Counting points of bounded height on del Pezzo surfaces

by

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I hereby declare that I am the sole author of this thesis.

This is a true copy of the thesis, including any required final revisions, as accepted by my examiners. I understand that my thesis may be made electronically available to the public.

Abstract

del Pezzo surfaces are isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ blown up a times, where $0 \leq a \leq 8$. We will look at lines on del Pezzo surfaces isomorphic to $\mathbb{P}^2$ blown up a times with $0 \leq a \leq 6$. We will show that when we count points of bounded height on one of these surfaces, the number of points on lines give us the primary growth order, but the secondary growth order calculates the number of points on the rest of the surface and hence is a better representation of the geometry of the surface.

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

Bombieri, Lang, and Vojta have conjectured that for a variety V of general type over k, the set of k-rational points is not Zariski dense [8]. If the set of k'-rational points on a variety V is Zariski dense in a finite extension k' of k, then we say V has a "potentially dense" set of k-rational points. Fano varieties in general are conjectured to have a potentially dense set of k-rational points and, in particular, k-rational points on del Pezzo surfaces are known to be potentially dense [4]. This thesis looks at counting points of bounded height on rational del Pezzo surfaces. In particular, all del Pezzo surfaces in this thesis will already have a dense set of k-rational points so the counting will be done with respect to a height function, since there are only a finite number of points of bounded height.

Chapter 2 presents background material on heights, intersection numbers, and del Pezzo surfaces. Chapter 3 works through Manin and Tschinkel's paper "Points of bounded height on del Pezzo surfaces," [6]. They show that for $\mathbb{P}^2$ blown up at 5 or 6 points, the number of rational points of bounded height on lines corresponds to the primary growth order and the number of points of bounded height not on lines gives the error term. Chapter 4 directly calculates an upper bound for the number of points of bounded height on $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at one point. The calculations show that, as in the previous chapter, the number of points of bounded height on lines corresponds to the primary growth order and the number of points of bounded height not on lines gives the error term. In both cases, however, the "error term" coming from points not lying on lines is the more significant quantity since it reflects the arithmetic of the entire surface rather than that of a line.

Chapter 2

Background and Definitions

2.1 Heights

In order to count the number of points of bounded height on a surface over a number field k, we need to know how to calculate the height of a point.

We will first consider the case $k = \mathbb{Q}$. For a nonzero rational number $x \in \mathbb{Q}$, define the *archimedean absolute value of* x to be:

$$|x|_\infty = \max\{x, -x\}$$

Note that this is just the ordinary absolute value on $\mathbb{R}$ restricted to $\mathbb{Q}$. Now for each prime p define the *p-adic* or *finite absolute value of* x to be:

$$|x|_p = p^{-ord_p(x)}$$

where $x = p^{ord_p(x)}\frac{a}{b}$ with $a, b \in \mathbb{Z}$ and $p \nmid ab$; in particular, if $x \in \mathbb{Z}$ then $ord_p(x)$ is the highest power of p dividing x. Let $M_{\mathbb{Q}}$ represent the set consisting of the archimedean absolute value $|\cdot|_\infty$ and the p-adic absolute values $|\cdot|_p$ for each prime p.

For a general number field k, an absolute value is any real-valued function

$$|\cdot| : k \longrightarrow [0, \infty)$$

such that the following properties hold:

$$\begin{array}{ll} (1) & |x| = 0 \text{ if and only if } x = 0 \\ (2) & |xy| = |x| \cdot |y| \\ (3) & |x + y| \leq |x| + |y| \end{array}$$

In addition, we say the absolute value is *archimedean* if $|x+y| \geq \max\{|x|,|y|\}$ and *nonarchimedean* or *finite* otherwise. Let M_k represent the set containing all absolute values whose restriction to $\mathbb{Q}$ is in $M_{\mathbb{Q}}$. Let $M_{k,\infty}$ denote the set of archimedean absolute values in k and let $M_{k,f}$ denote the set of finite absolute values in k

We say a point $P = [x_0 : x_1 : \cdots : x_n]$ has homogeneous coordinates in k if $x_i \in \mathcal{O}_k$, where $\mathcal{O}_k$ is the ring of integers of k, and $\gcd(x_i) = 1$. Now define the (multiplicative) *height* $H_k(P)$ of a point $P = [x_0 : x_1 : \cdots : x_n]$ in $\mathbb{P}^n(k)$ with homogeneous coordinates in k to be:

$$H_k(P) = \prod_{v \in M_k} \max\{|x_0|_v, |x_1|_v, \ldots, |x_n|_v\}$$

In particular, when $k = \mathbb{Q}$,

$$H(P) = \max\{|x_0|, |x_1|, \ldots, |x_n|\}$$

In general, the height function measures the size of a rational point. However, it is often difficult to directly determine the height function for a general variety V, so we use ample divisors on V together with Weil's Height Machine to construct a height function on V. Weil's Height Machine maps each divisor D on a smooth projective variety V to a corresponding height function $H_{V,D}$. Recall that $Div(V)$ is the group of all divisors on V, that two divisors are linearly equivalent if there difference is the divisor of a rational function, and that $Pic(V)$ is $Div(V)$ modulo linear equivalence.

Theorem 2.1.1. *(Weil's Height Machine) Let k be a general number field. For every smooth projective variety V defined over k there exists a map*

$$D \mapsto H_{V,D}$$

such that:
(a) For any hyperplane $h \subset \mathbb{P}^n$, there exists a constant C such that for any point $P \in \mathbb{P}^n(\bar{k})$:

$$H_{\mathbb{P}^n,h}(P) = \exp(O(1))H(P)$$

(b) Let $\phi : V \to W$ be a morphism and let $D \in Div(W)$. Then for any point $P \in V(\bar{k})$:

$$H_{V,\phi^*D}(P) = \exp(O(1))H_{W,D}(\phi(P))$$

(c) For divisors $D, E \in Div(V)$ and any point $P \in V(\bar{k})$.

$$H_{V,D+E}(P) = \exp(O(1))H_{V,D}(P)\,H_{V,E}(P)$$

(d) If D is ample, then for every finite extension k' of k and every constant B, the set

$$N_{V(k')}(D,B) = \#\{P \in V(k') \mid H_{V,D}(P) \leq B\}$$

is finite. We call $N_{V(k')}(D,B)$ the counting function of V with respect to D.

Proof. See, for example, the proof of Theorem B.3.2 in [5]. □

Thus part (a) of Weil's Height machine lets us evaluate the height of a point by calculating the height on $\mathbb{P}^n$, (b) lets us embed an abstract variety into $\mathbb{P}^n$ and thus makes calculating the height much easier, (c) lets us calculate the height of a divisor in terms of its irreducible components, and (d) shows us that the counting function is well-defined. In the next two chapters we will use Weil's height machine to compute heights on del Pezzo surfaces.

When counting points of bounded height on a copy of $\mathbb{P}^n$, it is easiest to use Schanuel's theorem to evaluate the counting function.

Theorem 2.1.2. *(Schanuel) Let k be a number field of degree d over $\mathbb{Q}$ and let $n \geq 1$ be an integer. Then*

$$N_{\mathbb{P}^n(k)}(B) = C(k,n)B^{n+1} + \begin{cases} O(B \log B), & \text{if } k = \mathbb{Q} \text{ and } n = 1 \\ O(B^{n+1-1/d}), & \text{otherwise} \end{cases}$$

where $C(k,n)$ is a constant depending on k and n.

Proof. See the proof of Theorem B.6.2 in [5]. □

Note that this together with part (b) of the Height Machine gives

$$N_{\mathbb{P}^1(k)}(C,B) = \exp(O(1))B^{2/d}$$

for any rational curve C of degree d on $\mathbb{P}^1$. In order to calculate the degree of a curve, we need to know about intersection numbers.

2.2 Intersection numbers

We will assume in this section that all of our varieties are projective. Recall that a variety of dimension one is called a *curve* and a variety of dimension two is called a *surface*. Thus a divisor on a surface is a finite sum of curves.

Theorem 2.2.1. *Let V be a surface. Given two divisors $C, D \in Div(V)$, the* intersection number *$C.D$ is the unique function from $Div(V)$ to $\mathbb{N}$ that satisfies:*
(a) if C and D are nonsingular curves meeting transversally, then $C.D = \#\{P \in (C \cap D)\}$.
(b) $C.D = D.C$
(c) $(C_1 + C_2).D = C_1.D + C_2.D$
(d) if C_1 is linearly equivalent to C_2, then $C_1.D = C_2.D$

Proof. See proof of Theorem 1.1 in [3] □

For example, if C and D are two (distinct) lines on $\mathbb{P}^2$, $D.E = 1$.

The degree of a curve C is the number of times it is intersected by a line L, i.e. $C.L$. Note that the degree is independent of the line chosen.

2.3 del Pezzo surfaces

A *Fano variety* is any smooth projective variety V with an ample anticanonical divisor $-K_V$. A special class of these, called *del Pezzo surfaces*, consists of varieties that are either isomorphic to $\mathbb{P}^2$ blown up at no more than 8 points or to $\mathbb{P}^1 \times \mathbb{P}^1$.

A *line* on a del Pezzo surface V over k is any divisor (or curve) C defined over $\bar{k}$ such that $-K_V.C = 1$. We say a del Pezzo surface is *split* if all lines are defined over k. In the next two chapters, we will calculate the height of points on del Pezzo surfaces and we will find that the number of points of bounded height on lines is greater than the number of points of bounded height not on lines. In order to calculate the height of points not on lines, however, we need to know what are all the lines so that we can be sure not to include them.

As an example, we will find the 16 lines on the del Pezzo surface V_5 which is isomorphic to $\mathbb{P}^2$ blown up at 5 points $P_1, \ldots, P_5$.

First we need to determine $-K_{V_5}$. Let L be the divisor class of a hyperplane $\mathcal{O}(1)$ in $\mathbb{P}^n$. Let π be the blowing-down map $\pi : V_5 \to \mathbb{P}^2$. Recall that the anticanonical divisor $-K_{\mathbb{P}^n}$ is $(n+1)H$, where H is a hyperplane. Since V_5 is $\mathbb{P}^2$ blown up five times, we have

$$-K_{V_5} = 3\pi^* L - \sum_{i=1}^{5} E_i$$

where $3\pi^* L$ comes from $\mathbb{P}^2$ since a line L is a hyperplane in $\mathbb{P}^2$ and E_i is the exceptional line associated to the blow-up of the point P_i.

Now, before we can determine which curves C satisfy $-K_{V_5}.C = 1$, we will first compute some basic results to use later.

First note that in general, two lines L and L' on $\mathbb{P}^2$ intersect exactly once, so if we look at the pullbacks of these lines on V_5, we still get $\pi^*L.\pi^*L' = 1$ (Note that L is a general line and it is not fixed. Thus we may sometimes abuse notation and write $(\pi^*L)^2$ where we mean $\pi^*L.\pi^*L'$). On the other hand, the exceptional curves E_i do not intersect a line L in general position on $\mathbb{P}^2$ since we can choose L to be whatever line we want, so $\pi^*L.E_i = 0$. Also, the E_i's are all parallel, so they do not intersect and thus when $i \neq j$, $E_i.E_j = 0$. Finally, we look at the self-intersection number $(E_i)^2$. If L_{ij} is the line on $\mathbb{P}^2$ passing through P_i and P_j, then by above, we have $\pi^*L_{ij}.E_i = 0$. But we also know that $\pi^*L_{ij} = \Lambda_{ij} + E_i + E_j$, where Λ_{ij} is the strict transform of π^*L_{ij}. Thus we have $(\Lambda_{ij} + E_i + E_j).E_i = 0$. Now, $E_i.E_j = 0$ by the above, $E_i.\Lambda_{ij} = 1$ by construction, so we must have $(E_i)^2 = -1$.

Thus, using our above calculations, we get:

$$\begin{aligned} -K_{V_5}.E_i &= (3\pi^*L - \sum_{i=1}^{5} E_i).E_i \\ &= 1 \end{aligned}$$

So we have five lines E_i.

Now look at the pullback of the line L_{ij} on $\mathbb{P}^2$ passing through P_i and P_j for $i \neq j$. As we noted above, the pullback π^*L_{ij} is made up of three lines: E_i, E_j and a line Λ_{ij} that intersects both E_1 and E_2. We want the divisor class of Λ_{ij}, which is $\pi^*L_{ij} - E_i - E_j$. Now when we calculate the intersection multiplicity $-K_{V_5}.\Lambda_{ij}$, we get:

$$\begin{aligned} -K_{V_5}.(\pi^*L_{ij} - E_i - E_j) &= -K_{V_5}.\pi^*L_{ij} + K_{V_5}.E_i + K_{V_5}.E_j \\ &= (3\pi^*L - \sum_{i=1}^{5} E_i).\pi^*L_{ij} - 2 \\ &= 3\pi^*L.\pi^*L_{ij} - (\sum_{i=1}^{5} E_i).\pi^*L_{ij} - 2 \\ &= 3 - 0 - 2 \\ &= 1 \end{aligned}$$

So we have $\binom{5}{2} = 10$ lines Λ_{ij}.

To determine the last line, we look for a conic. Conics are determined by any five distinct points they pass through, so we will look at the conic passing through

$P_1, \dots, P_5$. By similar reasoning as in the previous calculation, we can write this conic as $2\pi^*L - \sum_{i=1}^5 E_i$. To verify this is the last line on V_5, we calculate the intersection multiplicity:

$$\begin{aligned}
-K_{V_5}.(2\pi^*L - \sum_{i=1}^{5} E_i) &= 2(-K_{V_5}).\pi^*L + K_{V_5}.\sum_{i=1}^{5} E_i \\
&= 2(3\pi^*L - \sum_{i=1}^{5} E_i).\pi^*L - 5 \\
&= 6(\pi^*L)^2 - (2\sum_{i=1}^{5} E_i).\pi^*L - 5 \\
&= 6 - 0 - 5 \\
&= 1
\end{aligned}$$

So we have found all 16 lines on V_5.

Similar calculations give us the lines on other del Pezzo surfaces. The following table shows the number of lines on V_a of each type for $1 \leq a \leq 6$.

a	E_i	Λ_{ij}	Conics	Total number of lines
1	1	0	0	1
2	2	1	0	3
3	3	3	0	6
4	4	6	0	10
5	5	10	1	16
6	6	15	6	27

Table 2.1: Lines on V_a

Note that the case $a = 2$ is discussed in the last chapter (where we call Λ_{12} E). The cases $3 \leq a \leq 6$ are discussed in the next chapter.

Chapter 3

The del Pezzo surfaces of degree 3 to 6

In this chapter, I will work through the paper *Points of bounded height on del Pezzo surfaces* by Manin and Tschinkel, [6].

Look at the del Pezzo surfaces that are isomorphic to $\mathbb{P}^2$ blown up at a points, $0 \leq a \leq 8$. Given a split del Pezzo surface V_a over k, the degree d is the self-intersection number of the (ample) canonical divisor. Calculating the degree (and letting π be the blow-down map and L a general line on $\mathbb{P}^2$) gives us:

$$\begin{aligned} d &= K_V^2 \\ &= (3\pi^* L - \sum_{i=1}^{a} E_i)^2 \\ &= 9(\pi^* L)^2 - 6\pi^* L.(\sum_{i=1}^{a} E_i) + (\sum_{i=1}^{a} E_i)^2 \\ &= 9 - 0 + (-a) \\ &= 9 - a \end{aligned}$$

Let E_a represent the set of exceptional curves (lines) on V_a and U_a the complement to the set of exceptional lines on V_a i.e. $U_a = V_a \setminus \bigcup_{l \in E_a} l$. The goal is to prove that for any $\varepsilon > 0$:

1. When $a \leq 3$:

$$N_{U_a}(-K, B) = \begin{cases} O(B(\log B)^5), & \text{if } k = \mathbb{Q} \\ O(B^{1+\varepsilon}), & \text{in general} \end{cases}$$

and when $a = 4$ over $\mathbb{Q}$:

$$N_{U_4}(-K, B) = O(B(\log B)^6)$$

This result will be used to prove the next two:

2. When $a = 5$:

$$N_{V_5}(\mathcal{O}(1), B) = cB^2 + \begin{cases} O(B^{5/4+\varepsilon}), & \text{if } k = \mathbb{Q} \\ O(B^{3/2+\varepsilon}), & \text{in general} \end{cases}$$

and when $a = 6$:

$$N_{V_6}(\mathcal{O}(1), B) = cB^2 + O(B^{5/3+\varepsilon}), \quad \text{if } k = \mathbb{Q}$$

$$c_1 B^2 \leq N_{V_6}(\mathcal{O}(1), B) \leq c_2 B^{2+\varepsilon}, \qquad \text{in general}$$

To prove these results, first look at the exponent on B. Begin with a general projective variety V over a (sufficiently large) number field k, an ample line bundle L on V, and an infinite quasiprojective subset U of V. Define

$$\beta_U(L) = \limsup \log N_U(L, B)/\log B$$

which is essentially the largest exponent with respect to B in $N_U(L, B)$; for example in result 2 above we find $\beta_{V_5}(L) = 2$ from the term cB^2.

There is a unique minimal Zariski closed subset Z in U such that

$$\beta_U(L) = \beta_Z(L) > \beta_{U\setminus Z}(L)$$

Note that this means that U is the smallest Zariski closed subset of Z that has the same growth order as Z and whose complement $U\setminus Z$ has a strictly smaller growth order than that of Z. So Z is the union of all the irreducible Zariski closed subsets of U that have the same growth order as U.

Note that $U\setminus Z$ is Zariski open. Call Z the (minimal) *accumulating subset* (in U with respect to L). Letting $V = V_0$, $Z = Z_0$, and $V_1 = V_0\setminus Z_0$, we see that by repeating this process, we construct a chain of open subsets

$$V = V_0 \supset V_1 \supset \cdots \supset V_n \supset \cdots$$

such that $Z_i = V_i\setminus V_{i+1}$ is the minimal accumulating subset in V_i. We call the sequence $\{V_i\}$ an *arithmetical stratification* with *growth orders* $\beta_i = \beta_{V_i}(L)$.

In particular, we will prove for del Pezzo surfaces over $\mathbb{Q}$ with $a \leq 6$ if the exceptional curves are defined over $\mathbb{Q}$, they form the first accumulating subset.

3.1 Proving Result 1

3.1.1 Finite heights

For a projective variety V over a field k and an ample sheaf L on V, we can decompose a given Weil height H_L into a product of archimedean and finite local heights via an isomorphism $L \simeq \mathcal{O}(D)$ for some divisor D. Thus for any $x \in V \backslash D$,

$$H_L(x) = H_{D,\infty}(x) H_{D,f}(x)$$

where $H_{D,\infty}(x)$ and $H_{D,f}(x)$ denote the product of archimedean and of finite local heights respectively.

Note that if we let D_i be the divisor $\{x_i = 0\}$ on $\mathbb{P}^n$, then:

$$H_{D_i,f}(x_0 : x_1 : \cdots : x_n) = \prod_{v \in M_{k,f}} \max_j (|x_j/x_i|_v)$$

To get an estimate for $H_{D_i,f}$, we look for a good way to represent points in $\mathbb{P}^n(k)$. Let A be the ring of integers in k, A^* the group of units. Choose a family of ideals $a_1, \ldots, a_m \subset A$ representing all ideal classes and put

$$A^{n+1}_{\text{prim}} = \{(x_0 : x_1 : \cdots : x_n) \in A^{n+1} | \exists i, \gcd(x_0, x_1, \cdots, x_n) = a_i\}$$

Note that A^* acts diagonally on A^{n+1}_{prim} i.e. for $u \in A^*$, $u(x_0 : x_1 : \cdots : x_n) = (ux_0 : ux_1 : \cdots : ux_n)$. Thus we can identify $\mathbb{P}^n(k)$ with $A^{n+1}_{\text{prim}}/A^*$. From now on when we represent a point by its coordinates we take coordinates in A^{n+1}_{prim}. Thus we can express the height function above as

$$H_{D_i,f}(x) = d_i(x) N_{k/\mathbb{Q}}(x_i) \tag{3.1}$$

where $d_i : \mathbb{P}^n(k) \to \mathbb{Q}_{>0}$ is a finite-valued function. Since $d_i(x)$ takes on only finitely many values y, the range of $H_{D_i,f}(x)$ is the union of a finite number of copies of $\mathbb{Z}$ with each copy multiplied by a different y. Thus we can think of finite heights as being almost integers.

Lemma 3.1.1. *Let*

$$\eta : (\mathbb{P}^n \backslash \bigcup_{i=0}^{n} D_i)(k) \to \mathbb{Q}^{n+1}_{>0}$$

be the map

$$\eta(x) = (H_{D_0,f}(x), \ldots, H_{D_n,f}(x))$$

Then the number of points x with $H_{\mathcal{O}(1)}(x) \leq B$ having the same image $\eta(x)$ is bounded by $O(1)$ if $k = \mathbb{Q}$ and by $O(B^\varepsilon)$ for any ε for general k.

Proof. When $k = \mathbb{Q}$, we get $H_{D_i,f}(x) = |x_i|$ so if we know $H_{D_i,f}$, then we can reconstruct projective coordinates of x up to a finite bounded ambiguity.

In general, if we know the norm $N_{k/\mathbb{Q}}(x_i)$, we can reconstruct the ideal of x_i in A since (x_i) divides $(N_{k/\mathbb{Q}}(x_i))$.

Now suppose that I is an ideal dividing $(N_{k/\mathbb{Q}}(x_i))$. Then $N_{k/\mathbb{Q}}(I)$ divides $(N_{k/\mathbb{Q}}(x_i))^d$, where d is the degree of k over $\mathbb{Q}$. Thus there are $O((N_{k/\mathbb{Q}}(x_i))^{\varepsilon})$ choices for $N_{k/\mathbb{Q}}(I)$. We know that $N_{k/\mathbb{Q}}(I)$ is an integer, so we can factor it in the integers as a product of primes p_i: $N_{k/\mathbb{Q}}(I) = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$. Over k, the ideal generated by each p_i factors as a product of prime ideals $P_{i1}P_{i2}\cdots P_{il_i}$. Thus we have $N_{k/\mathbb{Q}}(P_{ij}) = p_i^{a_{ij}}$ and $\sum_{j=1}^{l_i} a_j = d$.

Now the question becomes how many ways are there to write $e_i = \sum_{\alpha_{ij}a_{ij}}$ where α_{ij} is a non-negative integer? Consider the map $\phi_i : \mathbb{Z}_i^l \to \mathbb{Z}$ given by $\phi_i(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{il}) = \sum_{\alpha_{ij}a_{ij}}$. For each i, there are no more than $(\prod_j \frac{e_i}{a_{ij}})\frac{1}{e_i} = \frac{e_i^{l_i-1}}{\prod_j a_{ij}} \leq \frac{e_i^{l_i-1}}{(d/l_i)^{l_i}}$ such ways. Since d is fixed and l_i is bounded, we see that $\frac{e_i^{l_i-1}}{(d/l_i)^{l_i}}$ is $O(p_i^{e_i})$. Thus, there are no more than $O(N_{k/\mathbb{Q}}(I))$ divisors of $N_{k/\mathbb{Q}}(I)$. Thus the number of ideals dividing $(N_{k/\mathbb{Q}}(x_i))$ is bounded by $C(\varepsilon)(N_{k/\mathbb{Q}}(x_i))^{\varepsilon}$, which is $O(B^{\varepsilon})$.

Now take a family of ideals $\{(x_i)\}$ corresponding to a given $\eta(x)$. From (3.1) we see that a set of such points is a union of a bounded number of subsets $\{(x_0 : \varepsilon_1 x_1 : \varepsilon_2 x_2 : \ldots : \varepsilon_n x_n)\}$ where $x_0, \ldots, x_n$ are fixed, and $\varepsilon_i \in A^*$ are variable (ε_0 can be killed by the overall multiplication by A^*). Now:

$$\begin{aligned} & H_{\mathcal{O}(1)}(x_0 : \varepsilon_1 x_1 : \varepsilon_2 x_2 : \ldots : \varepsilon_n x_n) \\ = & \left(\prod_{v \in M_\infty} \max_{i \geq 1}(1, |\varepsilon_i x_i / x_0|_v) \right) H_{D_0,f}(x_0 : \cdots : x_n) \end{aligned}$$

Note that $\prod_{v\in M_\infty} \max_{i\geq 1}(1, |\varepsilon_i x_i/x_0|_v) \leq B$ only if $c_1 B^{-2(n-1)} \leq |\varepsilon_i|_v \leq c_2 B^2$ for all $i = 1, \ldots, n$, all $v \in M_\infty$, and some constants $c_1, c_2 > 0$. Let r_1 be the number of real embeddings of k and $2r_2$ be the number of complex embeddings of k and set $r = r_1 + r_2 - 1$. Then from the Dirichlet theorem (see, for example, Prop. VI.1.1 in [7]) it follows that there are no more than $O((\log B)^r)$ such units. □

3.1.2 Finite exceptional heights on del Pezzo surfaces

For each $l \in E_a$, choose an exceptional height function $H_{l,f}$. We may assume they take values in $\mathbb{Z}_{>0}$ since we already showed that finite heights are almost integers.

If E_a has e lines l_i, set

$$\widetilde{\eta}(x) = (H_{l_1,f}(x), \ldots, H_{l_e,f}(x)) \in \mathbb{Z}^e_{>0}$$

To compute $H_{l_j,f}(x)$, we represent l_j as an infimum (or gcd) of divisors D_{ij} for which $H_{D_{ij},f}$ are known, and then

$$H_{l_j,f}(x) = \gcd_i(H_{D_{ij},f})$$

Lemma 3.1.2. *If $a \geq 3$ then the number of points $x \in U_a(k)$ with $H_{-K}(x) \leq B$ having the same image $\eta(x)$ doesn't exceed $O(1)$ when $k = \mathbb{Q}$ and $O(B^\varepsilon)$ for any $\varepsilon > 0$.*

Proof. Look at the birational morphism $\pi : V_a \to \mathbb{P}^2$ that blows down pairwise disjoint lines on V_a. Choose three of these lines, call them l_1, l_2, l_3. Note that π takes each line l_i on V_a to a point P_i on $\mathbb{P}^2$. Choose the lines D_{12}, D_{13}, and D_{23} on $\mathbb{P}^2$ so that $l_i = \pi^*(D_{ij} \cap D_{ik}) = \pi^*(P_i)$ for $\{i, j, k\} = \{1, 2, 3\}$, i.e. so that the line D_{ij} goes through the points P_i and P_j. Define the line l_{ij} on V_a to be the (proper) inverse image of D_{ij}, $\pi^{-1}(D_{ij})$.

Then by Weil's Height Machine we have

$$\begin{aligned} H_{D_{ij},f}(\pi(x)) &= H_{\pi^* D_{ij},f}(x) \\ &= d'_{ij}(x) H_{l_i,f}(x) H_{l_{ij},f}(x) H_{l_j,f}(x) \end{aligned}$$

where $d'_{ij}(x)$ is a finite-valued function.

Thus if we are given $\widetilde{\eta}(x)$, we can determine $\eta(\pi(x))$ up to a constant, and thus by Lemma 3.1.1 we can determine $\pi(x)$ up to $O(1)$ if $k = \mathbb{Q}$ and up to $O(\widetilde{B}^\varepsilon)$ in general for any $\varepsilon > 0$, where $\widetilde{B}$ is a bound for $H_{\mathcal{O}(1)}(\pi(x))$. Since $-K$ is ample there exists a constant $c > 0$ such that $c(-K) - \pi^*\mathcal{O}(1)$ is effective. Then the Height Machine gives $H_{-K}(x)^c \gg H_{\pi^*\mathcal{O}(1)} = H_{\mathcal{O}(1)}(\pi(x))$. Thus we can take $\widetilde{B}$ to be a fixed positive power of B. □

Note that the last proof relied on knowing $\{H_{l_i,f}(x)\}$ and $\{H_{l_{ij},f}(x)\}$ for $i, j \in \{1, 2, 3\}$. So, in other words, if we look at the intersection graph of the l_i's with the l_{ij}'s and label each vertex with the integer value of the corresponding height function, then we can determine the number of points with the same representation up to $O(1)$ if $k = \mathbb{Q}$ and up to $O(B^\varepsilon)$ in general.

Thus when $a \geq 4$, we must have strong constraints on $\{H_{l_i,f}(x)\}$ and $\{H_{l_{ij},f}(x)\}$ since choosing any three $\{i, j, k\} \in \{1, 2, \ldots, a\}$ gives us enough information.

Lemma 3.1.3. *(a) If $l \cap l' = \varnothing$, then* $\gcd(H_{l,f}(x), H_{l',f}(x))$ *is a finite-valued function and we say $H_{l,f}(x)$ and $H_{l',f}(x)$ are* almost relatively prime.

(b) Consider a complete subgraph Δ of E_a of the form:

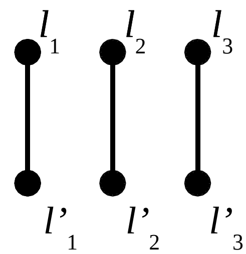

Figure 3.1: A subgraph of type Δ

Then for each $i \in \{1, 2, 3\}$, there is a function $\sigma_i : U_a(k) \to A^3_{prim}$ and finite-valued function $d_i : U_a(k) \to \mathbb{Q}^*$ such that

$$\sigma_1(x) + \sigma_2(x) + \sigma_3(x) = 0$$

$$N_{k/\mathbb{Q}}(\sigma_i(x)) = d_i(x) H_{l_i,f}(x) H_{l'_i,f}(x)$$

i.e. $\sigma_i(x)$ is the product of the finite heights of the intersecting lines l_i and l'_i shown in the graph Δ above.

Proof. Part (a) is classical and was first proved by A. Weil. The proof of (b) uses technology beyond the scope of this thesis, so the proof will not be included. □

Theorem 3.1.4. *For $V = V_3$ over an arbitrary number field k we have:*

$$N_{U_3}(-K, B) = \begin{cases} O(B(\log B)^5), & \text{for } k = \mathbb{Q} \\ O(B^{1+\varepsilon}), & \text{in general} \end{cases}$$

Proof. The intersection graph of E_3 is the hexagon: $(l_1, l_{12}, l_2, l_{23}, l_3, l_{13})$. Let Λ be the class of $\pi^*\mathcal{O}(1)$ in $Pic(V)$. Calculating the anticanonical divisor gives us:

$$\begin{aligned} -K_{V_3} &= 3\Lambda - \sum_{i=1}^{3} l_i \\ &= (l_1 + l_{12} + l_2) + (l_1 + l_{13} + l_3) + (l_2 + l_{23} + l_3) - l_1 - l_2 - l_3 \\ &= \sum_{1 \le i \le 3} l_i + \sum_{1 \le i < j \le 3} l_{ij} \end{aligned}$$

Thus for $x \in U_3(k)$,

$$\begin{aligned} H_{-K}(x) &= \exp(O(1)) \prod_{1\le i\le 3} H_{l_i}(x) \prod_{1\le i<j\le 3} H_{l_{ij}}(x) \\ &\ge C \prod_{1\le i\le 3} H_i(x) \prod_{1\le i<j\le 3} H_{ij}(x) \end{aligned}$$

Now considering $H_i(x), H_{ij}(x)$ as independent integer variables, we see there are $O(B(\log B)^5)$ different ways to label the hexagon and each labelling corresponds to $O(1)$ points if $k = \mathbb{Q}$ and to $O(B^\varepsilon)$ points in general. □

Note that this is an overestimate, since it was proven in [1] that for $k = \mathbb{Q}$:

$$N_{U_3}(-K, B) = \exp(O(1))B(\log B)^3$$

From the blow-down morphism $\pi : V_3 \to \mathbb{P}^2$ we see that $N_{U_3}(-K, B) \ge \exp(O(1))B$, so that $\beta_{U_3}(-K) = 1$.

Theorem 3.1.5. *For $V = V_4$ over $k = \mathbb{Q}$ we have:*

$$N_{U_4}(-K, B) = O(B(\log B)^6)$$

Proof. For each $\lambda \in E_4$ choose a finite height $H_{\lambda,f}(x) : U_4(\mathbb{Q}) \to \mathbb{Z}_{>0}$. Then define the subset $U^{(\lambda)}$ of $U_4(\mathbb{Q})$ to be the set

$$U^{(\lambda)} = \{x \in U_4(\mathbb{Q}) \mid H_{\lambda,f}(x) = \min_{\lambda' \in E_4}(H_{\lambda',f}(x))\}$$

Note that

$$U_4(\mathbb{Q}) = \bigcup_{\lambda \in E_4} U^{(\lambda)}$$

Thus to prove $N_{U_4}(-K, B) = O(B(\log B)^6)$, it suffices to show $N_{U^{(\lambda)}}(-K, B) = O(B(\log B)^6)$ for each $\lambda \in E_4$.

Represent V_4 by $\mathbb{P}^2$ blown up at points $P_0, \ldots, P_3$ and let π be the blow-down map. Then let λ_i denote the preimage of p_i and λ_{ij} denote the inverse image of the line joining p_i and p_j.

Look at $U^{(\lambda_0)}$ and consider the complete subgraph Γ of E_4:

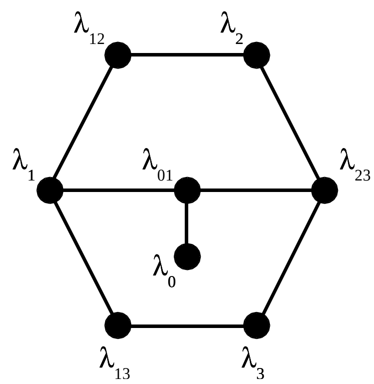

Figure 3.2: The subgraph Γ of E_4

Let Λ be the class of $\pi^*\mathcal{O}(1)$ in $Pic(V_4)$. Note that we can express the anticanonical divisor $-K$ as:

$$\begin{aligned} -K &= 3\Lambda - \sum_{i=0}^{3} \lambda_i \\ &= (\lambda_0 + \lambda_{01} + \lambda_1) + (\lambda_1 + \lambda_{12} + \lambda_2) + (\lambda_2 + \lambda_{23} + \lambda_3) - \sum_{i=0}^{3} \lambda_i \\ &= \lambda_{01} + \lambda_1 + \lambda_{12} + \lambda_2 + \lambda_{23} \end{aligned}$$

Thus, since $H_{-K}(x) \leq B$, we also have the weaker inequality (letting $H_i(x)$ represent $H_{\lambda_i,f}(x)$ for ease of notation):

$$H_{01}(x)H_1(x)H_{12}(x)H_2(x)H_{23}(x) \leq B$$

Considering each of the $H_i(x)$ as an independent variable and writing H_i for brevity, the following sum counts the number of solutions to the above inequality:

$$\begin{aligned}
\sum_{H_{01}H_1H_{12}H_2H_{23}\leq B} 1 &= \sum_{H_{01}H_1H_{12}H_2\leq B}\ \sum_{H_{23}\leq B/H_{01}H_1H_{12}H_2} 1 \\
&= \sum_{H_{01}H_1H_{12}H_2\leq B} \left[\frac{B}{H_{01}H_1H_{12}H_2}\right] \\
&\leq \sum_{H_{01}H_1H_{12}H_2\leq B} \frac{B}{H_{01}H_1H_{12}H_2} \\
&\leq B\sum_{H_{01}\leq B}\frac{1}{H_{01}}\sum_{H_{01}H_1\leq B}\frac{1}{H_1}\sum_{H_{01}H_1H_{12}\leq B}\frac{1}{H_{12}}\sum_{H_{01}H_1H_{12}H_2\leq B}\frac{1}{H_2} \\
&= O(B\log B)\sum_{H_{01}\leq B}\frac{1}{H_{01}}\sum_{H_{01}H_1\leq B}\frac{1}{H_1}\sum_{H_{01}H_1H_{12}\leq B}\frac{1}{H_{12}} \\
&= O(B(\log B)^4)
\end{aligned}$$

However, it is not clear how to reconstruct x with reasonable indeterminacy from $H_{01}(x), H_1(x), H_{12}(x), H_2(x), H_{23}(x)$.

Thus we will weaken the inequality further so that we can reconstruct x. Set $H_1'(x) = [H_1(x)/H_0(x)] \geq 1$ (since we're in $U^{(\lambda_0)}$). Then we get

$$H_{01}(x)H_0(x)H_1'(x)H_{12}(x)H_2(x)H_{23}(x) \leq B$$

Using the same method as above, we get an overestimate of the number of solutions to this of $O(B(\log B)^5)$. From now on, we will assume we know the heights $H_{01}(x), H_0(x), H_1'(x), H_{12}(x), H_2(x), H_{23}(x)$.

Calculating the (weaker) estimate $O(B^\varepsilon)$

(i) Reconstruction of $H_3(x), H_{13}(x)$. Consider the subgraph of Γ :

$$(\lambda_0, \lambda_{01}, \lambda_{12}, \lambda_2, \lambda_3, \lambda_{13})$$

and apply Lemma 3.1.2(b) to get:

$$d_1(x)H_0(x)H_{01}(x) + d_2(x)H_{12}(x)H_2(x) + d_3(x)H_3(x)H_{13}(x) = 0$$

Note that we are using the assumption $k = \mathbb{Q}$.

Thus we have:

$$O(B) \geq d_1(x)H_0(x)H_{01}(x) + d_2(x)H_{12}(x)H_2(x) = -d_3(x)H_3(x)H_{13}(x)$$

Knowing $H_0(x), H_{01}(x), H_{12}(x), H_2(x)$ and letting $a = O(B)$, we can reconstruct $H_3(x), H_{13}(x)$ in $O(\tau(a))$ ways, where $\tau(a)$ counts the number of divisors of a. Write $a = 2^{e_0}p_1^{e_1}\cdots p_n^{e_n}$, where $e_0 \geq 0$ and the p_i are distinct odd prime divisors of a and $e_i \geq 1$ for $i \geq 1$. If $a = 2^{e_0}p_1^{e_1}\cdots p_n^{e_n} < B$, then $p_i^{e_i} < B$ for each $i \geq 1$. Taking the logarithm of both sides gives us $e_i \log p_i < \log B$ and thus $e_i < \log B$ for $i \geq 1$ since $p_i > 2$. Also we have $2^{e_0} < B$ and taking the logarithm of both sides gives us $e_0 \log 2 < \log B$ which implies $e_0 < 2 \log B$. Thus we have

$$\begin{aligned}\tau(a) &= \prod_{i=0}^{n}(e_i + 1) \\ &< (2\log B + 1)(\log B + 1)^n \\ &= O((\log B)^{n+1})\end{aligned}$$

Let N be the smallest integer such that B is less than the product of the first N primes. Then $\max\{\tau(a) \mid a \leq B\} = O((\log B)^N) = O(B^{\varepsilon})$.

(ii) Reconstruction of $H_1(x) \mod H_0(x)$. Now consider a subgraph of E_4 (and not of Γ): $(\lambda_1, \lambda_{13}, \lambda_2, \lambda_{23}, \lambda_0, \lambda_{03})$ which is isomorphic to Γ. From this graph we get:

$$d_1'(x)H_1(x)H_{13}(x) + d_2'(x)H_2(x)H_{23}(x) + d_3'(x)H_0(x)H_{03}(x) = 0$$

Thus, knowing $H_{13}(x), H_2(x), H_{23}(x)$ and using Lemma 3.1.2(a) since $H_{13}(x)$ and $H_0(x)$ are almost relatively prime , we can reconstruct $H_1(x) \mod H_0(x)$ up to a finite ambiguity.

(iii) Reconstruction of x. Let $b = H_1(x) \mod H_0(x)$. We can reconstruct $H_1(x)$ since $H_1(x) = H_0(x)H_1'(x) + b$. Now if we look at the hexagon subgraph of $\Gamma : (\lambda_1, \lambda_{12}, \lambda_2, \lambda_{23}, \lambda_3, \lambda_{13})$, we know all of the associated heights and thus by the proof of Lemma 3.1.2, we can reconstruct x up to a finite ambiguity.

Calculating the estimate $O(B(\log B)^6)$

To get the sharper estimate, we normalize the points we blow up so that the function $d_i(x)$ in Lemma 3.1.2(b) only takes on the two values ± 1 and thus allows us to refine step (i) above.

Choose coordinates in $\mathbb{P}^2$ in such a way that V_4 is isomorphic to $\mathbb{P}^2$ blown up at $P_0 = (1:1:1)$, $P_1 = (1:0:0)$, $P_2 = (0:1:0)$, $P_3 = (0:0:1)$ and let $\pi : V_4 \to \mathbb{P}^2$ be the blow down map. Let λ_i and λ_{ij} be as before. Let $x \in U_4(\mathbb{Q})$. Then $\pi(x)$ can be represented by $(x_1, x_2, x_3) \in \mathbb{Z}^3_{\text{prim}}$.

Define the following ten integers for $\{i, j, k\} = \{1, 2, 3\}$:

$$
\begin{aligned}
d_i &= \gcd(x_j, x_k) \\
y_i &= x_i / d_j d_k \\
D &= \gcd_{i \neq k}\{y_i d_k - y_k d_i\} \\
z_j &= |y_i d_k - y_k d_i| / D
\end{aligned}
$$

Each of these integers is the finite height associated to a line $\lambda \in E_4$. The correspondence is shown in the following table.

$\lambda_i :$	λ_0	λ_{01}	λ_1	λ_{12}	λ_2	λ_{23}	λ_3	λ_{13}	λ_{02}	λ_{03}
$H_i(x) :$	D	z_1	d_1	$\|y_3\|$	d_2	$\|y_1\|$	d_3	$\|y_2\|$	z_2	z_3

Table 3.1: Lines on E_4 and their corresponding finite heights

To see the correspondence, note that d_i will be nonzero everywhere except on the corresponding line. But $d_i = \gcd(x_j, x_k)$ is only nonzero when $x_j = x_k = 0$, i.e. on λ_i. Similarly y_i is nonzero when $x_i = 0$ and $x_j x_k \neq 0$, i.e. on the line λ_{jk}. D is nonzero when $y_i d_k = y_k d_i$ for $i \neq k$ which happens when $x_1 = x_2 = x_3$; i.e. on λ_0. z_j is nonzero when $x_1 = x_2 = x_3$ or when $|y_i d_k - y_k d_i| = 0$ and $D \neq 0$, which happens on λ_{0j}.

The following figure shows all the subgraphs of type Δ with the vertices labelled by their respective finite heights.

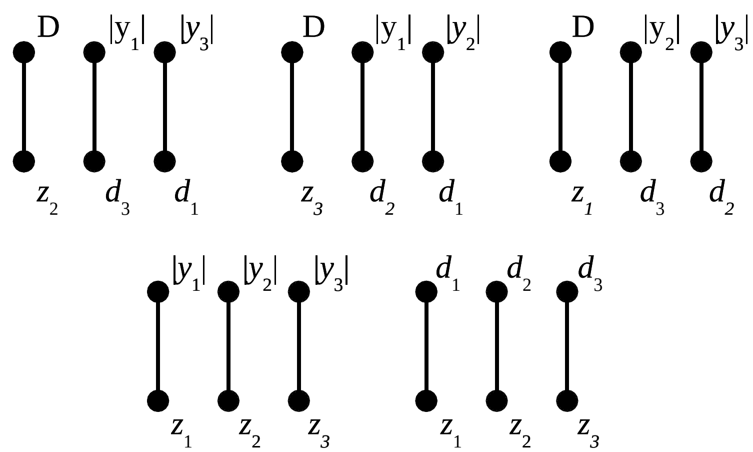

Figure 3.3: Subgraphs of E_4 of type Δ

For each subgraph, take the product of the labels of each pair of connected vertices. Note that one product always equals the sum of the other two:

So in the first case we have:

$$Dz_2 = |y_1 d_3 - y_3 d_1|$$

which implies

$$\begin{cases} Dz_2 = |y_1|d_3 + |y_3|d_1, & \text{if } y_1 > 0 > y_3 \text{ or } y_3 > 0 > y_1 \\ |y_3|d_1 = Dz_2 + |y_1|d_3, & \text{if } y_1, y_3 < 0 \text{ or } y_1 = 0 \\ |y_1|d_3 = Dz_2 + |y_3|d_1, & \text{if } y_1, y_3 > 0 \text{ or } y_3 = 0 \end{cases}$$

The second and third cases are similar.

In the fourth case we have:

$$\begin{aligned} |y_1|z_1 &= |y_1|\,|y_2 d_3 - y_3 d_2|/D \\ &= |y_1 y_2 d_3 - y_2 y_3 d_1 + y_2 y_3 d_1 - y_1 y_3 d_2|/D \\ &= |y_2(y_1 d_3 - y_3 d_1)/D + y_3(y_2 d_1 - y_1 d_2)/D| \\ &= \pm|y_2|z_2 \pm |y_3|z_3 \end{aligned}$$

In the fifth case we have:

$$\begin{aligned} d_1 z_1 &= d_1|y_2 d_3 - y_3 d_2|/D \\ &= |y_2 d_1 d_3 - y_3 d_2 d_3 + y_3 d_2 d_3 - y_3 d_1 d_2|/D \\ &= |d_3(y_2 d_1 - y_3 d_2)/D + d_2(y_3 d_3 - y_3 d_1)/D| \\ &= \pm d_3 z_3 \pm d_2 z_2 \end{aligned}$$

Now we find an upper bound for $H_{-K}(x)$ using the procedure from (i) - (iii) above. As before, for brevity, we will write H_i for $H_i(x)$. Thus we have:

$$N_{U^{(\lambda)}}(-K, B) \leq C \sum_{H_{01} H_0 H_1' H_{12} H_2 H_{23} \leq B} \tau(\delta_1 H_{01} H_0 + \delta_2 H_{12} H_2)$$

where $\delta_i = \pm 1$ by construction.

Then we have:

$$\begin{aligned}
&\sum_{H_{01}H_0H_1'H_{12}H_2H_{23}\le B} \tau(\delta_1 H_{01}H_0 + \delta_2 H_{12}H_2)\\
&= \sum_{H_{01}H_0H_{12}H_2\le B} \tau(\delta_1 H_{01}H_0 + \delta_2 H_{12}H_2) \sum_{H_1'H_{23}\le (B/H_{01}H_0H_{12}H_2)} 1\\
&\le \sum_{H_{01}H_0H_{12}H_2\le B} \tau(\delta_1 H_{01}H_0 + \delta_2 H_{12}H_2)\frac{B}{H_{01}H_0H_{12}H_2}(\log B + O(1))\\
&= \left(\sum_{ab\le B} \frac{\tau(a)\tau(b)\tau(\delta_1 a+\delta_2 b)}{ab}\right)(B\log B + O(B))
\end{aligned}$$

where $a = H_{01}H_0$ and $b = H_{12}H_2$. Now we just need to show:

$$\sum_{ab\le B} \frac{\tau(a)\tau(b)\tau(\delta_1 a+\delta_2 b)}{ab} = O((\log B)^5)$$

Step 1: Prove

$$g(n) := \sum_{ab\le n} \tau(a)\tau(b)\tau(\delta_1 a+\delta_2 b) = O(n(\log n)^4)$$

Recall that we are trying to reconstruct $H_3(x)$ and $H_{13}(x)$ from $H_{01}(x)$, $H_0(x)$, $H_1'(x)$, $H_{12}(x)$, $H_2(x)$, $H_{23}(x)$. By construction, we know that $H_3(x)H_{13}(x) = \pm H_{01}(x)H_0(x) \pm H_{12}(x)H_2(x) = \delta_1 a + \delta_2 b$. Thus δ_i depends on a, b. However, it suffices to prove this estimate for constant δ_i's in two separate cases:

(i) when $\delta_1 = 1$, $\delta_2 = -1$, and $a > b$

(ii) when $\delta_1 = \delta_2 = 1$ and $a \ge b$

Note that in the second case, if we let $a' = a + b$, $b' = b$, then $a' > b'$ and $a'b' = (a + b)b = ab + b^2 \le n + b^2 \le 2n$. Thus we can see that the second case reduces to the first one:

$$\sum_{\substack{ab\le n\\ b\le a}} \tau(a)\tau(b)\tau(a+b) = \sum_{\substack{a'b'\le 2n\\ b'<a'}} \tau(a'-b')\tau(b')\tau(a')$$

So from now on we will assume we are in the first case.

$$\sum_{\substack{ab\le n\\ b<a}} \tau(a)\tau(b)\tau(a-b) = \sum_{b\le n}\tau(b) \sum_{b<a\le (n/b)} \tau(a)\tau(a-b) \tag{3.2}$$

Note that for a fixed b:

$$\begin{aligned}
&\sum_{b<a\leq(n/b)} \tau(a)\tau(a-b) \\
&\leq \sum_{b<a\leq n/b} \tau(a) \sum_{0<a-b\leq n/b-b} \tau(a-b) \\
&\leq \sum_{b<a\leq n/b} \tau(a)(2 \sum_{\substack{0<rs=a-b\leq n/b-b \\ 0<s\leq r}} 1) \\
&\leq (\sum_{0<rs+b\leq n/b} \tau(rs+b))\,(2 \sum_{\substack{0<rs\leq n/b-b \\ 0<s\leq r}} 1) \\
&\leq (2 \sum_{\substack{0<pq\leq n/b \\ 0<q\leq p \\ pq=rs+b}} 1)(2 \sum_{\substack{0<rs\leq n/b-b \\ 0<s\leq r}} 1) \\
&= 4 \sum_{\substack{0<q\leq p \\ 0<s\leq r \\ rs+b=pq\leq n/b}} 1 \\
&\leq 4 \sum_{0<q,s\leq\sqrt{n/b}} \sum_{\substack{0<p\leq(n/bq) \\ pq\equiv b \mod s}} 1
\end{aligned}$$

Define the following function and recall that b is fixed:

$$\begin{aligned}
\omega(q,s) &= \#\{p \mod s \,|\, pq\equiv b \mod s\} \\
&= \begin{cases} d:=\gcd(q,s), & \text{if } d|b \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

Note that in the range $\{1,\dots,[\frac{n}{bq}]\}$ there are no more than $\left(\frac{n}{bqs}+1\right)$ distinct sets $\{c+1,c+2,\dots,c+s\}$ of length s. For each set $\{c+1,c+2,\dots,c+s\}$ we see that there are $\omega(q,s)$ number of p's. Continuing our approximation, we get:

$$\begin{aligned}
&\leq 4 \sum_{0<q,s\leq\sqrt{n/b}} \omega(q,s)\left(\frac{n}{bqs}+1\right) \\
&\leq 4\sum_{d|b} d \sum_{\substack{0<q,s\leq\sqrt{n/b} \\ (q,s)\equiv(0,0) \mod d}} \left(\frac{n}{bqs}+1\right)
\end{aligned}$$

Assuming $(q,s) \equiv (0,0) \mod d$, we can write $q = xd$ and $s = yd$. Then continuing the approximation, we get:

$$
\begin{aligned}
&\leq 4\sum_{d|b} d \sum_{0<x,y\leq \frac{\sqrt{n/b}}{d}} \left(\frac{n}{bd^2xy} + 1\right) \\
&\leq 4\sum_{d|b} d \sum_{0<x\leq \frac{\sqrt{n/b}}{d}} \left(\frac{n}{bd^2x}(\log(\frac{\sqrt{n/b}}{d}) + O(1)) + \frac{\sqrt{n/b}}{d} + 1\right) \\
&\leq 4\sum_{d|b} \frac{1}{d} \sum_{0<x\leq \frac{\sqrt{n/b}}{d}} \left(\frac{n}{bx}\log\frac{n}{b} + O\left(\frac{n}{bx}\right)\right) \\
&\leq 4\sum_{d|b} \frac{1}{d} \left(\frac{n}{b}\log\frac{n}{b}\left(\log(\frac{\sqrt{n/b}}{d}) + O(1)\right) + O\left(\frac{n}{b}(\log(\frac{\sqrt{n/b}}{d}) + O(1))\right)\right) \\
&\leq 4\sigma_{-1}(b)\left(\frac{n}{b}\log^2\frac{n}{b} + O(\frac{n}{b}\log\frac{n}{b})\right)
\end{aligned}
$$

where $\sigma_{-1}(b) = \sum_{d|b}\frac{1}{d}$

Returning to 3.2, we get:

$$
\begin{aligned}
\sum_{b\leq n}\tau(b)\sum_{b<a\leq(n/b)}\tau(a)\tau(a-b) &\leq 4\sum_{b\leq n}\tau(b)\sigma_{-1}(b)\left(\frac{n}{b}\log^2\frac{n}{b} + O(\frac{n}{b}\log\frac{n}{b})\right) \\
&\leq 4\left(n(\log n)^2 + O(n\log n)\right)\sum_{b\leq n}\frac{1}{b}\tau(b)\sigma_{-1}(b) \\
&= O(n(\log n)^4)
\end{aligned}
$$

Step 2: Let $c_n = \sum_{ab=n}\tau(a)\tau(b)\tau(\delta_1 a + \delta_2 b)$ and $f(n) = \frac{1}{n}$. Since we then have $g(m) = \sum_{n\leq m} c_n$, we can use Abel's summation to get:

$$
\begin{aligned}
\sum_{ab\leq B}\frac{\tau(a)\tau(b)\tau(\delta_1 a + \delta_2 b)}{ab} &= \frac{g(B)}{B} + \int_1^B g(u)\frac{1}{u^2}\,du \\
&= O((\log B)^4) + O\left(\int_1^B \frac{(\log u)^4}{u}\,du\right) \\
&= O((\log B)^4) + O((\log B)^5) \\
&= O((\log B)^5)
\end{aligned}
$$

□

3.2 Proving Result 2

Lemma 3.2.1. *Let $a \geq 2$. Let $\{l_1, l_2, \ldots, l_e\}$ be the set of all exceptional lines on a del Pezzo surface $V = V_{a+1}$ of degree $8 - a$. The class of their sum in $Pic(V)$ equals $\frac{e}{8-a}(-K_V)$.*

Proof. Let W_{a+1} represent the formal symmetry group of the configuration of lines l_i, in other words, W_{a+1} is the group of the permutations of lines that preserves their intersection indices. From the classical identification of W_{a+1} with the Weyl group, it follows that the subgroup of W_{a+1}-invariant elements is cyclic, with generator $-K_V$. Thus

$$\sum_{i=1}^{e} l_i = C(-K_V)$$

In order to determine C, we just intersect both sides of the above equation by $-K_V$. Note that $(-K_V)^2 = 8 - a$. And since $l_i.(-K_V) = 1$, we have $(\sum_{i=1}^{e} l_i).(-K_V) = e$. Thus $C = \frac{e}{8-a}$. □

Corollary 3.2.2. *Choose some Weil heights H_{-K} and H_{l_i} for all i. Then there exists a constant A such that for every $x \in U_{a+1}(k)$ one can find a line $l = l(x)$ with the property:*

$$H_l(x) \leq A(H_{-K}(x))^{1/(8-a)}$$

Proof. From Lemma 3.2.1 we get:

$$\sum_{i=1}^{e} l_i = \frac{e}{8-a}(-K_V)$$

Thus by Weil's height machine we get that:

$$\prod_{i=1}^{e} H_{l_i}(x) = \exp(O(1))\,(H_{-K}(x))^{e/(8-a)}$$

Since there are e lines l_i, we must have at least one line l with

$$H_l(x) \leq A(H_{-K}(x))^{1/(8-a)}$$

□

Note that the same argument shows we must have at least one line l' with

$$H_{l'}(x) \geq B(H_{-K}(x))^{1/(8-a)}$$

Thus since there are an infinite number of points $x \in U_{a+1}(k)$ and only a finite number of lines $l \in E_{a+1}$, there must be at least one line l' such that $H_{l'}(x) \geq B(H_{-K}(x))^{1/(8-a)}$ for infinitely many x. Thus the exceptional height has the same growth order as the ample height infinitely often.

Theorem 3.2.3. *Assume that for a given ground field k, some $a \geq 2$, and all split del Pezzo surfaces V_a of degree $9-a$ over k we have:*

$$\beta_{U_a}(-K) \leq \beta_a$$

Then for all split del Pezzo surfaces V_{a+1}, we have:

$$\beta_{U_{a+1}}(-K) \leq \beta_{a+1} = \frac{9-a}{8-a}\beta_a$$

Proof. Fix k, a split del Pezzo surface V_{a+1}, and some heights H_K, H_{l_i} on $V_{a+1}(k)$. By Corollary 3.2.2, we can partition $U(k)$ into a finite number of subsets U_l ordered by lines l such that for all $x \in U_l$, $H_l(x) \leq A(H_{-K}(x))^{1/(8-a)}$. It suffices to prove the number of points x with $H_{-K_{a+1}}(x) \leq B$ in U_l is $O(B^{\beta_{a+1}+\varepsilon})$. Embed l into a maximal system of pairwise disjoint lines on $V_{a+1} : l_1, l_2, \ldots, l_a, l_{a+1} = l$. Denote by $\pi : V_{a+1} \to \mathbb{P}^2$ the morphism that blows down this system. Let Λ be the class of $\pi^*\mathcal{O}(1)$ in $Pic(V_{a+1})$. Choosing all necessary Weil heights, we have for $x \in U_l$:

$$\begin{aligned} H_{-K_{a+1}}(x) &= \exp(O(1))H_{3\Lambda-l_1-\cdots-l_{a+1}}(x) \\ &= \exp(O(1))H_l(x)^{-1}H_{3\Lambda-l_1-\cdots-l_a}(x) \\ &\geq CH_{-K_{a+1}}(x)^{-1/(8-a)}H_{-K_a}(\sigma(x)) \end{aligned}$$

where $\sigma : V_{a+1} \to V_a$ blows down $l_{a+1} = l$.

Thus for $x \in U_l$:

$$H_{-K_{a+1}}(x)^{1+1/(8-a)} \geq CH_{-K_a}(\sigma(x))$$

which implies

$$H_{-K_{a+1}}(x) \geq C'(H_{-K_a}(\sigma(x)))^{(8-a)/(9-a)}$$

We assumed that the number of points $\sigma(x)$ with $H_{-K_a}(\sigma(x)) \leq B$ is $O(B^{\beta_a+\varepsilon})$. Thus when

$$B \geq H_{-K_{a+1}}(x) \geq C'(H_{-K_a}(\sigma(x)))^{(8-a)/(9-a)}$$

we have $\beta_{a+1} = \frac{9-a}{8-a}\beta_a + \varepsilon$.

$\square$

Now for $k = \mathbb{Q}$ and $a = 4$, Result 1 tells us that that $\beta_{U_4}(-K) \leq \beta_4 = 1 + \varepsilon$, and so applying Theorem 3.2.3 we get:

$$\begin{aligned} \beta_5 = \ & \tfrac{5}{4}\beta_4 & = \frac{5}{4} + \varepsilon \\ \beta_6 = \ & \tfrac{4}{3}\beta_5 & = \frac{5}{3} + \varepsilon \end{aligned}$$

Similarly, for a general field k with $a = 3$, Result 1 gives us $\beta_{U_3}(-K) \leq \beta_3 = 1 + \varepsilon$, and applying Theorem 3.2.3 gives us:

$$\begin{aligned} \beta_4 = \ & \tfrac{6}{5}\beta_3 & = \frac{6}{5} + \varepsilon \\ \beta_5 = \ & \tfrac{5}{4}\beta_4 & = \frac{3}{2} + \varepsilon \\ \beta_6 = \ & \tfrac{4}{3}\beta_5 & = 2 + \varepsilon \end{aligned}$$

Thus by considering V_5 and V_6 as blow-ups of V_3 and V_4 we get result 2. Note that when $k = \mathbb{Q}$, we use the smaller set of values for $\beta_4, \beta_5, \beta_6$ since both sets represent upper bounds on the actual growth order.

Chapter 4

The del Pezzo surface of degree 7

Let V be $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at $([0:1],[0:1])$. In this chapter, we compute the number of rational points P on V that have height $H_C(P)$ less than or equal to B, where C is an ample divisor. Since V is birational to $\mathbb{P}^2$, we know that the Néron-Severi group of V, $NS(V)$ is actually equal to $Pic(V)$. Thus $NS(V) \otimes \mathbb{R} \cong \mathbb{R}^n$, where n is an integer, has an intersection product. We will show that, as in the last chapter, although the number of points of bounded height on lines dominates our growth rate, the number of points of bounded height not on lines is a better representation of the geometry of the surface.

To calculate the height of a point we first determine what are the ample divisors on $\mathbb{P}^1 \times \mathbb{P}^1$. We look at the effective cone $NE(V)$, which is the cone generated by effective divisors on V. Its closure is denoted by $\overline{NE}(V)$. The cone of ample divisors is the dual of $NE(V)$ and if we take its closure, we get the nef cone, which is the set of divisors $\{D : D.C \geq 0, \forall C \in \overline{NE}(V)\}$.

Define π to be the blowing-down map $\pi : V \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and $\pi_i : V \rightarrow \mathbb{P}^1$ to be the composition of π with the projection onto the i^{th} factor. Then F_i is the class of $\pi_i^{-1}(P)$, where P is a generic point in $\mathbb{P}^1$. Thus we can write $F_i = \pi_i^*(P)$. Let E_1 represent the strict transform of the line $[0:1] \times \mathbb{P}^1$ and let E_2 represent the strict transform of the line $\mathbb{P}^1 \times [0:1]$. Observe that $Pic(V)$ is spanned by the divisor classes F_1, F_2, E, where E is the exceptional divisor of the blow-up. Thus any divisor D on $\mathbb{P}^1 \times \mathbb{P}^1$ has the form $D = \gamma_1 F_1 + \gamma_2 F_2 + \gamma_3 E$, where the γ_i are integers. We need to determine what are the restrictions on the γ_i so that D is ample, i.e. so that D is in the nef cone. To do this, we first calculate $\overline{NE}(V)$ and then take its dual.

We want to find a basis for $\overline{NE}(V)$, so we note that $E, F_1 - E, F_2 - E$ are all effective divisors since we can express F_1 as the sum of the line E_1 and E and

F_2 as the sum of the line E_2 and E. To determine whether they are a basis for $\overline{NE}(V)$, we compute the dual of the set they generate: $\{D : D.E \geq 0, D.(F_1 - E) \geq 0, D.(F_2 - E) \geq 0\}$.

To determine which divisors are in this set, we must calculate some intersection multiplicities. First note that F_1 and F_2 intersect at one point and so $F_1.F_2 = F_2.F_1 = 1$. Then note that in general, F_i does not intersect E, so $F_1.E = F_2.E = 0$. Also, if we choose two different representatives for F_i, they are parallel and do not intersect, so $F_1^2 = F_2^2 = 0$. To determine E^2, recall that we can write $F_1 = E_1 + E$. Thus we have $0 = F_1.E = (E_1 + E).E = E_1.E + E^2$. Thus $E^2 = -E_1.E = -1$ since E_1 and E intersect exactly once.

Using the intersection multiplicities calculated above we get:

$$\begin{aligned} -\gamma_3 &\geq 0 \\ \gamma_2 + \gamma_3 &\geq 0 \\ \gamma_1 + \gamma_3 &\geq 0 \end{aligned}$$

To determine the generators of the dual, we look at the condition $-\gamma_3 = \gamma_2 + \gamma_3 = 0$, which implies $\gamma_2 = \gamma_3 = 0$, so F_1 is a generator of the dual. Similarly F_2 is a generator of the dual. The condition $\gamma_1 + \gamma_3 = \gamma_2 + \gamma_3 = 0$ implies $\gamma_1 = \gamma_2 = -\gamma_3$, so $F_1 + F_2 - E$ also generates the dual.

If these three generators are all nef, then we know we chose the right basis and we're done. F_1 and F_2 are obviously nef. To see that $F_1 + F_2 - E$ is nef, note that we can write $F_1 + F_2 - E$ as $(F_1 - E) + F_2$ and as $F_1 + (F_2 - E)$. Thus $E.(F_1 + F_2 - E) = E.F_1 + E.F_2 - E^2 = 1$ and

$$\begin{aligned} (F_1 - E).(F_1 + F_2 - E) &= (F_1 - E).((F_1 - E) + F_2) \\ &= (F_1 - E).(F_1 - E) + (F_1 - E).F_2 \\ &= (F_1^2 - 2E.F_1 + E^2) + (F_1.F_2 - E.F_2) \\ &= -1 + 1 \\ &= 0 \end{aligned}$$

and similarly, $(F_2 - E).(F_1 + F_2 - E) = (F_2 - E).((F_2 - E) + F_1) = 0$, so $F_1 + F_2 - E$ is nef. Thus the ample divisors on V are of the form $C = \alpha_1 F_1 + \alpha_2 F_2 + \beta(F_1 + F_2 - E)$, where $\alpha_1, \alpha_2, \beta \geq 1$ are integers.

Note that there is a birational morphism $\psi : V \to \mathbb{P}^2$ that maps $([p : q], [s : t]) \mapsto [qs : pt : ps]$ for $(p, s) \neq (0, 0)$ and $E \mapsto \{z = 0\}$. Note that this map takes $[0 : 1] \times \mathbb{P}^1$ to $[1 : 0 : 0]$ and $\mathbb{P}^1 \times [0 : 1]$ to $[0 : 1 : 0]$. Note further that

$\psi^{-1}(\{z=0\}) = E \cup E_1 \cup E_2$ and $\psi^* L = E + E_1 + E_2$, where L is the class of a line in $\mathbb{P}^2$. Thus calculating the height of the point $([p:q],[s:t])$ on V with respect to $F_1 + F_2 - E = E + E_1 + E_2$ is equivalent to calculating the usual height of the point $[qs : pt : ps]$ on $\mathbb{P}^2$ by Weil's Height Machine.

Assume $\gcd(p,q) = gcd(s,t) = 1$. Set $g = \gcd(p,s)$ and write $p = gp'$ and $s = gs'$. Then

$$\begin{aligned} H_C([p:q],[s:t]) &= \max\{|p|,|q|\}^{\alpha_1} \max\{|s|,|t|\}^{\alpha_2} (\max\{|qs|,|pt|,|ps|\}/g)^{\beta} \\ &= \max\{|gp'|,|q|\}^{\alpha_1} \max\{|gs'|,|t|\}^{\alpha_2} \max\{|qs'|,|p't|,|gp's'|\}^{\beta}. \end{aligned}$$

Now we look at the possible cases:

Assuming $|p| \geq |q|$ and $|s| \geq |t|$ gives us

$$\begin{aligned} H_C([p:q],[s:t]) &= |gp'|^{\alpha_1} |gs'|^{\alpha_2} |gp's'|^{\beta} \\ &= g^{\alpha_1+\alpha_2+\beta} |p'|^{\alpha_1+\beta} |s'|^{\alpha_2+\beta} \end{aligned}$$

Assuming $|p| \geq |q|$ and $|s| \leq |t|$ gives us

$$\begin{aligned} H_C([p:q],[s:t]) &= |gp'|^{\alpha_1} |t|^{\alpha_2} |p't|^{\beta} \\ &= g^{\alpha_1} |p'|^{\alpha_1+\beta} |t|^{\alpha_2+\beta} \end{aligned}$$

Note that if we make the substitution $u = \frac{t}{g}$, we get:

$$H_C([p:q],[s:t]) = g^{\alpha_1+\alpha_2+\beta} |p'|^{\alpha_1+\beta} |u|^{\alpha_2+\beta}$$

Thus we can calculate the height in this case by using the same bounds as in the first case.

Assuming $|p| \leq |q|$ and $|s| \geq |t|$ gives us

$$\begin{aligned} H_C([p:q],[s:t]) &= |q|^{\alpha_1} |gs'|^{\alpha_2} |qs'|^{\beta} \\ &= g^{\alpha_2} |q|^{\alpha_1+\beta} |s'|^{\alpha_2+\beta} \end{aligned}$$

Note that if we make the substitution $v = \frac{q}{g}$, we get:

$$H_C([p:q],[s:t]) = g^{\alpha_1+\alpha_2+\beta} |v|^{\alpha_1+\beta} |s'|^{\alpha_2+\beta}$$

Thus we can calculate the height in this case by using the same bounds as in the first case.

Assuming $|p| \leq |q|$ and $|s| \leq |t|$, we have

$$H_C([p:q],[s:t]) = |q|^{\alpha_1} |t|^{\alpha_2} \max\{|qs'|,|p't|\}^{\beta}$$

Note that we cannot make a nice substitution as above, so we need to consider the two cases $|qs'| \leq |p't|$ and $|qs'| \geq |p't|$.

4.1 $|p| \geq |q|$ and $|s| \geq |t|$

Going back to our first assumption, $|p| \geq |q|$ and $|s| \geq |t|$, we calculate the number of points $([p:q],[s:t])$ such that $H_C([p:q],[s:t]) \leq B$. We fix g and count the number of points with $g^{\alpha_1+\alpha_2+\beta}|p'|^{\alpha_1+\beta}|s'|^{\alpha_2+\beta} \leq B$.

Note first that $|t|$ and $|q|$ both have a lower bound of 0 and upper bounds of $g|s'|$ and $g|p'|$ respectively. Taking $t = 0$ and $q = 0$ shows us that $|s'|$ and $|p'|$ both have lower bounds of 1. Further note that $|s'|$ has an upper bound of $|s'| \leq \left(\frac{B}{g^{\alpha_1+\alpha_2+\beta}|p'|^{\alpha_1+\beta}}\right)^{\frac{1}{\alpha_2+\beta}}$ from the inequality $g^{\alpha_1+\alpha_2+\beta}|p'|^{\alpha_1+\beta}|s'|^{\alpha_2+\beta} \leq B$. To get an upper bound for $|p'|$, note that if $|s'| = 1$, then $|p'| \leq B/g^{\alpha_1+\alpha_2+\beta}$.

The main term M in our calculations comes from the assumptions that $|p'| \geq 2$ and $|q|, |s'|, |t| \geq 1$. The error term Err comes from the cases where $|p'| = |q| = 1$, $|q| = 0$, or $|t| = 0$. Thus the total number of points for a fixed g with height less than or equal to B is $4M + 4Err$. The constant 4 takes into account that for each point of the form $([p:q],[s:t])$ with p, q, s, t positive there are three other points: $([-p:q],[s:t]), ([-p:q],[-s:t]), ([p:q],[-s:t])$.

The lattice of integer points under the constraints above (slightly) overestimates the number of points with height less than B. We count each lattice point $(P_g, P_q, P_{p'}, P_t, P_s')$ by associating to it the hypercube

$$\int_{g-1}^{g}\int_{q-1}^{q}\int_{p'-1}^{p'}\int_{t-1}^{t}\int_{s'-1}^{s'} 1 \, ds' \, dt \, dp' \, dq \, dg$$

We then integrate 1 over s', t, p', q, and g, using the bounds from above. Note that we should subtract 1 from each lower bound that to ensure that we include the entire hypercube.

Note that the following expression overestimates the number of possible values for q and t. To get a closer approximation we should integrate over q from 0 to $\varphi(p')$ and over t from 0 to $\varphi(s')$, where $\varphi(x)$ is Euler's function.

Calculating $4M$:

$$\begin{aligned}
4M &\leq 4\int_1^{\left(B/g^{\alpha_1+\alpha_2+\beta}\right)^{\frac{1}{\alpha_1+\beta}}} \int_0^{gp'} \int_0^{\left(B/g^{\alpha_1+\alpha_2+\beta}\right)^{\frac{1}{\alpha_2+\beta}} p'^{\,-\frac{\alpha_1+\beta}{\alpha_2+\beta}}} \int_0^{gs'} 1\, dt\, ds'\, dq\, dp' \\
&= 4g^2 \int_1^{\left(B/g^{\alpha_1+\alpha_2+\beta}\right)^{\frac{1}{\alpha_1+\beta}}} p' \int_0^{\left(B/g^{\alpha_1+\alpha_2+\beta}\right)^{\frac{1}{\alpha_2+\beta}} p'^{\,-\frac{\alpha_1+\beta}{\alpha_2+\beta}}} s'\, ds'\, dp' \\
&= 2B^{\frac{2}{\alpha_2+\beta}} g^{2\left(1-\frac{\alpha_1+\alpha_2+\beta}{\alpha_2+\beta}\right)} \int_1^{\left(B/g^{\alpha_1+\alpha_2+\beta}\right)^{\frac{1}{\alpha_1+\beta}}} p'^{\left(1-2\frac{\alpha_1+\beta}{\alpha_2+\beta}\right)}\, dp' \\
&= 2B^{\frac{2}{\alpha_2+\beta}} g^{-\frac{2\alpha_1}{\alpha_2+\beta}} \int_1^{\left(B/g^{\alpha_1+\alpha_2+\beta}\right)^{\frac{1}{\alpha_1+\beta}}} p'^{\left(1-2\frac{\alpha_1+\beta}{\alpha_2+\beta}\right)}\, dp'
\end{aligned}$$

Note that $\alpha_1 = \alpha_2$ if and only if $1 - 2\frac{\alpha_1+\beta}{\alpha_2+\beta} = -1$, so there are two cases to consider when we evaluate the remaining integral.

4.1.1 $\alpha_1 \neq \alpha_2$

We will start with the case $\alpha_1 \neq \alpha_2$ and evaluate the remaining integral in the above expression.

$$\begin{aligned}
&\int_1^{\left(B/g^{\alpha_1+\alpha_2+\beta}\right)^{\frac{1}{\alpha_1+\beta}}} p'^{\left(1-2\frac{\alpha_1+\beta}{\alpha_2+\beta}\right)}\, dp' \\
&= \frac{1}{2-2\frac{\alpha_1+\beta}{\alpha_2+\beta}} \left(B/g^{\alpha_1+\alpha_2+\beta}\right)^{\left(\frac{1}{\alpha_1+\beta}\right)\left(2-2\frac{\alpha_1+\beta}{\alpha_2+\beta}\right)} - \frac{1}{2-2\frac{\alpha_1+\beta}{\alpha_2+\beta}} \\
&= \frac{1}{2}\frac{\alpha_2+\beta}{\alpha_2-\alpha_1} \left(B/g^{\alpha_1+\alpha_2+\beta}\right)^{2\left(\frac{1}{\alpha_1+\beta}-\frac{1}{\alpha_2+\beta}\right)} - \frac{1}{2}\frac{\alpha_2+\beta}{\alpha_2-\alpha_1} \\
&= \frac{1}{2}\frac{\alpha_2+\beta}{\alpha_2-\alpha_1} B^{2\left(\frac{1}{\alpha_1+\beta}-\frac{1}{\alpha_2+\beta}\right)} g^{-2(\alpha_1+\alpha_2+\beta)\left(\frac{1}{\alpha_1+\beta}-\frac{1}{\alpha_2+\beta}\right)} - \frac{1}{2}\frac{\alpha_2+\beta}{\alpha_2-\alpha_1} \\
&= \frac{1}{2}\frac{\alpha_2+\beta}{\alpha_2-\alpha_1} B^{2\left(\frac{1}{\alpha_1+\beta}-\frac{1}{\alpha_2+\beta}\right)} g^{-2\left(\frac{\alpha_2}{\alpha_1+\beta}-\frac{\alpha_1}{\alpha_2+\beta}\right)} - \frac{1}{2}\frac{\alpha_2+\beta}{\alpha_2-\alpha_1}
\end{aligned}$$

Returning to our main term, we get:

$$\begin{aligned}
&4\int_1^{\left(B/g^{\alpha_1+\alpha_2+\beta}\right)^{\frac{1}{\alpha_1+\beta}}} \int_0^{gp'} \int_0^{\left(B/g^{\alpha_1+\alpha_2+\beta}\right)^{\frac{1}{\alpha_2+\beta}} p'^{-\frac{\alpha_1+\beta}{\alpha_2+\beta}}} \int_0^{gs'} 1\, dt\, ds'\, dq\, dp' \\
&= \frac{\alpha_2+\beta}{\alpha_2-\alpha_1} B^{\frac{2}{\alpha_1+\beta}} g^{-\frac{2\alpha_2}{\alpha_1+\beta}} - \frac{\alpha_2+\beta}{\alpha_2-\alpha_1} B^{\frac{2}{\alpha_2+\beta}} g^{-\frac{2\alpha_1}{\alpha_2+\beta}}
\end{aligned}$$

Now we sum over the possible values for g. Note that g goes from 1 to $B^{\frac{1}{\alpha_1+\alpha_2+\beta}}$.

If $g = 1$, then we have this many points (plus an error term which we will calculate later as part of Err) with height less than B:

$$\frac{\alpha_2+\beta}{\alpha_2-\alpha_1} B^{\frac{2}{\alpha_1+\beta}} - \frac{\alpha_2+\beta}{\alpha_2-\alpha_1} B^{\frac{2}{\alpha_2+\beta}}$$

Note that the following integral sums the number of points as g ranges from 2 to $B^{\frac{1}{\alpha_1+\alpha_2+\beta}}$.

$$\begin{aligned}
&\int_1^{B^{\frac{1}{\alpha_1+\alpha_2+\beta}}} \left(\frac{\alpha_2+\beta}{\alpha_2-\alpha_1} B^{\frac{2}{\alpha_1+\beta}} g^{-\frac{2\alpha_2}{\alpha_1+\beta}} - \frac{\alpha_2+\beta}{\alpha_2-\alpha_1} B^{\frac{2}{\alpha_2+\beta}} g^{-\frac{2\alpha_1}{\alpha_2+\beta}} \right) dg \\
&= \frac{\alpha_2+\beta}{\alpha_2-\alpha_1} B^{\frac{2}{\alpha_1+\beta}} \int_1^{B^{\frac{1}{\alpha_1+\alpha_2+\beta}}} g^{-\frac{2\alpha_2}{\alpha_1+\beta}}\, dg - \frac{\alpha_2+\beta}{\alpha_2-\alpha_1} B^{\frac{2}{\alpha_2+\beta}} \int_1^{B^{\frac{1}{\alpha_1+\alpha_2+\beta}}} g^{-\frac{2\alpha_1}{\alpha_2+\beta}}\, dg
\end{aligned}$$

Evaluating the first integral gives:

$$\begin{aligned}
&B^{\frac{2}{\alpha_1+\beta}} \int_1^{B^{\frac{1}{\alpha_1+\alpha_2+\beta}}} g^{-\frac{2\alpha_2}{\alpha_1+\beta}}\, dg \\
&= \begin{cases} B^{\frac{2}{\alpha_1+\beta}} \frac{1}{1-\frac{2\alpha_2}{\alpha_1+\beta}} \left(B^{\frac{1}{\alpha_1+\alpha_2+\beta}\left(1-\frac{2\alpha_2}{\alpha_1+\beta}\right)} - 1 \right), & \text{if } 2\alpha_2 \neq \alpha_1+\beta \\ B^{\frac{2}{\alpha_1+\beta}} \frac{1}{\alpha_1+\alpha_2+\beta} \log B, & \text{if } 2\alpha_2 = \alpha_1+\beta \end{cases} \\
&= \begin{cases} \frac{\alpha_1+\beta}{\alpha_1+\beta-2\alpha_2} B^{\left(\frac{2}{\alpha_1+\beta}+\frac{1}{\alpha_1+\alpha_2+\beta}\left(1-\frac{2\alpha_2}{\alpha_1+\beta}\right)\right)} - \frac{\alpha_1+\beta}{\alpha_1+\beta-2\alpha_2} B^{\frac{2}{\alpha_1+\beta}}, & \text{if } 2\alpha_2 \neq \alpha_1+\beta \\ \frac{1}{\alpha_1+\alpha_2+\beta} B^{\frac{2}{\alpha_1+\beta}} \log B, & \text{if } 2\alpha_2 = \alpha_1+\beta \end{cases} \\
&= \begin{cases} \frac{\alpha_1+\beta}{\alpha_1+\beta-2\alpha_2} B^{\frac{3(\alpha_1+\beta)}{(\alpha_1+\beta)(\alpha_1+\alpha_2+\beta)}} - \frac{\alpha_1+\beta}{\alpha_1+\beta-2\alpha_2} B^{\frac{2}{\alpha_1+\beta}}, & \text{if } 2\alpha_2 \neq \alpha_1+\beta \\ \frac{1}{\alpha_2+2\alpha_2} B^{\frac{2}{2\alpha_2}} \log B, & \text{if } 2\alpha_2 = \alpha_1+\beta \end{cases} \\
&= \begin{cases} \frac{\alpha_1+\beta}{\alpha_1+\beta-2\alpha_2} B^{\frac{3}{\alpha_1+\alpha_2+\beta}} - \frac{\alpha_1+\beta}{\alpha_1+\beta-2\alpha_2} B^{\frac{2}{\alpha_1+\beta}}, & \text{if } 2\alpha_2 \neq \alpha_1+\beta \\ \frac{1}{3\alpha_2} B^{\frac{1}{\alpha_2}} \log B, & \text{if } 2\alpha_2 = \alpha_1+\beta \end{cases}
\end{aligned}$$

Similarly, the second integral gives us:

$$\begin{aligned}
&B^{\frac{2}{\alpha_2+\beta}} \int_1^{B^{\frac{1}{\alpha_1+\alpha_2+\beta}}} g^{-\frac{2\alpha_1}{\alpha_2+\beta}}\, dg \\
&= \begin{cases} \frac{\alpha_2+\beta}{\alpha_2+\beta-2\alpha_1} B^{\frac{3}{\alpha_1+\alpha_2+\beta}} - \frac{\alpha_2+\beta}{\alpha_2+\beta-2\alpha_1} B^{\frac{2}{\alpha_2+\beta}}, & \text{if } 2\alpha_1 \neq \alpha_2+\beta \\ \frac{1}{3\alpha_1} B^{\frac{1}{\alpha_1}} \log B, & \text{if } 2\alpha_1 = \alpha_2+\beta \end{cases}
\end{aligned}$$

Now we substitute into our original equation and add in the term for g=1. If $2\alpha_2 \neq \alpha_1+\beta$ and $2\alpha_1 \neq \alpha_2+\beta$, then the number of points with height less than B is:

$$\tfrac{3(\alpha_1+\beta)(\alpha_2+\beta)}{(\alpha_1+\beta-2\alpha_2)(\alpha_2+\beta-2\alpha_1)} B^{\frac{3}{\alpha_1+\alpha_2+\beta}} - \tfrac{2\alpha_2}{\alpha_1+\beta-2\alpha_2} B^{\frac{2}{\alpha_1+\beta}} - \tfrac{2\alpha_1}{\alpha_2+\beta-2\alpha_1} B^{\frac{2}{\alpha_2+\beta}}$$

which is

$$\begin{cases} O(B^{\frac{3}{\alpha_1+\alpha_2+\beta}}), & \text{if } 2\alpha_2 < \alpha_1+\beta \text{ and } 2\alpha_1 < \alpha_2+\beta \\ O(B^{\max(\frac{2}{\alpha_i+\beta})}), & \text{otherwise} \end{cases}$$

Note that $2\alpha_2 = \alpha_1+\beta$ and $2\alpha_1 = \alpha_2+\beta$ are both true only when $\alpha_1 = \alpha_2 = \beta$, which is not possible since we're assuming $\alpha_1 \neq \alpha_2$.

Assume $2\alpha_2 = \alpha_1+\beta$. Then we cannot have $\beta = \alpha_1$ since then we would have $\alpha_1 = \alpha_2 = \beta$. The number of points with height less than B is:

$$\tfrac{\alpha_2+\beta}{3\alpha_2(\alpha_2-\alpha_1)} B^{\frac{1}{\alpha_2}} \log B - \tfrac{(\alpha_2+\beta)^2}{3(\alpha_2-\alpha_1)^2} B^{\frac{1}{\alpha_2}} + \tfrac{(\alpha_2+\beta)^2}{3(\alpha_2-\alpha_1)^2} B^{\frac{2}{\alpha_2+\beta}}$$

which is

$$\begin{cases} O(B^{\frac{1}{\alpha_2}} \log B), & \text{if } \alpha_2 > \alpha_1 \\ O(B^{\frac{2}{\alpha_2+\beta}}), & \text{if } \alpha_2 < \alpha_1 \end{cases}$$

If $2\alpha_1 = \alpha_2+\beta$ then the number of points with height less than B is:

$$-\tfrac{2\alpha_1(\alpha_1+\beta)}{3(\alpha_2-\alpha_1)^2} B^{\frac{1}{\alpha_1}} + \tfrac{2\alpha_1(\alpha_1+\beta)}{3(\alpha_2-\alpha_1)^2} B^{\frac{2}{\alpha_1+\beta}} - \tfrac{2}{3(\alpha_2-\alpha_1)} B^{\frac{1}{\alpha_1}} \log B$$

which is

$$\begin{cases} O(B^{\frac{1}{\alpha_1}} \log B), & \text{if } \alpha_1 > \alpha_2 \\ O(B^{\frac{2}{\alpha_1+\beta}}), & \text{if } \alpha_1 < \alpha_2 \end{cases}$$

4.1.2 $\alpha_1 = \alpha_2$

Consider the case $\alpha_1 = \alpha_2$. Set $\alpha = \alpha_1 = \alpha_2$ for ease of notation. Then

$$\begin{aligned} \int_1^{\left(B/g^{2\alpha+\beta}\right)^{\frac{1}{\alpha+\beta}}} & p'^{-1}\, dp' \\ &= \tfrac{1}{\alpha+\beta} \log\left(B/g^{2\alpha+\beta}\right) \\ &= \tfrac{1}{\alpha+\beta} \log B - \tfrac{2\alpha+\beta}{\alpha+\beta} \log g \end{aligned}$$

Returning to our main term, we get:

$$\begin{aligned}
&4\int_1^{\left(B/g^{2\alpha+\beta}\right)^{\frac{1}{\alpha+\beta}}}\int_0^{gp'}\int_0^{\left(B/g^{2\alpha+\beta}\right)^{\frac{1}{\alpha+\beta}}p'^{-1}}\int_0^{gs'} 1\,dt\,ds'\,dq\,dp' \\
&= \quad 2B^{\frac{2}{\alpha+\beta}}g^{-\frac{2\alpha}{\alpha+\beta}}\left(\tfrac{1}{\alpha+\beta}\log B - \tfrac{2\alpha+\beta}{\alpha+\beta}\log g\right)
\end{aligned}$$

Now we sum over the possible values for g. Note that g goes from 1 to $B^{\frac{1}{2\alpha+\beta}}$. If $g=1$, then we have this many points (plus an error term which we will calculate later) with height less than B:

$$\begin{aligned}
&4\int_1^{B^{\frac{1}{\alpha+\beta}}}\int_0^{p'}\int_0^{B^{\frac{1}{\alpha+\beta}}p'^{-\frac{\alpha+\beta}{\alpha+\beta}}}\int_0^{s'} 1\,dt\,ds'\,dq\,dp' \\
&= \quad \tfrac{2}{\alpha+\beta}B^{\frac{2}{\alpha+\beta}}\log B
\end{aligned}$$

Now we sum over g from 2 to $B^{\frac{1}{\alpha_1+\alpha_2+\beta}}$.

$$\begin{aligned}
&2\int_1^{B^{\frac{1}{2\alpha+\beta}}} B^{\frac{2}{\alpha+\beta}}g^{-\frac{2\alpha}{\alpha+\beta}}\left(\tfrac{1}{\alpha+\beta}\log B - \tfrac{2\alpha+\beta}{\alpha+\beta}\log g\right)\,dg \\
&= \quad \tfrac{2}{\alpha+\beta}B^{\frac{2}{\alpha+\beta}}\log B\int_1^{B^{\frac{1}{2\alpha+\beta}}} g^{-\frac{2\alpha}{\alpha+\beta}}\,dg - \tfrac{2(2\alpha+\beta)}{\alpha+\beta}B^{\frac{2}{\alpha+\beta}}\int_1^{B^{\frac{1}{2\alpha+\beta}}} g^{-\frac{2\alpha}{\alpha+\beta}}\log g\,dg
\end{aligned}$$

Now assume $\alpha \neq \beta$. Evaluating the first integral gives us:

$$\begin{aligned}
&\int_1^{B^{\frac{1}{2\alpha+\beta}}} g^{-\frac{2\alpha}{\alpha+\beta}}\,dg \\
&\quad = \quad \tfrac{1}{1-2\frac{\alpha}{\alpha+\beta}}\left(B^{\frac{1}{2\alpha+\beta}\left(1-2\frac{\alpha}{\alpha+\beta}\right)} - 1\right)
\end{aligned}$$

Then we look at the second integral:

$$\begin{aligned}
&\int_1^{B^{\frac{1}{2\alpha+\beta}}} g^{-\frac{2\alpha}{\alpha+\beta}}\log g\,dg \\
&= \quad \tfrac{1}{1-2\frac{\alpha}{\alpha+\beta}}B^{\frac{1}{2\alpha+\beta}\left(1-2\frac{\alpha}{\alpha+\beta}\right)}\tfrac{1}{2\alpha+\beta}\log B - \left(\tfrac{1}{1-2\frac{\alpha}{\alpha+\beta}}\right)^2 B^{\frac{1}{2\alpha+\beta}\left(1-2\frac{\alpha}{\alpha+\beta}\right)} + \left(\tfrac{1}{1-2\frac{\alpha}{\alpha+\beta}}\right)^2 \\
&= \quad \tfrac{\alpha+\beta}{\beta-\alpha}\left(\tfrac{1}{2\alpha+\beta}B^{\frac{1}{2\alpha+\beta}\left(1-2\frac{\alpha}{\alpha+\beta}\right)}\log B - \tfrac{\alpha+\beta}{\beta-\alpha}B^{\frac{1}{2\alpha+\beta}\left(1-2\frac{\alpha}{\alpha+\beta}\right)} + \tfrac{\alpha+\beta}{\beta-\alpha}\right)
\end{aligned}$$

Substituting back into the original equation, we get:

$$\begin{aligned}
&\tfrac{2}{(\alpha+\beta)(1-2\frac{\alpha}{\alpha+\beta})} B^{\frac{2}{\alpha+\beta}} \log B \left(B^{\frac{1}{2\alpha+\beta}\left(1-2\frac{\alpha}{\alpha+\beta}\right)} - 1 \right) \\
&\quad - \tfrac{2(2\alpha+\beta)}{\beta-\alpha} B^{\frac{2}{\alpha+\beta}} \left(\tfrac{1}{2\alpha+\beta} B^{\frac{1}{2\alpha+\beta}\left(1-2\frac{\alpha}{\alpha+\beta}\right)} \log B - \tfrac{\alpha+\beta}{\beta-\alpha} B^{\frac{1}{2\alpha+\beta}\left(1-2\frac{\alpha}{\alpha+\beta}\right)} + \tfrac{\alpha+\beta}{\beta-\alpha} \right) \\
= &\ \tfrac{2}{\beta-\alpha} \left(B^{\frac{3}{2\alpha+\beta}} \log B - B^{\frac{2}{\alpha+\beta}} \log B \right) \\
&\quad - \tfrac{2(2\alpha+\beta)}{\beta-\alpha} \left(\tfrac{1}{2\alpha+\beta} B^{\frac{3}{2\alpha+\beta}} \log B - \tfrac{\alpha+\beta}{\beta-\alpha} B^{\frac{3}{2\alpha+\beta}} + \tfrac{\alpha+\beta}{\beta-\alpha} B^{\frac{2}{\alpha+\beta}} \right) \\
= &\ -\tfrac{2}{\beta-\alpha} B^{\frac{2}{\alpha+\beta}} \log B + \tfrac{2(2\alpha+\beta)(\alpha+\beta)}{(\beta-\alpha)^2} B^{\frac{3}{2\alpha+\beta}} - \tfrac{2(2\alpha+\beta)(\alpha+\beta)}{(\beta-\alpha)^2} B^{\frac{2}{\alpha+\beta}}
\end{aligned}$$

Now when we add in the number of points with $g = 1$, we get:

$$-\tfrac{4\alpha}{(\beta-\alpha)(\alpha+\beta)} B^{\frac{2}{\alpha+\beta}} \log B + \tfrac{2(2\alpha+\beta)(\alpha+\beta)}{(\beta-\alpha)^2} B^{\frac{3}{2\alpha+\beta}} - \tfrac{2(2\alpha+\beta)(\alpha+\beta)}{(\beta-\alpha)^2} B^{\frac{2}{\alpha+\beta}}$$

which is

$$\begin{cases} O(B^{\frac{3}{2\alpha+\beta}}), & \text{if } \beta > \alpha \\ O(B^{\frac{2}{\alpha+\beta}} \log B), & \text{if } \beta < \alpha \end{cases}$$

If $\alpha = \beta$, we get:

$$\begin{aligned}
&2\int_1^{B^{\frac{1}{3\alpha}}} B^{\frac{1}{2\alpha}} g^{-1} \left(\tfrac{1}{\alpha} \log B - \tfrac{3}{2} \log g \right) \, dg \\
= &\ \tfrac{1}{3\alpha^2} B^{\frac{1}{\alpha}} (\log B)^2 - \tfrac{1}{6\alpha^2} B^{\frac{1}{\alpha}} (\log B)^2 \\
= &\ \tfrac{1}{6\alpha^2} B^{\frac{1}{\alpha}} (\log B)^2
\end{aligned}$$

Thus the total number of points we get if $\alpha = \beta$ is:

$$\tfrac{1}{6\alpha^2} B^{\frac{1}{\alpha}} (\log B)^2 + \tfrac{1}{\alpha} B^{\frac{1}{\alpha}} \log B$$

which is

$$O(B^{\frac{1}{\alpha}} (\log B)^2)$$

4.1.3 Error term

Now to calculate the error terms. We've ignored the cases where $|p'| = 1$, $|q| = 0$, or $|t| = 0$.

If $|p'| = 1$, then our points look like $([g : q], [gs' : t])$ and $H_C([g : q], [gs' : t]) \leq B$ means that we must have $g^{\alpha_1+\alpha_2+\beta}|s'|^{\alpha_2+\beta} \leq B$.

For a fixed g, the number of points with height less than or equal to B is:

$$\begin{aligned}
4\int_0^g \int_0^{B^{\frac{1}{\alpha_2+\beta}} g^{-\frac{\alpha_1+\alpha_2+\beta}{\alpha_2+\beta}}} \int_0^{gs'} 1\, dt\, ds'\, dq \\
&= 4g^2 \int_0^{B^{\frac{1}{\alpha_2+\beta}} g^{-\frac{\alpha_1+\alpha_2+\beta}{\alpha_2+\beta}}} s'\, ds' \\
&= 2B^{\frac{2}{\alpha_2+\beta}} g^{(2-\frac{2(\alpha_1+\alpha_2+\beta)}{\alpha_2+\beta})} \\
&= 2B^{\frac{2}{\alpha_2+\beta}} g^{-\frac{2\alpha_1}{\alpha_2+\beta}}
\end{aligned}$$

Note that if $2\alpha_1 = \alpha_2+\beta$, then $\frac{2\alpha_1}{\alpha_2+\beta} = -1$. First look at the case when $2\alpha_1 \neq \alpha_2+\beta$.

If we sum over the g's, the number of points of the form $([g : q], [gs' : t])$ with height less than B is:

$$\begin{aligned}
2B^{\frac{2}{\alpha_2+\beta}} \int_0^{B^{\frac{1}{\alpha_1+\alpha_2+\beta}}} g^{-\frac{2\alpha_1}{\alpha_2+\beta}}\, dg \\
&= \frac{2}{1-\frac{2\alpha_1}{\alpha_2+\beta}} B^{\frac{2}{\alpha_2+\beta}+\frac{1}{\alpha_1+\alpha_2+\beta}(1-\frac{2\alpha_1}{\alpha_2+\beta})} \\
&= \frac{2(\alpha_2+\beta)}{\alpha_2+\beta-2\alpha_1} B^{\frac{3}{\alpha_1+\alpha_2+\beta}}
\end{aligned}$$

which is:

$$O(B^{\frac{3}{\alpha_1+\alpha_2+\beta}})$$

Now look at the case $2\alpha_1 = \alpha_2 + \beta$. Then the number of points of the form $([g : q], [gs' : t])$ with height less than B is:

$$\begin{aligned}
2B^{\frac{1}{\alpha_1}} \int_1^{B^{\frac{1}{3\alpha_1}}} g^{-1}\, dg + 2B^{\frac{1}{\alpha_1}} \\
&= \frac{2}{3\alpha_1} B^{\frac{1}{\alpha_1}} \log B + 2B^{\frac{1}{\alpha_1}}
\end{aligned}$$

which is

$$O(B^{\frac{1}{\alpha_1}} \log B)$$

Now we count the number of points with $|q| = 0$, or $|t| = 0$. We will start with the case $|q| = 0$. Then our points look like $([1 : 0], [s : t])$ and $H_C([1 : 0], [s : t]) \leq B$ means that we must have $|s|^{\alpha_2+\beta} \leq B$.

The number of such points is given by:

$$\begin{aligned}
&\int_0^{B^{\frac{1}{\alpha_2+\beta}}} \int_0^s 1 \, dt \, ds \\
&= \int_0^{B^{\frac{1}{\alpha_2+\beta}}} s \, ds \\
&= \tfrac{1}{2} B^{\frac{2}{\alpha_2+\beta}}
\end{aligned}$$

Similarly, the number of points with $|t| = 0$ is:

$$\tfrac{1}{2} B^{\frac{2}{\alpha_1+\beta}}$$

Thus our error term is:

$$\begin{cases} O(B^{\max\{\frac{3}{\alpha_1+\alpha_2+\beta}, \frac{2}{\alpha_1+\beta}, \frac{2}{\alpha_2+\beta}\}}), & \text{if } 2\alpha_1 \neq \alpha_2 + \beta \\ O(B^{\frac{1}{\alpha_1}} \log B), & \text{if } 2\alpha_1 = \alpha_2 + \beta \text{ and } \alpha_1 \geq \alpha_2 \\ O(B^{\frac{2}{\alpha_1+\beta}}), & \text{if } 2\alpha_1 = \alpha_2 + \beta \text{ and } \alpha_1 < \alpha_2 \end{cases}$$

Compare this to our main term, which we calculated to be:

When $\alpha_1 \neq \alpha_2$, $2\alpha_1 \neq \alpha_2 + \beta$, and $2\alpha_2 \neq \alpha_1 + \beta$:

$$N_{V \setminus Z}(C, B) = O(B^{\max\{\frac{3}{\alpha_1+\alpha_2+\beta}, \frac{2}{\alpha_1+\beta}, \frac{2}{\alpha_2+\beta}\}})$$

When $\alpha_1 \neq \alpha_2$, $2\alpha_1 \neq \alpha_2 + \beta$, and $2\alpha_2 = \alpha_1 + \beta$:

$$N_{V \setminus Z}(C, B) = \begin{cases} O(B^{\frac{1}{\alpha_2}} \log B), & \text{if } \alpha_2 > \alpha_1 \\ O(B^{\frac{2}{\alpha_2+\beta}}), & \text{if } \alpha_2 < \alpha_1 \end{cases}$$

When $\alpha_1 \neq \alpha_2$, $2\alpha_1 = \alpha_2 + \beta$, and $2\alpha_2 \neq \alpha_1 + \beta$:

$$N_{V \setminus Z}(C, B) = \begin{cases} O(B^{\frac{1}{\alpha_1}} \log B), & \text{if } \alpha_1 > \alpha_2 \\ O(B^{\frac{2}{\alpha_1+\beta}}), & \text{if } \alpha_1 < \alpha_2 \end{cases}$$

When $\alpha := \alpha_1 = \alpha_2$ and $\alpha \neq \beta$:

$$N_{V\setminus Z}(C,B) = \begin{cases} O(B^{\frac{3}{2\alpha+\beta}}), & \text{if } \beta > \alpha \\ O(B^{\frac{2}{\alpha+\beta}}\log B), & \text{if } \beta < \alpha \end{cases}$$

And finally, when $\alpha := \alpha_1 = \alpha_2 = \beta$:

$$N_{V\setminus Z}(C,B) = O(B^{\frac{1}{\alpha}}(\log B)^2)$$

In each case, the error term is no larger than the main term, as desired.

4.2 $|p| \leq |q|$ and $|s| \leq |t|$

Recall that $\gcd(p,q) = gcd(s,t) = 1$, $g = \gcd(p,s)$ and $p = gp'$ and $s = gs'$. Then if $|p| \leq |q|$ and $|s| \leq |t|$, we have

$$\begin{aligned} H_C([p:q],[s:t]) &= \max\{|p|,|q|\}^{\alpha_1} \max\{|s|,|t|\}^{\alpha_2} (\max\{|qs|,|pt|,|ps|\}/g)^{\beta} \\ &= \max\{|gp'|,|q|\}^{\alpha_1} \max\{|gs'|,|t|\}^{\alpha_2} \max\{|qs'|,|p't|,|gp's'|\}^{\beta} \\ &= |q|^{\alpha_1} |t|^{\alpha_2} \max\{|qs'|,|p't|\}^{\beta} \end{aligned}$$

So we have two (symmetric) cases to consider: $|qs'| \leq |p't|, |qs'| \geq |p't|$.

Assume $|qs'| \leq |p't|$. Then

$$H_C([p:q],[s:t]) = |q|^{\alpha_1} |p'|^{\beta} |t|^{\alpha_2+\beta}$$

Note we get two upper bounds for $|s'|$: one from the inequality $|qs'| \leq |p't|$ and the other from the inequality $|gs'| \leq |t|$. Since $\left|\frac{t}{g}\right| \geq \left|\frac{p't}{q}\right|$ if and only if $|q| \geq |gp'|$, which is one of our assumptions, we have that $\left|\frac{p't}{q}\right|$ is the better upper bound for our integral. $|s'|$ has a lower bound of 0.

For $|t|$, we get an upper bound from the inequality $|q|^{\alpha_1}|p'|^{\beta}|t|^{\alpha_2+\beta} \leq B$ and a lower bound from the inequality $|qs'| \leq |p't|$, letting $|s'| = 1$ (We'll ignore the case $|s'| = 0$ for now and calculate it later as part of the error term). This means that we also must have $\left(\frac{B}{|q|^{\alpha_1}|p'|^{\beta}}\right)^{\frac{1}{\alpha_2+\beta}} \geq |\frac{q}{p'}|$ or, equivalently, $B|p'|^{\alpha_2} \geq |q|^{\alpha_1+\alpha_2+\beta}$.

For $|p'|$, we get an upper bound from the inequality $|gp'| \leq |q|$. Note that $|p'|$ has a lower bound from the inequality $B|p'|^{\alpha_2} \geq |q|^{\alpha_1+\alpha_2+\beta}$. Note that $\frac{|q|}{g} \geq (\frac{|q|^{\alpha_1+\alpha_2+\beta}}{B})^{\frac{1}{\alpha_2}}$ implies $B \geq g^{\alpha_2}|q|^{\alpha_1+\beta}$.

For $|q|$, we get an upper bound from the inequality $B \geq g^{\alpha_2}|q|^{\alpha_1+\beta}$. $|q|$ has a lower bound of g.

Thus, to count the number of points with height less than B, we will evaluate the following integral and add in some error terms later:

$$4\int_g^{\left(\frac{B}{g^{\alpha_2}}\right)^{\frac{1}{\alpha_1+\beta}}} \int_{\left(\frac{q^{\alpha_1+\alpha_2+\beta}}{B}\right)^{\frac{1}{\alpha_2}}}^{\frac{q}{g}} \int_{\frac{q}{p'}}^{\left(\frac{B}{p'^{\beta}q^{\alpha_1}}\right)^{\frac{1}{\alpha_2+\beta}}} \int_0^{\frac{p't}{q}} 1 \, ds' \, dt \, dp' \, dq$$

$$= 4\int_g^{\left(\frac{B}{g^{\alpha_2}}\right)^{\frac{1}{\alpha_1+\beta}}} \int_{\left(\frac{q^{\alpha_1+\alpha_2+\beta}}{B}\right)^{\frac{1}{\alpha_2}}}^{\frac{q}{g}} \int_{\frac{q}{p'}}^{\left(\frac{B}{p'^{\beta}q^{\alpha_1}}\right)^{\frac{1}{\alpha_2+\beta}}} \frac{p't}{q} \, dt \, dp' \, dq$$

$$= 4\int_g^{\left(\frac{B}{g^{\alpha_2}}\right)^{\frac{1}{\alpha_1+\beta}}} \int_{\left(\frac{q^{\alpha_1+\alpha_2+\beta}}{B}\right)^{\frac{1}{\alpha_2}}}^{\frac{q}{g}} \frac{p'}{2q} \left(\left(\frac{B}{p'^{\beta}q^{\alpha_1}}\right)^{\frac{2}{\alpha_2+\beta}} - \frac{q^2}{(p')^2} \right) dp' \, dq$$

$$= 2\int_g^{\left(\frac{B}{g^{\alpha_2}}\right)^{\frac{1}{\alpha_1+\beta}}} \int_{\left(\frac{q^{\alpha_1+\alpha_2+\beta}}{B}\right)^{\frac{1}{\alpha_2}}}^{\frac{q}{g}} \left(B^{\frac{2}{\alpha_2+\beta}} p'^{\,(1-\frac{2\beta}{\alpha_2+\beta})} q^{(-1-\frac{2\alpha_1}{\alpha_2+\beta})} - q(p')^{-1} \right) dp' \, dq$$

$$= 2\int_g^{\left(\frac{B}{g^{\alpha_2}}\right)^{\frac{1}{\alpha_1+\beta}}} \left(B^{\frac{2}{\alpha_2+\beta}} \frac{1}{2-\frac{2\beta}{\alpha_2+\beta}} \left(\frac{q}{g}\right)^{(2-\frac{2\beta}{\alpha_2+\beta})} q^{(-1-\frac{2\alpha_1}{\alpha_2+\beta})} \right.$$

$$- B^{\frac{2}{\alpha_2+\beta}} \frac{1}{2-\frac{2\beta}{\alpha_2+\beta}} \left(\frac{q^{\alpha_1+\alpha_2+\beta}}{B}\right)^{\frac{1}{\alpha_2}(2-\frac{2\beta}{\alpha_2+\beta})} q^{(-1-\frac{2\alpha_1}{\alpha_2+\beta})}$$

$$\left. - q\left(\log\frac{q}{g} - \log\left(\left(\frac{q^{\alpha_1+\alpha_2+\beta}}{B}\right)^{\frac{1}{\alpha_2}}\right)\right) \right) dq$$

$$= \int_g^{\left(\frac{B}{g^{\alpha_2}}\right)^{\frac{1}{\alpha_1+\beta}}} \left(\tfrac{\alpha_2+\beta}{\alpha_2} B^{\frac{2}{\alpha_2+\beta}} g^{-\frac{2\alpha_2}{\alpha_2+\beta}} q^{(1-\frac{2(\alpha_1+\beta)}{\alpha_2+\beta})} + \tfrac{2(\alpha_1+\beta)}{\alpha_2} q \log q \right.$$

$$\left. + \left(2\log g - \tfrac{2}{\alpha_2}\log B - \tfrac{\alpha_2+\beta}{\alpha_2}\right) q \right) dq$$

Now we must consider two cases: $\alpha_1 \neq \alpha_2$ and $\alpha_1 = \alpha_2$.

When $\alpha_1 \neq \alpha_2$, we get

$$\tfrac{\alpha_2+\beta}{\alpha_2} B^{\frac{2}{\alpha_2+\beta}} g^{-\frac{2\alpha_2}{\alpha_2+\beta}} \frac{1}{2-\frac{2(\alpha_1+\beta)}{\alpha_2+\beta}} \left(\left(\frac{B}{g^{\alpha_2}}\right)^{\frac{1}{\alpha_1+\beta}(2-\frac{2(\alpha_1+\beta)}{\alpha_2+\beta})} - g^{(2-\frac{2(\alpha_1+\beta)}{\alpha_2+\beta})} \right)$$

$$+ \tfrac{\alpha_1+\beta}{\alpha_2} \left(\left(\frac{B}{g^{\alpha_2}}\right)^{\frac{2}{\alpha_1+\beta}} \log\left(\frac{B}{g^{\alpha_2}}\right)^{\frac{1}{\alpha_1+\beta}} - \tfrac{1}{2}\left(\frac{B}{g^{\alpha_2}}\right)^{\frac{2}{\alpha_1+\beta}} - g^2\log g + \tfrac{1}{2}g^2 \right)$$

$$+ \left(\log g - \tfrac{1}{\alpha_2}\log B - \tfrac{\alpha_2+\beta}{2\alpha_2}\right) \left(\left(\frac{B}{g^{\alpha_2}}\right)^{\frac{2}{\alpha_1+\beta}} - g^2 \right)$$

Simplifying, we get that the total number of points with height less than B given g when $\alpha_1 \neq \alpha_2$ is

$$\begin{aligned}&\tfrac{(\alpha_1+\beta)^2}{2\alpha_2(\alpha_2-\alpha_1)}B^{\frac{2}{\alpha_1+\beta}}g^{-\frac{2\alpha_2}{\alpha_1+\beta}} - \tfrac{(\alpha_2+\beta)^2}{2\alpha_2(\alpha_2-\alpha_1)}B^{\frac{2}{\alpha_2+\beta}}g^{-\frac{2\alpha_1}{\alpha_2+\beta}}\\ &-\tfrac{\alpha_1+\alpha_2+\beta}{\alpha_2}g^2\log g + \tfrac{1}{\alpha_2}(\log B)g^2 + \tfrac{\alpha_1+\alpha_2+2\beta}{2\alpha_2}g^2\end{aligned}$$

And when $\alpha_1 = \alpha_2$, we get

$$\begin{aligned}&\tfrac{\alpha_2+\beta}{\alpha_2}B^{\frac{2}{\alpha_2+\beta}}g^{-\frac{2\alpha_2}{\alpha_2+\beta}}(\log\left(\frac{B}{g^{\alpha_2}}\right)^{\frac{1}{\alpha_1+\beta}} - \log g)\\ &+\tfrac{\alpha_1+\beta}{\alpha_2}\left(\left(\frac{B}{g^{\alpha_2}}\right)^{\frac{2}{\alpha_1+\beta}}\log\left(\frac{B}{g^{\alpha_2}}\right)^{\frac{1}{\alpha_1+\beta}} - \tfrac{1}{2}\left(\frac{B}{g^{\alpha_2}}\right)^{\frac{2}{\alpha_1+\beta}} - g^2\log g + \tfrac{1}{2}g^2\right)\\ &+\ \left(\log g - \tfrac{1}{\alpha_2}\log B - \tfrac{\alpha_2+\beta}{2\alpha_2}\right)\left(\left(\frac{B}{g^{\alpha_2}}\right)^{\frac{2}{\alpha_1+\beta}} - g^2\right)\end{aligned}$$

Simplifying for the case $\alpha_1 = \alpha_2$ (say $\alpha = \alpha_1 = \alpha_2$ for ease of notation) gives us:

$$\begin{aligned}&\tfrac{1}{\alpha}B^{\frac{2}{\alpha+\beta}}(\log B)g^{-\frac{2\alpha}{\alpha+\beta}} - \tfrac{2\alpha+\beta}{\alpha}B^{\frac{2}{\alpha+\beta}}g^{-\frac{2\alpha}{\alpha+\beta}}\log g\\ &-\tfrac{\alpha+\beta}{\alpha}B^{\frac{2}{\alpha+\beta}}g^{-\frac{2\alpha}{\alpha+\beta}} - \tfrac{2\alpha+\beta}{\alpha}g^2\log g + \tfrac{1}{\alpha}(\log B)g^2 + \tfrac{\alpha+\beta}{\alpha}g^2\end{aligned}$$

Now we sum over the possible values for g. Note that since we're assuming $|p| \leq |q|$ and $|s| \leq |t|$ and we've noted already that

$$H_C([p:q],[s:t]) = |q|^{\alpha_1}|p'|^{\beta}|t|^{\alpha_2+\beta}$$

we get

$$H_C([p:q],[s:t]) \geq |gp'|^{\alpha_1}|p'|^{\beta}|gs'|^{\alpha_2+\beta} = |p'|^{\alpha_1+\beta}|s'|^{\alpha_2+\beta}|g|^{\alpha_1+\alpha_2+\beta}$$

Thus if $H_C([p:q],[s:t]) \leq B$, there are no more than $B^{\frac{1}{\alpha_1+\alpha_2+\beta}}$ possible values for g.

We will look at the two cases separately:

4.2.1 $\alpha_1 \neq \alpha_2$

Assuming $\alpha_1 \neq \alpha_2$, we first calculate the number of points when $g = 1$:

$$\frac{(\alpha_1+\beta)^2}{2\alpha_2(\alpha_2-\alpha_1)} B^{\frac{2}{\alpha_1+\beta}} - \frac{(\alpha_2+\beta)^2}{2\alpha_2(\alpha_2-\alpha_1)} B^{\frac{2}{\alpha_2+\beta}} + \frac{1}{\alpha_2}\log B + \frac{\alpha_1+\alpha_2+2\beta}{2\alpha_2}$$

Before we calculate the next integral, note that since we're assuming $\alpha_1 \neq \alpha_2$, we cannot have both $2\alpha_2 = \alpha_1 + \beta$ and $2\alpha_1 = \alpha_2 + \beta$. Otherwise, subtracting the second equation from the first, we would have $2\alpha_2 - 2\alpha_1 = \alpha_1 - \alpha_2$, which implies $\alpha_1 = \alpha_2$.

First assume $2\alpha_2 \neq \alpha_1 + \beta$ and $2\alpha_1 \neq \alpha_2 + \beta$. Then

$$\begin{aligned}
&\int_1^{B^{\frac{1}{\alpha_1+\alpha_2+\beta}}} \Big(\tfrac{(\alpha_1+\beta)^2}{2\alpha_2(\alpha_2-\alpha_1)} B^{\frac{2}{\alpha_1+\beta}} g^{-\frac{2\alpha_2}{\alpha_1+\beta}} - \tfrac{(\alpha_2+\beta)^2}{2\alpha_2(\alpha_2-\alpha_1)} B^{\frac{2}{\alpha_2+\beta}} g^{-\frac{2\alpha_1}{\alpha_2+\beta}} \\
&\qquad -\tfrac{\alpha_1+\alpha_2+\beta}{\alpha_2} g^2 \log g + \tfrac{1}{\alpha_2}(\log B) g^2 + \tfrac{\alpha_1+\alpha_2+2\beta}{2\alpha_2} g^2 \Big)\; dg \\
= \;& \tfrac{(\alpha_1+\beta)^2(\alpha_1+\beta)}{2\alpha_2(\alpha_2-\alpha_1)(\alpha_1+\beta-2\alpha_2)} B^{\frac{2}{\alpha_1+\beta}} B^{\frac{1}{\alpha_1+\alpha_2+\beta}(1-\frac{2\alpha_2}{\alpha_1+\beta})} - \tfrac{(\alpha_1+\beta)^2(\alpha_1+\beta)}{2\alpha_2(\alpha_2-\alpha_1)(\alpha_1+\beta-2\alpha_2)} B^{\frac{2}{\alpha_1+\beta}} \\
&\qquad -\tfrac{(\alpha_2+\beta)^3}{2\alpha_2(\alpha_2-\alpha_1)(\alpha_2+\beta-2\alpha_1)} B^{\frac{2}{\alpha_2+\beta}} B^{\frac{1}{\alpha_1+\alpha_2+\beta}(1-\frac{2\alpha_1}{\alpha_2+\beta})} + \tfrac{(\alpha_2+\beta)^3}{2\alpha_2(\alpha_2-\alpha_1)(\alpha_2+\beta-2\alpha_1)} B^{\frac{2}{\alpha_2+\beta}} \\
&\qquad -\tfrac{1}{3\alpha_2} B^{\frac{3}{\alpha_1+\alpha_2+\beta}} \log B + \tfrac{\alpha_1+\alpha_2+\beta}{9\alpha_2} B^{\frac{3}{\alpha_1+\alpha_2+\beta}} - \tfrac{\alpha_1+\alpha_2+\beta}{9\alpha_2} \\
&\qquad +\tfrac{1}{3\alpha_2}(\log B)(B^{\frac{3}{\alpha_1+\alpha_2+\beta}} - 1) + \tfrac{\alpha_1+\alpha_2+2\beta}{6\alpha_2}(B^{\frac{3}{\alpha_1+\alpha_2+\beta}} - 1) \\
= \;& \tfrac{(\alpha_2+\beta)^3}{2\alpha_2(\alpha_2-\alpha_1)(\alpha_2+\beta-2\alpha_1)} B^{\frac{2}{\alpha_2+\beta}} - \tfrac{(\alpha_1+\beta)^2(\alpha_1+\beta)}{2\alpha_2(\alpha_2-\alpha_1)(\alpha_1+\beta-2\alpha_2)} B^{\frac{2}{\alpha_1+\beta}} \\
&\qquad +\tfrac{(\alpha_1+\beta)^2(\alpha_1+\beta)}{2\alpha_2(\alpha_2-\alpha_1)(\alpha_1+\beta-2\alpha_2)} B^{\frac{3}{\alpha_1+\alpha_2+\beta}} - \tfrac{(\alpha_2+\beta)^3}{2\alpha_2(\alpha_2-\alpha_1)(\alpha_2+\beta-2\alpha_1)} B^{\frac{3}{\alpha_1+\alpha_2+\beta}} \\
&\qquad +\tfrac{5(\alpha_1+\alpha_2+2\beta)}{18\alpha_2} B^{\frac{3}{\alpha_1+\alpha_2+\beta}} - \tfrac{1}{3\alpha_2}\log B - \tfrac{5(\alpha_1+\alpha_2+\beta)}{18\alpha_2}
\end{aligned}$$

So the total number of points with height less than B is:

$$\begin{aligned}
&\tfrac{(\alpha_1+\beta)^2}{2\alpha_2(\alpha_2-\alpha_1)} B^{\frac{2}{\alpha_1+\beta}} - \tfrac{(\alpha_2+\beta)^2}{2\alpha_2(\alpha_2-\alpha_1)} B^{\frac{2}{\alpha_2+\beta}} + \tfrac{1}{\alpha_2}\log B + \tfrac{\alpha_1+\alpha_2+2\beta}{2\alpha_2} \\
&\quad +\tfrac{(\alpha_2+\beta)^3}{2\alpha_2(\alpha_2-\alpha_1)(\alpha_2+\beta-2\alpha_1)} B^{\frac{2}{\alpha_2+\beta}} - \tfrac{(\alpha_1+\beta)^3}{2\alpha_2(\alpha_2-\alpha_1)(\alpha_1+\beta-2\alpha_2)} B^{\frac{2}{\alpha_1+\beta}} \\
&\quad +\tfrac{(\alpha_1+\beta)^3}{2\alpha_2(\alpha_2-\alpha_1)(\alpha_1+\beta-2\alpha_2)} B^{\frac{3}{\alpha_1+\alpha_2+\beta}} - \tfrac{(\alpha_2+\beta)^3}{2\alpha_2(\alpha_2-\alpha_1)(\alpha_2+\beta-2\alpha_1)} B^{\frac{3}{\alpha_1+\alpha_2+\beta}} \\
&\quad +\tfrac{5(\alpha_1+\alpha_2+2\beta)}{18\alpha_2} B^{\frac{3}{\alpha_1+\alpha_2+\beta}} - \tfrac{1}{3\alpha_2}\log B - \tfrac{5(\alpha_1+\alpha_2+\beta)}{18\alpha_2}
\end{aligned}$$

Combining like terms gives us:

$$\frac{\alpha_1(\alpha_2+\beta)^2}{\alpha_2(\alpha_2-\alpha_1)(\alpha_2+\beta-2\alpha_1)}B^{\frac{2}{\alpha_2+\beta}} - \frac{(\alpha_1+\beta)^2}{(\alpha_2-\alpha_1)(\alpha_1+\beta-2\alpha_2)}B^{\frac{2}{\alpha_1+\beta}}$$
$$+\frac{8(\alpha_1+\alpha_2)^3+20\beta(\alpha_1+\alpha_2)^2+27\beta^2(\alpha_1+\alpha_2)+18\alpha_1\alpha_2\beta+10\beta^3}{18\alpha_2(\alpha_1+\beta-2\alpha_2)(\alpha_2+\beta-2\alpha_1)}B^{\frac{3}{\alpha_1+\alpha_2+\beta}}$$
$$+\frac{2}{3\alpha_2}\log B + \frac{2(\alpha_1+\alpha_2+\beta)}{9\alpha_2}$$

which is

$$\begin{cases} O(B^{\frac{3}{\alpha_1+\alpha_2+\beta}}), & \text{if } 2\alpha_2 < \alpha_1+\beta \text{ and } 2\alpha_1 < \alpha_2+\beta \\ O(B^{\max(\frac{2}{\alpha_i+\beta})}), & \text{otherwise} \end{cases}$$

Now assume $2\alpha_2 = \alpha_1+\beta$ and $2\alpha_1 \neq \alpha_2+\beta$. Then

$$\int_1^{B^{\frac{1}{\alpha_1+\alpha_2+\beta}}} \Big(\frac{(\alpha_1+\beta)^2}{2\alpha_2(\alpha_2-\alpha_1)}B^{\frac{2}{\alpha_1+\beta}}g^{-\frac{2\alpha_2}{\alpha_1+\beta}} - \frac{(\alpha_2+\beta)^2}{2\alpha_2(\alpha_2-\alpha_1)}B^{\frac{2}{\alpha_2+\beta}}g^{-\frac{2\alpha_1}{\alpha_2+\beta}}$$
$$-\frac{\alpha_1+\alpha_2+\beta}{\alpha_2}g^2\log g + \frac{1}{\alpha_2}(\log B)g^2 + \frac{\alpha_1+\alpha_2+2\beta}{2\alpha_2}g^2\Big)\ dg$$
$$= \int_1^{B^{\frac{1}{3\alpha_2}}} \Big(\frac{2\alpha_2}{\beta-\alpha_2}B^{\frac{1}{\alpha_2}}g^{-1} - \frac{(\alpha_2+\beta)^2}{2\alpha_2(\beta-\alpha_2)}B^{\frac{2}{\alpha_2+\beta}}g^{-\frac{2(2\alpha_2-\beta)}{\alpha_2+\beta}}$$
$$-3g^2\log g + \frac{1}{\alpha_2}(\log B)g^2 + \frac{3\alpha_2+\beta}{2\alpha_2}g^2\Big)\ dg$$
$$= \frac{2}{3(\beta-\alpha_2)}B^{\frac{1}{\alpha_2}}\log B - \frac{(\alpha_2+\beta)^3}{6\alpha_2(\beta-\alpha_2)^2}B^{\frac{2}{\alpha_2+\beta}}(B^{\frac{(\beta-\alpha_2)}{\alpha_2(\alpha_2+\beta)}}-1)$$
$$-\frac{1}{3\alpha_2}B^{\frac{1}{\alpha_2}}\log B + \frac{1}{3}B^{\frac{1}{\alpha_2}} - \frac{1}{3} + \frac{1}{3\alpha_2}(\log B)(B^{\frac{1}{\alpha_2}}-1) + \frac{3\alpha_2+\beta}{6\alpha_2}(B^{\frac{1}{\alpha_2}}-1)$$
$$= \frac{2}{3(\beta-\alpha_2)}B^{\frac{1}{\alpha_2}}\log B - \frac{(\alpha_2+\beta)^3}{6\alpha_2(\beta-\alpha_2)^2}B^{\frac{1}{\alpha_2}} + \frac{(\alpha_2+\beta)^3}{6\alpha_2(\beta-\alpha_2)^2}B^{\frac{2}{\alpha_2+\beta}}$$
$$-\frac{1}{3\alpha_2}\log B + \frac{5\alpha_2+\beta}{6\alpha_2}B^{\frac{1}{\alpha_2}} - \frac{5\alpha_2+\beta}{6\alpha_2}$$

Now add in $g = 1$ terms:

$$\frac{2\alpha_2}{(\beta-\alpha_2)}B^{\frac{1}{\alpha_2}} - \frac{(\alpha_2+\beta)^2}{2\alpha_2(\beta-\alpha_2)}B^{\frac{2}{\alpha_2+\beta}} + \frac{1}{\alpha_2}\log B + \frac{3\alpha_2+\beta}{2\alpha_2}$$
$$+\frac{2}{3(\beta-\alpha_2)}B^{\frac{1}{\alpha_2}}\log B - \frac{(\alpha_2+\beta)^3}{6\alpha_2(\beta-\alpha_2)^2}B^{\frac{1}{\alpha_2}} + \frac{(\alpha_2+\beta)^3}{6\alpha_2(\beta-\alpha_2)^2}B^{\frac{2}{\alpha_2+\beta}}$$
$$-\frac{1}{3\alpha_2}\log B + \frac{5\alpha_2+\beta}{6\alpha_2}B^{\frac{1}{\alpha_2}} - \frac{5\alpha_2+\beta}{6\alpha_2}$$

Combining like terms gives us:

$$\frac{2}{3(\beta-\alpha_2)}B^{\frac{1}{\alpha_2}}\log B + \frac{(\alpha_2+\beta)^2(2\alpha_2-\beta)}{3\alpha_2(\beta-\alpha_2)^2}B^{\frac{2}{\alpha_2+\beta}} - \frac{\alpha_2(\alpha_2+3\beta)}{3(\beta-\alpha_2)^2}B^{\frac{1}{\alpha_2}} + \frac{2}{3\alpha_2}\log B + \frac{2\alpha_2+\beta}{3\alpha_2}$$

which is

$$\begin{cases} O(B^{\frac{2}{\alpha_2+\beta}}), & \text{if } \alpha_2 > \beta \\ O(B^{\frac{1}{\alpha_2}} \log B), & \text{if } \alpha_2 < \beta \end{cases}$$

Finally, assume $2\alpha_2 \neq \alpha_1 + \beta$ and $2\alpha_1 = \alpha_2 + \beta$. Then

$$\int_1^{B^{\frac{1}{\alpha_1+\alpha_2+\beta}}} \Big(\tfrac{(\alpha_1+\beta)^2}{2\alpha_2(\alpha_2-\alpha_1)} B^{\frac{2}{\alpha_1+\beta}} g^{-\frac{2\alpha_2}{\alpha_1+\beta}} - \tfrac{(\alpha_2+\beta)^2}{2\alpha_2(\alpha_2-\alpha_1)} B^{\frac{2}{\alpha_2+\beta}} g^{-\frac{2\alpha_1}{\alpha_2+\beta}}$$
$$- \tfrac{\alpha_1+\alpha_2+\beta}{\alpha_2} g^2 \log g + \tfrac{1}{\alpha_2}(\log B) g^2 + \tfrac{\alpha_1+\alpha_2+2\beta}{2\alpha_2} g^2 \Big) \; dg$$
$$= \int_1^{B^{\frac{1}{3\alpha_1}}} \Big(\tfrac{(\alpha_1+\beta)^2}{2(2\alpha_1-\beta)(\alpha_1-\beta)} B^{\frac{2}{\alpha_1+\beta}} g^{-\frac{2(2\alpha_1-\beta)}{\alpha_1+\beta}} - \tfrac{2(\alpha_1)^2}{(2\alpha_1-\beta)(\alpha_1-\beta)} B^{\frac{1}{\alpha_1}} g^{-1}$$
$$- \tfrac{3\alpha_1}{2\alpha_1-\beta} \, g^2 \log g + \tfrac{1}{2\alpha_1-\beta}(\log B) g^2 + \tfrac{3\alpha_1+\beta}{2(\alpha_1-\beta)} g^2 \Big) \; dg$$
$$= -\tfrac{(\alpha_1+\beta)^3}{6(2\alpha_1-\beta)(\alpha_1-\beta)^2} B^{\frac{2}{\alpha_1+\beta}} (B^{\frac{1}{3\alpha_1}(1-\frac{2(2\alpha_1-\beta)}{\alpha_1+\beta})} - 1) - \tfrac{2\alpha_1}{3(2\alpha_1-\beta)(\alpha_1-\beta)} B^{\frac{1}{\alpha_1}} \log B$$
$$- \tfrac{\alpha_1}{2\alpha_1-\beta} \, (\tfrac{1}{3\alpha_1} B^{\frac{1}{\alpha_1}} \log B + \tfrac{1}{3} B^{\frac{1}{\alpha_1}} - 1) + \tfrac{1}{3(2\alpha_1-\beta)} B^{\frac{1}{\alpha_1}} \log B - \tfrac{1}{3(2\alpha_1-\beta)} \log B$$
$$+ \tfrac{3\alpha_1+\beta}{6(\alpha_1-\beta)} (B^{\frac{1}{\alpha_1}} - 1)$$
$$= -\tfrac{(\alpha_1+\beta)^3}{6(2\alpha_1-\beta)(\alpha_1-\beta)^2} B^{\frac{1}{\alpha_1}} + \tfrac{(\alpha_1+\beta)^3}{6(2\alpha_1-\beta)(\alpha_1-\beta)^2} B^{\frac{2}{\alpha_1+\beta}} - \tfrac{2\alpha_1}{3(2\alpha_1-\beta)(\alpha_1-\beta)} B^{\frac{1}{\alpha_1}} \log B$$
$$- \tfrac{\alpha_1}{2\alpha_1-\beta} \, (\tfrac{1}{3} B^{\frac{1}{\alpha_1}} - 1) - \tfrac{1}{3(2\alpha_1-\beta)} \log B + \tfrac{3\alpha_1+\beta}{6(\alpha_1-\beta)} (B^{\frac{1}{\alpha_1}} - 1)$$

Now add in $g = 1$ terms:

$$\tfrac{(\alpha_1+\beta)^2}{2(2\alpha_1-\beta)(\alpha_1-\beta)} B^{\frac{2}{\alpha_1+\beta}} - \tfrac{2(\alpha_1)^2}{(2\alpha_1-\beta)(\alpha_1-\beta)} B^{\frac{1}{\alpha_1}} + \tfrac{1}{2\alpha_1-\beta} \log B + \tfrac{3\alpha_1+\beta}{2(2\alpha_1-\beta)}$$
$$- \tfrac{(\alpha_1+\beta)^3}{6(2\alpha_1-\beta)(\alpha_1-\beta)^2} B^{\frac{1}{\alpha_1}} + \tfrac{(\alpha_1+\beta)^3}{6(2\alpha_1-\beta)(\alpha_1-\beta)^2} B^{\frac{2}{\alpha_1+\beta}} - \tfrac{2\alpha_1}{3(2\alpha_1-\beta)(\alpha_1-\beta)} B^{\frac{1}{\alpha_1}} \log B$$
$$- \tfrac{\alpha_1}{2\alpha_1-\beta} \, (\tfrac{1}{3} B^{\frac{1}{\alpha_1}} - 1) - \tfrac{1}{3(2\alpha_1-\beta)} \log B + \tfrac{3\alpha_1+\beta}{6(\alpha_1-\beta)} (B^{\frac{1}{\alpha_1}} - 1)$$

Simplifying gives us:

$$\tfrac{2\alpha_1}{3(2\alpha_1-\beta)(\beta-\alpha_1)} B^{\frac{1}{\alpha_1}} \log B + \tfrac{(\alpha_1+\beta)^2}{3(\beta-\alpha_1)^2} B^{\frac{2}{\alpha_1+\beta}}$$
$$- \tfrac{\alpha_1((\alpha_1)^2+\alpha_1\beta+2\beta^2)}{3(2\alpha_1-\beta)(\alpha_1-\beta)^2} B^{\frac{1}{\alpha_1}} + \tfrac{2}{3(2\alpha_1-\beta)} \log B + \tfrac{9(\alpha_1)^2-11\alpha_1\beta-2\beta^2}{6(2\alpha_1-\beta)(\alpha_1-\beta)}$$

which is

$$\begin{cases} O(B^{\frac{2}{\alpha_1+\beta}}), & \text{if } \alpha_1 > \beta \\ O(B^{\frac{1}{\alpha_1}} \log B), & \text{if } \alpha_1 < \beta \end{cases}$$

4.2.2 $\alpha_1 = \alpha_2$

Set $\alpha = \alpha_1 = \alpha_2$. Note that if $\alpha = \beta$, then $-\frac{2\alpha}{\alpha+\beta} = -1$. So we evaluate the following integral, first assuming $\alpha \neq \beta$ and then assuming $\alpha = \beta$.

First we will count the number of point when $g = 1$:

$$\frac{1}{\alpha}B^{\frac{2}{\alpha+\beta}}\log B - \frac{\alpha+\beta}{\alpha}B^{\frac{2}{\alpha+\beta}} + \frac{1}{\alpha}\log B + \frac{\alpha+\beta}{\alpha}$$

Now counting the number of points for $g > 1$:

$$\begin{aligned}
&\int_1^{B^{\frac{1}{2\alpha+\beta}}} \Big(\frac{1}{\alpha}B^{\frac{2}{\alpha+\beta}}(\log B)g^{-\frac{2\alpha}{\alpha+\beta}} - \frac{2\alpha+\beta}{\alpha}B^{\frac{2}{\alpha+\beta}}g^{-\frac{2\alpha}{\alpha+\beta}}\log g \\
&\qquad -\frac{\alpha+\beta}{\alpha}B^{\frac{2}{\alpha+\beta}}g^{-\frac{2\alpha}{\alpha+\beta}} - \frac{2\alpha+\beta}{\alpha}g^2\log g + \frac{1}{\alpha}(\log B)g^2 + \frac{\alpha+\beta}{\alpha}g^2\Big)\ dg \\
= &\ \frac{\alpha+\beta}{\alpha(\beta-\alpha)}B^{\frac{2}{\alpha+\beta}}(\log B)(B^{\frac{1}{2\alpha+\beta}(1-\frac{2\alpha}{\alpha+\beta})} - 1) \\
&\ -\frac{2\alpha+\beta}{\alpha}B^{\frac{2}{\alpha+\beta}}\left(\frac{\alpha+\beta}{(2\alpha+\beta)(\beta-\alpha)}B^{\frac{1}{2\alpha+\beta}(1-\frac{2\alpha}{\alpha+\beta})}\log B - \left(\frac{\alpha+\beta}{\beta-\alpha}\right)^2 B^{\frac{1}{2\alpha+\beta}(1-\frac{2\alpha}{\alpha+\beta})}\right) \\
&\ -\frac{2\alpha+\beta}{\alpha}B^{\frac{2}{\alpha+\beta}}\left(\frac{\alpha+\beta}{\beta-\alpha}\right)^2 - \frac{\alpha+\beta}{\alpha}B^{\frac{2}{\alpha+\beta}}(B^{\frac{1}{2\alpha+\beta}(1-\frac{2\alpha}{\alpha+\beta})} - 1) + \frac{1}{3\alpha}(\log B)(B^{\frac{3}{2\alpha+\beta}} - 1) \\
&\ +\frac{\alpha+\beta}{3\alpha}(B^{\frac{3}{2\alpha+\beta}} - 1) - \frac{2\alpha+\beta}{\alpha}(\frac{1}{3(2\alpha+\beta)}B^{\frac{3}{2\alpha+\beta}}\log B - \frac{1}{9}B^{\frac{3}{2\alpha+\beta}} + \frac{1}{9}) \\
= &\ -\frac{\alpha+\beta}{\alpha(\beta-\alpha)}B^{\frac{2}{\alpha+\beta}}\log B + \frac{2\alpha+\beta}{\alpha}\left(\frac{\alpha+\beta}{\beta-\alpha}\right)^2 B^{\frac{3}{2\alpha+\beta}} - \frac{2\alpha+\beta}{\alpha}\left(\frac{\alpha+\beta}{\beta-\alpha}\right)^2 B^{\frac{2}{\alpha+\beta}} \\
&\ -\frac{\alpha+\beta}{\alpha}B^{\frac{3}{2\alpha+\beta}} + \frac{\alpha+\beta}{\alpha}B^{\frac{2}{\alpha+\beta}} - \frac{1}{3\alpha}\log B + \frac{5\alpha+4\beta}{9\alpha}B^{\frac{3}{2\alpha+\beta}} - \frac{5\alpha+4\beta}{9\alpha}
\end{aligned}$$

So the total number of points is:

$$\begin{aligned}
&\frac{1}{\alpha}B^{\frac{2}{\alpha+\beta}}\log B - \frac{\alpha+\beta}{\alpha}B^{\frac{2}{\alpha+\beta}} + \frac{1}{\alpha}\log B + \frac{\alpha+\beta}{\alpha} \\
&\quad -\frac{\alpha+\beta}{\alpha(\beta-\alpha)}B^{\frac{2}{\alpha+\beta}}\log B + \frac{2\alpha+\beta}{\alpha}\left(\frac{\alpha+\beta}{\beta-\alpha}\right)^2 B^{\frac{3}{2\alpha+\beta}} - \frac{2\alpha+\beta}{\alpha}\left(\frac{\alpha+\beta}{\beta-\alpha}\right)^2 B^{\frac{2}{\alpha+\beta}} \\
&\quad -\frac{\alpha+\beta}{\alpha}B^{\frac{3}{2\alpha+\beta}} + \frac{\alpha+\beta}{\alpha}B^{\frac{2}{\alpha+\beta}} - \frac{1}{3\alpha}\log B + \frac{5\alpha+4\beta}{9\alpha}B^{\frac{3}{2\alpha+\beta}} - \frac{5\alpha+4\beta}{9\alpha}
\end{aligned}$$

Simplifying gives us:

$$\begin{aligned}
&\frac{2}{\alpha-\beta}B^{\frac{2}{\alpha+\beta}}\log B + \frac{14\alpha^3+48\alpha_2\beta+42\alpha\beta^2+4\beta^3}{9\alpha(\beta-\alpha)^2}B^{\frac{3}{2\alpha+\beta}} \\
&\quad -\frac{(2\alpha+\beta)(\alpha+\beta)^2}{\alpha(\beta-\alpha)^2}B^{\frac{2}{\alpha+\beta}} + \frac{2}{3\alpha}\log B + \frac{4\alpha+5\beta}{9\alpha}
\end{aligned}$$

which is

$$\begin{cases} O(B^{\frac{2}{\alpha+\beta}}\log B), & \text{if } \alpha > \beta \\ O(B^{\frac{3}{2\alpha+\beta}}), & \text{if } \alpha < \beta \end{cases}$$

When $\alpha = \beta$, we get:

$$\begin{aligned} &\int_1^{B^{\frac{1}{2\alpha+\beta}}} \Big(\tfrac{1}{\alpha}B^{\frac{2}{\alpha+\beta}}(\log B)g^{-\frac{2\alpha}{\alpha+\beta}} - \tfrac{2\alpha+\beta}{\alpha}B^{\frac{2}{\alpha+\beta}}g^{-\frac{2\alpha}{\alpha+\beta}}\log g \\ &\quad -\tfrac{\alpha+\beta}{\alpha}B^{\frac{2}{\alpha+\beta}}g^{-\frac{2\alpha}{\alpha+\beta}} - \tfrac{2\alpha+\beta}{\alpha}g^2\log g + \tfrac{1}{\alpha}(\log B)g^2 + \tfrac{\alpha+\beta}{\alpha}g^2\Big)\ dg \\ =\ & \int_1^{B^{\frac{1}{3\alpha}}} \Big(\tfrac{1}{\alpha}B^{\frac{1}{\alpha}}(\log B)g^{-1} - 3B^{\frac{1}{\alpha}}g^{-1}\log g \\ &\quad -2B^{\frac{1}{\alpha}}g^{-1} - 3g^2\log g + \tfrac{1}{\alpha}(\log B)g^2 + 2g^2\Big)\ dg \\ =\ & \tfrac{1}{6\alpha^2}B^{\frac{1}{\alpha}}(\log B)^2 - \tfrac{2}{3\alpha}B^{\frac{1}{\alpha}}\log B + B^{\frac{1}{\alpha}} - \tfrac{1}{3\alpha}\log B - 1 \end{aligned}$$

Adding in the points when $g = 1$ we get:

$$\tfrac{1}{6\alpha^2}B^{\frac{1}{\alpha}}(\log B)^2 + \tfrac{1}{3\alpha}B^{\frac{1}{\alpha}}\log B - B^{\frac{1}{\alpha}} + \tfrac{2}{3\alpha}\log B + 1$$

which is

$$O(B^{\frac{1}{\alpha}}(\log B)^2)$$

4.2.3 Error term

Note that we need to calculate the number of points with $|q| = g$, $(\frac{|q|^{\alpha_1+\alpha_2+\beta}}{B})^{\frac{1}{\alpha_2}} \le |p'| < (\frac{|q|^{\alpha_1+\alpha_2+\beta}}{B})^{\frac{1}{\alpha_2}} + 1$, and $\frac{|q|}{|p'|} \le t < \frac{|q|}{|p'|} + 1$

So we count the number of points with $|q| = g$, $|p'| = 1, t = \frac{|q|}{|p'|}$.

Set $|q| = g$. Then $|p'| = 1$ and our constraints become $g|s'| \le t$ and $g^{\alpha_1}|t|^{\alpha_2+\beta} \le B$. Thus we calculate:

$$\begin{aligned} & 4\int_{g-1}^{(B/g^{\alpha_1})^{\frac{1}{\alpha_2+\beta}}} \int_0^{\frac{t}{g}} 1\, ds'\, dt \\ =\ & 4\int_{g-1}^{(B/g^{\alpha_1})^{\frac{1}{\alpha_2+\beta}}} \frac{t}{g}\, dt \\ =\ & \tfrac{2}{g}(B^{\frac{2}{\alpha_2+\beta}}g^{-\frac{2\alpha_1}{\alpha_2+\beta}} - g^2 + 2g - 1) \\ =\ & 2B^{\frac{2}{\alpha_2+\beta}}g^{-1-\frac{2\alpha_1}{\alpha_2+\beta}} - 2g + 4 - \tfrac{2}{g} \end{aligned}$$

When $g=1$, we have $2B^{\frac{2}{\alpha_2+\beta}}$ points. For $g>1$, we evaluate the following integral:

$$\begin{aligned}
&\int_1^{B^{\frac{1}{\alpha_1+\alpha_2+\beta}}} (2B^{\frac{2}{\alpha_2+\beta}} g^{-1-\frac{2\alpha_1}{\alpha_2+\beta}} - 2g + 4 - \tfrac{2}{g})\, dg \\
&= 2B^{\frac{2}{\alpha_2+\beta}} \frac{1}{-\frac{2\alpha_1}{\alpha_2+\beta}} (B^{-\frac{2\alpha_1}{(\alpha_1+\alpha_2+\beta)(\alpha_2+\beta)}} - 1) - (B^{\frac{2}{\alpha_1+\alpha_2+\beta}} - 1) \\
&\quad + 4(B^{\frac{1}{\alpha_1+\alpha_2+\beta}} - 1) - \frac{2}{\alpha_1+\alpha_2+\beta} \log B \\
&= \frac{\alpha_2+\beta}{\alpha_1} B^{\frac{2}{\alpha_2+\beta}} - \frac{\alpha_1+\alpha_2+\beta}{\alpha_1} B^{\frac{2}{\alpha_1+\alpha_2+\beta}} + 4B^{\frac{1}{\alpha_1+\alpha_2+\beta}} - \frac{2}{\alpha_1+\alpha_2+\beta} \log B - 3
\end{aligned}$$

Thus the total number of points in this case is

$$\frac{2\alpha_1+\alpha_2+\beta}{\alpha_1} B^{\frac{2}{\alpha_2+\beta}} - \frac{\alpha_1+\alpha_2+\beta}{\alpha_1} B^{\frac{2}{\alpha_1+\alpha_2+\beta}} + 4B^{\frac{1}{\alpha_1+\alpha_2+\beta}} - \frac{2}{\alpha_1+\alpha_2+\beta} \log B - 3$$

which is

$$O(B^{\frac{2}{\alpha_2+\beta}})$$

When $|p'| = (\frac{|q|^{\alpha_1+\alpha_2+\beta}}{B})^{\frac{1}{\alpha_2}} \geq 1$, the constraints become $g \leq |q|$, $B|s'|^{\alpha_2} \leq |q|^{\alpha_1+\beta}|t|^{\alpha_2}$, and $|q|^{\alpha_1+\beta}|t|^{\alpha_2} \leq B$. Thus we must have $|s'| \leq 1$.

If $|s'| = 0$, then $|t| = 1$ and so our final constraints are $1 \leq |q|$ and $|q|^{\alpha_1+\beta} \leq B$. The number of such points is:

$$\begin{aligned}
&4\int_0^{B^{\frac{1}{\alpha_1+\beta}}} 1\, dq \\
&= 4B^{\frac{1}{\alpha_1+\beta}}
\end{aligned}$$

If $|s'| = 1$, then $|q|^{\alpha_1+\beta}|t|^{\alpha_2} = B$ and so our final constraints are $g \leq |q|$ and $g^{\alpha_2}|q|^{\alpha_1+\beta} \leq B$.

$$\begin{aligned}
&4\int_{g-1}^{(B/g^{\alpha_2})^{\frac{1}{\alpha_1+\beta}}} 1\, dq \\
&= 4B^{\frac{1}{\alpha_1+\beta}} g^{-\frac{\alpha_2}{\alpha_1+\beta}} - 4g + 4
\end{aligned}$$

When $g=1$, we have $4B^{\frac{1}{\alpha_1+\beta}}$ points. For $g>1$, we evaluate the following integral and assume $\alpha_2 \neq \alpha_1 + \beta$:

$$\begin{aligned}
&\int_1^{B^{\frac{1}{\alpha_1+\alpha_2+\beta}}} \left(4B^{\frac{1}{\alpha_1+\beta}} g^{-\frac{\alpha_2}{\alpha_1+\beta}} - 4g + 4\right)\, dg \\
&= 4B^{\frac{1}{\alpha_1+\beta}} \frac{1}{1-\frac{\alpha_2}{\alpha_1+\beta}} (B^{(\frac{1}{\alpha_1+\alpha_2+\beta})(1-\frac{\alpha_2}{\alpha_1+\beta})} - 1) - 2(B^{\frac{2}{\alpha_1+\alpha_2+\beta}} - 1) + 4(B^{\frac{1}{\alpha_1+\alpha_2+\beta}} - 1) \\
&= 4\frac{\alpha_1+\beta}{\alpha_1+\beta-\alpha_2} (B^{\frac{2}{\alpha_1+\alpha_2+\beta}} - B^{\frac{1}{\alpha_1+\beta}}) - 2B^{\frac{2}{\alpha_1+\alpha_2+\beta}} + 4B^{\frac{1}{\alpha_1+\alpha_2+\beta}} - 2
\end{aligned}$$

which is

$$\begin{cases} O(B^{\frac{1}{\alpha_1+\beta}}), & \text{if } \alpha_2 > \alpha_1+\beta \\ O(B^{\frac{2}{\alpha_1+\alpha_2+\beta}}), & \text{if } \alpha_2 < \alpha_1+\beta \end{cases}$$

Now if we assume $\alpha_2 = \alpha_1 + \beta$, we get:

$$\begin{aligned} &\int_1^{B^{\frac{1}{2\alpha_2}}} \left(4B^{\frac{1}{\alpha_2}} g^{-1} - 4g + 4\right)\, dg \\ &= 4B^{\frac{1}{\alpha_2}} \frac{1}{2\alpha_2} \log B - 2(B^{\frac{1}{\alpha_2}} - 1) + 4(B^{\frac{1}{2\alpha_2}} - 1) \\ &= \frac{2}{\alpha_2} B^{\frac{1}{\alpha_2}} \log B - 2B^{\frac{1}{\alpha_2}} + 4B^{\frac{1}{2\alpha_2}} - 2 \end{aligned}$$

which is

$$O(B^{\frac{1}{\alpha_2}} \log B)$$

Set $|t| = |\frac{q}{p'}|$. Then $|s'| \leq 1$ and our constraints become $|gp'| \leq |q|$ and $|q|^{\alpha_1+\alpha_2+\beta} \leq |p'|^{\alpha_2} B$

$$\begin{aligned} &4\int_g^{\left(\frac{B}{g^{\alpha_2}}\right)^{\frac{1}{\alpha_1+\beta}}} \int_{(\frac{q^{\alpha_1+\alpha_2+\beta}}{B})^{\frac{1}{\alpha_2}}}^{\frac{q}{g}} dp'\, dq \\ &= 4\int_g^{\left(\frac{B}{g^{\alpha_2}}\right)^{\frac{1}{\alpha_1+\beta}}} \left(\frac{q}{g} - (\frac{q^{\alpha_1+\alpha_2+\beta}}{B})^{\frac{1}{\alpha_2}}\right) dq \\ &= 4\int_g^{\left(\frac{B}{g^{\alpha_2}}\right)^{\frac{1}{\alpha_1+\beta}}} (qg^{-1} - q^{(1+\frac{\alpha_1+\beta}{\alpha_2})} B^{-\frac{1}{\alpha_2}})\, dq \\ &= 2g^{-1}\left(\left(\frac{B}{g^{\alpha_2}}\right)^{\frac{2}{\alpha_1+\beta}} - g^2\right) - 4\frac{1}{2+\frac{\alpha_1+\beta}{\alpha_2}} B^{-\frac{1}{\alpha_2}} \left(\left(\frac{B}{g^{\alpha_2}}\right)^{\frac{1}{\alpha_1+\beta}(2+\frac{\alpha_1+\beta}{\alpha_2})} - g^{(2+\frac{\alpha_1+\beta}{\alpha_2})}\right) \\ &= \frac{2(\alpha_1+\beta)}{2\alpha_2+\alpha_1+\beta} B^{\frac{2}{\alpha_1+\beta}} g^{(-1-\frac{2\alpha_2}{\alpha_1+\beta})} - 2g + \frac{4\alpha_2}{2\alpha_2+\alpha_1+\beta} B^{-\frac{1}{\alpha_2}} g^{(2+\frac{\alpha_1+\beta}{\alpha_2})} \end{aligned}$$

When $g = 1$ we get: $\frac{2(\alpha_1+\beta)}{2\alpha_2+\alpha_1+\beta} B^{\frac{2}{\alpha_1+\beta}} - 2 + \frac{4\alpha_2}{2\alpha_2+\alpha_1+\beta} B^{-\frac{1}{\alpha_2}}$

For $g > 1$, we get:

$$\begin{aligned}
&\int_1^{B^{\frac{1}{\alpha_1+\alpha_2+\beta}}} \left(\tfrac{2(\alpha_1+\beta)}{2\alpha_2+\alpha_1+\beta} B^{\frac{2}{\alpha_1+\beta}} g^{(-1-\frac{2\alpha_2}{\alpha_1+\beta})} - 2g + \tfrac{4\alpha_2}{2\alpha_2+\alpha_1+\beta} B^{-\frac{1}{\alpha_2}} g^{(2+\frac{\alpha_1+\beta}{\alpha_2})}\right) dg \\
= \ & \tfrac{2(\alpha_1+\beta)}{2\alpha_2+\alpha_1+\beta} B^{\frac{2}{\alpha_1+\beta}} \frac{1}{-\frac{2\alpha_2}{\alpha_1+\beta}} (B^{-\frac{1}{\alpha_1+\alpha_2+\beta}\frac{2\alpha_2}{\alpha_1+\beta}} - 1) - B^{\frac{2}{\alpha_1+\alpha_2+\beta}} + 1 \\
& + \tfrac{4\alpha_2}{2\alpha_2+\alpha_1+\beta} B^{-\frac{1}{\alpha_2}} \frac{1}{3+\frac{\alpha_1+\beta}{\alpha_2}} (B^{\frac{1}{\alpha_1+\alpha_2+\beta}(3+\frac{\alpha_1+\beta}{\alpha_2})} - 1) \\
= \ & \tfrac{(\alpha_1+\beta)^2}{\alpha_2(2\alpha_2+\alpha_1+\beta)} B^{\frac{2}{\alpha_1+\beta}} - \tfrac{(\alpha_1+\beta)^2}{\alpha_2(2\alpha_2+\alpha_1+\beta)} B^{\frac{2}{\alpha_1+\alpha_2+\beta}} - B^{\frac{2}{\alpha_1+\alpha_2+\beta}} + 1 \\
& + \tfrac{4(\alpha_2)^2}{(2\alpha_2+\alpha_1+\beta)(3\alpha_2+\alpha_1+\beta)} B^{\frac{2}{\alpha_1+\alpha_2+\beta}} - \tfrac{4(\alpha_2)^2}{(2\alpha_2+\alpha_1+\beta)(3\alpha_2+\alpha_1+\beta)} B^{-\frac{1}{\alpha_2}}
\end{aligned}$$

So the total number of points is:

$$\tfrac{\alpha_1+\beta}{\alpha_2} B^{\frac{2}{\alpha_1+\beta}} + \tfrac{4(\alpha_2)^2}{(2\alpha_2+\alpha_1+\beta)(3\alpha_2+\alpha_1+\beta)} B^{\frac{2}{\alpha_1+\alpha_2+\beta}}$$
$$-\tfrac{(\alpha_1+\beta)^2}{\alpha_2(2\alpha_2+\alpha_1+\beta)} B^{\frac{2}{\alpha_1+\alpha_2+\beta}} - B^{\frac{2}{\alpha_1+\alpha_2+\beta}} + \tfrac{4\alpha_2}{3\alpha_2+\alpha_1+\beta} B^{-\frac{1}{\alpha_2}} - 1$$

which is

$$O(B^{\frac{2}{\alpha_1+\beta}})$$

Thus our error term is:

$$O(B^{\max\{\frac{2}{\alpha_1+\beta}, \frac{2}{\alpha_2+\beta}\}})$$

Compare this to our main term, which we calculated to be:

When $\alpha_1 \neq \alpha_2$, $2\alpha_1 \neq \alpha_2 + \beta$, and $2\alpha_2 \neq \alpha_1 + \beta$:

$$N_{V \setminus Z}(C, B) = O(B^{\max\{\frac{3}{\alpha_1+\alpha_2+\beta}, \frac{2}{\alpha_1+\beta}, \frac{2}{\alpha_2+\beta}\}})$$

When $\alpha_1 \neq \alpha_2$, $2\alpha_1 \neq \alpha_2 + \beta$, and $2\alpha_2 = \alpha_1 + \beta$:

$$N_{V \setminus Z}(C, B) = \begin{cases} O(B^{\frac{1}{\alpha_2}} \log B), & \text{if } \alpha_2 > \alpha_1 \\ O(B^{\frac{2}{\alpha_2+\beta}}), & \text{if } \alpha_2 < \alpha_1 \end{cases}$$

When $\alpha_1 \neq \alpha_2$, $2\alpha_1 = \alpha_2 + \beta$, and $2\alpha_2 \neq \alpha_1 + \beta$:

$$N_{V \setminus Z}(C, B) = \begin{cases} O(B^{\frac{1}{\alpha_1}} \log B), & \text{if } \alpha_1 > \alpha_2 \\ O(B^{\frac{2}{\alpha_1+\beta}}), & \text{if } \alpha_1 < \alpha_2 \end{cases}$$

When $\alpha := \alpha_1 = \alpha_2$ and $\alpha \neq \beta$:

$$N_{V\setminus Z}(C,B) = \begin{cases} O(B^{\frac{3}{2\alpha+\beta}}), & \text{if } \beta > \alpha \\ O(B^{\frac{2}{\alpha+\beta}} \log B), & \text{if } \beta < \alpha \end{cases}$$

And finally, when $\alpha := \alpha_1 = \alpha_2 = \beta$:

$$N_{V\setminus Z}(C,B) = O(B^{\frac{1}{\alpha}} (\log B)^2)$$

Thus we see the error term is no larger than the main term, as desired. Also note that the main term is the same in the case $|p| \leq |q|$ and $|s| \leq |t|$ as it was in the case $|p| \geq |q|$ and $|s| \geq |t|$.

4.3 $E_1, E_2,$ and E

Now we calculate the number of points on E_1, E_2, E with height less than or equal to B.

Note that since E_1, E_2, E are all isomorphic to $\mathbb{P}^1$, we can use Schanuel's theorem to get approximations for N_{E_1}, N_{E_2}, N_E. Recalling that an ample divisor C on $\mathbb{P}^1 \times \mathbb{P}^1$ has the form $C = \alpha_1 F_1 + \alpha_2 F_2 + \beta(F_1 + F_2 - E)$ and that we can write $F_1 = E_1 + E$ and $F_2 = E_2 + E$, we calculate the intersection multiplicities $C.E_1, C.E_2, C.E$ to get:

$$\begin{aligned} C.E_1 &= \alpha_2 \\ C.E_2 &= \alpha_1 \\ C.E &= \beta \end{aligned}$$

Thus by Schanuel's theorem we get:

$$\begin{aligned} N_{E_1} &\sim B^{\frac{2}{\alpha_2}} \\ N_{E_2} &\sim B^{\frac{2}{\alpha_1}} \\ N_E &\sim B^{\frac{2}{\beta}} \end{aligned}$$

The next section will relate these quantities to the counting function on $V \backslash (E \cup E_1 \cup E_2)$, which we computed in the previous sections.

4.4 Total number of points of bounded height

Let $Z = E_1+E_2+E$. We saw in the last section that $N_Z(C,B) = O(B^{\max\{\frac{2}{\alpha_1},\frac{2}{\alpha_2},\frac{2}{\beta}\}})$. Now we compare $N_Z(C,B)$ to $N_{V\setminus Z}(C,B)$.

Adding together all the terms calculated above (including error terms) we get:

When $\alpha_1 \neq \alpha_2$, $2\alpha_1 \neq \alpha_2+\beta$, and $2\alpha_2 \neq \alpha_1+\beta$, we have:

$$N_{V\setminus Z}(C,B) = O(B^{\max\{\frac{3}{\alpha_1+\alpha_2+\beta},\frac{2}{\alpha_1+\beta},\frac{2}{\alpha_2+\beta}\}})$$

When $\alpha_1 \neq \alpha_2$, $2\alpha_1 \neq \alpha_2+\beta$, and $2\alpha_2 = \alpha_1+\beta$, we have:

$$N_{V\setminus Z}(C,B) = \begin{cases} O(B^{\frac{1}{\alpha_2}}\log B), & \text{if } \alpha_2 > \alpha_1 \\ O(B^{\frac{2}{\alpha_2+\beta}}), & \text{if } \alpha_2 < \alpha_1 \end{cases}$$

When $\alpha_1 \neq \alpha_2$, $2\alpha_1 = \alpha_2+\beta$, and $2\alpha_2 \neq \alpha_1+\beta$, we have:

$$N_{V\setminus Z}(C,B) = \begin{cases} O(B^{\frac{1}{\alpha_1}}\log B), & \text{if } \alpha_1 > \alpha_2 \\ O(B^{\frac{2}{\alpha_1+\beta}}), & \text{if } \alpha_1 < \alpha_2 \end{cases}$$

When $\alpha := \alpha_1 = \alpha_2$ and $\alpha \neq \beta$, we have:

$$N_{V\setminus Z}(C,B) = \begin{cases} O(B^{\frac{3}{2\alpha+\beta}}), & \text{if } \beta > \alpha \\ O(B^{\frac{2}{\alpha+\beta}}\log B), & \text{if } \beta < \alpha \end{cases}$$

And finally, when $\alpha := \alpha_1 = \alpha_2 = \beta$, we have:

$$N_{V\setminus Z}(C,B) = O(B^{\frac{1}{\alpha}}(\log B)^2)$$

It is obvious that $B^{\frac{2}{\alpha_1+\beta}}$ and $B^{\frac{2}{\alpha_2+\beta}}$ are $O(B^{\frac{2}{\beta}})$. Now we'll show that $B^{\frac{3}{\alpha_1+\alpha_2+\beta}}$ is $O(B^{\max\{\frac{2}{\alpha_1},\frac{2}{\alpha_2},\frac{2}{\beta}\}})$. First note that for $\{\gamma_1,\gamma_2,\gamma_3\} = \{\alpha_1,\alpha_2,\beta\}$, if $\frac{3}{\gamma_1+\gamma_2+\gamma_3} > \frac{2}{\gamma_1}$, then $\gamma_1 > 2\gamma_2+2\gamma_3$. But then $\frac{3}{\gamma_1+\gamma_2+\gamma_3} < \frac{3}{3\gamma_2+3\gamma_3} = \frac{1}{\gamma_2+\gamma_3} < \frac{2}{\gamma_2}$.

Since the primary exponents in all of the other cases are the same as those in this case, we see that $N_{V\setminus Z}(C,B)$ is $O(N_Z(C,B))$.

Finally, we note that although this chapter only calculated an upper bound for $N_V(C,B)$, the upper bound is very close to the actual growth rate. See [2] for further details.

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