Multiple Criteria Decision Analysis: Classification Problems and Solutions

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Multiple criteria decision analysis (MCDA) techniques are developed to address challenging classification problems arising in engineering management and elsewhere. MCDA consists of a set of principles and tools to assist a decision maker (DM) to solve a decision problem with a finite set of alternatives compared according to two or more criteria, which are usually conflicting. The three types of classification problems to which original research contributions are made are

(1) Screening: Reduce a large set of alternatives to a smaller set that most likely contains the best choice.

(2) Sorting: Arrange the alternatives into a few groups in preference order, so that the DM can manage them more effectively.

(3) Nominal classification: Assign alternatives to nominal groups structured by the DM, so that the number of groups, and the characteristics of each group, seem appropriate to the DM.

Research on screening is divided into two parts: the design of a sequential screening procedure that is then applied to water resource planning in the Region of Waterloo, Ontario, Canada; and the development of a case-based distance method for screening that is then demonstrated using a numerical example.

Sorting problems are studied extensively under three headings. Case-based distance sorting is carried out with Model I, which is optimized for use with cardinal criteria only, and Model II, which is designed for both cardinal and ordinal criteria; both sorting approaches are applied to a case study in Canadian municipal water usage analysis. Sorting in inventory management is studied using a case-based distance method designed for multiple criteria ABC analysis, and then applied to a case study involving hospital inventory management. Finally sorting is applied to bilateral negotiation using a case-based distance model to assist negotiators that is then demonstrated on a negotiation regarding the supply of bicycle components.

A new kind of decision analysis problem, called multiple criteria nominal classification (MCNC), is addressed. Traditional classification methods in MCDA focus on sorting alternatives into groups ordered by preference. MCNC is the classification of alternatives into nominal groups, structured by the DM, who specifies multiple characteristics for each group. The features, definitions and structures of MCNC are presented, emphasizing criterion and alternative flexibility. An analysis procedure is proposed to solve MCNC problems systematically and applied to a water resources planning problem.

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I would like to give sincere thanks to my thesis supervisors, Drs. Keith W. Hipel and D. Marc Kilgour, for their guidance and financial support. Without their confidence in me, their continuing assistance and encouragement, constructive guidance, kind patience, and moral support, I would not have been able to accomplish this work. It is my great honour to work under their supervision and have my self-confidence strengthened through this experience.

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I cannot end without thanking my parents and my wife for their constant encouragement and love, on which I have relied throughout the period of this work.

Dedication

This is dedicated to Ms. Xin Su, my beloved wife.

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Chapter 1

Motivation and Objectives

The study of decision making is part of many of disciplines, including psychology, business, engineering, operations research, systems engineering, and management science. As society becomes more complex, the need for decisions that balance conflicting objectives (criteria) has grown. Government policy decisions, for example, which regulate growth, employment, and general welfare, have always faced this problem. Businesses encountering strategic decisions must consider multiple objectives as well; although short-run profit is important, long-run factors such as market position, product quality, and development of production capability often conflict with it.

Decision has attracted the attention of many thinkers since ancient times. The great philosophers Aristotle, Plato, and Thomas Aquinas, discussed the capacity of humans to decide and claimed that contemplation is what distinguishes humans from animals (Figueira et al., 2005). To illustrate some important aspects of decision, consider a quote a letter from Benjamin Franklin to Joseph Priestley which has been taken from a paper by MacCrimmon (1973).

London, Sept 19, 1772
Dear Sir,
In the affair of so much importance to you, wherein you ask my advice, I cannot, for what of sufficient premise, advise you want to determine, but if you please I will tell you how. [· · ·] When I have thus columns; writing over the one Pro, and over the other Con. [· · ·] When I have thus got them all together in one view, I endeavor to estimate their respective weights; and where I find two, one on each side, that seem equal, I strike them both out. If I find a reason pro equal to some two reasons con, I strike out the three. If I judge some two reasons con, equal to three reasons pro, I strike out the five; and thus proceeding I find at length where

the balance lies; and if, after a day or two of further consideration, nothing new that is of importance occurs on either side, I come to a determination accordingly. [$\cdots$] I have found great advantage from this kind of equation, and what might be called moral or prudential algebra. Wishing sincerely that you may determine for the best, I am ever, my dear friend, yours most affectionately.
B. Franklin

What is of interest in the above quotation is the fact that decision is strongly related to the comparison of different points of view, for which some are in favor and some are against. During the last forty years, systematic methodologies to look at such decision problems have caught the attention of many researchers. The approach recommended by Franklin, which explicitly takes into account the pros and the cons of different points of view, is the domain of multiple criteria decision analysis (MCDA).

Similar terms for describing this type of decision assistance include "multiple criteria decision aid" which comes from Europe (Roy, 1985; Vincke, 1992) and "multiple objective decision making" which is more widely used in the North America. The field of MCDA refers to the wide variety of tools and methodologies developed for the purpose of *helping a decision maker (DM) to select from finite sets of alternatives according to two or more criteria, which are usually conflicting.*

1.1 Motivation

The first and the most important step for studying a multiple criteria decision problem is the identification of a **problématique**, which was first introduced into MCDA by Roy (1985). The French word, "problématique" means fundamental problems and been translated as **problematics** in English by some researchers. The following was written by Roy (1996) as an explanation of problematics in MCDA.

The analyst must now determine in what terms he will pose the problem. What types of results does he envision and how does he see himself fitting into the decision process to aid in arriving at these results? Towards what will he direct his investigation? What form does he foresee his recommendation taking? ...We use the word problematic to describe the analyst's conception of the way he envisions the aid he will supply in the problem at hand based on answers to these questions.

Furthermore, Roy (1985, p.57) proposed four different kinds of problems as problématiques in MCDA — **P.** α, **P.** β, **P.** γ, **P.** δ.

Definition 1. • ***P.*** *α, **choice**. Choosing one alternative from a set of alternative,* **A**.

• ***P.*** *β, **sorting**. Sorting alternatives in predefined homogenous groups which are given in a preference order.*

• ***P.*** *γ, **ranking**. Ranking alternatives from best to worst.*

• ***P.*** *δ, **description**. Describing alternatives in terms of their major distinguishing features.*

Figure 1.1 provides an intuitive example of problématiques in MCDA. In the example, there are seven alternatives available for a particular multiple criteria decision analysis. In ranking analysis, the whole ordering sequence of alternatives A_1 to A_7, from most to least preferred, is listed as $A^2 \succ A^1 \succ A^6 \succ A^5 \succ A^4 \succ A^7 \succ A^3$, where $\succ$ means preferred to. For choice analysis, the best alternative is chosen as A^2. Under the category of description, one can describe features of alternatives. Within sorting analysis, one classifies all alternatives into two groups in which Group 1 (A^1, A^2, A^6) is preferred to Group 2 (A^3, A^4, A^5, A^7).

Three problématiques consisting of choice, sorting and ranking can lead to specific results implying regarding evaluations of alternatives. Some of these problématiques have been widely studied during the last thirty years. For example, methods for solving choice and ranking problems are so common that many researchers (for example, Olson (1996) in the book *Decision Aids for Selection Problems*) assume that they are the only problems of MCDA and do not distinguish problématiques explicitly, while substantial research on the sorting problem has not been carried out until recently.

Some new methods (Doumpos and Zopounidis, 1998; Slowinski and Zopounidis, 1995; Zopounidis and Doumpos, 2002) or revisions of well-known methods (Belacel, 2000; Yu, 1992) have recently been put forward to solve sorting problems. However, a systematic analysis of classification problems including sorting has not been well studied. For example, relationships among choice, sorting and ranking problems have never been investigated. In this thesis, a general research scheme involving classification problems in MCDA is systematically addressed and some practical applications are studied to demonstrate the proposed methodologies. The key objectives are outlined in the next section.

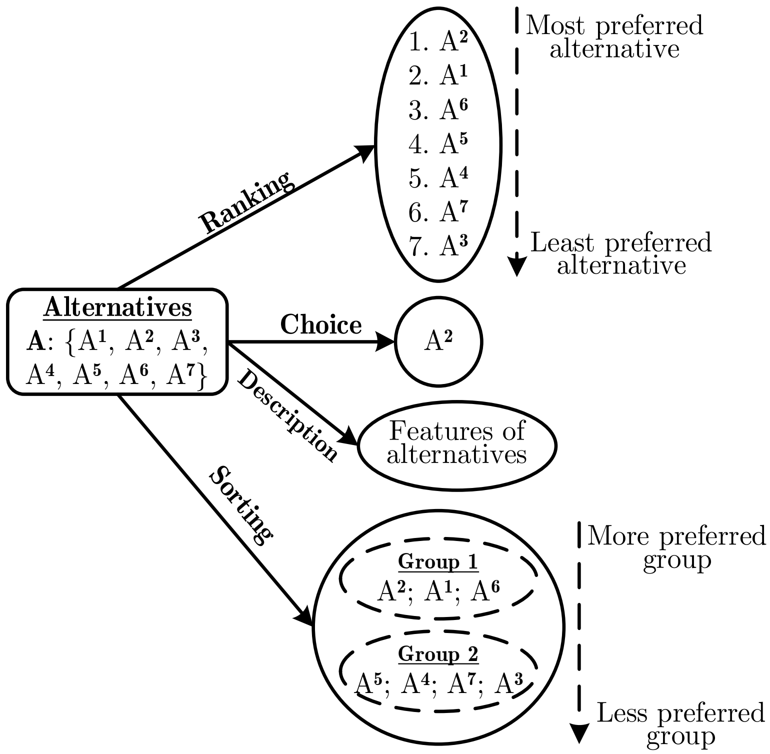


Figure 1.1: Problématiques in MCDA, adapted from Doumpos and Zopouidis (2002)

1.2 Objectives

The overriding objectives of this research are to develop new and improved tools and methodologies for classification problems in MCDA, multiple criteria classification (MCC), with application to challenging decision problems arising in engineering and other fields. In particular, the research topics described in the next three subsections are investigated thoroughly within this thesis.

1.2.1 Screening Problems in MCDA

Screening techniques in MCDA are studied to address a problem related to both choice and sorting. In practical applications of MCDA, it is common for a DM facing complex choice problems to *first identify those alternatives that do not appear to warrant further attention* (Bobbs and Meierm, 2000). A practical example can be seen recently in the popular TV program, *Bachelor*. When choosing one girl among several, a young man solves it by sequential elimination (screening) of one girl from further consideration. The choice is expressed by not assigning this girl a rose, while the others each get one.

Screening techniques can be regarded as useful MCDA methods, leading to a final choice. They apply when not enough information is available to reach a final choice directly, or too many alternatives must be considered. The number of alternatives to be taken into account further is dramatically reduced if screening is carried out properly reducing the work load for the DM. Screening techniques are designed to solve some sorting problems, and can help the DM simplify the final choice problem.

During the past few decades several different methods have been separately put forward to deal with screening problems. But there has been no systematic exploration of this topic in the literature, and researchers on sorting have paid little attention to it.

In summary, in this research topic the screening problem will be investigated systemically. Feasible screening methods are summarized according to the information provided by the DM, and are integrated into a unified framework, the *sequential screening procedure* (Chen et al., 2005b). The efficacy of this structure is demonstrated using a water resources planning problem. Also, a new and useful method, the case-based distance model, is proposed to solve screening problems and illustrated with a numerical example (Chen et al., 2005c).

1.2.2 Sorting Problems in MCDA

Practical applications of sorting include financial management such as business credit risk assessment; marketing analysis, such as customer satisfaction measurement; environmental and energy management, such as the analysis, measurement and classification of environmental impacts of different policies (Zopounidis and Doumpos, 2002). This rich range of potential real world applications has encouraged researchers to develop innovative methodologies for sorting. With the evolution of MCDA and the appearance of powerful new tools to deal with classification, research on sorting in MCDA is now receiving more attention. For example Doumpos and Zopouidis (2002) wrote the first book on sorting in MCDA. Kilgour et al. (2003) studied the problem of screening (two-group sorting) alternatives in subset selection problems. Zopounidis and Doumpos (2002) gave a comprehensive literature review of sorting in MCDA.

Generally speaking, there are two kinds of sorting methods: direct judgment methods and case-based reasoning methods. In direct judgement methods, a particular decision model is employed and the DM directly provides enough information to evaluate all preferential parameters in the model. In case-based reasoning methods, the DM furnishes decisions for selected cases, which determine preferential parameters to calibrate a chosen procedure as consistently as possible.

Direct judgement methods include ELECTRE TRI (Yu, 1992) and N-TOMIC (Massaglia and Ostanello, 1991). Both belong to the family of ELECTRE methods initially introduced by Roy (1968), but feature some theoretical modifications to address sorting. Case-based reasoning methods include UTADIS, MHDIS (Doumpos and Zopouidis, 2002) and the rough set method (Slowinski, 2001). UTADIS and MHDIS use the UTA (UTilités Additives) (Jacquet-Lagrèze and Siskos, 1982) technique to sort alternatives; the rough set method employs rough set theory, as developed by Slowinski (2001), for sorting.

In summary, within this research topic new techniques, *case-based distance methods for sorting*, are developed to solve problems (Chen et al., 2005d, 2004). The advantages of this method include (1) clear geometric meaning, so the DM can easily understand the method; (2) expeditious and accurate elicitation of the DM's preferences, which is much more efficient than direct inquiry. Then the applications of this method in inventory management (Chen et al., 2005e), and bilateral negotiation (Chen et al., 2005f) are investigated.

1.2.3 Multiple Criteria Nominal Classification

Current research on classification problems in MCDA mainly focuses on sorting, in which alternatives are assigned to groups defined ordinally by the DM. Another practical decision problem is to assign alternatives to homogeneous groups defined nominally. For example, in human resources management, some job applicants should be assigned to appropriate occupation groups according to their multiple qualifications (criteria). This kind of problem is called Multiple Criteria Nominal Classification (**MCNC**) to distinguish it from the sorting problem in MCDA.

To date, only a few papers are relevant to MCNC, such as those of Perny (1998), Scarelli and Narula (2000), and Malakooti and Yang (2004). One reason may be that distinctions between MCNC and other classification areas have not been clarified. Note that similar multiple criteria (multidimensional) classification problems (which can be termed multiple attribute classification, MAC) have been widely studied in other research areas such as statistical learning and pattern recognition, medical diagnosis, handwriting and voice recognition (Zervakis et al., 2004).

But there are great differences between sorting and MAC. As a decision analysis method, sorting is a prescriptive approach to assist individuals to make wise classification decisions, while MAC is a descriptive approach to detect and characterize general similarities within a large set of data. Sorting involves determining the DM's preferences in decision situations; MAC does not have this function.

Overall, this research area focuses on the theoretical extension of sorting problems (Chen et al., 2006), as follows: (1) the systematic modelling of the MCNC problems is presented including their features, definition and structures; and (2) the development of techniques to solve MCNC problems is addressed and a water resources planning problem is studied to demonstrate the proposed analysis procedure.

1.3 Overview of the Thesis

Figure 1.2 summarizes of the organization of the thesis. A detailed explanation follows:

- **Chapter 1** describes the motivation and objectives of this thesis, including a discussion of problématiques in MCDA and the organization of the thesis.
- **Chapter 2** is a background and literature review of MCDA, including the following: MCDA and relevant research topics, analysis procedures in MCDA, and a summary of MCDA methods.
- **Chapter 3** addresses screening problems in MCDA, including general descriptions of screening problem, a systematic sequential screening procedure, and a case study of water resource planning in the Regional Municipality of Waterloo.
- **Chapter 4** introduces a case-based distance model for screening and a numerical example to demonstrate the proposed procedure.
- **Chapter 5** focuses on sorting problems in MCDA, including general descriptions of the sorting problem, case-based distance model I for cardinal criteria, case-base distance model II for both cardinal and ordinal criteria, and a case study analyzing Canadian municipal water usage.

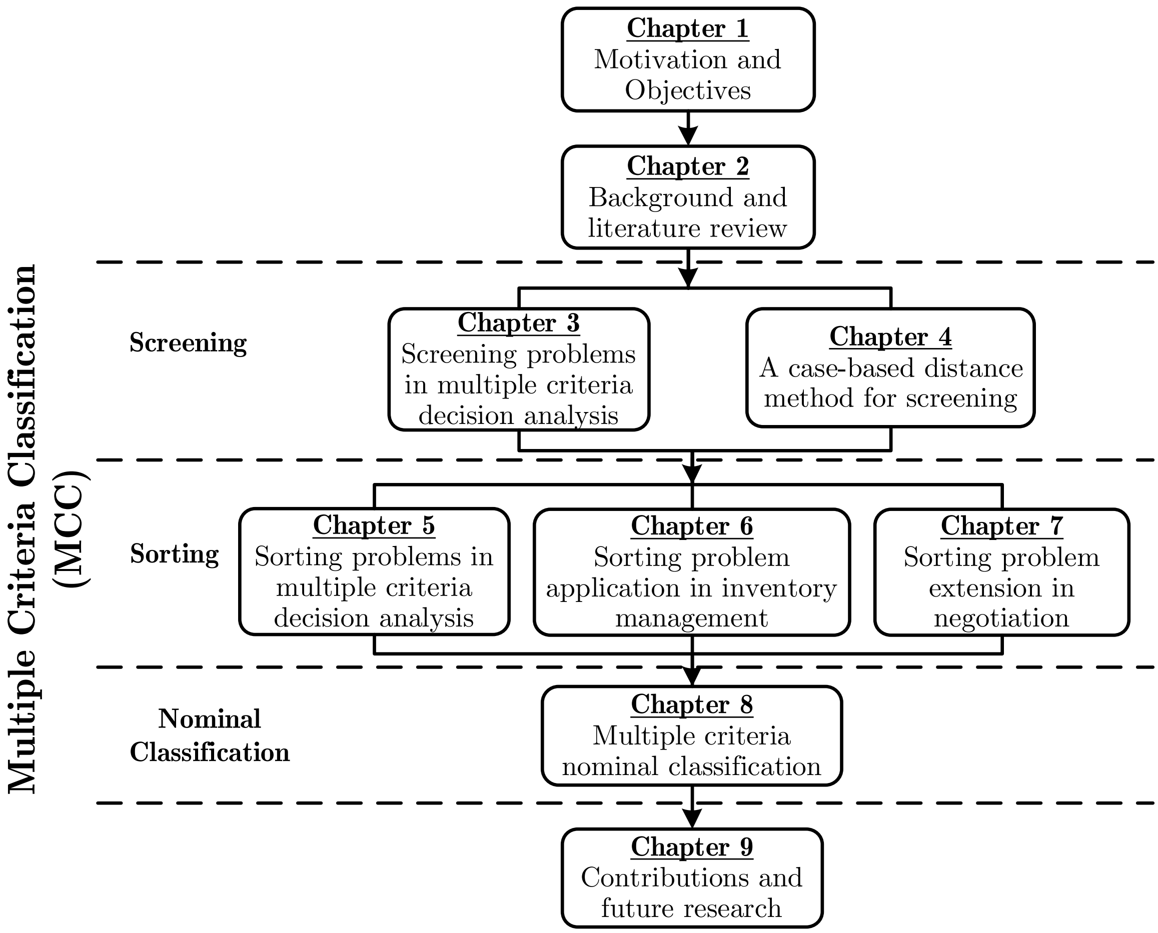


Figure 1.2: Contents of This Thesis

- **Chapter 6** contains a sorting problem application in inventory management, including an introduction of multiple criteria ABC analysis (MCABC), a case-based distance method for MCABC, and a case study of a hospital inventory management problem.

- **Chapter 7** is an extension of a sorting problem to negotiation, including an introduction to multiple issue bilateral negotiation, a case-based distance model for bilateral negotiation, and a case study in negotiation over the supply of bicycle components.

- **Chapter 8** proposes a new kind of decision analysis problem, multiple criteria nominal classification (MCNC), which includes an introduction to MCNC, an MCNC analysis procedure, a linear additive value function approach to MCNC, and a numerical example to demonstrate the procedure.

- **Chapter 9** contains a summary of the main contributions of the research and suggestions for future research.

Chapter 2

Background and Literature Review of Multiple Criteria Decision Analysis

2.1 Introduction

In this chapter, a background and literature review of MCDA are presented to provide a foundation for the research in this thesis. MCDA and relevant research topics are first explained briefly, and then an analysis procedure is proposed that provides a systematic framework for MCDA. This permits many approaches to MCDA to be summarized and integrated into one system. This chapter is based upon the research of Chen et al. (2004).

2.2 MCDA and Relevant Research Topics

Every decision situation exists within a context. This environment consists of a set of circumstances and conditions that affect the manner in which the decision making problem can be resolved. Radford (1989) and Hipel et al. (1993) suggested four major factors that determine the context, namely:

1. Whether or not uncertainty is present,
2. Whether or not the benefits and costs resulting from the implementation of potential courses of actions can be entirely assessed in quantitative terms,

3. Whether one criterion or multiple criteria must be taken into account,
4. Whether the power to make the decision is under the control of one organization, individual or group, or whether two or more of the participants have power to influence the decision.

Based on these factors, Radford (1989) and Hipel et al. (1993) focused on three decision analysis scenarios: single participant-multiple criteria, multiple participant-single criterion, and multiple participant-multiple criteria situations. Here, this classification is extended and a systematic discussion of the relationship of MCDA and its related research topics is presented.

Table 2.1: MCDA and Relevant Research Topics

	Single Criterion	Multiple Criteria
Single DM	SDSC	SDMC
Multiple DMs	MDSC	MDMC
Infinitely Many DMs	IDSC	IDMC
Finite Alternatives	FASC	FAMC
Infinitely Many Alternatives	IASC	IAMC

Table 2.1 presents the acronyms used for the various decision situations that arise in practice. The characteristics or commonalities of these decision situations are discussed below.

- **SDSC** and **FASC**: a single DM a single criterion, and finite alternatives problem. SDSC and FASC are simple decision problems, in which a DM solely considers one criterion to make decisions. SDSC related research topics include cost-benefit analysis in engineering economics and traditional one objective optimization based decision in operation research.
- **SDMC** and **FAMC**: a single DM with multiple criteria, and finite alternatives problem. Here one categorizes SDMC and FAMC as the MCDA studied in this thesis. The areas of SDMC and FAMC overlap with much of the MCDA research presented in this thesis.
- **MDSC** and **MDMC**: a multiple DM with single criterion or multiple criteria problem. MDSC and MDMC have a finite number of DMs who are involved in

conflict over one or multiple issues. For example, companies that manufacture television set, may be in competition to gain larger market share. Many game theory related conflict analysis methods focus on MDSC and MDMC problems. For instance, Howard (1971) developed metagame analysis with option form for structuring and modelling MDSC problems; Fraser and Hipel (1984) proposed conflict analysis to extend metagame analysis; Fang et al. (1993) designed the graph model for conflict resolution to analyze MDSC problems.

- **IDSC** and **IDMC**: an infinitely many DMs, single criterion or multiple criteria problem. IDSC and IDMC involve an infinite number of DMs' decisions based mainly on a single criterion. IDSC and IDMC related research includes statistical models and time series analysis for the estimation of aggregation effects of customer purchase behaviours. One important branch of this research, called discrete choice models, which was largely developed by Manski and MaFedden (1981), Moshe and Lerman (1985), and Train (2003). Based on the modeling of aggregation of individual behavior, discrete choice models can analyze and predict the impacts of various infinite DMs decision scenarios such as forecasting the ridership on a metropolitan transit service or predicting demand for a product under alternative pricing strategies.

- **IASC** and **IAMC**: an infinitely many alternative, single criterion or multiple criteria problem. IASC and IAMC related research includes single or multiple objective optimization techniques widely studied in operations research. IASC and IAMC focus on a a serial process of identifying decision variables, defining objectives, modeling constraints, and finding optimal solutions. For example Steuer (1986) discussed various approaches to generate Pareto Optimal solutions to multiple objective optimization problems.

2.3 Analysis Procedures in MCDA

2.3.1 The Structure of MCDA Problems

Multiple criteria decision analysis begins with a serial process of defining objectives, arranging them into criteria, identifying all possible alternatives, and then measuring consequences. A consequence is a direct measurement of the success of an alternative according to a criterion (e.g. cost in dollars, capacity in liters). Note that a consequence is a physical measurement related to a criterion and does not include preferential information.

The process of structuring MCDA problems has received a great deal of attention. von Winterfeldt (1980) called problem structuring the most difficult part of decision aid. Keeney (1992) and Hammond et al. (1999) proposed a systematic analysis method, **value-focused thinking**, which provided an excellent approach to this aspect of MCDA.

The basic structure of an MCDA problem established by carrying out the above steps is shown in Figure 2.1. In this figure, $\mathbf{A} = \{A^1, A^2, \cdots, A^i, \cdots, A^n\}$ is the set of alternatives, and $\mathbf{Q} = \{1, 2, \cdots, j, \cdots, q\}$ is the set of criteria. The consequence on criterion j of alternative A^i is expressed as $c_j(A^i)$, which can be shortened to c_j^i when there is no possibility of confusion. Note that there are n alternatives and q criteria altogether.

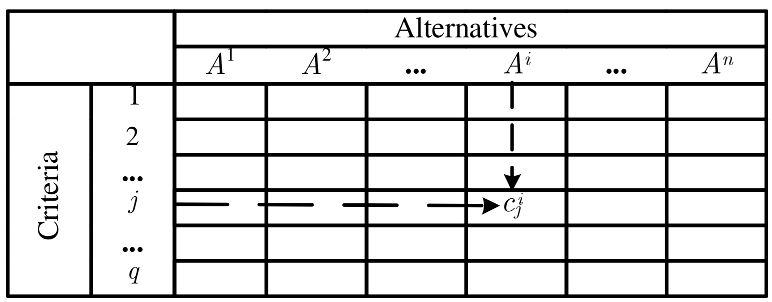


Figure 2.1: The Structure of an MCDA Problem

Criterion Definition

Defining criteria refer to the selection and specification of criteria, shown as the q criteria in Figure 2.1, to reflect multiple concerns or different objectives. For example, Keeney et al. (1987) identified eight criteria for German energy planning, including financial requirements, security of energy supplies and national economic impacts.

Definitions of criteria are important. Keeney (1992) suggested that there are three kinds of criteria, natural criteria, constructed criteria, and proxy criteria. Natural criteria are commonly understood by everyone. An example of a natural criterion for an objective of *minimizing loss of pine forest* would be the area of pine forest lost. Constructed criteria are constructed for a specific decision context. Keeney notes that the Dow Jones industrial average and the Richter scale started out as built criteria, although they have become so commonly used that they now are "natural criteria". Sometimes it is difficult to identify natural or constructed criteria for particular objectives; then proxy criteria or indirect measures are used. For instance, it is difficult to find a direct criterion for the objective *minimize damage to historic buildings from acid rain*, but a useful proxy criterion could be sulphur dioxide concentration in rain water (measured in parts per million). Of course the selection of criteria must satisfy some requirements; for example, Keeney and Raiffa (1976), Roy (1996) and Bobbs and Meierm (2000) have put forward different approaches.

Alternative Identification

Alternative identification means finding suitable alternatives to be modelled, evaluated and analyzed. The value-focused thinking approach of Keeney (1992) provides several valuable suggestions which can lead to the identification of decision opportunities and the creation of better alternatives. The **habitual domain theory** of Yu (1995) discusses the human decision mechanism from the psychology perspective, and proposed innovative ways to liberate thinking from the limitations of a rigid habitual domain and to find creative alternatives.

Most DMs would like to limit the number of alternatives for analysis. The number that is reasonable may vary greatly according to the circumstances. "Twenty may be too many, and two is likely to be too few" (Bobbs and Meierm, 2000). In fact, the number of alternatives to be identified may depend on the Problématique. For ranking and sorting problems, all possible alternatives within pre-specified boundaries should be considered. For example, in water resources planning, each lake within a specified area should be identified and classified. For choice problems, it may not be necessary to give comprehensive evaluations of all possible alternatives, because some inferior alternatives are not worth further consideration.

Consequence Measurement

Consequence measurement means measurement or estimation of the effect or consequence c_j^i of an alternative A^i on criterion j. Opinions or preferences of the DM play no role in consequences. The first task of analysts is to collect these data honestly and fairly, and not to evaluate immediately whether or not they are good or bad.

Different kinds of consequence data include:

• Cardinal data: The most common format for consequences is as cardinal data, for which c_j^i is a real number. For example, in a nuclear dump site selection, cardinal criteria can be *construction cost, expected lives lost, risk of catastrophe* and *civic improvement* (Olson, 1996). The consequence of construction cost can be expressed in monetary units like millions of US dollars.

• Ordinal data: In the above example, if DMs feel it is hard to obtain cardinal data or probabilistic data for civic improvement measurements, it may be measured ordinally. For example, linguistic grades can be used by DMs to assess alternatives for the criterion of civic improvement measurements.

• Interval data; • Probabilistic data; • Fuzzy data: Sometimes uncertainty must be considered in consequence measurement; the data may then be expressed as interval data, probabilistic data, fuzzy data or in some other suitable fashion reflecting uncertainty. In the above example, the criterion number of lives lost can be expressed probabilistically, since it is not easy to give precise data for mortality in nuclear dump accidents.

2.3.2 Decision Maker's Preference Expressions

An essential feature of decision problems is the DM's preferences. Different ways of expressing preferences may lead to different final results for the same MCDA problem. Generally speaking, there are two kinds of preference expressions: value data (preferences on consequence data) and weights (preferences on criteria).

Preferences on Consequence Data

A DM may have preferences based directly on consequence data, which can be expressed in several ways. Among them, the most famous are utility theory-based definitions (Fishburn, 1970; Keeney and Raiffa, 1976) and outranking-based definitions (Roy, 1968, 1985). Note that some MCDA methods do not distinguish consequence data from preferences on consequence data. For example, Nijkamp et al. (1983) gave criterion scores the same meaning as preferences on consequence data and do not explicitly differentiate consequence data from preferences on them. In fact, they used standardized methods for criterion scores, which were transformations from consequence data to preferences on that data. In this document, for the sake of easier modelling of the preferences of the DM, definitions of values as preferences on consequences are proposed.

Definition 2. *The DM's preference on consequence for criterion j and alternative A^i is a value datum $v_j\big(c_j(A^i)\big) = v_j(A^i)$, written v_j^i when no confusion can result.*

The DM's preference on consequences over all criteria for alternative A^i is the value vector $\mathbf{v}(A^i) = \big(v_1(A^i), v_2(A^i), ..., v_q(A^i)\big)$.

Values are refined data obtained by processing consequences according to the needs and objectives of the DM. The relationship between consequences and values can be expressed as

$$v_j^i = f_j(c_j^i) \tag{2.1}$$

where v_j^i and c_j^i are a value and a consequence, respectively, and $f_j(\cdot)$ is a mapping.

MCDA techniques may place different requirements on the preference information provided by the DM. Specifically in some situations it may be necessary to assume the following properties in order to be able to use certain MCDA methods:

- *Preference availability*: the DM can express which of two different consequence data on a criterion is preferred.
- *Preference independence*: the DM's preferences on one criterion have no relationship with preferences on any other criterion;
- *Preference monotonicity*: criterion j is a **positive preference criterion** iff larger consequences are preferred, i.e., $\forall A^l$ and $A^m \in \mathbf{A}$, $v_j(A^l) \geq v_j(A^m)$ for $c_j(A^l) > c_j(A^m)$; it is a **negative preference criterion** iff smaller consequences are preferred, i.e., $\forall A^l$ and $A^m \in \mathbf{A}$, $v_j(A^l) \geq v_j(A^m)$ for $c_j(A^l) < c_j(A^m)$; it is monotonic iff it is either positive or negative.

Many methods are available to obtain value transformation functions, such as multiattribute utility theory (MAUT) (Keeney and Raiffa, 1976) and the analytic hierarchy process (AHP) (Saaty, 1980). Two simple but frequently used transformation functions are linear normalization functions:

$$v_j^i = \frac{c_j^i}{\max\limits_{l=1,2,...,n} \{c_j^l\}} \tag{2.2}$$

for a positive preference criterion; and

$$v_j^i = \frac{\min\limits_{l=1,2,...,n} \{c_j^l\}}{c_j^i} \tag{2.3}$$

for a negative preference criterion. Both (2.2) and (2.3) assume that all consequences are positive real numbers.

Preferences on Criteria

Preferences on criteria refer to expressions of the relative importance of criteria. They are generally called **weights**; the weight for criterion $j \in \mathbf{Q}$ is $w_j \in \mathbb{R}$. It is usually assumed that $w_j > 0$ for all criteria, j. Usually weights are normalized to sum to 1, $\sum_{j \in \mathbf{Q}} w_j = 1$. Such normalization can help DMs to interpret the relative importance of each criterion. A weight vector is denoted $\mathbf{w} = (w_1, w_2, ..., w_j, ..., w_q)$, and the set of all possible weight vectors is denoted $\mathbf{W} \subseteq \mathbb{R}^q$. Other methods of expressing preferences over criteria include ranking criteria (from most to least preferred, with ties allowed) and determining intervals for weights. Still more methods based on probability or fuzzy sets, for example, are available if uncertainty is to be considered.

Aggregation Models in MCDA

After the construction of a basic MCDA problem and the acquisition of preferences from the DM, a global model to aggregate preferences and solve a specified problem (*choose*, *rank* or *sort*) may be constructed. For all $A^i \in \mathbf{A}$,

$$V(A^i) = F\big(\mathbf{v}(A^i), \mathbf{w}\big) \tag{2.4}$$

where $V(A^i) \in \mathbb{R}$ is the evaluation of alternative A^i (V^i when no confusion can result), and $F(\cdot)$ is a real function mapping the value vector $\mathbf{v}(A^i)$ and the weight vector $\mathbf{w}$ to the evaluation result. A typical example is the linear additive value function, which can be expressed as

$$V(A^i) = \sum_{j \in \mathbf{Q}} w_j \cdot v_j(A^i) \tag{2.5}$$

This step has been called **amalgamation** (Janssen, 1992), and **arithmetic multi-criteria evaluation** (Bobbs and Meierm, 2000). It is a necessary step for different methodologies in MCDA.

Most aggregation methods in MCDA require three steps:

1. Obtain values and weights.

2. Aggregate values using weights.

3. Apply the aggregate values to carry out the specified task (choice, ranking, or sorting).

2.4 Summary of MCDA Methods

During the last thirty years, a multitude of aggregation models has been developed, including MAUT, AHP and Outranking. New methods or improvements on existing methods continue to appear in international journals like the *Journal of Multi-Criteria Decision Analysis*, *European Journal of Operational Research*, and *Computer and Operations Research*. Olson (1996) provided a comprehensive bibliography for these methods. The purpose of this section is to classify and summarize popular MCDA methods, based on Chen et al. (2004).

2.4.1 Value Construction Methods

There are three common approaches to generating values based on consequences: *single alternative-based methods*, *binary alternative-based methods* and *linguistic rule-based methods*. These methods are elaborated in Figure 2.2.

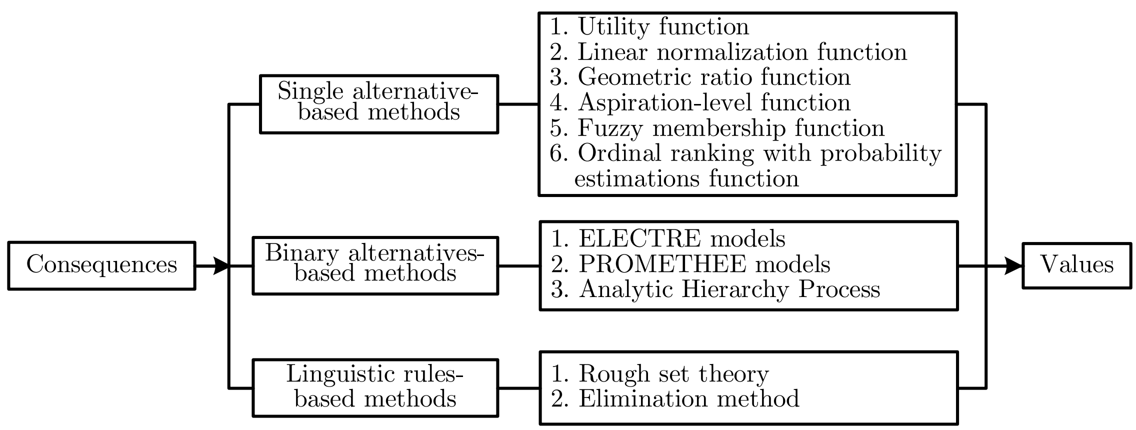


Figure 2.2: Different Approaches to Obtaining Values

Single alternative-based models focus on the expression of preference based only on the consequence data for a particular alternative; the consequence data for other alternatives is not considered. Models belonging to this class include *utility functions* (Keeney and Raiffa, 1976), *linear normalization functions* (Hwang and Yoon, 1981; Nijamp and Rietveld, 1990; Nijkamp et al., 1983), *geometric ratio functions* (Lootsma, 1999), *aspiration-level functions* (Korhonen, 1988; Lotfi et al., 1992) and *fuzzy membership functions* (Lootsma, 1997; Yager, 1977). These methods are mostly designed to deal with cardinal consequence data. The methods of *ordinal ranking with probability estimation* (Nijamp and Rietveld, 1990; Nijkamp et al., 1983) (since Nijkamp et al. (1983),and (Nijamp and Rietveld, 1990) did not give a clear name for this method, in this document it is named according to its most distinguishing feature) and *data envelopment analysis* (Cook and Kress, 1991) apply to ordinal consequence data.

Most single alternative-based methods represent preferences using real numbers. In some methods, such as fuzzy membership, an interval of real numbers is obtained first, and then aggregation methods are applied to obtain a single real number as representative of this interval.

Binary relation-based models focus on expressions of preferences on criteria via comparisons of two alternatives. They include *ELECTRE* (Roy, 1968, 1985), *PROMETHEE* (Brans et al., 1986; Brans and Vincke, 1985), and the *Analytic Hierarchy Process (AHP)* (Saaty, 1980). In AHP, binary relationships between alternatives are described by cardinal or ordinal numbers (usually relative scores on a 1-9 scale), which represent the degree of preference between the alternatives, while in Outranking methods (ELECTRE or PROMETHEE) binary relations of alternatives are represented by concordance and discordance matrices. Of course, difference exists between AHP and Outranking methods to obtain the final results.

Linguistic rules-based models focus on expressions of preferences on criteria via some linguistic rules, mostly expressed as "If ..., then ...". The advantage of this kind of preference data is that people make decisions by searching for *rules* that provide good justification of their choices. Rough set methods (Slowinski, 1992) and the elimination method (MacCrimmon, 1973; Radford, 1989) are based on this kind of preference representation.

2.4.2 Weighting Techniques

Belton and Stewart (2002) summarize two kinds of weights: tradeoff-based weights and non-tradeoff-based weights. Tradeoff-based weights emphasize the "compensation" of values across criteria, which permits preference data to be compared as they are aggregated into a single representative evaluation. Non-tradeoff-based weights do not permit direct tradeoffs across criteria; they are usually associated with outranking methods.

Figure 2.3 summarizes some methods for weight assessment in MCDA. Among the tradeoff-based weight methods, *AHP* and *geometric ratio weighting* are integrated methods, which means they proceed from preference data and weight assessments to aggregated preferences to final results. Some authors including von Winterfeldt (1986), Bobbs and Meierm (2000), and Belton and Stewart (2002))

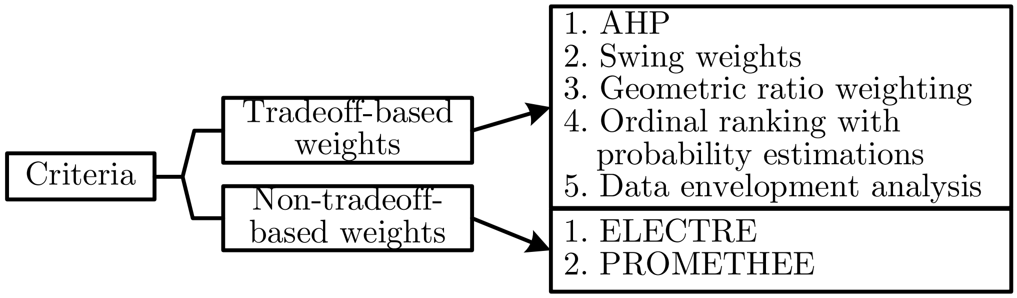


Figure 2.3: Methods of Weight Construction

prefer *swing weights* to other methods for direct estimation of weights. *Ordinal ranking with probability estimation* was introduced by Nijkamp et al. (1983) to express weights ordinally. *Data envelopment analysis*, proposed by Cook et al. (1996), possesses the unique feature that the values of weights are determined by preferences to optimize the measure of each alternative. Note that Outranking methods focus on the employment of weights and do not provide procedures to obtain weights while other methods can generate the weight information.

2.4.3 Aggregation Methods

Figure 2.4 shows these procedures and the relationships among them. Within this framework, the similarities and differences of methods for obtaining preferences and aggregating them can be clearly displayed.

Methods that employ cardinal preference data and tradeoff-based weights include the *aspiration-level interactive model (AIM)* (Lotfi et al., 1992), *Multiattribute Utility Theory (MAUT)* (Keeney and Raiffa, 1976), *Simple Multi-Attribute Rating Technique (SMART)* (von Winterfeldt, 1986), *Visual Interactive Method for Decision Analysis (VIMDA)* (Korhonen, 1988), and *Preference Cones* (Koksalan et al., 1984). Techniques that utilize binary preference data and tradeoff-based weights include *Analytic Hierarchy Process (AHP)* (Saaty, 1980), *Geometric Mean Technique* (Barzilai et al., 1987; Barzilai and Golany, 1994; Lootsma et al., 1990). Finally, methods that employ binary preference data and non-tradeoff-based weights include *ELECTRE* (Roy, 1968, 1985) and **PROMETHEE** (Brans et al., 1986; Brans and Vincke, 1985) which are *Outranking Methods*.

Linguistic aggregation methods use only linguistic preference data. Weights are assigned to rules rather than criteria. These methods include *Rough Set Method* (Slowinski, 1992) and *Elimination Method* (MacCrimmon, 1973; Radford, 1989).

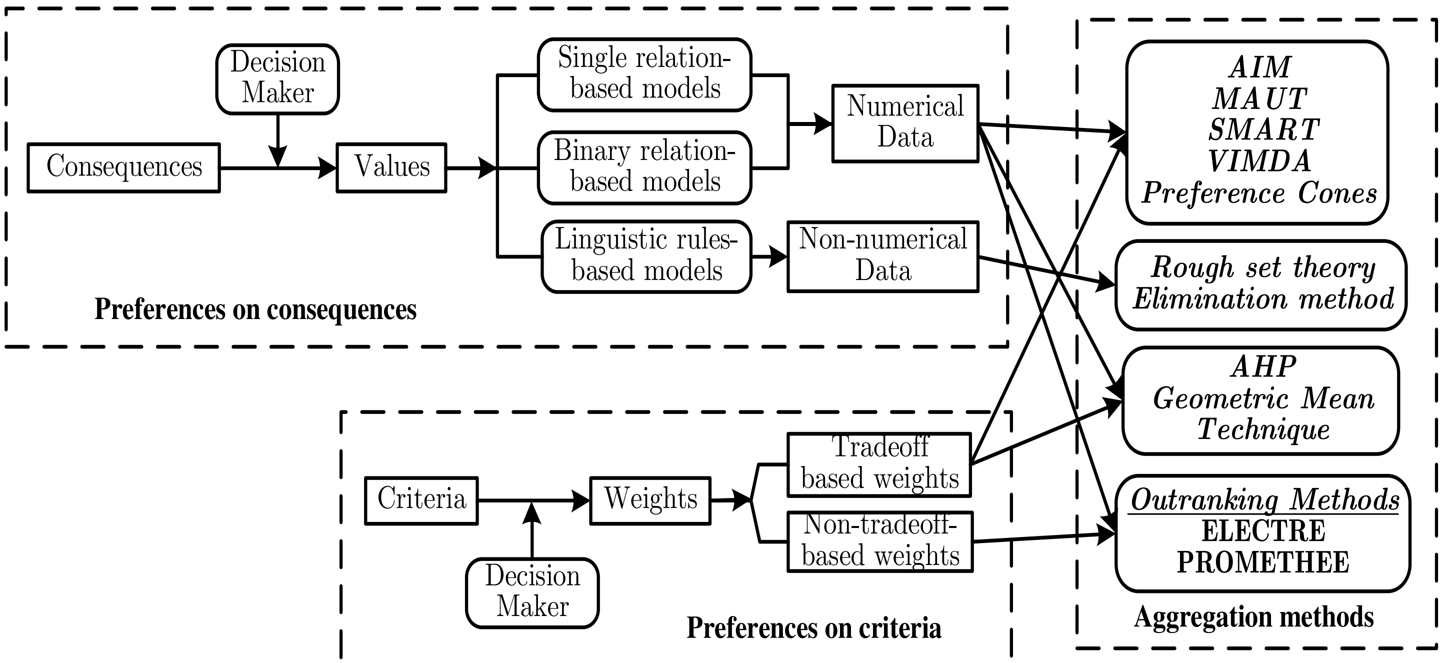


Figure 2.4: Preference Acquisition and Aggregation in MCDA

Greco et al. (2001) argue that "*The rules explain the preferential attitude of the decision maker and enable his/her understanding of the reasons of his/her preferences.*"

2.5 Conclusions

In this chapter, the basic context of MCDA is explained as follows:

- MCDA and relevant research topics: MCDA and its related research are discussed and summarized in detail.
- Analysis procedures in MCDA: An analysis procedure for MCDA, consequence based preference aggregation, is explained in detail to establish a general framework for MCDA.
- Summary of methods in MCDA: Based upon the aforementioned analysis procedure, many methods in MCDA are summarized and integrated into a systematic framework to demonstrate the essence of these different approaches.

Chapter 3

Screening Problems in Multiple Criteria Decision Analysis

3.1 Introduction

In this chapter, screening problems are systematically addressed. Firstly a general description of screening problems is presented. Next, a sequential screening procedure is proposed to solve screening problem and the properties of sequential screening are discussed; then several popular MCDA methods are applied to screening by using only partial decision information. Finally, the systematic use of this procedure is demonstrated in a case study of the Waterloo water supply planning in Southern Ontario, Canada. This Chapter is based on earlier research by Chen et al. (2005b).

3.2 General Description of Screening Problems

First a formal definition of screening for a decision problem in MCDA is presented.

Definition 3. *A screening procedure is any procedure Scr that always selects a non-empty subset of an alternative set* $\mathbf{A}$*,*

$$\varnothing \neq Scr(\mathbf{A}) \subset \mathbf{A}, \tag{3.1}$$

where Scr denotes a screening procedure (sometimes subscripts are used to distinguish among different types of screening procedures), and $Scr(\mathbf{A})$ denotes the screened set (the remaining alternatives) after the procedure Scr was applied to the alternative set $\mathbf{A}$.

Speaking more practically, screening is any process that reduces a larger set of alternatives to a smaller set that (most likely) contains the best choice. An illustration is shown in Figure 3.1.

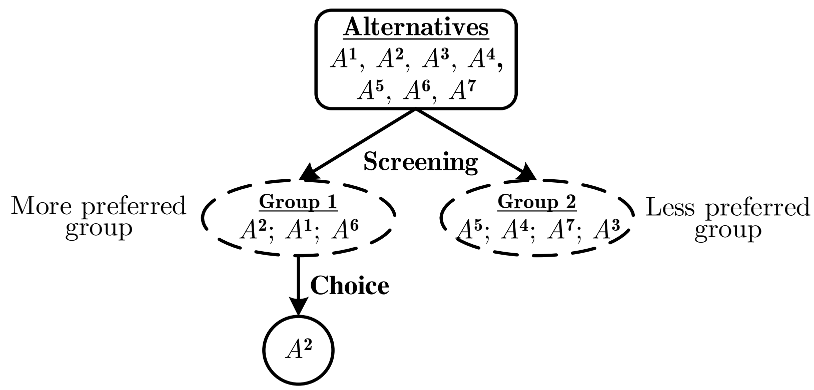


Figure 3.1: The Relationship among Screening, Sorting and Choice

Screening should accomplish the objective of reducing the number of alternatives to be considered. As Bobbs and Meierm (2000) put it, screening "should eliminate alternatives that are unlikely to be chosen, so that later effort can be focused on the more attractive options." With respect to problématiques in MCDA, we are focusing on a choice problématique(α), but it should be noted that screening can be interpreted as an application of the sorting problématique(β), since screening means arranging all alternatives into two groups, one of which is "screened out" from further consideration.

3.3 A Sequential Screening Procedure

First basic properties in the sequential screening procedure are defined as follows:

3.3.1 Basic Properties

Safety

A screening procedure for a choice problem, Scr, is *safe* iff whenever an alternative b is a best choice in $Scr(\mathbf{A})$, b is also a best alternative in $\mathbf{A}$.

Efficiency

If Scr_1 and Scr_2 are distinct screening procedures, then Scr_1 is more *efficient* than Scr_2, or a *refinement* of Scr_2, iff $Scr_1(\mathbf{A}) \subseteq Scr_2(\mathbf{A})$ is always true.

Information

In a screening procedure, information refers to preference information provided by the DM. For example, some aspects of the DM's preference may be confirmed without providing complete information. Generally speaking, the more preference information included in a screening procedure, the more alternatives it can screen out. In the extreme case, information may be so strong that only one alternative is left after screening.

Based on the description of consequences, values, weights and aggregation models in Chapter 2, there are four types of screening information as follows:

- $I1$: the validation of basic preference properties;
- $I2$: the application of preference information on consequences;
- $I3$: the application of preference information on criteria;
- $I4$: the integration of aggregation models.

It is assumed that if Scr_1 is a refinement of Scr_2, then Scr_1 is based on more information than Scr_2.

3.3.2 Sequential Screening

The DM may not be satisfied with the result of an initial screening. If so, other screening procedures may be applied to the screened set. Typically these follow-up procedures are based on more detailed preference information. This process is called *sequential screening*.

Definition 4. *Sequential screening is the application of a series of screening procedures in sequence to an alternative set* $\mathbf{A}$,

$$Scr_{h,h-1,\ldots,1}(\mathbf{A}) = Scr_h\Bigg(Scr_{h-1}\Big(\cdots\big(Scr_1(\mathbf{A})\big)\cdots\Big)\Bigg), \tag{3.2}$$

where $Scr_k, k = 1, 2, ..., h$ are screening procedures. Scr_h is the final screening procedure in this sequential screening.

Theorem 1. *Sequential screening $Scr_{h,h-1,...,1}$ is safe iff Scr_k is safe for $k = 1, 2, \cdots, h$.*

Proof: By hypothesis, Scr_1 is safe. Now assume $Scr_{k-1,k-2,...,1}$ is safe and consider $Scr_{k,k-1,...,1}$. Because Scr_k is safe, a best alternative in $Scr_{k,k-1,...,1}(\mathbf{A})$ is a best alternative in $Scr_{k-1,k-2,...,1}(\mathbf{A})$, which by assumption is a best alternative in $\mathbf{A}$. Therefore, Theorem 1 is true by induction. □

3.3.3 Decision Information Based Screening

As stated above, decision procedures using partial preference information can often be adapted for screening. Some popular MCDA methods can be modified for screening; the general relationship among them, grouped by decision information requirements, is shown in Figure 3.2.

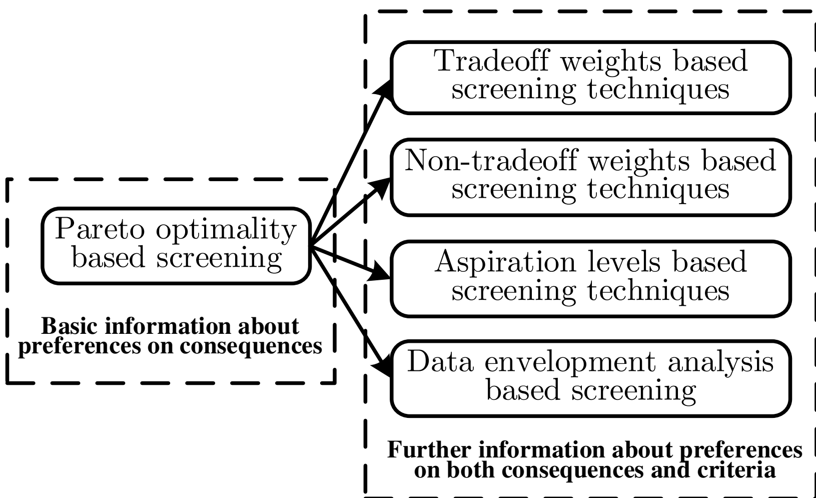


Figure 3.2: Screening Methods and Decision Information

As shown in Figure 3.2, *Pareto optimality based screening* is a basic screening technique, and we will discuss it first. Other screening techniques based on *tradeoff weights*, *non-tradeoff weights*, *aspiration levels*, and *data envelopment analysis (DEA)* can be carried out subsequent to Pareto optimality screening. Of course, these methods require more decision information and, when used in sequence, produce refined screening.

3.3.4 Pareto Optimality (PO) Based Screening

Pareto optimality is a famous concept put forward by Pareto (1909) and is widely used in economics and elsewhere. It provides a very useful definition of optimality in MCDA because it can be interpreted to take account of multiple aspects (criteria) for overall optimality. Definitions of Pareto optimality in *Multiple Objective Mathematical Programming (MOMP)* (Steuer, 1986) do not explain well the relations between consequence data and preference data and are not suitable as screening techniques in MCDA. So we re-define the concept of Pareto optimality in MCDA.

Domination and Pareto Optimality

Domination is a relation that might or might not hold between two alternatives.

Definition 5. *$B \in \mathbf{A}$ dominates $A \in \mathbf{A}$, denoted $B \succ A$, iff $\forall j \in \mathbf{Q}$, $v_j(B) \geq v_j(A)$ with at least one strict inequality.*

Using domination, we define *Pareto optimality.*

Definition 6. *$A \in \mathbf{A}$ is a Pareto Optimal (PO) alternative, also called an efficient alternative, iff $\nexists\ B \in \mathbf{A}$ such that $B \succ A$. The set of all Pareto optimal alternatives in $\mathbf{A}$ is denoted PO($\mathbf{A}$).*

Usually, the information provided by the DM during the screening phrase is limited and the DM may prefer not to spend too much energy on comprehensive preference expressions. Then the following theorem shows that by identifying *PO(*$\mathbf{A}$*)*, we carry out *PO* based direct screening in MCDA based on consequences, provided some properties of preference data hold.

Theorem 2. *Assume the preference directions over criteria are available. $A \notin PO(\mathbf{A})$ iff $\exists\ B \in \mathbf{A}$ such that $\forall j \in \mathbf{Q}$, $c_j(A) \leq c_j(B)$ when j is a positive preference criterion and $c_j(A) \geq c_j(B)$ when j is a negative preference criterion, with at least one strict inequality. For a choice problem, if $A \notin PO(\mathbf{A})$, then A can be safely screened out.*

Proof: Suppose that $A \in \mathbf{A}$ and $\exists\ B \in \mathbf{A}$ such that $\forall j \in \mathbf{Q}$, either $c_j(A) \leq c_j(B)$ and j is a positive preference criterion, or $c_j(A) \geq c_j(B)$ and j is negative preference criterion, with at least one strict inequality for some $j \in \mathbf{Q}$. Taking into account the available information on preference directions over criteria, $v_j(A) \leq v_j(B)$ whenever j is a positive preference criterion or a negative preference criterion. Therefore

$\exists\ B \in \mathbf{A}$ such that $v_j(A) \leq v_j(B)\ \forall j \in \mathbf{Q}$ with at least one strict inequality. Therefore, according to Definition 5, $B \succ A$ and by Definition 6, $A \notin PO(\mathbf{A})$. The reverse implication is easy to verify directly.

As stated, the choice problem is to select the best alternative from $\mathbf{A}$. If $A \notin PO(\mathbf{A})$, then $\exists B \succ A$. It is unknown whether B is the best alternative, but A can be safely screened out since B is better than A, so that A cannot be best. □

Theorem 2 clarifies both the relation between preference data and consequence data for the determination of Pareto optimality and the relation between Pareto optimality and screening for choice problems. We can safely use consequence data to screen out alternatives that are not Pareto optimal, as long as we are sure the basic preference properties are satisfied.

Although many researchers had assumed the idea of Theorem 2 (checking Pareto optimality by consequence data), it is important to clarify the relation between Pareto optimality in consequence data and preference data; otherwise improper screening of alternatives may result.

PO Based Screening Procedure

- Identify preference direction (positive or negative) for each criterion;
- Compare alternatives based on their consequence data;
- Determine dominated alternatives based on Theorem 2;
- Remove dominated alternatives and retain non-dominated alternatives.

3.3.5 Tradeoff Weights (TW) Based Screening

PO based screening removes some alternatives. But DMs may not be satisfied if many alternatives remain. Moreover, the power of PO based screening usually decreases as the number of criteria increases. Further information is needed in order that more efficient screening can be carried out.

TW based screening techniques are related to the research areas such as sensitivity analysis (Insua, 1990; Insua and French, 1991) and dominance and potential optimality (Athanassopoulos and Podinovski, 1997; Hazen, 1986). Although these methods focused on different questions, they all screen alternatives in choice problems since they classify alternatives as either dominated (and therefore screened out) or non-dominated. Here, we summarize these methods and put forward a systematic screening method, *tradeoff weights* (*TW*) *based screening*, that underlies the existence of *tradeoff weights*.

Basic Concepts and Information Requirements

With TW based screening, the following further information from the DM is needed:

(1) The preference vector $\mathbf{v}(A^i) \in \mathbb{R}^q$, $\forall A^i \in \mathbf{A}$.

(2) The final aggregation model.

Since most of these methods use a linear additive value function as the aggregation model, we also adopt it. The linear additive value function model, introduced in (2.5), is given by

$$V(A^i) = \left\langle \mathbf{v}(A^i), \mathbf{w} \right\rangle = \sum_{j \in \mathbf{Q}} w_j \cdot v_j(A^i)$$

where $V(A^i)$ is the evaluation of alternative A^i and $\mathbf{w}$ is the weight vector.

Potential Optimality (PotOp) and Screening

Based on the linear additive value model, we define *potential optimality* and consider the relationship between potential optimality and screening.

Definition 7. *An alternative* $A^i \in \mathbf{A}$ *is potentially optimal (PotOp) iff there exists* $\mathbf{w}$ *such that* $\left\langle \mathbf{v}(A^i), \mathbf{w} \right\rangle = \max_{A^k \in \mathbf{A}} \left\langle \mathbf{v}(A^k), \mathbf{w} \right\rangle$*. The set of all PotOp alternatives is denoted as* $Scr_{PotOp}(\mathbf{A}) = PotOp(\mathbf{A}) \subseteq \mathbf{A}$*.*

Theorem 3. *Suppose* $A^i \in PotOp(A)$ *and* $\mathbf{w} \in \mathbf{W}$*. If* $\left\langle \mathbf{v}(A^i), \mathbf{w} \right\rangle \geq \left\langle \mathbf{v}(A^k), \mathbf{w} \right\rangle$ *for all* $A^k \in \mathbf{PotOp}(\mathbf{A})$*, then* $\left\langle \mathbf{v}(A^i), \mathbf{w} \right\rangle \geq \left\langle \mathbf{v}(A^k), \mathbf{w} \right\rangle$ $\forall A^k \in \mathbf{A}$*.*

Proof: Assume that $\left\langle \mathbf{v}(A^i), \mathbf{w} \right\rangle \geq \left\langle \mathbf{v}(A^k), \mathbf{w} \right\rangle$ for all $A^k \in \mathbf{PotOp}(\mathbf{A})$. If the theorem fails, then there exists $A^j \notin \mathbf{PotOp}(\mathbf{A})$ such that $\left\langle \mathbf{v}(A^j), \mathbf{w} \right\rangle > \left\langle \mathbf{v}(A^i), \mathbf{w} \right\rangle$. Therefore, from Definition 7 there exists $A^h \in \mathbf{PotOp}(\mathbf{A})$ such that $\left\langle \mathbf{v}(A^h), \mathbf{w} \right\rangle \geq \left\langle \mathbf{v}(A^k), \mathbf{w} \right\rangle$ for all $A^k \in \mathbf{A}$. In particular, $\left\langle \mathbf{v}(A^h), \mathbf{w} \right\rangle \geq \left\langle \mathbf{v}(A^j), \mathbf{w} \right\rangle$, and hence, $\left\langle \mathbf{v}(A^h), \mathbf{w} \right\rangle > \left\langle \mathbf{v}(A^i), \mathbf{w} \right\rangle$, contradicting the hypothesis that $\left\langle \mathbf{v}(A^i), \mathbf{w} \right\rangle \geq \left\langle \mathbf{v}(A^k), \mathbf{w} \right\rangle$ for all $A^k \in \mathbf{PotOp}(\mathbf{A})$, which proves the theorem. □

From this theorem, it follows that if A^i is best with respect to $\mathbf{w}$ in **PotOp(A)**, then A^i is best with respect to $\mathbf{w}$ in $\mathbf{A}$. Therefore, PotOp screening is safe.

The following mathematical program can be used to determine whether alternative A^i is potentially optimal (Geoffrion, 1968).

$$\mathbf{SCR.1}(A^i) \qquad \textbf{Minimize}: \quad \delta$$
$$\textbf{Subject to}: \left\langle \mathbf{w}, \left(\mathbf{v}(A^i) - \mathbf{v}(B)\right) \right\rangle + \delta \geq 0, \ \forall\, B \in \mathbf{A}$$
$$\mathbf{w} \in \mathbf{W}$$
$$\delta \geq 0$$

The inputs of $\mathbf{SCR.1}(A^i)$ are $\mathbf{v}(A^i)$, $\mathbf{v}(B)$, and outputs are δ and $\mathbf{w}$. If $\delta^* = 0$, then $A^i \in \mathbf{PotOp}(\mathbf{A})$. If $\delta^* > 0$, then A^i is not potentially optimal. Unless computational considerations intervene, Scr_{PotOp} can be implemented using this program as the fundamental step.

TW Based Screening Procedure

- Check the validity of the linear additive preference function (SMART) assumption;
- Obtain the preference functions and get the preference data from the consequence data;
- For each $A^i \in \mathbf{A}$, apply $\mathbf{SCR.1}(A^i)$ to identify **PotOp(A)**;
- Retain only the potential optimal alternatives.

3.3.6 Non-tradeoff Weights (NTW) Based Screening

Since non-tradeoff weights based techniques belong to the family of outranking methods, they can also be called outranking based screening. Roy (1968) is considered to have originated these methods.

Definition 8. Vincke (1992, p.58) *An outranking relation is a binary relation* $\mathbf{S}$ *defined on* $\mathbf{A}$*, with the interpretation that* $A\mathbf{S}B$ *if, given what is known about the DM's preferences and values, the alternatives and the nature of the problem, there are enough arguments to decide that* A *is at least as good as* B*, while there is no essential reason to refute that statement.*

Many screening methods are based on an outranking relation.

Outranking methods are suitable for screening in MCDA, as Vincke (1992, p.57) notes: "*Considering a choice problem, for example: if it is known that some action* *(alternative) a is better than b and c, it becomes irrelevant to analyze preferences between b and c. Those two actions can perfectly remain incomparable without endangering the decision-aid procedure.*"

The major families of outranking methods are the ELECTRE methods and the PROME-THEE methods. ELECTRE I and PROMETHEE I are the best known and most widely used methods within their respective families.

ELECTRE Technique for Screening

ELECTRE seeks to reduce the set of nondominated alternatives. The DM is asked to provide a set of weights reflecting relative importance (but they are not for tradeoffs purposes). Alternatives are eliminated if they are dominated by other alternatives to a specified degree defined by the DM. The system uses a *concordance index* to measure the relative advantage of an alternative over all other alternatives and a *discordance index* to measure its relative disadvantage. These indices determine a dominated set, which is then screened out.

PROMETHEE Technique for Screening

PROMETHEE is an offshoot of ELECTRE. PROMETHEE methods begin with a valued outranking function, the outranking degree $\pi(A, B)$, defined for each ordered pair of alternatives (A, $B \in \mathbf{A} \times \mathbf{A}, A \neq B$). Hence $\pi(A, B)$ represents a measure of how much better A is than B; $0 \leq \pi(A, B) \leq 1$. For the details about the form of $\pi(A, B)$, see Vincke (1992).

The overall outranking degree for alternative A is defined by the values of two functions: $\Phi^+(A)$ (outgoing flow, which refers to the intensity of preference for A over other alternatives) and $\Phi^-(A)$ (incoming flow, the intensity of preference for other alternatives relative to A). The definitions of $\Phi^+(A)$ and $\Phi^-(A)$ are as follows:

$$\Phi^+(A) = \sum_{B \in \mathbf{A}} \pi(A, B), \tag{3.3}$$

$$\Phi^-(A) = \sum_{B \in \mathbf{A}} \pi(B, A) \tag{3.4}$$

The final step is the generation of the outranking relations over all alternatives.

Definition 9. *$A\mathbf{P}^+B$ iff $\Phi^+(A) > \Phi^+(B)$; $A\mathbf{P}^-B$ iff $\Phi^-(A) < \Phi^-(B)$.*

$A\mathbf{I}^+B$ iff $\Phi^+(A) = \Phi^+(B)$; $A\mathbf{I}^-B$ iff $\Phi^-(A) = \Phi^-(B)$.

$$A \text{ outranks } B \text{ if } \langle A\mathbf{P}^+B \text{ and } A\mathbf{P}^-B\rangle \text{ or } \langle A\mathbf{P}^+B \text{ and } A\mathbf{I}^-B\rangle \text{ or } \langle A\mathbf{I}^+B \text{ and } A\mathbf{P}^-B\rangle.$$

All alternatives that are outranked by any alternative are then screened out.

3.3.7 Aspiration Levels (AL) Based Screening

Aspiration-level based screening techniques are techniques employing desired or acceptable consequence levels of criteria to identify better alternatives (these methods are also called reference point approaches (Hanne, 2001), but we prefer the name aspiration levels based screening, as proposed by Lotfi et al. (1992). Techniques such as *Goal Programming* (Charnes and Cooper, 1961), *Compromise Programming* (Zeleny, 1982) and the *Reference Point Approach* (Wierzbicki, 1982) can be regarded as the inspirations of aspiration-based screening. To introduce these methods, we classify them into two categories, *simple linguistic methods* and *distance-based models.*

Simple Linguistic Screening

Simple linguistic screening techniques use linguistic expressions on criteria as constraints (standards), eliminating alternatives that do not satisfy these constraints. The advantage of this method is that expressions are simple and there are fewer model parameters to specify. We can further classify constraints into *lexicographic* and *conjunctive/disjunctive* (Hwang and Yoon, 1981), or *Elimination method* (MacCrimmon, 1973).

- **Lexicographic Constraints**: Criteria are ranked in order of the relative importance. For each criterion, a constraint (goal) is set as a standard. Then all alternatives are examined, one at a time, to assess whether the first criterion is satisfied. All alternatives that fail are screened out. Then proceed to the second criterion, *etc.*

- **Disjunctive and Conjunctive Constraints**: Disjunctive/conjunctive constraints express conditions involving more than one criterion. In conjunctive form, characterized by "and", all the constraints (goals) must be satisfied in order for an alternative not to be screened. In disjunctive form, characterized by "or", any alternative can remain as long as it meets at least one of the constraints. Conjunctive constraints are more powerful for screening since an alternative must pass all the standards, so relatively few alternatives will succeed unless the standards are set at a low level. In the disjunctive form, only one standard must be met, so most alternatives will pass unless all standards are set very high.

Distance-based Screening

Distance-based screening employs a measure of **distance** between an alternative and some reference point (ideal alternative or aspiration level) as an index to screen out the alternatives that are too far away. Here we briefly introduce the **aspiration-level interactive model (AIM) for screening**, proposed by Lotfi et al. (1992). Its main steps are listed below.

Problem Definition: Three attainment levels for consequences must be established for each criterion j:

- The "want" level AS_j expresses the aspiration level on each criterion;
- The "ideal" level (I_j) and "nadir" levels (N_j) express the largest and smallest consequence data for each criterion.

Solution Process:

- Order the consequence data, $\{c_j(A) : A \in \mathbf{A}\}$, from least to most preferred for each criterion j. Then inform the DM of the current aspiration level AS_j for each criterion, and the proportion of alternatives that achieve it;
- For every alternative, identify the "nearest nondominated alternative", defined to be the closest alternative according to a scalarizing function proposed by Wierzbicki (1986) (see Lotfi et al. (1992) for more details) with the weight on criterion j given by $w_j = (AS_j - N_j)/(I_j - N_j)$. The weights are set to reflect the increasing importance attached to criterion j as the aspiration level is moved closer to the ideal. Notice that these weights serve only for temporary rankings and not any other purpose.

Screening Process: To screen alternatives, two options are available to the DM.

- First, the DM can reset one or several aspiration levels. This is useful when the DM is not sure about his aspiration level. Updated nearest nondominated alternatives (perhaps the same, perhaps not) are obtained based on these aspiration levels. Then only these nearest alternatives remain for further consideration; all others are screened out.

- Second, if the DM prefers not to express aspiration levels, he or she can request a set of "neighboring" alternatives based on single aspiration levels. All alternatives other than these "neighboring" ones are then screened out. Lotfi et al. (1992) propose the use of ELECTRE-based outranking to find the neighbors of the nearest alternative.

3.3.8 Data Envelopment Analysis (DEA) Based Screening

Data Envelopment Analysis (DEA) is a technique used to measure the relative efficiency of a number of similar units performing essentially the same task. DEA was first put forward by Charnes et al. (1978) who proposed the basic DEA model, called CCR.

Within the past few decades, research into the relation between DEA and MCDA has carried out by authors including Belton and Vickers (1993), Stewart (1996), Cook and Kress (1991), and Cook et al. (1996). Stewart compared and contrasted the goals of DEA and MCDA: DEA arises from situations where the goal is to determine the productive efficiency of a system or 'decision making unit' (DMU) by comparing how well the unit converts inputs into outputs, while MCDA models have arisen from the need to analyze a set of alternatives according to conflicting criteria. A 'methodological connection' between MCDA and DEA is that if all criteria in an MCDA problem can be classified as either benefit criteria (benefits or output) or cost criteria (costs or inputs), then DEA is equivalent to MCDA using additive linear value functions (Stewart, 1996).

The basic function of DEA is to ascertain which units are efficient and which are not; in MCDA, these can be regarded as non-dominated and dominated alternatives, respectively. Cook and Kress (1991); Cook et al. (1996) applied some DEA-based models to deal with MCDA problems with both cardinal and ordinal criteria. All dominated alternatives identified by the DEA-based model can be screened out.

DEA Software: Frontier Analyst

Frontier Analyst is commercial software based on DEA theory to measure and improve the performance of organizations. Below, Frontier Analyst is employed for executing DEA computations in an illustrative example.

DEA Based Screening Procedure

- Identify preference direction (positive or negative) for each criterion;
- Apply a DEA model (usually CCR) to identify dominated alternatives;
- Remove dominated alternatives.

3.3.9 Implementation of Sequential Screening

The information requirements for each screening method, and an estimate of its efficiency, are summarized in Table 3.1. As stated before, $I1$ stands for the validation of basic preference properties, $I2$ is the application of preference information on consequences, $I3$ is the application of information on criteria and $I4$ is the integration of aggregation models.

Table 3.1: Information Requirements for Different Screening Methods

Screening Methods	*Information Requirements*					*Screening Efficiency*
	Consequence data	$I1$	$I2$	$I3$	$I4$	
Pareto optimality	√	√	×	×	×	Low
Tradeoff weights	√	√	√	×	√	Medium
Non-tradeoff weights	√	√	×	√	√	Medium
Aspiration levels	√	√	√	√	√	High
DEA	√	√	×	×	√	Low

Some typical sequential screening processes are $Scr_{\mathbf{TW}}\big(Scr_{\mathbf{PO}}(\mathbf{A})\big)$, $Scr_{\mathbf{NTW}}\big(Scr_{\mathbf{PO}}(\mathbf{A})\big)$, $Scr_{\mathbf{AL}}\big(Scr_{\mathbf{PO}}(\mathbf{A})\big)$, and $Scr_{\mathbf{DEA}}\big(Scr_{\mathbf{PO}}(\mathbf{A})\big)$. Note that Pareto optimality (PO) based screening is used as a primary screening and combined with another method to carry out a sequential screening.

A more sophisticated approach is $Scr_{\mathbf{TW}}\Big(Scr_{\mathbf{DEA}}\big(Scr_{\mathbf{PO}}(\mathbf{A})\big)\Big)$. DEA based screening and tradeoff weights (TW) based screening share the same aggregation model assumption (linear additive value function), and TW based screening can be applied after DEA screening. Combined with PO based screening, these screening methods together constitute a powerful sequential screening technique. Note that not all methods mentioned in Table 3.1 can be combined to carry out a sequential screening.

3.4 Case Study: Waterloo Water Supply Planning (WWSP)

The Regional Municipality of Waterloo, located in the southwestern part of Ontario, Canada, comprises the three cities of Kitchener, Waterloo, and Cambridge, plus adjacent rural municipalities. At present, the Waterloo region is one of the largest communities in Canada to rely almost exclusively on groundwater for its water supply. Due to increases in residential, industrial and commercial demand and decreases in the reliability of groundwater resources, the Regional Government developed over 1991-2000 a Long Term Water Strategy to the year 2041 (Rajabi, 1997; Waterloo Regional Council, 2000).

The overall purpose of this project was to design and implement the best water resources plan for the Waterloo Region. In light of this purpose, seven criteria were proposed to evaluate possible alternatives. The detailed meanings of the criteria are as follows: INVEST: project investment cost (millions of dollars); OPER: project operating cost (millions of dollars); INFRA: project negative infrastructure impact (0-100, greater values mean more negative impact); ENVIR: project negative environmental impact (0-100, greater values mean more negative impact); RISK: project implementation risk (0-100, greater values mean higher risk); SUPPLY: project supply capability (million imperial gallons per day, MIGD); QUAL: the quality of the water the project will deliver (0-100, greater values mean higher water quality). INFRA, ENVIR, RISK and QUAL are qualitative criteria; scores were assessed by the authors based on a report by Associated Engineering (1994).

Twelve alternatives were identified. Table 3.2 shows the MCDA problem constituted to represent the WWSP. The consequence data for water quality, environmental impacts and risk were estimated according to a preliminary evaluation by Associated Engineering (1994). SUPPLY and QUAL are identified as positive preference criteria (indicated by “+” in Table 2); others are negative preference criteria (indicated by “-” in Table 2).

Below is a brief explanation of each of the twelve alternatives.

- Groundwater, option 1 (GW1) - Develop additional groundwater sources in the vicinity of Kitchener-Waterloo.
- Groundwater, option 2 (GW2) - Develop groundwater sources in new fields, mainly in the south Woolwich area, the Roseville area, and the St. Agatha area.

Table 3.2: The Basic Structure of the WWSP

Criteria	*Alternatives*											
	GW1	*GW2*	*AQ1*	*AQ2*	*GR*	*LF1*	*LF2*	*LF3*	*PL1*	*PL2*	*PL3*	*PL4*
INVEST(-)	100	61	8.6	17	5	112	123.6	111.25	120.4	126	181	222
OPER(-)	4	2.4	5.9	8.8	2	6.2	6.6	6.7	4.2	3.4	2.3	2.5
INFRA(-)	30	30	40	50	30	60	60	60	60	65	60	60
ENVIR(-)	60	60	45	45	40	50	40	90	80	80	80	80
RISK(-)	80	80	50	50	80	60	70	70	30	30	30	30
SUPPLY(+)	29	20	40	40	5	50	80	80	80	80	80	80
QUAL(+)	50	50	70	70	30	60	60	60	70	70	80	70

- Aquifer Recharge, option 1 (AQ1) - Construct dual purpose recharge and recovery wells at the Mannheim site, with capacity of 10 MIGD.
- Aquifer Recharge, option 2 (AQ2) - Construct dual purpose recharge and recovery wells at the Mannheim site, with capacity of 20 MIGD.
- Grand River (GR) - Extract water from Grand River during times of peak demand.
- Grand River Low Flow Augmentation (LF1) - Augment Grand River water flow by implementing the West Montrose Dam project.
- Grand River Low Augment (LF2) - Augment Grand River water flow by constructing a pipeline from Georgian Bay.
- Grand River Low Flow Augmentation (LF3) - Augment Grand River water flow by constructing a pipeline from Lake Huron.
- Pipeline (PL1) - Transport water to the region via a high pressure pipeline from Lake Ontario.
- Pipeline (PL2) - Transport water to the region via a high pressure pipeline from Lake Erie, using the Nanticoke water treatment facility.
- Pipeline (PL3) - Transport water to the region via a high pressure pipeline from Lake Huron at Goderich.
- Pipeline (PL4) - Transport water to the region via high pressure pipeline from Georgian Bay via Thornbury.

3.4.1 Screening Procedure

First, each criterion was checked for reference availability, preference independence and preference monotonicity. Then a sequential screening, $Scr_{\mathbf{TW}}\Big(Scr_{\mathbf{DEA}}\big(Scr_{\mathbf{PO}}(\mathbf{A})\big)\Big)$, was applied. The results are as follows.

1. Pareto optimality based screening:

   AQ1≻AQ2 and PL3≻PL4, so alternatives AQ2 and PL4 can be screened out. Ten alternatives remain.

2. Data envelopment analysis based screening

   The software Frontier Analyst was used to analyze DEA efficiency based on the CCR model (Charnes et al., 1978). The result is shown in Figure 3.3. GW1 and LF3 are identified as inefficient and can be screened out. Eight alternatives remain.

3. Tradeoff weights Based Screening

   The linear normalization functions (2.2) and (2.3) are employed to generate value data based upon consequence data and listed in Table 3.3.

Table 3.3: Value Data of Reduced WWSP

Criteria	*Alternatives*							
	GW2	*AQ1*	*GR*	*LF1*	*LF2*	*PL1*	*PL2*	*PL3*
INVEST(+)	0.082	0.581	1.000	0.045	0.040	0.042	0.040	0.028
OPER(+)	0.833	0.339	1.000	0.323	0.303	0.476	0.588	0.870
INFRA(+)	1.000	0.750	1.000	0.500	0.500	0.500	0.462	0.500
ENVIR(+)	0.667	0.889	1.000	0.800	1.000	0.500	0.500	0.500
RISK(+)	0.375	0.600	0.375	0.500	0.429	1.000	1.000	1.000
SUPPLY(+)	0.250	0.500	0.063	0.625	1.000	1.000	1.000	1.000
QUAL(+)	0.625	0.875	0.375	0.750	0.750	0.875	0.875	1.000

   Based on **SCR.1**(A^i), the following program can be applied to assess alternative GW2:

   SCR.1(GW2)
   Minimize: δ

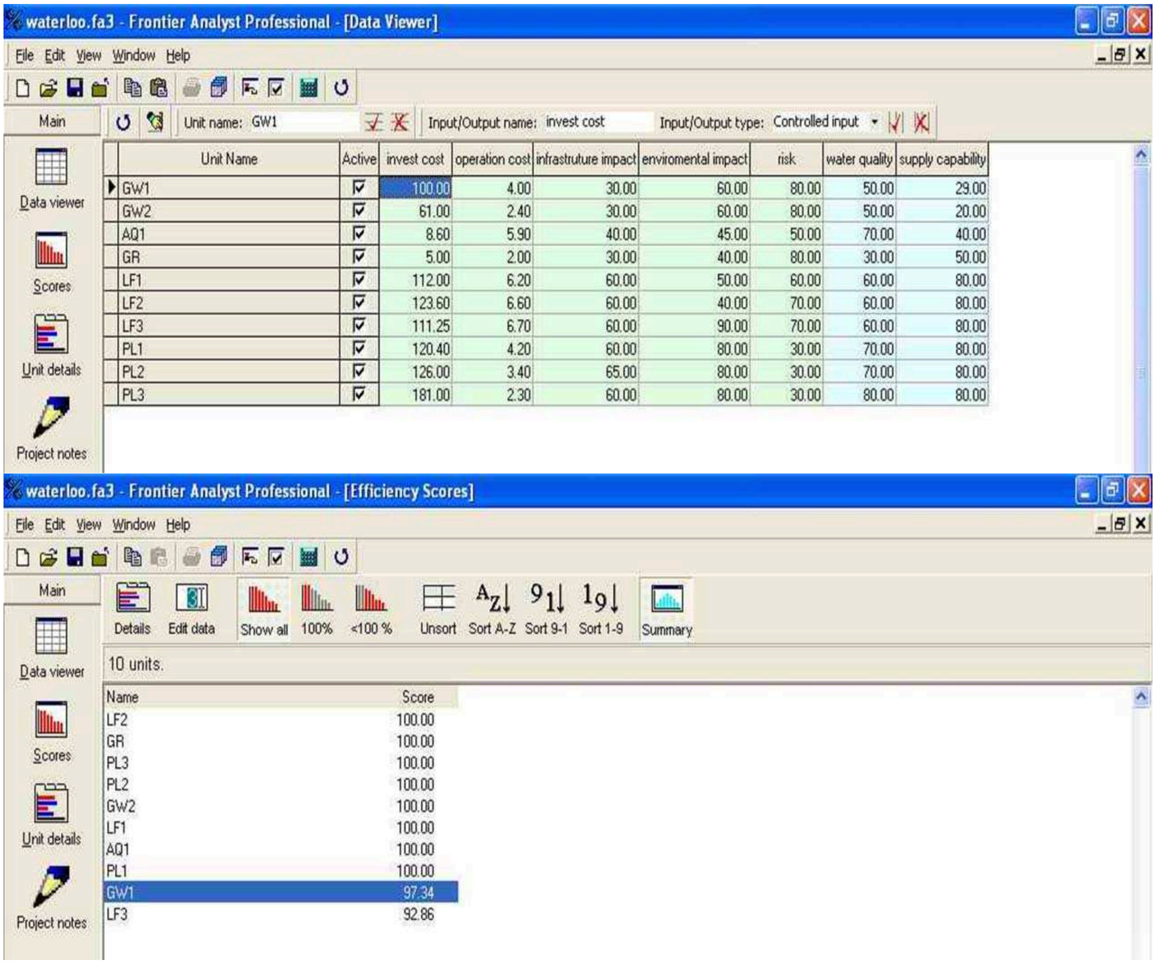


Figure 3.3: Frontier Analyst Based DEA Screening

Subject to:

$$-0.499w_1 + 0.494w_2 + 0.25w_3 - 0.222w_4 - 0.225w_5 - 0.25w_6 - 0.25w_7 + \delta \geq 0$$
$$-0.918w_1 - 0.167w_2 - 0.333w_4 + 0.188w_6 + 0.25w_7 + \delta \geq 0$$
$$0.037w_1 + 0.511w_2 + 0.5w_3 - 0.133w_4 - 0.125w_5 - 0.375w_6 - 0.125w_7 + \delta \geq 0$$
$$0.042w_1 + 0.53w_2 + 0.5w_3 - 0.333w_4 - 0.054w_5 - 0.75w_6 - 0.125w_7 + \delta \geq 0$$
$$0.04w_1 + 0.357w_2 + 0.5w_3 + 0.167w_4 - 0.625w_5 - 0.75w_6 - 0.25w_7 + \delta \geq 0$$
$$0.042w_1 + 0.245w_2 + 0.538w_3 + 0.167w_4 - 0.625w_5 - 0.75w_6 - 0.25w_7 + \delta \geq 0$$
$$0.054w_1 - 0.036w_2 + 0.5w_3 + 0.167w_4 - 0.625w_5 - 0.75w_6 - 0.375w_7 + \delta \geq 0$$
$$\sum_{j=1}^{7} w_j = 1$$
$$\delta \geq 0$$

$w_j > 0, j = 1, 2, 3, 4, 5, 6, 7.$
where w_j denotes the weight of each criteria in Table 3.2 sequentially (in practical computations using LINGO, $w_j \geq 0.01$ is set).

Using the software Lingo (2005), we get $\delta^*_{GW2} = 0$, so $GW2 \in$ **PotOp** and cannot be screened out. Other alternatives are checked similarly. The results, shown in Table 3.4, are that alternatives GW2, AQ1, GR, PL1, PL2, PL3 are retained for further consideration. The number of alternatives is decreased by half as a result of the three screening methods applied sequentially.

Table 3.4: Tradeoff Weights Based Screening

Screening steps	*Alternatives*							
	GW2	*AQ1*	*GR*	*LF1*	*LF2*	*PL1*	*PL2*	*PL3*
$\delta^*_{A^i}$	0	0	0	0.861	0.103	0	0	0
PotOp	√	√	√	×	×	√	√	√
Screening out	×	×	×	√	√	×	×	×

Figure 3.4 summarizes this application, and shows how each screening method works to screen out alternatives and what preference information is required from the DM.

3.4.2 Practical Implementation of WWSP

To ensure an adequate water supply to the region in the near future, Waterloo Regional council approved three alternatives with different construction schedules, as Waterloo's long term water strategy on May 10, 2000 (Waterloo Regional Council, 2000).

- *AQ1*: The immediate construction of a 5 MIGD Aquifer Storage and Recovery (ASR) facility, with an additional 5 MIGD ASR facility in 2007.
- *GW2*: 5 MIGD per day of additional groundwater facilities to be implemented between the years 2018 and 2020.
- *PL2/PL3*: 95 MIGD per day through a pipeline to either Lake Huron or Lake Erie by the year 2035.

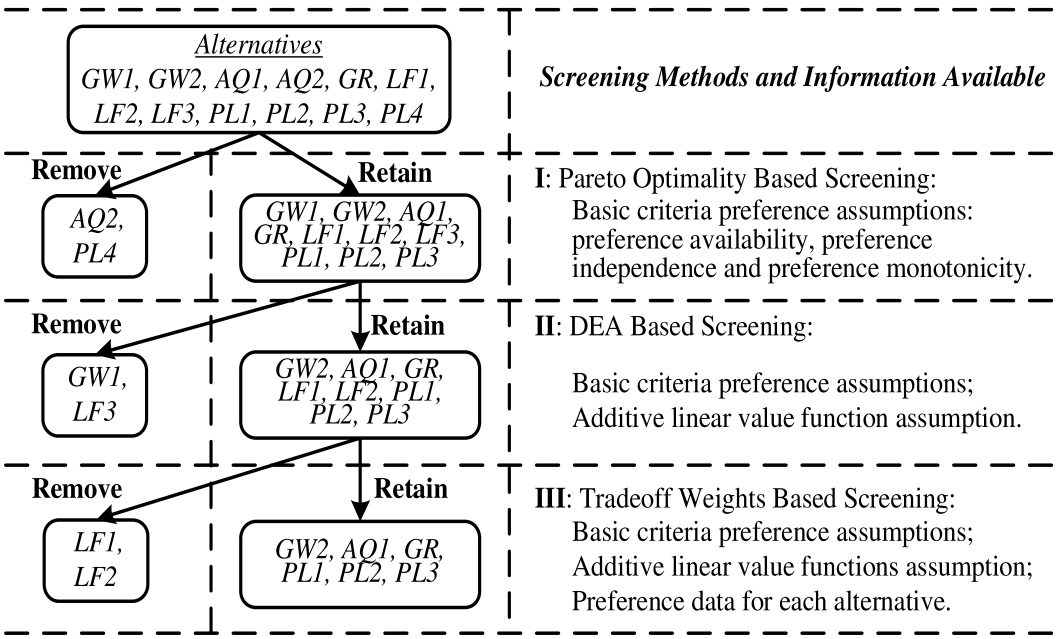


Figure 3.4: Screening Methods and Information Available

3.5 Conclusions

In this chapter, screening problems are discussed with respect to the following topics:

- General descriptions of screening problems: A formal definition of screening in MCDA is given and relationships among screening, sorting and choice are discussed.

- A sequential screening procedure: A sequential screening procedure is proposed to solve screening problems and several MCDA methods are systematically incorporated into this framework.

- Case study in water supply planning to the Region of Waterloo: The proposed sequential screening procedure is applied to the case study of Waterloo water supply planning problem.

Chapter 4

A Case-based Distance Method for Screening

4.1 Introduction

A case-based distance method for screening is proposed to solve screening problems. Firstly, research underlying case-based reasoning in MCDA is summarized. Then, the assumptions of the proposed case-based distance method are explained in detail. Subsequently, a quadratic program is constructed to find the best descriptive criterion weights and the screening thresholds. Based on this information, a procedure for carrying out a distance-based method for screening is proposed and a numerical example is utilized to demonstrate the proposed method. This chapter constitutes an expanded version of the research of Chen et al. (2005c).

4.2 Case-based Reasoning

The main difficulty in the application of many MCDA methods lies in the acquisition of the DM's preference information in the form of values or weights. Case-based reasoning is an approach to obtain preferential information from the DM's decisions on selected cases, as for example in UTilités Additives (UTA) (Jacquet-Lagrèze and Siskos, 1982) or rough set theory (Slowinski and Zopounidis, 1995). The test set of cases may include:

- Past decisions taken by the DM;

- Decisions taken for a limited set of fictitious but realistic alternatives;
- Decisions taken for a representative subset of the alternatives under consideration, which are sufficiently familiar to the DM that they are easy to evaluate.

The advantages of case-based information include that "decision makers may prefer to make exemplary decisions than to explain them in terms of specific functional model parameters" (Doumpos and Zopouidis, 2002). Meanwhile, some assumptions typical of functional models, for example preference monotonicity and preference independence, can be relaxed, simplifying the process of criterion construction. Here, we propose a case-based distance model for screening. Following the general concept of the aggregation-disaggregation approach to MCDA (Jacquet-Lagrèze and Siskos, 2001), we propose a specific case-based distance model for screening.

4.3 Model Assumptions

4.3.1 Case Set Assumptions

Assume a test set of alternatives $\mathbf{T}$. The alternatives in $\mathbf{T}$ may be, for example, fabricated by the DM or obtained by having the DM modify historical records. However all criteria in $\mathbf{Q}$ must apply and $c_j(A)$ must be measurable for all $A \in \mathbf{T}$ and all $j \in \mathbf{Q}$. Suppose that the DM specifies that all alternatives in $\mathbf{Z} = \{z^1, z^2, ..., z^r, ..., z^m\} \subseteq \mathbf{T}$ are **acceptable** for a choice problem. Let $\mid \mathbf{T} \mid = t$ and let $\mathbf{T} - \mathbf{Z} = \{z^{m+1}, z^{m+2}, ..., z^p, ..., z^t\}$ denote the **unacceptable** cases.

The idea of our model is that based on the "right" distance concept, the cases in $\mathbf{Z}$ should be close together, and the cases in $\mathbf{T} - \mathbf{Z}$ should be "outside" $\mathbf{Z}$ in some sense. We use $\mathbf{T}$ and $\mathbf{Z}$ to estimate criterion weights $\mathbf{w}$, and a basic distance threshold $R \in \mathbb{R}^+$, so that the distance of $z^r \in \mathbf{Z}$ from the "center" of $\mathbf{Z}$ is less or equal to R, and the distance of $z^p \notin \mathbf{Z}$ from this "center" is greater than R. Then we can apply $\mathbf{w}$ and S ($S = kR$, where $k \in \mathbb{R}^+$ is a controllable distance threshold for the DM) to screen out all "extreme" alternatives in $\mathbf{A}$, and thereby obtain $Scr(\mathbf{A})$. Figure 4.1 portrays this idea.

Given the acceptable case set $\mathbf{Z}$, $\overline{z}$, the **centroid** of $\mathbf{Z}$ is deemed to be a fictitious alternative at the center of $\mathbf{Z}$. By definition,

$$c_j(\overline{z}) = \frac{1}{m}\sum_{r=1}^{m} c_j(z^r). \tag{4.1}$$

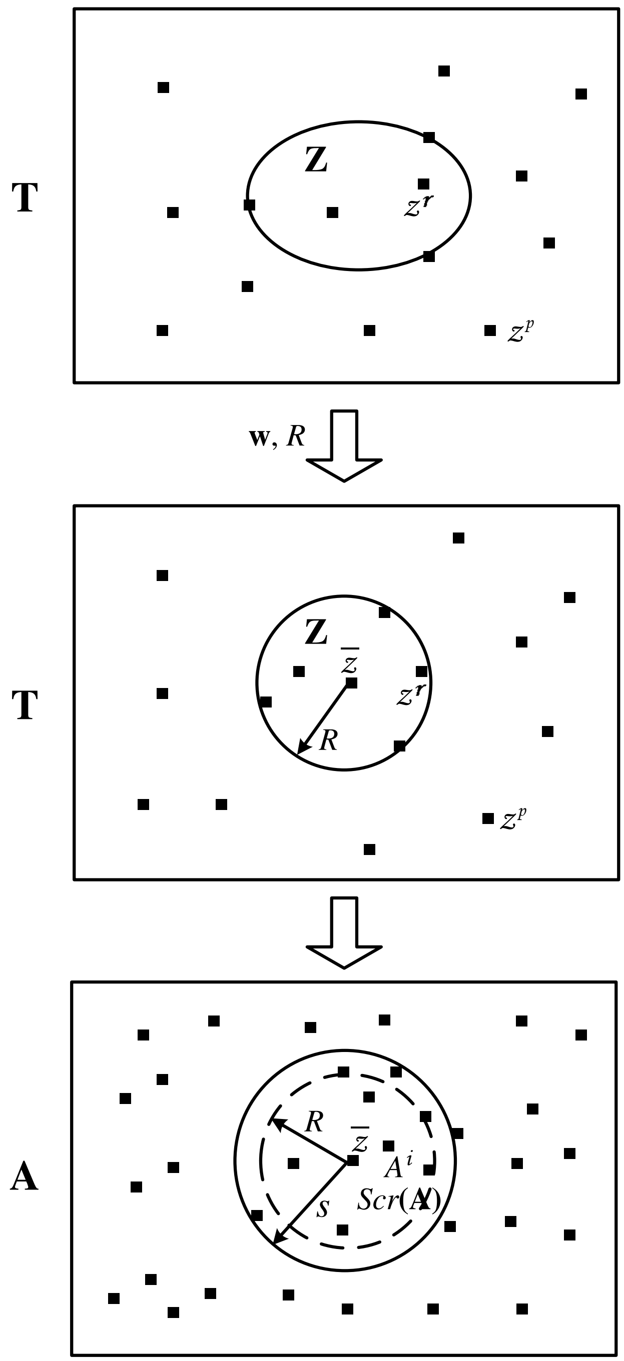


Figure 4.1: The Idea of Case-based Screening

Recall that $|\mathbf{Z}| = m$. Note that we assume that $\mathbf{Z}$ has the property that, for any criterion j, not all values of $c_j(z^r)$ are equal.

4.3.2 Distance Assumptions

For $j = 1, 2, ..., q$, define $d_j^{\max} = \max_{r=1,2,...,m} \big(c_j(z^r) - c_j(\overline{z})\big)^2$ to be the normalization factor for criterion j. The distance between $z^r \in \mathbf{Z}$ and the centroid $\overline{z}$ on criterion j is

$$d_j(z^r, \overline{z}) = \frac{\big(c_j(z^r) - c_j(\overline{z})\big)^2}{d_j^{\max}}. \tag{4.2}$$

Similarly, the distance between alternative $A^i \in \mathbf{A}$ and the centroid $\overline{z}$ on criterion j is

$$d_j(A^i, \overline{z}) = \frac{\big(c_j^i - c_j(\overline{z})\big)^2}{d_j^{\max}}. \tag{4.3}$$

Note that (4.3) applies if $A^i = z^p \in \{\mathbf{T} - \mathbf{Z}\}$, and $0 < d_j(z^r, \overline{z}) \leq 1$ for $z^r \in \mathbf{Z}$, but $d_j(z^p, \overline{z})$ or $d_j(A^i, \overline{z})$ could be larger than 1.

Weighted Euclidean distance has a clear geometric meaning, which can be easily understood and accepted to represent a DM's aggregated preference. Instead of weighted Euclidean distances, we use their squares, because they are easier to compute while preserving order.

The distance between alternative $z^r \in \mathbf{Z}$ and the case set $\mathbf{Z}$ is therefore identified with the distance between z^r and $\overline{z}$, which is

$$D(z^r) = D(z^r, \overline{z}) = \sum_{j \in \mathbf{Q}} w_j \cdot d_j(z^r, \overline{z}),$$

where $w_j \in \mathbf{w}$ is the weight (relative importance) of criterion j. The weight vector $\mathbf{w}$ is to be determined. The weights must satisfy $0 < w_j \leq 1$, and $\sum_{j \in \mathbf{Q}} w_j = 1$.

The distance of alternative A^i from the case set $\mathbf{Z}$ is

$$D(A^i) = D(A^i, \overline{z}) = \sum_{j \in \mathbf{Q}} w_j \cdot d_j(A^i, \overline{z}). \tag{4.4}$$

Note that (4.4) applies if $A^i = z^p \in \{\mathbf{T} - \mathbf{Z}\}$. The distance of an alternative from the case set is thus defined to be the distance between the alternative and the centroid of the case set. A similar idea has been widely used in cluster analysis:

cluster membership of a datum is determined by evaluation of a pre-defined distance between this datum and the cluster centroid. If the DM can further specify an ideal alternative (the most preferred alternative), z^* within $\mathbf{Z}$, then z^* can be used instead of the centroid to measure the distances and generate $\mathbf{w}$ and R. This modified procedure can be used for screening, exactly as carried out below. The details are omitted.

In terms of the aggregation approach to MCDA discussed in Chapter 2, $d_j(A^i, \overline{z})$ is analogous to v_j^i in (2.1), and $D(A^i)$ is analogous to $V(A^i)$ in (2.4). It is assumed that the closer A^i to $\overline{z}$, the greater the DM's preference. Therefore, smaller values of $D(A^i)$ indicate greater preference. The relative order of non-negative numbers (distances) is the same as the relative order of their squares, so the "order of elimination" of the screened set can be determined equally well using the squared distances.

Figure 4.2 shows the relationships of distances and the case set defined above. Taking $\overline{z}$ as the centroid, a compact ball (in q dimensions) with diameter R includes every case $z^r \in \mathbf{Z}$, and any case $z^p \notin \mathbf{Z}$ is (in principle) outside that ball. The DM can choose $S = kR$, $k \in \mathbb{R}^+$, so that $S > R, S = R$, or $S < R$, to control the screening process. If $D(A^i)$ is less than S, A^i can be regarded as having high preference and should be retained; if $D(A^i)$ is larger than S, A^i should be screened out.

4.4 Model Construction

For $j \in \mathbf{Q}$, w_j represents the DM's relative importance for criterion j, and R represents the threshold to distinguish cases in $\mathbf{Z}$ from cases in $\mathbf{T} - \mathbf{Z}$. Here we try to obtain $\mathbf{w} = (w_1, w_2, ..., w_q)$ and R by a case-based reasoning model based on $\mathbf{T}$.

Each alternative in the case set $\mathbf{Z}$ is assessed by the DM to be an acceptable alternative for a choice problem. So, based on induced preference between $Scr(\mathbf{A})$ and $\overline{Scr}(\mathbf{A})$ as defined above, these cases are more preferred than cases not in $\mathbf{Z}$. Therefore, the distance of $z^r \in \mathbf{Z}$ from the centroid $\overline{z}$ is less than R, and distance of $z^p \notin \mathbf{Z}$ from $\overline{z}$ is greater than R, provided there are no inconsistent judgements. Thus,

$$D(z^r) + \alpha^r \leq R,$$

or

$$\sum_{j \in \mathbf{Q}} w_j \cdot d_j(z^r, \overline{z}) + \alpha^r \leq R,$$

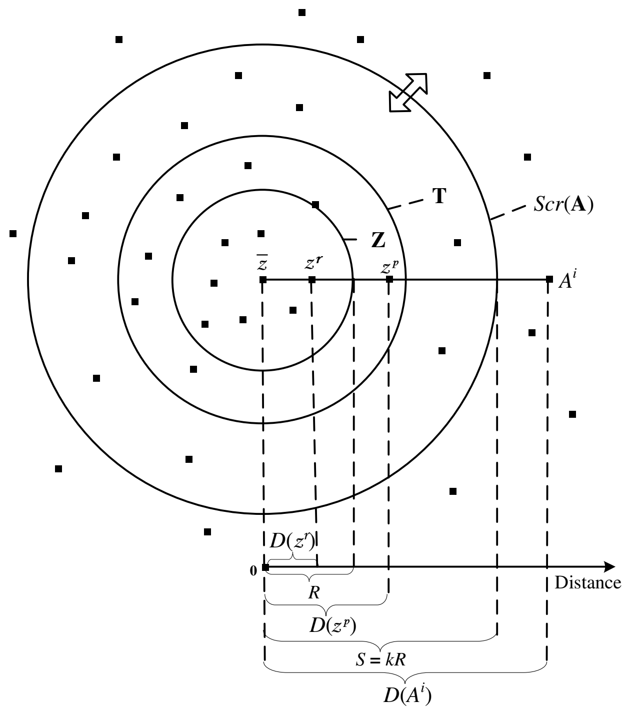


Figure 4.2: Distances and the Case Set

for $z^r \in \mathbf{Z}$. Here $-1 \leq \alpha^r \leq 0$ is an error adjustment parameter for the DM's inconsistent judgements on z^r. Also

$$D(z^p) + \alpha^p \geq R,$$

or

$$\sum_{j \in \mathbf{Q}} w_j \cdot d_j(z^p, \overline{z}) + \alpha^p \geq R,$$

for $z^p \in \{\mathbf{T} - \mathbf{Z}\}$. Again $0 \leq \alpha^p \leq 1$ is an error adjustment parameter for the DM's inconsistent judgements on z^p. Therefore, the overall squared error in $\mathbf{T}$ can be denoted as $ERR = \sum_{r=1}^{m} (\alpha^r)^2 + \sum_{p=m+1}^{t} (\alpha^p)^2$.

The following optimization model can be applied to find the most descriptive weight set $\mathbf{w}$ and the appropriate distance threshold R.

SCR.**2**(α) **Minimize**: $ERR = \sum_{r=1}^{m} (\alpha^r)^2 + \sum_{p=m+1}^{t} (\alpha^p)^2$

Subject to:

$$\sum_{j \in \mathbf{Q}} w_j \cdot d_j(z^r, \overline{z}) + \alpha^r \leq R, \; r = 1, 2, ..., m;$$
$$\sum_{j \in \mathbf{Q}} w_j \cdot d_j(z^p, \overline{z}) + \alpha^p \geq R, \; p = m+1, m+2, ..., t;$$
$$0 < R \leq 1;$$
$$w_j > 0, \; j = 1, 2, ..., q$$
$$\sum_{j \in \mathbf{Q}} w_j = 1;$$
$$-1 \leq \alpha^r \leq 0, \; r = 1, 2, ..., m;$$
$$0 \leq \alpha^p \leq 1, \; p = m+1, m+2, ..., t.$$

Note that optimization software such as Lingo (Lingo, 2005) automatically converts strict inequalities in constraints, like $w_j > 0$, to weak inequalities, like $w_j \geq \epsilon$, where ϵ is small positive number generated internally when optimization is carried out as in $\mathbf{SCR.2}(\alpha)$. Here, we retain zero in the constraints for easy understanding of the program. Similar settings are applied to the other optimization models addressed in the later chapters.

Theorem 4. $\mathbf{SCR.2}(\alpha)$ *has at least one optimal solution,* $\mathbf{w}^*$ *and* R^*.

Proof: The set of $\mathbf{w}$ satisfying the constraints in $\mathbf{SCR.2}(\alpha)$ is bounded. The objective function $ERR = \sum_{r=1}^{m} (\alpha^r)^2 + \sum_{p=m+1}^{t} (\alpha^p)^2$ is a quadratic function on this set. Based on the extreme value theorem of advanced calculus (Fitzpatrick, 1995, page 297), the function ERR is continuous and the set of all possible variables is bounded, ERR attains its minimum at least once. □

Two threshold parameters can be set by the DM to evaluate the errors generated by $\mathbf{SCR.2}(\alpha)$:

- The acceptable distance threshold, ΔD: ΔD is the threshold below which error generated by $\mathbf{SCR.2}(\alpha)$ is acceptable to the DM (Note that ΔD should be larger than ε^*). If $\varepsilon^* \leq ERR^* \leq \Delta D$, the errors still are acceptable; otherwise $\mathbf{T}$ should be reset. A suggested value of ΔD is $\frac{1}{10t}$, where $|\mathbf{T}| = t$.
- The indifference threshold (see Vincke (1992) for detailed discussion of indifference thresholds in MCDA), ε: ε is the threshold below which errors generated by $\mathbf{SCR.2}(\alpha)$ can be ignored. When $ERR^* \leq \varepsilon$, the case set $\mathbf{T}$ and $\mathbf{Z}$ are consistent; but when $ERR^* \geq \varepsilon$, the errors cannot be ignored and there is some inconsistency in $\mathbf{T}$. A value suggested for ε is $\frac{1}{10n}$, where $n = |\mathbf{A}|$ is the number of alternatives in $\mathbf{A}$. Of course,the DM can adjust the setting to keep the balance between indifference threshold and acceptable threshold.

When multiple optimal solutions appear in $\mathbf{SCR.2}(\alpha)$, different values of $\mathbf{w}$ and R can provide different screening abilities. Recall that $Scr(\mathbf{A})$ in Chapter 3 denotes the screened set (the remaining alternatives) after the procedure Scr is applied to the alternative set $\mathbf{A}$. When $Scr(\mathbf{A}) = \mathbf{A}$ or $Scr(\mathbf{A}) = \varnothing$, $\mathbf{T}$ is an unsatisfactory test set. Even if $\varnothing \subset Scr(\mathbf{A}) \subset \mathbf{A}$, the DM may not be fully satisfied with the results. In this thesis, the DM can control the screening process, as explained below.

4.5 Distance-based Screening

Assuming $ERR^* \leq \Delta D$, with $\mathbf{w}^* = (w_1^*, w_2^*, ..., w_q^*)$, the maximum error in $\mathbf{Z}$ is defined as $\alpha_{\max}^{\mathbf{Z}} = \max\limits_{r=1,2,...,m}\{|\alpha^r|\}$, the maximum error in $\mathbf{T}-\mathbf{Z}$ is defined as $\alpha_{\max}^{\mathbf{T}-\mathbf{Z}} = \max\limits_{p=m+1,m+2,...,t}\{\alpha^p\}$. Different policies to set the distance threshold, S, can be employed to carry out screening.

4.5.1 Different Screening Processes

Normal Screening

When $ERR^* \leq \varepsilon^*$, the errors can be ignored and $S = R^*$ is employed to screen alternatives.

- If $D(A^i) \leq R^*$, then $A^i \in Scr(\mathbf{A})$ and A^i should be retained;
- If $D(A^i) > R^*$, then $A^i \notin Scr(\mathbf{A})$ and A^i should be removed.

Conservative Screening

For $ERR^* \geq \varepsilon^*$, $\alpha_{\max}^{\mathbf{Z}}$ is taken into account as **error permission** in $\mathbf{T}$. A distance threshold from $\overline{z}$ is defined as $S = R^* + \alpha_{\max}^{\mathbf{Z}}$ to include all cases in $\mathbf{Z}$ and applied to screen alternatives.

- If $D(A^i) \leq (R^* + \alpha_{\max}^{\mathbf{Z}})$, then $A^i \in Scr(\mathbf{A})$ and A^i should be retained;;
- If $D(A^i) > (R^* + \alpha_{\max}^{\mathbf{Z}})$, then $A^i \notin Scr(\mathbf{A})$ and A^i should be removed.

Aggressive Screening

For $ERR^* \geq \varepsilon^*$, $\alpha_{\max}^{\mathbf{T}-\mathbf{Z}}$ is taken into account as **error deduction** in $\mathbf{T}$ and a distance threshold from $\overline{z}$ is defined as $S = R^* - \alpha_{\max}^{\mathbf{T}-\mathbf{Z}}$, so that all cases in $\mathbf{T} - \mathbf{Z}$ are "outside" the screening ball circle.

- If $D(A^i) \leq (R^* - \alpha_{\max}^{\mathbf{T}-\mathbf{Z}})$, then $A^i \in Scr(\mathbf{A})$ and A^i should be retained;
- If $D(A^i) > (R^* - \alpha_{\max}^{\mathbf{T}-\mathbf{Z}})$, then $A^i \notin Scr(\mathbf{A})$ and A^i should be removed.

It is easy to see that $(R^* - \alpha_{\max}^{\mathbf{T}-\mathbf{Z}}) \leq R^* \leq (R^* + \alpha_{\max}^{\mathbf{Z}})$. When $ERR^* \leq \varepsilon$, $R^* - \alpha_{\max}^{\mathbf{T}-\mathbf{Z}} = R^* = R^* + \alpha_{\max}^{\mathbf{Z}}$. Generally speaking, it can screen out more alternatives when a smaller value of distance threshold is applied to screening.

Controllable Screening

When the DM is not satisfied with the results generated by the above screening procedures, and would like to have ability to control the screening results, the DM could manually determine the distance threshold $S = kR^*$, $k \in \mathbb{R}^+$, to generate $Scr(\mathbf{A})$ instead of updating the the case set $\mathbf{Z}$ or $\mathbf{T}$. Hence,

- If $D(A^i) \leq S$, then $A^i \in Scr(\mathbf{A})$ and A^i should be retained;
- If $D(A^i) > S$, then $A^i \notin Scr(\mathbf{A})$ and A^i should be removed.

When $Scr(\mathbf{A})$ is too large, the DM should reduce S; when $Scr(\mathbf{A})$ is too small, the DM should increase S.

4.5.2 The Framework of Screening

A systematic procedure to analyze case-based distance screening problems is shown in Figure 4.3.

It includes the following steps:

1. *Identify the test set* $\mathbf{T}$ *and* $\mathbf{Z}$: Identify the test set $\mathbf{T}$ and ask the DM to select $\mathbf{Z} \subset \mathbf{T}$.
2. *Compute the centroid of* $\mathbf{Z}$: Find $\overline{z}$ using (4.1).

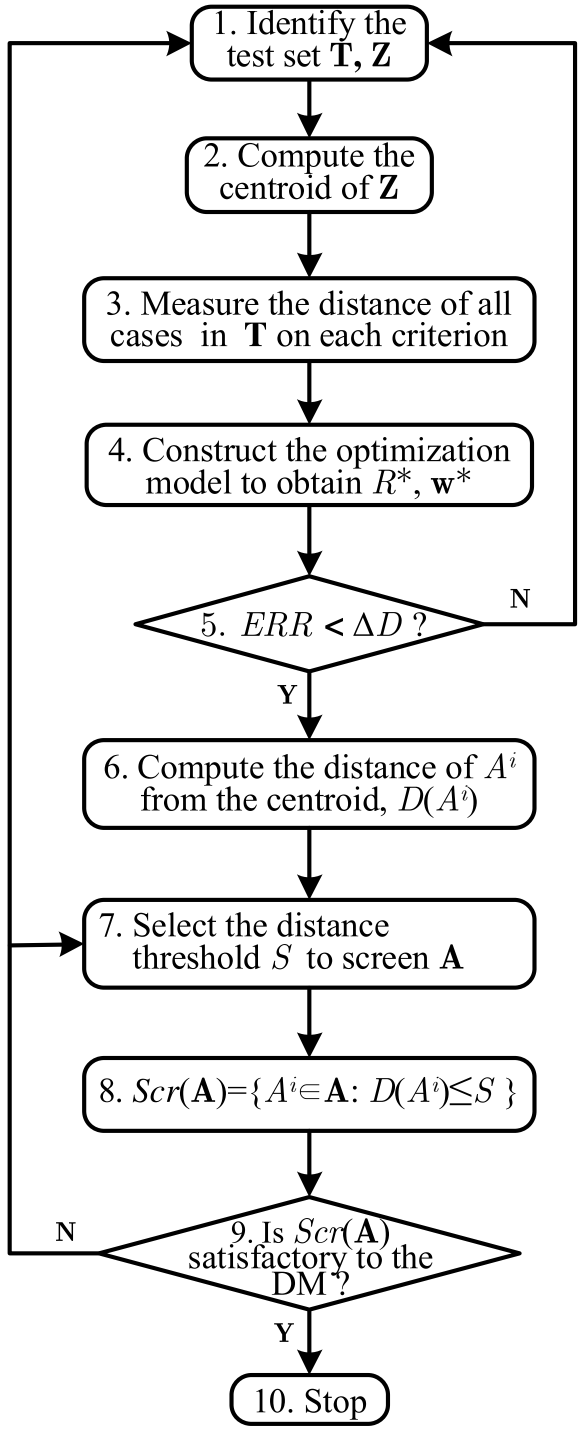


Figure 4.3: Analysis Procedure for Case-based Distance Screening

3. *Measure the distance of all cases on each criterion*: Compute $d_j(z^r, \overline{z})$ for $z^r \in \mathbf{Z}$ and $d_j(z^p, \overline{z})$ for $z^p \in \{\mathbf{T} - \mathbf{Z}\}$, using (4.2) and (4.3).

4. *Apply the optimization model*: The program $\mathbf{SCR.2}(\alpha)$ is used to obtain R^* and $\mathbf{w}^*$.

5. *Check the error ERR*: When $ERR > \triangle D$, ask the DM to repeat Step 1; if $ERR \leq \triangle D$, go to Step 6.

6. *Compute the distance of A^i from the centroid, $D(A^i)$*: Based on $\mathbf{w}^*$, obtain the distance of A^i from the centroid, $D(A^i)$ for $A^i \in \mathbf{A}$, using (4.4).

7. *Select the distance threshold S*: Initially select $S = R^*$.

8. *Screen using S*: If $D(A^i)$ is greater than S, then remove A^i; otherwise retain A^i. The remaining alternatives are $Scr(\mathbf{A})$.

9. *Check the screening result, $Scr(\mathbf{A})$*: If $Scr(\mathbf{A})$ is not satisfactory to the DM, either ask the DM to repeat Step 8 for fine tuning, or ask the DM to repeat Step 1 to re-assess the case set, etc.; otherwise stop the screening procedure. For example, when $Scr(\mathbf{A})$ is too large, the DM should first be asked to reduce S in Step 8, alternatively ask the DM to repeat Step 1 to re-assess the case set.

4.6 Numerical Example

4.6.1 Background

Similar to the WWSP problem described in Chapter 3, it assumed that twelve feasible alternatives were designed as shown in Table 4.1, and based upon them the best water resources project need to be identified and implemented for a region. Seven similar criteria, INVEST, OPER, INFRA, ENVIR, RISK, SUPPLY and QUAL, were proposed to evaluate possible alternatives.

4.6.2 Screening Procedure

The test set $\mathbf{T} = \{z^1, z^2, z^3, z^4, z^5, z^6, z^7\}$ shown in Table 4.2 is presented to the DM. Assume the DM chooses $\mathbf{Z} = \{z^1, z^2, z^3, z^4\}$ as acceptable cases and $\mathbf{T} - \mathbf{Z} =$

Table 4.1: The Basic Information of Alternatives

Criteria	*Alternatives*											
	A^1	A^2	A^3	A^4	A^5	A^6	A^7	A^8	A^9	A^{10}	A^{11}	A^{12}
INVEST	123.6	111.25	19.7	17.5	5	112	100	61	120.4	126	181	222
OPER	6.6	6.7	6.3	8.8	2	7.0	4	2.4	4.2	3.4	2.3	2.5
INFRA	60	60	40	50	30	60	30	30	60	65	60	60
ENVIR	40	90	45	45	40	50	60	60	80	80	80	80
RISK	70	70	50	50	80	75	80	80	30	30	30	30
SUPPLY	80	80	60	50	5	30	29	20	80	80	80	80
QUAL	60	60	73	70	30	50	50	50	70	70	80	70

$\{z^5, z^6, z^7\}$ as unacceptable cases. The acceptable distance error, $\triangle D$ is set as $\frac{1}{|\mathbf{T}|}$ $= \frac{1}{7}$ and the indifference threshold, ε^* is $\frac{1}{k\mathbf{T}} = \frac{1}{70}$.

The computed centroid $\overline{z}$ is given in Table 4.2.

Table 4.2: The case set for screening

Criteria	**Z**				Centroid	**T - Z**		
	z^1	z^2	z^3	z^4	$\overline{z}$	z^5	z^6	z^7
INVEST	90	100	110	120	105	100	80	90
OPER	5	8	6	4	5.75	20	25	15
INFRA	40	45	60	35	45	60	40	55
ENVIR	80	63	65	52	65	70	60	75
RISK	65	58	45	55	55.75	70	75	80
SUPPLY	70	80	80	85	78.75	50	50	70
QUAL	70	75	80	90	78.75	60	55	68

The computation results of the distances z^r and z^p from the centroid on each criterion are shown in Table 4.3.

The following quadratic program is applied to find R^* and $\mathbf{w}^*$.

SCR.2(α) **Minimize**: $ERR = (\alpha^1)^2+(\alpha^2)^2+(\alpha^3)^2+(\alpha^4)^2+(\alpha^5)^2+(\alpha^6)^2+(\alpha^7)^2$

Subject to:

Table 4.3: The Distances on Each Criterion in the Case Set

Criteria	*Distance between cases and centroid on each criterion*						
	$d(z^1, \overline{z})$	$d(z^2, \overline{z})$	$d(z^3, \overline{z})$	$d(z^4, \overline{z})$	$d(z^5, \overline{z})$	$d(z^6, \overline{z})$	$d(z^7, \overline{z})$
INVEST	1	0.1111	0.1111	1	0.1111	2.7778	1
OPER	0.1111	1	0.0123	0.6049	40.1111	73.1975	16.9012
INFRA	0.1111	0	1	0.4444	1	0.1111	0.4444
ENVIR	1	0.0178	0	0.7511	0.1111	0.1111	0.4444
RISK	0.7404	0.0438	1	0.0049	1.7572	3.2066	5.0887
SUPPLY	1	0.0204	0.0204	0.5102	10.7959	10.7959	1
QUAL	0.6049	0.1111	0.0123	1	2.7778	4.4568	0.9131

$$w_1 + 0.1111w_2 + 0.1111w_3 + w_4 + 0.7404w_5 + w_6 + 0.6049w_7 + \alpha^1 \leq R;$$
$$0.1111w_1 + w_2 + 0.0178w_4 + 0.0438w_5 + 0.0204w_6 + 0.1111w_7 + \alpha^2 \leq R;$$
$$0.1111w_1 + 0.0123w_2 + w_3 + w_5 + 0.0204w_6 + 0.0123w_7 + \alpha^3 \leq R;$$
$$w_1 + 0.6049w_2 + 0.4444w_3 + 0.7511w_4 + 0.0049w_5 + 0.5102w_6 + w_7 + \alpha^4 \leq R;$$
$$0.1111w_1+40.1111w_2+w_3+0.1111w_4+1.7572w_5+10.7959w_6+2.7778w_7+\alpha^5 \geq R;$$
$$2.7778w_1+73.1975w_2+0.1111w_3+0.1111w_4+3.2066w_5+10.7959w_6+4.4568w_7+\alpha^6 \geq R;$$
$$w_1 + 16.9012w_2 + 0.4444w_3 + 0.4444w_4 + 5.0887w_5 + w_6 + 0.9131w_7 + \alpha^7 \geq R;$$
$$w_1 + w_2 + w_3 + w_4 + w_5 + w_6 + w_7 = 1;$$
$$-1 \leq \alpha^1 \leq 0;\ -1 \leq \alpha^2 \leq 0;\ -1 \leq \alpha^3 \leq 0;\ -1 \leq \alpha^4 \leq 0;$$
$$0 \leq \alpha^5 \leq 1;\ 0 \leq \alpha^6 \leq 1;\ 0 \leq \alpha^7 \leq 1; 0 < R \leq 1.$$

The optimal solution found using Lingo Lingo (2005) is $ERR^* = 0 < \varepsilon = \frac{1}{120}$, $\mathbf{w}^* = (0.0935, 0.3199, 0.1316, 0.1022, 0.1370, 0.1138, 0.1020)$, and $R^* = 0.9029$. There is no inconsistent information in $\mathbf{T}$.

Using (4.3), $d_j(A^i, \overline{z})$ can be computed to obtain the results listed in Table 4.4.

Then, based on $\mathbf{w}^*$, the distance of each alternative from the centroid $\overline{z}$ can be calculated to produce the findings in Table 4.5.

For normal screening, $R^* = 0.9029$, and hence, the distances of GW_1, GW_2 and PL_1 are less than 0.9029. These three alternatives can be retained and the others should be removed. Assuming that the DM would like to retain more alternatives for further consideration, the screening threshold can be adjusted. For example, if $S = 4$ three more alternatives are retained, as shown in Table 4.6.

Table 4.4: Distances of Alternatives to the Centroid on Each Criterion

Alter-native	Criteria						
	INVEST	OPER	INFRA	ENVIR	RISK	SUPPLY	QUAL
A^1	0.1438	0.0457	0.1316	0.0521	0.0695	0.0023	0.2834
A^2	0.0162	0.0570	0.1316	0.0521	0.0695	0.0023	0.2834
A^3	3.0233	0.0191	0.0146	0.0334	0.0113	0.5225	0.0267
A^4	3.1813	0.5878	0.0146	0.00334	0.0113	1.2284	0.0617
A^5	4.1551	0.8886	0.1316	0.0521	0.2014	8.0831	1.9160
A^6	0.0204	0.0987	0.1316	0.0188	0.1269	3.5319	0.6664
A^7	0.0104	0.1935	0.1316	0.0021	0.2014	3.6782	0.6664
A^8	0.8044	0.7091	0.1316	0.0021	0.2014	5.1294	0.6664
A^9	0.0985	0.1518	0.1316	0.0188	0.2270	0.0023	0.0617
A^{10}	0.1832	0.3490	0.2340	0.0188	0.2270	0.0023	0.0617
A^{11}	2.4000	0.7521	0.1316	0.0188	0.2270	0.0023	0.0013
A^{12}	5.6880	0.6674	0.1316	0.0188	0.2270	0.0023	0.0617

Table 4.5: Distances of Alternatives from the Centroid

Alternative	$D(A^i)$	Alternative	$D(A^i)$
A^1	0.7285	A^2	0.6123
A^3	3.6509	A^4	5.1185
A^5	15.4280	A^6	4.5946
A^7	4.8836	A^8	7.6445
A^9	0.6918	A^{10}	1.0761
A^{11}	3.5331	A^{12}	6.7969

4.7 Conclusions

The following components for the case-based distance method for screening are discussed in this chapter.

- Case-based reasoning: Key in case-based reasoning in MCDA is summarized.
- Model assumptions: The assumptions of the proposed case-based distance

Table 4.6: Threshold Screening for Each Alternative

Alternative	$D(A^i)^*$, S	Status	Alternative	$D(A^i)^*$, S	Status
A^1	$0.7285 < 4$	$\surd$	A^2	$0.6123 < 4$	$\surd$
A^3	$3.6509 < 4$	$\surd$	A^4	$5.1185 > 4$	$\times$
A^5	$15.4280 > 4$	$\times$	A^6	$4.5946 > 4$	$\times$
A^7	$4.8836 > 4$	$\times$	A^8	$7.6445 > 4$	$\times$
A^9	$0.6918 < 4$	$\surd$	A^{10}	$1.0761 < 4$	$\surd$
A^{11}	$3.5331 < 4$	$\surd$	A^{12}	$6.7969 > 4$	$\times$

screening method are explained in detail including the case set and distance assumptions.

- Model construction: A quadratic program is constructed to find the most descriptive information of criterion weights and the screening thresholds.
- Distance-based screening: Distance-based screening is employed to screen alternatives in order to systematically explain how the screening procedure works.
- Numerical example: A numerical example is utilized to demonstrate how the screening method works in practice.

Chapter 5

Sorting Problems in Multiple Criteria Decision Analysis

5.1 Introduction

The case-based distance screening method is extended to solve sorting problems in MCDA, in order to arrange a set of alternatives into a few predefined groups in preference order so that the DM can manage them more efficiently and effectively. Firstly, a general description of sorting problems is presented in detail including a formal definition of a sorting procedure and a discussion of relationship between the alternative set and the sorting group set. Based on weighted Euclidean distance, two case-based distance models are developed for sorting using weights and group thresholds obtained by the assessment of a case set provided by a decision maker. Case-based sorting model I is designed for cardinal criteria; its extension, case-based sorting model II, can handle both cardinal and ordinal criteria. Optimization programs are employed to find the most descriptive weights and group thresholds. Finally, a case study on analyzing Canadian municipal water usage is presented. Earlier versions of the research contained in this chapter are provided by Chen et al. (2005a); Chen et al. (2005d).

5.2 General Description of Sorting Problems

A formal definition of a sorting procedure in MCDA is as follows:

Definition 10. *A sorting procedure is any procedure that always produces a ranking of a set* $\mathbf{A}$*,* $\mathbf{S} = (\mathbf{S}_1, \mathbf{S}_2, ..., \mathbf{S}_g, ..., \mathbf{S}_m)$*, that satisfies the following conditions:*

- $\varnothing \neq \mathbf{S}_g \subseteq \mathbf{A}\ \forall g$,
- $\forall\ g \neq h, \mathbf{S}_g \cap \mathbf{S}_h = \varnothing$,
- $\mathbf{S}_1 \cup \mathbf{S}_2 \cup ... \cup \mathbf{S}_g \cup ... \cup \mathbf{S}_m = \mathbf{A}$,

where $\mathbf{S}$ denotes the sorting, $\mathbf{S}_g$ denotes the g^{th} element (subset of $\mathbf{A}$) in $\mathbf{S}$ for $g = 1, 2, ..., m$. Note that $m \geq 2$ is the number of sorting groups, which is predetermined by the DM.

The preference and indifference relationships induced by the sorting on $\mathbf{A}$ include the following:

- If A^k, $A^l \in \mathbf{S}_g$, then $A^k \sim A^l$ ($\sim$ means the DM "equally prefers" A^k and A^l);
- If $A^k \in \mathbf{S}_g$ and $A^k \sim A^l$, then $A^l \in \mathbf{S}_g$;
- For $A^k \in \mathbf{S}_g$, and $A^l \in \mathbf{S}_h$, $A^k \succ A^l$ if $1 \leq g < h \leq m$ ($\succ$ means the DM "prefers to");

Thus, we say that if $g < h$, the DM prefers any alternative in $\mathbf{S}_g$ to any one in $\mathbf{S}_h$.

Figure 5.1 shows the relations between $\mathbf{A}$ and $\mathbf{S}$. In the figure n alternatives are sorted by the DM into m groups which are arranged from the most preferred group $\mathbf{S}_1$ to the least preferred group $\mathbf{S}_m$. Alternatives in the same group are equally preferred, for example $A^4, A^7 \in \mathbf{S}_2$, so $A^4 \sim A^7$; $A^2 \in \mathbf{S}_1$ and $A^{12} \in \mathbf{S}_2$, then $A^2 \succ A^{12}$.

5.3 Case-based Distance Sorting Model I: Cardinal Criteria

5.3.1 Case Set Assumptions

A sorting problem is to sort the alternatives in $\mathbf{A}$ ($|\mathbf{A}| = n$) into a ranked partition $\mathbf{S}$ ($|\mathbf{S}| = m$), based on criteria set $\mathbf{Q}$ ($|\mathbf{Q}| = q$). The goal of this research is to utilize the DM's views of the alternatives to create a good sorting and to do so as efficiently as possible. Hence, we present the DM with a case set of alternatives $\mathbf{T}$ ($|\mathbf{T}| = t$).

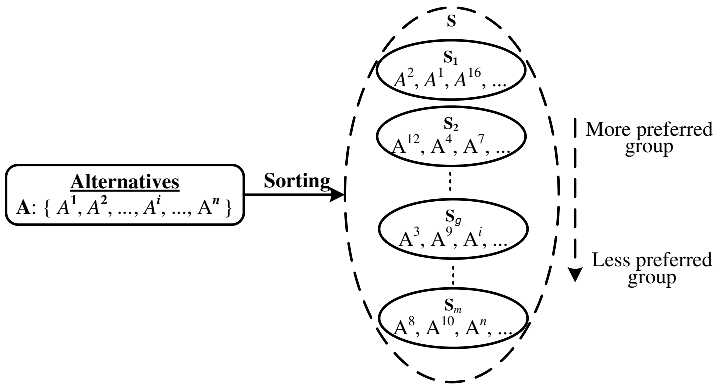


Figure 5.1: Relationships Among **A** and **S**

The alternatives in $\mathbf{T}$ may, for example, be fabricated by the DM or obtained by having the DM modify historical records. However, all criteria in $\mathbf{Q}$ must apply and $c_j(A^i)$ must be measurable for all $A^i \in \mathbf{T}$ and all $j \in \mathbf{Q}$. Suppose that the DM can specify m non-empty subsets of $\mathbf{T}$ so that $\mathbf{T}_1 \succ \mathbf{T}_2 \succ ... \succ \mathbf{T}_g \succ ... \succ \mathbf{T}_m$, $\mathbf{T}_1 \cup \mathbf{T}_2 \cup ... \cup \mathbf{T}_g \cup ... \cup \mathbf{T}_m = \mathbf{T}$ and $\forall\ g \neq h$, $\mathbf{T}_g \cap \mathbf{T}_h = \varnothing$. To say that $\mathbf{T}_g \succ \mathbf{T}_h$ is to say that the DM consider any alternative of $\mathbf{T}_g$ to have higher priority than any alternative of $\mathbf{T}_h$.

Thus $\mathbf{T}_1 = \{z_1^1, z_1^2, ..., z_1^r, ..., z_1^{t_1}\}$ is the most preferred group and $\mathbf{T}_g = \{z_g^1, z_g^2, ..., z_g^r, ..., z_g^{t_g}\}$ is the gth most preferred group. For $g = 1, 2, ..., m$, the number of alternatives in $\mathbf{T}_g$ is denoted $t_g > 0$, so $t_1 + t_2 + ... + t_g + ... + t_m = t$.

Our case-based reasoning idea is that based on the "right" distance concept, the cases in $\mathbf{T}_g$ should be approximately equally far from an "ideal" alternative, and closer to it than the alternatives in $\mathbf{T}_h$, when $h > g$. We use $\mathbf{T}_g$, $g = 1, 2, ..., m$ to estimate criterion weight $\mathbf{w}$, and a basic distance threshold vector $\mathbf{R} = (R_1, R_2, ..., R_g, ..., R_{m-1}) \in \mathbb{R}^{m-1}$, so that the distance of $z_g^r \in \mathbf{T}_g$ ($g \neq 1$ or m) from a central point "o" (the ideal alternative) is less than or equal to R_g, and greater than or equal to R_{g-1}. For $g = 1$, the distance of $z_1^r \in \mathbf{T}_1$ is less than or equal to R_1; for $g = m$, the distance of $z_m^r \in \mathbf{T}_m$ is greater than or equal to R_{m-1}. Then we can apply $\mathbf{w}$ and $\mathbf{R}$ to sort alternatives in $\mathbf{A}$ and thereby obtain $\mathbf{S}$.

This idea is illustrated in Figure 5.2 for a four-group sorting problem. A case set $\mathbf{T}$ is partitioned by the DM into four sub-case sets $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$ and $\mathbf{T}_4$. Three ellipses partition the alternatives into these four sub-case sets and represent preference sequences: ellipses closer to "o" represent more preferred groups. Then by a properly designed transformation from the original consequence data space to a weighted normalized consequence data space, ellipses can be transformed to circles, permitting distance information to be applied to sort all alternatives in **A**.

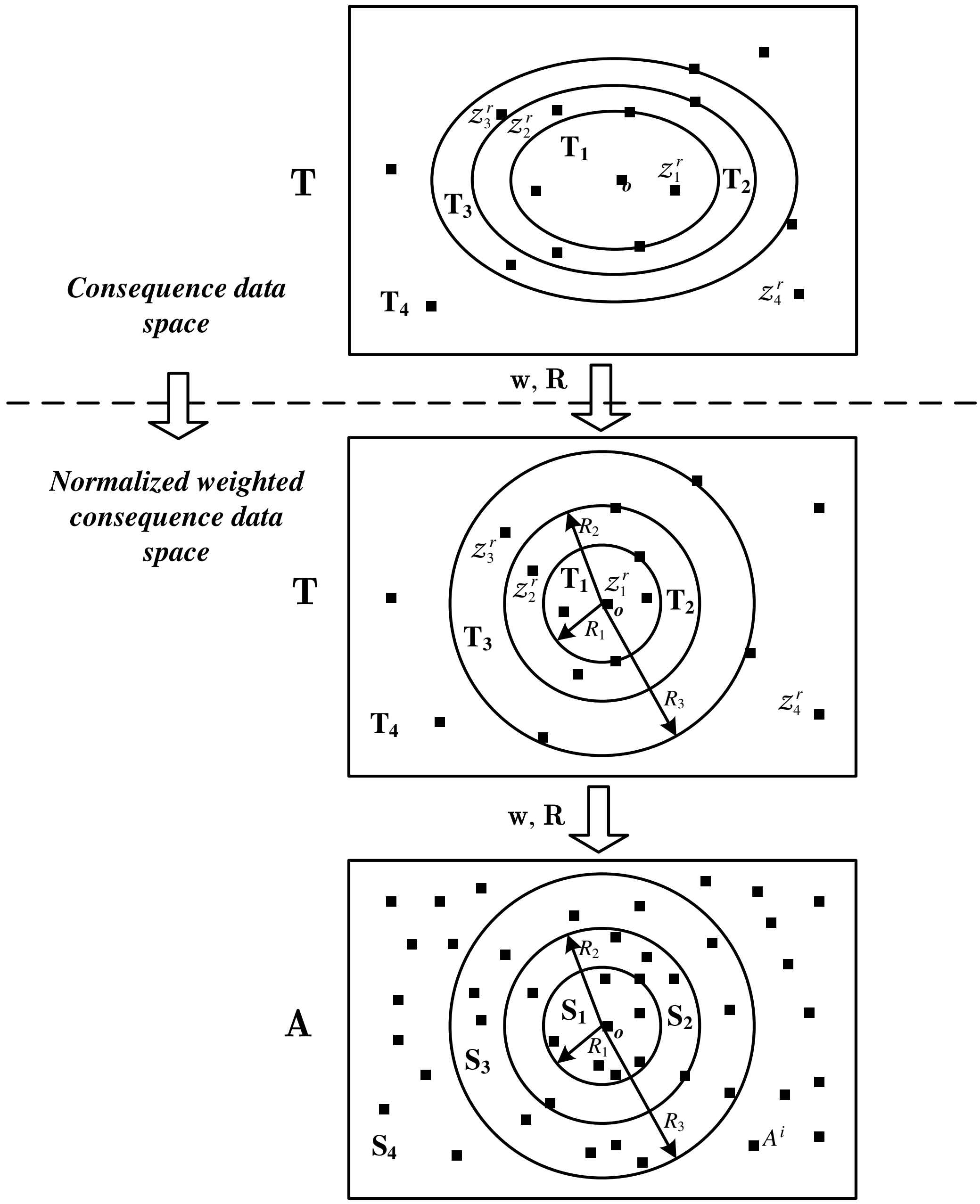


Figure 5.2: The Idea of Case-based Sorting

5.3.2 Distance Assumptions

Given the most preferred alternatives in the case set $\mathbf{T}_1$, the centroid of $\mathbf{T}_1$, $\overline{z}_+$, is chosen to be the fictitious "ideal" alternative, at the center of $\mathbf{T}_1$. By definition,

$$c_j(\overline{z}_+) = \frac{1}{t_1}\sum_{r=1}^{t_1} c_j(z_1^r). \tag{5.1}$$

Similarly, for the case set $\mathbf{T}_m$, a fictitious alternative, $\overline{z}_-$, the **centroid** of $\mathbf{T}_m$, is deemed to be a fictitious alternative at the center of $\mathbf{T}_m$.

$$c_j(\overline{z}_-) = \frac{1}{t_m}\sum_{r=1}^{t_m} c_j(z_m^r). \tag{5.2}$$

Note that an "ideal" point can also be based on $\mathbf{T}_2$, $\mathbf{T}_3$, ..., $\mathbf{T}_{m-1}$, but we use only the two extremes.

For $j = 1, 2, ..., q$, define $d_j^{\max +} = \max\limits_{g=1}^{m}\max\limits_{r=1}^{t_g}\left(c_j(z_g^r) - c_j(\overline{z}_+)\right)^2$ to be the normalization factor for criterion j. For $g = 1, 2, .., m$ and $r = 1, 2, ..., t_g$, the distance between $z_g^r \in \mathbf{T}_g$ and the centroid $\overline{z}_+$ on criterion j is

$$d_j(z_g^r, \overline{z}_+) = \frac{\left(c_j(z_g^r) - c_j(\overline{z}_+)\right)^2}{d_j^{\max +}}. \tag{5.3}$$

Similarly, the distance between alternative $A^i \in \mathbf{A}$ and the centroid $\overline{z}_+$ on criterion j is

$$d_j(A^i, \overline{z}_+) = \frac{\left(c_j^i - c_j(\overline{z}_+)\right)^2}{d_j^{\max +}}. \tag{5.4}$$

Note that $0 \leq d_j(z_g^r, \overline{z}_+) \leq 1$. Also $0 < d_j(A^i, \overline{z}_+)$, but $d_j(A^i, \overline{z}_+)$ could be larger than 1.

Weighted Euclidean distance is employed since it has a clear geometric meaning, which can be easily understood and accepted by the DM to represent his or her aggregated preference. The relative order of non-negative numbers (distances) is the same as the relative order of their squares. For this reason, instead of Euclidean distances we use their squares, because they are easier to compute while preserving order. The aggregated distance between alternatives $z_g^r \in \mathbf{T}_g$ and $\overline{z}_+$ over the criteria set $\mathbf{Q}$ is identified as

$$D(z_g^r)_+ = D(z_g^r, \overline{z}_+) = \sum_{j\in\mathbf{Q}} w_j \cdot d_j(z_g^r, \overline{z}_+),$$

where $w_j \in \mathbf{w}$ is the weight (relative importance) of criterion j. The weight vector $\mathbf{w}$ is to be determined subject to $0 < w_j \leq 1$ and $\sum_{j \in \mathbf{Q}} w_j = 1$. Similarly, the aggregate distance from alternative A^i to $\overline{z}_+$ is

$$D(A^i)_+ = D(A^i, \overline{z}_+) = \sum_{j \in \mathbf{Q}} w_j \cdot d_j(A^i, \overline{z}_+). \tag{5.5}$$

According to the above definition, the distance of an alternative from the centroid $\overline{z}_+$ is used to identify its group membership. A similar idea has been widely used in cluster analysis: cluster membership of a datum is determined by evaluation of a pre-defined distance between this datum and the cluster centroid. If the DM can in fact specify an ideal alternative (the most preferred alternative), z^*_+ within $\mathbf{T}_1$, then z^*_+ can be used instead of $\overline{z}_+$ to measure the distances and generate $\mathbf{w}$ and $\mathbf{R}$. The modified procedure can be used for sorting exactly as outlined below; the details are omitted.

Taking $\overline{z}_+$ as the centroid, a ball (in q dimensions) with radius R_g includes (in principle) every case $z^r_h \in \mathbf{T}_h$ for $h \leq g$ while any case $z^r_h \in \mathbf{T}_h$ for $h > g$ is in principle outside the ball. Therefore, if $R_{g-1} \leq D(A^i)_+ \leq R_g$, A^i can be regarded as having equal preference to $z^r_g \in \mathbf{T}_g$, and sorted into $\mathbf{S}_g$. For example, in Figure 5.3, A^i is sorted into $\mathbf{S}_3$, because $R_2 < D(A^i)_+ < R_3$.

Alternatively, it may be appropriate to have distance concepts between cases on $\overline{z}_-$ to sort alternatives. Then the closer A^i to $\overline{z}_-$, the less preferred for the DM. Therefore, greater distance of A^i indicates greater preference. Since the procedure is similar, the details are omitted. The $\overline{z}_+$ based model construction is explained in detail next.

5.3.3 Model Construction

As explained above, w_j refers to the DM's preference on criterion j, for $j \in \mathbf{Q}$, and represents the relative importance of criterion j within the aggregated distance. $\mathbf{R}$ represents thresholds to sort alternatives into different groups. Here, we obtain $\mathbf{w} = (w_1, w_2, ..., w_q)$ and $\mathbf{R}$ by a case-based reasoning model based on $\mathbf{T}$.

Each alternative in the case set $\mathbf{T}_g$ is evaluated by the DM belonging to the g^{th} group for a sorting problem, so based on the preference relationships as described above, these cases are more preferred than other cases in $\mathbf{T}_k$ when $g < k$, and less preferred than other cases in $\mathbf{T}_h$, when $h < g$. Therefore, based on distance

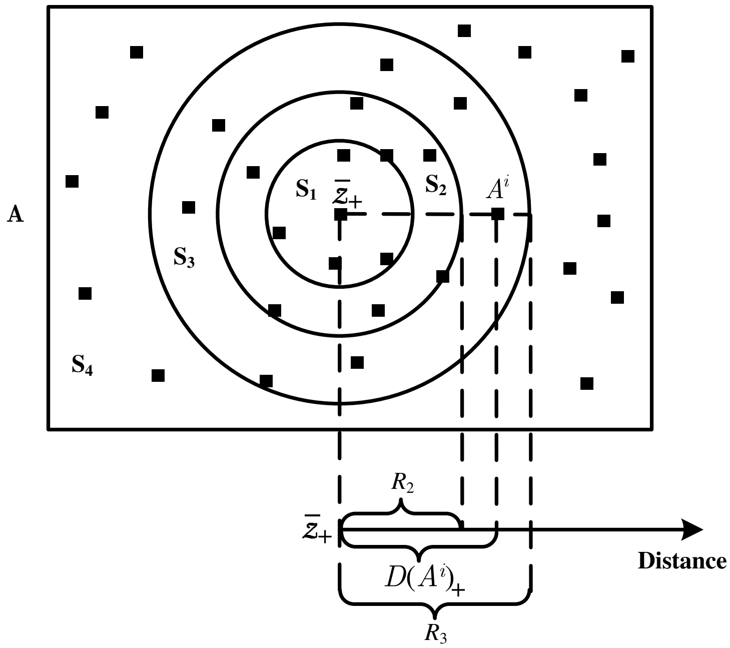


Figure 5.3: The Distances $\mathbf{R}$ and $D(A^i)_+$

measurement from $\overline{z}_+$, it is assumed that (provided there are no inconsistent judgements) (1) for $2 \leq g \leq m-1$, the distance of $z_g^r \in \mathbf{T}_g$ to $\overline{z}_+$ is larger than R_{g-1} and less than R_g; (2) the distance of $z_1^r \in \mathbf{T}_1$ to $\overline{z}_+$ is less than R_1; (3) the distance of $z_m^r \in \mathbf{T}_m$ to $\overline{z}_+$ is larger than R_{m-1}. Thus, for $g = 1, 2, ..., m-1$,

$$D(z_g^r)_+ + \alpha_g^r \leq R_g, \text{or} \sum_{j \in \mathbf{Q}} w_j \cdot d_j(z_g^r, \overline{z}_+) + \alpha_g^r \leq R_g,$$

where $z_g^r \in \mathbf{T}_g$ and $-1 \leq \alpha_g^r \leq 0$ is an upper-bound error adjustment parameter (so that the distance of z_g^r from $\overline{z}_+$ is less than the distance of any case in $\mathbf{T}_k$ for $k > g$), accounting for the DM's inconsistent judgement on z_g^r. Also, for $g = 2, 3, ..., m$,

$$D(z_g^r)_+ + \beta_g^r \geq R_{g-1}, \text{or} \sum_{j \in \mathbf{Q}} w_j \cdot d_j(z_g^r, \overline{z}_+) + \beta_g^r \geq R_{g-1},$$

where $z_g^r \in \mathbf{T}_g$ and $0 \leq \beta_g^r \leq 1$ is a lower-bound error adjustment parameter (so that the distance of z_g^r from $\overline{z}_+$ is larger than the distance of any case in $\mathbf{T}_h$ for $h < g$), accounting for the DM's inconsistent judgements on z_g^r.

Accordingly, the overall squared error in $\mathbf{T}$ is $ERR = \sum_{g=1}^{m-1} \sum_{r=1}^{t_g} \left(\alpha_g^r\right)^2 + \sum_{g=2}^{m} \sum_{r=1}^{t_g} (\beta_g^r)^2$.

Based on $\overline{z}_+$, the following optimization model can be used to find the most descriptive weight vector **w** and the distance threshold vector **R**.

$$\textbf{SOR.1}(\alpha,\beta) \qquad \textbf{Minimize}: ERR = \sum_{g=1}^{m-1}\sum_{r=1}^{t_g}\left(\alpha_g^r\right)^2 + \sum_{g=2}^{m}\sum_{r=1}^{t_g}(\beta_g^r)^2$$

Subject to:

$$\sum_{j\in\mathbf{Q}} w_j \cdot d_j(z_g^r, \overline{z}_+) + \alpha_g^r \le R_g,$$

$$r = 1, 2, ..., t_g,\ g = 1, 2, ..., m-1;$$

$$\sum_{j\in\mathbf{Q}} w_j \cdot d_j(z_g^r, \overline{z}_+) + \beta_g^r \ge R_{g-1},$$

$$r = 1, 2, ..., t_g,\ g = 2, 3, ..., m;$$

$$R_{g-1} < R_g,\ g = 2, ..., m-1;$$

$$-1 \le \alpha_g^r \le 0,\ g = 1, ..., m-1;$$

$$0 \le \beta_g^r \le 1,\ g = 2, ..., m;$$

$$w_j > 0;$$

$$\sum_{j\in\mathbf{Q}} w_j = 1.$$

The optimal solutions of $\textbf{SOR.1}(\alpha,\beta)$ are denoted by $\mathbf{w}^*$ and $\mathbf{R}^*$, respectively, and ERR^* is the minimal value of ERR.

Theorem 5. $\textbf{SOR.1}(\alpha,\beta)$ *has at least one optimal solution.*

Proof: The constraints in $\textbf{SOR.1}(\alpha,\beta)$ constitute a convex set. The objective function $ERR = \sum_{g=1}^{m-1}\sum_{r=1}^{t_g}\left(\alpha_g^r\right)^2 + \sum_{g=2}^{m}\sum_{r=1}^{t_g}(\beta_g^r)^2$ is a quadratic function on this set. Based on the extreme value theorem of advanced calculus (Fitzpatrick, 1995, page 297), the function ERR is continuous and the set of all possible variables is bounded, ERR attains its minimum at least once. □

Two threshold parameters can be set by the DM to evaluate the errors generated by $\textbf{SOR.1}(\alpha,\beta)$ as follows:

(1) The acceptable distance error, ΔD: ΔD is the threshold below which the error generated by $\textbf{SOR.1}(\alpha,\beta)$ is acceptable to the DM. If $ERR^* \le \Delta D$, the errors are acceptable; otherwise **T** should be reset. A suggested value of ΔD is $\frac{1}{10t}$, where $t = |\mathbf{T}|$.

(2) The indifference threshold (see Vincke (1992) for detailed discussion of indifference threshold in MCDA) ε: ε is the threshold below which errors generated by $\mathbf{SOR.1}(\alpha, \beta)$ can be ignored. When $ERR^* \leq \varepsilon$, the case set $\mathbf{T}$ holds consistent information; when $ERR^* \geq \varepsilon$, the errors cannot be ignored and there is some inconsistency in $\mathbf{T}$. A suggested value of ε is $\frac{1}{10n}$, where $n = |\mathbf{A}|$. Usually $\frac{1}{10n}$ is small enough to affect (but not strongly) the balance between the indifference threshold and acceptable error. Of course, in practice the DM should increase or decrease ε as desired.

5.3.4 Distance-based Sorting

Assuming $ERR^* \leq \Delta D$, with $\mathbf{w}^* = (w_1^*, w_2^*, ..., w_q^*)$ obtained from $\mathbf{SOR.1}(\alpha, \beta)$, the maximum upper-bound error in $\mathbf{T}_g$ is denoted as $\alpha_g^{\max} = \max_{r=1,2,...,t_g} \{|\alpha_g^r|\}$ and the maximum lower-bound error in $\mathbf{T}_g$ is denoted as $\beta_g^{\max} = \max_{r=1,2,...,t_g} \{\beta_g^r\}$. Different policies to set the sorting distance thresholds based on $\mathbf{R}$ can be employed to carry out sorting.

(1) *Normal Sorting*: When $ERR^* \leq \varepsilon$, the errors can be ignored and $\mathbf{R}^* = (R_1^*, R_2^*, ..., R_g^*, ..., R_{m-1}^*)$ is utilized to sort alternatives.

- If $D(A^i)_+ \leq R_1^*$, $A^i \in \mathbf{S}_1$;
- If $R_1^* < D(A^i)_+ \leq R_2^*$, $A^i \in \mathbf{S}_2$;

  ......
- If $R_{g-1}^* < D(A^i)_+ \leq R_g^*$, $A^i \in \mathbf{S}_g$;

  ......
- If $D(A^i)_+ > R_{m-1}^*$, $A^i \in \mathbf{S}_m$.

(2) *Conservative Sorting*: For $ERR^* \geq \varepsilon$, $\alpha_g^{\max}$ is taken into account as **error permission** in $\mathbf{T}$. The distance thresholds are defined as $(R_1^* + \alpha_1^{\max}, R_2^* + \alpha_2^{\max}, ..., R_g^* + \alpha_g^{\max}, ..., R_{m-1}^* + \alpha_{m-1}^{\max})$ and used to sort alternatives.

- If $D(A^i)_+ \leq R_1^* + \alpha_1^{\max}$, $A^i \in \mathbf{S}_1$;
- If $R_1^* + \alpha_1^{\max} < D(A^i)_+ \leq R_2^* + \alpha_2^{\max}$, $A^i \in \mathbf{S}_2$;

  ......
- If $R_{g-1}^* + \alpha_{g-1}^{\max} < D(A^i)_+ \leq R_g^* + \alpha_m^{\max}$, $A^i \in \mathbf{S}_g$;

  ......

- If $D(A^i)_+ > R^*_{m-1} + \alpha^{\max}_{m-1}$, $A^i \in \mathbf{S}_m$.

(3) *Aggressive Sorting*: For $ERR^* \geq \varepsilon$, $\beta^{\max}_g$ is taken into account as **error reduction** in $\mathbf{T}$. The distance thresholds are defined as $(R^*_1 - \beta^{\max}_1, R^*_2 - \beta^{\max}_2, ..., R^*_g - \beta^{\max}_g, ..., R^*_{m-1} - \beta^{\max}_{m-1})$ and employed to sort alternatives (We assume $R^*_g - \beta^{\max}_g > 0$ for $g = 1, 2, ..., m$).

- If $D(A^i)_+ \leq R^*_1 - \beta^{\max}_1$, $A^i \in \mathbf{S}_1$;
- If $R^*_1 - \beta^{\max}_1 < D(A^i)_+ \leq R^*_2 - \beta^{\max}_2$, $A^i \in \mathbf{S}_2$;
  ......
- If $R^*_{g-1} - \beta^{\max}_{g-1} < D(A^i)_+ \leq R^*_g - \beta^{\max}_m$, $A^i \in \mathbf{S}_g$;
  ......
- If $D(A^i)_+ > R^*_{m-1} - \beta^{\max}_{m-1}$, $A^i \in \mathbf{S}_m$.

Note that based on the setting of the acceptable distance error, $\triangle D$, $\alpha^{\max}_g$ and $\beta^{\max}_g$ are much smaller than any distance threshold, R^*_g. Hence, there is no reversing the ordering of classes.

5.3.5 Sorting Consistency

Based on the $\overline{z}_+$ sorting method, the procedures explained above can be implemented to obtain $\mathbf{w}^*$ and $\mathbf{R}^*$ and accordingly to generate a sorting $\mathbf{S}^+ = (\mathbf{S}^+_1, \mathbf{S}^+_2, ..., \mathbf{S}^+_g, ..., \mathbf{S}^+_m)$ for $\mathbf{A}$. Similar procedures can be carried out based on the $\overline{z}_-$ sorting method to obtain another sorting for $\mathbf{A}$ denoted as $\mathbf{S}^- = (\mathbf{S}^-_1, \mathbf{S}^-_2, ..., \mathbf{S}^-_g, ..., \mathbf{S}^-_m)$. The following method is proposed to compare the $\overline{z}_+$ and $\overline{z}_-$ sortings and provide the DM with more information to evaluate the sorting results.

The proportion of alternatives in $\mathbf{S}^+_g$ to alternatives in $\mathbf{A}$ is $\rho^+_g = \frac{1}{n}|\mathbf{S}^+_g|$ for $g = 1, 2, ..., m$, where n is the number of alternatives in $\mathbf{A}$ and . Similarly, $\rho^-_g = \frac{1}{n}|\mathbf{S}^-_g|$ is the proportion of alternatives in $\mathbf{S}^-_g$ to alternatives in $\mathbf{A}$.

Let Φ_+ represent the vector constituting of ρ^+_1, ρ^+_2, ..., ρ^+_m, sorted into decreasing order and similarly for Φ_-. When $\Phi_+ = \Phi_-$, we say that the case set $\mathbf{T}$ gives a **roughly consistent sorting** beginning with both the most preferred group $\mathbf{T}_1$ and the least preferred group $\mathbf{T}_m$. Let $\mathbf{C}_g$ denote the core alternatives for sorting group g, such that $\mathbf{C}_g = \{\mathbf{S}^+_g \cap \mathbf{S}^-_g\}$ for $g = 1, 2, ..., m$. The degree of consistency for $\mathbf{T}$ is defined as $\rho_{\mathbf{T}} = \frac{1}{n}\sum^m_{g=1}|\mathbf{C}_g|$.

5.4 Case-based Sorting Model II: Ordinal and Cardinal Criteria

5.4.1 Ordinal Criteria in MCDA

MCDA problems may involve the evaluation of ordinal criteria, such as customer satisfaction or priority. Consequence data cannot be naturally connected to ordinal criteria. Many researchers have studied how to handle ordinal criteria in MCDA, and some approaches are summarized next.

Many methods have been proposed to acquire directly a reasonable value datum (v_j^i) without measurement and transformation from a consequence (c_j^i). In the context of linear additive value (utility) functions, as in (2.5), Sage and White (1984) propose an imprecisely specified multiattribute utility theory (ISMAUT) model to handle ordinal criteria. Malakooti (2000) suggests a method to rank and screen alternatives based on partial preference information. Ahn (2003) extends Malakooti's work to include preference strength and partial information.

Several fuzzy set theory methods have been proposed to handle ordinal criteria, for example, Chen (1996); Herrera-Viedma et al. (2002). Most of them share a similar idea; instead of directly expressing preferences on ordinal criteria for each alternative, the DM asks experts to evaluate alternatives on each criterion using a pre-defined linguistic grade set, such as poor, neutral, good, very good. Then the DM assigns these ordinal statements fuzzy membership functions and employs aggregation models to obtain final conclusions. Similarly, based on the Dempster - Shafer theory of evidence, Yang and Xu (2002) develop an evidential reasoning approach to handle qualitative (ordinal) criteria.

Instead of the methods described above, which may involve complicated mathematical computations, one simple but efficient approach to deal with ordinal criteria is to ask a DM to consider whether there is cardinal information connected to ordinal criteria in an MCDA problem. By careful thinking, the DM may find proxy cardinal criteria representing ordinal criteria, permitting ordinal criteria to be transformed to relevant cardinal criteria. The advantages of such methods are that the DM's objectives for a decision problem are clarified and the DM may feel more confident of the final conclusions. Keeney (1992) proposes a systematic analysis method, value-focused thinking, which is an analytic approach to identify and define criteria properly. For example, Keeney recounts how it is difficult to find a direct criterion to represent the damage to historic buildings from acid rain, but a useful proxy criterion could be sulphur dioxide concentration in rain water.

Another application of value-focused thinking in cell phone plan design is given in Keeney (2004).

5.4.2 Ordinal Criteria Expressions

Sometimes the DM may not easily find suitable proxy cardinal criteria representing ordinal criteria for an MCDA problem. The following method is developed to incorporate ordinal criteria into our proposed case-based distance sorting model.

Recall that $\mathbf{Q}$ is the set of criteria. When ordinal criteria are involved, assume that $\mathbf{Q}$ can be partitioned into $\mathbf{Q}_{CRD}$ and $\mathbf{Q}_{ORD}$, which represent the cardinal and ordinal criteria sets, respectively. Obviously, $\mathbf{Q} = \mathbf{Q}_{CRD} \cup \mathbf{Q}_{ORD}$, and $\mathbf{Q}_{CRD} \cap \mathbf{Q}_{ORD} = \varnothing$.

For each ordinal criterion $j \in \mathbf{Q}_{ORD}$, we arrange a linguistic grade set, such as very good, good, fair, neutral, poor to directly evaluate the alternatives. The linguistic grade set is $\mathbf{L} = (1, 2, ..., l, ..., L)$, where 1 is the best linguistic evaluation and L is the worst. Then a value interval, v_j^i, can be assigned to A^i to represent its linguistic evaluation on each ordinal criterion. For example, with the $\overline{z}_+$ sorting method, $A^i \in \mathbf{A}$, if A^i is evaluated belonging to the best grade, then $v_j^i \in [0, \frac{1}{L}]$. The detailed relationships between linguistic evaluation grade and value interval for the $\overline{z}_+$ sorting method are listed in Table 5.1. For $\overline{z}_-$ sorting method, linguistic evaluation grade and value interval have the opposite relationship. Note that without loss of generality, we assume $v_j^i \in [\frac{l_i}{L}, \frac{l_i+1}{L}]$. When v_j^i is applied to $\mathbf{T}$, $v_j(z_g^r) \in [\frac{l_r}{L}, \frac{l_r+1}{L}]$ is used to represent the value interval of z_g^r on the ordinal criterion $j \in \mathbf{Q}_{ORD}$.

Table 5.1: Linguistic Evaluation Grade and Value Interval for $\overline{z}_+$ Sorting Method

linguistic evaluations	Value intervals
1	$[0, \frac{1}{L}]$
2	$[\frac{1}{L}, \frac{2}{L}]$
3	$[\frac{2}{L}, \frac{3}{L}]$
$\cdots$	$[\cdots, \cdots]$
l	$[\frac{l}{L}, \frac{l+1}{L}]$
$\cdots$	$[\cdots, \cdots]$
L	$[\frac{L-1}{L}, 1]$

5.4.3 Case-based Sorting Model Incorporating Ordinal Criteria

Assuming $\mathbf{Q} = \mathbf{Q}_{CRD} \cup \mathbf{Q}_{ORD}$, with the $\overline{z}_+$ sorting method, the cardinal criteria set, $\mathbf{Q}_{CRD}$ part of $\overline{z}_+$ can be calculated using (5.1). For the ordinal criteria set, $\mathbf{Q}_{ORD}$, there is no consequence centroid, but the best linguistic grade can be regarded as an ideal point and used to measure the distance. Therefore, we assign $d_j(z_g^r, \overline{z}_+) = (v_j(z_g^r))^2$ for $j \in \mathbf{Q}_{ORD}$. For $\mathbf{Q}_{CRD}$, Equation (4.3) is still applicable to obtain $d_j(z_g^r, \overline{z}_+)$. The aggregated distance of z_g^r under both cardinal and ordinal criteria is defined as

$$D(z_g^r)_+ = \sum_{j \in \mathbf{Q}_{CRD}} w_j \cdot d_j(z_g^r, \overline{z}_+) + \sum_{j \in \mathbf{Q}_{ORD}} w_j \cdot (v_j(z_g^r))^2. \tag{5.6}$$

Note that (5.6) can be applied to $\mathbf{A}$.

Similarly, the following optimization model can be used to find the most descriptive weight vector $\mathbf{w}$ and the distance threshold vector $\mathbf{R}$ for $\overline{z}_+$ sorting incorporated ordinal criteria.

$\mathbf{SOR.2}(\alpha, \beta)$ **Minimize** : $ERR = \sum_{g=1}^{m-1} \sum_{r=1}^{t_g} (\alpha_g^r)^2 + \sum_{g=2}^{m} \sum_{r=1}^{t_g} (\beta_g^r)^2$

Subject to:

$$\sum_{j \in \mathbf{Q}_{CRD}} w_j \cdot d_j(z_g^r, \overline{z}_+) + \sum_{j \in \mathbf{Q}_{ORD}} w_j \cdot (v_j(z_g^r))^2 + \alpha_g^r \leq R_g,$$

$$r = 1, 2, ..., t_g,\ g = 1, 2, ..., m-1;$$

$$\sum_{j \in \mathbf{Q}_{CRD}} w_j \cdot d_j(z_g^r, \overline{z}_+) + \sum_{j \in \mathbf{Q}_{ORD}} w_j \cdot (v_j(z_g^r))^2 + \beta_g^r \geq R_{g-1},$$

$$r = 1, 2, ..., t_g,\ g = 2, 3, ..., m;$$

$$R_{g-1} < R_g,\ g = 2, ..., m-1;$$

$$-1 \leq \alpha_g^r \leq 0,\ g = 1, ..., m-1;$$

$$0 \leq \beta_g^r \leq 1,\ g = 2, ..., m;$$

$$w_j > 0;\ w_j > 0;$$

$$\sum_{j \in \mathbf{Q}_{CRD}} w_j + \sum_{j \in \mathbf{Q}_{ORD}} w_j = 1;$$

$$v_j(z_g^r) \in [\tfrac{l_r}{L}, \tfrac{l_r+1}{L}],\ r = 1, 2, ..., t_g,\ g = 1, 2, 3, ..., m.$$

Similarly $\mathbf{SOR.2}(\alpha, \beta)$ generates at least one optimal solution $\mathbf{w}^*$ and $\mathbf{R}^*$. The suggested value of the indifference threshold, ε, and the acceptable distance error, ΔD remain the same as the ones described in Section 5.3.

5.4.4 Distance-based Sorting

In Table 5.1, the value of A^i on ordinal criterion $j \in \mathbf{Q}_{ORD}$, v_j^i is assigned an interval data, $[\frac{l_i}{L}, \frac{l_i+1}{L}]$. The following methods are designed to estimate a discrete value for v_j^i:

- The aggressive estimation when v_j^i is estimated using $\frac{l_i}{L}$;
- The normal estimation when v_j^i is estimated using $\frac{2l_i+1}{2L}$;
- The conservative estimation when v_j^i is estimated using $\frac{l_i+1}{L}$.

Then, based on policies described in Section 5.3, normal, conservative or aggressive sorting can be taken to carry out sorting depending on whether ERR^* can be ignored. Also $\overline{z}_-$ sorting incorporated ordinal criteria can be designed similarly. The details are omitted.

5.4.5 Analysis Procedure for Case-based Distance Sorting

To summarize all the contents addressed above, a systematic procedure to analyze multiple criteria sorting problems based on the proposed case-based distance models is shown in Figure 5.4. It includes the following steps:

- *Identify the alternative set* **A**: All possible alternatives within appropriate boundaries should be considered.
- *Construct the criteria set* **Q**: Build a criteria set **Q** to reflect the DM's concerns and objectives.
- *Check whether all criteria are cardinal*: For each criterion, the DM must identify whether it can be connected to cardinal consequence information.
- *Sort alternatives using Case-based Sorting Model I*: Case-based Sorting Model I described in Section 5.3 is applied to sort alternatives when all criteria in **Q** are cardinal.
- *Sort alternatives using Case-based Sorting Model II*: Case-based Sorting Model II described in Section 5.4 is used to sort alternatives when both cardinal and ordinal criteria are contained in **Q**.
- *Check sorting consistency*: Check the sorting consistency when both $\overline{z}_+$ and $\overline{z}_-$ sorting methods are carried out for sorting alternatives.

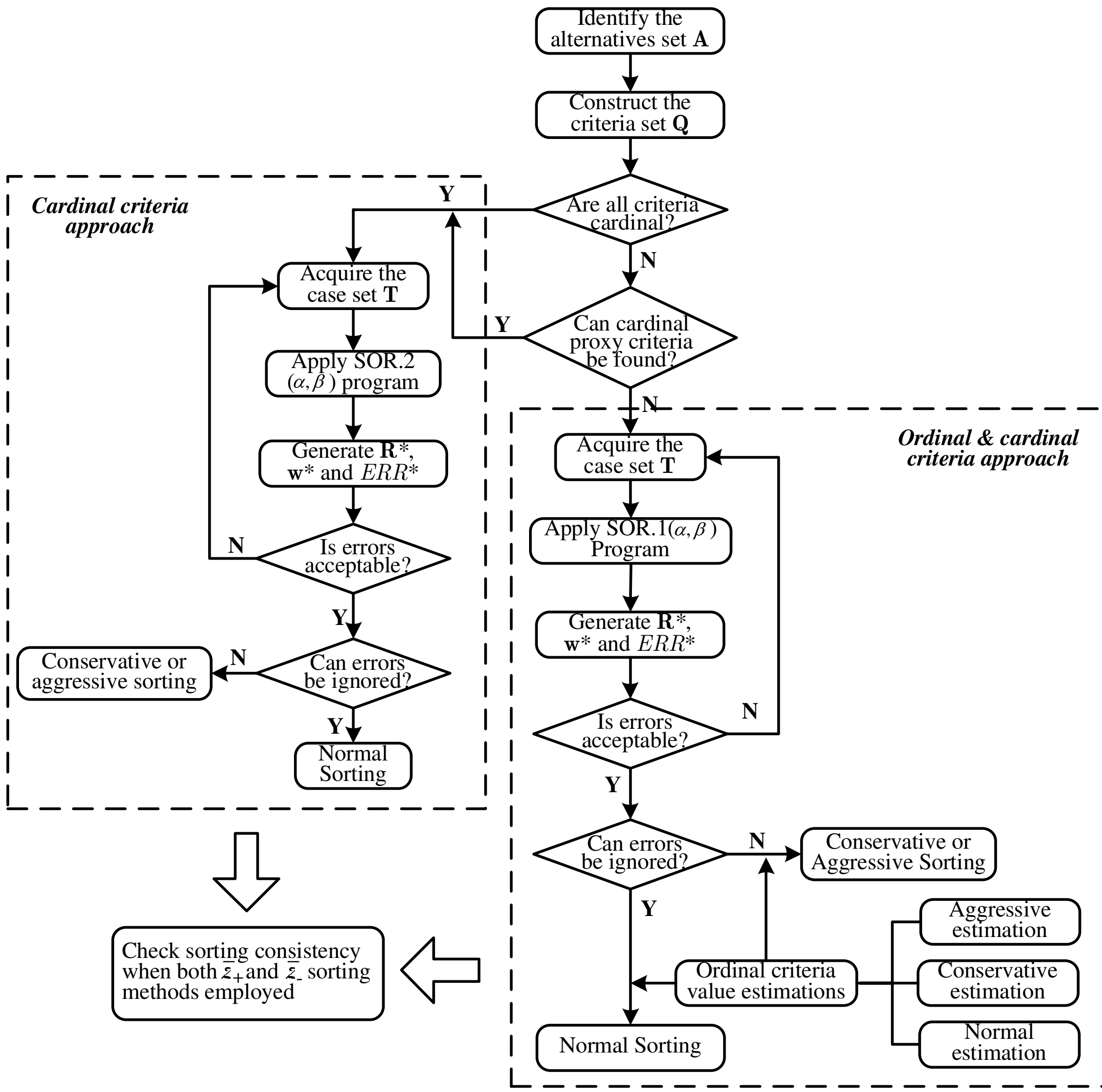


Figure 5.4: Analysis Procedure for Case-based Distance Sorting

5.5 Case Study: Canadian Municipal Water Usage Analysis

MCDA has been widely employed to solve water resources management problems. For example, Haimes et al. (1975) discussed multiple objective optimization techniques in water resources systems; Hipel (1992) edited a special issue on multiple objective decision making in water resources. Bobbs and Meierm (2000) reviewed the literature on MCDA methods and presented applications of MCDA to energy and environment decisions in North America. In terms of case-based approaches to water resources management, Greco et al. (1999) used rough set theory to solve a problem of programming water supply systems for use in a rural area; neural network techniques have been utilized to solve multiple objective optimization problems in water resources management - for example, Wen and Lee (1998) proposed a neural network-based multiple objective optimization approach to address water quality management for water pollution control and river basin planning. They suggested that case-based methods could overcome the difficulty of acquiring DM's preferences and provide direct help to analysts in real applications. The application of our proposed method to Canadian municipal water usage analysis is explained next.

5.5.1 Background

Water problems are mounting worldwide, and even water-rich countries such as Canada are not exempt. Increasing per capita consumption, population growth, urbanization, and pollution all put increasing pressure on this finite resource and result in both regional and global scarcities. With less than one percent of the world's population, Canada possesses almost 20 percent of global freshwater, so it seems to be in an enviable position. However, much of Canada's fresh water is found in distant glaciers, ice caps and remote water bodies, and is not easily accessible to a population concentrated in a relatively small number of urban areas. Therefore, Canadian cities, like many around the world, are beginning to confront the strains that growing water use places on the environment and on the financial capacity of local governments (Brandes and Ferguson, 2003).

The average Canadian used a total of 4,400 liters water per day in 1999 which took into account all uses of water: agriculture, manufacturing, mining and municipal uses. Municipal water use including residential and commercial uses is about 12% of all water withdrawals, the third among water use sectors in Canada. As a critical component of urban life, municipal water uses are diverse: drinking, cooking, bathing and sewerage. Residential use is the most significant part of municipal water use, representing around 52% of the total volume (OECD, 1999).

Many factors influence the demands of municipal water use: (1) population growth has significant impact on water resources - a decrease in per capita availability of water. It is predicted that population growth over the next 25 years will reduce renewable water resources from 6,600 to 4,800 cubic metres per person in

Canada; (2) pollution of surface and groundwater pose serious threats to freshwater quality and quantity; (3) the presence of meters and volume-based pricing are negatively correlated with domestic per capita water use; (4) urban water uses will have increasing impact on freshwater ecosystem, for example, industry and agriculture are often concentrated in and around urban areas and affect local watercourses through withdrawals and pollution. These cumulative negative impacts on urban watercourses are significant and could undermine water system function and limit its ability to provide sustainable water supplies in the future (Brandes and Ferguson, 2003).

To gain a better understanding of the overall situation of municipal water uses in Canada, we apply our case-based models to analyze water usage in Canadian cities. Some data are obtained from Environment Canada's Municipal Water Use Database (MUD), which provides basic data on municipal water systems. It currently contains water and sewage system information from 1285 Canadian municipalities (Environment Canada, 2005). We employ 1999 data, the most recent available.

In light of this purpose, six cardinal criteria from the MUD database and one ordinal criterion are selected to evaluate the water use situations in cities. The detailed descriptions of criteria with index numbers are listed as follows: 1. municipal population (POPU); 2. domestic per capita daily water use in litres (DWU); 3. total per capita daily water use in litres (TWU); 4. degree of domestic water metering (DWM); 5. the number of years since 1970 in which the municipality has experienced supply quantity problems (QUAN) ; 6. the number of years since 1970 in which the municipality has experienced supply quality problems (QUAL) ; 7. ecological impacts (ECO), an ordinal criterion that evaluates the ecological negative impacts of urban water use.

Twelve representative Canadian cities (Table 5.2) are selected as the case set **T**. Based on previous experience in water resources management, the authors place them into three classes: **High risk system $\mathbf{T}_1$**, **Low risk system $\mathbf{T}_2$** and **Robust system $\mathbf{T}_3$**. The consequences of these alternatives over six cardinal criteria are listed as Table 5.3. A linguistic evaluation grade set, which includes very low, low, fair, high and very high is set to evaluate alternatives on the ordinal criterion, ECO. The evaluation results are shown in Table 5.4. Then, case-based sorting model II is employed.

Table 5.2: Case Set for MWUA

Twelve Cities in Canada		
Robust system $\mathbf{T}_1$	Low risk system $\mathbf{T}_2$	High risk system $\mathbf{T}_3$
Fredericton (z_1^1)	Calgary (z_2^1)	Edmonton (z_3^1)
Hamilton (z_1^2)	Ottawa (z_2^2)	St. John's (z_3^2)
Waterloo (z_1^3)	Victoria (z_2^3)	Vancouver (z_3^3)
Whitehorse (z_1^4)	Yellowknife (z_2^4)	Winnipeg (z_3^4)

Table 5.3: Consequences of Cardinal Criteria in the Case Set

Cases	Cardinal criteria					
	POPU	DWU	TWU	DWM	QUAN	QUAL
Fredericton	45000	278	505	0.98	1	0
Hamilton	322252	470	921	0.65	0	0
Waterloo	78000	215	359	1	0	0
Whitehorse	20000	519	775	0.5	0	0
Calgary	819334	339	566	0.57	2	0
Ottawa	336269	259	563	1	3	0
Victoria	86000	340	519	1	1	3
Yellowknife	17250	164	406	1	0	3
Edmonton	636000	195	406	1	20	22
St. John's	106000	659	878	0	2	20
Vancouver	554000	357	650	0.01	22	23
Winnipeg	620000	190	403	1	1	12

Table 5.4: Linguistic Evaluations in the Case Set

Cases	Ordinal criterion, ECO	
	Linguistic evaluations	Value intervals
Fredericton	Very Low	$[0, \frac{1}{5}]$
Hamilton	Low	$[\frac{1}{5}, \frac{2}{5}]$
Waterloo	Low	$[\frac{1}{5}, \frac{2}{5}]$
Whitehorse	Very Low	$[0, \frac{1}{5}]$
Calgary	Low	$[\frac{1}{5}, \frac{2}{5}]$
Ottawa	Fair	$[\frac{2}{5}, \frac{3}{5}]$
Victoria	Fair	$[\frac{2}{5}, \frac{3}{5}]$
Yellowknife	Low	$[\frac{1}{5}, \frac{2}{5}]$
Edmonton	High	$[\frac{3}{5}, \frac{4}{5}]$
St. John's	High	$[\frac{3}{5}, \frac{4}{5}]$
Vancouver	Very High	$[\frac{4}{5}, 1]$
Winnipeg	High	$[\frac{3}{5}, \frac{4}{5}]$

5.5.2 Sorting Procedures

Firstly the centroid of $\mathbf{T}_1$, $\overline{z}_+$ and the centroid of $\mathbf{T}_3$, $\overline{z}_-$ are calculated by Excel using (5.1) and (5.2), and the results are shown in Table 5.5. The distance between $z_g^r(g = 1, 2, 3; r = 1, 2, 3, 4)$ and $\overline{z}_+$ on each criterion are listed in Table 5.6.

Table 5.5: Centroid of the Case Set

Centroid	Criteria					
	POPU	DWU	TWU	DWM	QUAN	QUAL
$\overline{z}_+$	116313	370.5	640	0.7825	0.25	0
$\overline{z}_-$	479000	350.25	584.25	0.5025	11.25	19.25

The program $\mathbf{SOR.2}(\alpha, \beta)$ is now applied to find $\mathbf{R}^*$ and $\mathbf{w}^*$ for $\overline{z}_+$ sorting. Note that $\mathbf{w} = (w_1, ..., w_7)$ is the weight vector of the criteria set, POPU, DWU, TWU, DWU, QUAN, QUAL, ECO and $\mathbf{R} = (R_1, R_2)$ is the distance threshold vector.

Table 5.6: The Distances on Each Criterion

Cases	Criteria					
	POPU	DWU	TWU	DWM	QUAN	QUAL
Fredericton	0.0103	0.1028	0.2308	0.0637	0.0012	0
Hamilton	0.0858	0.1189	1.0000	0.0287	0.0001	0
Waterloo	0.0030	0.2905	1.0000	0.0773	0.0001	0
Whitehorse	0.0188	0.2649	0.2308	0.1303	0.0001	0
Calgary	1.0000	0.0119	0.0694	0.0737	0.0065	0
Ottawa	0.0979	0.1494	0.0751	0.0773	0.0160	0
Victoria	0.0019	0.0112	0.1854	0.0773	0.0012	0.0170
Yellowknife	0.0199	0.5123	0.6935	0.0773	0.0001	0.0170
Edmonton	0.5464	0.3701	0.6935	0.0773	0.8245	0.9149
St. John's	0.0002	1.0000	0.7174	1.0000	0.0065	0.7561
Vancouver	0.3876	0.0022	0.0013	0.9746	1.0000	1.0000
Winnipeg	0.5133	0.3914	0.7114	0.0773	0.0012	0.2722

SOR.2(α, β) **Minimize**: $ERR = (\alpha_1^1)^2 + (\alpha_1^2)^2 + (\alpha_1^3)^2 + (\alpha_1^4)^2 + (\alpha_2^1)^2 + (\alpha_2^2)^2 + (\alpha_2^3)^2 + (\alpha_2^4)^2 + (\beta_2^1)^2 + (\beta_2^2)^2 + (\beta_2^3)^2 + (\beta_2^4)^2 + (\beta_3^1)^2 + (\beta_3^2)^2 + (\beta_3^3)^2 + (\beta_3^4)^2$

Subject to:

$0.0103w_1 + 0.1028w_2 + 0.2308w_3 + 0.0637w_4 + 0.0012w_5 + (v_7(z_1^1))^2w_7 + \alpha_1^1 \le R_1;$

$0.0858w_1 + 0.1189w_2 + w_3 + 0.0287w_4 + 0.0001w_5 + (v_7(z_1^2))^2w_7 + \alpha_1^2 \le R_1;$

$0.0030w_1 + 0.2905w_2 + w_3 + 0.0773w_4 + 0.0001w_5 + (v_7(z_1^3))^2w_7 + \alpha_1^3 \le R_1;$

$0.0188w_1 + 0.2649w_2 + 0.2308w_3 + 0.1303w_4 + 0.0001w_5 + (v_7(z_1^4))^2w_7 + \alpha_1^4 \le R_1;$

$w_1 + 0.0119w_2 + 0.0694w_3 + 0.0737w_4 + 0.0065w_5 + (v_7(z_2^1))^2w_7 + \alpha_2^1 \le R_2;$

$w_1 + 0.0119w_2 + 0.0694w_3 + 0.0737w_4 + 0.0065w_5 + (v_7(z_2^1))^2w_7 + \beta_2^1 \ge R_1;$

$0.0979w_1 + 0.1494w_2 + 0.0751w_3 + 0.0773w_4 + 0.0160w_5 + (v_7(z_2^2))^2w_7 + \alpha_2^2 \le R_2;$

$0.0979w_1 + 0.1494w_2 + 0.0751w_3 + 0.0773w_4 + 0.0160w_5 + (v_7(z_2^2))^2w_7 + \beta_2^2 \ge R_1;$

$0.0019w_1 + 0.0112w_2 + 0.1854w_3 + 0.0773w_4 + 0.0012w_5 + 0.0170w_6 + (v_7(z_2^3))^2w_7 + \alpha_2^3 \le R_2;$

$0.0019w_1 + 0.0112w_2 + 0.1854w_3 + 0.0773w_4 + 0.0012w_5 + 0.0170w_6 + (v_7(z_2^3))^2w_7 + \beta_2^3 \ge R_1;$

$0.0199w_1 + 0.5123w_2 + 0.6935w_3 + 0.0773w_4 + 0.0001w_5 + 0.0170w_6 + (v_7(z_2^4))^2 w_7 + \alpha_2^4 \le R_2;$

$0.0199w_1 + 0.5123w_2 + 0.6935w_3 + 0.0773w_4 + 0.0001w_5 + 0.0170w_6 + (v_7(z_2^4))^2 w_7 + \beta_2^4 \ge R_1;$

$0.5464w_1 + 0.3701w_2 + 0.6935w_3 + 0.0773w_4 + 0.8245w_5 + 0.9149w_6 + (v_7(z_3^1))^2 w_7 + \beta_3^1 \ge R_2;$

$0.0002w_1 + w_2 + 0.7174w_3 + w_4 + 0.0065w_5 + 0.7561w_6 + (v_7(z_3^2))^2 w_7 + \beta_3^2 \ge R_2;$

$0.3876w_1 + 0.0022w_2 + 0.0013w_3 + 0.9746w_4 + w_5 + w_6 + (v_7(z_3^3))^2 w_7 + \beta_3^3 \ge R_2;$

$0.5133w_1 + 0.3914w_2 + 0.7114w_3 + 0.0773w_4 + 0.0012w_5 + 0.2722w_6 + (v_7(z_3^4))^2 w_7 + \beta_3^4 \ge R_2;$

$0 < R_1 < R_2 < 1;$

$-1 \le \alpha_1^1 \le 0;\ -1 \le \alpha_1^2 \le 0;\ -1 \le \alpha_1^3 \le 0;\ -1 \le \alpha_1^4 \le 0;$

$-1 \le \alpha_2^1 \le 0;\ -1 \le \alpha_2^2 \le 0;\ -1 \le \alpha_2^3 \le 0;\ -1 \le \alpha_2^4 \le 0;$

$0 \le \beta_2^1 \le 1;\ 0 \le \beta_2^2 \le 1;\ 0 \le \beta_2^3 \le 1;\ 0 \le \beta_2^4 \le 1;$

$0 \le \beta_3^1 \le 1;\ 0 \le \beta_3^2 \le 1;\ 0 \le \beta_3^3 \le 1;\ 0 \le \beta_3^4 \le 1;$

$0 \le v_7(z_1^1) \le \frac{1}{5};\ \frac{1}{5} \le v_7(z_1^2) \le \frac{2}{5};$

$\frac{1}{5} \le v_7(z_1^3) \le \frac{2}{5};\ 0 \le v_7(z_1^4) \le \frac{1}{5};$

$\frac{1}{5} \le v_7(z_2^1) \le \frac{2}{5};\ \frac{2}{5} \le v_7(z_2^2) \le \frac{3}{5};$

$\frac{2}{5} \le v_7(z_2^3) \le \frac{3}{5};\ \frac{1}{5} \le v_7(z_2^4) \le \frac{2}{5};$

$\frac{3}{5} \le v_7(z_3^1) \le \frac{4}{5};\ \frac{3}{5} \le v_7(z_3^2) \le \frac{4}{5};$

$\frac{4}{5} \le v_7(z_3^3) \le 1;\ \frac{3}{5} \le v_7(z_3^4) \le \frac{4}{5};$

$w_1 > 0;\ w_2 > 0;\ w_3 > 0;\ w_4 > 0;\ w_5 > 0;\ w_6 > 0;\ w_7 > 0;$

$w_1 + w_2 + w_3 + w_4 + w_5 + w_6 + w_7 = 1;$

Because of its ability to handle nonlinear programs, Lingo software was used to find the optimal solution. The results are $ERR^* = 2.13 \times 10^{-8}$, $\mathbf{R}^* = (0.0263, 0.1138)$ and $\mathbf{w}^* = (0.036, 0.001, 0.001, 0.1928, 0.168, 0.3509, 0.2503)$. Since $ERR^* \ll \varepsilon = \frac{1}{10\times1285}$, we assess the errors as small and can be ignored. Then the $\overline{z}_+$ normal sorting procedure is adopted to check all 1285 cities in MUD and sort them into three groups defined above. Note that for the ordinal criterion ECO, the aggressive estimation method was employed to estimate the value for each alternative. Because of space limitations, we list the proportions of different groups in Figure 5.5.

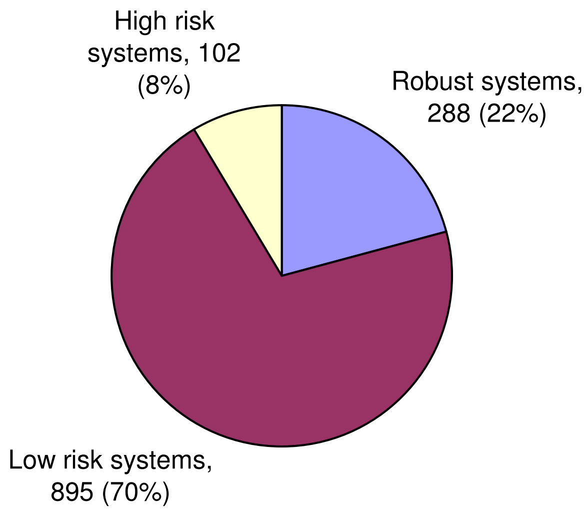


Figure 5.5: $\overline{z}_+$ Method Results

Based on the centroid of $\overline{z}_-$, similar computations can be carried out and the optimal solution is $ERR^* = 7.9602 \times 10^{-7}$, $\mathbf{R}^* = (0.4658, 0.7847)$, and $\mathbf{w}^* = (0.1232, 0.0503, 0.1337, 0.0815, 0.0965, 0.3827, 0.1321)$. As before, errors can be ignored. Based on this information, the sorting of 1285 Canadian cities was carried out and the proportions in different groups shown in Figure 5.6. (Similarly the aggressive estimation method is adopted to estimate the values for ordinal criterion, ECO).

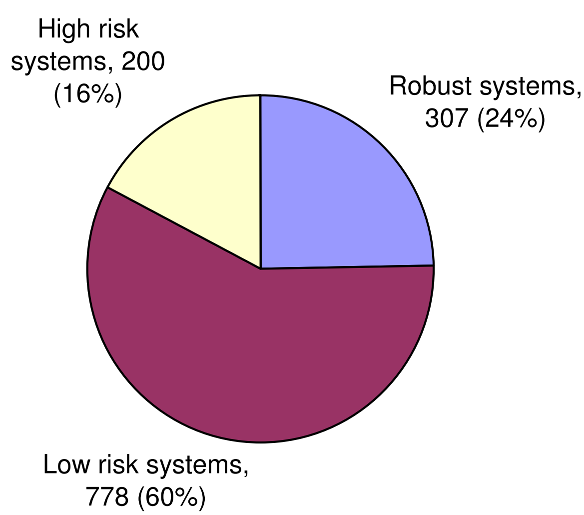


Figure 5.6: $\overline{z}_-$ Method Results

As seen in Figures 5.5 and 5.6, ρ_1, ρ_2, ρ_3 for the $\overline{z}_+$ sorting method are $22\%, 70\%$, 8% and ρ_1, ρ_2, ρ_3 for $\overline{z}_-$ sorting method are $24\%, 60\%, 16\%$, respectively. Therefore, $\Phi_+ = (\rho_2, \rho_1, \rho_3) = \Phi_-$, thus $\mathbf{T}$ provides roughly consistent results by both $\overline{z}_+$ and $\overline{z}_-$ sorting methods. We conclude that based on our current information (knowledge), the water supply systems in most Canadian cities are robust or low risk, so the overall situation is still acceptable. However, water use in Canadian cities is typically twice as great as European cities, which suggests that great potential exists to reduce the volume of water use by Canadian cities with minimal impacts on quality of life standard. It has been argued that curbing the trend of increasing water use can provide significant ecological, social and economic benefits (Brandes and Ferguson, 2003).

5.6 Conclusions

Case-based distance methods are proposed in this chapter to solve sorting problems in MCDA. The main components of the chapters are:

- General description of sorting problems: A formal definition of sorting problems is presented and the relationship between $\mathbf{A}$, the alternative set, and $\mathbf{S}$, the sorting group, is discussed.
- Case-based sorting model I: The method is designed for sorting problems having cardinal criteria.
- Case-based sorting model II: This method is an extension of case-based sorting model I and can handle both cardinal and ordinal criteria.
- Case study: An application to municipal water usage in Canada is presented.

Chapter 6

Sorting Problem Application in Inventory Management

6.1 Introduction

In ABC analysis, a well-known inventory planning and control technique, stock-keeping units (SKUs) are sorted into three categories. Traditionally, the sorting is based solely on annual dollar usage. The aim of this chapter is to introduce a case-based multiple criteria ABC analysis (MCABC) that improves on this approach by accounting for additional criteria, such as lead time and criticality of SKUs, thereby providing more managerial flexibility. Firstly, the motivation of MCABC is explained briefly. Next, research on ABC analysis and its extension, MCABC is explained in detail. Then, a case-based distance method is proposed to solve MCABC problems. Finally, a case study is developed to illustrate how the procedure can be applied; the results demonstrate that this approach is robust and can produce sound classifications of SKUs when multiple criteria are to be considered. The contents of this chapter is based on research by Chen et al. (2005e).

6.2 Motivation

Efficient and effective inventory management assumes increasing significance in maintaining a firm's competitive advantage with the acceleration of globalization of businesses (Silver et al., 1998). Generally, the number of SKUs in a firm can easily go up to tens of thousands or even more. Therefore, it is not economically feasible to design an inventory management policy for each individual SKU. In addition, different SKUs may play quite different roles in contributing to the firm's business and, hence, necessitate different levels of management attention. In order to implement any sound inventory control scheme, it becomes necessary to group SKUs into a manageable and meaningful number of categories first, and then design different policies for distinct groups according to the group's levels of importance to the firm (Chakravarty, 1981). In so doing, a generic set of inventory management policies requiring certain level of effort and control from management is applied to all members falling into this category. It is expected that this aggregation process will dramatically reduce the number of SKUs that require extensive attention from the management.

ABC analysis is the most frequently used approach to classifying SKUs into groups. A traditional method of aggregating SKUs is solely based on their annual dollar usage. The underlying principle of this approach is the fact that a small proportion of SKUs accounts for a majority fraction of the dollar usage. Classical ABC analysis roots from Pareto's famous observations on the uneven distribution of incomes (Pareto, 1971), and hence is sometimes referred to as Pareto analysis. Because of its easy-to-implementation nature and remarkable effectiveness in many inventory systems, this approach is still widely used in practice.

However, although the annual dollar usage is a crucial dimension to measure the importance of SKUs in the inventory system, many other criteria may also contribute to determining management's attention for a particular SKU, and hence affect the aggregation of SKUs. For instance, in the high technology industry, some parts may become obsolete in a very short period, therefore, should be closely monitored by inventory managers. In this case, the obsolescence becomes a critical criterion to classifying SKUs. Other factors, such as length and variability of lead time, substitutability, reparability, criticality, may also affect the management's decision (Flores and Whybark, 1986). Therefore, various multiple criteria ABC analysis (MCABC) models have been developed to complement the classical ABC analysis, including AHP (analytic hierarchy process) method (Flore et al., 1992; Partovi and Hopton, 1994), statistical method Cohen and Ernst (1988), artificial neural network approach (Partovi and Anadarajan, 2002) and genetic algorithm method (Guvenir and Erel, 1998), to name a few.

The approach presented in this thesis is motivated by the work of (Flores and Whybark, 1986), where the dollar usage is combined with another criterion that is relevant to a firm's inventory system. But their approach cannot handle situations that three or more criteria have to be taken into account at the same time in order to classify the inventory SKUs. Our research aims to lift this restriction and allow any finite number of criteria to be considered simultaneously.

6.3 Multiple Criteria ABC Analysis (MCABC)

Classical ABC analysis aggregates SKUs into different groups based solely on the annual dollar usage. The most important SKUs in terms of dollar usage are placed in group $\mathcal{A}$, which demand the greatest effort and attention from the management; the least important SKUs are aggregated into group $\mathcal{C}$, which are given the minimal control; other SKUs are categorized into group $\mathcal{B}$, on which medium level of control is exerted. The "80-20 (or 90-10) Rule" — 80% (or 90%) of the total annual usage comes from 20% (or 10%) of SKUs — constitutes the basis of the classical ABC analysis. The rule guarantees that the number of $\mathcal{A}$ SKUs is substantially smaller than the total number of SKUs in the inventory system. Although the exact values vary from industry to industry, the 80-20 rule can be applied to many real-world situations. Figure 6.1 captures the essence of this rule. Note that **A** represents the set of alternatives, $\{A^1, A^2, \cdots, A^i, \cdots, A^n\}$. $\mathbf{Q} = \{1, 2, \cdots, j, \cdots, q\}$ is the set of criteria. The consequence on criterion j of alternative A^i is expressed as $c_j(A^i)$.

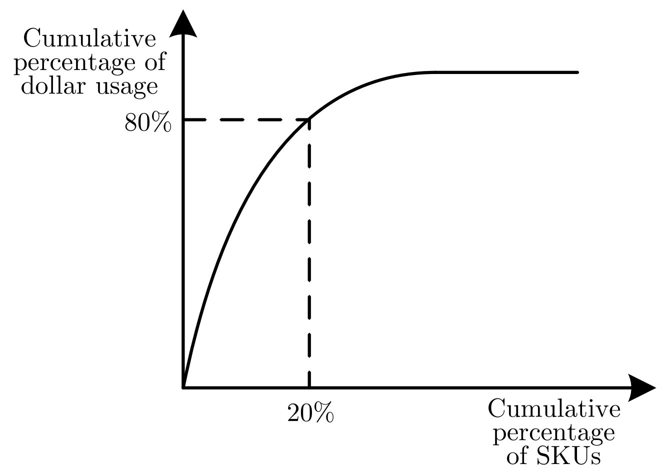


Figure 6.1: Example of Dollar Usage Distribution Curve

The classification obtained from the ABC analysis is sometimes subject to further adjustments, for example, some SKUs' dollar usages may not be so significant, but their stock-out cost can be unbearably high; while some other SKUs may have high dollar usages, but their supply is sufficient and consistent. In these cases, SKUs may have to be switched among the three groups. The relevance of this reclassification process is that some criteria, other than the dollar usage, may come into play in determining how much attention should be paid for them.

Flores and Whybark (1986) proposed a multiple criteria framework to handle ABC analysis, and applied it to a service organization and a manufacturing firm (Flores and Whybark, 1987). This approach begins with selecting another critical criterion, in addition to the dollar usage. This criterion depends on the nature of industries. Some examples are obsolescence, lead times, substitutability, reparability, criticality and commonality (Flores and Whybark, 1986). Next, the model requires that SKUs be divided into three levels of importance, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ against the two criteria, respectively. The model then reclassifies SKUs into three categories, $\mathcal{AA}$, $\mathcal{BB}$, and $\mathcal{CC}$, which represent the three new groups, according to some rules determined by the new criterion other than the dollar usage. The structure of the model can be conveniently represented as a joint criteria matrix as shown in Figure 6.2, adapted from Flores and Whybark (1986). A general guideline as indicated by the arrows is to regroup $\mathcal{AB}$ and $\mathcal{BA}$ as $\mathcal{AA}$, $\mathcal{AC}$ and $\mathcal{CA}$ as $\mathcal{BB}$, and $\mathcal{BC}$ and $\mathcal{CB}$ as $\mathcal{CC}$.

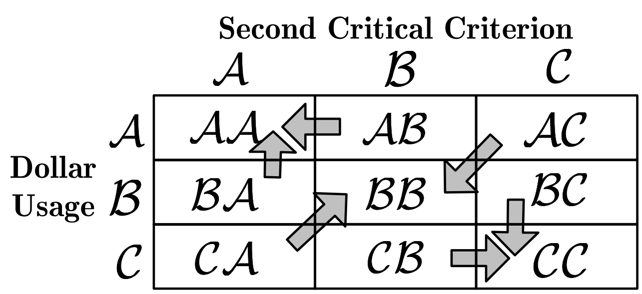


Figure 6.2: The Joint Matrix for Two Criteria

6.4 A Case-based Distance Model for MCABC

6.4.1 Case Set Definitions for MCABC

MCABC can be regarded as a three-group sorting problem, which arranges SKUs into group $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{C}$ with the preference order $\mathcal{A} \succ \mathcal{B} \succ \mathcal{C}$, where $\succ$ denotes the strict preference relation and all SKUs in a certain group are indifferent to the DM.

Assume a MCABC problem is to classify SKUs in $\mathbf{A}$ ($|\mathbf{A}| = n$) into group $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ based on the criteria set $\mathbf{Q}$ ($|\mathbf{Q}| = q$). Several representative SKUs for the

MCABC problem are available and they are partitioned into three categories $\mathbf{T}_{\mathcal{A}}$, $\mathbf{T}_{\mathcal{B}}$, and $\mathbf{T}_{\mathcal{C}}$ which represent case sets for groups $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, respectively. The number of SKUs in group g ($g = \mathcal{A}, \mathcal{B}, \mathcal{C}$) is denoted n_g and z_g^r is a representative SKU in $\mathbf{T}_g$. The SKUs in case sets may, for example, be fabricated by the DM or obtained by having the DM modify historical records and the case sets are comprehensive representative for the DM. Note that all criteria in $\mathbf{Q}$ must apply and $c_j(A^i)$ must be measurable for all SKUs in the case sets and all $j \in \mathbf{Q}$. Preference and indifference of SKUs are induced in the case sets. For two SKUs in the same group, they are equally preferred by the DM. For example, $z_{\mathcal{B}}^k$, $z_{\mathcal{B}}^l \in \mathbf{T}_{\mathcal{B}}$, $z_{\mathcal{B}}^k \sim z_{\mathcal{B}}^l$ ($\sim$ means the DM equally prefers z_g^k and z_g^l); any SKU in a more important group is more preferred to the one in a less important group. For example, $z_{\mathcal{A}}^k \in \mathbf{T}_{\mathcal{A}}$, and $z_{\mathcal{B}}^l \in \mathbf{T}_{\mathcal{B}}$, $z_{\mathcal{A}}^k \succ z_{\mathcal{B}}^l$.

Our case-based reasoning idea is based on the "right" distance based preference expression: the distances of cases to a pre-defined point in the same group should be close together within a range and farther from the distances of cases in other groups in some sense. We use $\mathbf{T}_g$, $g = \mathcal{A}, \mathcal{B}, \mathcal{C}$, to estimate criterion weight $\mathbf{w}$, and a distance threshold vector $\mathbf{R}$, so that this information can be applied to classify (sort) SKUs in $\mathcal{A}$. Figure 6.3 demonstrates one situation of this idea. Two ellipses partition the SKUs into three case sets and represent preference sequences: ellipses closer to "o" represent more preferred groups. For $z_{\mathcal{A}}^r \in \mathbf{T}_{\mathcal{A}}$, $z_{\mathcal{B}}^r \in \mathbf{T}_{\mathcal{B}}$ and $z_{\mathcal{C}}^r \in \mathbf{T}_{\mathcal{C}}$, $z_{\mathcal{A}}^r \succ z_{\mathcal{B}}^r \succ z_{\mathcal{C}}^r$. Then by a properly designed transformation from the original consequence data space to a weighted normalized consequence data space (preference space), ellipse-based distances can be transformed to circle-based distances and accordingly this information can be applied to classify SKUs in $\mathbf{A}$. Notice that in Figure 6.3 the distance of $z_{\mathcal{A}}^r$ from "o" is less than $R_{\mathcal{A}}$, the distance of $z_{\mathcal{B}}^r$ is greater than $R_{\mathcal{A}}$ and less than $R_{\mathcal{B}}$, and the distance of $z_{\mathcal{C}}^r$ is greater than $R_{\mathcal{B}}$.

6.4.2 Distance Assumptions

Assuming the DM's preferences over $\mathbf{Q}$ are monotonic and two kinds of criteria are defined as follows: (1) benefit criteria, $\mathbf{Q}^+$, which mean the greater value (consequences) the more important (preference) for the DM; (2) cost criteria, $\mathbf{Q}^-$, which mean the less value (consequences) the more important (preference) for the DM. Thus, $\mathbf{Q} = \mathbf{Q}^+ \cup \mathbf{Q}^-$. For example, the manager in a manufacturing company may set the criterion of dollar usage as $\mathbf{Q}^+$ while the criterion of lead time as $\mathbf{Q}^-$. Furthermore, the DM can identify the maximum consequence on criterion j ($j \in \mathbf{Q}$), $c_j^{\max} \in \mathbb{R}^+$ and the minimum consequence, $c_j^{\min} \in \mathbb{R}^+$, where $c_j^{\max} > c_j^{\min}$. Notice

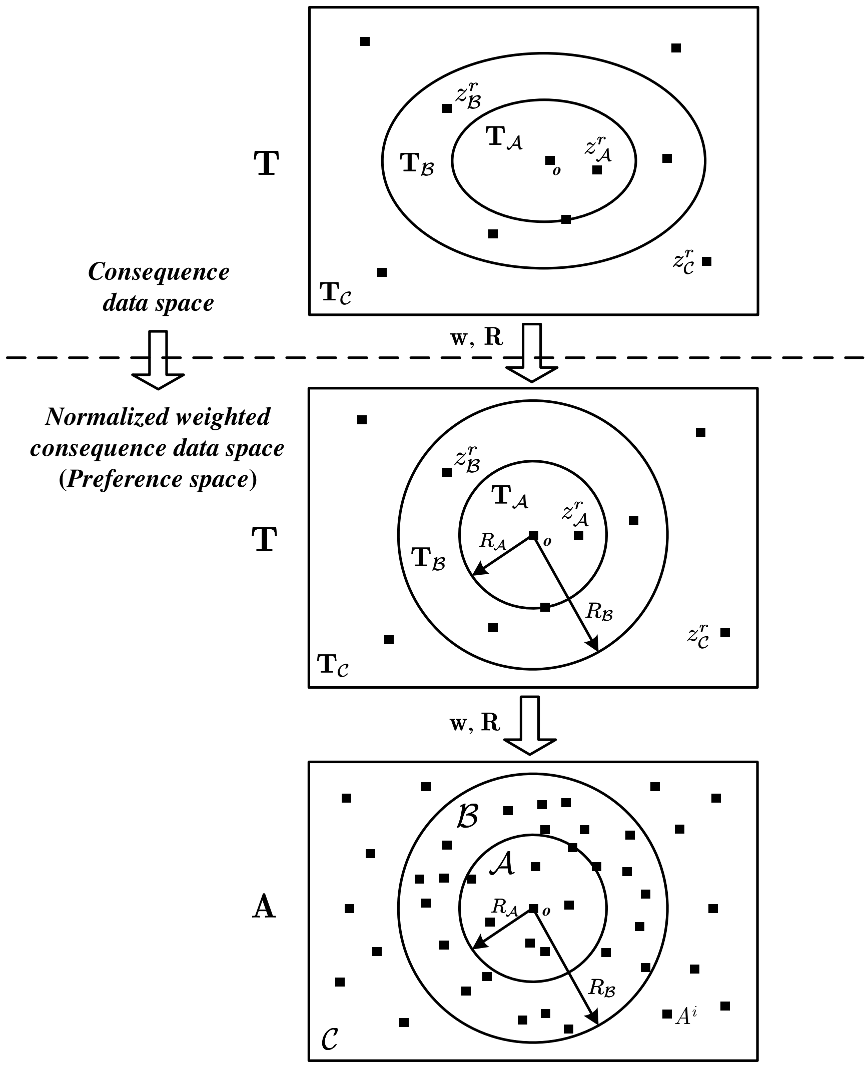


Figure 6.3: The Idea of Case-based Distance model

that $c_j^{\max}$ and $c_j^{\min}$ are extreme values for criterion j, so that the consequence of any SKU, $c_j(A^i)$ satisfies $c_j^{\min} \leq c_j(A^i) \leq c_j^{\max}$.

Two fictitious SKUs are set, the ideal SKU, A^+ and the anti-ideal SKU, A^-. By definitions,

$$c_j(A^+) = \begin{cases} c_j^{\max} & \text{if } j \in \mathbf{Q}^+; \\ c_j^{\min} & \text{if } j \in \mathbf{Q}^-. \end{cases}$$

and

$$c_j(A^-) = \begin{cases} c_j^{\min} & \text{if } j \in \mathbf{Q}^+; \\ c_j^{\max} & \text{if } j \in \mathbf{Q}^-. \end{cases}$$

For $j = 1, 2, ..., q$, define $d_j^{\max} = \left(c_j^{\max} - c_j^{\min}\right)^2$ to be the normalization factor for criterion j. For $g = \mathcal{A}, \mathcal{B}, \mathcal{C}$ and $r = 1, 2, ..., n_g$, the normalized distance between $z_g^r \in \mathbf{T}_g$ and A^+ on criterion j is

$$d_j(z_g^r, A^+) = d_j(z_g^r)^+ = \frac{\left(c_j(z_g^r) - c_j(A^+)\right)^2}{d_j^{\max}}. \tag{6.1}$$

Note that (6.1) defines $d_j(A^i)^+$ if $A^i = z_g^r$, $A^i \in \mathbf{A}$. Similarly, the distance between any SKU $z_g^r \in \mathbf{T}_g$ and A^- on criterion j is

$$d_j(z_g^r, A^-) = d_j(z_g^r)^- = \frac{\left(c_j(z_g^r) - c_j(A^-)\right)^2}{d_j^{\max}}. \tag{6.2}$$

Note that (6.2) defines $d_j(A^i)^-$ if if $A^i = z_g^r$, $A^i \in \mathbf{A}$. It is easy to verify that $d_j(z_g^r)^+ \in [0, 1]$, $d_j(A^i)^+ \in [0, 1]$, $d_j(z_g^r)^- \in [0, 1]$, and $d_j(A^i)^- \in [0, 1]$.

The aggregated distance between $z_g^r \in \mathbf{T}_g$ and A^+ over the criteria set $\mathbf{Q}$ is identified as

$$D(z_g^r, A^+) = D(z_g^r)^+ = \sum_{j \in \mathbf{Q}} w_j^+ \cdot d_j(z_g^r)^+, \tag{6.3}$$

where $w_j^+ \in \mathbf{w}^+$ is the A^+-based weight (relative importance) of criterion j. The weight vector $\mathbf{w}^+$ is to be determined. It is assumed that $0 < w_j^+ \leq 1$ and $\sum_{j \in \mathbf{Q}} w_j^+ = 1$. Similarly, the aggregated distance from alternative z_g^r to A^- is

$$D(z_g^r, A^-) = D(z_g^r)^- = \sum_{j \in \mathbf{Q}} w_j^- \cdot d_j(z_g^r)^-, \tag{6.4}$$

where $w_j^- \in \mathbf{w}^-$ is the A^--based weight of criterion j. Note that (6.3) and (6.4) define $D(A^i)^+$ and $D(A^i)^-$, respectively, if $A^i = z_g^r$, $A^i \in \mathbf{A}$.

6.4.3 Model Construction

Taking A^+ as the original point "o", A^+MCABC analysis is explained as follows: a compact ball (in q dimensions) with radius of $R_{\mathcal{A}}^+ \in \mathbb{R}^+$ includes (in principle) every case in $\mathbf{T}_{\mathcal{A}}$ and any case that is not in $\mathbf{T}_{\mathcal{A}}$ is (in principle) outside that ball.

Similarly, a compact ball with radius of $R_{\mathcal{B}}^{+} \in \mathbb{R}^{+}$ includes every case in $\mathbf{T}_{A}$ and $\mathbf{T}_{\mathcal{B}}$ and any case in $\mathbf{T}_{\mathcal{C}}$ is outside. Therefore, $R_{\mathcal{A}}^{+}$ and $R_{\mathcal{B}}^{+}$ can be employed to classify SKUs in **A** and Figure 6.4 demonstrates this idea. Similarly, A^{-}MCABC analysis can be developed taking A^{-} as the original point in which a greater distance indicates a greater preference. A ball with radius of $R_{\mathcal{C}}^{-}$ includes $z_{\mathcal{C}}^{r} \in \mathbf{T}_{\mathcal{C}}$, and a ball with radius of $R_{\mathcal{B}}^{-}$ includes $z_{\mathcal{B}}^{r} \in \mathbf{T}_{\mathcal{B}}$ and $z_{\mathcal{C}}^{r} \in \mathbf{T}_{\mathcal{C}}$. The A^{+}MCABC based model construction is explained in detail next. Since the procedure of A^{-}MCABC is similar, the details are omitted.

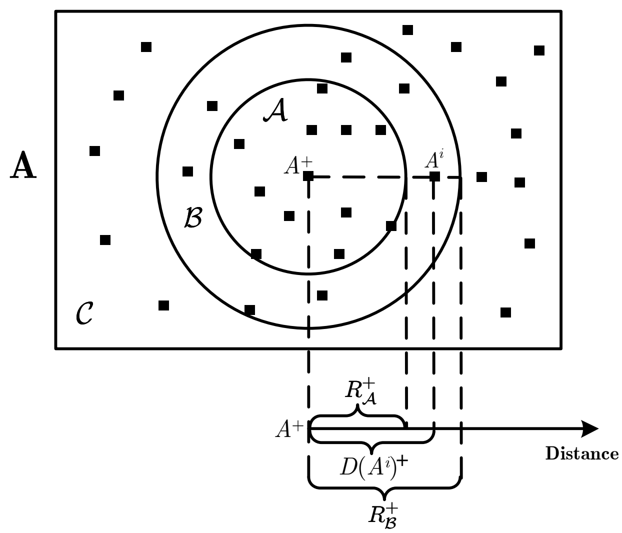


Figure 6.4: Relationships Among $R_{\mathcal{A}}^{+}$, $R_{\mathcal{B}}^{+}$ and $D(A^{i})^{+}$

For $j \in \mathbf{Q}$, w_j refers to the DM's preference on criterion j, and represents the relative importance of criterion j within the aggregated distance. $R_{\mathcal{A}}^{+}$, $R_{\mathcal{B}}^{+}$ represent thresholds to classify SKUs into different groups. Here, we obtain $\mathbf{w}^{+} = (w_1^{+}, w_2^{+}, ..., w_q^{+})$, and $R_{\mathcal{A}}^{+}$, $R_{\mathcal{B}}^{+}$ by a case-based reasoning model based on $\mathbf{T}_{\mathcal{A}}$, $\mathbf{T}_{\mathcal{B}}$ and $\mathbf{T}_{\mathcal{C}}$.

For an MCABC problem SKUs in the case set $\mathbf{T}_{\mathcal{A}}$ (belonging to $\mathcal{A}$) is assessed by the DM, with the preference relationships described above, they are more preferred than other cases in $\mathbf{T}_{\mathcal{B}}$ and $\mathbf{T}_{\mathcal{C}}$. Therefore, based on distance measurement from A^{+}, the following constraints are set:

(1) The distance of $z_{\mathcal{A}}^{r} \in \mathbf{T}_{\mathcal{A}}$ to A^{+} is less than $R_{\mathcal{A}}^{+}$, provided that there are no inconsistent judgements. Thus, for $r = 1, 2, ..., n_{\mathcal{A}}$,

$$D(z_{\mathcal{A}}^{r})^{+} + \alpha_{\mathcal{A}}^{r} \leq R_{\mathcal{A}}^{+}, \text{ or} \sum_{j \in \mathbf{Q}} w_j^{+} \cdot d_j(z_{\mathcal{A}}^{r})^{+} + \alpha_{\mathcal{A}}^{r} \leq R_{\mathcal{A}}^{+}, \tag{6.5}$$

where $-1 \leq \alpha_{\mathcal{A}}^r \leq 0$ is an upper-bound error adjustment parameter (keeping the distance of $z_{\mathcal{A}}^r$ less than $R_{\mathcal{A}}^+$).

(2) The distance of $z_{\mathcal{B}}^r \in \mathbf{T}_{\mathcal{B}}$ to A^+ is larger than $R_{\mathcal{A}}^+$ and less than $R_{\mathcal{B}}^+$ provided that there are no inconsistent judgements. Thus, for $r = 1, 2, ..., n_{\mathcal{B}}$,

$$D(z_{\mathcal{B}}^r)^+ + \alpha_{\mathcal{B}}^r \leq R_{\mathcal{B}}^+, \text{ or} \sum_{j \in \mathbf{Q}} w_j^+ \cdot d_j(z_{\mathcal{B}}^r)^+ + \alpha_{\mathcal{B}}^r \leq R_{\mathcal{B}}^+, \tag{6.6}$$

$$D(z_{\mathcal{B}}^r)^+ + \beta_{\mathcal{B}}^r \geq R_{\mathcal{A}}^+, \text{ or} \sum_{j \in \mathbf{Q}} w_j^+ \cdot d_j(z_{\mathcal{B}}^r)^+ + \beta_{\mathcal{B}}^r \geq R_{\mathcal{A}}^+, \tag{6.7}$$

where $-1 \leq \alpha_{\mathcal{B}}^r \leq 0$ is an upper-bound error adjustment parameter (keeping the distance of $z_{\mathcal{B}}^r$ less than $R_{\mathcal{B}}^+$) and $0 \leq \beta_{\mathcal{B}}^r \leq 1$ is a lower-bound error adjustment parameter (keeping the distance of $z_{\mathcal{B}}^r$ larger than $R_{\mathcal{A}}^+$).

(3) The distance of $z_{\mathcal{C}}^r \in \mathbf{T}_{\mathcal{C}}$ to A^+ is larger than $R_{\mathcal{B}}^+$ provided that there are no inconsistent judgements. Thus, for $r = 1, 2, ..., n_{\mathcal{C}}$,

$$D(z_{\mathcal{C}}^r)^+ + \beta_{\mathcal{C}}^r \geq R_{\mathcal{B}}^+, \text{ or} \sum_{j \in \mathbf{Q}} w_j^+ \cdot d_j(z_{\mathcal{C}}^r)^+ + \beta_{\mathcal{C}}^r \geq R_{\mathcal{B}}^+, \tag{6.8}$$

where $0 \leq \beta_{\mathcal{C}}^r \leq 1$ is a lower-bound error adjustment parameter (keeping the distance of $z_{\mathcal{C}}^r$ larger than $R_{\mathcal{B}}^+$).

Accordingly, the overall squared error in all case sets is denoted as $ERR = \sum_{r=1}^{n_{\mathcal{A}}} (\alpha_{\mathcal{A}}^r)^2 + \sum_{r=1}^{n_{\mathcal{B}}} [(\alpha_{\mathcal{B}}^r)^2 + \beta_{\mathcal{B}}^r)^2] + \sum_{r=1}^{n_{\mathcal{C}}} (\beta_{\mathcal{C}}^r)^2$. Then the following optimization model can be adopted to find the most descriptive weight vector $\mathbf{w}^+$, and the distance thresholds $R_{\mathcal{A}}^+$ and $R_{\mathcal{B}}^+$.

SOR.3(α, β) **Minimize** : $ERR = \sum_{r=1}^{n_{\mathcal{A}}} (\alpha_{\mathcal{A}}^r)^2 + \sum_{r=1}^{n_{\mathcal{B}}} [(\alpha_{\mathcal{B}}^r)^2 + \beta_B^r)^2] + \sum_{r=1}^{n_{\mathcal{C}}} (\beta_{\mathcal{C}}^r)^2$

Subject to:

$\sum_{j \in \mathbf{Q}} w_j^+ \cdot d_j(z_{\mathcal{A}}^r)^+ + \alpha_{\mathcal{A}}^r \leq R_{\mathcal{A}}^+$, $r = 1, 2, ..., n_{\mathcal{A}}$;

$\sum_{j \in \mathbf{Q}} w_j^+ \cdot d_j(z_{\mathcal{B}}^r)^+ + \alpha_{\mathcal{B}}^r \leq R_{\mathcal{B}}^+$, $r = 1, 2, ..., n_{\mathcal{B}}$;

$\sum_{j \in \mathbf{Q}} w_j^+ \cdot d_j(z_{\mathcal{B}}^r)^+ + \beta_{\mathcal{B}}^r \geq R_{\mathcal{A}}^+$, $r = 1, 2, ..., n_{\mathcal{B}}$;

$\sum_{j \in \mathbf{Q}} w_j^+ \cdot d_j(z_{\mathcal{C}}^r)^+ + \beta_{\mathcal{C}}^r \geq R_{\mathcal{B}}^+$, $r = 1, 2, ..., n_{\mathcal{C}}$;

$0 < R_{\mathcal{A}}^+ < 1$, $0 < R_{\mathcal{B}}^+ < 1$, $R_{\mathcal{A}}^+ < R_{\mathcal{B}}^+$;

$-1 \leq \alpha_g^r \leq 0,\ g = \mathcal{A}, \mathcal{B};$

$0 \leq \beta_g^r \leq 1,\ g = \mathcal{B}, \mathcal{C};$

$w_j^+ > 0,\ \sum_{j \in \mathbf{Q}} w_j^+ = 1;$

Theorem 6. **SOR.3**(α, β) *has at least one optimal solution.*

Proof: The constraints in **SOR.3**(α, β) constitute a convex set. The objective function ERR is a quadratic function on this set. Based on the extreme value theorem of advanced calculus (Fitzpatrick, 1995, page 297), the function ERR is continuous and the set of all possible variables is bounded, ERR attains its minimum at least once. □

An indifference distance threshold, ε is set to evaluate the error, ERR, generated by **SOR.3**(α, β): When $ERR \leq \varepsilon$, the error is small and can be ignored, so the information in the case sets provided by the DM is considered to be consistent; when $ERR > \varepsilon$, the error cannot be ignored and there is some inconsistency in the case sets. The DM should reconsider them. A suggested value of ε is $\frac{1}{kn}$, $n = |\mathbf{A}|$, where $k \in \mathbb{R}^+$ is the adjustment parameter. When the case set is large and the likelihood of the error is high, then $k < 1$; when the case set is small and the likelihood of the error is small, then $k \geq 1$.

Furthermore, to make the optimal results more closely reflect a DM's intrinsic preferences, the DM could provide some rough information about weights and incorporate it into **SOR.3**(α, β). The imprecise preference expressions proposed by Sage and White (1984), and Eum et al. (2001) can be used for this purpose. Some imprecise weight preference expressions are listed below:

- Weak ranking: $w_1 \geq w_2 \geq \cdots \geq w_q > 0$;
- Strict ranking: $w_j - w_{j+1} \geq \epsilon_j, j \in \mathbf{Q}$, where ϵ_j is a small positive value;
- Difference ranking: $w_1 - w_2 \geq \cdots \geq w_{q-1} - w_q \geq 0$;
- Fixed bounds: $L_j \leq w_j \leq U_j, j \in \mathbf{Q}$, where L_j and U_j are lower and upper bounds for w_j, respectively;

The following imprecise weight preference expressions are proposed to align with the MCABC scenario:

$$w_d \geq w_k, \forall\ k \in \mathbf{Q}, k \neq d \tag{6.9}$$

$$L_j \leq w_j \leq U_j,\ \forall\ j \in \mathbf{Q} \tag{6.10}$$

where w_d represents the weight of annual dollar usage, and L_j and U_j are, respectively, lower and upper bounds for w_j. For simplicity, we could set $L_j = L$ and $U_j = U$ for all criteria. (Setting $0 < L < U < 1$ ensures that all specified criteria count in the final classification—no criterion can be discarded. In particular, the value of L should be set to some non-negligible amount to ensure that no criterion is effectively dropped from the model.) If constraints (6.9) and (6.10) can be incorporated directly into $\mathbf{D}(\alpha, \beta)$, the program will still have at least one optimal solution. Alternatively, when (6.9) and (6.10) are not included in $\mathbf{D}(\alpha, \beta)$, they can guide the DM in selecting the most suitable solutions when multiple optimal results are identified in $\mathbf{D}(\alpha, \beta)$, as will be explained in Section 6.4.5, Post-optimality analyses.

6.4.4 Distance-based Sorting

Assuming $ERR \leq \varepsilon$, and $\mathcal{A}^+$, $\mathcal{B}^+$, and $\mathcal{C}^+$ denote A^+MCABC-based group $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, respectively. With $\mathbf{w}^+ = (w_1^+, w_2^+, ..., w_q^+)$, $R_{\mathcal{A}}^+$ and $R_{\mathcal{B}}^+$ obtained from $\mathbf{D}(\alpha, \beta)$, A^+MCABC can be carried out to classify SKUs in $\mathbf{A}$ as follows:

- If $D(A^i)^+ \leq R_{\mathcal{A}}^+$, $A^i \in \mathcal{A}^+$;
- If $R_{\mathcal{A}}^+ < D(A^i)^+ \leq R_{\mathcal{B}}^+$, $A^i \in \mathcal{B}^+$;
- If $D(A^i)^+ > R_{\mathcal{B}}^+$, $A^i \in \mathcal{C}^+$.

Employing similar procedures $\mathbf{w}^- = (w_1^-, w_2^-, ..., w_q^-)$, R_B^- and R_C^- can be calculated and A^-MCABC is thus carried out to classify SKUs in $\mathbf{A}$ as follows:

- If $D(A^i)^- \leq R_{\mathcal{C}}^-$, $A^i \in \mathcal{C}^-$;
- If $R_C^- < D(A^i)^- \leq R_{\mathcal{B}}^-$, $A^i \in \mathcal{B}^-$;
- If $D(A^i)^- > R_{\mathcal{B}}^-$, $A^i \in \mathcal{A}^-$.

Note that $\mathcal{A}^-$, $\mathcal{B}^-$ and $\mathcal{C}^-$ denote A^-MCABC-based groups $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, respectively.

Next, a process similar to Flores and Whybark (1986) is designed to finalize the classification of SKUs in $\mathbf{A}$ to different groups as shown in Figure 6.5.

Based on the classification results of A^+MCABC and A^-MCABC, nine combination groups, $\mathcal{A}^-\mathcal{A}^+$, $\mathcal{A}^-\mathcal{B}^+$, $\mathcal{A}^-\mathcal{C}^+$, $\mathcal{B}^-\mathcal{A}^+$, $\mathcal{B}^-\mathcal{B}^+$, $\mathcal{B}^-\mathcal{C}^+$, $\mathcal{C}^-\mathcal{A}^+$, $\mathcal{C}^-\mathcal{B}^+$ and $\mathcal{C}^-\mathcal{C}^+$,

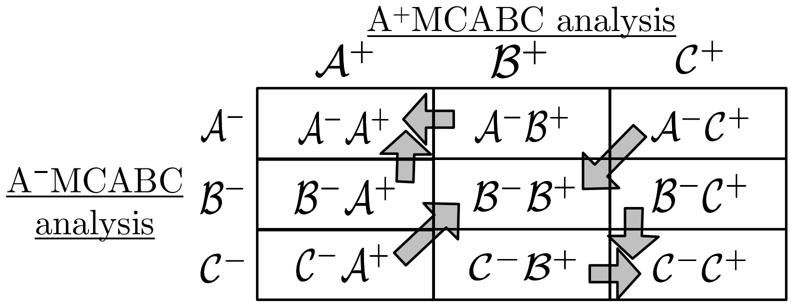


Figure 6.5: The Joint Matrix for Two MCABC Methods

are identified. Then these combination groups are reclassified into three categories, $\mathcal{A}^-\mathcal{A}^+$, $\mathcal{B}^-\mathcal{B}^+$ and $\mathcal{C}^-\mathcal{C}^+$, which represent the most important, the medium-level important and the least important groups, respectively. The guideline as indicated by the arrows is to regroup $\mathcal{A}^-\mathcal{B}^+$ and $\mathcal{B}^-\mathcal{A}^+$ as $\mathcal{A}^-\mathcal{A}^+$, $\mathcal{A}^-\mathcal{C}^+$ and $\mathcal{C}^-\mathcal{A}^+$ as $\mathcal{B}^-\mathcal{B}^+$, and $\mathcal{B}^-\mathcal{C}^+$ and $\mathcal{C}^-\mathcal{B}^+$ as $\mathcal{C}^-\mathcal{C}^+$.

6.4.5 Post-optimality Analyses

Because $\mathbf{SOR.3}(\alpha,\beta)$ may have many sets of criterion weights and distance thresholds that are optimal or near-optimal, we discuss how the robustness of each solution can be examined using post-optimality analysis. There are several ways to assess whether multiple or near-optimal solutions of $\mathbf{SOR.3}(\alpha,\beta)$ exist.

- *Programming-based Near-optimality Analyses*

  Assuming the optimal objective value of $\mathbf{SOR.3}(\alpha,\beta)$ is ERR^*, some suggestions of Jacquet-Lagrèze and Siskos (1982) can be adapted for use in post-optimality analysis of w_j (for each $j \in \mathbf{Q}$) using the following programs:

  $\mathbf{SOR.3}'(\alpha,\beta,w_j)$ **Maximize** : w_j

  Subject to:

$$\begin{cases} ERR \leq \max\{\eta, (1+\eta)ERR^*\}, \\ \text{all constraints of } \mathbf{D}(\alpha,\beta), \\ \text{constraints (6.9) and (6.10) as applicable,} \end{cases}$$

SOR.3$''(\alpha, \beta, w_j)$ **Minimize** : w_j

Subject to:

$$\begin{cases} ERR \leq \max\{\eta, (1+\eta)ERR^*\}, \\ \text{all constraints of } \mathbf{D}(\alpha, \beta), \\ \text{constraints (6.9) and (6.10) as applicable,} \end{cases}$$

In both programs, η is a small positive number.

These programs obtain maximum and minimum values for w_j. Similarly, the programs **SOR.3**$'(\alpha, \beta, R^+_{\mathcal{A}})$, **SOR.3**$'(\alpha, \beta, R^+_{\mathcal{B}})$, **SOR.3**$''(\alpha, \beta, R^+_{\mathcal{A}})$ and **SOR.3**$''(\alpha, \beta, R^+_{\mathcal{B}})$ yield, respectively, maximum and minimum values for $R^+_{\mathcal{A}}$ and $R^+_{\mathcal{B}}$. The difference between the generated minimum and maximum values for a criterion weight or distance threshold is a measure of the robustness of the initial solution.

There are two ways to use this robustness information to determine a final sorting.

1. *Average Value Method*: Based on the suggestions from Jacquet-Lagrèze and Siskos (1982) and Siskos et al. (2005), the averages of the initial solutions, and the maximum and minimum values for each criterion or distance threshold generated from the above procedures may be considered as a more representative solution of $\mathbf{D}(\alpha, \beta)$, and used to sort SKUs.

2. *Percentage Value Method*: Each of **SOR.3**(α, β), **SOR.3**$'(\alpha, \beta, R^+_{\mathcal{A}})$, **SOR.3**$'(\alpha, \beta, R^+_{\mathcal{B}})$, **SOR.3**$''(\alpha, \beta, R^+_{\mathcal{A}})$ and **SOR.3**$''(\alpha, \beta, R^+_{\mathcal{B}})$ generates a vector of solutions for all criterion weights and distance thresholds. Each of these vectors implies a different sorting of the SKUs. For each SKU, the frequency of sorting into $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ can be calculated. (A sharply peaked distribution is another indicator of robustness.) Then each SKU is assigned to the group where it appears most often, for both A^+MCABC and A^-MCABC. Finally, the procedure explained in Figure 6.5 is applied to generate the final sorting result.

- *Multiple Optimal Solution Identification and Selection*

Another way to conduct the post-optimality analyses is to employ optimization software packages, such as LINGO and Matlab, in which varying the initialization of the optimization algorithm can identify multiple optimal solutions. When $\mathbf{SOR.3}(\alpha, \beta)$ has multiple solutions, the DM could select a solution which is in some sense "closest" to the imprecise weight information he or she has supplied, as in (6.9) and (6.10). For example, solutions that do not satisfy the constraint (6.9) can simply be screened out. Also, the mean of the two parameters of (6.10), $(L+U)/2$, can be employed as a centroid—in which case, the solution at minimum distance from the centroid should be regarded as best. Notice that (6.9) and (6.10) are not incorporated into $\mathbf{SOR.3}(\alpha, \beta)$ as constraints.

- *Transformation of the Objective Function*

  The sum of squared errors, $\sum_{r=1}^{n_{\mathcal{A}}}(\alpha_{\mathcal{A}}^r)^2 + \sum_{r=1}^{n_{\mathcal{B}}}[(\alpha_{\mathcal{B}}^r)^2 + (\beta_{\mathcal{B}}^r)^2] + \sum_{r=1}^{n_{\mathcal{C}}}(\beta_{\mathcal{C}}^r)^2$, the objective function in $\mathbf{SOR.3}(\alpha, \beta)$, measures the overall error in representation of the entire case set. Because it is similar to linear regression in statistics, this representation may be easily understood by DMs. Nevertheless, there are other ways to express the overall error. For instance, following the example of Siskos and Yannacopoulos (1985) and Siskos et al. (2005), $\sum_{r=1}^{n_{\mathcal{A}}}(-\alpha_{\mathcal{A}}^r) + \sum_{r=1}^{n_{\mathcal{B}}}[(-\alpha_{\mathcal{B}}^r) + \beta_{\mathcal{B}}^r] + \sum_{r=1}^{n_{\mathcal{C}}}(\beta_{\mathcal{C}}^r)$ also measures the overall error. In other words, this procedure transforms $\mathbf{SOR.3}(\alpha, \beta)$ is transformed into a linear rather than a quadratic program. The same procedures described by Siskos et al. (2005) can then be employed to carry out post-optimality analyses. The constraints (6.9) and (6.10) may be incorporated in $\mathbf{SOR.3}(\alpha, \beta)$, depending on the DM's available information.

6.5 A Case Study in Hospital Inventory Management

6.5.1 Background

A case study to demonstrate the proposed procedure is carried out based upon data in Flore et al. (1992) on a hospital inventory management. In that example, 47 disposable SKUs used in a hospital-based respiratory therapy unit are classified using the AHP (Saaty, 1980) based MCABC. Table 6.1 lists the 47 disposable SKUs referred to as S1 through S47. Four criteria are defined for MCABC analysis: (1) average unit cost ($), which ranges from a low $5.12 to a high of $210.00; (2) annual dollar usage ($), which ranges from $25.38 to a high of $5840.64; (3) critical factor, 1, 0.50, or 0.01 is assigned to the 47 disposable SKUs. A value of 1 indicates very critical, a value of 0.50 indicates moderately critical and a value of 0.01 for non-critical; (4) lead time (week) is the time it takes to receive replenishment after it is ordered, ranging from 1 to 7 weeks.

6.5.2 Selection of Case Sets

In this case study, all criteria are assumed to be positive criteria, which means the greater value of the consequence, the more important it is for the DM. Note that for a product buyer, like hospitals, lead time is a positive criterion while for a producer it may be a negative criterion. The settings of A^+, A^- are listed in Table 6.2. It is assumed that the DM would like to provide the case information, three representative SKUs for $\mathcal{A}$, four for $\mathcal{B}$, and four for $\mathcal{C}$ among those 47 SKUs. We start with the most representative case set as shown in Table 6.2. Based on this information, the normalized consequence data of case sets for A^+MCABC and A^-MCABC are calculated using (6.1) and (6.2), and listed in Tables 6.3 and 6.4, respectively.

Based on this information, the normalized consequence data of case sets for A^+MCABC and A^-MCABC are calculated using (6.1) and (6.2), and listed in Table 6.3, and 6.4, respectively.

6.5.3 Model Construction

First, **SOR.3**(α, β) is employed to find $\mathbf{w}^+ = \{w_1^+, w_2^+, w_3^+, w_4^+\}$ as well as $R_{\mathcal{A}}^+$ and $R_{\mathcal{B}}^+$, which represent the weights for the average unit cost, the annual dollar usage, the critical factor, the lead time, and the distance thresholds for **A** and **B** in A^+ABC, respectively. The imprecise weight information is assumed to be as follows: for $j = 1, 2, 3, 4$, $0.01 \leq w_j^+ \leq 0.9$; $w_2^+ \geq w_1^+$; $w_2^+ \geq w_3^+$; $w_2^+ \geq w_4^+$. These two groups of constraints guarantee that each weight is positive and hence that each criterion contributes to the classification, and that the dollar usage is the most important criterion.

The results found using Lingo software (Lingo, 2005) are $ERR = 1.1878 \times 10^{-9}$; $\mathbf{w}^+ = (0.0915, 0.3118, 0.2987, 0.2980)$, $R_{\mathcal{A}}^+ = 0.2700$ and $R_{\mathcal{B}}^+ = 0.6717$. Assuming $\varepsilon = \frac{1}{n} = \frac{1}{47}$, since $\varepsilon \ll ERR$, the error can be ignored.

The optimization problem is:
Minimize : $ERR = (\alpha_{\mathcal{A}}^1)^2 + (\alpha_{\mathcal{A}}^2)^2 + (\alpha_{\mathcal{A}}^3)^2 + (\alpha_{\mathcal{B}}^1)^2 + (\alpha_{\mathcal{B}}^2)^2 + (\alpha_{\mathcal{B}}^3)^2 + (\alpha_{\mathcal{B}}^4)^2 + (\beta_{\mathcal{B}}^1)^2 +$

Table 6.1: Listing of SKUs with Multiple Criteria, adapted from Flore et al. (1992)

SKUs	Criteria			
	Average unit cost ($)	Annual dollar usage ($)	Critical factor	Lead time (week)
S1	49.92	5840.64	1	2
S2	210.00	5670.00	1	5
S3	23.76	5037.12	1	4
S4	27.73	4769.56	0.01	1
S5	57.98	3478.80	0.5	3
S6	31.24	2936.67	0.5	3
S7	28.20	2820.00	0.5	3
S8	55.00	2640.00	0.01	4
S9	73.44	2423.52	1	6
S10	160.50	2407.50	0.5	4
S11	5.12	1075.20	1	2
S12	20.87	1043.50	0.5	5
S13	86.50	1038.00	1	7
S14	110.40	883.20	0.5	5
S15	71.20	854.40	1	3
S16	45.00	810.00	0.5	3
S17	14.66	703.68	0.5	4
S18	49.50	594.00	0.5	6
S19	47.50	570.00	0.5	5
S20	58.45	467.60	0.5	4
S21	24.40	463.60	1	4
S22	65.00	455.00	0.5	4
S23	86.50	432.50	1	4
S24	33.20	398.40	1	3
S25	37.05	370.50	0.01	1
S26	33.84	338.40	0.01	3
S27	84.03	336.12	0.01	1
S28	78.40	313.60	0.01	6
S29	134.34	268.68	0.01	7
S30	56.00	224.00	0.01	1
S31	72.00	216.00	0.5	5
S32	53.02	212.08	1	2
S33	49.48	197.92	0.01	5
S34	7.07	190.89	0.01	7
S35	60.60	181.80	0.01	3
S36	40.82	163.28	1	3
S37	30.00	150.00	0.01	5
S38	67.40	134.80	0.5	3
S39	59.60	119.20	0.01	5
S40	51.68	103.36	0.01	6
S41	19.80	79.20	0.01	2
S42	37.70	75.40	0.01	2
S43	29.89	59.78	0.01	5
S44	48.30	48.30	0.01	3
S45	34.40	34.40	0.01	7
S46	28.80	28.80	0.01	3
S47	8.46	25.38	0.01	5

Table 6.2: The Basic Information Settings

SKUs		Criteria			
		Average unit cost ($)	Annual dollar usage ($)	Critical factor	Lead time (week)
A^+		250.00	6000.00	1	7
A^-		1.00	10.00	0	1
$d_j^{\max}$		62001.00	35880100.00	1.00	36.00
$\mathbf{T}_{\mathcal{A}}$	S1	49.92	5840.64	1.00	2.00
	S2	210.00	5670.00	1.00	5.00
	S13	86.50	1038.00	1.00	7.00
$\mathbf{T}_{\mathcal{B}}$	S10	160.50	2407.50	0.50	4.00
	S29	134.34	268.68	0.01	7.00
	S36	40.82	163.28	1.00	3.00
	S45	34.40	34.40	1.00	7.00
$\mathbf{T}_{\mathcal{C}}$	S4	27.73	4769.56	0.01	1.00
	S25	37.50	370.50	0.01	1.00
	S27	84.03	336.12	0.01	1.00
	S34	7.07	190.89	0.01	7.00

Table 6.3: The Normalized Consequence Data of Case Sets for A^+MCABC

SKUs		Criteria			
		Average unit cost ($)	Annual dollar usage ($)	Critical factor	Lead time (week)
$\mathbf{T}_{\mathcal{A}}$	S1	0.6457	0.0007	0.0000	0.6944
	S2	0.0258	0.0030	0.0000	0.1111
	S13	0.4312	0.6862	0.0000	0.0000
$\mathbf{T}_{\mathcal{B}}$	S10	0.1292	0.3597	0.2500	0.2500
	S29	0.2158	0.9155	0.9801	0.0000
	S36	0.7057	0.9495	0.0000	0.4444
	S45	0.7497	0.9919	0.9801	0.0000
$\mathbf{T}_{\mathcal{C}}$	S4	0.7968	0.0422	0.9801	1.0000
	S25	0.7314	0.8833	0.9801	1.0000
	S27	0.4443	0.8941	0.9801	1.0000
	S34	0.9518	0.9405	0.9801	0.0000

Table 6.4: The Normalized Consequence Data of Case Sets for A^{-}MCABC

SKUs		Criteria			
		Average unit cost ($)	Annual dollar usage ($)	Critical factor	Lead time (week)
	S1	0.0386	0.9475	1.0000	0.0278
$\mathbf{T}_{\mathcal{A}}$	S2	0.7045	0.8929	1.0000	0.4444
	S13	0.1179	0.0295	1.0000	1.0000
	S10	0.4103	0.1602	0.2500	0.2500
	S29	0.2868	0.0019	0.0001	1.0000
$\mathbf{T}_{\mathcal{B}}$	S36	0.0256	0.0007	1.0000	0.1111
	S45	0.0180	0.0000	0.0001	1.0000
	S4	0.0115	0.6314	0.0001	0.0000
$\mathbf{T}_{\mathcal{C}}$	S25	0.0210	0.0036	0.0001	0.0000
	S27	0.1112	0.0030	0.0001	0.0000
	S34	0.0006	0.0009	0.0001	1.0000

$(\beta_{\mathcal{B}}^{2})^2 + (\beta_{\mathcal{B}}^{3})^2 + (\beta_{\mathcal{B}}^{4})^2 + (\beta_{\mathcal{C}}^{1})^2 + (\beta_{\mathcal{C}}^{2})^2 + (\beta_{\mathcal{C}}^{3})^2 + (\beta_{\mathcal{C}}^{4})^2$

Subject to:

$0.6457w_1^+ + 0.0007w_2^+ + 0.6944w_4^+ + \alpha_{\mathcal{A}}^1 \leq R_{\mathcal{A}}^+;$
$0.0258w_1^+ + 0.0030w_2^+ + 0.1111w_4^+ + \alpha_{\mathcal{A}}^2 \leq R_{\mathcal{A}}^+;$
$0.4312w_1^+ + 0.6862w_2^+ + \alpha_{\mathcal{A}}^3 \leq R_{\mathcal{A}}^+;$
$0.1292w_1^+ + 0.3597w_2^+ + 0.2500w_3^+ + 0.2500w_4^+ + \alpha_{\mathcal{B}}^1 \leq R_{\mathcal{B}}^+;$
$0.2158w_1^+ + 0.9155w_2^+ + 0.9801w_3^+ + \alpha_{\mathcal{B}}^2 \leq R_{\mathcal{B}}^+;$
$0.7057w_1^+ + 0.9495w_2^+ + 0.4444w_4^+ + \alpha_{\mathcal{B}}^3 \leq R_{\mathcal{B}}^+;$
$0.7497w_1^+ + 0.9919w_2^+ + 0.9801w_3^+ + \alpha_{\mathcal{B}}^4 \leq R_{\mathcal{B}}^+;$
$0.1292w_1^+ + 0.3597w_2^+ + 0.2500w_3^+ + 0.2500w_4^+ + \beta_{\mathcal{B}}^1 \geq R_{\mathcal{A}}^+;$
$0.2158w_1^+ + 0.9155w_2^+ + 0.9801w_3^+ + \beta_{\mathcal{B}}^2 \geq R_{\mathcal{A}}^+;$
$0.7057w_1^+ + 0.9495w_2^+ + 0.4444w_4^+ + \beta_{\mathcal{B}}^3 \geq R_{\mathcal{A}}^+;$
$0.7497w_1^+ + 0.9919w_2^+ + 0.9801w_3^+ + \beta_{\mathcal{B}}^4 \geq R_{\mathcal{A}}^+;$
$0.7968w_1^+ + 0.0422w_2^+ + 0.9801w_3^+ + 1.0000w_4^+ + \beta_{\mathcal{C}}^1 \geq R_{\mathcal{B}}^+;$
$0.7314w_1^+ + 0.8833w_2^+ + 0.9801w_3^+ + 1.0000w_4^+ + \beta_{\mathcal{C}}^2 \geq R_{\mathcal{B}}^+;$
$0.4443w_1^+ + 0.8941w_2^+ + 0.9801w_3^+ + 1.0000w_4^+ + \beta_{\mathcal{C}}^3 \geq R_{\mathcal{B}}^+;$
$0.9518w_1^+ + 0.9405w_2^+ + 0.9801w_3^+ + \beta_{\mathcal{C}}^4 \geq R_{\mathcal{B}}^+;$
$0 \leq R_{A}^+ \leq 1,\ 0 \leq R_{\mathcal{B}}^+ \leq 1,\ R_{\mathcal{A}}^+ < R_{\mathcal{B}}^+;$
$-1 \leq \alpha_{\mathcal{A}}^1 \leq 0,\ -1 \leq \alpha_{\mathcal{A}}^2 \leq 0,\ -1 \leq \alpha_{\mathcal{A}}^3 \leq 0;$
$-1 \leq \alpha_{\mathcal{B}}^1 \leq 0,\ -1 \leq \alpha_{\mathcal{B}}^2 \leq 0,\ -1 \leq \alpha_{\mathcal{B}}^3 \leq 0,\ -1 \leq \alpha_{\mathcal{B}}^4 \leq 0;$
$0 \leq \beta_{\mathcal{B}}^1 \leq 1,\ 0 \leq \beta_{\mathcal{B}}^2 \leq 1,\ 0 \leq \beta_{\mathcal{B}}^3 \leq 1,\ 0 \leq \beta_{\mathcal{B}}^4 \leq 1;$
$0 \leq \beta_{\mathcal{C}}^1 \leq 1,\ 0 \leq \beta_{\mathcal{C}}^2 \leq 1,\ 0 \leq \beta_{\mathcal{C}}^3 \leq 1,\ 0 \leq \beta_{\mathcal{C}}^4 \leq 1;$

$0.01 \leq w_1^+ \leq 0.9$, $0.01 \leq w_2^+ \leq 0.9$, $0.01 \leq w_3^+ \leq 0.9$, $0.01 \leq w_4^+ \leq 0.9$;
$w_2^+ \geq w_1^+$; $w_2^+ \geq w_3^+$; $w_2^+ \geq w_4^+$;
$w_1^+ + w_2^+ + w_3^+ + w_4^+ = 1$.

Similar procedures are carried out for A^-MCABC. The details are omitted, and the results obtained are listed as: $ERR = 9.9018 \times 10^{-10}$; $\mathbf{w}^- = (0.2039, 0.3138, 0.2639, 0.2184)$; $R_{\mathcal{C}}^- = 0.2205$ and $R_{\mathcal{B}}^- = 0.4502$. As $\varepsilon \ll ERR$, the error is ignored. Then, both A^+ABC and A^-ABC methods are applied to classify the 47 SKUs into $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. The re-classification procedures shown in Figure 6.5 are implemented and the results are shown in Table 6.5.

Table 6.5: The Results of A^+ABC and A^-ABC Classification

SKUs	$D(A^i)^+$	A^+ABC results	$D(A^i)^-$	A^-ABC results	Final results
S1	0.2662	$\mathcal{A}^+$	0.5751	$\mathcal{A}^-$	$\mathcal{A}$
S2	0.0364	$\mathcal{A}^+$	0.7848	$\mathcal{A}^-$	$\mathcal{A}$
S3	0.1581	$\mathcal{A}^+$	0.5412	$\mathcal{A}^-$	$\mathcal{A}$
S4	0.6768	$\mathcal{C}^+$	0.2005	$\mathcal{C}^-$	$\mathcal{C}$
S5	0.3168	$\mathcal{B}^+$	0.2062	$\mathcal{C}^-$	$\mathcal{C}$
S6	0.3593	$\mathcal{B}^+$	0.1682	$\mathcal{C}^-$	$\mathcal{C}$
S7	0.3676	$\mathcal{B}^+$	0.1617	$\mathcal{C}^-$	$\mathcal{C}$
S8	0.5215	$\mathcal{B}^+$	0.1247	$\mathcal{C}^-$	$\mathcal{C}$
S9	0.1654	$\mathcal{A}^+$	0.4837	$\mathcal{A}^-$	$\mathcal{A}$
S10	0.2731	$\mathcal{B}^+$	0.2545	$\mathcal{B}^-$	$\mathcal{B}$
S11	0.5062	$\mathcal{B}^+$	0.2799	$\mathcal{B}^-$	$\mathcal{B}$
S12	0.3988	$\mathcal{B}^+$	0.1737	$\mathcal{C}^-$	$\mathcal{C}$
S13	0.2534	$\mathcal{A}^+$	0.5156	$\mathcal{A}^-$	$\mathcal{A}$
S14	0.3641	$\mathcal{B}^+$	0.2091	$\mathcal{C}^-$	$\mathcal{C}$
S15	0.4097	$\mathcal{B}^+$	0.3106	$\mathcal{B}^-$	$\mathcal{B}$
S16	0.5032	$\mathcal{B}^+$	0.1022	$\mathcal{C}^-$	$\mathcal{C}$
S17	0.4747	$\mathcal{B}^+$	0.1254	$\mathcal{C}^-$	$\mathcal{C}$
S18	0.3963	$\mathcal{B}^+$	0.2284	$\mathcal{B}^-$	$\mathcal{B}$
S19	0.4245	$\mathcal{B}^+$	0.1729	$\mathcal{C}^-$	$\mathcal{C}$
S20	0.4693	$\mathcal{B}^+$	0.1333	$\mathcal{C}^-$	$\mathcal{C}$
S21	0.4160	$\mathcal{B}^+$	0.3221	$\mathcal{B}^-$	$\mathcal{B}$
S22	0.4669	$\mathcal{B}^+$	0.1358	$\mathcal{C}^-$	$\mathcal{C}$
S23	0.3833	$\mathcal{B}^+$	0.3441	$\mathcal{B}^-$	$\mathcal{B}$
S24	0.4745	$\mathcal{B}^+$	0.2929	$\mathcal{B}^-$	$\mathcal{B}$
S25	0.9331	$\mathcal{C}^+$	0.0054	$\mathcal{C}^-$	$\mathcal{C}$
S26	0.7727	$\mathcal{C}^+$	0.0288	$\mathcal{C}^-$	$\mathcal{C}$
S27	0.9102	$\mathcal{C}^+$	0.0236	$\mathcal{C}^-$	$\mathcal{C}$
S28	0.6255	$\mathcal{B}^+$	0.1722	$\mathcal{C}^-$	$\mathcal{C}$
S29	0.5980	$\mathcal{B}^+$	0.2775	$\mathcal{B}^-$	$\mathcal{B}$
S30	0.9362	$\mathcal{C}^+$	0.0104	$\mathcal{C}^-$	$\mathcal{C}$
S31	0.4453	$\mathcal{B}^+$	0.1800	$\mathcal{C}^-$	$\mathcal{C}$
S32	0.5553	$\mathcal{B}^+$	0.2792	$\mathcal{B}^-$	$\mathcal{B}$
S33	0.6778	$\mathcal{C}^+$	0.1051	$\mathcal{C}^-$	$\mathcal{C}$
S34	0.6731	$\mathcal{C}^+$	0.2188	$\mathcal{C}^-$	$\mathcal{C}$
S35	0.7723	$\mathcal{C}^+$	0.0362	$\mathcal{C}^-$	$\mathcal{C}$
S36	0.4931	$\mathcal{B}^+$	0.2936	$\mathcal{B}^-$	$\mathcal{B}$
S37	0.6947	$\mathcal{C}^+$	0.1000	$\mathcal{C}^-$	$\mathcal{C}$
S38	0.5553	$\mathcal{B}^+$	0.1049	$\mathcal{C}^-$	$\mathcal{C}$
S39	0.6799	$\mathcal{C}^+$	0.1085	$\mathcal{C}^-$	$\mathcal{C}$
S40	0.6612	$\mathcal{B}^+$	0.1602	$\mathcal{C}^-$	$\mathcal{C}$
S41	0.8825	$\mathcal{C}^+$	0.0073	$\mathcal{C}^-$	$\mathcal{C}$
S42	0.8712	$\mathcal{C}^+$	0.0106	$\mathcal{C}^-$	$\mathcal{C}$
S43	0.7040	$\mathcal{C}^+$	0.0999	$\mathcal{C}^-$	$\mathcal{C}$
S44	0.7931	$\mathcal{C}^+$	0.0317	$\mathcal{C}^-$	$\mathcal{C}$
S45	0.6706	$\mathcal{B}^+$	0.2221	$\mathcal{B}^-$	$\mathcal{B}$
S46	0.8073	$\mathcal{C}^+$	0.0268	$\mathcal{C}^-$	$\mathcal{C}$
S47	0.7222	$\mathcal{C}^+$	0.0973	$\mathcal{C}^-$	$\mathcal{C}$

6.5.4 Post-optimality Analyses

The percentage value method, one of the techniques described in Section 6.4.5, is chosen to demonstrate post-optimality analysis.

- The post-optimality programs for A^+MCABC, $\mathbf{SOR.3}'(\alpha, \beta, w_j)$ and $\mathbf{SOR.3}''(\alpha, \beta, w_j)$ are formulated for each criterion weight, w_j, and distance threshold, $R^+_{\mathcal{A}}$ and $R^+_{\mathcal{B}}$. The minimum threshold η is fixed at 0.01. The results are listed in Table 6.6.

Table 6.6: Post-optimality Analyses and Final Solutions for A^+MCABC

	Criterion weights				Distance thresholds	
	w_1^+	w_2^+	w_3^+	w_4^+	$R^+_{\mathcal{A}}$	$R^+_{\mathcal{B}}$
1. Initial solution	0.0915	0.3118	0.2987	0.2980	0.2700	0.6717
2. max(w_1^+)	0.1141	0.3022	0.3022	0.2816	0.2694	0.6814
3. min(w_1^+)	0.0779	0.3074	0.3074	0.3074	0.2696	0.6645
4. max(w_2^+)	0.0861	0.3395	0.2562	0.3183	0.2768	0.6523
5. min(w_2^+)	0.0962	0.3013	0.3013	0.3013	0.2715	0.6681
6. max(w_3^+)	0.0794	0.3134	0.3134	0.2938	0.2673	0.6775
7. min(w_3^+)	0.0861	0.3394	0.2562	0.3183	0.2768	0.6523
8. max(w_4^+)	0.0813	0.3206	0.2774	0.3206	0.2754	0.6509
9. min(w_4^+)	0.1141	0.3022	0.3022	0.2816	0.2694	0.6814
10. max($R^+_{\mathcal{A}}$)	0.0861	0.3395	0.2561	0.3183	0.2769	0.6523
11. min($R^+_{\mathcal{A}}$)	0.0794	0.3134	0.3134	0.2938	0.2555	0.6775
12. max($R^+_{\mathcal{B}}$)	0.1083	0.3019	0.3019	0.2879	0.2701	0.6829
13. min($R^+_{\mathcal{B}}$)	0.0813	0.3206	0.2774	0.3206	0.2754	0.6509

Based on the information in Table 6.6, all 13 sortings of the 47 SKUs were generated; for each SKU, the percentage of sortings into $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are shown in Table 6.7. Table 6.7 also shows the final sorting for the A^+MCABC method, based on the rule that the group with the largest percentage is used to represent the sorting result for an SKU. Most of the sorting results are quite robust; only $S4$ is ambiguous, in that the percentages in $\mathcal{B}$ and $\mathcal{C}$ are close.

- The post-optimality programs for A^-MCABC were solved similarly; the results are shown in Tables 6.8 and 6.9. In this case, only $S5$, $S10$ $S18$ and $S45$ do not produce robust sortings. In this case study, the A^+MCABC method is more robust than the A^-MCABC method.

Table 6.7: Percentage Value Method Based Post-optimality Analyses for A^+ABC

SKUs	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$	Final results
S1	100.00%	0.00%	0.00%	$\mathcal{A}^+$
S2	100.00%	0.00%	0.00%	$\mathcal{A}^+$
S3	100.00%	0.00%	0.00%	$\mathcal{A}^+$
S4	0.00%	46.15%	53.85%	$\mathcal{C}^+$
S5	0.00%	100.00%	0.00%	$\mathcal{B}^+$
S6	0.00%	100.00%	0.00%	$\mathcal{B}^+$
S7	0.00%	100.00%	0.00%	$\mathcal{B}^+$
S8	0.00%	100.00%	0.00%	$\mathcal{B}^+$
S9	100.00%	0.00%	0.00%	$\mathcal{A}^+$
S10	38.46%	61.54%	0.00%	$\mathcal{B}^+$
S11	0.00%	100.00%	0.00%	$\mathcal{B}^+$
S12	0.00%	100.00%	0.00%	$\mathcal{B}^+$
S13	92.31%	7.69%	0.00%	$\mathcal{A}^+$
S14	0.00%	100.00%	0.00%	$\mathcal{B}^+$
S15	0.00%	100.00%	0.00%	$\mathcal{B}^+$
S16	0.00%	100.00%	0.00%	$\mathcal{B}^+$
S17	0.00%	100.00%	0.00%	$\mathcal{B}^+$
S18	0.00%	100.00%	0.00%	$\mathcal{B}^+$
S19	0.00%	100.00%	0.00%	$\mathcal{B}^+$
S20	0.00%	100.00%	0.00%	$\mathcal{B}^+$
S21	0.00%	100.00%	0.00%	$\mathcal{B}^+$
S22	0.00%	100.00%	0.00%	$\mathcal{B}^+$
S23	0.00%	100.00%	0.00%	$\mathcal{B}^+$
S24	0.00%	100.00%	0.00%	$\mathcal{B}^+$
S25	0.00%	0.00%	100.00%	$\mathcal{C}^+$
S26	0.00%	0.00%	100.00%	$\mathcal{C}^+$
S27	0.00%	0.00%	100.00%	$\mathcal{C}^+$
S28	0.00%	100.00%	0.00%	$\mathcal{B}^+$
S29	0.00%	100.00%	0.00%	$\mathcal{B}^+$
S30	0.00%	0.00%	100.00%	$\mathcal{C}^+$
S31	0.00%	100.00%	0.00%	$\mathcal{B}^+$
S32	0.00%	100.00%	0.00%	$\mathcal{B}^+$
S33	0.00%	15.38%	84.62%	$\mathcal{C}^+$
S34	0.00%	30.77%	69.23%	$\mathcal{C}^+$
S35	0.00%	0.00%	100.00%	$\mathcal{C}^+$
S36	0.00%	100.00%	0.00%	$\mathcal{B}^+$
S37	0.00%	7.69%	92.31%	$\mathcal{C}^+$
S38	0.00%	100.00%	0.00%	$\mathcal{B}^+$
S39	0.00%	15.38%	84.62%	$\mathcal{C}^+$
S40	0.00%	92.31%	7.69%	$\mathcal{B}^+$
S41	0.00%	0.00%	100.00%	$\mathcal{C}^+$
S42	0.00%	0.00%	100.00%	$\mathcal{C}^+$
S43	0.00%	0.00%	100.00%	$\mathcal{C}^+$
S44	0.00%	0.00%	100.00%	$\mathcal{C}^+$
S45	0.00%	100.00%	0.00%	$\mathcal{B}^+$
S46	0.00%	0.00%	100.00%	$\mathcal{C}^+$
S47	0.00%	0.00%	100.00%	$\mathcal{C}^+$

Table 6.8: Post-optimality Analyses and Final Solutions for A^{-}MCABC

	Criterion weights				Distance thresholds	
	w_1^-	w_2^-	w_3^-	w_4^-	$R_{\mathcal{C}}^-$	$R_{\mathcal{B}}^-$
Initial solution	0.2039	0.3138	0.2639	0.2184	0.2205	0.4502
$\max(w_1^-)$	0.3153	0.3153	0.1725	0.1970	0.2027	0.3786
$\min(w_1^-)$	0.0567	0.3584	0.3584	0.2266	0.2269	0.5208
$\max(w_2^-)$	0.1676	0.3813	0.2115	0.2396	0.2427	0.4308
$\min(w_2^-)$	0.2500	0.2500	0.2500	0.2500	0.2523	0.4445
$\max(w_3^-)$	0.0571	0.3585	0.3585	0.2259	0.2270	0.5206
$\min(w_3^-)$	0.3138	0.3138	0.1711	0.2013	0.2017	0.3806
$\max(w_4^-)$	0.2416	0.2646	0.2292	0.2646	0.2650	0.4415
$\min(w_4^-)$	0.2759	0.2759	0.2759	0.1723	0.1774	0.4305
$\max(R_{\mathcal{C}}^-)$	0.2463	0.2616	0.2305	0.2616	0.2661	0.4403
$\min(R_{\mathcal{C}}^-)$	0.2746	0.2746	0.2746	0.1761	0.1765	0.4316
$\max(R_{\mathcal{B}}^-)$	0.0567	0.3584	0.3584	0.2266	0.2269	0.6022
$\min(R_{\mathcal{B}}^-)$	0.2729	0.3005	0.2387	0.1879	0.1929	0.2667

- The final sorting, based on the re-arrangement procedure described in Figure 6.5, is shown in Table 6.10.

Table 6.9: Percentage Value Method Based Post-optimality Analyses for A^{-}ABC

SKUs	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$	Final results
S1	100.00%	0.00%	0.00%	$\mathcal{A}^{-}$
S2	100.00%	0.00%	0.00%	$\mathcal{A}^{-}$
S3	100.00%	0.00%	0.00%	$\mathcal{A}^{-}$
S4	0.00%	7.69%	92.31%	$\mathcal{C}^{-}$
S5	0.00%	53.85%	46.15%	$\mathcal{B}^{-}$
S6	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S7	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S8	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S9	84.62%	15.38%	0.00%	$\mathcal{A}^{-}$
S10	0.00%	53.85%	46.15%	$\mathcal{B}^{-}$
S11	0.00%	61.54%	38.46%	$\mathcal{B}^{-}$
S12	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S13	100.00%	0.00%	0.00%	$\mathcal{A}^{-}$
S14	0.00%	30.77%	69.23%	$\mathcal{C}^{-}$
S15	7.69%	92.31%	0.00%	$\mathcal{B}^{-}$
S16	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S17	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S18	0.00%	53.85%	46.15%	$\mathcal{B}^{-}$
S19	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S20	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S21	7.69%	92.31%	0.00%	$\mathcal{B}^{-}$
S22	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S23	7.69%	92.31%	0.00%	$\mathcal{B}^{-}$
S24	0.00%	69.23%	30.77%	$\mathcal{B}^{-}$
S25	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S26	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S27	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S28	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S29	0.00%	100.00%	0.00%	$\mathcal{B}^{-}$
S30	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S31	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S32	0.00%	61.54%	38.46%	$\mathcal{B}^{+}$
S33	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S34	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S35	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S36	0.00%	61.54%	38.46%	$\mathcal{B}^{-}$
S37	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S38	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S39	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S40	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S41	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S42	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S43	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S44	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S45	0.00%	53.85%	46.15%	$\mathcal{B}^{-}$
S46	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$
S47	0.00%	0.00%	100.00%	$\mathcal{C}^{-}$

Table 6.10: The Final Sorting Results for the Percentage Value Method

SKUs	A^{+}ABC	A^{-}ABC	Final results
S1	$\mathcal{A}^{+}$	$\mathcal{A}^{-}$	$\mathcal{A}$
S2	$\mathcal{A}^{+}$	$\mathcal{A}^{-}$	$\mathcal{A}$
S3	$\mathcal{A}^{+}$	$\mathcal{A}^{-}$	$\mathcal{A}$
S4	$\mathcal{C}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S5	$\mathcal{B}^{+}$	$\mathcal{B}^{-}$	$\mathcal{B}$
S6	$\mathcal{B}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S7	$\mathcal{B}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S8	$\mathcal{B}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S9	$\mathcal{A}^{+}$	$\mathcal{A}^{-}$	$\mathcal{A}$
S10	$\mathcal{B}^{+}$	$\mathcal{B}^{-}$	$\mathcal{B}$
S11	$\mathcal{B}^{+}$	$\mathcal{B}^{-}$	$\mathcal{B}$
S12	$\mathcal{B}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S13	$\mathcal{A}^{+}$	$\mathcal{A}^{-}$	$\mathcal{A}$
S14	$\mathcal{B}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S15	$\mathcal{B}^{+}$	$\mathcal{B}^{-}$	$\mathcal{B}$
S16	$\mathcal{B}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S17	$\mathcal{B}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S18	$\mathcal{B}^{+}$	$\mathcal{B}^{-}$	$\mathcal{B}$
S19	$\mathcal{B}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S20	$\mathcal{B}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S21	$\mathcal{B}^{+}$	$\mathcal{B}^{-}$	$\mathcal{B}$
S22	$\mathcal{B}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S23	$\mathcal{B}^{+}$	$\mathcal{B}^{-}$	$\mathcal{B}$
S24	$\mathcal{B}^{+}$	$\mathcal{B}^{-}$	$\mathcal{B}$
S25	$\mathcal{C}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S26	$\mathcal{C}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S27	$\mathcal{C}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S28	$\mathcal{B}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S29	$\mathcal{B}^{+}$	$\mathcal{B}^{-}$	$\mathcal{B}$
S30	$\mathcal{C}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S31	$\mathcal{B}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S32	$\mathcal{B}^{+}$	$\mathcal{B}^{+}$	$\mathcal{B}$
S33	$\mathcal{C}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S34	$\mathcal{C}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S35	$\mathcal{C}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S36	$\mathcal{B}^{+}$	$\mathcal{B}^{-}$	$\mathcal{B}$
S37	$\mathcal{C}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S38	$\mathcal{B}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S39	$\mathcal{C}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S40	$\mathcal{B}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S41	$\mathcal{C}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S42	$\mathcal{C}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S43	$\mathcal{C}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S44	$\mathcal{C}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S45	$\mathcal{B}^{+}$	$\mathcal{B}^{-}$	$\mathcal{B}$
S46	$\mathcal{C}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$
S47	$\mathcal{C}^{+}$	$\mathcal{C}^{-}$	$\mathcal{C}$

6.5.5 Comparisons and Explanation

Table 6.11: Comparison of Results with the Flores et al. Method (Flore et al., 1992)

		Case based distance model			
		$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$	Total
The	$\mathcal{A}$	5	5	0	10
AHP	$\mathcal{B}$	0	7	7	14
method	$\mathcal{C}$	0	0	23	23
	Total	5	12	30	47

Table 6.11 shows a comparison of the classification outcomes in Table 6.10 with the AHP findings of Flore et al. (1992). Some of the main results are explained below:

- There are no inconsistent classifications in the most important group, $\mathcal{A}$. In the AHP method, there are ten SKUs in $\mathcal{A}$ while our method produces five SKUs, which are all included in the top AHP group.
- There are five different classifications in group $\mathcal{B}$ and seven in group $\mathcal{C}$. The proportions of the number of SKUs in the two groups are 14/23 for the AHP method and 12/30 for our method. Both methods contain roughly consistent information: assign a larger number of SKUs to group $\mathcal{C}$, similar to the traditional ABC analysis.
- The weight generation mechanisms are different: the AHP method estimates a weight set by subjective judgements to suit all situations, while our method uses quadratic programming to estimate the weights. Based on the distance to an ideal SKU and an anti-ideal SKU, different weights are obtained. In our method, a weight for a criterion is connected with value (preference on consequences) in that when the definitions of values change, the weight sets are different. Because of its clear geometric meaning, our method can be readily understood and may thereby be more easily accepted by a DM.
- It is worth mentioning that the classification results in Flore et al. (1992) do not necessarily provide a benchmark to evaluate the merits or limitations of other methods. Because the proportions of SKUs in groups $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are 5/47, 12/47, and 30/47, respectively, which are close to the 80-20 rule that

is observed in many practical inventory systems, our model provides a sound classification result.

6.6 Conclusions

The classical ABC analysis is a straightforward approach that assists a DM in achieving cost-effective inventory management by arranging SKUs according to their annual dollar usages. However, in many situations, the DM should consider other criteria, such as lead time and criticality, in addition to annual dollar usage. MCABC procedures furnish an inventory manager with additional flexibility to account for more factors in classifying SKUs. This chapter proposes a case-based distance model to handle MCABC problems under the umbrella of MCDA theory. A case study is developed to illustrate how the procedure can be applied; the results demonstrate that this approach is robust and can produce sound classifications of SKUs when multiple criteria are to be considered.

Chapter 7

Sorting Problem Extension in Negotiation

7.1 Introduction

A case-based distance model founded on multiple criteria decision analysis theory is proposed for bilateral negotiations (BN) involving multiple issues. The unique feature of this negotiation model is that weighted Euclidean distance is employed to represent the negotiators' preferences; a case-based distance algorithm then helps negotiators express their preferences over different offers (alternatives) and suggests how to find better outcomes. The procedure takes advantage of the easily understood geometric meaning of Euclidean distance. The remainder of this chapter is organized as follows. Section 2 provides a brief literature review of group negotiation and decision. Next, Section 3 gives a basic background of multiple criteria decision analysis and defines a BN problem. Section 4 proposes a case-based distance model to solve this problem, while Section 5 presents a case study of BN in a business context. Finally, some conclusions are presented in Section 6. The contributions of this chapter are based upon research by Chen et al. (Chen et al., 2005f).

7.2 Motivation

Negotiation is an important research topic in many disciplines including social sciences, economics, game theory, decision support systems, engineering, and multiagent theory. In practice, people negotiate on a very broad range of subjects including diplomatic issues, international conflicts, meeting schedules, production plans and purchases. Within the few last decades, many methodologies have been proposed to study negotiations. Some of them are summarized next.

Pruitt (1981) studies negotiations from the point of view of social psychology, emphasizing cognitive processes. Many explanations are given for the motives, perceptions, and other micro-processes driving a negotiator's behavior, supported by evidence from laboratory experiments. Raiffa's influential book (Raiffa et al., 2002) divides negotiations into several classes according to the number of negotiators and the number of issues involved: two negotiators with one issue, two negotiators with many issues, or many negotiators with many issues. Here we address a two-negotiator multi-issue problem. Game theory provides a mathematical study of rational behavior in conflicts: models of negotiation address whether an agreement can be reached and how protocols that achieve Pareto optimal solutions should be designed (Brams, 2003). For example, based on multiple attribute utility theory, Ehtamo et al. (1999) design an interactive method to assist negotiators in moving from an initial inefficient point to an efficient solution. This method may be applicable to negotiations on political issues. Some approaches in multiple criteria decision analysis (MCDA) address group decision making and negotiations. For example, using distance minimization (related to goral programming), Kersten (1985) developed NEGO, a group decision support system, and Kersten and Szapiro (1986) introduced a general approach for structuring and modelling negotiations based on a concept of pressure.

Recently, with the development of internet-based electronic businesses, much research has focused on the application of computers and the internet to support or even automate negotiation processes (Kersten, 2004). New computing and communication technologies have introduced new opportunities for the design and deployment of software capable of supporting negotiations. Negotiations conducted over the web are commonly called *e-negotiations*, have an important role in e-marketplaces especially in personalizing and customizing processes. A few implementations of e-negotiation systems are available on the internet; for example, the special purpose system, Inspire, has been in operation since 1996 (http://interneg.org/inspire) (Kersten, 2004).

Most current e-negotiation systems employ simple additive value (utility) functions to evaluate the preferences of negotiators. But in practice many factors influence the effectiveness of human negotiations. Vetschera (2004) checked the analysis of about 4,700 multi-attribute utility functions elicited by Inspire systems and reported that negotiators' behavior during the negotiation process contradicted their imputed preferences in about 25% of all cases. He concluded that a simple additive utility function is probably not an adequate method to evaluate the performance of negotiators in such negotiations.

7.3 Multiple Issue Bilateral Negotiations

The bilateral negotiation model presented in this thesis is described in detail as follows:

(1) There are two DMs, DM_1 and DM_2, who are jointly making a decision (negotiation) in a decision problem with multiple criteria (issues), $\mathbf{Q} = \{1, 2, ..., q\}$, such as price, quantity and delivery time.

(2) Consequences on every criterion $j \in \mathbf{Q}$ are unambiguously measurable. Moreover, on each criterion $j \in \mathbf{Q}$, an indifference threshold ε_j is agreed upon by DM_1 and DM_2 (Vincke, 1992), such that differences in values of $c_j(A)$ less than ε_j are not meaningful and can be ignored. For example, differences in delivery time less than one day may not be meaningful because of the production cycle or, in the purchase of a car, people might bargain over prices in \$ 10 intervals (while, in the purchase of a television, their threshold might equal \$ 1).

(3) The maximum consequence on criterion j is $c_j^{\max} \in \mathbb{R}^+$ and the minimum consequence is $c_j^{\min} \in \mathbb{R}^+$, where $c_j^{\max}$ and $c_j^{\min}$ are known to both DM_1 and DM_2, and $c_j^{\max} > c_j^{\min}$. The interval $[c_j^{\min},\ c_j^{\max}]$ is the consequence interval for criterion j. For example, for the criterion of price, $c_j^{\max}$ and $c_j^{\min}$ might be a ceiling and floor that are determined external to the negotiation. We assume that $\frac{1}{\varepsilon_j}(c_j^{\max} - c_j^{min})$ is a positive integer for each criterion $j \in \mathbf{Q}$.

It follows from (3) that, for each criterion j, $j = 1, 2, ..., q$, the number of possible consequence values is $n_j = 1 + \frac{1}{\varepsilon_j}(c_j^{\max} - c_j^{min})$. In fact, the consequence values on criterion j (in increasing order) are $c_j^{min} < c_j^{min} + \varepsilon_j < ... < c_j^{min} + (n_j - 2)\varepsilon_j < c_j^{\max}$. Thus, the number of feasible alternatives for the BN problem is $|\mathbf{A}| = n = n_1 \cdot n_2 \cdot ... \cdot n_q$.

7.4 A Case-based Distance Model for Bilateral Negotiation

7.4.1 Case Set Assumptions

To assist DMs in providing a case set, we assume that a set of ordinally defined linguistic grades, such as {excellent, very good, good, fair, bad} can be used to assess alternatives. The linguistic grade set is denoted as $\mathbf{L} = \{1, 2, ..., g, ..., L\}$, where 1 is the best grade and L is the worst grade. For $A, B \in \mathbf{A}$, we say that $A \sim_k B$ (DM_k equally prefers A and B) whenever A and B are assigned the same linguistic grade by DM_k, and that $A \succ_k B$ (DM_k prefers A to B) whenever A is assigned a higher linguistic grade than B by DM_k.

Within the alternative set $\mathbf{A}$, suppose that DM_k ($k = 1, 2$) specifies a representative case set $\mathbf{Z}_k = \{Z_k^1, Z_k^2, ..., Z_k^r, ..., Z_k^m\} \subset \mathbf{A}$, in which each alternative is assigned a linguistic grade. Let m_k denote the number of alternatives in $\mathbf{Z}_k$. Suppose that $G_k : \mathbf{Z}_k \to \mathbf{L}$ is DM_k's assignment of grades, so that $G_k(Z_k^r) = g_k^r \in \mathbf{L}$ is the grade assigned by DM_k to Z_k^r. Furthermore, assume that at least one case is assigned to the best grade, so that $|\{Z_k^r : g_k^r = 1\}| = m_k^1 \geq 1$, for $k = 1, 2$. Assume further (without loss of generality) that $r < s$ implies $G_k(Z_k^r) \leq G_k(Z_k^s)$, i.e. that $\mathbf{Z}_k$ is listed in decreasing order of linguistic grade.

Our idea, from case-based reasoning, is that $\mathbf{Z}_k$ should enable us to estimate criterion weights for DM_k, which would simplify the process of acquisition of precise weights, enabling DM_k to express his or her preferences more easily. Weighted Euclidean distance is employed since it has a clear geometric meaning, which can be easily understood and accepted by a DM to represent his or her aggregated preference. The details are explained next.

7.4.2 Distance Assumptions

Given the representative case set $\mathbf{Z}_k$, $\overline{Z}_k$, the *centroid* of $\mathbf{Z}_k$ is deemed to be a fictitious alternative at the center of all cases assigned to the best grade. Thus,

$$c_j(\overline{Z}_k) = \frac{1}{m_k^1} \sum_{r=1}^{m_k^1} c_j(Z_k^r), \tag{7.1}$$

For each $j = 1, 2, ..., q$, define $d_j^{\max} = \max_{r=1,2,...,m_k} \left(c_j(Z_k^r) - c_j(\overline{Z}_k)\right)^2$; then $d_j^{\max}$ is the normalization factor for criterion j. For DM_k ($k = 1, 2$) the distance between

$A^i \in \mathbf{A}$ and $\overline{Z}_k$ on criterion j is

$$d_j(A^i, \overline{Z}_k) = d_j^k(A^i) = \frac{\left(c_j(A^i) - c_j(\overline{Z}_k)\right)^2}{d_j^{\max}}. \tag{7.2}$$

In particular, if $A^i = Z_k^r$, then (7.2) defines $d_j^k(Z_k^r)$. For DM_k $(k = 1, 2)$ the distance between alternatives A^i and $\overline{Z}_k$ is then

$$D_k(A^i) = D_k(A^i, \overline{Z}_k) = \left\{ \sum_{j \in \mathbf{Q}} w_j^k \cdot d_j^k(A^i) \right\}^{1/2}, \tag{7.3}$$

where $\mathbf{w}^k = (w_1^k, w_2^k, ..., w_q^k)$ is a weight vector for DM_k (later we will determine an appropriate weight vector for DM_k). Note that (7.3) defines $D_k(Z_k^r)$ if $A^i = Z_k^r$. It is easy to verify that $0 \leq D_k(A^i) \leq 1$ for $k = 1, 2$. Thus, the distances from $\overline{Z}_k$ to all alternatives are normalized for easy comparison.

In terms of the aggregation approach to MCDA discussed above, $d_j^k(A^i)$ is analogous to v_j^i in (2.1), and $D_k(A^i)$ is analogous to $V(A^i)$ in (2.5) for DM_k. It is assumed that $\overline{Z}_k$ is a good estimate of the ideal alternative (the most preferred alternative) so that the closer A^i to $\overline{Z}_k$, the greater DM_k's preference; in other words, smaller values of $D_k(A^i)$ indicate greater preference. The distance of an alternative is thus defined to be the distance between the alternative and the centroid of the case set.

A similar idea has been used in cluster analysis: cluster membership of a datum is determined by evaluation of a pre-defined distance between this datum and the cluster centroid. If the DM can explicitly specify the ideal alternative, Z^*, then Z^* could be used instead of the centroid to measure the distances. Yet another approach is to define the centroid of $\overline{Z}_k$ to be the center of the cases assigned to the worst grade, so that DM_k's preference increases with distance from $\overline{Z}_k$.

7.4.3 Distance Intervals for Linguistic Grades

Based on the distance definitions above and assuming a specific weight vector, a distance interval can be assigned to $Z_k^r \in \mathbf{Z}_k$ based on its linguistic grade as assessed by DM_k. For example, if Z_k^r is evaluated as belonging to the highest grade, i.e. $G_k(Z_k^r) = 1$, then $D_k(Z_k^r) \in [0, \frac{1}{L}]$. More generally, if the linguistic grade for Z_k^r is g_k^r, then $D_k(Z_k^r) \in [\frac{g_k^r - 1}{L}, \frac{g_k^r}{L}]$ for $r = 1, 2, ..., m_k$. Note that the gaps between adjacent linguistic grades are represented by equal spacing. We believe this setting is easily understood by a DM. Of course, a DM can adjust it to fit in with his and her instinctive preference; for example, the ordinal preference expressions discussed in Cook and Kress (1991) could be employed for this purpose.

7.4.4 Weight Determination and Distance Construction

For $j \in \mathbf{Q}$, the weight w_j^k presents DM_k's relative importance for criterion j compared with other criteria in the criteria set $\mathbf{Q}$. Here we try to obtain $\mathbf{w}^k = (w_1^k, w_2^k, ..., w_q^k)$ by a case-based reasoning model based on the information $\mathbf{Z}_k$ provided by DM_k. The details are explained below.

Assuming that $Z_k^r \in \mathbf{Z}_k$ is assigned the linguistic grade $G_k(Z_k^*) = g_k^r$, the distance from $\overline{Z}_k$ to Z_k^r, $D_k(Z_k^r)$, must satisfy $D_k(Z_k^r) \in [\frac{g_k^r-1}{L}, \frac{g_k^r}{L}]$, provided there are no inconsistent judgements. So the following constraints are associated with Z_k^r:

$$D_k(Z_k^r) + \alpha_k^r \leq \frac{g_k^r}{L}; \tag{7.4}$$

$$D_k(Z_k^r) + \beta_k^r \geq \frac{g_k^r - 1}{L}. \tag{7.5}$$

where $-1 \leq \alpha_k^r \leq 0$ is an upper error bound for the DM's inconsistent judgements when $D_k(Z_k^r) > \frac{g_k^r}{L}$ and $0 \leq \beta_k^r \leq 1$ is a lower error bound for the DM's inconsistent judgements when $D_k(Z_k^r) < \frac{g_k^r-1}{L}$. Clearly, (7.4) and (7.5) are equivalent to

$$\left\{\sum_{j\in\mathbf{Q}} w_j^k d_j^k(Z_k^r)\right\}^{1/2} + \alpha_k^r \leq \frac{g_k^r}{L}; \tag{7.6}$$

$$\left\{\sum_{j\in\mathbf{Q}} w_j^k d_j^k(Z_k^r)\right\}^{1/2} + \beta_k^r \geq \frac{g_k^r - 1}{L}. \tag{7.7}$$

Similarly, upper and lower error bounds exist for all alternatives in $\mathbf{Z}_k$, so that the overall error in $\mathbf{Z}_k$ can be measured as $ERR_k = \sum_{r=1}^{m_k} \left[(-\alpha_k^r) + (\beta_k^r)\right]$.

Figure 7.1 shows the relationships between Z_k^r and $\overline{Z}_k$ when there are two criteria (dimensions). The same settings are used in Figures 7.3 to 7.8. Note that in Figure 7.1, $L = 4$ and Z_k^r is assessed at the 3^{rd} linguistic grade ($G_k(Z_k^r) = 3$). Taking $\overline{Z}_k$ as the center, three concentric ellipses (depending on the weight vector, they must be concentric circles) partition the alternative set $\mathbf{A}$ into four parts representing the

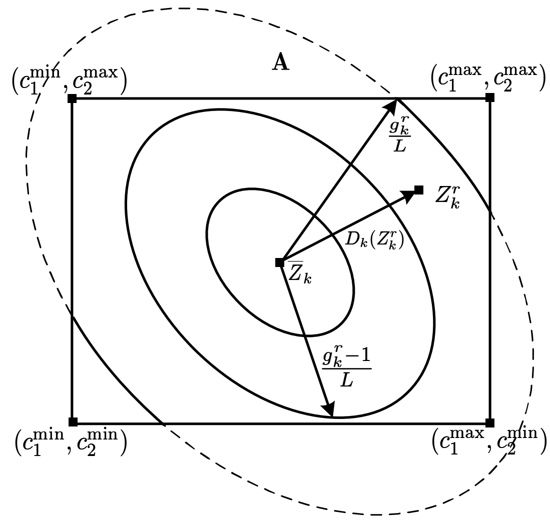


Figure 7.1: Relationships Among Z_k^r, A_k^+ and A_k^-.

different linguistic grades.The broken lines continue an ellipse where the alternatives are infeasible. As mentioned, the distances of all alternative lies between 0 and 1. Hence, $\frac{2}{4} \leq D_k(Z_k^r) \leq \frac{3}{4}$ provided the judgement on Z_k^r is consistent.

The following optimization model can be applied to find the most descriptive weight vector $\mathbf{w}^k$ for DM_k.

SOR.**4**(α, β, k) **Minimize** : $ERR_k = \sum_{r=1}^{m_k} \left[(-\alpha_k^r) + (\beta_k^r)\right]$

Subject to:

$$\left\{\sum_{j\in\mathbf{Q}} w_j^k d_j^k(Z_k^r)\right\}^{1/2} + \alpha_k^r \leq \frac{g_k^r}{L},\ -1 \leq \alpha_k^r \leq 0,\ r = 1, 2, ..., m_k;$$

$$\left\{\sum_{j\in\mathbf{Q}} w_j^k d_j^k(Z_k^r)\right\}^{1/2} + \beta_k^r \geq \frac{g_k^r - 1}{L},\ 0 \leq \beta_k^r \leq 1,\ r = 1, 2, ..., m_k;$$

$$w_j^k > 0,\ \sum_{j\in\mathbf{Q}} w_j^k = 1;$$

Theorem 7. **SOR**.**4**(α, β, k) *has at least one optimal solution.*

Proof: The constraints in **SOR**.**4**(α, β, k) constitute a convex set. The objective function ERR_k is a linear additive function on this set. Based on the extreme value theorem of advanced calculus (Fitzpatrick, 1995, page 297), the function

ERR is continuous and the set of all possible variables is bounded, ERR attains its minimum at least once. □

An indifference distance threshold, ε should be set to evaluate the error, ERR_k, generated by $\mathbf{D}(\alpha, \beta, k)$: When $ERR_k \leq \varepsilon$, the error is small and can be ignored, so the information in the case set $\mathbf{Z}_k$ provided by DM_k is considered to be consistent; when $ERR_k > \varepsilon$, the error cannot be ignored and there is some inconsistency in $\mathbf{Z}_k$. If so, DM_k should reconsider the case set $\mathbf{Z}_k$ and the linguistic grades assigned to it. Recall that ε_j is the indifference threshold on criterion $j \in \mathbf{Q}$, which suggests an overall indifference measure of $\sum_{j \in \mathbf{Q}} \frac{1}{d_j^{\max}}(\varepsilon_j)$. However, a typical indifference distance threshold is $\frac{1}{10n}$, where n is the number of alternatives in $\mathbf{A}$. Therefore, a good standard for the indifference distance threshold is $\varepsilon = \min\{\sum_{j \in \mathbf{Q}} \frac{1}{10 d_j^{\max}}(\varepsilon_j), \frac{1}{10n}\}$.

Similar to the discussion in Section 6.4.3, a DM may wish to provide information to make the results reflect more accurately his or her intrinsic preferences, which may be approximate, about weights for incorporation into $\mathbf{SOR.4}(\alpha, \beta, k)$. The imprecise preference expressions proposed by Sage and White (1984), and Eum et al. (2001) fulfill this purpose. Some imprecise weight preference expressions are listed below:

- Weak ranking:
$$w_1 \geq w_2 \geq \cdots \geq w_q > 0; \tag{7.8}$$
- Strict ranking:
$$w_j - w_{j+1} \geq \epsilon_j, j \in \mathbf{Q}, \tag{7.9}$$
where ϵ_j is a small positive value;
- Difference ranking:
$$w_1 - w_2 \geq \cdots \geq w_{q-1} - w_q \geq 0; \tag{7.10}$$
- Fixed bounds:
$$L_j \leq w_j \leq U_j, j \in \mathbf{Q}, \tag{7.11}$$
where L_j and U_j are lower and upper bounds for w_j, respectively;

Obviously, when any of these constraints is incorporated into $\mathbf{SOR.4}(\alpha, \beta, k)$, the program still has at least one optimal solution.

7.4.5 Post-optimality Analyses

Because $\mathbf{SOR.4}(\alpha, \beta, k)$ may have multiple optimal solutions for criterion weights, it is necessary to examine the robustness of the optimal solution via post-optimality analyses. There are several ways to handle the existence of multiple or near optimal solutions of $\mathbf{SOR.4}(\alpha, \beta, k)$:

- Assuming the optimal objective value of $\mathbf{SOR.4}(\alpha, \beta, k)$ is ERR^*, a post-optimal analysis of w_j can be carried out based on the suggestions of Jacquet-Lagrèze and Siskos (1982) through the following programs:

  $\mathbf{SOR.4}'(\alpha, \beta, k, w_j)$ **Maximize** : w_j

  Subject to:

$$\begin{cases} ERR \leq \max\{\eta, (1+\eta)ERR^*\}, \\ \text{all the constraints of } \mathbf{SOR.4}(\alpha, \beta, k), \\ \text{constraints (7.8) - (7.11) as applicable.} \end{cases}$$

  where η is a positive number.

  $\mathbf{SOR.4}''(\alpha, \beta, k, w_j)$ **Minimize** : w_j

  Subject to:

$$\begin{cases} ERR \leq \max\{\eta, (1+\eta)ERR^*\}, \\ \text{all the constraints of } \mathbf{SOR.4}(\alpha, \beta, k), \\ \text{constraints (7.8) - (7.11) as applicable.} \end{cases}$$

  Similar post-optimality analyses can be carried out for each $j \in \mathbf{Q}$. Then the average weight for each criterion from the procedures may be considered as the final result This procedure is adapted from the suggestions of Jacquet-Lagrèze and Siskos (1982).

- Another way to conduct post-optimality analysis is to employ different optimization software packages, such as LINGO or Matlab, or to carry out the optimization algorithms with different initializations to identify multiple optimal solutions, if any. When $\mathbf{SOR.4}(\alpha, \beta, k)$ has multiple solutions, the DM can select a solution which is in some way "closest" to his or her intuition — for example, most consistent with any imprecise weight information given in (7.8) - (7.11).

7.4.6 Distance-based Bilateral Negotiation Support System

Based on the distance representation of preference developed above, a distance-based negotiation support system is proposed to solve BN problems. It involves three phases as shown in Figure 7.2: pre-negotiation, negotiation and adjustment. In the pre-negotiation phase, the BN problem is constructed and parameters are set as in Section 2.2. Then using the linguistic grade vector $\mathbf{L}$, each DM selects a case set $\mathbf{Z}_k \in \mathbf{A}$ $(k = 1, 2)$ and assigns linguistic grades. During the negotiation phase the system provides some suggested alternatives based on available preference information from both sides. With the assistance of computer techniques, the DMs can easily evaluate different alternatives based upon their distances, especially in two-criterion consequence space. Interactive decision maps, developed by Lotov et al. (2004) for MCDA, could be employed to carry out this procedure. Since each DM's preferences can be represented geometrically, the DMs can evaluate alternatives (offers) and carry out negotiations.

The adjustment phase is used if the DMs appear to achieve an inefficient compromise; the system presents some efficient alternatives and invites the DMs to continue the negotiation until an efficient compromise alternative is reached.

The detailed procedure follows:

(1) Construct the negotiation problem: DM_1 and DM_2 must determine unambiguously the issue (criterion) set $\mathbf{Q}$ on which they are negotiating.

(2) Set the initial negotiation parameters. First DM_1 and DM_2 must select $c_j^{\max}$, $c_j^{\min}$ and ε_j for each criterion $j \in \mathbf{Q}$. Based on these settings, the alternative set, $\mathbf{A}$, can be identified and presented to DM_1 and DM_2. Then a set of linguistic grades, $\mathbf{L}$, to enable DMs to assess case sets, must be prepared.

(3) Acquire a case set $\mathbf{Z}_k$ from DM_k for $k = 1, 2$ with settings given by $G_k : Z_k \to L$. Note that it is required that at least one case is assigned to the top grade.

(4) Compute $\overline{Z}_k$ and $d_j^k(Z_k^r)$ for $r = 1, 2, ..., m_k$ and $k = 1, 2$ using (7.1) and (7.2). Then for each k, apply $\mathbf{SOR.4}(\alpha, \beta, k)$ to generate the weight vector $\mathbf{w}^k$ and present it to DM_k privately. Check the error ERR_k. If $ERR_k > \varepsilon$, ask DM_k to repeat step (3); if $ERR_k \leq \varepsilon$, continue. Post-optimality analyses can also be carried out to confirm the robustness of the results.

(5) Generate a suggested alternative set $\mathbf{T} \subset \mathbf{A}$ and present it to DM_1 and DM_2 publicly. Initially, $\mathbf{T}$ is constructed as described in Section 7.4.7 using equal values of the acceptable distance thresholds (explained later) for both DMs. Then,

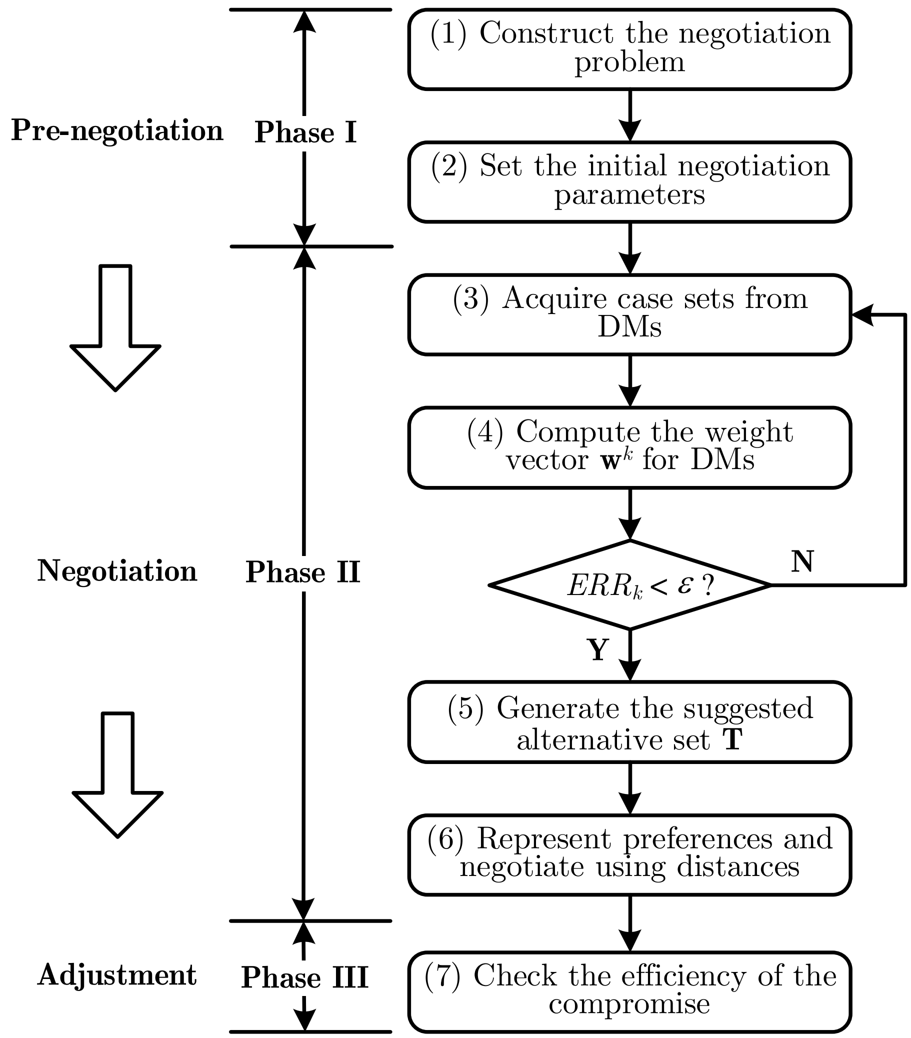


Figure 7.2: The Procedure of Negotiation Support System.

each DM is allowed to decrease or increase the acceptable distance threshold. The system updates $\mathbf{T}$ accordingly.

(6) Represent preferences and negotiate using distances, as explained in Section 7.4.8.

(7) Check the efficiency of the compromise. If it appears to be an inefficient compromise, identify efficient alternatives and update $\mathbf{T}$. DMs can share all weight information, $\mathbf{w}^k$, for $k = 1, 2$, when they both agree. The negotiations may continue until an efficient compromise alternative is found. The details are

explained in Section 7.4.9.

7.4.7 Generating the Suggested Alternative Set

The Edgeworth box (Rudy, 2002) describes interactive trading of two individuals trading two commodities. A procedure to generate a suggested alternative set based on the Edgeworth box, is explained next.

Using the preferences generated from the case sets provided by DM_1 and DM_2, a suggested alternative set $\mathbf{T}$ can be identified and presented to DM_1 and DM_2. The purpose of $\mathbf{T}$ is to assist DM_1 and DM_2 to focus on reasonable alternatives, rather than the entire alternative set $\mathbf{A}$, and thereby negotiate more efficiently. Let $\overline{\mathbf{A}}_k$ denote an acceptable alternative set for DM_k, $k = 1, 2$. Define the set $\{A^i \in \overline{\mathbf{A}}_k : D_k(A^i) \leq Y_k\}$, where Y_k is an acceptable distance threshold for DM_k. The suggested alternative set is $\mathbf{T} = \overline{\mathbf{A}}_1 \cap \overline{\mathbf{A}}_2$.

A natural principle of equality states that the negotiators should receive rewards of equal value (Pruitt, 1981). Accordingly the distance of 0.5 is set as the initial threshold for both DM_1 and DM_2: $\{A^h \in \overline{\mathbf{A}}_1 : D_1(A^h) \leq 0.5\}$ and $\{A^i \in \overline{\mathbf{A}}_2 : D_2(A^i) \leq 0.5\}$. A useful idea is that $\mathbf{T}$ should contain a few distinctive alternatives for DMs to compare. We recommend that $8 \leq |\mathbf{T}| \leq 10$, based on Miller (1956)'s famous observation that people can best handle comparisons around seven plus or minus two items. The following procedure produces a suitable $\mathbf{T}$ based on $Y_1 = Y_2 = 0.5$:

(1) When $8 \leq |\overline{\mathbf{A}}_1 \cap \overline{\mathbf{A}}_2| \leq 10$, $|\mathbf{T}| = \overline{\mathbf{A}}_1 \cap \overline{\mathbf{A}}_2$ as shown in Figure 7.3.

(2) When $|\overline{\mathbf{A}}_1 \cap \overline{\mathbf{A}}_2| < 8$, Y_1 and Y_2 are reset as $Y_1 = Y_2 = p_1 > 0.5$, to increase $|\overline{\mathbf{A}}_1 \cap \overline{\mathbf{A}}_2|$, as shown in Figure 7.4.

(3) When $|\overline{\mathbf{A}}_1 \cap \overline{\mathbf{A}}_2| > 10$, Y_1 and Y_2 are reset as $Y_1 = Y_2 = p_2 < 0.5$, to decrease $8 \leq |\overline{\mathbf{A}}_1 \cap \overline{\mathbf{A}}_2|$, as shown in Figure 7.5.

7.4.8 Representing Preferences and Negotiating Using Distances

A DM's preferences over alternatives are represented as distances from his or her ideal point. This allows the DM to more easily evaluate the counter-offers and adjust his or her offers as the negotiation proceeds, improving negotiation efficiency. For example, at a deadlock in the negotiation, the system can identify all non-dominated alternatives and present them to DMs, thereby helping them to break

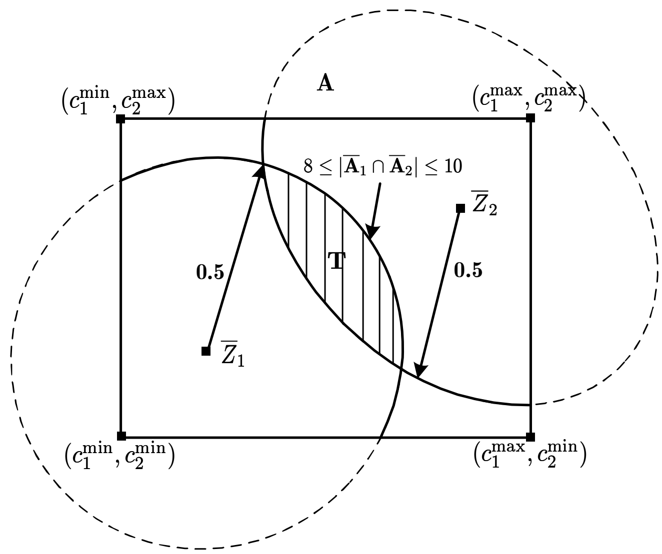


Figure 7.3: Obtaining Alternatives When $8 \le |\overline{\mathbf{A}}_1 \cap \overline{\mathbf{A}}_2| \le 10$.

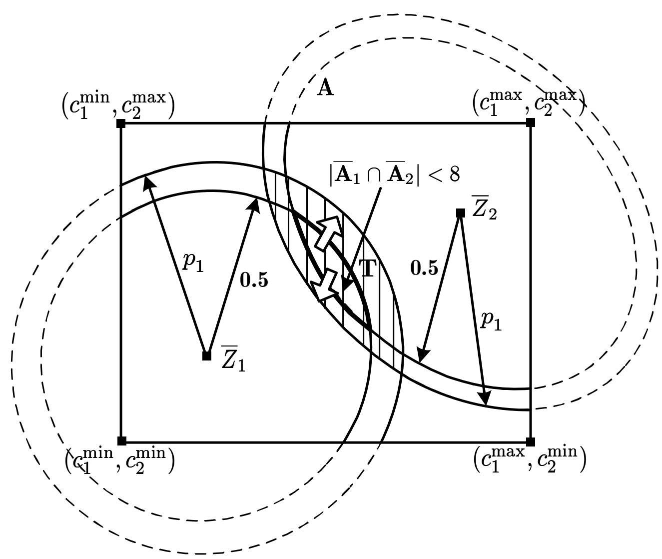


Figure 7.4: Obtaining Alternatives When $|\overline{\mathbf{A}}_1 \cap \overline{\mathbf{A}}_2| < 8$.

the deadlock. Also, DM_k can revise his or her acceptable alternative set, $\overline{\mathbf{A}}_k$, individually by increasing or decreasing the acceptable distance threshold, Y_k. The system then updates the suggested alternative set, $\mathbf{T}$ and presents it to the DMs. Because preferences are represented as geometric distances from ideal points, DMs can more easily negotiate and reach a compromise. Figure 7.6 demonstrates this

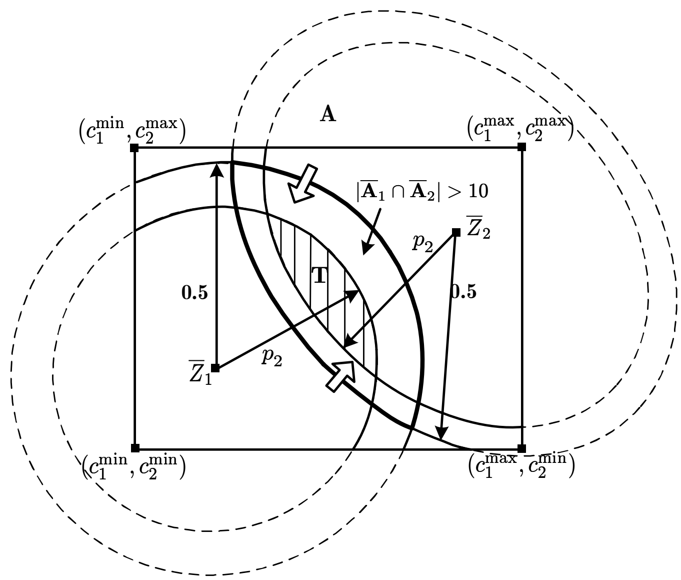


Figure 7.5: Obtaining Alternatives When $|\overline{\mathbf{A}}_1 \cap \overline{\mathbf{A}}_2| > 10$.

procedure.

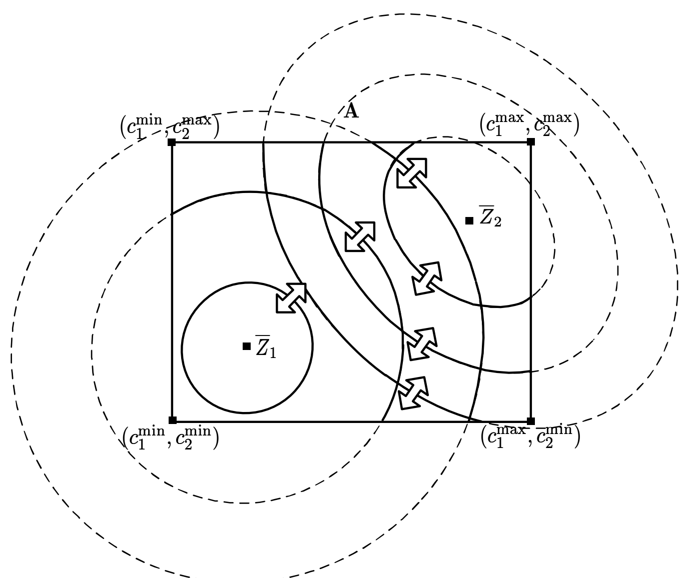


Figure 7.6: Negotiating Using Distances.

7.4.9 Checking the Efficiency of the Compromise

After a compromise alternative, A^*, is agreed upon by DM_1 and DM_2, the system can check its efficiency. If A^* is inefficient, the system can identify efficient alternatives and update $\mathbf{T}$. The negotiation may continue until an efficient compromise alternative is found. Let D_k^* denote the distance of A^* for DM_k $(k = 1, 2)$. The detailed procedure for checking the efficiency A^* and updating $\mathbf{T}$ is as follows:

(1) When $\min\{D_1(A^h) : D_2(A^h) \leq D_2^*\} \geq D_1^*$ and $\min\{D_2(A^i) : D_1(A^h) \leq D_1^*\} \geq D_2^*$, A^* is an efficient alternative, as shown in Figure 7.7.

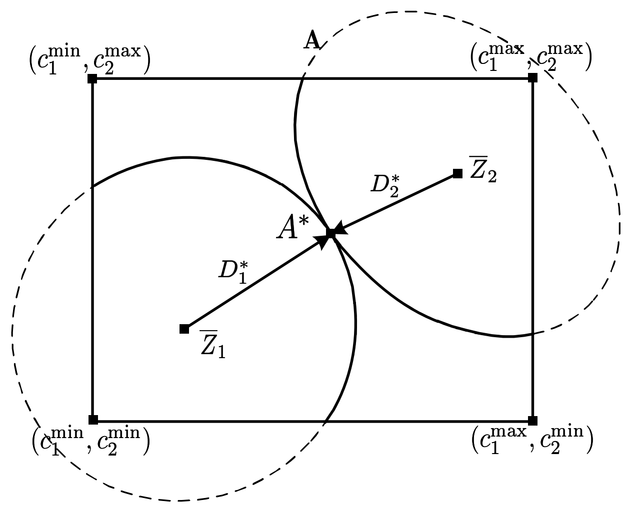


Figure 7.7: A^* is An Efficient Alternative.

(2) When $\min\{D_1(A^h) : D_2(A^h) \leq D_2^*\} < D_1^*$ and $\min\{D_2(A^i) : D_1(A^h) \leq D_1^*\} < D_2^*$, A^* is not efficient. Here, $\mathbf{T}$ can be reset as $\mathbf{T} = \{A^h : D_1(A^h) < D_2^*\} \cap \{A^i : D_2(A^i) < D_1^*\}$, as shown in Figure 7.8. The updated $\mathbf{T}$ is presented to the negotiators to assist them to reach an efficient alternative.

7.5 Case Study: A Bicycle Component Negotiations

7.5.1 Background

A case study similar to the Itex-Cypress negotiation example in Inspire (Kersten, 2004) is designed to demonstrate the algorithms described above. Two decision

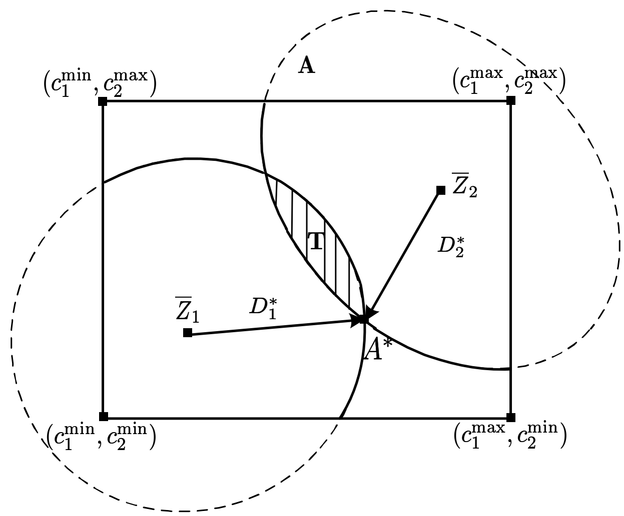


Figure 7.8: A^* is An Inefficient Alternative.

makers, DM_1 and DM_2, are negotiating the purchase of bicycle components. Both sides agree to negotiate over four criteria: price (dollars), quantity, delivery (days), and warranty (years).

7.5.2 Initial Parameter Settings

Assume that DM_1 and DM_2 agree upon the initial parameter settings as listed in Table 7.1, which include the lowest and highest consequence values for each criterion and the indifference threshold ε_j, for $j = 1, 2, 3, 4$, as shown in Table 7.1.

Table 7.1: Negotiation Initial Settings

	$c_j^{\min}$	$c_j^{\max}$	ε_j
Price	10	16	2
Quantity	500	560	20
Delivery	20	40	10
Warranty	1	3	1

Based on this information, the number of consequence values on the first criterion, price is $n_1 = 1 + \frac{1}{2}(16 - 10) = 4$. Similarly, $n_2 = 4$, $n_3 = 3$, $n_4 = 3$. The total number of feasible alternatives is $n = n_1 \cdot n_2 \cdot n_3 \cdot n_4 = 144$. The vector $\big(c_1(A), c_2(A), c_3(A), c_4(A)\big)$ is used to represent alternative $A \in \mathbf{A}$, where $c_j(A)$

for $j = 1, 2, 3, 4$ denotes the consequence value of A on criterion j. For example, $(10, 500, 20, 3)$ represents an alternative with price of \$10, quantity of 500, delivery of 20 days and warranty of 3 years.

7.5.3 Case Set Acquisition and Computations for DM_1

A linguistic grade set $\mathbf{L} = \{1, 2, 3\}$, representing {good, fair, poor}, is used to assist DMs to evaluate alternatives in their case sets. Suppose that DM_1 provides the case set, $\mathbf{Z}_1$, provided in Table 7.2. Table 7.2 also shows the consequence values defining $\overline{Z}_1$, the centroid of the top set.

Table 7.2: The Case Set Provided by DM_1

	Good			Centroid	Fair		Poor	
$\mathbf{Z}_1$	Z_1^1	Z_1^2	Z_1^3	$\overline{Z}_1$	Z_1^4	Z_1^5	Z_1^6	Z_1^7
Price	10	12	12	11.3333	12	14	16	14
Quantity	540	520	500	520	520	540	540	540
Delivery	20	20	30	23.3333	40	30	30	40
Warranty	3	2	2	2.3333	1	2	1	2

Then the normalized distances are computed using (7.2) and the results are listed in Table 7.3.

Table 7.3: Normalized Distances for DM_1

	Good			Fair		Poor	
$\mathbf{Z}_1$	Z_1^1	Z_1^2	Z_1^3	Z_1^4	Z_1^5	Z_1^6	Z_1^7
Price	0.0816	0.0204	0.0204	0.0204	0.3265	1.0000	0.3265
Quantity	0.2500	0.0000	0.2500	0.0000	0.2500	0.2500	0.2500
Delivery	0.0400	0.0400	0.1600	1.0000	0.1600	0.1600	1.0000
Warranty	0.2500	0.0625	0.0625	1.0000	0.0625	1.0000	0.0625

For illustration, suppose that DM_1 provides the imprecise weight information as follows: for $j = 1, 2, 3, 4$, $0.1 \leq w_j^1 \leq 0.9$. These constraints guarantee that each criterion has at least some relevance in the negotiation. The optimization program, **SOR.4**$(\alpha, \beta, 1)$, is applied to find $\mathbf{w}^1$ for DM_1.

Minimize : $ERR_1 = -(\alpha_1^1) - (\alpha_1^2) - (\alpha_1^3) - (\alpha_1^4) - (\alpha_1^5) + (\beta_1^4) + (\beta_1^5) + (\beta_1^6) + (\beta_1^7)$

Subject to:

$$\{0.0816w_1^1 + 0.2500w_2^1 + 0.0400w_3^1 + 0.2500w_4^1\}^{1/2} + \alpha_1^1 \leq 1/3;$$
$$\{0.0204w_1^1 + 0.0400w_3^1 + 0.0625w_4^1\}^{1/2} + \alpha_1^2 \leq 1/3;$$
$$\{0.0204w_1^1 + 0.2500w_2^1 + 0.1600w_3^1 + 0.0625w_4^1\}^{1/2} + \alpha_1^3 \leq 1/3;$$
$$\{0.0204w_1^1 + w_3^1 + w_4^1\}^{1/2} + \alpha_1^4 \leq 2/3;$$
$$\{0.0204w_1^1 + w_3^1 + w_4^1\}^{1/2} + \beta_1^4 \geq 1/3;$$
$$\{0.3265w_1^1 + 0.2500w_2^1 + 0.1600w_3^1 + 0.0625w_4^1\}^{1/2} + \alpha_1^5 \leq 2/3;$$
$$\{0.3265w_1^1 + 0.2500w_2^1 + 0.1600w_3^1 + 0.0625w_4^1\}^{1/2} + \beta_1^5 \geq 1/3;$$
$$\{w_1^1 + 0.2500w_2^1 + 0.1600w_3^1 + w_4^1\}^{1/2} + \beta_1^6 \geq 2/3;$$
$$\{0.3265w_1^1 + 0.2500w_2^1 + w_3^1 + 0.0625w_4^1\}^{1/2} + \beta_1^7 \geq 2/3;$$
$$-1 \leq \alpha_1^1 \leq 0;\ -1 \leq \alpha_1^2 \leq 0;\ -1 \leq \alpha_1^3 \leq 0;\ -1 \leq \alpha_1^4 \leq 0;\ -1 \leq \alpha_1^5 \leq 0;$$
$$0 \leq \beta_1^4 \leq 1;\ 0 \leq \beta_1^5 \leq 1;\ 0 \leq \beta_1^6 \leq 1;\ 0 \leq \beta_1^7 \leq 1;$$
$$0.1 \leq w_1^1 \leq 0.9;\ 0.1 \leq w_2^1 \leq 0.9;\ 0.1 \leq w_3^1 \leq 0.9;\ 0.1 \leq w_4^1 \leq 0.9;$$
$$w_1^1 + w_2^1 + w_3^1 + w_4^1 = 1;$$

The results of **SOR.4**$(\alpha, \beta, 1)$, are $ERR_1 = 0$ and $\mathbf{w}^1 = (0.5743, 0.1000, 0.2257, 0.1000)$, which were found using Lingo (2005). Note that the error can be ignored.

7.5.4 Post-optimality Analyses for DM_1

Based upon the first method described in Section 3.4, post-optimality analysis for DM_1 was carried out according to the following steps:

- The post-optimality programs, **SOR.4′**$(\alpha, \beta, 1, w_j)$ and **SOR.4″**$(\alpha, \beta, 1, w_j)$ were formulated for each criterion weight, w_j^1. The small positive number, η, was set as $\frac{1}{n} = \frac{1}{144} \approx 0.0069$. The results are listed in Table 7.4.
- From Table 7.4, it is easy to see that the initial optimal solution (row 1) and the average values of the post-optimality analyses (last row) are close, though small variations exist. We conclude that there is no good reason to doubt the results of the program **SOR.4**$(\alpha, \beta, 1)$. Based upon the average values of weights, the distances of all 144 alternatives in **A** were determined. To save space, we show in Figure 7.9 only the numbers and proportions of alternatives in the three linguistic grades.

Table 7.4: Post-optimality Analyses and Final Solutions DM_1

	Criterion weights			
	w_1^1	w_2^1	w_3^1	w_4^1
Initial solution	0.5743	0.1000	0.2257	0.1000
$\max(w_1^1)$	0.5879	0.1000	0.2121	0.1000
$\min(w_1^1)$	0.3855	0.1779	0.3366	0.1000
$\max(w_2^1)$	0.3977	0.1820	0.3203	0.1000
$\min(w_2^1)$	0.5860	0.1000	0.2126	0.1014
$\max(w_3^1)$	0.3937	0.1606	0.3457	0.1000
$\min(w_3^1)$	0.5879	0.1000	0.2121	0.1000
$\max(w_4^1)$	0.4650	0.1000	0.2663	0.1686
$\min(w_4^1)$	0.5863	0.1015	0.2123	0.1000
Average	0.5072	0.1247	0.2604	0.1078

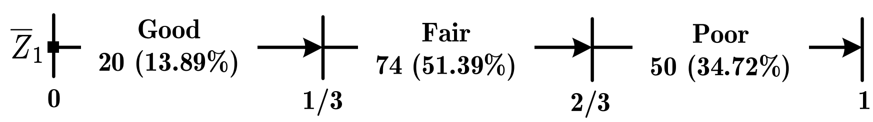


Figure 7.9: The Proportions of Different Alternatives for DM_1.

7.5.5 Case Set Acquisition and Computations for DM_2

Similar procedures are carried out for DM_2, assuming that DM_2 provides the case set, $\mathbf{Z}_2$, listed in Table 7.5. The normalized distances are listed in Table 7.6.

Table 7.5: The Case Set Provided by DM_2

	Good			Centroid	Fair		Poor	
$\mathbf{Z}_2$	Z_2^1	Z_2^2	Z_2^3	$\overline{Z}_2$	Z_2^4	Z_2^5	Z_2^6	Z_2^7
Price	14	16	16	15.3333	12	14	10	12
Quantity	540	560	540	546.6667	520	540	520	500
Delivery	40	40	30	36.6667	20	30	30	20
Warranty	2	1	2	1.6667	2	1	3	2

Similarly, imprecise weight information is assumed as follows: for $j = 1, 2, 3, 4$, $0.1 \leq w_j^2 \leq 0.9$. The optimization program, $\mathbf{SOR.4}(\alpha, \beta, 2)$, is applied to find $\mathbf{w}^2$

Table 7.6: Normalized Distances for DM_2

	Good			Fair		Poor	
$\mathbf{Z}_2$	Z_2^1	Z_2^2	Z_2^3	Z_2^4	Z_2^5	Z_2^6	Z_2^7
Price	0.0625	0.0156	0.0156	0.3906	0.0625	1.0000	0.3906
Quantity	0.0204	0.0816	0.0204	0.3265	0.0204	0.3265	1.0000
Delivery	0.0400	0.0400	0.1600	1.0000	0.1600	0.1600	1.0000
Warranty	0.0625	0.2500	0.0625	0.0625	0.2500	1.0000	0.0625

for DM_2.

Minimize : $ERR_2 = -(\alpha_2^1) - (\alpha_2^2) - (\alpha_2^3) - (\alpha_2^4) - (\alpha_2^5) + (\beta_2^4) + (\beta_2^5) + (\beta_2^6) + (\beta_2^7)$

Subject to:

$\{0.0625w_1^2 + 0.0204w_2^2 + 0.0400w_3^2 + 0.0625w_4^2\}^{1/2} + \alpha_2^1 \leq 1/3;$

$\{0.0156w_1^2 + 0.0816w_2^2 + 0.0400w_3^2 + 0.2500w_4^2\}^{1/2} + \alpha_2^2 \leq 1/3;$

$\{0.0156w_1^2 + 0.0204w_2^2 + 0.1600w_3^2 + 0.0625w_4^2\}^{1/2} + \alpha_2^3 \leq 1/3;$

$\{0.3906w_1^2 + 0.3265w_1^2 + w_3^2 + 0.0625w_4^2\}^{1/2} + \alpha_2^4 \leq 2/3;$

$\{0.3906w_1^2 + 0.3265w_1^2 + w_3^2 + 0.0625w_4^2\}^{1/2} + \beta_2^4 \geq 1/3;$

$\{0.0625w_1^2 + 0.0204w_2^2 + 0.1600w_3^2 + 0.2500w_4^2\}^{1/2} + \alpha_2^5 \leq 2/3;$

$\{0.0625w_1^2 + 0.0204w_2^2 + 0.1600w_3^2 + 0.2500w_4^2\}^{1/2} + \beta_2^5 \geq 1/3;$

$\{w_1^2 + 0.3265w_2^2 + 0.1600w_3^2 + w_4^2\}^{1/2} + \beta_2^6 \geq 2/3;$

$\{0.3906w_1^2 + w_2^2 + w_3^2 + 0.0625w_4^2\}^{1/2} + \beta_2^7 \geq 2/3;$

$-1 \leq \alpha_1^1 \leq 0;\ -1 \leq \alpha_1^2 \leq 0;\ -1 \leq \alpha_1^3 \leq 0;\ -1 \leq \alpha_1^4 \leq 0;\ -1 \leq \alpha_1^5 \leq 0;$

$0 \leq \beta_1^4 \leq 1;\ 0 \leq \beta_1^5 \leq 1;\ 0 \leq \beta_1^6 \leq 1;\ 0 \leq \beta_1^7 \leq 1;$

$0.1 \leq w_1^1 \leq 0.9;\ 0.1 \leq w_2^1 \leq 0.9;\ 0.1 \leq w_3^1 \leq 0.9;\ 0.1 \leq w_4^1 \leq 0.9;$

$w_1^1 + w_2^1 + w_3^1 + w_4^1 = 1;$

The results are $ERR_2 = 0$ and $\mathbf{w}^2 = (0.1000, 0.4489, 0.1898, 0.2613)$, and again the error can be ignored. Similar post-optimality analyses were carried out and the average values for the weights are obtained as: $(0.2719, 0.2573, 0.1898, 0.2810)$. The details are omitted. Based upon this information, the distances of all 144 alternatives in $\mathbf{A}$ for DM_2 were generated and the numbers and proportions in three linguistic grades were found to be as shown in Figure 7.10.

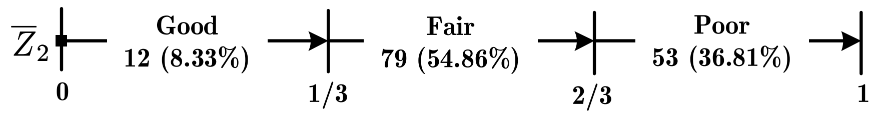


Figure 7.10: The Proportions of Different Alternatives for DM_2.

7.5.6 Generating Suggested Alternative Set

A suggested alternative set **T** can be presented to DM_1 and DM_2 as worthwhile alternatives for negotiation. When $Y_1 = Y_2 = 0.5$, $|\overline{\mathbf{A}}_1 \cap \overline{\mathbf{A}}_2| = 7$, so $Y_1 = Y_2 = 0.5555$ was used, yielding $|\overline{\mathbf{A}}_1 \cap \overline{\mathbf{A}}_2| = 9$ as shown in Table 7.7.

Table 7.7: Initial Suggested Alternative Set **T**

	Price	Quantity	Delivery	Warranty
T^1	12	520	30	1
T^2	12	520	30	2
T^3	12	540	30	1
T^4	12	540	30	2
T^5	12	560	30	2
T^6	14	520	30	2
T^7	14	540	20	2
T^8	14	540	30	2
T^9	14	560	20	2

7.5.7 Checking the Efficiency of the Compromise Alternative

Assume DM_1 and DM_2 find a compromise alternative, $(12, 520, 30, 1)$, where the distance for DM_1 is 0.3997 and the distance for DM_2 is 0.5393. Hence, DM_1 seems to have more negotiation advantages than DM_2. This compromise may not be efficient, since some alternatives dominating it can be identified based on the calculated distances. Accordingly, **T** is then updated as given in Table 7.8. The negotiators are invited to negotiate within this suggested alternative set, where a better compromise may be found.

DM_1 and DM_2 may finally reach an efficient compromise alternative $(12, 540, 30, 2)$, where the distance for DM_1 decreases from 0.3997 to 0.2999 and the distance

Table 7.8: Updated Suggested Alternative Set **T**

	Price	Quantity	Delivery	Warranty
T^1	12	520	30	2
T^2	12	540	30	2

for DM_2 decreases from 0.5393 to 0.3993. Notice that compared with the previous compromise, the quantity increases from 520 to 540 and the warranty increases from 1 year to 2 years. Both DMs benefit from this shift.

7.6 Conclusions

Bilateral negotiations are studied within the context of multiple criteria decision analysis within this chapter. A case-based distance model is developed to assist negotiators to reach an efficient compromise. Based on weighted Euclidean distance, an optimization program is employed to generate descriptive criterion weights by assessing case sets provided and rated by each individual. Then, a negotiator's preference for each alternative can be represented as the distance from an ideal alternative. A negotiation support system using these distances is designed to assist negotiators in eventually reaching an efficient compromise. A case study is presented to illustrate how the proposed negotiation procedure can work in practice.

Chapter 8

Multiple Criteria Nominal Classification

8.1 Introduction

A new kind of MCDA problem, multiple criteria nominal classification (MCNC), is studied in this chapter. Traditional classification methods in MCDA focus on sorting alternatives into groups ordered by preference. MCNC is the classification of alternatives into nominal groups, structured by the DM, who specifies multiple characteristics for each group. Starting with illustrative examples, the features, definition and structures of MCNC are presented, emphasizing criterion and alternative flexibility. Then, an analysis procedure is proposed to solve MCNC problems systematically. Assuming additive value functions, an optimization model with constraints that incorporate various classification strategies is constructed to solve MCNC problems. An application of MCNC in water resources planning is carried out and some future extensions are suggested. The research contained in this chapter is founded by the work of Chen et al. (2006).

8.2 Motivation

An important variant on screening or sorting problems is the classification of alternatives into nominal groups, as opposed to groups ordered by preference. Such an extension has both theoretical interest and practical applicability. For example, in human resources management, job applicants must be assigned to appropriate jobs, or rejected, according to their multiple qualifications (criteria). Further evaluations may refine an initial assignment. We call this problem *Multiple Criteria Nominal Classification (MCNC)* to distinguish it from the standard (ordered) sorting problem, *Multiple Criteria Sorting*. Other applications of MCNC include business management, environmental management and resource allocation.

So far as we know, traditional MCDA methods do not address MCNC problems efficiently and would not be suitable without modifications. Only a few published papers, such as those by Malakooti and Yang (2004); Perny (1998); Scarelli and Narula (2000) are relevant to MCNC. They apply outranking methods to solve special types of MCNC problems, but give no systematic analysis. Many issues remain to be investigated; for example, there has been no detailed discussion of the relationship between alternatives and nominal groups. In this thesis, a systematical analysis procedure is proposed to help the DM to better understand and solve MCNC problems. Also, we believe our work will enrich research in the MCDA area and inspire more research on multiple criteria nominal classification topics.

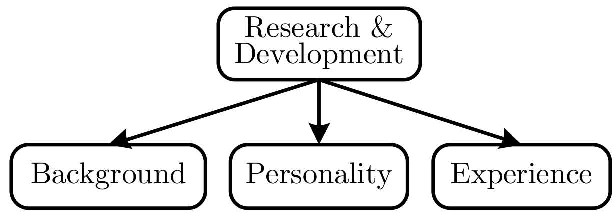
8.3 Features, Definition, Structures, and Properties of MCNC

8.3.1 Illustrative Examples of MCNC

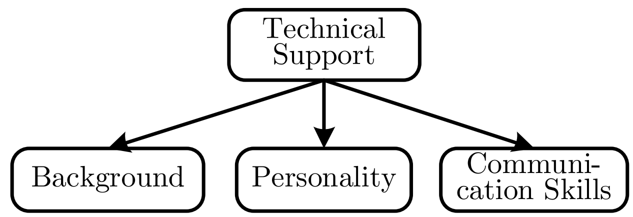
It is easiest to explain MCNC using an example.

Example 1. *An industrial company needs to recruit employees for positions in research and development, and technical support. The Human Resources Department received ten applications expressing interest in both positions. To improve his decision making, the manager decides to assign the applicants to two groups, those suitable for research development positions, and those suitable for technical support. He uses the following criteria to evaluate applicants for each group as shown in Figure 8.1.*

Based on the above criteria, the ten applicants, $\{A^1, A^2, ..., A^{10}\}$ *are evaluated and classified as shown in Figure 8.2, where* $\mathbf{A}$ *denotes the set of all applicants,* $\mathbf{A}_1$ *the set of applicants suitable for research and development, and* $\mathbf{A}_2$ *the set of applicants suitable for technical support. Note that the manager wants to screen out less qualified applicants and may (temporarily) assign highly qualified applicants to both positions. So* A^4 *does not appear in either group, while* A^6 *and* A^{10} *appear in both.*



(a) Criteria Set: Research and Development



(b) Criteria Set: Technical Support

Figure 8.1: Criteria Sets for the Two Groups

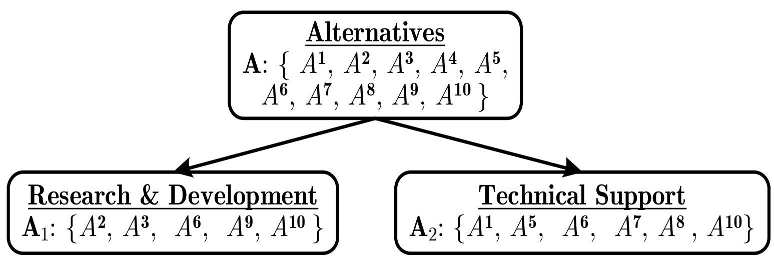


Figure 8.2: The Classification of Ten Applicants

Example 2. *In water resources management, three groups of rivers and lakes can be defined for development planning:* reserved for drinking water, reserved for wildlife refuge, *and* reserved for tourism. *Some rivers or lakes may be such important resources that they are assigned to more than one group, while others may have only one assignment, or none.*

8.3.2 Features of MCNC

Unlike other problems in MCDA, MCNC has the following unique features:

Criterion Flexibility for Groups

In traditional MCDA problems, all alternatives are evaluated according to the same criteria. This is not necessarily the case for MCNC problems, where it may not be appropriate to apply the same criteria set to different groups. In Example 1, some criteria like Background and Personality apply to both groups, while others like Experience and Communication Skills are special requirements for particular groups. Thus, MCNC can be regarded as a *multiple-MCDA problem* in which a set of alternatives is to be organized into groups; each group is its own MCDA problem, with its own criteria. Of course, some criteria may be common across groups, while others are unique to particular groups.

Alternative Flexibility for Groups

Unlike sorting problems in which each alternative belongs to only one group, in MCNC problems some alternatives may be assigned to more than one group and some may not be assigned to any group.

8.3.3 Preliminary Definitions

Let $P = (p^i_j)_{n\times m}$ be an $n \times m$ matrix. Denote the transpose of P by P^T. Let the row m-vector P^i denote the i^{th} row of P. Regarding P as a column of n (row) m-vectors, we write $P = (P^1, P^2, ..., P^i, ..., P^n)^T$. Let the column n-vector P_j indicate the j^{th} column of P. Similarly, P can be regarded as a row of m (column) n-vectors, and can be written as $P = (P_1, P_2, ..., P_j, ..., P_m)$. Let $I_m = (1, 1, ..., 1)^T$ be the identity column m-vector. For a row m-vector $A = (a_1, a_2, ..., a_i, ..., a_m)$, and a column m-vector $B = (b_1, b_2, ..., b_i, ..., b_m)^T$, $\langle A, B\rangle = \sum_{i=1}^{m} a_i \cdot b_i$ denotes the dot product (scalar product or inner product) of A and B. Finally, define e^g as the row m-vector with 1 in the g^{th} position and 0 elsewhere.

8.3.4 Definition and Structures of MCNC

First a formal definition of MCNC is proposed.

Multiple Criteria Classification: Definition

Definition 11. *Multiple Criteria Nominal Classification (MCNC) is the assignment of a finite set of alternatives to nominally defined groups (subsets). Condi-*

tions for group membership are based on sets of criteria that may overlap. Any alternative may be assigned to one, several, or no groups.

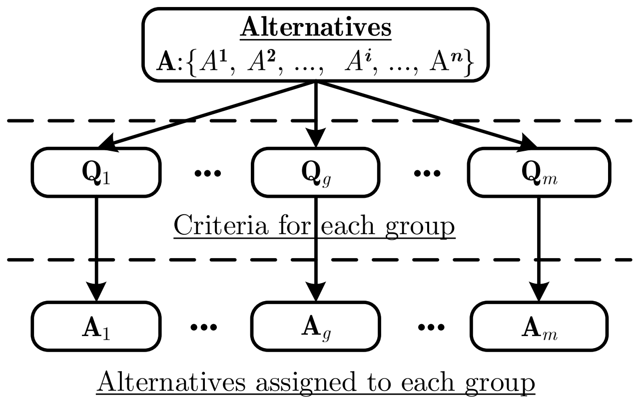


Figure 8.3: The Structures of MCNC

Figure 8.3 shows the structure of MCNC with the following notation: $\mathbf{A} = \{A^1, A^2, ..., A^i, ..., A^n\}$ denotes the alternative set, and $|\mathbf{A}| = n$. Let $\mathbf{A}_g$ denote the g^{th} nominal group, $1 \leq g \leq m$, where m is the total number of nominal groups. We assume that $n \geq m$, which fits most practical applications. Let $\mathbf{Q}$ denote the set of criteria covering all groups, and $|\mathbf{Q}| = q$. Let $\mathbf{Q}_g$ denote the subset of criteria for nominal group g, $1 \leq g \leq m$. We assume that $\mathbf{Q} = \mathbf{Q}_1 \cup \mathbf{Q}_2 \cup ... \cup \mathbf{Q}_m$, which means that all criteria not applicable to any group are discarded.

In Example 1, the alternative set $\mathbf{A} = \{A^1, A^2, ..., A^{10}\}$ consists of the job candidates; $\mathbf{A}_1$ denotes the candidates for the research and development position, $\mathbf{A}_2$ denotes the candidates for the technical support position. In the typical solution, $\mathbf{A}_1 = \{A^2, A^3, A^6, A^9, A^{10}\}$ and $\mathbf{A}_2 = \{A^1, A^5, A^6, A^7, A^8, A^{10}\}$. The criteria set is $\mathbf{Q} = \{B, P, E, C\}$, where B, P, E, C correspond to the criteria Background, Personality, Experience, and Communication Skills respectively. As shown in Figure 8.1, $\mathbf{Q}_1 = \{B, P, E\}$ and $\mathbf{Q}_2 = \{B, P, C\}$.

MCNC Criterion Structure

For $j = 1, 2, ..., q$ and $g = 1, 2, ..., m$, r_g^j is an indicator variable indicating whether criterion j applies to group $\mathbf{A}_g$,

$$r_g^j = \begin{cases} 1, & \text{if } j \in \mathbf{Q}_g; \\ 0, & \text{if } j \notin \mathbf{Q}_g. \end{cases} \tag{8.1}$$

$R = \left(r_g^j\right)_{q\times m}$ defines *the MCNC criterion structure*, and therefore represents the MCNC problem. Note that the row index $j = 1, 2, ..., q$ refers to criteria, and the column index $g = 1, 2, ..., m$ refers to groups. R^j, the j^{th} row of R, indicates *the group affiliation of criterion* j, i.e. the groups to which criterion j applies. R_g, the g^{th} column of R, indicates *the criteria that apply to* $\mathbf{A}_g$. The number of criteria associated with group $\mathbf{A}_g$ is denoted q_g, $g = 1, 2, ..., m$. (Note that $0 < q_g \leq q$, since typically only a few criteria in $\mathbf{Q}$ may apply to any group.). In Example 1,

$$R = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},\ R_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix},\ R_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix},$$

$R^1 = (1, 1)$, $R^2 = (1, 1)$, $R^3 = (1, 0)$, $R^4 = (0, 1)$, $q_1 = q_2 = 3$, and $q = 4$.

MCNC Alternative Structure

For $i = 1, 2, ..., n$ and $g = 1, 2, ..., m$, s_g^i is an indicator variable indicating whether alternative A^i belongs to group $\mathbf{A}_g$:

$$s_g^i = \begin{cases} 1, & \text{if } A^i \in \mathbf{A}_g; \\ 0, & \text{if } A^i \notin \mathbf{A}_g. \end{cases} \tag{8.2}$$

Thus, $S = \left(s_g^i\right)_{n\times m}$ is the *solution* of an MCNC problem with criterion structure R. Note that the row index $i = 1, 2, ..., n$ refers to alternatives and the column index $g = 1, 2, ..., m$ to groups.

Note that for a classification problem of n alternatives and m groups, the number of mathematically possible solutions is 2^{mn} (when there are no constraints). S^i, the i^{th} row of S, indicates *the group affiliations of alternative* A^i, while S_g, the g^{th} column of S is *the alternative structure for group* $\mathbf{A}_g$. The number of alternatives in group $\mathbf{A}_g$ is denoted by n_g, $g = 1, 2, ..., m$. (Note that $0 \leq n_g \leq n$, since typically each group contains some but not all alternatives in $\mathbf{A}$.)

In Example 1, $$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix},\ S_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix},\ S_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix},$$

$S^1 = (0, 1)$, $S^2 = (1, 0)$, $S^3 = (1, 0)$, $S^4 = (0, 0)$, $S^5 = (0, 1)$, $S^6 = (1, 1)$, $S^7 = (0, 1)$, $S^8 = (0, 1)$, $S^9 = (1, 0)$, $S^{10} = (1, 1)$, $n_1 = 5$ and $n_2 = 6$.

8.3.5 Properties of MCNC

Criterion Overlap Properties

The following definitions formalize the property of criterion flexibility.

(1) Criterion Overlap

In MCNC, $R = (r_g^j)_{q\times m}$, is an *overlapping criterion classification* for criterion j, where $j = 1, 2, ..., q$, iff $\langle R^j, I_m\rangle > 1$. If R is not overlapping for any j, it is a *non-overlapping criterion classification.*

In Example 1, $\langle R^1, I_2\rangle = 2$, $\langle R^2, I_2\rangle = 2$, $\langle R^3, I_2\rangle = 1$, $\langle R^4, I_2\rangle = 1$, so criteria 1 and 2 are shared by both groups and criteria 3 and 4 apply uniquely to groups 1 and 2, respectively. Thus, Example 1 is an overlapping classification for criteria 1 and 2.

(2) Identity of Criteria

Groups g and h have *identical classification criteria* iff $R_g = R_h \neq 0$, where $g, h \in \{1, 2, ..., m\}$, $g \neq h$, and 0 is the column vector of all 0's. Clearly, if there are two groups with identical classification criteria, the classification is an overlapping criteria classification for at least one criterion j. An MCNC has *completely identical classification criteria* iff $R_g = R_h$ for all groups g and h. In Example 1, $R_1 \neq R_2$, so R does not have identical classification criteria for groups 1 and 2. Note that two group with identical classification criteria are generally different in criterion weights, in other words, the DM may have different weights for the same criterion in different groups.

(3) Degree of Criterion Overlap

The degree of criterion overlap, d_r^o for an MCNC, R, is defined as follows:

$$d_r^o = \frac{\sum\limits_{g,h\in\{1,2,...,m\};\ g\neq h} \left\langle R_g^T, R_h\right\rangle}{q \cdot m \cdot (m-1)}, \tag{8.3}$$

where $0 \leq d_r^o \leq 1$. If $d_r^o = 0$, then $R = (r_g^j)_{q\times m}$, is a non-overlapping criterion classification. If $d_r^o > 0$, then R is an overlapping criterion classification for at least one criterion. If $d_r^o = 1$, then R has completely identical classification criteria.

In general, greater values of d_r^o indicate higher levels of overall criteria overlap. In Example 1, $d_r^o = \frac{\langle R_1^T,\ R_2\rangle + \langle R_2^T,\ R_1\rangle}{q \cdot m \cdot (m-1)} = \frac{2\Big\langle (1,1,1,0),\ (1,1,0,1)^T \Big\rangle}{4 \cdot 2 \cdot 1} = \frac{4}{8} = 0.5$, which implies that half of the criteria are shared.

(4) Maximum Number of Groups for a Criterion

The maximum number of groups for a criterion in an MCNC, n_r^o is defined as follows

$$n_r^o = \max_{j=1,2,\ldots,q} \langle R^j, I_m \rangle. \tag{8.4}$$

In Example 1, $n_r^o = \max\Big\{\Big\langle(1,1),(1,1)^T\Big\rangle, \Big\langle(1,1),(1,1)^T\Big\rangle, \Big\langle(1,0),(1,1)^T\Big\rangle, \Big\langle(1,0),$ $(1,1)^T\Big\rangle\Big\} = \max\{2,2,1,1\} = 2$, so the maximum overlap is 2, achieved by criteria 1 and 2.

(5) Criterion Correlation Between Groups

The criterion correlation of groups g and h $(g, h \in \{1,2,...,m\}, g \neq h)$ for an MCNC is ρ_{gh}^r and defined as follows

$$\rho_{gh}^r = \frac{\langle R_g^T, R_h \rangle}{\Big\langle (R_g - R_h)^T, (R_g - R_h) \Big\rangle + \langle R_g^T, R_h \rangle}. \tag{8.5}$$

Clearly, $\rho_{gh}^r = \rho_{hg}^r$ and $0 \leq \rho_{gh}^r \leq 1$. If $\rho_{gh}^r = 0$, group g and h have no overlapping criteria. If $\rho_{gh}^r > 0$, groups g and h have overlap on at least one criterion. If $\rho_{gh}^r = 1$, groups g and h have the same criteria set.

The average criterion correlation for $\mathbf{Q}$ is defined as

$$\rho_{\mathbf{Q}}^r = \frac{\sum\limits_{g,h=1,2,\ldots,m;\ g \neq h} \rho_{gh}^r}{m \cdot (m-1)}, \tag{8.6}$$

where $0 \leq \rho_{\mathbf{Q}}^r \leq 1$. If $\rho_{\mathbf{Q}}^r = 0$, then R is a non-overlapping criterion classification. If $\rho_{\mathbf{Q}}^r > 0$, R is an overlapping criterion classification for at least one criterion. If $\rho_{\mathbf{Q}}^r = 1$, R has completely identical classification criteria.

In general, greater values of $\rho_{\mathbf{Q}}^r$ indicate higher overall criterion overlap. In Example 1, $\rho_{12}^r = \rho_{21}^r = \frac{\Big\langle (1,1,1,0),(1,1,0,1)^T \Big\rangle}{\Big\langle \Big((1,1,1,0)-(1,1,0,1)\Big), \Big((1,1,1,0)-(1,1,0,1)\Big)^T \Big\rangle + \Big\langle (1,1,1,0),(1,1,0,1)^T \Big\rangle} = \frac{2}{4} = 0.5$. Since there are only two groups, $d_r^o = \rho_{gh}^r = \rho_{\mathbf{Q}}^r = 0.5$.

Alternative Overlap Properties

Similar definitions formalize the properties of alternative flexibility.

(1) Alternative Overlap

In MCNC, R is an *overlapping alternative classification* for alternative $i \in \{1, 2, ..., n\}$ iff $\langle S^i, I_m \rangle > 1$, where S^i is i^{th} row of S. If R is not overlapping for any i, it is a *non-overlapping alternative classification*.

In Example 1, $\langle S^6, I_m \rangle = \langle S^{10}, I_m \rangle = \left\langle (1,1), (1,1)^T \right\rangle = 2$, so alternative 6 and 10 are assigned to both groups. Therefore, Example 1 is an overlapping alternative classification for alternatives 6 and 10.

(2) Identity of Groups

Groups g and h are *identical classifications* iff $S_g = S_h \neq 0$, where $g, h \in \{1, 2, ..., m\}$, $g \neq h$ and 0 is the column vector of all 0's. Clearly, if there are two groups with identical classification alternatives, R is an overlapping alternative classification for at least one alternative. R is a *completely identical classification* iff $S_g = S_h$ for all groups g and h. In Example 1, $S_1 \neq S_2$, so R does not exhibit any identical classification.

(3) Average Degree of Alternative Overlap

The average degree of alternatives overlap, d_s^o, is defined as

$$d_s^o = \frac{\sum\limits_{g,h=1,2,...,m;\ g \neq h} \left\langle S_g^T, S_h \right\rangle}{n \cdot m \cdot (m-1)}, \tag{8.7}$$

where $0 \leq d_s^o \leq 1$. If $d_s^o = 0$, then R, is a non-overlapping alternative classification. If $d_s^o > 0$, then R is an overlapping classification for some alternatives. If $d_s^o = 1$, then R is a completely identical classification.

In general, greater values of d_s^o indicate high levels of alternative overlap. In Example 1,

$d_s^o = \frac{\langle S_1^T, S_2 \rangle + \langle S_2^T, S_1 \rangle}{n \cdot m \cdot (m-1)} = \frac{2\left\langle (0,1,1,0,0,1,0,0,1,1),\ (1,0,0,0,1,1,1,1,0,1)^T \right\rangle}{10 \cdot 2 \cdot 1} = 0.2.$

(4) Maximum Number of Groups for An Alternatives

The maximum number of groups for an alternative for an MCNC, n_s^o, is defined as

$$n_s^o = \max_{i=1,2,...,n} \langle S^i, I_m \rangle. \tag{8.8}$$

In Example 1, $n_s^o = \max\Big\{\big\langle(0,1),(1,1)^T\big\rangle, \big\langle(1,0),(1,1)^T\big\rangle, \big\langle(1,0),(1,1)^T\big\rangle, \big\langle(0,0),(1,1)^T\big\rangle, \big\langle(0,1),(1,1)^T\big\rangle, \big\langle(1,1),(1,1)^T\big\rangle, \big\langle(0,1),(1,1)^T\big\rangle, \big\langle(0,1),(1,1)^T\big\rangle, \big\langle(1,0),(1,1)^T\big\rangle, \big\langle(1,1),(1,1)^T\big\rangle\Big\} = \max\{1,1,1,0,1,2,1,1,1,2\} = 2$, so the maximum overlap is 2, which is achieved by alternatives 6 and 10.

(5) Complete Classification of Alternatives

In MCNC, R is a *complete classification of alternatives* iff $\langle S^i, I_m\rangle \geq 1$, for all alternatives $i = 1,2,...,n$. Otherwise, R is an *incomplete classification of alternatives.* In Example 1, $\langle S^4, I_m\rangle = 0$, so Example 1 is an incomplete classification of alternatives.

(6) Alternative Deficiency Degree

First M^i, $i = 1,2,...,n$, an indicator variable, is defined as

$$M^i = \begin{cases} 1, & \text{if } A^i \text{ belongs to at least one group;} \\ 0, & \text{if } A^i \text{ does not belong to any group.} \end{cases} \tag{8.9}$$

Thus, $M^i = \max_{g=1}^m s_g^i$.

Then the degree of alternative deficiency, d_s^d, is defined as follows.

$$d_s^d = 1 - \frac{\sum\limits_{i=1,2,...,n} M^i}{n}, \tag{8.10}$$

where n is the size of the alternative set $\mathbf{A}$. Here, $0 \leq d_s^d \leq 1$. If $d_s^d = 0$, then R is a complete classification of alternatives. If $d_s^d > 0$, then R is an incomplete classification of alternatives.

In general, greater values of d_s^d indicate more unassigned alternatives. In Example 1, $M^1 = M^2 = M^3 = M^5 = M^6 = M^7 = M^8 = M^9 = M^{10} = 1$, $M^4 = 0$, $d_s^d = 1 - \frac{\sum_{i=1}^{10} M^i}{10} = 0.1$.

(7) Correlation of Alternatives Across Groups

In an MCNC, the correlation of alternatives of groups g and h ($g, h \in \{1,2,...,m\}$, $g \neq h$) is ρ_{gh}^s, and defined as follows

$$\rho_{gh}^s = \frac{\langle S_g^T, S_h\rangle}{\big\langle(S_g - S_h)^T, (S_g - S_h)\big\rangle + \langle S_g^T, S_h\rangle}. \tag{8.11}$$

Clearly, $\rho_{gh}^s = \rho_{hg}^s$ and $0 \leq \rho_{gh}^s \leq 1$. If $\rho_{gh}^s = 0$, groups g and h have no overlapping alternatives. If $\rho_{gh}^s > 0$, groups g and h have at least one overlapping alternative. If $\rho_{gh}^s = 1$, groups g and h are identical classifications.

In general, greater values indicate higher overlap between two groups. Moreover, the average alternative correlation within $\mathbf{A}$ is defined as

$$\rho^s_{\mathbf{A}} = \frac{\sum\limits_{g,h=1,2,\ldots,m;\ g\neq h} \rho^s_{gh}}{m \cdot (m-1)}, \tag{8.12}$$

where $0 \leq \rho^s_{\mathbf{A}} \leq 1$. If $\rho^s_{\mathbf{A}} = 0$, R has no overlapping alternatives. If $\rho^s_{\mathbf{A}} > 0$, R is an overlapping alternatives classification for at least one alternative. If $\rho^s_{\mathbf{A}} = 1$, then R is a completely identical classification.

In general, greater values of $\rho^s_{\mathbf{A}}$ indicate higher overall alternative overlap. In Example 1, $\rho^s_{12} = \rho^s_{21} = \frac{\langle S_1^T, S_2\rangle}{\left\langle (S_1-S_2)^T,\ (S_1-S_2)\right\rangle + \langle S_1^T, S_2\rangle} = \frac{2}{8+2} = 0.2$. Since there are only two groups, $d^o_s = \rho^s_{gh} = \rho^s_{\mathbf{A}} = 0.2$.

8.3.6 Types of Classification

Based on alternative flexibility features, four types of multiple criteria classification can be distinguished, as shown in Table 8.1.

Table 8.1: MCNC Problems Classification

	NOVLP	*OVLP*
CMPL	$\mathbf{MCNC_1}$	$\mathbf{MCNC_2}$
INCMPL	$\mathbf{MCNC_3}$	$\mathbf{MCNC_4}$

$\mathbf{MCNC_1}$ is an MCNC situation with non-overlapping alternatives classification (NOVLP) and complete classification of alternatives (CMPL). In this type of classification, each alternative is assigned to one group only. For example, in a sport competition each athlete may be constrained to participate in exactly one sport.

$\mathbf{MCNC_2}$ is an MCNC problem with overlapping alternatives classification (OVLP) and CMPL. This type of classification requires each alternative to be assigned to at least one group. For instance, each athlete on the team can take part in one or more sports (so that excellent athletes may compete in more than one sport).

$\mathbf{MCNC_3}$ is an MCNC situation with NOVLP and incomplete classification of alternatives (INCMPL). This type of classification refers to the assignment of each alternative to at most one group. For example, there may be a regulation that an athlete can compete in only one sport, and some athletes may not be assigned to any sport.

MCNC$_4$ is an MCNC problem with OVLP and INCMPL. In this type of classification, one alternative to be assigned to more than one group, while another alternative may not belong to any group. For example, good athletes may represent a team in more than one competition, and weak athletes in none.

8.4 MCNC Analysis Procedure

8.4.1 Analysis Procedure for MCNC

A systematic procedure to analyze MCNC problems is shown in Figure 8.4. It includes the following steps.

- *Identify the alternatives set* **A**: The alternative set **A** must be identified at the start.
- *Construct the groups and their criteria sets*: Find **Q** and R.
- *Post-Criteria Assessment*: Some indices, like the degree of criteria overlap (d_r^o) and maximum number of groups for a criterion (n_r^o), can provide some descriptive information about the criterion construction and help the DM to assess the criterion construction. If the DM is not satisfied with these information, then the group criterion construction needs to be modified. Otherwise continue.
- *Express the decision maker's preferences*: Determine the DM's alternative preferences (values) and preferences on criteria (weights) for each group.
- *Optimize the alternative classification* S: An optimization model must be applied to obtain an optimal classification S. The DM's classification strategies, such as the permission of alternative overlap and the maximum number of alternatives for a group, can be incorporated into the optimization model as constraints.
- *Post-Optimization Assessment*: After completing the optimization procedure, the DM should assess the results. If the DM is not satisfied with the findings, the DM's classification strategies can be modified and new classification results are presented, until the DM is satisfied with the results.

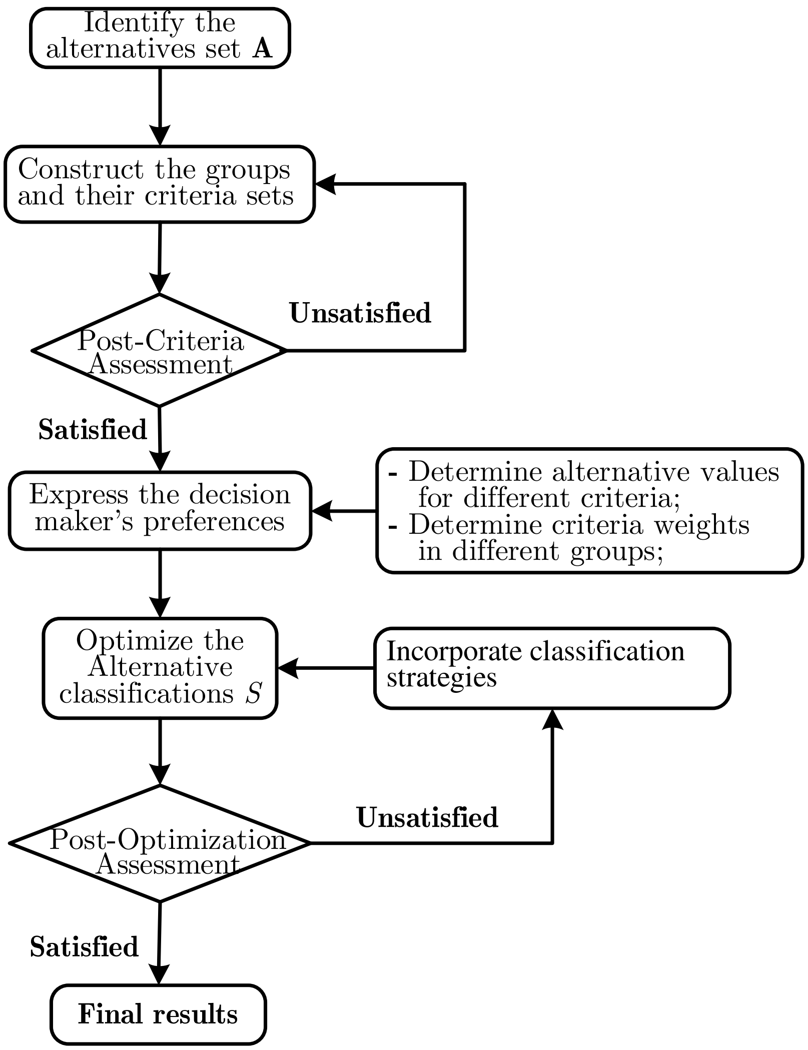


Figure 8.4: Analysis Procedure for MCNC Problems

8.5 A Linear Additive Value Function Approach to MCNC

8.5.1 Model Assumptions

The linear additive value function or SMART (Simple Multi-Attribute Rating Technique) is a well-known approach in MCDA. Stewart (1996) argues that additive value functions can provide reasonable preference orderings. Hence, we propose a SMART-based optimization model to solve MCNC problems. First, some notation must be introduced, as follows:

For $g = 1, 2, ..., m$, $\mathbf{w}^g = (w_1^g, w_2^g, ..., w_j^g, ..., w_q^g)$ is the weight vector for group g. Note that if $r_g^j = 0$, then $w_j^g = 0$. The weights for the same criterion in different groups may be the same ($w_j^g = w_j^h$) or not ($w_j^g \neq w_j^h$), depending on the DM's preferences. In Example 1, the manager may place different relative importance on the criterion of *Personality* in two groups. $\mathbf{w} = (\mathbf{w}^1, \mathbf{w}^2, ..., \mathbf{w}^g, ..., \mathbf{w}^m)^T$ is the weight matrix covering all the groups.

For $g = 1, 2, ..., m$ and $i = 1, 2, ..., n$, $\mathbf{v}_g^i = (v_{1g}^i, v_{2g}^i, ..., v_{jg}^i, ..., v_{qg}^i)$ is the value vector of alternative A^i for group g. Note that if $r_g^j = 0$, then $v_{jg}^i = 0$. The extension of value transformation function (2.1) for MCNC problem can be set as:

$$v_{jg}^i = f_{jg}(c_j^i) \tag{8.13}$$

where $f_{jg}(\cdot)$ is a mapping from consequences to values for criterion j in group g. Note that for an alternative, the values of the same criterion in different groups may be the same ($v_{jg}^i = v_{jh}^i$) or not ($v_{jg}^i \neq v_{jh}^i$), depending on how the DM constructs the value transformation function (8.13). We allow for the same alternative to have different values on the same criterion in different groups. For instance, the criterion "abundance of aquatic species" can be applied to rivers; a high level of consequence on this criterion is valuable in a recreation grouping, but a detriment in a hydropower grouping.

For all alternatives, $i = 1, 2, ..., n$, all criteria $j = 1, 2, ..., q$, and all groups $g = 1, 2, ..., m$,

$$\sum_{j \in \mathbf{Q}} w_j^g = 1, w_j^g \geq 0; \tag{8.14}$$

$$0 \leq v_{jg}^i \leq 1. \tag{8.15}$$

Many methods are available to obtain weights and values. With the Analytic Hierarchy Process (AHP) (Saaty, 1980), weights or values can be obtained by the calculation of eigenvalues. Swing weights is a method for direct estimation of weights preferred by Belton and Stewart (2002); von Winterfeldt (1986). UTA (UTilités Additives) is a regression method using case studies to estimate values (Jacquet-Lagrèze and Siskos, 1982). In this thesis, we assume that weights and values can be obtained for each group without uncertainty.

8.5.2 Objective Function

Based on the SMART model assumption, the following definitions are given.

The aggregation value of alternative A^i for group g can be expressed as

$$V_g^i = \langle \mathbf{w}^g, {\mathbf{v}_g^i}^T \rangle = \sum_{j \in \mathbf{Q}} w_j^g \cdot v_{jg}^i. \tag{8.16}$$

The aggregation value vector for group g is denoted $\mathbf{V}_g = (V_g^1, V_g^2, ..., V_g^i, ..., V_g^n)$. The aggregation value matrix for all groups is denoted $\mathbf{V} = (\mathbf{V}_1, \mathbf{V}_2, ..., \mathbf{V}_g, ..., \mathbf{V}_m)^T$.

The DM's whole value for the classification solution S is denoted V_S. Since groups are nominally defined, we assume that each group has equal importance for the DM. Adding the aggregation values of all alternatives in all groups produces the DM's whole value V_S. The computation is implemented by the following function

$$V_S = \sum_{g=1,2,...,m} e^g \cdot (\mathbf{V} \cdot S) \cdot (e^g)^T. \tag{8.17}$$

We take V_S to be the objective function, so we find the best classification result using

$$\max_{S \in \mathbb{S}} \{V_S\}, \tag{8.18}$$

where $\mathbb{S}$ denotes the set of all possible solutions of an MCNC problem.

8.5.3 Constraints

Based on the type classification chosen and some more elaborate strategies made by the DM, some constraints need to be set for the above objective function. The detailed explanations are as follows.

Constraints for Different MCNC Types

First recall that the meanings of the following formula: (1) $\langle S^i, I_m \rangle$ equals the number of groups A^i belongs to; (2) $\sum_{i=1}^{n} M^i$ equals the number of all alternatives have been assigned to at least one group; (3) $\frac{\sum_{\{g,h=1,2,\ldots,n;\ g\neq h\}} \left\langle S_g^T, S_h \right\rangle}{n \cdot m \cdot (m-1)}$ equals the average degree of alternative overlap; and (4) $\langle S_g^T, I_m \rangle$ equals the number of alternatives assigned to group g.

(1) **MCNC**$_1$, the combination of NOVLP and CMPL, implies each alternative need to be assigned to one group only and the following constraints are set:

$$\text{for all } i = 1, 2, ..., n,\ \langle S^i, I_m \rangle = 1 \tag{8.19}$$

(2) **MCNC**$_2$, the combination of OVLP and CMPL, implies each alternative to be assigned to at least one group and the following constraints are set:

$$\text{for all } i = 1, 2, ..., n,\ \langle S^i, I_m \rangle \geq 1, \tag{8.20}$$

$$\sum_{i=1}^{n} M^i = n. \tag{8.21}$$

(3) **MCNC**$_3$, the combination of NOVLP and INCMPL, implies each alternative to be assigned to at most one group and some alternatives are not assigned to any group. The following constraints are set:

$$\text{for all } i = 1, 2, ..., n,\ 0 \leq \langle S^i, I_m \rangle \leq 1, \tag{8.22}$$

$$0 < \sum_{i=1}^{n} M^i < n. \tag{8.23}$$

(4) **MCNC**$_4$, the combination of OVLP and INCMPL implies some alternatives to be assigned to more than one group, while other alternatives may not belong to any group. The following constraints are set:

$$0 < \sum_{i=1}^{n} M^i < n, \tag{8.24}$$

$$\frac{\sum_{\{g,h=1,2,\ldots,n;\ g\neq h\}} \left\langle S_g^T, S_h \right\rangle}{n \cdot m \cdot (m-1)} > 0. \tag{8.25}$$

Classification Strategies

Moreover, the DM can implement some classification strategies as constraints to elaborate upon the classification.

(1) The maximum number of groups for an alternative, $1 \leq n_s^o \leq m$, can be set with the following constraints:

$$\text{for all } i = 1, 2, ..., n, \ \langle S^i, I_m \rangle \leq n_s^o. \tag{8.26}$$

(2) The average degree of alternative overlap, $0 \leq d_s^o \leq 1$, can be specified using the following constraint:

$$\frac{\sum_{\{g,h=1,2,...,m;\ g \neq h\}} \left\langle S_g^T, S_h \right\rangle}{n \cdot m \cdot (m-1)} \leq d_s^o. \tag{8.27}$$

(3) The minimum deficiency degree of alternatives, $0 \leq d_s^d \leq 1$, can be specified using the following constraint:

$$\sum_{i=1}^{n} M^i \geq n(1 - d_s^d). \tag{8.28}$$

(4) The maximum number of alternatives for group g, n_g can be specified as follows:

$$\langle S_g^T, I_m \rangle \leq n_g. \tag{8.29}$$

8.6 Numerical Example: Water Supply Planning

8.6.1 Problem Descriptions

Due to increases in residential, industrial, and commercial demand for water and decreases in the reliability of groundwater resources, a city needs to develop a long term water supply planning. The following decision process was stipulated by the Region Council to design and implement the best resources plan:

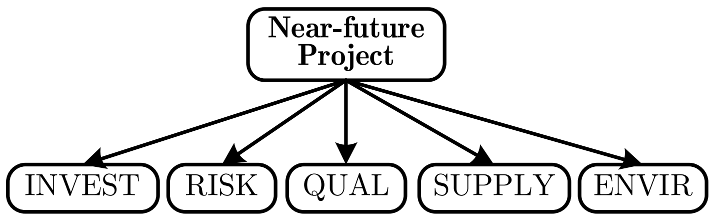
1. *Near-future Project*: an immediate construction project to be completed to meet water demands in the near future.

2. *Mid-term Project*: another construction project to be built between the years 2018 and 2020 to meet water demands over the middle term.

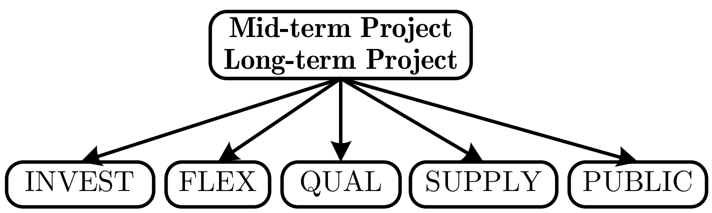
3. *Long-term Project*: a final construction project to be finished by the year 2035 to meet long-term demands.

This decision process can be regarded as a MCNC problem with three nominal groups consisting of the near-future, mid-term and long-term projects. By the application of the proposed MCNC model, we carry out the following decision analysis.

First several criteria are proposed for measuring the effectiveness of possible alternatives for each group as shown in Figure 8.5. The groups of near-future projects, mid-term projects and long-term projects are named $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$, respectively.



(a) The Criteria Set for Near-future Projects



(b) The Criteria Set for Mid-term and Long-term Projects

Figure 8.5: The Criteria Set of WWSP

The detailed meanings of the criteria with the index numbers in the criteria set $\mathbf{Q}$ are described as follows: 1. INVEST: the project investment cost (millions of dollars); 2. RISK: the project implementation risk; 3. QUAL: the water quality of the project; 4. SUPPLY: the project supply capability (million imperial gallons); 5. ENVIR: the project environmental impacts; 6. FLEX: the project ability to react to changes in demand; 7. PUBLIC: the community perception of the acceptability of the project.

8.6.2 Post-criteria Assessment

The post-criteria analysis is carried out to provide some general criteria information.

$|\mathbf{A}| = 10$, and $m = 3$ for the set $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$.

$\mathbf{Q} = \{1, 2, ..., j, ..., 7\}$, $|\mathbf{Q}| = 7$.

$$R = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, R_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, R_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, R_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

$R^1 = (\ 1 \ \ 1 \ \ 1\ )$, $R^2 = (\ 1 \ \ 0 \ \ 0\ )$, $R^3 = (\ 1 \ \ 1 \ \ 1\ )$, $R^4 = (\ 1 \ \ 1 \ \ 1\ )$, $R^5 = (\ 1 \ \ 0 \ \ 0\ )$, $R^6 = (\ 0 \ \ 1 \ \ 1\ )$, $R^7 = (\ 0 \ \ 1 \ \ 1\ )$.

$d_r^o = \frac{2\left(\langle R_1^T, R_2\rangle + \langle R_1^T, R_3\rangle + \langle R_2^T, R_3\rangle\right)}{7 \cdot 3 \cdot 2} = \frac{11}{21} = 52.4\%$.

$n_r^o = \max\left(\langle R^1, I_3\rangle, \langle R^2, I_3\rangle, \langle R^3, I_3\rangle, \langle R^4, I_3\rangle, \langle R^5, I_3\rangle, \langle R^6, I_3\rangle, \langle R^7, I_3\rangle\right) = 3$.

$\rho_{12}^r = \frac{\langle R_1^T, R_2\rangle}{\left\langle (R_1 - R_2)^T,\ (R_1 - R_2)\right\rangle + \langle R_1^T, R_2\rangle} = \frac{3}{7} = 42.9\% = \rho_{13}^r,\ \ \rho_{23}^r = 100\%$.

$\rho_{\mathbf{Q}}^r = \frac{\sum\limits_{\{g,h=1,2,...,7; g \neq h\}} \rho_{gh}^r}{m \cdot (m-1)} = \frac{2 \cdot [\frac{3}{7} \cdot 2 + 1]}{3 \cdot 2} = \frac{13}{21} = 61.9\%$.

The criteria overlapping in this MCNC is high since $d_r^o = 52.5\%$, $\rho_{\mathbf{Q}}^r = 61.9\%$, and $n_r^o = 3$. The largest contribution of overlapping comes from Groups 2 and 3, since $\rho_{23}^r = 100\%$ (they have identical classification criteria). Although there is high overlapping, the criterion structure is reasonable, since we believe that such criterion structure can represent most of the concerns of both local residents and water resource management experts. Note that we will show later that the difference between a mid-term project and long-term project is caused by different criterion weights. Different weights assigned to the same criterion in these two projects represents different concerns regarding the criteria.

8.6.3 Alternative Identification

Ten alternatives have been identified. Table 8.2 shows the consequences for the MCDA problem. SUPPLY, QUAL, PUBLIC, and FLEX are identified as positive preference criteria (the greater the consequence, the better, as indicated by "+" in Table 8.2); others are negative preference criteria (the less the consequence, the better, as indicated by "-" in Table 8.2).

Table 8.2: The Basic Structure of the WWSP

Criteria	*Alternatives*									
	A^1	A^2	A^3	A^4	A^5	A^6	A^7	A^8	A^9	A^{10}
INVEST(-)	100	55	8.6	5	112	123.6	111.25	120.4	126	181
RISK(-)	40	40	50	80	60	70	70	60	70	70
QUAL(+)	75	80	70	30	60	60	60	70	60	55
SUPPLY(+)	29	25	40	5	50	80	80	80	80	80
ENVIR(-)	40	40	45	40	50	40	90	80	80	80
FLEX(+)	70	85	50	70	60	60	65	70	80	90
PUBLIC(+)	60	70	60	60	70	65	80	80	85	85

8.6.4 Decision Maker's Preferences

(1) Preference on Consequences

For the more (consequences) the better criterion, $j \in \mathbf{Q}_g$, equation (2.2) is employed to estimate the values of each alternative for criterion j in all three groups. Similarly, for the less (consequences) the better criterion, $k \in \mathbf{Q}_g$, equation (2.3) is employed to obtain the values of each alternative for criterion k in all three groups. As noted earlier, an alternative can have different values on the same criterion in different groups. In this example, we assume they are the same. The values for the ten alternatives are as listed in Table 8.3 and 8.4.

Table 8.3: The Values of Alternatives for Near-future project

Criteria	*Alternatives*									
	A^1	A^2	A^3	A^4	A^5	A^6	A^7	A^8	A^9	A^{10}
INVEST(-)	0.050	0.091	0.581	1.000	0.045	0.040	0.045	0.042	0.040	0.028
RISK(-)	1.000	1.000	0.800	0.500	0.667	0.571	0.571	0.667	0.571	0.571
QUAL(+)	0.938	1.000	0.875	0.375	0.750	0.750	0.750	0.875	0.750	0.688
SUPPLY(+)	0.363	0.313	0.500	0.063	0.625	1.000	1.000	1.000	1.000	1.000
ENVIR(-)	1.000	1.000	0.889	1.000	0.800	1.000	0.444	0.500	0.5000	0.500
FLEX(+)	0	0	0	0	0	0	0	0	0	0
PUBLIC(+)	0	0	0	0	0	0	0	0	0	0

(2) Preferences on Criteria

Table 8.4: The Values of Alternatives for Mid-term and Long-term project

Criteria	*Alternatives*									
	A^1	A^2	A^3	A^4	A^5	A^6	A^7	A^8	A^9	A^{10}
INVEST(-)	0.050	0.091	0.581	1.000	0.045	0.040	0.045	0.042	0.040	0.028
RISK(-)	0	0	0	0	0	0	0	0	0	0
QUAL(+)	0.938	1.000	0.875	0.375	0.750	0.750	0.750	0.875	0.750	0.688
SUPPLY(+)	0.363	0.313	0.500	0.063	0.625	1.000	1.000	1.000	1.000	1.000
ENVIR(-)	0	0	0	0	0	0	0	0	0	0
FLEX(+)	0.824	1.000	0.588	0.824	0.706	0.706	0.765	0.824	0.941	0.941
PUBLIC(+)	0.706	0.824	0.706	0.706	0.824	0.765	0.941	0.941	1.000	1.000

Preferences on criteria are explained as weights, which reflect the relative importance of the criteria for all three groups. Based on previous experience in water resources management, the authors estimate that the weight vector of all three groups are $\mathbf{w}^1 = (0.25, 0.15, 0.15, 0.25, 0.20, 0, 0)$, $\mathbf{w}^2 = (0.20, 0, 0.30, 0.10, 0, 0.25, 0.15)$, and $\mathbf{w}^3 = (0.15, 0, 0.20, 0.30, 0, 0.15, 0.20)$. Then the weight matrix $\mathbf{w} = (\mathbf{w}^1, \mathbf{w}^2, \mathbf{w}^3)^T$ is

$$\mathbf{w} = \begin{pmatrix} 0.25 & 0.15 & 0.15 & 0.25 & 0.20 & 0 & 0 \\ 0.20 & 0 & 0.30 & 0.10 & 0 & 0.25 & 0.15 \\ 0.15 & 0 & 0.2 & 0.30 & 0 & 0.15 & 0.2 \end{pmatrix}$$

By comparing $\mathbf{w}^1$, $\mathbf{w}^2$ and $\mathbf{w}^3$, it can be seen that the near-future project reflects more concerns about the criteria of INVEST and SUPPLY, mid-term project about the criteria of QUAL and FLEX, while long-term project about the criteria of SUPPLY and PUBLIC.

(3) Aggregation Values of the Groups and Aggregation Value Matrix

Using (8.16), the DM's aggregation value for each of the three groups is found to be:
$\mathbf{V}_1 = \big(\langle \mathbf{w}^1, \mathbf{v}_1^{1^T}\rangle, \langle \mathbf{w}^1, \mathbf{v}_1^{2^T}\rangle, \langle \mathbf{w}^1, \mathbf{v}_1^{3^T}\rangle, \langle \mathbf{w}^1, \mathbf{v}_1^{4^T}\rangle, \langle \mathbf{w}^1, \mathbf{v}_1^{5^T}\rangle, \langle \mathbf{w}^1, \mathbf{v}_1^{6^T}\rangle, \langle \mathbf{w}^1, \mathbf{v}_1^{7^T}\rangle, \langle \mathbf{w}^1, \mathbf{v}_1^{8^T}\rangle, \langle \mathbf{w}^1, \mathbf{v}_1^{9^T}\rangle, \langle \mathbf{w}^1, \mathbf{v}_1^{10^T}\rangle\big) = (0.594, 0.601, 0.699, 0.597, 0.540, 0.658, 0.548, 0.592, 0.558, 0.546)$.
$\mathbf{V}_2 = (0.639, 0.723, 0.682, 0.631, 0.596, 0.624, 0.666, 0.718, 0.718, 0.697)$, and
$\mathbf{V}_3 = (0.568, 0.622, 0.642, 0.508, 0.615, 0.715, 0.760, 0.793, 0.797, 0.783)$.

Therefore,

$$\begin{pmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{V}_3 \end{pmatrix} = \begin{pmatrix} 0.594 & 0.601 & 0.699 & 0.597 & 0.540 & 0.658 & 0.548 & 0.592 & 0.558 & 0.546 \\ 0.639 & 0.723 & 0.682 & 0.631 & 0.596 & 0.624 & 0.666 & 0.718 & 0.718 & 0.697 \\ 0.568 & 0.622 & 0.642 & 0.508 & 0.615 & 0.715 & 0.760 & 0.793 & 0.797 & 0.783 \end{pmatrix}$$

$$\text{and}\mathbf{V} = \begin{pmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{V}_3 \end{pmatrix}$$

$\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{V}_3$ capture the relative contribution of the ten alternatives to the three groups. One can see, for example, that A^3, A^2, and A^9 make the largest contributions to group 1, group 2, and group 3 respectively. Therefore, the DM may directly determine the best solution under simple constraints. Our proposed model uses a systematic analysis to find the best solutions incorporating different classification types and DM's strategies.

8.6.5 Value Function

Using (8.18), the DM's whole value function is established as

$$V_S = (1,0,0)\cdot(\mathbf{V}\cdot S)\cdot\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (0,1,0)\cdot(\mathbf{V}\cdot S)\cdot\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (0,0,1)\cdot(\mathbf{V}\cdot S)\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where $S = (s_g^i)$, $i = 1, 2, ..., 10$ and $g = 1, 2, 3$, is an alternatives classification solution, and s_g^i is a binary variable.

8.6.6 Classification Analysis

(1) $\mathbf{MCNC_1}$ Result Without the DM's Strategy Information

The optimization model is set as follows:

Maximize : V_S

Subject to: $\langle S^i, I_3 \rangle = 1, \forall\ S^i \in \mathbf{S}$

The following result is obtained.

$$V_S^* = 7.14,\ S^* = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

So A^1, A^2, A^3, A^4 are assigned to group 2; A^5, A^6, A^7, A^8, A^9, A^{10} are assigned to group 3. Likely, the DM will not be satisfied with this findings, since no alternative is assigned to group 1. Hence, this $\mathbf{MCNC}_1$ problem requires further strategic information and this is investigated next.

(2) $\mathbf{MCNC_1}$ Situation With the DM's Strategy Information

The numbers of alternatives in groups 1 and 2, n_1 and n_2, are set to be 3 each. The corresponding optimization model is as follows:

Maximize : V_S

$$\textbf{Subject to}: \quad \langle S^i, I_3 \rangle = 1, \forall\ S^i \in \mathbf{S}$$
$$\langle S_1^T, I_3 \rangle = 3$$
$$\langle S_2^T, I_3 \rangle = 3$$

for which the following result is obtained.

$$V_S^* = 7.02,\ S^* = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

So A^3, A^4, A^6 are assigned to group 1; A^1, A^2, A^5 are assigned to group 2; A^7, A^8, A^9 and A^{10} are assigned to group 3.

8.7 Conclusions

This research is concerned with investigating a new kind of MCDA problem, called Multiple Criteria Nominal Classification. MCNC aims to assign alternatives to nominal groups without preferences, based on sets of criteria that may be the same or different. The definitions, features, and properties of MCNC are stated and explained in detail. A systematic procedure is proposed to solve MCNC problems and an optimization model is designed that takes into account the DM's strategies for the solution of MCNC problems. An application of MCNC in water supply planning is presented.

Chapter 9

Contributions and Future Research

Several comprehensive and flexible procedures for classification problems in MCDA have been defined and analyzed in this thesis and many practical applications have been described. The main contributions of the thesis and suggestions for future research are summarized in the next two subsections.

9.1 Main Contributions of the Thesis

The main contributions of this thesis are the following:

1. In Chapter 2, an analysis procedure, consequence-based preference aggregation, is proposed to provide a systematic framework of MCDA. Many approaches to MCDA are summarized and integrated into a meaningful structure.

2. In Chapter 3, screening problems are systematically addressed: a general description of a screening problem is provided, and a sequential screening procedure is designed to integrate several popular MCDA methods and adapt them to solve screening problems. A case study in water supply planning in the Regional Municipality of Waterloo is carried out to demonstrate the proposed method.

3. In Chapter 4, a case-based distance method is proposed to solve screening problems. Research on case-based reasoning in MCDA is summarized. Then,

the assumptions of the proposed case-based distance method are explained in detail. Next, a quadratic program is constructed to find the best descriptive criterion weights and the screening thresholds. Based on this information, a distance-based method is proposed for screening and a numerical example is used to demonstrate how the proposed method works.

4. In Chapter 5, the case-based distance method is extended to solve sorting problems in MCDA. A general description of sorting problems is given in detail including a formal definition of sorting procedure and a relationship discussion between the alternative set and the sorting group set. Based on weighted Euclidean distance, two case-based distance methods are developed for sorting using weights and group thresholds obtained by assessment of a case set provided by a DM. Case-based sorting method I is designed for use with cardinal criteria; its extension, case-based sorting method II, can handle both cardinal and ordinal criteria. Optimization programs are employed to find the most descriptive weights and group thresholds. Finally, a case study on Canadian municipal water usage analysis is presented.

5. In Chapter 6, a case-based multiple criteria ABC analysis is proposed to improve the traditional ABC analysis in inventory management and provide more managerial flexibility by accounting for additional criteria, such as lead time and criticality. A case study is developed to illustrate how the procedure can be applied; the results demonstrate that this approach is robust and can produce sound classifications when multiple criteria are considered.

6. In Chapter 7, a case-based distance model founded on multiple criteria decision analysis theory is proposed for bilateral negotiations (BN) involving multiple issues. The unique feature of this negotiation model is that weighted Euclidean distance is employed to represent the negotiators' preferences; a case-based distance algorithm then helps negotiators express their preferences over different offers (alternatives) and suggests how to find better outcomes. The procedure takes advantage of the easily understood geometric meaning of Euclidean distance.

7. In Chapter 8, a new kind of MCDA problem, multiple criteria nominal classification (MCNC), is studied. Traditional classification methods in MCDA focus on sorting alternatives into groups ordered by preference. MCNC is the classification of alternatives into nominal groups, structured by the DM, who specifies multiple characteristics for each group. Starting with illustrative examples, the features, definition and structures of MCNC are presented, emphasizing criterion and alternative flexibility. Then an analysis procedure is

proposed to solve MCNC problems systematically. Assuming additive value functions, an optimization model with constraints that incorporate various classification strategies is constructed to solve MCNC problems. An application of MCNC in water resources planning is carried out and some future extensions are suggested.

As a result of both the theoretical developments and the real world applications, it is believed that this thesis has laid down the foundation of a new methodology for MCDA, namely, multiple criteria classification (MCC).

9.2 Suggestions for Future Research

The development of this thesis not only provides several approaches to modelling and analyzing classification problems in MCDA, but it also opens up new avenues to further research in MCDA. Below are some possible directions for future research.

1. Further research is required to refine the case-based distance approach to MCDA. Following are some important topics:
   (a) The problem of handling or avoiding inconsistencies in the case set, which is of crucial importance to the final result;
   (b) The classification abilities of the case-based distance method with other techniques, such as proposals of Doumpos and Zopouidis (2002).
   (c) The comparison of the significance of various distance definitions, such as city block distance (L_1) versus Euclidean distance (L_2), particularly in an experimental setting such as multiple-issue negotiations.
   (d) The expansion of the case-based distance approach to accommodate a multiple stakeholder scenario. For example, the case set sorting provided by different stakeholders might be aggregated so that final results reflect all stakeholders' preferences.
2. In terms of the nominal classification problem, MCNC, procedures could be designed to take into account the effects of uncertainty in weights and values. Another research topic is the analysis of the DM's strategies, since different strategic requirements may be incompatible. The DM may require assistance to define the problem and construct his or her strategies. Other methods, for example, distance-based methods, may be designed to solve MCNC problems.

3. A software-based decision support system (DSS) could help a DM implement this approach easily and expeditiously. Hence, a computer-based DSS should be developed to integrate the classification procedures discussed in the thesis and assist in practical applications.

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