The Valuation and Risk Management of a DB Underpin Pension Plan

by

Kai Chen

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Kai Chen

Abstract

Hybrid pension plans offer employees the best features of both defined benefit and defined contribution plans. In this work, we consider the hybrid design offering a defined contribution benefit with a defined benefit guaranteed minimum underpin. This study applies the contingent claims approach to value the defined contribution benefit with a defined benefit guaranteed minimum underpin. The study shows that entry age, utility function parameters and the market price of risk each has a significant effect on the value of retirement benefits.

We also consider risk management for this defined benefit underpin pension plan. Assuming fixed interest rates, and assuming that salaries can be treated as a tradable asset, contribution rates are developed for the Entry Age Normal (EAN), Projected Unit Credit(PUC), and Traditional Unit Credit (TUC) funding methods. For the EAN, the contribution rates are constant throughout the service period. However, the hedge parameters for this method are not tradable. For the accruals method, the individual contribution rates are not constant. For both the PUC and TUC, a delta hedge strategy is derived and explained.

The analysis is extended to relax the tradable assumption for salaries, using the inflation as a partial hedge. Finally, methods for incorporating volatility reducing and risk management are considered.

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Chapter 1

Introduction

In recent years, the design of pension plans has become an important topic for pension fund managers. There are two basic kinds of pension plans: Defined Benefit(DB) and Defined Contribution(DC).

In a defined contribution plan, the total rate of contribution is fixed in advance, sometimes at a set rate, such as 7% of annual salary. The level of pension benefits is unpredictable; the amount of pension employees receive depends on the investment experience of their own pension accounts and the cost of annuitization at retirement. Depending on the long-term investment results that employees achieve, the defined contribution pension could be significantly higher, or significantly lower, than the pension under a comparable defined benefit plan. Examples of defined contribution plans include 401(k) plans, 403(b) plans, employee stock ownership plans, and profit-sharing plans.

In a defined benefit plan, employees receive a pension based on a formula. The plan may state this promised benefit as an exact dollar amount, such as $1000 per month at retirement. More commonly, the benefit is calculated through a specified formula that includes such factors as years of service and salary. For example, a defined benefit pension might be calculated as 1.5% of average salary for the final 5 years of employment, for every year of service with an employer. The benefits in most traditional DB plans may be protected, within certain limitations, by federal insurance. For example, in the U.S., such insurance is provided through the Pension Benefit Guaranty Corporation (PBGC). Nevertheless, it is the employers' responsibility to ensure that contributions and investment earnings are sufficient to provide the employees' pension benefits. To compare DB plans with DC plans, we assume there is a defined benefit account equal to the market value of the DB benefits.

To mitigate the effect of adverse investment experience on a defined contribution plan, some employers use hybrid pension plans to support employees after retirement. Hybrid pension plans have advantages of both the DB plan and the DC plan. For example, a common hybrid pension plan is the cash balance plan. It defines the promised benefit in terms of a stated account balance. So it is a defined benefit plan with defined contribution characteristics, as the post-retirement risk is transferred to the employee. Another kind of hybrid pension plan has been designed to offer "greater of" retirement, resignation, and death benefits that are the maximum of two different benefit accounts. One benefit account is based on the accumulation of defined contributions, the other one is regular defined benefit plan, based on the employee's final salary and years of service. The defined contribution account accumulates contributions paid by employees and also their employers. When an employee reaches a special settlement, such as death, disability or retirement, the maximum value of two accounts will be paid to the employee, and his or her spouse. This pension plan guarantees minimum DB benefits to protect employees against adverse investment experiences. The features and more details of this pension plan in Australian retirement funds are discussed in Britt(1991). The retirement benefits provided by a number of large public employers in Canada are also of this form, such as York University and McGill University(2006).

The payoff of this hybrid pension plan can be decomposed into two parts. The first part is the value of the defined contribution account. The other part is the maximum of the value of the DB account minus the DC account and zero. The payoff of the second part is similar to an exchange option, developed by Margrabe(1978). The exchange option gives holder the right to give up an underlying asset worth S_2 and receive in return another underlying asset worth S_1, which has the payoff $\max\{S_1 - S_2, 0\}$. Boyle and Schwartz(1976) extend option pricing techniques to price the benefits of life insurance products, by assuming mortality is independent of the risky assets, and is fully diversifiable. In this case the risk-neutral mortality measure is the same as the real world measure.

Sherris(1995) applies a contingent claims approach to the case of guaranteed minimum DB retirement benefits. He demonstrates that lattice models are not computationally feasible for retirement fund benefits, although they are commonly used in the valuation of financial options. Sherris(1995) prices the option under a number of different assumptions, but does not add in wider risk management questions. In this work, we start from the work of Sherris and develop pricing and risk management models for the DB guarantee. We first consider different measures and different pricing models using Monte Carlo simulation. Secondly, we consider the risk management of hybrid pension plans. We have researched the relationships between salaries, bonds, stocks and inflation. We find salary growth rates and inflation rates have very high correlation. Implementing ideas from exchange option valuation, we propose four funding strategies and analyze sensitivities of monthly hedging costs to different assumptions.

We consider the current situation for pension plans in Chapter 2. In this chapter, we illustrate some pension plan designs and describe the current scientific literature on pricing and hedging pension risks. In Chapter 3, we use a traditional actuarial method to calculate the expected cost of a particular hybrid plan under the nature's measure. We also consider the equilibrium pricing model in the incomplete market. We propose four funding strategies using different cost methods with tradable salary assumption in Chapter 4. We show the numerical results by simulation in Chapter 5. In Chapter 6, we analyze the relationship between salary and other financial indexes, such as bonds, stocks and inflation. In Chapter 7, we assume that the salary is no longer tradable. We introduce the stochastic interest rate in our model in Chapter 8. To smooth the hedging cash flows, we consider alternative cost control approaches in Chapter 9. Chapter 10 describes our further work.

Chapter 2

Current Pension Systems and Pension Fund Risk Management

In the current pension system, there are two basic kinds of pension plans: Defined Benefit (DB) and Defined Contribution (DC). In the last two decades, many countries have put reform of the existing pension systems on the political agenda, because the global aging problem together with adverse investment experience has resulted in widespread underfunding of DB plans. As the contribution period in a pension fund may be very long, generally from 20 to 40 years, pension fund managers need to consider a long-term investment strategy, as well as uncertainties caused by the economy, legal reforms, and of course the aging problem. In this chapter, we introduce some popular pension plans in detail. We analyze their important properties and illustrate some current risk management methods.

2.1 Defined Benefit Plan

In a defined benefit plan, benefits are defined in advance by a formula, and employer contributions are treated as the variable factor. Employees will earn defined benefits at retirement without investment risk (though there is a default risk). The formula for establishing the benefits may vary, but generally may be classified into two categories: flat benefit or unit benefit.

In a flat benefit formula, retirement benefits are independent on the length of service under the plan beyond a minimum period of service. The compensation base is normally the average earnings during a specified period before retirement. Another type of flat benefit formula ignores differences in both compensation and length of service. It only provides a flat dollar amount of benefits for all qualified employees at retirement. This type of formula was typical of the early negotiated plans but is rare today.

In a unit benefit formula, an explicit unit of benefit is credited for each year of recognized service with the employer. The credit benefits earned each year can be fixed, or determined by the final salary, the final average salary or the whole career average salary. There are various forms of limit of plans and ancillary benefits. The most common limits exclude (1) all service performed before a specified age, (2) the first year or few years of service, or (3) all service over a maximum number of years, such as 30 or 35. The withdrawal benefits, death benefits and disability benefits are also an important consideration in the design of pension plans.

2.2 Funding Methods for DB Plans

There are many cost methods, as described by Aitken(1996), to calculate the actuarial value of pension benefits. They are generally acceptable to the supervisory authorities for funding purposes. The two main categories of cost methods are the accruals method and the level premium method. We describe three most common approaches here. These will be emphasized again in Chapter 4.

2.2.1 Terminology

We illustrate some notation that we will use in our work. Given an employee who enters the plan at age xe, and the normal retirement age (NRA) xr, we let $T = (xr - xe)$ denote the time to the normal retirement age, which is also the maximum years of the membership. We use the following notation for the DB benefit.

S_t is the employee's salary at age $xe + t$. For simplicity, we assume that this increases monthly, which is the frequency of the valuation and hedging process.

α is the DB accrual rate, which defines how much benefit accrues with each year of service.

${}_{T-t|}\ddot{a}^{(12)}_{xe+t}$ is the value at exit time t of a deferred annuity of 1/12 per month according to the pension plan rules. At the normal retirement age (NRA), where $t = T$, $\ddot{a}^{(12)}_{xr}$ denotes the value at retirement time T of an annuity of 1/12 per month.

A_t is an index representing the return of the DC account. A_t is the accumulation at t of \$1 invested in the underlying DC fund at time 0.

To be clarified, the word 'projected' will be used to denote a random variable in later chapters. For example, $DB(t, T)$ and $DC(t, T)$ denote the projected value of the DB account and DC account at time T with the information available at current time t. That means $DB(t,T)$ and $DC(t,T)$ are two random variables. $DB^i(t,T)$ and $DC^i(t,T)$ denote the projected value of DB account and DC account at time T with the current time t according to different cost methods, where $i = T$ denotes the traditional unit credit (TUC) cost method, $i = P$ denotes the projected unit credit (PUC) cost method, and $i = E$ denotes the entry age normal (EAN) cost method.

2.2.2 Traditional Unit Credit Cost Method

The accruals method includes the traditional unit credit, and the projected unit credit cost method. The traditional unit credit (TUC) actuarial liability is the value of pension benefits accrued from the entry to the valuation date. It is always used for flat benefit pension plans. Under the traditional unit credit cost method, an employee's given credited years of service is determined by the past service. It is often used when the annual benefit accrual is expressed as a flat dollar amount or a specified percentage of the employee's current salary for each year.

For a final salary plan, where the final benefit is $DB^T(T,T) = \alpha T S_T \ddot{a}^{(12)}_{xr}$, the TUC value of retirement benefit at the valuation time t with an exit time s can be expressed as,

$$DB^T(t,s) = \alpha t S_t \cdot {}_{T-s|}\ddot{a}^{(12)}_{xe+s} | S_t \tag{2.1}$$

where $t \leq s \leq T$, α is the accrual rate, S_t is the salary at time t, and ${}_{T-s|}\ddot{a}^{(12)}_{xe+s}$ is the deferred annuity rate with the exit time s. Then, the actual benefit value, $\alpha t S_t$, is known at time t.

Under the traditional unit credit (TUC) cost method, the retirement benefit is defined as the expected increase in the employee's accumulated plan benefit during the year. Under a final salary plan, the increase in an employee's accumulated benefit is a combination of that year's benefit accrual and the adjustment of the salary base.

2.2.3 Projected Unit Credit Cost Method

The traditional unit credit (TUC) cost method fails to satisfy the criteria of an "ideal" actuarial cost method, because the normal cost of the plan is volatile. It is likely to rise from year to year, and the actuarial liability may fall short of the plan termination liability. Both of these shortcomings are modified by the projected unit credit (PUC) cost method.

The projected unit credit (PUC) cost method adds the salary scale to the traditional unit credit cost method. The current salary is projected to retirement by a salary scale in the unit benefit calculation. The PUC method is very commonly used for final salary plans.

For a final salary plan, where the final benefit is $DB^P(T,T) = \alpha T S_T \ddot{a}_{xr}^{(12)}$, the PUC value of retirement benefit at the valuation time t with the exit time s can be mathematically expressed as,

$$DB^P(t,s) = \alpha t S_s \cdot {}_{T-s|}\ddot{a}_{xe+s}^{(12)} | S_t \tag{2.2}$$

where $t \le s \le T$, α is the accrual rate, S_s is the salary at time s, and ${}_{T-s|}\ddot{a}_{xe+s}^{(12)}$ is the deferred annuity rate with the exit time s.

Under the projected unit credit (PUC) credit method, the retirement benefit is expected to increase because of the increase of years of service, since the salary scale has been projected to the exit time. So, although both TUC and PUC generate increasing contributions over time, the PUC benefits start higher and increase less.

2.2.4 Entry Age Normal Cost Method

The entry age normal (EAN) cost method funds the retirement benefit using level annual contributions, unlike unit credit methods which are based on accruals. A salary-increase assumption in used when the pension benefit is based on career average or final average salary. Under the entry age normal (EAN) cost method for conventional final salary DB benefits, the value of DB benefits is calculated by the projected years of service and the projected final salary.

$$DB^E(t,s) = \alpha s S_s \cdot {}_{T-s|}\ddot{a}^{(12)}_{xe+s}|S_t \tag{2.3}$$

where $t \leq s \leq T$, α is the accrual rate, S_s is the salary at time s, and ${}_{T-s|}\ddot{a}^{(12)}_{xe+s}$ is the deferred annuity rate with the exit time s.

The entry age normal (EAN) cost method treats earned years of service and unearned years of service equally. Table 2.1 summarizes three valuation formulas of the DB benefit at the normal retirement age. We will show it in Chapter 4 again.

Table 2.1: DB Valuation Formulas Based on Different Cost Methods

	DB Account ($DB(t,T)$)
EAN	$DB^E(t,T) = kTS_T$
PUC	$DB^P(t,T) = ktS_T$
TUC	$DB^T(t,T) = ktS_t$

Where we define $k = \alpha \times \ddot{a}^{12}_{xr}$.

2.2.5 Other Cost Methods

Besides the individual cost method, there are some aggregate methods, such as individual aggregate cost method, aggregate method, frozen initial liability(entry age normal), frozen initial liability(attained age normal) and aggregate entry age normal cost method. Aggregate cost methods usually consider value of the future benefits and the future liabilities for all participants, active, deferred vested, and retired. The portion of the total projected cost to be allocated to each plan year is generally expressed as a percentage of covered payroll if benefits are related to pay. It is one of peculiarities of the aggregate cost method that no actuarial liability every directly emerges. The normal cost is defined and derived in such a way that the present value of future benefits, less plan assets and any unfunded liability, is always fully and precisely offset by the actuarial value of future normal cost accruals.

2.3 Defined Contribution Plan

A pure defined contribution plan is a pension scheme where only contributions are fixed and benefits therefore depend on returns on the assets of the fund. Pension benefits are totally defined by the investment performance and employees make investment decisions. All investment risk is transferred to employees. The pension benefits may be very low when employees make bad, or unlucky investment decisions or the financial market is poor. Defined contribution plans have been far more popular recently for two main reasons. First, an employee knows the value of his or her retirement account at any time; his or her plan is then more easily portable from a company to another one. Moreover, employers do not bear any risk linked with the retirement system of companies. The problem here is the real need for a downside protection for employees. The ultimate aim of a pension plan is to finance retirement and it usually provides the most important source of employees' incomes after retirement. To provide some down side protection, some plans incorporate guarantees or top-ups for when the benefit is very low. The main problem considered in this thesis is the DB minimum, which has not been much written about in the scientific literature. However, there are other forms of DC guarantees, with some relevant academic research.

2.3.1 Defined Contribution Plan with Guaranteed Rate

Defined contribution pension guarantees resemble minimum cash values for equity-linked life insurance policies in some ways. Brennan and Schwartz(1976), Boyle and Schwartz(1977) and Banicello and Ortu(1993) have discussed such insurance policy guarantees. However guarantees on defined contribution plans are more complicated, since there are a series of sequential guarantees instead of a single guarantee at the maturity.

Pennacchi(1999), extending Zarita(1994), values the guaranteed rate of return on the defined contribution plan by contingent claims analysis. He illustrates that the martingale pricing technique for calculating contingent claims values can be a unifying framework for valuing many kinds of guarantees. He allows for employees' salaries to be stochastic, and thus the monthly contributions follow a random process. Real interest rates also follow a stochastic process. This adds uncertainty in the cost at employees' retirement annuities. Under the restriction that equilibrium asset prices do not allow for arbitrage opportunities, the martingale pricing approach can be applied to value a variety of guarantees on pension fund returns.

Boulier et al.(2001) consider pension fund management of protected defined contribution plans where a guarantee is given on pension benefits, and the guarantee depends on the level of the stochastic interest rate when the employee retires. They assume that there are three different assets in the market: cash, bonds and stocks.

The pension fund will be invested in a portfolio which is constructed by these three assets to guarantee the retirement benefit. Boulier et al.(2001) assume that the guarantee $G(T)$ at retirement is a function of the short interest rate and the wealth process $X(t)$ at time t is invested in the risk-free asset, the stock index and the rolling bond. They propose four steps to maximize the expectation of the utility of the surplus between the wealth process and the guarantee. This can be solved by numerical methods. There are two important features of this strategy. First, the model introduces a stochastic interest rate process. For any movement of the rate, the manager exactly knows how to react and rebalance the portfolio. Second, this strategy is described by the wealth invested in the three classes of assets: cash, long-term bonds and stocks. So a practical tool can be easily implemented to help employers in choosing their hedging portfolios.

2.4 Pension Reform

The contemporary discussion of pension reforms has been initiated mainly by concern for the long-term financial viability of existing pension systems. In some countries, particularly in Latin America and Eastern Europe, such systems have more or less broken down. In developed OECD (Organization for Economic Co-operation and Development) countries, this problem is less dramatic but still urgent. Given the anticipated developments in demography and productivity growth, pension reform has become a serious global problem. For example, the average contribution rate in the European Union is 16 percent today. A report by the EU commission(2001) estimates that it has to be increased to 27 percent in 2050 if the present rules are kept unchanged. The Social Security Administration(2001) shows that the average contribution rate in the U.S. is 12.4 percent and is expected to increase to 17.8 percent in 2050 with unchanged rules. Pension reforms are necessary based on those predictions.

Comparisons of pension systems and discussions of pension reforms are usually between defined benefit and defined contribution systems. Lindbeck and Persson(2003) propose a three-dimensional classification: actuarial versus non-actuarial, funded versus unfunded, and defined benefit versus defined contribution pension system. Each of these dimensions is associated with a special aspect of pension reform: labor market distortions, aggregate saving, and considerations of risk, respectively.

Bader and Gold(2003) discuss the actuarial and financial economic valuation models. They illustrate many principles that are universally accepted in financial economics and almost as universally violated in the actuarial model. They propose that a new model should rely on market values and reject the use of expected returns on assets for discount rates.

Sinn(1999) and Miles(2000) focus on the second dimension and analyze pension reform and the demographic crisis under the funded and unfunded pension schemes. Miles(2000) uses stochastic simulation on calibrated models to assess the optimal degree of reliance on funded pensions and on the pay-as-you-go(PAYGO) system. Sinn(1999) also discusses the transition from pay-as-you-go(PAYGO) system to a funded system. Both of them agree that the pay-as-you-go(PAYGO) system does not waste economic resources and there is no Pareto improving way of making this transition although there is a higher rate of return relative to sustainable GDP growth. A combination of these two systems could be used to optimize the expected welfare of employees. Miles(2000) tests various combination of assumptions about the distribution of rates of return and pension generosity and concludes that the optimal size of unfunded pensions is highly sensitive to both of the distribution of rates of return and the efficiency of annuities contracts.

In our research, we focus on the last dimension: defined benefit versus defined contribution pension systems. In the United States private pension market, defined contribution pension plans have been growing over the last two decades. Some recent research also shows that many public pension plans are converting from defined benefit to defined contribution. A defined contribution plan can offer employees flexibility, portability, and investment portfolio choice, with investment risks transferred to employees. In an interesting approach to balancing DC and DB plan benefits, the State of Florida implemented state-wide pension reform in 2002. In the new Public Employee Optional Retirement Program(PEORP), the State of Florida granted each and every employee who wants to convert from the traditional defined benefit plan to the self-managed defined contribution plan, the right but not the obligation, to switch back into the defined benefit plan at any time prior to retirement. The strike price of this DB buy-back option is the employee's accumulated benefit obligation(ABO) in the defined benefit plan. Lachance et al.(2003) and Milevsky and Promislow(2004) evaluatee this guaranteed defined contribution pension, and came up with very different conclusions.

Lachance et al.(2003) developed a theoretical framework to analyze the option design and illustrated how employee characteristics influence the option's cost. They adopted the risk neutral valuation technique based on no-arbitrage arguments and calculated the employee's optimal time of exercise by maximizing the employee's expected utility function. They showed that offering employees an opportunity to buy back the DB benefit requires balancing participant protection and employer costs.

Milevsky and Promislow(2004) also considered the State of Florida pension reform. Their conclusions were different from the Lachance et al.(2003) results.

They thought Lachance et al.(2003) overestimated the incremental liability created by exercising the option. However, they only used a strictly deterministic model in which all parameters are fixed with certainty. Although this assumption helps us to concentrate on the time of exercising the buy-back option, a more refined model with stochastic factors should be introduced. Ignoring mortality and early termination probabilities in their model, they concluded that nobody should retire from the DC plan; rather, they all eventually return to the DB plan. Since they did not consider the random factors to valuate an option, their results are very doubtful.

2.5 Hybrid Pension Plans

In general, defined benefit plans provide a specific benefit at retirement for each eligible employee, while defined contribution plans specify the amount of contributions to be made by the employer toward an employee's retirement account. In a defined contribution plan, the actual amount of retirement benefits provided to an employee depends on the amount of the contributions as well as the gains or losses of the account. Each plan type has advantages and disadvantages, so an employer or a plan sponsor may want to combine the advantages of each type of plan, such as the ease of communication of a defined contribution plan coupled with employer's assumption of investment risks and rewards in the defined benefit plan. Hybrid plans attempt to combine the advantages of each of pure types of plans into a single plan. For example, the guaranteed defined contribution plan with a buy-back option in the Florida State is a kind of hybrid pension plan.

2.5.1 Cash Balance Plan

A significant and common hybrid pension plan is the cash balance plan. A cash balance plan is actually a defined benefit plan that defines the benefit and also has the characteristic of a defined contribution plan. In other words, a cash balance plan defines the promised benefit in terms of a stated account balance. The cash balance plan was introduced by Bank of America in the early 1980s. In contrast to pure defined benefit plans, a cash balance plan defines a lump-sum account at retirement, but not a payable annuity. This is similar to the pure defined contribution pension plan. But cash balance plan accounts grow by a predetermined formula, where the pure defined contribution plan accounts grow by the actual earnings of the plan.

In a typical cash balance plan, an employee's account is credited each year with a "pay credit" (such as 5 percent of compensation from his or her employer) and an "interest credit" (either a fixed rate or a variable rate that is linked to an index such as the one-year treasury bill rate). Increases and decreases in the value of the plan's investments do not directly affect the benefit amounts promised to participants. Thus, the investment risks and rewards on plan assets are borne solely by the employer. When a participant becomes entitled to receive benefits under a cash balance plan, the benefits that are received are defined in terms of an account balance. The benefits in most of the U.S. cash balance plans are also protected by federal insurance provided through the Pension Benefit Guaranty Corporation (PBGC).

2.5.2 Defined Contribution Plan with Minimum Benefit Guaranteed Rates

The reformed defined contribution plans sometimes include a guaranteed minimum benefit(DC-MB). Many countries have converted their public pension systems from a pay-as-you-go(PAYGO) defined benefit plan to a defined contribution plan with minimum benefit guarantees. Such plans offer a guaranteed rate of return to employees. The cost of benefit guarantee is becoming a more common topic in the academic literature. The cost of pension guarantees has been analyzed in papers by Pesando(1982), Marcus(1985,1987), Bodie and Merton(1993), Smetters(2001,2002), and Bodie(2001). Smetters(2001) considers recent privatization plans, such as the Feldstein-Samwick(1997) plan and the Gramm(1998) plan. He shows that unfunded minimum benefit guarantees can be costly enough to undo most of the salutary long-run benefit typically associated with funded private accounts.

Bodie and Merton(1993) explain an effective method of managing the risk associated with fixed guaranteed benefits inside traditional defined benefit plans. A common technique to control guarantees costs is over-funding. This creates a buffer against shocks on the investment performance. A higher contribution rate could be used to over-fund the private pension fund. The mandatory contribution rate in Chile, for example, is 10% of payroll, which could produce large enough benefits to cover the minimum benefits. The same method has been used in Argentina and is the dominant strategy in most countries. This over-funding technique does not work well for the DC-MB, although it is very effective for defined benefits plans.

Smetters(1998), and Feldstein and Ranguelova(2000) suggest that participants in DC-MB plans could sell part of their upside potential for downside protection. It is like going short a call option and long a put option. Smetters(2002) considers two approaches of controlling the cost of DC-MB. In the first one, Smetters suggests constructing a standardized portfolio consisting of some bonds. Investors or employees bear basis risk if they choose a portfolio different from the standardized portfolio. However, this approach is only effective for some smaller DB to DC-MB conversions, since agents might anticipate an implicit guarantee and assume a lot of basis risk, a so-called Samaritan's Dilemma. The second approach considers a little more brute force and allows for a fair amount of portfolio selection. It taxes assets in DC-MB accounts in good states of the world and subsidizes assets in bad states of the world. These two alternative methods are both more effective than the over-funding method, since they shift resources from the good states of the world to the bad states. As a result, unfunded liabilities are lower under these two compared to the traditional over-funding method which does not shift.

Chapter 3

The Valuation of a DB Underpin Pension

3.1 Introduction

In this chapter, we will consider a particular DB underpin pension plan which offers "greater of" benefits and which has not been discussed extensively in the academic literature. This DB underpin pension plan offers a defined contribution benefit with a guaranteed defined benefit minimum underpin. Employees in this plan have their own defined benefit and defined contribution accounts. The pension benefit at exit, such as retirement, death and disability, is determined by the maximum of DB and DC accounts. Britt(1991) discusses the features and more details of this pension plan in Australian retirement funds. Sherris(1995), and Lin and Chang(2004) evaluate the cost of this plan. This DB underpin plan has been also provided by a number of large public employers in Canada, such as York University and McGill University.

There are many approaches to valuing pension liabilities. Using option pricing techniques to value actuarial liabilities is gaining acceptance. The initial application of such techniques was to investment guarantees provided in maturity benefits of life insurance products as first discussed in Boyle and Schwartz(1976). In 1990's, option pricing techniques had been adopted or proposed to value a range of actuarial liabilities. For instance, Wilkie(1989) discusses the use of these approaches in valuation of pension payments from U.K. pension schemes. The contingent claims framework based on arbitrage free pricing has also been applied to the valuation of life insurance policy cash flows. In this chapter, we use the contingent claims approach to calculate the expected cost of a DB underpin pension plan. This approach allows the calculation of a market value for these pension liabilities. Since these pension benefits are non-tradable, we can not replicate pension liabilities and the financial market is incomplete.

3.2 The Model and Assumptions

Traditional actuarial valuation techniques are based on deterministic assumptions for the interest rate, the salary growth rate and the crediting rate for the defined benefit account. Details of the traditional actuarial valuation approach for defined benefit pensions are found in Bowers et al.(1986). This approach can not be used when we have embedded options involved. The valuation of these benefits requires the use of a stochastic model. We assume in the chapter a constant interest rate and that salary growth rate and defined contribution crediting rate are stochastic. Stochastic interest rates will be considered in Chapter 8.

The rate of salary growth is denoted by $s(t)$ and the crediting rate is denoted by $f(t)$ at time t. They are assumed to follow two stochastic differential equations of the form

$$ds(t) = \mu_1(s(t), t)dt + \sigma_1(s(t), t)dZ_s(t), \tag{3.1}$$

$$df(t) = \mu_2(f(t), t)dt + \sigma_2(f(t), t)dZ_f(t), \tag{3.2}$$

with $dZ_s(t)dZ_f(t) = \rho_{fs}dt$, where ρ_{fs} denotes the instantaneous correlation coefficient between the standardized Wiener increments $dZ_s(t)$ and $dZ_f(t)$. So we can rewrite the stochastic process for $f(t)$ as

$$df(t) = \mu_2(f(t), t)dt + \sigma_2(f(t), t)\left(\rho_{fs}dZ_s(t) + \sqrt{(1-(\rho_{fs})^2)}dZ_f^*(t)\right) \tag{3.3}$$

where $dZ_s(t)$ and $dZ_f^*(t)$ are independent Wiener increments.

This result is used for the numerical evaluation of the pension plan with minimum defined benefit guarantee and for simulation of the processes. For this DB underpin pension plan, the early exercise time is determined by the mortality and service table. We assume withdrawal or retirement is independent of the option value. We define the termination value of the DB account at the exit time t with entry age xe and normal retirement age xr as

$$DB(t, t) = \alpha \cdot t \cdot S_t \cdot {}_{T-t|}\ddot{a}_{xe+t}^{(12)} \tag{3.4}$$

where t is the years of service, $T = xr - xe$ is the normal retirement time, α is the accrual rate, ${}_{T-t|}\ddot{a}_{xe+t}^{(12)}$ is the deferred annuity value at age $xe + t$, and S_t is the annual salary which is equal to the amount that the member earns in the whole year before time t. We first assume the mortality rate and the interest rate are constant. So the deferred annuity can be considered as deterministic.

At time t, the annual salary S_t is a function of the salary growth rate variable $s(t)$:

$$dS_t = s(t)S_tdt, \tag{3.5}$$

where $s(t)$ is defined from equation (3.1).

Another account $DC(t,t)$, the defined contribution account, is constructed by accumulating monthly contributions with the contribution rate c. The contributions are accumulated using the crediting rate. So this account is analogous to a notional security that has a negative continuous dividend equal to the contribution rate times salary. The value of $DC(t,t)$ is determined from

$$dDC(t,t) = [f(t)DC(t,t) + cS_t]dt, \tag{3.6}$$

where:

$f(t)$ is defined in equation (3.2),

S_t is defined in equation (3.5),

c is the assumed contribution rate.

In general, the maximum value of the defined benefit account and the accumulated account is paid at the end of month when members die or retire. On withdrawal, the benefit may be the defined contribution account. In some DB underpin pension plans which are used in the U.S., the maximum value of two accounts is only paid when employees retire or die. At the moment of withdrawal, the plan pays the value of the defined contribution account to employees. However, under others, such as McGill university's pension plan, the maximum value of two accounts is paid as long as members leave the plan for any reason (i.e. retirement, withdrawal, disability or death). Pension benefits can be expressed as $\max(DB(T,T), DC(T,T)) = DC(T,T) + \max(DB(T,T) - DC(T,T), 0)$ at retirement time T.

In practice, we need a flexible and efficient way to calculate the value of benefits. In the next section, we will use numerical techniques to price this pension plan with minimum DB guarantee.

3.3 Numerical Techniques

Following Sherris(1995), we simulate variables using the following equations. The rate of salary growth and the crediting rate at time $t+h$, s_{t+h} and f_{t+h} are derived from the risk-adjusted stochastic difference equation as Sherris(1995)

$$s(t+h) = s(t) + (\alpha_s + b_s s(t) - \lambda_s \sigma_s s(t)^{\gamma_s - 1.0})h + \sigma_s s(t)^{\gamma_s - 1.0}\sqrt{h}Z_s(t) \qquad (3.7)$$

$$f(t+h) = f(t) + (\alpha_f + b_f f(t) - \lambda_f \sigma_f f(t)^{\gamma_f - 1.0})h + \sigma_f f(t)^{\gamma_f - 1.0}\sqrt{h}(\rho Z_s(t) + \sqrt{1-\rho^2}Z_f^*(t)) \qquad (3.8)$$

where $Z_s(t) \sim N(0,1)$, $Z_f^*(t) \sim N(0,1)$, and $Z_s(t)$ and $Z_f^*(t)$ are independent; h is the time step. We use a monthly time interval. The correlation coefficient between the rate of salary growth and the crediting rate is ρ.

This formulation allows several alternative distributions to be generated for the rate of salary growth and the crediting rate depending on the parameter γ_s. If $\gamma_s = 1$, then the unconditional distribution of s_t is normal; if $\gamma_s = 1.5$, then the unconditional distribution of s_t is gamma; and if $\gamma_s = 2$, then the unconditional distribution of s_t is lognormal.[1]

There are many discussions about how to model the salary growth better, since salary is not tradable and is not a continuous process. Equation (3.7) may not be

[1]This model has been used for interest rates with γ_s equal to one in Vasicek(1997) and with γ_s equal to 1.5 in Cox, Ingersoll, and Ross(1985).

the best choice. We retain it at this stage to reproduce Sherris'(1995) results as a starting point for our own work. Sherris(1995) uses equation (3.7) and equation (3.8) to generate the salary growth rate and the crediting rate and value the whole benefit, $DC(T,T) + \max(DB(T,T) - DC(T,T), 0)$. However, we only consider the cost of the DB guarantee, $\max(DB(T,T) - DC(T,T), 0)$. To compare with his results, we adopt the same processes in this section. Although parameters in Sherris' model are not appropriate for current situations, we use the same ones as a starting point for comparison with Sherris'(1995) results. We use the same parameters and same mortality table, which are summarized in Table 3.1 and Table 3.2, respectively. Sherris(1995) shows the costs of the DB underpin pension plan. We use the same model and same values of parameters to analyze the difference between costs of the pension plan and costs of the DB guarantee. In Section 3.5, we will use more reasonable values of parameters to analyze sensitivities of the DB underpin guarantee to parameters.

Six sets of parameters in stochastic difference equations for the rate of salary growth and the crediting rate are given by Table 3.1. We assume the DC contribution rate c is equal to 12.5%, the discount rate is 0.1, and $h = 1/12$. Here we use the risk-free zero coupon bond as the discount rate. We also assume the risk free rate and the annuity are constant and the mortality table is given. Let the accrual rate be 1.5% and the life annuity be 10, we have $k = 15\%$, which is equal to the accrual rate times the annuity. The decrement rate is given by Table 3.2, where the normal retirement age is 65.

Table 3.1: Assumptions for Stochastic Simulation Valuations

Parameter Values							
α_s	b_s	σ_s	γ_s	α_f	b_f	σ_f	γ_f
0.072	-1.2	0.08	1.0	0.1296	-0.96	0.2	1.0

Scenario	1	2	3	4	5	6
λ_s	0.0	0.0	0.0	0.0	-0.1	0.0
λ_f	0.168	0.168	0.168	0.000	0.000	0.100
ρ	0.0	0.5	-0.5	0.0	0.0	0.0
$\mu_s = \frac{\alpha_s - \lambda_s \sigma_s}{-b_s}$	0.06	0.06	0.06	0.06	0.067	0.06
$\mu_f = \frac{\alpha_f - \lambda_f \sigma_f}{-b_f}$	0.10	0.10	0.10	0.135	0.135	0.114

3.4 Results

In this section, we present some numerical results and investigate the sensitivity of the results to the various parameters. We assume that the maximum value of two accounts is paid at any exit, including retirement, withdrawal, death and disability. If the employee dies before the retirement, we assume the DB benefit is calculated in the same way as the early retirement benefit with the deferred annuity. We consider the difference between the defined benefit account and the accumulated account when the plan ends. This is expressed as $\max(DB(t,t) - DC(t,t), 0)$ at the exit time t. We use 10,000 paths to simulate the rate of salary growth and the crediting rate, using six scenarios to indicate the sensitivity to the assumption. The results for entry age 20, 30, 40 and 50 are shown in Table 3.3.

Sherris(1995) analyzes the cost of the whole DB underpin pension plan, where the payoff is $\max(DB(t,t), DC(t,t)) = \max(DB(t,t) - DC(t,t), 0) + DC(t,t)$ at time t. However, our objective is to value the DB underpin guarantee, where the payoff is $\max(DB(t,t) - DC(t,t), 0)$ at the exit time t. We start with comparing

Table 3.2: Annual Decrement Rates for Retirement Benefit Valuations

Age	Withdrawal	Death and Disability	Age	Withdrawal	Death and Disability	Retirement
20	0.1665	0.0009	43	0.0396	0.00231	
21	0.1602	0.00089	44	0.03555	0.00262	
22	0.1539	0.00083	45	0.0315	0.00297	
23	0.1476	0.00075	46	0.02745	0.00338	
24	0.1413	0.00067	47	0.0234	0.00385	
25	0.135	0.00063	48	0.01935	0.00439	
26	0.1287	0.00063	49	0.0153	0.00501	
27	0.1224	0.00064	50	0.01125	0.00572	
28	0.1161	0.00066	51	0.009	0.00655	
29	0.1098	0.00068	52	0.00675	0.00751	
30	0.1035	0.00072	53	0.0045	0.0086	
31	0.0972	0.00077	54	0.00225	0.00983	
32	0.0909	0.00082	55	0	0.01123	
33	0.0846	0.00087	56	0	0.01282	
34	0.0783	0.00093	57	0	0.01462	
35	0.072	0.00101	58	0	0.01665	
36	0.06795	0.0011	59	0	0.01895	
37	0.0639	0.00121	60	0	0.02154	
38	0.05985	0.00132	61	0	0.02443	
39	0.0558	0.00146	62	0	0.02784	
40	0.05175	0.00163	63	0	0.03185	
41	0.077	0.00183	64	0	0.03659	
42	0.04365	0.00205	65	0	0.04215	0.95785

with Sherris'(1995) results and our results in this section. In the following sections, we will improve and update values of parameters to test sensitivities.

Table 3.3: The Lump Sum Mean Cost and Standard Error of the Guarantee as a Percent of Salary

Entry Age	20	30	40	50
Scenario	Mean(StErr)	Mean(StErr)	Mean (StErr)	Mean (StErr)
1	0.2116	0.8700	1.795	1.809
	(0.0431)	(0.1661)	(0.2946)	(0.2446)
2	0.1190	0.5345	1.184	1.378
	(0.0245)	(0.1035)	(0.1951)	(0.1838)
3	0.3063	1.239	2.096	2.325
	(0.0626)	(0.2349)	(0.2903)	(0.3816)
4	0.0486	0.2267	0.6541	0.8801
	(0.01698)	(0.07474)	(0.1728)	(0.1644)
5	0.0767	0.3536	0.8939	1.060
	(0.0263)	(0.1098)	(0.2178)	(0.1903)
6	0.1159	0.5315	1.168	1.354
	(0.0306)	(0.1279)	(0.2331)	(0.2110)

The unconditional or long-term mean salary growth rate is given by $\mu_s = -(\alpha_s - \lambda_s \sigma_s)/b_s$, and the long-term mean crediting rate is given by $\mu_f = -(\alpha_f - \lambda_f \sigma_f)/b_f$. In scenario 1, the salary growth rates and crediting rates are uncorrelated and normally distributed. Scenarios 2 and 3 examine the sensitivity of results to a variation in the correlation between the salary growth rate and the crediting rate. In all simulations, the value for λ_f is chosen to produce a risk-adjusted long-term crediting rate equal to the risk-free rate discount. In the first three scenarios, the long-term crediting rate μ_f is equal to the risk free rate. In scenarios 4, 5, and 6, we assume there is no risk-adjusted factor. So the cost is calculated under the nature's measure.

Sherris(1995) considers the maximum value of two pension accounts, but we focus on the difference between values of two accounts. The expected value of the defined contribution account plus the expected cost of guarantee will be the expected cost of the DB underpin pension plan, which is Sherris'(1995) result. This change gives us different results. Sherris'(1995) results indicate that the correlation coefficient has very little effect on the aggregate cost of benefit, but our results show it has a important effect. Results are very sensitive to values of parameters. When the correlation coefficient between salary growth rate and crediting rate is negative, the cost is higher. The cost decreases when the correlation coefficient is positive.

In the last three scenarios, we calculate the cost under the nature's measure. Scenario 4 indicates the effect of not making a risk adjustment to the crediting rate. This creates an inconsistency between the discount rate and the crediting rate. In scenario 5, we use the same parameters, except that the market price of risk for the salary growth rate is equal to -0.1. We set the market price of risk for the crediting rate as +0.1 in scenario 6. This has a major effect on the expected cost; clearly, the determination of market price of risk should be made carefully.

We also consider different entry ages. The expected cost increases when the entry age increases. This is because younger members have higher probability of each withdrawal and lower death and disability probabilities. So the expected cost will be lower overall.

Table 3.3 shows us the lump sum costs of the DB underpin guarantee. In Section 3.5, we will test the sensitivity of the DB underpin guarantee to changes in the values of parameters. The market is incomplete here. Both results are calculated under the nature's measure.

3.5 Scenario Test

In the previous sections, we started with Sherris' model and assumptions, since there are only few literatures about the DB underpin pension plan. Sherris(1995) only considered the valuation of the DB underpin pension plan. His model[2] was too complicated and had too many parameters. However, he did not really justify his model and parameters. In this section, we will start with two geometric Brownian motions and change the parameters in our model to move appropriate values.

We assume salary growth rate and crediting rate follow two stochastic differential equations as follows

$$ds(t) = \mu_s dt + \sigma_s dZ_s(t), \tag{3.9}$$

$$df(t) = \mu_f dt + \sigma_f dZ_f(t), \tag{3.10}$$

with $dZ_s(t)dZ_f(t) = \rho_{fs}dt$, where ρ_{fs} denotes the instantaneous correlation coefficient between the standardized Wiener increments $dZ_s(t)$ and $dZ_f(t)$.

We first test the sensitivity of the results to a change in the discount rate and in the contribution rate. Intuitively, a higher discount rate or a lower contribution rate represents a lower expected value of the DB underpin guarantee. Sherris(1995) used a discount rate of 10% p.y., which gives a low expected cost for the guarantee. The retirement benefit at the normal retirement age has a significant effect on the expected cost. Clearly, a lower, more realistic discount rate will generate a considerably higher overall value.

[2] See equation (3.7) and equation (3.8)

If employees (or employers) are willing to pay more contributions to the DC account during the working period, the value of their defined contribution account increases, while the value of the defined benefit pension does not change. This increases the probability that the value of DC account is greater than the value of the DB account. In other words, there is more chance that the value of the guarantee is zero. Hence, the expected cost is lower.

We fix the salary growth rate and the crediting rate and consider the following four scenarios, as in given Table 3.4.

Table 3.4: Change in Discount Rate and Contribution Rate*

Scenario	Discount Rate	Contribution Rate
1	0.05	0.125
2	0.05	0.15
3	0.07	0.125
4	0.07	0.15

*Mean of Salary Growth Rate: 5.5%, StD of Salary Growth Rate: 4%, Mean of Crediting Rate: 7%, StD of Crediting Rate: 10%.

Table 3.5: Expected Lump-Sum Cost of the Guarantee as Percentage of Initial Salary with Fixed Salary Growth and Crediting Rate

Entry Age	20	30	40	50
Scenario 1	0.2699	0.4884	0.5698	0.3875
Scenario 2	0.1347	0.2639	0.3055	0.1859
Scenario 3	0.1612	0.2852	0.3555	0.2965
Scenario 4	0.0714	0.1511	0.1933	0.1333

From Table 3.5, the expected cost of the guarantee decreases by almost 50% when the contribution rate increases from 12.5% to 15%. This result shows that employer can reduce the guarantee cost significantly by increasing the contribution rate as we expect. As the discount rate changes from 5% to 7%, the expected cost decreases, and the effect is, not surprisingly, greatest for younger lives. The expected cost is clearly very sensitive to the change of discount rate.

We now consider the sensitivities of the results to the salary growth rate and the crediting rate for the DC fund. A higher crediting rate represents a good investment performance by the DC fund leading to a higher value of the defined contribution account which leads to a lower cost for the guarantee.

Since the salary growth rate affects both the defined benefit account and the defined contribution account, it is not immediately obvious how the salary growth rate assumption affects the price. As the salary growth rate increases, the final salary increases and the value of defined benefit account increases. Meanwhile, contributions invested to the defined contribution account are higher, since the salary is higher, so the value of defined contribution account increases too.

To explore the sensitivity of the results to the salary growth rate and the crediting rate, we test several scenarios with different combinations, as shown in Table 3.6 and Table 3.7. We fix the salary growth rate and change the mean and standard deviation of the crediting rate first. The results are shown on Table 3.8. If we compare scenario 1 and scenario 5, which have the same contribution rate and discount rate, the expected cost decreases as the mean of the crediting rate increases as we expect. Also, we find that the expected cost increases when the standard deviation of the crediting rate increases from 0.15 to 0.2. When the crediting rate is more volatile, the probability that the value of DB account is greater than the value of DC account increases. Hence, the expected cost is higher.

Next, we fix the crediting rate and change the mean and the standard deviation of salary growth. From Table 3.9, the expected cost is lower when salary growth decreases, showing that the effect on the DB value is greater than the effect on the DC value.

Since this expected cost is the lump sum at the entry age, we want to amortize it into employees' monthly payroll and get the monthly expected cost as Table 3.10. We divide the expected lump sum cost by the value of salary indexed annuity to get the amortized cost. It represents an additional monthly cost rate of the salary. Given the fixed mortality table, the salary indexed annuity only depends on the salary scale and the discount rate. We find the sensitivity of the amortized cost is similar to the sensitivity of the lump sum cost.

Table 3.6: Change in Crediting Rate**

Scenario	Discount Rate	Mean of Crediting Rate	StD of Crediting Rate	Contribution Rate
5	0.05	0.1	0.15	0.125
6	0.05	0.1	0.15	0.15
7	0.07	0.1	0.15	0.125
8	0.07	0.1	0.15	0.15
9	0.05	0.1	0.2	0.125
10	0.05	0.1	0.2	0.15
11	0.07	0.1	0.2	0.125
12	0.07	0.1	0.2	0.15

**Mean of Salary Growth Rate: 5.5%, StD of Salary Growth Rate: 4%.

Table 3.7: Change in Salary Growth Rate

Scenario	Discount Rate	Mean of Salary Growth	StD of Salary Growth	Contribution Rate
13	0.05	0.04	0.02	0.125
14	0.05	0.04	0.02	0.15
15	0.07	0.04	0.02	0.125
16	0.07	0.04	0.02	0.15
17	0.05	0.045	0.04	0.125
18	0.05	0.045	0.04	0.15
19	0.07	0.045	0.04	0.125
20	0.07	0.045	0.04	0.15

***Mean of Crediting Rate: 10%, StD of Crediting Rate: 20%,

Table 3.8: Expected Lump-Sum Cost of the Guarantee as Percentage of Initial Salary with Fixed Salary Growth

Entry Age	20	30	40	50
Scenario 5	0.1572	0.2692	0.3430	0.2671
Scenario 6	0.0754	0.1457	0.1857	0.1342
Scenario 7	0.1005	0.1609	0.2155	0.2109
Scenario 8	0.0432	0.0813	0.1168	0.0984
Scenario 9	0.2353	0.4322	0.5018	0.3692
Scenario 10	0.1485	0.2899	0.3267	0.2188
Scenario 11	0.1447	0.2472	0.3217	0.2771
Scenario 12	0.0832	0.1602	0.2077	0.1659

Table 3.9: Expected Lump-Sum Cost of the Guarantee as Percentage of Initial Salary with Fixed Crediting Rate

Entry Age	20	30	40	50
Scenario 13	0.1143	0.1884	0.2423	0.2209
Scenario 14	0.0600	0.1117	0.1431	0.1206
Scenario 15	0.0800	0.1251	0.1581	0.1668
Scenario 16	0.0388	0.0658	0.0933	0.0919
Scenario 17	0.1488	0.2487	0.3212	0.2651
Scenario 18	0.0841	0.1518	0.1961	0.1558
Scenario 19	0.0969	0.1565	0.2048	0.2023
Scenario 20	0.0518	0.0925	0.1303	0.1153

Table 3.10: Amortization to Monthly Expected Cost of the Guarantee as Percentage of Salary(%)

Entry Age	20	30	40	50
Scenario 1	2.93	3.15	3.17	2.79
Scenario 2	1.46	1.72	1.70	1.34
Scenario 3	2.20	2.40	2.47	2.46
Scenario 4	0.97	1.27	1.34	1.11
Scenario 5	1.71	1.75	1.91	1.92
Scenario 6	0.82	0.95	1.03	0.97
Scenario 7	1.37	1.37	1.50	1.75
Scenario 8	0.59	0.69	0.81	0.82
Scenario 9	2.55	2.81	2.79	2.66
Scenario 10	1.61	1.78	1.82	1.58
Scenario 11	1.97	2.08	2.23	2.30
Scenario 12	1.13	1.35	1.44	1.38
Scenario 13	1.45	1.46	1.58	1.76
Scenario 14	0.76	0.87	0.93	0.96
Scenario 15	1.22	1.22	1.25	1.52
Scenario 16	0.59	0.64	0.74	0.85
Scenario 17	1.78	1.81	1.97	2.04
Scenario 18	1.01	1.10	1.20	1.20
Scenario 19	1.41	1.44	1.54	1.77
Scenario 20	0.75	0.85	0.98	1.01

Chapter 4

Funding Strategies with Two Traded Assets

4.1 Introduction to Risk Management

In the last chapter, we developed the valuation approach for the DB underpin pension guarantee. In this chapter, we focus on the funding of the pension guarantee using financial engineering principles, and we propose four funding strategies to manage the risk.

As we have discussed, the DB underpin pension benefit is the maximum of the DB and DC accounts. The payoff of the guarantee of this pension plan is defined as the maximum of zero and the difference of the DB and DC accounts. Given entry age xe, it can be expressed as

$$\max(0, DB(0,T) - DC(0,T)) \tag{4.1}$$

where T is the time at retirement, and $DB(0,T)$ and $DC(0,T)$ are the values of the DB and DC accounts given the information at $t = 0$, corresponding to the entry age xe. As both DB and DC involve uncertainty, this is similar to the Margrabe(1978) option, which is an option to exchange two risky tradable assets. We will describe the Margrabe option later in Section 4.3, and will use a similar pricing and hedging approach to construct the hedging portfolio, where it is feasible.

There are many cost methods to calculate the actuarial value of pension benefits. They are generally acceptable to the supervisory authorities for different funding purposes. We have discussed some of them in Chapter 2. According to different cost methods, the valuation formulas of the DB and DC account are different. That difference affects our funding strategies as well.

In Chapter 3, we have analyzed three valuation formulas based on the Entry Age Normal (EAN) cost method, Projected Unit Credit (PUC) cost method, and Traditional Unit Credit (TUC) cost method. Here, we emphasize those cost methods again. In the Entry Age Normal cost method, the value of DB account is calculated using the projected salary and all years of service. The value of DB account also includes all past and future contributions. The Projected Unit Credit method and the Traditional Unit Credit method both consider the known information at given time. Only earned years of service and paid contributions are credited to calculate the value of DB and DC accounts. The only difference is that PUC cost method uses the projected final salary and TUC cost method uses the current salary. For each strategy, we will explain the valuation formula in detail.

4.2 Assumptions

Assumption 1: No early retirement.

In the following chapters, we consider the cost of the guarantee assuming no exits before retirement. This is a common assumption in pension plan funding calculation. Generally, the pension plan cost is lower if employees leave the plan before the normal retirement age, so the 'no exits' assumption is conservative. The calculation of the present value of the pension guarantee allowing for exits before retirement is a straightforward extension of the 'no exits' case.

Assuming that the mortality table is given, the expected present value of pension guarantee can be considered as a weighted sum of benefits paid at each exit time. Under nature's measure, the weights are determined by the discount rate and the probability that employees quit the plan for any reason, such as death, retirement and resignation. The expected present value of pension guarantee can be expressed as

$$Cost_{xe} = \sum_{j=1}^{12T} {}_{\frac{j-1}{12}|\frac{1}{12}}q_{xe}^{(\tau)} e^{-rj/12} \max(0, DB(j/12, j/12) - DC(j/12, j/12)) \qquad (4.2)$$

where

xe is the entry age;

$DB(j/12, j/12)$ is the value of the defined benefit account at time $j/12$, given the exit time $j/12$, which is defined by equation(3.4);

$DC(j/12, j/12)$ is the value of the defined contribution account at time $j/12$, given the exit time $j/12$, which is defined by equation(3.6);

${}_{\frac{j-1}{12}|\frac{1}{12}}q_{xe}^{(\tau)}$ is the probability that the employee quits the plan at time $j/12$.

With no early retirement assumption, employees will not quit the pension plan before the normal retirement age. The present value of this guarantee is $e^{-rT} \max(0, DB(T, T) - DC(T, T))$, where $T = xr - xe$ is the time at retirement.

Assumption 2: Salary is tradable.

Given the mortality rate and interest rate, there are two random variables in our model, the salary growth rate and the crediting rate. In Section 3.5, we will consider the salary growth rate and the crediting rate as two geometric Brownian motions. Let S_t denote the employee's salary at age $xe+t$ and A_t denote an index representing the return on the DC funds. For simplicity, we assume that these two increase monthly, which is the frequency of the hedging and valuation process. So S_t is accumulated by the salary growth rate, and A_t is the accumulation at t of \$1 invested in the underlying DC fund at time 0. Given the initial salary S_0, we can express S_t and A_t mathematically,

$$dS_t = s(t)S_t dt \tag{4.3}$$

$$dA_t = f(t)A_t dt \tag{4.4}$$

where $s(u)$ and $f(u)$ are the salary growth rate and the crediting rate defined by equation (3.9) and equation (3.10), respectively.

In this chapter, we assume the interest rate is deterministic and two processes S_t and A_t are tradable. In this case, S_t and A_t are two random processes and under the 'tradable' assumption, those two processes can be perfectly replicated in the financial market. We use these hypothetical assets to construct the hedging portfolio and show numerical results to analyze hedging costs and risks of our funding strategies.

However, S_t is not tradable in practice. Employers and pension fund managers may use traded assets to hedge it to some extent. A_t can be easily replicated as it represents the accumulation of the traded asset in the DC account. Employees' DC contributions are usually invested in a mix of fixed income securities and stocks.

But it is hard to find financial securities to replicate the salary. In Chapter 6, we discuss the relationship between the inflation and salaries. In Chapter 7, we relax the assumption that S_t is tradable by considering the use of inflation as a partial salary hedge because of a high correlation between salaries and inflation.

4.3 Margrabe Option

Margrabe(1978) develops an equation for the value of this kind of option to exchange one risky asset for another. Margrabe(1978) shows the valuation of the European-type exchange option is an extension of the Black-Scholes equation.

Let Y_1 and Y_2 be the prices of assets one and two. Assume there are no dividends: all returns come from capital gains. Assume also that the rate of return on each asset is given by

$$dY_i = Y_i[\alpha_i dt + \sigma_i dZ_i] \qquad i = 1, 2$$

with $dZ_1 dZ_2 = \rho dt$, where ρ denotes the instantaneous correlation coefficient between the standardized Wiener increments dZ_1 and dZ_2.

A European-type option can be only exercised at maturity date T. It will yield $Y_1 - Y_2$ if exercised or nothing if not exercised. This implies the initial condition

$$w(Y_1, Y_2, T) = \max(0, Y_1 - Y_2). \tag{4.5}$$

Assume Y_1 and Y_2 are non-negative, the option value $w(Y_1, Y_2, t)$ at time t is worth at least zero, and no more than Y_1. If assets one and two are worth at least zero, the boundary condition is

$$0 \leq w(Y_1, Y_2, t) \leq Y_1. \tag{4.6}$$

Margrabe(1978) shows that the function $w(Y_1, Y_2, t)$ is the solution to the differential equation:

$$w_t + \frac{1}{2}\left[w_{11}\sigma_1^2 Y_1^2 + 2w_{12}\sigma_1\sigma_2 Y_1 Y_2 + w_{22}\sigma_2^2 Y_2^2\right] = 0. \tag{4.7}$$

which is subject to the initial condition (4.5) and the boundary condition (4.6).

The closed form solution is quite similar to the Black-Scholes formula:

$$\begin{aligned} w(Y_1, Y_2, t) &= Y_1 N(d_1) - Y_2 N(d_2) \\ d_1 &= \frac{\ln(Y_1/Y_2) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned} \tag{4.8}$$

Here, $N(\cdot)$ is the cumulative standard normal density function and $\sigma^2 = \sigma_1^2 - 2\sigma_1\sigma_2\rho + \sigma_2^2$ is the variance of $(Y_1/Y_2)^{-1}d(Y_1/Y_2)$.

Margrabe(1978) also proves this problem can be transformed into the Black-Scholes problem by letting asset two be the numeraire.

In following sections, we consider different approaches to the DB underpin guarantee valuation. One object is to be able to express the option in term of the Margrabe option so that we can apply the pricing formula, i.e. Equation 4.8.

4.4 Strategy 1: EAN Cost Method

Given the 'no early exits' assumption, at the valuation time t, the present value of the DB underpin option is $e^{-r(T-t)}\max(0, DB(t,T) - DC(t,T))$, where $T = (xr - xe)/12$ is the time on retirement. In strategy 1, we consider an Entry Age Normal (EAN) cost method. The EAN cost method calculates future liabilities first, then finds the appropriate level contribution rate to match liabilities. It is no longer very popular in pension valuation problems, because all years of service and contributions are considered at any given time before retirement. Employers take all liabilities into consideration at the beginning of the plan and there is substantial pre-funding-that is, funding benefits before service occurs. This unnecessary ties up company capital. It also ignores the important option to terminate the plan. The option to terminate means there is no need to pre-fund. Later, we will consider unit credit methods, such as PUC and TUC. Those methods are based on the past years of service and contributions, and avoid the pre-funding issue.

In the EAN cost method, we assume that the value of DB account is calculated using the projected final salary and all earned and unearned years of service. The projected value at time t of DB account at retirement time T can be expressed as

$$DB^E(t,T) = \alpha T S_T \ddot{a}^{(12)}_{xr} | S_t \tag{4.9}$$

where S_T is the projected final salary based on current salary S_t at the valuation time t, and credited years of service T includes all earned (t) and unearned ($T-t$) years of service.

Since the value of the DB account includes all years of service and the final salary, the value of the DC account should be consistent. We assume the value of the DC account is projected to include all contributions, both past and future. Given the monthly contribution rate c, the projected value at time t of DC account at retirement time T can be expressed as

$$\begin{aligned}
DC^E(t,T) &= DC^E(t,t)\frac{A_T}{A_t} + \sum_{j=12t}^{12T-1} \frac{cS_{j/12}}{12}\frac{A_T}{A_{j/12}} \\
&= \sum_{j=0}^{12t-1} \frac{cS_{j/12}}{12}\frac{A_t}{A_{j/12}} \cdot \frac{A_T}{A_t} + \sum_{j=12t}^{12T-1} \frac{cS_{j/12}}{12}\frac{A_T}{A_{j/12}} \\
&= DC^E(t,t)\frac{A_T}{A_t} + \sum_{j=12t}^{12T-1} \frac{cS_t}{12}\frac{S_{j/12}}{S_t}\frac{A_T}{A_{j/12}}
\end{aligned} \tag{4.10}$$

where the current value of the DC account at time t, $DC^E(t,t)$, and the salary S_t at time t are known; $s(u)$ and $f(u)$ are the salary growth rate and crediting rate, respectively.

The projected final value of DC account $DC^E(t,T)$ comprises the accumulation of past contributions $DC^E(t,t) = \sum_{j=0}^{12t-1} \frac{cS_{j/12}}{12}\frac{A_t}{A_{j/12}}$ and the accumulation of future contributions $\sum_{j=12t}^{12T-1} \frac{cS_{j/12}}{12}\frac{A_T}{A_{j/12}}$. In equation (4.9) and equation (4.10), we treat all earned and unearned years of service and all past and future contributions equally. At the entry age, we have the projected value of DB and DC accounts at time 0.

$$DB^E(0,T) = \alpha T S_T \ddot{a}_{xr}^{(12)} | S_0 \tag{4.11}$$

$$DC^E(0,T) = \sum_{j=0}^{12T-1} \frac{cS_{j/12}}{12}\frac{A_T}{A_{j/12}} | S_0 \tag{4.12}$$

Then, the expected value of the DB underpin option at time 0 (entry) is

$$\begin{aligned} C(S_0, A_0, T) &= E_0^Q\left[e^{-rT}\left(DB^E(0,T) - DC^E(0,T)\right)^+\right] \\ &= E_0^Q\left[e^{-rT}\left(\alpha T S_T \ddot{a}_{xr}^{(12)} - \sum_{j=0}^{12T-1} \frac{cS_{j/12}}{12}\frac{A_T}{A_{j/12}}\right)^+\right] \\ &= E_0^Q\left[e^{-rT}\left(\alpha T S_T \ddot{a}_{xr}^{(12)} - \frac{c}{12}A_T \sum_{j=0}^{12T-1} \frac{cS_{j/12}}{A_{j/12}}\right)^+\right] \end{aligned} \tag{4.13}$$

We see that the exchange is more complicated than the regular exchange option through the factor $(\sum_{j=0}^{12T-1} \frac{cS_{j/12}}{A_{j/12}})$. Even though each term in the sum is lognormal, the sum of dependent lognormal random variables is not tractable. This term indicates that option is path dependent, since different paths for the salary to DC fund return ratio will give different payoffs for the guarantee.

Although the guarantee is not analytically tractable, we can use Monte Carlo simulation to estimate the value.

In Figure 4.1, we show the value at entry of the guarantee accrued over the entire working lifetime, for ages at entry from 25 to 64. We assume the monthly salaries follow a lognormal process with volatility $\sigma_s = 0.02$. We assume the monthly DC fund returns follow a lognormal process with volatility $\sigma_f = 0.2$. The processes are dependent, through the correlation coefficient $\rho = -0.15$, of the underlying normal processes. We assume a constant risk free rate of 5% per year continuously compounded. The estimates under a risk neutral measure are determined using 10,000 simulations. We assume the contribution rate is 12.5% of salary, paid monthly, the accrual rate is $\alpha = 1.5\%$ per year of service, and the annuity factor at retirement is 10.0.

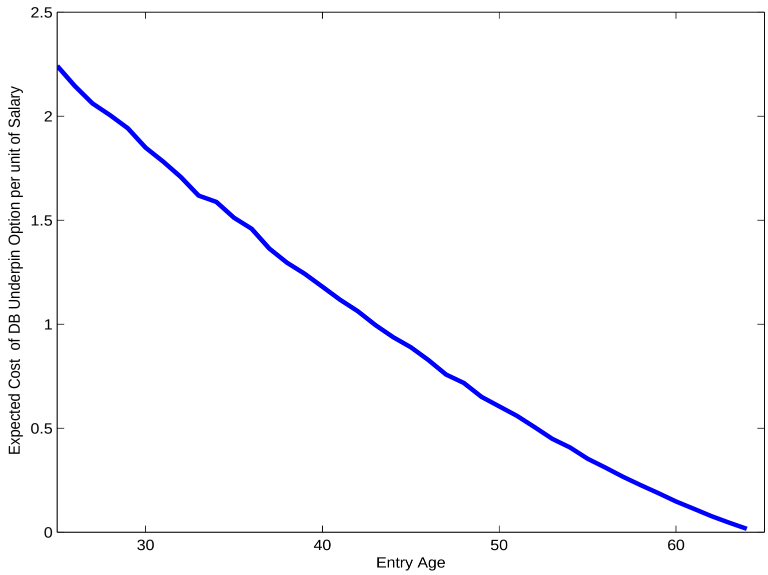


Figure 4.1: Value at Entry of the DB Underpin Option under a Risk Neutral Measure, per unit of Initial Salary

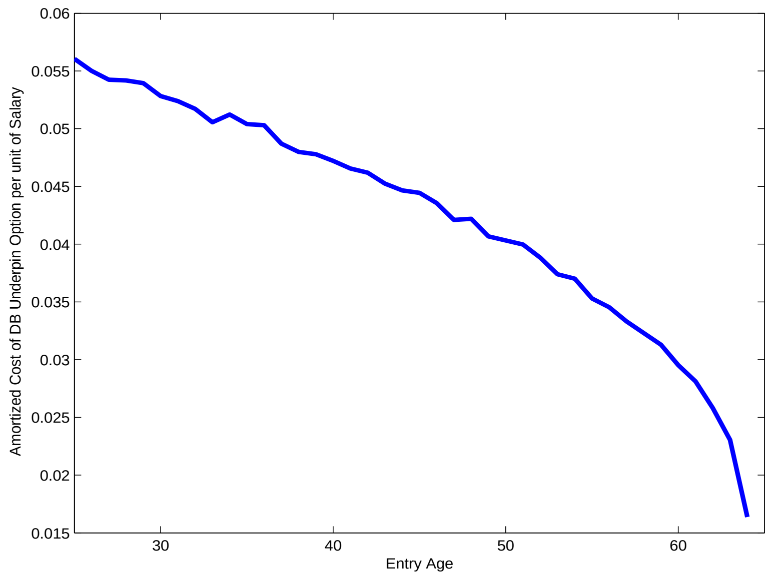


Figure 4.2: Amortized Value of the DB Underpin Option under a Risk Neutral Measure, % of Salary, Monte Carlo Valuation using 10,000 Sample Paths

The guaranteed cost is expressed as a multiple of the initial salary. Figure 4.1 shows that the option is valuable for younger employees, with a value of 2.25 times the starting salary at age 25, reducing to 0.0164 times the starting salary for a life entering the plan at age 64. The standard errors for the estimates of the guarantee cost at entry range from 0.0181 for age 25 to 0.00013 at age 64.

Following the EAN approach, we can also amortize the expected cost like in Chapter 3. For this we must divide the lump sum value by the value of a salary indexed annuity. This annuity is also valued under the risk neutral measure, and also assumes no exits, giving a value:

$$E_0^Q\left[\sum_{j=1}^{12T} e^{-rj/12}\frac{S_{j/12}}{12}\right] = TS_0 \tag{4.14}$$

because under the risk neutral measure the discounted value at t of a risky asset due at $t+u$ is always simply the current value at t of the asset.

The amortized cost of DB underpin guarantee at different entry ages is given in Figure 4.2. This shows that the amortized cost also decreases with the entry age. For more common entry ages, from 30 to 45, say, the amortized cost is between 5.3% and 4.4% of salary. These rates may be economically feasible, but are considerably higher than current funding ratio for these plans. However, the 'no exits' assumption is conservative, so the cost is probably over stated.

We can also calculate the delta hedging portfolio for the pension plan using Monte Carlo simulation. The expected value of the guarantee at any time t is determined by two processes S_t and A_t at time t. At time t, if the salary S_t and the value of the underlying fund assets S_t are changed, the expected value of the guarantee in equation (4.13) is also changed. So we can estimate the first partial derivatives of the guarantee with respect to the salary and the underlying fund assets individually.

In strategy 1, we treat all earned and unearned years of service in DB account and all past and future contributions in DC account equally. We estimate hedging parameters by simulating the effect of the value of accounts on the DB underpin option. The expected present value of the DB underpin pension guarantee, $C(S_0, A_0, T)$ defined by equation (4.13), is evaluated using simulation under the risk-neutral measure as given by equation (4.15).

$$\begin{aligned} C(S_0, A_0, T) &= e^{-rT} E_0^Q[(DB^E(0,T) - DC^E(0,T))^+] \\ &\approx \hat{C}(S_0, A_0, T) \\ &\approx \frac{1}{N} e^{-rT} \sum_{i=1}^{N} \left(\alpha T \hat{S}_{i,T} \ddot{a}_{xr}^{(12)} - \frac{c}{12} \hat{A}_{i,T} \sum_{j=0}^{12T-1} \frac{c\hat{S}_{i,j/12}}{\hat{A}_{i,j/12}} \right)^+ \end{aligned} \tag{4.15}$$

where $\hat{S}_{i,j/12}$ and $\hat{A}_{i,j/12}$ are the simulated value of the salary and the underlying assets for i-th simulation at time $j/12$; and $\hat{C}(S_0, A_0, T)$ is the simulated present value of the DB underpin pension guarantee at time 0.

Equation (4.15) shows that this cost is the present value of the expected value of the expected payoff of the plan, where the expectation is taken with respect to the probability distribution of terminal value of two accounts. Given that the contribution rate and initial salary are fixed, $\hat{S}_{i,j/12}$ and $\hat{A}_{i,j/12}$ are determined by the salary growth rate and the crediting rate.

In equation (4.16) and (4.17), the hedging parameters at time t can be calculated by a finite difference estimate of the first derivatives of the cost,

$$\begin{aligned} \Delta_{S_t,T} &= \frac{\partial C(S_t, A_t, T)}{\partial S} \\ &\approx \frac{\hat{C}(S_t + \varepsilon, A_t, T) - \hat{C}(S_t - \varepsilon, A_t, T)}{2\varepsilon} \end{aligned} \tag{4.16}$$

$$\begin{aligned} \Delta_{A_t,T} &= \frac{\partial C(S_t, A_t, T)}{\partial A} \\ &\approx \frac{\hat{C}(S_t, A_t + \varepsilon, T) - \hat{C}(S_t, A_t - \varepsilon, T)}{2\varepsilon} \end{aligned} \tag{4.17}$$

where S_t and A_t are values of salary and underlying assets at time t, and ε is close to zero.

At time t between the entry age xe and the retirement age xr, we will rebalance the hedging portfolio. The shares of $\Delta_{D,t}$ and $\Delta_{A,t}$ will be held in long position of the salary and the underlying fund assets, respectively. Since $\Delta_{A,t}$ is negative in fact, we actually short the share of $-\Delta_{A,T}$ in the underlying fund assets.

This strategy works well for regular options. However, in our DB underpin pension plan, we have two random variables, the salary growth rate and the crediting rate. The crediting rate only affects the terminal value of the DC account. The values of the DB and DC accounts both depend on salaries. The value of the DB account is proportional to salaries and the value of the DC account is accumulated by monthly contributions which are proportional to salaries. Once salaries change, both the expected terminal values of DB and DC account change and the expected value of the guarantee changes. This means that we need to take a long position in salaries to partially hedge the DB account, and take a short position in salaries to partially hedge the DC account at the same time. If the salary increases, both the DB and DC account values increase. The projected terminal value of pension guarantee can either increase or decrease. Hence, the change of the projected value is very volatile. This causes significant variance of the delta with respect to the salary. In Figure 4.3, we plot the convergence of delta with increasing number of simulations. The value of delta with respect to salary will converge with a higher number of simulations. The figure shows that around 10,000 simulations might be required each month. is a computationally intensive hedging problem.

In strategy 2, we split the DB underpin pension guarantee and implement the valuation of the Margrabe option. Strategy 2 avoids the computational problem for the calculation of delta values.

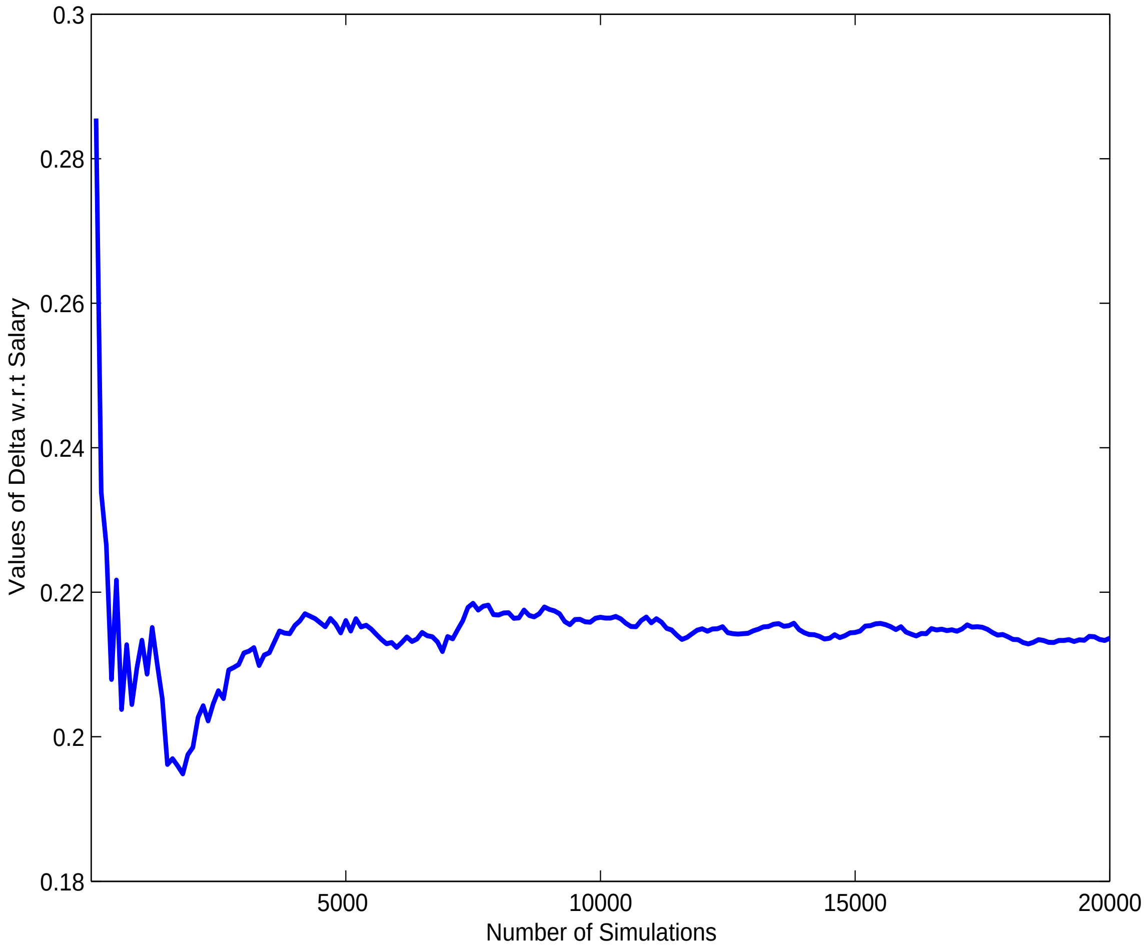


Figure 4.3: Convergence of Delta

4.5 Strategy 2: EAN Cost Method

In equation (4.13), the exchange part is more complicated than the regular exchange option through the factor $(\sum_{j=0}^{12T-1} \frac{cS_{j/12}}{A_{j/12}})$. This causes the hedging problem since the sum of lognormal random variable is not tractable. In strategy 2, we propose another funding strategy to avoid the relation between the DC account and the future salary. Since the value of DB account depends on the years of service and the number of contribution payments is equal to twelve times years of service[1], we split the DB and DC accounts into the individual monthly parts and value and hedge each part separately. We then calculate the total hedging cash flows. In this strategy, we still treat all the earned and unearned years of service and all paid and unpaid contributions equally. The only difference is that we split the whole DB underpin guarantee into several parts and hedge them individually.

Mathematically, our hedging object is to hedge the final payment $(DB(0,T) - DC(0,T))^+$, where T is the time at retirement. In this section, we still use the Entry Age Normal (EAN) cost method. At any valuation time, the DB underpin guarantee still includes all earned and unearned years of service and past and future contributions. That means that values of DB and DC accounts are calculated as same as the last section. At time 0, the projected values of DB and DC accounts are expressed as

$$DB^E(0,T) = \alpha T S_T \ddot{a}_{xr}^{(12)} | S_0 \tag{4.18}$$

$$DC^E(0,T) = \sum_{j=0}^{12T-1} \frac{cS_{j/12}}{12} \frac{A_T}{A_{j/12}} | S_0 \tag{4.19}$$

In this strategy 2, we split the DB underpin guarantee to avoid this problem. In equation (4.19), the summation includes $12T$ parts, $\frac{cS_{j/12}}{12}\frac{A_T}{A_{j/12}}$. At time $j/12$,

[1]Because we assume monthly contributions.

$\frac{cS_{j/12}}{12A_{j/12}}$ is known. The only random process here is A_T. Comparatively, we also split the projected value of DB account into $12T$ parts. In equation (4.18), S_T is the projected final salary at retirement based on current salary. T is the retirement time. Since the contribution is paid monthly, we re-write $T \cdot S_T$ as $\sum_{j=0}^{12T-1} S_T/12$. Then, equation (4.18) is expressed as

$$DB^E(0,T) = \sum_{j=0}^{12T-1} \frac{\alpha \ddot{a}_{xr}^{(12)}}{12} S_T | S_0 \tag{4.20}$$

The present value of the DB underpin option is

$$\begin{aligned}
& e^{-rT} E_0^Q[(DB^E(0,T) - DC^E(0,T)^+] \\
= \; & e^{-rT} E_0^Q \left[\left(\sum_{j=0}^{12T-1} \frac{\alpha \ddot{a}_{xr}^{(12)}}{12} S_T - \sum_{j=0}^{12T-1} \frac{cS_{j/12}}{12} \frac{A_T}{A_{j/12}} \right)^+ \right] \\
= \; & e^{-rT} E_0^Q \left[\left(\sum_{j=0}^{12T-1} \frac{\alpha \ddot{a}_{xr}^{(12)} S_{j/12}}{12} \frac{S_T}{S_{j/12}} - \sum_{j=0}^{12T-1} \frac{cS_{j/12}}{12} \frac{A_T}{A_{j/12}} \right)^+ \right] \\
= \; & e^{-rT} E_0^Q \left[\left(\sum_{j=0}^{12T-1} \frac{S_{j/12}}{12} \left(\alpha \ddot{a}_{xr}^{(12)} \frac{S_T}{S_{j/12}} - c \frac{A_T}{A_{j/12}} \right) \right)^+ \right]
\end{aligned} \tag{4.21}$$

It can be easily shown that the maximum of sums is less than the sum of maximums, i.e. the option on the sum is cheaper than the sum of the options, so

$$\begin{aligned}
& \max \left(0, \sum_{j=0}^{12T-1} \frac{S_{j/12}}{12} \left(\alpha \ddot{a}_{xr}^{(12)} \frac{S_T}{S_{j/12}} - c \frac{A_T}{A_{j/12}} \right) \right) \\
\leq & \sum_{j=0}^{12T-1} \frac{S_{j/12}}{12} \left(\max(0, \alpha \ddot{a}_{xr}^{(12)} \frac{S_T}{S_{j/12}} - c \frac{A_T}{A_{j/12}}) \right) \\
= & \sum_{j=0}^{12T-1} \frac{S_{j/12}}{12} \left(\max(0, \alpha \ddot{a}_{xr}^{(12)} Y_{1,j/12}(T) - cY_{2,j/12}(T)) \right)
\end{aligned} \tag{4.22}$$

where $Y_{1,j/12}(T)$ and $Y_{2,j/12}(T)$ are values at time T of two assets:

$$dY_{1,j/12}(u) = Y_{1,j/12}(u)[\mu_s du + \sigma_s dZ_s(u)] \quad (4.23)$$

$$dY_{2,j/12}(u) = Y_{2,j/12}(u)[\mu_f du + \sigma_f dZ_f(u)] \quad (4.24)$$

where $j/12 \leq u \leq T$. Initial conditions are $Y_{i,j/12}(j/12) = 1$, $i = 1, 2$.

Equation (4.22) shows that we can hedge $\max(0, \alpha\ddot{a}_{xr}^{(12)} Y_{1,j/12}(T) - cY_{2,j/12}(T))$ at each time $j/12$ between 0 and T and sum them together. The total amount will overhedge the amount of $\max(0, DB^E(0,T) - DC^E(0,T))$. The excess can be simulated, and will be released at the end of the contract.

Using a similar approach to Margrabe(1978), for each individual "option", we have

$$w^*(Y_{1,t}, Y_{2,t}, T) = \max(0, \alpha\ddot{a}_{xr}^{(12)} Y_{1,t}(T) - cY_{2,t}(T)) \quad 0 \leq t \leq T \quad (4.25)$$

and boundary conditions

$$0 \leq w^*(Y_{1,t}, Y_{2,t}, u) \leq \alpha\ddot{a}_{xr}^{(12)} Y_{1,t} \quad (4.26)$$

The function $w^*(Y_{1,t}, Y_{2,t}, u)$ is the solution to the differential equation:

$$w_u^* + \frac{1}{2}\left[(\alpha\ddot{a}_{xr}^{(12)})^2 w_{11}^* \sigma_s^2 Y_{1,t}^2 + 2\alpha\ddot{a}_{xr}^{(12)} c w_{12}^* \sigma_s \sigma_f Y_{1,t} Y_{2,t} + c^2 w_{22}^* \sigma_f^2 Y_{2,t}^2\right] = 0. \quad (4.27)$$

Equation (4.27) can be solved, subject to initial conditions (4.25) and boundary conditions (4.26):

$$w^*(Y_{1,t}, Y_{2,t}, u) = \alpha\ddot{a}_{xr}^{(12)} Y_{1,t} N(d_{1,t}) - cY_{2,t} N(d_{2,t})$$

$$d_{1,t} = \frac{\ln(\alpha\ddot{a}_{xr}^{(12)} Y_{1,t}/(cY_{2,t})) + \frac{1}{2}\sigma^2(T-u)h}{\sigma\sqrt{(T-u)h}} \quad (4.28)$$

$$d_{2,x+ih} = d_{1,x+ih} - \sigma\sqrt{(T-u)h}. \quad 0 \le t \le u \le T$$

Here, $N(\cdot)$ is the cumulative standard normal density function and $\sigma^2 = \sigma_s^2 - 2\sigma_s\sigma_f\rho_{sf} + \sigma_f^2$ is the variance of $(Y_{1,t}/Y_{2,t})^{-1}d(Y_{1,t}/Y_{2,t})$.

Equation (4.28) gives the closed form solution. We can calculate the delta hedging parameters. The first derivatives with respect to $Y_{1,t}$ and $Y_{2,t}$ are:

$$\begin{aligned}
\Delta_{1,t,u} &= \frac{\partial w^*(Y_{1,t}, Y_{2,t}, u)}{\partial Y_{1,t}} \\
&= \alpha \ddot{a}_{xr}^{(12)} \cdot N(d_{1,t}) + \alpha \ddot{a}_{xr}^{(12)} \cdot Y_{1,t}\frac{\partial N(d_{1,t})}{\partial Y_{1,t}} - c \cdot Y_{2,t}\frac{\partial N(d_{2,t})}{\partial Y_{1,t}} \\
&= \alpha \ddot{a}_{xr}^{(12)} \cdot N(d_{1,t}) \qquad (4.29) \\
\Delta_{2,t,u} &= \frac{\partial w^*(Y_{1,t}, Y_{2,t}, u)}{\partial Y_{2,t}} \\
&= \alpha \ddot{a}_{xr}^{(12)} \cdot Y_{1,t}\frac{\partial N(d_{1,t})}{\partial Y_{2,t}} - c \cdot Y_{2,t}\frac{\partial N(d_{2,t})}{\partial Y_{2,t}} - c \cdot N(d_{2,t}) \\
&= -c \cdot N(d_{2,t}) \qquad (4.30)
\end{aligned}$$

The notation $\Delta_{1,t,u}(\Delta_{2,t,u})$ denotes the delta hedging amount at time u for the salary and the DC underlying asset, and it is hedged by the contribution at time t, where $0 \le t \le u \le T$.

According to this funding strategy, we divide the final pension benefits into 12 parts. We propose that each part of the final benefit is hedged by the single contribution at time t, where $0 \le t \le T$. This means that we will have $12t$ parts in the hedging fund at time t, where $t = 1, 1\frac{1}{12}, \ldots, T = xr - xe$. We need to sum $12t$ parts together to get the following,

$$\Delta_{1,t}^* = \sum_{j=1}^{12t} \Delta_{1,j/12,t} \cdot \frac{S_{j/12}}{12} \qquad (4.31)$$

$$\Delta_{2,t}^* = \sum_{j=1}^{12t} \Delta_{2,j/12,t} \cdot \frac{A_{j/12}}{12} \qquad (4.32)$$

At time t, we will invest $\Delta^*_{1,t}$ into the salary S_t and $\Delta^*_{2,t}$ into the underlying assets of the DC account A_t. The value of hedging portfolio is equal to

$$H(t) = \Delta^*_{1,t} + \Delta^*_{2,t} \tag{4.33}$$

where $\Delta^*_{1,t}$ and $\Delta^*_{2,t}$ are defined by equations (4.31) and (4.32). One month later, at time $t + \frac{1}{12}$, the hedging portfolio established at t and brought forward to $t + \frac{1}{12}$ has value:

$$Hbf(t + \frac{1}{12}) = \Delta^*_{1,t}\frac{S_{t+\frac{1}{12}}}{S_t} + \Delta^*_{2,t}\frac{A_{t+\frac{1}{12}}}{A_t} \tag{4.34}$$

The new hedge costs $H(t + \frac{1}{12})$, so the cash flow for the guarantee at $t + \frac{1}{12}$ is

$$CF(t + \frac{1}{12}) = H(t + \frac{1}{12}) - Hbf(t + \frac{1}{12}) \tag{4.35}$$

and the monthly hedging cash flow as a proportion of salary is

$$cf(t + \frac{1}{12}) = \frac{CF(t + \frac{1}{12})}{S_{t+\frac{1}{12}}} \tag{4.36}$$

In this strategy, we split the DB underpin pension guarantee. That solves the difficulty of hedging the complicated exchange part $(\sum_{j=0}^{12T-1} \frac{cS_{j/12}}{A_{j/12}})$. However, this hedge will, as noted above, exceed the true cost. Employers need more money to construct such a hedging portfolio.

4.6 Strategy 3: PUC Cost Method

In our first two strategies, we adopt the Entry Age Normal (EAN) cost method and treat all earned and unearned years of service and past and future contributions equally. Philosophically, the EAN approach ignores the fact that the DB benefit is accruing over time. The employer maintains an option to close the plan to new accruals, or to close it altogether. It is therefore inconsistent with the nature of the developing liability to treat accrued benefits, which are a debt on the plan assets, on an equal footing with future service benefits, which are not.

For this reason, most pension actuaries now use an accruals approach to funding, whereby the liability is not recognized until the relevant service has been completed. For a final salary DB plan, there are two accruals approaches. The first one we discuss is the Projected Unit Credit (PUC) approach. In a conventional DB valuation, the PUC valuation uses the projected final salary, but only the past service at the valuation is included in the liability.

Using the PUC cost method, we can use past service and future salary to express the projected value at time t of DB account at retirement time T as

$$DB^P(t,T) = \alpha t S_T \ddot{a}_{xr}^{(12)} | S_t \tag{4.37}$$

The main difference between equation (4.37) for PUC and (4.9) for EAN is that the PUC only counts the earned years of service. To be consistent, we consider the DC account using past paid contributions only, that is

$$DC^P(t,T) = DC^P(t,t)\frac{A_T}{A_t} | A_t \tag{4.38}$$

The projected value of DC account is accumulated by the current value of DC account. So the projected value at retirement age T of the pension guarantee with entry age xe and the current age $xe+t$ can be expressed as

$$\begin{aligned}
&\max(0, DB^P(t,T) - DC^P(t,T))|S_t, A_t \\
&= \max(0, \alpha t S_t e^{\int_t^T s(u)du} \ddot{a}_{xr}^{(12)} - DC^P(t,T) e^{\int_t^t f(u)du})|S_t, A_t \\
&= \max(0, \alpha t S_T \ddot{a}_{xr}^{(12)} - DC^P(t,t)\frac{A_T}{A_t})|S_t, A_t
\end{aligned} \tag{4.39}$$

The present value of the pension guarantee is then $e^{-r(T-t)}\max(0, DB^P(t,T) - DC^P(t,T))$.

At time t, the salary S_t and the value of the DC account $DC^P(t,t)$ is known. There is no allowance for future contributions into the DC account. There are two stochastic processes S_T and A_T in equation (4.39). We can adopt the valuation of the Margrabe option again. Similar to equation (4.28), the value of the guarantee at time t with retirement age T is

$$\begin{aligned} w^*(S_t, A_t, u) &= c_{1,t}S_tN(d_{1,t}) - c_{2,t}A_tN(d_{2,t}) \\ d_{1,t} &= \frac{\ln(c_{1,t}S_t/(c_{2,t}A_t)) + \frac{1}{2}\sigma^2(T-u)}{\sigma\sqrt{(T-u)}} \\ d_{2,t} &= d_{1,t} - \sigma\sqrt{(T-u)h}. \quad 0 \le t \le u \le T \end{aligned} \tag{4.40}$$

where $c_{1,t} = \alpha\ddot{a}_{xr}^{(12)}t$ and $c_{2,t} = DC^P(t,t)$ is the value at t of accumulated past contributions. $N(\cdot)$ is the cumulative standard normal density function and $\sigma^2 = \sigma_s^2 - 2\sigma_s\sigma_f\rho_{sf} + \sigma_f^2$ is the variance of $(S_t/A_t)^{-1}d(S_t/A_t)$.

We can calculate the delta hedging parameters. The first partial derivatives at time u with respect to S_t and A_t are $\Delta_{1,t,u} = c_{1,t}N(d_{1,t}), \Delta_{2,t,u} = -c_{2,t}N(d_{2,t})$, respectively.

We also need to rebalance the hedging portfolio at each period. At time $(t+1)^-$, the value of the hedging portfolio is accumulated. We then rebalance the hedging portfolio at time $t+1$ and get the cash flow at time $t+1$.

This strategy also solves the problem that the value of DC account depends on the future salaries. We can also apply the exchange option valuation approach to calculate delta hedging cash flows.

4.7 Strategy 4: TUC Cost Method

In the projected unit credit cost method, given the current salary S_t, the projected DB benefit is dependent on the projected future salary, which is $S_t \cdot e^{\int_t^T s(u)du}$. In this section, we consider the traditional unit credit cost method(TUC) instead of PUC cost method. TUC is always used for flat benefit pension plans. It also uses past years of service and past contributions. However, it assumes future salaries will not change. So the only difference between TUC and PUC is that the projected DB benefit is dependent on the current salary at time t under TUC cost method. As shown in Table 4.1, the projected DB benefit using TUC cost method, $DB^T(t,T)$, is deterministic at time t, since the current salary is given.

$$DB^T(t,T) = \alpha t S_t \ddot{a}_{xr}^{(12)} \tag{4.41}$$

The DC account also counts past contributions.

$$DC^T(t,T) = DC^T(t,t)\frac{A_T}{A_t}|A_t \tag{4.42}$$

The projected payoff of the guarantee, which is $\max(0, DB^T(t,T) - DC^T(t,T))$, is no longer the payoff of a Margrabe option, it is the exact form of the European type put option, since $DB^T(t,T)$ is a constant.

From equation (4.39), the projected value at retirement age T of the pension guarantee with entry age xe and the current age $xe+t$ can be expressed as

$$\begin{aligned}
&\max(0, DB^T(t,T) - DC^T(t,T))|A_t \\
&= \max(0, \alpha t S_t \ddot{a}_{xr}^{(12)} - DC^T(t,t)\frac{A_T}{A_t})|A_t
\end{aligned} \tag{4.43}$$

The only stochastic part in equation (4.43) is A_T. This form is similar to the payoff of an European type put option. We can apply the Black-Scholes formula to get the value of the pension guarantee at current age $x + th$

$$\begin{aligned} w^*(S_t, A_t, u) &= c_{1,t}e^{-r(T-t)}N(-d_{2,t}) - c_{2,t}A_tN(-d_{1,t}) \\ d_{1,t} &= \frac{\ln(c_{2,t}A_t/c_{1,t}e^{-r(T-u)}) + \frac{1}{2}\sigma^2(T-u)}{\sigma\sqrt{(T-u)}} \\ d_{2,t} &= d_{1,t} - \sigma\sqrt{(T-u)}. \quad 0 \le t \le u \le T \end{aligned} \tag{4.44}$$

where $c_{1,t} = \alpha t S_t \ddot{a}_{xr}^{(12)}$ and $c_{2,t} = DC^T(t, T)$.

The first derivative with respect to A_t is $-c_{2,t}N(-d_{1,t})$, which is the delta parameter at time t.

Under the TUC cost method, we also rebalance the hedging portfolio every month. There is only one random variable in this hedging problem. The payoff of the guarantee is similar to the payoff of the regular European put option. The hedging portfolio is constructed by the stock, since the salary is assumed as a constant. So the assumption on salaries will not affect the result too much. Later, we will show some numerical results to compare with different salary assumptions in Chapter 7.

4.8 Summary

In this chapter, we propose four strategies according to different cost methods. Strategy 1 and strategy 2 use the Entry Age Normal (EAN) cost method. The EAN cost method considers all past and future service equally. Strategy 3 adopts the Projected Unit Credit (PUC) cost method. The valuation under the PUC cost method includes the past service and the projected final salary. Strategy 4 considers the Traditional Unit Credit (TUC) cost method, which only includes the past service and the current salary.

Table 4.1 illustrates three valuation formulas of DB and DC accounts according to the different cost methods.

Table 4.1: Valuation Formulas Based on Different Cost Methods

	DB Account ($DB(t,T)$)	DC Account ($DC(t,T)$)
EAN	$DB^E(t,T) = kTS_T$	$DC^E(t,T) = DC^E(t,t)\frac{A_T}{A_t} + \sum_{j=12t}^{12T-1} \frac{cS_{j/12}}{12}\frac{A_T}{A_{j/12}}$
PUC	$DB^P(t,T) = ktS_T$	$DC^P(t,T) = DC^P(t,t)\frac{A_T}{A_t}$
TUC	$DB^T(t,T) = ktS_t$	$DC^T(t,T) = DC^T(t,t)\frac{A_T}{A_t}$

Adopting different cost methods, four funding strategies are different in term of credited years of service, salary projection and counted contributions. Table 4.2 briefly describes the differences in the funding strategies according to different cost methods. Because of different valuation approaches, results are also different. The following chapter will show same numerical results under different funding strategies.

Table 4.2: Funding Strategies According to Cost Mehods

	Cost Method	Service	Salary Projection	Contributions
Strategy 1 & 2	EAN	Past+Future	To Retirement	Paid+Unpaid
Strategy 3	PUC	Past	To Retirement	Paid
Strategy 4	TUC	Past	No Projection	Paid

Chapter 5

Numerical Examples of Hedging Costs

5.1 Introduction

In the last chapter, we proposed four funding strategies according to three cost methods. In this chapter, we will illustrate some numerical results on the other three funding strategies. We use the same assumptions as in the last chapter: no early retirement, and salary is tradable. In our three funding strategies, the value of the DC account at a given time t only depends on the future crediting rate. Therefore, the salary growth rate only affects the DB account and the first derivative of the pension underpin guarantee with respect to salary is more stable.

5.2 Numerical Simulation

To implement the funding strategies proposed in the last chapter, we use Monte Carlo simulation to generate the salary process and the crediting rate process under P-measure. We then estimate the future salary and the value of DC account. The hedge parameters are found using Q-measure.

We simulate the salary as following,

$$S_{t+h} = S_t \cdot e^{(\mu_s - \frac{1}{2}\sigma_s^2)h + \sigma_s \sqrt{h} W_s(t)} \tag{5.1}$$

So the value of DC account can be generated by

$$DC^i(t+h, t+h) = DC^i(t,t) \cdot e^{(\mu_f - \frac{1}{2}\sigma_f^2)h + \sigma_f \sqrt{h} W_f(t)} + c \cdot hS_{t+h} \tag{5.2}$$

where $W_s(t) \sim N(0,1)$, $W_f(t) \sim N(0,1)$, and the correlation coefficient is ρ_{sf}. The value of the DC account is calculated using the three different hedging cost methods. $i = E, P, T$ denotes different cost methods, $h = 1/12$ denotes the monthly interval.

We generate 10,000 paths of the salary growth rate and the crediting rate. On each path, we dynamically hedge the pension fund every month using the last three strategies. We test monthly hedging costs for different entry ages.

Hedging this hybrid pension plan by those funding strategies, we assume the fund is rebalanced at the end of each month, leading to a monthly hedging cost cash flow at each period. Using different funding strategies, we can calculate the hedge portfolio of the salary and the underlying DC fund at time t, $\Delta_S(t)$ and $\Delta_A(t)$, respectively. Then, the value of hedging portfolio is equal to

$$H(t) = \Delta_S(t)S_t + \Delta_A(t)DC(t,t) \tag{5.3}$$

Table 5.1: Parameters for the Scenario Test

	Mean of Salary Growth	StD of Salary Growth	Discount Rate
Scenario 1	0.04	0.02	0.05
Scenario 2	0.04	0.04	0.05
Scenario 3	0.05	0.02	0.05
Scenario 4	0.04	0.02	0.05
Scenario 5	0.04	0.02	0.05
Scenario 6	0.04	0.02	0.05
Scenario 7	0.04	0.02	0.035
	Mean of Crediting Rate	StD of Crediting Rate	Contribution Rate
Scenario 1	0.1	0.2	0.125
Scenario 2	0.1	0.2	0.125
Scenario 3	0.1	0.2	0.125
Scenario 4	0.15	0.2	0.125
Scenario 5	0.1	0.15	0.125
Scenario 6	0.1	0.2	0.10
Scenario 7	0.1	0.2	0.125

One month later, at time $t + \frac{1}{12}$, the hedging portfolio established at t and brought forward to $t + \frac{1}{12}$ has value:

$$Hbf(t + \frac{1}{12}) = \Delta_S(t) S_t \frac{S_{t+\frac{1}{12}}}{S_t} + \Delta_A(t) DC(t,t) \frac{A_{t+\frac{1}{12}}}{A_t} \tag{5.4}$$

The new hedge costs $H(t + \frac{1}{12})$, so the cash flow for the guarantee at $t + \frac{1}{12}$ is

$$CF(t + \frac{1}{12}) = H(t + \frac{1}{12}) - Hbf(t + \frac{1}{12}) \tag{5.5}$$

and the monthly hedging cash flow as a proportion of salary is

$$cf(t + \frac{1}{12}) = \frac{CF(t + \frac{1}{12})}{S_{t+\frac{1}{12}}} \tag{5.6}$$

We assume the accrual rate is 1.5%, which is close to the value used, for example, in the McGill plan. The other parameters are given by scenario 1 in Table 5.1.

5.3 Hedging Costs

Table 5.2: Lump Sum Hedging Costs as Percentage of Salary Under Scenario 1

Entry Age	Strategy 2(%)	Strategy 3(%)	Strategy 4(%)
20	216.12	191.60	51.26
	(0.5837)	(0.3110)	(0.5015)
25	197.22	168.06	52.36
	(0.5549)	(0.2664)	(0.4643)
30	158.11	144.60	52.95
	(0.2394)	(0.2248)	(0.4215)
35	131.04	120.86	50.64
	(0.1978)	(0.1870)	(0.3660)
40	104.06	97.20	46.78
	(0.1600)	(0.1482)	(0.3002)
45	78.51	74.72	41.55
	(0.1230)	(0.1086)	(0.2267)
50	54.66	52.82	33.98
	(0.0872)	(0.0752)	(0.1566)
55	32.89	32.39	22.40
	(0.0516)	(0.0433)	(0.0876)

Table 5.2 shows the mean and standard error (in brackets) of lump sum hedging costs for different entry ages and three strategies with 10,000 simulations. The lump sum hedging costs are calculated by the accumulation of all hedging cash flows, i.e. $\sum_{j=1}^{12T} e^{-r\,j/12} CF(j/12)$. The costs for funding strategy 2 are always higher than costs for funding strategy 3 and 4, since it overhedges the DB underpin pension

plan. We assume no early retirement. The hedging cost low entry ages is very high. For example, the highest mean of lump sum hedging costs is 216.12% at entry age 20 in funding strategy 2.

At employees' retirement, employers pay the guaranteed retirement benefits. Employers either receive the excess of their investment, or pay extra costs to the pension plan. Because the retirement benefit is known at the last payment, the last cash flow is the amount overhedged or underhedged in the funding strategy. Employers guarantee employees the pension benefits at retirement. They do not want to overhedge or underhedge the pension benefits too much. Table 5.3 shows the last payment for employers by those three funding strategies.

Table 5.3: The Last Payment for Employers as Percentage of the Final Salary Under Scenario 1

Entry Age	Strategy 2(%)	Strategy 3(%)	Strategy 4(%)
20	-282.01	0.72	4.62
25	-263.06	0.51	4.64
30	-231.75	0.32	4.26
35	-203.01	0.73	4.97
40	-170.87	0.76	4.74
45	-135.62	0.43	3.97
50	-101.46	0.56	3.97
55	-61.01	0.86	3.80

In strategy 2, we overhedge the DB underpin pension fund. The last cash flow is negative and high. Employers will receive a huge amount of money when employees retire. In other words, employers must have paid too much before employees' retirement. In strategy 2, hedging costs are over-estimated. Later, we will mainly focus on funding strategy 3 and 4. In strategy 3, the last payment for employers is

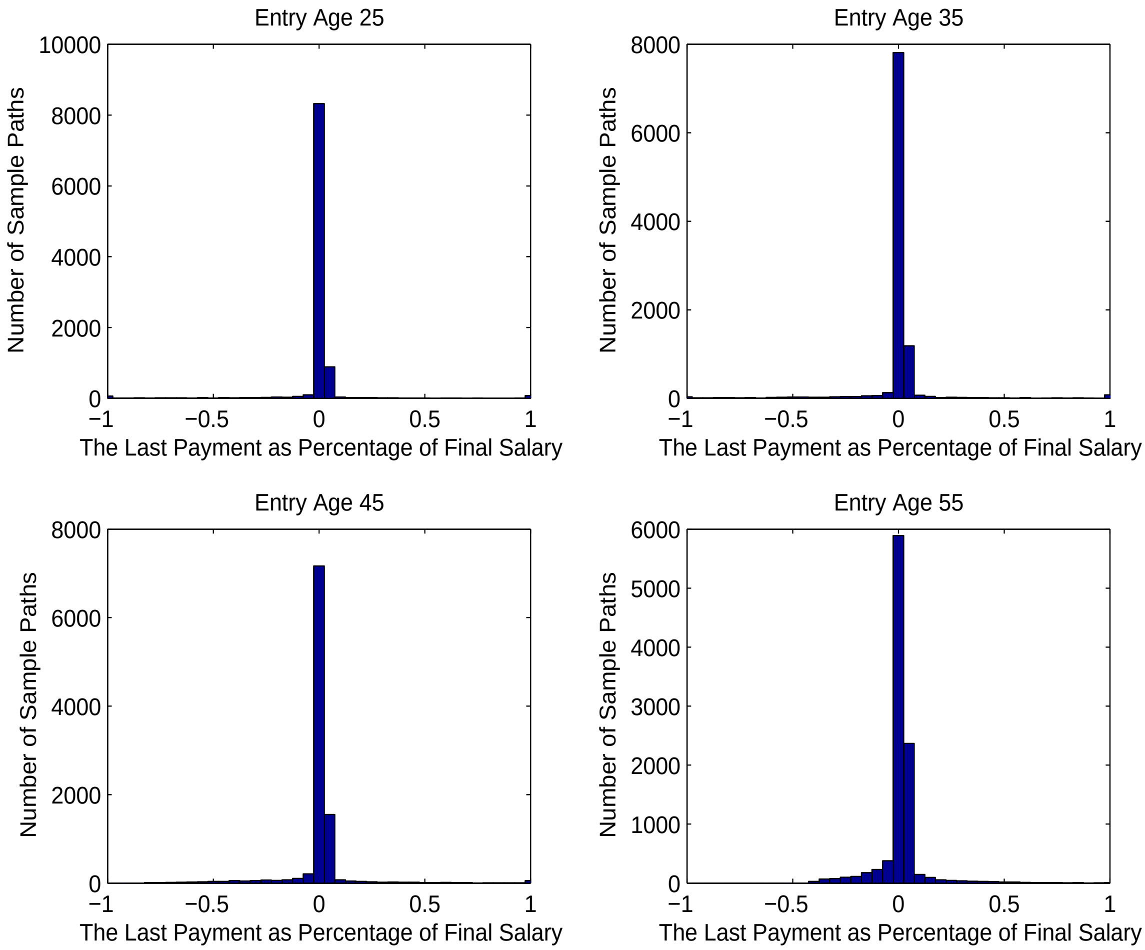


Figure 5.1: Histogram for the Last Payment in Strategy 3

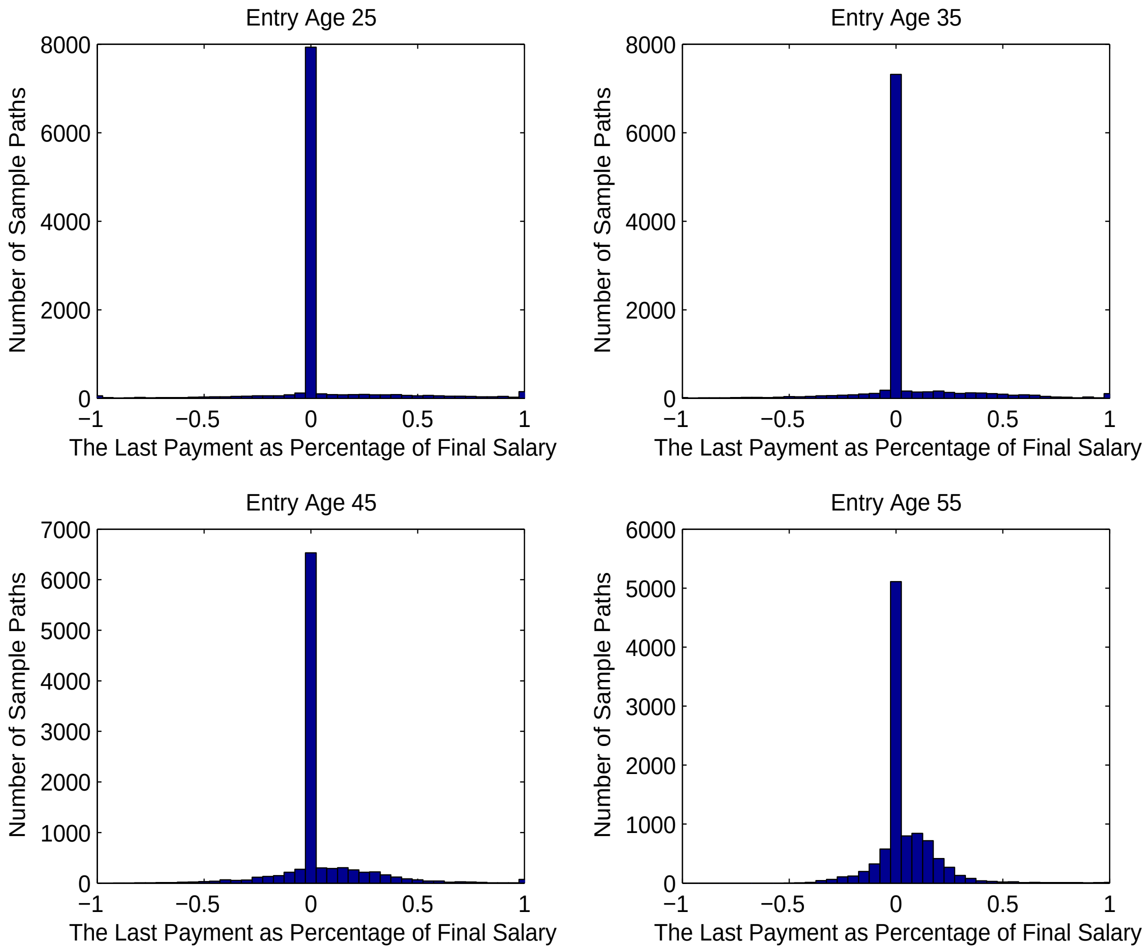


Figure 5.2: Histogram for the Last Payment in Strategy 4

Table 5.4: The Mass Probability that the Last Payment is Equal to Zero with 10,000 Simulations

Entry Age	Strategy 3	Strategy 4
20	0.7587	0.7625
25	0.7194	0.7248
30	0.6784	0.6828
35	0.6294	0.6350
40	0.5706	0.5859
45	0.5049	0.5121
50	0.4015	0.4083
55	0.2748	0.2839

very close to zero since we use the projected final salary. In strategy 4, the value of DB underpin guarantee is calculated by the current salary. Employers underestimate the salary. So the last payments for employers are around 4%, which is approximately the salary growth rate. However, the last payments in strategy 3 and strategy 4 are very volatile. We will show more figures later to illustrate the growing volatility. From employers' view, the last payment should, ideally, be close to zero. Then, employers do not have to pay extra money at employee's retirement or spend too much before the retirement to construct the hedging portfolio. Figure 5.1 and Figure 5.2 show the histogram for the last payment in strategy 3 and strategy 4 with 10,000 simulations. The mass probability that the last payment is equal to zero is shown in Table 5.4.

Table 5.2 shows us the lump sum hedging costs. Next, we amortize this lump sum cost into a monthly contribution rate, i.e. $\sum_{j=1}^{12T} e^{-r\frac{j}{12}} CF(\frac{j}{12}) / \sum_{j=1}^{12T} e^{-r\frac{j}{12}} \frac{S_{j/12}}{12}$. and get amortized monthly hedging costs as Table 5.5,

After the amortization, expected values of hedging costs in strategy 2 and 3

Table 5.5: Amortized Monthly Hedging Costs as Percentage of Monthly Salary Under Scenario 1

Entry Age	Strategy 2(%)	Strategy 3(%)	Strategy 4(%)
20	6.33	5.10	1.36
	(0.0044)	(0.0083)	(0.0133)
25	6.05	4.92	1.54
	(0.0042)	(0.0079)	(0.0135)
30	5.76	4.71	1.69
	(0.0038)	(0.0074)	(0.0135)
35	5.45	4.50	1.88
	(0.0035)	(0.0069)	(0.0135)
40	5.12	4.25	2.05
	(0.0032)	(0.0064)	(0.0131)
45	4.74	3.99	2.25
	(0.0029)	(0.0059)	(0.0124)
50	4.32	3.69	2.39
	(0.0026)	(0.0053)	(0.0111)
55	3.84	3.33	2.50
	(0.0024)	(0.0047)	(0.0093)

decrease as the entry age increases. The expected value of hedging costs in strategy 4 increases when the entry age increases. This result is sensitive to the discount rate. We discounted all hedging cash flows back to calculate the present value of the hedging cost, and divided by the salary-related annuity which is also determined using the discount rate. In scenario 7, we keep all parameters same except that the discount rate changes to 3.5%. When the discount rate decreases, the annuity rate increases. However, we do not change it in scenario 7 since it is hard to estimate the change of annuity rates in the discount rate.

Table 5.6 shows that the expected values of amortized monthly hedging costs

Table 5.6: Amortized Monthly Hedging Costs as Percentage of Monthly Salary Under Scenario 7

Entry Age	Strategy 2(%)	Strategy 3(%)	Strategy 4(%)
20	5.93	4.61	2.60
	(0.0048)	(0.0091)	(0.0191)
25	5.72	4.51	2.76
	(0.0045)	(0.0087)	(0.0208)
30	5.49	4.38	2.89
	(0.0041)	(0.0081)	(0.0198)
35	5.23	4.25	3.06
	(0.0036)	(0.0075)	(0.0188)
40	4.95	4.05	3.12
	(0.0034)	(0.0069)	(0.0172)
45	4.62	3.85	3.19
	(0.0031)	(0.0063)	(0.0154)
50	4.25	3.60	3.21
	(0.0027)	(0.0056)	(0.0131)
55	3.81	3.29	3.12
	(0.0025)	(0.0049)	(0.0105)

decrease with age at entry in strategy 2 and 3. In strategy 4, the expected value of amortized monthly hedging cost increases when the interest rate decreases from 5% to 3.5%. Although we amortize the monthly hedging costs by dividing the salary-related annuity factor, this cost is still very sensitive to the discount rate.

To avoid the effect on the discount rate, we propose another way to calculate the monthly hedging cost. We already have dynamic hedging cash flows at each period. Instead of calculating the present value of cash flows and amortizing it, we calculate the monthly cost by using the monthly cash flow as a proportion of the salary in that month. And then, we find the average of monthly costs.

Table 5.7: Average Monthly Hedging Costs Under Scenario 1

Entry Age	Strategy 2(%)	Strategy 3(%)	Strategy 4(%)
20	6.13	4.77	1.50
	(0.0017)	(0.0081)	(0.0151)
25	5.88	4.65	1.66
	(0.0018)	(0.0078)	(0.0150)
30	5.61	4.49	1.80
	(0.0019)	(0.0073)	(0.0148)
35	5.33	4.32	1.97
	(0.0019)	(0.0068)	(0.0144)
40	5.02	4.12	2.12
	(0.0020)	(0.0063)	(0.0138)
45	4.67	3.90	2.31
	(0.0020)	(0.0058)	(0.0128)
50	4.26	3.63	2.42
	(0.0020)	(0.0052)	(0.0113)
55	3.79	3.30	2.52
	(0.0020)	(0.0046)	(0.0093)

The first amortized monthly hedging cost is calculated by the expected present value of future cash flows over the expected present value of future monthly salaries, i.e. $\sum_{j=1}^{12T} e^{-r\,j/12} CF(j/12)/\sum_{j=1}^{12T} e^{-r\,j/12}\frac{S_{j/12}}{12}$, where $CF(j/12)$ and $S_{j/12}$ are the cash flow and the monthly salary at time $j/12$, respectively, and r is the discount rate. The second average monthly hedging cost is calculated by the expected cost of the cash flow over the monthly salary at each period, i.e. $\sum_{j=1}^{12T} cf(j/12) = \sum_{j=1}^{12T} CF(j/12)/S_{j/12}$. There is no discount rate in the second one.

The average monthly hedging costs are different with the amortized monthly hedging costs. The amortized monthly hedging costs can be implicitly considered as a monthly contribution rate, which is even for each month. However, the average

Table 5.8: Monthly Hedging Costs as Percentage of Salary(%) Under Strategy 2

Entry Age	Beginning	Middle	Retirement
20	8.59	6.42	-282.01
25	8.24	6.15	-263.06
30	7.86	5.90	-231.75
35	7.43	5.55	-203.01
40	6.96	5.22	-170.87
45	6.42	4.85	-135.62
50	5.80	4.42	-101.46
55	5.05	3.93	-61.01

monthly hedging costs are time-dependent. It changes from time to time. Table 5.8, Table 5.9, and Table 5.10 show expected hedging costs at different time period in the whole plan.

Given the entry age, we show the hedging cost flows in Figure 5.3. Figure 5.3 does not include the last payment.

We first consider the sample distribution of the average hedging cost. Given the entry age, we show the histogram for the distribution of average hedging costs in strategy 3 and 4. Figure 5.4 and Figure 5.5 show the distributions of 10,000 paths in funding strategy 3 and 4 for entry ages 25, 35, 45 and 55. We calculate the probability density estimate of the 10,000 sample paths in Figure 5.6 and Figure 5.7. The estimate is based on a normal kernel function, using a window parameter which is optimal for estimating normal density. The density is evaluated at 100 equally-spaced points covering the range of the data. In Figure 5.6, density functions for earlier entry age have lower summit and heavier tail. All 4 estimated density functions have the similar shape to the normal distribution. The earlier the entry age, the higher the mean. In Figure 5.7, we find strategy 4 has the opposite

Table 5.9: Monthly Hedging Costs as Percentage of Salary(%) Under Strategy 3

Entry Age	Beginning	Middle	Retirement
20	8.59	5.11	0.72
25	8.24	4.97	0.51
30	7.85	4.83	0.32
35	7.42	4.61	0.73
40	6.95	4.38	0.76
45	6.41	4.15	0.43
50	5.79	3.86	0.56
55	5.04	3.51	0.86

Table 5.10: Monthly Hedging Costs as Percentage of Salary(%) Under Strategy 4

Entry Age	Beginning	Middle	Retirement
20	0.15	1.15	4.62
25	0.21	1.26	4.64
30	0.29	1.47	4.26
35	0.41	1.68	4.97
40	0.56	1.88	4.74
45	0.78	2.12	3.97
50	1.08	2.29	3.97
55	1.50	2.43	3.80

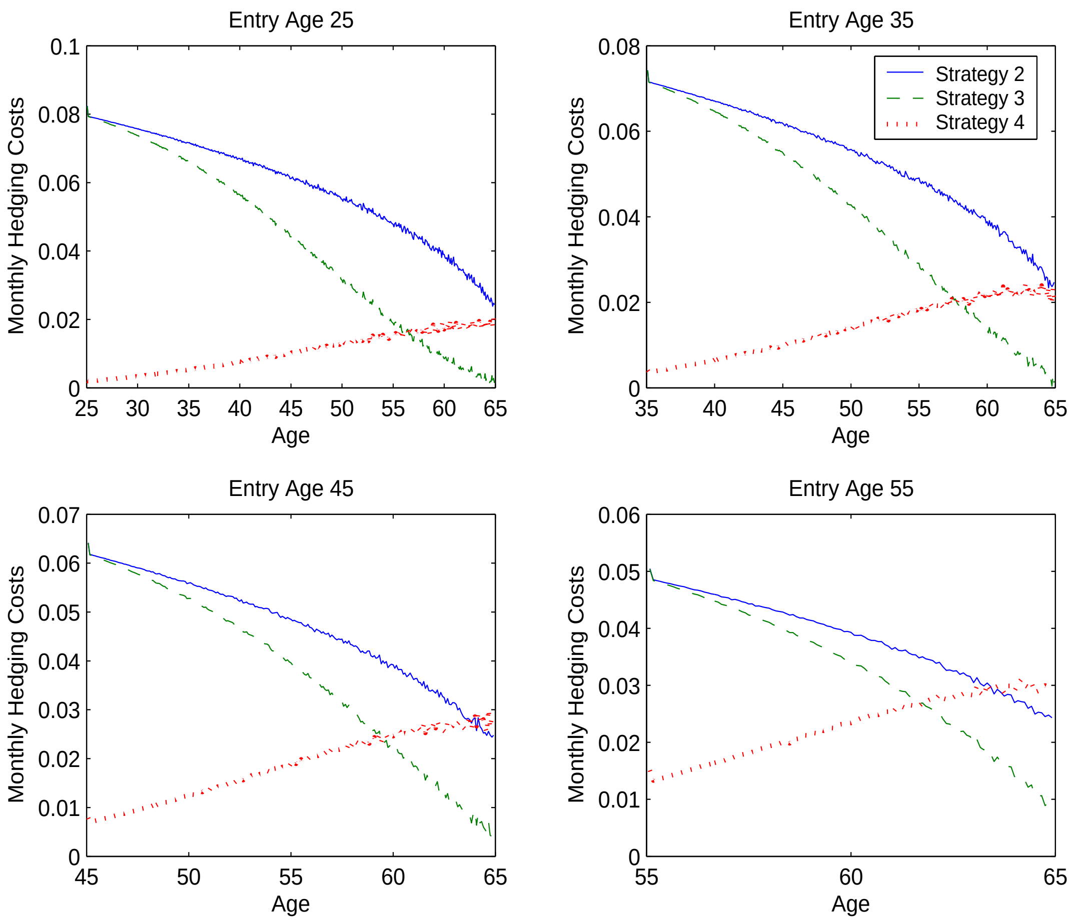


Figure 5.3: Hedging Cost Flows for Average Monthly Hedging Cost

properties. The earlier the entry age, the lower the mean and the lighter tail. These properties are identical to results from Table 5.7.

Next, we test the scheme of monthly hedging costs. Given the entry age, we use Monte Carlo simulation with 10,000 paths to calculate monthly hedging cash flows and monthly hedging costs. We pick 4 different entry ages here.

Figure 5.8 and Figure 5.9 illustrate 20 sample paths for each entry age in strategy 3 and 4, respectively. The lower bounds are shown very clearly on Figure 5.8 and Figure 5.9. The volatilities close to the retirement are very high. We draw the 90%, 50%, and 10% quantiles, and mean of monthly hedging costs on the same graph. Figure 5.10 and Figure 5.11 show that the mean and the median of monthly hedging costs are close, and they are closer to the 10% quantile than 90% quantile. Hence, the mean of monthly hedging costs are always above the median.

5.4 Scenario Tests

As shown in Table 5.1, we have seven scenarios. In this section, we will run 10,000 simulations on each scenario and test the sensitivity of hedging costs to each variable. We change the assumptions for the contribution rate, the crediting rate and the salary growth rate. The results are given in Table 5.11 and 5.12. The numbers in brackets are the standard errors.

Table 5.1 shows all parameters for seven scenarios. Considering scenario 1 as the benchmark, we calculate the average monthly hedging costs for different parameters. Since only the discount rate is changed in scenario 7, the average monthly hedging costs will not change.

Scenario testing results for funding strategy 3 are shown in Table 5.11. In

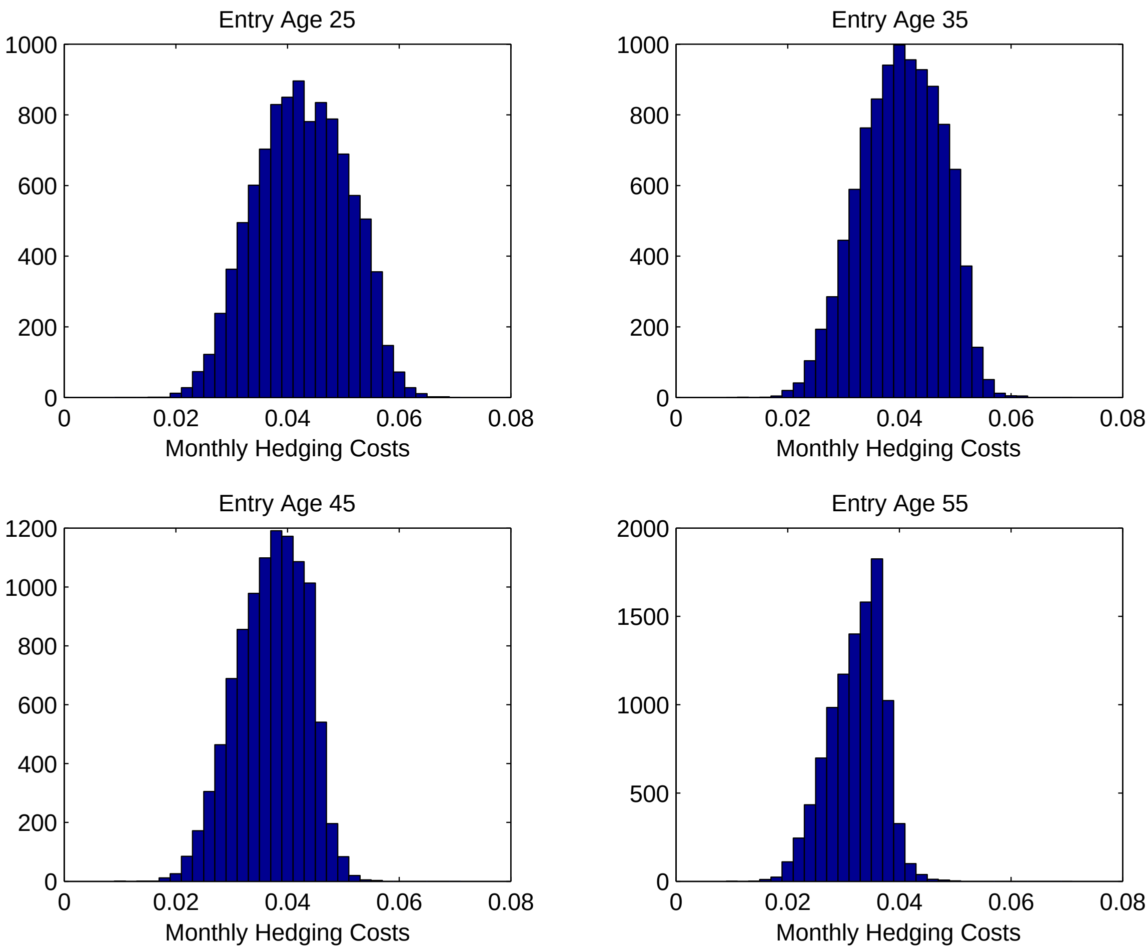


Figure 5.4: Histogram for Average Monthly Hedging Cost in Strategy 3

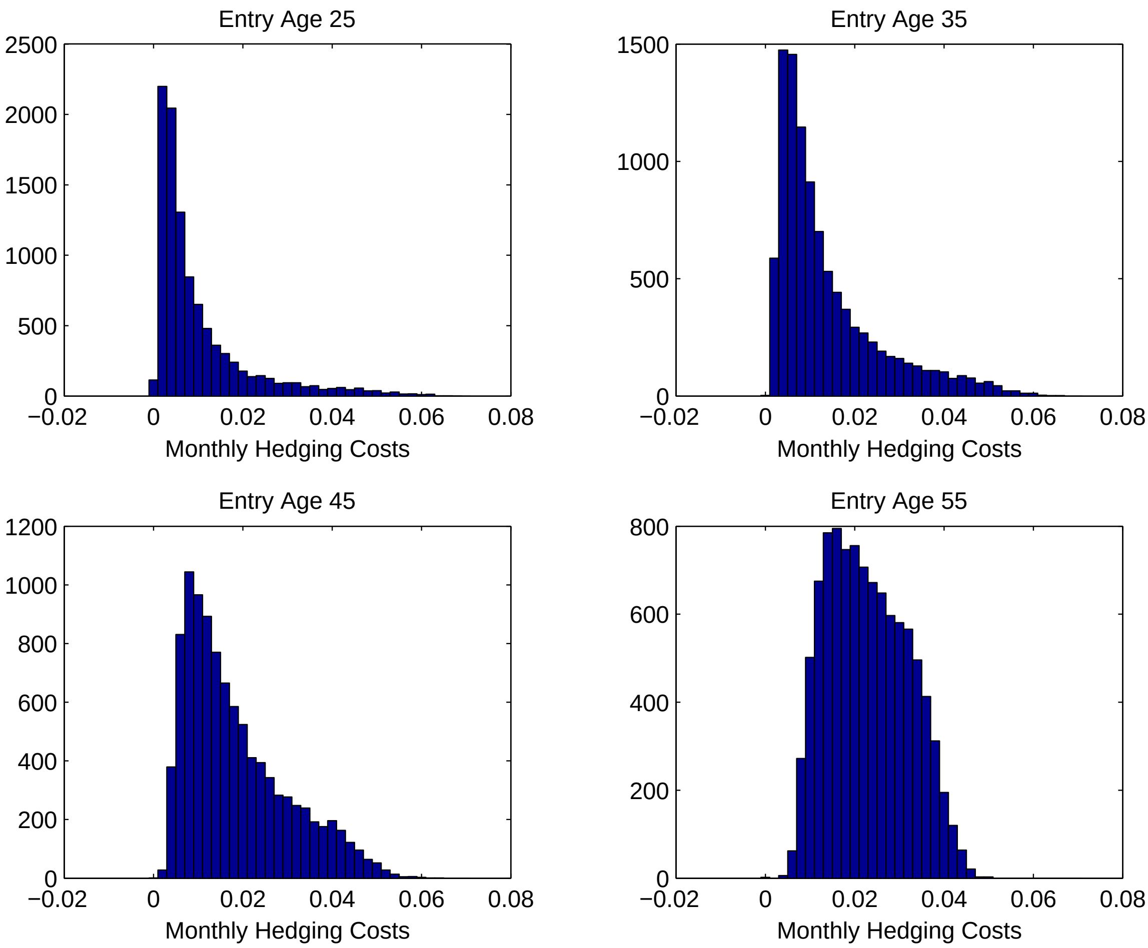


Figure 5.5: Histogram for Average Monthly Hedging Cost in Strategy 4

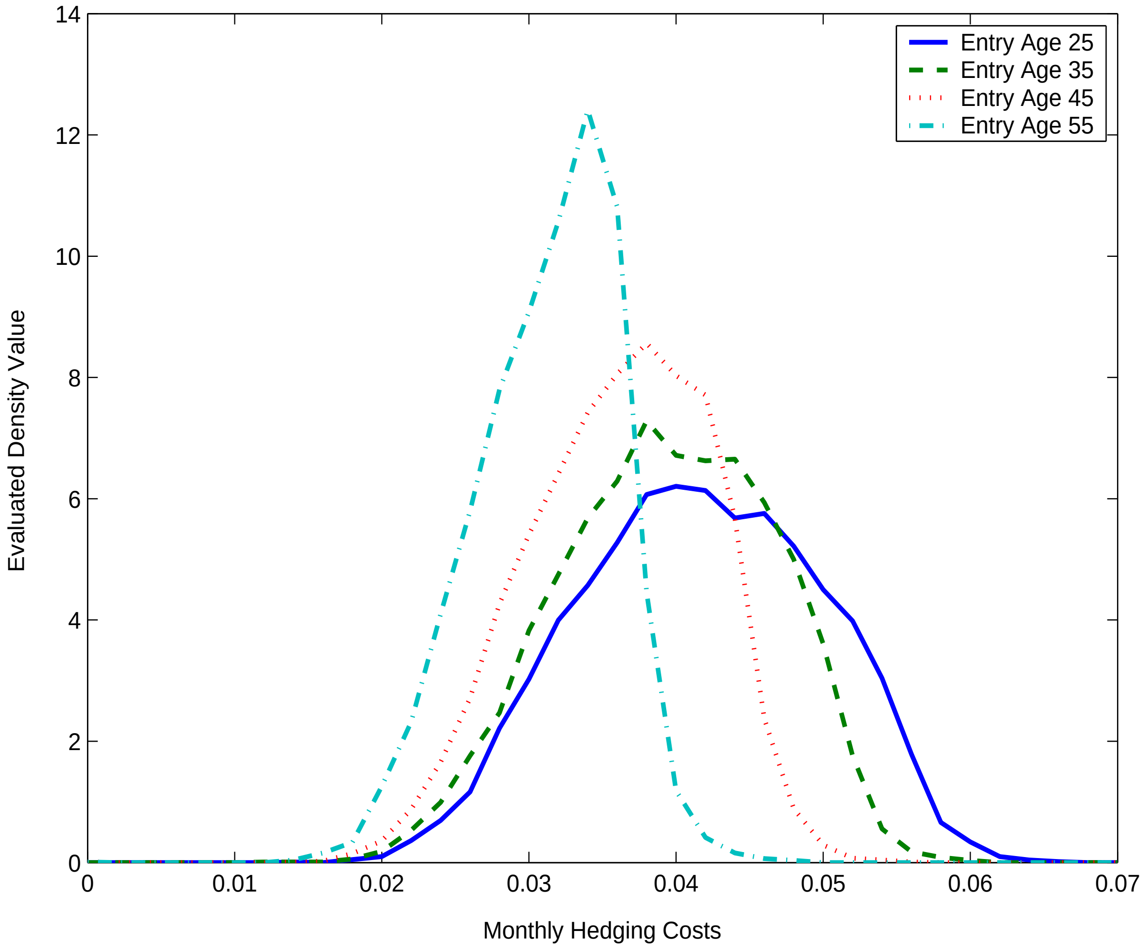


Figure 5.6: Estimated Density for Average Monthly Hedging Cost in Strategy 3

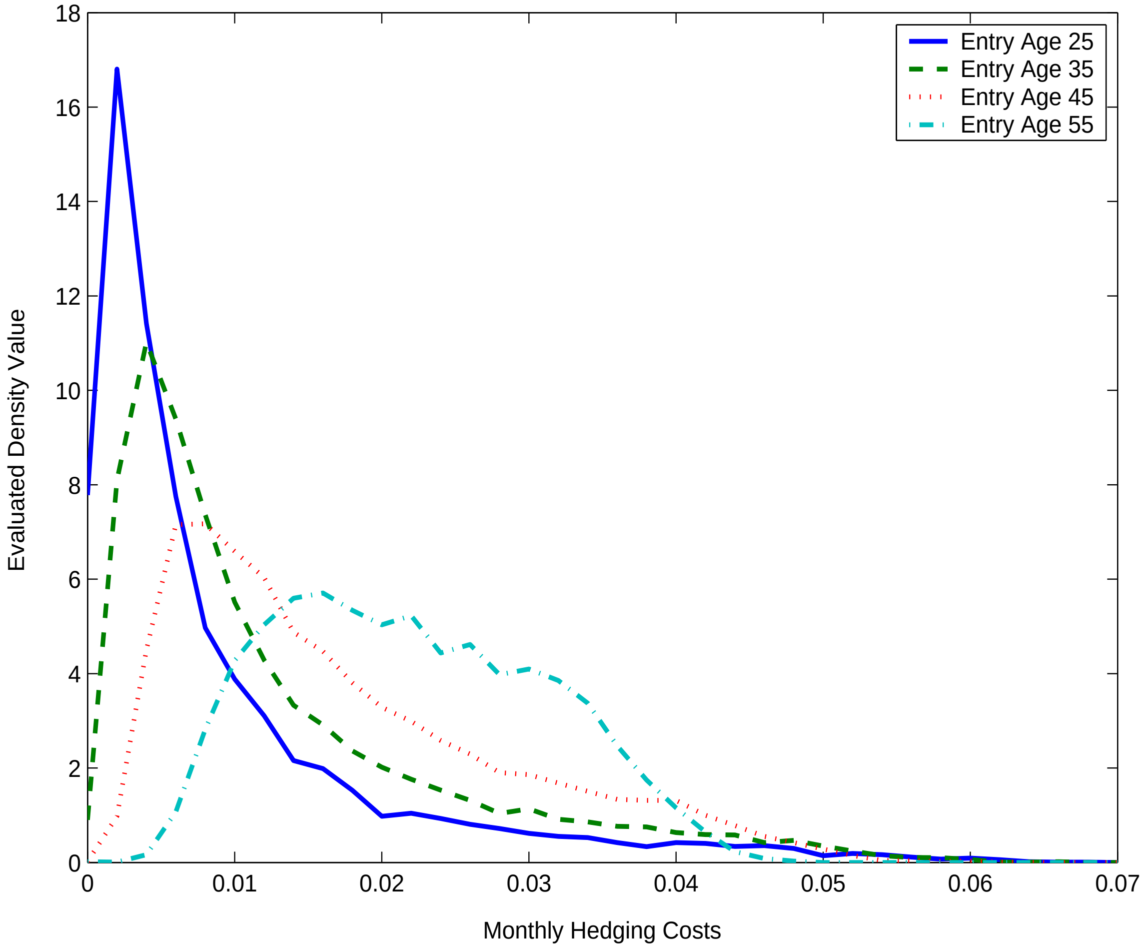


Figure 5.7: Estimated Density for Average Monthly Hedging Cost in Strategy 4

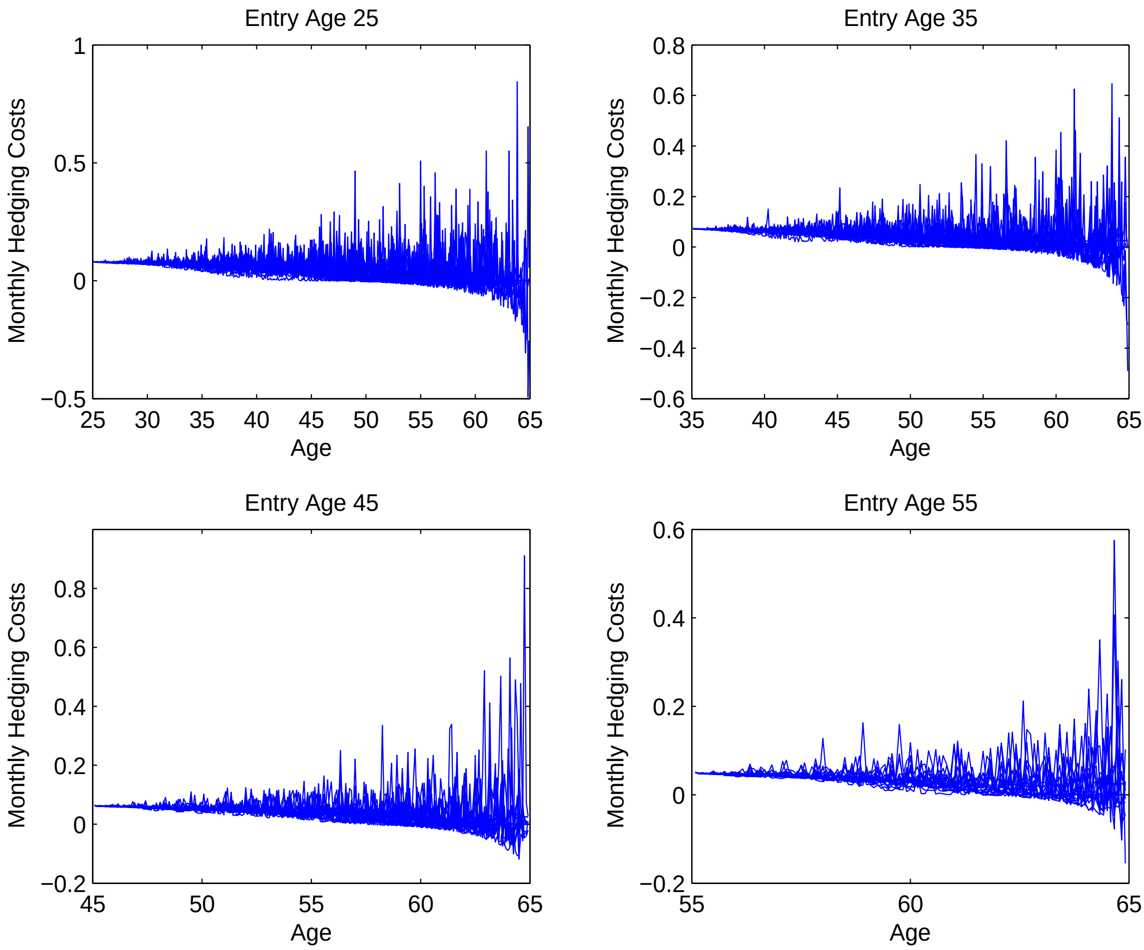


Figure 5.8: Monthly Hedging Costs on 20 Sample Paths in Strategy 3, as % of Salary

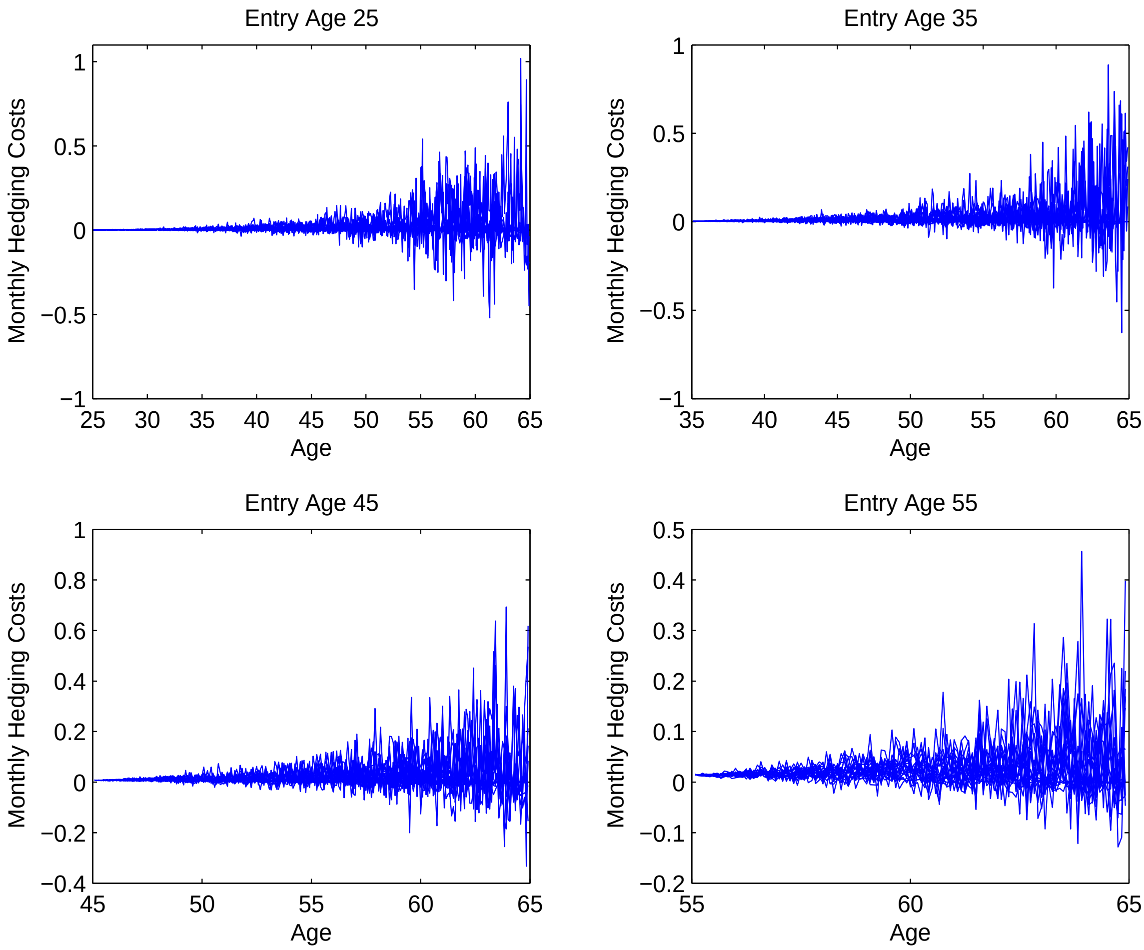


Figure 5.9: Monthly Hedging Costs on 20 Sample Paths in Strategy 4, as % of Salary

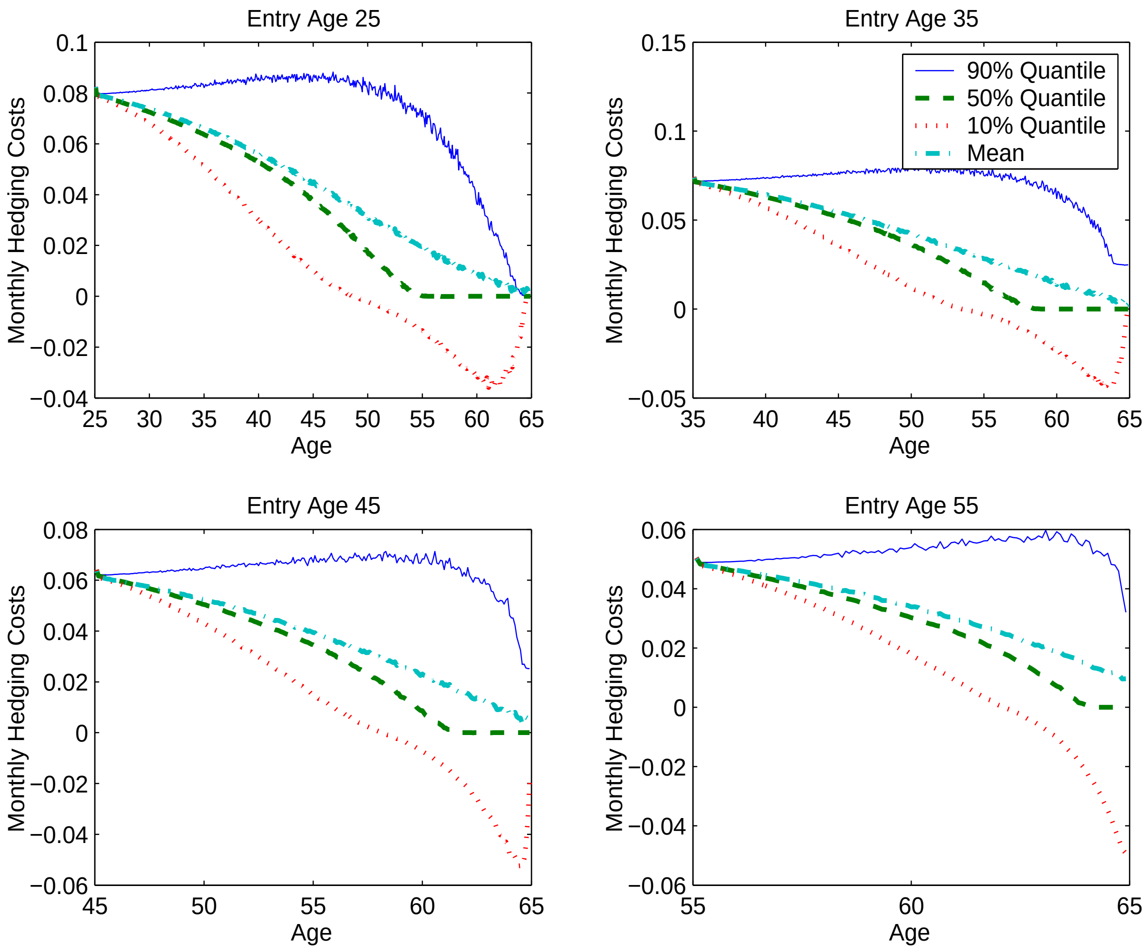


Figure 5.10: Quantile of Monthly Hedging Costs in Strategy 3

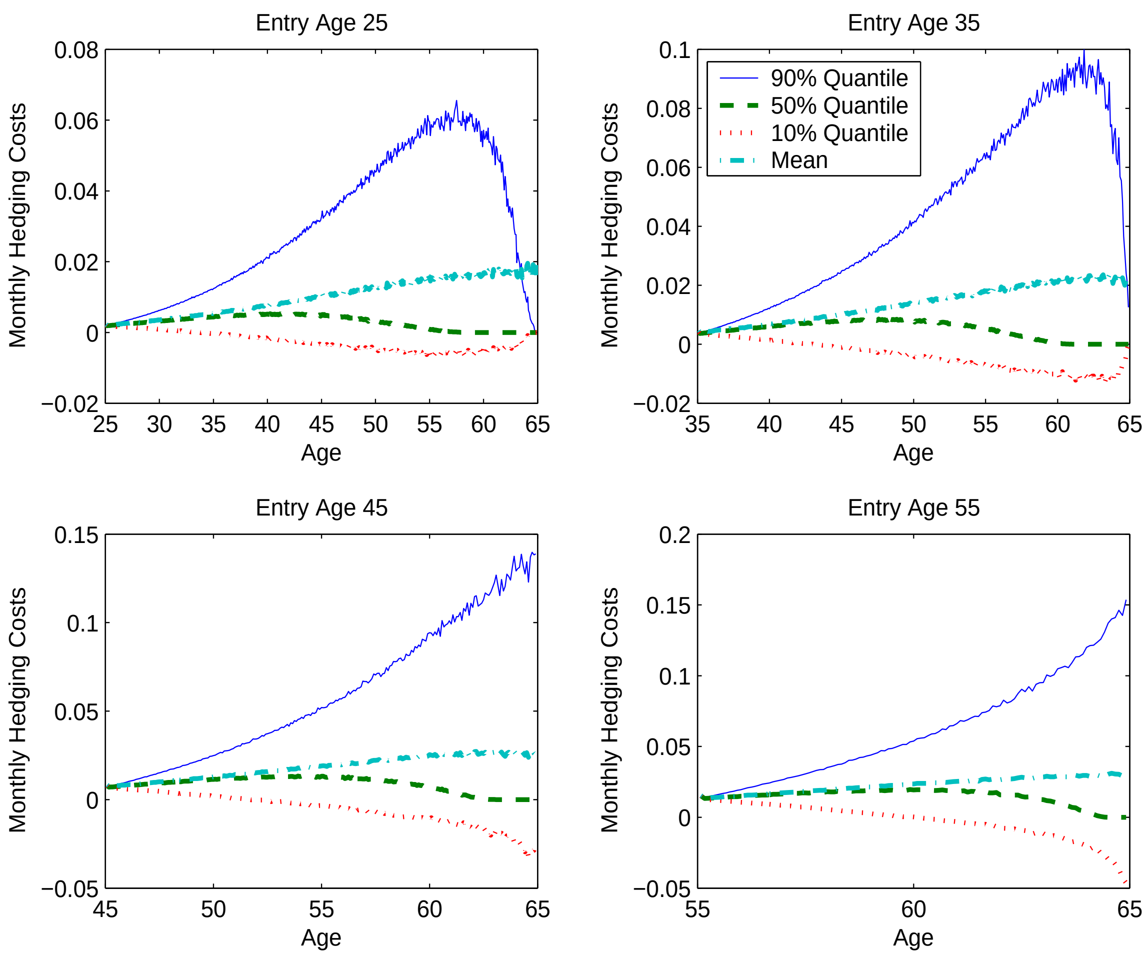


Figure 5.11: Quantile of Monthly Hedging Costs in Strategy 4

Table 5.11: Average Monthly Hedging Costs for the Scenario Test as Percent of Monthly Salary in Funding Strategy 3

Entry Age	Scenario 1	Scenario 2	Scenario 3	Scenario 4	Scenario 5	Scenario 6
20	4.77	4.82	4.99	3.69	3.54	5.74
	(0.0081)	(0.0080)	(0.0078)	(0.0077)	(0.0067)	(0.0101)
25	4.65	4.68	4.84	3.66	3.47	5.68
	(0.0078)	(0.0076)	(0.0073)	(0.0076)	(0.0065)	(0.0096)
30	4.49	4.52	4.65	3.61	3.40	5.58
	(0.0073)	(0.0073)	(0.0068)	(0.0074)	(0.0063)	(0.0093)
35	4.32	4.34	4.45	3.56	3.33	5.49
	(0.0068)	(0.0067)	(0.0064)	(0.0073)	(0.0060)	(0.0088)
40	4.12	4.15	4.23	3.47	3.23	5.39
	(0.0063)	(0.0063)	(0.0060)	(0.0069)	(0.057)	(0.0083)
45	3.90	3.93	3.99	3.38	3.11	5.28
	(0.0062)	(0.0062)	(0.0055)	(0.0064)	(0.0053)	(0.0077)
50	3.63	3.65	3.70	3.23	2.98	5.16
	(0.0052)	(0.0055)	(0.0050)	(0.0059)	(0.0048)	(0.0069)
55	3.30	3.33	3.35	3.029	2.80	5.05
	(0.0046)	(0.0048)	(0.0045)	(0.0052)	(0.0042)	(0.0060)

scenario 2 and scenario 3, we change the mean and the standard deviation of the salary growth rate. The results are similar to scenario 1. The average monthly hedging costs and standard errors are almost same. This is because we take the ratio of monthly hedging cash flows and monthly salaries, so the effect of the salary growth rate is cancelled. When the mean of salary growth rate increases or the standard deviation increases, the average monthly hedging costs increase slightly. Since the salary growth rate is objectively determined by employers, this shows that a salary policy change will not cause significant effect on the funding strategy. We change the mean and the standard deviation of the crediting rate in scenario 4 and scenario 5, respectively. In scenario 4, the mean crediting rate increases from 10% to 15%. The average monthly hedging costs and standard errors decrease. In scenario 5, the standard deviation of the crediting rate goes down from 20% to 15%. The average monthly hedging costs and standard errors go down too. The results from those two scenarios are intuitive. When the crediting rate increases or the investment returns are better, on average, the DC benefit will be higher, on average, so the hedging costs should be lower. When the investment market is less volatile, the investment risk also decreases. This is the reason that the average and standard error of monthly hedging costs are both decreasing. On the opposite side, the average and standard error of monthly hedging costs will go up when the investment market is poor or more volatile. The contribution rate is changed from 0.125 to 0.1 in scenario 6. The less contribution to the defined contribution plan, the higher the guarantee risk and the higher hedging costs. The marginal costs are higher for higher entry ages, since there is less time to accumulate contributions into the DC account.

In Table 5.12, we show scenario testing results for funding strategy 4. In scenario 2, the standard deviation of the salary growth rate goes up. Monthly hedging costs also increase slightly as in Table 5.11. However, when the mean of the salary growth rate grows up in scenario 3, monthly hedging costs change a lot, especially for lower entry ages. Since we apply the current salary instead of projected final salary to predict the future retirement benefit, the monthly hedging cost is very sensitive to the change in salary growth rate. In scenario 4 and 5, the changes in the crediting rate cause the same results as in funding strategy 3. Monthly hedging costs decrease when the mean of crediting rate increases or the standard deviation decreases. When the contribution rate changes in scenario 6, monthly hedging costs do not change a lot. For example, the contribution goes down from 12.5% to 10%. Average monthly hedging costs only change about 1.5% for entry age 55 and 0.4% for entry age 20.

Table 5.12: Average Monthly Hedging Costs for the Scenario Test as Percent of Monthly Salary in Funding Strategy 4

Entry Age	Scenario 1	Scenario 2	Scenario 3	Scenario 4	Scenario 5	Scenario 6
20	1.50	1.51	2.34	0.35	0.48	2.17
	(0.0151)	(0.0162)	(0.0210)	(0.0046)	(0.0081)	(0.0194)
25	1.66	1.66	2.50	0.47	0.58	2.40
	(0.0150)	(0.0163)	(0.0207)	(0.0055)	(0.0087)	(0.0192)
30	1.80	1.80	2.58	0.61	0.73	2.65
	(0.0148)	(0.0161)	(0.0196)	(0.0064)	(0.0096)	(0.0191)
35	1.97	1.95	2.71	0.80	0.92	2.92
	(0.0144)	(0.0155)	(0.0186)	(0.0075)	(0.0101)	(0.0185)
40	2.12	2.14	2.82	1.02	1.11	3.21
	(0.0138)	(0.0152)	(0.0173)	(0.0069)	(0.0105)	(0.0175)
45	2.31	2.32	2.91	1.31	1.34	3.53
	(0.0128)	(0.0142)	(0.0155)	(0.0091)	(0.0105)	(0.0162)
50	2.42	2.44	2.98	1.61	1.63	3.87
	(0.0113)	(0.0128)	(0.0134)	(0.0092)	(0.0101)	(0.0142)
55	2.52	2.54	2.94	1.94	1.89	4.23
	(0.0093)	(0.0108)	(0.0106)	(0.0088)	(0.0088)	(0.0114)

Chapter 6

Salary, Inflation, and Equity Returns

6.1 Objectives

To reduce risks in the pension plan, both financial economists and traditional actuaries start to hedge the pension plan liabilities. Many models have been proposed to help pension fund managers match the future asset and liability cash flows.

Pension fund managers construct portfolios to hedge liabilities based on financial assets, such as stocks, bonds, options, and swaps. For example, to hedge their pension liabilities, the Boots Company invested all the pension plan fund in bonds (Ralfe et al. (2002)). One problem with this method is that there are sometimes not enough eligible bonds, since the pension plan liabilities are generally very long term. Also, the nature of the liabilities in a final salary plan is that they are increasing with salaries. We will explore whether one of the major asset classes offers a salary hedge that replicates salary, at least to some extent.

This is also relevant for the DB underpin guarantee. In Chapter 4, we assumed the salary is tradable. However, it is hard to replicate salaries in practice. In this chapter, we first consider the relationship between salaries and three other financial indexes: bonds, stocks, and inflation. Then, we construct a tractable model for salary and inflation.

6.2 Data Analysis

The data we consider in this section is Canadian annual data from 1954 to 2003, from *2002 Report on Canadian Economic Statistics 1924-2003* (2003). We use the Canadian data to determine the correlations between salary growth and three other asset classes: stock indexes, long bond indexes and the inflation index.

Assuming the values of four indexes were 100 each in 1954, we can draw a graph to see the accumulated values of these four indexes in the latest 50 years, from 1954 to 2003.

From Figure 6.1, the salary and inflation indexes have similar shapes, but it is hard to visualize a relationship between the stock index, long bond index and the salary index. We can see that stocks have increased more than bonds, and that salaries and inflation rates have increased the least. Also, the mean increase rate of salaries is higher than inflation rates.

To see the relationship more clearly, we compare the annual increase rate of these four indexes. Since the stock index and the long bonds are more volatile than the salary and the inflation, we use a 7-year average value of stock index and long bonds.

Salary growth and inflation appear quite similar in Figure 6.2. The correlation

Figure 6.1: The accumulated values of four indexes in the latest 50 years

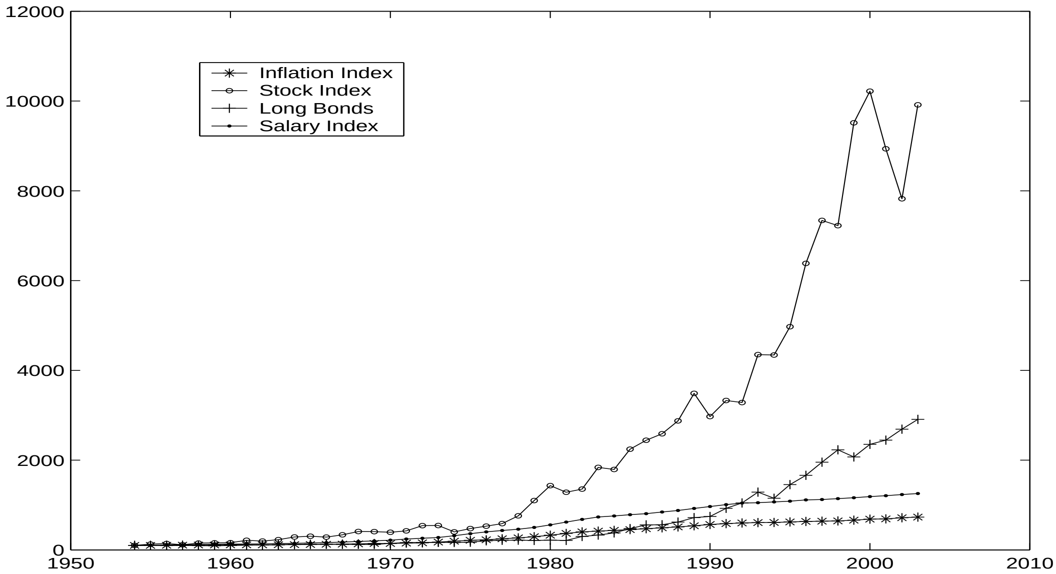
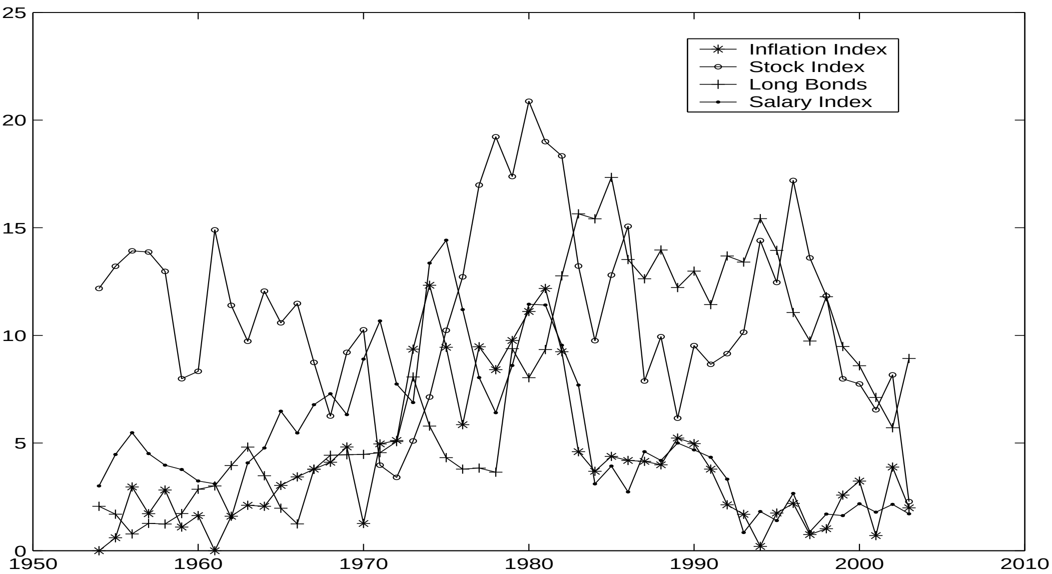


Figure 6.2: The annual increase rates of four indexes in the latest 50 years



between the salary growth rates and the inflation rate is 0.7905.

We also consider changes of these four assets chosen in the latest 30 years, which is from 1974 to 2003. Figure 6.3 and Figure 6.4 show the accumulated values and the annual increase rates of these four indexes, respectively.

In these two figures, we can see more recent relationship between the salary and the inflation clearly. The correlation coefficient between the increase rate of these two indexes is 0.9041.

We also consider correlations between the salary and another two assets: stocks and long bonds. We find that stocks also have a positive correlation with salaries, though much lower than the inflation rate. Long bonds have a negative correlation with salaries.

Table 6.1: The correlation between salaries and inflation, stocks and long bonds in the latest 50 years

Corr Coefficient	Inflation Index	Stock Index	Long Bonds
Salary Index	0.7905	0.1540	-0.2441

Table 6.2: The correlation between salaries and inflation, stocks and long bonds in the latest 30 years

Corr Coefficient	Inflation Index	Stock Index	Long Bonds
Salary Index	0.9041	0.3758	-0.4631

Figure 6.3: The accumulated values of four indexes in the latest 30 years

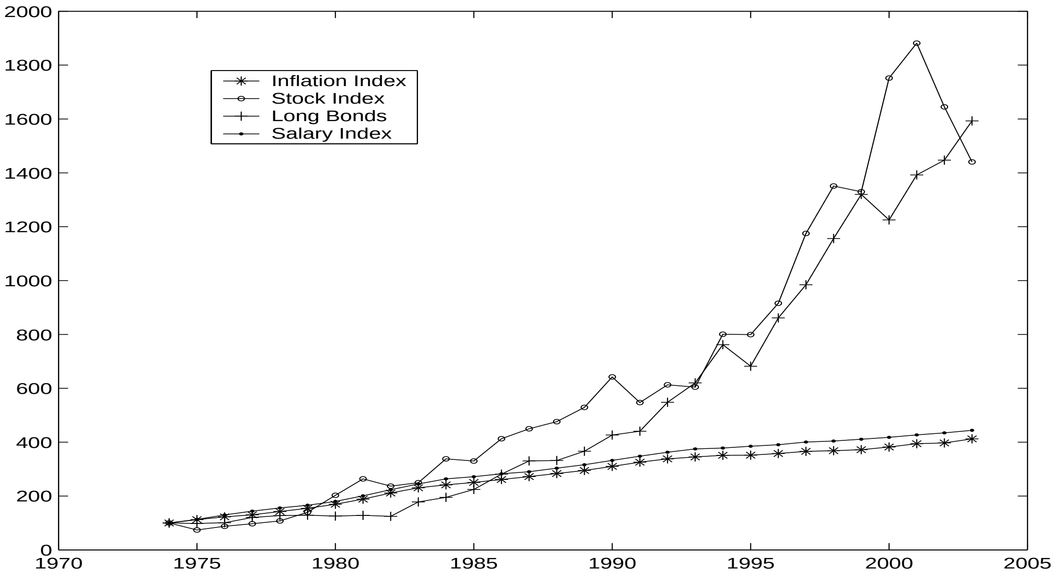
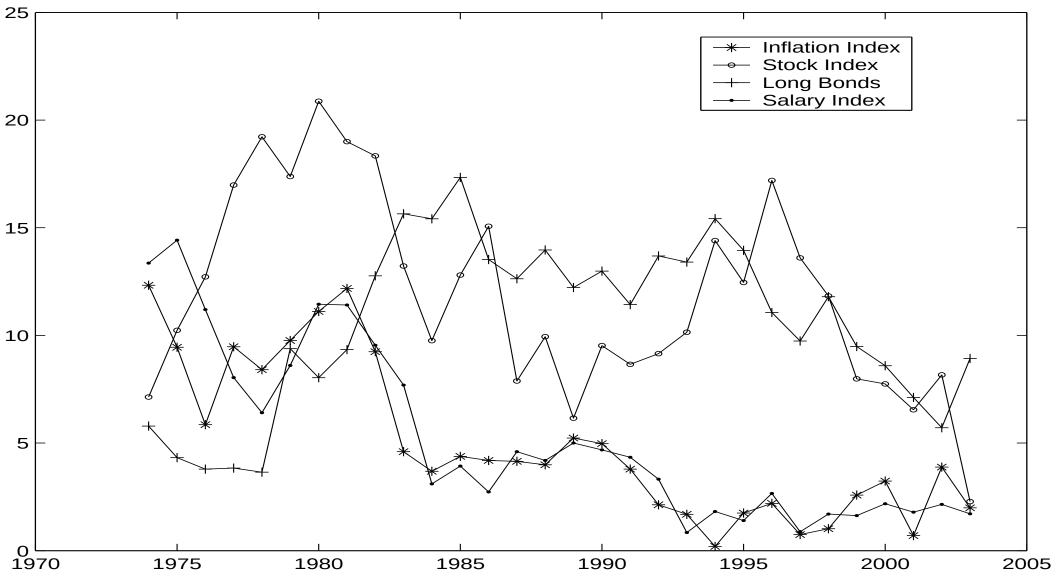


Figure 6.4: The annual increase rates of four indexes in the latest 30 years



6.3 Selection of Hedging Assets

Traditionally, actuaries do not really try to hedge. Due to lower interest rates and the growing longevity, actuaries start to worry more about their future liabilities. In current approaches for making provision for pension plan liabilities, almost every kind of financial asset is used to construct the pension plan asset portfolio. The selection of the hedging assets should be determined by characteristics of the benefit cash flows.

If the amount of future benefits is fixed, it is easy (at least in principle) to hedge the liabilities with long bonds. The Boots Company moved their pension plan fund to a 100% bond portfolio in July 2001, even though Boots plan was a final salary plan. They used bonds to match salary related benefits. (Subsequently, they closed the DB plan and opened a DC plan.) Since the default risk of high grade bonds is low, the value of assets at retirement is almost certain. The fund managers can match values of assets and expected value of liabilities by using government bonds, if sufficient long term bonds exist. If the value of future benefits is variable, bonds with fixed return can not be used to hedge the liabilities perfectly. A combination of risky assets and risk-free assets may offer a better hedge.

In the DB underpin pension plan, pension benefits depend on two accounts and both are related to the salary, especially the defined benefit account. The DB account is defined by a formula with salaries. The DC account is accumulated by the monthly contribution, which is proportional to the monthly salary. We therefore require the increase rate of the hedging portfolio to be highly correlated with the salary growth. The inflation rate is highly correlated to salary growth. It is a good choice to use the inflation to hedge the salary. In the financial market, it is not possible to trade the inflation. However, inflation-linked bonds, which are indexed with the inflation index, have been introduced in the U.K., Canada, and the U.S., and can be used to hedge the inflation risk, in principle.

However, it is hard to use the inflation-linked bond in our model. First, inflation-linked bonds have been introduced recently, especially in the North America. Second, inflation-linked bonds are more volatile than inflation, so using inflation introduces basis risk in our model. In practice, we can not buy and sell inflation. But we will consider inflation to construct the model, and assume that pension fund managers can use inflation-linked bonds to construct the hedging portfolio. Next, we will consider two possible models for salary and inflation.

6.3.1 A Vector Autoregressive Model

Wilkie(1995) present a vector autoregressive(VAR) model to connect salary and inflation. A first order VAR model would be:

$$
\begin{aligned}
l_t &= \mu_l + A_{11}(l_t - \mu_l) + A_{12}(s_t - \mu_s) + \sigma_l(t)W_l(t) && (6.1)\\
s_t &= \mu_s + A_{21}(l_t - \mu_l) + A_{22}(s_t - \mu_s) + \sigma_s(t)W_s(t) && (6.2)
\end{aligned}
$$

where $W_l(t)$ and $W_s(t)$ follow on standard Brownian distribution, $N(0,1)$. They are correlated, with correlation coefficient ρ_{ls}. Alternatively, we can replace $W_s(t)$ by $\rho_{ls}W_l(t) + \sqrt{1-\rho_{ls}^2}W(t)$, where $W_l(t)$ and $W(t)$ are independent.

Let $\beta = \rho_{ls} \cdot \sigma_s(t)/\sigma_l(t)$, we have

$$
\begin{aligned}
\sigma_s(t)W_s(t) &= \sigma_s(t) \cdot \rho_{ls}W_l(t) + \sigma_s(t) \cdot \sqrt{1-\rho_{ls}^2}W(t)\\
&= \sigma_l(t) \cdot \frac{\rho_{ls}\sigma_s(t)}{\sigma_l(t)}W_l(t) + \sigma_s(t) \cdot \sqrt{1-\rho_{ls}^2}W(t)\\
&= \sigma_l(t) \cdot \beta W_l(t) + \sigma_s(t) \cdot \sqrt{1-\rho_{ls}^2}W(t) && (6.3)
\end{aligned}
$$

From equation (6.1), we have

$$\sigma_l(t)W_l(t) = (l_t - \mu_l) - A_{11}(l_t - \mu_l) - A_{12}(s_t - \mu_s) \tag{6.4}$$

Plug this and equation (6.3) into equation (6.2), we can rearrange l_t to get

$$s_t - \mu_s = \beta(l_t - \mu_l) + (A_{21} - A_{11})(l_t - \mu_l) + (A_{22} - \beta A_{12})(s_t - \mu_s) + \sigma_s(t) \cdot \sqrt{1 - \rho_{ls}^2} W(t) \tag{6.5}$$

Empirical studies indicate that the volatility of the inflation is not deterministic. The weakness of this model is that the volatilities of the inflation rate and the salary growth rate are assumed to be constant.

6.3.2 Connection between Inflation and Salary

The last two models for inflation and salary both give us unsatisfactory results. We need to find a practicable and feasible model to connect stochastic inflation and salary growth. Wilkie(1995) finds that logarithms of salary inflation and the salary are indeed cointegrated, that is, it is reasonable to model the difference in logarithms as a stationary series. Cairns et al.(2006) also consider the incomplete market issue. They assume the salary is not fully tradable. It can be split into the hedgeable part and non-hedgeable part.

Following Cairns et al.(2006), we assume the pension plan member has a salary at time t of S_t, which is governed by equation (6.6).

$$S_t = S_t^H \cdot S_t^N \tag{6.6}$$

where

$$S_t^H = S_0^H \cdot e^{(\mu_s - \frac{1}{2}\sigma_s^2)t + \sigma_s Z_s(t)} \tag{6.7}$$

and

$$S_t^N = \frac{S_0}{S_0^H} e^{-\frac{1}{2}\sigma_0^2 t + \sigma_0 Z_0(t)} \tag{6.8}$$

S_t^H is the hedgeable component of S_t and S_t^N is non-hedgeable component. μ_s and σ_s are the mean and the standard deviation of the salary. $Z_S(t)$ is a standard Brownian motion. For the non-hedgeable part, we introduce σ_0 to allow for possible correlation between the salary and equity returns and $Z_0(t)$ is a second standard Brownian motion independent of $Z_s(t)$.

Equation (6.6) is also equivalent to Wilkie(1995)'s assumption. We assume the mean and the standard deviation of the inflation index are μ_I and σ_I, respectively. The inflation index I_t and the salary S_t at time t satisfies equation (6.9).

$$S_t = I_t \cdot S_t^{SI} \tag{6.9}$$

where

$$I_t = I_0 \cdot e^{(\mu_I - \frac{1}{2}\sigma_I^2)t + \sigma_I Z_I(t)} \tag{6.10}$$

and

$$S_t^{SI} = \frac{S_0}{I_0} e^{(\mu_{SI} - \frac{1}{2}\sigma_{SI}^2)t + \sigma_{SI} Z_{SI}(t)} \tag{6.11}$$

$Z_I(t)$ and $Z_{SI}(t)$ are two independent standard Brownian motions. Then, S_t/I_t follows a log-normal distribution. If the only hedgeable part of the salary is the inflation index, the Brownian motions $Z_s(t)$ in equation (6.7) and $Z_I(t)$ in equation (6.10) are equivalent. Matching parameters in equation (6.6) and equation (6.9), we have $\mu_s = \mu_I$, $\sigma_s = \sigma_I$, $\mu_{SI} = 0$, and $\mu_0 = \sigma_{SI}$. We assume the risk free asset is always available in the market. Therefore, we can construct the hedging portfolio to match the mean of the salary regardless of the standard deviation of the salary.

Chapter 7

Hedging Costs

7.1 Introduction

In Chapter 5, we illustrated numerical results of three funding strategies. The most important assumption we used in Chapter 4 and Chapter 5 is that salary is fully tradable. In this chapter, we assume salary is no longer tradable. In Chapter 6, we have analyzed the relationship between salary and other financial indexes, such as long-term bonds, stocks, and inflations. We first want to connect the salary and the inflation and make the model tractable. Second, we assume the hedging asset for DB account is invested in inflation-linked bonds and the hedging asset for DC account is still invested in stock indexes. We consider three models and use one of them to test our funding strategies.

7.2 The Model for Salary and Inflation

In this chapter, we assume the salary is not fully tradable. We have analyzed three models for salary and inflation in Chapter 6. The third model is more reasonable and tractable. Here, we will assume the salary is partially tradable. We split the salary process into two parts.

We assume the pension plan member has a salary at time t of S_t, which is governed by equation (7.1).

$$S_t = S_t^H \cdot S_t^N \tag{7.1}$$

where

$$S_t^H = S_0^H \cdot e^{(\mu_s - \frac{1}{2}\sigma_s^2)t + \sigma_s Z_s(t)} \tag{7.2}$$

and

$$S_t^N = \frac{S_0}{S_0^H} e^{-\frac{1}{2}\sigma_0^2 t + \sigma_0 Z_0(t)} \tag{7.3}$$

S_t^H is the hedgeable component of S_t and S_t^N is non-hedgeable component. μ_s and σ_s are the mean and the standard deviation of the salary. $Z_S(t)$ is a standard Brownian motion. For the non-hedgeable part, we introduce σ_0 to allow for possible correlation between the salary and inflation and $Z_0(t)$ is a second standard Brownian motion independent of $Z_s(t)$.

We assume the hedgable part S_t^H can be replicated by the inflation and risk-free bond. At time t, with all known information, we can calculate the value of the hedging portfolio $H(t)$,

$$H(t) = \Delta_S(t) S_t^H + \Delta_A(t) DC(t,t) \tag{7.4}$$

One month later, at time $t + \frac{1}{12}$, the hedging portfolio established at t and brought forward to $t + \frac{1}{12}$ has value:

$$Hbf(t + \frac{1}{12}) = \Delta_S(t) S_t^H \frac{S^H_{t+\frac{1}{12}}}{S_t^H} + \Delta_A(t) DC(t,t) \frac{A_{t+\frac{1}{12}}}{A_t} \tag{7.5}$$

The new hedging portfolio $H(t + \frac{1}{12})$ is calculated by the information available at time $t + \frac{1}{12}$, so the cash flow for the guarantee at time $t + \frac{1}{12}$ is

$$CF(t + \frac{1}{12}) = H(t + \frac{1}{12}) - Hbf(t + \frac{1}{12}) \tag{7.6}$$

and the monthly hedging cash flow as a proportion of salary is

$$cf(t + \frac{1}{12}) = \frac{CF(t + \frac{1}{12})}{S_{t+\frac{1}{12}}} \tag{7.7}$$

The hedge portfolio share of the salary and the underlying fund at time t, $\Delta_S(t)$ and $\Delta_A(t)$, are calculated by the current salary, inflation, years of service, and the DC underlying asset. However, only part of salary can be replicated. $\Delta_S(t)$ and $\Delta_A(t)$ are invested into the inflation and the underlying asset, respectively. This will cause more risks because of the volatility of the non-hedgable part, σ_0. Since we modify the salary assumption and keep original hedging strategies. This hedging maybe only suboptimal. In the following sections, we will use some numerical results to show hedging costs will not change too much when σ_0 is small, and hedging costs are very volatile when σ_0 is large.

7.3 Numerical Results

Since salary is assumed not fully hedgeable in this chapter, we assume part of the salary is hedgeable and the process of the salary follows equation (7.1). Given the salary at time t, the salary at time $t+h$ can be calculated recursively by the following equation(7.8).

$$S_{t+h} = S_t \cdot e^{(\mu_s - \frac{1}{2}\sigma_s^2 - \frac{1}{2}\sigma_0^2)h + \sigma_s \sqrt{h} \cdot Z_s(t) + \sigma_0 \sqrt{h} \cdot Z_0(t)} \tag{7.8}$$

We generate the salary and crediting rate processes by Monte Carlo simulation and implement funding strategy 3 and 4 in this section with 10,000 simulations. Therefore, at any time t, we can calculate the value of the hedging portfolio. The growth of the hedging portfolio is dependent on the hedgeable part of the salary and the crediting rate because salary is not fully tradable here. We consider the non-hedgable part as a hedging risk. Equation (7.9) shows that the hedgeable part of the salary is defined as:

$$S_{t+h}^H = S_t^H \cdot e^{(\mu_s - \frac{1}{2}\sigma_s^2)h + \sigma_s \sqrt{h} \cdot Z_s(t)} \tag{7.9}$$

We also rebalance the hedging portfolio every period and calculate the ratio of hedging cash flow and current salary. Similar to the last section, we can find the average monthly hedging cost for funding strategy 3 and 4 by equation (7.7).

Based on the last 30 years Canadian inflation index and salary index data, we estimate that the value of parameter σ_0 is approximately equal to 0.016. The value of parameter σ_0 expresses the standard deviation of the log-ratio of the salary and the inflation index. For the hedgeable part of the salary, the hedging portfolio can be constructed by the inflation and the risk-free bond. To be consistent with results in Chapter 5, the mean and standard deviation of hedgeable salary growth rate which are defined by equation (7.9) are given in Table 7.1.

Since salary satisfies equation (7.8) and is not fully tradable, we only consider the tradable part when we apply the exchange option valuation to calculate hedging cash flows. At any time t, the hedgeable part of salary satisfies equation (7.9).

Table 7.1: Parameters for the Scenario Test

	Mean of Hedgeable Salary Growth	StD of Hedgeable Salary Growth	Discount Rate
Scenario 1	0.04	0.012	0.05
Scenario 2	0.04	0.037	0.05
Scenario 3	0.05	0.012	0.05
Scenario 4	0.04	0.012	0.05
Scenario 5	0.04	0.012	0.05
Scenario 6	0.04	0.012	0.05
	Mean of Crediting Rate	StD of Crediting Rate	Contribution Rate
Scenario 1	0.1	0.2	0.125
Scenario 2	0.1	0.2	0.125
Scenario 3	0.1	0.2	0.125
Scenario 4	0.15	0.2	0.125
Scenario 5	0.1	0.15	0.125
Scenario 6	0.1	0.2	0.10

We can implement funding strategy 3 and 4 on this hedgeable part of salary to calculate the value of hedging portfolio. Then, we rebalance the hedging portfolio in the next period and find hedging cash flows. Table 7.2 and Table 7.3 show results in 6 scenarios. Standard errors are shown in brackets.

We also draw the histogram of 10,000 paths and estimated density for average monthly hedging costs to observe the distribution of simulated hedging costs. Given the entry age, we compare the distribution of monthly hedging costs. Results show us that, in Figure 7.3, the density functions for earlier entry ages have lower summit and heavier tail. In Figure 7.4, the shapes of the estimated density function are almost identical to Figure 5.7.

Because of the path dependence of hedging costs, we draw the quantile of

Table 7.2: Average Monthly Hedging Costs for the Scenario Test as Percent of Monthly Salary in Funding Strategy 3

Entry Age	Scenario 1	Scenario 2	Scenario 3	Scenario 4	Scenario 5	Scenario 6
20	4.83	4.83	4.99	3.70	3.55	5.76
	(0.0085)	(0.0081)	(0.0079)	(0.0079)	(0.0068)	(0.0103)
25	4.65	4.68	4.83	3.66	3.48	5.70
	(0.0082)	(0.0077)	(0.0075)	(0.0077)	(0.0068)	(0.0099)
30	4.51	4.54	4.66	3.63	3.44	5.61
	(0.0078)	(0.0073)	(0.0072)	(0.0075)	(0.0065)	(0.0095)
35	4.30	4.36	4.46	3.57	3.35	5.52
	(0.0072)	(0.0068)	(0.0066)	(0.0074)	(0.0063)	(0.0090)
40	4.13	4.16	4.25	3.48	3.25	5.40
	(0.0068)	(0.0065)	(0.0062)	(0.0070)	(0.060)	(0.0084)
45	3.94	3.95	4.01	3.39	3.13	5.31
	(0.0062)	(0.0059)	(0.0055)	(0.0067)	(0.0055)	(0.0078)
50	3.66	3.67	3.72	3.24	3.01	5.17
	(0.0057)	(0.0054)	(0.0052)	(0.0063)	(0.0052)	(0.0074)
55	3.34	3.33	3.36	3.04	2.83	5.06
	(0.0049)	(0.0047)	(0.0045)	(0.0057)	(0.0044)	(0.0061)

monthly hedging costs in strategy 3 and 4 in Figure 7.5 and Figure 7.6, respectively. In funding strategy 3, 90% quantile lines in Figure 7.5 are higher than in Figure 5.10 and 10% quantile lines are lower for given entry ages. This means the difference between 10% quantile and 90% quantile becomes wider and monthly hedging costs are more volatile. In funding strategy 4, we can not see too much difference in Figure 5.11 and Figure 7.6.

Comparing Table 7.2 and 7.3 with Table 5.11 and 5.12, we find there is no significant difference under fully hedgeable assumption and not fully hedgeable assumption. In funding strategy 3, monthly hedging costs with not fully hedgeable assumption in Table 7.2 are slightly higher than monthly hedging costs with hedge-

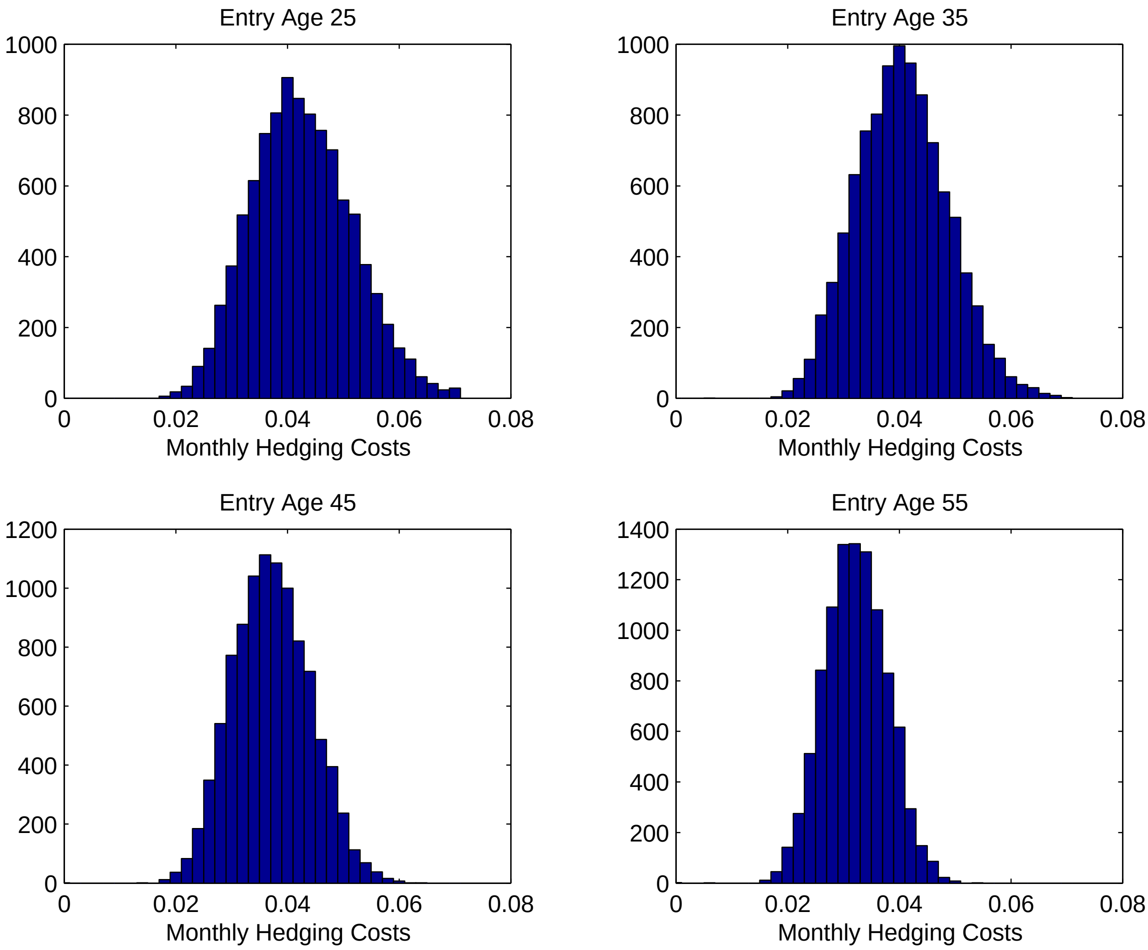


Figure 7.1: Histogram for Average Monthly Hedging Cost in Strategy 3

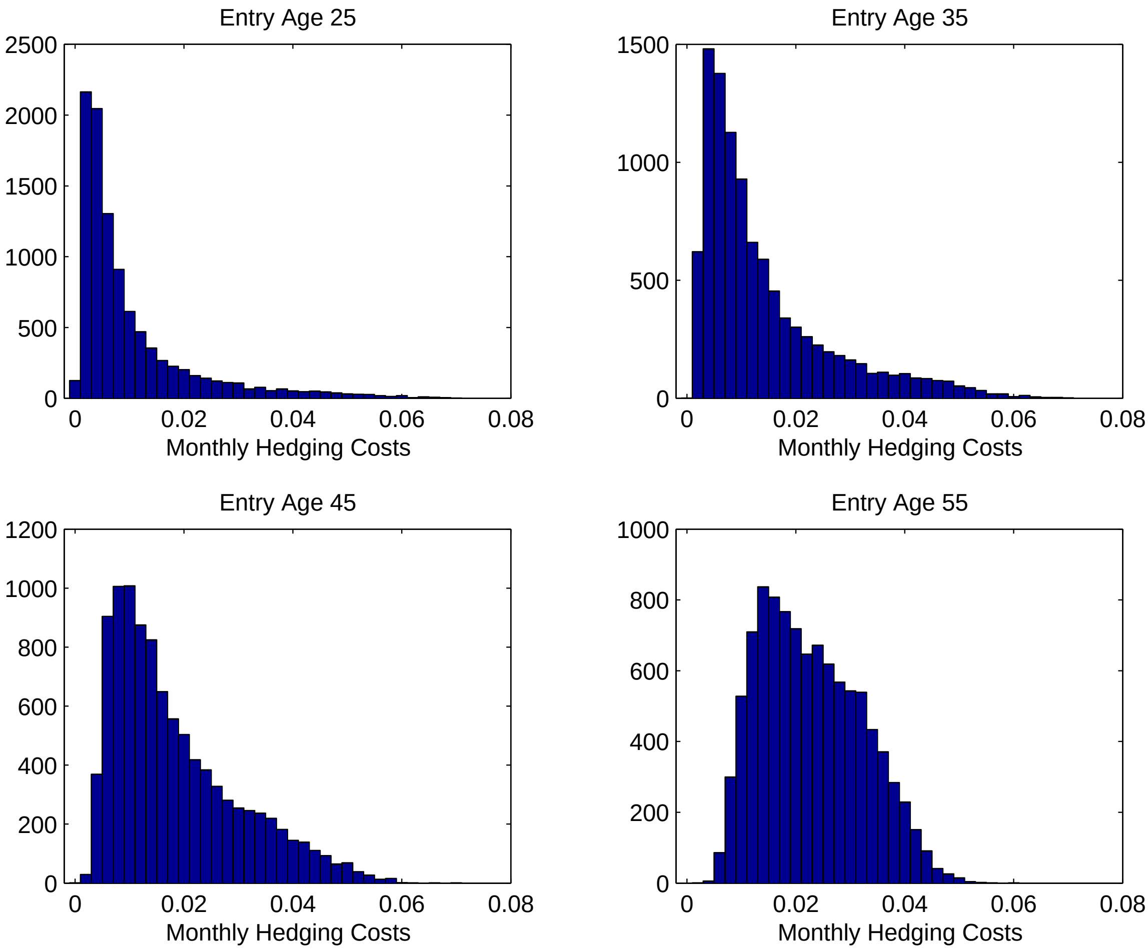


Figure 7.2: Histogram for Average Monthly Hedging Cost in Strategy 4

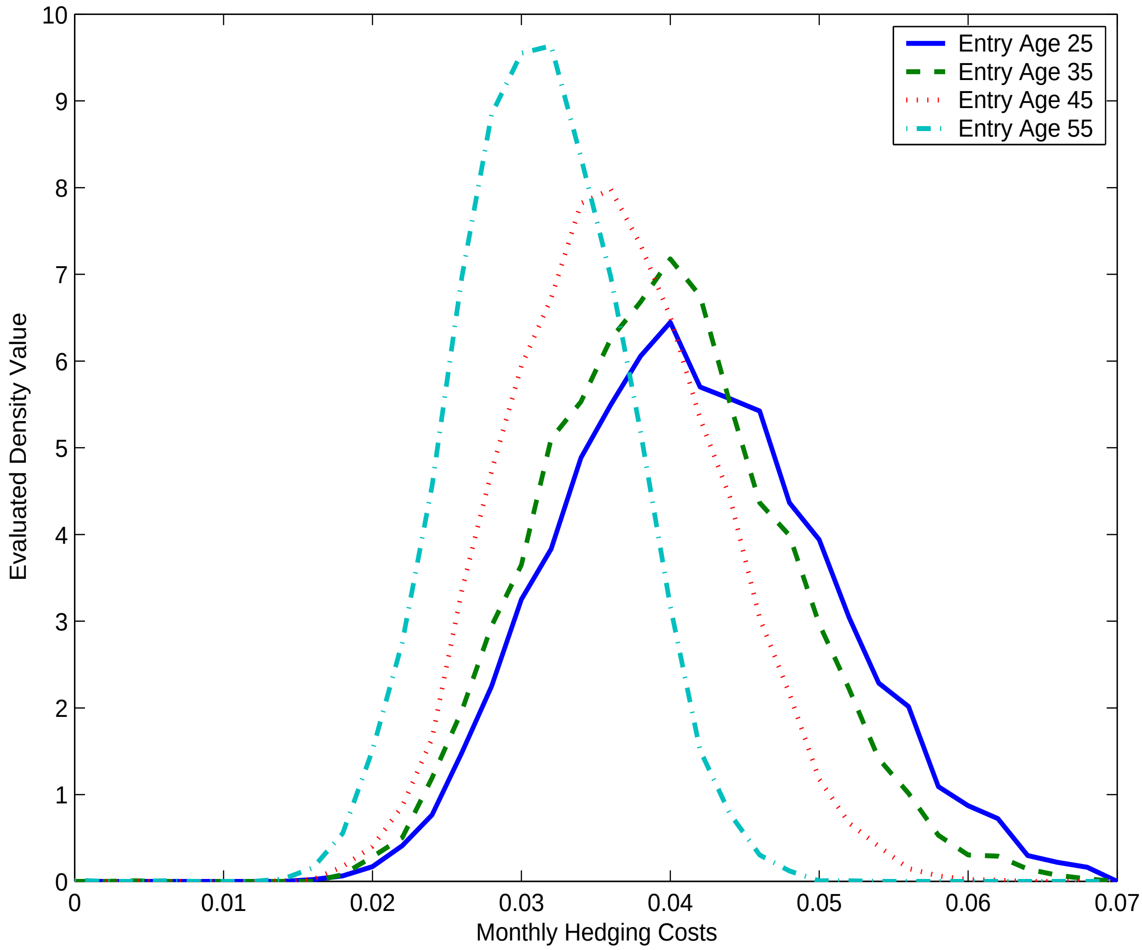


Figure 7.3: Estimated Density for Average Monthly Hedging Cost in Strategy 3

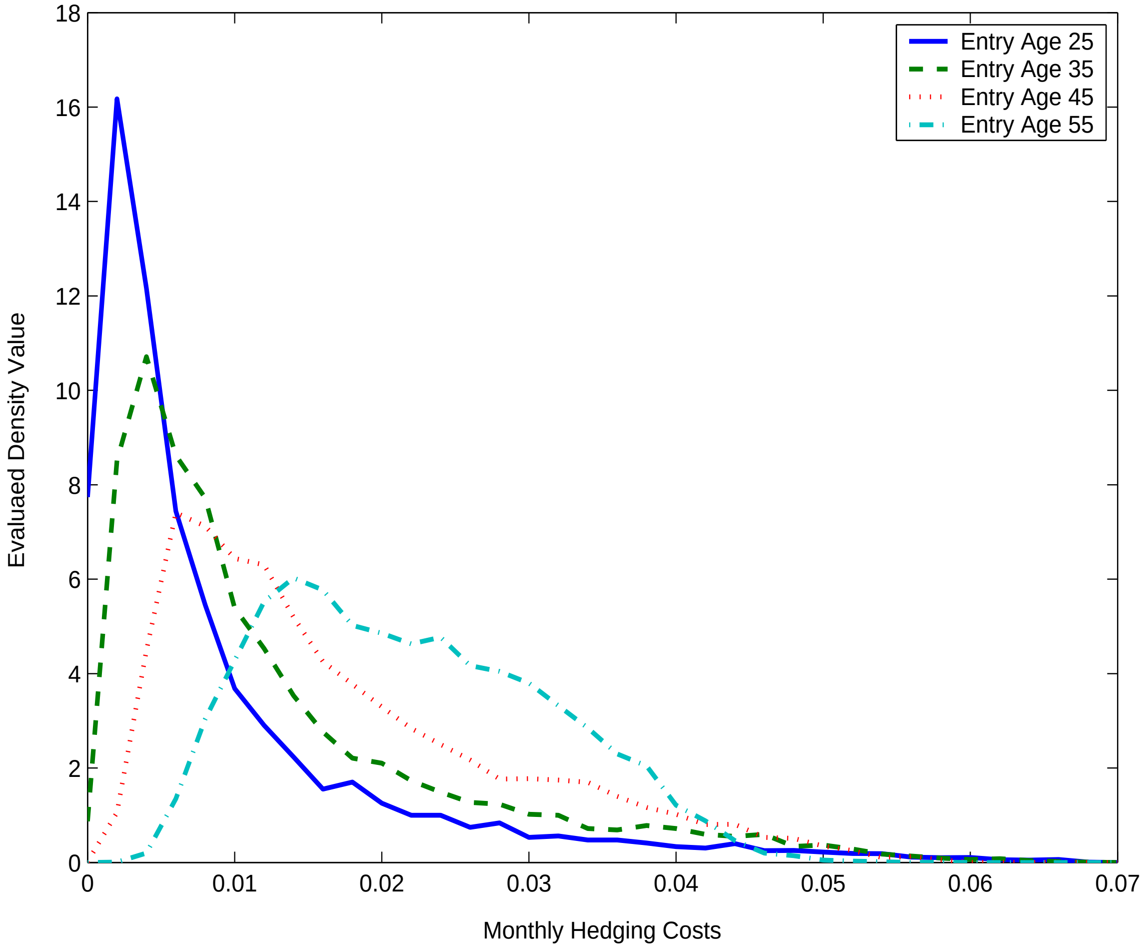


Figure 7.4: Estimated Density for Average Monthly Hedging Cost in Strategy 4

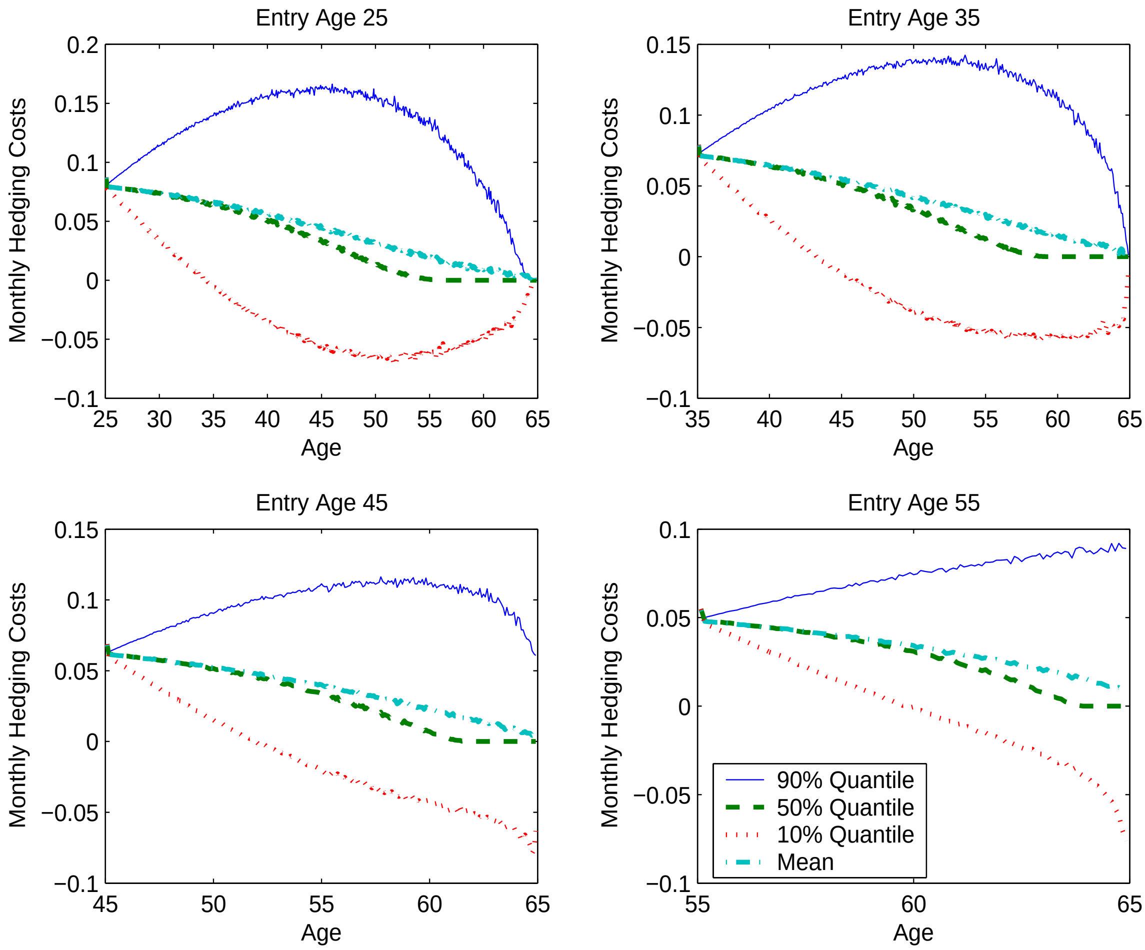


Figure 7.5: Quantile of Monthly Hedging Costs in Strategy 3

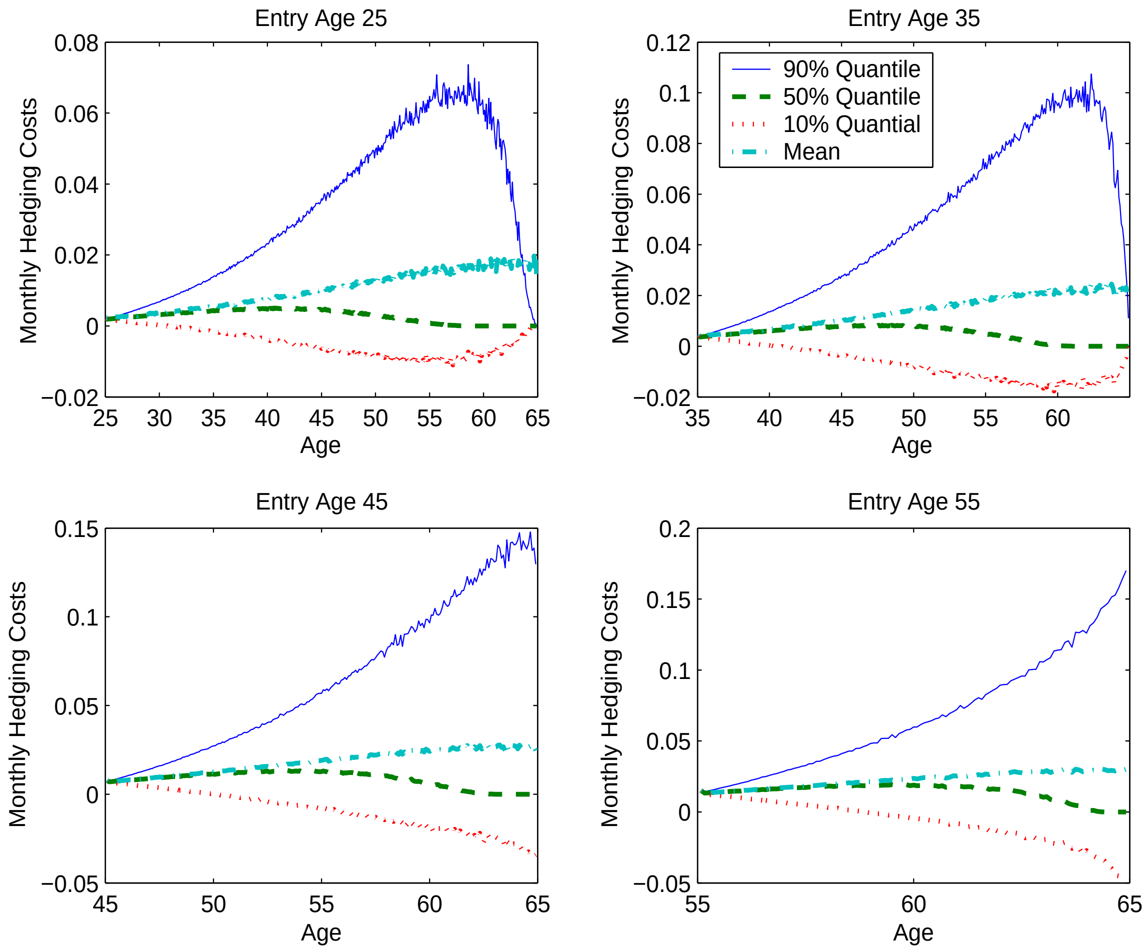


Figure 7.6: Quantile of Monthly Hedging Costs in Strategy 4

Table 7.3: Average Monthly Hedging Costs for the Scenario Test as Percent of Monthly Salary in Funding Strategy 4

Entry Age	Scenario 1	Scenario 2	Scenario 3	Scenario 4	Scenario 5	Scenario 6
20	1.51	1.52	2.35	0.37	0.49	2.18
	(0.0151)	(0.0161)	(0.0211)	(0.0047)	(0.0082)	(0.0195)
25	1.65	1.67	2.52	0.48	0.60	2.40
	(0.0150)	(0.0162)	(0.0208)	(0.0056)	(0.0089)	(0.0192)
30	1.79	1.82	2.60	0.63	0.75	2.65
	(0.0148)	(0.0161)	(0.0197)	(0.0066)	(0.0098)	(0.0191)
35	1.95	1.99	2.72	0.83	0.94	2.93
	(0.0144)	(0.0156)	(0.0188)	(0.0077)	(0.0104)	(0.0185)
40	2.12	2.12	2.84	1.04	1.13	3.22
	(0.0138)	(0.0149)	(0.0175)	(0.0085)	(0.0105)	(0.0174)
45	2.32	2.32	2.92	1.32	1.36	3.53
	(0.0129)	(0.0142)	(0.0157)	(0.0093)	(0.0107)	(0.0163)
50	2.43	2.43	3.00	1.63	1.64	3.87
	(0.0114)	(0.0125)	(0.0133)	(0.0094)	(0.0103)	(0.0143)
55	2.52	2.50	2.95	1.95	1.91	4.24
	(0.0094)	(0.0107)	(0.0108)	(0.0089)	(0.090)	(0.0115)

able assumption in Table 5.11. The higher the volatility, the more risks and the more hedging costs. To be consistent, we consider the same value of parameters for the fully hedgeable assumption and not fully hedgeable assumption. Sensitivities of hedging costs to parameters, such as the entry age, the salary growth rate, the crediting rate, and the contribution rate, are similar under fully hedgeable assumption and not fully hedgeable assumption.

Although the average monthly hedging costs do not change too much when $\sigma_0 = 0.016$, the standard errors of monthly hedging costs increase. Since the non-hedgable part of salary causes more risks, results are more volatile. When the non-hedgable part volatility σ_0 increases, we can see the average monthly hedging costs and volatilities increase even more. In scenario 2, the volatility of the salary is 4%. Table 7.4 and Table 7.5 show us that there are more risks in our problem when σ_0 is high. When $\sigma_0 = 0$, it means that the non-hedgable part is a constant. That case is equivalent to this situation where salary is fully tradable. In strategy 3, standard errors of hedging costs increase when the non-hedgabale part volatility increases. Since the non-hedgable part of salary is not hedged using strategy 3, the hedging cost is more volatile when σ_0 is high. In strategy 4, we consider the current salary instead of the projected final salary for the DB underpin guarantee. Therefore, the assumption on salaries does not cause too much difference on the monthly hedging costs. Table 7.5 shows the hedging cost is very close when σ_0 increases.

Table 7.4: Average Monthly Hedging Costs with Different σ_0 under Scenario 2 in Strategy 3

Entry Age	$\sigma_0 = 0$ $\sigma_s = 0.04$	$\sigma_0 = 0.016$ $\sigma_s = 0.037$	$\sigma_0 = 0.03$ $\sigma_s = 0.026$	$\sigma_0 = 0.039$ $\sigma_s = 0.009$
20	4.82	4.83	4.84	4.84
	(0.0080)	(0.0081)	(0.0119)	(0.0142)
25	4.65	4.68	4.70	4.72
	(0.0076)	(0.0077)	(0.0116)	(0.0138)
30	4.51	4.53	4.55	4.56
	(0.0073)	(0.0073)	(0.0110)	(0.0129)
35	4.30	4.33	4.38	4.40
	(0.0067)	(0.0068)	(0.0105)	(0.0124)
40	4.15	4.16	4.20	4.18
	(0.0063)	(0.0065)	(0.0100)	(0.0115)
45	3.93	3.95	3.97	4.01
	(0.0062)	(0.0059)	(0.0091)	(0.0107)
50	3.65	3.67	3.70	3.73
	(0.0055)	(0.0054)	(0.0081)	(0.0098)
55	3.33	3.33	3.34	3.35
	(0.0048)	(0.0047)	(0.0071)	(0.0096)

Table 7.5: Average Monthly Hedging Costs with Different σ_0 under Scenario 2 in Strategy 4

Entry Age	$\sigma_0 = 0$ $\sigma_s = 0.04$	$\sigma_0 = 0.016$ $\sigma_s = 0.037$	$\sigma_0 = 0.03$ $\sigma_s = 0.026$	$\sigma_0 = 0.039$ $\sigma_s = 0.009$
20	1.51	1.52	1.52	1.53
	(0.0162)	(0.0161)	(0.0160)	(0.0161)
25	1.66	1.67	1.67	1.68
	(0.0163)	(0.0162)	(0.0161)	(0.0162)
30	1.80	1.82	1.82	1.83
	(0.0161)	(0.0161)	(0.0162)	(0.0162)
35	1.95	1.99	2.00	2.01
	(0.0155)	(0.0156)	(0.0156)	(0.0156)
40	2.14	2.12	2.15	2.14
	(0.0152)	(0.0149)	(0.0150)	(0.0150)
45	2.32	2.32	2.33	2.33
	(0.0142)	(0.0142)	(0.0142)	(0.0143)
50	2.44	2.43	2.44	2.44
	(0.0128)	(0.0125)	(0.0125)	(0.0125)
55	2.54	2.50	2.53	2.55
	(0.0108)	(0.0107)	(0.0109)	(0.0110)

Chapter 8

Hedging with Stochastic Interest Rates

8.1 Introduction

In Chapter 4, we propose four different funding strategies for the minimum DB guarantee. In Chapter 7, we have assumed that the salary is not completely hedgeable and showed that the monthly hedging costs are higher when the non-hedgable volatility, σ_0, is high. In previous chapters, we assumed the annuity rate at retirement is a constant and that employers can buy the annuity product from insurance companies at the given rate. Then, the value of the DB account only depends on the salary scale at retirement because other factors are all assumed to be constant. In general, pension fund liabilities involve very long term guarantees. If the mortality and interest rate are all stochastic, the real value of the annuity can be very high or low at retirement. When we construct our hedging portfolio, we need to consider both the mortality risk and the interest risk. In this chapter, we assume the interest rate is no longer a constant, but instead follows a stochastic process.

Since we are considering average monthly hedging costs, the effect of the interest rate is not significant when we implement funding strategy 3. This is because that the interest rate is not involved in the calculation of the average monthly hedging costs. In strategy 3, only the annuity factor is affected by the interest rate. At retirement, the value of DB account depends on both he final salary and the annuity rate. When the interest rate goes up, the annuity cost goes down. Employers pay less for the DB retirement benefits. As the interest rate goes down, the annuity rate at retirement goes up. The value of the hedging portfolio may not be enough to pay the DB retirement benefits to employees. Employers will spend more money and hedging costs also increase. In strategy 4, the stochastic interest rate causes more trouble, since it also affects monthly hedging cash flows. Strategy 4 is based on the Traditional Unit Credit (TUC) cost method. The values of the salary and the DC underlying asset are assumed known at the valuation time t. The interest rate affects the roll-up of the delta hedging portfolio. The hedging cost in strategy 4 is, therefore, more sensitive to the interest rate than in strategy 3. We will discuss both strategies separately.

8.2 Affine Term Structures

Before we discuss hedging the DB underpin pension guarantee when the interest rate is random, we will clarify some definitions and assumptions. The primary objects of investigation are **zero coupon bonds**, also known as **pure discount bonds**, of various maturities.

Definition 8.2.1. A **zero coupon bond** with **maturity date** T is a contract which guarantees the holder 1 dollar to be paid on the date T. The price at time t of a bond with maturity date T is denoted by $p(t,T)$.

Now, we make an assumption to guarantee the existence of a sufficiently rich and regular bond market.

Assumption 8.2.2. *We assume the following.*

- *There exists a (frictionless) market for T-bonds for every $T > 0$.*
- *The relation $p(t,t) = 1$ holds for all t.*
- *For each fixed t, the bond price $p(t,T)$ is differentiable with respect to time of maturity T.*

The bond price $p(t,T)$ is a stochastic object with two variables, t and T, and, for each outcome in the underlying sample space, the dependence upon these variables is very different. For a hybrid pension guarantee at time t, employers consider the long-term interest rate from retirement time T to death time S, where $T < S$. Hence, we need a contract guaranteeing a riskless rate of interest over the future interval $[T,S]$. In the financial market, such an interest rate is called a **forward rate**.

At time t, assume there are two zero coupon bonds in the market with maturity time T and S, with $t < T < S$. Then, the price of those two bonds are $p(t,T)$ and $p(t,S)$, respectively. To guarantee one dollar at time S, we can spend $p(t,S)$ dollars to purchase the S-year bond or we can purchase some T-year bonds and switch to the (S-T)-year bond at the maturity time T. Under the no arbitrage assumption, the money we spend should be equal, which is $p(t,S)$ dollars at time t. Using $p(t,S)$ dollars at time t, we can purchase $p(t,S)/p(t,T)$ shares of T-bonds. At time T, we receive $p(t,S)/p(t,T)$ dollars since the maturity of T-bonds. In other words, the price of S-bonds at time T will be $p(t,S)/p(t,T)$. The net effect is that, based on a contract at time t, an investment of one dollar at time T has yielded $p(t,T)/p(t,S)$ dollars at time S. We call the resulting annualized rate the forward rate for $[T,S]$.

Definition 8.2.3.

1. The **continuously compounded forward rate for [T,S] contracted at** t is defined as
$$R(t;T,S) = -\frac{\log p(t,S) - \log p(t,T)}{S-T}$$
2. The **instantaneous forward rate with maturity** S**, contracted at** t, is defined as
$$f(t,S) = -\frac{\partial \log p(t,S)}{\partial S}.$$
3. The instantaneous **short rate at time** t is defined as
$$r(t) = f(t,t).$$

The instantaneous forward rate is the limit of the continuously compounded forward rate when $T \to S$. It is very important, since it can be interpreted as the riskless rate of interest over the infinitesimal interval $[S, S+dS]$, when it is contracted at time t. From this definition, it is easy to find the following useful results.

Lemma 8.2.4. *For $t \leq s \leq T$, we have*
$$p(t,T) = p(t,s) \cdot \exp\{-\int_s^T f(t,u)du\}$$
and in particular
$$p(t,T) = \exp\{-\int_t^T f(t,u)du\}$$

With these definitions, we go back to discuss the DB underpin pension guarantee with stochastic interest rates.

The continuous-time short-term interest rate is one of the most fundamental and important prices determined in financial market. Many popular models are currently used by academic researcher and industrial practitioners. There are many well-known papers about these interest rate models by Merton(1973), Vasicek(1977), Dothan(1978), Cox, Ingersoll, and Ross(1980,1985), Longstaff(1989), Hull and White(1990), Black and Karasinski(1991), Longstaff and Schwartz(1992), and Cairns(2004).

The general form of the short-term interest rate model is given by

$$dr = \mu(t,r)dt + \sigma(t,r)dZ \tag{8.1}$$

Where Z is a Wiener process, $\mu(t,r)$ is the drift, and $\sigma(t,r)$ is the diffusion.

We illustrate a (far from complete) list of the most popular short interest models with time-independent drift and diffusion.

1. Merton $dr = \alpha dt + \sigma dZ$
2. Vasicek $dr = (\alpha + \beta r)dt + \sigma dZ$
3. CIR SR $dr = (\alpha + \beta r)dt + \sigma r^{1/2} dZ$
4. Dothan $dr = \sigma r dZ$
5. GBM $dr = \beta r dt + \sigma r dZ$
6. Brennan-Schwartz $dr = (\alpha + \beta r)dt + \sigma r dZ$
7. CIR VR $dr = \sigma r^{3/2} dZ$
8. CEV $dr = \beta r dt + \sigma r^{\gamma} dZ$

There are also some short rate models with time dependent parameters.

1. Black-Derman-Toy $dr = \Theta(t)rdt + \sigma(t)rdZ$
2. Ho-Lee $dr = \Theta(t)dt + \sigma dZ$
3. Hull-White (extended Vasiček) $dr = (\Theta(t) - a(t)r)dt + \sigma(t)dZ \ \ (a(t) > 0)$
4. Hull-White (extended CIR) $dr = (\Theta(t) - a(t)r)dt + \sigma(t)\sqrt{r}dZ \ \ (a(t) > 0)$

Among those models, we want to pick some reasonable models to model interest rates properly. It turns out that there are some models are much easier to deal with analytically than the others. So we give the definition of such interest rates processes, which are called **affine term structures**.

Definition 8.2.5. If the term structure $\{p(t,T); 0 \le t \le T, T > 0\}$ has the form

$$p(t,T) = F(t, r(t); T) \tag{8.2}$$

where F has the form

$$F(t, r; T) = e^{C(t,T) - B(t,T)r} \tag{8.3}$$

and where C and B are deterministic functions, then the model is said to possess an **affine term structure** (ATS).

The functions C and B are functions of two real variables t and T, but conceptually it is easier to think of C and B as being functions of t, while T is treated as a fixed value. The existence of an affine term structure is extremely convenient from an analytical and a computational point of view. Another advantage is that many interest models have an affine term structure. For example, Vasiček, Ho-Lee and Hull-White (extended Vasiček) all have an ATS. Björk(1998) presents a probabilistic reason why many models have such a property. He also illustrates some popular affine one factor models.

Proposition 8.2.6. ***(Affine term structure)*** *Assume that the short rate of interest is given by*

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dZ.$$

where $\mu(t, r(t))$ and $\sigma(t, r(t))$ are of the form

$$\begin{cases} \mu(t,r) & = & \alpha(t)r + \beta(t), \\ \sigma(t,r) & = & \sqrt{\gamma(t)r + \delta(t)}. \end{cases} \tag{8.4}$$

Then the model admits an affine term structure of the form (8.3), where C and B satisfy that the equations

$$\begin{cases} B_t(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) & = & -1, \\ B(T,T) & = & 0. \end{cases} \tag{8.5}$$

$$\begin{cases} C_t(t,T) & = & \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T), \\ C(T,T) & = & 0. \end{cases} \tag{8.6}$$

Equation (8.5) determines the function B which does not involve C. Having solved equation (8.5), we may then insert the solution B into equation (8.6) to obtain C.

8.3 Estimated Annuity Rates

Boyle and Hardy(2003) implement a numeraire approach to calculate the market value of guaranteed annuity options. They use the zero coupon bond which matures at retirement as the numeraire and find the corresponding risk neutral measure, which is often called the 'forward measure'. Since the guaranteed annuity rate is fixed in a GAO contract, there is a closed form of the value of a GAO contract if the interest rate follows a one-factor process. However, in the DB underpin pension guarantee case, the payoff at retirement is much more complicated. Therefore, we will solve this problem numerically.

To be simple, we use the Vasiček model since it is a mean reverting model with an ATS. For an affine term structure, we can find the price of zero coupon bonds at time t with given maturity. That means at any rebalance time we are able to calculate the forward rate from the retirement time T to the death time S. Given the mortality table from Table 8.1 and no expense assumption, we can calculate the projected annuity rate at retirement.

Assume the annuity is paid at the begin of the month and the normal retirement age is 65, the annuity rate at retirement T can be expressed as

$$a_{65}(T) = \frac{1}{12} \sum_{s=0}^{12\cdot(\omega-65)} {}_{s/12}p_{65} \cdot p(T, T+s) \tag{8.7}$$

where ${}_{s/12}p_{65}$ is the survival probability that insured survives in the s-th month after age 65, which is linear interpolated from Table 8.1; $p(T, T+s)$ is the price at time T of zero coupon bonds with maturity $T+s$; ω is the limiting life age, which is 110 here.

Equation (8.7) describes the annuity rate at retirement T. This annuity contract pays 1/12 per month until the insured dies. Before retirement, we assume there is no exit and the annuity rate can be calculated by the forward rate. At the valuation time t, the annuity rate is equal to

$$a_{65}(t) = \frac{1}{12} \sum_{s=0}^{12\cdot(\omega-65)} {}_{s/12}p_{65} \cdot \frac{p(t, T+s)}{p(t, T)} \tag{8.8}$$

The short rate of interest follows a Vasiček model, which is given by

$$dr = (b - ar)dt + \sigma dZ$$

Table 8.1: Annual Decrement Rates for Annuity Rate Valuations

Age	Death Probability	Age	Death Probability
65	0.02059	88	0.13708
66	0.02216	89	0.14728
67	0.02389	90	0.15868
68	0.02585	91	0.17169
69	0.02806	92	0.18570
70	0.03052	93	0.20023
71	0.03315	94	0.21495
72	0.03593	95	0.22976
73	0.03882	96	0.24338
74	0.04184	97	0.25637
75	0.04507	98	0.26868
76	0.04867	99	0.28030
77	0.05274	100	0.29120
78	0.05742	101	0.30139
79	0.06277	102	0.31089
80	0.06882	103	0.31970
81	0.07552	104	0.32786
82	0.08278	105	0.33539
83	0.09041	106	0.34233
84	0.09842	107	0.34870
85	0.10725	108	0.35453
86	0.11712	109	0.35988
87	0.12717	110	1

*Extracted from Life Table: United States, 1979-81

Table 8.2: Values of Parameters in the Vasiček Model

$dr(t) = (b - ar(t))dt + \sigma dZ$

Initial Rate $r(0)$	b	a	Volatility σ
0.05	0.02	0.35	0.025

The Vasiček model has the property of being mean reverting in the sense that it will tend to revert to the mean level b/a. Equations (8.5) and (8.6) become

$$\begin{cases} B_t(t,T) - aB(t,T) = -1, \\ B(T,T) = 0. \end{cases} \tag{8.9}$$

$$\begin{cases} C_t(t,T) = bB(t,T) - \frac{1}{2}\sigma^2 B^2(t,T), \\ C(T,T) = 0. \end{cases} \tag{8.10}$$

Proposition 8.3.1. ***(The Vasiček term structure)*** *In the Vasiček model, bond prices are given by*

$$p(t,T) = e^{C(t,T) - B(t,T)r(t)} \tag{8.11}$$

where

$$B(t,T) = \frac{1}{a}(1 - e^{-a(T-t)}) \tag{8.12}$$

$$C(t,T) = \frac{(B(t,T) - T + t) \cdot (ab - \frac{1}{2}\sigma^2)}{a^2} - \frac{\sigma^2 B^2(t,T)}{4a} \tag{8.13}$$

Using this proposition and equation (8.8), we can calculate the annuity rate given the short rate at time t. We assume values of parameters in the Vasiček model as given in Table 8.2.

Assume the mortality table and values of parameters are given, we can combine equations (8.8) and (8.11) to simulate the annuity rate at t.

Based on the specific mortality table, we pick up the appropriate parameters in the Vasiček model to determine the mean of annuity rates. Figure 8.1 illustrates

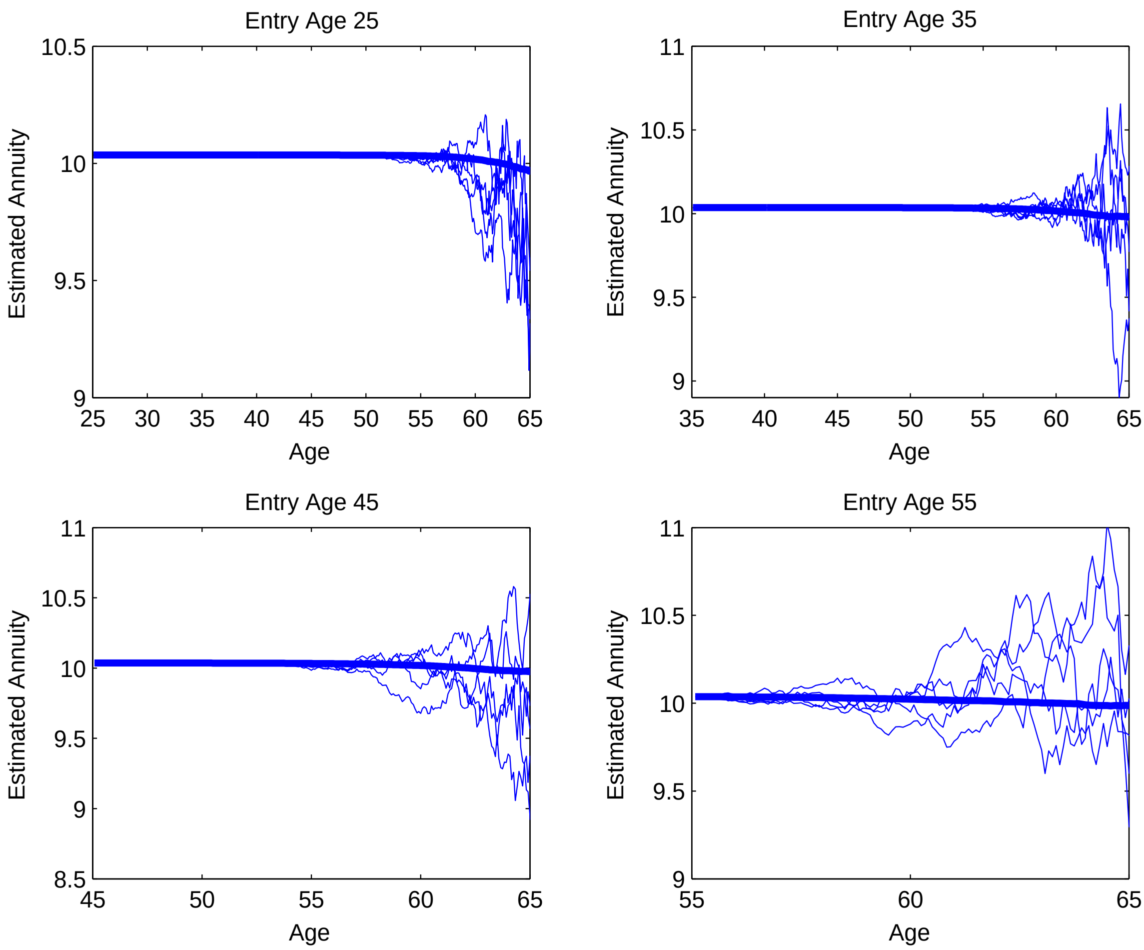


Figure 8.1: Simulated Annuity Rates

the mean of annuity rates and five sample paths. The mean of annuity rates is around 10, which is the same as the fixed annuity rate we used before. Then, we can compare our results between the fixed interest case and the stochastic interest case. Given the entry age, the volatility of the projected annuity rate goes up when the employee is close to his retirement.

8.4 Numerical Results for Strategy 3

Under funding strategy 3, the effect of the stochastic interest rates only affects the annuity rate. We keep the same assumptions as last chapter except that the interest rate follows a Vasiček process with parameters given in Table 8.2. For simplicity, we assume the interest rate is independent of the salary growth rate and the crediting rate.

At time t, the projected payoff at retirement age T of the pension guarantee can be expressed as

$$\begin{aligned} & \max(0, DB^P(t,T) - DC^P(t,T)) \\ = & \max(0, \alpha S_T a_{65}(t)t - DC^P(t,T)) \\ = & \max(0, \alpha S_t \frac{S_T}{S_t} a_{65}(t)t - DC^P(t,t)\frac{A_T}{A_t}) \end{aligned} \tag{8.14}$$

At rebalance time t, the salary S_t, the value of the DC account $DC^P(t,t)$ and the current short rate $r(t)$ are known. We can calculate the price of the annuity rate $a_{65}(t)$. Hence, we implement funding strategy 3 to construct the hedging portfolio in order to match the projected liabilities in equation (8.14).

Table 8.3 gives us monthly hedging costs as a percentage of salaries. The standard errors are shown in brackets. Based on parameters of the Vasiček model given

Table 8.3: Average Monthly Hedging Costs with Stochastic Interest Rates as Percent of Monthly Salary under Funding Strategy 3

Entry Age	Scenario 1	Scenario 2	Scenario 3	Scenario 4	Scenario 5	Scenario 6
25	4.68	4.69	4.84	3.67	3.49	5.69
	(0.0086)	(0.0085)	(0.0087)	(0.0078)	(0.0072)	(0.0106)
30	4.50	4.55	4.69	3.63	3.42	5.62
	(0.0083)	(0.0084)	(0.0085)	(0.0076)	(0.0069)	(0.0106)
35	4.32	4.38	4.45	3.57	3.33	5.53
	(0.0082)	(0.0080)	(0.0081)	(0.0074)	(0.0068)	(0.0103)
40	4.14	4.18	4.26	3.49	3.25	5.40
	(0.0079)	(0.0079)	(0.0081)	(0.0073)	(0.0068)	(0.0101)
45	3.96	3.94	4.01	3.38	3.14	5.31
	(0.0079)	(0.0078)	(0.0080)	(0.0070)	(0.0068)	(0.0100)
50	3.67	3.67	3.71	3.24	3.00	5.17
	(0.0077)	(0.0076)	(0.0079)	(0.0069)	(0.0069)	(0.0098)
55	3.35	3.34	3.36	3.06	2.84	5.03
	(0.0078)	(0.0076)	(0.0079)	(0.0070)	(0.0074)	(0.0098)

in Table 8.2, the mean of annuity rates is around 10, which is comparable to the fixed annuity as we assumed before. If the annuity rates decrease, the projected DB benefits decrease. So the value of guarantee also goes down and hedging costs decrease. Compared with Table 7.2, the monthly average hedging cost is similar, since the annuity rate is very close to 10 that we used before. However, the standard error is much higher for higher entry age. Table 8.4 shows the monthly hedging cost and standard errors given scenario 1.

From Table 8.3, we can also see that the sensitivities of hedging costs to other assumptions, such as salary growth rates, crediting rates, and the contribution rate, are pretty much similar. The hedging costs are not sensitive to the mean and the standard deviation of salary growth rates. The changes of salary growth rates only cause a slight difference since we calculate the proportion of monthly

Table 8.4: Average Monthly Hedging Costs as Percent of Monthly Salary with Deterministic/Stochastic Interest Rates Under Scenario 1

Entry Age	Deterministic Interest Rate(5%)	Stochastic Interest Rate
25	4.65	4.68
	(0.0082)	(0.0086)
30	4.51	4.55
	(0.0078)	(0.0084)
35	4.30	4.32
	(0.0072)	(0.0082)
40	4.13	4.14
	(0.0068)	(0.0079)
45	3.94	3.96
	(0.0062)	(0.0079)
50	3.66	3.67
	(0.0057)	(0.0077)
55	3.34	3.35
	(0.0049)	(0.0078)

salaries at the valuation time. When the mean of crediting rates goes up or the standard deviation of crediting rates goes down, employers have less risk in their DC accounts, so hedging costs decrease. If employees and employers are willing to invest more contributions each month, the hedging costs also decrease.

We use parameters in scenario 1 here to draw two figures. Figure 8.2 exhibits the histogram of monthly hedging costs with different entry ages. As the entry age increases, the mean hedging cost moves to the left of the figure, and sample paths are closer to the mean as the entry age increases. This means the monthly hedging cost increases as the entry age increases. Figure 8.3 displays the quantile of monthly hedging costs with different entry ages. Compared with Figure 7.5, there is no significant difference. This is because we assume that interest rates follow a stochastic process with an affine term structure. When we consider a long-term

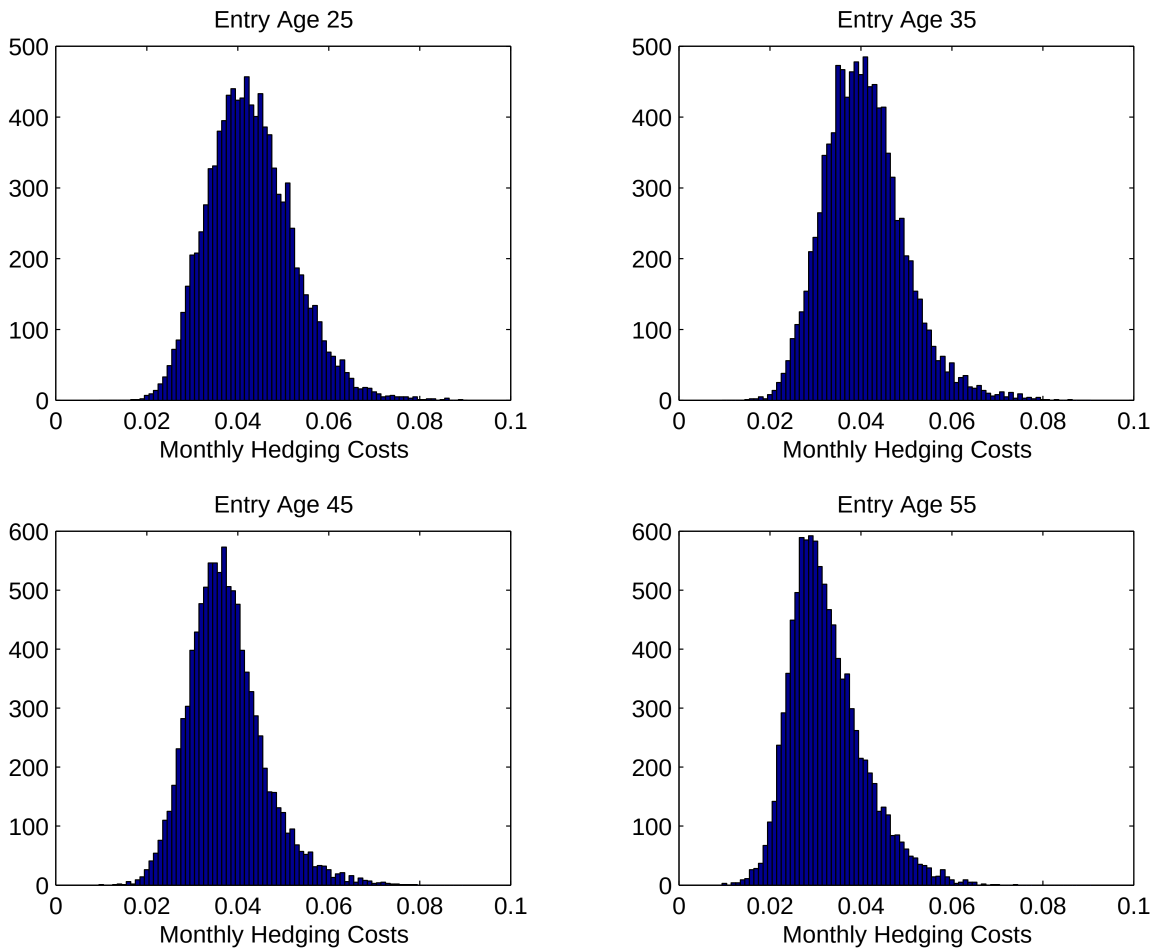


Figure 8.2: Histogram for Average Monthly Hedging Cost in Strategy 3

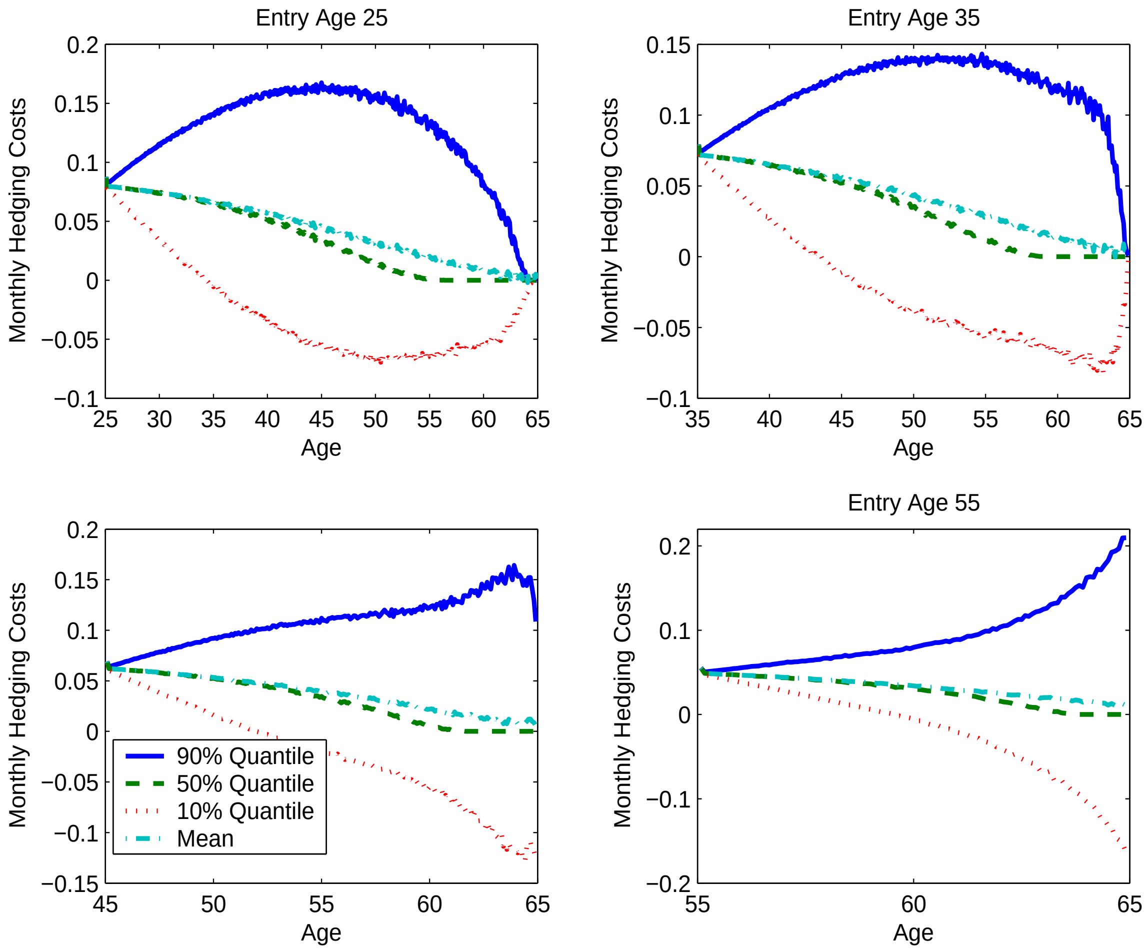


Figure 8.3: Quantile of Average Monthly Hedging Cost in Strategy 3

case, the forward rate is almost fixed. Hence the stochastic interest rate assumption does not cause much problem.

The hedging costs are very sensitive to the annuity rate. If values of parameters are changed in the interest rate model, the estimated annuity rate also changes. As an illustration, we change the mean reverting rate in the Vasiček model.

Table 8.5: Values of Parameters in the Vasiček Model

$dr(t) = (b - ar(t))dt + \sigma dZ$

Initial Rate $r(0)$	b	a	Volatility σ
0.05	0.028	0.35	0.025

In Table 8.5, we change the value of parameter b to increase the mean reverting rate from $0.02/0.35 \approx 0.057$ to $0.028/0.35 = 0.08$. Then, the estimated annuity rate is around 8.6. Assume all other assumptions are the same in scenario 1, we find the average monthly hedging costs as following.

Table 8.6: Average Monthly Hedging Costs with Stochastic Interest Rates as Percent of Monthly Salary in Funding Strategy 3 under Scenario 1

Entry Age	20	25	30	35
Hedging Costs	3.42	3.30	3.14	2.95
(Standard Errors)	(0.0064)	(0.0059)	(0.0056)	(0.0054)
Entry Age	40	45	50	55
Hedging Costs	2.75	2.50	2.21	1.86
(Standard Errors)	(0.0051)	(0.0049)	(0.0049)	(0.0048)

We find that the average monthly hedging cost in Table 8.6 is much lower than the first column in Table 8.3. Mean reversion is a tendency for a stochastic process to remain, or tend to return over time to a long-run average value. Since we reduce the mean reversion rate from 8% to 5.7%, the expected annuity rate goes up. Also, the interest rate is more unstable and is hard to reach the long term average. This also causes more risks and more hedging costs for employers.

8.5 Numerical Results for Strategy 4

Under funding strategy 4, the effect of stochastic interest rates is more complicated. We have discussed in Section 8.1 that hedging costs using strategy 4 are sensitive to the short interest rates. We keep the same assumptions as the last chapter except the interest rate follows a Vasiček process with parameters given in Table 8.2. We assume the interest rate is independent of the salary growth rate and the crediting rate.

At time t, the projected payoff at retirement age T of the pension guarantee can be expressed as

$$\begin{aligned} & \max(0, DB^T(t,T) - DC^T(t,T)) \\ = \; & \max(0, \alpha S_t a_{65}(t)t - DC^T(t,t)e^{\int_t^T f(u)du}) \\ = \; & \max(0, \alpha S_t a_{65}(t)t - DC^T(t,t)\frac{A_T}{A_t}) \end{aligned} \tag{8.15}$$

Equation (8.15) is calculated by adopting the traditional unit credit (TUC) cost method. The salary S_t, the value of the DC underlying asset and the current short rate $r(t)$ are known at time t. We can calculate the price of the annuity rate $a_{65}(t)$ using the forward risk-neutral valuation argument. Hence, equation (8.15) is similar to the payoff of a regular European put option. We implement funding strategy 4 to construct the hedging portfolio. The difference is that the interest rate is no longer deterministic. So the present value of the pension guarantee should be

$$\begin{aligned} & p(t,T)\max(0, DB^T(t,T) - DC^T(t,T)) \\ = \; & p(t,T)\max(0, \alpha S_t a_{65}(t)t - DC^T(t,t)\frac{A_T}{A_t}) \end{aligned} \tag{8.16}$$

where $p(t,T)$ is the price at time t of a zero-coupon bond with maturity time T.

Implementing the Black-Scholes valuation, we have the value of the pension guarantee at time t,

$$\begin{aligned} & c_{1,t}p(t,T)N(-d_{2,t}) - c_{2,t}A_tN(-d_{1,t}) \\ d_{1,t} = \; & \frac{\ln(c_{2,t}A_t/(c_{1,t}p(t,T))) + \frac{1}{2}\sigma^2(T-u)}{\sigma\sqrt{(T-u)}} \\ d_{2,t} = \; & d_{1,t} - \sigma\sqrt{(T-u)}. \quad 0 \le t \le u \le T \end{aligned} \tag{8.17}$$

where $c_{1,t} = \alpha \cdot S_t \cdot a_{65}(t) \cdot t$ and $c_{2,t} = \frac{DC^T(t,t)}{A_t}$.

Table 8.7 shows us monthly hedging costs under strategy 4 as a percentage of salaries and standard errors. The parameters of each scenario are given by Table 7.1 and parameters of the Vasiček model are given by Table 8.2. The stochastic interest rate affects not only the estimated annuity rate, but also the price of the zero-coupon bond $p(t,T)$. That causes higher and more volatile hedging costs. Since the mean reverting rate is approximately 5.7%, we also assume the discount rate is 5.7% and keep all other parameters same. Table 8.8 compares the monthly hedging cost and standard errors for both deterministic interest rate and stochastic interest rate. The hedging cost using strategy 4 is very sensitive to the interest rate. When the interest rate is stochastic, both hedging costs and standard errors are higher, especially for higher entry ages.

Table 8.7: Average Monthly Hedging Costs with Stochastic Interest Rates as Percent of Monthly Salary under Funding Strategy 4

Entry Age	Scenario 1	Scenario 2	Scenario 3	Scenario 4	Scenario 5	Scenario 6
25	1.56	1.53	2.29	0.45	0.56	2.25
	(0.0163)	(0.0172)	(0.0216)	(0.0057)	(0.0098)	(0.0210)
30	1.68	1.71	2.46	0.59	0.69	2.48
	(0.0164)	(0.0174)	(0.0214)	(0.0069)	(0.0102)	(0.0213)
35	1.83	1.87	2.50	0.77	0.86	2.77
	(0.0166)	(0.0174)	(0.0205)	(0.0078)	(0.0113)	(0.0211)
40	2.03	2.04	2.69	0.99	1.06	3.02
	(0.0166)	(0.0175)	(0.0200)	(0.0091)	(0.0123)	(0.0207)
45	2.20	2.19	2.78	1.25	1.29	3.38
	(0.0160)	(0.0170)	(0.0190)	(0.0099)	(0.0129)	(0.0204)
50	2.29	2.36	2.82	1.57	1.57	3.69
	(0.0152)	(0.0163)	(0.0172)	(0.0111)	(0.0132)	(0.0197)
55	2.44	2.47	2.85	1.94	1.88	4.07
	(0.0145)	(0.0152)	(0.0157)	(0.0118)	(0.0136)	(0.0183)

Figure 8.4 shows us histograms for average monthly hedging cost under scenario 1. The tails of histogram are very heavy in all four figures. This means the volatilities of hedging costs are very high. Figure 8.5 shows us quantile average monthly hedging cost under scenario 1. Since the TUC cost method only considers the current salary, employers underestimate the value of the hedging portfolio in the beginning of the plan. Hedging costs are very high and unstable when employees are close to the retirement age.

Given the stochastic interest rate assumption, results are different in strategy 3 and strategy 4. In strategy 3, the only factor which is affected by stochastic interest rates is the annuity rate. With the mean-reverting assumption, hedging costs do not change too much since we consider a long-term forward valuation. However, hedging costs are very sensitive to the interest rate in strategy 4, as the monthly

Table 8.8: Average Monthly Hedging Costs as Percent of Monthly Salary with Deterministic/Stochastic Interest Rates Under Scenario 1

Entry Age	Deterministic Interest Rate(5.7%)	Stochastic Interest Rate
25	1.32	1.56
	(0.0136)	(0.0163)
30	1.49	1.68
	(0.0137)	(0.0164)
35	1.66	1.83
	(0.0134)	(0.0166)
40	1.80	2.03
	(0.0128)	(0.0166)
45	1.98	2.20
	(0.0120)	(0.0160)
50	2.13	2.29
	(0.0107)	(0.0152)
55	2.27	2.44
	(0.0090)	(0.0145)

hedging cash flows are highly related to the price of zero-coupon bond.

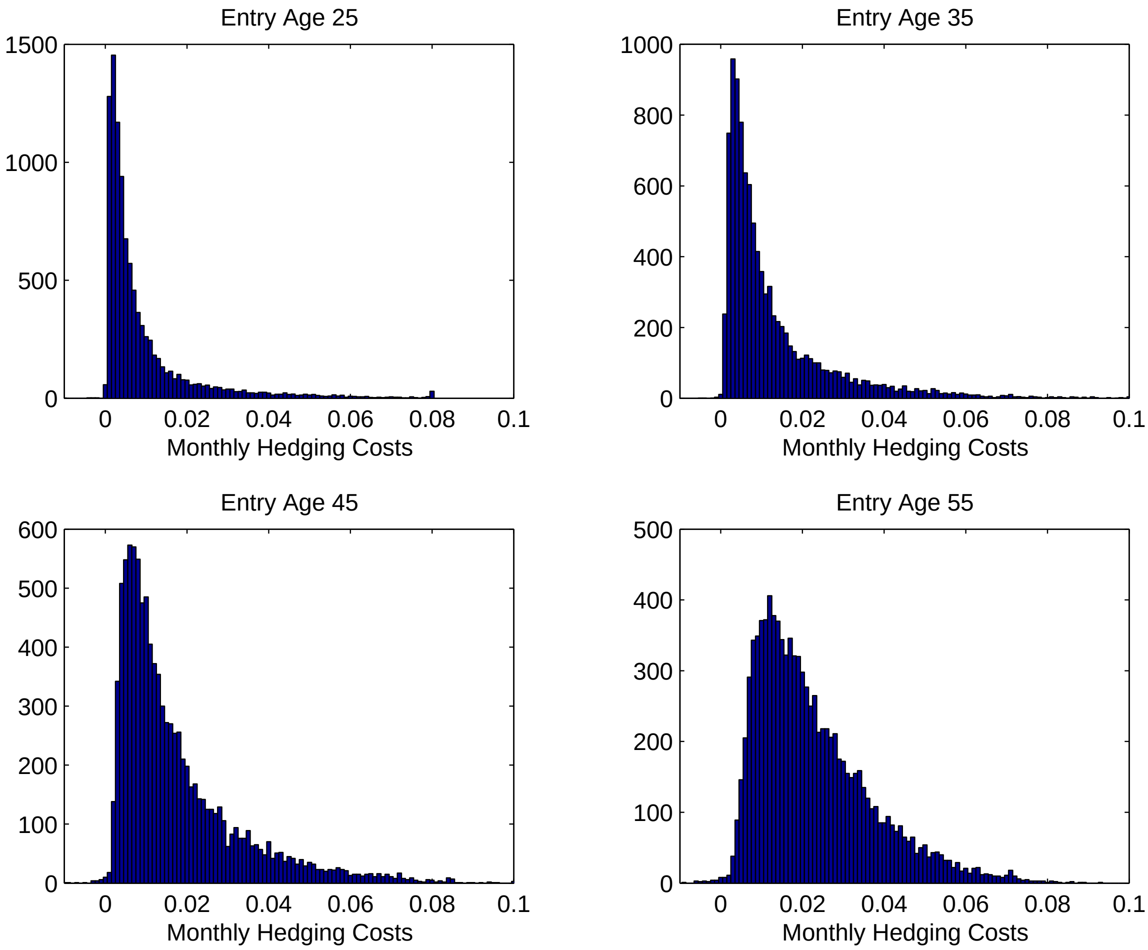


Figure 8.4: Histogram for Average Monthly Hedging Cost in Strategy 4

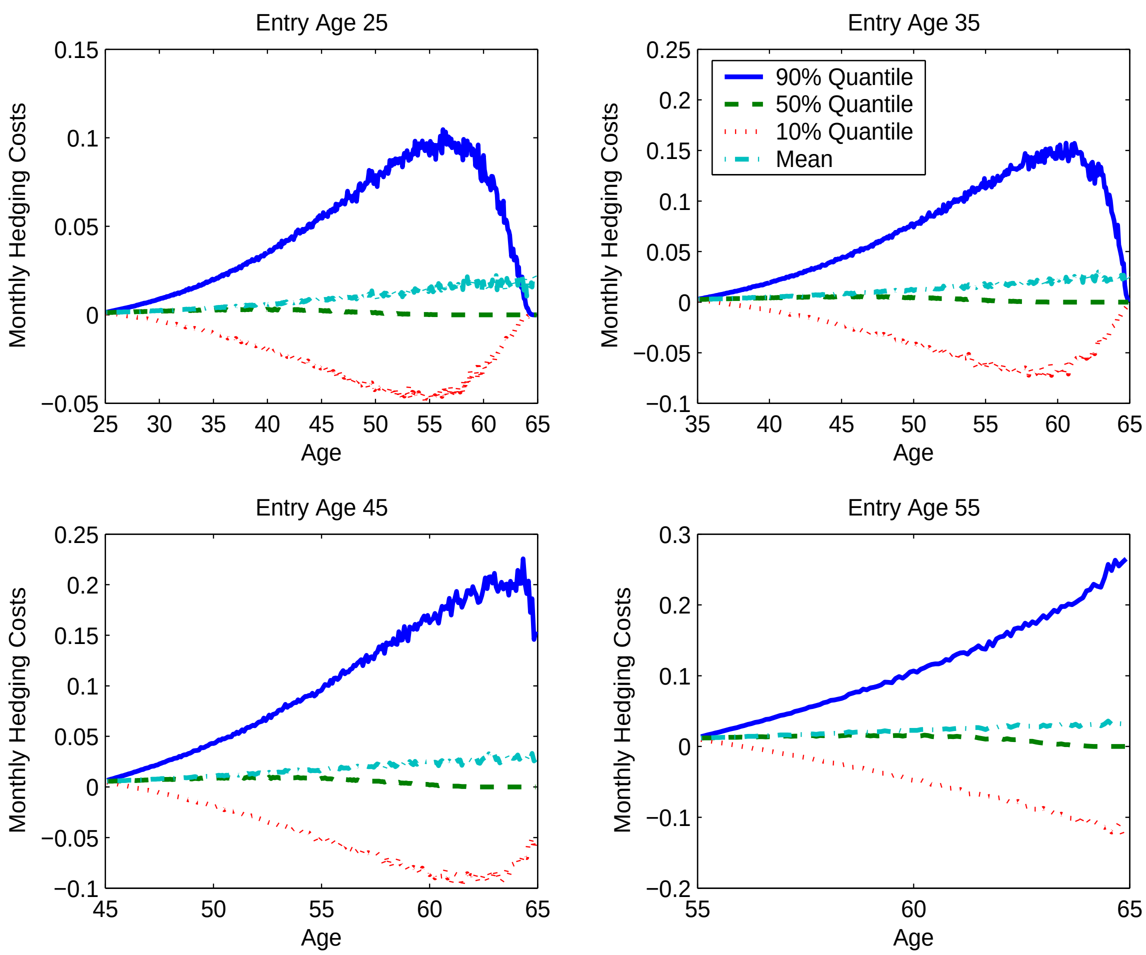


Figure 8.5: Quantile of Average Monthly Hedging Cost in Strategy 4

Chapter 9

Costs Control

9.1 Introduction

In the last three chapters we have shown that hedging cash flows can be very volatile, even though the expected average hedging costs may be reasonable. However, the average monthly hedging costs are time dependent. The results demonstrate hedging costs are particularly unstable when employees are close to retirement. Obviously, employers want to reduce their risks and stabilize their hedging cash flows. In this chapter, we will propose some alternative ways to smooth hedging cash flows and to reduce the monthly hedging costs.

9.2 Unstable Hedging Cash Flows and Hedging Cost Spikes

Figure 9.1 and Figure 9.2 show us five sample paths of monthly hedging costs with different entry ages under strategies 3 and 4. The parameters here are given by scenario 1 in Table 5.1 and interest rates follow a Vasiček model.

From these figures, we see that there is a significant chance that monthly hedging costs are very high. In this chapter, we will discuss some approaches for costs control. When the hedging cash flows are very high, we use alternative ways to reduce the spiked cash flows. We note that in a DB underpin pension plan, employers can control salaries and they can also determine how to construct the hedging portfolio.

Given the mortality rate, there are three random factors in our model: salary growth rate, crediting rate, and interest rate. Generally, employers or pension sponsors can control the first two factors. The salary growth rate is determined by employers. If the cost is exorbitant, employers may choose to give smaller increases in employees' salaries. Although the crediting rate is affected by the financial market, employers may have the right to construct the investment portfolio by choosing either low-risk low-return securities or high-risk high-return securities. The selection of securities determines the expected return and volatility of the DC account.

Figure 9.1 and Figure 9.2 show us sample paths of monthly hedging cash flows. First, we explore what causes the volatility in the hedging costs, using a scatter plot. In Figure 9.3 and Figure 9.4 we plot the salary growth rate and the crediting rate for a new entrant age 35. We have superimposed crosses where the monthly hedging costs spikes over 50%. We can see there are more spikes in Figure 9.4 than

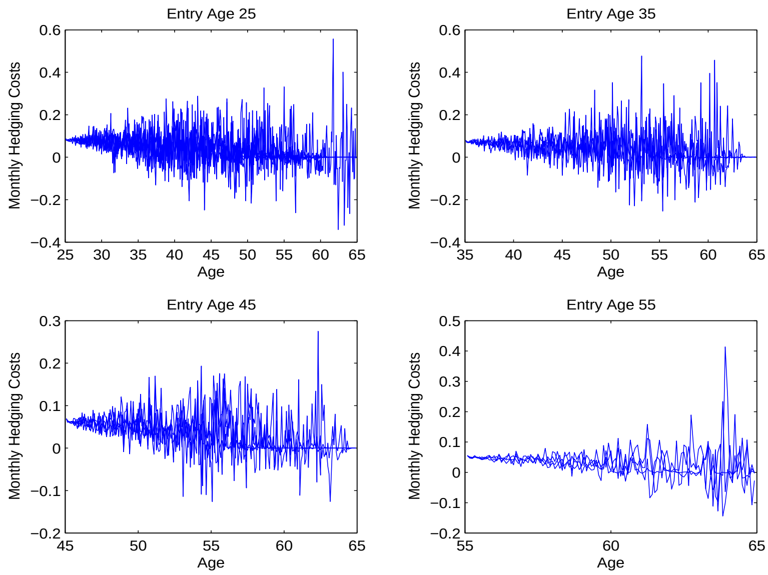


Figure 9.1: 5 Sample Paths of Monthly Hedging Cost under Strategy 3

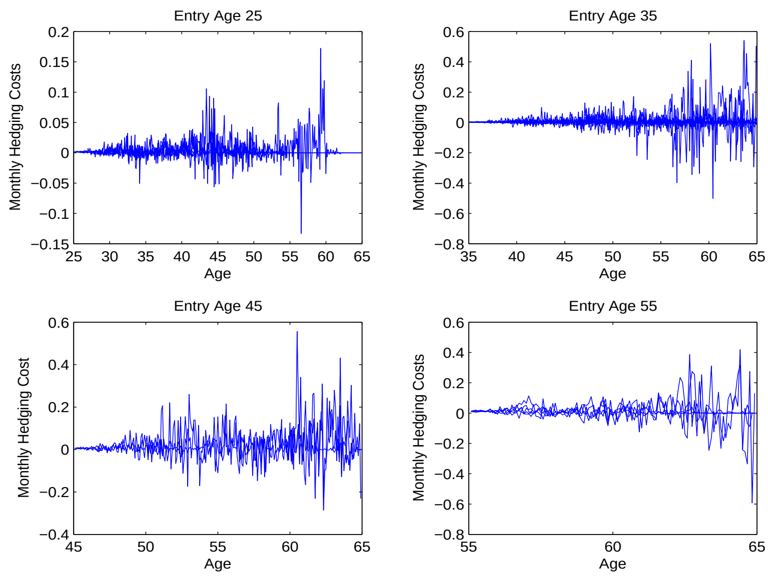


Figure 9.2: 5 Sample Paths of Monthly Hedging Cost under Strategy 4

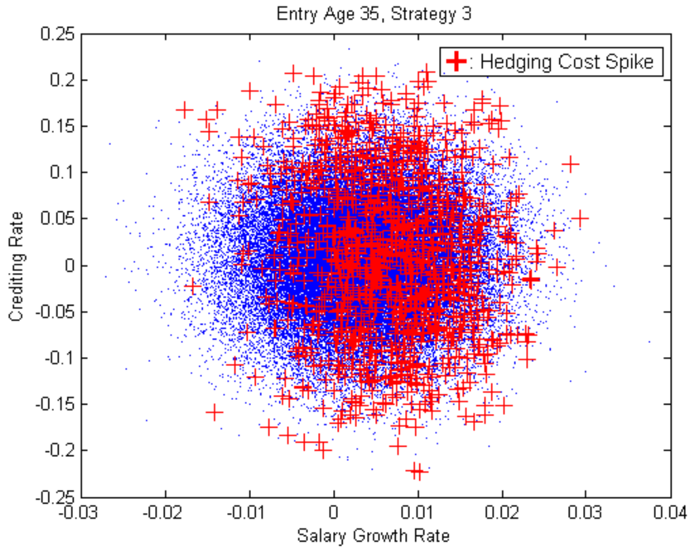


Figure 9.3: Scatter Plot of Salary Growth Rates and Crediting Rates, Strategy 3, Spike Level:50%

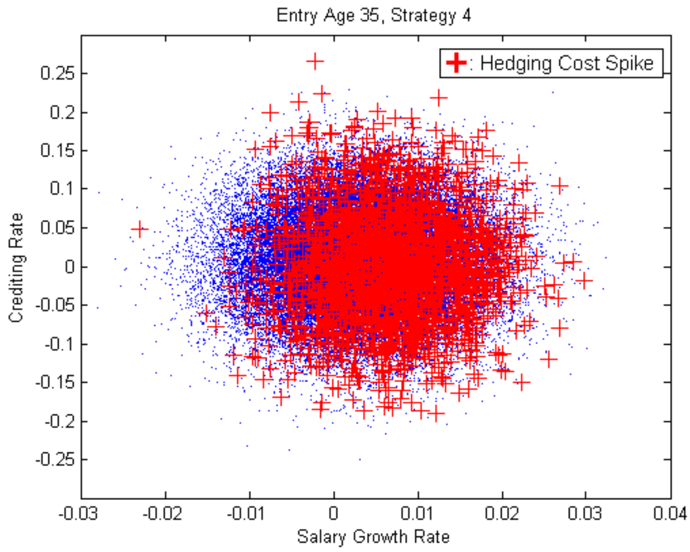


Figure 9.4: Scatter Plot of Salary Growth Rates and Crediting Rates, Strategy 4, Spike Level:50%

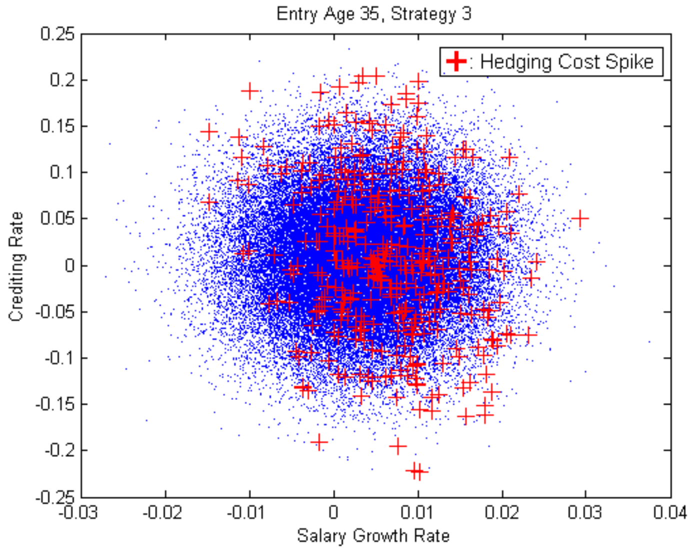


Figure 9.5: Scatter Plot of Salary Growth Rates and Crediting Rates, Strategy 3, Spike Level:80%

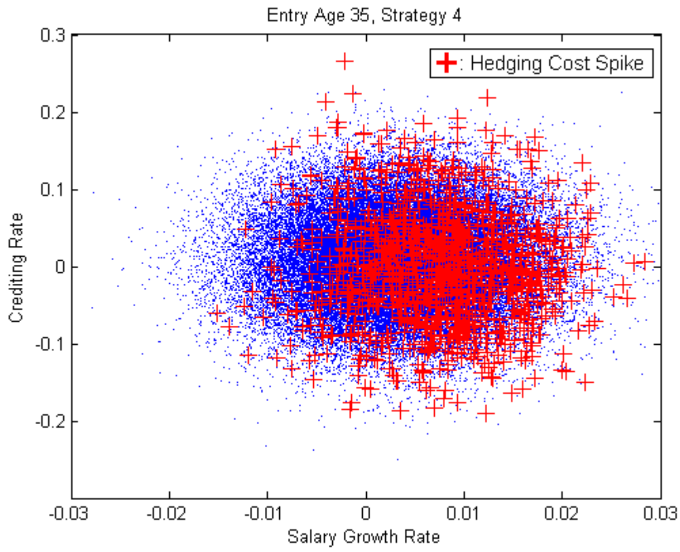


Figure 9.6: Scatter Plot of Salary Growth Rates and Crediting Rates, Strategy 4, Spike Level:80%

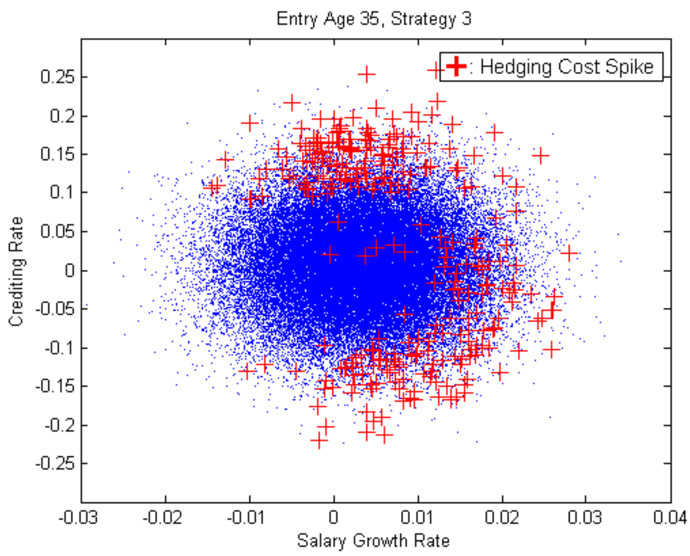


Figure 9.7: Scatter Plot of Salary Growth Rates and Crediting Rates, Strategy 3, Spike Level:50%

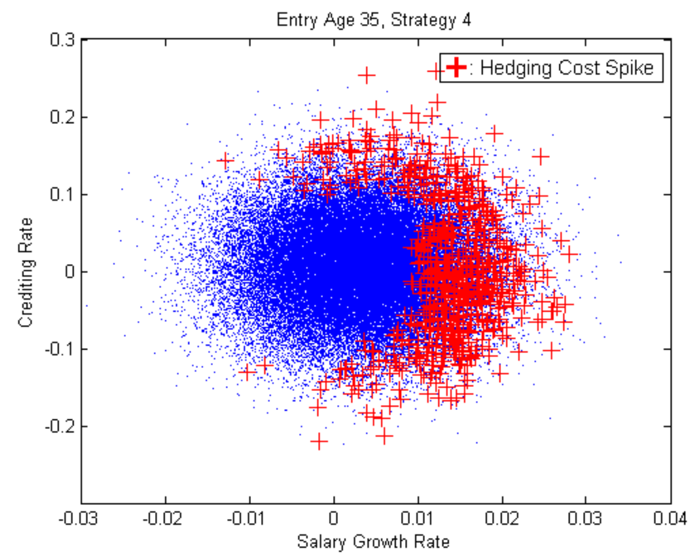


Figure 9.8: Scatter Plot of Salary Growth Rates and Crediting Rates, Strategy 4, Spike Level:50%

in Figure 9.3. Since funding strategy 4 is highly related to the interest rate, the randomness of the interest rate causes higher volatilities and more hedging costs spikes. Moreover, it seems there are more spikes on the right side of figures where the salary growth rate is high, though this is not entirely clear for these plots. In Figure 9.5 and Figure 9.6, we have superimposed crosses where the monthly hedging costs spikes over 80%. Some spikes disappear when salary growth rates are lower. To see this clearly, we ignore the effect of stochastic interest rates and assume the interest rate is constantly equal to 5%. In Figure 9.7 and Figure 9.8, we see that most spikes are associated with increase of more than around 1% per month in salary. There are still some spikes associated with values more than around 10% per month or less than -10% per month in crediting rate. Besides the effect of stochastic interest rates, spikes can be caused by high salary growth rates and by volatile crediting rates. In Section 9.3 and Section 9.4, we will discuss two alternative approaches to control the hedging costs. We will also consider how to reduce the interest rate risk as further work.

9.3 Salary Growth Rate Control

Although the salary affects both the DB and DC accounts, it causes more significant effects on the DB guarantee cost when the security market goes down. When the salary is high and crediting rate is low, the projected guarantee moves further into-the-money. So employers are requested to deposit more money into the hedging portfolio. That causes a spike in the hedging cash flows. However, employers want their hedging costs to be smooth. We propose two ways to control the impact of salary increase. Both salaries and pension are benefits provided by employers. They do not have to offer both high salaries and high retirement benefits.

Table 9.1: Average Monthly Hedging Costs with Stochastic Interest Rates as Percent of Monthly Salary in Funding Strategy 3

Entry Age	No Costs	Salary Growth Control			
	Control	1 Month	3 Months	6 Months	12 Months
25	4.68	4.41	4.35	4.21	4.08
	(0.0086)	(0.0082)	(0.0082)	(0.0079)	(0.0080)
30	4.50	4.33	4.27	4.15	4.05
	(0.0083)	(0.0081)	(0.0079)	(0.0077)	(0.0077)
35	4.32	4.18	4.13	4.02	3.94
	(0.0082)	(0.0079)	(0.0078)	(0.0073)	(0.0075)
40	4.14	4.15	4.03	3.99	3.84
	(0.0079)	(0.0077)	(0.0074)	(0.0071)	(0.0073)
45	3.93	3.82	3.79	3.72	3.68
	(0.0079)	(0.0076)	(0.0073)	(0.0071)	(0.0070)
50	3.63	3.59	3.57	3.53	3.50
	(0.0077)	(0.0075)	(0.0075)	(0.0071)	(0.0071)
55	3.31	3.29	3.29	3.28	3.27
	(0.0078)	(0.0078)	(0.0077)	(0.0075)	(0.0076)

Table 9.2: Average Monthly Hedging Costs with Stochastic Interest Rates as Percent of Monthly Salary in Funding Strategy 4

Entry Age	No Costs	Salary Growth Control			
	Control	1 Month	3 Months	6 Months	12 Months
25	1.56	1.21	1.14	1.00	0.94
	(0.00163)	(0.0121)	(0.0109)	(0.0093)	(0.0087)
30	1.68	1.37	1.29	1.14	1.07
	(0.0164)	(0.0126)	(0.0116)	(0.0100)	(0.0094)
35	1.83	1.57	1.48	1.33	1.25
	(0.0166)	(0.0133)	(0.0123)	(0.0106)	(0.0100)
40	2.03	1.76	1.69	1.53	1.45
	(0.0166)	(0.0136)	(0.0127)	(0.0113)	(0.0106)
45	2.20	2.01	1.94	1.80	1.71
	(0.0160)	(0.0143)	(0.0135)	(0.0122)	(0.0115)
50	2.29	2.22	2.17	2.06	1.98
	(0.0152)	(0.0144)	(0.0139)	(0.0128)	(0.0122)
55	2.44	2.43	2.42	2.37	2.32
	(0.0145)	(0.0141)	(0.0139)	(0.0133)	(0.0129)

The most simple control for the employer is to stop increasing the salary for a period, such as one month or half a year. Employees still pay monthly contributions as a proportion of their salaries. This control does not only affect the value of DB benefit, but also the value of DC account. It will decrease the value of both benefits, but will have a more significant effect on the DB benefit.

We explore a strategy whereby, if the monthly hedging cash flow is higher than 50% of the monthly salary, employers will stop increasing the salary for a while. That is employers set a 0% salary cap when there is a hedging cost spike. Tables 9.1 and 9.2 show the average hedging costs after applying the cap for both strategy 3 and strategy 4. We can see that the salary limit control reduces both monthly hedging costs and standard errors using strategy 3 and strategy 4. The longer the salary limiting period, the more reduction on hedging costs.

In strategy 3, we adopt the projected unit credit cost method. The DB guarantee is valued using the projected final salary. So the salary limit control does not cause a significant reduction in the monthly hedging costs. However, we use the traditional unit credit cost method in strategy 4. The DB guarantee is valued using the current salary. Employers underestimate the value of DB guarantee in the beginning of the plan and the monthly hedging cash flows include the impact of salary increases. After we implement the salary limit control, Table 9.2 shows us that monthly hedging costs with salary limit are much lower than those without salary limit.

The monthly hedging costs volatility can be significantly reduced, by smoothing out the salary growth rate. Figure 9.9 and Figure 9.10 demonstrate the difference in strategy 3 of 95% quantile and mean of monthly hedging costs among no salary limit, one-month limit and twelve-month limit. We still generate 10,000 paths. The value of parameters are given by scenario 1 in Table 7.1. The stochastic interest rate follows a Vasiček model with parameters in Table 8.2. We find that the only

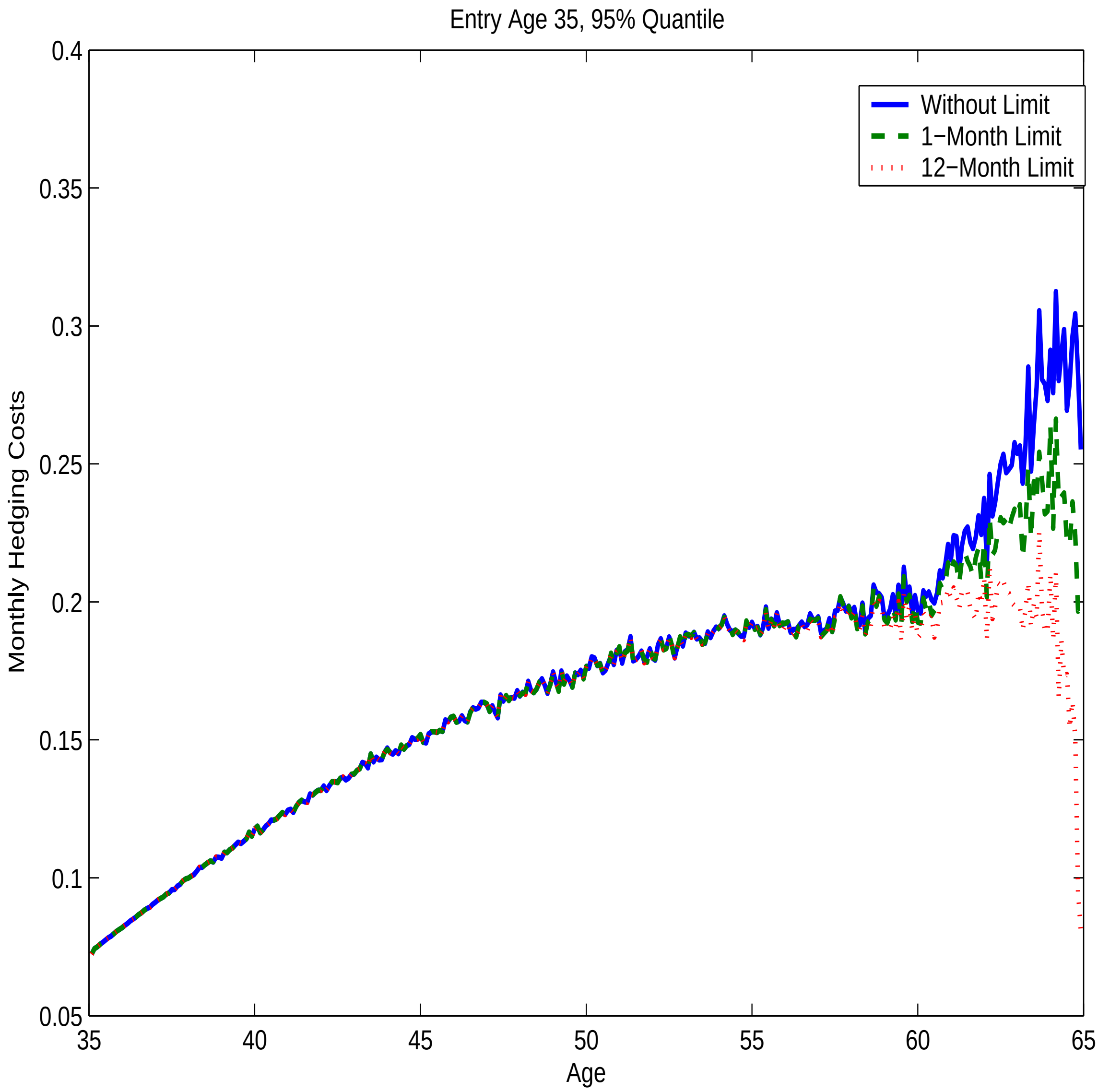


Figure 9.9: Comparison of 95% Monthly Hedging Costs Quantiles using Strategy 3 with Stochastic Interest Rate Assumption, Without and With Salary Limit

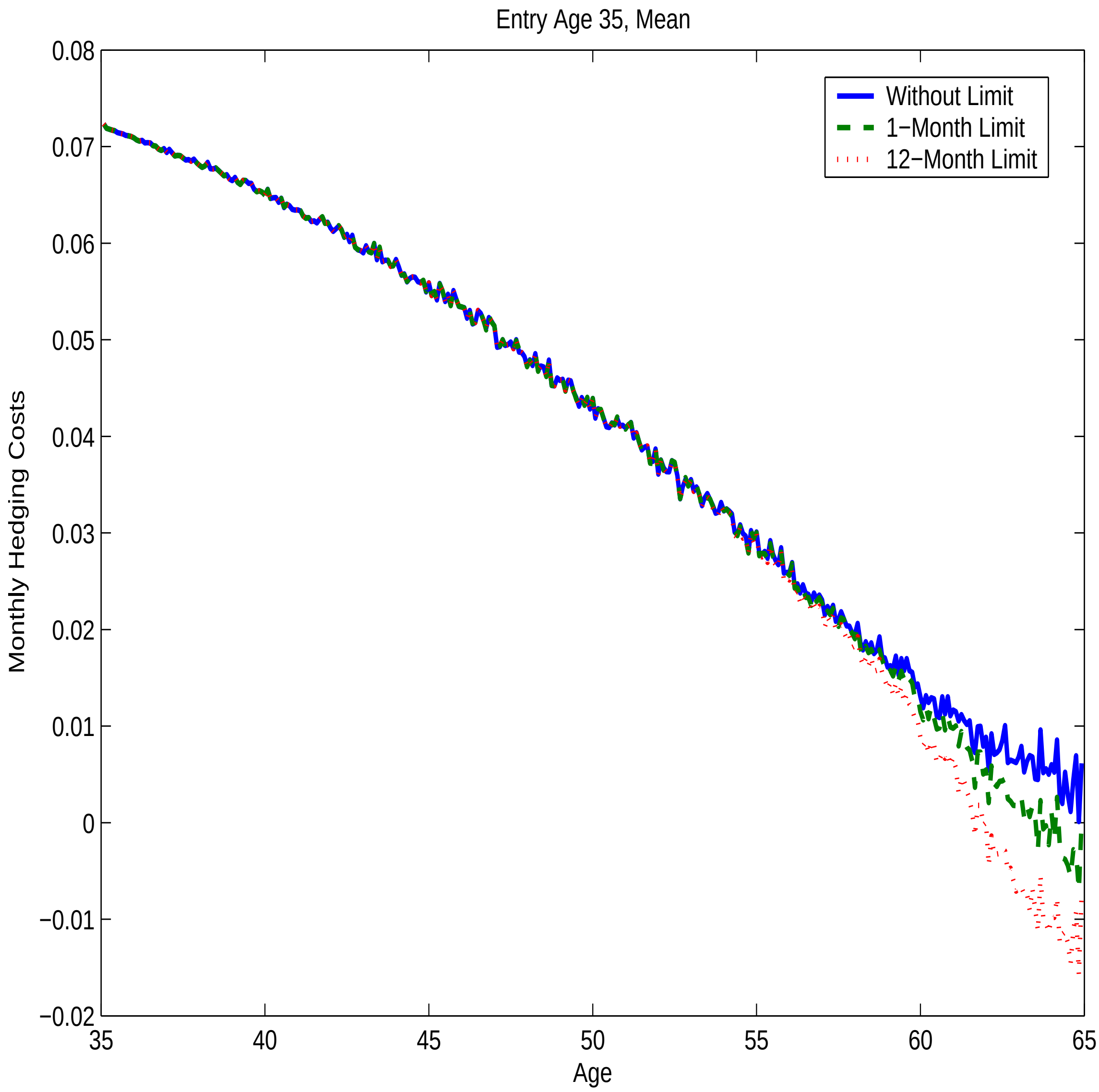


Figure 9.10: Comparison of Mean of Monthly Hedging Costs using Strategy 3 with Stochastic Interest Rate Assumption, Without and With Salary Limit

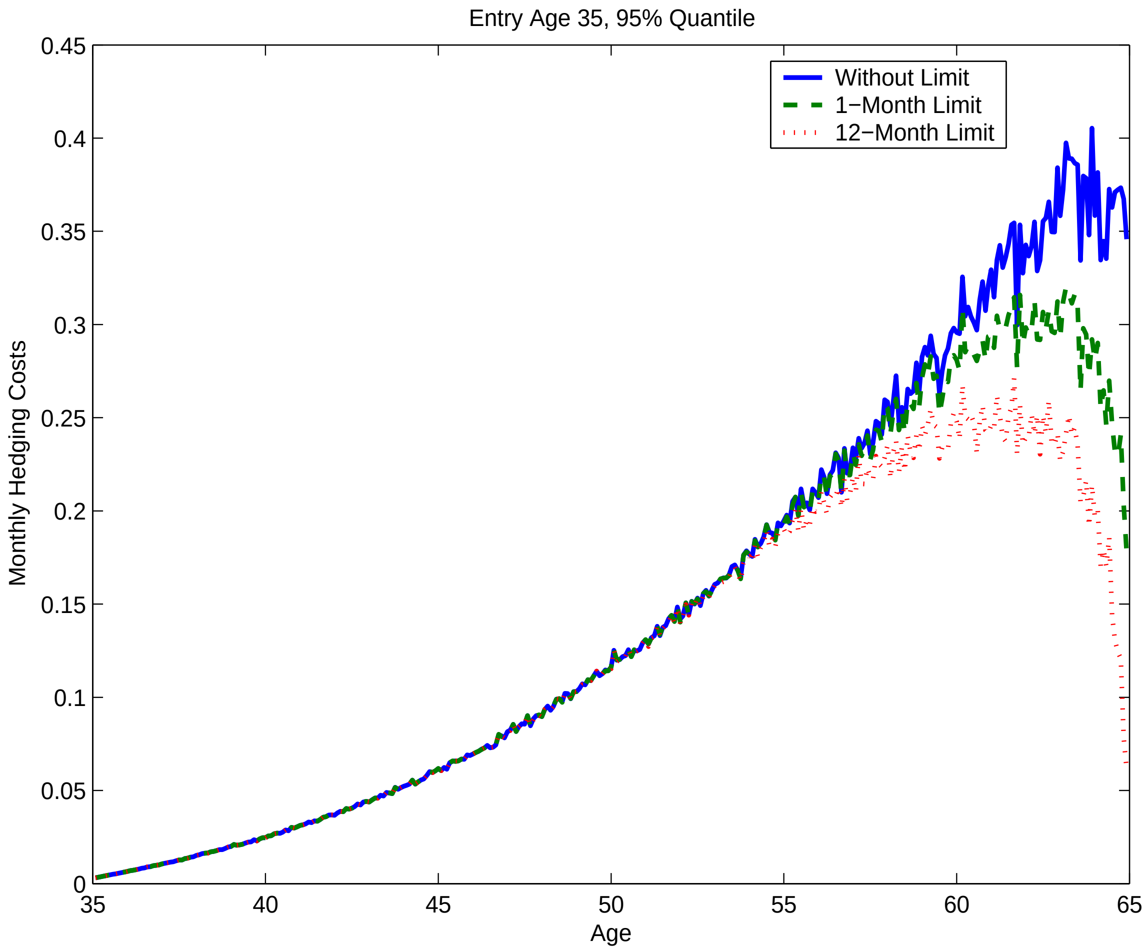


Figure 9.11: Comparison of 95% Monthly Hedging Costs Quantiles using Strategy 4 with Stochastic Interest Rate Assumption, Without and With Salary Limit

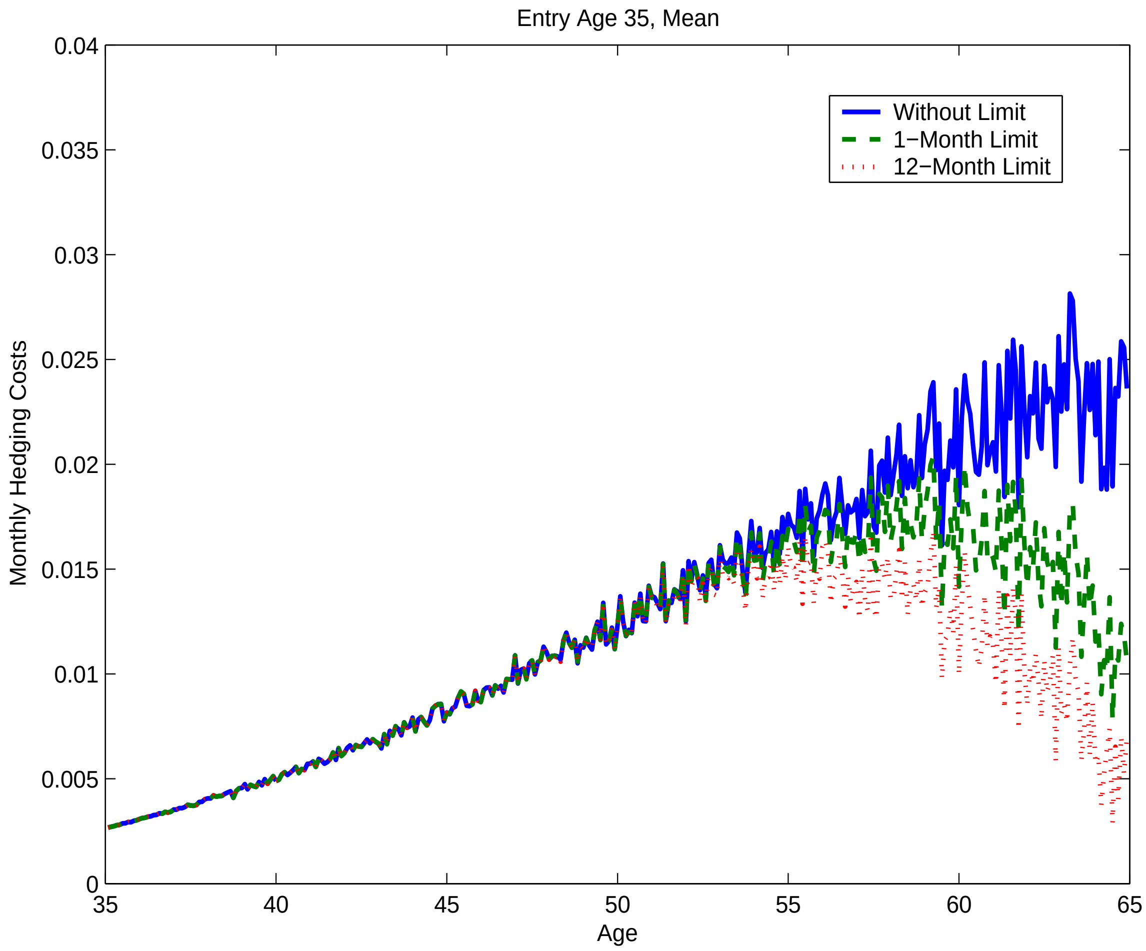


Figure 9.12: Comparison of Mean of Monthly Hedging Costs using Strategy 4 with Stochastic Interest Rate Assumption, Without and With Salary Limit

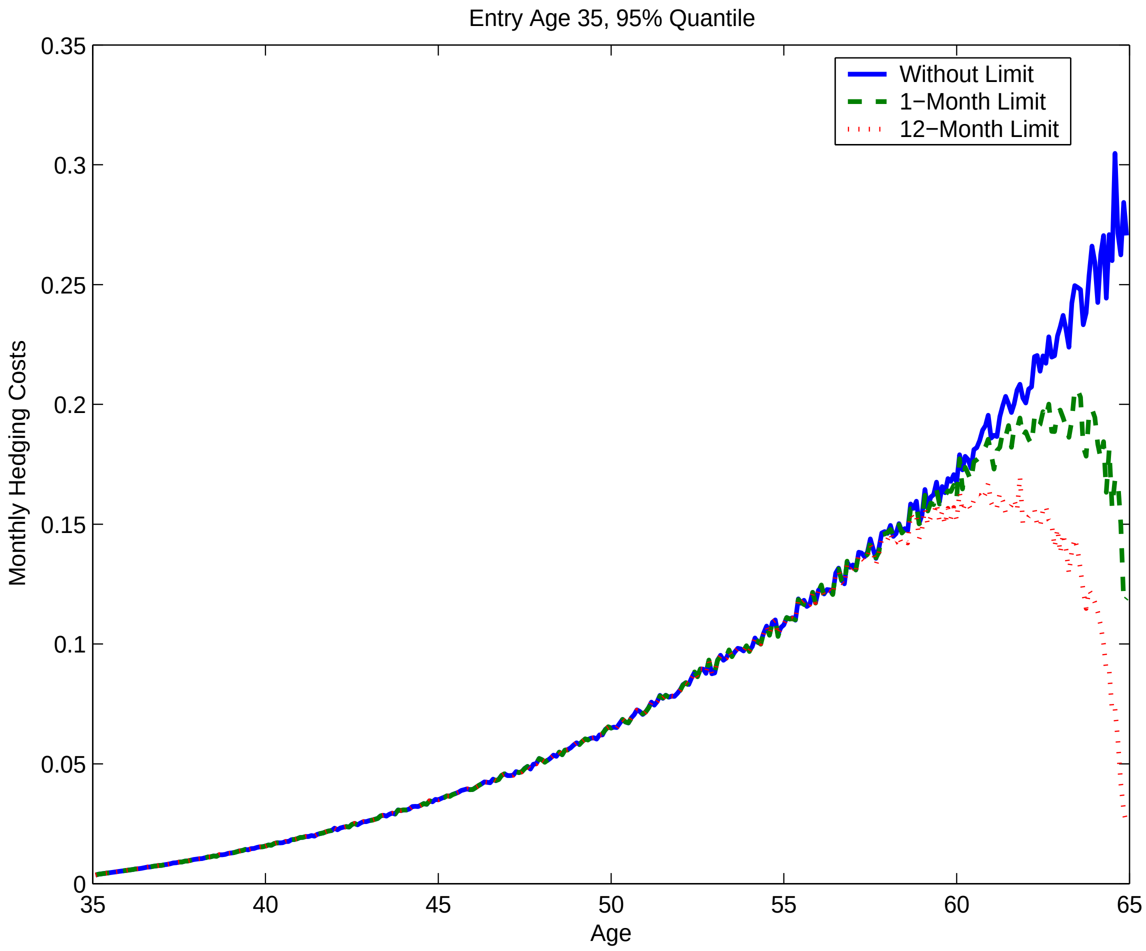


Figure 9.13: Comparison of 95% Monthly Hedging Costs Quantiles using Strategy 4 with Deterministic Interest Rate Assumption, Without and With Salary Limit

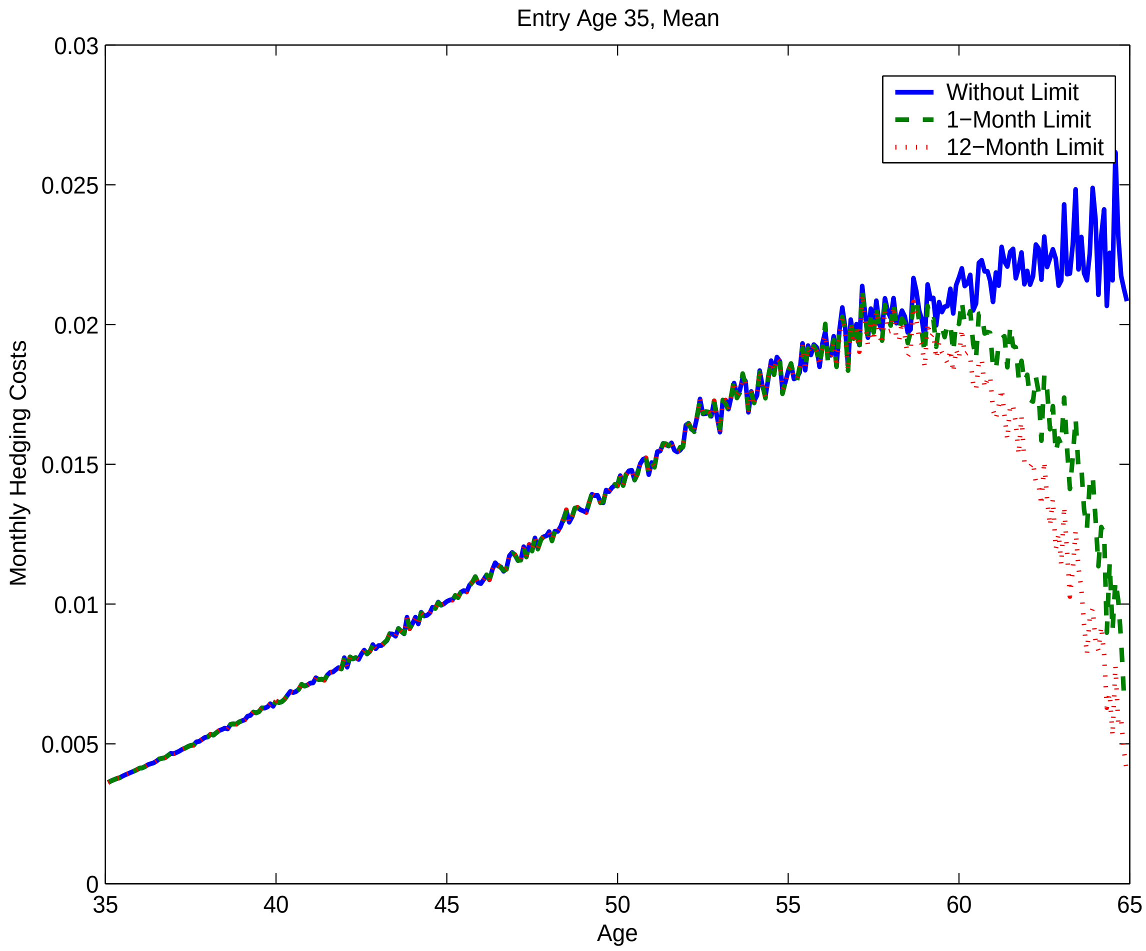


Figure 9.14: Comparison of Mean of Monthly Hedging Costs using Strategy 4 with Deterministic Interest Rate Assumption,Without and With Salary Limit

difference occurs approximately after age 60. Although the difference in the mean of hedging costs is small, the reduction in the 95% quantile means employers face less risks with salary limit control.

In Figure 9.11 and Figure 9.12, we can compare the change from using a salary cap in strategy 4, considering the 95% quantile and the mean of monthly hedging costs. As we analyzed before, the monthly hedging costs in strategy 4 are very volatile at the end of the plan. The salary limit control works here even better than in strategy 3. Both the 95% quantile and the mean of hedging costs drop a lot near retirement. If we assume no interest risk, Figure 9.13 and Figure 9.14 demonstrate results of 95% quantile and mean under the deterministic interest rate assumption. The mean of monthly hedging costs is very smooth. This shows that the salary limit control can reduce the salary risk and the volatility of monthly hedging costs.

We note that by changing the salary process to introduce dynamic cost control, the real world process is no longer consistent with the risk neutral measure we need, which assumes unconstrained salaries.

9.4 Arithmetic Average on Salaries

In all final salary pension plans, the benefit will actually be based on the average salary in the period before retirement. In such a plan, the final DB benefit is calculated by the average of the final 3 or 5 years salary. This policy is usually specified in the pension contract at the beginning of the plan. There are two benefits of using final average salary for our model. First, using the average of salaries can reduce the volatility. Second, our model considers a DB pension guarantee. Since employees' salaries usually increase all the time, the final average salary is less than the final salary. This causes a lower DB benefit, and therefore a lower DB pension guarantee.

When we consider the final average salary, our model is similar to an asian option problem. There are some numerical approaches to calculating the regular European type asian option, such as Kemna and Vorst(1990), and Turnbull and Wakeman(1991). First Kemna and Vorst(1990) use Monte-Carlo simulation. However, it is time-consuming and does not lead to the hedging portfolio. A faster approach based on an Edgeworth expansion around lognormal distribution has been given by Turnbull and Wakeman(1991). Levy(1992) also remarks that the higher terms in the Edgeworth expansion have a negligible numerical value for reasonable values of the parameters of the model and proposes simply to use a Lognormal approximation for pricing, giving a closed-form formulae for the approximate price.

However, there are three random variables in our model, the salary, the DC underlying asset, and the interest rate. For simplicity, we consider a constant interest rate case. In strategy 3, the DB benefit is calculated by the projected salary. It is like a asian exchange option. In strategy 4, the DB benefit is calculated by the current salary. When we consider the average salary plan, all past years salaries are known. So it is still a European option problem in strategy 4 with different strike price.

Here, we only use strategy 4 to consider a simple model. Figure 9.15 shows the mean of the monthly hedging cost with entry age 35. The 1-year average and 5-year average hedging costs are lower than the original one. Since the average salary is usually less than the current salary, the expected value of DB pension guarantee is also lower. The average salary plan also reduces the volatility of the monthly hedging cost. Figure 9.16 shows the 95% quantile of the monthly hedging cost with

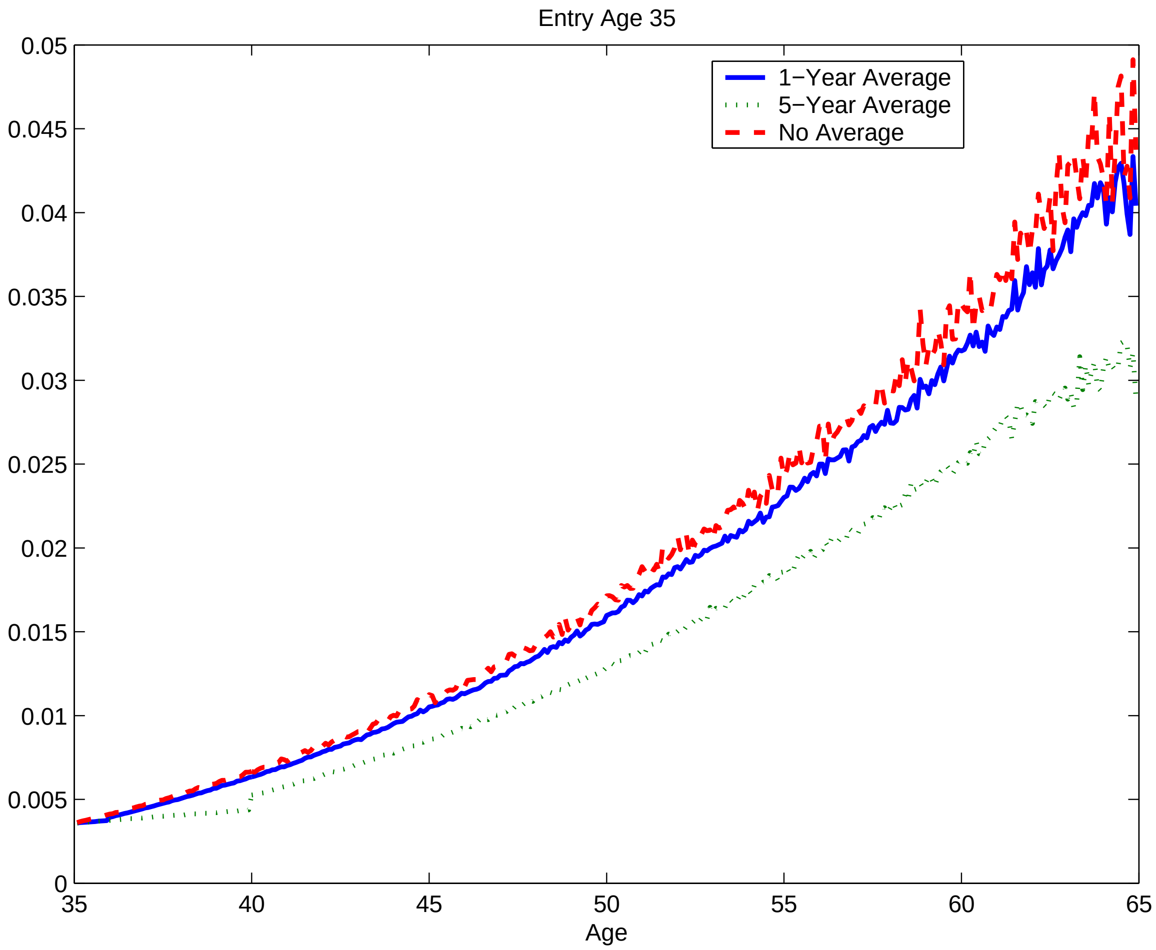


Figure 9.15: Comparison of Mean of Monthly Hedging Costs using Strategy 4 with Different Average Period for the final average salary

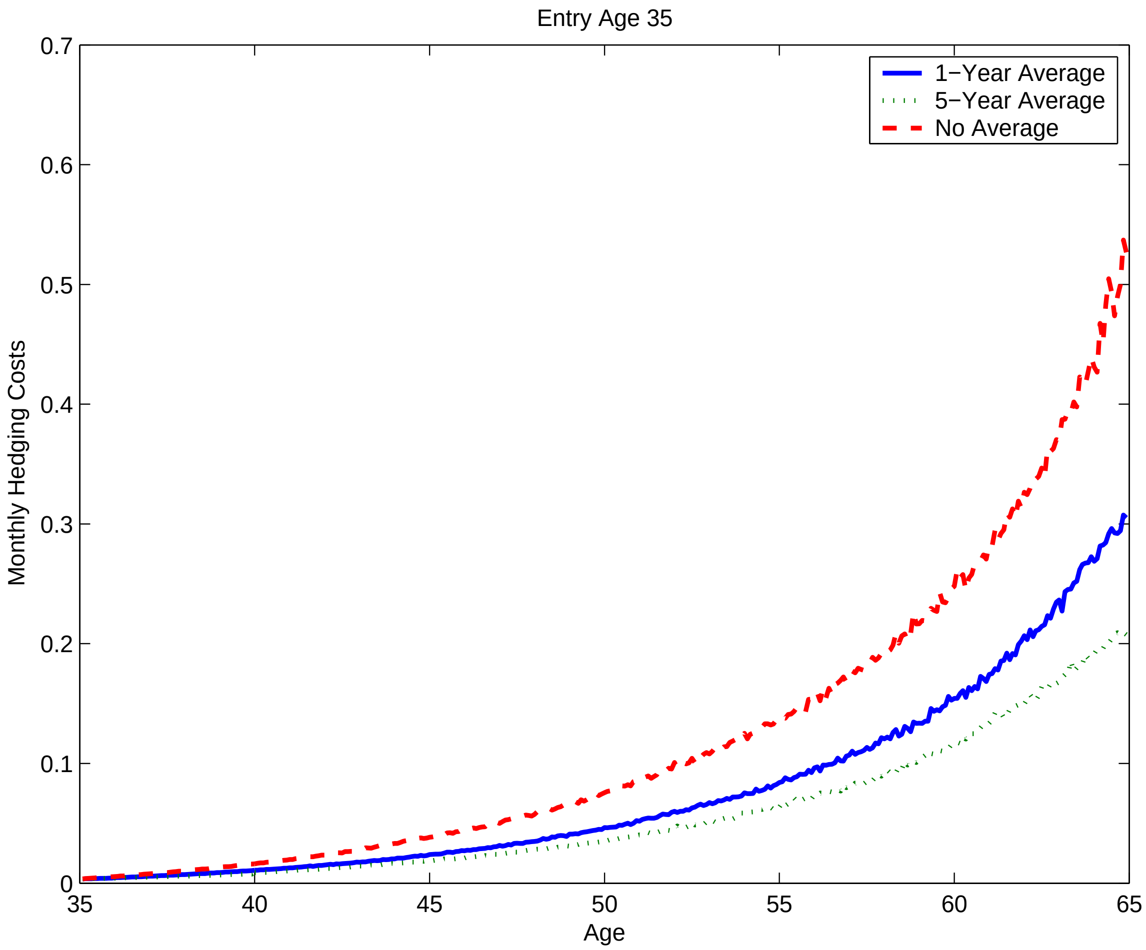


Figure 9.16: Comparison of 95% Monthly Hedging Costs Quantiles using Strategy 4 with Different Average Period for the final average salary

entry age 35. The 1-year average and 5-year average quantiles are significantly lower than the no average one. This is also because the volatility of the average salary is lower than the volatility of the salary. Hence, the average pension plan has both lower expected costs and lower volatilities.

In strategy 3, the calculation is more complicated, since both salaries and DC underlying assets are projected to the retirement. In the further work, we will use the Edgeworth expansion to approximate the average salary process to apply strategy 3.

9.5 Other Cost Control Methods

In the last two sections, we proposed cost control approaches by limiting the salary increase dynamically to smooth the spike of hedging cash flows and introducing the average salary. The salary cap stabilizes the hedging cash flows, especially when employers are getting older. We also consider other cost control methods based on the crediting rate or the interest rate.

9.5.1 Cap and Floor on the DC Account

The value of DC account depends on the amount of contributions and the investment performance. Usually, the contribution rate is specified by the pension policy. Some employers, such as York University, allow voluntary contributions. However, the extra contributions will not be used to compare with the DB benefits at retirement. Therefore, employers' investment choices significantly affect the value of DC account. In most DB underpin pension plans, DC accounts are managed by professional investment companies. They construct the investment portfolio by equities, bonds, and other financial securities, but the overall strategy and risk tolerance is determined by the plan sponsor.

The estimated value of DB guarantee is negatively correlated to the value of DC account. If the crediting rate is high, the projected DB guarantee is lower. For a higher value of the DC account, the estimation of the DB guarantee is usually out-of-the-money. So the value of hedging portfolio is also lower. However, due to the volatility, the difference between the projected value and the actual value can be huge at the next valuation time. The DB guarantee easily changes from out-of-the-money to in-the-money. So the hedging cash flow may be huge because of the volatility of the crediting rate.

On the opposite side, the projected DB guarantee is higher, if there is a lower crediting rate. The estimation of the DB guarantee is in-the-money. This causes a higher value of the hedging portfolio. If the market goes up, the projected value of DB guarantee decreases at the next valuation time. There exists a lower hedging cash flow, or even negative cash flow.

Usually, the pension fund is managed by the Pension Administration Committee, such as at Mcgill University. The DC underlying portfolio is constructed by equities and fixed income securities. A higher return or a lower volatility can reduce the monthly hedging cost. However, due to the risk-return trade off, it will be difficult to determine an optimal balance. We leave this for future work.

9.5.2 Interest Rate Derivatives

Another random factor in our model is the interest rate. In strategy 3, the interest rate will only affect the annuity factor. Figure 9.3 and Figure 9.7 show spikes of hedging cash flows with/without stochastic interest rates. In strategy 4, the

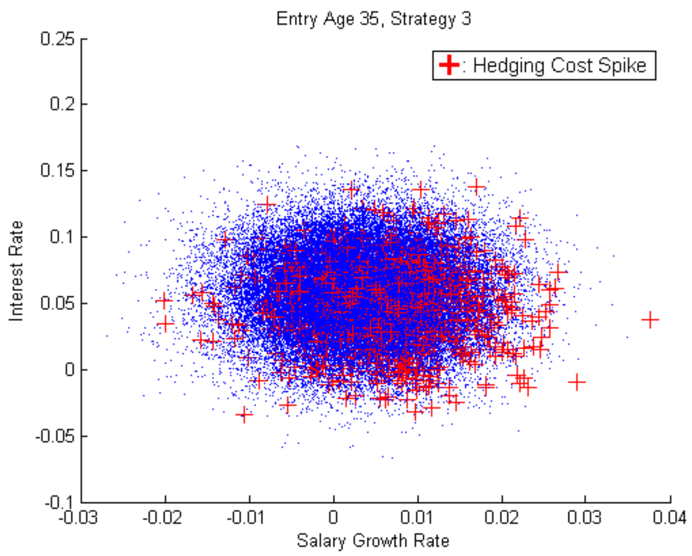


Figure 9.17: Scatter Plot of Salary Growth Rates and Interest Rates, Strategy 3, Spike Level:50%

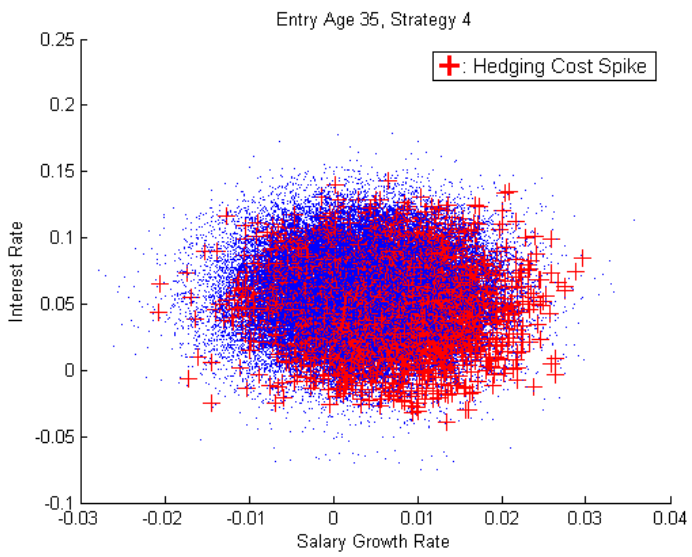


Figure 9.18: Scatter Plot of Salary Growth Rates and Interest Rates, Strategy 4, Spike Level:50%

interest rate affect more than in strategy 3. Figure 9.4 and Figure 9.8 show spikes of hedging cash flows with/without stochastic interest rates. In Chapter 8, we have compared hedging costs with/without stochastic interest rates.

Figure 9.17 and Figure 9.18 plot scatters of the salary growth rate and the interest rate. It is hard to say how stochastic interest rates affect the hedging cash flow from two figures. Since the DB underpin guarantee is usually exercised for a long term, some interest rate derivatives may be used to reduce the volatility arising from the interest volatility. In further work, we will consider using such derivatives to control interest risk.

Chapter 10

Comments and Further Work

10.1 Salary Growth Rate

Since the DB underpin pension guarantee is highly related to the salary, the assumption of the salary growth rate becomes a very important issue. It is hard to predict the future salary change. In our model, we assume salary follows a Brownian geometric motion. This is a simple but not necessarily practical assumption, since it may generate negative salary growth. There are many studies about how to model salary growth. We have analyzed the high correlation between salary and inflation in Chapter 3. This may help us to predict the salary growth rate. In future work, we will consider more extensive integrated models for the salary growth and inflation rates.

10.2 Other Risk Management Approaches

Pension fund actuaries are interested in risk management. They are worry about how much risk they should take if they offer the pension plan. We have moved our interest and research focus to the risk management and the pension fund hedging. For the DB underpin guarantee we have calculated the first order derivative of and implement delta hedging. In the future, we will consider more complicated hedging approaches and introduce some derivatives to hedge the risk of the DB guarantee.

We have shown the relationship between salary and financial indices in Chapter 6. We used inflation and risk-free bonds to partially hedge the salary in Chapter 7 because of the high correlation between salary and inflation. We assumed that inflation was tradable through inflation-linked bonds. These bonds have not been widely used in the U.S. and Canada. A problem is that inflation-linked bonds are rather more volatile than inflation, but also give a real return, typically of 1-2%. We plan to introduce inflation-linked bonds into our model. This will introduce higher hedging volatility, but more realistically represent the traded assets available.

10.3 Costs Control

In Chapter 9, we discussed several cost control methods. There are three random variables in our model, the salary growth rate, the crediting rate, and the interest rate. We have proposed two ways to reduce the hedging costs and the volatility using salary control. We first limit the salary growth rate when there is a hedging cash flow spike. This is an ex-post control, we also consider a modified pension design using the final average salary. This is an ex ante control. Both approaches reduce the volatility of monthly hedging costs.

Also in future work, we will consider the other two random factors: the crediting rate and the interest rate. The DC underlying portfolio is constructed by a mix of equities and fixed income securities. A higher equity proportion should decrease the cost of the guarantee on average, but at the cost of higher volatility. An interesting open problem is the determination of an optimal balance in the benchmark DC fund.

In our model, the DB underpin guarantee is positively related to the annuity factor. It is hard to say how stochastic interest rates affect the hedging cash flow from two figures. In further work, we plan to explore the use of interest rate derivatives to manage the interest rate risk in future work.

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