On Diagonal Acts of Monoids

by

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Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.
I understand that my thesis may be made electronically available to the public.

Abstract

In this paper we discuss what is known so far about diagonal acts of monoids. The first results that will be discussed comprise an overview of some work done on determining whether or not the diagonal act can be finitely generated or cyclic when looking at specific classes of monoids. This has been a topic of interest to a handful of semigroup theorists over the past seven years. We then move on to discuss some results pertaining to flatness properties of diagonal acts. The theory of flatness properties of acts over monoids has been of major interest over the past two decades, but so far there are no papers published on this subject that relate specifically to diagonal acts. We attempt to shed some light on this topic as well as present some new problems.

Acknowledgements

I would like to thank my supervisor, Professor Sydney Bulman-Fleming for introducing me to this interesting area of mathematics as well as for all of his guidance over both my graduate and undergraduate careers. I would also like to thank Professors Ross Willard and John Lawrence for being part of my thesis committee.

Dedication

This thesis is dedicated to my parents, Charles and Sharon, as well as to all of my classmates.

Contents

Chapter 1

Introduction and Preliminaries

1.1 Introduction

The study of flatness properties of acts over monoids was begun in the early 1970's by Mati Kilp and Bo Stenström as a way to generalise the notions of flatness of modules to the non-additive setting. Since then there has been a large body of research in the area that all culminated in the year 2000 with the appearance of the monograph [11]. This text introduces the various flatness properties of acts and then discusses in detail the results pertaining to homological classification of monoids, which explores the connections between flatness properties of acts over a monoid S and properties of S itself.

For the purpose of motivation, we will begin by discussing some of the results about flatness of modules. In this study, there are four properties that we are concerned with: free, projective, flat and torsion free. The definitions of free and projective are the same for any category. A module M_R is *flat* if the functor $M_R \otimes -$ from the category of left R-modules to the category of abelian groups preserves embeddings, and, if R is an integral domain then M_R is *torsion free* if $mr = 0$ for $m \in M$, $r \in R$, implies that $m = 0$ or $r = 0$. We can now state some equivalent definitions of free, projective and flat.

Theorem 1.1. *A module M_R is free if and only if M_R is a direct sum of a collection of isomorphic copies of R_R.*

Theorem 1.2. *A module M_R is projective if and only if M_R is a direct summand of a free module.*

Theorem 1.3. *The following conditions are equivalent for a module M_R.*

1) M_R is flat;

2) The functor $M_R \otimes -$ preserves embeddings of left ideals ${}_RJ$ into ${}_RR$;

3) The functor $M_R \otimes -$ preserves embeddings of finitely generated left ideals ${}_RJ$ into ${}_RR$;

4) For every relation

$$\sum_{j=1}^{n} v_j r_j = 0,$$

for $v_j \in M, r_j \in R$, there exist elements $u_1, \dots, u_m \in M$ and $c_{ij} \in R$, for $i = 1, \dots, m$, $j = 1, \dots, n$, such that

$$v_j = \sum_{i=1}^{m} u_i c_{ij} \ (j = 1, \dots, n)$$

and

$$\sum_{j=1}^{n} c_{ij} r_j = 0 \ (i = 1, \dots, m);$$

5) M_R is the direct limit of a family of finitely generated free right R-modules.

We may then wonder what conditions a ring R must satisfy in order for all R-modules to be free, projective, or flat. The following is a quick summary which may be found in [2].

Theorem 1.4. *All R-modules are free if and only if R is a division ring.*

Theorem 1.5. *All R-modules are projective if and only if R is a semisimple ring.*

Theorem 1.6. *All R-modules are flat if and only if R is (von Neumann) regular.*

Finally, we can see what conditions R must satisfy in order for distinct flatness properties to collapse.

Theorem 1.7. *Let R be an integral domain in which every finitely generated right ideal is principal. Then every torsion free right R-module is flat.*

When it comes to acts, we can ask if the equivalent conditions stated in Theorem 1.3 have analogues, and if they are all equivalent to flatness. It turns out that each of them can be adapted to the acts case; however they are not all equivalent to each other. Condition (2) will be our definition of *weak flatness*, a version of condition (4) will later on appear as *condition (P)* and will be shown to be stronger than flatness, and condition (5) is an equivalent definition to what we will call *strong flatness*, which is even stronger than condition (P). In fact, we can obtain even more flatness properties by requiring that tensoring preserve principal left ideals, pullbacks, or equalizers. All of these properties and the major results concerning them are discussed in [11]. Since the release of [11], the study of flatness of acts has evolved to studying flatness of S-*posets*, which is simply an ordered version of S-acts. Most of the results about the unordered case transfer over to the ordered case. However, some do not, and we can actually obtain new results about ordered acts that do not apply to unordered acts (see [3] for example). The study of flatness properties of *diagonal acts* (both ordered and unordered) is a new topic in this area. The goal of this paper is to determine what conditions a monoid must satisfy in order for its diagonal act to have certain flatness properties. We will concentrate entirely on the unordered case, and leave working on the ordered case for the future.

1.2 Acts Over Monoids

In this chapter we lay down the fundamentals of acts over monoids. It is assumed that the reader has a basic understanding of semigroup theory. For an exhaustive treatment of the subject the reader is referred to [9]. All of the concepts and proofs of results from this chapter may be found in [11]. Throughout the entire document we will always let S denote a monoid.

Definition 1.8. *Let $A \neq \phi$ be a set. We call A a **right S-act**, and denote it by A_S, if there exists a mapping from $A \times S$ to A that sends the pair (a, s) to as such that*
1) $a1 = a$ and
2) $a(st) = (as)t$ for all $a \in A,\ s, t \in S$.
*We may define **left S-acts** analogously, and we denote them by ${}_SA$. A non-empty subset B of A is called a **subact** of A_S if $bs \in B$ for all $b \in B$ and $s \in S$.*

We will often not specify the side on which the monoid acts and instead rely on the notation used.

Examples: 1) Consider the set $S \times S$ and let S act on $S \times S$ on the right by componentwise multiplication. That is, $(x, y)s = (xs, ys)$, for $x, y, s \in S$. Then $S \times S$ with this action is a right S-act called the **right diagonal act of** S and is denoted by $(S \times S)_S$. It is this specific example that will be the main focus of this paper.

2) We call a non-empty subset K of S a **right ideal** of S if for any $x \in K$ and $s \in S$, $xs \in K$. It is clear from this definition that any right ideal of S can be

considered as a right S-act.

Notation: Let A_S be an act and let U be a non-empty subset of A. Then

$$US := \{us : u \in U, s \in S\}.$$

We also use the notation aS instead of $\{a\}S$ whenver dealing with a singleton subset.

The following definition will be the main focus of Chapter 2.

Definition 1.9. *An act A_S is said to be* ***finitely generated*** *if there exists a finite subset U of A such that $US = A_S$ We call A_S* ***cyclic*** *if $aS = A_S$ for some $a \in A_S$.*

Example: If S is a non-trivial finite monoid then $(S \times S)_S$ cannot be cyclic. Indeed, if $|S| = n$, then $|S \times S| = n^2$, but $|(a,b)S| \leq n$ for any $(a,b) \in S \times S$.

Next, we give a definition that will be useful when we discuss projective acts.

Definition 1.10. *Let A_S be an S-act. If there exist subacts B_S, C_S of A_S such that*

$$B_S \cap C_S = \phi \text{ and } A_S = B_S \cup C_S$$

then we call A_S ***decomposable****. Otherwise we say A_S is* ***indecomposable****.*

It may be easily shown that every cyclic act is indecomposable and that every act has a unique decomposition as a disjoint union (or *coproduct* using categorial terminology) of indecomposable subacts.

It is often difficult to determine whether or not a certain act is indecomposable. The following definition will give us a more applicable approach to this problem.

Definition 1.11. *Let A_S be an S-act. We define the* ***connectedness relation*** *on A_S by saying $a, b \in A_S$ are* ***connected*** *if there exist $a_1, \ldots, a_n \in A_S$, $s_1, \ldots, s_n, t_1, \ldots, t_n \in S$ such that*

$$\begin{aligned} a &= a_1 s_1 \\ a_1 t_1 &= a_2 s_2 \\ &\vdots \\ a_n t_n &= b. \end{aligned}$$

It may be easily shown that the connectedness relation is an equivalence relation on A_S and we call the equivalence classes the *connected components*.

Theorem 1.12. *Let $a, b \in A_S$. Then a and b are connected if and only if they are in the same indecomposable component of A_S.*

Thus, the indecomposable components of A_S are the connected components.

1.3 Tensor Products of Acts

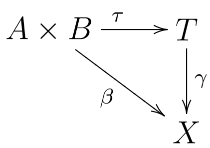
In module theory we learn that the tensor product of two R-modules consists of an abelian group and a group homomorphism that is universal among all balanced maps into arbitrary abelian groups. In this section we will construct the tensor product of two S-acts. The construction is essentially the same as in the module setting, but simpler because addition need not be considered.

Definition 1.13. *Let A_S be a right S-act, ${}_SB$ be a left S-act and X a set. We say a mapping $\beta : A_S \times {}_SB \longrightarrow X$ is S-**balanced** (or simply **balanced**) if*

$$\beta(as, b) = \beta(a, sb)$$

for all $a \in A$, $b \in B$ and $s \in S$.

Definition 1.14. *Let A_S and ${}_SB$ be S-acts. A set T together with a balanced mapping $\tau : A \times B \longrightarrow T$ is called a **tensor product** of A_S and ${}_SB$ if for every set X and balanced mapping $\beta : A \times B \longrightarrow X$, there exists a unique (set) mapping $\gamma : T \longrightarrow X$ such that $\beta = \gamma\tau$ (i.e. such that the following diagram commutes).*



It can easily be shown that any two tensor products of A_S and ${}_SB$ are in fact isomorphic. We may thus speak of *the* tensor product of A_S and ${}_SB$. To construct the tensor product, let τ be the equivalence relation on the set $A \times B$ generated by the set of all pairs $((as, b), (a, sb))$ for $a \in A_S, b \in {}_SB, s \in S$. We call the relation τ the *tensor relation*. Then, we define

$$A_S \otimes {}_SB := (A_S \times {}_SB)/\tau, \text{ and } a \otimes b := [(a, b)]_\tau \in A_S \otimes {}_SB$$

for $a \in A, b \in B$. The next proposition has a straightforward proof and says that the set just constructed is the tensor product that we are looking for.

Proposition 1.15. *Let A_S and ${}_SB$ be S-acts and let $A_S \otimes {}_SB$ be defined as above. Then the set $A_S \otimes {}_SB$ together with the canonical surjection $\tau : A \times B \longrightarrow A_S \otimes {}_SB$ is the tensor product of A_S and ${}_SB$.*

The following lemma will be used throughout the remainder of this paper to determine when two elements are equal in a tensor product.

Lemma 1.16. *Let A_S and ${}_SB$ be S-acts and let $a, a' \in A_S,\ b, b' \in {}_SB$. Then $a \otimes b = a' \otimes b'$ in $A_S \otimes {}_SB$ if and only if there exist $a_1, \ldots, a_n \in A_S,\ b_2, \ldots, b_n \in {}_SB,\ s_1, \ldots, s_n, t_1, \ldots, t_n \in S$ such that*

$$\begin{array}{rclrcl} a & = & a_1 s_1 & & & \\ a_1 t_1 & = & a_2 s_2 & s_1 b & = & t_1 b_2 \\ a_2 t_2 & = & a_3 s_3 & s_2 b_2 & = & t_2 b_3 \\ & \vdots & & & \vdots & \\ a_n t_n & = & a' & s_n b_n & = & t_n b' \end{array}$$

We call an arrangement above a *scheme* of length n connecting (a, b) to (a', b'). The study of flatness properties of acts relies on the analysis and manipulation of these schemes.

1.4 Flatness Properties of Acts

We now introduce the various so-called "flatness" properties of S-acts. These properties are similar to those of modules; however there are many more properties to discuss when it comes to S-acts. We begin with a property that is similar to the corresponding module property and is the "weakest" of all the flatness properties.

Definition 1.17. *An S-act A_S is called **torsion free** if for any $a, b \in A_S$ and any right cancellable element $c \in S$ the equality $ac = bc$ implies $a = b$.*

Any right ideal of S is an example of a torsion free act. However, as we will see later, not all acts are torsion free.

Definition 1.18. *An S-act A_S is called **principally weakly flat** if the functor $A_S \otimes -$ from the category of left S-acts to the category of sets preserves embeddings of principal left ideals of S into S.*

This definition says that whenever $a \otimes s = a' \otimes s$ in the tensor product $A_S \otimes {}_SS$ for $a, a' \in A_S,\ s \in S$, then the equality also holds in the tensor product $A_S \otimes {}_S(Ss)$. However, it can be easily shown that $A_S \otimes {}_SS$ is isomorphic to A_S by means of the mapping $a \otimes s \mapsto as$, and so we have the following more applicable definition of principally weak flatness.

Lemma 1.19. *An S-act, A_S is principally weakly flat if and only if $as = a's$ for $a, a' \in A_S,\ s \in S$ implies $a \otimes s = a' \otimes s$ in the tensor product $A_S \otimes {}_S(Ss)$.*

In the language of schemes, the latter part of the lemma means that there exist $a_1, \ldots, a_n \in A_S,\ s_1, \ldots, s_n, t_1, \ldots, t_n \in S$ such that

$$\begin{array}{rclrcl} a & = & a_1 s_1 & & & \\ a_1 t_1 & = & a_2 s_2 & s_1 s & = & t_1 s \\ a_2 t_2 & = & a_3 s_3 & s_2 s & = & t_2 s \\ & \vdots & & & \vdots & \\ a_n t_n & = & a' & s_n s & = & t_n s. \end{array}$$

By manipulating this scheme above we may easily obtain the following.

Proposition 1.20. *If an S-act A_S is principally weakly flat then A_S is torsion free.*

We will later see an example of a right act that is torsion free but is not principally weakly flat. If we strengthen the notion of principally weak flatness to allow embeddings of all left ideals then we obtain the following.

Definition 1.21. *An S-act A_S is called **weakly flat** if the functor $A_S \otimes -$ preserves embeddings of arbitrary left ideals of S into S.*

Again, what this means is that whenever $a \otimes s = a' \otimes t$ in the tensor product $A_S \otimes {}_S S$, for $a, a' \in A_S,\ s, t \in {}_S K$ where ${}_S K$ is a left ideal of S, then the equality also holds in the tensor product $A_S \otimes {}_S K$. As with principal weak flatness above, we may give an alternative, more useful description of weak flatness.

Lemma 1.22. *An S-act A_S is weakly flat if and only if $as = a't$ for $a, a' \in A_S,\ s, t \in S$ implies $a \otimes s = a' \otimes t$ in the tensor product $A_S \otimes {}_S(Ss \cup St)$.*

It is easily seen by the above lemma that weak flatness implies principal weak flatness. In Chapter 3 we will give an example to show that these two notions are in fact distinct. We may now move on to flatness; the definition here is the same as in the module case.

Definition 1.23. *An S-act A_S is called **flat** if the functor $A_S \otimes -$ preserves monomorphisms between arbitrary left S-acts.*

That is, if A_S and ${}_S B$ are S-acts and ${}_S C$ is a subact of ${}_S B$ then we say that A_S is flat if whenever $a \otimes b = a' \otimes b'$ in the tensor product $A_S \otimes {}_S B$, for $a, a' \in A_S,\ b, b' \in {}_S C$, then the equality also holds in the tensor product $A_S \otimes {}_S C$.

Lemma 1.24. *An S-act A_S is flat if and only if for any left S-act ${}_S B$, if $a \otimes b = a' \otimes b'$ in the tensor product $A_S \otimes {}_S B$, for $a, a' \in A_S,\ b, b' \in {}_S B$, then the equality already holds in the tensor product $A_S \otimes {}_S(Sb \cup Sb')$.*

In fact, we may improve upon the schemes obtained in the previous lemma.

Proposition 1.25. *A right S-act A_S is flat if and only if for any left S-act ${}_SB$ and all $a, a' \in A,\ b, b' \in B$, if $a \otimes b = a' \otimes b'$ in $A_S \otimes {}_SB$ then*

$$\begin{aligned} a &= a_1 s_1 & & \\ a_1 t_1 &= a_2 s_2 & s_1 b &= t_1 b_2 \\ &\vdots & &\vdots \\ a_n t_n &= a' & s_n t_n &= t_n b', \end{aligned}$$

where each $s_i, t_i \in S,\ a_i \in A$ and $b_i \in \{b, b'\}$.

Perhaps a word of warning is needed here. We use the letter 'n' in the above proposition loosely. From the equality $a \otimes b = a' \otimes b'$ in $A_S \otimes {}_SB$ we will get a scheme of some length, but the new scheme we obtain that replaces all of the b_i's from our original schemes with either b or b' may be of a different length. The next proposition should come as no surprise simply from the terminology that we have defined.

Proposition 1.26. *If an S-act A_S is flat then it is also weakly flat.*

For an example of an act that is weakly flat but not flat the reader is referred to [11]. The next two properties have to do with pullbacks and equalizers, which have the same definition in this context as in any other category. We say that A_S is *pullback flat* if the functor $A_S \otimes -$ preserves pullbacks and *equalizer flat* if it preserves equalizers. In 1971, Bo Stenström studied these two properties and also discovered the following two interpolation conditions, which were given their present names in [14] by Peeter Normak.

Definition 1.27. *An S-act A_S satisfies* ***Condition (P)*** *if whenever $as = a's'$, for $a, a' \in A_S,\ s, s' \in S$, there exist $a'' \in A_S,\ u, v \in S$ such that*

$$a = a''u,\ a' = a''v,\ us = vs'.$$

Definition 1.28. *An S-act A_S satisfies* ***Condition (E)*** *if whenever $as = as'$, for $a \in A_S,\ s, s' \in S$, there exist $a' \in A_S,\ u \in S$ such that*

$$a = a'u,\ us = us'.$$

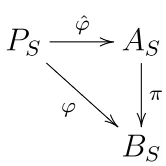
In general, these two properties are unrelated. It can be shown though that if an act is pullback flat then it satisfies condition (P) and also that equalizer flat implies condition (E). We say that A_S is *strongly flat* if $A_S \otimes -$ preserves pullbacks and equalizers. Although condition (P) does not imply pullback flatness in general, and condition (E) does not imply equalizer flatness, it was shown in Stenström's paper that strongly flat is in fact equivalent to the conjunction of conditions (P) and (E). We will discuss these two conditions as well as pullback flatness in more detail as they relate to diagonal acts later in the paper. For now we continue with establishing the relations between all of these properties in general.

Proposition 1.29. *If an S-act satisfies Condition (P) then it is flat.*

Example: Let $S = \{1, e, 0\}$ where $e^2 = e$. Then S is an *inverse monoid*: that is, a monoid S where for each $s \in S$ there exists a unique $t \in S$ such that $sts = s$ and $tst = t$. Consider the subset $K = \{e, 0\}$. Then K is a right ideal of S. We define the *Rees congruence* on S by $x\rho_K y$ iff $x = y$ or $x, y \in K$. We then denote the resulting factor act by S/K_S. It is shown in [4] that all acts over inverse monoids are flat, and so in particular S/K_S is flat. However, in S/K_S we have $\bar{1}e = \bar{1}0$, and so if condition (P) were satisfied then we would get $\bar{1} = \bar{1}u = \bar{1}v$ for some $u, v \in S$ with $ue = v0$. But then it must be that $u = v = 1$, and so $e = 0$, which is a contradiction. Thus S/K_S does not satisfy condition (P).

The final two properties are once again identical to the module case because they are category-theoretical in nature.

Definition 1.30. *Let A_S, B_S, P_S be S-acts. Then P_S is* ***projective*** *if for any surjective homomorphism $\pi : A_S \longrightarrow B_S$ and any homomorphism $\varphi : P_S \longrightarrow B_S$ there exists a homomorphism $\hat{\varphi} : P_S \longrightarrow A_S$ such that $\varphi = \pi\hat{\varphi}$; that is, such that the following diagram commutes.*



We call an element $z \in S$ *left e-cancellable* for an idempotent element $e \in S$ if $zx = zy$ for $x, y \in S$ implies that $ze = z$ and $ex = ey$. The following result gives a nice description of projective S-acts.

Theorem 1.31. *Let P_S be a right S-act. Then the following are equivalent,*

1) P_S is projective,

2) $P_S \cong \dot{\cup}_{i\in I} e_i S$*, for idempotents* $e_i \in S$*,*

3) $P_S \cong \dot{\cup}_{i\in I} z_i S$*, for left* e_i*-cancellable elements* $z_i \in S$*, where each* e_i *is idempotent.*

Proposition 1.32. *If an* S*-act is projective then it is strongly flat.*

The one element S act, Θ_S, where $S = (\mathbb{N}, \max)$ is an example of an act that is strongly flat but not projective. We now finally arrive at the "strongest" flatness property.

Definition 1.33. *Let* A_S *be an* S*-act and let* B *be a subset of* A_S*. We say* B *is a* ***basis*** *for* A_S *if every element* $a \in A_S$ *can be expressed uniquely in the form* $a = bs$ *where* $b \in B,\ s \in S$*. If such a basis exists we say that* A_S *is* ***free****.*

The following theorem describes the structure of free S-acts.

Theorem 1.34. *An* S*-act* F_S *is free if and only if* $F_S \cong \dot{\cup}_{i\in I} S_i$*, for some non-empty set* I*, where* $S_i \cong S_S$ *for each* $i \in I$*.*

This theorem should come as no surprise to readers who are familiar with category theory. The disjoint union is simply the coproduct of objects in the category of S-acts, much the same as the direct sum being the coproduct in the category of R-modules. With this theorem it is easy to see that all free S-acts are projective. The one element S-act, Θ_S, where S is any non-trivial monoid that contains a left zero element, is an example of an act that is projective but not free.

We may now summarize all of these properties with the following diagram, where each implication is indeed strict:

free $\Rightarrow$ projective $\Rightarrow$ strongly flat $\Rightarrow$ condition (P) $\Rightarrow$ flat $\Rightarrow$ weakly flat
$\Rightarrow$ principally weakly flat $\Rightarrow$ torsion free

Finally, just as we did for modules, we can establish connections between the strucure of the monoid S and flatness properties of S-acts. For the complete details the reader is referred to [11], but some of the highlights of these problems are: all S-acts are principally weakly flat if and only if S is regular, and all S-acts satisfy condition (P) if and only if S is a group. Notice now that this gives another distinction between the studies of modules and acts; for modules regularity was sufficient for flatness, but for acts it is not.

Chapter 2

Finitely Generated and Cyclic Diagonal Acts

The interest of studying diagonal acts over monoids was sparked in 1989 when Sydney Bulman-Fleming and Ken McDowell posed a problem in the American Mathematical Monthly (see [5]). What they asked was the following:

Let S be a monoid such that $(S \times S)_S$ is cyclic. Show that S must be a singleton if S is finite, commutative or inverse and show that S need not be a singleton in general.

The second question was answered by letting $S = \mathcal{T}_{\mathbb{N}}$, the full transformation monoid on the natural numbers. Over a decade later, M.R. Thomson (see [17]), together with E.F. Robertson and N. Ruškuc, studied diagonal acts in relation to wreath products. Since then, the previously mentioned researchers, along with P. Gallagher, have further developed this field of study by answering more questions regarding the finite generation of diagonal acts. This chapter will be devoted to displaying some of these results. Because the theme of this paper focuses on diagonal *right* acts, we will concentrate mostly on results pertaining to this, and briefly mention results about left or bi-acts. Also, in this chapter we will not require that we are dealing with monoids, rather simply semigroups, but we will go back to monoids in subsequent chapters.

We begin by looking at specific classes of semigroups and determining if it is possible for their diagonal acts to be finitely generated. Most of the proofs in this section will not be given; however we will present the proof of the first result simply to show the strategy of the original authors.

Theorem 2.1. *Let S be an infinite left cancellative semigroup. Then $(S \times S)_S$ is not finitely generated.*

Proof. Assume to the contrary that there exists a finite set $A \subseteq S \times S$ such that $(S \times S)_S = AS$. Select and fix an arbitrary element $y \in S$. Now consider the mapping

$$\varphi : A \longrightarrow S$$
$$(a, b) \mapsto z$$

if there exists $q \in S$ such that

$$(y, z) = (a, b)q,$$

and $(a, b) \mapsto z_0$ otherwise, for some other arbitrary fixed $z_0 \in S$. Then this mapping is well-defined. Indeed, if we had

$$(y, z_1) = (a, b)u, \quad (y, z_2) = (a, b)v$$

with $(a, b) \in A$ and $z_1, z_2, u, v \in S$, then $(a, b)\varphi = z_1$ and $(a, b)\varphi = z_2$. However, $y = au = av$, and so by left cancellativity we get $u = v$. Hence, $z_1 = bu = bv = z_2$. Also, the mapping φ is clearly surjective because A generates $(S \times S)_S$. But then this implies that S is finite, which is a contradiction. □

It turns out that right cancellative semigroups are no better in that their right diagonal acts are not finitely generated either. Recall that a semigroup S is said to be *regular* if for every $s \in S$ there exists a $t \in S$ such that $sts = s$ and S is *inverse* if for every $s \in S$, there exists a unique $t \in S$ such that $sts = s$ and $tst = t$. It may be shown that S being inverse is equivalent to S being regular with commuting idempotents. The next theorem gives the result for diagonal acts over inverse semigroups, which will lead to several corollaries.

Theorem 2.2. *Let S be an infinite inverse semigroup. Then $(S \times S)_S$ is not finitely generated.*

Corollary 2.3. *$(S \times S)_S$ is not finitely generated if S is an infinite group.*

A semigroup S is said to be *simple* if it has no proper ideals, that is there does not exist a proper subset K of S such that $SK \subseteq K$ and $KS \subseteq K$. If S has a zero element (i.e. an element $z \in S$ such that $zs = z$ and $sz = z$ for all $s \in S$, usually just denoted 0) then S is called *0-simple* if S has no non-trivial ideals and $S^2 \neq 0$. S is called *completely simple* if S is simple and has no infinite descending chains of left or of right ideals. One may define completely 0-simple in a similar manner. The following theorem may be found in [9] and gives a classification of completely simple and completely 0-simple semigroups. In order to state the theorem we need to introduce the construction of a Rees matrix semigroup:

Let G be a group, I and Λ be non-empty sets and let P be a $\Lambda \times I$ matrix with entries from G^0 (where G^0 denotes G with a zero adjoined) such that every row and every column of P contains at least one nonzero entry (this is called the *regularity condition* on P). Consider now the set

$$\mathcal{M}^0(I, G, \Lambda; P) = \{(i, g, \lambda) : i \in I, g \in G, \lambda \in \Lambda\} \cup \{0\},$$

with multiplication defined by

$$(i, g, \lambda)(j, h, \mu) = \begin{cases} (i, gP_{\lambda j}h, \mu) \text{ if } P_{\lambda j} \neq 0 \\ 0 \text{ otherwise.} \end{cases}$$

The resulting structure is a semigroup called the *Rees matrix semigroup with zero with sandwich matrix* P. If we do the same construction leaving out the adjoined zero in G, then we get a *Rees matrix semigroup without zero.*

Theorem 2.4. *Let S be a semigroup. Then,*
(1) S is completely simple if and only if S is isomorphic to a Rees matrix semigroup without zero.
(2) S is completely 0-simple if and only if S is isomorphic to a regular Rees matrix semigroup with zero.

Theorem 2.2 can now be used, together with some lemmas shown in [6], to prove the following corollary.

Corollary 2.5. $(S \times S)_S$ *is not finitely generated if S is an infinite completely simple or completely 0-simple semigroup.*

A semigroup S is called *completely regular* if for every $s \in S$ there exists $t \in S$ such that $sts = s$ and $st = ts$. In [9], the completely regular semigroups are classified as being exactly semilattices of completely simple semigroups, and they are also exactly the unions of groups. This gives us one more corollary.

Corollary 2.6. $(S \times S)_S$ *is not finitely generated if S is an infinite completely regular semigroup.*

S is said to be *locally finite* if every finite subset of S generates a finite subsemigroup of S.

Theorem 2.7. *$(S \times S)_S$ is not finitely generated if S is an infinite locally finite semigroup.*

So far we have not seen much hope as to when the diagonal right act is finitely generated. It turns out, however, that when considering the diagonal *bi-act* of S, we can get an affirmative answer for some of the classes of semigroups mentioned above. The diagonal bi-act of S is defined analogously to the right act, with action given by $s(x, y)t = (sxt, syt)$, for $s, t, x, y \in S$. The following theorems will summarize the results related to diagonal bi-acts.

Theorem 2.8. *There exist infinte semigroups S such that the diagonal bi-act of S is finitely generated if S is:*
i) inverse,
ii) left cancellative,
iii) right cancellative,
iv) cancellative,
v) completely simple,
vi) completely 0-simple,
vii) completely regular.

Theorem 2.9. *Let S be an infinite semigroup. Then the diagonal bi-act of S is not finitely generated if S is commutative or idempotent.*

Once we have found out that a certain class of semigroup has a finitely generated diagonal bi-act we can ask if the bi-act is actually cyclic. Recall that the diagonal bi-act of S is cyclic if there exists some $(a, b) \in S \times S$ such that $S \times S = S(a, b)S$.

Theorem 2.10. *There exist infinte inverse semigroups S such that the diagonal bi-act of S is cyclic.*

Theorem 2.11. *Let S be an infinite semigroup. Then the diagonal bi-act of S is not cyclic if S is:*
i) cancellative,
ii) completely simple,
iii) completely 0-simple,
iv) completely regular.

It is actually unknown so far as to whether or not the diagonal bi-act of S is cyclic if S is just left or right cancellative. It is also unknown about the finite generation at all of the diagonal bi-act of a locally finite semigroup. We will now summarize the results given so far in the following table:

Summary of Results

Property of S	Fin. Gen. right act	Fin. Gen. bi-act	Cyclic bi-act
Commutative	No	No	No
Inverse	No	Yes	Yes
Idempotent	No	No	No
Left Cancellative	No	Yes	??
Right Cancellative	No	Yes	??
Cancellative	No	Yes	No
Completely Simple	No	Yes	No
Completely 0-Simple	No	Yes	No
Completely Regular	No	Yes	No
Locally Finite	No	??	??

As was explained earlier, the monoid of all full transformations on the natural numbers has a cyclic diagonal right act (using the natural numbers is completely arbitrary; any infinite set will do).
To conclude this chapter we will briefly mention some of the examples of cyclic diagonal right acts which are given in [7].

Theorem 2.12. *Let $S = \mathcal{B}(X)$ be the semigroup of binary relations on an infinite set, X. Then the diagonal right act of S is cyclic.*

Proof. We will simply sketch the proof. Because X is infinite, we may partition X into disjoint subsets X_1 and X_2 with $|X| = |X_1| = |X_2|$. Fix bijections $\alpha : X \longrightarrow X_1$ and $\beta : X \longrightarrow X_2$ and consider them as binary relations on X. Then it can be shown that $(S \times S)_S = (\alpha, \beta)S$. □

The same argument may be used for each of the following corollaries.

Corollary 2.13. *The diagonal right act of $\mathcal{P}(X)$, the semigroup of partial transformations on an infinite set X, is cyclic.*

Corollary 2.14. *The diagonal right act of $\mathcal{F}(X)$, the semigroup of full finite-to-one transformations on an infinite set X, is cyclic.*

However, this argument will not work when considering $\mathcal{I}(X)$, the semigroup of partial injective transformations on an infinite set X. This is due to the fact that $\mathcal{I}(X)$ is inverse, and so from Theorem 2.2 we know that its diagonal right act is not even finitely generated. This, of course, implies that the diagonal right act of $\mathcal{S}(X)$, the group of bijections on an infinite set X, is not finitely generated.

Chapter 3

Flatness Properties of Diagonal Acts

There have been many papers published on the topic of homological classification of monoids by flatness properties of acts which deal with questions such as "What conditions must a monoid S possess in order for all right S-acts to have certain flatness properties?". However, none so far have been concerned with flatness of diagonal acts. So, in this chapter we attempt to give answers to the following questions:

- What conditions on a monoid must we impose in order for its diagonal act to have certain flatness properties?

- Can we distinguish among the various properties using only diagonal acts as examples?

- Are there any adjacent properties that are in fact equivalent when considering only diagonal acts?

For the remainder of the text we will assume that we are working with right acts. Also, unless otherwise stated, all results in the remainder of the paper are new results obtained by the author and Sydney Bulman-Fleming.

3.1 Principally Weak Flatness

First, it is clear that all diagonal acts are torsion-free. Indeed, if $(a, b)c = (a', b')c$ for $a, a', b, b', c \in S$ where c is right cancellable, then $ac = a'c$ and $bc = b'c$ imply $a = a'$

and $b = b'$, hence $(a,b) = (a',b')$. Unfortunately this convenience does not continue as we move on to principally weak flatness. It is shown in [11] that all acts over a monoid S are principally weakly flat if and only if S is regular. The next result gives another sufficient condition for $(S \times S)_S$ to be principally weakly flat.

Proposition 3.1. *If the monoid S is right cancellative then the right act $(S \times S)_S$ is principally weakly flat.*

Proof. Suppose we have $(a,b)s = (a',b')s$ for elements $a, b, a', b', s \in S$. Then since S is right cancellative, $a = a'$ and $b = b'$. Hence, $(a,b) \otimes s = (a',b') \otimes s$ in $(S \times S)_S \otimes {}_S(Ss)$, showing that $(S \times S)_S$ is principally weakly flat. □

To distinguish between torsion-freeness and principal weak flatness using diagonal acts we consider the following example.

Example: Let $S = \{0, x, 1\}$ where $x^2 = 0$. We see that $(1,0)x = (1,x)x$, but we show by induction on the length of schemes that $(1,0) \otimes x$ and $(1,x) \otimes x$ cannot be equal in $(S \times S) \otimes Sx$. So, assume we have a scheme of length one connecting $((1,0),x)$ and $((1,x),x)$. That is, suppose there exist $u, v, s, t \in S$ such that

$$\begin{array}{rcl} (1,0) & = & (u,v)s \\ (u,v)t & = & (1,x) \quad sx \;=\; tx. \end{array}$$

Then necessarily $u = s = t = 1$. Hence, $v = 0$ and $v = x$, a contradiction. Now we show that if we have a scheme of length $n > 1$ that it can be shortened to a scheme of length $n - 1$. So, assume we have

$$\begin{array}{rclrcl} (1,0) & = & (u_1,v_1)s_1 & & & \\ (u_1,v_1)t_1 & = & (u_2,v_2)s_2 & s_1x & = & t_1x \\ & \vdots & & & \vdots & \\ (u_n,v_n)t_n & = & (1,x) & s_nx & = & t_nx. \end{array}$$

Then $u_1 = s_1 = 1$ and so $v_1 = 0$ and $t_1 = 1$ because $x = t_1x$. Hence, we may replace our original scheme with

$$\begin{array}{rclrcl} (1,0) & = & (u_2,v_2)s_2 & & & \\ (u_2,v_2)t_2 & = & (u_3,v_3)s_3 & s_2x & = & t_2x \\ & \vdots & & & \vdots & \\ (u_n,v_n)t_n & = & (1,x) & s_nx & = & t_nx \end{array}$$

which is a scheme of length $n-1$. Thus, we cannot find a scheme of any length and so $(S \times S)_S$ is not principally weakly flat.

After looking at this example one might think that regularity might be a necessary condition in order for the diagonal act to be principally weakly flat. However, if we consider the monoid $\mathbb{N}$, of all natural numbers, we see that $\mathbb{N}$ is also not regular but it does have a principally weakly flat diagonal act since it is right cancellative. Thus, regularity is not a necessary condition, and in fact we do not have a nice characterization of monoids with principally weakly flat diagonal acts. The best we can do then is to limit our attention to certain classes of monoids and see if we can obtain any results about them. For example, we say that a monoid S is *left PSF* if all principal left ideals of S are strongly flat as left S-acts. In [13], the authors were able to classify such monoids in terms of semi-cancellable elements. That is, an element $x \in S$ is called *right semi-cancellable* if whenever $sx = tx$ for $s, t \in S$, there exists $r \in S$ such that $rx = x$ and $sr = tr$. The result then states that a monoid is left PSF if and only if each of its elements is right semi-cancellable. We can now give a nice description of PSF monoids whose diagonal acts are principally weakly flat. To do so, we will call an element $x \in S$ *right bi-semi-cancellable* if whenever $sx = tx$ and $s'x = t'x$ for $s, s', t, t' \in S$, there exists $r \in S$ such that $rx = x,\ sr = tr$, and $s'r = t'r$. Clearly if an element $x \in S$ is right bi-semi-cancellable then it is right semi-cancellable. Indeed, if $sx = tx$, then take also the equality $1x = 1x$ and apply the definition.

Proposition 3.2. *Let S be a left PSF monoid. Then the following are equivalent:*
1) $(S \times S)_S$ is principally weakly flat;
2) Every element of S is right bi-semi-cancellable.

Proof. For (1) $\Rightarrow$ (2), let $x \in S$ such that $ax = a'x,\ bx = b'x$ for $a, a', b, b' \in S$. Then $(a, b)x = (a', b')x$ and so by assumption, $(a, b) \otimes x = (a', b') \otimes x$ in $(S \times S)_S \otimes {}_S Sx$. So we have a scheme

$$\begin{array}{rclrcl} (a,b) & = & (a_1,b_1)s_1 & & & \\ (a_1,b_1)t_1 & = & (a_2,b_2)s_2 & s_1x & = & t_1x \\ (a_2,b_2)t_2 & = & (a_3,b_3)s_3 & s_2x & = & t_2x \\ & \vdots & & & \vdots & \\ (a_n,b_n)t_n & = & (a',b') & s_nx & = & t_nx. \end{array}$$

Since x is right semi-cancellable, from $s_1x = t_1x$, we get an element $v_1 \in S$ such that $v_1x = x$ and $s_1v_1 = t_1v_1$. Then

$$s_2v_1x = s_2x = t_2x = t_2v_1x,$$

so again we get an element $v_2 \in S$ such that $v_2x = x$ and $s_2v_1v_2 = t_2v_1v_2$. Let $v_1' = v_1v_2$. Then we see that

$$v_1'x = x,\ s_1v_1' = t_1v_1',\ s_2v_1' = t_2v_2'.$$

Similarly, from $s_3x = t_3x$ we can replace x with $v_1'x$ to get $s_3v_1'x = t_3v_1'x$. So there exists $v_3 \in S$ such that $v_3x = x,\ s_3v_1'v_3 = t_3v_1'v_3$. Let $v_2' = v_1'v_3$. Then we see that

$$v_2'x = x,\ s_1v_2' = t_1v_2',\ s_2v_2' = t_2v_2',\ s_3v_2' = t_3v_2'.$$

By induction, we can find an element $v \in S$ such that $vx = x$ and $s_iv = t_iv$ for all $i = 1, 2, \ldots, n$. Thus we have,

$$av = a_1s_1v = a_1t_1v = a_2s_2v = a_2t_2v = a_3t_3v = \\ \cdots = a_ns_nv = a_nt_nv = a'v,$$

and similarly $bv = b'v$. Hence, x is right bi-semi-cancellable.

For (2) $\Rightarrow$ (1), suppose we have $(a, b)x = (a', b')x$. Then $ax = a'x$ and $bx = b'x$, so by assumption there exists $r \in S$ such that $rx = x,\ ar = a'r,\ br = b'r$. Then in $(S \times S)_S \otimes {}_SSx$ we have

$$(a, b) \otimes x = (a, b) \otimes rx = (ar, br) \otimes x \\ = (a'r, b'r) \otimes x = (a', b') \otimes rx = (a', b') \otimes x.$$

So $(S \times S)_S$ is principally weakly flat. $\square$

A condition on a monoid that is slightly stronger than left PSF is that of *left PP* which is defined as all principal left ideals being projective. It is shown in [11] that S being left PP is equivalent to the condition that every element of S is right e-cancellable for an idempotent $e \in S$. That is, for every element $x \in S$, there exists an idempotent $e \in S$ such that whenever $sx = tx$ for $s, t \in S$, we get that $ex = x$ and $se = te$. So, we can see that, because of the location of the quantifiers in the definition of being right e-cancellable, this condition is much stronger that that of being right bi-semi-cancellable, hence we have the following corollary.

Corollary 3.3. *If S is a left PP monoid then $(S \times S)_S$ is principally weakly flat.*

3.2 Weak Flatness

In section III. 11 of [11] the following condition is defined:

(W) If $as = a't$ for $a, a' \in A_S$, $s, t \in S$, then there exist $a'' \in A_S$, $u \in Ss \cap St$ such that $as = a't = a''u$.

The classification of weakly flat acts involving this condition is due to Bulman-Fleming and McDowell and appears as Theorem III. 11.4 in [11].

Theorem 3.4. *A right S-act A_S is weakly flat if and only if it is principally weakly flat and satisfies condition (W).*

If ${}_SB$ is a left S-act we say that it is *locally cyclic* if any finite subset of elements from ${}_SB$ are contained in some cyclic subact of ${}_SB$. That is, for any $b, b' \in {}_SB$, there exists $b'' \in {}_SB$ such that $\{b, b'\} \subseteq Sb''$. This definition can also be applied to left ideals of S; in this case though we use the term *locally principal*. When considering only diagonal acts, condition (W) may be interpreted as the following.

Theorem 3.5. *The diagonal act $(S \times S)_S$ is weakly flat if and only if it is principally weakly flat and the intersection of any two principal left ideals of S is locally principal or else empty.*

Proof. ($\Longrightarrow$) Let $s, t \in S$ and assume $Ss \cap St \neq \phi$. Note that it suffices to show that any pair of elements contained in $Ss \cap St$ is contained in some principal left ideal contained in $Ss \cap St$ and the general case will hold by induction. Let $w, v \in Ss \cap St$. So we may write $w = as = a't$ and $v = bs = b't$ for some $a, a', b, b' \in S$. Then we have $(a, b)s = (a', b')t$. Because $S \times S$ is weakly flat we know that $(a, b) \otimes s = (a', b') \otimes t$ in $(S \times S)_S \otimes {}_S(Ss \cup St)$. So we have a scheme:

$$\begin{array}{rclrcl}
(a, b) & = & (a_1, b_1)u_1 & & & \\
(a_1, b_1)v_1 & = & (a_2, b_2)u_2 & u_1 s & = & v_1 x_2 \\
(a_2, b_2)v_2 & = & (a_3, b_3)u_3 & u_2 x_2 & = & v_2 x_3 \\
& \vdots & & & \vdots & \\
(a_n, b_n)v_n & = & (a', b') & u_n x_n & = & v_n t
\end{array}$$

where each $x_i \in \{s, t\}$. We wish to find $a'', b'' \in S$ and $u \in Ss \cap St$ such that $w = a''u$, $v = b''u$. If $s \in St$, then let $u = s$, $a'' = a$, $b'' = b$ and we are done. If not, then let j be the first index such that $x_j = t$ and let $u = v_{j-1}x_j$. Then $u \in Ss \cap St$ since $v_{j-1}x_j = u_{j-1}x_{j-1} \in Ss$ by definition of j. Then we have,

$$as = a_1u_1s = a_1v_1x_2 = a_2u_2x_2 = \cdots = a_{j-1}v_{j-1}x_j = a_{j-1}u$$

and similarly $bs = b_{j-1}u$. So, letting $a'' = a_{j-1}$ and $b'' = b_{j-1}$ we are done. Thus, $Ss \cap St$ is locally principal.

($\Longleftarrow$) Assume now that $(S \times S)_S$ is principally weakly flat and that the intersection of any two principal left ideals of S is either empty or else locally principal. We wish to show that $S \times S$ satisfies condition (W). So, suppose $(a, b)s = (a', b')t$. Then $as = a't$ and $bs = b't$. If we denote $x = as = a't$, $y = bs = b't$ we see that $x, y \in Ss \cap St$. So by assumption, there exists $u \in Ss \cap St$, $a'', b'' \in S$ such that $x = a''u, y = b''u$. Hence, $(a, b)s = (a'b')t = (a'', b'')u$, i.e. $S \times S$ satisfies condition (W), showing that it is weakly flat. □

Now we wish to distinguish between principal weak flatness and weak flatness using only diagonal acts. Recall first that $\mathcal{T}_{\mathbb{N}}$ denotes the full transformation monoid on $\mathbb{N}$, where the function symbols are written to the right of their arguments. We now give the definition of the Baer-Levi semigroup:

$$S = \{\alpha \in \mathcal{T}_{\mathbb{N}} : \alpha \text{ is injective and } \mathbb{N} \setminus \mathbb{N}\alpha \text{ is infinite}\}.$$

It can be shown that this semigroup is right cancellative and idempotent-free, and it also turns out to be the example that we are looking for, as illustrated by the following few results.

Lemma 3.6. *Let S be the Baer-Levi semigroup and let $\alpha, \beta \in S$. Then $\alpha \in S\beta$ if and only if $\mathbb{N}\alpha \subseteq \mathbb{N}\beta$ and $\mathbb{N}\beta \setminus \mathbb{N}\alpha$ is infinite.*

Proof. ($\Longrightarrow$) Assume $\alpha \in S\beta$. Then $\alpha = \gamma\beta$ for some $\gamma \in S$. Then clearly $\mathbb{N}\alpha \subseteq \mathbb{N}\beta$. Moreover,

$$\mathbb{N}\beta \setminus \mathbb{N}\alpha = \mathbb{N}\beta \setminus \mathbb{N}\gamma\beta = (\mathbb{N} \setminus \mathbb{N}\gamma)\beta$$

which is infinite since $\mathbb{N} \setminus \mathbb{N}\gamma$ is infinite and β is injective.

($\Longleftarrow$) Now assume that $\mathbb{N}\alpha \subseteq \mathbb{N}\beta$ and $\mathbb{N}\beta \setminus \mathbb{N}\alpha$ is infinite. Then we know that $\alpha = \gamma\beta$ for some $\gamma \in \mathcal{T}_{\mathbb{N}}$. Also, γ is unique since S is right cancellative and since α is injective it follows that γ is also injective. Finally,

$$(\mathbb{N} \setminus \mathbb{N}\gamma)\beta = \mathbb{N}\beta \setminus \mathbb{N}\alpha,$$

which is infinite by assumption. It follows that $\mathbb{N} \setminus \mathbb{N}\gamma$ is infinite. Thus, $\gamma \in S$ and so $\alpha \in S\beta$. □

Example: Let S^1 be the Baer-Levi semigroup with an adjoined identity. Consider the elements $(n)\alpha = 2n, (n)\beta = 3n$. Then $S^1\alpha$ and $S^1\beta$ are incomparable, and clearly $S^1\alpha \cap S^1\beta \neq \phi$. Now choose $\gamma \neq \delta \in S$ such that $\mathbb{N}\gamma = \mathbb{N}\delta = 6\mathbb{N}$ and note that

$\mathbb{N}\alpha \setminus (\mathbb{N}\alpha \cap \mathbb{N}\beta)$ and $\mathbb{N}\beta \setminus (\mathbb{N}\alpha \cap \mathbb{N}\beta)$ are both infinite. Then by our choice of α and β and by the lemma above we have,

$$\begin{array}{rc} & \gamma, \delta \in S^1\alpha \cap S^1\beta \\ \iff & \gamma, \delta \in S\alpha \cap S\beta \\ \iff & \mathbb{N}\gamma, \mathbb{N}\delta \subseteq \mathbb{N}\alpha \cap \mathbb{N}\beta = 6\mathbb{N}. \end{array}$$

Now suppose $\gamma, \delta \in S^1\mu \subseteq S^1\alpha \cap S^1\beta$. Then

$$\mathbb{N}\alpha \cap \mathbb{N}\beta = \mathbb{N}\gamma = \mathbb{N}\delta \subseteq \mathbb{N}\mu \subseteq \mathbb{N}\alpha \cap \mathbb{N}\beta,$$

and so all of these sets are equal. Hence, $\mathbb{N}\mu \setminus \mathbb{N}\gamma$ and $\mathbb{N}\mu \setminus \mathbb{N}\delta$ cannot be infinite. The only possibility is that $\gamma = \mu = \delta$, which is a contradiction. Hence, $(S^1 \times S^1)_{S^1}$ is not weakly flat by Theorem 3.5, but $(S^1 \times S^1)_{S^1}$ is principally weakly flat since S^1 is right cancellative.

3.3 Conditions (P) and (E)

The reader may be wondering something at this point: What happened to flatness? Well, unfortunately we do not know any new results that pertain to flatness of diagonal acts. All we have to work with in this situation are results that we already know for general acts. For instance, as we have seen previously in Chapter 1, we know that all acts over an inverse monoid are flat; so if we ever come across a diagonal act of an inverse monoid we know it will be flat. In this section, we discover necessary and sufficient conditions on S so that its diagonal act will satisfy each of the interpolation conditions (P) and (E). First we must define some new notation.

Definition 3.7. *Let S be a monoid and let I be any nonempty set. Then for any element $\vec{a} \in S^I$ we define the following sets:*
$L(\vec{a}) = \{\vec{s} \in S^I : s_i a_i = s_j a_j$ *for all* $i, j \in I\}$
$l(\vec{a}) = \{s \in S : s a_i = s a_j$ *for all* $i, j \in I\}$.

In particular, for $S \times S$ we have $L(a, b) = \{(s, t) \in S \times S : sa = tb\}$ and $l(a, b) = \{s \in S : sa = sb\}$, for any $(a, b) \in S \times S$. Note also that, for any nonempty set I, the sets $L(\vec{a})$ and $l(\vec{a})$ are left sub-acts of S^I and left ideals of S, respectively, if they are nonempty. We are now ready to characterize conditions (P) and (E) for $S \times S$ in terms of these new sets. The result is actually simply a weakened condition from [8] that dealt with arbitrary products.

Theorem 3.8. *Let S be a monoid. Then the following are equivalent:*

(1) $(S \times S)_S$ satisfies condition (P) (resp. condition (E));

(2) If A_S and B_S are S-acts that satisfy condition (P) (resp. condition (E)), then so does $A_S \times B_S$;

(3) S^n satisfies condtion (P) (resp. condition (E)) for every $n \in \mathbb{N}$;

(4) If $A_1, \ldots, A_n$ are all right S-acts that satisfy condition (P) (resp. condition (E)), then so does $A_1 \times \cdots \times A_n$;

(5) Each set $L(a, b)$, if nonempty, is a locally cyclic left S-act (resp. each set $l(a, b)$, if non-empty, is a locally principal left ideal of S).

Proof. We present the details for condition (P) and remark that a similar strategy may be followed for condition (E). Notice first of all that (4) clearly implies (3) and (2), each of which imply (1). So it suffices to show that (1) implies (5) and that (5) implies (4).

((1)$\Longrightarrow$(5)) Assume that $S \times S$ satisfies condition (P) and let $(u, v),\ (u', v') \in L(a, b)$. Then $ua = vb$ and $u'a = v'b$, giving us the equality

$$(u, u')a = (v, v')b.$$

By condition (P) for $S \times S$, there exist $(w, w') \in S \times S,\ p, q \in S$ such that

$$\begin{aligned} (u, u') &= (w, w')p \\ (v, v') &= (w, w')q \\ pa &= qb. \end{aligned}$$

From the last equality we see that $(p, q) \in L(a, b)$. Moreover,

$$\begin{aligned} (u, v) &= w(p, q) \\ (u', v') &= w'(p, q) \end{aligned}$$

and so $(u, v),\ (u', v') \in S(p, q)$. Thus, $L(a, b)$ is locally cyclic.

((5)$\Longrightarrow$(4)) Assume that each set $L(a, b)$, if nonempty, is locally cyclic and that $A_1, \ldots, A_n$ are right S-acts satisfying condition (P). Assume further that we have $(a_1, \ldots, a_n)s = (a'_1, \ldots, a'_n)s'$ for elements $a_i, a'_i \in A_i,\ s, s' \in S$. If we apply condition (P) to each equality $a_i s = a'_i s'$, we obtain elements $a''_i \in A_i,\ u_i, u'_i \in S$ such that

$$\begin{aligned} a_i &= a''_i u_i \\ a'_i &= a''_i u'_i \\ u_i s &= u'_i s' \end{aligned}$$

for each i. By assumption, each set $L(s, s')$ is locally cyclic. Hence, there exists $(u, v) \in L(s, s')$ such that

$$(u_1, u_1'), \ldots, (u_n, u_n') \in S(u, v).$$

So, we may write $(u_i, u_i') = p_i(u, v)$ for some $p_i \in S$, for each i, and let $b_i'' = a_i'' p_i$. Then we have

$$\begin{array}{rcl} (b_1'', \ldots, b_n'')u &=& (a_1''p_1, \ldots, a_n''p_n)u = (a_1''u_1, \ldots, a_n''u_n) = (a_1, \ldots, a_n) \\ (b_1'', \ldots, b_n'')v &=& (a_1''p_1, \ldots, a_n''p_n)v = (a_1''u_1', \ldots, a_n''u_n') = (a_1', \ldots, a_n') \\ us &=& vs'. \end{array}$$

Thus, we have shown that $A_1 \times \cdots \times A_n$ satisfies condition (P), which completes the proof. $\square$

We can actually distinguish among flatness and condition (P) with the following example.

Example: Let $S = \{0, 1\}$. Then since S is an inverse monoid, $(S \times S)_S$ is flat. But, $L(0, 0) = S \times S$, which cannot be (locally) cyclic because it has four elements and any cyclic act in this case will have at most two elements. Hence, by the previous theorem, $S \times S$ does not satisfy condition (P).

3.4 Projectivity and Freeness

In this section we discuss what is known so far about projectivity of diagonal acts. In 1991, Bulman-Fleming studied products of projective S-acts and was able to give a characterization for the case of general products. However, disappointingly we do not have as nice a result for the case of diagonal acts. This is because in Bulman-Fleming's work, he was able to show that if a monoid S has the property that S^I is projective for any set I then S must be right cancellative, but when we limit our attention to just $S \times S$ then we do not have this result. Consequently, the main result of this section will give a description of projective diagonal acts in the case that our monoid is right cancellative. We will then see an example that shows that right cancellativity is not even a necessary condition. The first result we state is Lemma 1.1 of [1] and will be used in the proof of our main theorem.

Lemma 3.9. *Let S be a monoid and I any non-empty set. Then the following are equivalent:*

1) S^I is a projective right S-act;

2) If $\{A_i : i \in I\}$ is a family of projective right S-acts, then $A_S = \prod_{i\in I} A_i$ is projective.

The next proposition is a simple adaptation of Proposition 2.2 of [1]. We include the proof here for completeness.

Proposition 3.10. *Let S be a monoid such that the right diagonal act $S \times S$ is projective. Then each non-empty $L(\vec{a})$ is locally cyclic and each non-empty $l(\vec{a})$ is locally principal for each $\vec{a} \in S^I$ where I is a non-empty set.*

Proof. First we will consider the case of $L(\vec{a})$. Suppose $\vec{b}, \vec{c} \in L(\vec{a})$, so that $b_i a_i = b_k a_k$ and $c_i a_i = c_k a_k$ for all $i, k \in I$. Then we have

$$\begin{aligned}(b_i, c_i) &= (b_i, c_i)1\\ (b_i, c_i)a_i &= (b_k, c_k)a_k\\ (b_k, c_k)1 &= (b_k, c_k),\end{aligned}$$

and so all of the elements (b_k, c_k) belong to a single connected component of $S \times S$. Because $S \times S$ is projective, this component is of the form $(p, q)S$ where (p, q) is left e-cancellable for some idempotent $e \in S$. That is, $(p, q) = (p, q)e$ and $(p, q)u = (p, q)u$, for any $u, v \in S$, implies $eu = ev$. So we may write

$$(b_i, c_i) = (p, q)z_i$$

for some $z_i \in S$. We will show that the element $e\vec{z}$ belongs to $L(\vec{a})$ and that $\vec{b}, \vec{c} \in Se\vec{z}$. First, notice that

$$(p, q)z_i a_i = (b_i, c_i)a_i = (b_k, c_k)a_k = (p, q)z_k a_k$$

giving us $ez_i a_i = ez_k a_k$ for all $i, k \in I$, i.e. $e\vec{z} \in L(\vec{a})$. Finally,

$$\begin{aligned}\vec{b} &= p\vec{z} = pe\vec{z} \in Se\vec{z} \text{ and}\\ \vec{c} &= q\vec{z} = qe\vec{z} \in Se\vec{z}.\end{aligned}$$

Hence, $L(\vec{a})$ is locally cyclic.

Next we deal with $l(\vec{a})$. Suppose $s, t \in l(\vec{a})$. Because $S \times S$ is projective, $(s, t) \in (p, q)S$ where (p, q) is left e-cancellable. Then,

$$(s, t) = (p, q)z = (p, q)ez$$

for some $z \in S$. So, for each $i, k \in I$ we have

$$(p, q)za_i = (sa_i, ta_i) = (sa_k, ta_k) = (p, q)za_k,$$

and so $s, t \in Sez$. Thus, $l(\vec{a})$ is locally principal. □

On the monoid S we define the relation $\mathcal{L}$ by $s\mathcal{L}t$ iff $Ss = St$. It is easily seen that $\mathcal{L}$ is an equivalence relation on S. (Note: This is one of the relations known as Green's relations, which play a fundamental role in semigroup theory. See [9] for more information on them.) Consider now the $\mathcal{L}$-preorder on S given by $s \leq_{\mathcal{L}} t$ iff $s \in St$. For any subset X of S, by a *least upper bound (lub)* of X we mean an upper bound of X in the $\mathcal{L}$-preorder which lies below all other upper bounds of X. It is easy to see that least upper bounds of X are determined up to $\mathcal{L}$-equivalence. The next result is an analogue of Proposition 2.4 of [1] in the case that S is right cancellative.

Proposition 3.11. *Let S be a right cancellative monoid. Then the following are equivalent.*

(1) For each non-empty set I and each $\vec{a} \in S^I$, the set $L(\vec{a})$ is either empty or else a locally cyclic left S-act.

(2) For each non-empty set I and each $\vec{a} \in S^I$, $\bigcap_{i \in I} Sa_i$ is either empty or else a locally principal left ideal of S.

(3) Every non-empty finite subset of S has a least upper bound in the $\mathcal{L}$-preorder.

Proof. First, to show that (1) and (2) are equivalent, notice that the emptiness cases match up. So assume now that both of $L(\vec{a})$ and $\bigcap_{i \in I} Sa_i$ are non-empty. Define a mapping

$$\varphi : L(\vec{a}) \longrightarrow \bigcap_{i \in I} Sa_i$$

by $\varphi(\vec{b}) = b_i a_i$ where i is any fixed element of I. This mapping is clearly well-defined and surjective. The injectivity of φ follows from the right cancellativity of S. Finally, it is clearly an S-homomorphism. Hence, $L(\vec{a})$ and $\bigcap_{i \in I} Sa_i$ are isomorphic as left S-acts.

Next we show that (2) implies (3). So assume (2) holds and let F be a non-empty finite subset of S. Let X denote the set of upper bounds of F in the $\mathcal{L}$-preorder. Then X is non-empty as it contains 1. Then the set $\bigcap_{x \in X} Sx$ is also non-empty since it contains F. By assumption, there exists $u \in \bigcap_{x \in X} Sx$ such that $F \subseteq Su$. Then this element u is a least upper bound of F in the $\mathcal{L}$-preorder.

Finally, assume condition (3) holds. We show that (3) implies (2). Let F be a non-empty finite subset of $\bigcap_{i \in I} Sa_i$, and let u be a least upper bound of F in the $\mathcal{L}$-preorder. Then $F \subseteq Su$ and u belongs to $\bigcap_{i \in I} Sa_i$. Notice also that $F \subseteq Sa_i$ for each $i \in I$, so every a_i is an upper bound for F in the $\mathcal{L}$-preorder. Therefore, $u \leq_{\mathcal{L}} a_i$ for all i, which means that $u \in \bigcap_{i \in I} Sa_i$. □

In the following few results we describe the connected components of $S \times S$. The approach is much the same as that in [1], and we include a proof that is not provided in that paper.

Lemma 3.12. *Let S be a right cancellative monoid where every pair of elements has a least upper bound in the $\mathcal{L}$-preorder. For each pair $a, b \in S$ let $[a, b]$ denote some fixed lub of a and b and let (a', b') be the unique element of $S \times S$ such that $(a, b) = (a', b')[a, b]$. Then for any $s \in S$, we have $[as, bs]\mathcal{L}[a, b]s$, and $[a', b']$ is a unit is S for every $a, b \in S$.*

Proof. Because $a, b \in S[a, b]$, it follows that $as, bs \in S[a, b]s$ for every $s \in S$, and so $[as, bs] \in S[a, b]s$. Suppose $[as, bs] = v[a, b]s$ for some $v \in S$. Then

$$\begin{aligned} (a, b)s &= (a'', b'')[as, bs] = (a'', b'')v[a, b]s \text{ implies} \\ (a, b) &= (a'', b'')v[a, b], \end{aligned}$$

and so $a, b \in Sv[a, b]$. It follows that $[a, b] \in Sv[a, b]$ and so $1 = uv$ for some $u \in S$. Therefore, $u[as, bs] = [a, b]s$ and so $[a, b]s \in S[as, bs]$, as required.

Finally, we have $[a', b'][a, b]\mathcal{L}[(a', b')[a, b]] = [a, b]$, and so it follows easily that $[a', b']$ is a unit. □

The next proposition, which finally gives our description of the connected components, is a special case of Proposition 1.3 of [1] and the proof given there works here. For completeness of the section, the proof will be presented here this case. First though, note the following remark.

Remark: If S is a right cancellative monoid, then $s\mathcal{L}t$, for $s, t \in S$, if and only if $s = ut$ for some unit $u \in S$.

Proof: If $s\mathcal{L}t$ then $Ss = St$ and so $s = xt$ and $t = ys$ for some $x, y \in S$. Then $s = xt = xys$, so by right cancellativity, $xy = 1$. Hence x is a unit. Conversely, if $s = ut$ where u is a unit then $t = u^{-1}s$, implying $Ss = St$, i.e. $s\mathcal{L}t$. ♢

Proposition 3.13. *Let S be a right cancellative monoid where every pair of elements has a least upper bound in the $\mathcal{L}$-preorder. Then (in the notation of the previous lemma) the connected component of $(S \times S)_S$ containing (a, b) is $(a', b')S$.*

Proof. We show by induction on n that if $(c, d) \in S \times S$ is connected to (a, b) by a scheme

$$\begin{aligned} (c, d) &= (c_1, d_1)s_1 \\ (c_1, d_1)t_1 &= (c_2, d_2)s_2 \\ &\cdots \\ (c_n, d_n)t_n &= (a, b) \end{aligned}$$

then $(c', d') = (a', b')u$ for some unit $u \in S$.

First, suppose that

$$\begin{aligned} (c, d) &= (c_1, d_1)s_1 \\ (c_1, d_1)t_1 &= (a, b). \end{aligned}$$

By Lemma 3.12 we have that $[c_1s_1, d_1s_1]\mathcal{L}[c_1, d_1]s_1$, and so by the remark, $[c_1s_1, d_1s_1] = u[c_1, d_1]s_1$ for some unit $u \in S$. Then $(c, d) = (c_1, d_1)s_1$ implies

$$[c, d] = [c_1s_1, d_1s_1] = u[c_1, d_1]s_1,$$

and similarly $[a, b] = v[c_1, d_1]t_1$ for some unit $v \in S$. Thus, we have $u^{-1}[c, d] = [c_1, d_1]s_1$, and so

$$(c_1', d_1')u^{-1}[c, d] = (c_1', d_1')[c_1, d_1]s_1 = (c_1, d_1)s_1 = (c, d) = (c', d')[c, d],$$

giving us $(c_1', d_1')u^{-1} = (c', d')$ by right cancellativity. Similarly, $(c_1', d_1')v^{-1} = (a', b')$, and so $(a', b')v = (c_1', d_1')$. Thus, $(c', d') = (a', b')vu^{-1}$, which completes the base case as vu^{-1} is a unit.

Now assume the result holds for connections to (a, b) via a scheme of length $n - 1$ and let (c, d) be connected to (a, b) as at the start of the proof. Then exactly as before, we get $(c_1', d_1') = (c', d')u$ for a unit $u \in S$. We see that (c_1t_1, d_1t_1) is connected to (a, b) via a scheme of length $n-1$, so by induction $((c_1t_1)', (d_1t_1)') = (a', b')x$ for some unit $x \in S$. Thus,

$$\begin{aligned} (c_1', d_1')[c_1, d_1]t_1 &= (c_1, d_1)t_1 \\ &= (c_1t_1, d_1t_1) \\ &= ((c_1t_1)', (d_1t_1)')[c_1t_1, d_1t_1] \\ &= (a', b')xy[c_1, d_1]t_1 \end{aligned}$$

for some unit $y \in S$ since $[c_1t_1, d_1t_1]\mathcal{L}[c_1, c_1]t_1$. By right cancellativity, $(c_1', d_1') = (a', b')xy$. Then $(c', d') = (c_1', d_1')u^{-1} = (a', b')u^{-1}xy$, completing the inductive step as $u^{-1}xy$ is a unit.

Finally,

$$\begin{aligned}(c', d') = (a', b')u \quad &\Rightarrow \quad (c', d')[c, d] = (a', b')u[c, d] \\ &\Rightarrow \quad (c, d) = (a', b')u[c, d] \\ &\Rightarrow \quad (c, d) \in (a', b')S.\end{aligned}$$

Hence, we have shown that if (c, d) is connected to (a, b), then $(c, d) \in (a', b')S$, which completes the proof. □

We are now ready to give our main result concerning the projectivity of $(S \times S)_S$ in the case when S is right cancellative.

Theorem 3.14. *Let S be a right cancellative monoid. Then the following are equivalent:*

1) $(S \times S)_S$ is free;

2) $(S \times S)_S$ is projective;

3) S^n is projective for every $n \in \mathbb{N}$;

4) The direct product $A_S \times B_S$ is projective whenever A_S and B_S are projective;

5) The direct product of any non-empty finite family of projective right S-acts is projective;

6) For every non-empty set I and every $\vec{a} \in S^I$ the sets $L(\vec{a})$ and $l(\vec{a})$ are either empty or else locally cyclic left S-acts;

7) Every non-empty finite subset of S has a least upper bound in the $\mathcal{L}$-preorder and each non-empty $l(a, b)$ is locally principal left ideal of S, for $a, b \in S$.

Proof. Firstly, we see that clearly (1) implies (2), and in the right cancellative case it is not hard to see that, by Theorem 1.31, (2) also implies (1). Second, (2) and (4) are equivalent by Lemma 3.9, (4) and (5) can easily be shown to be equivalent by induction, and (5) implies (3) which in turn implies (2). Thus, (1)-(5) are all equivalent. Next, (2) implies (6) by Proposition 3.10 and (6) implies (7) by Proposition 3.11. Hence, all we need to show is (7) implies (2).

Assume (7). Then clearly every pair of elements has a least upper bound in the $\mathcal{L}$-preorder, so by Proposition 3.13 we know that each connected component of $S \times S$ is equal to some $(a, b)S$ where $[a, b]$ is a unit. We will show that $(a, b)S_S \cong S_S$ to show that $(S \times S)_S$ is projective. (Note that we are actually showing that $S \times S$ is free, which will then imply projectivity.) Define a mapping

$$\varphi : S_S \longrightarrow (a,b)S_S$$
$$s \mapsto (a,b)s.$$

Then φ is clearly a surjective S-homomorphism. For injectivity, suppose that $(a,b)s = (a,b)t$. Then $as = at$, $bs = bt$ and so $a, b \in l(s,t)$. By assumption there exists $u \in l(s,t)$ such that $a, b \in Su$. Then $a, b \leq_{\mathcal{L}} u$ and so $[a,b] \leq_{\mathcal{L}} u$, i.e. $[a,b] = xu$ for some $x \in S$. Now, because $[a,b]$ is a unit

$$y[a,b] = yxu = 1$$

for some $y \in S$ and hence u is also a unit. Thus, $l(s,t) = Su = S$, implying that $s = t$ since $1 \in l(s,t)$. So φ is indeed an isomorphism, which completes the proof. □

Of course, when looking at this result one may wonder if right cancellativity is in fact necessary. The next result gives an example to show that right cancellativity is actually far from being necessary.

Proposition 3.15. *Let $S = \mathcal{T}_{\mathbb{N}}$ be the full transformation monoid on the natural numbers. Then S is not right cancellative. However, $(S \times S)_S$ is isomorphic to S as a right S-act, so $(S \times S)_S$ is free and hence also projective.*

Proof. Let $\alpha, \beta \in S$ be defined by $n\alpha = 2n - 1$, $n\beta = 2n$, for $n \in \mathbb{N}$. We claim that $(S \times S)_S$ is isomorphic to $(\alpha, \beta)S$. Indeed, let $(\gamma, \delta) \in S \times S$. Consider $\mu \in S$ defined by

$$n\mu = \begin{cases} (\frac{n+1}{2})\gamma & \text{if } n \text{ is odd} \\ (\frac{n}{2})\delta & \text{if } n \text{ is even} \end{cases}$$

Then,

$$\begin{aligned} n\alpha\mu &= (2n-1)\mu = n\gamma, \text{ and} \\ n\beta\mu &= (2n)\mu = n\delta, \end{aligned}$$

showing that $(\gamma, \delta) = (\alpha, \beta)\mu$. Moreover, $(\alpha, \beta)S$ is isomorphic to S via the mapping $(\alpha, \beta)\tau \mapsto \tau$. The injectivity of this map follows from the fact that (α, β) is "left cancellable" in the sense that if $(\alpha, \beta)\lambda = (\alpha, \beta)\nu$, then λ and μ agree on both all odd and all even elements of $\mathbb{N}$ and hence are equal. The surjectivity of this map is clear as well as it being an S-homomorphism. □

We may use this theorem though for the case of groups because groups are (right) cancellative. In fact, what we get for the case of diagonal acts of groups is the strongest type of flatness we could want.

Proposition 3.16. *Let G be a group. Then the diagonal act $(G \times G)_G$ is free.*

Proof. First we will use part (7) of our theorem above to show that $(G \times G)_G$ is free. It is clear that in the group situation, the set $l(a, b)$ is either empty or all of G depending on if $a \neq b$ or $a = b$ respectively. Thus, whenever it is nonempty it is (locally) cyclic. Next, for any $g \in G$, clearly $Gg = G$, and so $g \leq_{\mathcal{L}} h$ for any $g, h \in G$, and so any (finite) subset of G will have a least upper bound in the $\mathcal{L}$-preorder. Hence, $(G \times G)_G$ is free. $\square$

To end the section, we give a partial answer to the third question asked at the beginning of the chapter. Specifically, sufficient conditions are discovered in order for a diagonal act that satisfies condition (P) to be projective. In order to set it up we first give a lemma that explains the relation between irreducible and prime elements of a monoid that satisfies our criteria. Here, of course, the notions of irreducible and prime are precisely the same as in the ring sense.

Lemma 3.17. *Let S be a commutative monoid such that $(S \times S)_S$ satisfies condition (P). Then irreducible elements of S are also prime.*

Proof. Suppose $p \in S$ is irreducible and suppose that $p|rs$ for some $r, s \in S$, that is $rs = pq$ for some $q \in S$. Then because S is commutative, $(s, q)r = (p, r)q$. So by condition (P) on $S \times S$, we get

$$\begin{aligned} (s, q) &= (x, y)u \\ (p, r) &= (x, y)v \\ ur &= vq \end{aligned}$$

for some $x, y, u, v \in S$. Now, from the equality $p = xv$ and the assumption that p is irreducible, it must be that either x or v is a unit. If x is a unit, then from $r = yv$ we get that $r = yx^{-1}p$, i.e. $p|r$. Otherwise, v is a unit, and from $s = xu$ we get $s = pv^{-1}u$, i.e. $p|s$. Hence, p is prime. $\square$

Now we wish to show that if we impose more conditions on S then we get that $(S \times S)_S$ is in fact projective. To do so we will apply part (6) of Theorem 3.14. Firstly, note that if S is a cancellative monoid then the sets $l(a, b)$ for $a, b \in S$ are trivially (locally) cyclic. Indeed, say $s \in l(a, b)$. Then $sa = sb$, so by cancellativity $a = b$. Hence, for any $a, b \in S$, the set $l(a, b)$ is simply S itself, which is clearly cyclic ($S = S1$). Secondly, in the commutative case, the notions of least upper bound in the $\mathcal{L}$-preorder and greatest common divisor coincide. Indeed, $s \leq_{\mathcal{L}} t$ means that $s = xt$ for some $x \in S$, which implies that $t|s$ because S is commutative. Hence, if an element $u \in S$ is a least upper bound of a subset X of S in the $\mathcal{L}$-preorder, then it is actually a greatest common divisor of X. Finally, we say that a monoid S satisfies the *divisor chain condition* if S contains no infinite sequence of elements $s_1, s_2, \ldots$ such that each s_{i+1} is a proper factor of s_i. So to get what we want, all we need to do is use the following result which may be found in [10].

Theorem 3.18. *Let S be a commutative, cancellative monoid with the divisor chain condition and such that irreducibles are prime. Then every pair of elements of S has a greatest common divisor.*

Proposition 3.19. *Let S be a commutative, cancellative monoid that satisfies the divisor chain condition and such that $(S \times S)_S$ satisfies condition (P). Then $(S \times S)_S$ is projective.*

Proof. As mentioned above, we will apply theorem 3.14. So by the notes above, all we need to show is that greatest common divisors exist for any non-empty finite subset of S. But by the previous theorem we have that a greatest common divisor of any pair of elements exists. Hence we have by induction that a greatest common divisor of any finite subset of S exists. □

Chapter 4

Other Results

4.1 Pullbacks and Other Flatness Properties

Around the beginning of the new millennium, a student of Mati Kilp's, Valdis Laan, began looking at flatness properties from a different angle. He asked if one could obtain known flatness properties and acquire new ones by investigating the extent to which the functor $A_S \otimes -$ preserves pullbacks. Specifically, whenever we have a pullback diagram in the category of left S-acts and we tensor by a right S-act, A_S, we get a canonical mapping $\varphi : A_S \otimes {}_SP \longrightarrow P'$ where ${}_SP$ is the pullback from the original diagram and P' is the pullback from the new diagram. We say that A_S is *pullback flat* if this mapping is bijective for any such diagram. What Laan was interested in was restricting the diagrams to certain situations involving only S itself or (principal) right ideals of S and then also considering when φ is injective or surjective. In this chapter we will display some of these results and then discuss a few new results that relate specifically to diagonal acts. First, we will make clear the notion of pullbacks in the category of left S-acts. Consider the following pullback situation:

$$\begin{array}{ccc} {}_SM & & \\ {\scriptstyle f}\big\downarrow & & \\ {}_SQ & \overset{g}{\longleftarrow} & {}_SN \end{array}$$

The pullback of the pair (f, g) may be concretely realized as a pair $({}_SP, (p_1, p_2))$ where

$${}_SP = \{(m, n) \in ({}_SM \times {}_SN) : f(m) = g(n)\}$$

and $p_1,\ p_2$ are the projections from ${}_SM \times_S N$ onto ${}_SM$ and ${}_SN$ respecively. Note that, to guarantee the existence of all pullbacks, empty left acts must be allowed. For notational convenience, the pullback diagram

$$\begin{array}{ccc} {}_SP & \xrightarrow{p_1} & {}_SM \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ {}_SN & \xrightarrow{g} & {}_SQ \end{array}$$

will be denoted by $P(M, N, f, g, Q)$. Now, if we tensor this diagram by a right S-act A_S, it produces the following commutative diagram

$$\begin{array}{ccc} A_S \otimes {}_SP & \xrightarrow{id_A \otimes p_1} & A_S \otimes {}_SM \\ {\scriptstyle id_A \otimes p_2}\downarrow & & \downarrow{\scriptstyle id_A \otimes f} \\ A_S \otimes {}_SN & \xrightarrow{id_A \otimes g} & A_S \otimes {}_SQ \end{array}$$

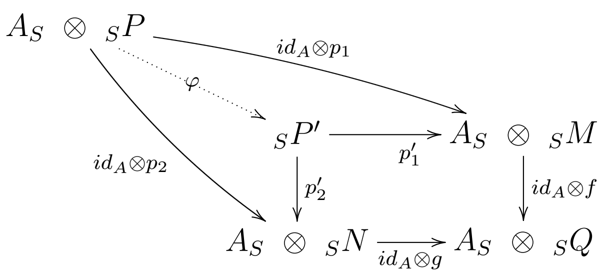
in the category of sets. For the pullback of the pair $(id_A \otimes f, id_A \otimes g)$ we take

$$P' = \{(a \otimes m, a' \otimes n) \in (A_S \otimes {}_SM) \times (A_S \otimes {}_SN) : a \otimes f(m) = a' \otimes g(n)\}$$

with p_1' and p_2' being the corresponding projections. So from the definition of pullbacks, there exists a unique mapping $\varphi : A_S \otimes {}_SP \longrightarrow {}_SP'$ given by

$$\varphi(a \otimes (m, n)) = (a \otimes m, a \otimes n)$$

that makes the diagram



commute. To give the flavour of the types of results shown in [12] we give the following proposition without proof.

Proposition 4.1. *Let A_S be a right S-act. Then,*

1) A_S is torsion-free if and only if the corresponding φ is surjective for every pullback diagram $P(S, S, \iota, \iota, S)$ where $\iota : {}_S S \longrightarrow {}_S S$ is a monomorphism of left S-acts.

2) A_S is principally weakly flat if and only if the corresponding φ is surjective for every pullback diagram $P(Ss, Ss, \iota, \iota, S)$ where $s \in S$ and $\iota : {}_S(Ss) \longrightarrow {}_S S$ is a monomorphism of left S-acts.

3) A_S is weakly flat if and only if the corresponding φ is surjective for every pullback diagram $P(I, I, \iota, \iota, S)$ where I is a left ideal of S and $\iota : {}_S I \longrightarrow {}_S S$ is a monomorphism of left S-acts.

4) A_S is flat if and only if the corresponding φ is surjective for every pullback diagram $P(M, M, \iota, \iota, Q)$ where $\iota : {}_S M \longrightarrow {}_S Q$ is a monomorphism of left S-acts.

5) A_S satisfies condition (P) if and only if the corresponding φ is surjective for every pullback diagram $P(M, M, f, f, Q)$ where $f : {}_S M \longrightarrow {}_S Q$ is any homomorphism of left S-acts.

Consider now part (5) of the above proposition. If we restrict our attention to pullback diagrams of the form $P(Ss, Ss, f, f, S)$ for $s \in S$ then we should get a "principally weak" version of condition (P). This inspires the next definition.

Definition 4.2. *We say a right S-act A_S satifies* ***condition (PWP)*** *if the corresponding φ is surjective for every pullback diagram $P(Ss, Ss, f, f, S)$, for $s \in S$.*

The next theorem, whose proof may be found in [12], gives an equivalent interpolation condition for (PWP).

Theorem 4.3. *Let A_S be a right S-act. Then the following are equivalent:*

1) A_S satisfies condition (PWP);

2) The corresponding φ is surjective for pullback diagram $P(S, S, f, f, S)$.

3) For any $a, a' \in A_S,\ s \in S$ such that $as = a's$ there exist $a'' \in A_S,\ u, v \in S$ such that $a = a''u,\ a' = a''v,\ us = vs$.

We can now give an alternative description for condition (PWP) when it comes to diagonal acts.

Proposition 4.4. *Let S be a monoid. Then its diagonal act $(S \times S)_S$ satisfies condition (PWP) if and only if the sets $L(s, s)$ are either empty or else locally cyclic left S-acts, for any $s \in S$.*

Proof. ($\Rightarrow$) Suppose that $(S \times S)_S$ satisfies condition (PWP) and let $(a, b),\ (a', b') \in L(s, s)$. Then $as = bs,\ a's = b's$, i.e. $(a, a')s = (b, b')s$. By condition (PWP), there exist $a'', b'', u, v \in S$ such that

$$\begin{aligned} (a,a') &= (a'',b'')u \\ (b,b') &= (a'',b'')u \\ us &= vs. \end{aligned}$$

Then $(u,v) \in L(s,s)$ and $(a,b) = a''(u,v),\ (a',b') = b''(u,v)$. Thus, $L(s,s)$ is locally cyclic.

($\Leftarrow$) Suppose we have $(a,b)s = (a',b')s$. Then $as = a's$ and $bs = b's$ and so $(a,a'),\ (b,b') \in L(s,s)$. By assumption, there exists $(u,v) \in L(s,s)$ such that $(a,a'),\ (b,b') \in S(u,v)$. Say $(a,a') = a''(u,v)$ and $(b,b') = b''(u,v)$. Then we have

$$us = vs,\ (a,b) = (a'',b'')u,\ (a',b') = (a'',b'')v.$$

Hence, $(S \times S)_S$ satisfies condition (PWP). $\square$

It is clear that condition (P) implies condition (PWP). Using the above result we may now give an example of a diagonal act which satisfies condition (PWP) but not (P).

Example: Let $S = \langle x,y : xy = x^2, yx = y^2\rangle \cup \{1\}$. Then S is a right cancellative monoid. So, for any $t \in S$,

$$\begin{aligned} L(t,t) &= \{(s,r) \in S \times S : st = rt\} \\ &= \{(s,s) : b \in S\} \\ &= S(1,1), \end{aligned}$$

and so $(S \times S)_S$ satisfies condition (PWP). Now consider the set $L(x,y)$. We can easily show that $L(x,y) = \{(x^n,x^n) : n \in \mathbb{N}\} \cup \{(y^n,y^n) : n \in \mathbb{N}\}$. We claim that this set cannot be a locally cyclic left S-act. Indeed, consider the elements $(x,x),(y,y)$ of $L(x,y)$. Suppose there existed an element (s,t) of $L(x,y)$ such that $(x,x),(y,y) \in S(s,t)$. Then we have the following two cases:

<u>Case 1</u> $(s,t) = (x^r,x^r)$: Then if $s(x^r,x^r) = (x,x)$ for some $s \in S$ then $r = 1$ and $s = 1$. But then it is clearly impossible for $t(x,x) = (y,y)$ for any $t \in S$.

<u>Case 2</u> $(s,t) = (y^r,y^r)$: Similar to case 1.

Thus, $L(x,y)$ is not locally cyclic, so we conclude that $(S \times S)_S$ does not satisfy condition (P) by Theorem 3.8.

The next few facts lead to a result that gives a nice characterisation of inverse monoids whose diagonal acts satisfy condition (PWP). First we recall a basic fact about inverse monoids.

Lemma 4.5. *Let S be an inverse monoid. Then right inverses, if they exist, are unique.*

Proof. Let $s, x, y \in S$ such that $sx = sy = 1$. Then $xsx = x$ and $sxs = s$, so x is an inverse of s. On the other hand, $ysy = y$ and $sys = s$, so y is also an inverse of s. By uniqueness of inverses in an inverse monoid, we have $x = y$. □

Proposition 4.6. *Let S be an inverse monoid such that $(S \times S)_S$ satisfies condition (PWP). Then S is right cancellative.*

Proof. Suppose we have $sx = tx$ for $s, t, x \in S$. Then $(s, 1)x = (t, 1)x$. By condition (PWP), there exist $s', t', u, v \in S$ such that

$$\begin{aligned} (s,1) &= (s',t')u \\ (t,1) &= (s',t')v \\ ux &= vx. \end{aligned}$$

So $t'u = 1 = t'v$ and since S is an inverse monoid, by Lemma 4.5 we get that $u = v$. Hence, $s = s'u = s'v = t$, proving that S is right cancellative. □

Corollary 4.7. *Let S be an inverse monoid such that $S \times S$ satisfies condition (PWP). Then S is a group.*

The next little result will come in handy in the section where we discuss completely (0-)simple semigroups.

Proposition 4.8. *Let S be a monoid such that the identity of S is isolated (that is, $st = 1$ if and only if $s = t = 1$). If $(S \times S)_S$ satisfies condition (PWP) then the semigroup $S \setminus \{1\}$ is idempotent-free.*

Proof. Suppose there existed $1 \neq e \in S$ such that e is idempotent. Then $(e, 1)e = (1, e)e$. So if $(S \times S)_S$ satisfied condition (PWP) then we could find elements $(a, b) \in S \times S$ and $u, v \in S$ such that

$$\begin{aligned} (e,1) &= (a,b)u, \\ (1,e) &= (a,b)v, \\ ue &= ve. \end{aligned}$$

So we have that $bu = 1$ and then because 1 is isolated, $b = u = 1$. But if $b = 1$, then to get the second equality we must have that $v = e$, which would make it impossible for $av = 1$. Thus, $(S \times S)_S$ fails to have condition (PWP). □

Recall from the first chapter that we said an act A_S was strongly flat if the functor $A_S\otimes -$ preserved pullbacks and equalizers, and that this is equivalent to A_S satisfying both conditions (P) and (E). In 1991, however, Bulman-Fleming showed that only preserving pullbacks was in fact enough for strong flatness. He did this through use of a new interpolation condition called (PF) (sometimes called (BF) as in [11] for instance).

Definition 4.9. *Let A_S be a right S-act. We say that A_S satisfies* ***condition (PF)*** *if for any $a, a' \in A_S,\ s, s', t, t' \in S$ such that $as = a's',\ at = a't'$ there exist $a'' \in A_S,\ u, v \in S$ such that*

$$a = a''u,\ a' = a''v,\ us = vs',\ ut = vt'.$$

Theorem 4.10. *For a right S-act A_S the following are equivalent:*
1) $A_S \otimes -$ preserves pullbacks and equalizers;
2) $A_S \otimes -$ is strongly flat (that is, satisfies conditions (P) and (E));
3) $A_S \otimes -$ preserves pullbacks;
4) A_S satisfies condition (PF);
5) A_S is the direct limit of a family of finitely generated free right S-acts.

We have already seen previously how to classify conditions (P) and (E) when referring to diagonal acts. The following result gives another criterion that further specifies the above theorem to diagonal acts.

Proposition 4.11. *Let S be a monoid. Then the following are equivalent:*
1) $(S \times S)_S$ is strongly flat;
2) The sets $L(a, b)$ and $l(a, b)$ are either empty or else locally cyclic left S-acts;
3) The intersection of any pair of sets $L(a, b)$ and $L(a', b')$ is either empty or else a locally cyclic left S-act.

Proof. We know from Chapter 3 that (1) and (2) are equivalent. So to complete the proof we will use theorem 4.10 and show that part (3) is equivalent to $(S \times S)_S$ satisfying condition (PF).

First assume that $(S \times S)_S$ satisfies condition (PF). Let $(s, s'),\ (t, t') \in L(a, b) \cap L(a', b')$. Then

$$\begin{aligned} sa &= s'b, & sa' &= s'b', \\ ta &= t'b, & ta' &= t'b'. \end{aligned}$$

Hence we get

$$\begin{aligned} (s, t)a &= (s', t')b, \\ (s, t)a' &= (s', t')b'. \end{aligned}$$

So by condition (PF), there exist $s'', t'', u, v \in S$ such that

$$\begin{array}{rclrcl} (s,t) & = & (s'',t'')u, & (s',t') & = & (s'',t'')v, \\ ua & = & vb, & ua' & = & vb'. \end{array}$$

Thus, $(u,v) \in L(a,b) \cap L(a',b')$ and $(s,s') = s''(u,v),\ (t,t') = t''(u,v)$, and so $L(a,b) \cap L(a',b')$ is locally cyclic.

Conversely, suppose we have the equalities

$$\begin{array}{rcl} (s,t)a & = & (s',t')b, \\ (s,t)a' & = & (s',t')b'. \end{array}$$

Then $(s,s'), (t,t') \in L(a,b) \cap L(a',b')$. By assumption, there exists $(u,v) \in L(a,b) \cap (a',b')$ such that $(s,s'), (t,t') \in S(u,v)$. Say $(s,s') = s''(u,v),\ (t,t') = t''(u,v)$. Then

$$\begin{array}{rclrcl} (s,t) & = & (s'',t'')u, & (s',t') & = & (s'',t'')v, \\ ua & = & vb, & ua' & = & vb'. \end{array}$$

So $(S \times S)_S$ satisfies condition (PF), which completes the proof. □

4.2 Diagonal Acts of Completely (0-)Simple Semigroups

In this section we will investigate propeties of the diagonal act when we start with a completely (0-) simple semigroup. Recall the structure theorem stated in the second chapter that states that completely simple semigroups are exactly the Rees matrix semigroups, and completely 0-simple semigroups are exactly Rees matrix semigroups with zero. Because we prefer to work with acts over monoids, we will always insist that the monoid S^1, where S is a completely (0-) simple semigroup, acts on the set $S \times S$, and for some results $S^1 \times S^1$. We will examine the various flatness properties in increasing order of strength. The first two properties, namely torsion-freeness and principally weak flatness, are clear because all diagonal acts are torsion-free and because all completely (0-) simple semigroups are regular.

Proposition 4.12. *Let S be a completely simple semigroup. Then $(S \times S)_{S^1}$ is always weakly flat.*

Proof. We will show that the intersection of any two principal left ideals of S is either empty or else locally principal, and then the result will follow by Theorem 3.5. Let $(i, g, \lambda) \in S$. Then without too much trouble we can see that

$$S^1(i, g, \lambda) = \{(j, h, \lambda) : j \in I, h \in G\}.$$

So, the intersection of two principal left ideals will either be the above set if the third components of the elements are equal, or else empty if they are not equal. In the first case it is clearly (locally) principal. $\square$

This result can also be easily adapted to the completely 0-simple case. Flatness continues to give us trouble. We do not have anything new to report here, so we will just promptly move on to condition (P). In order to state the result we will have to define left groups. A *left group* is a semigroup of the form $L \times G$ where L is a left zero semigroup (that is, $st = s$ for all $s, t \in L$) and G is a group. It is not hard to check that if $S = M(I, G, \Lambda; P)$ is a completely simple semigroup and $|\Lambda| = 1$ then S is in fact a left group.

Theorem 4.13. *Let S be a completely simple semigroup. Then $(S \times S)_{S^1}$ satisfies condition (P) if and only if S is a left group.*

Proof. ($\Longrightarrow$) Assume that $(S \times S)_{S^1}$ has condition (P), and let $i \in I,\ \lambda, \mu \in \Lambda$. Then we get the equality

$$((i, e, \lambda), (i, e, \mu))(i, e, \mu) = ((i, P_{\lambda i} P_{\mu i}^{-1}, \mu), (i, P_{\mu i} P_{\lambda i}^{-1}, \lambda))(i, e, \mu),$$

where $e \in G$ is the identity element (both sides are equal to $((i, P_{\lambda i}, \mu), (i, P_{\mu i}, \mu))$). So by condition (P), we know that we can find elements $a \in S \times S,\ u, v \in S^1$ such that

$$\begin{gathered} ((i, e, \lambda), (i, e, \mu)) = au, \\ ((i, P_{\lambda i} P_{\mu i}^{-1}, \mu), (i, P_{\mu i} P_{\lambda i}^{-1}, \lambda)) = av, \\ u(i, e, \mu) = v(i, e, \mu). \end{gathered}$$

Now, if $\lambda \neq \mu$, then necessarily $u = v = 1$, which would then imply $((i, e, \lambda), (i, e, \mu)) = a = ((i, P_{\lambda i} P_{\mu i}^{-1}, \mu), (i, P_{\mu i} P_{\lambda i}^{-1}, \lambda))$, which is a contradiction. Hence $\lambda = \mu$, implying $|\Lambda| = 1$ and so S is a left group.

($\Longleftarrow$) Now suppose $S = L \times G$ where L is a left zero semigroup and G is a group. Suppose also that we have

$$((a, x), (b, y))(c, z) = ((a', x'), (b', y'))(c', z'),$$

for $a, a', b, b', c, c' \in L,\ x, x', y, y', z, z' \in G$. Then it must be that $xz = x'z'$, and $yz = y'z'$. So, in $(G \times G)_G$, $(x, y)z = (x', z')z'$. Because all G-acts, where G is a group, satisfy condition (P), there exist $(g, h) \in G \times G,\ u, v \in G$ such that

$$\begin{aligned} (x, y) &= (g, h)u, \\ (x', y') &= (g, h)v, \\ uz &= vz'. \end{aligned}$$

Choose any $i \in L$. Then

$$\begin{aligned} ((a,x),(b,y)) &= ((a,g),(b,h))(i,u), \\ ((a',x'),(b',y')) &= ((a',g),(b',h))(i,v), \\ (i,u)(c,z) &= (i,v)(c',z'). \end{aligned}$$

Thus, $(S \times S)_{S^1}$ satisfies condition (P). □

As soon as we add the 0, though, we hit a problem. Recall Proposition 4.8 that stated that if a monoid S with an isolated identity has any non-trivial idempotents then $(S \times S)_S$ will fail to have condition (PWP). Completely 0-simple semigroups surely contain a non-trivial idempotent, namely 0, and condition (P) is stronger than condition (PWP). Thus, all completely 0-simple semigroups with an identity adjoined fail to have diagonal acts satisfying condition (P). This same strategy will work for showing that if S is completely simple then the diagonal act $(S^1 \times S^1)_{S^1}$ will also fail to have condition (P) because any element of the form $(i, P_{\lambda i}^{-1}, \lambda)$ is idempotent.

Now let us consider condition (E). As we will see, condition (E) depends on the structure of the sandwich matrix P.

Theorem 4.14. *Let $S = \mathcal{M}(I, G, \Lambda; P)$ be a completely simple semigroup. Then $(S \times S)_{S^1}$ satisfies condition (E) if and only if P satisfies the following condition:*

($*$) *for all $i, j \in I$ and all $\lambda, \mu \in \Lambda$, if $i \neq j$ and $\lambda \neq \mu$ then $P_{\lambda i}^{-1} P_{\lambda j} \neq P_{\mu i}^{-1} P_{\mu j}$*

Proof. ($\Longrightarrow$) Assume that P fails to satisfy condition ($*$). Then there exist $i \neq j \in I$ and $\lambda \neq \mu \in \Lambda$ such that $P_{\lambda i}^{-1} P_{\lambda j} = P_{\mu i}^{-1} P_{\mu j}$. For ease of notation, let us call this element α. Then we get the equality

$$((j,e,\lambda),(i,e,\mu))(j,e,\lambda) = ((j,e,\lambda),(i,e,\mu))(i,\alpha,\lambda)$$

(both sides are equal to $((i, P_{\lambda i}, \lambda), (j, P_{\mu i}, \lambda)))$. So, if $(S \times S)_{S^1}$ satisfied condition (E) then there would be elements $a \in S \times S$, $u \in S^1$ such that

$$\begin{gathered} ((j,e,\lambda),(i,e,\mu)) = au, \\ u(i,e,\lambda) = u(j,\alpha,\lambda). \end{gathered}$$

If $u \neq 1$, then that would force $\lambda = \mu$, which is a contradiction. Hence $u = 1$. But $(i,e,\lambda) \neq (j,\alpha,\lambda)$, and so $(S \times S)_{S^1}$ fails to have condition (E).

($\Longleftarrow$) Now suppose that P satisfies condition ($*$) and suppose we have the equality

$$((i,g,\lambda),(j,h,\mu))(k,x,\delta) = ((i,g,\lambda),(j,h,\mu))(l,y,\eta).$$

Then necessarily it must be that $\delta = \eta$, $P_{\lambda k}x = P_{\lambda l}y$ and $P_{\mu k}x = P_{\mu l}y$. Then after some simple rearrangment of the latter two equalities we get that $P_{\lambda k}^{-1}P_{\lambda l} = P_{\mu k}^{-1}P_{\mu l}$. So by condition $(*)$ we know that either $\lambda = \mu$ or $k = l$. If $\lambda = \mu$, then we can set

$$a = ((i, gP_{\lambda i}^{-1}, \lambda), (j, hP_{\lambda i}^{-1}, \lambda)),$$
$$u = (i, e, \lambda).$$

Then it is easily checked that

$$((i, g, \lambda), (j, h, \lambda) = au,$$
$$u(k, x, \delta) = u(l, y, \delta),$$

so we get condition (E). If $k = l$, then it follows that $x = y$ also. So by setting $u = 1$ and $a = ((i, g, \lambda), (j, h, \mu))$ we get condition (E) again. □

Theorem 4.15. *Let $S = \mathcal{M}(I, G, \Lambda; P)$ be a completely 0-simple semigroup. Then $(S \times S)_{S^1}$ satisfies condition (E) if and only if P satisfies the following conditions:*

i) each column of P contains at most one zero entry,

ii) each row of P contains at most one zero entry,

iii) for all $i \neq j \in I$, $\lambda \neq \mu \in \Lambda$, if each of $P_{\lambda i}, P_{\lambda j}, P_{\mu i}, P_{\mu j}$ are nonzero, then $P_{\lambda i}^{-1}P_{\lambda j} \neq P_{\mu i}^{-1}P_{\mu j}$.

Proof. ($\Longrightarrow$) Firstly, assume P fails to satisfy condition (i) above. That is, there is some column, say column i, that contains at least two zero entries, say $P_{\lambda i} = P_{\mu i} = 0$, where $\lambda \neq \mu$. Then we get the equality

$$((i, e, \lambda), (i, e, \mu))(i, e, \lambda) = ((i, e, \lambda), (i, e, \mu))0.$$

So if $(S \times S)_{S^1}$ satisfied condition (E), then we would be able to find elements $a \in S \times S$, $u \in S^1$ such that

$$((i, e, \lambda), (i, e, \mu)) = au,$$
$$u(i, e, \lambda) = u0.$$

In order to satisfy the first equation it must be that $u = 1$ (or else $\lambda = \mu$), but then we get $(i, e, \lambda) = 0$, which is a contradiction. Thus, $(S \times S)_{S^1}$ does not satisfy condition (E). A similar approach can be used if P fails condition (ii), and if P fails condition (iii) then we can take the same strategy as in the proof of the previous theorem.

($\Longleftarrow$) Now assume that P satisfies conditions (i), (ii) and (iii) of the theorem and suppose we have an equality of the form

$$((i, g, \lambda), (j, h, \mu)), (k, x, \delta) = ((i, g, \lambda), (j, h, \mu))(l, y, \eta).$$

If $P_{\lambda k}, P_{\lambda l}, P_{\mu k}$ and $P_{\mu l}$ are all nonzero then $\delta = \eta$ and also $P_{\lambda k}x = P_{\lambda l}y$ and $P_{\mu k}x = P_{\mu l}y$. Again, after some rearrangement we can get $P_{\lambda k}^{-1}P_{\lambda l} = P_{\mu k}^{-1}P_{\mu l}$, and so by condition (iii) either $\lambda = \mu$ or $k = l$. If $\lambda = \mu$, then we may use the regularity of the matrix P to choose a nonzero entry of column i, say it is $P_{\alpha i}$. Then by setting $a = ((i, gP_{\alpha i}^{-1}, \alpha), (j, hP_{\alpha i}^{-1}, \alpha))$ and $u = (i, e, \lambda)$ we get

$$((i, g, \lambda), (j, h, \lambda)) = au,$$
$$u(k, x, \delta) = u(l, y, \delta),$$

and thus condition (E) is satisfied. If $k = l$, then $x = y$ also, and we may choose the same a and u as above to work.

Now, if $P_{\lambda k} = 0$, then $P_{\lambda l} = 0$ also. So by condition (ii) we must have that $k = l$. But if $k = l$ then $x = y$ also. Thus, simply setting $a = ((i, g, \lambda), (j, h, \mu))$ and $u = 1$ will suffice.

Next, suppose $P_{\lambda k} = P_{\lambda l} = P_{\mu k} = P_{\mu l} = 0$. Then by conditions (i) and (ii) we get $\lambda = \mu$ and $k = l$. Using regularity of P, choose $m \in I$ such that $P_{\lambda m} \neq 0$. Then set $a = ((i, g, \lambda), (j, h, \lambda))$ and $u = (m, P_{\lambda m}^{-1}, \lambda)$.

Finally, we'll look at some situations where 0 is involved. Say we have

$$((i, g, \lambda), (j, h, \mu))0 = ((i, g, \lambda), (j, h, \mu))(l, y, \delta).$$

Then $P_{\lambda l} = P_{\mu l} = 0$, and so by condition (i) we get $\lambda = \mu$. Using regularity again, choose $m \in I$ such that $P_{\lambda m} \neq 0$. Then we can set $a = ((i, g, \lambda), (j, h, \lambda))$, $u = (m, P_{\lambda m}^{-1}, \lambda)$.

Suppose

$$(0, (j, h, \mu))(k, x, \delta) = (0, (j, h, \mu))(l, y, \eta).$$

If $P_{\mu k}$ and $P_{\mu l}$ are both nonzero, then $\delta = \eta$ and $P_{\mu k}x = P_{\mu l}y$. So setting $a = (0, (j, h, \mu))$ and $u = (k, P_{\mu k}^{-1}, \mu)$ does the job. On the other hand, if $P_{\mu k} = 0 = P_{\mu l}$ then by condition (ii) $k = l$. Choose $m \in I$ such that $P_{\mu m} \neq 0$, and let $a = (0, (j, h, \mu))$, $u = (i, P_{\mu m}^{-1}, \mu)$.

All other cases are either trivial or else symmetric of the above cases. So in any case $(S \times S)_{S^1}$ satisfies condition (E). □

It follows easily from Theorem 4.14 that all left groups have diagonal acts satisfying condition (E), and thus they are always strongly flat. So we may then wonder if they are always projective also.

Proposition 4.16. *Let $S = L \times G$ be a left group. Then $(S \times S)_{S^1}$ is projective.*

Proof. Note first that although left groups are right cancellative, we cannot use Theorem 3.14 because S is not a monoid. So instead we must examine the connected components of $(S \times S)_{S^1}$. Firstly, if $((a,g),(a',g'))$ is connected to $((b,h),(b',h'))$ then $a = b$ and $a' = b'$. Indeed, suppose we had an array of the form

$$\begin{array}{rcl} ((a,g),(a',g')) & = & ((c_1,x_1),(d_1,y_1))(e_1,z_1) \\ ((c_1,x_1),(d_1,y_1))(f_1,w_1) & = & ((c_2,x_2),(d_2,y_2))(e_2,z_2) \\ & \vdots & \\ ((c_n,x_n),(d_n,y_n))(f_n,w_n) & = & ((b,h),(b',h')). \end{array}$$

Then $a = c_1e_1 = c_1 = c_2e_2 = c_2 = \cdots = c_nf_n = c_n = b$, and similarly $a' = b'$. Next, we claim that $((a,g),(a',g'))$ is connected to $((a,h),(a',h'))$ if and only if

$$((a,h),(a',h')) \in ((a,g),(a',g'))S.$$

To show this, note that it suffices to show that (g,g') is connected to (h,h') in $(G \times G)_G$ if and only if $(h,h') \in (g,g')G$. The sufficiency is obvious. For the converse, suppose we have an array

$$\begin{array}{rcl} (g,g') & = & (x_1,y_1)z_1 \\ (x_1,y_1)w_1 & = & (x_2,y_2)z_2 \\ & \vdots & \\ (x_n,y_n)w_n & = & (h,h'). \end{array}$$

Then we can compute

$$\begin{array}{rcl} h & = & x_nw_n \\ & = & x_{n-1}w_{n-1}z_n^{-1}w_n \\ & = & \cdots \\ & = & x_1w_1z_2^{-1}w_2z_3^{-1}w_3\cdots w_{n-1}z_n^{-1}w_n \\ & = & gz_1^{-1}w_1z_2^{-1}w_2z_3^{-1}w_3\cdots w_{n-1}z_n^{-1}w_n, \end{array}$$

and similarly $h' = g'z_1^{-1}w_1z_2^{-1}w_2z_3^{-1}w_3\cdots w_{n-1}z_n^{-1}w_n$. Thus,

$$(h,h') = (g,g')z_1^{-1}w_1z_2^{-1}w_2z_3^{-1}w_3\cdots w_{n-1}z_n^{-1}w_n,$$

and so $(h,h') \in (g,g')G$.

So what we have shown is that the connected component of $((a,g),(a',g'))$ is the set $((a,g),(a',g'))S^1$. Now consider the mapping

$$\begin{array}{rcl} \varphi : ((a,g),(a',g'))S^1 & \longrightarrow & ((a,e),(a',e))S^1 \\ ((a,g),(a',g'))s & \mapsto & ((a,e),(a',e))s. \end{array}$$

It is routine to show that φ is an isomorphism. Finally, we claim $((a,e),(a',e))$ is left (a,e)-cancellative. Indeed, clearly $((a,e),(a',e))(a,e) = ((a,e),(a',e))$ and suppose $((a,e),(a',e))(c,x) = ((a,e),(a',e))(d,y)$. Then $x = y$ and

$$(a,e)(c,x) = (a,x) = (a,e)(d,x).$$

Hence, we have that each connected component is isomorphic to a cyclic right S^1-act, zS^1, where z is left e-cancellative for an idempotent e, showing that $(S \times S)_{S^1}$ is projective by Theorem 1.31. □

Proposition 4.17. *Let $S = L \times G$ be a left group where $|L| > 1$. Then $(S \times S)_{S^1}$ is not free.*

Proof. Suppose $(S \times S)_{S^1}$ was free. Then for any $((i,g),(j,h)) \in S \times S$, we could write $((i,g),(j,h)) = b(k,x)$ uniquely, where $b \in S \times S$ is a basis element and $(k,x) \in S$. But because L is left zero, we could replace k with anything we want from L and still get $((i,g),(j,h))$. Thus, $(S \times S)_{S^1}$ cannot be free. □

Corollary 4.18. *Let $S = \mathcal{M}(I,G,\Lambda;P)$ be a completely simple semigroup where $|I| > 1$. Then $(S \times S)_{S^1}$ is not free.*

Proof. If $(S \times S)_{S^1}$ was free, then in particular it would satisfy condition (P), and so S is a left group. But that would contradict Proposition 4.17. □

Chapter 5

Conclusion and Further Research

We have made a good start in studying the flatness properties of acts as they relate to diagonal acts. However, there are still many open problems, such as:
1) Classify all monoids whose diagonal acts are (principally weakly) flat, projective and free.
2) Find examples of monoids whose diagonal acts are weakly flat but not flat, or which satisfy condition (P) but are not projective.
3) Find conditions on a completely (0-)simple semigroup in order for its diagonal act to be flat.

For another idea, recall the section at the start of this paper that briefly mentioned the study of flatness of S-posets. The study of diagonal acts can be extended to this by simply equipping S with a partial ordering and then studying $(S \times S)_S$ as an S-poset. We can then examine which of the results obtained so far can be carried over to the ordered case, and which can not. There are also many more flatness properties to consider in the ordered case, depending on whether you want the functor $A_S \otimes -$ to preserve order embeddings or simply injective maps, but this is beyond the scope of the present paper.

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