

THE CLARK-OCONE FORMULA AND OPTIMAL
PORTFOLIOS

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In this thesis we propose a new approach to solve single-agent investment problems with deterministic coefficients. We consider the classical Merton's portfolio problem framework, which is well-known in the modern theory of financial economics: an investor must allocate his money between one riskless bond and a number of risky stocks. The investor is assumed to be "small" in the sense that his actions do not affect market prices and the market is complete. The objective of the agent is to maximize expected utility of wealth at the end of the planning horizon. The optimal portfolio should be expressed as a "feedback" function of the current wealth. Under the so-called complete market assumption, the optimization can be split into two stages: first the optimal terminal wealth for a given initial endowment is determined, and then the strategy is computed that leads to this terminal wealth. It is possible to extend this martingale approach and to obtain explicit solution of Merton's portfolio problem using the Malliavin calculus and the Clark-Ocone formula.

Keywords

Optimal portfolio, Malliavin Calculus, Clark-Ocone formula

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List of Symbols

- \mathbb{R} real line;
- \perp independence;
- D_t Malliavin derivative;
- $E[x]$ expectation;
- \square end of proof;
- $\mathbb{I}_G(t)$ indicator function: $\mathbb{I}_G(t) = 1$ if $t \in G$, $\mathbb{I}_G(t) = 0$ if $t \notin G$;
- \triangleq equality by definition;
- (Ω, \mathcal{F}, P) probability space;
- ω sample taken from sample space Ω ;
- $C^{1,2}$ stands for the space of all twice continuously differentiable functions;
- $\bar{1}$ N-dimensional vector with all components equal to one;
- a.s. almost surely.

Chapter 1

Introduction

In this chapter we describe the optimal portfolio allocation problem. We also describe the main objective of the thesis which is to solve the portfolio allocation problem using Malliavin calculus and the Clark-Ocone formula. Further we provide a brief overview of the thesis.

1.1 Portfolio Allocation Problem

This thesis considers a classical problem in mathematical finance, that is, how to select an optimal investment strategy in a securities market. Speaking in terms of a utility function, the problem is to maximize the expected utility from consumption or terminal wealth. The problem was first presented and investigated by Merton(1969, 1971) [M1]. Using a dynamic programming approach he derived a nonlinear partial differential equation for the value function of this stochastic control problem and obtained closed-form solutions for different specifications of the agent's utility function.

Under the so-called complete market assumption, the optimization can be split into two stages: first the optimal terminal wealth for a given initial endowment is determined, and then the strategy is computed that leads to this terminal wealth. This martingale approach was developed to obtain a general formula for the portfolio process (Cox and Huang [CH]). Specifically for deterministic coefficients r, b, θ , the standard approach can be found in Karatzas and Shreve [KS]. They consider the associated Cauchy problem and solve a PDE to obtain the result. We do not provide the full derivation, since the reader can refer to [KS] (Chapter 3.8).

The main idea of this work is that the portfolio can be computed directly. The martingale approach establishes the relation between a portfolio process and the *integrand* ψ' in the stochastic integral representation of

$$M(t) = x + \int_0^t \psi'(u) dW(u), \quad (1.1)$$

where M is the nonnegative martingale

$$M(t) = E[\textit{terminal wealth} | \mathcal{F}(t)], \quad 0 \leq t \leq T \quad (1.2)$$

The Clark-Ocone formula allows us to compute the integrand by means of a Malliavin derivative. A thorough derivation will be given in the main result of the work.

The thesis thus merely gives a second more compact solution and does not state any new results.

1.2 Thesis Overview

The thesis is structured as follows. In the second chapter we present a model setup and formulate an investment problem. We will consider a small investor who has initial endowment x . He is acting in a standard complete market of n risky assets and one non-risky (bond). We will formulate the problem for this investor in terms of utility maximization of the terminal wealth. We will consider the case where the investor is restricted by a finite-time horizon $[0, T]$ and the market coefficients are non-random.

In the third chapter we give a short introduction to the Malliavin calculus along with some examples of its financial applications. We provide some notations and basic properties of the Malliavin derivative. In the last part of the chapter we provide the Clark-Ocone (CO) formula, which we use to obtain the main result.

In chapter 4 we present the main result - the detailed computation of the optimal portfolio π under deterministic coefficients. The computation is split into several phases: the optimal terminal wealth for a given initial endowment is determined. Then the general expression for the portfolio process π is determined using the martingale approach. Lastly, we apply CO formula and basic facts from Malliavin calculus from chapter 3 to obtain the expression for the portfolio.

We conclude the thesis by discussing the extension of the approach and give some possible directions of future work.

Chapter 2

Problem Formulation

2.1 Complete Financial Market

In this section we present a complete standard market framework with N tradeable stocks and one risk-free asset(bond). We will talk about an ideal market meaning that there are no transaction costs, amounts of stock are infinitely divisible and there are no portfolio constraints. We assume that a share of a money-market asset has price $S_0(t)$ at time t , with $S_0(0) = 1$. We have

$$S_0(t) = \exp\left\{\int_0^t r(u)du\right\} \quad (2.1)$$

where r is an instantaneous (risk-free) rate. Since it does not contain random parameters, it represents a riskless investment. Next we have N stocks with price-per-share $S_1(t), \dots, S_N(t)$ at time t . These processes are continuous, strictly positive, and satisfy stochastic differential

equations

$$dS_n(t) = S_n(t) \left[b_n(t)dt + \sum_{d=1}^D \sigma_{nd}(t)dW^{(d)}(t) \right], \quad (2.2)$$

where $b(t)$ is a progressively measurable, N -dimensional mean rate of return satisfying $\int_0^T \|b(t)\|dt < \infty$ a.s., $\sigma(t)$ is a progressively measurable, $(N \times D)$ -matrix-valued volatility process satisfying

$$\sum_{n=1}^N \sum_{d=1}^D \int_0^T \sigma_{nd}^2(t)dt < \infty \quad \text{a.s.},$$

and W is a D -dimensional Brownian motion defined on (Ω, \mathcal{F}, P) . We assume that no risky asset can be obtained by a linear combination of the other assets, implying that the $(N \times D)$ matrix σ is nonsingular, thus $N = D$

For each stock we also associate a dividend rate process $\delta_n(t)$ which is the rate of dividend payment per dollar invested in the stock at time t .

The solution to the equation (2.2) (see [Ø2] for details):

$$S_n(t) = S_n(0) \exp \left\{ \int_0^t \sum_{d=1}^D \sigma_{nd}(s)dW^{(d)}(s) + \int_0^t \left[b_n(s) - \frac{1}{2} \sum_{d=1}^D \sigma_{nd}^2(s) \right] ds \right\} \quad (2.3)$$

We say that our agent acts in a complete financial market in sense that:

- (a) No arbitrage possibility exists.
- (b) The number N of stocks is not greater than the dimension D of the underlying Brownian motion.

(c) The D-dimensional, progressively measurable market price of risk process θ satisfies

$$\int_0^T \|\theta\|^2 dt \leq \infty, \quad (2.4)$$

where

$$\theta \triangleq \sigma^{-1}(t)[b(t) + \delta(t) - r(t)\bar{1}], \quad 0 \leq t < \infty \quad (2.5)$$

and $\bar{1}$ is N -dimensional unit vector.

(d) For every $\mathcal{F}(T)$ -measurable random variable B , with $\frac{B}{S_0(T)}$ bounded from below and satisfying

$$x \triangleq E_0 \left[\frac{B}{S_0(T)} \right] < \infty. \quad (2.6)$$

we can find processes $(\pi_0(\cdot), \pi(\cdot))$ such that

$$\frac{B}{S_0(T)} = x + \int_0^T \frac{1}{S_0(u)} \pi'(u) \sigma(u) dW_0(u), \quad \text{almost surely.} \quad (2.7)$$

The arbitrage-free condition (a) means that there is no possibility for the investor to obtain only positive earnings and strictly positive earnings with strictly positive probability. By using martingale theory the no arbitrage assumption is equivalent to the existence of a risk-free probability measure under which every price process is a martingale [HP].

In the last condition $\pi(\cdot)$ represents the dollar amount invested in each stock, which is called a *portfolio process* under some constraints: $\pi(\cdot)$ is an $\mathcal{F}(t)$ -progressively measurable

\mathbb{R}^N -valued process such that

$$\int_0^T |\pi_0(t) + \pi'(t)1|r(t)dt < \infty \quad (2.8)$$

$$\int_0^T |\pi'(t)(b(t) + \delta(t) - r(t)1)|dt < \infty \quad (2.9)$$

$$\int_0^T \|\sigma'(t)\pi(t)\|^2 dt < \infty \quad (2.10)$$

The condition (d) also states that for any random variable $B(T)$ we can find some initial wealth x and some portfolio process $\pi(\cdot)$ that at time T we can achieve wealth $B(T)$ almost surely. Speaking in proper terms it provides the existence of a replicating portfolio for any contingent claim.

2.2 Utility Function

To formulate an optimization problem we define the utility function and some miscellaneous derivatives of it (to be used in the derivation of the main result).

Every investor can be described in terms of his wealth preference structure. Psychologically, for most people desirability for more money decays as the wealth grows. For example, there would not be a big difference in terms of investor satisfaction between 1 billion and 2 billion dollars, whereas the difference between thousand and million is huge. Although absolute difference in the first case is bigger, satisfaction gain is smaller than in the latter case. We provide a formal definition of the utility function:

Definition 1. We call a concave, nondecreasing, upper semicontinuous function $U : \mathbb{R} \rightarrow [-\infty, \infty)$ a **utility function** if:

1. the half-line $dom(U) \triangleq \{x \in \mathbb{R}; U(x) > -\infty\}$ is a nonempty subset of $[0, \infty)$;
2. U' is continuous, positive, and strictly decreasing on the interior of $dom(U)$, and $U'(\infty) \triangleq \lim_{x \rightarrow \infty} U'(x) = 0$.

If we put

$$\bar{x} \triangleq \inf\{x \in \mathbb{R}; U(x) > -\infty\}, \quad (2.11)$$

the strictly decreasing continuous function $U' : (\bar{x}, \infty) \rightarrow (0, U'(\bar{x}+))$ has a strictly decreasing, continuous inverse

$$I : (0, U'(\bar{x}+)) \rightarrow (\bar{x}, \infty). \quad (2.12)$$

Some commonly used utility functions are logarithmic and power functions:

$$U_p(x) \triangleq \begin{cases} x^p/p, & x \geq 0, \\ -\infty, & x < 0 \end{cases}$$

for $p \in (0, 1)$.

$$U_{\log}(x) \triangleq \begin{cases} \log(x), & x > 0, \\ -\infty, & x \leq 0. \end{cases}$$

Negatively infinite utility function at $x < 0$ represents the least desirable amount of wealth for the investor. For example, it might be the case of bankruptcy.

2.3 Model Setup, Problem Formulation

If the investor possesses an initial endowment x and follows the strategy π his wealth $X^{x,\pi}$ can be expressed as:

$$\frac{X^{x,\pi}(t)}{S_0(t)} = x + \int_0^t \frac{1}{S_0(u)} \pi'(u) \sigma(u) (dW + \theta(u) du), \quad (2.13)$$

see (B.5) from Appendix B for the derivation details. We define a so-called "state price density" process $H_0(t)$ which will be useful for later computations:

$$H_0(t) = \frac{Z_0(t)}{S_0(t)}, \quad (2.14)$$

where

$$Z_0(t) \triangleq \exp\left\{-\int_0^t \theta'(s) dW(s) - \frac{1}{2} \int_0^t \|\theta(s)\|^2 ds\right\} \quad (2.15)$$

is a positive local martingale.

All previous assumptions about the market give us that $Z_0(t)$ is in fact a P-martingale ([KS], chapter 1, eq. 7.1). The property is useful in the derivatives pricing, because the discounted payoff process of a derivative on the stock is a martingale. It allows us to apply martingale representation theorem to find a replicating strategy to hedge the derivative. However we do not require $Z_0(t)$ to be a martingale, whereas the approach given in [KS] does that.

We impose the following assumption on $H_0(t)$:

$$E \left[\int_0^T H_0(t) dt + H_0(T) \right] < \infty \quad (2.16)$$

Clearly, if $S_0(t)$ is bounded away from zero, the assumption holds, but we do not need to impose such a strict condition.

The existence of a state price density process is closely related to the concept of arbitrage free prices. Harrison and Kreps (1979, [HK]) showed that the state price process exists if the market is arbitrage free.

Given $x \geq 0$, we say that a portfolio process π is admissible at x and write $\pi \in \mathcal{A}(x)$, if the wealth process $X^{x,\pi}(\cdot)$ corresponding to x, π satisfies

$$X^{x,\pi}(t) \geq 0, \quad 0 \leq t \leq T, \quad (2.17)$$

almost surely. For $x < 0$, we set $\mathcal{A}(x) = \emptyset$.

This non-negative constraint on wealth is posed to rule out the strategies that lead to an arbitrage. Here we give an example of a so-called "doubling strategy" - a discrete case strategy which leads to a sure profit given the fact that the wealth can be infinitely negative.

Consider a game where the player can bet on one of the two equally likely outcomes; the number of rounds is infinite. The player either wins and is given $2X$ from the initial bet X , or he loses his bet. The idea of the doubling strategy is that the player should double his bet if he lost his previous bet:

- the player bets one dollar for the first round;
- if he wins, he takes the win and exits the game with a total profit of one dollar;
- if he loses, he bets two dollars; if he loses again he bets four dollars, etc., i.e. he keeps on betting until the first win, when he takes the money and leaves the game.

In the limit the win should occur with probability 1, so with probability 1 the player will make a profit of one dollar.

The strategy can be extended to a continuous case, see appendix C. Having assumption (2.17), we reject any strategy that allows the wealth to be negative. In other words, it is impossible to continue trading when the initial capital is lost and the current wealth is below zero, which is bankruptcy.

By applying Ito formula for the product of $Z_0(t)$ and $\frac{X(t)}{S_0(t)}$ we obtain from (2.14)

$$H_0(t)X^{x,\pi}(t) = x + \int_0^t H_0(u)[\sigma'(u)\pi(u) - X^{x,\pi}(u)\theta(u)]'dW(u), \quad 0 \leq t \leq T. \quad (2.18)$$

If π is admissible the left-hand side of (2.18) is nonnegative. Thus by Fatou's lemma a local martingale on the right-hand side is also a supermartingale. This implies a so-called *budget constraint*:

$$E[H_0(T)X(T)] \leq x \quad (2.19)$$

Now we formulate an optimization problem for an agent. This agent is sometimes called a small investor because his actions do not affect the prices of financial assets. The case when there are large investors in the game and thus the market may not be liquid was studied by

Bank and Baum (2004) [BB].

We need to find an admissible portfolio $\pi \in \mathcal{A}$ for the problem

$$V(x) = \sup_{\pi \in \mathcal{A}_1(x)} E[U(X^{x,\pi}(T))] \quad (2.20)$$

of maximizing expected utility from terminal wealth, where

$$\mathcal{A}_1(x) = \{\pi - \text{admissible}; E \min[0, U(X^{x,\pi}(T))] > -\infty\} \quad (2.21)$$

We restrict ourselves to the case when \mathcal{F}_t measurable processes $r(t), b(t), \sigma(t)$ are in fact deterministic.

For this model a general result for the optimal portfolio is known (see, e.g., [KS], Chapter 3, Corollary 6.5). Our objective will be to use this general result to obtain a closed-form solution for the case of deterministic coefficients (chapter 4).

Note that we consider a set of unconstrained portfolios, i.e. short selling is allowed, it is possible to buy any small amount of stock, there is no upper bound for the number of stocks bought. The incomplete market model with constraints was studied by Schachermayer ([Sch1]). The problem is reduced to a dual optimization problem using completion of the market by introducing "fictitious securities".

More realistic models exist, where an investor can only observe the price process (the case of partial information) and cannot observe the Brownian motion and the drift process. These models are solving the investor's objective of maximizing the utility of the terminal wealth

under only partial market information. Lakner (1998)[L1] gives the general expression for the optimal terminal wealth and shows the existence of an optimal strategy.

Chapter 3

Malliavin Calculus and Clark-Ocone Formula

3.1 A Short Introduction to the Malliavin Calculus

The purpose of this section is to familiarize the reader with the Malliavin Calculus and introduce some basic results from it. We refer the reader to the lecture notes by B. Øksendal [Ø1].

Let \mathbb{P} denote the family of all random variables $F : \Omega \rightarrow \mathbb{R}$ of the form

$$F(\omega) = \phi(\theta_1, \dots, \theta_n)$$

where $\phi(x_1, \dots, x_n) = \sum_{\alpha} a_{\alpha} x^{\alpha}$ is a polynomial in n variables x_1, \dots, x_n and $\theta_i = \int_0^T f_i(t) dW(t)$

for some $f_i \in L^2([0, T])$ (deterministic). For \mathbb{P} functionals we define the Malliavin derivative:

$$DF = \sum_i \partial_i \phi(\theta_1, \dots, \theta_n) f_i(t)$$

The domain \mathbb{P} for the operator D can be extended to all functions $F(\omega)$ for which there exists a sequence of smooth functions F_m such that $F_m \rightarrow F$ in L_2 and DF_m is Cauchy in $L_2()$. In this case set DF to be the L_2 -limit of DF_m . The extended domain is denoted by $\mathbb{D}_{1,2}$ and is exactly the closure of \mathbb{P} with respect to $\|\cdot\|_{1,2}$ where

$$\|F\|_{1,2}^2 = \int_{\Omega} |F|^2 dW + \int_{\Omega} \|DF\|_H^2 dW = E|F|^2 + E\|DF\|_H^2$$

We give some basic properties of Malliavin derivative under the following framework: we are given a Wiener process $\{W_t, 0 \leq t \leq T\}$ on (Ω, \mathcal{F}, P) , and we put $\mathcal{G}_t^W = \sigma\{W_s, 0 \leq s \leq t\}$.

(a) Suppose $F \in \mathbb{D}_{1,2}$, $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ is C^1 function. Put $G = \Psi(F)$. Then $G \in \mathbb{D}_{1,2}$ and

$$D_t G = (\partial \Psi)(F)(D_t F), 0 \leq t \leq T. \tag{3.1}$$

This is a so-called "chain rule" for Malliavin derivatives.

(b) Suppose $\{X_t\}$ is a \mathcal{G}_t^W -adapted process. Put

$$F = \int_0^T X_s ds$$

If $F \in \mathbb{D}_{1,2}$ then for each $0 \leq t \leq T$ we have

$$D_t F = \int_t^T D_t X_s ds$$

(c) Malliavin derivative of Itô integral: suppose $\{X_t\}$ is a \mathcal{G}_t^W -adapted process. Put

$$F = \int_0^T X_s dW_s$$

If $F \in \mathbb{D}_{1,2}$ then for each $0 \leq t \leq T$ we have

$$D_t F = \int_t^T (D_t X_s) dW_s + X_t$$

(d) Interchange of Expectation and Malliavin derivative operators: if $F \in \mathbb{D}_{1,2}$, and G is a Borel subset of $[0, T]$, then $E[F|\mathcal{F}_G] \in \mathbb{D}_{1,2}$ and

$$D_t(E[F|\mathcal{F}_G]) = E[D_t F|\mathcal{F}_G] \cdot \mathbb{I}_G(t), \tag{3.2}$$

where \mathbb{I} is the indicator function: $\mathbb{I}(t) = 1$ if $t \in G$, $\mathbb{I}(t) = 0$ if $t \notin G$.

Malliavin calculus is extensively used to solve a variety of financial problems. Using the celebrated integration by part formula one can obtain a method for numerical computation of price sensitivities (Greeks), see [F1]. Imkeller [I1] used Malliavin calculus to study additional utility of the insider traders. A series of studies ([T1], [CMZ], [B1]) have been done to apply

Malliavin calculus to hedging.

3.2 Clark-Ocone formula

This section presents the Clark-Ocone formula which gives an explicit expression of the density in the martingale representation theorem in terms of Malliavin derivatives. From martingale representation theorem (e.g., Karatzas and Shreve (1991), see Appendix A) we know that for any Brownian Functional $F(W)$ there exists a unique \mathcal{F}_t -progressively measurable process ψ such that

$$F(W) = E[F(W)] + \int_0^T \psi_s dW_s \quad (3.3)$$

Theorem 1. *Clark-Ocone formula: For any Brownian functional (3.3), if $F \in \mathbb{D}_{1,2}$ we have that*

$$\psi(t) = E \left[DF(W; (t, T]) \middle| \mathcal{F}_t \right] \quad (3.4)$$

where DF is Malliavin derivative.

Proof. We refer the reader to the appendix E in [KS] for a rigorous proof. □

Chapter 4

Main Result

In this chapter we find an explicit formula for the optimal portfolio process $\pi(t)$ in the case where all coefficients are deterministic. We will obtain the expression for the portfolio using the Clark-Ocone formula and elements of Malliavin calculus.

The portfolio represents an investor's strategy that evolves over time. The problem to be solved is to express quantitatively this strategy at time t given some known information. The investor's current wealth at time t is such a known process. Thus we should be able to construct a portfolio process as a "feedback" function of the current wealth. Basically, if the investor follows optimal strategy, his wealth process will follow an *optimal wealth process*. Further in this chapter we state some important theorems that give expressions for the optimal wealth process. Finally, we will find the optimal portfolio as a feedback from the optimal wealth process.

4.1 Optimal Wealth Process

We start the section with the following useful theorem.

Theorem 2. *Given a starting wealth x and $\mathcal{F}(T)$ -measurable, nonnegative random variable β such that*

$$E\left[H_0(T)\beta\right] = x$$

there exists a portfolio process $\pi(\cdot)$ such that it is admissible at x and $\beta = X^{x,\pi}(T)$.

Proof. See Karatzas et. al [KLS], Chapter 3, Theorem 3.5 for the proof. □

Now we define a starting wealth expressed as:

$$\mathcal{X}(\infty) \triangleq E[H_0(T)\bar{x}],$$

where \bar{x} is defined in (2.11), $H_0(T)$ is from (2.14). Clearly, if $x < \mathcal{X}(\infty)$ then from budget constraint $E[X^{x,\pi}(T)H_0(T)] \leq x < \mathcal{X}(\infty) = E[H_0(T)\bar{x}] \Rightarrow P\{X^{x,\pi}(T) \leq \bar{x}\} > 0$ and $U(X^{x,\pi}(T)) = -\infty$. If $x = \mathcal{X}(\infty)$ then from theorem 2 there exists portfolio process $\bar{\pi}$ such that $X^{\mathcal{X}(\infty),\bar{\pi}} = \bar{x}$. Thus

$$V(x) = \begin{cases} U(\bar{x}), & x = \mathcal{X}(\infty), \\ -\infty, & x < \mathcal{X}(\infty) \end{cases}$$

Hence $\mathcal{X}(\infty)$ is a minimal starting wealth required to avoid expected terminal utility of $-\infty$, forcing the constraint:

$$X^{x,\pi}(T) \geq \bar{x}. \tag{4.1}$$

In addition, condition (2.16) implies that $\mathcal{X}(\infty)$ is finite.

Now we determine the expression for the optimal terminal wealth that can be achieved from starting wealth x . First we define a process $\mathcal{X}_w(y)$:

$$\mathcal{X}_w(y) \triangleq E \left[H_0(T) I(y H_0(T)) \right], \quad 0 < y < \infty.$$

Lemma 3. $\mathcal{X}_w(y)$ has a strictly decreasing inverse function

$$\mathcal{Y}_w : (\mathcal{X}_w(\infty), \infty) \rightarrow (0, r), \tag{4.2}$$

where

$$r = \sup\{y > 0; \mathcal{X}_w(y) > \mathcal{X}_w(\infty)\} > 0.$$

Thus $\mathcal{X}_w(\mathcal{Y}_w(x)) = x, \quad \forall x \in (\mathcal{X}_w(\infty), \infty)$.

Proof. See Karatzas et. al [KLS], Chapter 3, Lemma 6.2 for the proof. □

Theorem 4. A unique value ζ for the optimal terminal wealth is achieved:

$$\zeta = I(\mathcal{Y}_w(x) H_0(T)), \tag{4.3}$$

where $\mathcal{Y}_w(x), I(\cdot), H_0(T)$ are defined in (4.2), (2.12) and (2.14) respectively.

Proof. See Karatzas et. al [KLS], Chapter 3, Theorem 6.3 for the proof. □

Now we are able to find the expression for the optimal wealth process.

Theorem 5. *The optimal wealth process $X(t)$ satisfies*

$$X(t) = \frac{1}{H_0(t)} E \left[H_0(T) \zeta \middle| \mathcal{F}(t) \right], \quad (4.4)$$

The optimal portfolio π is given by

$$\sigma'(t)\pi(t) = \frac{\psi}{H_0(t)} + X(t)\theta(t), \quad (4.5)$$

in which ψ is the integrand in the stochastic integral representation

$$M(t) = x + \int_0^t \psi'(u) dW(u)$$

of the martingale

$$M(t) = E \left[H_0(T) \zeta \middle| \mathcal{F}(t) \right], \quad (4.6)$$

Proof. From Theorem 2 and Theorem 4 the result immediately follows. □

4.2 Unconstrained Portfolio Optimization under Deterministic Coefficients.

Theorem 6. *Consider the case when we have non-random coefficients $r(t), \sigma(t)$ and $\delta(t)$; the market is complete and the investor is small in the sense that his trades do not affect the market. Then the the optimal portfolio $\pi(t)$ which solves the optimization problem (2.20)*

can be expressed as a feedback function of the current wealth process $X(t)$:

$$\pi'(t)\sigma(t) = -\theta'(t)\frac{\mathcal{Y}(t, X(t))}{\mathcal{Y}_2(t, X(t))}, \quad (4.7)$$

where $\mathcal{Y}(\cdot, \cdot)$ is some strictly decreasing $C^{1,2}$ and \mathcal{Y}_2 is a partial derivative with respect to the second parameter. The value of $\mathcal{Y}(\cdot, \cdot)$ will be given in the following Lemma 8.

Proof. The idea is to find $\psi(t)$ explicitly using the Clark-Ocone formula (C.6) and Malliavin calculus. First compute Malliavin derivative $D_t M(t)$:

$$\begin{aligned} D_t M(t) &= D_t \left(E \left[H_0(T) \zeta(T) \middle| \mathcal{F}(t) \right] \right) \quad (\text{from (4.6)}) \\ &= D_t \left(E \left[H_0(T) I(\mathcal{Y}_w(x) H_0(T)) \middle| \mathcal{F}(t) \right] \right) \quad (\text{from (4.3)}) \\ &= E \left[D_t \left(H_0(T) I(\mathcal{Y}_w(x) H_0(T)) \right) \middle| \mathcal{F}(t) \right] \mathbb{I}_{[0,t]}(t) \quad (\text{by [?], Prop. 5.6}) \\ &= E \left[D_t \left(H_0(T) I(\mathcal{Y}_w(x) H_0(T)) \right) \middle| \mathcal{F}(t) \right] \end{aligned} \quad (4.8)$$

By Clark formula and \mathcal{F}_t -measurability of $D_t M(t)$:

$$\psi(t) = E \left[D_t M(t) \middle| \mathcal{F}(t) \right] = D_t M(t), \quad (4.9)$$

Compute Malliavin derivative inside (4.8). By chain rule

$$D_t \left(H_0(T) I(\mathcal{Y}_w(x) H_0(T)) \right) = \left[I(\mathcal{Y}_w(x) H_0(T)) + H_0(T) \mathcal{Y}_w(x) \partial I(\mathcal{Y}_w(x) H_0(T)) \right] D_t H_0(T) \quad (4.10)$$

Compute $D_t H_0(T)$:

$$D_t H_0(T) = \frac{1}{S_0(T)} D_t Z(T) \quad (4.11)$$

By chain rule:

$$\begin{aligned} D_t Z(T) &= D_t \exp \left\{ - \int_0^T \theta'(s) dW(s) - \frac{1}{2} \int_0^T \|\theta(s)\|^2 ds \right\} \\ &= \exp \left\{ - \int_0^T \theta'(s) dW(s) - \frac{1}{2} \int_0^T \|\theta(s)\|^2 ds \right\} D_t \left\{ - \int_0^T \theta'(s) dW(s) \right\} \\ &= - \exp \left\{ - \int_0^T \theta'(s) dW(s) - \frac{1}{2} \int_0^T \|\theta(s)\|^2 ds \right\} \theta(t) \\ &= -\theta(t) Z(T) \end{aligned} \quad (4.12)$$

From (2.14), (4.11), (4.12):

$$D_t H_0(T) = -\theta(t) H_0(T). \quad (4.13)$$

Thus from (4.8), (4.9), (4.10), (4.13)

$$\begin{aligned} \psi(t) &= E \left[- \left\{ I(\mathcal{Y}_w(x) H_0(T)) + H_0(T) \mathcal{Y}_w(x) \partial I(\mathcal{Y}_w(x) H_0(T)) \right\} \theta(t) H_0(T) \middle| \mathcal{F}(t) \right] \\ &= -H_0(t) \theta(t) X(t) - E \left[\left\{ H_0(T) \mathcal{Y}_w(x) \partial I(\mathcal{Y}_w(x) H_0(T)) \right\} \theta(t) H_0(T) \middle| \mathcal{F}(t) \right], \end{aligned} \quad (4.14)$$

where (4.14) comes from (4.4).

Now we use the following Basic Fact 1 to rewrite the expectation part of (4.14).

Basic Fact 7. *Suppose that:*

(1) (Ω, \mathcal{F}, P) is a probability space,

(2) $\mathcal{G} \subset \mathcal{F}$ is a sub- σ algebra,

(3) $\rho: \Omega \rightarrow \mathbb{R}$, $\eta: \Omega \rightarrow \mathbb{R}$ are random variables such that $\rho \perp \mathcal{G}$ and η is \mathcal{G} -measurable.

(4) $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is Borel measurable such that $E[|f(\rho, \eta)|] < \infty$.

Then

$$E[f(\rho, \eta)|\mathcal{G}] = Q(\eta) \quad p\text{- a.s.}$$

where

$$Q(y) = E[f(\rho, y)] \quad \text{for all } y \in \mathbb{R}.$$

Proof. We start with the simple case when $f(\rho, \eta) = \mathbb{I}_{B_1 \times B_2}(\rho, \eta)$ is an indicator function.

Then we verify the statement

$$E[\mathbb{I}_{B_1 \times B_2}(\rho, \eta)|\mathcal{G}] = E[\mathbb{I}_{B_1 \times B_2}(\rho, \eta)|\eta = y] = E[\mathbb{I}_{B_1 \times B_2}(\rho, y)] \quad (4.15)$$

Indeed,

$$\int_{\omega: \eta \in A} \mathbb{I}_{B_1 \times B_2}(\rho, \eta) P(d\omega) = \int_{y \in A} E[\mathbb{I}_{B_1 \times B_2}(\rho, y) P_\eta(dy)] \quad (4.16)$$

Lefthand side of the equation is $P\{\rho \in B_1, \eta \in A \cap B_2\}$ and the right hand side is $P(\rho \in B_1) \times P(\eta \in A \cup B_2)$. The equality follows from the independence of ρ and η . The extension to the general case can be obtained from the monotone class theorem (see Shiryaev [Sh1]) \square

From (2.14) we have:

$$H_0(T) = H_0(t) \exp \left\{ - \int_t^T r(u) du - \int_t^T \theta'(u) dW(u) - \frac{1}{2} \int_t^T \|\theta(u)\|^2 du \right\} \quad (4.17)$$

for $0 \leq t \leq T$ i.e.

$$H_0(T) = H_0(t) \rho(t) \quad (4.18)$$

where

$$\rho(t) = \exp \left\{ - \int_t^T r(u) du - \int_t^T \theta'(u) dW(u) - \frac{1}{2} \int_t^T \|\theta(u)\|^2 du \right\} \quad (4.19)$$

Then from the right hand side of (4.14):

$$\begin{aligned} & E \left[\left\{ H_0(T) \mathcal{Y}_w(x) \partial I(\mathcal{Y}_w(x) H_0(T)) \right\} \theta(t) H_0(T) \middle| \mathcal{F}(t) \right] \\ &= E \left[H_0^2(T) \mathcal{Y}_w(x) \partial I(\mathcal{Y}_w(x) H_0(T)) \middle| \mathcal{F}(t) \right] \theta(t) \\ &\stackrel{\text{by(4.18)}}{=} E \left[H_0^2(t) \rho^2(t) \mathcal{Y}_w(x) \partial I(\mathcal{Y}_w(x) H_0(t) \rho(t)) \middle| \mathcal{F}(t) \right] \theta(t) \\ &= H_0^2(t) \mathcal{Y}_w(x) E \left[\rho^2(t) \partial I(\mathcal{Y}_w(x) H_0(t) \rho(t)) \middle| \mathcal{F}(t) \right] \theta(t) \quad (4.20) \\ &\quad \left(\text{since } H_0(t), \mathcal{Y}_w(x) \in \mathcal{F}_t \right) \end{aligned}$$

Now from (4.19)

$$\rho \perp \mathcal{F}_t, \quad H_0(t) \in \mathcal{F}_t \quad (4.21)$$

Thus from (4.21) and Basic Fact 1:

$$E\left[\rho^2(t)\partial I(\mathcal{Y}_w(x)H_0(t)\rho(t))|\mathcal{F}_t\right] = Q(t, H_0(t)), \quad (4.22)$$

where

$$Q(t, y) = E\left[\rho^2(t)\partial I(\mathcal{Y}_w(x)\rho(t)y)\right] \quad (4.23)$$

From (4.20), (4.22):

$$E\left[\left\{H_0(T)\mathcal{Y}_w(x)\partial I(\mathcal{Y}_w(x)H_0(T))\right\}\theta(t)H_0(T)\middle|\mathcal{F}(t)\right] = H_0^2(t)\mathcal{Y}_w(x)Q(t, H_0(t))\theta(t) \quad (4.24)$$

Now put

$$\kappa(t, y) = E[\rho(t)I(y\rho(t))] \quad (4.25)$$

Taking derivative with respect to y gives

$$\partial_y\kappa(t, y) = E[\rho^2(t)\partial I(y\rho(t))] \quad (4.26)$$

Now from (4.23) and (4.26)

$$Q(t, y) = \partial_y\kappa(t, y\mathcal{Y}_w(x)) \quad (4.27)$$

Now combine (4.22) and (4.27):

$$E\left[\rho^2(t)\partial I(\mathcal{Y}_w(x)H_0(t)\rho(t))|\mathcal{F}_t\right] = \partial_y\kappa(t, \mathcal{Y}_w(x)H_0(t)), \quad (4.28)$$

From (4.20) and (4.28)

$$\begin{aligned}
& E\left[\left\{H_0(T)\mathcal{Y}_w(x)\partial I(\mathcal{Y}_w(x)H_0(T))\right\}\theta(t)H_0(T)\middle|\mathcal{F}(t)\right] \\
&= \mathcal{Y}_w(x)H_0^2(t)(\partial_y\kappa)(t,\mathcal{Y}_w(x)H_0(t))\theta(t) \\
&= H_0(t)\theta(t)\left[\mathcal{Y}_w(x)H_0(t)(\partial_y\kappa)(t,\mathcal{Y}_w(x)H_0(t))\right]
\end{aligned} \tag{4.29}$$

Now we evaluate the expression $\left[\mathcal{Y}_w(x)H_0(t)(\partial_y\kappa)(t,\mathcal{Y}_w(x)H_0(t))\right]$ of (4.29).

Lemma 8. *For each $t \in [0, T)$, $\kappa(t, \cdot)$ is strictly monotone and hence has a strictly monotone inverse function $\mathcal{Y}(t, \cdot)$, i.e.*

$$\mathcal{Y}(t, \kappa(t, y)) = y, \quad \forall y > 0, t \in [0, T]$$

Proof. Follows from the definition of $\kappa(t, y)$:

$$\kappa(t, y) = E[\rho(t)I(y\rho(t))]$$

Since $I(\cdot)$ is a strictly decreasing (2.12) function of x , $\kappa(t, y)$ is a strictly decreasing function of y . □

From lemma 8 we have:

$$(\partial_2\mathcal{Y})(t, \kappa(t, y)) \cdot (\partial_2\kappa)(t, y) = 1, \quad \text{i.e. } (\partial_2\kappa)(t, y) = \frac{1}{(\partial_2\mathcal{Y})(t, \kappa(t, y))} \tag{4.30}$$

From (4.30):

$$\mathcal{Y}_w(x)H_0(t)(\partial_y\kappa)(t, \mathcal{Y}_w(x)H_0(t)) = \frac{\mathcal{Y}_w(x)H_0(t)}{(\partial_2\mathcal{Y})(t, \kappa(t, \mathcal{Y}_w(x)H_0(t)))} \quad (4.31)$$

From (4.4)

$$\begin{aligned} X(t) &= \frac{1}{H_0(t)} E \left[H_0(T) I(\mathcal{Y}_w(x)H_0(T)) | \mathcal{F}_t \right] \\ &= \frac{1}{H_0(t)} E \left[H_0(t) \rho(t) I(\mathcal{Y}_w(x)H_0(t)\rho(t)) | \mathcal{F}_t \right] \\ &= E \left[\rho(t) I(\mathcal{Y}_w(x)H_0(t)\rho(t)) | \mathcal{F}_t \right] \\ &= \kappa(t, \mathcal{Y}_w(x)H_0(t)), \end{aligned} \quad (4.32)$$

where the last equality comes from the Basic Fact 1 and (4.25), (4.21). Using (4.31) and (4.32):

$$\mathcal{Y}_w(x)H_0(t)(\partial_y\kappa)(t, \mathcal{Y}_w(x)H_0(t)) = \frac{\mathcal{Y}_w(x)H_0(t)}{(\partial_2\mathcal{Y})(t, X(t))} \quad (4.33)$$

Moreover for the numerator of (4.33):

$$\mathcal{Y}_w(x)H_0(t) = \mathcal{Y}(t, \kappa(t, \mathcal{Y}_wH_0(t))) = \mathcal{Y}(t, X(t)) \quad (4.34)$$

Combine (4.33) and (4.34):

$$\mathcal{Y}_w(x)H_0(t)(\partial_y\kappa)(t, \mathcal{Y}_w(x)H_0(t)) = \frac{\mathcal{Y}(t, X(t))}{(\partial_2\mathcal{Y})(t, X(t))} \quad (4.35)$$

and from (4.35),(4.29)

$$E \left[\left\{ H_0(T) \mathcal{Y}_w(x) \partial I(\mathcal{Y}_w(x) H_0(T)) \right\} \theta(t) H_0(T) \middle| \mathcal{F}(t) \right] = H_0(t) \theta(t) \frac{\mathcal{Y}(t, X(t))}{(\partial_2 \mathcal{Y})(t, X(t))} \quad (4.36)$$

Thus

$$\psi(t) = -H_0(t) \theta(t) X(t) - H_0(t) \theta(t) \frac{\mathcal{Y}(t, X(t))}{\mathcal{Y}_2(t, X(t))} \quad (4.37)$$

Plug $\psi(t)$ from (4.37) into (4.5):

$$\sigma'(t) \pi(t) = \frac{-H_0(t) \theta(t) X(t) - H_0(t) \theta(t) \frac{\mathcal{Y}(t, X(t))}{\mathcal{Y}_2(t, X(t))}}{H_0(t)} + X(t) \theta(t)$$

justifying the result. □

Theorem 1 leads to Merton's mutual fund theorem. It states that it is possible to construct a synthetic stock such that the investor can only trade this stock and risk-less bonds. In other words there exists a mutual fund such that every individual, regardless of the preference structure (utility function), is indifferent between investing in the mutual fund or directly purchasing the individual assets.

The same argument holds when we add consumption to the model. If we are given an optimization problem to find an optimal consumption-portfolio pair $(c(t), \pi(t))$:

$$V_2(x) = \sum_{(c, \pi) \in \mathcal{A}(x)} E \left[\int_0^T U_1(t, c(t)) dt + U_2(X^{x, c, \pi}(T)) \right],$$

where $U_1(\cdot)$ and $U_2(\cdot)$ are utility functions for consumption and terminal wealth respectively.

A general expression for the optimal wealth process can be given

$$X_2(t) = \frac{1}{H_0(t)} E \left[\int_t^T H_0(u) c(u) du + H_0(T) \zeta | \mathcal{F}(t) \right]$$

where optimal terminal wealth and optimal consumption process are:

$$c(t) = I_1(t, \mathcal{Y}(x) H_0(t))$$

$$\zeta(x) = I_2(\mathcal{Y}(x) H_0(T)),$$

and I_1 and I_2 are defined as in (2.12).

We still can compute the portfolio process π from

$$\sigma'(t) \pi_2(t) = \frac{\psi_2(t)}{H_0(t)} + X_2(t) \theta(t)$$

by finding the integrand ψ_2 in the representation of the martingale M_2 :

$$M_2(t) = E \left[\int_0^T H_0(u) c(u) du + H_0(T) \zeta | \mathcal{F}(t) \right]$$

Chapter 5

Conclusions and Future Work

An interesting opportunity for future work is to extend the calculation by adapting more complex models of price process, such as *fractional Brownian Motion*(fBm) driven models.

The important assumption of the Merton's model is that the price processes are driven by geometric Brownian motion. The consequence of this fact is that returns are log-normally distributed. Empirical studies of historical price dynamics show that in fact bigger price movements occur with higher probability than that of a log-normal distribution, i.e returns exhibit higher kurtosis or "fatter tails".

B. Mandelbrot was the first who extensively researched this issue. The extensive reliance on the normal distribution for much of the body of modern finance and investment theory is also a serious flaw of any other related models (Black-Scholes option model developed by Myron Scholes and Fischer Black, and the Capital Asset Pricing Model developed by William Sharpe). (Ideas of Mandelbrot and his argument that the Gaussian models for financial risk used by economists should be discarded can be found in his recent book *The Misbehavior of*

Markets [MH]).

The alternative to Brownian motion as a price driving process is fractional Brownian motion (fBm). fBm is a Gaussian process with correlation function of the following form:

$$E[B_H(t)B_H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \quad (5.1)$$

where H is the Hurst parameter, a factor of self-similarity:

$$[X(ct)] \approx [c^H X(t)] \quad (5.2)$$

Analogously to Brownian motion model, the SDE for the price process will take the following form:

$$dS(t) = S(t) \left[b(t)dt + \sigma(t)dB_H(t) \right], \quad (5.3)$$

where B_H is a fBm with Hurst parameter H . Empirical studies gave the consistent results, with Hurst parameter to be in the range $(\frac{1}{2}, 1)$ for different stocks.

Since fBm process is not a semimartingale a special integration framework was introduced by Duncan et. al [D1], where Malliavin calculus and the Wick-Ito product was used.

To date, optimal portfolio is determined for power and logarithmic utilities ([HOS], [VZS]). The question is, whether it is possible to find the general expression for the portfolio for *any* utility function U .

Appendix A

Martingale representation theorem

Theorem 9. *Let $W(t)$ be a Brownian motion on a standard filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and let \mathcal{G}_t be the augmentation of the filtration generated by W . If X is a square integrable G_∞ -measurable random variable, then there exists a G_t -adapted process ψ such that*

$$X = E[X] + \int_0^\infty \psi(t) dW(t) \tag{A.1}$$

Proof. See Øksendal ([Ø2]) for the proof. □

Appendix B

Wealth Process Equation

We consider the price processes for N risky assets and one non-risky asset (bond):

$$\begin{aligned}dS_0(t) &= S_0(t)r(t)dt, \\dS_n(t) &= S_n(t)\left[b_n(t)dt + \sum_{d=1}^D \sigma_{nd}(t)dW^{(d)}(t)\right].\end{aligned}$$

Since each stock may yield some dividend payment, we define *yield per share* process for each asset by adding a dividend payment for price process:

$$\begin{aligned}dY_n(t) &= S_n(t)\left[b_n(t)dt + \sum_{d=1}^D \sigma_{nd}(t)dW^{(d)}(t) + \delta_n(t)dt\right], \\Y_0(t) &= S_0(t)\end{aligned}\tag{B.1}$$

Consider $\eta_n(t)$ is number of bought shares of the asset n at time t , the following holds:

$$\pi_n(t) = \eta_n(t)S_n(t) \quad (\text{B.2})$$

If we take $G(t)$ as a total gain process from trading, we have

$$\begin{aligned} G(t_{m+1}) - G(t_m) &= \sum_{n=0}^N \eta_n(t_m)(Y_n(t_{m+1}) - Y_n(t_m)) \\ G(0) &= 0. \end{aligned}$$

If we trade continuously

$$dG(t) = \sum_{n=0}^N \eta_n(t)dY_n(t)$$

Combination of (B.1) and (B.2) gives

$$dG(t) = \left(\pi_0(t) + \pi'(t)\bar{1}\right)r(t)dt + \pi'(t)\left(b(t) + \delta(t) - r(t)\bar{1}\right)dt + \pi'(t)\sigma(t)dW(t) \quad (\text{B.3})$$

A wealth process $X(\cdot)$ consists of initial wealth x and the time continuous gains process $G(t)$:

$$X(t) = x + G(t)$$

If the whole amount of money is involved in trading,

$$X(t) = \pi_0(t) + \pi(t)\bar{1}$$

Rewrite the expression in differential form and expand:

$$dX(t) = \frac{X(t)}{S_0(t)} dS_0(t) + \pi'(t)(b(t) + \delta(t) - r(t)\bar{1})dt + \pi'\sigma(t)dW(t) \quad (\text{B.4})$$

Solution of (B.4) yields

$$\frac{X^{x,\pi}(t)}{S_0(t)} = x + \int_0^t \frac{1}{S_0(u)} \pi'(u) \sigma(u) (dW + \theta(u)du) \quad (\text{B.5})$$

Note, that the formula does not depend on the risk-less invested amount $\pi_0(t)$. The value of $\pi_0(t)$ can be obtained from $X(\cdot)$ and $\pi(\cdot)$.

Appendix C

Continuous version of the doubling strategy

For simplicity we assume a financial market with one stock ($r(t) = 0, b = 0, \sigma = 1, \delta = 0$).

The gains process (B.4) will take the following form

$$G(t) = \int_0^t \pi(s) dW(s) \tag{C.1}$$

We need to construct such a portfolio $\pi(\cdot)$ that $G(T) = \alpha$ - some positive constant, almost surely.

We consider a stochastic integral

$$M(t) = \int_0^t \frac{1}{T-u} dW(u), \tag{C.2}$$

which is a martingale on $[0, T)$. It is possible to show that

$$\overline{\lim}_{t \rightarrow T} M(t) = \infty, \quad \underline{\lim}_{t \rightarrow T} M(t) = -\infty. \quad (\text{C.3})$$

Thus for

$$\tau_\alpha \triangleq \inf\{t \in [0, T); \quad M(t) = \alpha\} \wedge T, \quad (\text{C.4})$$

$0 < \tau_\alpha < T$ a.s. If we define $\pi(t) = (\frac{1}{T-t} \mathbb{I}_{\{t \leq \tau_\alpha\}})$ and $\pi(t)_0 = M(t \wedge \tau_\alpha) - \pi(t)$ for all $t \in [0, T]$

then the gains process

$$G(t) = \int_0^{t \wedge \tau_\alpha} \frac{1}{T-u} dW(u) = M(t \wedge \tau_\alpha), \quad \forall t \in [0, T]. \quad (\text{C.5})$$

From (C.5) we obtain that $G(T) = \alpha$ almost surely.

$$\psi(t) = E \left[D_t F(W) \Big| \mathcal{F}_t \right] \quad (\text{C.6})$$

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