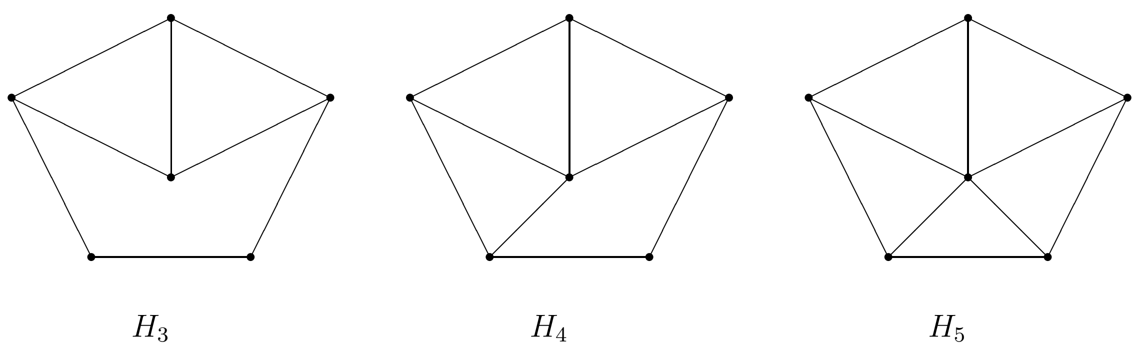

Figure 4.3: The three non-isomorphic imperfect graphs on 6 nodes that has N_0-rank 2

With that, there are 12 non-isomorphic cases for G, as listed in Figure 4.4. Either both $(G-6)$ and $(G-7)$ are isomorphic to H_3, (G_1, G_2 and G_3), both $(G-6), (G-7)$ are isomorphic to H_4, (G_4, G_5 and G_6), $(G-7) \cong H_3$ and $(G-6) \cong H_4$ (G_7, G_8 and G_9), or $(G-6) \cong H_5$ (G_{10}, G_{11} and G_{12}).

We let $\alpha(G)$ denote the stability number (i.e. the size of the largest stable set) of G. Notice that for any graph G, $\sum_{i \in V(G)} x_i \leq \alpha(G)$ is valid for $STAB(G)$.

Now we show exhaustively that none of these 12 graphs have N_0-rank 3 and N-rank 2.

For G_1, the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$ is in $N(G_1)$. Then $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^T \in N^2(G_1)$ by Proposition 53. Since this point has weight $9/4 \geq 2 = \alpha(G_1)$, we have $r(G_1) > 2$. The same argument also applies for G_2, G_4 and G_7, since node 4 is not in a 3-cycle and $\alpha(G) = 2$ for all these graphs.

$(G_3 - 1)$ is series-parallel, so by Lemma 52 we have $r_0(G_3) \leq 2$. The same argument also shows that $r_0(G_9) \leq 2$.

For G_5, it is not hard to see that

$$STAB(G_5) = OC(G_5) \cap \left\{ x : \sum_{i \in [7]} x_i \leq 2 \right\}.$$

We know all facets of $OC(G)$ have N_0-rank 1. For the extra facet $\sum_{i \in [7]} x_i \leq 2$, we see that both deletion and destruction of node 2 from it give an inequality that is valid for $OC(G_5)$. Hence, the facet $\sum_{i \in [7]} x_i \leq 2$ has N_0-rank at most 2, and therefore $r_0(G_5) = 2$.

$(G_6 - 3)$ can be expressed at a union of a 4-wheel induced (induced by $\{1, 4, 5, 6, 7\}$) and a 3-cycle (induced by $\{1, 2, 6\}$) that intersect at a 2-clique (the edge $\{1, 6\}$). Therefore, by Lemma 18 $r_0(G_6 - 3) = 1$. Hence, $r_0(G_6) \leq 2$. Similarly, $(G_8 - 1)$ can also be expressed as a union of two rank-1 graphs intersecting at the 2-clique $\{4, 6\}$. Therefore, by Lemma 18 again, $r_0(G_8) \leq 2$ as well.

For G_{10}, we consider the matrix

$$\frac{1}{40}\begin{pmatrix} 40 & 4 & 8 & 14 & 8 & 14 & 16 & 16 \\ 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 8 & 0 & 0 & 0 & 3 & 3 \\ 14 & 0 & 0 & 14 & 0 & 4 & 10 & 0 \\ 8 & 0 & 0 & 0 & 8 & 0 & 3 & 3 \\ 14 & 0 & 0 & 4 & 0 & 14 & 0 & 10 \\ 16 & 0 & 3 & 10 & 3 & 0 & 16 & 0 \\ 16 & 0 & 3 & 0 & 3 & 10 & 0 & 16 \end{pmatrix}.$$

Since each column and the difference of each column and the first column is in $OC(G_{10})$ and the matrix is symmetric, it is in $M^2(G_{10})$. However, the first column violates the inequality $x_1+x_2+x_3+x_4+x_5+x_6+2x_7 \leq 2$ which is a valid inequality for $STAB(G_{10})$, so $r(G_{10}) = 3$.

The same argument also shows that $r(G_{11}) = 3$, but instead of the matrix above, we consider this following matrix

$$\frac{1}{7}\begin{pmatrix} 7 & 1 & 2 & 2 & 2 & 2 & 3 & 2 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 & 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 & 2 & 0 & 1 & 1 \\ 2 & 0 & 0 & 1 & 0 & 2 & 0 & 1 \\ 3 & 0 & 1 & 1 & 1 & 0 & 3 & 0 \\ 2 & 0 & 0 & 0 & 1 & 1 & 0 & 2 \end{pmatrix}.$$

Each column and the difference of each column and the first column is in $OC(G_{11})$, so the matrix is in $M^2(G_{11})$. However $x_1+x_2+x_3+x_4+x_5+x_6+2x_7 \leq 2$ is a valid inequality for $STAB(G_{11})$ and is violated by the first column of the matrix, hence $r(G_{11}) = 3$.

For G_{12}, the point $\frac{1}{4}\bar{e}$ is in $N^2(G_{12})$. However, this point violates the inequality $x_1 + x_2 + x_3 + x_4 + x_5 + 2x_6 + 2x_7 \leq 2$, which is a valid inequality for $STAB(G_{12})$. Hence $r(G_{12}) = 3$.

This completes the proof. □

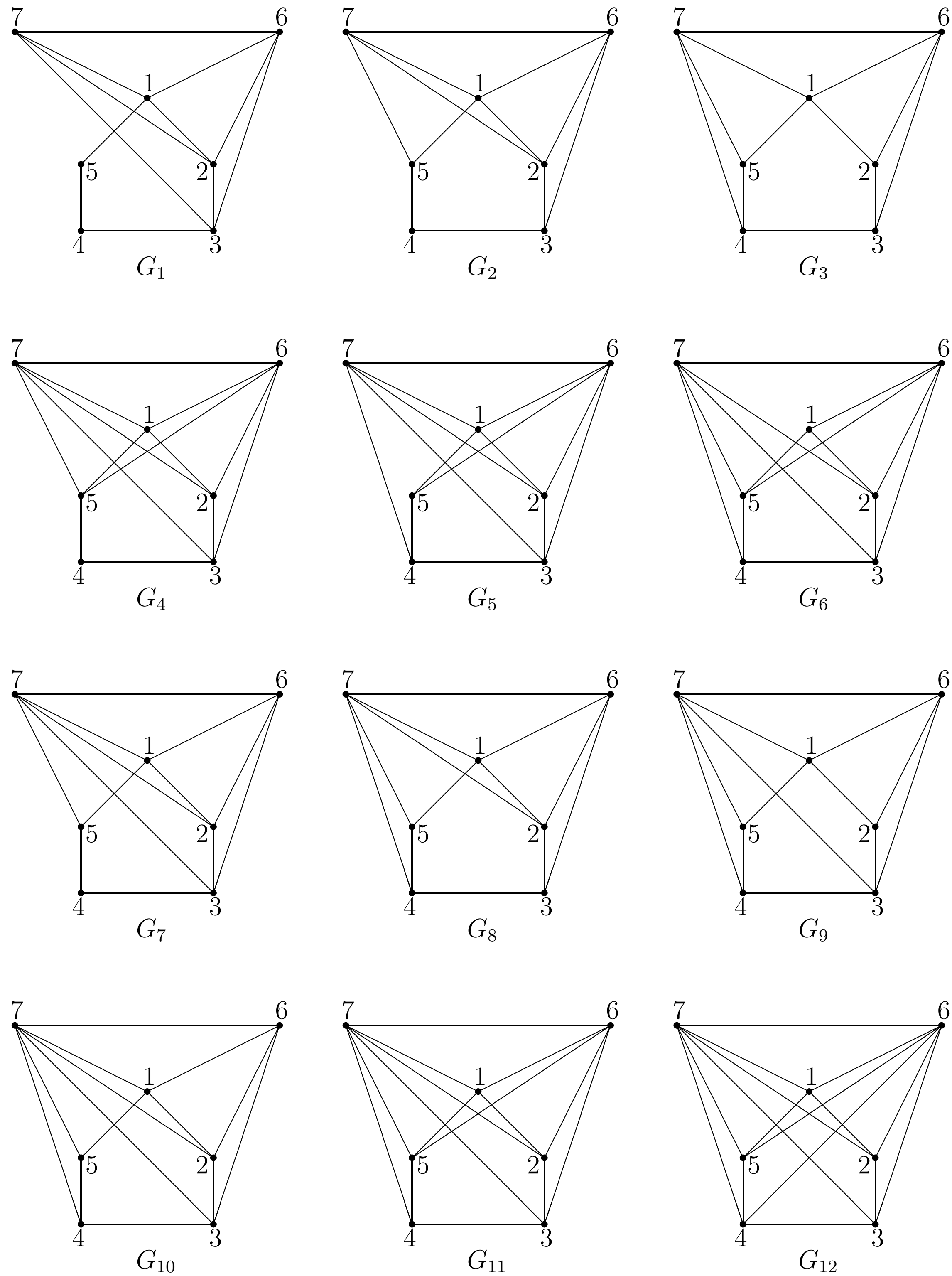

Figure 4.4: The 12 non-isomorphic cases for $[C_5, S_1, S_2]$

Chapter 5

Verifying the Rank Conjecture on graphs with no more than 8 nodes

We proved in Chapter 4 that the Rank Conjecture holds for all graphs with no more than 7 nodes. In this chapter, we extend our result to all 8-node and some 9-node graphs.

We first give some general results about the N- and N_0-ranks of certain families of graphs in Section 5.1. Using those tools and some computerized checking, we show that the Rank Conjecture holds for all graphs on 8-nodes in Section 5.2. In Section 5.3, we wrap up the chapter by showing that the Rank Conjecture also holds for 9-node graphs that contain a 7-hole or a 7-antihole as an induced subgraph.

5.1 General facts applicable to the 8-node case

If a counterexample to the Rank Conjecture exists, it has to be imperfect. Hence, we may assume that G contains either an odd-hole or an odd-antihole.

If G is an odd-hole, we know that $r_0(G) = r(G) = 1$. Its N- and N_0-rank are also the same if G is an odd-antihole, due to the following well known result.

Proposition 57. *Let G be an odd-antihole on $2k+1$ nodes. Then*

$$r_0(G) = r(G) = k - 1$$

Proof. Notice that $(G-v)$ is perfect for any $v \in V(G)$ (this is more apparent by looking at the complement of $(G-v)$). Also, it is easy to see that the largest clique in $(G-v)$ has size k, hence $r_0(G-v) = k-2$.

Also, observe that $\sum_{i\in[2k+1]} x_i = 2$ is a facet for $STAB(G)$ and is violated by the point $\frac{1}{k}\bar{e}$. Since $\frac{1}{k}\bar{e} \in N^{k-2}(G)$ by Corollary 54, we see that $r(G) > k-2$.

Since $r_0(G-v) = k-2, r_0(G) \leq k-1$, and hence $r_0(G) = r(G) = k-1$. $\square$

Now we look into graphs that consist of a "core" that an odd-hole or an odd-antihole, plus a few nodes. Recall that, for any graph G and $S \subseteq V(G)$, G_S denotes the subgraph of G induced by the nodes in S. Also, for any odd integer n, we let C_n denote the n-hole and $\bar{C}_n$ denote the n-antihole.

Suppose we are given a graph G with more than n nodes, and let S_i denote the set $\{j : j \sim n+i, j \in [n]\}$. We define the *weakness of G with respect to S_i* as

$$\mu(S_i) := \alpha(G_{[n]}) - \alpha(G_{[n]\setminus S_i}).$$

In many cases, $\mu(S_i)$ is closely related to the coefficient of node $n+i$ in certain facets of $STAB(G)$, and that sometimes lead to the knowledge of the N- and N_0-rank of the graph. The following result is an example of that:

Proposition 58. *Suppose $G = (C_n, S)$ for some odd n. Then the inequality*

$$\sum_{i=1}^{n} x_i + \mu(S)x_{n+1} \leq \frac{n-1}{2} \tag{5.1}$$

is a facet of $STAB(G)$. Moreover, $r_0(G) = 2$ if $\mu(S) \geq 1$, and $r_0(G) = 1$ otherwise.

Before we prove Proposition 58, we first take a look at what the operator $\mu(\cdot)$ does when $G_{[n]}$ is an odd-hole.

Given a non-empty set $T := \{t_i : i \in [k]\}$ such that $1 \leq t_1 < t_2 < \ldots < t_k \leq n$, we call T an *odd partition* of $[n]$ if $t_{i+1} - t_i$ is odd for all $i \in [k-1]$, and $t_1 - t_k \mod n$ is odd. For example $\{1,2,3\}, \{1,2,5\}$ and $\{1,2,3,4,5\}$ are all odd partitions of $[7]$. Notice that an odd partition must have odd size. Then we know that

Lemma 59. *Suppose $G = (C_n, S)$. Then*

$$\mu(S) = \frac{\max\{|T| : T \subseteq S, T \text{ an odd partition of } [n]\} - 1}{2}.$$

Proof. Suppose $T := \{t_1, \ldots, t_{2d+1}\}$ is the largest subset of S that is an odd partition of $[n]$. We want to show that $\mu(S) = d$. Notice that if there exist i, p, q such that p is odd, q is even, $0 < p < q < t_{i+1} - t_i$, and $t_i + p, t_i + q \in S$, then $T \cup \{t_i + p, t_i + q\}$ is a larger odd partition in S, contradicting the maximality assumption on T.

Therefore, given any i, if there does not exist an odd p such that $p < t_{i+1} - t_i$ and $t_i + p \in S$, then $\left\{t_i + 2k : k \in [\frac{t_{i+1}-t_i-1}{2}]\right\}$ is a stable set in $G_{[n]\setminus S}$. If there does exist an odd p, we choose the smallest such p, and see that $\left\{t_i + 2k - 1 : 1 \leq k \leq \frac{p-1}{2}\right\} \cup \left\{t_i + 2k : \frac{p+1}{2} \leq k \leq \frac{t_{i+1}-t_i-1}{2}\right\}$ is a stable set in $G_{[n]\setminus S}$. In both cases, the stable set we found have size $\frac{t_{i+1}-t_i-1}{2}$.

We can do this for every $i \in [2d+1]$, Observe that the $2d+1$ stable sets we found in $G_{[n]\setminus S}$ each belongs to a different component in that graph, and hence their union is also a stable set in $G_{[n]\setminus S}$. Moreover, it is obvious that each of the stable set we found is maximal within its corresponding component, so the union of them has to be a maximal stable set in $G_{[n]\setminus S}$.

Therefore, we have determined that

$$\alpha(G_{[n]\setminus S}) = \sum_{i=1}^{2d+1} \frac{t_{i+1} - t_i - 1}{2} = \frac{n - 2d - 1}{2} = \frac{n-1}{2} - d,$$

which implies that $\mu(S) = d$. $\square$

Also, since many graphs G (odd-holes and odd-antiholes, among others) have the facet $\bar{e}^T x \leq \alpha(G)$ as a facet of $STAB(G)$, the following lemma is useful.

Lemma 60. *Suppose $G = [H, S_1, \ldots, S_k]$ and that $\bar{e}^T x \leq \alpha(H)$ is a facet of $STAB(H)$. Then*

$$\sum_{i\in[n]} x_i + \sum_{i\in[k]} \mu(S_i) x_{n+i} \leq \alpha(H) \tag{5.2}$$

is a facet of $STAB(G)$.

Proof. First we show that (5.2) is valid for $STAB(G)$. Let T be a stable set in G. If $n + i \notin T$ for any $i \in [k]$, then obviously χ_T does not violate (5.2). On the other hand, if $n + i \in T$ for some i (there could only be one such i, since $\{n+1, \ldots, n+k\}$ induce a clique), then we know that $|T \setminus \{n+i\}\,| \leq \alpha(H) - \mu(S_i)$, and hence (5.2) is valid for $STAB(G)$.

Now we show that it is indeed a facet. $\sum_{i=1}^{n} x_i \leq \alpha(H)$ is a facet for $STAB(H)$ implies that there exist n linearly independent vectors in $STAB(H)$, $u^{(1)}, \ldots, u^{(n)}$, that satisfy $\bar{e}^T u^{(i)} = \alpha(H)$. Therefore, for every $i \in [n]$ we know that $\begin{pmatrix} u^{(i)} \\ 0 \end{pmatrix}$ is in $STAB(G)$ and satisfies (5.2) with equality. Now for every $i \in [k]$, let T_i be a stable set formed by $n+i$ and $\alpha(H) - \mu(S_i)$ nodes in H, and we see that χ_{T_i} satisfies (5.2) with equality for every i. It is obvious that these points are linearly independent with all of the previous points. So, (5.2) is a facet of $STAB(G)$. $\square$

Now we are ready to prove Proposition 58.

Proof of Proposition 58. (5.1) is a facet of $STAB(G)$ by Lemma 60. For the N- and N_0-rank, notice that if we delete node $n+1$ from (5.1), then we get an odd cycle inequality, which is valid for $N_0(G)$. Also, the inequality obtained from (5.1) by destroying $n+1$ is a sum of edge inequalities. Therefore, by Proposition 23, (5.1) is valid for $N_0^2(G)$.

We see that if $\mu(S) \geq 1$, then (5.1) is not valid for $N_0(G)$, hence (5.1) has N_0-rank 2. Also, since $r_0(G - (n+1)) = 1$, it follows that $r_0(G) = 2$.

On the other hand, if $\mu(S) = 0$, then other than C_n, there is only one other chordless odd cycle in G. We delete any node on that cycle that is not $n+1$ and see that the resulting graph is bipartite. Therefore, we have $r_0(G) = 1$. $\square$

Remark 61. *The fact that $r_0(G) = 1$ when $\mu(S) = 0$ also follows from Proposition 49, since in such case, G only has n maximal stable sets.*

We call a graph $G = (C_n, S)$ a *partial wheel* if $\mu(S) \geq 1$, and (5.1) the *partial wheel inequality* of G.

Next, we attempt to determine the ranks for graphs $G = (\bar{C}_{2k+1}, S)$. First, we show an extremely simple fact that will be called upon numerous times later in this chapter:

Lemma 62. *Suppose we have two graphs $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$. If $V(G_1) = V(G_2)$ and $E(G_1) \supseteq E(G_2)$, then*

$$N_0^k(G_1) \subseteq N_0^k(G_2) \quad \forall k \geq 0.$$

Analogous containment holds for N.

Proof. This follows directly from the fact $FRAC(G_1) \subseteq FRAC(G_2)$, and that both the N_0 and N operators preserve containment. $\square$

Then we have the following:

Proposition 63. *Suppose $G = (\bar{C}_{2k+1}, S)$ for some k, S. Then*

$$\sum_{i \in [2k+1]} x_i + \mu(S) x_{2k+2} \leq 2 \tag{5.3}$$

is a facet of $STAB(G)$. Moreover, $r_0(G) = r(G) = k$ if $\mu(S) \geq 1$.

Proof. The facet that (5.3) is a facet follows from Lemma 60.

Now suppose $\mu(S) \geq 1$. Let $x := \left(\frac{2}{2k+1}, \frac{2}{2k+1}, \ldots, \frac{2}{2k+1}, \frac{1}{2k+1}\right)^T$. Notice that x violates (5.3). Also, since $r_0(G) \leq k$ follows from the fact that $r_0(\bar{C}_{2k+1}) = k-1$, it suffices to show that $x \in N^{k-1}(G)$ for every $k \geq 1$ to show that $r_0(G) = r(G) = k$. Moreover, by Lemma 62, we only have to verify our claim for the case when $S = [2k+1]$.

We define a $2k+2$ by $2k+2$ matrix Y such that:

$$Y_{i,j} = \begin{cases} x_i & \text{if } i = j; \\ \frac{1}{2k+1} & \text{if } i, j \in [2k+1] \text{ and } j - i \equiv 1 \mod 2k+1; \\ 0 & \text{otherwise.} \end{cases}$$

So Y is the matrix

$$\frac{1}{2k+1} \begin{pmatrix} 2 & 1 & 0 & \ldots & 0 & 1 & 0 \\ 1 & 2 & 1 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 2 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2 & 1 & 0 \\ 1 & 0 & 0 & \ldots & 1 & 2 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 1 \end{pmatrix}.$$

We show that $\begin{pmatrix} 1 & x^T \\ x & Y \end{pmatrix} \in M^{k-1}(G)$. First of all, it is apparent that $Y_i \in x_i STAB(G)$ for every $i \in [2k+2]$.

Now for any fixed $i \in [2k+1]$,

$$(x - Y_i)_j = \begin{cases} 0 & \text{if } j = i; \\ \frac{1}{2k+1} & \text{if } j - i \equiv 1 \mod 2k+1 \text{ or } j = 2k+2; \\ \frac{2}{2k+1} & \text{otherwise.} \end{cases}$$

We show that $y := \frac{1}{1-x_i}(x - Y_i) \in STAB(G)$, which implies that $(x - Y_i) \in (1-x_i)N^{k-2}(G)$. First we notice that $(G - i)$ is perfect, so $STAB(G - i)$ is defined by the clique constraints in $(G - i)$.

We see that $y_j \leq \frac{2}{2k-1}$ $\forall j \in [2k+2]$, and hence does not violate any clique constraints of size $k-1$ or less. Also, any k-clique in $(G - i)$ must include the node $2k+2$. Since $y_{2k+2} = \frac{1}{2k-1}$, the sum of it with any other $k-1$ coordinates of y does exceed 1. Since there are no cliques of size larger than k in $(G - i)$ and that $y_i = 0$, we conclude that $y \in STAB(G)$.

Finally,

$$x - Y_{2k+2} \leq \frac{2}{2k+1}\bar{e} = (1 - x_{2k+2})\frac{1}{k}\bar{e} \in (1 - x_{2k+2})N^{k-2}(G),$$

thus we have $x \in N^{k-1}(G)$, and our claim follows. $\square$

In general for a graph $G = (\bar{C}_{2k+1}, S)$, unlike the case when the core of the graph is an odd-hole, it is possible that $r(G) > k-1$ while $\mu(S) = 0$. For an example, the graph $G = (\bar{C}_9, \{3, 4, 6, 8, 9\})$ has N- and N_0-rank 4.

Now we look into the facets of $STAB(G)$ for graphs G that are an odd-hole plus two nodes. First, we focus on the case when the two nodes are not adjacent to each other.

Suppose $G = (C_n, S_1, S_2)$. We define the quantity

$$\lambda(S_1, S_2) := \max \left\{ \mu(S_1' \cup S_2') : S_i' \text{ an odd partition of } [n], S_i' \subseteq S_i \ \forall i \in [2] \right\}.$$

The following lemma is useful for the subsequent proposition:

Lemma 64. *Given $G = (C_n, S_1, S_2)$, we have*

$$\lambda(S_1, S_2) \leq \min \left\{ \mu(S_1) + \mu(S_2), \mu(S_1 \cup S_2) \right\}.$$

Proof. Let S_1', S_2' be the subsets in S_1, S_2 such that $\mu(S_1' \cup S_2') = \lambda(S_1, S_2)$. From Lemma 59 we know that $|S_1'| = 2\mu(S_1) + 1$ and $|S_2'| = 2\mu(S_2) + 1$.

Now we observe that if P, Q are odd partitions of $[n]$, then the largest possible size of an odd partition in $P \cup Q$ is $(|P| + |Q| - 1)$. Therefore, the largest odd partition that lies in $S_1' \cup S_2'$ has size at most $2(\mu(S_1) + \mu(S_2)) + 1$, and so by Lemma 59, $\lambda(S_1, S_2) = \mu(S_1' \cup S_2') \leq \mu(S_1) + \mu(S_2)$.

For the second claim, it is obvious that $P \subseteq Q \Rightarrow \mu(P) \leq \mu(Q)$, hence we have $\lambda(S_1, S_2) = \mu(S_1' \cup S_2') \leq \mu(S_1 \cup S_2)$. □

Now we show two classes of facets of $STAB(G)$ we discovered for the graphs $G = (C_n, S_1, S_2)$: the *double partial wheel inequalities* and the *P-augmented odd cycle inequalities.*

Proposition 65. *Suppose $G = (C_n, S_1, S_2)$ and $\lambda(S_1, S_2) > 0$. Then the double partial wheel inequalities*

$$\sum_{i \in [n]} x_i + \mu(S_1)x_{n+1} + (\lambda(S_1, S_2) - \mu(S_1))x_{n+2} \leq \frac{n-1}{2} \tag{5.4}$$

and

$$\sum_{i \in [n]} x_i + (\lambda(S_1, S_2) - \mu(S_2))x_{n+1} + \mu(S_2)x_{n+2} \leq \frac{n-1}{2} \tag{5.5}$$

are both valid inequalities of $STAB(G)$ of N_0-rank 2. Moreover, they are both facets of $STAB(G)$ if $\lambda(S_1, S_2) = \mu(S_1 \cup S_2)$.

Proof. It suffices to prove the claims for (5.4), as those for (5.5) follow by symmetry.

We first show that (5.4) is valid for $STAB(G)$ by showing that all incidence vectors of maximal stable sets in G satisfy the inequality. Let S be a maximal stable set in G. If $n+2 \notin S$, then it is obvious that χ_S satisfies (5.4) because the inequality obtained by deleting $n+2$ from (5.4) is valid for $STAB(G-(n+2))$. Similarly, this is true if $n+1 \notin S$, since $\lambda(S_1, S_2) - \mu(S_1) \leq \mu(S_2)$. If $n+1, n+2 \in S$, then $|S \cap [n]| = \frac{n-1}{2} - \mu(S_1 \cup S_2) \leq \frac{n-1}{2} - \lambda(S_1, S_2)$, hence the inequality holds as well.

Now we determine the N_0-rank of (5.4). It is easy to see that it is at least 2 if $\lambda(S_1, S_2) > 0$, as it implies the partial wheel inequalities, which have N_0-rank 2. To show that (5.4) has

N_0-rank no higher than 2, we show that there exists a node whose deletion and destruction from (5.4) both give an inequality that is valid for $OC(G)$.

Suppose we have $S_1' \subseteq S_1, S_2' \subseteq S_2$ such that $\lambda(S_1, S_2) = \mu(S_1' \cup S_2')$. We let $T_1 := S_1'$, and we define

$$T_2 := \begin{cases} S_2' & \text{if } S_1' \cap S_2' = \emptyset; \\ S_2' \setminus S_1' \cup \{i\} & \text{otherwise, for any } i \in S_1' \cap S_2'. \end{cases}$$

Notice that T_2 is an odd partition of $[n]$, $\mu(T_1 \cup T_2) = \mu(S_1' \cup S_2')$ and $|T_1 \cap T_2| \leq 1$.

First we show that if $k \in T_1 \cup T_2$, then destroying k yields a valid inequality for $OC(G)$. Without loss of generality, we assume that $k = 1$ (we can re-label the nodes on C_n to make that happen). If $1 \in T_1 \cap T_2$, then both $n+1, n+2$ are removed in the destruction and the resulting inequality is implied by edge constraints. If $1 \in T_1 \setminus T_2$, then the inequality after destruction is

$$\sum_{i=3}^{n-1} x_i + (\lambda(S_1, S_2) - \mu(S_1))x_{n+2} \leq \frac{n-3}{2}. \tag{5.6}$$

Since $1 \notin T_2$ and T_2 is an odd partition, we know that $|\{n, 1, 2\} \cap T_2| \leq 1$, and hence T_2 has at least $2(\lambda(S_1, S_2) - \mu(S_1))$ neighbours in $(G \ominus 1)$. Then we can find $\{t_i \in T_2 : i \in [2(\lambda(S_1, S_2) - \mu(S_1))\}$ such that $1 < t_1 < t_2 < \ldots < t_{2(\lambda(S_1,S_2)-\mu(S_1))} < n$ and $t_{i+1} - t_i$ is odd for every $i \in [2(\lambda(S_1, S_2) - \mu(S_1)) - 1]$.

For every $i \in [\lambda(S_1, S_2) - \mu(S_1)]$, we let K_i denote the odd cycle formed by the path from t_{2i-1} to t_{2i} on C_n, and the two edges, $\{t_{2i-1}.n+2\}$ and $\{t_{2n}, n+2\}$. Then we see that (5.6) is the sum of the odd cycle inequalities of K_i's and edge constraints. For the case when $1 \in T_2 \setminus T_1$, the argument is similar (with $\mu(S_1)$'s replacing the $(\lambda(S_1, S_2) - \mu(S_1))$'s).

Now we find a node from $T_1 \cup T_2$ whose deletion from (5.4) gives a valid inequality for $OC(G)$.

If $T_1 \cap T_2 \neq \emptyset$, then let $s_1 \in T_1 \cap T_2$. We assume without loss of generality that $s_1 = 1$ (again, we can achieve this by shifting indices on C_n cyclically). Then we let $\{s_i : i = [2\lambda(S_1, S_2) + 2]\}$ denote the indices in $T_1 \cup T_2$ and order the indices s_i's such that $s_1 = 1 < s_2 < s_3 < \ldots < s_{2\lambda(S_1,S_2)+1} \leq n$. We know that $s_{i+1} - s_i$ is odd for every $i \in [2\lambda(S_1, S_2)]$.

If $T_1 \cap T_2 = \emptyset$, then $|T_1 \cup T_2| = 2\lambda(S_1, S_2) + 2$. Since $T_1 \cup T_2$ has even cardinality, there exist $s_0, s_1 \in T_1 \cup T_2$ such that $s_1 - s_0 \mod n$ is even and $\{s_0 + 1, s_0 + 2, \ldots, s_1 - 2, s_1 - 1\} \cap (T_1 \cup T_2) = \emptyset$. Again, we assume without loss of generality that $s_0 = 1$, and order the rest of the indices in $T_1 \cup T_2$ such that $s_0 = 1 < s_1 < \ldots < s_{2\lambda(S_1,S_2)+1} \leq n$. We again have $s_{i+1} - s_i$ is odd for every $i \in [2\lambda(S_1, S_2)]$ (and even for $s_1 - s_0$).

In either case, we delete s_1 from the inequality. We see that if $s_2 \in T_1$, then $s_3 \in T_1$ as well. This is because we know that one of s_0 (if defined) and s_1 is in T_2, and s_3 minus either of s_0, s_1 is even. Since T_2 is an odd partition, we know that s_3 has to belong to T_1. By the same rationale, we have $s_{2i} \in T_1 \iff s_{2i+1} \in T_1 \; \forall i \in [\lambda(S_1, S_2)]$, and same for T_2.

We let K_i denote the odd cycle formed by the $s_{2i}s_{2i+1}$ -path on C_n, plus the edges $\{s_{2i}, n+1\}$ and $\{s_{2i+1}, n+1\}$ if $s_{2i} \in T_1$, or $\{s_{2i}, n+2\}$ and $\{s_{2i+1}, n+2\}$ if $s_{2i} \in T_2$. Notice that exactly $\mu(S_1)$ of these cycles pass through $n+1$, and exactly $\lambda(S_1, S_2) - \mu(S_1)$ of them pass through $n+2$. Also, every node on C_n appear in at most one of the K_i's, and we see that the inequality obtained by deleting 1 from (5.4) is implied by the odd cycle inequalities of K_i's and edge constraints.

Since every valid inequality of $OC((C_n, T_1, T_2))$ is valid for $OC(G)$, it follows that (5.4) has N_0-rank 2.

To show that (5.4) is a facet of $STAB(G)$ when $\lambda(S_1, S_2) = \mu(S_1 \cup S_2)$, we let $P_1, \ldots, P_n$ be the n maximal stable sets of C_n, P_{n+1} be the set that contains node $n+1$ and a maximal stable set in $G_{[n]\setminus S_1}$ and P_{n+2} be the set that contains node $n+1, n+2$ and a maximal stable set in $G_{[n]\setminus(S_1 \cup S_2)}$. Then we see that the incidence vectors of the P_i's are linearly independent and all satisfy 5.4 with equality. Hence, 5.4 is a facet of $STAB(G)$ in this case. $\square$

Proposition 66. *Given $G = (C_n, S_1, S_2)$. Suppose $\beta := \mu(S_1) + \mu(S_2) - \mu(S_1 \cup S_2) \geq 0$ and there exist $p, q \in [n]$ such that*

- *$q - p \mod n$ is odd;*
- *$p \in S_1 \setminus S_2, q \in S_2 \setminus S_1$ or $q \in S_1 \setminus S_2, p \in S_2 \setminus S_1$; and*
- *$\{p+1, p+2, \ldots, q-2, q-1\} \cap (S_1 \cup S_2) = \emptyset$.*

Let $P := \{p, p+1, \ldots, q-1, q\}$. *If* $\mu(S_1) + \mu(S_2) \leq \frac{n+\beta|P|-1}{2}$, *then the* P*-augmented odd cycle inequality*

$$\sum_{i \in P} (\beta+1)x_i + \sum_{i \in [n] \setminus P} x_i + \mu(S_1)x_{n+1} + \mu(S_2)x_{n+2} \leq \frac{n+\beta|P|-1}{2} \tag{5.7}$$

is a facet of $STAB(G)$.

Proof. As in the proof of Proposition 65, we first show that (5.7) is valid for $STAB(G)$. First, we assume without loss of generality that $q \in S_1 \setminus S_2, p \in S_2 \setminus S_1$. Notice that (5.7) can be rewritten as

$$\sum_{i \in P} \beta x_i + \sum_{i \in [n]} x_i + \mu(S_1)x_{n+1} + \mu(S_2)x_{n+2} \leq \frac{\beta|P|}{2} + \frac{n-1}{2}. \tag{5.8}$$

Since the nodes in P induce an odd path, $\sum_{i \in P} \beta x_i \leq \frac{\beta|P|}{2}$ is the sum of edge constraints, and hence the inequality results from deleting $n+i$ from (5.7) is valid for $STAB(G-(n+i))$, for $i \in [2]$. Note that we have used the conditions $\beta \geq 0$ and $\mu(S_1) + \mu(S_2) \leq \frac{n+\beta|P|-1}{2}$ here.

If S is a maximal stable set in G and $n+1, n+2 \in S$, then we know that $|S \cap [n]| = \frac{n-1}{2} - \mu(S_1 \cup S_2)$. By assumptions on nodes p and q, we know that $|S \cap P| = \frac{|P|}{2} - 1$. Therefore, checking the inequality in the form (5.8), we get

$$\begin{aligned} & \beta\left(\frac{|P|}{2} - 1\right) + \left(\frac{n-1}{2} - \mu(S_1 \cup S_2)\right) + \mu(S_1) + \mu(S_2) \\ = \;& \frac{\beta|P|}{2} - \beta + \frac{n-1}{2} + \beta \\ = \;& \frac{\beta|P|}{2} + \frac{n-1}{2}. \end{aligned} \tag{5.9}$$

Hence, the inequality is valid for $STAB(G)$.

To show that it is a facet, we need the following claims.

Claim 67. *Given* $|P| = 2k$, *there are* $n-k$ *distinct stable sets that contain neither of* $n+1, n+2$, *and each has* k *nodes whose indices are in* P. *Moreover, the incidence vector of such a stable set satisfies* (5.7) *with equality.*

Proof. Without loss of generality assume that $P = [2k]$. For any $i \in [n]$, define $T_i := \left\{i + 2(j-1) \mod n : j \in [\frac{n-1}{2}]\right\}$. We know that $T_1, \dots, T_n$ are the n distinct maximal stable sets in C_n.

We see that

$$|T_i \cap P| = \begin{cases} k-1 & \text{if } i \in \{2j+1 : j \in [k]\}; \\ k & \text{otherwise.} \end{cases}$$

Therefore, our first claim follows. To check the second claim, we evaluate the incidence vector of any T_i on the left side of (5.7) and get

$$(\beta+1)k + \left(\frac{n-1}{2} - k\right) = \frac{n-1}{2} + \beta k = \frac{n + \beta|P| - 1}{2}$$

which is exactly the right side of (5.7). □

Claim 68. *Given* $|P| = 2k$*, there are* k *distinct stable sets that contain both of* $n+1, n+2$*, and each has* $k-1$ *nodes whose indices are in* P*. Moreover, the incidence vector of such a stable set satisfies* (5.7) *with equality.*

Proof. If $k = 1$, the claim is obviously true, so we assume that $k \geq 2$. Let S be a maximal stable set in G such that $n+1, n+2 \in T$. By assumptions on the nodes, we know that S contains exactly $k-1$ nodes whose indices are in P. Let S' be the set T after removing those $k-1$ nodes.

Now we assume again that $P = [2k]$, and for $i \in [k]$, define the set

$$T_i := \{2j+1 : j \in [i-1]\} \cup \{2j : j \in [k-1] \setminus [i-1]\},$$

where we defined $[0] := \emptyset$. We see that the T_i's all have size $k-1$, and are all distinct. Moreover, $S' \cup T_i$ is a maximal stable set in G. The fact that the incidence vector of $S' \cup T_i$ satisfies (5.7) with equality follows from the string of equalities (5.9). □

So given P of any size, we can find n stable sets whose incidence vectors satisfy (5.7) with equality. We can also find stable set S such that $n+2 \notin S, n+1 \in S$ and $|S \setminus \{n+1\}| = \frac{n-1}{2} - \mu(S_1)$. Since we know that $|S \cap P| = \frac{|P|}{2}$, it is easy to check that equality holds in (5.7) for χ_S. We can similarly find another stable set that contains $n+2$ but not $n+1$. These $n+2$ vectors are linearly independent, and hence (5.7) is a facet of $STAB(G)$. □

Remark 69. *For a graph G, there can be more than one P that satisfies the conditions in the statement of Proposition 66. For an example, Consider $G = (C_{21}, S_1, S_2)$ where*

$$S_1 := \{7i + j : i \in \{0,1,2\}\, j \in \{1,2,3,4,5\}\},$$
$$S_2 := \{7i + j : i \in \{0,1,2\}, j \in \{3,4,5,6,7\}\}$$

Then the sets $\{7,8\}, \{14,15\}$ and $\{1,21\}$ all satisfy the conditions, and hence there are 3 different P-augmented odd cycle facets for $STAB(G)$.

Also, the N_0-rank and N-rank of the (5.7) *can be* 2 *or* 3*, depending on the graph. For example, for $G = (C_7, \{1,2,3,4,5\}, \{3,4,5,6,7\})$, the inequality $2x_1 + \sum_{i=2}^{6} x_i + \sum_{i=7}^{9} 2x_i \leq 4$ is a facet of $STAB(G)$ and has N_0-rank* 2*. On the contrary, for $G = (C_9, \{1,2,3,4,5,8\}, \{3,4,5,6,7,9\})$, the inequality $\sum_{i\in[9]} x_i + 2x_{10} + 2x_{11} \leq 4$ is a facet of $STAB(G)$ and has N-rank* 3*.*

Now we consider the graphs of the form $G = [C_n, S_1, S_2]$. The following fact follows directly from Lemma 60.

Proposition 70. *Suppose $G = [C_n, S_1, S_2]$. Then the inequality*

$$\sum_{i\in[n]} x_i + \mu(S_1)x_{n+1} + \mu(S_2)x_{n+2} \leq \frac{n-1}{2} \tag{5.10}$$

is a facet of $STAB(G)$.

The N- and N_0-rank of above facet can be 2 or 3. It is not yet known if its N_0-rank always coincides with its N-rank. We summarize in the next proposition some instances in which we know the N- and N_0-rank of this facet.

Proposition 71. *Let $G = [C_n, S_1, S_2]$. Suppose $T \subset [n]$ induces a stable set in C_n and $T \cap (S_1 \cup S_2) = \emptyset$. If either*

1. *$n - 2|T| \geq 5$ and $\mu(S_1) + \mu(S_2) + |T| > \frac{n-1}{2}$, or*
2. *$n - 2|T| \geq 3$ and $\mu(S_1) + \mu(S_2) + 2|T| > n - 2$,*

then the N-rank of (5.10) *is 3.*

Proof. We first prove (1). It is obvious that for any S_1, S_2, $r(G) \leq 3$, so we only have to show that the facet is not valid for $N^2(G)$. Also, by Lemma 62 it suffices to verify the result for $S_1 = S_2 = [n] \setminus T$ for any given T.

Define $x(n,T) \in \mathbb{R}^{n+2}$,

$$x(n,T)_i := \begin{cases} \frac{n-2|T|-1}{2n-4|T|+2} & \text{if } i \in [n] \setminus T; \\ \frac{n-2|T|+3}{2n-4|T|+2} & \text{if } i \in T; \\ \frac{1}{n-2|T|+1} & \text{if } i \in \{n+1, n+2\}. \end{cases}$$

We prove by induction on $|T|$ that $x(n,T) \in N^2(G)$.

When $T = \emptyset$, we consider $Y(n) \in \mathbb{R}^{n+2 \times n+2}$,

$$Y(n)_{ij} := \begin{cases} x(n,\emptyset)_i & \text{if } i = j; \\ \frac{1}{n+1}\left(\frac{n-1}{4} + (-1)^l\left(\frac{n-1}{4} - \lfloor \frac{l}{2} \rfloor\right)\right) & \text{if } i,j \leq n \text{ , } l \leq \frac{n-1}{2} \text{ and } i - j \equiv \pm l \mod n; \\ 0 & \text{otherwise.} \end{cases}$$

To give some intuition to the somewhat complicated formula above, here are the first columns of $Y(n)$ for some small values of n.

n	$Y(n)_1$
5	$\frac{1}{6}(2,0,1,1,0,0,0)^T$
7	$\frac{1}{8}(3,0,2,1,1,2,0,0,0)^T$
9	$\frac{1}{10}(4,0,3,1,2,2,1,3,0,0,0)^T$
11	$\frac{1}{12}(5,0,4,1,3,2,2,3,1,4,0,0,0)^T$

Obviously, $Y(n) = (Y(n))^T$ for any n. It is also clear that $Y(n)_i \in x(n,\emptyset)_i STAB(G)$ for $i \in \{n+1, n+2\}$. Now suppose $i \in [n]$. We see that $Y(n)_i$ is exactly $\frac{1}{n+1}$ times the sum of the incidence vectors of the maximal stable sets in C_n that contains i. For example, $Y(5)_1 = \frac{1}{6}\left(\chi_{\{1,3\}} + \chi_{\{1,4\}}\right)$, $Y(7)_1 = \frac{1}{8}\left(\chi_{\{1,3,5\}} + \chi_{\{1,3,6\}} + \chi_{\{1,4,6\}}\right)$, and so on. Since, for any fixed i, there are $\frac{n-1}{2}$ maximal stable sets in C_n that contain i and $\left(\frac{1}{n+1}\right)\left(\frac{n-1}{2}\right) = \frac{n-1}{2n+2} = x(n,\emptyset)_i$, it follows that $Y(n)_i \in x(n,\emptyset)_i STAB(G)$.

Then we show that $(x(n,\emptyset) - Y(n)_i) \in (1 - x(n,\emptyset)_i)OC(G)$ for every $i \in [n+2]$. The claim is easy to see for $i \in \{n+1, n+2\}$. For $i \in [n]$, we see that all triangle inequalities and C_n inequalities are satisfied because $Y_{pi} + Y_{qi} \geq \frac{n-3}{2n+2}$ for every edge $\{p,q\}$ on C_n. Since those are the only chordless odd cycles in G, our claim follows.

If $\mu(S_1) + \mu(S_2) > \frac{n-1}{2}$, then (5.10) is violated by $x(n,T)$, hence the case when $T = \emptyset$ is justified.

Now we assume $|T| > 0$ and $n - 2|T| \geq 5$. Since T is a stable set in C_n, there exists $t \in T$ such that either $t-2$ or $t+2$ is not in T. We assume without loss of generality that it is $t-2$. We also assume without loss of generality that $t = k$.

Let $T' = T \cap [n-2]$ and consider the graph $G := [C_{n-2}, T', T']$. Since T is a stable set in G, we know that $n-1 \notin T$, so $|T'| = |T| - 1$. Also, since $(n-2) - 2|T'| = n - 2|T| \geq 5$, we know by inductive hypothesis that $x(n-2, T') \in N^2(G')$.

Now consider $G'' = [C_n, T', T']$. It can be seen as the graph G' with the edge $\{n-2, 1\}$ subdivided into the odd path $(n-2)$-$(n-1)$-n-1 (and the two nodes not on the cycle are re-labelled from $n-1, n$ in G' to $n+1, n+2$ in G''). We can derive from $x(n-2, T')$ a point in $N^2(G'')$ by the construction given in the proof of Theorem 16 in [16]. Moreover, if we use $v = 1$ and $w = n$, then the derived point we get is exactly $x(n, T)$.

Observe that the only difference between G and G'' is the presence of the edges $\{n+1, n-1\}$ and $\{n+2, n-1\}$, and the only chordless odd cycle containing these edges in G are the triangles $(n-2)$-$(n-1)$-$(n+1)$-$(n-2)$ and $(n-2)$-$(n-1)$-$(n+2)$-$(n-2)$. Therefore, we know that

$$OC(G) = OC(G'') \cap \left\{x \in \mathbb{R}^{n+2} : x_{n-2} + x_{n-1} + x_{n+1} \leq 1, x_{n-2} + x_{n-1} + x_{n+2} \leq 1\right\}.$$

Let Y', Y be the matrices that prove $x(n-2, T') \in N^2(G')$ and $x(n, T) \in N^2(G'')$ respectively. We see that $Y_{i,n-2} + Y_{i,n-1} + Y_{i,n+1} = Y'_{i,n-2} + Y'_{i,1} + Y'_{i,n-1} \leq x(n-2, T')_i = x(n, T)_i$ for every $i \in [n-1]$. By symmetry, $Y_{i,n-2} + Y_{i,n-1} + Y_{i,n+2} \leq x(n, T)_i$ and it follows that $Y_i \in OC(G)$. It can be checked similarly that $Y_i \in x(n, T)_i OC(G)$ and $x(n, T) - Y_i \in (1 - x(n, T)_i) OC(G)$ for all $i \in [n+2]$. The key facts required are the symmetries between columns Y_{n+1} and Y_1, symmetries between columns Y_{n+2} and $x(n, T) - Y_1$, and the presence of the 3-cycles $(n-2)$-$(n+1)$-1-$(n-2)$ and $(n-2)$-$(n+2)$-1-$(n-2)$ in G'.

Now we substitute $x(n, T)$ into (5.10), and see that if $\mu(S_1) + \mu(S_2) + |T| > \frac{n-1}{2}$, then $x(n, T)$ violates (5.10).

For (2), we see that if we have $n - 2|T| \geq 3$, the point $\frac{1}{3}\bar{e} + \frac{1}{3}\sum_{i \in T} e_i$ is in $N(G)$. By Proposition 53, the point $\frac{1}{4}\bar{e} + \frac{1}{2}\sum_{i \in T} e_i$ is in $N^2(G)$, and if $\mu(S_1) = \mu(S_2)$ and $2|T| + \mu(S_1) + \mu(S_2) > n - 2$, then this point violates (5.10). $\square$

When $T = \emptyset$, we can generalize (1) above to graphs with more than 2 nodes on top of C_n. First we have the following lemma:

Lemma 72. *Suppose G is a graph on $n+k$ nodes such that n is odd and $G_{[n]}$ is a cycle. Define $x(n,l) \in \mathbb{R}^{n+k}$ such that*

$$x(n,l)_i := \begin{cases} \frac{n-1}{2n+2l-2} & \text{if } i \in [n]; \\ \frac{1}{n+l-1} & \text{if } i \in \{n+1, n+2, \ldots, n+k\}. \end{cases}$$

Then $x(n,l) \in N^l(G)$.

Proof. We fix n, k and prove our claim by induction on l. First, we define $Y(n,l) \in \mathbb{R}^{(n+k)\times(n+k)}$ such that

$$Y(n,l)_{ij} = \begin{cases} x(n,l)_i & \text{if } i = j; \\ \frac{1}{n+l-1}\left(\frac{n-1}{4} + (-1)^l\left(\frac{n-1}{4} - \lfloor \frac{l}{2} \rfloor\right)\right) & \text{if } i,j \leq n \text{ , } l \leq \frac{n-1}{2} \text{ and } i-j \equiv \pm l \bmod n; \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 62, we only have to prove our claim for the graph $G = [C_n, S_1, \ldots, S_k]$ with $S_1, \ldots, S_k = [n]$.

First, we see that $Y(n,l)_i \in x(n,l)_i STAB(G)$ is equivalent to $Y(n) \in x(n,\emptyset)_i STAB(G)$ in the proof of Lemma 71, hence is true for all n, k and l. Also, observe that $x(n,l) - Y(n,l)_1$ can be written as $\frac{l+1}{n+l-1}(0, \frac{1}{l+1}, \frac{1}{l+1}, \ldots, \frac{1}{l+1})^T$ plus $\frac{1}{n+l-1}$ times the sum of the incidence vectors of the maximal stable sets in C_n that contain nodes 2 and n. Since these stable sets have a one-to-one correspondence with the maximal stable sets in C_{n-2} that contain 2 (namely, S is a maximal stable set in C_{n-2} that contains 2 if and only if $S \cup \{n\}$ is a maximal stable set in C_n that contains both 2 and n), there are $\frac{n-3}{2}$ of those stable sets in C_n. Now we see that $x(n,l) - Y(n,l)_1 \in (1 - x(n,l)_1)N^l(G)$ because it can be written as a convex combination of points in $N^{l-1}(G)$ and $STAB(G)$. By symmetry, it follows that $x(n,l) - Y(n,l)_i \in (1 - x(n,l)_i)N^{l-1}(G)$ for every $i \in [n]$.

Now we show that $x(n,l) - Y(n,l)_i \in (1 - x(n,l)_i)N^{l-1}(G)$ when $i \in \{n+1, \ldots, n+k\}$, and this is the only part of the proof in which we use our inductive hypothesis. First of all, it is clear that $x(n,0) - Y(n,0)_i \leq \frac{n-1}{2n+2}\bar{e} = (1 - \frac{1}{n+1})\frac{1}{2}\bar{e}$, hence is in $(1 - x(n,0)_i)FRAC(G)$. Now for $l \geq 1$, we see that $x(n,l) - Y(n,l)_i \leq \frac{n+l-2}{n+l-1}x(n,l-1)$. Therefore, $x(n,l) - Y(n,l)_i \in (1 - \frac{1}{n+l-1}N^{l-1}(G)$ by inductive hypothesis, and we are finished. $\square$

Then we have the following:

Corollary 73. *Suppose $G = [C_n, S_1, \ldots, S_k]$. Then*

$$r(G) \geq \left\lceil \frac{2\sum_{i=1}^{k} \mu(S_i)}{n-1} \right\rceil + 1$$

Moreover, if $\sum_{i=1}^{k} \mu(S_i) > \frac{(k-1)(n-1)}{2}$, then $r(G) = r_0(G) = k+1$.

Proof. By Lemma 60, we know that $\sum_{i=1}^{n} x_i + \sum_{i=1}^{k} \mu(S_i) x_{n+i} \leq \frac{n-1}{2}$ is a facet of $STAB(G)$. Since we know from Lemma 72 that $x(n,l) \in N^l(G)$, we have

$$r(G) \geq \min \left\{ l : \sum_{i=1}^{n} \frac{n-1}{2n+2l-2} + \sum_{i=1}^{k} \frac{\mu(S_i)}{n+l-1} \leq \frac{n-1}{2} \right\}.$$

After some algebraic manipulations, we see that the expression on the right side of the inequality amounts to $\left\lceil \frac{\sum_{i=1}^{k} 2\mu(S_i)}{n-1} \right\rceil + 1$, and our first claim follows.

For our second claim, if $\sum_{i=1}^{k} \mu(S_i) > \frac{(k-1)(n-1)}{2}$, then $r(G) \geq k+1$ from above. It is obvious that $r_0(G) \leq k+1$ (since G is a cycle plus k nodes), hence $r(G) = r_0(G) = k+1$. $\square$

Finally, since the proof of our 8-node result requires computerized assistance, we want first to establish methods that minimize over-generating isomorphic graphs when we check them one by one. For example, when we check the ranks of the graphs that are of the form (C_5, S_1, S_2), it is clear that we do not have to check both $(C_5, \{1,2,3\}, \{2,3,4\})$ and $(C_5, \{2,3,4\}, \{3,4,5\})$, as they are isomorphic to each other.

We now give the method we use to eliminate redundant pairs and keep the number of graphs to check to a minimum. This method works especially well when G is an odd-hole or odd-antihole plus two nodes.

Given S a subset of $[k]$ and $p \in [k]$, we define the shift function $s_p(S)$ to be the set $\{p + i \mod k : i \in S\}$. We also define the flip function f so that $f_0(S) = S$ and $f_1(S) = \{k+1-i : i \in S\}$. Furthermore, we call a set S *symmetric in* $[k]$ if there exists $i \in [k]$ such that $s_i(f_1(S)) = S$. For example, all subsets of [5] are symmetric in [5], and $\{1,2,4\}$ is not symmetric in [7].

For any odd number n, we call $\mathcal{T} = \{T_1, \ldots, T_d\}$ a *minimal collection of* $[n]$ if all of the following are satisfied.

1. $\emptyset \subset T_i \subseteq [n],\ \forall i \in [d]$;
2. $\forall S \subseteq [n], S \neq \emptyset,\ \exists i \in [d], p \in [k], q \in \{0,1\}$ such that $s_p(f_q(T_i)) = S$;
3. $\forall i,j \in [d] i \neq j,\quad \nexists p \in [k], q \in \{0,1\}$ such that $s_p(f_q(T_i)) = T_j$;
4. if T_i is symmetric in $[n]$, then $f_1(T_i) = T_i$;
5. $T_d = [n]$.

The first 3 rules require that every subset $S \subseteq [n]$ has exactly one corresponding $T_i \in \mathcal{T}$ such that T_i can be obtained from S by flipping and shifting operations. The last two rules are more for convenience purposes.

Then we have the following:

Proposition 74. *Given $G = (H, S_1, S_2)$, where H is either an n-hole or an n-antihole and $S_1, S_2 \neq \emptyset$. If $\mathcal{T} = \{T_1, \ldots, T_d\}$ is a minimal collection of $[n]$, Then G is isomorphic to one of the graphs in the following set:*

$$
\begin{aligned}
&\{(H, T_u, [n]) : u \in [d]\} \cup \\
&\{(H, T_u, s_i(f_j(T_v))) : S_u, S_v \in \mathcal{T}, u \leq v < d) \\
&i \in \left\{0, 1, \ldots, \frac{n-1}{2}\right\}, j \in \{0\} \textit{ if } T_u, T_v \textit{ both symmetric}; \\
&i \in \{0, 1, \ldots, n-1\}, j \in \{0\} \textit{ if exactly one of } T_u, T_v \textit{ is symmetric}; \\
&i \in \left\{0, 1, \ldots \frac{n-1}{2}\right\}, j \in \{0\},\ \textit{ or} \\
&i \in \{0, \ldots, n-1\}, j \in \{1\} \textit{ if } T_u \textit{ is not symmetric and } u = v; \\
&i \in \{0, 1, \ldots, n-1\}, j \in \{0,1\} \textit{ if } u \neq v \textit{ and neither } T_u, T_v \textit{ are symmetric}\}.
\end{aligned}
$$

Proof. First we find i, j, i', j' such that there are elements $s_i(f_j(S_1)) = T_u, s_{i'}(f_{j'}(S_2)) = T_v$ for some elements $T_u, T_v \in \mathcal{T}$. Assume without loss of generality that $u \leq v$. If $T_v = [n]$, then G is isomorphic to a graph in $\{(H, T_u, [n]) : u \in [d]\}$. Otherwise, we know G is isomorphic to a graph $(H, T_u, s_p(f_q(T_v)))$ for some p, q and such that $p \leq q < d$.

First assume that both T_u, T_v are symmetric. Then we may assume that $q = 0$ because $f_1(T_v) = T_v$. If $p > \frac{n-1}{2}$, then we see that

$$(H, T_u, s_p(T_v)) \cong (H, f_1(T_u), f_1(s_p(T_v))) = (H, T_u, s_{n-p}(f_1(T_v))) = (H, T_u, s_{n-p}(T_v)).$$

By the assumption on p, $n-p \leq \frac{n-1}{2}$. The case when exactly one of T_u, T_v is symmetric is similar.

For the last case when neither T_u, T_v is symmetric, $u = v$ and $q = 0$, we see that $(H, T_u, s_p(f_q(T_v))) \cong (H, s_{n-p}(T_u), T_v) \cong (H, T_u, s_{n-p}(T_v))$ (since $u = v$). If $p > \frac{n-1}{2}, n - p \leq \frac{n-1}{2}$. ☐

Obviously, the above result extends to the case when $(n+1) \sim (n+2)$.

5.2 Specialization to the 8-node case

In this section, we verify (somewhat exhaustively) that the Rank Conjecture holds for all 8-node graphs.

Proposition 75. *Suppose G is a graph with no more than 8 nodes. Then $r_0(G) = r(G)$.*

Proof. Again, we may assume that G is imperfect. We know that the Rank Conjecture holds for the cases when G is an odd-hole or an odd-antihole, and also when $G = (C_5, S)$ or (C_7, S) (by Proposition 58). When $G = (\bar{C}_7, S)$, we know from Proposition 63 that $r_0(G) = r(G) = 3$ if $\mu(S) > 0$. For the case when $\mu(S) = 0$, we have the following:

Claim 76. *Suppose $G = (\bar{C}_7, S)$ and $\mu(S) = 0$. Then $r_0(G) = 2$.*

Proof. Since $\mu(S) = 0$, we may assume without loss of generality that $1, 2 \notin S$. By Proposition 49, if there does not exist $i \in [7]$ such that $i-1, i+1 \in S$ and $i \notin S$, then G only has 7 maximal stable sets and $r_0(G) = 2$.

If such i exists, then S has to be one of $\{3,5,7\}, \{3,5,6,7\}, \{3,4,5,7\}$ and $\{3,4,6,7\}$. In the first 3 cases, we delete any node other than 8 from G. In the last case when $S = \{3,4,6,7\}$, we delete node 5 from G.

In any case, we see that after the removal of the selected node, the remaining graph is perfect and does not contain a K_4. This is easier checked by looking at the equivalent condition, that its complement is perfect and does not contain a stable set of size 4.

Therefore our claim follows. ☐

Now all it remains is to show that the Rank Conjecture holds for graphs that has a core of a 5-hole plus 2 or 3 nodes, and these cases will be settled in the next several claims.

Claim 77. *If $G = (C_5, S_1, S_2)$ for some $S_1, S_2 \subseteq [5]$. Then $r_0(G) \leq 2$.*

Proof. First we let $\mathcal{T}$ be the following minimal collection:

i	T_i	$\mu(T_i)$
1	$\{3\}$	0
2	$\{1,5\}$	0
3	$\{2,4\}$	0
4	$\{1,3,5\}$	0
5	$\{2,3,4\}$	1
6	$\{1,2,4,5\}$	1
7	$\{1,2,3,4,5\}$	2

Notice that all of the above subsets are symmetric in [5].

Then we wrote a program in java, compiled it using Java 2 JDK Standard Edition version 1.3.0_02, and generated the input files for Qhull (version 2003.1, can be found on http://www.qhull.org/). Each of the input files refers to one graph contained in the set in the statement of Proposition 74, and contains the number of nodes, the number of stable sets, and all incidence vectors of stable sets in the graph. The input files are then processed by Qhull to produce text files that contain the facets of the stable set polytope of the corresponding graphs. Finally, we wrote another java program that takes in all the output files created by Qhull, and returns one text file that lists the graphs and the facets they had that are full (we call a facet $a^T x \leq \alpha$ *full* if $a_i \neq 0\ \forall i \in V(G)$). All of the programming, compiling and processing mentioned above are performed on a regular household computer (Pentium 4 2.66GHz, 512MB RAM, Windows XP Professional with Service Pack 2).

We are only interested in full facets because otherwise, the facet corresponds to a proper subgraph of (C_5, S_1, S_2), which we already know has N_0-rank at most 2.

With that, we have found that all the full facets found are double partial wheel inequalities, which we know have N_0-rank 2. ☐

Claim 78. *Suppose $G = [C_5, S_1, S_2]$ for some $S_1, S_2 \subseteq [5]$. Then $r_0(G) = r(G)$.*

Proof. We can use the same minimal collection $\mathcal{T}$ given in the previous claim. Also, we may assume that $\mu(S_1), \mu(S_2) \geq 1$. Otherwise, there exists a node in G whose deletion results in a rank-1 graph.

If $\lambda(S_1, S_2) = \mu(S_1) + \mu(S_2)$, then (5.10) is really a double partial wheel inequality, which has N- and N_0-rank 2. Also, if either of the conditions in the statement of Proposition 71 is satisfied, then $r_0(G) = r(G)$. We see that the above observations take care of all 12 non-isomorphic cases, and hence our claim follows. □

Claim 79. *Suppose $G = (C_5, S_1, S_2, S_3) + S'$, where $S_1, S_2, S_3 \subseteq [5], S' \subseteq \{67, 68, 78\}$. Then $r_0(G) = r(G)$.*

Proof. Let $\mathcal{T}$ be the following collection:

i	T_i	$\mu(T_i)$
1	$\emptyset$	0
2	$\{3\}$	0
3	$\{1,5\}$	0
4	$\{2,4\}$	0
5	$\{1,3,5\}$	0
6	$\{2,3,4\}$	1
7	$\{1,2,4,5\}$	1
8	$\{1,2,3,4,5\}$	2

Notice that every set in $\mathcal{T}$ is symmetric at 3. Therefore, given any graph G on 8 nodes, 5 of which induce a 5-hole, we can find $p, q, r \in [8], i, j \in [5], S' \subseteq \{67, 68, 78\}$ such that G is isomorphic to $(C_5, T_p, s_i(T_q), s_j(T_r)) + S'$.

Using the same simple tricks used in the proof of Proposition 74, it is not hard to see that we may further assume that

1. $p \leq q \leq r$;
2. $i \leq 2$
3. If $T_p = \emptyset$, then $i = 0, j \leq 2$;

4. If $T_q = [5]$, then $i = j = 0$;

5. if $T_r = [5]$, then $j = 0$;

6. if $T_q = T_r$, then $i \leq j$;

7. if $i = 0$, then $j < 2$.

Also, in finding a counterexample to the Rank Conjecture, we may assume that $(G-6), (G-7)$ and $(G-8)$ all have N_0-rank 2 or all have N_0-rank 3, or otherwise we know that $r_0(G) = r(G)$.

Let $G = (C_5, S_1, S_2)$ or $[C_5, S_1, S_2]$ for some $S_1, S_2 \subseteq [5]$. We have seen in the last two claims the N- and N_0-rank of G when $\mu(S_1), \mu(S_2) > 0$. If exactly one of $\mu(S_1), \mu(S_2)$ is zero, then we know $r_0(G) = r(G) = 2$. For the case when $\mu(S_1) = \mu(S_2) = 0$, we have found that $r_0(G) = 2$ if and only of G contains a K_4, a star-subdivision of K_4, or a partial wheel as a subgraph.

Finally, since we already know that all graphs with 7 nodes of less hold for the Rank Conjecture, we are again only interested in graphs whose stable set polytope has a full facet. We found that no graphs in our consideration have more than one full facet.

Given a graph G such that $r_0(G-6) = r_0(G-7) = r_0(G-8) = k, k \in \{2,3\}$ and $STAB(G)$ has a full facet, we either show that there is a node whose deletion and destruction from the full facet both result in an inequality of N_0-rank k to show that $r_0(G) = r(G) = k$, or give a vector $x \in N^k(G) \setminus STAB(G)$ to show that $r_0(G) = r(G) = k+1$.

The complete list of the graphs we checked can be found in the Appendix. $\square$

This completes the proof. $\square$

It should be noted that while verifying the Rank Conjecture for the 8-node graphs, we discovered that the graph $((C_5, \{2,3,4\}, \{1,2,5\}, \{1,2,3,4\}) + \{67\})$ that is planar and has N- and N_0-rank 3. This defies the pattern suggested by many known results that K_n is the critical structure of a graph that has N- and N_0-rank $n-2$.

5.3 Verifying the Rank Conjecture for some 9-node graphs

Here we take a step further and verify the Rank Conjecture for some 9-node graphs.

Proposition 80. *Suppose $G = (C_7, S_1, S_2)$ for some $S_1, S_2 \subseteq [7]$. Then $r_0(G) = r(G) = 2$.*

Proof. The minimal collection we used is

i	T_i	$\mu(T_i)$	Symmetric
1	$\{3,4,5\}$	1	Yes
2	$\{1,4,7\}$	1	Yes
3	$\{1,2,6,7\}$	1	Yes
4	$\{1,2,3,5\}$	1	No
5	$\{2,3,5,6\}$	1	Yes
6	$\{1,2,4,6,7\}$	1	Yes
7	$\{1,3,4,5,7\}$	1	Yes
8	$\{2,3,4,5,6\}$	2	Yes
9	$\{1,2,3,5,6,7\}$	2	Yes
10	$\{1,2,3,4,5,6,7\}$	3	Yes

Notice that we have omitted the T_i's that give $\mu(T_i) = 0$.

We checked all 221 non-isomorphic cases, and found that all the full facets of such graphs are either double partial wheel inequalities or P-augmented odd cycle inequalities.

We already know that double partial wheel inequalities have N- and N_0-rank 2. Now we show that, if $G = (C_7, S_1, S_2)$ and $STAB(G)$ has a P-augmented odd cycle facet, then it has to have N_0-rank 2 as well. Suppose $\mu(S_1) + \mu(S_2) - \mu(S_1 \cup S_2) \geq 0$ and there exist $p, q \in [7]$ that satisfy the hypothesis in the statement of Proposition 66. We also assume that $\lambda(S_1, S_2) < \mu(S_1) + \mu(S_2)$, for otherwise the facet can be viewed as a double partial wheel inequality.

Also, we assume without loss of generality that $q = 1$, and $\mu(S_1) \leq \mu(S_2)$. Since $q - p$ mod 7 is even, we know $p \in 3, 5, 7$. $\mu(S_2) \geq 1$ rules out $p = 3$, and $p = 5$ implies that $S_1 \subseteq \{2,3,4,5\}$ and $S_2 \subseteq \{1,2,3,4\}$, which in turn implies that $\lambda(S_1, S_2) = 2 = \mu(S_1) + \mu(S_2)$. Therefore we may assume that $p = 7$.

Suppose $\mu(S_1) = \mu(S_2) = \lambda(S_1, S_2) = 1$. That implies that $S_1 = \{3,4,7\}$ or $\{3,4,5,7\}$ and $S_2 = \{1,4,5\}$ or $\{1,3,4,5\}$. In each of the 4 cases, $(G - \{4,7\})$ is bipartite. Hence, we know that the facet (and the graph) has N_0-rank 2.

If $\mu(S_1) = 1$ and $\mu(S_2) = 2$, then we know that $S_2 = \{1,2,3,4,5\}$ or $\{1,2,3,4,5,6\}$. If $6 \in S_1$, then $\lambda(S_1, S_2) = 3$. Therefore, we may assume that $S_1 \in \{\{3,4,7\}, \{3,4,5,7\}, \{2,3,4,7\}, \{2,3,4,5,7\}\}$. In all 8 cases, $\lambda(S_1, S_2) = 2$, and we see that removing node 7 from the graph results in a perfect graph that does not contain a K_4. Therefore $r_0(G) \leq 2$.

If $\mu(S_1) = \mu(S_2) = 2$, then $S_1 \in \{\{3,4,5,6,7\}, \{2,3,4,5,6,7\}\}$ and $S_2 \in \{\{1,2,3,4,5\}, \{1,2,3,4,5,6\}\}$. In each of the 4 cases, deleting 7 from the graph yields a perfect graph that does not contain a K_4, and again $r_0(G) \leq 2$.

Since we know that $\mu(S_1) + \mu(S_2) - \mu(S_1 \cup S_2) \geq 0$, $\mu(S_1) \geq 1$ and $\mu(S_1 \cup S_2) \leq 3$, we need not consider any S_2 such that $\mu(S_2) = 3$, so we are finished. ☐

Proposition 81. *Suppose $G = [C_7, S_1, S_2]$ for some $S_1, S_2 \subseteq [7]$, then $r_0(G) = r(G)$.*

Proof. As in the proof of Claim 78, we only have to consider the graphs for which $\lambda(S_1, S_2) < \mu(S_1) + \mu(S_2)$, and that neither of the conditions in the statement of Proposition 71 is satisfied. By computerized checking again, we found that there are only 3 such graphs: $(S_1, S_2) = (\{1,4,7\}, \{1,2,3,5,6,7\})$, $(\{1,2,4,6,7\}, \{1,2,3,5,6,7\})$ or $(\{1,2,4,7\}, \{1,2,3,5,6,7\})$. In all 3 cases, deleting and destroying the node 4 both yield an inequality of N_0-rank 1 (contraction inequality being the sum of the triangle constraints for 2-3-9-2 and 5-6-9-5, deletion inequality being that plus the triangle inequality for 1-7-8-1). Therefore all 3 graphs have N_0-rank 2. ☐

We have also exhaustively verified that all graphs that are a 7-antihole plus two nodes satisfy $r_0(G) = r(G)$. The complete list of graphs can be found in the Appendix.

We see that verifying the Rank Conjecture gets difficult very quickly when the number of nodes in the graph goes up from 7 to 8 or 9. The only 9-node case that is not verified here is the 5-hole plus 4 nodes case, which contains more than 10^5 non-isomorphic graphs, and would be very time consuming to check exhaustively.

Chapter 6

On possible counterexamples to the Rank Conjecture

After showing that the Rank Conjecture holds for all graphs with no more than 8 nodes, we conclude our thesis by considering the properties of graphs that would potentially disprove the Rank Conjecture.

First of all, if a counterexample to the Rank Conjecture does exist, we know from results in Section 3.1 and Chapter 5 that the graph has to be imperfect, and it must have more than 8 nodes. Also, we may assume that the graph does not satisfy any of the decomposition criteria mentioned in Section 3.5. Moreover, we may assume that our counterexample is very "critical" in N_0-rank and "loose" in N-rank, as more formally stated in the next proposition.

Proposition 82. *If the Rank Conjecture is false, then there exist an integer k_0 and a graph G such that*

1. $r_0(H) \leq k_0 \Rightarrow r(H) = r_0(H) \quad \forall$ *graphs* H;
2. $r_0(G) = k_0 + 1, r(G) = k_0$;
3. $r_0(G - i) = r(G - i) = k_0 \quad \forall i \in V(G)$.

Proof. First, we choose G so that it is a counterexample to the Rank Conjecture of the lowest N-rank. Moreover, we choose G such that it has the fewest number of nodes among such graphs. Now if we let $k_0 := r(G)$, then condition (1) is satisfied. Also, by the choice of G, we know that $(G-i)$ is not a counterexample to the Rank Conjecture for any $i \in V(G)$. Therefore, we know that $r_0(G) > r(G) \geq r(G-i) = r_0(G-i)$. Combining this with the fact that $r_0(G) \leq r_0(G-i)+1$, we see that G satisfies both conditions (2) and (3). $\square$

Given a graph G, we want to consider a "certificate" (a set of necessary and sufficient conditions) for G to be a counterexample to the Rank Conjecture. A simple certificate is as follows:

Proposition 83. *A graph G is a counterexample to the Rank Conjecture if and only if there exist a vector x, an integer k and a facet of $STAB(G)$ $a^T y \leq b$ such that*

1. $x \in N_0^k(G)$,
2. $a^T x > b$, *and*
3. $N^k(G) = STAB(G)$.

We will show that, with the assumption that the Rank Conjecture holds for all proper induced subgraphs of G, Proposition 83 can be slightly improved. First we need two lemmas. Recall that given a graph G, $i \in V(G)$ and a vector $x \in \mathbb{R}^{V(G)}$, we let $\Phi_i(x)$ and $\Psi_i(x)$ denote the vectors that are x restricted to the subgraphs $(G-i)$ and $(G \ominus i)$ respectively. Then we have the following:

Lemma 84. *Given a graph G and $z \in [0,1]^{V(G)}$, if $z_i = 1$, and $z_j = 0$ $\forall j \in \mathcal{N}(i)$, then $z \in N_0^k(G) \iff \Psi_i(z) \in N_0^k(G \ominus i)$, for every $k \geq 0$. Same for N.*

Proof. "$\Rightarrow$" is true in general, without the assumption on z_i's. We now prove "$\Leftarrow$" for N_0 by induction on k.

When $k=0$, $N_0^k(G) = FRAC(G)$. Suppose $z_i = 1$, $z_j = 0$ $\forall j \in \mathcal{N}(i)$, and $\Psi_i(z) \in FRAC(G \ominus i)$. Then first of all, z satisfies all edge constraints in $FRAC(G)$ that does not involve i or its neighbours. Also, since $z_j = 0$ $\forall j \in \mathcal{N}(i)$ and $z_j \leq 1$ $\forall j \in V(G)$, all new edge constraints will be satisfied (because each new edge constraint involves at least one $j \in \mathcal{N}(i)$).

For the inductive step, we assume that $z_i = 1, z_j = 0$ $\forall j \in \mathcal{N}(i), \Psi_i(z) \in N_0^{k-1}(G \ominus i) \Rightarrow z \in N_0^{k-1}(G)$.

Now suppose we are given z such that $z_i = 1$. We order the coordinates of z so that all nodes in $(G \ominus i)$ come first, followed by nodes in $\mathcal{N}(i)$, with node i being the last coordinate. So we know $z = \begin{pmatrix} \Psi_i(z) \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$.

Assume that $z \in \Psi_i(z) \in N_0^k(G \ominus i)$. This implies that there exists Y such that

$$Y' := \begin{pmatrix} 1 & \Psi_i(z)^T \\ \Psi_i(z) & Y \end{pmatrix} \in M_0^k(G \ominus i).$$

We consider the following matrix Y'', where

$$Y'' := \begin{pmatrix} 1 & \Psi_i(z)^T & 0 & 1 \\ \Psi_i(z) & Y & 0 & \Psi_i(z) \\ 0 & 0 & 0 & 0 \\ 1 & \Psi_i(z)^T & 0 & 1 \end{pmatrix}.$$

Each column from 1 to $n-1$ is in $N_0^{k-1}(G)$ by inductive hypothesis. Column n is exactly z. Since $\Psi_i(z) \in N_0^k(G \ominus i) \subseteq N_0^{k-1}(G \ominus i)$, we can apply the inductive hypothesis again and claim that z is in $N_0^{k-1}(G)$.

Now we look at the difference of the columns with the z. For the first $n - |\mathcal{N}(i)| - 1$ columns, their differences with z are in $N_0^{k-1}(G)$ follows from the fact that $Y' \in N_0^{k-1}(G \ominus i)$. The subsequent columns are either 0 or z, and we know that both $z - 0$ and $z - z$ is in $N_0^{k-1}(G)$.

Therefore, $Y'' \in M_0^k(G)$, and that $z \in N_0^k(G)$.

Since Y'' is symmetric as long as Y' is, the argument above applies for N^k as well. $\square$

We include the assumption that $z_j = 0 \; \forall j \in \mathcal{N}(i)$ because otherwise z would definitely not be in $FRAC(G)$, so the discussion about whether it is in $N^k(G)$ and such would be meaningless.

Not surprisingly, we have an analogous result for the case when $z_i = 0$.

Lemma 85. *Given a graph G and $z \in [0,1]^{V(G)}$, if $z_i = 0$, then $z \in N_0^k \iff \Phi_i(z) \in N_0^k(G-i)$, for every $k \geq 0$. Same for N.*

Proof. The result follows directly from the fact that

$$N^k(\{x \in FRAC(G) : x_i = 0\}) = N^k(G) \cap \{x : x_i = 0\}\,. \qquad \square$$

Then Proposition 83 can be evolved into the following:

Proposition 86. *Suppose we have a graph G such that $r_0(G_S) = r(G_S)$, $\forall S \subset V(G)$. Then G is a counterexample to the Rank Conjecture if and only if there exist a vector x, an integer k and a facet of $STAB(G)$ $a^T y \leq b$ such that*

1. *$x \in N_0^k(G) \cap (0,1)^{V(G)}$,*
2. *$a \in \mathbb{Z}_{++}^{V(G)}, b \in \mathbb{Z}_{++}, a^T x > b$, and*
3. *$N^k(G) = STAB(G)$.*

Proof. It is clear that the above conditions are sufficient. Therefore it suffices to show that they are necessary.

Given G a counterexample to the Rank Conjecture of N-rank k, if we have an incidence vector $x \in N_0^k(G) \setminus STAB(G)$ and $x_i = 1$ for some i, then we know from Lemma 84 that $\Psi_i(x) \in N_0^k(G \ominus i) \setminus STAB(G \ominus i)$. Hence, $(G \ominus i)$ is also a counterexample to the Rank Conjecture (since $r(G \ominus i) \leq r(G) < r_0(G) = r_0(G \ominus i)$), which is a contradiction. Similarly, If some $x_i = 0$ for some i, then Lemma 85 implies that $(G - i)$ is also a counterexample to the Rank Conjecture. Therefore, we may assume that $0 < x_i < 1$, $\forall i \in V(G)$.

Also, we may assume that a, b are integral because all extreme points of $STAB(G)$ are incidence vectors of stable sets of G, which are integral. We can assume that $a > 0$ because, if any of the a_i's is 0, then the facet $a^T y \leq b$ corresponds to a proper induced subgraph of G, which contradicts our assumption that the Rank Conjecture holds for all proper induced subgraphs of G. Combining with the fact that $STAB(G)$ is lower-comprehensive, we can assume that $a_i > 0$, $\forall i \in V(G)$ (and hence $b > 0$). $\square$

As it now stands, more must be done before we can settle the Rank Conjecture either way. One of the possible research directions we can take from here is to look into $N_0^2(G)$

and $N^2(G)$ more closely, and find out precisely which inequalities are valid for one but not the other. Another approach is to construct and study counterexamples to the N-N_0 Conjecture, and examine the gaps between the polytopes $N^k(G)$ and $N_0^k(G)$ for different values of k. Understanding the behaviour of the gaps between the polytopes can potentially help us construct a graph with a large enough gap between $N_0^k(G)$ and $N^k(G)$ that it takes N_0 more steps than N to reach $STAB(G)$.

Appendix A

Verifying the ranks of graphs

Here we show the complete lists of graphs we checked and the detailed methods of how we verified their ranks.

A.1 The graphs $(C_5, S_1, S_2, S_3) + S'$

Here we show the 8-node graphs that we verified the ranks for. Again, we only have to check graphs that satisfy both of the following properties:

- $r_0(G-6) = r_0(G-7) = r_0(G-8) = 2$ or 3;
- $STAB(G)$ has a full facet.

Here is the list of graphs such that $r_0(G-6) = r_0(G-7) = r_0(G-8) = 2$ whose full facet is of N_0-rank 2 (hence, $r_0(G) = r(G) = 2$). Under the "Node" column, we give the node whose deletion and destruction from the facet both yield an inequality of N_0-rank 1.

S_1	S_2	S_3	S'	The full facet	Node
$\{3\}$	$\{1,5\}$	$\{2,3,4\}$	$\{67\}$	$(11211111)^T x \leq 3$	3
$\{3\}$	$\{1,5\}$	$\{1,2,3\}$	$\{67\}$	$(11211111)^T x \leq 3$	3
$\{3\}$	$\{1,5\}$	$\{2,3,4\}$	$\{67,68\}$	$(11211112)^T x \leq 3$	3

S_1	S_2	S_3	S'	The full facet	Node
$\{3\}$	$\{1,5\}$	$\{1,2,3\}$	$\{67,68\}$	$(11211111)^T x \leq 3$	3
$\{3\}$	$\{1,5\}$	$\{2,3,4\}$	$\{67,78\}$	$(11211111)^T x \leq 3$	3
$\{3\}$	$\{1,5\}$	$\{1,2,3\}$	$\{67,78\}$	$(11211111)^T x \leq 3$	3
$\{3\}$	$\{1,5\}$	$\{2,3,4\}$	$\{67,68,78\}$	$(11211112)^T x \leq 3$	3
$\{3\}$	$\{1,5\}$	$\{1,4,5\}$	$\{67,68,78\}$	$(11211111)^T x \leq 3$	3
$\{3\}$	$\{1,5\}$	$\{3,4,5\}$	$\{67,68,78\}$	$(11211112)^T x \leq 3$	3
$\{3\}$	$\{1,5\}$	$\{1,3,4,5\}$	$\{67\}$	$(11211111)^T x \leq 3$	3
$\{3\}$	$\{1,5\}$	$\{2,3,4,5\}$	$\{67\}$	$(11211111)^T x \leq 3$	3
$\{3\}$	$\{1,5\}$	$\{1,3,4,5\}$	$\{67,68\}$	$(11211111)^T x \leq 3$	3
$\{3\}$	$\{1,5\}$	$\{2,3,4,5\}$	$\{67,68\}$	$(11211112)^T x \leq 3$	3
$\{3\}$	$\{1,5\}$	$\{1,2,4,5\}$	$\{67,78\}$	$(11211111)^T x \leq 3$	3
$\{3\}$	$\{1,5\}$	$\{1,3,4,5\}$	$\{67,78\}$	$(11211111)^T x \leq 3$	3
$\{3\}$	$\{1,5\}$	$\{2,3,4,5\}$	$\{67,78\}$	$(11211111)^T x \leq 3$	3
$\{3\}$	$\{1,5\}$	$\{1,2,4,5\}$	$\{67,68,78\}$	$(11211111)^T x \leq 3$	3
$\{3\}$	$\{1,5\}$	$\{1,3,4,5\}$	$\{67,68,78\}$	$(11211112)^T x \leq 3$	3
$\{3\}$	$\{1,5\}$	$\{2,3,4,5\}$	$\{67,68,78\}$	$(11211112)^T x \leq 3$	3
$\{3\}$	$\{1,5\}$	$\{1,2,3,4,5\}$	$\{67\}$	$(11211112)^T x \leq 3$	3
$\{3\}$	$\{1,5\}$	$\{1,2,3,4,5\}$	$\{67,68\}$	$(11211112)^T x \leq 3$	3
$\{3\}$	$\{1,5\}$	$\{1,2,3,4,5\}$	$\{67,78\}$	$(11211112)^T x \leq 3$	3
$\{3\}$	$\{1,5\}$	$\{1,2,3,4,5\}$	$\{67,68,78\}$	$(11211113)^T x \leq 3$	3
$\{3\}$	$\{1,2,5\}$	$\{1,2,3,5\}$	$\{67\}$	$(11211111)^T x \leq 3$	3
$\{3\}$	$\{1,2,5\}$	$\{1,2,3,4,5\}$	$\{67\}$	$(11211112)^T x \leq 3$	3
$\{3\}$	$\{1,2,4,5\}$	$\{1,2,3,4,5\}$	$\{67\}$	$(11211112)^T x \leq 3$	3
$\{1,5\}$	$\{1,3\}$	$\{1,2,5\}$	$\{67,68\}$	$(11211111)^T x \leq 3$	3
$\{1,5\}$	$\{1,3\}$	$\{3,4,5\}$	$\{67,68,78\}$	$(11211112)^T x \leq 3$	3
$\{1,5\}$	$\{1,2,3\}$	$\{1,2,3,5\}$	$\{67\}$	$(21112111)^T x \leq 3$	5
$\{1,5\}$	$\{1,2,3\}$	$\{1,2,3,4,5\}$	$\{67\}$	$(21112112)^T x \leq 3$	5
$\{1,5\}$	$\{2,3,4,5\}$	$\{1,2,3,4,5\}$	$\{67\}$	$(21112112)^T x \leq 3$	1
$\{2,4\}$	$\{1,2,3\}$	$\{1,3,4,5\}$	$\{78\}$	$(11112112)^T x \leq 3$	4
$\{1,3,5\}$	$\{1,2,5\}$	$\{1,2,4,5\}$	$\emptyset$	$(21112111)^T x \leq 3$	1
$\{1,3,5\}$	$\{1,2,3\}$	$\{1,2,3,5\}$	$\{67\}$	$(21112111)^T x \leq 3$	5

S_1	S_2	S_3	S'	The full facet	Node
$\{1,3,5\}$	$\{1,2,3\}$	$\{1,2,3,4,5\}$	$\{67\}$	$(21112112)^T x \leq 3$	5
$\{1,3,5\}$	$\{1,2,4,5\}$	$\{1,2,3,4,5\}$	$\{68\}$	$(21112121)^T x \leq 3$	5
$\{1,3,5\}$	$\{2,3,4,5\}$	$\{1,2,3,4,5\}$	$\{67\}$	$(21112112)^T x \leq 3$	1

Now we turn to the graphs that satisfy $r_0(G-6) = r_0(G-7) = r_0(G-8) = 2$ whose full facet is of N-rank 3 (which implies that $r_0(G) = r(G) = 3$). First we list the graphs for which $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})^T$ violates the full facet of $STAB(G)$, and hence G has N-rank 3 by Lemma 72.

S_1	S_2	S_3	S'	The full facet
$\{2,3,4\}$	$\{1,2,5\}$	$\{1,3,4,5\}$	$\{67\}$	$(11111111)^T x \leq 2$
$\{2,3,4\}$	$\{1,2,5\}$	$= \{1,3,4,5\}$	$\{67,78\}$	$(11111111)^T x \leq 2$
$\{2,3,4\}$	$\{1,2,5\}$	$\{1,2,3,4,5\}$	$\{67\}$	$(11111111)^T x \leq 2$
$\{2,3,4\}$	$\{1,2,4,5\}$	$\{1,3,4,5\}$	$\{78\}$	$(11111111)^T x \leq 2$
$\{2,3,4\}$	$\{1,2,4,5\}$	$\{1,2,3,5\}$	$\{78\}$	$(11111111)^T x \leq 2$
$\{2,3,4\}$	$\{1,3,4,5\}$	$\{1,2,3,5\}$	$\{78\}$	$(11111111)^T x \leq 2$
$\{2,3,4\}$	$\{1,2,4,5\}$	$\{1,3,4,5\}$	$\{67,78\}$	$(11111111)^T x \leq 2$
$\{2,3,4\}$	$\{1,3,4,5\}$	$\{1,2,3,5\}$	$\{67,78\}$	$(11111111)^T x \leq 2$
$\{2,3,4\}$	$\{1,2,4,5\}$	$\{1,3,4,5\}$	$\{68,78\}$	$(11111111)^T x \leq 2$
$\{2,3,4\}$	$\{1,3,4,5\}$	$\{1,2,3,5\}$	$\{68,78\}$	$(11111111)^T x \leq 2$
$\{2,3,4\}$	$\{1,2,4,5\}$	$\{1,2,3,4,5\}$	$\{67\}$	$(11111111)^T x \leq 2$
$\{2,3,4\}$	$\{1,3,4,5\}$	$\{1,2,3,4,5\}$	$\{67\}$	$(11111111)^T x \leq 2$
$\{1,2,4,5\}$	$\{1,3,4,5\}$	$\{1,2,3,4,5\}$	$\{67\}$	$(11111111)^T x \leq 2$
$\{1,2,4,5\}$	$\{2,3,4,5\}$	$\{1,2,3,4,5\}$	$\{67\}$	$(11111111)^T x \leq 2$

Here are the other graphs such that $r_0(G-6) = r_0(G-7) = r_0(G-8) = 2$, $STAB(G)$ has a full facet, and $r_0(G) = r(G) = 3$. We give the justification of $r(G) \geq 3$ under the "Proof" column, which could be one of the following:

- A matrix in $M^2(G)$ whose first column violates the full facet. If we let $a^T x \leq \alpha$ denote the full facet of $STAB(G)$, then such a matrix is found by solving the following LP:

$$\begin{array}{lrcl}
\max & a^T x & & \\
\text{s.t.} & x_i & = & Y_{ii} \\
 & Y_i & \in & x_i OC(G) \\
 & x_i - Y_i & \in & (1 - x_i) OC(G) \\
 & x & \in & [0,1]^8 \\
 & Y & \in & [0,1]^{8\times 8} \\
 & i & \in & [8].
\end{array}$$

We programmed the LP in GAMS and solved it using the MOSEK solver on the NEOS server (http://www-neos.mcs.anl.gov/). If the optimal value of the LP is strictly greater than α, then we know that our optimal solution x^* is in $N^2(G) \setminus STAB(G)$, which shows that $r(G) \geq 3$.

- "See Below": If a "See below" appears next to a graph G_1, and the first matrix that appears under "See Below" corresponds to the graph G_2, this means that G_1 is a subgraph of G_2, and by Lemma 62, the matrix given in $M^2(G_2)$ is also in $M^2(G_1)$. Sometimes a suitable permutation of the rows and the columns is needed.

S_1, S_2, S_3, S' and the full facet	Proof
$S_1 = \{1,5\}$ $S_2 = \{1,3,5\}$ $S_3 = \{1,2,4,5\}$ $S' = \{67,78\}$ $(21112112)^T x \leq 3$	$\frac{1}{8}\begin{pmatrix} 8 & 2 & 3 & 2 & 3 & 2 & 2 & 2 & 2 \\ 2 & 2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 3 & 0 & 2 & 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 3 & 0 & 1 & 1 & 0 & 2 \\ 3 & 1 & 2 & 0 & 3 & 0 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 & 0 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

S_1, S_2, S_3, S' and the full facet	Proof
$S_1 = \{1,5\}$ $S_2 = \{2,3,4\}$ $S_3 = \{1,2,3,4,5\}$ $S' = \{67\}$ $(21112112)^T x \leq 3$	$\frac{1}{20}\begin{pmatrix} 20 & 6 & 6 & 3 & 6 & 6 & 6 & 8 & 5 \\ 6 & 6 & 0 & 0 & 3 & 0 & 0 & 3 & 0 \\ 6 & 0 & 6 & 0 & 3 & 3 & 2 & 0 & 0 \\ 3 & 0 & 0 & 3 & 0 & 0 & 1 & 0 & 0 \\ 6 & 3 & 3 & 0 & 6 & 0 & 2 & 0 & 0 \\ 6 & 0 & 3 & 0 & 0 & 6 & 0 & 3 & 0 \\ 6 & 0 & 2 & 1 & 2 & 0 & 6 & 0 & 3 \\ 8 & 3 & 0 & 0 & 0 & 3 & 0 & 8 & 2 \\ 5 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & 5 \end{pmatrix}$
$S_1 = \{1,3,5\}$ $S_2 = \{1,3,5\}$ $S_3 = \{1,2,4,5\}$ $S' = \{67\}$ $(21112112)^T x \leq 3$	See Below
$S_1 = \{1,3,5\}$ $S_2 = \{1,3,5\}$ $S_3 = \{1,2,4,5\}$ $S' = \{67,68\}$ $(21112112)^T x \leq 3$	See Below
$S_1 = \{1,3,5\}$ $S_2 = \{1,3,5\}$ $S_3 = \{1,2,4,5\}$ $S' = \{67,78\}$ $(21112112)^T x \leq 3$	$\frac{1}{10}\begin{pmatrix} 10 & 3 & 4 & 4 & 4 & 3 & 2 & 2 & 2 \\ 3 & 3 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 0 & 2 & 2 & 1 & 1 & 0 \\ 4 & 1 & 0 & 4 & 0 & 1 & 0 & 0 & 2 \\ 4 & 2 & 2 & 0 & 4 & 0 & 1 & 1 & 0 \\ 3 & 0 & 2 & 1 & 0 & 3 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

S_1, S_2, S_3, S' and the full facet	Proof
$S_1 = \{1,3,5\}$ $S_2 = \{2,3,4\}$ $S_3 = \{1,2,4,5\}$ $S' = \{67\}$ $(21112112)^T x \leq 3$	$\frac{1}{13}\begin{pmatrix} 13 & 4 & 3 & 4 & 3 & 4 & 3 & 4 & 4 \\ 4 & 4 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 3 & 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & 1 & 0 & 4 & 0 & 1 & 0 & 0 & 1 \\ 3 & 2 & 1 & 0 & 3 & 0 & 1 & 0 & 0 \\ 4 & 0 & 2 & 1 & 0 & 4 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 & 3 & 0 & 2 \\ 4 & 1 & 0 & 0 & 0 & 1 & 0 & 4 & 1 \\ 4 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 4 \end{pmatrix}$
$S_1 = \{1,3,5\}$ $S_2 = \{1,2,3\}$ $S_3 = \{1,2,4,5\}$ $S' = \{67\}$ $(21112112)^T x \leq 3$	See Below
$S_1 = \{1,3,5\}$ $S_2 = \{1,2,3\}$ $S_3 = \{1,2,4,5\}$ $S' = \{78\}$ $(21112112)^T x \leq 3$	See Below
$S_1 = \{1,3,5\}$ $S_2 = \{1,2,3\}$ $S_3 = \{1,2,4,5\}$ $S' = \{67,78\}$ $(21112112)^T x \leq 3$	$\frac{1}{20}\begin{pmatrix} 20 & 5 & 6 & 8 & 6 & 6 & 6 & 3 & 6 \\ 5 & 5 & 0 & 2 & 3 & 0 & 0 & 0 & 0 \\ 6 & 0 & 6 & 0 & 2 & 3 & 3 & 0 & 0 \\ 8 & 2 & 0 & 8 & 0 & 3 & 0 & 0 & 3 \\ 6 & 3 & 2 & 0 & 6 & 0 & 2 & 1 & 0 \\ 6 & 0 & 3 & 3 & 0 & 6 & 0 & 0 & 0 \\ 6 & 0 & 3 & 0 & 2 & 0 & 6 & 0 & 3 \\ 3 & 0 & 0 & 0 & 1 & 0 & 0 & 3 & 0 \\ 6 & 0 & 0 & 3 & 0 & 0 & 3 & 0 & 6 \end{pmatrix}$

S_1, S_2, S_3, S' and the full facet	Proof
$S_1 = \{1,3,5\}$ $S_2 = \{2,3,4\}$ $S_3 = \{1,2,3,4,5\}$ $S' = \{67\}$ $(21112112)^T x \leq 3$	$\frac{1}{8}\begin{pmatrix} 7 & 2 & 2 & 1 & 2 & 2 & 2 & 3 & 2 \\ 2 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 1 \\ 3 & 1 & 0 & 0 & 0 & 1 & 0 & 3 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}$
$S_1 = \{1,3,5\}$ $S_2 = \{2,3,4\}$ $S_3 = \{1,2,4,5\}$ $S' = \{78\}$ $(21112112)^T x \leq 3$	See Below
$S_1 = \{1,3,5\}$ $S_2 = \{2,3,4\}$ $S_3 = \{1,2,4,5\}$ $S' = \{67, 78\}$ $(21112112)^T x \leq 3$	$\frac{1}{7}\begin{pmatrix} 7 & 2 & 2 & 3 & 2 & 2 & 2 & 1 & 2 \\ 2 & 2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 0 \\ 3 & 1 & 0 & 3 & 0 & 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 2 \end{pmatrix}$
$S_1 = \{1,3,5\}$ $S_2 = \{1,2,4,5\}$ $S_3 = \{1,3,4,5\}$ $S' = \{78\}$ $(21112121)^T x \leq 3$	$\frac{1}{10}\begin{pmatrix} 10 & 2 & 4 & 4 & 2 & 3 & 4 & 3 & 2 \\ 2 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 0 & 1 & 2 & 2 & 0 & 1 \\ 4 & 2 & 0 & 4 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 1 & 0 & 3 & 0 & 0 & 0 \\ 4 & 0 & 2 & 0 & 1 & 0 & 4 & 2 & 1 \\ 3 & 0 & 0 & 1 & 0 & 0 & 2 & 3 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 2 \end{pmatrix}$

S_1, S_2, S_3, S' and the full facet	Proof
$S_1 = \{1,3,5\}$ $S_2 = \{1,2,4,5\}$ $S_3 = \{1,2,3,4\}$ $S' = \{78\}$ $(21112121)^T x \leq 3$	See Below
$S_1 = \{1,3,5\}$ $S_2 = \{1,2,4,5\}$ $S_3 = \{1,2,3,4\}$ $S' = \{68,78\}$ $(21112121)^T x \leq 3$	$\frac{1}{7}\begin{pmatrix} 7 & 2 & 2 & 3 & 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 0 \\ 3 & 1 & 0 & 3 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$S_1 = \{1,3,5\}$ $S_2 = \{1,2,4,5\}$ $S_3 = \{1,3,4,5\}$ $S' = \{68,78\}$ $(21112122)^T x \leq 3$	$\frac{1}{20}\begin{pmatrix} 20 & 6 & 6 & 8 & 6 & 5 & 6 & 6 & 3 \\ 6 & 6 & 0 & 3 & 3 & 0 & 0 & 0 & 0 \\ 6 & 0 & 6 & 0 & 2 & 3 & 2 & 0 & 1 \\ 8 & 3 & 0 & 8 & 0 & 2 & 0 & 3 & 0 \\ 6 & 3 & 2 & 0 & 6 & 0 & 3 & 0 & 0 \\ 5 & 0 & 3 & 2 & 0 & 5 & 0 & 0 & 0 \\ 6 & 0 & 2 & 0 & 3 & 0 & 6 & 3 & 0 \\ 6 & 0 & 0 & 3 & 0 & 0 & 3 & 6 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$
$S_1 = \{2,3,4\}$ $S_2 = \{1,2,5\}$ $S_3 = \{1,2,3,4\}$ $S' = \{67\}$ $(22111121)^T x \leq 3$	$\frac{1}{20}\begin{pmatrix} 20 & 6 & 5 & 3 & 8 & 6 & 6 & 6 & 6 \\ 6 & 6 & 0 & 0 & 3 & 0 & 3 & 0 & 0 \\ 5 & 0 & 5 & 0 & 2 & 3 & 0 & 0 & 0 \\ 3 & 0 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 8 & 3 & 2 & 0 & 8 & 0 & 0 & 3 & 0 \\ 6 & 0 & 3 & 1 & 0 & 6 & 2 & 0 & 2 \\ 6 & 3 & 0 & 0 & 0 & 2 & 6 & 0 & 3 \\ 6 & 0 & 0 & 0 & 3 & 0 & 0 & 6 & 3 \\ 6 & 0 & 0 & 0 & 0 & 2 & 3 & 3 & 6 \end{pmatrix}$

S_1, S_2, S_3, S' and the full facet	Proof
$S_1 = \{2,3,4\}$ $S_2 = \{1,2,5\}$ $S_3 = \{1,2,3,5\}$ $S' = \{67\}$ $(12211211)^T x \le 3$	$\frac{1}{20}\begin{pmatrix} 20 & 3 & 5 & 6 & 6 & 8 & 6 & 6 & 6 \\ 3 & 3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 5 & 0 & 5 & 0 & 3 & 2 & 0 & 0 & 0 \\ 6 & 0 & 0 & 6 & 0 & 3 & 0 & 3 & 0 \\ 6 & 1 & 3 & 0 & 6 & 0 & 0 & 2 & 2 \\ 8 & 0 & 2 & 3 & 0 & 8 & 3 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 3 & 6 & 0 & 3 \\ 6 & 0 & 0 & 3 & 2 & 0 & 0 & 6 & 3 \\ 6 & 0 & 0 & 0 & 2 & 0 & 3 & 3 & 6 \end{pmatrix}$

Now we turn to the 8-node graphs that satisfy $r_0(G-6) = r_0(G-7) = r_0(G-8) = 3$. First, here are the list of those whose stable set polytope has a full facet of N_0-rank 3. And again, we give under the "Node" column, the node whose deletion and destruction from the full facet both result in inequalities of N_0-rank 2.

S_1	S_2	S_3	S'	The full facet	Node
$\{1,2,4,5\}$	$\{1,2,4,5\}$	$\{1,2,3,4,5\}$	$\{67,68,78\}$	$(11111112)^T x \le 2$	3
$\{2,3,4\}$	$\{1,2,3\}$	$\{1,2,3,4,5\}$	$\{67,68,78\}$	$(11111112)^T x \le 2$	5
$\{2,3,4\}$	$\{2,3,4,5\}$	$\{1,2,3,4,5\}$	$\{67,68,78\}$	$(11111112)^T x \le 2$	1

Here are the graphs that satisfy $r_0(G-6) = r_0(G-7) = r_0(G-8) = 3$, and the point $(\frac{2}{7}, \frac{2}{7}, \frac{2}{7}, \frac{2}{7}, \frac{2}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7})^T$ violates the full facet of $STAB(G)$, showing that $r_0(G) = r(G) = 4$.

S_1	S_2	S_3	S'	The full facet
$\{2,3,4\}$	$\{1,2,3,4,5\}$	$\{1,2,3,4,5\}$	$\{67,68,78\}$	$(11111122)^T x \le 2$
$\{1,2,4,5\}$	$\{1,2,3,4,5\}$	$\{1,2,3,4,5\}$	$\{67,68,78\}$	$(11111122)^T x \le 2$
$\{1,2,3,4,5\}$	$\{1,2,3,4,5\}$	$\{1,2,3,4,5\}$	$\{67,68,78\}$	$(11111222)^T x \le 2$

There is one other 8-node graph that satisfies $r_0(G-6) = r_0(G-7) = r_0(G-8) = 3$ whose stable set polytope has a full facet of N- and N_0-rank 4. The matrix under the "Proof" column is a matrix in $M^3(G)$ whose first column violates the full facet.

S_1, S_2, S_3, S' and the full facet	Proof
$S_1 = \{2,3,4\}$ $S_2 = \{2,3,4\}$ $S_3 = \{1,2,3,4,5\}$ $S' = \{67, 68, 78\}$ $(11111112)^T x \leq 2$	$\frac{1}{12}\begin{pmatrix} 12 & 4 & 3 & 3 & 3 & 4 & 3 & 3 & 3 \\ 4 & 4 & 0 & 1 & 2 & 0 & 1 & 1 & 0 \\ 3 & 0 & 3 & 0 & 0 & 2 & 0 & 0 & 0 \\ 3 & 1 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 4 & 0 & 2 & 1 & 0 & 4 & 1 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 & 3 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 & 0 & 3 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$

We mention here how we obtained the above matrix. We will use the same method to obtain matrices for other graphs to show that they are of N-rank at least 4. Given a graph G and its full facet $a^T x \leq \alpha$, we obtain a matrix in $M^3(G)$ as follows. First, we solve the following LP:

$$
\begin{array}{lrcl}
\max & a^T x & & \\
\text{s.t.} & x_i & = & Y_{ii} \\
 & Y_{ij} & = & U_{ii}^{(j)} \\
 & x_i - Y_{ij} & = & V_{ii}^{(j)} \\
 & Y & = & Y^T \\
 & U^{(i)} & = & (U^{(i)})^T \\
 & V^{(i)} & = & (V^{(i)})^T \\
 & U_i^{(j)} & \in & Y_{ij} OC(G) \\
 & Y_j - U_i^{(j)} & \in & (x_j - Y_{ij}) OC(G) \\
 & V_i^{(j)} & \in & (x_i - Y_{ij}) OC(G) \\
 & (x - Y_j) - V_i^{(j)} & \in & [(1 - x_j) - (x_i - Y_{ij})] OC(G) \\
 & x & \in & [0,1]^8 \\
 & Y, U^{(i)}, V^{(i)} & \in & [0,1]^{8 \times 8} \\
 & i, j & \in & [8].
\end{array}
$$

Alternatively, we are finding $x, Y, U^{(1)}, \ldots, U^{(8)}, V^{(1)}, \ldots, V^{(8)}$ such that

$$\begin{pmatrix} x_i & Y_i^T \\ Y_i & U^{(i)} \end{pmatrix}, \begin{pmatrix} 1-x_i & (x-Y_i)^T \\ x-Y_i & V^{(i)} \end{pmatrix} \in M(G) \quad \forall i \in [8],$$

which implies that $Y_i \in x_i N^2(G), (x - Y_i) \in (1 - x_i)N^2(G)$ $\forall i \in [8]$. That together with the constraints $x_i = Y_{ii}$ $\forall i \in [8]$ and $Y = Y^T$ imply that $x \in N^3(G)$. Then again, we programmed the LP in GAMS and solved it using the MOSEK solver on the NEOS server.

A.2 The graphs $(\bar{C}_7, S_1, S_2)$

Similar to the above, when verifying the N- and N_0-rank for a graph that is a 7-antihole plus two nodes, we only need to check those that satisfy $r_0(G-8) = r_0(G-9)$. From Proposition 63 and Claim 76, we know that $(\bar{C}_7, S)$ has N- and N_0-rank 3 if and only if $\mu(S) > 0$, and has N- and N_0-rank 2 otherwise. Also, we only need to check those whose stable set polytope have a full facet.

Here is the list of graphs such that $r_0(G-8) = r_0(G-9) = 2$ and $STAB(G)$ has a full facet of N- and N_0-rank 2:

S_1	S_2	The full facet	Node
$\{1,2,4\}$	$\{2,3,6,7\}$	$(212111111)^T x \leq 3$	1
$\{2,3,6,7\}$	$\{1,3,4,7\}$	$(121111211)^T x \leq 3$	2

The list of graphs such that $r_0(G-8) = r_0(G-9) = 2$ and $STAB(G)$ has a full facet of N- and N_0-rank 3:

S_1, S_2 and the full facet	Proof
$S_1 = \{1, 7\}$ $S_2 = \{1, 2, 4, 5\}$ $(211111111)^T x \leq 3$	$\frac{1}{6}\begin{pmatrix} 6 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 5 & 2 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 2 & 1 \\ 2 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 2 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 5 & 0 & 1 & 2 & 2 & 2 & 2 & 0 & 5 & 2 \\ 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 2 & 2 \end{pmatrix}$
$S_1 = \{1, 2, 4\}$ $S_2 = \{2, 3, 5, 6\}$ $(121211111)^T x \leq 3$	$\frac{1}{9}\begin{pmatrix} 9 & 2 & 2 & 3 & 3 & 2 & 2 & 3 & 3 & 3 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 2 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 & 3 & 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 & 1 & 3 & 2 & 2 \\ 3 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 3 & 1 \\ 3 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 3 \end{pmatrix}$
$S_1 = \{1, 2, 4\}$ $S_2 = \{1, 4, 5, 7\}$ $(211211111)^T x \leq 3$	$\frac{1}{9}\begin{pmatrix} 9 & 2 & 2 & 3 & 2 & 2 & 3 & 2 & 4 & 4 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 2 & 2 \\ 2 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 2 & 2 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 4 & 0 & 0 & 2 & 0 & 0 & 2 & 1 & 4 & 3 \\ 4 & 0 & 1 & 2 & 0 & 0 & 2 & 0 & 3 & 4 \end{pmatrix}$

S_1, S_2 and the full facet	Proof
$S_1 = \{1,4,7\}$ $S_2 = \{1,2,4,5\}$ $(211211111)^T x \leq 3$	$\frac{1}{8}\begin{pmatrix} 8&2&2&2&2&2&3&2&3&3\\ 2&2&1&0&0&0&0&1&0&0\\ 2&1&2&0&0&0&0&0&1&0\\ 2&0&0&2&1&0&0&0&1&1\\ 2&0&0&1&2&1&0&0&0&0\\ 2&0&0&0&1&2&1&0&0&0\\ 3&0&0&0&0&1&3&1&2&2\\ 2&1&0&0&0&0&1&2&0&1\\ 3&0&1&1&0&0&2&0&3&2\\ 3&0&0&1&0&0&2&1&2&3 \end{pmatrix}$
$S_1 = \{1,2,6,7\}$ $S_2 = \{1,3,4,7\}$ $(121111211)^T x \leq 3$	$\frac{1}{7}\begin{pmatrix} 7&2&2&1&2&2&2&2&2&2\\ 2&2&1&0&0&0&0&1&0&0\\ 2&1&2&0&0&0&0&0&0&1\\ 1&0&0&1&0&0&0&0&0&0\\ 2&0&0&0&2&1&0&0&1&0\\ 2&0&0&0&1&2&1&0&0&0\\ 2&0&0&0&0&1&2&1&0&1\\ 2&1&0&0&0&0&1&2&1&0\\ 2&0&0&0&1&0&0&1&2&0\\ 2&0&1&0&0&0&1&0&0&2 \end{pmatrix}$
$S_1 = \{2,3,5,6\}$ $S_2 = \{2,3,6,7\}$ $(222213111)^T x \leq 4$	$\frac{1}{9}\begin{pmatrix} 7&3&2&2&3&2&2&2&4&4\\ 3&3&1&0&0&0&0&0&2&2\\ 2&1&2&1&0&0&0&0&0&0\\ 2&0&1&2&1&0&0&0&0&0\\ 3&0&0&1&3&0&0&0&2&2\\ 2&0&0&0&0&2&1&0&0&1\\ 2&0&0&0&0&1&2&1&0&0\\ 2&0&0&0&0&0&1&2&1&0\\ 4&2&0&0&2&0&0&1&4&3\\ 4&2&0&0&2&1&0&0&3&4 \end{pmatrix}$

The list of graphs such that $r_0(G-8) = r_0(G-9) = 3$ and $STAB(G)$ has a full facet of N- and N_0-rank 3:

S_1	S_2	The full facet	Node
$\{1,2,4,6,7\}$	$\{2,3,4,6,7\}$	$(212112111)^T x \leq 3$	1
$\{1,2,4,6,7\}$	$\{1,3,5,6,7\}$	$(111111111)^T x \leq 2$	7
$\{1,2,4,6,7\}$	$\{1,3,4,5,6\}$	$(111111111)^T x \leq 2$	7
$\{1,2,4,6,7\}$	$\{1,3,4,5,7\}$	$(111111111)^T x \leq 2$	7
$\{1,2,4,6,7\}$	$\{1,2,3,5,6\}$	$(111111111)^T x \leq 2$	7
$\{1,2,4,6,7\}$	$\{1,2,3,5,6,7\}$	$(111111111)^T x \leq 2$	7
$\{1,2,4,6,7\}$	$\{1,3,4,5,6,7\}$	$(111111111)^T x \leq 2$	7
$\{1,2,4,6,7\}$	$\{2,3,4,5,6,7\}$	$(111111111)^T x \leq 2$	1
$\{1,2,4,6,7\}$	$\{1,2,3,4,5,6,7\}$	$(111111111)^T x \leq 2$	7
$\{1,3,4,5,7\}$	$\{2,3,4,6,7\}$	$(111111111)^T x \leq 2$	7
$\{1,3,4,5,7\}$	$\{1,2,4,5,6,7\}$	$(111111111)^T x \leq 2$	7
$\{1,3,4,5,7\}$	$\{2,3,4,5,6,7\}$	$(111111111)^T x \leq 2$	7
$\{1,3,4,5,7\}$	$\{1,2,3,4,5,6,7\}$	$(111111111)^T x \leq 2$	7
$\{1,2,3,5,6,7\}$	$\{1,2,4,5,6,7\}$	$(111111111)^T x \leq 2$	7
$\{1,2,3,5,6,7\}$	$\{1,3,4,5,6,7\}$	$(111111111)^T x \leq 2$	4
$\{1,2,3,5,6,7\}$	$\{2,3,4,5,6,7\}$	$(111111111)^T x \leq 2$	1
$\{1,2,3,5,6,7\}$	$\{1,2,3,4,5,6,7\}$	$(111111111)^T x \leq 2$	4

The list of graphs such that $r_0(G-8) = r_0(G-9) = 3$ and $STAB(G)$ has a full facet of N- and N_0-rank 4:

S_1, S_2 and the full facet	Proof
$S_1 = \{1,3,4,5,7\}$ $S_2 = \{1,2,3,5,6\}$ $(111111111)^T x \leq 2$	See Below

S_1, S_2 and the full facet	Proof
$S_1 = \{1,3,4,5,7\}$ $S_2 = \{1,2,3,5,6,7\}$ $(111111111)^T x \leq 2$	$\frac{1}{56}\begin{pmatrix} 56 & 12 & 15 & 12 & 14 & 12 & 15 & 12 & 14 & 11 \\ 12 & 12 & 5 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 15 & 5 & 15 & 6 & 0 & 0 & 0 & 0 & 4 & 0 \\ 12 & 0 & 6 & 12 & 5 & 0 & 0 & 0 & 0 & 0 \\ 14 & 0 & 0 & 5 & 14 & 5 & 0 & 0 & 0 & 4 \\ 12 & 0 & 0 & 0 & 5 & 12 & 6 & 0 & 0 & 0 \\ 15 & 0 & 0 & 0 & 0 & 6 & 15 & 5 & 4 & 0 \\ 12 & 6 & 0 & 0 & 0 & 0 & 5 & 12 & 0 & 0 \\ 14 & 0 & 4 & 0 & 0 & 0 & 4 & 0 & 14 & 6 \\ 11 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 6 & 11 \end{pmatrix}$

A.3 The graphs $[\bar{C}_7, S_1, S_2]$

The list of graphs such that $r_0(G-8) = r_0(G-9) = 2$ and $STAB(G)$ has a full facet of N- and N_0-rank 2:

S_1	S_2	The full facet	Node
$\{4\}$	$\{3,5\}$	$(111211111)^T x \leq 3$	4
$\{4\}$	$\{2,3,5\}$	$(111211111)^T x \leq 3$	4
$\{4\}$	$\{1,3,5\}$	$(111211111)^T x \leq 3$	4
$\{4\}$	$\{1,2,3,5\}$	$(111211111)^T x \leq 3$	4
$\{4\}$	$\{2,3,5,6\}$	$(111211111)^T x \leq 3$	4
$\{2,6\}$	$\{1,3\}$	$(121111111)^T x \leq 3$	2
$\{2,6\}$	$\{1,3,6\}$	$(121112111)^T x \leq 3$	2

S_1	S_2	The full facet	Node
$\{2,6\}$	$\{1,3,7\}$	$(131212211)^T x \leq 4$	2
$\{2,6\}$	$\{1,3,6,7\}$	$(121112111)^T x \leq 3$	2
$\{3,5\}$	$\{3,4,6\}$	$(112121111)^T x \leq 3$	5
$\{3,5\}$	$\{2,3,4,6\}$	$(112121111)^T x \leq 3$	5
$\{3,5\}$	$\{1,2,4,5\}$	$(112121111)^T x \leq 3$	3
$\{3,4,5\}$	$\{3,4,6\}$	$(112121111)^T x \leq 3$	5
$\{3,4,5\}$	$\{2,3,4,6\}$	$(112121111)^T x \leq 3$	5
$\{3,4,5\}$	$\{1,2,4,5\}$	$(112121111)^T x \leq 3$	3
$\{1,2,4\}$	$\{1,3,7\}$	$(212111111)^T x \leq 3$	3
$\{1,2,4\}$	$\{1,2,3,7\}$	$(212111111)^T x \leq 3$	3
$\{1,2,4\}$	$\{2,3,6,7\}$	$(212111111)^T x \leq 3$	1

The list of graphs such that $r_0(G-8) = r_0(G-9) = 2$ and $STAB(G)$ has a full facet of N- and N_0-rank 3:

S_1, S_2 and the full facet	Proof
$S_1 = \{3,5\}$ $S_2 = \{2,4,6\}$ $(112121111)^T x \leq 3$	See Below
$S_1 = \{3,4,5\}$ $S_2 = \{2,4,6\}$ $(112121111)^T x \leq 3$	$\frac{1}{8}\begin{pmatrix} 8 & 3 & 2 & 2 & 1 & 2 & 2 & 3 & 3 & 3 \\ 3 & 3 & 1 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \\ 2 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 & 1 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 & 0 & 0 & 1 & 2 & 3 & 0 \\ 3 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 3 \end{pmatrix}$

S_1, S_2 and the full facet	Proof
$S_1 = \{1,2,4\}$ $S_2 = \{2,3,7\}$ $(212111111)^T x \leq 3$	$\frac{1}{13}\begin{pmatrix} 13 & 4 & 3 & 4 & 3 & 4 & 4 & 3 & 4 & 4 \\ 4 & 4 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 1 \\ 3 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 & \\ 3 & 0 & 0 & 2 & 3 & 0 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 & 4 & 3 & 0 & 2 & 2 \\ 4 & 0 & 0 & 0 & 0 & 3 & 4 & 0 & 2 & 2 \\ 3 & 2 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 \\ 4 & 0 & 0 & 1 & 0 & 2 & 2 & 1 & 4 & 0 \\ 4 & 1 & 0 & 0 & 1 & 2 & 2 & 0 & 0 & 4 \end{pmatrix}$
$S_1 = \{2,4,6\}$ $S_2 = \{2,3,5,6\}$ $(121212111)^T x \leq 3$	See Below
$S_1 = \{1,2,3,5\}$ $S_2 = \{1,2,4,5\}$ $(212121111)^T x \leq 3$	See Below
$S_1 = \{2,3,5,6\}$ $S_2 = \{2,3,5,6\}$ $(121212111)^T x \leq 3$	See Below
$S_1 = \{2,3,5,6\}$ $S_2 = \{2,3,4,5,6\}$ $(121212111)^T x \leq 3$	$\frac{1}{8}\begin{pmatrix} 8 & 3 & 2 & 2 & 2 & 2 & 2 & 3 & 2 & 2 \\ 3 & 3 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 1 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \end{pmatrix}$

S_1, S_2 and the full facet	Proof
$S_1 = \{2,3,5,6\}$ $S_2 = \{1,3,4,7\}$ $(131312211)^T x \leq 4$	$\frac{1}{7}\begin{pmatrix} 7 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}$

The list of graphs such that $r_0(G-8) = r_0(G-9) = 3$ and $STAB(G)$ has a full facet of N- and N_0-rank 3:

S_1	S_2	The full facet	Node
$\{1,2,4,6,7\}$	$\{1,3,5,6,7\}$	$(111111111)^T x \leq 2$	1
$\{1,2,4,6,7\}$	$\{2,4,5,6,7\}$	$(111111111)^T x \leq 2$	3
$\{1,2,4,6,7\}$	$\{1,3,4,5,6\}$	$(111111111)^T x \leq 2$	7
$\{1,2,4,6,7\}$	$\{1,3,4,5,7\}$	$(111111111)^T x \leq 2$	2
$\{1,2,4,6,7\}$	$\{1,2,3,5,6\}$	$(111111111)^T x \leq 2$	7
$\{1,2,4,6,7\}$	$\{1,2,4,5,7\}$	$(111111111)^T x \leq 2$	3
$\{1,2,4,6,7\}$	$\{1,2,3,5,6,7\}$	$(111111111)^T x \leq 2$	1
$\{1,2,4,6,7\}$	$\{1,2,4,5,6,7\}$	$(111111111)^T x \leq 2$	3
$\{1,2,4,6,7\}$	$\{1,3,4,5,6,7\}$	$(111111111)^T x \leq 2$	2
$\{1,2,4,6,7\}$	$\{2,3,4,5,6,7\}$	$(111111111)^T x \leq 2$	2
$\{1,3,4,5,7\}$	$\{2,3,4,6,7\}$	$(111111111)^T x \leq 2$	1
$\{1,3,4,5,7\}$	$\{1,2,4,5,7\}$	$(111111111)^T x \leq 2$	6
$\{1,3,4,5,7\}$	$\{1,2,4,5,6,7\}$	$(111111111)^T x \leq 2$	3
$\{1,3,4,5,7\}$	$\{1,3,4,5,6,7\}$	$(111111111)^T x \leq 2$	2
$\{1,3,4,5,7\}$	$\{2,3,4,5,6,7\}$	$(111111111)^T x \leq 2$	1

S_1	S_2	The full facet	Node
$\{1,2,3,5,6,7\}$	$\{1,2,3,5,6,7\}$	$(111111111)^T x \leq 2$	4
$\{1,2,3,5,6,7\}$	$\{1,2,4,5,6,7\}$	$(111111111)^T x \leq 2$	4
$\{1,2,3,5,6,7\}$	$\{1,3,4,5,6,7\}$	$(111111111)^T x \leq 2$	4
$\{1,2,3,5,6,7\}$	$\{2,3,4,5,6,7\}$	$(111111111)^T x \leq 2$	4

The list of graphs such that $r_0(G-8) = r_0(G-9) = 3$ and $STAB(G)$ has a full facet of N- and N_0-rank 4:

S_1, S_2 and the full facet	Proof
$S_1 = \{1,2,4,6,7\}$ $S_2 = \{1,2,4,6,7\}$ $(111111111)^T x \leq 2$	$\frac{1}{120}\begin{pmatrix} 120 & 24 & 24 & 42 & 24 & 42 & 24 & 24 & 24 & 24 \\ 24 & 24 & 10 & 0 & 0 & 0 & 0 & 13 & 0 & 0 \\ 24 & 10 & 24 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 42 & 0 & 12 & 42 & 10 & 0 & 0 & 0 & 10 & 10 \\ 24 & 0 & 0 & 10 & 24 & 10 & 0 & 0 & 0 & 0 \\ 42 & 0 & 0 & 0 & 10 & 42 & 12 & 0 & 10 & 10 \\ 24 & 0 & 0 & 0 & 0 & 12 & 24 & 11 & 0 & 0 \\ 24 & 13 & 0 & 0 & 0 & 0 & 11 & 24 & 0 & 0 \\ 24 & 0 & 0 & 10 & 0 & 10 & 0 & 0 & 24 & 0 \\ 24 & 0 & 0 & 10 & 0 & 10 & 0 & 0 & 0 & 24 \end{pmatrix}$

S_1, S_2 and the full facet	Proof
$S_1 = \{1,3,4,5,7\}$ $S_2 = \{1,3,4,5,7\}$ $(111111111)^T x \leq 2$	$\frac{1}{156}\begin{pmatrix} 156 & 30 & 60 & 24 & 24 & 24 & 60 & 30 & 39 & 39 \\ 30 & 30 & 18 & 0 & 0 & 0 & 0 & 10 & 0 & 0 \\ 60 & 18 & 60 & 6 & 0 & 0 & 0 & 0 & 18 & 18 \\ 24 & 0 & 6 & 24 & 12 & 0 & 0 & 0 & 0 & 0 \\ 24 & 0 & 0 & 12 & 24 & 12 & 0 & 0 & 0 & 0 \\ 24 & 0 & 0 & 0 & 12 & 24 & 6 & 0 & 0 & 0 \\ 60 & 0 & 0 & 0 & 0 & 6 & 60 & 18 & 18 & 18 \\ 30 & 10 & 0 & 0 & 0 & 0 & 18 & 30 & 0 & 0 \\ 39 & 0 & 18 & 0 & 0 & 0 & 18 & 0 & 39 & 0 \\ 39 & 0 & 18 & 0 & 0 & 0 & 18 & 0 & 0 & 39 \end{pmatrix}$
$S_1 = \{1,2,4,6,7\}$ $S_2 = \{2,3,4,6,7\}$ $(111111111)^T x \leq 2$	See Below
$S_1 = \{1,3,4,5,7\}$ $S_2 = \{1,2,3,5,6\}$ $(111111111)^T x \leq 2$	See Below
$S_1 = \{1,3,4,5,7\}$ $S_2 = \{1,2,3,5,6,7\}$ $(111111111)^T x \leq 2$	$\frac{1}{14}\begin{pmatrix} 14 & 4 & 4 & 2 & 3 & 2 & 4 & 4 & 4 & 2 \\ 4 & 4 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 4 & 2 & 4 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 3 & 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 4 & 2 & 2 & 0 \\ 4 & 2 & 0 & 0 & 0 & 0 & 2 & 4 & 0 & 0 \\ 4 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 4 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

S_1, S_2 and the full facet	Proof
$S_1 = \{1,2,4,6,7\}$ $S_2 = \{1,2,3,4,5,6,7\}$ $(111111112)^T x \leq 2$	See Below
$S_1 = \{1,3,4,5,7\}$ $S_2 = \{1,2,3,4,5,6,7\}$ $(111111112)^T x \leq 2$	See Below
$S_1 = \{1,2,3,5,6,7\}$ $S_2 = \{1,2,3,4,5,6,7\}$ $(111111112)^T x \leq 2$	See below
$S_1 = \{1,2,3,4,5,6,7\}$ $S_2 = \{1,2,3,4,5,6,7\}$ $(111111122)^T x \leq 2$	$\frac{1}{8}\begin{pmatrix} 8 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Appendix B

Symbol Index

Here we give a list of the symbols we have used throughout this thesis. The page number next to the symbol indicates where it was first introduced and defined.

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