On-Shell Recursion Relations in General Relativity

by

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Abstract

This thesis is a study of the validity and application of the on-shell recursion relations within the theory of General Relativity. These relations are also known as the Britto-Cachazo-Feng-Witten (BCFW) relations. They reduce the calculation of a tree-level graviton scattering amplitude into the evaluation of two smaller physical amplitudes and of a propagator. With multiple applications of the recursion relations, amplitudes can be uniquely constructed from fundamental three-graviton amplitudes.

The BCFW prescriptions were first applied to gauge theory. We thus provide a self-contained description of their usage in this context. We then generalize the proof of their validity to include gravity. The BCFW recursion relations can then be used to reconstruct the full theory from cubic vertices. We finally describe how these three-graviton vertices can be determined uniquely from Poincare symmetries.

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Contents

Chapter 1

Introduction

The Standard Model is a quantum field theory of the electroweak and strong forces. Very accurate predictions can be made using perturbation theory. In quantum electrodynamics, the anomalous magnetic moment of the electron ($a_e = (g-2)/2$) has been recently calculated at the four-loop level[1]. The following values were obtained:

$$\begin{aligned} a_e(\mathrm{Rb}) &= 1\ 159\ 652\ 182.78\ (7.72)(0.11)(0.26) \times 10^{-12}, \\ a_e(\mathrm{Cs}) &= 1\ 159\ 652\ 172.98\ (9.33)(0.11)(0.26) \times 10^{-12} \end{aligned} \tag{1.1}$$

where the elements in parenthesis denote which atom was used in order to determine the fine structure constant that comes into the calculations. The first uncertainty comes from this value of α, while the second and third are from the theoretical computation. The anomalous moment was also determined experimentally[2]:

$$a_e = 1\ 159\ 652\ 180.85\ (0.76) \times 10^{-12}. \tag{1.2}$$

The values obtained are in great agreement, making $(g-2)$ measurements one of the best proof of perturbative QED. Precision tests of QED were also performed by measuring transition energies, decay rates and through condensed matter experiments. Perturbation theory obviously cannot be applied to QCD at low energies since, at this scale, the coupling constant becomes very large. However, because of asymptotic freedom, perturbation theory has been successfully applied at high energies.

The other force, gravity, is classically described by General Relativity through the Einstein-Hilbert action:

$$S = \int d^4x \sqrt{-g} R \tag{1.3}$$

where g is the determinant of the metric $g_{\mu\nu}$ and R is the Ricci scalar, constructed by contracting the Riemann tensor $R^{\alpha}{}_{\beta\mu\nu}$. This theory was very successful in predicting or explaining many phenomena such as the precession of the perihelion of Mercury, gravitational redshift, light bending and gravitational lensing, frame dragging and black hole formation.

Following the Standard Model success, we now try to quantize General Relativity using perturbation theory. We expand the metric $g_{\mu\nu}$ as a perturbation on the flat background:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{1.4}$$

$h_{\mu\nu}$, a symmetric tensor field, represents the gravitational field. As a special relativistic field, it lives in Minkowski space-time. Thus, its indices are raised and lowered using the flat space metric $\eta_{\mu\nu}$.

However, in contrast with the electroweak and strong forces, it is now well known that perturbative quantum gravity is non-renormalizable at two loops. In string theory, this is resolved by seeing perturbative quantum gravity as an effective theory. GR is the low energy limit of the more general theory of string theory. In loop quantum gravity, non-perturbative methods of quantization are attempted such that the Einstein-Hilbert Lagrangian remains fundamental. These questions will not be addressed in this thesis since we will only consider tree-level amplitudes. No matter which theory describes the full behavior of gravitational interactions, tree-level amplitudes provide a valid approximation for small curvatures.

We thus proceed with the derivation of the Lagrangian for $h_{\mu\nu}$. We expand the

Einstein-Hilbert Lagrangian density perturbatively using (1.4). First consider the term $\sqrt{-g}$. Using the identity $\sqrt{-g} = \exp(\frac{1}{2}\mathrm{Tr}\ln(g_{\mu\nu}))$, one gets the following infinite series in powers of $h_{\mu\nu}$:

$$\sqrt{-g} = 1 + \frac{1}{2}h_\mu{}^\mu + \frac{1}{8}(h_\mu{}^\mu)^2 - \frac{1}{4}h_{\mu\nu}h^{\mu\nu} + \text{cubic and higher order terms.} \tag{1.5}$$

We now look at the Ricci scalar. The metric enters through the Christoffel symbols $\Gamma^\rho_{\mu\nu}$ as follows:

$$\Gamma^\rho_{\mu\nu} = g^{\rho\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}) \tag{1.6}$$

where $g^{\mu\nu}$ is the metric inverse. We can also expand it as an infinite power series:

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + h^\mu{}_\sigma h^{\sigma\nu} + \text{cubic and higher order terms.} \tag{1.7}$$

One then substitutes (1.7) into (1.6). This perturbative expansion of the Christoffel symbols is then substituted in the usual Ricci tensor expression. Contracting the Ricci tensor into the Ricci scalar and combining its expression with (1.5), one gets the perturbative version of the Einstein Lagrangian. Keeping terms up to second order we obtain the Fierz-Pauli Lagrangian:

$$L = \frac{1}{4}\partial^\mu h^{\nu\rho}\partial_\mu h_{\nu\rho} - \frac{1}{2}\partial^\mu h^{\nu\rho}\partial_\nu h_{\mu\rho} + \frac{1}{2}\partial^\mu h \partial^\lambda h_{\lambda\mu} - \frac{1}{4}\partial^\mu h \partial_\mu h \tag{1.8}$$

where h is the trace of $h_{\mu\nu}$. We can notice that (1.8) is invariant up to total derivatives under the following gauge transformation:

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \eta_\nu + \partial_\nu \eta_\mu. \tag{1.9}$$

We now compute the propagator which is the inverse of the Lagrangian quadratic term coefficient. Following [3], we add the gauge breaking term $-\frac{1}{2}C_\mu^2$, where $C_\mu = \partial^\nu h_{\mu\nu} - \frac{1}{2}\partial_\mu h$, to the Lagrangian in order to simplify the calculations. In this gauge, $C_\mu = 0$, the Lagrangian becomes:

$$L = -\frac{1}{4}\partial^\mu h^{\nu\rho}\partial_\mu h_{\nu\rho} + \frac{1}{8}\partial^\mu h \partial_\mu h. \tag{1.10}$$

This can be reexpressed as:

$$L = -\frac{1}{2}\partial_\mu h_{\nu\rho} V^{\nu\rho\alpha\beta} \partial^\mu h_{\alpha\beta} \tag{1.11}$$

where $V^{\nu\rho\alpha\beta} = \frac{1}{2}\eta^{\nu\alpha}\eta^{\rho\beta} - \frac{1}{4}\eta^{\nu\rho}\eta^{\alpha\beta}$. Inverting $V_{\nu\rho\alpha\beta}$ we get the propagator:

$$P^{\nu\rho\alpha\beta} = \frac{\eta^{\nu\alpha}\eta^{\rho\beta} + \eta^{\alpha\rho}\eta^{\beta\nu} - \eta^{\nu\rho}\eta^{\alpha\beta}}{p^2 - i\epsilon} \tag{1.12}$$

where p is the graviton momentum. Thus, in this gauge the propagator is inversely quadratic in momenta.

We now focus on the vertices. General Relativity only contains two derivatives of $h_{\mu\nu}$. Thus, even though the perturbative expansion of the Lagrangian contains an infinite number of vertices, they are at most quadratic in momenta.

Using solely the fact that the propagators are proportional to $1/p^2$ and that the vertices are proportional to p^2, we will show how to compute all graviton tree-level amplitudes. They will be computed recursively, using unique 3-particle amplitudes as building blocks.

Recursion relations have been used in gauge theory since the 80's. In QCD, the Berends-Giele recursion relations [4] lead to analytic formulas for specific types of amplitudes [5, 6, 7, 8]. In 2005, the Britto-Cachazo-Feng-Witten (BCFW) recursion relations were introduced in [9] and fully proven in [10]. These relations reduce the calculation of a gluon tree-level scattering amplitude into the calculation of two smaller *physical* amplitudes and of a propagator. The BCFW recursion relations were soon generalized to include other processes and to one-loop calculations [11, 12, 13, 14, 15, 16, 17, 18, 19]. Early on, it was also attempted to use them in General Relativity [20, 21].

We now briefly describe the BCFW techniques in GR in order to compute recursively an n-graviton scattering amplitude M_n. We use the spinor-helicity formalism which is described in section 2.1. A pair of spinors is assigned to each graviton. For the i^{th} graviton, they are denoted by $\lambda^{(i)}$ and $\tilde{\lambda}^{(i)}$ which are respectively in the $(1/2,\ 0)$ and

(0, 1/2) representations of the Lorentz group. Along with the graviton helicities, these are the only quantities required to compute the amplitude[1]: $M_n = M_n(\{\lambda^{(i)}, \tilde{\lambda}^{(i)}, h_i\})$. M_n will then be function of the spinor inner products:

$$< \lambda^{(1)}, \lambda^{(2)} >\equiv \varepsilon^{ab}\lambda^{(1)}_a\lambda^{(2)}_b, \qquad [\tilde{\lambda}^{(1)}, \tilde{\lambda}^{(2)}] \equiv \varepsilon^{\dot{a}\dot{b}}\tilde{\lambda}^{(1)}_{\dot{a}}\tilde{\lambda}^{(2)}_{\dot{b}}. \tag{1.13}$$

We now apply a one complex parameter deformation to the i^{th} and j^{th} gravitons spinors as follows:

$$\lambda^{(i)}(z) = \lambda^{(i)} + z\lambda^{(j)}, \qquad \tilde{\lambda}^{(j)}(z) = \tilde{\lambda}^{(j)} - z\tilde{\lambda}^{(i)}, \tag{1.14}$$

the momenta all remain on-shell and momentum conservation is preserved. Since the unshifted amplitude M_n is a rational function of the spinor products, $M_n(z)$ is a rational function of z. If $M_n(z)$ contains only simple poles and vanishes at infinity, it can be reconstructed from its singularity structure as follows:

$$M_n(z) = \sum_{k\in poles} \frac{\mathrm{Res}(z_k)}{z - z_k}. \tag{1.15}$$

Physically, we will see that the poles come from propagators going on shell. The amplitude is then split into two smaller amplitudes denoted by $M_{\mathcal{I}}$ and $M_{\mathcal{J}}$. The residue is simply the product of these amplitudes. To get the physical amplitude one simply sets $z = 0$ and we have:

$$M_n = \sum_{\mathcal{I},\mathcal{J}} M_{\mathcal{I}} M_{\mathcal{J}} \frac{1}{P^2_{\mathcal{I}}} \tag{1.16}$$

where $P_{\mathcal{I}}$ is the sum of the momenta in $\mathcal{I}$ and we sum over all possible partitions of the gravitons into the sets $\mathcal{I}$ and $\mathcal{J}$ with the i^{th} graviton in $\mathcal{I}$ and the j^{th} graviton in $\mathcal{J}$ and over the helicity of the on-shell propagator $1/P^2_{\mathcal{I}}$.

The hardest part of the proof of BCFW recursion relations validity in GR is showing that $M_n(z)$ vanishes at infinity. In fact, we will see that individual diagrams behave

[1] As we will see later, the momenta and polarization tensors are easily constructed from these spinors.

as z^{n-5} in the large z limit. However, in [20, 21], it was shown that for specific types of amplitudes, $M_n(z)$ vanishes at infinity for more than four gravitons. This was done by first reexpressing the amplitudes with the Berends-Giele-Kuijf (BGK) [22] or Kawai-Lewellen-Tye (KLT) [23] formulas. The resummed diagrams then showed good behavior at infinity.

This led us to introduce an auxiliary recursion relation in order to resum the diagrams into better behaved objects at infinity [24]. This method allowed us to prove the good behavior of graviton scattering amplitudes with any number of legs.

With the BCFW recursion relations, one can reconstruct the full theory of General Relativity from three-graviton amplitudes. Moreover, the 3-particle spin-2 amplitudes are determined uniquely using Poincaré symmetry [25]. Thus, one can reconstruct General Relativity solely from Poincaré symmetry, without considering a complicated Lagrangian and vertex structure.

This thesis is organized as follows. In chapter 2, we discuss Feynman amplitudes in QCD since these bear many resemblances with graviton amplitudes. Simpler, they also serve as a warm up. The spinor-helicity formalism and the color ordering technique are introduced. We then proof the validity of the BCFW recursion relations in gauge theory and provide a simple example of their application.

Chapter 3 is the core of this thesis. It exposes the proof of the validity of the BCFW recursion relations in GR. As previously mentioned, the good behavior of $M_n(z)$ at infinity is the main part of the proof. In chapter 4, we show how the 3-vertex can be derived from the Poincaré group and prove the fact that the coupling constants do not depend on any other quantum number (such as color, charge, etc.).

Chapter 2

Calculation Techniques for Feynman Amplitudes in QCD

2.1 Spinor-Helicity Formalism

This formalism was introduced in [26, 27, 28] in order to greatly simplify the Feynman amplitude expressions for massless particles (or massive particles in the high energy limit). The idea is to start with external legs of fixed helicity. Note that we will always refer to the helicities of outgoing particles. Incoming particles are transformed into equivalent outgoing ones using crossing symmetry (an incoming particle of momentum p becomes an outgoing particle of momentum $-p$). We will see that, using gauge freedom, one can cleverly choose the polarization vectors, thus leading to many cancelations. In many cases, the cancelations are important, making the extra complication of summing over the helicities afterwards worth it.

We will first apply this formalism to QCD, looking into gluon scattering amplitudes. More than a warmup, this will have direct applications to the subject at hand: graviton scattering amplitudes. For instance, as we will later discuss, the graviton polarization tensor can be expressed as the product of two gluon polarization vectors.

In the following, we will use the Van der Waerden formalism [29](for reviews see [30, 31]). Its main advantage is that it expresses both the momenta and the polarization vectors in terms of spinors. The amplitudes will then only be a function of the helicities (implicitly) and of spinor products. We will use Weyl spinors and will denote by λ the two component spinor transforming as (1/2,0) (also called holomorphic or negative helicity spinor) and by $\tilde{\lambda}$ the two component spinor transforming as (0,1/2) (also called antiholomorphic or positive helicity spinor).

We raise and lower the indices of holomorphic spinors using the antisymmetric Lorentz invariant tensor ε^{ab} defined as:

$$\varepsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.1}$$

We thus have the following inner product:

$$< \lambda^{(1)}, \lambda^{(2)} >\equiv \lambda^{(1)b}\lambda^{(2)}_b = \varepsilon^{ab}\lambda^{(1)}_a\lambda^{(2)}_b \tag{2.2}$$

where $a, b = 1, 2$ and the repeated indices are summed according to the usual convention. One can see that the antisymmetry of ε_{ab} gives rise to an antisymmetric inner product: $< \lambda^{(1)}\lambda^{(2)} >= - < \lambda^{(2)}\lambda^{(1)} >$ and $< \lambda^{(1)}\lambda^{(1)} >= 0$. The inner product is similarly defined for antiholomorphic spinors:

$$[\tilde{\lambda}^{(1)}, \tilde{\lambda}^{(2)}] \equiv \tilde{\lambda}^{(1)\dot{b}}\tilde{\lambda}^{(2)}_{\dot{b}} = \varepsilon^{\dot{a}\dot{b}}\tilde{\lambda}^{(1)}_{\dot{a}}\tilde{\lambda}^{(2)}_{\dot{b}} \tag{2.3}$$

where $\dot{a}, \dot{b} = 1, 2$.

Our next task is to express the momentum vector in terms of spinors. Knowing that the vector representation is isomorphic to the (1/2, 1/2) representation, we will express the momentum $p_\mu \equiv (E, \vec{p})$ as a 2 by 2 "matrix" $p_{a\dot{b}}$. To do so we will use the only other Lorentz invariant tensors: the Pauli matrices.

$$p_{a\dot{b}} = \sigma^\mu_{a\dot{b}} p_\mu \tag{2.4}$$

where σ^0 is the identity matrix and the σ^i's, $i = 1, 2, 3$ are the Pauli matrices. Raising indices with ε_{ac} and $\varepsilon_{\dot{b}\dot{d}}$ and performing the calculation, we get $p_{a\dot{b}}p^{a\dot{b}} = 2\ \det(p_{a\dot{b}})$. Using the identity: $\sigma^{\mu}_{a\dot{b}}\sigma^{a\dot{b}}_{\nu} = 2\delta^{\mu}_{\nu}$, we get:

$$p_\mu p^\mu = \det(p_{a\dot{b}}). \tag{2.5}$$

Thus, a null vector $p_{a\dot{b}}$ has determinant 0. It is then a rank one matrix and can be expressed as:

$$p_{a\dot{b}} = \lambda_a \tilde{\lambda}_{\dot{b}}. \tag{2.6}$$

One can then compute the product of two null vectors, $p_{a\dot{b}}(\lambda_a\tilde{\lambda}_{\dot{b}})$ and $q_{a\dot{b}}(\lambda'_a\tilde{\lambda}'_{\dot{b}})$,

$$p^\mu q_\mu = \frac{1}{2}\lambda^a\tilde{\lambda}^{\dot{b}}\lambda'_a\tilde{\lambda}'_{\dot{b}} = \frac{1}{2} < \lambda, \lambda' > [\tilde{\lambda}, \tilde{\lambda}']. \tag{2.7}$$

Using (2.4) and (2.6), one can see that in order to have real momenta, we need $\tilde{\lambda} = \pm\lambda$ (thus the names holomorphic and antiholomorphic spinors). So, for real momenta we have: $< \lambda^{(1)}, \lambda^{(2)} >= [\tilde{\lambda^{(1)}}, \tilde{\lambda^{(2)}}]$. It will be needed to use complex momenta when introducing the recursion relation techniques. However, until specified, all momenta will be real.

We can already use what we've learned to prove that the 3-gluon amplitude vanishes. Before we do so, we introduce the convenient notation $< \lambda^{(1)}, \lambda^{(2)} >\equiv< 1, 2 >$. For the 3-gluon amplitudes, the momentum conservation condition leads to:

$$p_1 = -p_2 - p_3. \tag{2.8}$$

Squaring on both sides one gets:

$$0 = 2p_2 \cdot p_3 =< 2, 3 > [2, 3]. \tag{2.9}$$

Similarly, one also gets:

$$< 1, 2 > [1, 2] = 0, \qquad < 1, 3 > [1, 3] = 0. \tag{2.10}$$

For real momenta $< i,j >= [i,j]$, thus all spinor products vanish and since the amplitude is a function of spinor inner products, we find that the 3-gluon amplitude vanishes.

Obviously, in order to perform more complicated calculations, we will need to introduce a spinor representation of the polarization vector. We are looking for null-vector $\epsilon_\mu(p)$ that satisfies the condition $\epsilon_\mu p^\mu = 0$. For the negative helicity vector, we have:

$$\epsilon^-_{a\dot{b}}(p_{a\dot{b}} = \lambda_a \tilde{\lambda}_{\dot{b}}) = \frac{\lambda_a \tilde{\mu}_{\dot{b}}}{[\tilde{\lambda}, \tilde{\mu}]} \tag{2.11}$$

where $\tilde{\mu}$ is an arbitrary spinor. This arbitrary spinor will come in handy in Feynman amplitudes calculations. We will soon demonstrate in an example how to cleverly choose it in order to yield cancelations. We have a similar expression for the positive helicity polarization vector:

$$\epsilon^+_{a\dot{b}}(p_{a\dot{b}} = \lambda_a \tilde{\lambda}_{\dot{b}}) = \frac{\mu_a \tilde{\lambda}_{\dot{b}}}{< \mu, \lambda >} \tag{2.12}$$

where μ is again an arbitrary spinor[2]. One can now wonder if we could have used expression (2.11) for positive helicity and expression (2.12) for negative helicity instead. We have already mentioned that λ has negative helicity (-1/2) and that $\tilde{\lambda}$ has positive helicity (+1/2). We apply the transformation $\lambda \to t\lambda, \tilde{\lambda} \to t^{-1}\tilde{\lambda}$. From (2.6), we see that the momentum is invariant under that transformation. We would expect that a negative helicity vector (helicity -1) would transform as $\epsilon^- \to t^2 \epsilon^-$. We see from (2.11) that this is indeed the case. Also, we see from (2.12) that ϵ^+ transforms like $\tilde{\lambda}^2$ as expected.

It is worth mentioning here that since a Feynman amplitude can be expressed as a function of the polarization and momentum vectors, one can see that it needs to be

[2] Throughout this thesis, we will use this spinor representation of the polarization vectors since it is more convenient for our calculations. However, it is completely equivalent to the usual vector choice: $\epsilon^+_\mu = \frac{1}{\sqrt{2}}(0,1,i,0)$, $\epsilon^-_\mu = \frac{1}{\sqrt{2}}(0,1,-i,0)$ (for momentum along the z-axis).

invariant under the transformation:

$$\lambda \rightarrow t\lambda, \qquad \tilde{\lambda} \rightarrow t^{-1}\tilde{\lambda} \tag{2.13}$$

where t is a root of unity.

As an application, we will calculate $M_n(1^-, 2,^+, 3^+, ..., n^+)$: the scattering amplitude of one negative helicity and $n-1$ positive helicity gluons where $n \geq 4$. In QCD, trivalent vertices are linear in momenta and 4-vertices are momentum independent. So, there will be at most $n-2$ momenta available for contraction with the polarization vectors. Thus, M will always contain a product of the type $\epsilon_i \cdot \epsilon_j$, where i, j are in $[1, n]$. By cleverly choosing the polarization vectors reference spinors we can make all such products vanish. For all the positive helicity spinors, we pick $\mu = \lambda_1$:

$$\epsilon^+_{ia\dot{b}} = \frac{\lambda^{(1)}_a \tilde{\lambda}^{(i)}_{\dot{b}}}{< \lambda^{(1)}, \lambda^{(i)} >} \tag{2.14}$$

for $i \in [2, n]$. Physically, this means that all positive helicity polarization vectors are chosen perpendicular to p_1. One can easily check that all products of polarization vectors vanish no matter what we pick for the ϵ^-_1 reference spinor. Obviously, $M_n(1^+, 2^-, 3^-, ..., n^-)$ vanishes when we pick $\tilde{\lambda}^{(1)}$ as reference spinor for all the negative helicity gluons. This can also be seen directly from parity invariance.

Also, one can easily see that $M_n(1^+, 2^+, ..., n^+)$ vanishes. All $\epsilon_i \cdot \epsilon_j$ are zero when all the gluons have the same reference spinor μ. One simply has to be careful not to pick μ proportional to any of the $\lambda^{(i)}$'s since this would yield a division by zero. Similarly, $M_n(1^-, 2^-, \ldots, n^-)$ vanishes.

2.2 Color Ordering

Color ordering [30, 5, 32] is another way to simplify Feynman amplitude calculations in QCD (for a review see [34, 6]). In the following, we will focus on pure gluon scattering. The idea is to decompose the amplitude into a color factor (or group theory factor) times a subamplitude that only depends on the kinematical invariants (momenta and polarization vectors). The main advantage is that the subamplitudes are color-ordered, meaning that one only has to take into account diagrams with a definite gluon ordering up to cyclical permutation. So, we want:

$$M_n = \sum \mathrm{Tr}(T^{a_1}T^{a_2}\dots T^{a_n})M_n'(1,2,...,n) \tag{2.15}$$

where we sum over all non-cyclical permutation of $\{1, 2, \dots, n\}$.

We now show how to combine all color information into a single trace factor. The gauge group of QCD is SU(3) and its generators are the traceless 3×3 hermitian matrices $T^a, a = 1, 2, \dots, 8$. They obey the following properties:

$$\mathrm{Tr}(T^a, T^b) = \delta^{ab}, \tag{2.16}$$

$$[T^a, T^b] = if^{abc}T^c. \tag{2.17}$$

Multiplying by T^c on both sides of (2.17), taking the trace, one gets using (2.16):

$$f^{abc} = -i(\mathrm{Tr}(T^aT^bT^c) - \mathrm{Tr}(T^aT^cT^b)). \tag{2.18}$$

We now show how to express the 4-gluon amplitude in the form of (2.15). First look at a diagram with two trivalent vertices (gluons 1 and 2 connected via a propagator to gluons 3 and 4). We have the following color factor: $f^{a_1a_2b}\delta^{bc}f^{ca_3a_4}$. We then use (2.18) and (2.17) to obtain:

$$\begin{aligned} f^{a_1a_2b}\delta^{bc}f^{ca_3a_4} &= -i\mathrm{Tr}((T^{a_1}T^{a_2} - T^{a_2}T^{a_1})T^c)f^{ca_3a_4} \\ &= -\mathrm{Tr}((T^{a_1}T^{a_2} - T^{a_2}T^{a_1})(T^{a_3}T^{a_4} - T^{a_4}T^{a_3})) \end{aligned} \tag{2.19}$$

which are four single trace factors. One can apply the same process to obtain single trace factors for the two other diagrams with two 3-vertices. The process is also the same for the diagram with one 4-vertex since its color factor is $f^{a_1a_2b}\delta^{bc}f^{ca_3a_4} + f^{a_1a_3b}\delta^{bc}f^{ca_2a_4} + f^{a_1a_4b}\delta^{bc}f^{ca_2a_3}$. Obviously, one can similarly express amplitudes of more than 4 gluons as (2.15) using the same process. From now on, "amplitude" will always mean color-ordered subamplitude. We can mention that color ordering obviously does not apply to graviton scattering. This will be one reason why the later are more complicated to compute.

2.3 The Berends-Giele Recursion Relations

We now introduce techniques to calculate scattering amplitudes recursively on the number of legs. The first such recursion relation was proposed by Berends and Giele in the 80's [4]. To compute the n-gluon scattering amplitudes, one replaces the n^{th} gluon polarization vector $\epsilon^\mu(p_n)$ by the propagator allowing it to be off-shell. On the Feynman diagram, one replaces the n^{th} leg by an off-shell dotted line. This newly formed object is called gluonic current and denoted by $J_\mu(1, 2, \dots, n-1)$.

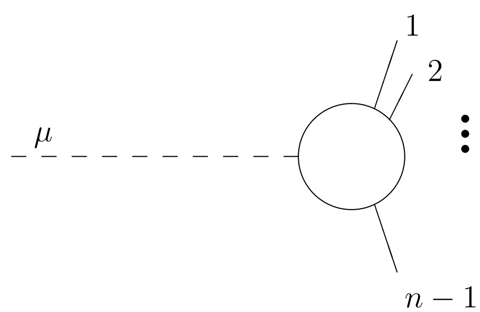

Figure 1: The Gluonic Current

Once its expression has been found recursively, the n-particle amplitude can be reconstructed [6]:

$$M_n(1, 2, \dots, n) = \{\epsilon^\mu(p_n) i (P(1, n-1))^2 J_\mu(1, \dots, n-1)\} \,|_{P(1,n-1)=-p_n} \tag{2.20}$$

where $P(1, n-1) = \sum_{i=1}^{n-1} p_i$.

The way to compute the gluonic current is very simple although it might be quite complicated to carry out. The off mass shell leg is attached to the rest of the diagram either through a 3-vertex or through a 4-vertex. Consider the case where it is attached to two off-shell propagators through a 3-vertex. That means that the (n-1) on-shell gluons are split into two groups, forming two gluonic currents with fewer external legs. Similarly, if it is attached through a 4-vertex, we have 3 gluonic currents with less than (n-1) external leg to compute. Thus we can write:

$$\begin{aligned} J_\mu(1,\ldots,n-1) = \quad & \frac{-i}{P(1,n-1)^2}\{\sum_{i=1}^{n-2} V3^{\mu\nu\rho}(P(1,i),P(i+1,n-1)) \times \\ & J_\nu(1,\ldots,i)J_\rho(i+1,\ldots,n-1) \\ & + \sum_{j=i+1}^{n-2}\sum_{i=1}^{n-3} V4^{\mu\nu\rho\sigma} J_\nu(1,\ldots,i)J_\rho(i+1,\ldots,j)J_\sigma(j+1,\ldots,n-1)\} \end{aligned} \tag{2.21}$$

where $V3^{\mu\nu\rho}(P,Q)$ and $V4^{\mu\nu\rho\sigma}$ are the usual color-ordered 3 and 4 QCD vertex functions.

2.4 The BCFW Recursion Relations

The BCFW recursion relations were first introduced in [9] and were later proven in [10]. These recursion relations reduce the calculation of an amplitude into the calculation of two smaller amplitudes (with fewer external legs) and of a propagator. To do so, we will first need to pick two gluons and apply the following deformation:

$$\tilde{\lambda}^{(k)}(z) = \tilde{\lambda}^{(k)} - z\tilde{\lambda}^{(n)}, \qquad \lambda^{(n)}(z) = \lambda^{(n)} + z\lambda^{(k)} \tag{2.22}$$

where z is a complex number. Since λ_k and $\tilde{\lambda}_n$ are not deformed, one can see that the condition for real momenta in Minkowski space ($\tilde{\lambda} = \pm\lambda$) is no longer satisfied. For instance, we now have $< i, j > \neq [i, j]$.

Now consider the shifted amplitude $M_n(z)$. It is on-shell since $p_k(z)^2 = p_n(z)^2 = 0$, and still satisfies momentum conservation since $p_k(z)+p_n(z) = p_k+p_n$. The deformed amplitude is a rational function of z because it is a rational function of the spinor products. The z-deformation of the latter is as follows:

$$
\begin{aligned}
&[\tilde{\lambda}^{(i)}, \tilde{\lambda}^{(k)}(z)] \equiv [i, \hat{k}] = [i, k] - z[i, n], \\
&< \lambda^{(n)}(z), \lambda^{(j)} > \equiv < \hat{n}, j > = < n, j > + z < k, j >
\end{aligned} \tag{2.23}
$$

for $i, j \;\epsilon\; [1, n-1] \backslash \{k\}$.

We will now analyze the singularity structure of $M_n(z)$. We can pick the polarization vectors reference spinors in order to insure that the former are not singular. It will later be convenient to pick a negative helicity k^{th} gluon and a positive helicity n^{th} gluon. One then picks $\tilde{\lambda}^{(n)}$ as the k^{th} gluon reference spinor and $\lambda^{(k)}$ as the n^{th} gluon reference spinor. Since the vertex functions are obviously singularity-free, the only possible singularity sources are the propagators. A given propagator will split the diagram in two parts: $\mathcal{I}$ and $\mathcal{J}$. Since the gluons are cyclically ordered we can say without loss of generality that $\mathcal{I}$ contains the gluons $\{i, i+1, i+2, ..., j\}$. The propagator, denoted by $1/P_{ij}(z)^2$ where $P_{ij} = p_i + p_{i+1} + ... + p_j$, will be z-dependant if n is in $\mathcal{I}$ and k is in $\mathcal{J}$ or vice versa. If n is in $\mathcal{I}$, we have:

$$
P_{ij}(z) = P_{ij} + z\lambda_k\tilde{\lambda}_n. \tag{2.24}
$$

Then,

$$
P_{ij}^2(z) = \frac{1}{2}P_{ija\dot{a}}(z)P_{ij}^{a\dot{a}}(z) = P_{ij}^2 - z < k|P_{ij}|n] \tag{2.25}
$$

where we define $-\lambda_a^{(k)}P_{ij}^{a\dot{a}}\tilde{\lambda}_{\dot{a}}^{(n)} \equiv < k|P_{ij}|n]$. We can then conclude that $M_n(z)$ only has simple poles located at

$$
z_{ij} = \frac{P_{ij}^2}{< k|P_{ij}|n]}. \tag{2.26}
$$

We now prove that $M_n(z)$ vanishes at infinity. This can most easily be done by picking a negative helicity gluon for the k^{th} gluon and a positive helicity gluon for the n^{th} gluon. In a given Feynman diagram, the z-dependance flows in a unique path between the k^{th} and the n^{th} gluon leading to s deformed vertices and $(s-1)$ deformed propagators. Since 3-vertices are linear in momenta and 4-vertices are momentum independent, the deformed vertices will be at most linear in z. As we just saw, the deformed propagators will go as $1/z$. Also, because of our helicity choices, the polarization vector of the k^{th} and n^{th} gluons both go as $1/z$ while the other vectors can be made z-independent. Thus, we have:

$$M_n(z) \sim z^s \cdot \frac{1}{z^{s-1}} \cdot \frac{1}{z^2} = \frac{1}{z} \tag{2.27}$$

which vanishes at infinity.

Since $M_n(z)$ is a rational function that vanishes at infinity and only contains single poles, it can be expressed as follows:

$$M_n(z) = \sum_{poles} \frac{\text{Res}(z_{ij})}{z - z_{ij}} \tag{2.28}$$

where we sum over the poles z_{ij}.

We pause here in order to prove (2.28). We consider the following contour integral on the Riemann sphere ($\mathbb{C} \cup \{\infty\}$):

$$\oint_C \frac{M_n(y)}{y-z} dy \tag{2.29}$$

where C is a circle enclosing all the poles of $M_n(y)$ and the point $y = z$. We can first deform this contour by sending it to infinity. In this limit, $M_n(y)$ vanishes since it is behaving as $1/y$. Thus, we have:

$$\oint_C \frac{M_n(y)}{y-z} dy \to \oint_C \frac{1}{y^2} dy \to 0. \tag{2.30}$$

We now deform the contour inward, encircling the poles of $M_n(y)$ and $y = z$ such that

$$\oint_C \frac{M_n(y)}{y-z} dy = \sum_{z_{ij} \in poles} \oint_{|y-z_{ij}|=r} \frac{M_n(y)}{y-z} dy + \oint_{|y-z|=r} \frac{M_n(y)}{y-z} dy = 0 \tag{2.31}$$

where the z_{ij}'s are the poles of $M_n(y)$. The left hand side is zero as we just saw. We can simplify the equation as follows:

$$\sum_{z_{ij} \in poles} \frac{\text{Res}(z_{ij}))}{z_{ij} - z} + M_n(z) = 0 \tag{2.32}$$

which is exactly (2.28).

We now discuss the physical meaning of these residues. When $P_{ij}(z)$ goes on shell near $z = z_{ij}$, the diagrams that are split into two parts $\mathcal{I}$ and $\mathcal{J}$ by $P_{ij}(z)$ become dominant. This is simply because since $1/P_{ij}^2$ goes to infinity, any diagram without this propagator is negligible. One has near $z = z_{ij}$:

$$M_n(z) = M_{\mathcal{I}}(z) M_{\mathcal{J}}(z) \frac{1}{P_{ij}^2(z)}. \tag{2.33}$$

Thus, the residue at $z = z_{ij}$ is:

$$\mathrm{Res}(z_{ij}) = \lim_{z \to z_{ij}} (z - z_{ij}) \frac{M_{\mathcal{I}}(z) M_{\mathcal{J}}(z)}{P_{ij}^2(z)}. \tag{2.34}$$

Using (2.25) and (2.26) we find:

$$\mathrm{Res}(z_{ij}) = \frac{-M_{\mathcal{I}}(z_{ij}) M_{\mathcal{J}}(z_{ij})}{< k|P_{ij}|n]}. \tag{2.35}$$

Substituting into (2.28) using again (2.26), we finally obtain:

$$M(z) = \sum \frac{M_{\mathcal{I}}(z_{ij}) M_{\mathcal{J}}(z_{ij})}{P_{ij}^2(z)}. \tag{2.36}$$

To take into account all poles, we need to sum over all possible partitions $\mathcal{I}$ and $\mathcal{J}$ of the cyclically ordered gluons and over the two possible helicities of P_{ij}. To recover the unshifted amplitude, one simply sets z=0:

$$M_n = \sum \frac{M_{\mathcal{I}}(z_{ij}) M_{\mathcal{J}}(z_{ij})}{P_{ij}^2}. \tag{2.37}$$

2.5 A Simple Application of the BCFW Recursion Relations

As an example, we will compute $M_4(1^-, 2^-, 3^+, 4^+)$. To simplify the calculation, we will shift two adjacent gluons: gluon 2 and gluon 3:

$$\tilde{\lambda}^{(2)} \to \tilde{\lambda}^{(2)} - z\tilde{\lambda}^{(3)} \qquad \lambda^{(3)} \to \lambda^{(3)} + z\lambda^{(2)}. \tag{2.38}$$

The only way to split our 4 gluons into two groups $\mathcal{I}$ and $\mathcal{J}$ with the shifted gluon $\hat{3}$ in $\mathcal{I}$ is to have $\mathcal{I} = \{\hat{3}^+, 4^+\}$ and $\mathcal{J} = \{1^-, \hat{2}^-\}$. Since amplitudes of 3 negative helicity gluons vanish, we need to have the positive helicity side of $\hat{P}_{34}$ next to the group $\mathcal{J}$. Thus we have the following 3-gluon amplitudes to compute: $M_{\mathcal{I}}(\hat{3}^+, 4^+, \hat{P}^-)$ and $M_{\mathcal{J}}(1^-, \hat{2}^-, \hat{P}^+)$.

We have previously shown that 3-gluon amplitudes vanish in the case of real momenta. However, in the complex case ($< i, j > \neq [i, j]$), (2.9) and (2.10) instead mean that the amplitudes are expressed only in terms of holomorphic or antiholomorphic spinors. From (2.9) and (2.10), we see that at least two holomorphic (or antiholomorphic) spinor products will vanish. For instance, consider $< 1, 2 > = < 1, 3 > = 0$. Thus, $\lambda^{(1)}$ is proportional to both $\lambda^{(2)}$ and $\lambda^{(3)}$. Since we are in a two dimensional space, this yields that $\lambda^{(2)}$ is also proportional to $\lambda^{(3)}$ ($< 2, 3 >=0$).

We have already discussed that under the transformation

$$\lambda^{(i)} \to t^{-1/2}\lambda^{(i)}, \ \tilde{\lambda}^{(i)} \to t^{1/2}\tilde{\lambda}^{(i)} \tag{2.39}$$

the momenta are invariant and the polarization vectors go to $\epsilon_i^+ \to t\epsilon_i^+, \ \epsilon_i^- \to t^{-1}\epsilon_i^-$. Thus, the amplitude would scale as follows under the rescaling of the i^{th} gluon:

$$M_n \to t^{h_i} M_n \tag{2.40}$$

where h_i is the i^{th} gluon helicity. We now compute $M_3(1^-, 2^-, 3^+)$. We first suppose it only contains holomorphic spinors:

$$M^H = \sum a_{n_1,n_2,n_3} < 1, 2 >^{n_3} < 2, 3 >^{n_1} < 1, 3 >^{n_2} . \tag{2.41}$$

We now rescale every spinor according to (2.39) and get from (2.40) the following 3 conditions:

$$\begin{aligned} t^{-1/2(n_3+n_2)} &= t^{-1}, \\ t^{-1/2(n_3+n_1)} &= t^{-1}, \\ t^{-1/2(n_1+n_2)} &= t^{+1}. \end{aligned} \tag{2.42}$$

This yields the unique solution:

$$M^H(1^-, 2^-, 3^+) = a^H \frac{< 1,2 >^3}{< 2,3 >< 3,1 >}. \tag{2.43}$$

One can repeat the same process assuming M is only function of the antiholomorphic spinors. Then one would get:

$$M^A(1^-, 2^-, 3^+) = a^A \frac{[2,3][1,3]}{[1,3]^3}. \tag{2.44}$$

We know that, in the real momenta case, $< i, j >= [i, j] = 0$ and the amplitude vanishes. However, in the limit $[i, j] \to 0$ M^A blows up. Thus, $a^A = 0$ and:

$$M_3(1^-, 2^-, 3^+) = \frac{< 1,2 >^3}{< 2,3 >< 3,1 >} \tag{2.45}$$

up to a constant. One can show similarly that

$$M_3(1^+, 2^+, 3^-) = \frac{[1,2]^3}{[2,3][3,1]} \tag{2.46}$$

up to a constant. The constants can be determined from the Lagrangian. For the 3-gluon amplitudes, the constant is the gauge coupling g since we have only one vertex.

We pause here to make some important remarks. Using only the fact that the amplitude is a function of the spinor inner products and the amplitude properties under the helicity operator, we have found unique forms for the 3-gluon amplitudes $M_3(1^-, 2^-, 3^+)$ and $M_3(1^+, 2^+, 3^-)$. This, however, is not a proof that QCD is a consistent theory. This is irrelevant here since we already know this is the case and one can check (2.45) and (2.46) against the color-ordered amplitudes computed using the Feynman rules.

A theory is constructible if the 4-particle amplitudes can be computed from the 3-particle amplitudes using the BCFW recursion relations. As we just saw, this is possible if the 4-particle amplitudes vanish at infinity. We can apply the 4-particle test to constructible theories[25]. Basically, the 4-particle test requires that a 4-particle amplitude be the same no matter which pair of particles one deforms. We will discuss the 4-particle test in more detail and give examples in chapter 4, but we can briefly mention here that higher spin (spin higher than 2) theories are constructible, but only satisfy the 4-particle test if the coupling constants are zero. This means we only have free theories for higher spins.

One might also remember that we proved in the last section that $M_n(1^-, 2,^+, 3^+, ..., n^+) = 0$ and that proof is applicable to complex momenta. One might wonder why this argument is not valid for 3-gluon amplitudes. For instance, consider $M_3(1^+, 2^+, 3^-)$ which, as we just saw, is only function of the antiholomorphic spinors. In order to make every polarization vector dot product vanish, we would need to pick $\mu = \lambda_3$ as reference spinor for ϵ_1^+ and ϵ_2^+. However we can see from (2.12) that this is impossible as $< 1, 2 >=< 1, 3 >= 0$.

We are now finally ready to substitute (2.45) and (2.46) into (2.37):

$$M(1^-, \hat{2}^-, \hat{3}^+, 4^+) = \frac{[\hat{3}, 4]^3}{[4, \hat{P}_{34}][\hat{P}_{34}, \hat{3}]} \frac{1}{P_{34}^2} \frac{< 1, \hat{2} >^3}{< \hat{2}, \hat{P}_{34} >< \hat{P}_{34}, 1 >} \tag{2.47}$$

where both amplitudes are evaluated at $z = z_{34}$. We will now evaluate $< i, \hat{P}_{34} >$. Since, $\hat{P}_{34}^2(z_{34}) = 0$, we can define $\hat{P}_{34}(z_{34}) \equiv \lambda^{(\hat{P})}\tilde{\lambda}^{(\hat{P})}$. Thus:

$$< i|\hat{P}|\ 3] =< \lambda^{(i)}, \lambda^{(\hat{P})} > [\tilde{\lambda}^{(\hat{P})}, \tilde{\lambda}^{(3)}] =< i|P|\ 3] \tag{2.48}$$

where the last equality comes from (2.24). Then:

$$< \lambda^{(i)}, \lambda^{(\hat{P})} >= \frac{< i|P|\ 3]}{[\tilde{\lambda}^{(\hat{P})}, \tilde{\lambda}^{(3)}]}. \tag{2.49}$$

Similarly:

$$[\tilde{\lambda}^{(\hat{P})}, \tilde{\lambda}^{(i)}] = \frac{< 2|P|\ i]}{< \lambda^{(2)}, \lambda^{(\hat{P})} >}. \tag{2.50}$$

Finally, one can see that:

$$< \lambda^{(2)}, \lambda^{(\hat{P})} > [\tilde{\lambda}^{(\hat{P})}, \tilde{\lambda}^{(3)}] =< 2|\hat{P}|\ 3] =< 2|P|\ 3]. \tag{2.51}$$

Substituting everything into (2.47) while keeping in mind that $\tilde{\lambda}^{(3)}$ and $\lambda^{(2)}$ are not deformed, we get:

$$
\begin{aligned}
M_4 = & \frac{[3,4]^3}{<2,3>[3,4]<2,4>[4,3]} \frac{1}{<3,4>[3,4]} \times \\
& \frac{<1,2>^3}{<2,4>[4,3]<1,4>[4,3]} <2,4>^2 [4,3]^2
\end{aligned}
\tag{2.52}
$$

which simplifies to the known result (up to a constant):

$$
M_4(1^-,2^-,3^+,4^+) = \frac{<1,2>^3}{<2,3><3,4><4,1>}. \tag{2.53}
$$

Chapter 3

BCFW Recursion Relations in GR

As with QCD, scattering amplitudes in General Relativity are traditionally calculated using Feynman diagrams. We however run into the same problems as in QCD such as the factorial growth in the number of diagrams. We now wish to compute scattering amplitudes in GR using the BCFW recursion relations. In the following we will focus on pure graviton scattering.

We will again use the Van der Waerden formalism. The momenta are still defined as in (2.4, 2.5, 2.6). The graviton polarization tensor can be expressed as the product of two polarization vectors:

$$\epsilon^-_{a\dot{a},b\dot{b}} = \epsilon^-_{a\dot{a}}\epsilon^-_{b\dot{b}} = \frac{\lambda_a\tilde{\mu}_{\dot{a}}\lambda_b\tilde{\mu}_{\dot{b}}}{[\tilde{\lambda},\tilde{\mu}]^2}, \tag{3.1}$$

$$\epsilon^+_{a\dot{a},b\dot{b}} = \epsilon^+_{a\dot{a}}\epsilon^+_{b\dot{b}} = \frac{\mu_a\tilde{\lambda}_{\dot{a}}\mu_b\tilde{\lambda}_{\dot{b}}}{<\mu,\lambda>^2}. \tag{3.2}$$

We now proceed to expand the graviton scattering amplitude M_n in a way similar to what was done in section 2.4. We deform two gravitons, i^+ and j^- as follows:

$$\lambda^{(i)}(z) = \lambda^{(i)} + z\lambda^{(j)}, \qquad \tilde{\lambda}^{(j)}(z) = \tilde{\lambda}^{(j)} - z\tilde{\lambda}^{(i)}. \tag{3.3}$$

We can again pick reference spinors in order to make the polarization tensors singularity free. Thus, the pole structure only comes from propagators. Again, a propagator splits the diagram in two parts $\mathcal{I}$ and $\mathcal{J}$ and is z-dependant if i^+ or j^- (but not both) is in $\mathcal{I}$. Without loss of generality, we put i^+ in $\mathcal{I}$. One can notice that we do not have color ordering in GR so we will need to take into account all possible partitions of $[1, \ldots, n]$ with i in $\mathcal{I}$ (not only cyclical ordered ones). Thus, we denote the propagator as $1/P_{\mathcal{I}}^2$ and we have:

$$P_{\mathcal{I}}(z) = \sum_{k \epsilon I} p_k(z) = P_{\mathcal{I}} + z\lambda^{(i)}\tilde{\lambda}^{(j)}. \tag{3.4}$$

Again, the poles are located at $z_{\mathcal{I}}$ where $P_{\mathcal{I}}^2(z_I) = 0$. Thus,

$$z_{\mathcal{I}} = \frac{P_{\mathcal{I}}^2}{< j|P_{\mathcal{I}}|i]}. \tag{3.5}$$

We now have to prove that M_n vanishes at infinity. This ended up being much more complicated to achieve than in the QCD case. We will assume it for now as the proof is detailed in the next section.

Since we claim that $M_n(z)$ only has simple poles and that it vanishes as z is taken to infinity, we have:

$$M_n(z) = \sum_{\alpha} \frac{c_\alpha}{z - z_\alpha} \tag{3.6}$$

where the sum is over all poles of $M_n(z)$.

The final step is the computation of the residues $c_{\mathcal{I}}$. This is easily done since close to the region where a given propagator goes on-shell the amplitude factorizes as the product of lower amplitudes. Collecting all these results one finds that

$$M_n(z) = \sum_{\mathcal{I},\mathcal{J}} \sum_{h=\pm} M_{\mathcal{I}}\left(\{K_{\mathcal{I}}\}, p_i(z_{\mathcal{I}}), -P_{\mathcal{I}}^h(z_{\mathcal{I}})\right) \frac{1}{P_{\mathcal{I}}(z)^2} M_{\mathcal{J}}\left(\{K_{\mathcal{J}}\}, p_j(z_{\mathcal{I}}), P_{\mathcal{I}}^{-h}(z_{\mathcal{I}})\right) \tag{3.7}$$

where $\{\mathcal{I}, \mathcal{J}\}$ is a partition of the set of all gravitons such that $i \in \mathcal{I}$ and $j \in \mathcal{J}$, $K_{\mathcal{I}}$ ($K_{\mathcal{J}}$) is the collection of all gravitons in $\mathcal{I}$ ($\mathcal{J}$) except for i (j) and h is the helicity of the internal graviton.

The BCFW recursion relation is obtained by setting $z = 0$ in (3.7). It is important to mention that the value of $z_{\mathcal{I}}$ was determined by requiring $P_{\mathcal{I}}(z_{\mathcal{I}})$ be a null vector. Therefore the BCFW recursion relations only involve physical on-shell amplitudes.

3.1 Vanishing of M_n(z) at infinity

In the previous section we showed that the validity of the BCFW recursion relations for gravity amplitudes simply follows from the vanishing of $M_n(z)$ at infinity. In this section we provide a proof of this statement.

We now try to do that using a Feynman diagram argument similar to the one of section 2.4.We can pick reference spinors in order to make the polarization tensors of the i^{th} and j^{th} gravitons both go like $1/z^2$. As we have seen in section one, the 3-vertex function contains two momentum operators. The most dangerous n-graviton Feynman diagrams contains $(n-2)$ vertices that are all z-dependant, each contributing by a factor z^2 to the behavior at infinity. In that case we have $(n-3)$ propagator that go like $1/z$ as in the QCD case. Putting everything together we get the following behavior at infinity:

$$M_n(z) \sim \frac{1}{z^4} \cdot z^{2(n-2)} \cdot \frac{1}{z^{(n-3)}} = z^{n-5}. \tag{3.8}$$

Thus, from this analysis, the amplitude blows up for more than 4 gravitons. However, we can mention here that we only took into account the behavior of individual Feynman diagrams. Cancelations between different diagrams can, and in fact do, dramatically improve the behavior at infinity. For instance, one can see from the BGK relations (that have been proven numerically for $n < 11$) that maximally helicity violating (MHV) amplitudes vanish at infinity under the BCFW deformation. We wish to resum the diagrams into objects that are better behaved at infinity. Since the way to achieve this is a little intricate and can be confusing with its many subcases, we will first provide an outline of the proof.

3.1.1 Outline Of The Proof

We start by finding a convenient representation of $M_n(z)$. The new representation comes from some auxiliary recursion relations. The auxiliary recursion relations are obtained using a BCFW-like construction but with a deformation under which individual Feynman diagrams vanish at infinity. The way we achieve this is by making as many polarization tensors go to zero at infinity as possible.

Let us denote the new deformation parameter w. Then one has that $M_n(w) \to 0$ as $w \to \infty$. The recursion relations are schematically of the form

$$M_n = \sum_{\mathcal{I},\mathcal{J}} \sum_{h=\pm} M_{\mathcal{I}}^{h}(w_{\mathcal{I}}) \frac{1}{P_{\mathcal{I}}^2} M_{\mathcal{J}}^{-h}(w_{\mathcal{I}}) \tag{3.9}$$

where the sum is over some sets $\mathcal{I}, \mathcal{J}$ of gravitons. These auxiliary recursion relations actually provide the first example of recursion relations valid for all physical amplitudes of gravitons. However, the price one pays for being able to prove that $M_n(w) \to 0$ as $w \to \infty$ directly from Feynman diagrams is that the number of terms in (3.9) is very large and many of the gravitons depend on $w_{\mathcal{I}}$. These features make (3.9) not very useful for actual computations.

The next step in our proof is to apply the BCFW deformation to M_n now given by (3.9). Then we have

$$M_n(z) = \sum_{\{i,j\}\subset\mathcal{J}} \sum_{h=\pm} M_{\mathcal{I}}^{h}(w_{\mathcal{I}}) \frac{1}{P_{\mathcal{I}}^2} M_{\mathcal{J}}^{-h}(w_{\mathcal{I}}, z) + \sum_{i\in\mathcal{I}, j\in\mathcal{J}} \sum_{h=\pm} M_{\mathcal{I}}^{h}(w_{\mathcal{I}}(z), z) \frac{1}{P_{\mathcal{I}}^2(z)} M_{\mathcal{J}}^{-h}(w_{\mathcal{I}}(z), z) \tag{3.10}$$

where the z dependence on the right hand side can appear implicitly through $w_{\mathcal{I}}(z)$ as well as explicitly. The first set of terms on the right hand side of (3.10) has both deformed gravitons in $\mathcal{J}$. Therefore, all the z dependence is confined to $M_{\mathcal{J}}$. We then show that $M_{\mathcal{J}}$ is a physical amplitude with less than n gravitons under a BCFW deformation. Therefore, we can use an induction argument to prove that it vanishes as $z \to \infty$.

For the second set of terms the z dependence appears not only explicitly but also implicitly via $w_{\mathcal{I}}$ in many gravitons. Quite nicely, it turns out that one can show that each one of those terms vanishes as z goes to infinity by using a Feynman diagram analysis similar to the one done at the beginning of this section. The reason for this is again the large number of polarization tensors that pick up a z dependence.

There is a special case that has to be considered separately. This is when there is only one positive helicity graviton in $\mathcal{I}$, i.e., the i^{th} graviton. We prove the desired behavior at infinity in this case at the end of this section.

3.1.2 Auxiliary Recursion Relation

The auxiliary recursion relations we need are obtained by using a composition of BCFW deformations introduced in [21] and which was used to prove the vanishing of $M_n(z)$ for next-to-MHV amplitudes. The basic idea comes form the analysis of Feynman diagrams we performed above. It is clear that the reason individual Feynman diagrams diverge as $z \to \infty$ for $n \geq 5$ is that the number of propagators and vertices grow in the same way but vertices give an extra power of z which can be compensated by two polarization tensors that depend on z only if n is not too large. The key is then to perform a deformation that will make more polarization tensors contribute.

Recall from the outline of the proof that the deformation parameter is denoted by w. The simplest choice is to deform the λ's of all positive helicity gravitons and the $\tilde{\lambda}$'s of all negative helicity gravitons. This choice will give $1/w^{2n}$ from the polarization tensors. This makes $M_n(w)$ go at most as $1/w^4$ even without taking into account the propagators. Propagators are now quadratic functions of w and therefore they contribute $1/w^2$ each. This last feature is what makes this choice very inconvenient since every multi-particle singularity of the amplitude will result in two simple poles rather than one.

We are then looking for a deformation that gives a w dependence to the largest number of gravitons and at the same time keeps all propagators at most linear functions of w. The most general such deformation depends on the number of plus and minus helicity gravitons in the amplitude. Let $\{r^-\}$ and $\{k^+\}$ denote the sets of negative and positive helicity gravitons in the amplitude respectively. Also let m and p be the number of elements in each. Then if $p \geq m$ the deformation is

$$\tilde{\lambda}^{(j)}(w) = \tilde{\lambda}^{(j)} - w \sum_{s\in\{k^+\}} \alpha^{(s)}\tilde{\lambda}^{(s)}, \qquad \lambda^{(k)}(w) = \lambda^{(k)} + w\alpha^{(k)}\lambda^{(j)}, \qquad \forall\, k \in \{k^+\} \tag{3.11}$$

where j is a negative helicity graviton and $\alpha^{(k)}$'s can be arbitrary rational functions of kinematical invariants.

If $m \geq p$ the deformation is

$$\lambda^{(i)}(w) = \lambda^{(i)} + w \sum_{s\in\{r^-\}} \alpha^{(s)}\lambda^{(s)}, \qquad \tilde{\lambda}^{(k)}(w) = \tilde{\lambda}^{(k)} - w\alpha^{(k)}\tilde{\lambda}^{(i)}, \qquad \forall\, k \in \{r^-\} \tag{3.12}$$

where i is a positive helicity graviton.

The deformation introduced in [21] to prove the case of next-to-MHV amplitudes corresponds to taking all $\alpha^{(s)} = 1$ in (3.11). It turns out that not all choices of $\alpha^{(s)}$ lead to the desired behavior of individual Feynman diagrams at infinity. For example, any choice that removes the w dependence on any single spinor or even on any linear combination of subsets of them will fail. This is usually due to some subtle Feynman diagrams. It is interesting that one has to use precisely the maximal choice. In other words, we have to choose all $\alpha^{(s)} = 1$. Given that this is the choice we use in the rest of the proof, we rewrite (3.11) and (3.12) with $\alpha^{(k)} = 1$ for later reference.

For $p \geq m$:

$$\tilde{\lambda}^{(j)}(w) = \tilde{\lambda}^{(j)} - w \sum_{s\in\{k^+\}} \tilde{\lambda}^{(s)}, \qquad \lambda^{(k)}(w) = \lambda^{(k)} + w\lambda^{(j)}, \qquad \forall\, k \in \{k^+\} \tag{3.13}$$

and j a negative helicity graviton.

If $m \geq p$ the deformation is

$$\lambda^{(i)}(w) = \lambda^{(i)} + w \sum_{s\in\{r^-\}} \lambda^{(s)}, \qquad \tilde{\lambda}^{(k)}(w) = \tilde{\lambda}^{(k)} - w\tilde{\lambda}^{(i)}, \qquad \forall\, k \in \{r^-\} \tag{3.14}$$

and i a positive helicity graviton.

The proof that this choice gives $M_n(w) \to 0$ as $w \to \infty$ and more details are given in the appendix. The proof involves a careful analysis of when the w can possibly drop out of propagators. This is basically the point where all other deformations fail.

Here we simply give the final form of the auxiliary recursion relations. Again we have to distinguish cases. If $p \geq m$ we write M_n as sums of products of amplitudes with less than n gravitons as follows:

$$M_n(\{r^-\},\{k^+\}) \ = \\ = \sum_{\mathcal{I}} \sum_{h=\pm} M_{\mathcal{I}}\left(\left\{r^-_{\mathcal{I}}\right\}, \left\{k^+_{\mathcal{I}}(w_{\mathcal{I}})\right\}, -P^h_{\mathcal{I}}(w_{\mathcal{I}})\right) \frac{1}{P^2_{\mathcal{I}}} M_{\mathcal{J}}\left(\left\{r^-_{\mathcal{J}}(w_{\mathcal{I}})\right\}, \left\{k^+_{\mathcal{J}}(w_{\mathcal{I}})\right\}, P^{-h}_{\mathcal{I}}(w_{\mathcal{I}})\right) \tag{3.15}$$

where:

- $\mathcal{I}$ and $\mathcal{J}$ are subsets of the set $\{1,\ldots,n\}$ such that $\mathcal{I} \cup \mathcal{J} \ = \ \{1,\ldots,n\}$. The sum is over all partitions $\{\mathcal{I},\mathcal{J}\}$ of $\{1,\ldots,n\}$ such that at least one positive helicity graviton is in $\mathcal{I}$ and $j \in \mathcal{J}$.
- $P_{\mathcal{I}}$ is the sum of all the momenta of gravitons in $\mathcal{I}$;
- $\left\{r^-_{\mathcal{I}}\right\} \equiv \mathcal{I}^-$ is the set of negative helicity gravitons in $\mathcal{I}$;
- $\left\{r^-_{\mathcal{J}}(w_{\mathcal{I}})\right\}$ is the set of negative helicity gravitons in $\mathcal{J}$. The $w_{\mathcal{I}}$ dependence is only through $\tilde{\lambda}^{(j)}(w_{\mathcal{I}})$;
- $\left\{k^+_{\mathcal{I}}(w_{\mathcal{I}})\right\} \equiv \mathcal{I}^+$ is the set of positive helicity gravitons in $\mathcal{I}$. All of them have been deformed and their dependence on $w_{\mathcal{I}}$ is only through

$$\lambda^{(k)}(w_{\mathcal{I}}) \ = \ \lambda^{(k)} + w_{\mathcal{I}}\lambda^{(j)}; \tag{3.16}$$

- $\left\{k^+_{\mathcal{J}}(w_{\mathcal{I}})\right\}$ is the set of positive helicity gravitons in $\mathcal{J}$. All of them have also been deformed via (3.16).

- The deformation parameter is given by

$$w_{\mathcal{I}} \; = \; \frac{P_{\mathcal{I}}^2}{\sum_{k\in\mathcal{I}^+} \langle j|P_{\mathcal{I}}|k]} \; . \tag{3.17}$$

This definition ensures that the momentum

$$P_{\mathcal{I}}(w_{\mathcal{I}})_{a\dot{a}} \; = \; P_{\mathcal{I}\,a\dot{a}} + w_{\mathcal{I}}\lambda_a^{(j)} \sum_{k\in\mathcal{I}^+} \tilde{\lambda}_{\dot{a}}^{(k)} \tag{3.18}$$

is a null vector, i.e., $P_{\mathcal{I}}(w_{\mathcal{I}})^2 = 0$.

Now, if $m \geq p$ then we write M_n as a sum over terms involving the product of amplitudes with less than n gravitons as follows:

$$\begin{aligned} &M_n(\{r^-\},\{k^+\}) \; = \\ &\quad = \sum_{\mathcal{I}} \sum_{h=\pm} M_{\mathcal{I}}\left(\left\{r_{\mathcal{I}}^-(w_{\mathcal{I}})\right\}, \left\{k_{\mathcal{I}}^+(w_{\mathcal{I}})\right\}, -P_{\mathcal{I}}^h(w_{\mathcal{I}})\right) \frac{1}{P_{\mathcal{I}}^2} M_{\mathcal{J}}\left(\left\{r_{\mathcal{J}}^-(w_{\mathcal{I}})\right\}, \left\{k_{\mathcal{J}}^+\right\}, P_{\mathcal{I}}^{-h}(w_{\mathcal{I}})\right) \end{aligned} \tag{3.19}$$

where most definitions are as in the $p \geq m$ case except that the sets $\mathcal{I}$ and $\mathcal{J}$ are such that $i \in \mathcal{I}$ and all the negative helicity gravitons and the i^{th} positive helicity graviton are deformed via (3.14) instead of (3.13).

The two rules, (3.15) and (3.19), provide a full set of recursion relations for gravity amplitudes. To see this note that using them one can express any n-graviton amplitude as the sum of products of two amplitudes with less than n gravitons. The smaller amplitudes which depend on deformed spinors and the intermediate null vector $P(w_{\mathcal{I}})$ are completely "physical" in the sense that by construction their momenta are on-shell and satisfy momentum conservation. Therefore they admit a definition in terms of Feynman diagrams again and can serve as a starting point to apply either (3.15) or (3.19), depending on the new number of plus and minus helicity gravitons.

3.1.3 Induction And Feynman Diagram Argument

Consider any n-graviton amplitude under the BCFW deformation (3.3) on gravitons i^+ and j^-:

$$\lambda^{(i)}(z) = \lambda^{(i)} + z\lambda^{(j)}, \qquad \tilde{\lambda}^{(j)}(z) = \tilde{\lambda}^{(j)} - z\tilde{\lambda}^{(i)}. \tag{3.20}$$

Without loss of generality we can assume that M_n has $p \geq m$ and use (3.15) as our starting point. If $m \geq p$ we use (3.19) and everything that follows applies equally well.

Note that the choice of deformed gravitons in (3.20) is correlated to that in (3.15) or (3.19).

Our goal now is to prove that by using (3.20) on (3.15) the function $M_n(z)$ vanishes as z is taken to infinity.

Let us consider each term in the sum of (3.15) individually. There are two classes of terms. The first kind is when $\{i, j\} \subset \mathcal{J}$. The second kind is when $i \in \mathcal{I}$ and $j \in \mathcal{J}$.

Consider a term of the first kind,

$$\sum_{h=\pm} M_{\mathcal{I}} \left(\left\{ r_{\mathcal{I}}^- \right\}, \left\{ k_{\mathcal{I}}^+(w_{\mathcal{I}}) \right\}, -P_{\mathcal{I}}^h(w_{\mathcal{I}}) \right) \frac{1}{P_{\mathcal{I}}^2} M_{\mathcal{J}} \left(\left\{ r_{\mathcal{J}}^-(w_{\mathcal{I}}, z) \right\}, \left\{ k_{\mathcal{J}}^+(w_{\mathcal{I}}, z) \right\}, P_{\mathcal{I}}^{-h}(w_{\mathcal{I}}) \right). \tag{3.21}$$

Since both i^+ and j^- belong to $\mathcal{J}$, the momentum $P_{\mathcal{I}}$ does not depend on z. Likewise from the definition of $w_{\mathcal{I}}$ in (3.17) one can see that it does not depend on z. Therefore, the z dependence is confined to the second amplitude in (3.21) which we can write more explicitly as

$$M_{\mathcal{J}} \Big(\left\{ r_{\mathcal{J}'}^- \right\}, \left\{ k_{\mathcal{J}'}^+(w_{\mathcal{I}}) \right\}, \{\lambda^{(i)}(w_{\mathcal{I}}, z), \tilde{\lambda}^{(i)}\}, \{\lambda^{(j)}, \tilde{\lambda}^{(j)}(w_{\mathcal{I}}, z)\}, P_{\mathcal{I}}^{-h}(w_{\mathcal{I}}) \Big) \tag{3.22}$$

where the set $\mathcal{J}' = \mathcal{J} \setminus \{i, j\}$. It is straightforward to show that

$$\lambda^{(i)}(w_{\mathcal{I}}, z) = \lambda^{(i)}(w_{\mathcal{I}}) + z\lambda^{(j)}, \qquad \tilde{\lambda}^{(j)}(w_{\mathcal{I}}, z) = \tilde{\lambda}^{(j)}(w_{\mathcal{I}}) - z\tilde{\lambda}^{(i)}. \tag{3.23}$$

The fact that $\lambda^{(i)}(w_{\mathcal{I}})$ and $\tilde{\lambda}^{(j)}(w_{\mathcal{I}})$ get deformed exactly in the same way as $\lambda^{(i)}$ and $\tilde{\lambda}^{(j)}$ do is what allows us to use induction for these terms. Note that the amplitude

(3.22) is therefore a physical amplitude with a BCFW deformation. The number of gravitons is less than n and by our induction hypothesis it vanishes as z goes to infinity.

To complete the induction argument it suffices to note that the auxiliary recursion relations we are using can reduce any amplitude to products of three graviton amplitudes. Finally, recall that the Feynman diagram argument at the beginning of this section showed that amplitudes with less than five gravitons vanish at infinity under the BCFW deformation.

Consider now a term of the second kind,

$$\begin{aligned}
&\sum_{h=\pm} M_{\mathcal{I}}\left(\left\{r_{\mathcal{I}}^{-}\right\},\left\{k_{\mathcal{I}}^{+}(w_{\mathcal{I}}(z),z)\right\},-P_{\mathcal{I}}^{h}(w_{\mathcal{I}}(z),z)\right)\frac{1}{P_{\mathcal{I}}^{2}(z)} \\
&\qquad M_{\mathcal{J}}\left(\left\{r_{\mathcal{J}}^{-}(w_{\mathcal{I}}(z),z)\right\},\left\{k_{\mathcal{J}}^{+}(w_{\mathcal{I}}(z))\right\},P_{\mathcal{I}}^{-h}(w_{\mathcal{I}}(z),z)\right).
\end{aligned} \tag{3.24}$$

Recall that for these terms $i^+ \in \mathcal{I}$ while $j^- \in \mathcal{J}$. The z dependence we have displayed in (3.24) looks complicated at first since

$$w_{\mathcal{I}}(z) \;=\; \frac{P_{\mathcal{I}}(z)^2}{\sum_{k\in\mathcal{I}^+}\langle j|P_{\mathcal{I}}(z)|k]} \tag{3.25}$$

appears to be a rational function of z since $P_{\mathcal{I}}(z)_{a\dot{a}} = P_{\mathcal{I}\,a\dot{a}} + z\lambda_a^{(j)}\tilde{\lambda}_{\dot{a}}^{(i)}$. Note, however, that $\tilde{\lambda}^{(k)}$'s with $k \in \mathcal{I}^+$ do not depend on z and that the z dependence $z\lambda_a^{(j)}\tilde{\lambda}_{\dot{a}}^{(i)}$ in $P_{\mathcal{I}}(z)$ drops out of the denominator thanks to the contraction with $\langle j|$.

Then we find that $w_{\mathcal{I}}(z)$ is simply a linear function of z:

$$w_{\mathcal{I}}(z) \;=\; w_{\mathcal{I}} - z\left(\frac{\langle j|P_{\mathcal{I}}|i]}{\sum_{k\in\mathcal{I}^+}\langle j|P_{\mathcal{I}}|k]}\right) \tag{3.26}$$

where $w_{\mathcal{I}}$ is just the undeformed one, i.e., $w_{\mathcal{I}}(0)$.

The final step before we proceed to study the behavior for $z \to \infty$ using Feynman diagrams is to determine the properties of the internal graviton that enters with opposite helicities in the amplitudes of (3.24). The momentum of the internal graviton is given by

$$P_{\mathcal{I}}(w_{\mathcal{I}}(z),z) = \sum_{k\in\mathcal{I}^-} p_k + p_i(w_{\mathcal{I}}(z),z) + \sum_{s\in\mathcal{I}^+,\, s\neq i} p_s(w_{\mathcal{I}}(z)). \tag{3.27}$$

The important observation is that the z-dependence can be fully separated as follows

$$P_{\mathcal{I}}(w_{\mathcal{I}}(z),z) = P_{\mathcal{I}}(w_{\mathcal{I}}) + z\lambda^{(j)}\left(-\left(\frac{\langle j|P_{\mathcal{I}}|i]}{\sum_{k\in\mathcal{I}^+}\langle j|P_{\mathcal{I}}|k]}\right)\sum_{s\in\mathcal{I}^+}\tilde{\lambda}^{(s)}+\tilde{\lambda}^{(i)}\right) \tag{3.28}$$

where $P_{\mathcal{I}}(w_{\mathcal{I}})$ is the z-undeformed one, i.e., $P_{\mathcal{I}}(w_{\mathcal{I}}(0),0)$.

Note that we have written $P_{\mathcal{I}}(w_{\mathcal{I}}(z),z)$, which is a null vector, as the sum of two null vectors. For real momenta, this would imply that all three vectors are proportional. However, in this case all three vectors are complex and all that is required is that either all λ's or all $\tilde{\lambda}$'s be proportional. We claim that in this particular case all $\tilde{\lambda}$'s are proportional. To see this note that if we write $P_{\mathcal{I}}(w_{\mathcal{I}})_{a\dot{a}} = \lambda_a^{(P)}\tilde{\lambda}_{\dot{a}}^{(P)}$, then $\tilde{\lambda}_{\dot{a}}^{(P)}$ is proportional to $\zeta_{\dot{a}} = \eta^a P_{\mathcal{I}}(w_{\mathcal{I}})_{a\dot{a}}$ for some arbitrary spinor η^a.

We claim that the $\tilde{\lambda}$ spinor of the vector multiplying z in (3.28) is also proportional to $\zeta^{\dot{a}}$ if $\eta_a = \lambda_a^{(j)}$. In this case, $\zeta_{\dot{a}} = \lambda^{(j)\,a}P_{\mathcal{I}}(w_{\mathcal{I}})_{a\dot{a}} = \lambda^{(j)\,a}P_{\mathcal{I}\,a\dot{a}}$. To prove our claim consider the inner product of the two spinors

$$\left(-\left(\frac{\langle j|P_{\mathcal{I}}|i]}{\sum_{k\in\mathcal{I}^+}\langle j|P_{\mathcal{I}}|k]}\right)\sum_{s\in\mathcal{I}^+}\tilde{\lambda}_{\dot{a}}^{(s)}+\tilde{\lambda}_{\dot{a}}^{(i)}\right)\zeta^{\dot{a}} = \left(\left(\frac{\langle j|P_{\mathcal{I}}|i]}{\sum_{k\in\mathcal{I}^+}\langle j|P_{\mathcal{I}}|k]}\right)\sum_{s\in\mathcal{I}^+}\langle j|P_{\mathcal{I}}|s]-\langle j|P_{\mathcal{I}}|i]\right). \tag{3.29}$$

The right hand side of (3.29) vanishes trivially showing that the two spinors are proportional.

Therefore, it follows that we can write $P_{\mathcal{I}}(w_{\mathcal{I}}(z),z)_{a\dot{a}} = \lambda_a(z)\tilde{\lambda}_{\dot{a}}^P$ where $\lambda_a(z) = \lambda_a^{(P)} + z\beta\lambda_a^{(j)}$ for some β which is z independent. Note that if $z=0$ we recover $P_{\mathcal{I}}(w_{\mathcal{I}})_{a\dot{a}} = \lambda_a^{(P)}\tilde{\lambda}_{\dot{a}}^{(P)}$.

Let us turn to the analysis of the amplitudes in (3.24) to show that their product vanishes as z is taken to infinity. In other words, we will see that $M_{\mathcal{I}}$ and $M_{\mathcal{J}}$ may not vanish simultaneously but their product together with the propagator always does.

Consider the first amplitude $M_{\mathcal{I}}\left(\{r_{\mathcal{I}}^-\},\{k_{\mathcal{I}}^+(w_{\mathcal{I}}(z),z)\},-P_{\mathcal{I}}^h(w_{\mathcal{I}}(z),z)\right)$. Let the number of particles in the sets $\{r_{\mathcal{I}}^-\}$ and $\{k_{\mathcal{I}}^+\}$ be $m_{\mathcal{I}}$ and $p_{\mathcal{I}}$ respectively[3].

[3]Note that if $h=+$ this is a physical amplitude where only the λ's of positive helicity gravitons

The Feynman diagram analysis is very similar to that performed at the beginning of section III. The leading Feynman diagram is again one with only cubic vertices that posses a quadratic dependence on momenta. The number of cubic vertices is the total number of particles[4] minus two, i.e, $m_{\mathcal{I}} + p_{\mathcal{I}} - 1$. Therefore the contribution from vertices gives at most a factor of $z^{2(m_{\mathcal{I}}+p_{\mathcal{I}}-1)}$. There are $p_{\mathcal{I}} + 1$ polarization vectors that depend on z, giving a total contribution of $1/z^{2(p_{\mathcal{I}}+h)}$. Here we have used that since z enters in $-P^h_{\mathcal{I}}(w_{\mathcal{I}}(z), z)$ only through $\lambda(z)$, its polarization tensor gives a contribution of $1/z^{2h}$. Finally, we need to count the number of propagators that depend on z. It turns out that there are exactly $m_{\mathcal{I}} + p_{\mathcal{I}} - 2$ of them giving a contribution of $1/z^{m_{\mathcal{I}}+p_{\mathcal{I}}-2}$. This last statement is not obvious since there could be accidental cancelations of the z dependence. Let us continue with the argument here and we will prove that there is no accidental cancelations within the propagators in the next subsection[5]. Collecting all factors we get

$$M_{\mathcal{I}}\left(\left\{r^-_{\mathcal{I}}\right\}, \left\{k^+_{\mathcal{I}}(w_{\mathcal{I}}(z), z)\right\}, -P^h_{\mathcal{I}}(w_{\mathcal{I}}(z), z)\right) \sim \frac{1}{z^{p_{\mathcal{I}}-m_{\mathcal{I}}+2h}}. \tag{3.30}$$

The propagator $1/P^2_{\mathcal{I}}(z)$ in (3.24) goes as $1/z$.

The reader might have noticed that in this argument special care is required when $\mathcal{I}^+ = \{i\}$. We postpone the study of this case to the end of the section. Until then we simply assume that $i \in \mathcal{I}^+$ but $\mathcal{I}^+ \neq \{i\}$.

Consider now the second amplitude in (3.24),

$$M_{\mathcal{J}}\left(\left\{r^-_{\mathcal{J}}(w_{\mathcal{I}}(z), z)\right\}, \left\{k^+_{\mathcal{J}}(w_{\mathcal{I}}(z))\right\}, P^{-h}_{\mathcal{I}}(w_{\mathcal{I}}(z), z)\right). \tag{3.31}$$

Let the number of gravitons in $\{r^-_{\mathcal{J}}\}$ and $\{k^+_{\mathcal{J}}\}$ be $m_{\mathcal{J}}$ and $p_{\mathcal{J}}$ respectively.

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have been deformed. It is interesting to note that this deformation is basically the one introduced by Risager in [35] and later in [36] to construct an MHV diagram expansion for gravity amplitudes.

[4] The total number of gravitons in $M_{\mathcal{I}}$ is $m_{\mathcal{I}} + p_{\mathcal{I}} + 1$ since $-P^h_{\mathcal{I}}(w_{\mathcal{I}}(z), z)$ should also be included.

[5] More precisely, what we prove in the next subsection is that trivial cancelations in which neither propagators nor vertices depend on z are the only ones that can occur.

The cubic vertices give again a factor of $z^{2(p_{\mathcal{J}}+m_{\mathcal{J}}-1)}$. The polarization tensors give a factor of $1/z^{2(p_{\mathcal{J}}-h+1)}$. Here we have taken into account the contribution from the z dependent negative helicity graviton, i.e, the j^{th} graviton, and from the internal graviton, $P_{\mathcal{I}}^{-h}(w_{\mathcal{I}}(z), z)$. Finally, the propagators contribute again a factor of $1/z^{p_{\mathcal{J}}+m_{\mathcal{J}}-2}$. Collecting all factors we get

$$M_{\mathcal{J}}\left(\left\{r_{\mathcal{J}}^{-}(w_{\mathcal{I}}(z), z)\right\}, \left\{k_{\mathcal{J}}^{+}(w_{\mathcal{I}}(z))\right\}, P_{\mathcal{I}}^{-h}(w_{\mathcal{I}}(z), z)\right) \sim \frac{1}{z^{p_{\mathcal{J}}-m_{\mathcal{J}}-2(h-1)}}. \tag{3.32}$$

Combining all contributions from (3.30), the propagator and (3.32), the leading z behavior of (3.24) is $1/z^{p-m+3}$.

This shows that all the amplitudes with $p \geq m$ vanish at infinity.

As stated at the beginning of this subsection, a similar discussion holds for the case of amplitudes with $m \geq p$: by repeating the same counting starting from relation (3.19), the behavior at infinity of terms of the second kind turns out to be $1/z^{m-p+3}$. Terms of the first kind can again be treated by induction.

It is important to mention that the way amplitudes vanish at infinity is generically only as $1/z^2$. This is because terms of the first kind which are treated by induction vanish as three-graviton amplitudes do, i.e, as $1/z^2$.

This completes our proof of the vanishing of $M_n(z)$ as z goes to infinity up to the claim made about the number of propagators that contribute a $1/z$ factor and the exceptional case when $\mathcal{I}^+ = \{i\}$. We now turn to these crucial steps of our proof.

3.1.4 Analysis Of The Contribution From Propagators

One thing left to prove is that in the leading Feynman diagrams contributing to the first amplitude, $M_{\mathcal{I}}$, there are exactly $m_{\mathcal{I}}+p_{\mathcal{I}}-2$ propagators giving a $1/z$ contribution at infinity while in the second amplitude, $M_{\mathcal{J}}$, there are exactly $m_{\mathcal{J}}+p_{\mathcal{J}}-2$ of them.

Propagators In Leading Feynman Diagrams Of $M_{\mathcal{I}}$

Let us start with $M_{\mathcal{I}}$. The argument here uses similar elements to the ones given in the appendix where we provided a proof of the auxiliary recursion relations.

Consider a given Feynman diagram. A propagator naturally divides the diagram into two subdiagrams. Let use denote them by $\mathcal{L}$ and $\mathcal{R}$. Without loss of generality, we can always take the graviton with momentum $-P^h_{\mathcal{I}}(w_{\mathcal{I}}(z), z)$ to be in $\mathcal{R}$. In the set of positive helicity gravitons, $\{k^+_{\mathcal{I}}(w_{\mathcal{I}}(z), z)\}$, there is one that is special; the $i^{\rm th}$ graviton. We consider two cases, the first is when $i \in \mathcal{L}^+$ and the second when $i \in \mathcal{R}^+$.

Case A: $i \in \mathcal{L}^+$

Let $i \in \mathcal{L}^+$, then the propagator under consideration has the form

$$P_{\mathcal{L}}(w_{\mathcal{I}}(z), z) = P_{\mathcal{L}}(w_{\mathcal{I}}(0)) + z\lambda^{(j)} \left(-\frac{\langle j|P_{\mathcal{I}}|i]}{\sum_{k\in\mathcal{I}^+} \langle j|P_{\mathcal{I}}|k]} \sum_{s\in\mathcal{L}^+} \tilde{\lambda}^{(s)} + \tilde{\lambda}^{(i)} \right). \tag{3.33}$$

We are interested in asking when

$$P_{\mathcal{L}}(w_{\mathcal{I}}(z), z)^2 = P_{\mathcal{L}}(w_{\mathcal{I}}(0))^2 + z \left(\frac{\langle j|P_{\mathcal{I}}|i] \sum_{k\in\mathcal{L}^+} \langle j|P_{\mathcal{L}}|k]}{\sum_{k\in\mathcal{I}^+} \langle j|P_{\mathcal{I}}|k]} - \langle j|P_{\mathcal{L}}|i] \right) \tag{3.34}$$

can be z independent. Therefore we have to analyze under which conditions the factor multiplying z can be zero for a generic choice of momenta and polarization tensors of the physical gravitons subject only to the overall momentum conservation constrain.

Let us write the factor of interest as follows

$$\langle j|P_{\mathcal{I}}|i] \sum_{k\in\mathcal{L}^+} \langle j|P_{\mathcal{L}}|k] - \langle j|P_{\mathcal{L}}|i] \sum_{k\in\mathcal{I}^+} \langle j|P_{\mathcal{I}}|k] = \lambda^{(j)\,a}\lambda^{(j)\,b} P_{\mathcal{L}\,a\dot{a}} P_{\mathcal{I}\,b\dot{b}} T^{\dot{a}\dot{b}} \tag{3.35}$$

with

$$T^{\dot{a}\dot{b}} = \tilde{\lambda}^{(i)\,\dot{a}} \sum_{k\in\mathcal{L}^+} \tilde{\lambda}^{(k)\,\dot{b}} - \tilde{\lambda}^{(i)\,\dot{b}} \sum_{k\in\mathcal{I}^+} \tilde{\lambda}^{(k)\,\dot{a}}. \tag{3.36}$$

Here we have to consider two different cases[6]:

[6]There are actually three cases. The third is when $\mathcal{I}^+ = \mathcal{L}^+ = \{i\}$ but this is part of the special case that is considered at then end of the section.

- $\mathcal{I}^+ \setminus \mathcal{L}^+ \neq \emptyset$.

- $\mathcal{I}^+ = \mathcal{L}^+$ and $\mathcal{L}^+ \neq \{i\}$.

Let us start by assuming that $\mathcal{I}^+ \setminus \mathcal{L}^+$ is non-empty and that, say, $s \in \mathcal{I}^+ \setminus \mathcal{L}^+$. The space of kinematical invariants we consider is determined by the momentum and polarization tensors of each of the original gravitons. Consider both objects for the s^{th} graviton

$$\epsilon^{+\,(s)}_{a\dot{a},b\dot{b}} = \frac{\mu_a \tilde{\lambda}^{(s)}_{\dot{a}} \mu_b \tilde{\lambda}^{(s)}_{\dot{b}}}{\langle \mu, \lambda^{(s)} \rangle^2}, \qquad p^{(s)}_{a\dot{a}} = \lambda^{(s)}_a \tilde{\lambda}^{(s)}_{\dot{a}}. \tag{3.37}$$

It is clear that if we take $\{\lambda^{(s)}_a, \tilde{\lambda}^{(s)}_{\dot{a}}\}$ to $\{t^{-1}\lambda^{(s)}_a, t\tilde{\lambda}^{(s)}_{\dot{a}}\}$ with t a fourth root of unity, i.e, $t^4 = 1$ then (3.37) is invariant. Therefore, any quantity that vanishes for $t = 1$ must also vanish for all four values of t. In particular, it must be the case that (3.35) must vanish for all four values of t. Since momentum is not affected only the tensor $T^{\dot{a}\dot{b}}$ changes. Taking the difference between two values of t, say $t = 1$ and $t = i$, we find that $T^{\dot{a}\dot{b}}|_{t=1} - T^{\dot{a}\dot{b}}|_{t=i} \sim \tilde{\lambda}^{(i)\,\dot{b}} \tilde{\lambda}^{(s)\,\dot{a}}$. Therefore, the vanishing of (3.35) implies that of

$$\langle j|P_{\mathcal{L}}|i] \langle j|P_{\mathcal{I}}|s] = 0. \tag{3.38}$$

This condition is then equivalent to

$$\mathrm{tr}\,(\not{p}_j \not{P}_{\mathcal{L}}\, \not{p}_i \not{P}_{\mathcal{L}}) = 0 \qquad \text{or} \qquad \mathrm{tr}\,(\not{p}_j \not{P}_{\mathcal{I}}\, \not{p}_s \not{P}_{\mathcal{I}}) = 0 \tag{3.39}$$

but these are constraints on the kinematical space which are not satisfied at generic points.

The second case we have to consider is when $\mathcal{I}^+ = \mathcal{L}^+$ and $\mathcal{L}^+ \neq \{i\}$. Let us introduce the notation $\tilde{\mu}_{\dot{a}} = \sum_{k \in \mathcal{I}^+} \tilde{\lambda}^{(k)}_{\dot{a}}$. Therefore the condition we want to exclude is

$$\langle j|P_{\mathcal{I}}|i] \langle j|P_{\mathcal{L}}|\tilde{\mu}] - \langle j|P_{\mathcal{L}}|i] \langle j|P_{\mathcal{I}}|\tilde{\mu}] = 0. \tag{3.40}$$

Using Schouten's identity we can write this as

$$\langle j|P_{\mathcal{I}} P_{\mathcal{L}}|j\rangle [i, \tilde{\mu}] = 0 \tag{3.41}$$

where $\langle j|P_{\mathcal{I}}P_{\mathcal{L}}|j\rangle = \sum_{k\in\mathcal{I}}\sum_{s\in L}\langle\lambda^{(j)},\lambda^{(k)}\rangle[\tilde{\lambda}^{(k)},\tilde{\lambda}^{(s)}]\langle\lambda^{(s)},\lambda^{(j)}\rangle$. The vanishing of either factor implies a constraint for the space of kinematical invariants. In the case of the second factor this can easily be seen by choosing $s\in\mathcal{I}^{+}$ and $s\neq i$, then using the scaling by t with $t^4=1$ to conclude that $(p_s+p_i)^2=0$.

This completes the proof that the z dependence cannot drop out of any propagator and therefore all $m_{\mathcal{I}}+p_{\mathcal{I}}-2$ of them give a $1/z$ factor in $M_{\mathcal{I}}$ if $i\in\mathcal{L}$.

Case B: $i\in\mathcal{R}^{+}$

The analysis when $i\in\mathcal{R}$ is completely analogous except for the fact that there is one case that was not possible before. As we will show, this will correspond to diagrams which give a non-leading contribution.

Consider the analog of (3.34)

$$P_{\mathcal{L}}(w_{\mathcal{I}}(z),z)^2 = P_{\mathcal{L}}(w_{\mathcal{I}}(0))^2 + z\langle j|P_{\mathcal{I}}|i]\left(\frac{\sum_{k\in\mathcal{L}^{+}}\langle j|P_{\mathcal{L}}|k]}{\sum_{k\in\mathcal{I}^{+}}\langle j|P_{\mathcal{I}}|k]}\right). \tag{3.42}$$

The new case is when $\mathcal{L}^{+}=\emptyset$, then the z dependence drops out. Of course, this is not a problem because if the set $\mathcal{L}^{+}$ is empty it means that nothing on the subdiagram $\mathcal{L}$ depends on z, including the cubic vertices. Therefore, neither propagators nor cubic vertices contribute. One can then concentrate on the subdiagram $\mathcal{R}$, but this subdiagram has less particles than the total diagram and the same number of z-dependent polarization tensors. Therefore these diagrams go to zero even faster than diagrams where $\mathcal{L}^{+}$ is not empty.

Propagators In Leading Feynman Diagrams Of $M_{\mathcal{J}}$

Let us now study the leading Feynman diagrams contributing to $M_{\mathcal{J}}$. Again, the propagator divides the diagram in two subdiagrams that we denote $\mathcal{L}$ and $\mathcal{R}$. Without loss of generality, we can always take the graviton with momentum $P_{\mathcal{I}}^{-h}(w_{\mathcal{I}}(z),z)$ to be in $\mathcal{R}$. As in the previous discussion we have a special graviton, i.e, the j^{th} graviton. Therefore we have to consider two cases, $j\in\mathcal{L}$ and $j\in\mathcal{R}$.

Case A: $j \in \mathcal{L}^-$

Let us first consider the case $j \in \mathcal{L}$. The z dependence of $\tilde{\lambda}^{(j)}(w_{\mathcal{I}}(z), z)$ is the most complicated of all. This is why we write it explicitly

$$\tilde{\lambda}^{(j)}(w_{\mathcal{I}}(z), z)_{\dot{a}} = \tilde{\lambda}^{(j)}(w_{\mathcal{I}})_{\dot{a}} + z\left(-\tilde{\lambda}^{(i)}_{\dot{a}} + \frac{\langle j|P_{\mathcal{I}}|i]}{\sum_{k\in\mathcal{I}^+}\langle j|P_{\mathcal{I}}|k]}\sum_{s\in\{k^+\}}\tilde{\lambda}^{(s)}_{\dot{a}}\right). \tag{3.43}$$

Using this and the fact that the set of labels of all positive helicity gravitons $\{k^+\}$ must be equal to $\mathcal{I}^+ \cup \mathcal{J}^+$, we find that the propagator of interest has a momentum dependence of the form

$$P_{\mathcal{L}}(w_{\mathcal{I}}(z), z)^2 = P_{\mathcal{L}}(w_{\mathcal{I}}(0))^2 + z\left(\langle j|P_{\mathcal{L}}|i] - \langle j|P_{\mathcal{I}}|i]\frac{\sum_{k\in\mathcal{I}^+\cup(\mathcal{J}^+\setminus\mathcal{L}^+)}\langle j|P_{\mathcal{L}}|k]}{\sum_{k\in\mathcal{I}^+}\langle j|P_{\mathcal{I}}|k]}\right). \tag{3.44}$$

We are then interested in asking when this expression can be z independent.

The analysis is similar to the one given for $M_{\mathcal{I}}$ so we will be brief. The factor of interest is now

$$\langle j|P_{\mathcal{L}}|i]\sum_{k\in\mathcal{I}^+}\langle j|P_{\mathcal{I}}|k] - \langle j|P_{\mathcal{I}}|i]\sum_{k\in\mathcal{I}^+\cup(\mathcal{J}^+\setminus\mathcal{L}^+)}\langle j|P_{\mathcal{L}}|k]. \tag{3.45}$$

We have to consider two cases:

- $\mathcal{J}^+ \setminus \mathcal{L}^+ \neq \emptyset$.
- $\mathcal{J}^+ = \mathcal{L}^+$ and $\mathcal{I}^+ \neq \{i\}$.

In the first case we can assume that, say, the s^{th} graviton is in $\mathcal{J}^+ \setminus \mathcal{L}^+$. Then by using the argument that any statement about $\{\lambda^{(s)}, \tilde{\lambda}^{(s)}\}$ must also be true for $\{t^{-1}\lambda^{(s)}, t\tilde{\lambda}^{(s)}\}$ with $t^4 = 1$ one can show that the vanishing of (3.45) implies a nontrivial constraint on kinematical invariants that is not generically satisfied.

The second case is also similar to one we considered in the analysis of $M_{\mathcal{I}}$. Here we have that $\mathcal{I}^+ \cup (\mathcal{J}^+ \setminus \mathcal{L}^+) = \mathcal{I}^+ \cup \emptyset = \mathcal{I}^+$. Therefore (3.45) becomes

$$\langle j|P_{\mathcal{L}}|i]\langle j|P_{\mathcal{I}}|\tilde{\mu}] - \langle j|P_{\mathcal{I}}|i]\langle j|P_{\mathcal{L}}|\tilde{\mu}] \tag{3.46}$$

where $\tilde{\mu}_{\dot{a}} = \sum_{s\in\mathcal{I}^+\setminus\{i\}} \tilde{\lambda}^{(s)}$. Since by assumption $\mathcal{I}^+ \setminus \{i\} \neq \emptyset$ we can use Schouten's identity to derive non-trivial constraints on the kinematical invariants which are not satisfied for generic momenta.

Recall that the case when $\mathcal{I}^+ = \{i\}$ is special and will be treated separately.

Case B: $j \in \mathcal{R}^-$

In this case, the propagator of interest can be written as

$$P_{\mathcal{L}}(w_{\mathcal{I}}(z), z)^2 = P_{\mathcal{L}}(w_{\mathcal{I}}(0))^2 + z\langle j|P_{\mathcal{I}}|i] \left(\frac{\sum_{k\in\mathcal{L}^+}\langle j|P_{\mathcal{L}}|k]}{\sum_{k\in\mathcal{I}^+}\langle j|P_{\mathcal{I}}|k]} \right). \tag{3.47}$$

This is again similar to the corresponding case in $M_{\mathcal{I}}$. The only new case compared to when $j \in \mathcal{L}^-$ is when $\mathcal{L}^+$ is empty. Then nothing in $\mathcal{L}$ depends on z and we can consider a Feynman diagram that has less minus helicity gravitons than the original one and therefore it goes faster to zero at infinity than the leading diagrams obtained when $\mathcal{L}^+ \neq \emptyset$.

This conclude our discussion about the contribution of the propagators.

3.1.5 Analysis Of The Special Case $\mathcal{I}^+ = \{i\}$

Let us now consider the final case. This is when $\mathcal{I}^+ = \mathcal{L}^+ = \{i\}$. This case is quite interesting since several unexpected cancelations take place. Consider $w_{\mathcal{I}}(z)$ given in (3.26). In this case, it is easy to check that $w_{\mathcal{I}}(z) = w_{\mathcal{I}}(0) - z$. A consequence of this is that $\lambda^{(i)}(w_{\mathcal{I}}(z), z) = \lambda^{(i)}(z) + w_{\mathcal{I}}(z)\lambda^{(j)}$ becomes z-independent. To see this recall that $\lambda^{(i)}(z) = \lambda^{(i)} + z\lambda^{(j)}$. Therefore $\lambda^{(i)}(w_{\mathcal{I}}(z), z) = \lambda^{(i)}(w_{\mathcal{I}})$. This also implies that $P^h_{\mathcal{I}}(w_{\mathcal{I}}(z), z)$ is z independent. Therefore, the full amplitude $M_{\mathcal{I}}$ is z independent.

Recall that we are interested in the behavior of

$$\sum_{h=\pm} M^h_{\mathcal{I}} \frac{1}{P^2_{\mathcal{I}}(z)} M^{-h}_{\mathcal{J}}(z). \tag{3.48}$$

The propagator $1/P^2_{\mathcal{I}}(z)$ contributes a factor of $1/z$.

Now we have to look at

$$M_{\mathcal{J}}(z) = M_{\mathcal{J}}\left(\left\{r^-_{\mathcal{J}}(w_{\mathcal{I}}(z),z)\right\},\left\{k^+_{\mathcal{J}}(w_{\mathcal{I}}(z))\right\},P^{-h}_{\mathcal{I}}(w_{\mathcal{I}}(z),z)\right).$$

Let us study the z dependence of each graviton carefully. We have that the j^{th} graviton (which has negative helicity) and all positive helicity gravitons in $\mathcal{J}^+ = \left\{k^+_{\mathcal{J}}(w_{\mathcal{I}}(z))\right\}$ behave as

$$\tilde\lambda^{(j)}(w_{\mathcal{I}}(z),z) = \tilde\lambda^{(j)}(w_{\mathcal{I}}) + z\sum_{s\in\mathcal{J}^+}\tilde\lambda^{(s)}, \qquad \lambda^{(s)}(w_{\mathcal{I}}(z)) = \lambda^{(s)}(w_{\mathcal{I}}) - z\lambda^{(j)} \qquad \forall\, s\in\mathcal{J}^+. \tag{3.49}$$

Close inspection of (3.49) shows a striking fact. This deformation is exactly the same as the one that led to the auxiliary recursion relations in the first place, i.e, the deformation given in (3.13) but using z instead of w as deformation parameter and $\tilde\lambda^{(j)}(w_{\mathcal{I}})$ and $\lambda^{(s)}(w_{\mathcal{I}})$ as undeformed spinors. Finally, recall that $P_{\mathcal{I}}(w_{\mathcal{I}}(z),z)$, which also appears in $M_{\mathcal{J}}$, was shown to be z independent.

Now, if $h=+$ we have $P^-_{\mathcal{I}}(w_{\mathcal{I}})$ and therefore, $M_{\mathcal{J}}(z)$ is nothing but a physical amplitude under the maximal deformation (3.13). In the appendix, we showed that amplitudes vanish as the deformation parameter, which in this case is z, is taken to infinity if the number of pluses is greater than or equal to the number of minuses minus two. To see that this condition is satisfied in $M_{\mathcal{J}}$ note that since $\mathcal{I}^+ = \{i\}$ we have that the total number of positive helicity gravitons in $M_{\mathcal{J}}$ is $p-1$ while that of negative helicity gravitons is $m - m_{\mathcal{I}} + 1$. Since the number of external negative helicity gravitons in $M_{\mathcal{I}}$ must be at least one, i.e, $m_{\mathcal{I}} \geq 1$ and recalling that we are studying the case when $p \geq m$, we get the desired result.

The next case to consider is when $h=-$. Since $P^+_{\mathcal{I}}(w_{\mathcal{I}})$ is z independent, the deformation (3.49) of $M_{\mathcal{J}}$ is no longer maximal. However, it is possible to show that these terms are identically zero. This is obvious when the on-shell physical amplitude $M_{\mathcal{I}}$, which has only one positive helicity graviton, has more than two negative helicity gravitons.

Consider now the case when $M_{\mathcal{I}}$ has precisely two negative helicity gravitons. A three-graviton on-shell amplitude need not vanish if momenta are complex therefore this is a potentially dangerous case. Three-graviton amplitudes are given as the square of the gauge theory ones. Therefore we have

$$M_{\mathcal{I}}(i^{+}(w_{\mathcal{I}}), s^{-}, -P_{\mathcal{I}}^{-}(w_{\mathcal{I}})) \; = \; \left(\frac{\langle \lambda^{(s)}, \lambda^{(P)} \rangle^{3}}{\langle \lambda^{(P)}, \lambda^{(i)}(w_{\mathcal{I}}) \rangle \langle \lambda^{(i)}(w_{\mathcal{I}}), \lambda^{(s)} \rangle} \right)^{2} \tag{3.50}$$

where as in section III.C we have defined $P_{\mathcal{I}}(w_{\mathcal{I}})_{a\dot{a}} = \lambda_{a}^{(P)} \tilde{\lambda}_{\dot{a}}^{(P)}$.

Since this is a physical amplitude, momentum is conserved which means

$$\lambda^{(i)}(w_{\mathcal{I}})_{a} \tilde{\lambda}_{\dot{a}}^{(i)} + \lambda_{a}^{(s)} \tilde{\lambda}_{\dot{a}}^{(s)} \; = \; \lambda_{a}^{(P)} \tilde{\lambda}_{\dot{a}}^{(P)}. \tag{3.51}$$

For real momenta, this equation implies that all $\lambda's$ and all $\tilde{\lambda}'s$ are proportional. Therefore three-graviton amplitudes must vanish. For complex momenta, this need not be the case and one can have all $\tilde{\lambda}'s$ be proportional with the λ's unconstrained. In such a case (3.50) would not vanish.

We claim that, luckily in our case of interest, all $\lambda's$ are proportional and (3.50) vanishes. To see this note that $w_{\mathcal{I}} = -\langle i, s \rangle / \langle j, s \rangle$ and $\lambda^{(i)}(w_{\mathcal{I}})_{a} = \lambda_{a}^{(i)} + w_{\mathcal{I}} \lambda_{a}^{(j)}$, therefore $\langle \lambda^{(i)}(w_{\mathcal{I}}), \lambda^{(s)} \rangle = 0$. Contracting (3.51) with $\lambda^{(s)\,a}$ we find $\langle \lambda^{(P)}, \lambda^{(s)} \rangle \tilde{\lambda}_{\dot{a}}^{(P)} = 0$. Therefore we must have $\langle \lambda^{(P)}, \lambda^{(s)} \rangle = 0$ which completes the proof of our claim.

From (3.50), this condition implies that $M_{\mathcal{I}}$ is identically zero. Thus, we can conclude that the cases of $M_{\mathcal{J}}$ with a non-maximal deformation are not there.

This is the end of our proof. We now turn to some extensions and applications of the BCFW recursion relations that can be obtained by using Ward identities.

3.2 Ward Identities

Our proof of the BCFW recursion relations was based on deforming two gravitons of opposite helicities, i^{+} and j^{-}, in the following way:

$$\lambda^{(i)}(z) = \lambda^{(i)} + z\lambda^{(j)}, \qquad \tilde{\lambda}^{(j)}(z) = \tilde{\lambda}^{(j)} - z\tilde{\lambda}^{(i)}. \tag{3.52}$$

However, it is known that in gauge theory, deformed amplitudes also vanish at infinity if the helicities (h_i, h_j) of the deformed gluons are $(-,-)$ or $(+,+)$ [10]. It would be interesting to prove a similar statement for General Relativity. Here we show that this is indeed very straightforward in the case of MHV scattering amplitudes if one uses Ward identities.

The Ward identity of relevance for our discussion can be found for example in [37] and it is given by

$$\frac{M^{\text{MHV}}_{l,m}}{\langle\lambda^{(l)}, \lambda^{(m)}\rangle^8} = \frac{M^{\text{MHV}}_{s,q}}{\langle\lambda^{(s)}, \lambda^{(q)}\rangle^8}, \tag{3.53}$$

where the notation $M^{\text{MHV}}_{a,b}$ indicates that the gravitons a and b in this amplitude are the ones with negative helicity.

Consider first the $(+,+)$ case. We use the Ward identity (3.53) to relate it to the usual $(+,-)$ case. For clarity purposes, we explicitly exhibit the dependence of the amplitudes on only four gravitons: $\{l, m, i, j\}$. The dependence on the rest of the gravitons (all of which have positive helicity) will be implicit. Then we have

$$M^{\text{MHV}}_n(i^+(z), j^+(z), l^-, m^-) \;=\; \left(\frac{\langle\lambda^{(l)}, \lambda^{(m)}\rangle}{\langle\lambda^{(j)}, \lambda^{(l)}\rangle}\right)^8 M^{\text{MHV}}_n(i^+(z), j^-(z), l^-, m^+). \tag{3.54}$$

The MHV amplitude on the right hand-side is deformed as in (3.52), thus it vanishes at infinity by our proof. Since both inner products expressed explicitly in (3.54) do not depend on z, the amplitude on the left hand side of (3.54), where $(h_i, h_j) = (+,+)$, will vanish as z goes to infinity.

Consider now the $(-,-)$ case. Using again the Ward identity (3.53) we have

$$M^{\text{MHV}}_n(i^-(z), j^-(z), l^+, m^+) \;=\; \left(\frac{\langle\lambda^{(i)}(z), \lambda^{(j)}\rangle}{\langle\lambda^{(j)}, \lambda^{(l)}\rangle}\right)^8 M^{\text{MHV}}_n(i^+(z), j^-(z), l^-, m^+) \tag{3.55}$$

Note that $\langle\lambda^{(i)}(z), \lambda^{(j)}\rangle$ does not depend on z since $\lambda^{(i)}(z) = \lambda^{(i)} + z\lambda^{(j)}$. Therefore, the amplitude still vanishes in this case.

In [20], a very nice compact formula was conjectured for MHV amplitudes of gravitons by assuming the validity of BCFW recursion relations obtained via a deformation of the two negative helicity gravitons. Our proof and the discussion in this section validates the recursion relations used to construct the all multiplicity ansatz. It would be highly desirable to show that the formula proposed by Bedford et al. [20] does indeed satisfy the recursion relations. The formula is explicitly given by

$$M_n(1^-,2^-,i_1^+,\ldots,i_{n-2}^+) = \frac{\langle 1,2\rangle^6[1,i_{n-2}]}{\langle 1,i_{n-2}\rangle}G(i_1,i_2,i_3)\prod_{s=3}^{n-3}\frac{\langle 2|i_1+\ldots+i_{s-1}|i_s]}{\langle i_s,i_{s+1}\rangle\langle 2,i_{s+1}\rangle}+\mathcal{P}(i_1,\ldots,i_{n-2}) \tag{3.56}$$

where $\mathcal{P}(i_1,\ldots,i_{n-2})$ indicates a sum over all permutations of $(i_1,\ldots,i_{n-2})$ and

$$G(i_1,i_2,i_3) = \frac{1}{2}\left(\frac{[i_1,i_2]}{\langle 2,i_1\rangle\langle 2,i_2\rangle\langle i_1,i_2\rangle\langle i_2,i_3\rangle\langle i_1,i_3\rangle}\right). \tag{3.57}$$

It is also interesting to show why the case $(h_i,h_j) = (-,+)$ does not lead to recursion relations. Using the Ward identity (3.53) once again we have

$$M_n^{\text{MHV}}(i^-(z),j^+(z),l^+,m^-) = \left(\frac{\langle\lambda^{(i)}(z),\lambda^{(m)}\rangle}{\langle\lambda^{(i)}(z),\lambda^{(j)}\rangle}\right)^8 M_n^{\text{MHV}}(i^-(z),j^-(z),l^+,m^+) \tag{3.58}$$

The amplitude on the right hand-side vanishes as z goes to infinity. However, $\langle\lambda^{(i)}(z),\lambda^{(m)}\rangle^8$ contributes with a factor of z^8 while $\langle\lambda^{(i)}(z),\lambda^{(j)}\rangle$ is z independent. Either using BGK (together with (3.55)) or directly (3.56), one can show that $M_n^{\text{MHV}}(i^-(z),j^-(z),l^+,m^+)$ goes like $1/z^2$, therefore the amplitude with $(h_i,h_j) = (-,+)$ behaves as z^6 at infinity.

Chapter 4

General Relativity as a fully-constructible theory

As we have just proven, using the BCFW deformation, graviton scattering amplitudes can be expressed as the product of 3-particle amplitudes and propagators. Once these 3-graviton amplitudes are known, the theory is fully determined! The reader might remember that we obtained the form of the 3-gluon amplitude in section 2.5 by solely assuming Poincaré symmetry.

We will now generalize this to spin 2 theories allowing us to construct GR without recurring to a specific interaction Lagrangian nor to the energy-momentum tensor [7]. Again, momentum conservation yields that the 3-amplitude can only be function of the holomorphic or of antiholomorphic spinors:

$$\begin{aligned} M_3 = \sum_{n_1,n_2,n_3} \quad & (a^H_{n_1,n_2,n_3} < 1,2 >^{n_3} < 2,3 >^{n_1} < 1,3 >^{n_2} \\ & + a^A_{n'_1,n'_2,n'_3} [1,2]^{n'_3} [2,3]^{n'_1} [1,3]^{n'_2}). \end{aligned} \tag{4.1}$$

[7] The reader might remember that we used the fact that the 3-vertices are quadratic in momenta in order to prove the validity of the BCFW recursion relations in GR and that this fact was deduced from the Lagrangian. However, in the following, we will derive the 3-vertex power of momenta again by solely using the Poincaré group

Rescaling the spinors as (2.39) we get using (2.40) $n'_i = -n_i$ and

$$n_1 = h_1 - h_2 - h_3, \tag{4.2}$$
$$n_2 = h_2 - h_3 - h_1,$$
$$n_3 = h_3 - h_1 - h_2$$

where h_i is the helicity of the i^{th} graviton. In the case of gravity, $h = \pm 2$. As in section 2.5, we now take the real momentum limit ($< i, j >, [i, j] \to 0$)and get that $a^A = 0$ if $h_1 + h_2 + h_3 < 0$ and $a^H = 0$ if $h_1 + h_2 + h_3 > 0$.

4.1 The Vertex Functions

We can use (4.1) to deduce the 3-vertex function power of momenta. The 3-graviton amplitude is the product of 3 dimensionless polarization tensors and the 3-vertex function. Looking at (4.1), we see by dimensional analysis that its power of momenta needs to be $L_3 = |n_1 + n_2 + n_3| \ = \ |h_1 + h_2 + h_3|$.

We call constructible a theory which 4-particle amplitudes can be determined from its 3-particle amplitudes using the BCFW recursion relations. This can be achieve if the 4-particle amplitude M_4 vanishes at infinity. As we saw, this is the case with GR since GR is even fully constructible (M_n can be constructed from 3-graviton amplitudes). We will now use that fact to find some constraints on the vertices power of momenta.

We consider the amplitude $M_4(1^+, 2^-, 3^+, 4^-)$, where we apply the usual shift (3.3) with $i^+ = 1^+$ and $j^- = 2^-$. We consider the diagram with 1^+ and 3^+ at one 3-vertex and 2^- and 4^- at a second 3-vertex.

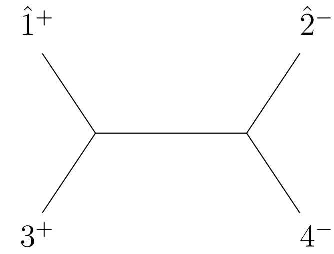

Figure 2: This diagram would go as z^7 is all same helicity cubic interactions were allowed.

If we allowed for all plus or all minus helicity interactions, we could have both vertex go as momentum to the power 6. Thus, with the contribution of the polarization tensors and propagator:

$$M_4(z) \sim \frac{1}{z^4} \cdot z^{12} \cdot \frac{1}{z} = z^7 \tag{4.3}$$

which blows up dramatically. Although there could be some magical cancelations between diagrams or even within the diagram (coming from the precise vertex structure), we choose that three same helicity gravitons do not interact. Thus, in all cases, $L_3 = 2$.

In order to have the diagram with one 4-vertex vanish at infinity, we need $L_4 < 4$, meaning less than four derivatives. Thus, we can't have R^2 or higher order terms.

4.2 Many Spin 2 Particles, The 4 Particle Test

As we briefly mentionned previously, the 4-particle amplitude cannot depend on which pair of particles we deform. This is the 4-particle test for a constructible theory [25]. We will use this test to prove that the cubic spin 2 couplings do not depend one the type of particle (independant of color, charge, etc.). We now assume that each graviton i has an additionnal quantum number a_i and that the gravity couplings depend on this number. Recall that all same helicity 3-amplitudes vanish. We thus have from

(4.1) and (4.2) the following 3-amplitudes (up to a constant), our building blocks for the 4-graviton amplitude:

$$M_3(1^-,2^-,3^+) = f_{a_1a_2a_3}\frac{<1,2>^6}{<2,3>^2<3,1>^2}, \quad (4.4)$$

$$M_3(1^+,2^+,3^-) = f_{a_1a_2a_3}\frac{[1,2]^6}{[2,3]^2[3,1]^2}. \quad (4.5)$$

We can mention here that $f_{a_1a_2a_3}$ needs to be symmetric under label exchange since the amplitude itself needs to be invariant under the exchange of labels.

We want to calculate $M_4(1^+,2^-,3^+,4^-)$. We first do so by deforming the first and second gravitons. We will do the calculation in detail as it also serves as an example of the BCFW recursion relations in General Relativity. We have to compute three diagrams:

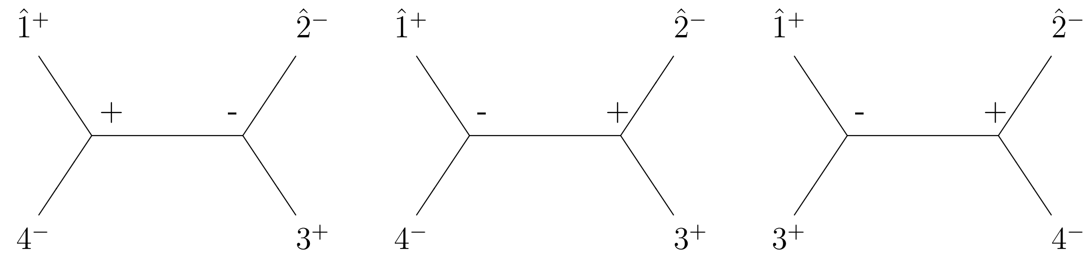

Figure 3: The 3 diagrams entering in the computation of $M_4(1^+,2^-,3^+,4^-)$.

Using the same notation as in section 2.5, we have for the first diagram:

$$M_4 = \sum_{a_I} f_{a_1a_Ia_4}\frac{[1,\hat{P}_{14}]^6}{[\hat{P}_{14},4]^2[4,1]^2} f_{a_Ia_2a_3}\frac{<\hat{P}_{14},2>^2}{<2,3>^2<3,\hat{P}_{14}>^2}\frac{1}{P_{14}^2}. \quad (4.6)$$

We now have to express $\hat{P}_{14}$ as a $\lambda^{(\hat{P})}\tilde{\lambda}^{(\hat{P})}$. We first get from:

$$\hat{P}_{14}^2 =<1,4>[1,4] + z_{14}<2,4>[1,4] = 0 \quad (4.7)$$

that $z_{14} = -<1,4>/<2,4>$. Thus, we have for $\hat{\lambda}^{(1)}$:

$$\hat{\lambda}^{(1)} = \lambda^{(1)} - \frac{<1,4>}{<2,4>}\lambda^{(2)}. \quad (4.8)$$

One sees that $<\hat{\lambda}^{(1)},\lambda^{(4)}>=0$. To find the constant of proportionality α, we contract (4.8) with $\lambda^{(2)}$:

$$<\hat{\lambda}^{(1)},\lambda^{(2)}>=\alpha<4,2>=<1,2>\,. \tag{4.9}$$

Thus,

$$\hat{\lambda}^{(1)}=\frac{<2,1>}{<2,4>}\lambda^{(4)}. \tag{4.10}$$

Now, subbing into $\hat{P}_{14}$, we get:

$$\hat{P}_{14}=(\tilde{\lambda}^{(1)}\frac{<2,1>}{<2,4>}+\tilde{\lambda}^{(4)})\lambda^{(4)}. \tag{4.11}$$

We then pick $\lambda^{(\hat{P})}=\lambda^{(4)}$. From momentum conservation,$(<2,1>/<2,4>)=(-[4,3]/[1,3])$. One can use this relation to see that $\tilde{\lambda}^{(\hat{P})}$ is proportional to $\tilde{\lambda}^{(3)}$. The proportionality constant is easily found to be $([1,4]/[1,3])$. Thus:

$$\hat{P}_{14}=\frac{[1,4]}{[1,3]}\lambda^{(4)}\tilde{\lambda}^{(3)}. \tag{4.12}$$

The reader might have noticed that we can rescale $\lambda^{(\hat{P})}$ and $\tilde{\lambda}^{(\hat{P})}$ while keeping $\hat{P}_{14}$ invariant. However, that won't be a problem since, as we will soon see, the amplitude will always contain the same number of $\lambda^{(\hat{P})}$'s and $\tilde{\lambda}^{(\hat{P})}$'s. We are now ready to substitute (4.12) into (4.6). Using momentum conservation, one gets for the first diagram:

$$\sum_{a_I} f_{a_1a_Ia_4}f_{a_Ia_2a_3}\frac{<2,4>^7\,[1,3]}{<1,2>^2<2,3><3,4>^2<4,1>}. \tag{4.13}$$

Since, the second diagram contains the factors $<4,4>$ and $[3,3]$ in the numerator, it does not contribute. We can now follow exactly the same procedure with the third diagram and find its amplitude:

$$\sum_{a_I} f_{a_Ia_3a_1}f_{a_2a_4a_I}\frac{<2,4>^6\,[1,3]}{<1,2>^2<4,3><3,1><3,4>}. \tag{4.14}$$

Thus adding (4.13) and (4.14), we find the following 4-particle amplitude:

$$
\begin{aligned}
M_4 = & \sum_{a_I} f_{a_1 a_I a_4} f_{a_I a_2 a_3} \frac{< 2,4 >^7 [1,3]}{< 1,2 >^2 < 2,3 >< 3,4 >^2 < 4,1 >} \\
& + \sum_{a_I} f_{a_I a_3 a_1} f_{a_2 a_4 a_I} \frac{< 2,4 >^6 [1,3]}{< 1,2 >^2 < 4,3 >< 3,1 >< 3,4 >}.
\end{aligned} \tag{4.15}
$$

We can now repeat the procedure, but this time deforming the first and fourth graviton. This basically simply corresponds to exchanging the labels 2 and 4 in (4.15). We get:

$$
\begin{aligned}
M_4' = & \sum_{a_I} f_{a_1 a_I a_2} f_{a_I a_4 a_3} \frac{< 4,2 >^7 [1,3]}{< 1,4 >^2 < 4,3 >< 3,2 >^2 < 2,1 >} \\
& + \sum_{a_I} f_{a_I a_3 a_1} f_{a_4 a_2 a_I} \frac{< 2,4 >^6 [1,3]}{< 1,4 >^2 < 2,3 >< 3,1 >< 3,2 >}.
\end{aligned} \tag{4.16}
$$

The 4-particle test requires that $M_4' - M_4 = 0$. At first glance, M_4 and M_4' look radically different since, for instance, the denominator factors are not the same. However, one can use Schouten identity,

$$
< i,j >< k,l > = < i,k >< j,l > + < i,l >< k,j >, \tag{4.17}
$$

to relate the expressions. Using (4.17) repetitively one can find that M_4 and M_4' are equal only if all the $\sum_{a_I} f_{a_i a_j a_k} f_{a_i' a_j' a_k'}$ are equal. We now define an algebra with the $f_{a_i a_j a_k}$'s as structure constants:

$$
\varepsilon_a * \varepsilon_b = f_{abc} \varepsilon_c. \tag{4.18}
$$

Since the all structure constant products are equal, we see that our algebra is commutative and associative. Thus, it is reducible and the couplings are independent of the quantum number a_i.

4.3 Explicit Results

As an application of the BCFW recursion relations in GR, we will express explicitly some graviton scattering amplitudes. We can first simplify (4.15) using Schouten identity and momentum conservation. Including all constants and the momentum conservation delta function, we have:

$$M_4(1^+, 2^-, 3^+, 4^-) = \kappa^2 \delta^{(4)}(\sum_{i=1}^{4} \lambda^{(i)} \tilde{\lambda}^{(i)}) \frac{< 2,4 >^8}{< 1,2 >^2 < 2,3 >^2 < 3,4 >^2 < 4,1 >^2} \quad (4.19)$$

where κ is the gravity coupling constant ($\kappa^2 = 8\pi G$ with G the Newtonian constant of gravitation). In order to obtain the conjugate amplitude, one simply replaces the holomorphic products by antiholomorphic products. Thus, we can easily obtain the cross-section from the previous equation:

$$\begin{aligned} |M_4(1^+, 2^-, 3^+, 4^-)|^2 = \quad & \kappa^4 \delta^{(4)}(\sum_{i=1}^{4} \lambda^{(i)} \tilde{\lambda}^{(i)}) \times \\ & \frac{< 2,4 >^8 \ [2,4]^8}{< 1,2 >^2 \ [1,2]^2 < 2,3 >^2 \ [2,3]^2 < 3,4 >^2 \ [3,4]^2 < 4,1 >^2 \ [4,1]^2}. \end{aligned} \quad (4.20)$$

This can be reexpressed in terms of the momentum vectors:

$$|M_4(1^+, 2^-, 3^+, 4^-)|^2 = \kappa^4 \delta^{(4)}(\sum_{i=1}^{4} p_i) \frac{(p_2 \cdot p_4)^8}{(p_1 \cdot p_2)^2 (p_2 \cdot p_3)^2 (p_3 \cdot p_4)^2 (p_4 \cdot p_1)^2}. \quad (4.21)$$

Closed expressions for all amplitudes up to six gravitons have been calculated using the BCFW prescription [21]. Also, as mentioned in section 3.2, a compact formula for graviton MHV amplitudes has been proposed by Bedford et al. [20] by assuming the validity of the BCFW recursion relations in GR. Explicitly, the formula is as follows:

$$M_n(1^-, 2^-, i_1^+, \ldots, i_{n-2}^+) = \frac{\langle 1,2 \rangle^6 [1, i_{n-2}]}{\langle 1, i_{n-2} \rangle} G(i_1, i_2, i_3) \prod_{s=3}^{n-3} \frac{\langle 2 | i_1 + \ldots + i_{s-1} | i_s]}{\langle i_s, i_{s+1} \rangle \langle 2, i_{s+1} \rangle} + \mathcal{P}(i_1, \ldots, i_{n-2}) \quad (4.22)$$

where $\mathcal{P}(i_1, \ldots, i_{n-2})$ indicates a sum over all permutations of $(i_1, \ldots, i_{n-2})$ and

$$G(i_1, i_2, i_3) = \frac{1}{2} \left(\frac{[i_1, i_2]}{\langle 2, i_1 \rangle \langle 2, i_2 \rangle \langle i_1, i_2 \rangle \langle i_2, i_3 \rangle \langle i_1, i_3 \rangle} \right). \quad (4.23)$$

This formula is more compact than the one usually used in MHV amplitudes calculations: the BGK formula [22]. Moreover, the BGK formula has only been proven numerically for $n < 11$. Now that the validity of the BCFW prescription has been proven, it would be interesting to show that the conjecture (4.22) satisfies the recursion relations for any number of gravitons.

Chapter 5

Conclusions and Future Directions

In this thesis, we have proven the validity of the BCFW recursion relations in General relativity. We recall that in BCFW, the positive helicity i^{th} graviton holomorphic spinor and negative helicity j^{th} graviton antiholomorphic spinor are deformed as follows:

$$\lambda^{(i)}(z) = \lambda^{(i)} + z\lambda^{(j)}, \qquad \tilde{\lambda}^{(j)}(z) = \tilde{\lambda}^{(j)} - z\tilde{\lambda}^{(i)}. \tag{5.1}$$

The deformed amplitude $M_n(z)$ is a rational function. It was shown that it only contains simple poles, located when the internal propagators (denoted by $1/P_{\mathcal{I}}^2$) go on shell at $z = z_{\mathcal{I}}$. Thus, if $M_n(z)$ vanishes as z goes to infinity, it can be expressed as a sum over its residues. Proving it was the case and setting $z = 0$ we obtained our recursion relation for the physical amplitude M_n:

$$M_n = \sum_{\mathcal{I},\mathcal{J}} \sum_{h=\pm} M_{\mathcal{I}}^h(z_{\mathcal{I}}) \frac{1}{P_{\mathcal{I}}^2} M_{\mathcal{J}}^{-h}(z_{\mathcal{I}}) \tag{5.2}$$

where we sum over all allowed partitions of the n gravitons into the sets $\mathcal{I}$ and $\mathcal{J}$ and over the helicity h of the on-shell propagator $1/P_{\mathcal{I}}^2$.

The main difficulty we encountered was to prove that $M_n(z)$ vanishes at infinity. We can mention that the deformed amplitude behavior at infinity is also an interesting problem by itself. It represents the UV limit of the theory in some direction. Let us review here the main steps we followed in order to demonstrate that the theory is well behaved as z goes to infinity. First, we recalled that individual Feynman diagrams behave like z^{n-5} in the large z limit. We thus used an auxiliary recursion relation (parameter w) to reexpress M_n. The deformation applied depends on the relative numbers of positive versus negative helicity gravitons. We denoted by $\{r^-\}$ the set containing the m negative helicity gravitons and by $\{k^+\}$ the set with the p positive helicity gravitons. If $p \geq m$, we apply the following deformation[8]:

$$\tilde{\lambda}^{(j)}(w) = \tilde{\lambda}^{(j)} - w \sum_{s \in \{k^+\}} \tilde{\lambda}^{(s)}, \qquad \lambda^{(k)}(w) = \lambda^{(k)} + w\lambda^{(j)}, \qquad \forall\, k \in \{k^+\}. \tag{5.3}$$

We then proceeded to show that $M_n(w)$ vanishes at infinity directly from Feynman diagram analysis. We analyzed the singularity structure of $M_n(w)$ and noticed that it only contains simple poles located at $w = w_I$. We could then expand $M_n(w)$ as a sum over its residues at the poles. Setting $w = 0$, we expressed M_n as follows:

$$M_n = \sum_{\mathcal{I},\mathcal{J}} \sum_{h=\pm} M_{\mathcal{I}}^h(w_{\mathcal{I}}) \frac{1}{P_{\mathcal{I}}^2} M_{\mathcal{J}}^{-h}(w_{\mathcal{I}}). \tag{5.4}$$

One can see that this already provides a valid method for computing recursively graviton scattering amplitudes. The terms in (5.4) contain physical amplitudes with less than n legs. However, this first recursion relations can be impractical since it contains a lot of terms caused by the large number of deformed spinors.

We thus applied the the usual BCFW deformation to (5.4) and obtained the following expression:

$$M_n(z) = \sum_{\{i,j\} \subset \mathcal{J}} \sum_{h=\pm} M_{\mathcal{I}}^h(w_{\mathcal{I}}) \frac{1}{P_{\mathcal{I}}^2} M_{\mathcal{J}}^{-h}(w_{\mathcal{I}}, z) + \sum_{i \in \mathcal{I}, j \in \mathcal{J}} \sum_{h=\pm} M_{\mathcal{I}}^h(w_{\mathcal{I}}(z), z) \frac{1}{P_{\mathcal{I}}^2(z)} M_{\mathcal{J}}^{-h}(w_{\mathcal{I}}(z), z). \tag{5.5}$$

where the z-dependance enters explicitly as well as through $w_{\mathcal{I}}(z)$. We showed by induction that the first term vanishes at infinity. For the second term we had to

[8]For $m \geq p$ we apply the deformation (3.14); the proof is completely analogous to the $p \geq m$ case.

carefully consider the contribution of each term and noticed that although $M_{\mathcal{I}}$ or $M_{\mathcal{J}}$ might not vanish individually, their product with the propagator would. Thus, $M_n(z)$ vanishes at infinity and the BCFW prescription is valid in General Relativity.

General Relativity is then fully constructible. All tree-level amplitudes can be computed from 3-graviton amplitudes. We also showed that these 3-graviton amplitudes can be uniquely determined using Poincaré symmetry.

The main result of this thesis, the proof that $M_n(z)$ vanishes at infinity in GR, inspires interesting questions. As previously mentioned, the z goes to infinity limit corresponds to the UV behavior of the theory in some direction. At tree level, we have seen that the behavior of individual Feynman diagrams does not represent the behavior of the full theory. Individual diagrams were diverging as z^{n-5} while the full theory is well defined at infinity!

One might then wonder if the same phenomenon arises at loop level. However, it is well known that pure General Relativity is already ill defined at two loops. $\mathcal{N} = 8$ supergravity had for a long time been thought to diverge at 3 loops. This belief came from power counting arguments although the precise counter term had not been computed. However, some new evidence points to possible surprising cancelations. It has been hypothesized that $\mathcal{N} = 8$ SUGRA exhibits the same behavior as $\mathcal{N} = 4$ super-Yang-Mills at infinity (see [38] and references therein). Thus, it seems possible that, for $\mathcal{N} = 8$ SUGRA, the behavior of individual Feynman diagrams does not represent the full behavior of the theory. If that is the case, one could speculate that $\mathcal{N} = 8$ supergravity is possibly finite to all orders in perturbation theory.

It would also be interesting to see whether recursion relations similar to the ones described in this thesis could be applied at strong coupling. Recently, Alday and Maldacena have given a prescription for computing scattering amplitudes of gluons in $\mathcal{N} = 4$ super-Yang-Mills using the AdS/CFT correspondance[39]. They have also noticed similarities between the computation of scattering amplitudes at strong coupling and the calculation of lightlike Wilson loops. Their work leads to many interesting avenues. For instance, very recently, Wilson loops were used to compute MHV amplitudes in $\mathcal{N} = 4$ super-Yang-Mills [40]. Thus, in light of these new developments, it would be very interesting to find recursion relations valid at strong coupling.

Appendix A

Proof Of Auxiliary Recursion Relations

In the main part of the proof of the validity of the BCFW recursion relations in GR, we used certain auxiliary recursion relations to prove that $M_n(z)$ vanishes as z is taken to infinity under the BCFW deformation. It is therefore very important to establish the validity of the auxiliary recursion relations.

Consider the case when then number of positive helicity gravitons is larger or equal than the number of negative helicity ones, i.e, $p \geq m$. The case when $m \geq p$ is completely analogous. Let us start by constructing a rational function $M_n(w)$ of a complex variable w via the deformation (3.13), i.e,

$$\tilde{\lambda}^{(j)}(w) = \tilde{\lambda}^{(j)} - w \sum_{s\in\{k^+\}} \tilde{\lambda}^{(s)}, \qquad \lambda^{(k)}(w) = \lambda^{(k)} + w\lambda^{(j)}, \qquad \forall k \in \{k^+\} \tag{A.1}$$

where j is a negative helicity graviton and $\{k^+\}$ is the set of all positive helicity gravitons in M_n.

The claim is that $M_n(w)$ vanishes as w is taken to infinity and its only singularities are simple poles at finite values of w.

A.1 Vanishing Of $M_n(w)$ At Infinity

Let us prove that $M_n(w)$ vanishes as $w \to \infty$. Consider the leading Feynman diagram that contributes to $M_n(w)$. Such a diagram has $n-2$ cubic vertices each contributing a factor of w^2. It also has $p+1$ polarization tensors that depend on w and give $1/w^2$ each. Finally, we claim that all $n-3$ propagators that can possibly depend on w actually do giving each a contribution of $1/w$. Putting all contributions together we find that the leading Feynman diagrams go like $1/w^{p-m+3}$. Therefore, if $p \geq m$ then $M_n(w) \to 0$ as $w \to \infty$.

We are only left to prove that $n-3$ propagators depend on w. A similar statement has to be proven in section III.D. The proof there is more involved since it requires the study of many cases. The discussion that follows can be thought of as a warm up for that in section III.D.

Consider a given Feynman diagram. A propagator naturally divides the diagram into two sub-diagrams. Let us denote them by $\mathcal{I}$ and $\mathcal{J}$. Without loss of generality, we can always take the j^{th} graviton to be in $\mathcal{J}$. Let us denote the set of positive helicity gravitons in $\mathcal{I}$ by $\mathcal{I}^+$.

The propagator under consideration has the form $1/P^2_{\mathcal{I}}(w)$ with

$$P^2_{\mathcal{I}}(w) = P^2_{\mathcal{I}} - w \sum_{k \in \mathcal{I}^+} \langle j|P_{\mathcal{I}}|k] \tag{A.2}$$

where $P_{\mathcal{I}} = P_{\mathcal{I}}(0)$.

The only way the w dependence can drop out of the propagator is that $\sum_{k\in\mathcal{I}^+}\langle j|P_{\mathcal{I}}|k] = 0$.

Since the j^{th} graviton belongs to $\mathcal{J}$, the condition $\sum_{k\in\mathcal{I}^+}\langle j|P_{\mathcal{I}}|k] = 0$ can only be satisfied if the vector $\sum_{k\in\mathcal{I}^+} P_{\mathcal{I}\, a\dot{a}}\tilde{\lambda}^{(k)\,\dot{a}}$ vanishes. To see this note that there must be at least two gravitons in $\mathcal{J}$, one of them j. Therefore we can use momentum conservation to determine the other one in terms of the other $n-1$ gravitons. This allows us to consider all the remaining $n-1$ gravitons as independent. In particular, the $j^{\rm th}$ graviton is independent from the ones in $\mathcal{I}$.

Our goal is then to prove that the combination $P_{\mathcal{I}\,a\dot{a}}(\sum_{k\in\mathcal{I}^+}\tilde{\lambda}^{(k)}_{\dot{a}})$ cannot vanish for generic choice of momenta and polarization tensors.

Consider first the case when the set $\mathcal{I}^+$ has only one element, say the $s^{\rm th}$ graviton. Then the vanishing of $P_{\mathcal{I}\,a\dot{a}}\tilde{\lambda}^{(s)\,\dot{a}}$ implies that of $\sum_{k\in\mathcal{I}} s_{k,s}$, where $s_{k,s} = (p_k+p_s)^2$. Since $\mathcal{I}$ must have at least two gravitons, the vanishing of $\sum_{k\in\mathcal{I}} s_{k,s}$ is a constraint on the kinematical invariants which is not satisfied for generic momenta.

Consider the case when $\mathcal{I}^+$ has at least two elements. Let one of them be the $s^{\rm th}$ graviton. Since our starting point is a physical on-shell amplitude, the dependence of the amplitude on the $s^{\rm th}$ graviton can only be through its polarization tensor and its momentum vector,

$$\epsilon^{+\,(s)}_{a\dot{a},b\dot{b}} = \frac{\mu_a\tilde{\lambda}^{(s)}_{\dot{a}}\mu_b\tilde{\lambda}^{(s)}_{\dot{b}}}{\langle\mu,\lambda^{(s)}\rangle^2}, \qquad p^{(s)}_{a\dot{a}} = \lambda^{(s)}_a\tilde{\lambda}^{(s)}_{\dot{a}}. \tag{A.3}$$

If we transform $\{\lambda^{(s)},\tilde{\lambda}^{(s)}\}$ into $\{t^{-1}\lambda^{(s)}, t\tilde{\lambda}^{(s)}\}$ with $t^4=1$, i.e., t is any $4^{\rm th}$ root of unity, then both $\epsilon^{+\,(s)}_{a\dot{a},b\dot{b}}$ and $p^{(s)}_{a\dot{a}}$ are invariant. This means that any statement we make for $t=1$ must be true for the other three possible values of t. In particular, it must be the case that $P_{\mathcal{I}\,a\dot{a}}(\sum_{k\in\mathcal{I}^+,\,k\neq s}\tilde{\lambda}^{(k)}_{\dot{a}} + t\tilde{\lambda}^{(s)}_{\dot{a}})$ vanishes for all four values of t. Since $P_{\mathcal{I}\,a\dot{a}}$ does not depend on t the only way to satisfy this condition is if $P_{\mathcal{I}}\cdot p^{(s)} = 0$. This is clearly a condition that is not satisfied for generic momenta and therefore this possibility is also excluded.

Finally, there is one more possibility to consider. If the set $\mathcal{I}^+$ is empty then the w dependence drops out. Of course, this is not a problem because if the set $\mathcal{I}^+$ is empty it means that nothing on the subdiagram $\mathcal{I}$ depends on w, including the cubic vertices. Therefore, neither propagators nor cubic vertices contribute. One can then concentrate on the subdiagram $\mathcal{J}$, but this subdiagram has less particles than the total diagram and the same number of w-dependent polarization tensor. Therefore these diagrams go to zero even faster than diagrams where $\mathcal{I}^+$ is not empty.

A.2 Location Of Poles And Final Form Of The Auxiliary Recursion Relations

Having proven that $M_n(w)$ vanishes at infinity, we turn to the question of the singularity structure. We claim that it has only simple poles coming from propagators in Feynman diagrams. Again as in section II where we discussed the BCFW deformation, one has that the poles generated by the w dependence in the polarization tensors can be eliminated by a gauge choice. We pick the reference spinor of each of the polarization tensors of the positive helicity gravitons to be $\mu_a = \lambda_a^{(j)}$ and that of the j^{th} helicity graviton to be $\tilde{\mu}_{\dot{a}} = \sum_{k\in\{k^+\}} \tilde{\lambda}^{(k)}$.

We have already given the structure of propagators in (A.2) from where we can immediately read off the location of the poles to be

$$w_{\mathcal{I}} = \frac{P_{\mathcal{I}}^2(0)}{\sum_{k\in\mathcal{I}^+}\langle j|P_{\mathcal{I}}(0)|k]}. \tag{A.4}$$

Finally, we need the fact that a rational function that vanishes at infinity and only has simple poles can be written as $M_n(w) = \sum_\alpha c_\alpha/(w-w_\alpha)$ where the sum is over the poles and c_α are the residues. The residues in this case can be determined from factorization limits since all poles come from physical propagators.

Collecting all results we arrive at the final form of the auxiliary recursion relation used in the text (3.15):

$$\begin{aligned} &M_n(\{r^-\},\{k^+\}) \;= \\ &\quad = \sum_{\mathcal{I}}\sum_{h=\pm} M_{\mathcal{I}}\left(\left\{r_{\mathcal{I}}^-\right\}, \left\{k_{\mathcal{I}}^+(w_{\mathcal{I}})\right\}, -P_{\mathcal{I}}^h(w_{\mathcal{I}})\right)\frac{1}{P_{\mathcal{I}}^2} M_{\mathcal{J}}\left(\left\{r_{\mathcal{J}}^-(w_{\mathcal{I}})\right\}, \left\{k_{\mathcal{J}}^+(w_{\mathcal{I}})\right\}, P_{\mathcal{I}}^{-h}(w_{\mathcal{I}})\right). \end{aligned} \tag{A.5}$$

Bibliography

[1] T. Aoyama, M. Hayakawa, T. Kinoshita and M. Nio, "Revised value of the eighth-order electron g-2," arXiv:0706.3496 [hep-ph].

[2] G. Gabrielse, D. Hanneke, T. Kinoshita, M. Nio, B. Odom, "New Determination of the Fine Stucture Constant from the Electron g value and QED," Phys. Rev. Lett. **97**, 030802 (2007).

[3] M. J.G. Veltman, "Quantum Theory of Gravitation," Les Houches, Session XXVIII (1976).

[4] F. A. Berends and W. T. Giele, "Recursive Calculations for Processes with n Gluons," Nucl. Phys. B **306**, 759 (1988).

[5] M. L. Mangano, S. J. Parke and Z. Xu, "Duality and multi-gluon scattering," Nucl. Phys. B **298**, 653 (1988).

[6] M. L. Mangano and S. J. Parke, "Multiparton amplitudes in gauge theories," Phys. Rept. **200**, 301 (1991) hep-th/0509223.

[7] F. A. Berends, W. T. Giele and H. Kuijf, "Exact and approximate expressions for multi-gluon scattering," Nucl. Phys. B **333**, 120 (1990).

[8] D. A. Kosower, "Light cone recurrence relations for QCD amplitudes," Nucl. Phys. B **335**, 23 (1990).

[9] R. Britto, F. Cachazo and B. Feng, "New recursion relations for tree amplitudes of gluons," Nucl. Phys. B **715**, 499 (2005) [arXiv:hep-th/0412308].

[10] R. Britto, F. Cachazo, B. Feng and E. Witten, "Direct proof of tree-level recursion relation in Yang-Mills theory," Phys. Rev. Lett. **94**, 181602 (2005) [arXiv:hep-th/0501052].

[11] M. x. Luo and C. k. Wen, "Recursion relations for tree amplitudes in super gauge theories," JHEP **0503**, 004 (2005) hep-th/0501121.

[12] M. x. Luo and C. k. Wen, "Compact formulas for all tree amplitudes of six partons," Phys. Rev. D **71**, 091501 (2005) hep-th/0502009.

[13] Z. Bern, L. J. Dixon and D. A. Kosower, "The last of the finite loop amplitudes in QCD," Phys. Rev. D **72**, 125003 (2005) hep-ph/0505055.

[14] Z. Bern, L. J. Dixon and D. A. Kosower, "Bootstrapping multi-parton loop amplitudes in QCD," Phys. Rev. D **73**, 065013 (2006) hep-ph/0507005.

[15] C. F. Berger, Z. Bern, L. J. Dixon, D. Forde and D. A. Kosower, "Bootstrapping one-loop QCD amplitudes with general helicities," Phys. Rev. D **74**, 036009 (2006) hep-ph/0604195.

[16] C. F. Berger, Z. Bern, L. J. Dixon, D. Forde and D. A. Kosower, "All one-loop maximally helicity violating gluonic amplitudes in QCD," hep-ph/0607014.

[17] Z. Bern, N. E. J. Bjerrum-Bohr, D. C. Dunbar and H. Ita, "Recursive calculation of one-loop QCD integral coefficients," JHEP **0511**, 027 (2005) hep-ph/0507019.

[18] A. Brandhuber, S. McNamara, B. Spence and G. Travaglini, "Recursion relations for one-loop gravity amplitudes," hep-th/0701187.

[19] S. D. Badger, E. W. N. Glover, V. V. Khoze and P. Svrcek, “Recursion relations for gauge theory amplitudes with massive particles,” JHEP **0507**, 025 (2005) hep-th/0504159.

[20] J. Bedford, A. Brandhuber, B. J. Spence and G. Travaglini, “A recursion relation for gravity amplitudes,” Nucl. Phys. B **721**, 98 (2005) hep-th/0502146.

[21] F. Cachazo and P. Svrcek, “Tree level recursion relations in general relativity,” hep-th/0502160.

[22] F. A. Berends, W. T. Giele and H. Kuijf, “On relations between multi - gluon and multigraviton scattering,” Phys. Lett. B **211** (1988) 91.

[23] H. Kawai, D. C. Lewellen and S. H. H. Tye, “A Relation Between Tree Amplitudes Of Closed And Open Strings,” Nucl. Phys. B **269** (1986) 1.

[24] P. Benincasa, C. Boucher-Veronneau and F. Cachazo, “Taming tree amplitudes in general relativity,” arXiv:hep-th/0702032.

[25] P. Benincasa and F. Cachazo, “Consistency Conditions on the S-Matrix of Massless Particles,” arXiv:0705.4305 [hep-th].

[26] F. A. Berends, R. Kleiss, P. De Causmaecker, R. Gastmans and T. T. Wu, “Single Bremsstrahlung Processes In Gauge Theories,” Phys. Lett. B **103**, 124 (1981).

[27] P. De Causmaecker, R. Gastmans, W. Troost and T. T. Wu, “Multiple Bremsstrahlung In Gauge Theories At High-Energies. 1. General Formalism For Quantum Electrodynamics,” Nucl. Phys. B **206**, 53 (1982).

[28] R. Kleiss and W. J. Stirling, “Spinor Techniques For Calculating P Anti-P $\rightarrow$ W+- / Z0 + Jets,” Nucl. Phys. B **262**, 235 (1985).

[29] H. Weyl, “Gruppentheorie und quantenmechanik” (Leipzig, 1928); B.L. van der Waerden, “Goettinger Nachrichten” (1929), 100; O. Laporte and G.E. Uhlenbeck, Phys. Rev. 37 (1931), 1380; E.M. Corson, “Introduction to tensors, spinors and relativistic wave equations” (Blackie, London, 1953).

[30] F. A. Berends and W. Giele, ‘The Six Gluon Process As An Example Of Weyl-Van Der Waerden Spinor Calculus,” Nucl. Phys. B **294**, 700 (1987).

[31] E. Witten, “Perturbative gauge theory as a string theory in twistor space,” Commun. Math. Phys. **252**, 189 (2004) [arXiv:hep-th/0312171].

[32] M. L. Mangano, “The Color Structure Of Gluon Emission,” Nucl. Phys. B **309**, 461 (1988).

[33] Z. Bern and D. A. Kosower, “Color Decomposition Of One Loop Amplitudes In Gauge Theories,” Nucl. Phys. B **362**, 389 (1991).

[34] L. J. Dixon, “Calculating scattering amplitudes efficiently,” arXiv:hep-ph/9601359.

[35] K. Risager, “A direct proof of the CSW rules,” JHEP **0512**, 003 (2005) hep-th/0508206.

[36] N. E. J. Bjerrum-Bohr, D. C. Dunbar, H. Ita, W. B. Perkins and K. Risager, “MHV-vertices for gravity amplitudes,” JHEP **0601**, 009 (2006) hep-th/0509016.

[37] Z. Bern, “Perturbative quantum gravity and its relation to gauge theory,” Living Rev. Rel. **5**, 5 (2002) gr-qc/0206071.

[38] Z. Bern, L. J. Dixon and R. Roiban, ‘Is N = 8 supergravity ultraviolet finite?,” Phys. Lett. B **644**, 265 (2007) [arXiv:hep-th/0611086].

[39] L. F. Alday and J. Maldacena, "Gluon scattering amplitudes at strong coupling," arXiv:0705.0303 [hep-th].

[40] A. Brandhuber, P. Heslop and G. Travaglini, "MHV Amplitudes in N=4 Super Yang-Mills and Wilson Loops," arXiv:0707.1153 [hep-th].