A Fast Sphere Decoding Algorithm for Rank Deficient MIMO Systems

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The problem of rank deficient multiple input multiple out (MIMO) systems arises when the number of transmit antennas M is greater than number of receive antennas N or when the channel gains are strongly correlated.

Most of the optimal algorithms that deal with uncoded rank-deficient (under-determined) V-BLAST MIMO systems (e.g. Damen ,Meraim and Belfiore) suffer from high complexity and large processing time. Recently, some new optimal algorithms were introduced with low complexity for small constellations like 4-QAM yet they still suffer from very high complexity and processing time with large constellations like the 16 QAM.

In order to reduce the complexity and the processing time of the decoding algorithms, some suboptimal algorithms were introduced. One of the most efficient suboptimal solutions for this problem is based on the Minimum mean square error decision-feedback equalizer (MMSE-DFE) followed by either sphere decoder or fano decoder. The performance of these algorithms is shown to be a fraction of dB from the maximum likelihood decoders while offering outstanding reduction in complexity compared to the most efficient ML algorithms (e.g. Cui and Tellambura algorithm).

These suboptimal algorithms employ a two stage approach. In the first stage, the channel is pre-processed to transform the original decoding problem into a simpler form which facilitates the search decoding step. The second stage is basically the application of the sphere decoding search algorithm in the case of MMSE-DFE sphere decoding step or Fano decoder in the case of MMSE-DFE Fano decoder.

In this study, various algorithms which deal with rank deficient MIMO systems such as Damen,Meraim and Belfiore algorithm ,Dayal and Varansi algorithm, and Cui and Tellambura algorithm are discussed and compared. Moreover, the MMSE-DFE sphere decoding algorithm and MMSE-DFE fano decoding algorithm are applied on uncoded V-BLAST rank deficient MIMO systems. The optimality of MMSE-DFE sphere decoding algorithm is analyzed in the case of V-BLAST 4-QAM. Furthermore, Simulation results show that when these algorithms are extended to cover large constellations, their performance falls within a fraction of dB behind the ML while achieving a significant decrease in the processing time by more than an order of magnitude when compared to the least complex optimal algorithms.

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Dedication

To my parents and the memory of my beloved uncle Mohamed shamy

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Chapter 1

Introduction

Wireless communication systems with multiple input multiple output (MIMO) antennas provide very high data rate with low error probability. A lot of decoders were developed for these systems. Maximum-likelihood (ML) decoding of the MIMO systems is known to be NP-hard. However, some reduced complexity decoding techniques like V-BLAST[20] were developed but their performance is much worse than ML detection. New set of decoders called sphere decoders[14][9] were proposed to achieve ML performance where the complexity is observed to be polynomial in the number of unknowns [18] for systems designed for Rayleigh fading channels with additive white Gaussian noise.

The under-determined or rank deficient MIMO systems are a special case where the number of transmit antennas M is greater than the number of receive antennas N or when the channel gains are strongly correlated. The rule of the decoder in these systems becomes crucial lately because of it wide applications in modern mobile communication systems. The standard sphere decoders fail to decode the rank deficient MIMO systems as the rank of the channel matrix is N and the number of unknowns is higher than the number of equations. A generalized sphere decoding algorithm for this case was developed by Damen et al. in [9] [8]. The complexity of this algorithms is exponential in $(M-N)$, regardless of the SNR. Dayal and Varansi [11] introduced an algorithm that attains less complexity than that of Damen et al. Cui and Tellambura presented in [7] a special algorithm for constant modulus constellations that reduces the complexity for decoding these constellations when compared with the previous algorithms, however it still suffers from high complexity with non-constant modulus constellations. Murugan et al. in [27] suggested the idea of using suboptimal sphere decoder that is based on minimum mean square error decision feedback equalizers (MMSE-DFE). They applied it on decoders of coded rank deficient MIMO systems and proved through numerical simulations that the performance of this decoder is just a fraction of dB behind the ML. also, they introduced MMSE-DFE Fano decoder with over-determined systems that proved very good performance with outstanding reduction in complexity with this type of systems. This motivated us to investigate the performance and the processing time of MMSE-DFE sphere decoder algorithm versus the optimal decoders. Finally, we introduce the application of the MMSE-DFE Fano decoder for uncoded rank deficient MIMO systems to analyze its performance and complexity when applied to these systems.

1.0.1 Notations

Bold symbols denote matrices or column vectors. $(.)^T$ and $(.)^\dagger$ denote transpose and conjugate transpose. $\mathbb{Z}$ is the ring of integers. $\mathbb{R}$ is the field of real numbers. I is the identity matrix. For matrix M, the element (i,j) is denoted by $a(i,j)$. For vector $\boldsymbol{m}$ the entry is represented as m_i.

1.0.2 Thesis outline

A literature review of ML algorithms for normal and rank deficient MIMO systems is presented in Chapter 2. In Chapter 3 details of the MMSE-DFE sphere decoding suboptimal algorithm is outlined. Furthermore, the simulation and numerical results are presented and analyzed in Chapter 4. Chapter 5 gives the conclusions of this study and suggestions for future work.

Chapter 2

Literature Review

In this chapter, the necessary background to understand the sphere decoding algorithms and the pre-processing methods for the normal MIMO systems is covered first. Then, for the under-determined systems, the details of some optimal algorithms are introduced with comparison between them.

2.1 Search stage (Sphere decoding)

The multi-input multi-output (MIMO) system can be modeled with this linear real model

$$\boldsymbol{y} = H\boldsymbol{x} + \boldsymbol{z} \tag{2.1}$$

Where $\boldsymbol{x} \in \Re^m, \boldsymbol{y}, \boldsymbol{z} \in \Re^n$ denote the channel input, channel output, and the noise(whose components are chosen from independent and identically distributed zero-mean Gaussian random distribution), and $H \in \Re^{n\times m}$ is the channel matrix which is often assumed to be full column rank.

In order to get the unknown information symbols $\boldsymbol{x}$, the receiver has to solve the following minimization problem

$$\min_{\boldsymbol{x}\in\mathbb{Z}^m} \parallel \boldsymbol{y} - H\boldsymbol{x} \parallel^2 \tag{2.2}$$

This minimization problem for arbitrary $\boldsymbol{y}$ and H is known to be NP-hard. However, it has been proved that for certain ranges of parameters like SNR, m, and n the complexity can be reduced [26] [33]. In lattice theory, H is called the generator matrix of the m-dimensional lattice $\{\Lambda(H) = H\boldsymbol{x} : x \in \mathbb{Z}^m\}$, and the problem of 2.2 is actually to find a vector $\hat{\boldsymbol{c}} \in \Lambda(H)$ such that

$$\parallel \boldsymbol{x} - \hat{\boldsymbol{c}} \parallel \leq \parallel \boldsymbol{x} - \boldsymbol{c} \parallel, \qquad \forall \boldsymbol{c} \in \Lambda(H) \tag{2.3}$$

The problem of searching for this vector is called closest lattice point search problem (CLPS) [6],[1]. In communication it is called decoding. One method to solve CLPS consists of two stages

1. pre-processing stage

2. search stage

The first step is always done to convert 2.2 into a new form which makes the decoding process easier and more efficient[27], some of the pre-processing methods will be discussed in this chapter later. The goal of the search stage is mainly to find the optimal point $\hat{\boldsymbol{x}}$ in the corresponding hyper-ellipsoid. The receiver that uses the former method in decoding is referred to as sphere decoder.

Sphere decoding algorithms [1][9] are based originally on Phost enumeration strategy[14][28] and on Schnorr-Euchner enumeration method [30].

2.1.1 Pohst enumeration method [9]

Viterbo and Biglieri [13] were the first to apply this method in communication, then Viterbo and Boutros applied it for ML detection of multidimensional constellations transmitted over single antenna fading channels[32].

Pohst enumeration can be summarized as follows. Let C_0 to be the squared radius of an n-dimensional sphere $S(y, \sqrt{C_0})$ centered at $\boldsymbol{y}$. The goal of this method is to produce a list of all points inside the lattice that belong to this sphere. The reduction pre-processing step is done here by applying the QR decomposition to the channel matrix H,

$$H = \begin{bmatrix} Q & \grave{Q} \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix} \tag{2.4}$$

Then try to solve

$$|\grave{\boldsymbol{y}} - Rx|^2 \leq \grave{C}_0 \tag{2.5}$$

Where $\grave{\boldsymbol{y}} = Q^T \boldsymbol{y}$ and $\grave{C}_0 = C_0 - |(\grave{Q})^T \boldsymbol{y}|^2$. The solution of this inequality produces the range of values at each level starting from the level m. More explicitly, let $\boldsymbol{x}_l^m = (x_l, x_{l+1}, \ldots, x_m)^T$ denote the last $m - l + 1$ components of the vector

$\mathbf{x}$. For fixed $\boldsymbol{x}_{i+1}^m$ the component x_i can take any values in the range of integers $I_i(\boldsymbol{x}_{i+1}^m) = [A_i(\boldsymbol{x}_{i+1}^m), B_i(\boldsymbol{x}_{i+1}^m)]$.where

$$A_i(\boldsymbol{x}_{i+1}^m) = \left\lceil \frac{1}{r_{i,i}} \left(\grave{y}_i - \sum_{j=i+1}^{m} r_{i,j} x_j - \sqrt{\grave{C}_0 - \sum_{j=i+1}^{m} |\grave{y}_j - \sum_{l=j}^{m} r_{j,l} x_l|^2} \right) \right\rceil \tag{2.6}$$

$$B_i(\boldsymbol{x}_{i+1}^m) = \left\lfloor \frac{1}{r_{i,i}} \left(\grave{y}_i - \sum_{j=i+1}^{m} r_{i,j} x_j + \sqrt{\grave{C}_0 - \sum_{j=i+1}^{m} |\grave{y}_j - \sum_{l=j}^{m} r_{j,l} x_l|^2} \right) \right\rfloor \tag{2.7}$$

At each level, these intervals are computed depending on the lower levels, the $\boldsymbol{x}$ vectors within these intervals are declared as possible solutions. After getting all the nominated vectors, the one with least square Euclidean distance from the received vectors $\boldsymbol{y}$ is chosen as the estimated vector. If at any level no points were found inside the sphere, the sphere is declared empty, then the radius C_0 is increased, and the search is restarted.

2.1.2 Schnorr-Euchner enumeration[9]

It was first introduced by Agrell[1]. The numerical results proved that it is more efficient than Pohst enumeration [30]. In Schnorr-Euchner enumeration the spanning of the intervals is not natural spanning like Pohst enumeration that starts from the first value in the range and then go to the next one etc. till the last value in the interval. In Schnorr-Euchner, the spanning starts from the central value which is

$$S_i(\boldsymbol{x}_{i+1}^{m+1}) = \left\lfloor \frac{1}{r_{i,i}} (\grave{y}_i - \sum_{j=i+1}^{m} r_{i,j} x_j) \right\rceil \tag{2.8}$$

Then go as zig zag from this central value so that at this level
$x_i \in \{S_i(\boldsymbol{x}_{i+1}^{m+1}), S_i(\boldsymbol{x}_{i+1}^{m+1}) + 1, S_i(\boldsymbol{x}_{i+1}^{m+1}) - 1, S_i(\boldsymbol{x}i + 1^{m+1}) + 2, \ldots\} \cap I_i(\boldsymbol{x}_{i+1})$
If

$$\grave{y}_i - \sum_{j=i+1}^{m} r_{i,j}x_j - r_{i,i}S_i(\boldsymbol{x}_{i+1}^{m}) \geq 0$$

Or the direction of the sequence has to be

$x_i \in \{S_i(\boldsymbol{x}_{i+1}^{m+1}), S_i(\boldsymbol{x}_{i+1}^{m+1}) - 1, S_i(\boldsymbol{x}_{i+1}^{m+1}) + 1, S_i(\boldsymbol{x}_{i+1}^{m+1}) - 2, \ldots\} \cap I_i(\boldsymbol{x}_{i+1})$

If

$$\grave{y}_i - \sum_{j=i+1}^{m} r_{i,j}x_j - r_{i,i}S_i(\boldsymbol{x}_{i+1}^{m}) < 0$$

A modification of Schnorr-Euchner was introduced in [9] (Algorithm 2), which took into account the finite signal set boundary. This algorithm will be explained in details in next chapter as it is a basic part in the suboptimal algorithm that is used in this study.

2.2 Pre-processing stage

The performance of the sphere decoding search stage depends on the ordering over which the elements of $\boldsymbol{x}$ is to be decoded and the initial search radius C_0. The pre-processing stage was introduced in order to overcome these two problems:

1. The case of rank deficient MIMO system where the rank of the channel matrix H is less than m and the number of unknowns will be larger than the number equations, or even when the rank of $H = m$ but it is ill-conditioned. In theses two cases the values of the diagonal elements of the matrix R resulting from the QR decomposition of the channel matrix H will be near or equal to zero making the decoding using the ordinary sphere decoder so complex because the decoding depends on the diagonal elements of R, also when the channel matrix is ill-conditioned this will result in a very skewed lattice for which some of the points $\{H\boldsymbol{x} : x \in \mathbb{Z}^m\}$ are very close and difficult to be decoded.

2. Since the sphere decoder at each level depends on the lower levels, then intuitively the better the quality of the first point found the better the error

rate will become, so the order of the columns of R is an important factor to enhance the performance of the decoder, the sparsity of the matrix R can lead to a reduction on the complexity of the decoders. One can argue that the sparser the matrix R the faster the closest point is found. [9].

Left pre-processing was developed in order to overcome the first problem by modifying the channel matrix and the noise vectors so that the new CLPS problem that results is not ML or in another word it is suboptimal. Examples of left pre-processing include the use of the zero-forcing decision feedback equalizer(ZFE-DFE)[15] or application of the minimum mean square error decision feedback equalizer(MMSE-DFE). The sparsity of the matrix R can be increased using what is called right pre-processing and can be done in many ways like column permutation ,lattice reduction or combination of them [27].

Although the use of some of the previous pre-processing techniques may lead to sub optimality of the new minimization problem. It turns out that the the performance is so close to the ML, while the complexity is highly reduced, the information theoretic aspects of this arguments can be found in [18][17].

2.2.1 V-BLAST ZF-DFE pre-processing and ordering

ZF-DFE pre-processing [15] is done by just applying QR decomposition on the channel matrix so that 2.2 becomes

$$\min_{\boldsymbol{x}\in\mathbb{Z}^m} \parallel \grave{\boldsymbol{y}} - R\boldsymbol{x} \parallel^2 \tag{2.9}$$

Such that$\grave{\boldsymbol{y}} = \boldsymbol{y}Q^T$and Q is the feed- forward matrix of the ZF-DFE.

ZF-DFE ordering is used to produce a permutation matrix Π such that the QR decomposition of this new permutated channel matrix $H\Pi$ has the property that $\min_{1\leq i\leq m} r_{i,i}$ is maximized overall column computations. ZF-DFE ordering algorithm proceeds as follows :

1. for $k = m, m-1, m-1, \ldots, \ldots, 2, 1$

2. Let A_k be the set of columns still not chosen by the algorithm and initiated by the values from $1 : m$ and we consider $j \in A_k$

3. For each j we compute

$$\boldsymbol{h}_j^T[I - H_{k,j}(H_{k,j}^T H_{k,g})^{-1} H_{k,j}^T]\boldsymbol{h}_j \tag{2.10}$$

   where

   $H_{k,j}$ is the $n \times (k-1)$ matrix formed by the columns $\boldsymbol{h}_i$ of the channel matrix where $i \in A_k - \{j\}$, and I is the identity matrix.

4. At each iteration we compute the columns of the permutation matrix such that

$$\pi(k) = \arg\max_{j \in A_k} \boldsymbol{h}_j^T[I - H_{k,j}(H_{k,j}^T H_{k,g})^{-1} H_{k,j}^T]h_j \tag{2.11}$$

5. We remove from A_k the resulting $\pi(k)$ value and then repeats

6. The column ordering is given by $\pi(m), \pi(m-1), \ldots, \pi(1)$

In communication the first point obtained by Schnorr-Euchner enumeration is called ZF-DFE point [15] and the complexity of the sphere decoding algorithm depends on how much this solution is close to the ML solution. It has been proved that ZF-DFE ordering enhance the quality of this point , hence the complexity of the algorithm will be also improved.

2.2.2 MMSE-DFE pre-processing and ordering

The left pre-processing using MMSE-DFE can be done by making the QR decomposition of the augmented channel matrix

$$\tilde{H} = \begin{bmatrix} H \\ I \end{bmatrix} = \tilde{Q}R_1 \tag{2.12}$$

Where $\tilde{Q} \in \Re^{(n+m)\times m}$ has orthonrmal columns and R_1 is upper triangular matrix. The MMSE-DFE forward filter Q_1 is obtained by taking the upper $n \times m$

part of $\tilde{Q}$, where the MMSE-DFE backward filter will be R_1 [5]. Hence the original CLPS problem 2.2 can be transformed into

$$\min_{\boldsymbol{x} \in \mathbb{Z}^m} \parallel \grave{\boldsymbol{y}} - R_1 \boldsymbol{x} \parallel^2 \tag{2.13}$$

Where $\grave{\boldsymbol{y}} = Q_1^T \boldsymbol{y}$. There is an approximation done over here as the columns of Q_1 are generally not orthonormal so this new problem will not be equivalent to 2.2, instead it will be suboptimal. The noise $\boldsymbol{w}$ in this new problem consists of one Gaussian term $Q^T \boldsymbol{z}$ and none Gaussian signal dependent term $(Q_1^T H - R_1)x$. However, this noise component is still white [17] so the minimum distance rule in 2.13 is expected to be slightly suboptimal. The augmented channel matrix will have the rank of m and it is well conditioned, so the diagonal values of the matrix R becomes larger and the complexity of the decoding algorithm will become less.

The MMSE-DFE V-BLAST ordering is used to replace the decoding of all the transmitted signals at once by decoding the strongest signal first, then cancels its effect from the received signal and then proceeds to decode the strongest of the remaining and so on. This technique is referred to as Nulling and cancellation. Let The MMSE filter G to be

$$G = (\alpha I + H^H H)^{-1} H^H \tag{2.14}$$

The least mean square estimate of the channel will be

$$\hat{\boldsymbol{x}} = G\boldsymbol{x} \tag{2.15}$$

The covariance matrix of the estimation error $\boldsymbol{x} - \hat{\boldsymbol{x}}$ will be

$$Q = (\alpha I + H^H H)^{-1} \tag{2.16}$$

It is obvious that the strongest signal will be the one with smallest error covariance which is the one with smallest Q_{ii}. V-BLAST ordering algorithm based on this MMSE-DFE criteria [4] will be explained in details in chapter 3.

2.3 Underdetermined MIMO systems decoders

The standard sphere decoding explained previously in this chapter works for over-determined MIMO systems but when the system is rank deficient it fails. In this section several algorithms for dealing with the rank deficient MIMO systems will be reviewed.

The system that will be of consideration will be

$$\boldsymbol{y} = H\boldsymbol{x} + \boldsymbol{z} \tag{2.17}$$

Where $\boldsymbol{x} \in \Re^m$,$\boldsymbol{y}$,$\boldsymbol{z} \in \Re^n$ denote the channel input, channel output, and the noise(whose components are chosen from independent and identically distributed zero-mean Gaussian random distribution)signal, and $H \in \Re^{n\times m}$ is the channel matrix with $m > n$.

2.3.1 Damen,Abed-Meraim and Belfiore(DAB) algorithm [8]

The algorithm proceeds as follows:

1. In this algorithm the pre-processing of the channel matrix is done by using general Cholesky factorization(GCF) such that $H^T H = R^T R$, with R an upper triangular matrix.

2. The vector $\boldsymbol{x}$ and the matrix R are partitioned as

$$R = \begin{bmatrix} R_1^{n\times n} & R_2^{n\times(m-n)} \\ 0 & 0 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_1^n \\ \boldsymbol{x}_2^{m-n} \end{bmatrix} \tag{2.18}$$

3. The value of $\boldsymbol{x}_2$ is fixed, then the conventional sphere decoder is applied just to decode $\boldsymbol{x}_1$, the problem now is converted from

$$\min_{\boldsymbol{x}\in\mathbb{Z}^m} \parallel \grave{\boldsymbol{y}} - R\boldsymbol{x} \parallel^2 \tag{2.19}$$

Where $\grave{\boldsymbol{y}} = RH^T(HH^T)^{-1}\boldsymbol{y}$. Into

$$\min_{\boldsymbol{x}_1 \in \mathbb{Z}^m} \parallel (\grave{\boldsymbol{y}} - R_2 x_2) - R_1 \boldsymbol{x}_1 \parallel^2 \tag{2.20}$$

4. If the algorithm succeeds in finding valid $\boldsymbol{x}_1$ the value of the search radius is updated, a new value of $\boldsymbol{x}_2$ is fixed and step 3 is repeated. Otherwise, the new value of $\boldsymbol{x}_2$ is tested with the original search radius of the sphere decoder.

5. Step 4 is repeated for an exhaustive search of $\boldsymbol{x}_2$ until all it is 2^{m-n} values are tested with recording the value of the metric2.20 each time.

6. The value of $\boldsymbol{x}_2$ and $\boldsymbol{x}_1$ that achieves the minimum value of the metric 2.20 is declared as the decoded symbol

The disadvantage of this algorithm comes from its high complexity (exponential in $m - n$) independent of signal to noise ratio.

2.3.2 Damen, El. Gamal, and Caire (DEC) [9]

This algorithm is an extension to DAB algorithm. Since the number of flops of QR decomposition is less than that of Cholesky decomposition, this algorithm uses QR decomposition of the channel matrix which is consedered as the ZF-DFE instead of the Cholesky decomposition as a pre-processing stage, this results in reduction of the complexity of the pre-processing stage. The algorithm steps are:

1. The channel matrix is partitioned as $H = [H_1, H_2]$ where H_1 is $n \times n$ matrix

2. QR decomposition is applied on H_1, the minimization problem becomes

$$\parallel (\grave{\boldsymbol{y}} - R_2 x_2) - R_1 \boldsymbol{x}_1 \parallel^2 \tag{2.21}$$

Where $\grave{\boldsymbol{y}} = Q^T\boldsymbol{y}$ and $R_2 = Q^T H_2$

3. The algorithm proceeds as DAB algorithm from step 3

2.3.3 Dayal and Varanasi (DV) algorithm[11]

At the beginning Cholesky decomposition of [8] is applied on H^TH to produce upper triangular $m \times m$ matrix R such that

$$R = \begin{pmatrix} r_{1,1} & r_{2,2} & \dots & r_{1,n} & \dots & r_{1,m} \\ 0 & r_{2,2} & \dots & r_{2,n} & \dots & r_{1,m} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & r_{n,n} & \dots & r_{n,m} \\ 0 & 0 & 0 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & \dots & 0 \end{pmatrix} = \begin{bmatrix} R_1 & R_2 \\ 0 & \boldsymbol{r}^T \end{bmatrix} \tag{2.22}$$

Where $R_1 \in \Re^{n-1 \times n-1}$,$R_2 \in \Re^{n-1 \times m-n+1}$,and $\boldsymbol{r}^T \in \Re^{m-n+1}$. The minimization problem now is

$$\min_{\boldsymbol{x} \in \mathbb{Z}^m} \| R(\grave{\boldsymbol{y}} - \boldsymbol{x}) \|^2 \tag{2.23}$$

Where $\grave{\boldsymbol{y}} = H^T(HH^T)^{-1}\boldsymbol{y}$. From sphere decoding principles [13][32]

$$\| R(\grave{\boldsymbol{y}} - \boldsymbol{x}) \|^2 \leq C_0 \tag{2.24}$$

But

$$\| R(\grave{\boldsymbol{y}} - \boldsymbol{x}) \|^2 = \| \begin{bmatrix} R_1 & R_2 \end{bmatrix} (\grave{\boldsymbol{y}} - \boldsymbol{x}) \|^2 + \| \boldsymbol{r}^T(\grave{\boldsymbol{y}}_g - \boldsymbol{x}_g) \|^2 \tag{2.25}$$

Where $\grave{\boldsymbol{y}}_g$ and $\boldsymbol{x}_g$ refers to the last $m - n + 1$ elements. So 2.24 can be applied on each element of 2.25

$$\| \boldsymbol{r}^T(\grave{\boldsymbol{y}}_g - \boldsymbol{x}_g) \|^2 \leq C_0 \tag{2.26}$$

$$-\sqrt{C_0} \leq \sum_{j=n}^{m} r_{n,j}(\grave{y}_j - x_j) \leq \sqrt{C_0} \tag{2.27}$$

The main idea of DV algorithm is to search for the values of $\boldsymbol{x}_g$ that satisfy 2.27. This algorithm replaces the exhaustive search for all the 2^{m-n+1} possible values of $\boldsymbol{x}_g$ in DAB by introducing a way to select among them those which satisfy 2.27 in order to reduce the time taken by this search. Then after finding them the problem will be the same as in DAB where the normal sphere decoding applied to get the remaining elements of $\boldsymbol{x}$. For simplicity ,we assume x_j takes only the values $\{\pm 1\}$.

Define the bijective transformation for $\boldsymbol{x}_g$

$$b_j = \frac{1+x_j}{2} = \begin{cases} 0, & x_j = -1 \\ 1, & x_j = 1 \end{cases} \qquad n \leq j \leq m \tag{2.28}$$

Let $a_j = 2r_{n,j}, n \leq j \leq m$ so 2.27 becomes

$$\grave{y}_n + \sum_{j=n}^{m} r_{n,j} - \sqrt{C_0} \leq \sum_{j=n}^{m} a_j b_j \leq \grave{y}_n + \sum_{j=n}^{m} r_{n,j} + \sqrt{C_0} \tag{2.29}$$

Define the vectors $\boldsymbol{a} = [a_n, a_{n+1}, \ldots, a_m]^T$, $\boldsymbol{b} = [b_n, b_{n+1}, \ldots, b_m]^T$, and $k = m - n + 1$. Then 2.29 can be rewritten as

$$LB \leq \boldsymbol{a}^T \boldsymbol{b} \leq UB \tag{2.30}$$

where
$LB = \grave{y}_n + \sum_{j=n}^{m} r_{n,j} - \sqrt{C_0}$,and
$UB = \grave{y}_n + \sum_{j=n}^{m} r_{n,j} + \sqrt{C_0}$
Let $\mathbb{S}$ be the set of all possible 2^k binary sequences for $\boldsymbol{b}$. Let $2^{\mathbb{S}}$ denotes the set of all subsets $\mathbb{S}$. For every set $\beta \in 2^{\mathbb{S}}$, there is a lower bound $lb(\beta)$ and an upper bound $ub(\beta)$ such that

$$lb(\beta) \leq \boldsymbol{a}^T \boldsymbol{b} \leq ub(\beta) \qquad , \forall \boldsymbol{b} \in \beta \tag{2.31}$$

If $lb(\beta) > UB$, no binary sequence on that set β is valid. Moreover, if there is another set $\grave{\beta}$ such the $lb(\grave{\beta}) \geq lb(\beta)$, no elements in that set $lb(\grave{\beta})$ will satisfy 2.30 as well. On the other hand, if $ub(\beta) < LB$ or $ub(\grave{\beta}) \leq ub(\beta)$, then no elements in either β or $\grave{\beta}$ are valid.

Dayal and Varansi in [11] introduced a disjoint partition of $\mathbb{S}$ as follows: let $w(.)$ denote the Hamming weight (i.e number of ones in binary sequence)

$$\mathbb{S}_{0,1} = \{\boldsymbol{0}_n\}, \quad \mathbb{S}_{1,1} = \{\boldsymbol{b} \in \mathbb{S} | w(\boldsymbol{b}) = 1\} \tag{2.32}$$

$$\mathbb{S}_{d,l} = \{\boldsymbol{b} \in \mathbb{S} | w(\boldsymbol{b}) = d, \boldsymbol{b}_l = 1 \text{and} \boldsymbol{b}_j = 0, 1 \leq j \leq l-1\}, \tag{2.33}$$

For $2 \leq d \leq k, 1 \leq l \leq k-d+1$.

Let $\grave{\boldsymbol{a}}$ be a sorted version from $\boldsymbol{a}$ in increasing order so that

$$\grave{\boldsymbol{a}}_1 \leq \grave{\boldsymbol{a}}_2 \leq \ldots \leq \grave{\boldsymbol{a}}_k \tag{2.34}$$

define σ to be the permutation that brings $\grave{\boldsymbol{a}}$ back to$\boldsymbol{a}$ then $\sigma(\grave{\boldsymbol{a}}) = \boldsymbol{a}$.Furthermore ,the following optimal upper (ub) and lower(lb) bounds for the disjoint sets of $\mathbb{S}$ is defined

$$lb(\sigma(\mathbb{S}_{1,1})) = \grave{\boldsymbol{a}}_1, \qquad ub(\sigma(\mathbb{S}_{1,1})) = \grave{\boldsymbol{a}}_k \tag{2.35}$$

$$lb(\sigma(\mathbb{S}_{d,l})) = \sum_{j=l}^{d+l-1} \grave{\boldsymbol{a}}_j \tag{2.36}$$

$$ub(\sigma(\mathbb{S}_{d,l})) = \sum_{j=k-d+1}^{k} \grave{\boldsymbol{a}}_j \tag{2.37}$$

By taking this optimal upper and lower bound for the disjoint sets, these sets will have the following two important properties

$$lb(\sigma(\mathbb{S}_{d,l_1})) \leq lb(\sigma(\mathbb{S}_{d,l_2})), \quad \forall d \geq 2, l_2 \geq l_1 \tag{2.38}$$

$$ub(\sigma(\mathbb{S}_{d,l_1})) \geq ub(\sigma(\mathbb{S}_{d,l_2})), \quad \forall d \geq 2, l_2 \geq l_1 \tag{2.39}$$

Based on 2.38 and 2.39, the DV algorithm Proceed as follows:

1. For the head of each disjoint sets $(\sigma\mathbb{S}_{d,1})$, two conditioned are tested if

   $$lb(\sigma(\mathbb{S}_{d,1})) > UB \tag{2.40}$$

   and

   $$ub(\sigma(\mathbb{S}_{d,1})) < LB \tag{2.41}$$

   If any of them were satisfied, then $(\sigma\mathbb{S}_{d,1})$ and $\{\sigma(\mathbb{S}_{d,2}), \sigma(\mathbb{S}_{d,3}), \ldots, \sigma(\mathbb{S}_{d,k-d+1})\}$ will not be valid and they will be discarded from the search. Otherwise, if both of the inequalities were satisfied, then for each $\mathbf{b} \in \sigma(\mathbb{S}_{d,l})$ the algorithm has to check if 2.30 is valid.

2. If step 1 was able to get valid vectors $\boldsymbol{b}$, then the bijective transformation 2.28 has to be made to get $\grave{\boldsymbol{x}}_g$, which is the last $m - n + 1$ elements of the estimated vector $\grave{\boldsymbol{x}}$

3. Now, the sub problem can be solved using conventional sphere decoding like the case of DAB, from 2.22

$$\min_{\boldsymbol{x}\in\{\pm 1\}} \parallel (\boldsymbol{y}_1 - R_2\boldsymbol{x}_g) - R_1\boldsymbol{x}_1 \parallel^2 \tag{2.42}$$

Where $\boldsymbol{x}_1$ and $\boldsymbol{y}_1$ are the upper $n-1$ part from the received vector $\boldsymbol{y}$ and the transmitted signal $\boldsymbol{x}$.

In the case of nonbinary signals, let $q = 2^m$ for some positive integer m. The bigicitve transformation at 2.28 will be replaced by

$$b_j = \frac{q-1+x_j}{2}, \quad n \le j \le m \tag{2.43}$$

Such that $0 \le b_j \le q-1$. The binary representation $b_j = \sum_{i=0}^{v-1} 2^i b_{i,j}$ is considered where $b_{i,j} \in \{0,1\}$, then the value of b_j in 2.29 has to be replaced by this new value. Furthermore, these two vectors have to be defined
$\boldsymbol{a} = [a_n, 2a_n, \dots, 2^{v-1}a_n, \dots, a_m, 2a_m, \dots, 2^{v-1}a_m]$ and
$\boldsymbol{b} = [b_{0,n}, \dots, b_{v-1,n}, \dots, b_{0,m}, \dots, b_{v-1,m}]$

When n becomes large, this leads to increase in the cardinality of the subsets of $\mathbb{S}$, as a result the algorithm may not be able to discard many candidates of $\boldsymbol{x}_g$ because the upper and lower bound of the sets become very week. So by using the same idea of dividing a group of sequences into ordered subgroups, multi-depth DV algorithm can be obtained[11]. For example $\mathbb{S}_{d,l}$, where $d \ge 3$. The subsets $\mathbb{S}_{d,l,v}$ can be obtained as

$$\mathbb{S}_{d,l,v} = \{\boldsymbol{b} \in \mathbb{S}_{d,l} | \boldsymbol{b}_{l+j} = 0, 1 \le j \le v-1\}, \tag{2.44}$$

for $1 \le v \le k-l-d+2$, the upper and lower bounds can be defined as

$$lb(\sigma(\mathbb{S}_{d,l,v})) = \grave{\boldsymbol{a}}_l + \sum_{j=l+v}^{d+l+v-2} \grave{\boldsymbol{a}}_j \tag{2.45}$$

$$ub(\sigma(\mathbb{S}_{d,l,v})) = \grave{\boldsymbol{a}}_l + \sum_{j=k-d+2}^{k} \grave{\boldsymbol{a}}_j \tag{2.46}$$

Taking this ub and lb will produce sets with

$$lb(\sigma(\mathbb{S}_{d,l,1})) \leq lb(\sigma(\mathbb{S}_{d,l,2})), \leq \ldots \leq lb(\sigma(\mathbb{S}_{d,l,k-l-d+2})) \tag{2.47}$$

$$ub(\sigma(\mathbb{S}_{d,l,1})) \geq ub(\sigma(\mathbb{S}_{d,l,2})) \geq \ldots \geq ub(\sigma(\mathbb{S}_{d,l,k-l-d+2})) \tag{2.48}$$

All the elements of $\sigma(\mathbb{S}_{d,l,v})$ will be discarded only if

$$lb(\sigma(\mathbb{S}_{d,l,v})) > UB \tag{2.49}$$

or

$$ub(\sigma(\mathbb{S}_{d,l,v})) < LB \tag{2.50}$$

When $\mathbb{S}$ is partitioned into just $\mathbb{S}_{d,l}$, the DV algorithm is called of Depth 1 GSD, and when it involves the sets $\mathbb{S}_{d,l,v}$, it will be Depth 2 GSD. If it involves more partitions it will be of higher GSD Depth. The increase in the algorithm depth will decrease the max cardinality of the sets while at the same time increase the accuracy of the upper and lower bounds of the sets, hence the complexity of the sphere decoder algorithm will be reduced more than that of the DAB.

2.3.4 Yang ,Liu,and He (YLH) algorithm [35]

This algorithm was based on DV algorithm with some modifications in how to get the candidates for $\boldsymbol{x}_g$. Define the following

$$A^+ = \{j|r_{n,j} \geq 0, n \leq j \leq m\}, \quad A^- = \{j|r_{n,j} < 0, n \leq j \leq m\} \tag{2.51}$$

Assume this bijective transformation

$$b_j == \begin{cases} \frac{1+x_j}{2}, & j \in A^+ \\ \frac{1-x_j}{2}, & j \in A^- \end{cases} \tag{2.52}$$

If $a_j = 2|r_{n,j}|$ for $j = n, n+1, n+2, \ldots, m$ and $c_j = a_j b_j$,then 2.29 can be rewritten as

$$\grave{y}_n + \sum_{j=n}^{m} |r_{n,j}| - \sqrt{C_0} < \sum_{j=n}^{m} c_j < \grave{y}_n + \sum_{j=n}^{m} |r_{n,j}| + \sqrt{C_0} \tag{2.53}$$

If it is assumed that $a_j > 0$ and $x_j \in \{\pm 1\}$, so this will be valid

$$b_j = \begin{cases} 1, & \text{in this case} c_j > 0 \\ 0, & \text{in this case} c_j = 0 \end{cases} \tag{2.54}$$

For that jth element,

$$c_j \in \{0, a_j\} \cap \left[V - \sqrt{C_0} - \sum_{j=d+1}^{m} c_j - \sum_{j=n}^{d-1} a_j, V + \sqrt{C_0} - \sum_{j=d+1}^{m} c_j \right] \tag{2.55}$$

Where $V = y_n + \sum_{j=n}^{m} |r_{n,j}|$ and $d = m, m-1, \ldots, n$. As the intervals of $\boldsymbol{c} = [c_n, c_{n+1}, \ldots, c_m]$ is known, so conventional sphere decoding can be used to obtain $\boldsymbol{c}$. Then from 2.54 b_j can be obtained. The rest of the algorithm is like the corresponding part of the DV algorithm.

2.3.5 Cui and Tellambura (CT) algorithm [7]

This algorithm is used with constant modulus constellations, an example of this is any 2^q-ary phase shift keying (PSK) constellation set ζ_{2^q}. The minimization problem is defined as

$$\hat{\boldsymbol{x}} = \arg\min_{x \in \zeta^n} \parallel \boldsymbol{y} - H\boldsymbol{x} \parallel^2 + \alpha \boldsymbol{x}^\dagger \boldsymbol{x} \tag{2.56}$$

Define the positive definite matrix $G = H^\dagger H + \alpha I_n$ and $D^\dagger D$ to be its Cholesky decomposition, where D is an upper triangular matrix. So 2.56 will be changed to

$$\hat{\boldsymbol{x}} = \arg\min_{x \in \zeta^n} \parallel D(\rho - \boldsymbol{x}) \parallel^2 \tag{2.57}$$

Where $\rho = G^{-1} H^\dagger \boldsymbol{y}$. Now D becomes of rank n and its diagonal elements are higher than zero, so the problem becomes an over-determined rather than under-determined and the normal sphere decoding [9] can be applied.

Although this algorithm is used for constant modulus signals, it can be used for non constant modulus by using linear combinations of constant modulus constellations like the 4 QAM. In General

$$\boldsymbol{x} = \sum_{i=1}^{(k/2)-1} 2^i \boldsymbol{x}_i \tag{2.58}$$

Where $\boldsymbol{x}$ is the M-QAM ($M = 2^k$) vector to be transmitted and $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are chosen from 4QAM constellation. As an example is the 16 QAM transmitted vector $\boldsymbol{x}$, it can be expressed as

$$\boldsymbol{x} = \boldsymbol{x}_1 + 2\boldsymbol{x}_2 \tag{2.59}$$

The received vector in this case will be

$$\begin{aligned} \boldsymbol{y} &= \begin{bmatrix} H & 2H \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} + \boldsymbol{n} \\ &= \tilde{H}\tilde{\boldsymbol{x}} + \boldsymbol{n} \end{aligned} \tag{2.60}$$

So it can be seen that the MIMO (M, N) system with M-QAM non constant modulus constellation is equivalent to the increased dimensions $(k/2)M, N$ MIMO system of constant modulus constellation. So this algorithm increases the complexity for the case of non-constant modulus constellations if compared with other algorithms like DAB algorithm.

Chapter 3

The Algorithm

The main contribution of this work is the application of fast sub-optimal MMSE-DFE sphere decoder and MMSE-DFE Fano decoder on an uncoded V-BLAST rank deficient MIMO system. In this chapter, system model is given, outline of the steps of these algorithms is then introduced, and finally the details of each step are presented.

3.1 System model

The proposed algorithms deal with the case of under-determined MIMO systems that have M number of transmit antennas and N number of receive antennas. Perfect channel state information is assumed at the receiver. The input-output model of the system is described by

$$\boldsymbol{y}^c = \sqrt{\frac{\rho}{M}} H^c \boldsymbol{x}^c + \boldsymbol{w}^c \tag{3.1}$$

Where $\boldsymbol{x}^c$ is the input complex signal that has components chosen from unit energy Q^2-QAM, $H^c \in \mathbb{C}^{N\times M}$ is the complex channel matrix that contains independent and identically distributed (i.i.d) elements $h^c_{i,j} \sim N_c(0,1)$, the noise has i.i.d components $w^c_i \sim N_c(0,I)$, and ρ denotes the signal to noise ratio (SNR)observed per received antenna. The system can be expressed in real form by using vector and matrix transformation defined by

$$u^c \longrightarrow u = [\mathrm{Re}\{u^c\}^T, \quad \mathrm{Im}\{u^c\}^T]^T$$

$$H^c \longrightarrow H = \begin{bmatrix} \mathrm{Re}\{H^c\} & -\mathrm{Im}\{H^c\} \\ \mathrm{Im}\{H^c\} & \mathrm{Re}\{H^c\} \end{bmatrix}$$

The resulting MIMO real model will be

$$\boldsymbol{y} = \sqrt{\frac{\rho}{M}} H\boldsymbol{x} + \boldsymbol{w} \tag{3.2}$$

Where $\boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{x} \in \mathbb{R}^n$, $H \in \mathbb{R}^{n\times m}$, $\boldsymbol{w} \sim N_c(0,I)$, $m = 2M$, and $n = 2N$. For the sake of simplicity, each element of $\boldsymbol{x} \in C$, where C is pulse amplitude modulation

(PAM) signal set of size Q i.e

$$C = \{x = 2q - Q + 1, q \in \mathbb{Z}_Q\} \tag{3.3}$$

With $\mathbb{Z}_Q = \{0, 1, \ldots, Q-1\}$. The minimization Problem is described as

$$\hat{x} = \arg\min_{\boldsymbol{x} \in \mathbb{Z}_Q^m} |\boldsymbol{y} - H\boldsymbol{x}|^2 \tag{3.4}$$

3.2 Steps of the algorithm

The algorithms proceed as follows:

- Pre-processing stage

  1. V-BLAST ordering algorithm [4]

  2. MMSE-DFE filtering of the received signal [27]

- In the case of MMSE-DFE sphere decoding algorithm, sphere decoding based on Schnorr-Euchner enumeration with a finite radius as in [9] is applied. For MMSE-DFE Fano decoder, the pre-processing stage is followed by Fano decoder.

3.2.1 V-BLAST ordering algorithm

In V-BLAST algorithm, the order of detecting the symbols is based on starting with the strongest signal which has the highest SNR or smallest diagonal entry of the error covariance matrix Q so that

$$\mathrm{p}_1 = \arg\min_k q_{k,k} \tag{3.5}$$

where $q_{k,k}$ denotes the diagonal elements of the matrix Q, and p_1 is the index of the first element to be decoded. Then, the effect of this symbol is cancelled from the received vector, and 3.5 is repeated for the new Q matrix. The details of the algorithm are in table 3.1.

Table 3.1: V-BLAST ordering algorithm

Initialization	$H^m = H = [\boldsymbol{h}^m_{:,1} \quad \boldsymbol{h}^m_{:,2} \quad \ldots \boldsymbol{h}^m_{:,m}]$	
	$Q^m = [(H^m)^\dagger H^m + \alpha I^{m\times m}]^{-1} = [\boldsymbol{q}^m_{:,1} \quad \boldsymbol{q}^m_{:,2} \quad \boldsymbol{q}^m_{:,m}]$	
	$\boldsymbol{f} = [1 \quad 2 \quad \ldots M]^T$	
	$\tau_1 = \arg\min_k q_{m,kk} \quad \pi(1) = f_{\tau_1}$	
	Move the τ_1-th entry of vector $\boldsymbol{f}$ to the end	
Recursion	for $i = 1, 2, \ldots, m-1$	
	(a)	compute H^{m-i} by removing the τ_m-th column of the H^{m-i+1}
	(b)	$Q^{m-i} = [(H^{m-i})^\dagger H^{m-i} + \alpha I^{(m-i)\times(m-i)}]^{-1}$
	(c)	$\tau_{i+1} = \arg\min_k q_{m-i,kk}, \pi(i+1) = f_{\tau_{i+1}}$
	(d)	Move the τ_1-th entry of $\boldsymbol{f}$ to be behind the $(m-i)$-th entry
Output	The decoding order $\boldsymbol{f} = [\pi(m) \quad \pi(m-1) \quad \ldots \quad \pi(1)]^T$	

3.2.2 MMSE-DFE pre-processing

After having the proper order through which the receiver vector is to be decoded, the pre-processing step of the channel is done in order to regularize the channel and have it in better well-conditioned form if it is not [27]. This can be done by having this augmented channel matrix

$$\tilde{H} = \begin{bmatrix} H \\ \sqrt{\alpha} I \end{bmatrix} = \tilde{Q} R_1 \tag{3.6}$$

$\alpha = 1/SNR$ is used to regularize the channel when it is ill-conditioned as it will increase the eigen values for $\tilde{H}$ while its rank becomes m. Consequently, the under-determined system is transformed to over-determined.

The MMSE-DFE forward filter Q_1 is obtained by taking the upper $n \times m$ part of $\tilde{Q}$. The columns of Q_1 are not orthonormal, therefore the new problem is suboptimal. The MMSE-DFE backward filter is R_1 [5]. Hence the original CLPS problem 2.2 is transformed into

$$\min_{\boldsymbol{x} \in C} \parallel \grave{\boldsymbol{y}} - R_1 \boldsymbol{x} \parallel^2 \tag{3.7}$$

Where $\grave{\boldsymbol{y}} = Q_1^T \boldsymbol{y}$.

3.2.3 The sphere decoder [9]

. This algorithm is a modification of the Schnorr-Euchner enumeration method in order to take into account the finite signal set boundary. The description of the algorithm is shown in table 3.2.

Table 3.2: The sphere decoding (SD) algorithm

Step 1(Initialization)	Set $i = m, w_m, \xi_m = 0$, and $rad = C_0$(initial sphere radius)
Step 2(DFE on x_i)	Set $x_i = \lfloor (\grave{y}_i - \xi_i)/r_{i,i} \rceil$ and $\Delta_i = \text{sign}(r_{i,i}) \cdot \text{sign}(\grave{y}_i - \xi_i - r_{i,i})$
Step 3(Decoding Step)	
if	$rad < w_i + \lvert \grave{y}_i - \xi_i - r_{i,i} x_{i,i} \rvert^2$ then GoTo
	Step 4 (i.e we are outside the sphere)
Else	
If	$x_i \notin C$(i.e boundary range of values of x_i),GoTo Step 6
	(i.e inside the sphere, outside the signal set boundaries)
Else	(i.e inside the sphere and signal set boundaries)
If	$i > 1$ then { let $\xi_{i-1} = \sum_{j=i}^{m} r_{i-1,j} x_j$,
	$w_{i-1} = w_i + \lvert \grave{y}_i - \xi_i - r_{i,i} x_i \rvert^2, i = i-1$,GoTo Step 2
Else	GoTo Step 5
End	
End	
End	
Step 4	If $i = m$, terminate
	Else set $i = i + 1$, GoTo Step 6
Step 5	Valid point is found, let $rad = w_1 + \lvert \grave{y}_1 - \xi_1 - r_{1,1} x_1 \rvert^2$
	Save $\hat{\boldsymbol{x}} = \boldsymbol{x}$. Then $i = i + 1$, GoTo Step 6
Step 6	(Schnorr-Euchner enumeration of Level i)
	Let $x_i = x_i + \Delta_i, \Delta_i = -\Delta_i - \text{sign}(\Delta_i)$ GoTo Step 3

Where $r_{i,j}$ represents the components of R_1. The resulting estimated vector has to be multiplied by the permutation matrix Π in order to get the exact estimation.

3.2.4 Fano Decoder

Table 3.3: The Fano decoder algorithm

Step 1(Initialization)	Set $k \leftarrow 0, T \leftarrow 0, \boldsymbol{x} \leftarrow x_0$
Step 2(Look forward)	Set $\boldsymbol{x}_1^{k+1} \leftarrow (\boldsymbol{x}_1^k, x_{k+1})$, where x_{k+1} is the $(k+1)$th component of the best child node of $\boldsymbol{x}_1^k$
Step 3(Decoding Step)	
if	$f(\boldsymbol{x}_1^{k+1}) \leq T$
If	$k+1 = m$(Leaf node), then $\hat{\boldsymbol{x}} = \boldsymbol{x}_1^m$, Exit
Else	Move forward, $k \leftarrow k+1$
If	$f(\boldsymbol{x}_1^{k+1}) > T - \triangle$ (i.e visit for the first time)
While	$f(\boldsymbol{x}_1^k) \leq T - \triangle, T \leftarrow T - \triangle$
	(Tighten Threshold)
End	
	GoTo step 2
End	
End	
Else	
If	$(k = 0$ or $f(\boldsymbol{x}_1^{k-1}) > T)$,$T \leftarrow T + \triangle$
	cannot move back, relax the threshold
	GoTo step 2
Else	move back and look forward to the next best node
	$\boldsymbol{x}_1^k \leftarrow \{\boldsymbol{x}_1^{k-1}, x_k\}$, where x_k is
	the last component of the next best child node of $\boldsymbol{x}_1^{k-1}$
	$k \leftarrow k-1$
	GoTo step 3
End	
End	

The Fano decoder is one technique to search thorough the tree of the possible coded symbols. The tree is of depth m. The branches of the candidates are generated according to Shnorr-Euchner enumeration method. At each node, the algorithm has to check the cost of that node against a certain Threshold, if the cost is less than the Threshold, then it is valid forward move otherwise the algorithm looks backward and check again for valid backward move. With each valid forward move, the Threshold is updated. If the algorithm makes a backward move, the next best mode has to be checked for the next forward move. The details of the algorithm are given in the table 3.3.

The cost function $f(.)$ of each node $\boldsymbol{x}_1^k = (x_1, x_2, \ldots, x_k)$ at level k is associated with the squared distance $\sum_{i=1}^{k} = w_i(\boldsymbol{x}_1^i)$, where

$$w_i(\boldsymbol{x}_1^i) = \left| y_i - \sum_{j=1}^{i} r_{i,j} x_j \right|^2 \tag{3.8}$$

The cost function in this case will be

$$f(\boldsymbol{x}_1^k) = \sum_{j=1}^{k} w_j(x_1^j) - bk \tag{3.9}$$

Where $b \in \mathbb{R}^+$ is called the bias

Chapter 4

Simulation Results

This chapter includes the application of the MMSE-DFE SD algorithm and MMSE-DFE Fano decoding algorithm on the uncoded V-BLAST rank deficient MIMO systems with comparison of the performance and processing time of these algorithms versus that of the DEC decoding algorithm [9] and CT algorithm [7]. The simulations show the cases of uncoded rank deficient V-BLAST MIMO systems with 4-QAM, 16-QAM, and 64-QAM constellations. Moreover, the MSE-DFE SD algorithm is implemented on highly correlated channels in order to study the effect of changing the value of MMSE-DFE coefficient(α) on the performance.

4.1 Simulation configurations

All simulations deal with the transmission of multidimensional square QAM constellations over flat Rayleigh-channel. The channel matrix changes randomly over each frame iteration. Perfect channel state information at the receiver is assumed. The CPU time was taken as a measure of the processing time for each algorithm. The value of CT algorithm coefficient is taken as 1 while the coefficient of MMSE-DFE is $\sqrt{1/\mathrm{SNR}}$.

All simulations were done for at least 10 000 channel realizations. The algorithms were simulated on MATLAB 7 environment, the function of {**tic,toc**} is used to measure the CPU processing time of just the search stage without counting the pre-processing stage.

4.2 V-BLAST uncoded under-determined MIMO systems

The system model 3.2 of chapter 3 is used to conduct all the simulations. Fig. 4.1 compares the Performance of DEC decoding algorithm , and CT algorithm with the proposed algorithms MMSE-DFE SD algorithm, and the MMSE-DFE Fano algorithm when decoding 4×3 uncoded V-BLAST MIMO system with 16 QAM. It is seen that the suboptimal algorithm of MMSE-DFE SD and MMSE-DFE Fano is only 0.2 dB away from the two others optimal algorithms at high SNR like 25dB, while in Fig. 4.2 that compares the CPU time for the same scenario, the MMSE-DFE sphere decoding algorithm processing time is one degree of magnitude less than that of CT and DEC algorithm. It can be also seen that the MMSE-DFE Fano decoder is achieving outstanding reductions in processing time (degree of magnitudes) when compared with other decoders.

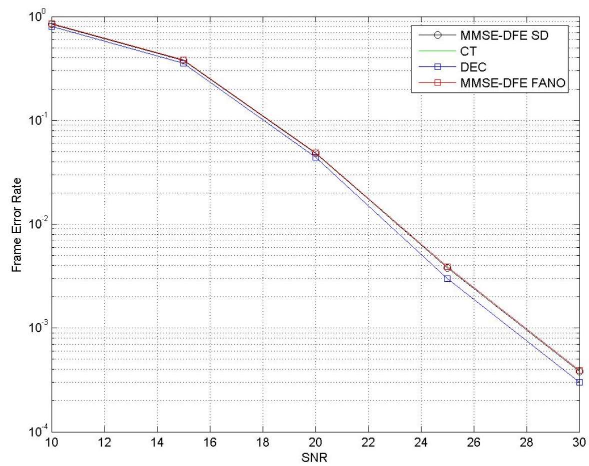

Figure 4.1: Performance of DEC, CT, MMSE-DFE SD, and MMSE-DFE Fano decoders of an uncoded 4×3 system with a 16-QAM constellation

These results match the intuition that the processing time of CT algorithm is high, as it has to increase the dimension of the MIMO system for non-constant modulus constellations like the 16-QAM [7].

Fig. 4.3 and Fig. 4.4 show the performance and processing time for 4×2 16-QAM MIMO system. It is observed that, as the difference between the number of transmit antennas and receive antennas increased, the processing time of the proposed decoders increased and at high SNR(between 30 and 35 dB)it approaches that of CT algorithm.

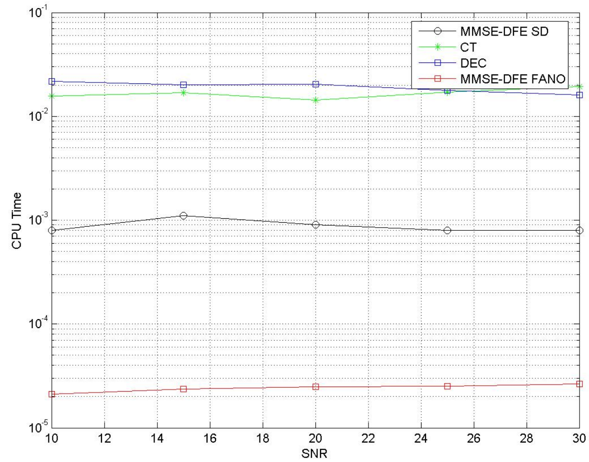

Figure 4.2: CPU time of DEC, CT, MMSE-DFE SD, and MMSE-DFE Fano decoders of an uncoded 4×3 system with a 16-QAM constellation

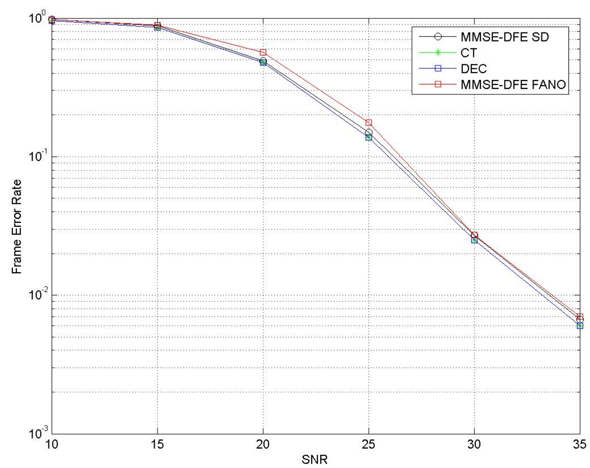

Figure 4.3: Performance of DEC, CT, MMSE-DFE SD, and MMSE-DFE Fano decoders of an uncoded 4×2 system with a 16-QAM constellation

The optimality of the MMSE-DFE sphere decoding algorithm with the case of 4-QAM while achieving outstanding processing time is demonstrated in Fig. 4.5 and Fig. 4.6, where the frame error rate and CPU time of CT, DEC, MMSE-DFE SD, and MMSE-DFE Fano decoders for 4×2 MIMO system are simulated.

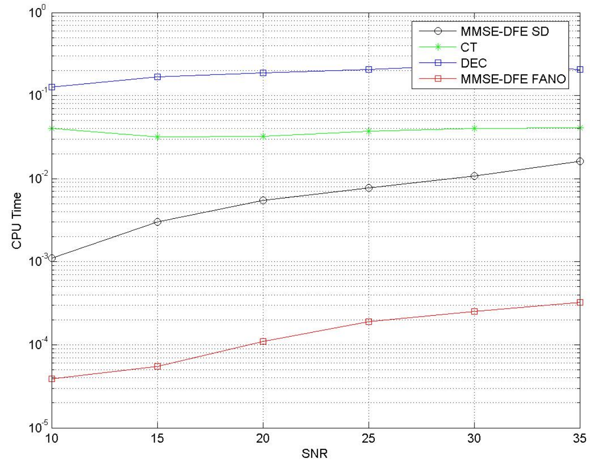

Figure 4.4: CPU time of DEC, CT, MMSE-DFE SD, and MMSE-DFE Fano decoders of an uncoded 4×2 system with a 16-QAM constellation

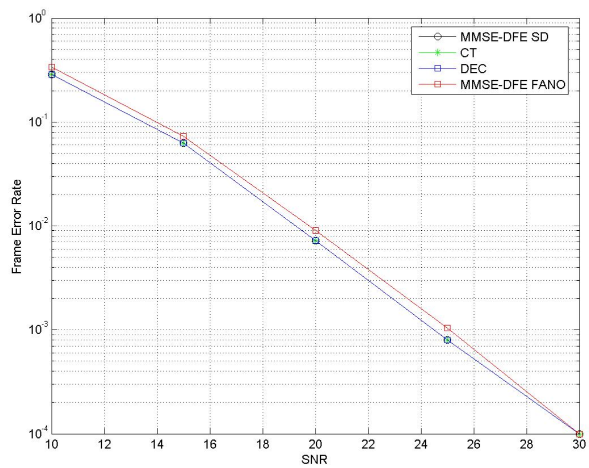

Figure 4.5: Performance of DEC, CT, MMSE-DFE SD, and MMSE-DFE Fano decoders of an uncoded 4×2 system with a 4-QAM constellation

The reduction in the processing time of the MMSE-DFE sphere decoding algorithm is at least 50% as compared to the CT algorithm that has the least processing time among all known optimal algorithms for constant modulus constellations like 4-QAM. The MMSE-DFE SD and MMSE-DFE Fano decoders performs better when the number of transmitters are increased like the case of 4×3 4-QAM shown in Fig. 4.7 and Fig. 4.8 where the optimality is still maintained.

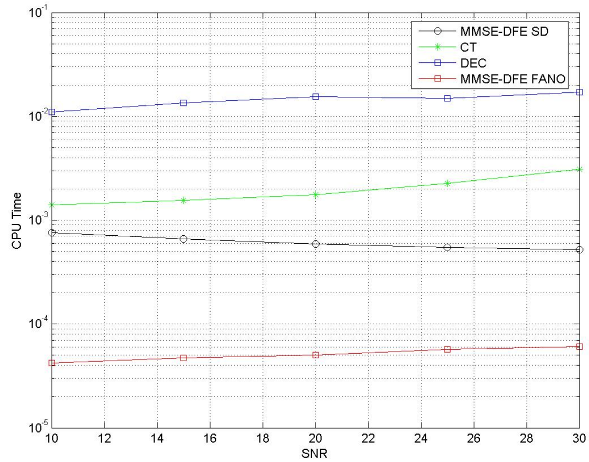

Figure 4.6: CPU time of DEC, CT, MMSE-DFE SD, and MMSE-DFE Fano decoders of an uncoded 4×2 system with a 4-QAM constellation

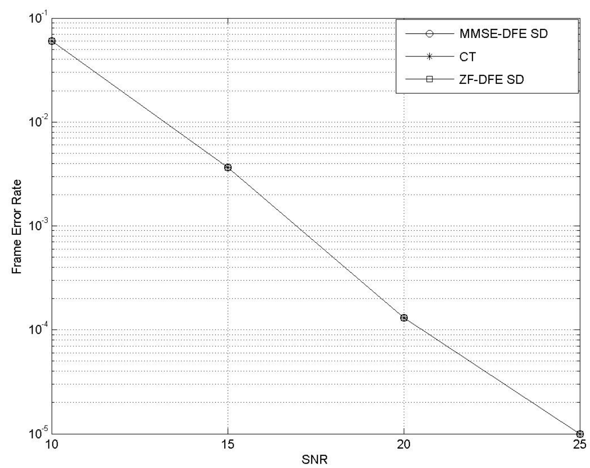

Figure 4.7: Performance of ZF-DFE SD , CT, and MMSE-DFE SD decoders of an uncoded 4×3 system with a 4-QAM constellation

Not only the performance of the introduced decoders is exceptional for small constellations, they also perform the same for large dimension constellations like 64-QAM. This is demonstrated in Fig. 4.9 and Fig. 4.10, where both new decoders are orders of magnitude lower than that of CT algorithm while the performance still fraction of dB away from the ML for 64-QAM 4×3 MIMO system.

The ratio between the processing time of the MMSE-DFE sphere decoding al-

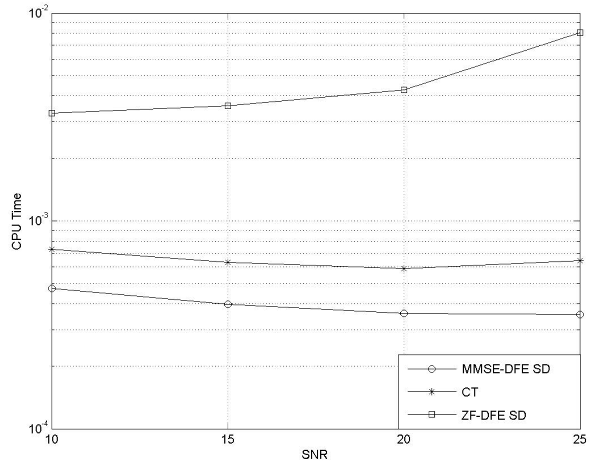

Figure 4.8: CPU time of ZF-DFE SD ,CT algorithm, and MMSE-DFE SD decoders of an uncoded 4×3 system with a 4-QAM constellation

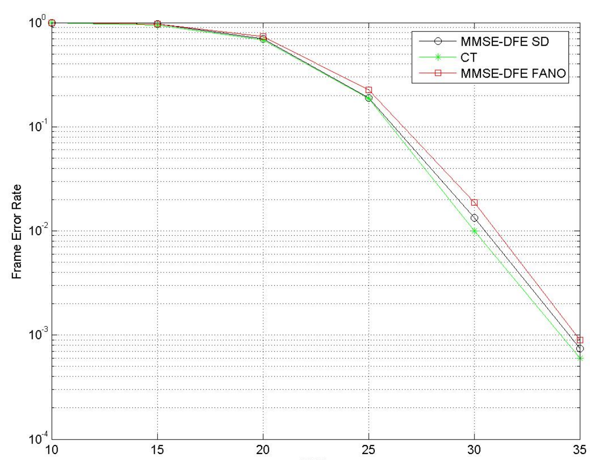

Figure 4.9: Performance of DEC, CT, MMSE-DFE SD, and MMSE-DFE Fano decoders of an uncoded 4×3 system with a 64-QAM constellation

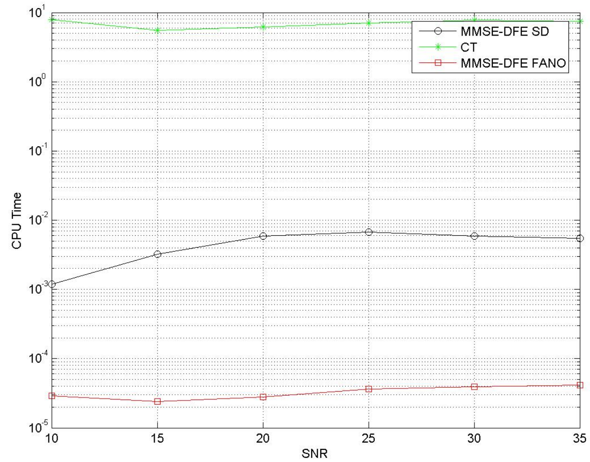

Figure 4.10: CPU time of DEC, CT, MMSE-DFE SD, and MMSE-DFE Fano decoders of an uncoded 4×3 system with a 64-QAM constellation

gorithm and that of the CT algorithm for 4-QAM with fixed number of transmit antennas $M = 10$, increasing number of receive antennas $N = 5, \ldots, M$, and SNR=20dB is illustrated in Fig. 4.11. The MMSE-DFE decoder processing time decrease with the increasing of the difference between the M and N.

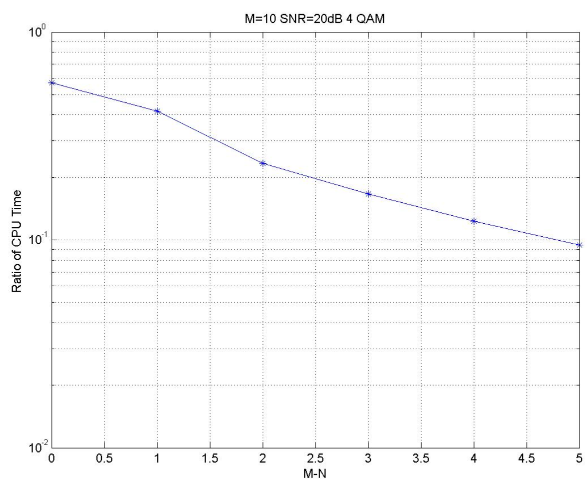

Figure 4.11: Ratio of CPU time= $\frac{\text{CPU time of MMSE-DFE sphere decoding algorithm}}{\text{CPU time of CT algorithm}}$ versus M-N for fixed M and SNR with 4-QAM constellation

The CPU time ratio between MMSE-DFE Fano decoder and MMSE-DFE SD is considered for 16-QAM and fixed SNR=20dB with M= $4, \ldots, 8$ and N=$M-1, \ldots, 2$

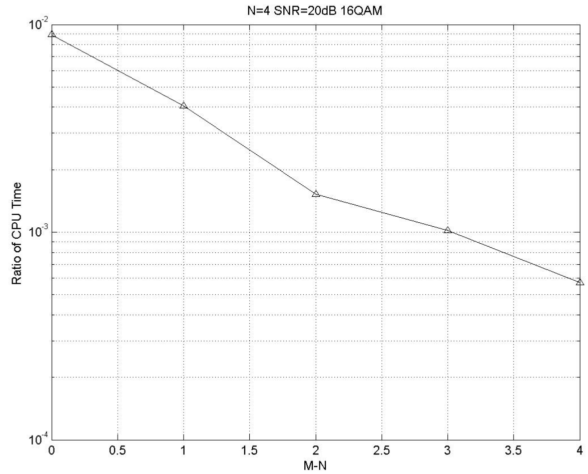

Figure 4.12: Ratio of CPU time= $\frac{\text{CPU time of MMSE-DFE Fano decoding algorithm}}{\text{CPU time of MMSE-DFE sphere decoding algorithm}}$ versus M-N for fixed N, fixed SNR, and with 16-QAM

in Fig. 4.12. It can be noticed that the performance of the MMSE-DFE Fano decoder becomes better then MMSE-DFE SD decoder when the number of tranmit antennas increased .

4.3 Channels with high correlation coefficients

An interesting trend is observed when MMSE-DFE sphere decoding algorithm is used in the case of highly correlated channel gains. In Fig. 4.13 MMSE-DFE sphere decoding algorithm is used with flat fading channel that has correlation matrix with correlation factor of .3. Despite, it is expected that the higher the value of MMSE-DFE coefficient α, the better the performance [4], the simulation shows it is not the case. The best performance is achieved at $\alpha = 1/\text{SNR}$. When α is increased to be proportional to the average value of the condition number (*cond*) of the channel(see table 4.1), the performance deteriorate. The performance is enhanced when α became proportional to half the *cond* and when $\alpha = 1/100$ of *cond*.

Table 4.1: Average condition number of the channel matrix for different SNR

SNR	Condition Number
10	358.4883
15	755.7155
20	1030.1129
25	1424.2338

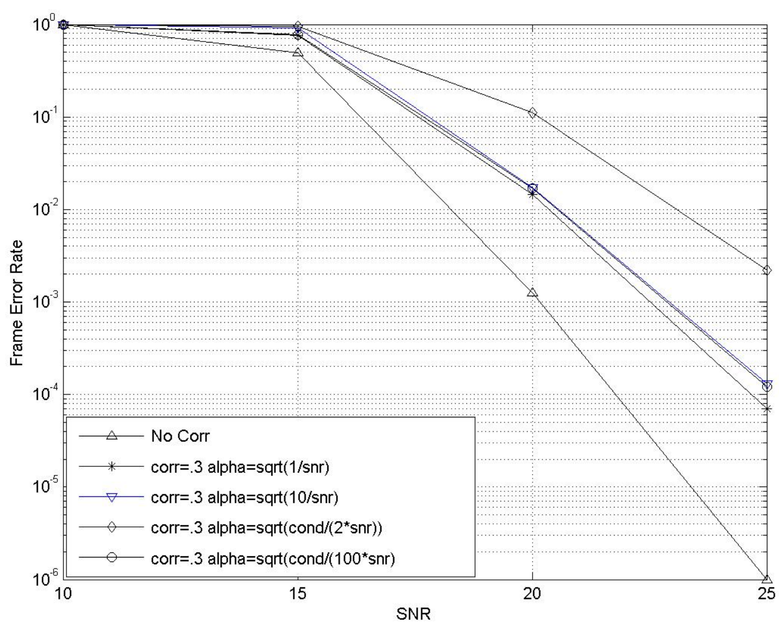

Figure 4.13: Performance of ML detection, CT algorithm, and MMSE-DFE Lattice algorithm of an uncoded, correlated, and non correlated 4×4 system with a 16-QAM constellation with different values of alpha

Chapter 5

Conclusion and Comments

Earlier studies on rank deficient MIMO systems concentrated on optimal decoders and used complex algorithms to decode the transmitted symbols. The sub-optimal decoders haven't received much attention in the literature. This work presented the application of two sub-optimal decoders on uncoded rank deficient MIMO system with comprehensive comparison of the performance and the processing time of these algorithms with some other optimal algorithms.

The use of these sub-optimal decoders offers several improvements over traditional optimal decoders. Most importantly, these decoding methods eliminate the need for performing complex pre-processing on the channel matrix. Furthermore, the processing time of these decoders is highly improved while achieving outstanding performance.

These sub-optimal decoders was based on the work of Damen et al. in [9] and Murugan et al. in [27], that simulated the case of coded rank deficient MIMO systems and was limited to lattice decoding where the signal set boundary control is not introduced. This work presented the use of the MMSE-DFE Fano decoder of Murugan [27] on the uncoded rank deficient MIMO systems. Moreover, this work included the application of MMSE-DFE SD decoder on uncoded systems with introducing the signal set boundary control inside the decoder by using the sphere decoder of Damen et al. in [9]. Both decoders employed the MMSE-DFE as a pre-processing stage which was the source of sub-optimality, in addition V-BLAST ordering was included.

The simulations conducted in this work demonstrated that the performance of these decoders is just a fraction of dB behind the optimal solution, while achieving exceptional reduction in processing time sometimes orders of magnitude below the optimal algorithms known in the literature for the case of 16-QAM and 64-QAM. More interesting, the MMSE-DFE SD algorithm was able to achieve the optimal solution for the case of 4-QAM and be more efficient in processing time by more than 50% than the CT algorithm [7] which is the least complex algorithm for constant modulus constellations. Finally, it was shown that increasing the MMSE-

DFE coefficient α does not enhance the performance of MMSE-DFE SD decoder for the channels with high correlation coefficients.

5.1 Future work

Although the outstanding reduction in complexity of the proposed decoders was shown for flat fading channels, it will be of interest to analyse the performance and complexity of these decoders on other fading channels(like ISI channels). Also, the possibility for practical applications of this work has to be investigated specially that some of the systems considered here are already used in some cell phone systems like the 64-QAM 4×4 MIMO system, which is used in WI-MAX of the 4th generation mobile systems.

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