Boundary Integral Equation Method in Elasticity with Microstructure

by

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Abstract

Problems involving mechanical behavior of materials with microstructure are receiving an increasing amount of attention in the literature. First of all, it can be attributed to the fact that a number of recent experiments shows a significant discrepancy between results of the classical theory of elasticity and the actual behavior of materials for which microstructure is known to be significant (e.g. synthetic polymers, human bones). Second, materials, for which microstructure contributes significantly in the overall deformation of a whole body, are becoming more and more important for applications in different areas of modern day mechanics, physics and engineering.

Since the classical theory is not adequate for modeling the elastic behavior of such materials, a new theory, which allows us to incorporate microstructure into a classical model, should be used.

The foundations of a theory allowing to account for the effect of material microstructure were developed in the beginning of the twentieth century and is known as the theory of Cosserat (micropolar, asymmetric) elasticity. For the last forty years significant results have been accomplished leading to a better understanding of processes occurring in Cosserat continuum. In particular, significant progress has been achieved in the investigation of three-dimensional problems of micropolar elasticity, plane and anti-plane problems, bending of micropolar plates. These problems can be effectively solved in a very elegant manner using the boundary integral equation method.

However, the boundary integral equation method imposes significant restrictions on properties of boundaries of domains under consideration. In particular, it requires that the boundary be represented by a twice differentiable curve which makes it impossible to apply the method for domains with reduced boundary smoothness or domains containing cuts or cracks. Therefore, the rigorous treatment of boundary value problems of Cosserat elasticity for domains with irregular boundaries has remained untouched until today.

A mathematically sophisticated, but very effective approach which allows to overcome the difficulty relating to the boundary requirement consists of the formulation of the corresponding boundary value problems in terms of the distributional setting in Sobolev spaces. In this case the appropriate weak solution may be found in terms of the corresponding integral potentials which perfectly works for domains with reduced boundary smoothness.

The objective of this work is to develop such a method that allows us to describe and solve the boundary value problems of plane Cosserat elasticity for domains with non-smooth boundaries and for domains weakened by cracks. We illustrate the method by establishing the analytical solutions for boundary value problems of plane Cosserat elasticity, which plays an important role as a pilot problem for the investigation of more challenging problems of three-dimensional theory of micropolar elasticity. We show that the analytical solutions derived in this work may be successfully approximated numerically using the boundary element method and that these solutions can be extremely important for applications in engineering science.

One of the important applied problems considered herein is the problem of stress distribution around a crack in a human bone. The bone is modeled under assumptions of plane Cosserat elasticity and the solution derived on the basis of the method developed in this dissertation shows that material microstructure does indeed have a significant effect on stress distribution around a crack.

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This thesis is dedicated to my parents

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Chapter 1

Introduction

The classical theory of elasticity is based on an ideal model of the elastic continuum in which the transfer of loading through any interior surface element occurs only by means of the (force) stress vector. This assumption leads to a description of the strain of the body in terms of symmetric strain and stress tensors.

Results from analytical models derived on the basis of classical elasticity are in good agreement with experiments performed within the elastic range on numerous structural materials for example such as concrete, steel or aluminium.

The classical theory of elasticity, however, fails to produce acceptable results when the microstructure of the material contributes significantly to the overall deformation of the body, for example, in the case of granular bodies with large molecules (e.g. polymers) or human bones (see, for example, Kruyt [1], Rothenburg [2], [3], and Lakes [4] and [5]). These cases are becoming increasingly important in the design and manufacture of modern day advanced materials as small-scale effects become paramount in the prediction of the overall mechanical behavior of these materials.

An attempt to eliminate these shortcomings was first made by Voigt [6] in 1886 who assumed that the transfer of the interaction between two elements of the body through a surface element occurs not only by means of a force (stress) vector but also by means of an independent moment (couple-stress) vector.

However, the first more or less harmonious theory was introduced only in 1909 by the brothers E. and F. Cosserat [7-9]. In their theory, the Cosserat brothers made further developments of the Voigt's theory. They suggested that deformation of the body should be described by the displacement vector $u(x,t)$ and independent microrotation vector $\phi(x,t)$. The assumption that a material element has six degrees of freedom leads to the description of deformation of the body in terms of asymmetric strain and stress tensors unlike the classical theory of elasticity in which deformations can be described by only one symmetric stress tensor.

In spite of these new ideas Cosserats' work remained unnoticed for a significant period of time. The major drawbacks of the theory could have been that first, the theory had already been non-linear, secondly it was formulated in a very unclear manner and thirdly and probably the most important reason, the theory contained many problems lying very far from the framework of elasticity theory. In addition to problems related to the theory of elasticity the authors considered problems of non-ideal fluids, some problems related to electrodynamics and magnetism, in other words, they made an attempt to create a unified theory containing mechanics, optics and electrodynamics. No wonder that the theory was found to be very complicated and was not heeded.

However, the investigations in the area of solid mechanics and mechanics of fluids in the middle of the twentieth century demonstrated that the behavior of certain classes of materials and fluids cannot be described in terms of classical theory, hence the Cosserat theory was rediscovered and drew attention of many workers.

The investigations first have been concentrated on the simplified Cosserat theory, i.e. on the asymmetric elasticity in so-called Cosserat pseudocontinuum, sometimes this theory is also called couple-stress elasticity. In a Cosserat pseudocontinuum there is still a possibility of the generation of asymmetric stresses and couple stresses during deformation of the body, but at the same time the whole deformation of the body is described only by the displacement field. In other words, for reasons of simplicity it is assumed that the microrotation vector ϕ and the displacement vector u are dependent as in the classical theory of elasticity (see for example [10]) by means of the following relation

$$\phi = \frac{1}{2}\text{curl } u.$$

Among the papers on the couple-stress theory of elasticity, first of all it is necessary to notice the works of Toupin [11], [12] and Truesdell and Toupin [13], on the linear and non-linear elasticity of Cosserat pseudocontinuum. This work was further developed by Grioli [14], Mindlin [15-17] and Mindlin and Thiersten [18].

However, like almost any simplified theory, the couple-stress theory of elasticity could not entirely and precisely describe the deformations of granular bodies. The series of more recent experiments [19-23] clearly confirmed this fact once again. Mostly, the theory was represented only for the reason of simplicity, for example, the governing equations of the theory are just Navier's equations with respect to three unknown displacements - exactly as in the classical theory of elasticity. No wonder, that soon after appearance of the first papers on couple-stress theory, the general stipulations of the mathematically rigorous more general theory of Cosserat elasticity were introduced.

The foundations of the theory of a Cosserat continuum, when the microrotations and displacements are no longer dependent, were formulated by Gunther [24] and Schaefer [25], [26] in the late fifties and early sixties of the twentieth century. The first author examined the three-dimensional model of the Cosserat continuum and emphasized the importance of the Cosserat continuum for dislocation theory. The second author rediscovered the foundations of Cosserat theory for the state of plane strain. Then, several years later Aero and Kuvshinsky [27] and Palmov [28], [29] presented constitutive relations and governing equations of the general theory of Cosserat elasticity.

The interesting exposition of the theory of Cosserat elasticity was given by Eringen and his colleagues [30-32] who introduced the new name for the theory – the theory of micropolar or asymmetric elasticity. Eringen has also formulated the general provisions of the theory of micropolar plates [32]. An extensive description of the work that has been done in this field including the extensive bibliography, may be found in the book by Nowacki [33].

Parallel to the works in the area of Cosserat elasticity, investigations have also been conducted in the area of Cosserat fluids. A relatively complete bibliography in this field can be found, for example, in [34] and [35].

However, in spite of the importance of all afore-mentioned work, none of these papers or monographs dealt with both the mathematically rigorous formulation of the boundary value problems arising in the theory of micropolar elasticity and the methods of their solutions. Mostly it can be explained by the fact that methodology, methods and approaches of the classical theory of elasticity (for example, theory of analytical functions, Fredholm's theory of integral equations, theory of one-dimensional singular integral equations) are inadequate for the rigorous mathematical analysis of the governing equations and boundary conditions of such complicated structure. Fortunately, this situation is now changing mostly due to the important work in the area of three-dimensional classical elasticity carried out in the last 40 years.

Three-dimensional problems of classical elasticity can be worked out by a variety of means. Some of these approaches may be further successfully applied to the analysis of the boundary value problems of micropolar elasticity. The first possibility is the theory of multidimensional singular potentials and singular integral equations. The second one is the modern theory of generalized solutions of differential equations (the method of Hilbert spaces, variational methods).

The first direction based on the rapidly developing theory of singular integrals and integral equations is a direct extension of the concepts of the theory of potentials and Fredholm equations which are, as known, the prevailing concepts of the classical mechanics. This approach, being not so general as the one to be discussed below, allows to investigate in detail cases most important for the theory and applications, retaining the efficiency of the methods of the classical mechanics of continua. The breakthrough in this direction occurred after the pioneering work of Muskhelishvili on singular integral equations [36]. Further this approach has been extensively developed and applied to the rigorous investigation of the boundary value problems of three-dimensional theory of elasticity in the works by Kupradze and his colleagues [37], [38] and to the analysis of the bending of plates with transverse shear deformation in the work by Constanda [39].

The work of Kupradze has provided researchers with effective tools for investigations in the micropolar theory of elasticity. Iesan [40], using the approach proposed by Kupradze for the treatment of three-dimensional problems of micropolar elasticity, formulated uniqueness and existences theorems for the boundary-value problems of a micropolar state of plain strain. However, the analysis presented in [40] overlooks certain differentiability requirements to establish the rigorous solution to the problem. In a series of recent works by Schiavone and Constanda [41-44] and Potapenko [45-51] the framework of singular integral equations has been successfully adapted for establishing analytical solutions and analysis of boundary value problems of the theory of micropolar plates, plane micropolar elasticity and to the problem of torsion of cylindrical beams with microstructure. In addition, in [44] the boundary integral equation method was extended for the rigorous treatment of plane strain problems of micropolar elasticity allowing to overcome deficiency represented in [40].

The second direction – based on the ideas of the modern functional analysis which are novel to the classical mechanics – is characterized by the distributional approach. The distributional approach is preferable to the classical one because it allows the technique to work in domains with a relatively low degree of smoothness - for example, domains containing cuts or cracks. In addition, it facilitates the close monitoring of the performance of numerical schemes in such domains. Thus, to find out how fast the rate of convergence deteriorates near a corner, one needs to apply the Bramble-Hilbert lemma, which is formulated in terms of distributional solutions [52]. Generally, error bounds are expressed in a natural way by means of Sobolev space norms.

Since this approach is relatively new in the classical mechanics, there are very few works in this area that consider fundamental boundary value problems in conjunction with both variational and potential methods for finite and infinite domains. There should be mentioned here works by Dafermos [53] and Fichera [54], [55] and certainly a very recent book by Chudinovich and Constanda [56] that gives a very interesting exposition of the treatment of boundary value problems of the theory of plates in terms of weak solutions in Sobolev spaces.

In the present work we apply the distributional approach to the investigation of the boundary value problems of plane Cosserat elasticity. In [44] such solutions have been obtained in terms of integral potentials using boundary integral equation method in L^2-space. However, in L^2 such solutions can be found only if the boundary is sufficiently smooth and cannot be obtained in the case of the reduced boundary smoothness or if the domain contains cracks. Consideration of the case when the boundary of the domain is not smooth enough is, to the author's knowledge, still absent from the literature. Since the classical approach demonstrated in [44] does not allow to accommodate the problems with reduced boundary smoothness we reformulate these boundary value problems in a Sobolev space setting and employ the distributional approach to obtain exact analytical solutions in the closed form.

Plane strain deformation is a very important particular case of deformations solids can undergo. In the case of plain deformations the field quantities e.g. displacements, stresses depend only on two coordinates (x_1, x_2) and the boundary conditions are imposed on a curve $f(x_1, x_2) = 0$ in the x_1x_2- plane. Consequently, we assume that the solid body is a surface bounded by a curve $f(x_1, x_2) = 0$ with a thickness that can be neglected.

In this sense, certainly, there are no strictly two-dimensional problems in elasticity i.e. there are circumstances in which the stresses or displacements are independent of the x_3 coordinate but all real bodies must have some bounding surfaces which are not represented by a line in the plane.

There is however a distinct class of problems which can be dealt with in the context of plane strain elasticity. Such a simplification of a three-dimensional problem is possible when the dimension of the body in the x_3-direction is large in comparison with the dimensions of coordinates (x_1, x_2). For example, if a long cylindrical body is loaded by forces that are perpendicular to the longitudinal direction and do not vary along the length, it may be assumed that all cross-sections behave in the same way. It is simplest to suppose at first that the end sections of the cylinder are confined between fixed smooth rigid planes – so that displacement in the x_3-direction is not allowed – then, since there is no axial displacement at the ends and, by symmetry, at the midsection, it may be assumed that the same holds at every cross-section i.e. the vertical displacement is equal to zero or constant e.g. a retaining wall with lateral pressure, a cylindrical tube with internal pressure etc., see, for example, [57].

In recent years, considerable attention has been paid to the analysis of plane deformations within the context of various constitutive theories (linear and non-linear) of solid mechanics. Such studies were largely motivated by the promise of relative analytic simplicity compared with the three-dimensional problems since the governing equations are a system of two second-order partial differential equations instead of a system of three equations in the three-dimensional case. Thus the plane problem plays a useful role as a pilot problem, within which various aspects of solutions in solid mechanics may be examined in a particularly simple setting.

Unlike its classical counterpart, however, the theory describing plane deformations of a linearly elastic Cosserat solid is not marked by its relative analytic simplicity. The governing equations and fundamental boundary value problems describing the plane deformations of a linearly elastic, homogeneous and isotropic Cosserat elastic solid have been formulated by Nowacki [33] and later analyzed by Iesan [40] and Schiavone [44] in a classical L^2-setting by means of the boundary integral equation method. In fact, in the case of a Cosserat solid, the theory reduces to a coupled system of three partial differential equations for three unknowns: two describing the in-plane displacements and one more representing the microrotation.

The importance of studying this system is rigorously twofold. In the first place, its 'hybrid' nature (three equations for three unknown functions depending on only two independent variables) makes it a desirable object of analytic investigation in its own right. Second, a full mathematical analysis is necessary to answer the question of existence, uniqueness and stability of the solution of the model before numerical approximation algorithms are constructed. Such questions are handled particularly well through the use of variational and potential methods, which are general, powerful and elegant.

In this work we consider a number of important problems arising in the theory of plane Cosserat elasticity such as interior and exterior Dirichlet and Neumann boundary value problems and problems for domains weakened by cracks. In each case we give a variational formulation of the problems and discuss their solvability in spaces of distributions. Then we seek the solutions in the form of integral potentials and reduce the problems to integral equations on the contour of the domain. These equations are solved by means of specially constructed boundary operators, whose mapping properties in Sobolev spaces are studied in details.

The approach we use in this thesis has a number of certain advantages over that applied in [40] and [44]. First of all, it allows to extend the boundary integral equation method to the consideration of problems with irregular boundaries, second it makes it possible to obtain closed form analytical solutions for domains containing cracks and notches and third it offers a very efficient fast converging numerical scheme that provides an opportunity to use the obtained solutions in numerous engineering applications.

As a result of the investigation performed in this thesis, the following works [58-61] of the author have been recently published.

The thesis is organized as follows.

In Chapter 2 we provide a brief overview of the three-dimensional theory of micropolar elasticity, presented in details in [33]. The purpose of this chapter is to introduce the governing equations describing three-dimensional deformations of a linearly elastic Cosserat solid and to formulate the basic constitutive and kinematic relations that will be used for derivation of corresponding relations of the theory of plane Cosserat elasticity in the subsequent chapters.

Chapter 3 is devoted entirely to the plane problems of micropolar elasticity. On the basis of the governing equations and constitutive relations of the three-dimensional Cosserat theory, we derive the governing equations and formulate interior and exterior Dirichlet and Neumann problems of plane micropolar elasticity in Sobolev spaces. We prove the existence, uniqueness and continuous dependence on the data of weak solutions to these problems.

In Chapter 4 we show that weak solutions can be found in terms of specially constructed integral potentials with distributional densities and we prove the existence and uniqueness of solutions of corresponding boundary integral equations. The approach we introduce in Chapters 3 and 4 allows to extend the area of applications of the boundary integral equation method to domains with non-smooth boundaries. However, it is not the only case and the method can be further applied to the consideration of problems for domains containing cracks.

In Chapter 5 we formulate the boundary value problem for a domain weakened by a crack . We show that the solution to this problem exists and can be found in terms of integral potentials with distributional densities by means of the boundary integral equation method.

Since the solutions in the form of integral potentials may not be convenient for applications, in Chapter 6 we introduce the boundary element method by means of which we can approximate the solutions obtained in Chapter 5 numerically. We provide the evidence that the constructed solutions converge rapidly to the exact solutions of the corresponding boundary-value problem and demonstrate importance of the presented theory for applications. As an example, we consider the problem of stress distribution around a crack in a human bone. Such problem is very important for applications in orthopedic biomechanics. It is well-known that a human bone (both trabecular and cortical) is a highly organized composite material in which various geometrical features appear in a wide range of length scales and work jointly to give a bone its mechanical properties. Accurate mechanical models should include an interaction between different length scales. It has been shown in a series of works [4], [5], [62-64] that Cosserat theory can be advantageously applied to the investigation of deformations in a bone. In addition, the recent studies by Bouyge, Jasiuk and Ostoja-Starzewski [65-67] have demonstrated that elastic behavior of a bone may be very well modelled under assumptions of the plane strain theory. Using theory developed in the previous chapters, in Chapter 6 we construct the approximate solution using boundary element method and obtain the stress distribution around a crack. This example confirms the importance of the effect of material microstructure on the stress distribution around a crack.

Finally, in Chapter 7 we make several important conclusions and recommendations for future work.

Chapter 2

The Basic Foundations of Three-Dimensional Theory of Cosserat Elasticity

The purpose of this chapter is to present a brief overview of the general provisions of the three-dimensional theory of Cosserat elasticity. Since, this chapter summarizes only what has been done before, we skip certain details related to the derivation of constitutive equations and methods of solutions of the system of governing equations. Detailed description of three-dimensional theory of Cosserat elasticity can be found in [33].

In this chapter, Latin indices take the values 1, 2, 3, the convention of summation over repeated indices is understood.

Let an elastic isotropic Cosserat body occupy a domain V in $\mathbb{R}^3$ and be bounded by surface S. Assume that the body undergoes deformation due to the action of external forces $\mathbf{X} = (X_1, X_2, X_3)^T$ and external moments $\mathbf{Y} = (Y_1, Y_2, Y_3)^T$. The elastic properties of the body can be characterized by elastic constants $\lambda, \mu, \alpha, \beta, \gamma, \varepsilon$, where λ and μ are classical Lame coefficients as in the classical theory of elasticity and α, β, γ, and ε are micropolar elastic constants, representing the contribution of material microstructure to the elastic properties of the body. The state of deformation is characterized by a displacement field

$$u(x) = (u_1(x), u_2(x), u_3(x))^T$$

and a microrotation field

$$\phi(x) = (\phi_1(x), \phi_2(x), \phi_3(x))^T,$$

where $x = (x_1, x_2, x_3)$ is a generic point in $\mathbb{R}^3$. This leads to the description of deformation of the body in terms of asymmetric strain , torsion, stress and couple-stress tensors [33] of the form

$$\varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}, \quad \varkappa = \begin{bmatrix} \varkappa_{11} & \varkappa_{12} & \varkappa_{13} \\ \varkappa_{21} & \varkappa_{22} & \varkappa_{23} \\ \varkappa_{31} & \varkappa_{32} & \varkappa_{33} \end{bmatrix}, \tag{2.1}$$

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}, \quad \varrho = \begin{bmatrix} \varrho_{11} & \varrho_{12} & \varrho_{13} \\ \varrho_{21} & \varrho_{22} & \varrho_{23} \\ \varrho_{31} & \varrho_{32} & \varrho_{33} \end{bmatrix}, \tag{2.2}$$

where ε is the strain tensor, $\varkappa$ the torsion tensor, σ the stress tensor, ϱ the couple-stress tensor.

Following the procedure given in detail in [33] and [31] we can derive the equations of equilibrium in terms of stresses and couple stresses of the form

$$\begin{aligned} \sigma_{ji,j} + X_i &= 0, \\ \epsilon_{ijk}\sigma_{jk} + \varrho_{ji,j} + Y_i &= 0. \end{aligned} \tag{2.3}$$

where $\epsilon_{ijk}-$ alternating symbol.

Note, that in case of micropolar media, the equilibrium equations are more complicated than in the classical case because of the appearance of the extra system of equations due to the presence of couple stresses. It leads to a description of the elastic behavior of a Cosserat solid in terms of asymmetric stress and couple-stress tensors. It can be easily shown that if we set all couple stresses equal to zero we again obtain a symmetric stress tensor as in the classical case. Consequently, the presence of couple stresses prevents the symmetry of the stress tensor.

Using the constitutive relations [33]

$$\sigma_{ji} = (\mu + \alpha)\varepsilon_{ji} + (\mu - \alpha)\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ji} \tag{2.4}$$

$$\varrho_{ji} = (\gamma + \varepsilon)\varkappa_{ji} + (\gamma - \varepsilon)\varkappa_{ij} + \beta\varkappa_{kk}\delta_{ji}, \tag{2.5}$$

where δ_{ji} is the Kronecker symbol,

and the kinematic relations

$$\varepsilon_{ji} = u_{i,j} - \epsilon_{kji}\phi_k, \quad \varkappa_{ji} = \phi_{i,j}, \tag{2.6}$$

we can formulate the equilibrium equations in terms of displacements and microrotations in the following vector form

$$
\begin{aligned}
(\mu+\alpha)\Delta u+(\lambda+\mu-\alpha)\ grad\ div\ u+2\alpha\ curl\ \phi+X &= 0,\\
[(\gamma+\varepsilon)-4\alpha]\,\Delta\phi+(\beta+\gamma-\varepsilon)\ grad\ div\ \phi+2\alpha\ curl\ u+Y &= 0, \quad (2.7)
\end{aligned}
$$

where Δ is the Laplace operator.

As it can be seen from (2.7), the governing equations of micropolar elasticity have much more complicated structure than those of classical elasticity. In the case of a Cosserat solid, the system of governing equations is a system of coupled partial differential equations with six unknowns: three usual displacements as in the classical theory of elasticity and three more representing independent microrotations. We can conclude that the theory of Cosserat elasticity is much more general in comparison with the classical one. If we assume, that micropolar elastic constants $\alpha, \beta, \gamma, \varepsilon$ are equal to zero, we can easily see that the governing equations are reduced to the well-known Navier's equations : the governing equations of classical theory of elasticity.

In [37] the boundary value problems corresponding to (2.3) and (2.7) were shown to be well-posed and solved rigorously by means of the boundary integral equation method. In [33], [68-71] system (2.7) was integrated by means of the method of potentials under different sets of boundary conditions. Since our goal is to investigate plane problems of Cosserat elasticity we will not pay significant attention to the integration of equations (2.7) but we will use them to derive the governing equations for the plane-strain state.

Chapter 3

Weak solutions to the Boundary Value Problems of Plane Cosserat Elasticity

In this chapter we first derive the governing equations of plane micropolar elasticity on the basis of the general three-dimensional theory presented in Chapter 2. After that we will formulate the corresponding Dirichlet and Neumann boundary value problems in the Sobolev space setting and show that they are well-posed by means of establishing existence and uniqueness theorems.

3.1 Basic definitions

In what follows we assume that the convention of summation over repeated indices is understood, Greek and Latin indices take the values 1, 2 and 1, 2, 3, respectively, $\mathcal{M}_{m\times n}$ is the space of $(m\times n)$- matrices, E_n is the identity element in $\mathcal{M}_{n\times n}$, the columns of a (3×3)-matrix P are denoted by $P^{(i)}$, a superscript T indicates matrix transposition, the generic symbol c denotes various strictly positive constants, and $(...)_{,\alpha} \equiv \partial(...)/\partial x_\alpha$. Also, if X is a space of scalar functions and v is a matrix, $v \in X$ means that every component of v belongs to X.

Let S be a domain in $\mathbb{R}^2$ occupied by a homogeneous and isotropic linearly elastic micropolar material with elastic constants λ, μ, α, γ and ε, and ∂S is its boundary. We use the notations $\|\cdot\|_{0;S}$ and $\langle\cdot,\cdot\rangle_{0;S}$ for the norm and inner product in $L^2(S)\cap\mathcal{M}_{m\times 1}$ for any $m\in\mathbb{N}$. When $S=\mathbb{R}^2$, we write $\|\cdot\|_0$ and $\langle\cdot,\cdot\rangle_0$.

The state of plane micropolar strain is characterized by a displacement field

$$u(x') = (u_1(x'), u_2(x'), u_3(x'))^T$$

and a microrotation field

$$\phi(x') = (\phi_1(x'), \phi_2(x'), \phi_3(x'))^T$$

of the form

$$\begin{aligned} u_\alpha(x') &= u_\alpha(x), \qquad u_3(x') = 0, \qquad (3.1)\\ \phi_\alpha(x') &= 0, \qquad \phi_3(x') = \phi_3(x), \end{aligned}$$

where $x' = (x_1, x_2, x_3)$ and $x = (x_1, x_2)$ are generic points in $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively. The equilibrium equations of plane micropolar strain written in terms of displacements and microrotations are given by [40], [44]

$$L(\partial_x)u(x) + q(x) = 0, \qquad x \in S, \tag{3.2}$$

in which now, denoting ϕ_3 by u_3, we have $u(x) = (u_1, u_2, u_3)^T$, the matrix partial differential operator $L(\partial_x) = L(\partial/\partial x_\alpha)$ is defined by

$$L(\xi) = L(\xi_\alpha)$$

$$= \begin{pmatrix} (\mu+\alpha)\Delta + (\lambda+\mu-\alpha)\xi_1^2 & (\lambda+\mu-\alpha)\xi_1\xi_2 & 2\alpha\xi_2 \\ (\lambda+\mu-\alpha)\xi_1\xi_2 & (\mu+\alpha)\Delta + (\lambda+\mu-\alpha)\xi_2^2 & -2\alpha\xi_1 \\ -2\alpha\xi_2 & 2\alpha\xi_1 & (\gamma+\varepsilon)\Delta - 4\alpha \end{pmatrix},$$

where $\Delta = \xi_\alpha\xi_\alpha$, and vector $q = (q_1, q_2, q_3)^T$ represents body forces and body couples.

Together with L we consider the boundary stress operator $T(\partial_x) = T(\partial/\partial x_\alpha)$ defined by

$$T(\xi) = T(\xi_\alpha)$$

$$= \begin{pmatrix} (\lambda+2\mu)\,\xi_1 n_1 + (\mu+\alpha)\,\xi_2 n_2 & (\mu-\alpha)\xi_1 n_2 + \lambda\xi_2 n_1 & 2\alpha n_2 \\ (\mu-\alpha)\xi_2 n_1 + \lambda\xi_1 n_2 & (\lambda+2\mu)\,\xi_2 n_2 + (\mu+\alpha)\,\xi_1 n_1 & -2\alpha n_1 \\ 0 & 0 & (\gamma+\varepsilon)\xi_\alpha n_\alpha \end{pmatrix},$$

where $n = (n_1, n_2)^T$ is the unit outward normal to ∂S. To guarantee the ellipticity of system (3.2), in what follows we assume that

$$\lambda + \mu > 0, \quad \mu > 0, \quad \gamma + \varepsilon > 0, \quad \alpha > 0.$$

The internal energy density is given by

$$
\begin{aligned}
2E\,(u,v) &= 2E_0\,(u,v) \\
&\quad + \mu(u_{1,2}+u_{2,1})(v_{1,2}+v_{2,1}) \\
&\quad + \alpha(u_{1,2}-u_{2,1}+2u_3)(v_{1,2}-v_{2,1}+2v_3) \\
&\quad + (\gamma+\varepsilon)(u_{3,1}v_{3,1}+u_{3,2}v_{3,2}),
\end{aligned}
$$

$$
2E_0\,(u,v) = (\lambda+2\mu)\,(u_{1,1}v_{1,1}+u_{2,2}v_{2,2}) + \lambda(u_{1,1}v_{2,2}+u_{2,2}v_{1,1}).
$$

Clearly, $E(u,u)$ is a positive quadratic form.

The space of rigid displacements and microrotations $\mathcal{F}$ is spanned by the columns of the matrix

$$
\mathbb{F} = \begin{pmatrix} 1 & 0 & -x_2 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix}
$$

from which it can be seen that $L\mathbb{F}=0$ in $\mathbb{R}^2$, $T\mathbb{F}=0$ on ∂S and a general rigid displacement can be written as $\mathbb{F}k$, where $k \in \mathcal{M}_{3\times 1}$ is constant and arbitrary.

Let S^+ be a domain in $\mathbb{R}^2$ bounded by a closed curve ∂S, and $S^- = \mathbb{R}^2 \backslash \overline{S^+}$. Using the same technique as in the derivation of the Betti formula [44], it is easy to show that if u is a solution of (3.2) in S^+, then for any $v \in C^2(S^+) \cap C^1(\overline{S}^+)$

$$
\int_{S^+} v^T q dx = -\int_{S^+} v^T L u dx = 2\int_{S^+} E(u,v)\; dx - \int_{\partial S} v^T T u\; ds, \tag{3.3}
$$

A Galerkin representation for the solution of (3.2) when $q(x) = -\delta(|x-y|)$, where δ is the Dirac delta distribution, yields the matrix of fundamental solutions [44]

$$D(x,y) = L^*(\partial x)t(x,y), \tag{3.4}$$

where L^* is the adjoint of L,

$$t(x,y) = \frac{a}{8\pi k^4}\left\{[k^2\,|x-y|^2+4]\ln|x-y| + 4K_0(k|x-y|)\right\}, \tag{3.5}$$

K_0 is the modified Bessel function of order zero and the constants a, k^2 are defined by

$$a^{-1} = (\gamma+\varepsilon)(\lambda+2\mu)(\mu+\alpha), \quad k^2 = \frac{4\mu\alpha}{(\gamma+\varepsilon)(\mu+\alpha)}.$$

In view of (3.4) and (3.5)

$$D(x,y) = D^T(x,y) = D(y,x).$$

Along with matrix $D(x,y)$ we consider the matrix of singular solutions

$$P(x,y) = (T(\partial y)D(y,x))^T. \tag{3.6}$$

It is easy to verify that $D^{(i)}(x,y)$ and $P^{(i)}(x,y)$ satisfy (3.2) with $q(x)=0$ at all $x \in \mathbb{R}^2, x \neq y$.

We introduce class $\mathcal{A}$ of vectors $u \in \mathcal{M}_{3\times 1}$ whose components in terms of polar coordinates, as $r = |x| \to \infty$, are of the form

$$\begin{aligned}
u_1(r,\theta) &= r^{-1}(\beta m_0 \sin\theta + m_1\cos\theta + m_0\sin 3\theta + m_2\cos 3\theta) + O(r^{-2}),\\
u_2(r,\theta) &= r^{-1}(m_3 \sin\theta + \beta m_0\cos\theta + m_4\sin 3\theta - m_0\cos 3\theta) + O(r^{-2}),\\
u_3(r,\theta) &= r^{-2}(m_5\sin 2\theta + m_6\cos 2\theta) + O(r^{-3}),
\end{aligned}$$

where

$$\beta = \frac{3\mu+\lambda}{\lambda+\mu},$$

and $m_0, \ldots, m_6$ are arbitrary constants. Also, let

$$\mathcal{A}^* = \left\{ u : u = \mathcal{F}c + \sigma^{\mathcal{A}} \right\},$$

where $c \in \mathcal{M}_{3\times 1}$ is constant and arbitrary and $\sigma^{\mathcal{A}} \in \mathcal{M}_{3\times 1} \cap \mathcal{A}$.

For exterior domain the Betti formula [44] is as follows. If u is a solution of (3.2) in S^-, then for any $v \in C^2(S^-) \cap C^1(\overline{S}^-) \cap \mathcal{A}^*$

$$\int_{S^-} v^T q dx = -\int_{S^-} v^T L u dx = 2\int_{S^-} E(u,v)\, dx + \int_{\partial S} v^T T u\, ds. \tag{3.7}$$

Further, we introduce the corresponding area, single layer, and double layer potentials given respectively by

$$(U\varphi)(x) = \int_{\mathbb{R}^2} D(x,y)\varphi(y)\, dy,$$

$$(V\varphi)(x) = \int_{\partial S} D(x,y)\varphi(y)\, ds(y),$$

$$(W\varphi)(x) = \int_{\partial S} P(x,y)\varphi(y)\, ds(y),$$

where $\varphi \in \mathcal{M}_{3\times 1}$ is an unknown density matrix.

It is not difficult to check that $L(Uq) = q$ in $\mathbb{R}^2$.

We recall the properties of single and double layer integral potentials in the following theorem, which has been proved in [44].

Theorem 1 *(i) If $\varphi \in C(\partial S)$, then $V\varphi$, $W\varphi$ are analytic and satisfy $L(V\varphi) = L(W\varphi) = 0$ in $S^+ \cup S^-$.*

(ii) If $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0,1)$, then the direct values $V_0\varphi, W_0\varphi$ of $V\varphi, W\varphi$ on ∂S exist (the latter as principal value), the functions

$V^+(\varphi) = (V\varphi)|_{\overline{S}^+}$, $V^-(\varphi) = (V\varphi)|_{\overline{S}^-}$ are of class $C^{1,\alpha}(\overline{S}^+)$ and $C^{1,\alpha}(\overline{S}^-)$, respectively and

$TV^+(\varphi) = (W_0^* + \frac{1}{2}I)\varphi, \qquad TV^-(\varphi) = (W_0^* - \frac{1}{2}I)\varphi$ on ∂S, where W_0^* is the adjoint of W_0 and I - the identity operator.

(iii) If $\varphi \in C^{1,\alpha}(\partial S),\ \ \alpha \in (0,1)$, then the functions

$$W^+(\varphi) = \begin{cases} (W\varphi)|_{S^+}, & \text{in } S^+, \\ (W_0 - \frac{1}{2}I)\varphi, & \text{on } \partial S, \end{cases} \qquad W^-(\varphi) = \begin{cases} (W\varphi)|_{S^-}, & \text{in } S^+, \\ (W_0 + \frac{1}{2}I)\varphi, & \text{on } \partial S, \end{cases}$$

are of class $C^{1,\alpha}(\overline{S}^+)$ and $C^{1,\alpha}(\overline{S}^-)$, respectively, and $TW^+(\varphi) = TW^-(\varphi)$ on ∂S.

For any $m \in \mathbb{R}$, let $H_m(\mathbb{R}^2)$ be the standard real Sobolev space of three-component distributions, equipped with the norm

$$\| u \|_m^2 = \int_{\mathbb{R}^2} (1 + |\xi|^2)^m |\widetilde{u}(\xi)|^2 d\xi,$$

where $\widetilde{u}$ is the Fourier transform of u. In what follows we do not distinguish between equivalent norms and denote them by the same symbol; thus, the norm in $H_1(\mathbb{R}^2)$ can be defined by

$$\| u \|_1^2 = \| u \|_0^2 + \sum_{i=1}^{3} \| \nabla u_i \|_0^2 .$$

The spaces $H_m(\mathbb{R}^2)$ and $H_{-m}(\mathbb{R}^2)$ are dual with respect to duality induced by $\langle \cdot, \cdot \rangle_0$.

We introduce the space $L^2_\omega(\mathbb{R}^2)$ of (3×1)-vector functions $u = (\overline{u}, u_3)^T$, where $\overline{u} = (u_1, u_2)^T$, such that

$$\| u \|_{0,\omega}^2 = \int_{\mathbb{R}^2} \frac{|\overline{u}(x)|^2}{(1+|x|)^2(1+\ln|x|)^2} dx + \int_{\mathbb{R}^2} \frac{|u_3(x)|^2}{(1+|x|)^4(1+\ln|x|)^2} dx < \infty.$$

We consider the bilinear form $b(u,v) = 2\int_{\mathbb{R}^2} E(u,v)\ dx$. Let $H_{1,\omega}(\mathbb{R}^2)$ be the space of three-component distributions on $\mathbb{R}^2$ for which

$$\| u \|_{1,\omega}^2 = \| u \|_{0,\omega}^2 + b(u,u) < \infty,$$

$H_{-1,\omega}(\mathbb{R}^2)$ is dual to $H_{1,\omega}(\mathbb{R}^2)$ with respect to duality generated by $\langle \cdot,\cdot \rangle_0$. The norm in $H_{-1,\omega}(\mathbb{R}^2)$ is denoted by $\| \cdot \|_{-1,\omega}$.

Let $\overset{\circ}{H}_m(S^+)$ be the subspace of $H_m(\mathbb{R}^2)$ consisting of all u which have a compact support in S^+. $H_m(S^+)$ is the space of the restrictions to S^+ of all $u \in H_m(\mathbb{R}^2)$. Denoting by $\pi^\pm$ the operators of restrictions from $\mathbb{R}^2$ to $S^\pm$, respectively, we introduce the norm of $u \in H_m(S^+)$ by $\| u \|_{m;S^+} = \inf_{v \in H_m(\mathbb{R}^2):\pi^+ v = u} \| v \|_m$. If $m = 1$, then the norms of $u \in \overset{\circ}{H}_1(S^+)$ and $u \in H_1(S^+)$ are equivalent to

$$\left\{ \| u \|_{0;S^+}^2 + \sum_{i=1}^{3} \int_{S^+} |\nabla u_i(x)|^2 dx \right\}^{1/2}.$$

The spaces $\overset{\circ}{H}_m(S^+)$ and $H_{-m}(S^+)$ are dual with respect to duality induced by $\langle \cdot,\cdot \rangle_{0;S^+}$.

Let $\overset{\circ}{H}_{1,\omega}(S^-)$ be the subspace of $H_{1,\omega}(\mathbb{R}^2)$ consisting of all u which have compact support in S^-. $H_{1,\omega}(S^-)$ is the space of the restrictions to S^- of all $u \in H_{1,\omega}(\mathbb{R}^2)$. The norm in $H_{1,\omega}(S^-)$ is defined by

$$\| u \|_{1,\omega;S^-} = \inf_{v \in H_{1,\omega}(\mathbb{R}^2):\pi^- v = u} \| v \|_{1,\omega}.$$

From the definition it follows that $H_{1,\omega}(S^-)$ is isometric to $H_{1,\omega}(\mathbb{R}^2) \backslash \overset{\circ}{H}_1(S^+)$. It can be shown that the norm of $u \in H_{1,\omega}(S^-)$ is equivalent to

$$\left\{ \| u \|_{0,\omega;S^-}^2 + b_-(u,u) \right\}^{1/2},$$

where

$$\| u \|^2_{0,\omega;S^-} = \int_{S^-} \frac{|\overline{u}(x)|^2}{(1+|x|)^2(1+\ln|x|)^2} dx + \int_{S^-} \frac{|u_3(x)|^2}{(1+|x|)^4(1+\ln|x|)^2} dx$$

and $b_\pm(u,v) = 2\int_{S^\pm} E(u,v)\, dx$. This norm is compatible with asymptotic class $\mathcal{A}$.

The dual of $\overset{\circ}{H}_{1,\omega}(S^-)$ with respect to the duality generated by $\langle\cdot,\cdot\rangle_{0;S^-}$ is the space $H_{-1,\omega}(S^-)$, with norm $\|\cdot\|_{-1,\omega;S^-}$; the dual of $H_{1,\omega}(S^-)$ is $\overset{\circ}{H}_{-1,\omega}(S^-)$, which can be identified with a subspace of $H_{-1,\omega}(\mathbb{R}^2)$. It can be shown that if $u \in \overset{\circ}{H}_{-1}(S^-)$ and has compact support in S^-, or if

$$\int_{S^-} |\overline{u}(x)|^2\,(1+|x|)^2(1+\ln|x|)^2 dx + \int_{S^-} |u_3(x)|^2\,(1+|x|)^4(1+\ln|x|)^2 dx < \infty,$$

then $u \in \overset{\circ}{H}_{-1,\omega}(S^-)$.

Let $H_m(\partial S)$ be the standard Sobolev space of distributions on ∂S, with norm $\|\cdot\|_{m;\partial S}$. $H_m(\partial S)$ and $H_{-m}(\partial S)$ are dual with respect to the duality generated by the inner product $\langle\cdot,\cdot\rangle_{0;\partial S}$ in $L^2(\partial S)$.

We denote by γ^+ and γ^- the trace operators defined first on $C_0^\infty(S^\pm)$ and then extended by continuity to the surjections $\gamma^+ : H_1(S^+) \longrightarrow H_{1/2}(\partial S)$, $\gamma^- : H_{1,\omega}(S^-) \longrightarrow H_{1/2}(\partial S)$. This is possible because of the local equivalence of $H_{1,\omega}(S^-)$ and $H_1(S^-)$. We also consider a continuous extension operators $l^+ : H_{1/2}(\partial S) \longrightarrow H_1(S^+)$, $l^- : H_{1/2}(\partial S) \longrightarrow H_1(S^-)$, the latter, since norm in $H_1(S^-)$ is stronger than that in $H_{1,\omega}(S^-)$, can also be regarded as a continuous operator from $H_{1/2}(\partial S)$ into $H_{1,\omega}(S^-)$.

To proceed further we will need the following well-known fact from the functional analysis.

Theorem 2 *(Lax-Milgram Lemma) Let H be a Hilbert space and $b(u,v)$ be a bilinear functional defined for every ordinate pair $u, v \in H$, for which there exist two constants h and k such that:*

$$|b(u,v)| \leqslant h \|u\| \|v\|, \quad \|u\|^2 \leqslant k |b(u,u)| \quad \forall u, v \in H,$$

in this case we say that $b(u,v)$ is coercive. Then however we assign the bounded linear functional $\mathcal{L}(v)$ on H there exists one and only one u such that

$$b(u,v) = \mathcal{L}(v), \quad \forall v \in H, \quad \|u\| \leqslant c \|\mathcal{L}\|_*,$$

where $\|\cdot\|_*$ is the norm on the dual H' of H.

The proof of this lemma can be found in [72].

3.2 Interior boundary value problems

We consider Dirichlet and Neumann interior boundary value problems.

The (interior) Dirichlet problem is formulated as follows.

Find $u \in C^2(S^+) \cap C^1(\overline{S}^+)$ satisfying (3.2) such that $u|_{\partial S} = f$, $\qquad$ (D^+)

where f is prescribed on ∂S.

Let (D_0^+) be the interior Dirichlet problem with $f = 0$. From (3.3) we see that a solution u of (D_0^+) satisfies

$$b_+(u,v) = \langle q, v \rangle_{0,S^+} \quad \forall v \in C_0^\infty(S^+), \tag{3.8}$$

Since $C_0^\infty(S^+)$ is dense in $\mathring{H}_1(S^+)$, it is clear that (3.8) holds for any $v \in \mathring{H}_1(S^+)$. Obviously, any $u \in C^2(S^+) \cap C^1(\overline{S}^+)$ satisfying (3.8) for any $v \in \mathring{H}_1(S^+)$

and $u|_{\partial S} = 0$ is a classical (regular) solution of (D_0^+). Hence, the variational formulation of (D_0^+) is as follows.

Find $u \in \overset{\circ}{H}_1(S^+)$ such that

$$b_+(u,v) = \langle q, v\rangle_{0,S^+} \quad \forall v \in \overset{\circ}{H}_1(S^+). \tag{3.9}$$

Theorem 3 *There exists a constant $c = c(S^+) > 0$ such that*

$$b_+(u,u) + \| u \|^2_{0;S^+} \geqslant c \| u \|^2_{1;S^+} \; \forall u \in H_1(S^+). \tag{3.10}$$

Proof. In view of the condition on α, λ, γ, ε and μ, $E(u,u)$ is a positive quadratic form. Consequently, we may introduce the space G of all (3×1)-vector functions u on S^+ with norm

$$\| u \|^2_G = b_+(u,u) + \| u \|^2_{0;S^+} .$$

Let $\{u^{(n)}\}$ be a Cauchy sequence in G. From the definition of $b_+(u,v)$ it follows that there are β_{11}, β_{22}, β_{12}, β_{13}, β_{31}, β_{32}, $\beta \in L^2(S^+)$ such that

$$\begin{aligned} u^{(n)}_{1,1} &\to \beta_{11},\; u^{(n)}_{2,2} \to \beta_{22},\; u^{(n)}_{1,2} + u^{(n)}_{2,1} \to \beta_{12}, \\ u^{(n)}_{2,1} - u^{(n)}_{1,2} - 2u^{(n)}_3 &\to \beta_{13},\; u^{(n)}_{3,1} \to \beta_{31},\; u^{(n)}_{3,2} \to \beta_{32},\; u^{(n)} \to \beta \end{aligned}$$

in $L^2(S^+)$. Then

$$\begin{aligned} u^{(n)}_{1,1} &\to \beta_{11} = \beta_{1,1}, \quad u^{(n)}_{2,2} \to \beta_{22} = \beta_{2,2}, \\ u^{(n)}_{1,2} + u^{(n)}_{2,1} &\to \beta_{12} = \beta_{1,2} + \beta_{2,1}, \\ u^{(n)}_{2,1} - u^{(n)}_{1,2} - 2u^{(n)}_3 &\to \beta_{13} = \beta_{2,1} - \beta_{1,2} - 2\beta_3, \\ u^{(n)}_{3,1} &\to \beta_{31} = \beta_{3,1}, \quad u^{(n)}_{3,2} \to \beta_{32} = \beta_{3,2} \end{aligned}$$

in the sense of distributions, hence, also in $L^2(S^+)$. This implies that β, $\beta_{1,1}$, $\beta_{2,2}$, $\beta_{1,2}+\beta_{2,1}$, $\beta_{2,1}-\beta_{1,2}-2\beta_3$, $\beta_{3,1}$, $\beta_{3,2} \in L^2(S^+)$ and $\| u^{(n)} - \beta \|_G \to 0$, which means that G is complete. For any $u \in G$ we have

$$\begin{aligned} u_{1,21} &= (u_{1,1})_{,2} \in H_{-1}(S^+), \\ u_{1,22} &= (u_{1,2}+u_{2,1})_{,2} - (u_{2,2})_{,1} \in H_{-1}(S^+), \\ u_{1,2} &\in H_{-1}(S^+), \end{aligned}$$

so $u_{1,2} \in L^2(S^+)$ [73]. Since $u_1 \in L^2(S^+)$, $u_{1,1} \in L^2(S^+)$ and, as we have just seen, $u_{1,2} \in L^2(S^+)$, the same argument yields $u_1 \in H_1(S^+)$. The fact that $u_2 \in H_1(S^+)$ is shown similarly. Finally,

$$\begin{aligned} u_{3,1} &\in L^2(S^+), \quad u_{3,2} \in L^2(S^+), \\ u_3 &= \frac{1}{2}\left(u_{2,1} - u_{1,2} - (u_{2,1} - u_{1,2} - 2u_3)\right) \in L^2(S^+), \end{aligned}$$

so $u_3 \in H_1(S^+)$. This indicates that G is a subset of $H_1(S^+)$. The converse statement being obvious, we conclude that G and $H_1(S^+)$ coincide as sets. The imbedding operator $\mathcal{I} : H_1(S^+) \to G$ is bijective and continuous, therefore, by Banach's theorem [72] on the inverse operator $\mathcal{I}^{-1}$ is continuous; in other words, $\| u \|_G^2 \geqslant c \| u \|_{1,S^+}^2$, which is the same as (3.10). ■

Theorem 4 *There exists a constant* $c = c(S^+) > 0$ *such that*

$$b_+(u,u) \geqslant c \| u \|_1^2 \ \forall u \in \overset{\circ}{H}_1(S^+). \tag{3.11}$$

Proof. We claim that there is a $c = c(S^+) > 0$ such that

$$b_+(u,u) \geqslant c \| u \|_{0,S^+}^2 \ \forall u \in \overset{\circ}{H}_1(S^+). \tag{3.12}$$

Indeed, if the opposite is true, then we can construct a sequence $\{u^{(n)}\}$ in $\overset{\circ}{H}_1(S^+)$ such that

$$b_+(u^{(n)}, u^{(n)}) \to 0, \quad \| u^{(n)} \|_{0,S^+} = 1 \text{ for all } n. \tag{3.13}$$

By (3.10), $\{u^{(n)}\}$ is bounded in $H_1(S^+)$ so, by Rellich's lemma, it contains a convergent subsequence (again denoted by $\{u^{(n)}\}$, for simplicity); that is, there is a $u \in L^2(S^+)$ such that $u^{(n)} \to u$ in $L^2(S^+)$. This means that, in view of (3.13),

$$\begin{aligned} u_{1,1}^{(n)} &\to 0 = u_{1,1}, \quad u_{2,2}^{(n)} \to 0 = u_{2,2}, \\ u_{1,2}^{(n)} + u_{2,1}^{(n)} &\to 0 = u_{1,2} + u_{2,1}, \\ u_{2,1}^{(n)} - u_{1,2}^{(n)} - 2u_3^{(n)} &\to 0 = u_{2,1} - u_{1,2} - 2u_3, \\ u_{3,1}^{(n)} &\to 0 = u_{3,1}, \quad u_{3,2}^{(n)} \to 0 = u_{3,2} \end{aligned}$$

in $L^2(S^+)$. These equalities imply that u is a rigid displacement. Since $u = 0$ on ∂S, it follows that $u = 0$ in $\overline{S^+}$ which contradicts the corollary $\| u \|_{0,S^+} = 1$ of (3.13). Hence, (3.12) holds, and the statement of the theorem is now obtained from (3.12) and (3.10). ■

Theorem 5 *Problem (3.9) has a unique solution* $u \in \overset{\circ}{H}_1(S^+)$ *for every* $q \in H_{-1}(S^+)$*, and this solution satisfies the estimate*

$$\| u \|_1 \leqslant c \| q \|_{-1,S^+} . \tag{3.14}$$

Proof. Since $H_{-1}(S^+)$ is the dual of $\overset{\circ}{H}_1(S^+)$ with respect to duality induced by $\langle \cdot, \cdot \rangle_{0,S^+}$, it follows that $\langle q, v \rangle_{0,S^+}$ is continuous linear functional on $\overset{\circ}{H}_1(S^+)$

for every $q \in H_{-1}(S^+)$. By Theorem 3, $b_+(u,v)$ is a continuous bilinear form on $\overset{\circ}{H}_1(S^+) \times \overset{\circ}{H}_1(S^+)$. By Theorem 4, $b_+(u,u)$ is coercive on $\overset{\circ}{H}_1(S^+)$. We now apply the Lax-Milgram lemma to complete the proof. ■

The variational formulation of (D^+) is as follows.

Find $u \in H_1(S^+)$ such that

$$b_+(u,v) = \langle q, v\rangle_{0,S^+} \, \forall v \in \overset{\circ}{H}_1(S^+) \tag{3.15}$$

and

$$\gamma^+ u = f. \tag{3.16}$$

Theorem 6 *Problem (3.15)-(3.16) has a unique solution $u \in H_1(S^+)$ for any $q \in H_{-1}(S^+)$ and any $f \in H_{\frac{1}{2}}(\partial S)$, and this solution satisfies the estimate*

$$\| u \|_{1,S^+} \leqslant c \left(\| q \|_{-1,S^+} + \| f \|_{\frac{1}{2},\partial S} \right). \tag{3.17}$$

Proof. The substitution $u = u_0 + l^+ f$ reduces (3.15)-(3.16) to a new variational problem, consisting in finding $u_0 \in \overset{\circ}{H}_1(S^+)$ such that

$$b_+(u_0,v) = \langle q, v\rangle_{0,S^+} - b_+(l^+ f, v) \quad \forall v \in \overset{\circ}{H}_1(S^+). \tag{3.18}$$

Clearly, $b_+(u,v)$ is continuous on $H_1(S^+) \times H_1(S^+)$, which implies that

$$\langle q, v\rangle_{0,S^+} - b_+(l^+ f, v)$$

is a continuous linear functional on $\overset{\circ}{H}_1(S^+)$. Also,

$$\begin{aligned} | \langle q, v\rangle_{0,S^+} - b_+(l^+ f, v) | &\leqslant \| q \|_{-1,S^+} \| v \|_1 + c \| l^+ f \|_{1,S^+} \| v \|_1 \qquad (3.19) \\ &\leqslant c \left(\| q \|_{-1,S^+} + \| l^+ f \|_{1,S^+} \right) \| v \|_1 . \end{aligned}$$

The statement now follows from Theorem 5, with (3.17) obtained from (3.19) and the continuity of l^+. ■

The (interior) Neumann problem is formulated as follows.

$$\text{Find } u \in C^2(S^+) \cap C^1(\overline{S}^+) \text{ satisfying (3.2) and } Tu = g \text{ on } \partial S, \qquad (\mathrm{N}^+)$$

where g is prescribed on ∂S.

In this case (3.3) leads to the following variational formulation.

Find $u \in H_1(S^+)$ such that

$$b_+(u,v) = \langle q, v\rangle_{0,S^+} + \langle g, \gamma^+ v\rangle_{0,\partial S} \quad \forall v \in H_1(S^+). \tag{3.20}$$

It is clear that, in view of the properties of rigid displacements,

$$\left\langle q, \mathbb{F}^{(i)}\right\rangle_{0,S^+} + \left\langle g, \gamma^+\mathbb{F}^{(i)}\right\rangle_{0,\partial S} = 0 \tag{3.21}$$

is a necessary solvability condition for (N^+). In what follows we assume (3.21) holds.

Theorem 7 *There is a $c = c(S^+) > 0$ such that for any $u \in H_1(S^+)$*

$$b_+(u,u) + \sum_{i=1}^{3} \left\langle u, \mathbb{F}^{(i)}\right\rangle_{0,S^+}^2 \geqslant c \parallel u \parallel_{1,S^+}^2, \tag{3.22}$$

$$b_+(u,u) + \sum_{i=1}^{3} \left\langle \gamma^+ u, \gamma^+\mathbb{F}^{(i)}\right\rangle_{0,\partial S}^2 \geqslant c \parallel u \parallel_{1,S^+}^2 . \tag{3.23}$$

Proof. If either (3.22) or (3.23) does not hold, then, by repeating the argument in the proof of Theorem 4, we find that there is a $u \in \mathcal{F}$ such that $\langle u, \mathbb{F}^{(i)}\rangle_{0,S^+} = 0$ in the case of (3.22) or $\langle \gamma^+ u, \gamma^+\mathbb{F}^{(i)}\rangle_{0,\partial S} = 0$ in the case of (3.23), while $\parallel u \parallel_{1,S^+} = 1$, which is an obvious contradiction. Inequalities (3.22) and (3.23) hold. ■

Theorem 8 *Problem (3.20) is solvable for any* $q \in \overset{\circ}{H}_{-1}(S^+)$ *and* $g \in H_{-\frac{1}{2}}(\partial S)$. *Any two solutions differ by a rigid displacement, and there is a solution* u_0 *that satisfies the estimate*

$$\| u_0 \|_{1,S^+} \leqslant c \left(\| q \|_{-1,S^+} + \| g \|_{-\frac{1}{2},\partial S} \right). \tag{3.24}$$

Proof. We introduce the factor space $\mathcal{H}_1(S^+) = H_1(S^+) \backslash \mathcal{F}$, the bilinear form

$$\mathcal{B}_+(U, V) = b_+(u, v) \text{ on } \mathcal{H}_1(S^+) \times \mathcal{H}_1(S^+),$$

and the linear functional

$$\mathcal{L}(V) = \langle q, v \rangle_{0,S^+} + \left\langle g, \gamma^+ v \right\rangle_{0,\partial S} \text{ on } \mathcal{H}_1(S^+),$$

where u and v are arbitrary representatives of the classes $U, V \in \mathcal{H}_1(S^+)$. We define the norm in $\mathcal{H}_1(S^+)$ by

$$\| U \|_{\mathcal{H}_1(S^+)} = \inf_{\substack{u \in H_1(S^+) \\ u \in U}} \| u \|_{1,S^+}.$$

Instead of (3.20) we now consider the new variational problem of finding $U \in \mathcal{H}_1(S^+)$ such that

$$\mathcal{B}_+(U, V) = \mathcal{L}(V) \quad \forall V \in \mathcal{H}_1(S^+). \tag{3.25}$$

In view of the definition of $\mathcal{B}_+(U, V)$, we see that for any U, $V \in \mathcal{H}_1(S^+)$ and any $u \in U$, $v \in V$

$$|\mathcal{B}_+(U, V)| = |b_+(u, v)| \leqslant c \| u \|_{1,S^+} \| v \|_{1,S^+},$$

therefore

$$|\mathcal{B}_+(U, V)| \leqslant c \inf_{\substack{u \in H_1(S^+) \\ u \in U}} \| u \|_{1,S^+} \inf_{\substack{v \in H_1(S^+) \\ v \in U}} \| v \|_{1,S^+} = c \| U \|_{\mathcal{H}_1(S^+)} \| V \|_{\mathcal{H}_1(S^+)},$$

which shows that $\mathcal{B}_+(U,V)$ is continuous on $\mathcal{H}_1(S^+)\times\mathcal{H}_1(S^+)$.

Next, we can choose $\widetilde{u}\in U$ such that $\left\langle \widetilde{u},\mathbb{F}^{(i)}\right\rangle_{0,S^+}=0$. Then, by Theorem 7,

$$\mathcal{B}_+(U,U)=b_+(\widetilde{u},\widetilde{u})\geqslant c\parallel \widetilde{u}\parallel^2_{1,S^+}\geqslant c\inf_{\substack{u\in H_1(S^+)\\ u\in U}}\parallel u\parallel_{1,S^+}=c\parallel U\parallel_{\mathcal{H}_1(S^+)},$$

so $\mathcal{B}_+(U,U)$ is coercive on $\mathcal{H}_1(S^+)$.

Finally, since γ^+ is continuous on $H_1(\partial S)$, for any $V\in\mathcal{H}_1(S^+)$

$$\begin{aligned}\mathcal{L}(V) &\leqslant \parallel q\parallel_{-1,S^+}\parallel v\parallel_{1,S^+}+\parallel g\parallel_{-\frac{1}{2},\partial S}\parallel \gamma^+v\parallel_{\frac{1}{2},\partial S}\\ &\leqslant c\left(\parallel q\parallel_{-1,S^+}+\parallel g\parallel_{-\frac{1}{2},\partial S}\right)\parallel v\parallel_{1,S^+},\end{aligned}$$

which shows that $\mathcal{L}$ is continuous linear functional on $\mathcal{H}_1(S^+)$.

By the Lax–Milgram lemma, problem (3.25) has a unique solution $U\in\mathcal{H}_1(S^+)$, and this solution satisfies the estimate

$$\parallel U\parallel_{\mathcal{H}_1(S^+)}\leqslant c\left(\parallel q\parallel_{-1,S^+}+\parallel g\parallel_{-\frac{1}{2},\partial S}\right).$$

Clearly, any $u\in U$ is a solution of (3.20), and $u_0\in U$ such that

$$\parallel u_0\parallel_{1,S^+}=\parallel U\parallel_{\mathcal{H}_1(S^+)}$$

satisfies (3.24). ■

3.3 Exterior boundary value problems

We consider Dirichlet and Neumann exterior boundary value problems.

The (exterior) Dirichlet problem is formulated as follows:

$$\text{Find } u\in C^2(S^-)\cap C^1(\overline{S}^-)\cap\mathcal{A}^* \text{ satisfying (3.2) such that } u\big|_{\partial S}=f, \quad (\mathrm{D}^-)$$

where f is prescribed on ∂S.

Let (D_0^-) be the exterior Dirichlet problem with $f = 0$. From (3.7) we see that a solution u of (D_0^-) satisfies

$$b_-(u, v) = \langle q, v \rangle_{0,S^-} \quad \forall v \in C_0^\infty(S^-). \tag{3.26}$$

Since $C_0^\infty(S^-)$ is dense in $\overset{\circ}{H}_{1,\omega}(S^-)$, it is clear that (3.26) holds for any $v \in \overset{\circ}{H}_{1,\omega}(S^-)$. Obviously, any $u \in C^2(S^-) \cap C^1(\overline{S}^-) \cap \mathcal{A}^*$ satisfying (3.26) for any $v \in \overset{\circ}{H}_{1,\omega}(S^-)$ and $u\,|_{\partial S} = 0$ is a classical (regular) solution of (D_0^-). Hence, the variational formulation of (D_0^-) is as follows.

Find $u \in \overset{\circ}{H}_{1,\omega}(S^-)$ such that

$$b_-(u, v) = \langle q, v \rangle_{0,S^-} \quad \forall v \in \overset{\circ}{H}_{1,\omega}(S^-). \tag{3.27}$$

Let $K_R^- = \left\{x \in \mathbb{R}^2 : |x| > R\right\}$, $R > 1$, and $\partial K_R = \left\{x \in \mathbb{R}^2 : |x| = R\right\}$.

Theorem 9 *There are $c_i(R) = const > 0$ such that*

$$\| u \|_{0,\omega;K_R^-}^2 \leqslant c_1 b_{K_R^-}(u, u) + c_2 \| \overline{u} \|_{1/2,\partial K_R}^2 + c_3 \| u_3 \|_{0,\partial K_R}^2 \quad \forall u \in H_{1,\omega}(K_R^-), \tag{3.28}$$

where $\| \cdot \|_{0,\partial K_R}$ and $\| \cdot \|_{1/2,\partial K_R}$ are the norms in $L^2(\partial K_R)$ and $H_{1/2}(\partial K_R)$, respectively.

Proof. Since $C_0^\infty(\overline{K}_R^-)$ is dense in $H_{1,\omega}(K_R^-)$, it suffices to consider $u \in C_0^\infty(\overline{K}_R^-)$. We write $u(x) = v(\rho, \varphi)$ in terms of polar coordinates with the pole in the centre of the circle ∂K_R.

We fix φ for the moment. We recall the generalized Hardy inequality [74]. If

$$\int_a^b |w(\rho)|^2 \lambda(\rho)\, d\rho < \infty, \quad \lambda(\rho) > 0, \quad \Lambda(\rho) = \int_a^\rho \frac{d\eta}{\lambda(\eta)}, \quad W(\rho) = \int_a^\rho w(\eta) d\eta,$$

then

$$\int_a^b \frac{|W(\rho)|^2}{\lambda(\rho)\Lambda^2(\rho)}\, d\rho \leqslant 4 \int_a^b |w(\rho)|^2 \lambda(\rho)\, d\rho.$$

We use the generalized Hardy inequality with $a = R$, $b = \infty$, $\lambda(\rho) = \rho$, and $w(\rho) = \partial_\rho v_3(\rho, \varphi)$, where $\partial_\rho(\dots) = \partial(\dots)/\partial\rho$, it follows that

$$\int_R^\infty \frac{|v_3(\rho,\varphi) - v_3(R,\varphi)|^2}{\rho \ln^2(\rho/R)}\, d\rho \leqslant 4 \int_R^\infty |\partial_\rho v_3(\rho,\varphi)|^2 \rho\, d\rho. \tag{3.29}$$

Thus, from (3.29) we find

$$\begin{aligned}
\int_R^\infty \frac{|v_3(\rho,\varphi)|^2}{(1+\rho)^4(1+\ln\rho)^2}\rho\, d\rho \quad &\leqslant \quad 2\int_R^\infty \frac{|v_3(\rho,\varphi) - v_3(R,\varphi)|^2}{(1+\rho)^4(1+\ln\rho)^2}\rho\, d\rho \\
&\quad +2\int_R^\infty \frac{|v_3(R,\varphi)|^2}{(1+\rho)^4(1+\ln\rho)^2}\rho\, d\rho \\
&\leqslant \quad 8\int_R^\infty |\partial_\rho v_3(\rho,\varphi)|^2 \rho\, d\rho \\
&\quad + \frac{2}{R^2 \ln R} |v_3(R,\varphi)|^2 .
\end{aligned}$$

Integration with respect to $\varphi \in (0, 2\pi)$ now yields

$$\begin{aligned}
\int_{K_R^-} \frac{|u_3|^2}{(1+|x|)^4(1+\ln|x|)^2}\, dx \quad &\leqslant \quad 8\int_{K_R^-} |\nabla u_3|^2\, dx \\
&\quad + \frac{2}{R^3 \ln R}\int_{\partial K_R} |u_3|^2\, ds.
\end{aligned} \tag{3.30}$$

We use the generalized Hardy inequality again with $a = R$, $b = \infty$, $\lambda(\rho) = \rho$, and $w(\rho) = \partial_\rho v_\alpha(\rho,\varphi)$, to obtain

$$\int_R^\infty \frac{|v_\alpha(\rho,\varphi) - v_\alpha(R,\varphi)|^2}{\rho \ln^2(\rho/R)}\, d\rho \leqslant 4 \int_R^\infty |\partial_\rho v_\alpha(\rho,\varphi)|^2 \rho\, d\rho,$$

from which

$$\int_R^\infty \frac{|v_\alpha(\rho,\varphi)-v_\alpha(R,\varphi)|^2}{(1+\rho)^2(1+\ln\rho)^2}\rho\ d\rho \leqslant 4\int_R^\infty |\partial_\rho v_\alpha(\rho,\varphi)|^2\,\rho\ d\rho.$$

Hence,

$$\begin{aligned}\int_R^\infty \frac{|v_\alpha(\rho,\varphi)|^2}{(1+\rho)^2(1+\ln\rho)^2}\rho\ d\rho \quad &\leqslant \quad 8\int_R^\infty |\partial_\rho v_\alpha(\rho,\varphi)|^2\,\rho\ d\rho \\ &\quad +2\,|v_\alpha(R,\varphi)|^2\int_R^\infty \frac{\rho}{(1+\rho)^2(1+\ln\rho)^2}\ d\rho \\ &\leqslant \quad 8\int_R^\infty |\partial_\rho v_\alpha(\rho,\varphi)|^2\,\rho\ d\rho + \frac{2}{\ln R}\,|v_\alpha(R,\varphi)|^2\,.\end{aligned}$$

Integrating this inequality with respect to $\varphi \in (0, 2\pi)$, we obtain

$$\int_{K_R^-} \frac{|\overline{u}_\alpha|^2}{(1+|x|)^2(1+\ln|x|)^2}\ dx \leqslant 8\int_{K_R^-} |\nabla u_\alpha|^2\ dx + \frac{2}{R\ln R}\int_{\partial K_R} |u_\alpha|^2\ ds \tag{3.31}$$

From (3.30) and (3.31), it follows that

$$\begin{aligned}\| \quad u \|^2_{0,\omega;K_R^-} \leqslant 8\sum_{\alpha=1}^{2}\int_{K_R^-} |\nabla u_\alpha|^2\ dx + \frac{2}{R\ln R}\sum_{\alpha=1}^{2}\int_{\partial K_R} |u_\alpha|^2\ ds \\ +8\int_{K_R^-} |\nabla u_3|^2\ dx + \frac{2}{R^3\ln R}\int_{\partial K_R} |u_3|^2\ ds.\end{aligned}$$

Next,

$$
\begin{aligned}
& \sum_{\alpha=1}^{2} \int_{K_R^-} |\nabla u_\alpha|^2 \, dx \\
= \; & \int_{K_R^-} \left(|u_{1,1}|^2 + |u_{1,2}|^2 + |u_{2,1}|^2 + |u_{2,2}|^2 \right) \, dx \\
= \; & \int_{K_R^-} \left(|u_{1,1}|^2 + |u_{2,2}|^2 + |u_{1,2} + u_{2,1}|^2 \right) \, dx - 2 \int_{K_R^-} u_{1,2} u_{2,1} \, dx \\
= \; & \int_{K_R^-} \left(|u_{1,1}|^2 + |u_{2,2}|^2 + |u_{1,2} + u_{2,1}|^2 \right) \, dx - 2 \int_{K_R^-} u_{1,1} u_{2,2} \, dx \\
& - \int_{K_R^-} \left[(u_2 u_{1,2} - u_1 u_{2,2})_{,1} + (u_1 u_{2,1} - u_2 u_{1,1})_{,2} \right] \, dx \\
\leqslant \; & 2 \int_{K_R^-} \left(|u_{1,1}|^2 + |u_{2,2}|^2 + |u_{1,2} + u_{2,1}|^2 \right) \, dx + 2 \int_{K_R^-} u_{1,1} u_{2,2} \, dx \\
& + \int_{\partial K_R} \left[n_1 (u_2 u_{1,2} - u_1 u_{2,2}) + n_2 (u_1 u_{2,1} - u_2 u_{1,1}) \right] \, ds \\
= \; & 2 \int_{K_R^-} \left(|u_{1,1}|^2 + |u_{2,2}|^2 + |u_{1,2} + u_{2,1}|^2 \right) \, dx + 2 \int_{K_R^-} u_{1,1} u_{2,2} \, dx \\
& + \int_{\partial K_R} (u_2 \partial_\tau u_1 - u_1 \partial_\tau u_2) \, ds,
\end{aligned}
$$

where $\tau = (-n_2, n_1)$ is the unit tangent to ∂K_R and $\partial_\tau(\ldots) = \partial(\ldots)/\partial\tau$. Consequently, taking into account the continuity of the operator $\partial_\tau : H_{1/2}(\partial K_R) \to$

$H_{-1/2}(\partial K_R)$, we obtain

$$
\begin{aligned}
\| \quad & u \|^2_{0,\omega;K_R^-} \leqslant 16 \int_{K_R^-} \left(|u_{1,1}|^2 + |u_{2,2}|^2 + |u_{1,2}+u_{2,1}|^2 \right) \, dx \\
& +16 \int_{K_R^-} u_{1,1} u_{2,2} \, dx + 8 \int_{\partial K_R} \left(u_2 \partial_\tau u_1 - u_1 \partial_\tau u_2 \right) ds \\
& + \frac{2}{R \ln R} \sum_{\alpha=1}^{2} \int_{\partial K_R} |u_\alpha|^2 \, ds + 8 \int_{K_R^-} |\nabla u_3|^2 \, dx + \frac{2}{R^3 \ln R} \int_{\partial K_R} |u_3|^2 \, ds \\
\leqslant \quad & \frac{16}{\lambda + 2\mu} (\lambda + 2\mu) \int_{K_R^-} \left(|u_{1,1}|^2 + |u_{2,2}|^2 \right) \, dx + \frac{16}{\mu} \mu \int_{K_R^-} |u_{1,2} + u_{2,1}|^2 \, dx \\
& + \frac{8}{\lambda} 2\lambda \int_{K_R^-} u_{1,1} u_{2,2} \, dx + \frac{8}{\gamma + \varepsilon} (\gamma + \varepsilon) \int_{K_R^-} |\nabla u_3|^2 \, dx \\
& + \alpha \int_{K_R^-} |u_{2,1} - u_{1,2} - 2u_3|^2 \, dx \\
& + 8a_1 (\|u_2\|_{1/2;\partial K_R} \|u_1\|_{1/2;\partial K_R} + \|u_1\|_{1/2;\partial K_R} \|u_2\|_{1/2;\partial K_R}) \\
& + \frac{2a_2^2}{R \ln R} \|\overline{u}\|^2_{1/2;\partial K_R} + \frac{2}{R^3 \ln R} \|u_3\|^2_{0;\partial K_R},
\end{aligned}
$$

where a_1 is the norm of ∂_τ in $H_{1/2;\partial K_R}$ and a_2 is the norm of (continuous) imbedding operator $\mathcal{I} : H_{1/2;\partial K_R} \to L^2(\partial K_R)$. Inequality (3.28) is obtained by setting

$$
c_1 = \max \left\{ \frac{16}{\lambda + 2\mu}, \frac{16}{\mu}, \frac{8}{\lambda}, \frac{8}{\gamma + \varepsilon}, 1 \right\}; \quad c_2 = 16a_1 + \frac{2a_2^2}{R \ln R}, \quad c_3 = \frac{2}{R^3 \ln R}.
$$

■

Theorem 10 *There is a* $c = c(S^-) = const > 0$ *such that any* $u \in H_{1,\omega}(S^-)$ *satisfies the estimates*

$$
\| u \|^2_{1,\omega;S^-} \leqslant c \left[b_-(u,u) + \left| \int_{\Gamma_0} u \, ds \right|^2 \right], \tag{3.32}
$$

$$
\| u \|^2_{1,\omega;S^-} \leqslant c \left[b_-(u,u) + \sum_{i=1}^{3} \left\langle u, \gamma^- \mathbb{F}^{(i)} \right\rangle^2_{0,\partial S} \right], \tag{3.33}
$$

where $\Gamma_0 \subseteq \partial S$, *measure of* Γ_0 *is larger than zero.*

Proof. We claim that for any $u \in H_{1,\omega;S^-}$

$$\| u \|_{0,\omega;S^-}^2 \leqslant c \left[b_-(u,u) + \left| \int_{\Gamma_0} u \ \mathrm{d}s \right|^2 \right], \tag{3.34}$$

$$\| u \|_{0,\omega;S^-}^2 \leqslant c \left[b_-(u,u) + \sum_{i=1}^{3} \left\langle u, \gamma^- \mathbb{F}^{(i)} \right\rangle_{0,\partial S}^2 \right], \tag{3.35}$$

First suppose that the opposite of formula (3.34) is true. Then we can construct a sequence $\{u^{(n)}\} \subset H_{1,\omega}(S^-)$ such that

$$b_-(u^{(n)}, u^{(n)}) \to 0, \qquad \int_{\Gamma_0} u^{(n)} ds \to 0 \tag{3.36}$$

while

$$\| u \|_{0,\omega;S^-}^2 = 1. \tag{3.37}$$

Let ∂K_R be a circle with the center at the origin and of radius $R > 1$ sufficiently large so that ∂S is contained inside ∂K_R. We write $S_R = S^- \cap K_R^-$. Since S_R is bounded, we may repeat the proof of Theorem 4 to deduce that there is a $c_R = const > 0$ such that

$$\| u \|_{1;S_R}^2 \leqslant c_R \left[b_{S_R}(u,u) + \left| \int_{\Gamma_0} u \ \mathrm{d}s \right|^2 \right] \quad \forall u \in H_1(S_R). \tag{3.38}$$

Then, by Theorem 9,

$$\begin{aligned}
\| \ u^{(n)} \|_{0,\omega;S^-}^2 &= \| u^{(n)} \|_{0,\omega;S_R}^2 + \| u^{(n)} \|_{0,\omega;K_R^-}^2 \\
&\leqslant \| u^{(n)} \|_{0,S_R}^2 + \| u^{(n)} \|_{0,\omega;K_R^-}^2 \\
&\leqslant c_R \left[b_{S_R}(u^{(n)}, u^{(n)}) + \left| \int_{\Gamma_0} u^{(n)} \ \mathrm{d}s \right|^2 \right] + c_1 b_{K_R^-}(u^{(n)}, u^{(n)}) \\
+c_2 \ \| \ \breve{u}^{(n)} \|_{1/2,\partial K_R}^2 &+ c_3 \| u_3^{(n)} \|_{0,\partial K_R}^2 .
\end{aligned}$$

From (3.38) for $u^{(n)}$ we now conclude that $u^{(n)} \to 0$ in $H_1(S_R)$. Then $u^{(n)} \to 0$ in $H_{1/2}(\partial K_R)$, hence in $L^2(\partial K_R)$. Consequently, from the last inequality we find that $\lim_{n\to\infty} \parallel u^{(n)} \parallel^2_{0,\omega;S^-} = 0$, which contradicts (3.37). Formula (3.35) is proved similarly. ■

Theorem 11 *The variational problem (3.26) has a unique solution* $u \in \overset{\circ}{H}_{1,\omega}(S^-)$ *for every* $q \in H_{-1,\omega}(S^-)$*, and this solution satisfies the estimate*

$$\parallel u \parallel_{1,\omega} \leqslant c \parallel q \parallel_{-1,\omega;S^-} .$$

Proof. By Theorem 10,

$$\parallel u \parallel^2_{1,\omega} \leqslant cb_-(u,u) \quad \forall u \in \overset{\circ}{H}_{1,\omega}(S^-),$$

which means that $b_-(u,u)$ is coercive on $\overset{\circ}{H}_{1,\omega}(S^-)$. Since $b_-(u,u)$ is clearly continuous on $\overset{\circ}{H}_{1,\omega}(S^-) \times \overset{\circ}{H}_{1,\omega}(S^-)$, we apply the Lax–Milgram lemma to complete the proof. ■

The variational formulation of (D$^-$) is as follows.

Find $u \in H_{1,\omega}(S^-)$ such that

$$b_-(u,v) = \langle q,v\rangle_{0,S^-} \; \forall v \in \overset{\circ}{H}_{1,\omega}(S^-) \tag{3.39}$$

and

$$\gamma^- u = f. \tag{3.40}$$

Theorem 12 *Problem (3.39)-(3.40) has a unique solution* $u \in H_{1,\omega}(S^-)$ *for any* $q \in H_{-1,\omega}(S^-)$ *and any* $f \in H_{\frac{1}{2}}(\partial S)$*, and this solution satisfies the estimate*

$$\parallel u \parallel_{1,\omega;S^-} \leqslant c \left(\parallel q \parallel_{-1,\omega;S^-} + \parallel f \parallel_{\frac{1}{2},\partial S} \right).$$

Proof. The substitution $u = u_0 + l^- f$ reduces (3.39)-(3.40) to a new variational problem, consisting in finding $u_0 \in \overset{\circ}{H}_{1,\omega}(S^-)$ such that

$$b_-(u_0,v) = \langle q,v\rangle_{0,S^-} - b_-(l^- f, v) \quad \forall v \in \overset{\circ}{H}_{1,\omega}(S^-). \tag{3.41}$$

Since for any $v \in \overset{\circ}{H}_{1,\omega}(S^-)$

$$\begin{aligned} | \langle q,v\rangle_{0,S^-} - b_-(l^- f,v) \, | &\leqslant \; \| \, q \, \|_{-1,\omega;S^-} \| \, v \, \|_{1,\omega} + \left[b_-(l^- f, l^- f)\right]^{1/2} \left[b_-(v,v)\right]^{1/2} \\ &\leqslant \; \left(\| \, q \, \|_{-1,\omega;S^-} + \| \, l^- f \, \|_{1,\omega;S^-}\right) \| \, v \, \|_{1,\omega} \\ &\leqslant \; c\left(\| \, q \, \|_{-1,\omega;S^-} + \| \, f \, \|_{1/2,\partial S}\right) \| \, v \, \|_{1,\omega}, \end{aligned}$$

the linear form $\langle q,v\rangle_{0,S^-} - b_-(l^- f,v)$ is a continuous linear functional on $\overset{\circ}{H}_{1,\omega}(S^-)$. The statement of the theorem now follows from the Lax–Milgram lemma applied to the auxiliary problem (3.41) and the estimates

$$\begin{aligned} \| \quad & u_0 \, \|_{1,\omega} \leqslant c\left(\| \, q \, \|_{-1,\omega;S^-} + \| \, f \, \|_{1/2,\partial S}\right) \\ \| \quad & u \, \|_{1,\omega;S^-} \leqslant \| \, u_0 \, \|_{-1,\omega;S^-} + \| \, l^- f \, \|_{1,\omega;S^-} \leqslant c\left(\| \, q \, \|_{-1,\omega;S^-} + \| \, f \, \|_{\frac{1}{2},\partial S}\right). \end{aligned}$$

■

The (exterior) Neumann problem is formulated as follows.

Find $u \in C^2(S^-) \cap C^1(\overline{S}^-) \cap \mathcal{A}$ satisfying (3.2) and $Tu = g$ on ∂S, $\quad$ (N$^-$)

where g is prescribed on ∂S.

In this case (3.7) leads to the following variational formulation.

Find $u \in H_{1,\omega}(S^-)$ such that

$$b_-(u,v) = \langle q,v\rangle_{0,S^-} - \left\langle g, \gamma^- v\right\rangle_{0,\partial S} \quad \forall v \in H_{1,\omega}(S^-). \tag{3.42}$$

In view of the properties of rigid displacements,

$$\left\langle q, \mathbb{F}^{(i)}\right\rangle_{0,S^-} - \left\langle g, \gamma^- \mathbb{F}^{(i)}\right\rangle_{0,\partial S} = 0 \tag{3.43}$$

is a necessary solvability condition for (3.42). In what follows we assume (3.43) holds.

Theorem 13 *Problem (3.42) is solvable for any* $q \in \overset{\circ}{H}_{-1,\omega}(S^-)$ *and* $g \in H_{-\frac{1}{2}}(\partial S)$. *Any two solutions differ by a rigid displacement, and there is a solution* u_0 *that satisfies the estimate*

$$\| u_0 \|_{1,\omega;S^-} \leqslant c \left(\| q \|_{-1,\omega} + \| g \|_{-\frac{1}{2},\partial S} \right). \tag{3.44}$$

Proof. We introduce the factor space $\mathcal{H}_{1,\omega}(S^-) = H_{1,\omega}(S^-) \backslash \mathcal{F}$, the bilinear form

$$\mathcal{B}_-(U,V) = b_-(u,v) \text{ on } \mathcal{H}_{1,\omega}(S^-) \times \mathcal{H}_{1,\omega}(S^-),$$

and the linear functional

$$\mathcal{L}(V) = \langle q, v \rangle_{0,S^-} - \left\langle g, \gamma^- v \right\rangle_{0,\partial S} \text{ on } \mathcal{H}_{1,\omega}(S^-),$$

where u and v are arbitrary representatives of the classes $U, V \in \mathcal{H}_{1,\omega}(S^-)$. We define the norm in $\mathcal{H}_{1,\omega}(S^-)$ by

$$\| U \|_{\mathcal{H}_{1,\omega}(S^-)} = \inf_{\substack{u \in H_{1,\omega}(S^-) \\ u \in U}} \| u \|_{1,\omega;S^-} .$$

Instead of (3.42) we now consider the new variational problem of finding $U \in \mathcal{H}_{1,\omega}(S^-)$ such that

$$\mathcal{B}_-(U,V) = \mathcal{L}(V) \quad \forall V \in \mathcal{H}_{1,\omega}(S^-). \tag{3.45}$$

In view of the definition of $\mathcal{B}_-(U,V)$, we see that for any U, $V \in \mathcal{H}_{1,\omega}(S^-)$ and any $u \in U$, $v \in V$

$$|\mathcal{B}_-(U,V)| = |b_-(u,v)| \leqslant c \| u \|_{1,\omega;S^-} \| v \|_{1,\omega;S^-},$$

therefore

$$\begin{aligned} |\mathcal{B}_-(U,V)| &\leqslant c \inf_{\substack{u\in H_{1,\omega}(S^-)\\ u\in U}} \| u \|_{1,\omega;S^-} \inf_{\substack{v\in H_{1,\omega}(S^-)\\ v\in U}} \| v \|_{1,\omega;S^-} \\ &= c \| U \|_{\mathcal{H}_{1,\omega}(S^-)} \| V \|_{\mathcal{H}_{1,\omega}(S^-)}, \end{aligned}$$

which shows that $\mathcal{B}_-(U,V)$ is continuous on $\mathcal{H}_{1,\omega}(S^-) \times \mathcal{H}_{1,\omega}(S^-)$.

Next, we can choose $\widetilde{u} \in U$ such that $\left\langle \gamma^- \widetilde{u}, \gamma^- \mathbb{F}^{(i)} \right\rangle_{0,\partial S} = 0$. Then, by (3.33),

$$\mathcal{B}_-(U,U) = b_-(\widetilde{u},\widetilde{u}) \geqslant c \| \widetilde{u} \|^2_{1,\omega;S^-} \geqslant c \inf_{\substack{u\in H_{1,\omega}(S^-)\\ u\in U}} \| u \|^2_{1,\omega;S^-} = k \| U \|^2_{\mathcal{H}_{1,\omega}(S^-)},$$

so $\mathcal{B}_-(U,U)$ is coercive on $\mathcal{H}_{1,\omega}(S^-)$.

Finally, since γ^- is continuous on $H_{1,\omega}(S^-)$, for any $V \in \mathcal{H}_{1,\omega}(S^-)$

$$\begin{aligned} \mathcal{L}(V) &\leqslant \| q \|_{-1,\omega} \| v \|_{1,\omega;S^-} + \| g \|_{-\frac{1}{2},\partial S} \| \gamma^- v \|_{\frac{1}{2},\partial S} \\ &\leqslant c \left(\| q \|_{-1,\omega} + \| g \|_{-\frac{1}{2},\partial S} \right) \| v \|_{1,\omega;S^-}, \end{aligned}$$

which shows that $\mathcal{L}$ is continuous linear functional on $\mathcal{H}_{1,\omega}(S^-)$.

By the Lax–Milgram lemma, problem (3.45) has a unique solution $U \in \mathcal{H}_{1,\omega}(S^-)$, and this solution satisfies the estimate

$$\| U \|_{\mathcal{H}_{1,\omega}(S^-)} \leqslant c \left(\| q \|_{-1,\omega} + \| g \|_{-\frac{1}{2},\partial S} \right).$$

Clearly, any $u \in U$ is a solution of (3.42), and $u_0 \in U$ such that

$$\| u_0 \|_{1,\omega;S^-} = \| U \|_{\mathcal{H}_{1,\omega}(S^-)}$$

satisfies (3.44). ■

3.4 Summary

In this chapter we have formulated interior and exterior Dirichlet and Neumann problems of plane Cosserat elasticity in Sobolev spaces and established existence, uniqueness and continuous dependence on the data results for these problems. This is a necessary step to deal with such problems from the practical point of view, since it validates the subsequent application of numerical procedures such as finite element method.

Chapter 4

Boundary Integral Equations

In this chapter we will show that the variational problems formulated in Chapter 3 can be solved using the boundary integral equation method and corresponding weak solutions can be represented in the form of specially constructed integral potentials with unknown distributional densities. First, we introduce boundary operators which associate with the displacement field u on ∂S the corresponding boundary moments and forces known as Poincaré-Steklov operators, and boundary operators corresponding to the integral potentials. Then we employ them to reduce boundary value problems to boundary integral equations with respect to the integral densities and show a unique solvability of boundary integral equations.

4.1 Poincaré–Steklov operators

Let $f \in H_{1/2}(\partial S)$, and let $u \in H_1(S^+)$ be the (unique) solution of the variational problem (D$^+$) (3.15)-(3.16) with $q = 0$

$$b_+(u, v) = 0 \;\; \forall v \in \overset{\circ}{H}_1(S^+), \quad \gamma^+ u = f.$$

We consider an arbitrary $\alpha \in H_{1/2}(\partial S)$ and write $w = l^+\alpha$. Using Riesz representation theorem [72], we can define an operator $\mathcal{T}^+$ on $H_{1/2}(\partial S)$ by

$$\left\langle \mathcal{T}^+ f, \alpha \right\rangle_{0;\partial S} = b_+(u, w). \tag{4.1}$$

The definition is consistent, for if $\breve{w} \in H_1(S^+)$ is another extension of α, then $w - \breve{w} \in \overset{\circ}{H}_1(S^+)$ and $b_+(w - \breve{w}, u) = 0$, $\forall \alpha \in H_{1/2}(\partial S)$.

Let $f \in H_{1/2}(\partial S)$, and let $u \in H_{1,\omega}(S^-)$ be the (unique) solution of the variational problem (D$^-$) (3.36)-(3.37) with $q = 0$

$$b_-(u, v) = 0 \;\; \forall v \in \overset{\circ}{H}_{1,\omega}(S^-), \quad \gamma^- u = f.$$

We consider an arbitrary $\alpha \in H_{1/2}(\partial S)$ and write $w = l^-\alpha$. Using Riesz representation theorem [72], we can define an operator $\mathcal{T}^-$ on $H_{1/2}(\partial S)$ by

$$\left\langle \mathcal{T}^- f, \alpha \right\rangle_{0;\partial S} = -b_-(u, w). \tag{4.2}$$

The definition is consistent, for if $\breve{w} \in H_{1,\omega}(S^-)$ is another extension of α, then $w - \breve{w} \in \overset{\circ}{H}_{1,\omega}(S^-)$ and $b_-(w - \breve{w}, u) = 0$, $\forall \alpha \in H_{1/2}(\partial S)$.

$\mathcal{T}^\pm$ are known as the Poincaré-Steklov operators corresponding to (3.2).

Denoting the space of the rigid displacements on ∂S by $\mathcal{F}(\partial S)$, let $\mathcal{H}_{1/2}(\partial S)$ be the subspace of $H_{1/2}(\partial S)$ of all u such that

$$\langle u, z \rangle_{0;\partial S} = 0 \;\; \forall z \in \mathcal{F}(\partial S),$$

and let $\mathcal{H}_{-1/2}(\partial S)$ be the subspace of $H_{-1/2}(\partial S)$ of all g such that

$$\langle g, z \rangle_{0;\partial S} = 0 \quad \forall z \in \mathcal{F}(\partial S).$$

Theorem 14 *(i) $\mathcal{T}^{\pm}$ are continuous operators from $H_{1/2}(\partial S)$ to $H_{-1/2}(\partial S)$.*

(ii) $\mathcal{T}^{\pm}$ are self-adjoint in the sense that

$$\left\langle \mathcal{T}^{\pm} f, v \right\rangle_{0;\partial S} = \left\langle f, \mathcal{T}^{\pm} v \right\rangle_{0;\partial S} \quad \forall f, v \in H_{1/2}(\partial S). \tag{4.3}$$

(iii) The kernels of $\mathcal{T}^{\pm}$ coincide with $\mathcal{F}(\partial S)$.

(iv) The ranges of $\mathcal{T}^{\pm}$ coincide with $\mathcal{H}_{-1/2}(\partial S)$.

Proof. (i) By the definition of $\mathcal{T}^{\pm}$, for $f, v \in H_{1/2}(\partial S)$

$$\left\langle \mathcal{T}^{\pm} f, v \right\rangle_{0;\partial S}^{2} = b_{\pm}(u, l^{\pm} v)^2 \leq b_{\pm}(u, u) b_{\pm}(l^{\pm} v, l^{\pm} v).$$

Since

$$b_{+}(l^{+} v, l^{+} v) \leq c \parallel l^{+} v \parallel_{1,S^{+}}^{2} \leq c \parallel v \parallel_{1/2;\partial S}^{2},$$

and

$$b_{-}(l^{-} v, l^{-} v) \leq c \parallel l^{-} v \parallel_{1,\omega;S^{-}}^{2} \leq c \parallel v \parallel_{1/2;\partial S}^{2},$$

it follows that

$$\left\langle \mathcal{T}^{\pm} f, v \right\rangle_{0;\partial S}^{2} \leq c b_{\pm}(u, u) \parallel v \parallel_{1/2;\partial S}^{2};$$

therefore, $\mathcal{T}^{\pm} f \in H_{-1/2}(\partial S)$ and

$$\begin{aligned} \parallel & \mathcal{T}^{\pm} f \parallel_{-1/2;\partial S}^{2} \leq c b_{\pm}(u, u) = \pm c \left\langle \mathcal{T}^{\pm} f, f \right\rangle_{0;\partial S} \\ \leq & \; c \parallel \mathcal{T}^{\pm} f \parallel_{-1/2;\partial S} \parallel f \parallel_{1/2;\partial S}, \end{aligned}$$

from which

$$\| \mathcal{T}^{\pm} f \|_{-1/2;\partial S} \leq c \| f \|_{1/2;\partial S} .$$

(ii) We take $l^{\pm}$ to be the operators that associate with $v \in H_{1/2}(\partial S)$ the solutions of the corresponding problems (D$^{\pm}$), and (4.3) follows from (4.1) and (4.2).

(iii) If $z \in \mathcal{F}(\partial S)$, then z is the solution of (D$^{\pm}$) and we have

$$\left\langle \mathcal{T}^{\pm} z, v \right\rangle_{0;\partial S} = \pm b_{\pm}(z, l^{\pm} v) = 0 \quad \forall z \in \mathcal{F}(\partial S);$$

hence,

$$\mathcal{T}^{\pm} z = 0.$$

Conversely, if $\mathcal{T}^{\pm} f = 0$, then

$$\left\langle \mathcal{T}^{+} f, f \right\rangle_{0;\partial S} = b_{+}(u, u) = 0;$$

therefore, $u \in \mathcal{F}$ and $f \in \mathcal{F}(\partial S)$.

(iv) By (4.2), for any $f \in H_{1/2}(\partial S)$ and $z \in \mathcal{F}(\partial S)$

$$\left\langle \mathcal{T}^{\pm} f, z \right\rangle_{0;\partial S} = \pm b_{\pm}(u, z) = 0,$$

so $\mathcal{T}^{\pm} f \in \mathcal{H}_{-1/2}(\partial S)$ for any $f \in H_{1/2}(\partial S)$. We define operators $\overset{\curlywedge}{\mathcal{T}}{}^{\pm}$ on the factor space $H_{1/2}(\partial S) \backslash \mathcal{F}(\partial S)$ by

$$\overset{\curlywedge}{\mathcal{T}}{}^{\pm} f_0 = \mathcal{T}^{\pm} f, \quad f_0 \in H_{1/2}(\partial S) \backslash \mathcal{F}(\partial S),$$

where f is any representative of the class f_0. Clearly, $\overset{\curlywedge}{\mathcal{T}}{}^{\pm}$ are injective and their ranges coincide with the ranges of $\mathcal{T}^{\pm}$, correspondingly. We now show that inverse operators $\left(\overset{\curlywedge}{\mathcal{T}}^{\pm}\right)^{-1}$ are continuous. Let $f_0 \in H_{1/2}(\partial S)\backslash\mathcal{F}(\partial S)$, let f be representative of f_0 such that

$$\langle f, z\rangle_{0;\partial S} = 0 \quad \forall z \in \mathcal{F}(\partial S),$$

and let u be the solution of (D^+) with boundary data f and $q = 0$. By Theorem 7, we have

$$\begin{aligned} \| \quad & f_0 \|^2_{H_{1/2}(\partial S)\backslash\mathcal{F}(\partial S)} \leq \| f \|^2_{1/2;\partial S} \leq c \| u \|^2_{1,S^+} \\ \leq \quad & cb_+(u,u) = c\left\langle \mathcal{T}^+ f, f\right\rangle_{0;\partial S} \leq c \| \mathcal{T}^+ f \|_{-1/2;\partial S} \| f \|_{1/2;\partial S} . \end{aligned}$$

Now let u be the solution of (D^-) with boundary data f and $q = 0$. By Theorem 10, we have

$$\begin{aligned} \| \quad & f_0 \|^2_{H_{1/2}(\partial S)\backslash\mathcal{F}(\partial S)} \leq \| f \|^2_{1/2;\partial S} \leq c \| u \|^2_{1,S^+} \\ \leq \quad & cb_-(u,u) = -c\left\langle \mathcal{T}^- f, f\right\rangle_{0;\partial S} \leq c \| \mathcal{T}^- f \|_{-1/2;\partial S} \| f \|_{1/2;\partial S} . \end{aligned}$$

From which it follows that

$$\| f_0 \|_{H_{1/2}(\partial S)\backslash\mathcal{F}(\partial S)} \leq \| f \|_{1/2;\partial S} \leq c \| \mathcal{T}^{\pm} f \|_{-1/2;\partial S} = c \| \overset{\curlywedge}{\mathcal{T}}^{\pm} f \|_{-1/2;\partial S} .$$

To prove that the ranges of $\mathcal{T}^{\pm}$ coincide with $\mathcal{H}_{-1/2}(\partial S)$, it suffices to establish that these ranges are dense in $\mathcal{H}_{-1/2}(\partial S)$. Suppose that this is not true. Then we can find non-zero Φ in the dual $H_{1/2}(\partial S)\backslash\mathcal{F}(\partial S)$ of $\mathcal{H}_{-1/2}(\partial S)$ such that

$$\left\langle \mathcal{T}^{\pm} f, \varphi\right\rangle_{0;\partial S} = 0 \quad \forall f \in H_{1/2}(\partial S),$$

where φ is any representative of Φ. Taking $f = \varphi$, we arrive at

$$\left\langle \mathcal{T}^{\pm} \varphi, \varphi\right\rangle_{0;\partial S} = 0;$$

therefore, $\varphi \in \mathcal{F}(\partial S)$ and $\Phi = 0$. This contradiction completes the proof. ■

In the proof of assertion (iv) the following result has been established.

Corollary 15 $\overset{\curlywedge}{\mathcal{T}}{}^{\pm}$ *are homeomorphisms from* $H_{1/2}(\partial S)\backslash\mathcal{F}(\partial S)$ *to* $\mathcal{H}_{-1/2}(\partial S)$.

Let $\mathcal{N}^{\pm}$ be the restriction of $\mathcal{T}^{\pm}$ to $\mathcal{H}_{1/2}(\partial S)$.

Theorem 16 *The operators* $\mathcal{N}^{\pm}$ *and* $\mathcal{N}^{+} - \mathcal{N}^{-}$ *are homeomorphisms from* $\mathcal{H}_{1/2}(\partial S)$ *to* $\mathcal{H}_{-1/2}(\partial S)$.

Proof. The bijectivity and continuity of $\mathcal{N}^{\pm}$ were established in the previous theorem, while the continuity of the inverse operators $\left(\mathcal{N}^{\pm}\right)^{-1}$ follows from Banach theorem [72]. The statement for $\mathcal{N}^{+} - \mathcal{N}^{-}$ is proved similarly. ■

4.2 Integral potentials

Theorem 17 $V\varphi \in H_{1,\omega}(S^{-})$ *if and only if* $\varphi \in \mathcal{H}_{-1/2}(\partial S)$.

Proof. The asymptotic behavior of $V\varphi(x)$ as $|x| \to \infty$ is given by the formula $V\varphi = M^{\infty}\Phi + \sigma^{\mathcal{A}}$, where $M^{\infty} \in \mathcal{M}_{3\times 3}$, $M^{\infty} = O(\ln|x|)$, $\Phi = \left(\left\langle F^{(1)}, \varphi\right\rangle_{0;\partial S}, \left\langle F^{(2)}, \varphi\right\rangle_{0;\partial S}, \left\langle F^{(3)}, \varphi\right\rangle_{0;\partial S}\right)^{T}$ (see [44]), and $\sigma^{\mathcal{A}} \in H_{1,\omega}(S^{-})$. In view of the definition of the norm in $H_{1,\omega}(S^{-})$, it is clear that $V\varphi \in H_{1,\omega}(S^{-})$ if and only if $\Phi = 0$, which, in turn, is equivalent to $\varphi \in \mathcal{H}_{-1/2}(\partial S)$. ■

Let $\left\{\tilde{z}{}^{(i)}\right\}$ be the basis for $\mathcal{F}$ obtained from $\left\{\mathbb{F}^{(i)}\right\}$ by orthonormalization in $L^{2}(\partial S)$. We define a modified single layer potential of density $\varphi \in C^{\infty}(\partial S)$ by

$$(\mathcal{V}\varphi)(x) = (V\varphi)(x) - \left\langle V_0\varphi, \tilde{z}{}^{(i)}\right\rangle_{0;\partial S} \tilde{z}{}^{(i)}(x), \quad x \in \mathbb{R}^2,$$

and its boundary operator $\mathcal{V}_0$ by $\mathcal{V}_0\varphi = (\mathcal{V}\varphi)|_{\partial S}$.

Theorem 18 *The operator $\mathcal{V}_0$, extended by continuity from $C^\infty(\partial S)$ to $\mathcal{H}_{-1/2}(\partial S)$, is a homeomorphism from $\mathcal{H}_{-1/2}(\partial S)$ to $\mathcal{H}_{1/2}(\partial S)$.*

Proof. If $\varphi \in C^\infty(\partial S) \cap \mathcal{H}_{-1/2}(\partial S)$, then $V\varphi$ belongs to both $H_1(S^+)$ and $H_{1,\omega}(S^-)$; consequently, so does $\mathcal{V}\varphi$. The jump formulae (Theorem 1) yield $\mathcal{N}^+\mathcal{V}_0\varphi - \mathcal{N}^-\mathcal{V}_0\varphi = (\mathcal{N}^+ - \mathcal{N}^-)\mathcal{V}_0\varphi = \varphi$, which means that $\mathcal{V}_0$ is injective and

$$\mathcal{V}_0^{-1} = \mathcal{N}^+ - \mathcal{N}^-. \tag{4.4}$$

Given that $\gamma^-\mathcal{V}\varphi = \mathcal{V}_0\varphi \in \mathcal{H}_{1/2}(\partial S)$, we see that

$$\| \mathcal{V}_0\varphi \|^2_{1/2;\partial S} \leqslant \| \mathcal{V}\varphi \|^2_{1,\omega;S^-} \leqslant cb_-(\mathcal{V}\varphi, \mathcal{V}\varphi) \leqslant c\,[b_+(\mathcal{V}\varphi, \mathcal{V}\varphi) + b_-(\mathcal{V}\varphi, \mathcal{V}\varphi)]\,.$$

Since $b_+(\mathcal{V}\varphi, \mathcal{V}\varphi) + b_-(\mathcal{V}\varphi, \mathcal{V}\varphi) = \langle(\mathcal{N}^+ - \mathcal{N}^-)\,\mathcal{V}_0\varphi, \mathcal{V}_0\varphi\rangle_{0;\partial S} = \langle\varphi, \mathcal{V}_0\varphi\rangle_{0;\partial S}$, we conclude that $\| \mathcal{V}_0\varphi \|^2_{1/2;\partial S} \leqslant c\,\langle\varphi, \mathcal{V}_0\varphi\rangle_{0;\partial S} \leqslant c\, \| \mathcal{V}_0\varphi \|_{1/2;\partial S} \| \varphi \|_{-1/2;\partial S}$, or

$$\| \mathcal{V}_0\varphi \|_{1/2;\partial S} \leqslant c\, \| \varphi \|_{-1/2;\partial S}\ . \tag{4.5}$$

Hence, $\mathcal{V}_0$ can be extended by continuity to $\mathcal{H}_{-1/2}(\partial S)$ with estimate (4.5) remains valid on the latter space. The continuity of $\mathcal{V}_0^{-1}$ follows from (4.4) and Theorem 16.

To complete the proof, it suffices to show that the range of $\mathcal{V}_0$ coincides with $\mathcal{H}_{1/2}(\partial S)$. Assuming the opposite, we can find $f \in \mathcal{H}_{1/2}(\partial S)$ that does not belong to the range of $\mathcal{V}_0$ and set $\varphi = (\mathcal{N}^+ - \mathcal{N}^-)f$, $\mathcal{V}_0\varphi = g$. Then from (4.4) it follows that $\varphi = (\mathcal{N}^+ - \mathcal{N}^-)g$, which contradicts Theorem 16. ■

We now turn to the properties of the double layer potential.

Since $L(\partial_y)D^{(i)}(y,x) = 0$ for $x \in S^+ \cup S^-$, $x \neq y$, we may write the double layer potential as

$$(W\psi)(x) = \begin{cases} \left\langle \mathcal{T}^+ D^{(i)}(\cdot, x), \psi(\cdot) \right\rangle_{0;\partial S} e_i, & x \in S^-, \\ \left\langle \mathcal{T}^- D^{(i)}(\cdot, x), \psi(\cdot) \right\rangle_{0;\partial S} e_i, & x \in S^+, \end{cases}$$

where e_i are the unit coordinate vectors.

We fix a point $x \in S^-$ and consider y to be the argument of the functions below. Let $u \in H_1(S^+)$ be the (unique) solution of variational problem

$$b_+(u, v) = 0, \quad \forall v \in \mathring{H}_1(S^+), \quad \gamma^+ u = \psi.$$

Then $b_+(u, D^{(i)}) = \left\langle \mathcal{T}^+\psi, D^{(i)} \right\rangle_{0;\partial S} = \left\langle \psi, \mathcal{T}^+ D^{(i)} \right\rangle_{0;\partial S}$, which leads to $V\mathcal{T}^+\psi = W\psi$ in S^-. From this we conclude that for smooth functions ψ on ∂S

$$W^- = V_0 \mathcal{T}^+, \tag{4.6}$$

where $W^\pm$ are the operators of the limiting values of the double layer potential on ∂S from within $S^\pm$ and V_0 is the boundary operator defined by $V_0\varphi = (V\varphi)|_{\partial S}$.

We now define a modified double layer potential by

$$(\mathcal{W}\psi)(x) = \begin{cases} (W\psi)(x) - \left\langle W^+\psi, \tilde{z}^{(i)} \right\rangle_{0;\partial S} \tilde{z}^{(i)}(x), & x \in S^+, \\ (W\psi)(x) - \left\langle W^-\psi, \tilde{z}^{(i)} \right\rangle_{0;\partial S} \tilde{z}^{(i)}(x), & x \in S^-, \end{cases}$$

and the operators $\mathcal{W}^\pm$ of its limiting values on ∂S by

$$\mathcal{W}^\pm \psi = W^\pm \psi - \left\langle W^\pm \psi, \tilde{z}^{(i)} \right\rangle_{0;\partial S} \tilde{z}^{(i)}.$$

Theorem 19 *$\mathcal{W}^\pm$ can be extended by continuity to $\mathcal{H}_{1/2}(\partial S)$ by setting*

$$\mathcal{W}^\pm \psi = \mathcal{V}_0 \mathcal{N}^\mp \psi, \quad \psi \in \mathcal{H}_{1/2}(\partial S),$$

and these extensions are homeomorphisms from $\mathcal{H}_{1/2}(\partial S)$ to $\mathcal{H}_{1/2}(\partial S)$.

Proof. The equality $\mathcal{W}^-\psi = \mathcal{V}_0\mathcal{N}^+\psi$ is proved for smooth densities ψ by means of (4.6). The full statement for $\mathcal{W}^-$ follows from the properties of $\mathcal{N}^+$ and $\mathcal{V}_0$ established in Theorems 16 and 18.

Using (4.4), we find that

$$\begin{aligned}\mathcal{W}^+\psi &= W^+\psi - \left\langle W^+\psi, \tilde{z}^{(i)}\right\rangle_{0;\partial S}\tilde{z}^{(i)} = -\psi + W^-\psi - \left\langle W^-\psi, \tilde{z}^{(i)}\right\rangle_{0;\partial S}\tilde{z}^{(i)}\\ &= -\psi + V_0\mathcal{T}^+\psi - \left\langle V_0\mathcal{T}^+\psi, \tilde{z}^{(i)}\right\rangle_{0;\partial S}\tilde{z}^{(i)} = -\psi + \mathcal{V}_0\mathcal{T}^+\psi = \mathcal{V}_0\mathcal{T}^-\psi,\end{aligned}$$

which completes the proof. ■

4.3 Solvability of boundary integral equations

We now show that ($\mathrm{D}^{\pm}$) can be reduced to similar problems for the homogeneous equilibrium equation by means of the area potential.

Let $\mathcal{H}_{-1,\omega}(\mathbb{R}^2)$ be the subspace of $H_{-1,\omega}(\mathbb{R}^2)$ consisting of all q such that $\langle q, z\rangle_0 = 0$ for all $z \in \mathcal{F}$. We can describe $\mathcal{H}_{-1,\omega}(\mathbb{R}^2)$ explicitly. If we define

$$\begin{aligned}Def\,\overline{u} &= (\partial_1 u_1, \partial_2 u_2, \partial_2 u_1 + \partial_1 u_2)^T, \quad Div\,u = (\partial_1 u_1 + \partial_2 u_3, \partial_2 u_2 + \partial_1 u_3)^T,\\ Grad\,\sigma &= (\partial_2\sigma, -\partial_1\sigma)^T, \text{ where } \sigma \in \mathcal{M}_{1\times 1},\\ R &= \begin{pmatrix}\lambda + 2\mu & \lambda & 0\\ \lambda & \lambda + 2\mu & 0\\ 0 & 0 & \mu\end{pmatrix},\end{aligned}$$

then we can write

$$Lu = \begin{pmatrix} Div\,(R\,Def\,\overline{u}) + \alpha\,Grad(2u_3 - (curl\,u)_3) \\ \\ (\gamma+\varepsilon)div\nabla u_3 - 2\alpha(2u_3 - (curl\,u)_3) \end{pmatrix} \tag{4.7}$$

and

$$b(u,v) = \langle R\,Def\,\overline{u}, Def\,\overline{v}\rangle_0 + \alpha\langle 2u_3 - (curl\,u)_3, 2v_3 - (curl\,v)_3\rangle_0 + (\gamma+\varepsilon)\langle\nabla u_3, \nabla v_3\rangle_0.$$

Theorem 20 *$\mathcal{H}_{-1,\omega}(\mathbb{R}^2)$ consists of all $q = (\overline{q}^T, q_3)^T$, where $\overline{q} = (q_1, q_2)^T$, of the form*

$$\overline{q} = Div\,P + Grad\,Q, \quad q_3 = div\,V - 2Q, \tag{4.8}$$

where $P \in L^2(\mathbb{R}^2) \cap \mathcal{M}_{3\times1}$, $Q \in L^2(\mathbb{R}^2) \cap \mathcal{M}_{1\times1}$, $V \in L^2(\mathbb{R}^2) \cap \mathcal{M}_{1\times1}$. Also there are constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \parallel q \parallel_{-1,\omega} \leqslant \parallel P \parallel_0 + \parallel Q \parallel_0 + \parallel V \parallel_0 \leqslant c_2 \parallel q \parallel_{-1,\omega} .$$

Proof. Let $q \in \mathcal{H}_{-1,\omega}(\mathbb{R}^2)$, and let $u_0 \in H_{1,\omega}(\mathbb{R}^2)$ be the solution of equilibrium problem in $\mathbb{R}^2$ which satisfies $\|u_0\|_{1,\omega} \leqslant c\,\|q\|_{-1,\omega}$. We set

$$\begin{aligned} -R\;Def\;\overline{u_0} &= P \in L^2(\mathbb{R}^2) \cap \mathcal{M}_{3\times1}, \\ -\alpha(2u_3 - (curl\,u)_3) &= Q \in L^2(\mathbb{R}^2) \cap \mathcal{M}_{1\times1}, \\ -(\gamma+\varepsilon)\nabla u_3 &= V \in L^2(\mathbb{R}^2) \cap \mathcal{M}_{2\times1}. \end{aligned}$$

From (4.7) it follows that

$$\overline{q} = Div\,P + Grad\,Q, \quad q_3 = div\,V - 2Q.$$

Also,

$$\parallel P \parallel_0^2 + \parallel Q \parallel_0^2 + \parallel V \parallel_0^2 \leqslant cb(u_0,u_0) \leqslant c \parallel u_0 \parallel_{1,\omega}^2 \leqslant \frac{1}{3}c_2^2 \parallel q \parallel_{-1,\omega}^2 .$$

Conversely, let q be of the form (4.8). We claim that $q \in \mathcal{H}_{-1,\omega}(\mathbb{R}^2)$. Indeed, for any $v \in C_0^\infty(\mathbb{R}^2)$

$$\begin{aligned} \langle q, v\rangle_0 &= \langle Div\ P + Grad\ Q, \overline{v}\rangle_0 + \langle div\ V - 2Q, v_3\rangle_0 \\ &= -\langle P, Def\ \overline{v}\rangle_0 - \langle Q, 2v_3 - (curl\ v)_3\rangle_0 - \langle V, \nabla v_3\rangle_0 ; \end{aligned} \tag{4.9}$$

therefore,

$$|\langle q, v\rangle_0| \leqslant c_1^{-1}(\| P \|_0 + \| Q \|_0 + \| V \|_0) \|v\|_{1,\omega} .$$

which shows that $q \in H_{-1,\omega}(\mathbb{R}^2)$ and

$$\| q \|_{-1,\omega} \leqslant c_1^{-1} (\| P \|_0 + \| Q \|_0 + \| V \|_0) .$$

Finally,for any $z \in \mathcal{F}$, form(4.9) it follows that

$$\langle q, z\rangle_0 = 0,$$

consequently, $q \in \mathcal{H}_{-1,\omega}(\mathbb{R}^2)$. ■

We choose an $L^2_\omega(\mathbb{R}^2)$-orthonormal basis $\{z^{(i)}\}_{i=1}^3$ for $\mathcal{F}$ and introduce a modified area potential of density $\varphi \in C_0^\infty(\mathbb{R}^2) \cap \mathcal{H}_{-1,\omega}(\mathbb{R}^2)$ by

$$(\mathcal{U}\varphi)(x) = (U\varphi)(x) - \left\langle U\varphi, z^{(i)} \right\rangle_{0,\omega} z^{(i)}(x), \quad x \in \mathbb{R}^2,$$

where $\langle \cdot, \cdot \rangle_{0,\omega}$ is inner product in $L^2_\omega(\mathbb{R}^2)$. Clearly, $\mathcal{U}\varphi \in H_{1,\omega}(\mathbb{R}^2)$ and satisfies

$$b(\mathcal{U}(-q), v) = \langle q, v\rangle_0 \quad \forall v \in H_{1,\omega}(\mathbb{R}^2). \tag{4.10}$$

The defined operator $\mathcal{U}$ can be extended by continuity from $C_0^\infty(\mathbb{R}^2) \cap \mathcal{H}_{-1,\omega}(\mathbb{R}^2)$ to $\mathcal{H}_{-1,\omega}(\mathbb{R}^2)$. The extended operator $\mathcal{U}$ is continuous from $\mathcal{H}_{-1,\omega}(\mathbb{R}^2)$ to $H_{1,\omega}(\Omega)$. For any $q \in \mathcal{H}_{-1,\omega}(\mathbb{R}^2)$, $\mathcal{U}(-q)$ is a solution of (4.10).We start with (D^+). Let

$u \in H_1(S^+)$ be the solution of the problem

$$\begin{aligned} b_+(u,v) &= \langle q, v\rangle_{0,S^+} \; \forall v \in \overset{\circ}{H}_1(S^+) \\ \gamma^+ u &= f, \end{aligned}$$

where $q \in H_{-1}(S^+)$ and $f \in H_{1/2}(\partial S)$ are given. Repeating the proof of Theorem 20, we see that any $q \in H_{-1}(S^+)$ can be represented in the form

$$\overline{q} = Div\, P + Grad\, Q, \quad q_3 = div\, V - 2Q, \tag{4.11}$$

where $P \in L^2(\mathbb{R}^2) \cap \mathcal{M}_{3\times 1}$, $Q \in L^2(\mathbb{R}^2) \cap \mathcal{M}_{1\times 1}$, $V \in L^2(\mathbb{R}^2) \cap \mathcal{M}_{2\times 1}$, and (4.11) is understood as an equality of distributions in $\mathcal{S}'(S^+)$. The norms $\|q\|_{-1;S^+}$ and $\| P \|_0 + \| Q \|_0 + \| V \|_0$ are equivalent. Let

$$\begin{aligned} \widehat{P}(x) &= \begin{cases} P(x), & x \in S^+, \\ 0, & x \in S^-, \end{cases} \\ \widehat{Q}(x) &= \begin{cases} Q(x), & x \in S^+, \\ 0, & x \in S^-, \end{cases} \\ \widehat{V}(x) &= \begin{cases} V(x), & x \in S^+, \\ 0, & x \in S^-, \end{cases} \end{aligned}$$

and

$$\overline{\widehat{q}} = Div\, \widehat{P} + Grad\, \widehat{Q}, \quad \widehat{q}_3 = div\, \widehat{V} - 2\widehat{Q}, \tag{4.12}$$

where (4.12) is understood as an equality of distributions in $\mathcal{S}'(\mathbb{R}^2)$. Clearly, $\widehat{q} = \left(\overline{\widehat{q}}^T, \widehat{q}_3\right)^T \in \mathcal{H}_{-1,\omega}(\mathbb{R}^2)$ and $\widehat{q} = q$ in $\mathcal{S}'(S^+)$.

We represent solution in the form

$$u = \mathcal{U}(-\widehat{q}) + w.$$

Then $w \in H_1(S^+)$ is a solution of the problem

$$\begin{aligned} b_+(w,v) &= 0 \quad \forall v \in \overset{\circ}{H}_1(S^+) \\ \gamma^+ w &= f + \gamma^+ (\mathcal{U}\widehat{q}) \,. \end{aligned}$$

Since $\gamma^+ (\mathcal{U}\widehat{q}) \in H_{1/2}(\partial S)$, we arrive at (D$^+$) for the homogeneous equilibrium equation. So, without loss of generality, instead of the original problem we may consider the problem (D$^+$) with $q = 0$ of finding $u \in H_1(S^+)$ such that

$$\begin{aligned} b_+(u,v) &= 0 \quad \forall v \in \overset{\circ}{H}_1(S^+) \\ \gamma^+ u &= f. \end{aligned} \tag{4.13}$$

The general problem (D$^-$) can similarly be reduced to the problem of finding $u \in H_{1,\omega}(S^-)$ such that

$$\begin{aligned} b_-(u,v) &= 0 \quad \forall v \in \overset{\circ}{H}_{1,\omega}(S^-) \\ \gamma^- u &= f. \end{aligned} \tag{4.14}$$

We represent the solution of (4.13) and (4.14) in the form

$$u = \mathcal{V}\varphi + z \quad \text{in } S^{\pm}, \tag{4.15}$$

where the density $\varphi \in \mathcal{H}_{-1/2}(\partial S)$ and $z \in \mathcal{F}$ are unknown. Representation (4.15) leads to the system of boundary equations

$$\mathcal{V}_0\varphi + z = f. \tag{4.16}$$

Representing the weak solution of (4.13) and (4.14) in the form

$$u = \mathcal{W}\psi + z \quad \text{in } S^{\pm}, \tag{4.17}$$

where $\psi \in \mathcal{H}_{1/2}(\partial S)$ and $z \in \mathcal{F}$ are unknown, we obtain the following system of boundary equations

$$\mathcal{W}^{\pm}\psi + z = f. \tag{4.18}$$

Theorem 21 *Systems (4.16) and (4.18) have unique solutions*

$$\begin{aligned} \{\varphi, z\} &\in \mathcal{H}_{-1/2}(\partial S) \times \mathcal{F}(\partial S), \\ \{\psi, z\} &\in \mathcal{H}_{1/2}(\partial S) \times \mathcal{F}(\partial S), \end{aligned}$$

respectively, for any $f \in H_{1/2}(\partial S)$, *and*

$$\begin{aligned} \| \quad \varphi \|_{-1/2;\partial S} &\leq c \parallel f \parallel_{1/2;\partial S}, \qquad (4.19) \\ \| \quad \psi \|_{1/2;\partial S} &\leq c \parallel f \parallel_{1/2;\partial S} . \end{aligned}$$

In this case, (4.15) and (4.17) are the solutions of problems ($D^{\pm}$) *with* $q = 0$, *and they satisfy the estimates*

$$\begin{aligned} \| \quad u \|_{1,S^+} &\leq c \parallel f \parallel_{1/2;\partial S}, \qquad (4.20) \\ \| \quad u \|_{1,\omega;S^-} &\leq c \parallel f \parallel_{1/2;\partial S} \end{aligned}$$

Proof. In both cases we choose $z \in \mathcal{F}(\partial S)$ defined by

$$z = -\left\langle f, \mathbb{F}^{(i)} \right\rangle_{0;\partial S} \mathbb{F}^{(i)}.$$

Then it is obvious that $f - z \in \mathcal{H}_{1/2}(\partial S)$ and

$$\| f - z \|_{1/2;\partial S} \leq c \parallel f \parallel_{1/2;\partial S} . \tag{4.21}$$

The solvability of systems (4.16) and (4.18), and estimates (4.19), now follow from the Theorems 18, 19 and (4.21). To prove uniqueness, let $\{\varphi_1, z_1\}$ and

$\{\psi_1, z_1\}$ be other solutions of (4.16) and (4.18), respectively. Then, writing

$$\breve{\varphi} = \varphi - \varphi_1, \quad \breve{\psi} = \psi - \psi_1, \quad \breve{z} = z - z_1,$$

we see that

$$\begin{aligned} \mathcal{V}_0\breve{\varphi} + \breve{z} &= 0, \\ \mathcal{W}^{\pm}\breve{\psi} + \breve{z} &= 0. \end{aligned}$$

Since $\mathcal{V}_0\breve{\varphi}$ and $\breve{z}$ belong to L^2-orthogonal subspaces of $\mathcal{H}_{1/2}(\partial S)$, it follows that

$$\mathcal{V}_0\breve{\varphi} = 0, \quad \breve{z} = 0;$$

therefore, we also have $\breve{\varphi} = 0$.

The proof that $\breve{\psi} = 0$ and $\breve{z} = 0$ in the second case is similar.

Since any rigid displacement belongs to both $H_1(S^+)$ and $H_{1,\omega}(S^-)$, to show that functions u constructed from (4.15) and (4.17) by means of the solutions of (4.16) and (4.18) are the solutions of ($\mathrm{D}^{\pm}$) with $q = 0$, it is enough to verify that $\mathcal{V}\varphi$ and $\mathcal{W}\psi$ also belong to both $H_1(S^+)$ and $H_{1,\omega}(S^-)$ for $\varphi \in \mathcal{H}_{-1/2}(\partial S)$ and $\psi \in \mathcal{H}_{1/2}(\partial S)$.

In the proof of Theorem 18, we showed that

$$\| \mathcal{V}\varphi \|_{1,\omega;S^-}^2 \leqslant c\,[b_+(\mathcal{V}\varphi, \mathcal{V}\varphi) + b_-(\mathcal{V}\varphi, \mathcal{V}\varphi)] = c\,\langle \varphi, \mathcal{V}_0\varphi \rangle_{0;\partial S}\,.$$

The same theorem gives

$$\| \mathcal{V}\varphi \|_{1,\omega;S^-} \leqslant c\, \| \varphi \|_{-1/2;\partial S}\,. \tag{4.22}$$

In the interior problem, since $\mathcal{V}_0\varphi \in \mathcal{H}_{1/2}(\partial S)$, formula (3.23) enables us to write

$$\begin{aligned} \| \quad \mathcal{V}\varphi \|^2_{1;S^+} &\leqslant cb_+(\mathcal{V}\varphi, \mathcal{V}\varphi) \leqslant c\,[b_+(\mathcal{V}\varphi, \mathcal{V}\varphi) + b_-(\mathcal{V}\varphi, \mathcal{V}\varphi)] \qquad (4.23) \\ &= \quad c\,\langle \varphi, \mathcal{V}_0\varphi\rangle_{0;\partial S} \leqslant c \,\|\, \varphi \,\|^2_{-1/2;\partial S}; \end{aligned}$$

consequently, $\mathcal{V}\varphi$ belongs to both $H_1(S^+)$ and $H_{1,\omega}(S^-)$ for $\varphi \in \mathcal{H}_{-1/2}(\partial S)$.

Let $\psi \in \mathcal{H}_{1/2}(\partial S)$ be a smooth density. Since $\mathcal{W}^{\pm}\psi \in \mathcal{H}_{1/2}(\partial S)$, it follows that $\| \,\mathcal{W}\psi\, \|^2_{1;S^+} \leqslant cb_+(\mathcal{W}\psi, \mathcal{W}\psi)$ and $\| \,\mathcal{W}\psi\, \|^2_{1,\omega;S^-} \leqslant cb_-(\mathcal{W}\psi, \mathcal{W}\psi)$; therefore,

$$\begin{aligned} \| \quad \mathcal{W}\psi \|^2_{1;S^+} + \| \,\mathcal{W}\psi\, \|^2_{1,\omega;S^-} &\leqslant c\,[b_+(\mathcal{W}\psi, \mathcal{W}\psi) + b_-(\mathcal{W}\psi, \mathcal{W}\psi)] \\ = \quad & c\left[\left\langle \mathcal{N}^+\mathcal{W}^+\psi, \mathcal{W}^+\psi\right\rangle_{0;\partial S} - \left\langle \mathcal{N}^-\mathcal{W}^-\psi, \mathcal{W}^-\psi\right\rangle_{0;\partial S}\right] \\ = \quad & c\left[\left\langle \mathcal{T}^+\mathcal{W}^+\psi, \mathcal{W}^+\psi\right\rangle_{0;\partial S} - \left\langle \mathcal{T}^-\mathcal{W}^-\psi, \mathcal{W}^-\psi\right\rangle_{0;\partial S}\right]. \end{aligned}$$

But from Theorem 1, it follows that $\mathcal{T}^+\mathcal{W}^+ = \mathcal{T}^-\mathcal{W}^-$, so

$$\| \,\mathcal{W}\psi\, \|^2_{1;S^+} + \| \,\mathcal{W}\psi\, \|^2_{1,\omega;S^-} \leqslant -\left\langle \mathcal{T}^+\mathcal{W}^+\psi, \psi\right\rangle_{0;\partial S} \leqslant c \,\|\, \psi \,\|^2_{1/2;\partial S}\,. \qquad (4.24)$$

Since this is valid for any $\psi \in \mathcal{H}_{1/2}(\partial S)$, we conclude that $\mathcal{W}\psi$ belongs to both $H_1(S^+)$ and $H_{1,\omega}(S^-)$.

Estimates (4.20) follow from (4.21)-(4.24). ■

We now proceed with Neumann problems. First, we show that ($\mathrm{N}^{\pm}$) can be reduced to similar problems for the homogeneous equilibrium equation by means of the area potential. We note that if $g \in H_{-1/2}(\partial S)$, then $\left|\langle g, \gamma^+ v\rangle_{0,\partial S}\right| \leqslant c\,\|g\|_{-1/2;\partial S}\,\|\gamma^+ v\|_{1/2;\partial S}$. But the trace theorem [56] implies that

$$\left|\langle g, \gamma^+ v\rangle_{0,\partial S}\right| \leqslant c\,\|g\|_{-1/2;\partial S}\,\|v\|_{1;S^+}\,.$$

Consequently, $\langle g, \gamma^- v\rangle_{0,\partial S}$ defines a bounded linear functional on $H_1(S^+)$ and can be written in the form $\langle \widetilde{q}^+, v\rangle_{0;S^+}$ with some $\widetilde{q}^+ \in \overset{\circ}{H}_{-1}(S^+)$, and let $Q^+ =$

$q+\widetilde{q}^+$. Then the problem (N^+) can be rewritten as follows. We seek $u \in H_1(S^+)$ such that

$$b_+(u,v) = \left\langle Q^+, v\right\rangle_{0,S^+} \qquad \forall v \in H_1(S^+),$$

where $Q^+ \in \overset{\circ}{H}_{-1}(S^+)$ satisfies

$$\left\langle Q^+, z\right\rangle_{0,S^+} = 0 \quad \forall z \in \mathcal{F}.$$

Since the norms on $H_1(S^+)$ and $H_{1,\omega}(S^+)$ are equivalent, it follows that so are the norms on their duals $\overset{\circ}{H}_{-1}(S^+)$ and $\overset{\circ}{H}_{-1,\omega}(S^+)$; hence, $Q^+ \in \mathcal{H}_{-1,\omega}(\mathbb{R}^2)$.

Similarly, $\langle g, \gamma^- v\rangle_{0,\partial S}$ defines a bounded linear functional on $H_{1,\omega}(S^-)$ and can be written in the form $\langle \widetilde{q}^-, v\rangle_{0;S^+}$ with some $\widetilde{q}^- \in \overset{\circ}{H}_{-1,\omega}(S^-)$, and let $Q^- = q - \widetilde{q}^-$. So, the solution $u \in H_{1,\omega}(S^-)$ of (N^-) satisfies

$$b_-(u,v) = \left\langle Q^-, v\right\rangle_{0,S^-} \qquad \forall v \in H_{1,\omega}(S^-),$$

where $Q^- \in \overset{\circ}{H}_{-1,\omega}(S^-)$ and such that $\langle Q^-, z\rangle_{0,S^-} = 0 \;\forall z \in \mathcal{F}$, in other words, $Q^- \in \mathcal{H}_{-1,\omega}(\mathbb{R}^2)$

We represent the solution of (N^+) in the form

$$u = w + \pi_{S^+}(\mathcal{U}Q^+),$$

where π_{S^+} is the operator of restriction from $\mathbb{R}^2$ to S^+. Clearly, $w \in H_1(S^+)$ satisfies

$$b_+(w,v) = \left\langle Q^+, v\right\rangle_{0,S^+} - b_+(\pi_{S^+}(\mathcal{U}Q^+), v) \qquad \forall v \in H_1(S^+).$$

We now show that

$$\mathcal{L}(v) = \left\langle Q^+, v\right\rangle_{0,S^+} - b_+(\pi_{S^+}(\mathcal{U}Q^+), v)$$

can be regarded as a bounded linear functional on $H_{1/2}(\partial S)$. Indeed, let $v_1, v_2 \in H_1(S^+)$ be such that $\gamma^+ v_1 = \gamma^+ v_2$. Since $v_1 - v_2 \in \mathring{H}_1(S^+) \subset H_{1,\omega}(\mathbb{R}^2)$ and $\mathcal{U}Q^+ \in H_{1,\omega}(\mathbb{R}^2)$ satisfies

$$b(\mathcal{U}(-Q^+), v) = \left\langle Q^+, v \right\rangle_0 \quad \forall v \in H_{1,\omega}(\mathbb{R}^2),$$

it follows that

$$\begin{aligned}
\mathcal{L}(v_1 - v_2) &= \left\langle Q^+, v_1 - v_2 \right\rangle_{0,S^+} - b_+(\pi_{S^+}(\mathcal{U}(-Q^+)), v_1 - v_2) \\
&= \left\langle Q^+, v_1 - v_2 \right\rangle_{0,S^+} - b((\mathcal{U}(-Q^+)), v_1 - v_2) \\
&= \left\langle Q^+, v_1 - v_2 \right\rangle_{0,S^+} - \left\langle Q^+, v_1 - v_2 \right\rangle_{0,S^+} = 0
\end{aligned}$$

This means that definition of $\mathcal{L}(v)$ on $H_{1/2}(\partial S)$ is consistent. If $f \in H_{1/2}(\partial S)$ and $v = l^+ f \in H_1(S^+)$, then

$$|\mathcal{L}(v)| \leq c \parallel Q^+ \parallel_{-1} \parallel v \parallel_{1,S^+} \leq c \parallel Q^+ \parallel_{-1} \parallel f \parallel_{1/2;\partial S} .$$

This shows that $\mathcal{L}(v)$ can be written in the form $\langle g, v \rangle_{0,\partial S}$ with some $g \in H_{-1/2}(\partial S)$.

The case of the exterior problem is treated similarly.

We remark that since $Q^\pm \in \mathcal{H}_{-1,\omega}(\mathbb{R}^2)$, from the definition of $\mathcal{L}(v)$ it follows that

$$\langle g, z \rangle_{0;\partial S} = 0 \quad \forall z \in \mathcal{F}(\partial S);$$

that is, the necessary solvability condition for ($\mathrm{N}^\pm$) is satisfied, so instead of ($\mathrm{N}^\pm$) we may consider without loss of generality their version for the homogeneous equilibrium equations.

In the problem (N^+) with $q = 0$ we seek $u \in H_1(S^+)$ such that

$$b_+(u, v) = \langle g, v \rangle_{0;\partial S} \quad \forall v \in H_1(S^+). \tag{4.25}$$

In the problem (N^-) with $q = 0$ we seek $u \in H_{1,\omega}(S^-)$ such that

$$b_-(u, v) = \langle g, v \rangle_{0;\partial S} \quad \forall v \in H_{1,\omega}(S^-). \tag{4.26}$$

We represent the solutions of problems (4.25) and (4.26) in the form

$$u = \mathcal{V}\varphi + z \ \text{ in } S^\pm, \tag{4.27}$$

where $z \in \mathcal{F}(\partial S)$. Representation (4.27) leads to the systems of boundary equations

$$\mathcal{T}^\pm \mathcal{V}_0 \varphi = g, \tag{4.28}$$

where g must satisfy the necessary solvability condition, $g \in \mathcal{H}_{-1/2}(\partial S)$.

If we represent the solutions as

$$u = \mathcal{W}\psi + z \ \text{ in } S^\pm, \tag{4.29}$$

then we arrive at the system of boundary equations

$$\mathcal{T}^\pm \mathcal{W}^\pm \psi = g. \tag{4.30}$$

Theorem 22 *Systems (4.28) and (4.30) have unique solutions* $\varphi \in \mathcal{H}_{-1/2}(\partial S)$ *and* $\psi \in \mathcal{H}_{1/2}(\partial S)$ *for any* $g \in H_{-1/2}(\partial S)$*, and*

$$\| \varphi \|_{-1/2;\partial S} \leq c \| g \|_{-1/2;\partial S}, \tag{4.31}$$

$$\| \psi \|_{1/2;\partial S} \leq c \| g \|_{-1/2;\partial S} . \tag{4.32}$$

In this case, (4.27) and (4.29) are the solutions of problems ($N^{\pm}$) with $q = 0$, and they satisfy the estimates

$$\| \quad u \|_{1,S^+} \leq c \| g \|_{-1/2;\partial S}, \tag{4.33}$$

$$\| \quad u \|_{1,\omega;S^-} \leq c \| g \|_{-1/2;\partial S} .$$

Proof. The unique solvability of (4.28) and (4.31) follows from properties of operators $\mathcal{N}^{\pm}$ and $\mathcal{V}_0$ established in the Theorems 16 and 18. The unique solvability of (4.30) and (4.32) follows from properties of $\mathcal{W}^{\pm}$ given by Theorem 19. Finally, estimates (4.33) are obtained from Theorems 7 and 10. ■

4.4 Summary

In this chapter we have shown that weak solutions of boundary value problems of plane Cosserat elasticity can be found in terms of integral potentials with distributional densities and the corresponding boundary integral equations are uniquely solvable with respect to these densities. Since it is very hard, if not impossible, to find the densities analytically they can be approximated numerically, for example, by means of generalized Fourier series following the procedure given in details in [39], [45]-[46], [49]-[50]. After the integral densities have been found the quantitative characteristics of the solution may be used for practical purposes.

The method introduced in Chapters 3 and 4 is a generalization of the boundary integral equation method in regular (L^2) form in Sobolev spaces. It is applicable to domains with irregular boundaries, for example, domains with cuts.

This result is extremely important for application itself. However, we further plan to extend this result to develop a method allowing us to find a solution to the boundary value problem for a domain weakened by a crack.

Chapter 5

Stress Distribution Around a Crack in Plane Micropolar Elasticity

In the case when a domain is weakened by a crack the nature of the boundary conditions across the crack region presents formidable difficulties in the boundary integral analysis in a classical setting. Several studies of a crack problem in two-dimensional Cosserat elasticity have been undertaken in the classical elastic setting using the finite element method [75] and under assumptions of a simplified theory of plane Cosserat elasticity when displacements and microrotations are constrained (couple-stress elasticity) [76-77]. Also there has been some activity in the area of crack analysis in three-dimensional Cosserat elasticity [78-81]. The rigorous analysis of the corresponding crack problem in plane Cosserat elasticity in the general case, to the author's knowledge, still remains absent from the literature.

Recently, Chudinovich and Constanda [56] used the boundary integral equation method in a weak (Sobolev) space setting to obtain the solution for several crack problems in a theory of bending of classical elastic plates. In spite of the fact that the methods used are extremely complicated (mathematically) they seem to be very effective and give very good results for applications. We continue to study the effectiveness of these methods with a view to the analysis and solution of the plane problems of Cosserat elasticity.

In this chapter we formulate boundary value problems for both finite and infinite domains weakened by a crack in the case of plane micropolar elasticity when displacements and microrotations or stresses and couple stresses are prescribed along two sides of the crack in Sobolev spaces and find the corresponding weak solutions in terms of integral potentials with distributional densities.

5.1 Basic definitions

First, we consider an infinite domain with a crack modelled by an open arc Γ_0. We assume that Γ_0 is a part of a simple closed C^2-curve Γ that divides $\mathbb{R}^2$ into interior and exterior domains Ω^+ and Ω^-. In what follows we denote by the superscripts $+$ and $-$ the limiting values of functions as $x \to \Gamma$ from within Ω^+ or Ω^-. We define $\Omega = \mathbb{R}^2 \backslash \overline{\Gamma_0}$ and $\Gamma_1 = \Gamma \backslash \overline{\Gamma_0}$. Regarding definition of Ω, we can also use $\| \cdot \|_0$ and $\langle \cdot, \cdot \rangle_0$ for the norm and inner product in $L^2(\Omega)$.

Let $H_m(\Gamma)$ be the standard Sobolev space of distributions on Γ, with norm $\| \cdot \|_{m;\Gamma}$. $H_m(\Gamma)$ and $H_{-m}(\Gamma)$ are dual with respect to the duality generated by the inner product $\langle \cdot, \cdot \rangle_{0;\Gamma}$ in $L^2(\Gamma)$. We denote by $\overset{\circ}{H}_m(\Gamma_0)$ the subspace of all $f \in H_m(\Gamma)$ with a compact support on Γ_0, and by $H_m(\Gamma_0)$ the space of the restrictions to Γ_0 of all $f \in H_m(\Gamma)$. Let π_0 and π_1 be the operators of restriction from Γ to Γ_0 and Γ_1. The norm of $f \in H_m(\Gamma_0)$ is defined by $\| f \|_{m;\Gamma_0} = \inf_{v \in H_m(\Gamma): \pi_0 v = f} \| v \|_{m;\Gamma}$. For any $m \in \mathbb{R}$, $\overset{\circ}{H}_m(\Gamma_0)$ and $H_{-m}(\Gamma_0)$ are dual with respect to the duality generated by the inner product $\langle \cdot, \cdot \rangle_{0;\Gamma_0}$ in $L^2(\Gamma_0)$.

Let γ^+ and γ^- be continuous trace operators from $H_1(\Omega^+)$ and $H_{1,\omega}(\Omega^-)$ to $H_{1/2}(\Gamma)$. Also, let $\gamma_i^{\pm} = \pi_i \gamma^{\pm}$, $i = 0, 1$. For any u defined in Ω (or $\mathbb{R}^2$) we write $u = \{u_+, u_-\}$, where $u_{\pm} = \pi^{\pm} u$.

Let $H_{1,\omega}(\Omega)$ be the space of all $u = \{u_+, u_-\}$ such that $u_+ \in H_1(\Omega^+)$, $u_- \in H_{1,\omega}(\Omega^-)$ and $\gamma_1^+ u_+ = \gamma_1^- u_-$. The norm in $H_{1,\omega}(\Omega)$ is defined by

$$\| u \|_{1,\omega;\Omega}^2 = \| u_+ \|_{1;\Omega^+}^2 + \| u_- \|_{1,\omega;\Omega^-}^2 .$$

$\overset{\circ}{H}_{1,\omega}(\Omega)$ is the subspace of $H_{1,\omega}(\Omega)$ consisting of all u such that $\gamma_0^+ u_+ = \gamma_0^- u_- = 0$; therefore, $\overset{\circ}{H}_{1,\omega}(\Omega)$ can be identified with a subspace of $H_{1,\omega}(\mathbb{R}^2)$.

We denote by $H_{-1,\omega}(\Omega)$ and $\overset{\circ}{H}_{-1,\omega}(\Omega)$ the duals of $\overset{\circ}{H}_{1,\omega}(\Omega)$ and $H_{1,\omega}(\Omega)$ with respect to the duality induced by $\langle \cdot, \cdot \rangle_0$. The norms in $H_{-1,\omega}(\Omega)$ and $\overset{\circ}{H}_{-1,\omega}(\Omega)$ are denoted by $\| \cdot \|_{-1,\omega;\Omega}$ and $\| \cdot \|_{-1,\omega}$.

5.2 Boundary value problems

We consider two types of boundary values problems: Dirichlet and Neumann boundary value problems. The first one consists of seeking $u \in C^2(\Omega) \cap C(\overline{\Omega})$, $u_- \in \mathcal{A}^*$ such that

$$\begin{aligned} Lu(x) + q(x) &= 0, \quad x \in \Omega, \\ u^+(x) &= f^+(x), \quad u^-(x) = f^-(x), \quad x \in \Gamma_0, \end{aligned} \tag{D}$$

where f^+ and f^- are prescribed on Γ_0.

The second problem consists of finding $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, $u_- \in \mathcal{A}$ such that

$$\begin{aligned} Lu(x) + q(x) &= 0, \quad x \in \Omega, \\ (Tu)^+(x) &= g^+(x), \quad (Tu)^-(x) = g^-(x), \quad x \in \Gamma_0, \end{aligned} \tag{N}$$

where g^+ and g^- are prescribed on Γ_0. Asymptotic classes $\mathcal{A}^*$ and $\mathcal{A}$ were introduced in Section 3.1.

The variational formulations are based on the Betti formulae (3.3) and (3.9). The variational formulation of (D) is as follows. We seek $u \in H_{1,\omega}(\Omega)$ such that

$$\begin{aligned} b(u,v) &= \langle q, v \rangle_0 \quad \forall v \in \mathring{H}_{1,\omega}(\Omega), \\ \gamma_0^+ u_+ &= f^+, \quad \gamma_0^- u_- = f^-, \end{aligned} \tag{5.1}$$

where $q \in H_{-1,\omega}(\Omega)$ and $f^+, f^- \in H_{1/2}(\Gamma_0)$ are given.

The variational formulation of (N) is as follows. We seek $u \in H_{1,\omega}(\Omega)$ such that

$$b(u,v) = \langle q, v \rangle_0 + \left\langle g^+, \gamma_0^+ v_+ \right\rangle_{0;\Gamma_0} - \left\langle g^-, \gamma_0^- v_- \right\rangle_{0;\Gamma_0} \quad \forall v \in H_{1,\omega}(\Omega), \tag{5.2}$$

where $q \in \overset{\circ}{H}_{-1,\omega}(\Omega)$ and $g^+, g^- \in H_{1/2}(\Gamma_0)$ are given.

In what follows we write $\delta f = f^+ - f^-$ and $\delta g = g^+ - g^-$ for the jump of these quantities across the crack.

Theorem 23 *Problem (5.1) has a unique solution $u \in H_{1,\omega}(\Omega)$ for any $q \in H_{-1,\omega}(\Omega)$ and any $f^+, f^- \in H_{1/2}(\Gamma_0)$ such that $\delta f \in \overset{\circ}{H}_{1/2}(\Gamma_0)$, and this solution satisfies the estimate*

$$\| u \|_{1,\omega;\Omega} \leqslant c \left(\| q \|_{-1,\omega;\Omega} + \| f^+ \|_{1/2;\Gamma_0} + \| \delta f \|_{1/2;\Gamma} \right). \tag{5.3}$$

Proof. Assume first that $f^+ = f^- = 0$. To prove this assertion it is sufficient to verify that $b(u,v)$ is coercive on $\overset{\circ}{H}_{1,\omega}(\Omega)$. In Theorems 4 and 10 it was shown that any $u = \{u_+, u_-\} \in \overset{\circ}{H}_{1,\omega}(\Omega)$ satisfies $\| u_+ \|^2_{1;\Omega^+} \leqslant cb_+(u_+, u_+)$ and $\| u_- \|^2_{1,\omega;\Omega^-} \leqslant cb_-(u_-, u_-)$, where $b_\pm(u,v) = 2\int_{\Omega^\pm} E(u,v)dx$; consequently,

$$\| u \|^2_{1,\omega;\Omega} = \| u_+ \|^2_{1;\Omega^+} + \| u_- \|^2_{1,\omega;\Omega^-} \leqslant c\,[b_+(u_+,u_+) + b_-(u_-,u_-)] = cb(u,u).$$

By the Lax-Milgram lemma, (D) with $f^+ = f^- = 0$ has a unique solution $u \in \overset{\circ}{H}_{1,\omega}(\Omega)$ and

$$\| u \|_{1,\omega} \leqslant c \| q \|_{-1,\omega;\Omega} . \tag{5.4}$$

In the full problem (D), we consider an operator l_0 of the extension from Γ_0 to Γ, which maps $H_{1/2}(\Gamma_0)$ continuously to $H_{1/2}(\Gamma)$. Let $F^+ = l_0 f^+$, and let F^- be the extension of f^- to Γ such that $\pi_1 F^+ = \pi_1 F^-$. We denote by $l_\pm$ operators of extension from Γ to $\Omega^\pm$, which map $H_{1/2}(\Gamma)$ continuously to $H_1(\Omega^+)$ and $H_{1,\omega}(\Omega^-)$, respectively. Let $w_+ = l_+ F^+ \in H_1(\Omega^+)$ and $w_- = l_- F^- \in H_{1,\omega}(\Omega^-)$. Clearly, $w = \{w_+, w_-\} \in H_{1,\omega}(\Omega)$. We seek a solution to

(D) of the form $u = u_0 + w$, where $u_0 \in \overset{\circ}{H}_{1,\omega}(\Omega)$ satisfies

$$b(u_0, v) = \langle q, v \rangle_0 - b(w, u) \quad \forall v \in \overset{\circ}{H}_{1,\omega}(\Omega). \tag{5.5}$$

Since for all $v \in \overset{\circ}{H}_{1,\omega}(\Omega)$

$$\begin{aligned} |b(w,v)| &\leqslant |b_+(w_+, v_+)| + |b_-(w_-, v_-)| \leqslant c(\| w_+ \|_{1;\Omega^+} + \| w_- \|_{1,\omega;\Omega^-}) \| v \|_{1,\omega} \\ &\leqslant c(\| F^+ \|_{1/2;\Gamma} + \| F^- \|_{1/2;\Gamma}) \| v \|_{1,\omega} \\ &\leqslant c(\| f^+ \|_{1/2;\Gamma_0} + \| f^- \|_{1/2;\Gamma_0}) \| v \|_{1,\omega} \\ &\leqslant c(\| f^+ \|_{1/2;\Gamma_0} + \| \delta f \|_{1/2;\Gamma}) \| v \|_{1,\omega}, \end{aligned}$$

the right-hand side $L(v) = \langle q, v \rangle_0 - b(w, u)$ in (5.5) defines the continuous linear functional on $\overset{\circ}{H}_{1,\omega}(\Omega)$, and $\| L \|_{-1,\omega;\Omega} \leqslant c \left(\| q \|_{-1,\omega;\Omega} + \| f^+ \|_{1/2;\Gamma_0} + \| \delta f \|_{1/2;\Gamma} \right)$; therefore (5.5) has a unique solution $u_0 \in \overset{\circ}{H}_{1,\omega}(\Omega)$, and

$$\| u_0 \|_{1,\omega;\Omega} \leqslant c \left(\| q \|_{-1,\omega;\Omega} + \| f^+ \|_{1/2;\Gamma_0} + \| \delta f \|_{1/2;\Gamma} \right).$$

The theorem now follows from this inequality and the estimate

$$\| w \|_{1,\omega;\Omega} \leqslant c(\| f^+ \|_{1/2;\Gamma_0} + \| \delta f \|_{1/2;\Gamma}).$$

■

We proceed with problem (5.2). It is clear that, in view of the properties of rigid displacements,

$$\langle q, z \rangle_0 + \left\langle g^+, z \right\rangle_{0;\Gamma_0} - \left\langle g^-, z \right\rangle_{0;\Gamma_0} = 0 \quad \forall z \in \mathcal{F} \tag{5.6}$$

is a necessary solvability condition for (N).

Theorem 24 *Problem (5.2) is solvable for any* $q \in \overset{\circ}{H}_{-1,\omega}(\Omega)$ *and any* $g^+, g^- \in H_{-1/2}(\Gamma_0)$ *such that* $\delta g \in \overset{\circ}{H}_{-1/2}(\Gamma_0)$ *satisfying (5.6). Any two solutions differ by a rigid displacement, and there is a solution* u_0 *that satisfies the estimate*

$$\| u_0 \|_{1,\omega;\Omega} \leqslant c \left(\| q \|_{-1,\omega} + \| \delta g \|_{-1/2;\Gamma} + \| g^- \|_{-1/2;\Gamma_0} \right). \tag{5.7}$$

Proof. We notice that the expression

$$\begin{aligned} L(v) &= \left\langle g^+, \gamma_0^+ v_+ \right\rangle_{0;\Gamma_0} - \left\langle g^-, \gamma_0^- v_- \right\rangle_{0;\Gamma_0} \\ &= \left\langle \delta g, \gamma_0^+ v_+ \right\rangle_{0;\Gamma_0} + \left\langle g^-, \delta v \right\rangle_{0;\Gamma_0}, \quad \forall v \in H_{1,\omega}(\Omega), \end{aligned}$$

where $\delta v = \gamma_0^+ v_+ - \gamma_0^- v_-$, defines a continuous linear functional on $H_{1,\omega}(\Omega)$. Consequently, there is $q_1 \in \overset{\circ}{H}_{-1,\omega}(\Omega)$ such that $L(v) = \langle q_1, v \rangle_0$ for all $v \in H_{1,\omega}(\Omega)$, and

$$\| q_1 \|_{-1,\omega} \leqslant c \left(\| g^- \|_{-1/2;\Gamma_0} + \| \delta g \|_{-1/2;\Gamma} \right). \tag{5.8}$$

We set $q + q_1 = \widetilde{q}$ and write (5.2) in the form $b(u,v) = \langle \widetilde{q}, v \rangle_0$, $v \in H_{1,\omega}(\Omega)$. We consider the factor space $\mathbb{H}_{1,\omega}(\Omega) = H_{1,\omega}(\Omega) \backslash \mathcal{F}$ with the norm $\| U \|_{\mathbb{H}_{1,\omega}(\Omega)} = \inf_{u \in H_{1,\omega}(\Omega), u \in U} \| u \|_{1,\omega;\Omega}$ and define on it a bilinear form $\mathcal{B}(U,V)$ and a linear functional $\mathcal{L}(V)$ by

$$\mathcal{B}(U,V) = b(u,v), \qquad \mathcal{L}(V) = L(v) = \langle \widetilde{q}, v \rangle_0, \tag{5.9}$$

where u and v are arbitrary representatives of the classes $U, V \in \mathbb{H}_{1,\omega}(\Omega)$. Since $b(z,z) = 0$ and $\langle \widetilde{q}, z \rangle_0 = 0$ for any $z \in \mathcal{F}$, definitions (5.9) are consistent.

We now consider the problem of finding $U \in \mathbb{H}_{1,\omega}(\Omega)$ such that

$$\mathcal{B}(U,V) = \mathcal{L}(V), \quad \forall V \in \mathbb{H}_{1,\omega}(\Omega). \tag{5.10}$$

We claim that (5.10) has a unique solution. First, from (5.8) it follows that

$$|\mathcal{L}(V)| \leqslant c \left(\| q \|_{-1,\omega} + \| \delta g \|_{-1/2;\Gamma} + \| g^- \|_{-1/2;\Gamma_0}\right) \| v \|_{1,\omega;\Omega} \quad v \in V,$$

which gives $|\mathcal{L}(V)| \leqslant c \left(\| q \|_{-1,\omega} + \| \delta g \|_{-1/2;\Gamma} + \| g^- \|_{-1/2;\Gamma_0}\right) \| V \|_{\mathbb{H}_{1,\omega}(\Omega)}$; this means that $\mathcal{L}(V)$ is continuous. The continuity of $\mathcal{B}$ is clear. In every class U we choose a representative u such that $\left\langle \gamma_0^+ u_+, z \right\rangle_{0;\Gamma_0} = 0$ for all $z \in \mathcal{F}$. By Theorems 7 and 10,

$$\begin{aligned} \| \quad u_- \|^2_{1,\omega;\Omega^-} &\leqslant c\left[b_-(u_-, u_-) + \| \gamma_1^- u_- \|^2_{0;\Gamma_1}\right] \\ &\leqslant c\left[b_-(u_-, u_-) + \| \gamma_1^+ u_+ \|^2_{0;\Gamma_1}\right] \\ &\leqslant c\left[b_-(u_-, u_-) + \| u_+ \|^2_{1;\Omega^+}\right] \end{aligned}$$

and $\| u_+ \|^2_{1;\Omega^+} \leqslant c b_+(u_+, u_+)$, where $\| \cdot \|_{0;\Gamma_1}$ is the norm in $L^2(\Gamma_1)$. Hence, $\| U \|_{\mathbb{H}_{1,\omega}(\Omega)} \leqslant \| u \|^2_{1,\omega;\Omega} \leqslant \mathcal{B}(U, V)$, which proves that $\mathcal{B}$ is coercive on $\mathbb{H}_{1,\omega}(\Omega)$. By the Lax-Milgram lemma, (5.10) has a unique solution $U \in \mathbb{H}_{1,\omega}(\Omega)$ and

$$\| U \|_{\mathbb{H}_{1,\omega}(\Omega)} \leqslant c \left(\| q \|_{-1,\omega} + \| \delta g \|_{-1/2;\Gamma} + \| g^- \|_{-1/2;\Gamma_0}\right).$$

Clearly, any element u in U is a solution of (5.2). If u_1 and u_2 are two solutions of (5.2), then $w = u_1 - u_2$ satisfies

$$b(w, w) = 0, \quad w \in H_{1,\omega}(\Omega).$$

We conclude that $w \in \mathcal{F}$. To complete the proof, we choose $u_0 \in U$ such that $\| u_0 \|_{1,\omega;\Omega} = \| U \|_{\mathbb{H}_{1,\omega}(\Omega)}$. ■

Theorem 25 *$H_{-1,\omega}(\Omega)$ consists of all $q = (\overline{q}^T, q_3)^T$, where $\overline{q} = (q_1, q_2)^T$, of the form*

$$\overline{q} = Div\, P + Grad\, Q, \quad q_3 = div\, V - 2Q, \tag{5.11}$$

where $P \in L^2(\mathbb{R}^2) \cap \mathcal{M}_{3\times 1}$, $Q \in L^2(\mathbb{R}^2) \cap \mathcal{M}_{1\times 1}$, $V \in L^2(\mathbb{R}^2) \cap \mathcal{M}_{2\times 1}$, *Div and Grad were introduced in Section 4.3. Also there are constants* $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$c_1 \parallel q \parallel_{-1,\omega;\Omega} \leqslant \parallel P \parallel_0 + \parallel Q \parallel_0 + \parallel V \parallel_0 \leqslant c_2 \parallel q \parallel_{-1,\omega;\Omega} .$$

The proof of this theorem repeats the proof of Theorem 20.

We now show that (D) and (N) can be reduced to similar problems for the homogeneous equilibrium equation by means of the area potential.

We start with (D). By Theorem 25, any $q \in H_{-1,\omega}(\Omega)$ can be represented in the form (5.11), where the equality is understood in $\mathcal{S}'(\Omega)$. Let $\widehat{q} \in \mathcal{H}_{-1,\omega}(\mathbb{R}^2)$ be defined by the same formula (5.11), in which the equality is understood in $\mathcal{S}'(\mathbb{R}^2)$. We represent solution of (D) in the form $u = \mathcal{U}(-\widehat{q}) + w$. Since $b(\mathcal{U}(-\widehat{q}), v) = \langle \widehat{q}, v\rangle_0 = \langle q, v\rangle_0$ for $v \in \overset{\circ}{H}_{1,\omega}(\Omega)$, we conclude that $w \in H_{1,\omega}(\Omega)$ satisfies

$$\begin{aligned} b(w,v) &= 0 \quad \forall v \in \overset{\circ}{H}_{1,\omega}(\Omega) \\ \gamma_0^+ w_+ &= f^+ - \gamma_0^+ (\mathcal{U}(-\widehat{q}))_+ , \quad \gamma_0^- w_- = f^- - \gamma_0^- (\mathcal{U}(-\widehat{q}))_- . \end{aligned}$$

Let γ_0 be the trace operator defined on $H_{1,\omega}(\Omega)$ by $\gamma_0 v = \{\gamma_0^+ v_+, \gamma_0^+ v_+ - \gamma_0^- v_-\}$. It is clear that γ_0 is continuous from $H_{1,\omega}(\Omega)$ to $H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$. Consequently, without loss of generality, in what follows we consider the problem (D) that consists in finding $u \in H_{1,\omega}(\Omega)$ such that

$$b(u,v) = 0 \quad \forall v \in \overset{\circ}{H}_{1,\omega}(\Omega), \qquad \gamma_0 u = \{f^+, \delta f\}. \tag{5.12}$$

In problem (N) we seek $u \in H_{1,\omega}(\Omega)$ such that

$$b(u,v) = \langle \widetilde{q}, v \rangle_0 , \quad \forall v \in H_{1,\omega}(\Omega), \tag{5.13}$$

where $\widetilde{q} \in \overset{\circ}{H}_{-1,\omega}(\Omega)$ was defined in Theorem 24 and satisfies

$$\langle \widetilde{q}, z \rangle_0 = 0 \quad \forall z \in \mathcal{F}. \tag{5.14}$$

Since $H_{1,\omega}(\mathbb{R}^2)$ is a subspace of $H_{1,\omega}(\Omega)$, we may consider $\widetilde{q}$ which belongs to $H_{-1,\omega}(\mathbb{R}^2)$; in addition, from (5.14) it follows that $\widetilde{q} \in \mathcal{H}_{-1,\omega}(\mathbb{R}^2)$. We represent the solution of (5.13) in the form $u = \mathcal{U}\widetilde{q} + w$, then (5.13) becomes

$$b(w,v) = \langle \widetilde{q}, v \rangle_0 - b(\mathcal{U}\widetilde{q}, v) \quad \forall v \in H_{1,\omega}(\Omega).$$

Lemma 26 *For all* $\widetilde{q} \in \overset{\circ}{H}_{-1,\omega}(\Omega)$ *satisfying (5.14), the expression*

$$\mathcal{L}(\gamma_0 v) = \langle \widetilde{q}, v \rangle_0 - b(\mathcal{U}\widetilde{q}, v), \quad v \in H_{1,\omega}(\Omega), \tag{5.15}$$

defines a continuous linear functional on $H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$*; therefore,* $\mathcal{L}(\gamma_0 v)$ *can be written in the form*

$$\langle \widetilde{q}, v \rangle_0 - b(\mathcal{U}\widetilde{q}, v) = \left\langle \delta g, \gamma_0^+ v_+ \right\rangle_{0;\Gamma_0} + \left\langle g^-, \delta v \right\rangle_{0;\Gamma_0} , \quad v \in H_{1,\omega}(\Omega),$$

where $\{\delta g, g^-\} \in \overset{\circ}{H}_{-1/2}(\Gamma_0) \times H_{-1/2}(\Gamma_0)$.

Proof. Let $v_1, v_2 \in H_{1,\omega}(\Omega)$ such that $\gamma_0 v_1 = \gamma_0 v_2$. The difference $v_1 - v_2 \in \overset{\circ}{H}_{1,\omega}(\Omega) \subset H_{1,\omega}(\mathbb{R}^2)$, and since $b(\mathcal{U}\widetilde{q}, v_1 - v_2) = \langle \widetilde{q}, v_1 - v_2 \rangle_0$, we find that $\mathcal{L}(\gamma_0 v_1) = \mathcal{L}(\gamma_0 v_2)$. This means that definition (5.15) of $\mathcal{L}$ on $H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$ is consistent. Let $\{f^+, \delta f\} \in H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$. Repeating the proof of Theorem 23, we choose $v \in H_{1,\omega}(\Omega)$ so that $\gamma_0 v = \{f^+, \delta f\}$ and

$\| v \|_{1,\omega;\Omega} \leqslant c(\| f^+ \|_{1/2;\Gamma_0} + \| \delta f \|_{1/2;\Gamma})$. We have

$$\left|\mathcal{L}\left(\{f^+, \delta f\}\right)\right| \leqslant c \| \widetilde{q} \|_{-1,\omega} \| v \|_{1,\omega;\Omega} \leqslant c \| \widetilde{q} \|_{-1,\omega} (\| f^+ \|_{1/2;\Gamma_0} + \| \delta f \|_{1/2;\Gamma}),$$

which shows that $\mathcal{L}$ is continuous on $H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$; since $\overset{\circ}{H}_{-1/2}(\Gamma_0) \times H_{-1/2}(\Gamma_0)$ is the dual of $H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$, this completes the proof. ■

Lemma 26 implies that, without loss of generality, we may consider (N) only for the homogeneous equilibrium equation; that is, we seek $u \in H_{1,\omega}(\Omega)$ such that

$$b(u,v) = \left\langle \delta g, \gamma_0^+ v_+ \right\rangle_{0;\Gamma_0} + \left\langle g^-, \delta v \right\rangle_{0;\Gamma_0} \quad \forall v \in H_{1,\omega}(\Omega). \tag{5.16}$$

We remark that (5.16) is solvable only if

$$\langle z, \delta g \rangle_{0;\Gamma_0} = 0 \ \forall z \in \mathcal{F}. \tag{5.17}$$

5.3 Poincaré–Steklov operator for the crack problem

For $F = \{f^+, \delta f\} \in H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$ and $G = \{\delta g, g^-\} \in \overset{\circ}{H}_{-1/2}(\Gamma_0) \times H_{-1/2}(\Gamma_0)$ we use the notation

$$[F,G]_{0;\Gamma_0} = \left\langle f^+, \delta g \right\rangle_{0;\Gamma_0} + \left\langle \delta f, g^- \right\rangle_{0;\Gamma_0}.$$

We define the Poincaré-Steklov operator $\mathcal{T}$ on $H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$ by

$$\begin{aligned} [\mathcal{T}F, \Psi]_{0;\Gamma_0} &= b(u,v) \quad \forall \Psi \in H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0), \qquad (5.18) \\ F &\in H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0), \end{aligned}$$

where u is the solution of (5.12) and v is any element in $H_{1,\omega}(\Omega)$ such that $\gamma_0 v = \Psi = \{\psi^+, \delta\psi\}$. The definition is independent of the choice of v. In particular, we may take $v = l\Psi$, where l is an operator of extension from Γ_0 to Ω which maps $H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$ continuously to $H_{1,\omega}(\Omega)$.

We identify $\mathcal{F}$ with the subspace of $H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$ consisting of all $Z = \{z, 0\}$, $z \in \mathcal{F}$. We also introduce the spaces

$$\begin{aligned}\widehat{\mathcal{H}}_{1/2}(\Gamma_0) &= \{F \in H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0) : \langle f^+, z\rangle_{0;\Gamma_0} = 0 \ \ \forall z \in \mathcal{F}\},\\ \widehat{\mathcal{H}}_{-1/2}(\Gamma_0) &= \{G \in \overset{\circ}{H}_{-1/2}(\Gamma_0) \times H_{-1/2}(\Gamma_0) : \langle \delta g, z\rangle_{0;\Gamma_0} = 0 \ \ \forall z \in \mathcal{F}\},\end{aligned}$$

Theorem 27 *(i)* $\mathcal{T} : H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0) \to \overset{\circ}{H}_{-1/2}(\Gamma_0) \times H_{-1/2}(\Gamma_0)$ *is self-adjoint and continuous.*

(ii) The kernel of $\mathcal{T}$ *coincides with* $\mathcal{F}$.

(iii) The range of $\mathcal{T}$ *coincides with* $\widehat{\mathcal{H}}_{-1/2}(\Gamma_0)$.

(iv) The restriction $\mathcal{N}$ *of* $\mathcal{T}$ *to* $\widehat{\mathcal{H}}_{1/2}(\Gamma_0)$ *is a homeomorphism from* $\widehat{\mathcal{H}}_{1/2}(\Gamma_0)$ *to* $\widehat{\mathcal{H}}_{-1/2}(\Gamma_0)$.

Proof. (i) If u is the solution of (5.12) and $v = l\Psi$, then, by the definition of $\mathcal{T}$, for $F, \Psi \in H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$

$$|[\mathcal{T}F, \Psi]|^2 = |b(u,v)|^2 \leq b(u,u)b(v,v) \leqslant cb(u,u) \parallel \Psi \parallel^2_{H_{1/2}(\Gamma_0)\times \overset{\circ}{H}_{1/2}(\Gamma_0)} .$$

Consequently, $\mathcal{T}F \in \overset{\circ}{H}_{-1/2}(\Gamma_0) \times H_{-1/2}(\Gamma_0)$ and

$$\begin{aligned}\parallel \ \ &\mathcal{T}f \parallel^2_{\overset{\circ}{H}_{-1/2}(\Gamma_0)\times H_{-1/2}(\Gamma_0)} \leq cb(u,u) = c\,[\mathcal{T}F, F]_{0;\Gamma_0} \qquad (5.19)\\ \leq \ \ & c \parallel \mathcal{T}f \parallel_{\overset{\circ}{H}_{-1/2}(\Gamma_0)\times H_{-1/2}(\Gamma_0)} \parallel F \parallel_{H_{1/2}(\Gamma_0)\times \overset{\circ}{H}_{1/2}(\Gamma_0)} .\end{aligned}$$

From (5.19) it follows that

$$\| \mathcal{T}F \|_{\overset{\circ}{H}_{-1/2}(\Gamma_0)\times H_{-1/2}(\Gamma_0)} \leq c \| F \|_{H_{1/2}(\Gamma_0)\times \overset{\circ}{H}_{1/2}(\Gamma_0)}, \tag{5.20}$$

which proves the continuity of $\mathcal{T}$. The definition of $\mathcal{T}$ shows that it is self-adjoint in the sense that

$$[\mathcal{T}F, \Psi]_{0;\Gamma_0} = [\Psi, \mathcal{T}F]_{0;\Gamma_0} \quad \forall F, \Psi \in H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0).$$

(ii) It is clear that $\mathcal{T}Z = 0$ for $Z \in \mathcal{F}$. If $F \in H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0), \mathcal{T}F = 0$ and u is the solution of (5.12), then $b(u, u) = 0$; therefore, $u \in \mathcal{F}$, which implies that $F = \gamma_0 u \in \mathcal{F}$. This also proves that $\mathcal{N}$ is injective.

(iii) By (5.20), the range of $\mathcal{T}$ is a subset of $\widehat{\mathcal{H}}_{-1/2}(\Gamma_0)$. Let $\{\widetilde{z}^{(i)}\}_{i=1}^{3}$ be an $L^2(\Gamma_0)$-orthonormal basis for $\mathcal{F}$. From Theorems 7 and 10 it follows that any $u \in H_{1,\omega}(\Omega)$ satisfies

$$\| u \|_{1,\omega;\Omega}^2 \leqslant c \left[b(u, u) + \sum_{i=1}^{3} \left\langle \gamma_0^+ u_+, \widetilde{z}^{(i)} \right\rangle_{0;\Gamma_0}^2 \right]. \tag{5.21}$$

Let $F \in \widehat{\mathcal{H}}_{1/2}(\Gamma_0)$. By the trace theorem [56] and (5.21),

$$\| F \|_{H_{1/2}(\Gamma_0)\times \overset{\circ}{H}_{1/2}(\Gamma_0)}^2 \leqslant c \| u \|_{1,\omega;\Omega}^2 \leqslant cb(u, u) = c\, [\mathcal{T}F, F]_{0;\Gamma_0};$$

hence,

$$\| F \|_{H_{1/2}(\Gamma_0)\times \overset{\circ}{H}_{1/2}(\Gamma_0)} \leqslant c \| \mathcal{T}F \|_{\overset{\circ}{H}_{-1/2}(\Gamma_0)\times H_{-1/2}(\Gamma_0)},$$

which shows that $\mathcal{N}^{-1}$ is continuous. If the range of $\mathcal{T}$ is not dense in $\widehat{\mathcal{H}}_{-1/2}(\Gamma_0)$, then there is a nonzero $\widehat{F}$ in the dual $\left[H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0) \right] \backslash \mathcal{F}$ of $\widehat{\mathcal{H}}_{-1/2}(\Gamma_0)$ such that $\langle \mathcal{T}F, \Psi \rangle_{0;\Gamma_0} = 0$ for all representatives F of the class $\widehat{F}$ and all $\Psi \in H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$. Taking $F \in \widehat{\mathcal{H}}_{1/2}(\Gamma_0)$ and $\Psi = F$, we find that

$[\mathcal{T}F, F]_{0;\Gamma_0} = 0$; therefore, $F \in \mathcal{F}$ and $\widehat{F} = 0$. This contradiction proves the third statement.

(iv) This assertion follows from preceding ones. ■

5.4 Boundary equations

We introduce single and double layer potentials on the crack defined by

$$(V\varphi)(x) = \int_{\Gamma_0} D(x,y)\varphi(y)\, ds(y),$$
$$(W\varphi)(x) = \int_{\Gamma_0} P(x,y)\varphi(y)\, ds(y),$$

Let $\overset{\circ}{\mathcal{H}}_{-1/2}(\Gamma_0)$ be the subspace of $\overset{\circ}{H}_{-1/2}(\Gamma_0)$ of all g such that $\langle g, z\rangle_{0;\Gamma_0} = 0$ for all $z \in \mathcal{F}$.

We define the modified single layer potential $\mathcal{V}$ of density $\varphi \in \overset{\circ}{\mathcal{H}}_{-1/2}(\Gamma_0)$ by

$$(\mathcal{V}\varphi)(x) = (V\varphi)(x) - \left\langle (V\varphi)_0, \tilde{z}^{(i)} \right\rangle_{0;\Gamma_0} \tilde{z}^{(i)}(x), \quad x \in \mathbb{R}^2,$$

where $V\varphi$ is the single layer potential, and V_0 is the boundary operator defined by $(V\varphi)_0 = \gamma_0^{\pm}\pi^{\pm}V\varphi$. Let $\mathcal{V}_0\varphi$ be the operator defined on $\overset{\circ}{\mathcal{H}}_{-1/2}(\Gamma_0)$ by $\varphi \to (\mathcal{V}\varphi)_0 = \gamma_0^{\pm}\pi^{\pm}\mathcal{V}\varphi$. From the results established in Section 4.2 $\mathcal{V}_0$ is continuous from $\overset{\circ}{\mathcal{H}}_{-1/2}(\Gamma_0)$ to the subspace $\mathcal{H}_{1/2}(\Gamma_0)$ of all $f^+ \in H_{1/2}(\Gamma_0)$ such that $\langle f^+, z\rangle_{0;\Gamma_0} = 0$ for all $z \in \mathcal{F}$. Let $\widetilde{\mathcal{V}}$ be the continuous operator from $\overset{\circ}{\mathcal{H}}_{-1/2}(\Gamma_0)$ to $\widehat{\mathcal{H}}_{1/2}(\Gamma_0)$ defined by $\widetilde{\mathcal{V}}\varphi = \{\mathcal{V}_0\varphi, 0\}$.

Theorem 28 *The operator $\mathcal{V}_0$ is a homeomorphism from $\overset{\circ}{\mathcal{H}}_{-1/2}(\Gamma_0)$ to $\mathcal{H}_{1/2}(\Gamma_0)$.*

Proof. The continuity of $\mathcal{V}_0$ is proved in Theorem 18. From the jump formula for the normal boundary stresses and couple stresses of the single layer potential (Theorem 1) it follows that the first component of $\mathcal{N}\widetilde{\mathcal{V}}\varphi \in \widehat{\mathcal{H}}_{-1/2}(\Gamma_0)$ is φ. By Theorem 27,

$$
\begin{aligned}
\| \quad \varphi \|_{-1/2;\Gamma_0} &\leqslant \| \mathcal{N}\widetilde{\mathcal{V}}\varphi \|_{\overset{\circ}{H}_{-1/2}(\Gamma_0)\times H_{-1/2}(\Gamma_0)} \\
\leqslant \quad c \, \| \widetilde{\mathcal{V}}\varphi \|_{H_{1/2}(\Gamma_0)\times \overset{\circ}{H}_{1/2}(\Gamma_0)} &= c \, \| \mathcal{V}_0\varphi \|_{1/2;\Gamma_0},
\end{aligned}
$$

which shows that $\mathcal{V}_0^{-1}$ is continuous. Next, we claim that the range of $\mathcal{V}_0$ is $\mathcal{H}_{1/2}(\Gamma_0)$. Let $f^+ \in \mathcal{H}_{1/2}(\Gamma_0)$ and $F = \{f^+, 0\} \in \widehat{\mathcal{H}}_{1/2}(\Gamma_0)$, and let $u \in H_{1,\omega}(\Omega)$ be the solution of (5.12) with $\delta f = 0$. We take $G = \{\delta g, g^-\} = \mathcal{N}F \in \widehat{\mathcal{H}}_{-1/2}(\Gamma_0)$ and $\varphi = \delta g \in \overset{\circ}{\mathcal{H}}_{-1/2}(\Gamma_0)$. Then $w = u - \mathcal{V}_0\varphi$ satisfies $\gamma_0 w = \{f^+ - \mathcal{V}_0\varphi, 0\} = \Psi$. By the jump formula, the first component of $\mathcal{N}\Psi$ is zero; consequently, $b(w,w) = [\mathcal{N}\Psi, \Psi]_{0;\Gamma_0} = 0$. This means that $w \in \mathcal{F}$, so $\gamma_0^+ w_+$ is a rigid displacement on Γ_0. Since $\gamma_0^+ w_+ = f^+ - \mathcal{V}_0\varphi \in \mathcal{H}_{1/2}(\Gamma_0)$, we have $f^+ = \mathcal{V}_0\varphi$, and the assertion is proved. ■

Also we introduce the modified double layer potential $\mathcal{W}$ of density $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0)$

$$
(\mathcal{W}\psi)(x) = (W\psi)(x) - \left\langle \pi_0 W^+\psi, \tilde{z}^{(i)} \right\rangle_{0;\Gamma_0} \tilde{z}^{(i)}(x), \quad x \in \Omega,
$$

Clearly, if $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0)$, then $\mathcal{W}\psi \in H_{1,\omega}(\Omega)$ and $\| \mathcal{W}\psi \|_{1,\omega;\Omega} \leqslant c \, \| \psi \|_{1/2;\Gamma}$. Hence, for $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0)$ we can define the operators $\mathcal{W}^\pm$ of the limiting values of the modified double layer potential on Γ from within $\Omega^\pm$, by writing $\mathcal{W}^\pm\psi = \gamma^\pm\pi^\pm\mathcal{W}\psi$. It is obvious that $\mathcal{W}^\pm$ are continuous from $\overset{\circ}{H}_{1/2}(\Gamma_0)$ to $H_{1/2}(\Gamma)$ and satisfy the jump formula

$$
\mathcal{W}^+\psi - \mathcal{W}^-\psi = -\psi. \tag{5.22}
$$

For $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0)$, we now define the operator $\mathcal{W}_0$ of the limiting values of the modified double layer potential on Γ_0 from within Ω, by writing

$$\mathcal{W}_0\psi = \left\{\pi_0\mathcal{W}^+\psi, \pi_0(\mathcal{W}^+\psi - \mathcal{W}^-\psi)\right\} = \left\{\pi_0\mathcal{W}^+\psi, -\psi\right\}.$$

Clearly, $\mathcal{W}_0$ is continuous from $\overset{\circ}{H}_{1/2}(\Gamma_0)$ to $\widehat{\mathcal{H}}_{1/2}(\Gamma_0)$.

Let $\widetilde{\mathcal{G}} = \mathcal{N}\mathcal{W}_0$. From the jump formula for the normal boundary stresses and couple stresses of the double layer potential it follows that the first component of $\widetilde{\mathcal{G}}\psi$ is zero for any $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0)$; therefore, we can write $\widetilde{\mathcal{G}}\psi = \{0, \mathcal{G}\psi\}$ for all $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0)$.

Theorem 29 *$\mathcal{G}$ is a homeomorphism from $\overset{\circ}{H}_{1/2}(\Gamma_0)$ to $H_{-1/2}(\Gamma_0)$.*

Proof. The continuity of $\mathcal{G}$ follows from the properties of $\mathcal{W}_0$ and $\mathcal{N}$. We claim that $\mathcal{G}^{-1}$ is continuous. Let $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0)$. By (5.22) and the trace theorem [56],

$$\begin{aligned}
\| \quad & \psi \|^2_{1/2;\Gamma} = \| \mathcal{W}^+\psi - \mathcal{W}^-\psi \|^2_{1/2;\Gamma} \leqslant c \| \mathcal{W}\psi \|^2_{1,\omega;\Omega} \\
\leqslant \quad & cb(\mathcal{W}\psi, \mathcal{W}\psi) = -c\left\langle \mathcal{G}\psi, \psi\right\rangle_{0;\Gamma_0} \\
\leqslant \quad & c \| \mathcal{G}\psi \|_{-1/2;\Gamma_0} \| \psi \|_{1/2;\Gamma};
\end{aligned}$$

consequently, $\| \psi \|_{1/2;\Gamma} \leqslant c \| \mathcal{G}\psi \|_{-1/2;\Gamma_0}$. If the range of $\mathcal{G}$ is not dense in $H_{-1/2}(\Gamma_0)$, then there is a nonzero ψ in the dual $\overset{\circ}{H}_{1/2}(\Gamma_0)$ such that $\langle \psi, \mathcal{G}\xi\rangle_{0;\Gamma_0} = 0$ for all $\xi \in \overset{\circ}{H}_{1/2}(\Gamma_0)$. We take $\xi = \psi$ and obtain $\langle \psi, \mathcal{G}\psi\rangle_{0;\Gamma_0} = 0$, which means that $\mathcal{W}\psi \in \mathcal{F}$; hence, $\psi = \mathcal{W}^-\psi - \mathcal{W}^+\psi = 0$. This contradiction completes proof. ■

We represent the solution of (5.12) in the form

$$u = (\mathcal{V}\varphi)_\Omega + W\psi + z, \tag{5.23}$$

where $\varphi \in \overset{\circ}{\mathcal{H}}_{-1/2}(\Gamma_0)$ and $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0)$ are unknown densities, $(\mathcal{V}\varphi)_\Omega$ is the restriction of $\mathcal{V}\varphi$ to Ω and

$$z = \left\langle f^+ - \pi_0 W^+\psi, \tilde{z}^{(i)} \right\rangle_{0;\Gamma_0} \tilde{z}^{(i)}.$$

Representation (5.23) leads to the system of boundary equations

$$\left\{\mathcal{V}_0\varphi + \pi_0 W^+\psi + \gamma_0^+ z, -\psi\right\} = \{f^+, \delta f\}. \tag{5.24}$$

Theorem 30 *For any* $\{f^+, \delta f\} \in H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$, *system (5.24) has a unique solution*

$$\{\varphi, \psi\} \in \overset{\circ}{\mathcal{H}}_{-1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$$

respectively, and

$$\| \{\varphi, \psi\} \|_{\overset{\circ}{H}_{-1/2}(\Gamma_0)\times \overset{\circ}{H}_{1/2}(\Gamma_0)} \leq c \| \{f^+, \delta f\} \|_{H_{1/2}(\Gamma_0)\times \overset{\circ}{H}_{1/2}(\Gamma_0)} .$$

In this case, (5.23) is the solution of problem (5.12).

Proof. From (5.24) $\psi = -\delta f \in \overset{\circ}{H}_{1/2}(\Gamma_0)$, consequently the equation for φ becomes

$$\mathcal{V}_0\varphi = f^+ + \pi_0 W^+\delta f - \left\langle f^+ + \pi_0 W^+\delta f, \tilde{z}^{(i)} \right\rangle_{0;\Gamma_0} \tilde{z}^{(i)}. \tag{5.25}$$

The right-hand side in (5.25) belongs to $\mathcal{H}_{1/2}(\Gamma_0)$. By Theorem 28, (5.25) has a unique solution $\varphi \in \overset{\circ}{\mathcal{H}}_{1/2}(\Gamma_0)$ and

$$\begin{aligned} \| \quad & \varphi \|_{-1/2;\Gamma} \leqslant c \left(\| f^+ \|_{1/2;\Gamma_0} + \| \pi_0 W^+\delta f \|_{1/2;\Gamma_0}\right) \\ \leqslant \quad & c \left(\| f^+ \|_{1/2;\Gamma_0} + \| \delta f \|_{1/2;\Gamma}\right) = c \| \{f^+, \delta f\} \|_{H_{1/2}(\Gamma_0)\times \overset{\circ}{H}_{1/2}(\Gamma_0)} . \end{aligned}$$

■

We represent the solution of problem (5.16) in the form

$$u = (\mathcal{V}\varphi)_\Omega + \mathcal{W}\psi + z, \tag{5.26}$$

where $\varphi \in \overset{\circ}{\mathcal{H}}_{-1/2}(\Gamma_0)$ and $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0)$ are unknown densities, and $z \in \mathcal{F}$ is arbitrary. Representation (5.26) leads to the systems of boundary equations

$$\mathcal{N}\widetilde{\mathcal{V}}\varphi + \widetilde{\mathcal{G}}\psi = \{\delta g, g^-\}. \tag{5.27}$$

Theorem 31 *For any* $\{\delta g, g^-\} \in \overset{\circ}{H}_{-1/2}(\Gamma_0) \times H_{-1/2}(\Gamma_0)$ *satisfying (5.17), system (5.27) has a unique solution* $\{\varphi, \psi\} \in \overset{\circ}{\mathcal{H}}_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$ *and*

$$\| \{\varphi, \psi\} \|_{\overset{\circ}{H}_{-1/2}(\Gamma_0)\times \overset{\circ}{H}_{1/2}(\Gamma_0)} \leq c \, \| \{\delta g, g^-\} \|_{\overset{\circ}{H}_{-1/2}(\Gamma_0)\times H_{-1/2}(\Gamma_0)} .$$

In this case, (5.26) is the solution of problem (5.16).

Proof. Comparing first components on both sides of (5.27), we see that $\varphi = \delta g$; therefore, (5.27) takes form

$$\mathcal{G}\psi = g^- - \left(\mathcal{N}\widetilde{\mathcal{V}}\delta g\right)^-, \tag{5.28}$$

where $\left(\mathcal{N}\widetilde{\mathcal{V}}\delta g\right)^-$ is the second component of $\mathcal{N}\widetilde{\mathcal{V}}\delta g$. By Theorems 27, 28 and 29, (5.28) has a unique solution $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0)$ and

$$\| \psi \|_{1/2;\Gamma} \leqslant c \left(\| g^- \|_{-1/2;\Gamma_0} + \| \delta g \|_{-1/2;\Gamma} \right).$$

■

5.5 The boundary equations for a finite domain

Let ∂S be a simple closed C^2-curve that divides $\mathbb{R}^2$ into interior and exterior domains S^+ and S^-. We assume that S^+ contains inside an auxiliary simple closed C^2-curve $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, where Γ_0 is an open arc modeling crack. We write $\Omega = S^+ \backslash \overline{\Gamma}_0$. Let Ω^+ be the interior domain bounded by Γ, and let $\Omega^- = S^+ \backslash \overline{\Omega}^+$.

If u is defined in Ω, then we denote by u_+ and u_- its restrictions to Ω^+ and Ω^-, respectively, and write $u = \{u_+, u_-\}$. The spaces $H_1(\Omega^\pm)$ are introduced in the usual way. The traces of the elements $u_\pm \in H_1(\Omega^\pm)$ on Γ are denoted by $\gamma^+ u_+$ and $\gamma^- u_-$.

We denote by π_i, $i = 0, 1$, the operators of restrictions from Γ to Γ_i and write $\gamma_i^\pm = \pi_i \gamma^\pm$, $i = 0, 1$. The space $H_1(\Omega)$ consists of all $u = \{u_+, u_-\}$ defined in Ω and such that $u_+ \in H_1(\Omega^+)$, $u_- \in H_1(\Omega^-)$ and $\gamma_1^+ u_+ = \gamma_1^- u_-$. The norm in $H_1(\Omega)$ is defined by $\| u \|^2_{1;\Omega} = \| u_+ \|^2_{1;\Omega^+} + \| u_- \|^2_{1;\Omega^-}$. Let γ_0 be the trace operator that acts on $u \in H_1(\Omega)$ according to the formula $\gamma_0 u = \{\gamma_0^+ u_+, \gamma_0^+ u_+ - \gamma_0^- u_-\}$. Clearly, γ_0 is continuous from $H_1(\Omega)$ to $H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$. The trace of $u \in H_1(\Omega)$ on ∂S is denoted by $\gamma^+_{\partial S} u$. $\overset{\circ}{H}_1(\Omega)$ is the subspace of $H_1(\Omega)$ consisting of all $u \in H_1(\Omega)$ such that $\gamma_0 u = \{0, 0\}$ and $\gamma^+_{\partial S} u = 0$.

Let $\widehat{\Gamma} = \Gamma_0 \cup \partial S$. In what follows we make use of spaces $H_{1/2}(\widehat{\Gamma}) = H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0) \times H_{1/2}(\partial S)$ of all $\widehat{F} = \{F, f_{\partial S}\}$, where $F = \{f^+, \delta f\}$, and $H_{-1/2}(\widehat{\Gamma}) = \overset{\circ}{H}_{-1/2}(\Gamma_0) \times H_{-1/2}(\Gamma_0) \times H_{-1/2}(\partial S)$ of all $\widehat{G} = \{G, g_{\partial S}\}$, where $G = \{\delta g, g^-\}$. It is clear that these spaces are dual with respect to the duality $\left[\widehat{F}, \widehat{G}\right]_{0;\widehat{\Gamma}} = [F, G]_{0;\Gamma_0} + \langle f_{\partial S}, g_{\partial S} \rangle_{0;\partial S}$, where $[F, G]_{0;\Gamma_0}$ is the form defined in Section 5.3. This duality is generated by the inner product $[\cdot, \cdot]_{0;\widehat{\Gamma}}$ in $L^2(\widehat{\Gamma}) = L^2(\Gamma_0) \times$

$L^2(\Gamma_0) \times L^2(\partial S)$.

We consider the following boundary value problems.

Given $\widehat{F} = \{F, f_{\partial S}\} \in H_{1/2}(\widehat{\Gamma})$, we seek $u \in H_1(\Omega)$ such that

$$b_\Omega(u,v) = 0 \ \ \forall v \in \overset{\circ}{H}_1(\Omega), \qquad \gamma_0 u = F, \ \ \gamma_{\partial S}^{+} u = f_{\partial S}, \tag{5.29}$$

where $b_\Omega(u,v) = \int_\Omega E(u,v)dx$.

Given $\widehat{G} = \{G, g_{\partial S}\} \in H_{-1/2}(\widehat{\Gamma})$, we seek $u \in H_1(\Omega)$ such that

$$b_\Omega(u,v) = [G, \gamma_0 v]_{0;\Gamma_0} + \langle g_{\partial S}, \gamma_{\partial S} v\rangle_{0;\partial S}\,, \quad \forall v \in H_1(\Omega). \tag{5.30}$$

Clearly, (5.30) is solvable only if

$$\langle \delta g, z\rangle_{0;\Gamma_0} + \langle g_{\partial S}, z\rangle_{0;\partial S} = 0, \quad \forall z \in \mathcal{F}. \tag{5.31}$$

In what follows we assume that (5.31) holds. The proofs of the unique solvability of (5.29) and (5.30) repeat those of Theorems 23 and 24 with the obvious changes, so we omit them.

We introduce the Poincaré-Steklov operator $\widehat{\mathcal{T}}$ by $\left[\widehat{\mathcal{T}}\widehat{F}, \widehat{\Psi}\right]_{0;\widehat{\Gamma}} = b_\Omega(u,v)$, where $\widehat{F}, \widehat{\Psi} \in H_{1/2}(\widehat{\Gamma})$ are arbitrary, u is a solution of (6.1) and $v \in H_1(\Omega)$ is any extension of $\widehat{\Psi}$ to Ω. Let $\mathcal{F}(\widehat{\Gamma})$ be the space of all $\widehat{Z} = \{Z, z\}$, $Z = \{z, 0\}$, where $z \in \mathcal{F}$ is arbitrary. We define the spaces

$$\begin{aligned} \mathcal{H}_{1/2}(\widehat{\Gamma}) &= \left\{\widehat{F} \in H_{1/2}(\widehat{\Gamma}) : \left[\widehat{F}, \widehat{Z}\right]_{0;\widehat{\Gamma}} = 0 \ \ \forall \widehat{Z} \in \mathcal{F}(\widehat{\Gamma})\right\}, \\ \mathcal{H}_{-1/2}(\widehat{\Gamma}) &= \left\{\widehat{G} \in H_{-1/2}(\widehat{\Gamma}) : \left[\widehat{G}, \widehat{Z}\right]_{0;\widehat{\Gamma}} = 0 \ \ \forall \widehat{Z} \in \mathcal{F}(\widehat{\Gamma})\right\}. \end{aligned}$$

Theorem 32 *(i) $\widehat{\mathcal{T}}$ is self-adjoint and continuous from $H_{1/2}(\widehat{\Gamma})$ to $H_{-1/2}(\widehat{\Gamma})$.*

(ii) The kernel of $\widehat{\mathcal{T}}$ coincides with $\mathcal{F}(\widehat{\Gamma})$.

(iii) The range of $\widehat{\mathcal{T}}$ coincides with $\mathcal{H}_{-1/2}(\widehat{\Gamma})$.

(iv) The restriction $\widehat{\mathcal{N}}$ of $\widehat{\mathcal{T}}$ from $H_{1/2}(\widehat{\Gamma})$ to $\mathcal{H}_{1/2}(\widehat{\Gamma})$ is a homeomorphism from $\mathcal{H}_{1/2}(\widehat{\Gamma})$ to $\mathcal{H}_{-1/2}(\widehat{\Gamma})$.

The proof of this theorem is identical to that of Theorem 27.

Let $\overset{\circ}{\mathcal{H}}_{-1/2}(\widehat{\Gamma})$ be the subspace of $\overset{\circ}{H}_{-1/2}(\Gamma_0)\times H_{-1/2}(\partial S)$ of all $\varphi = \{\varphi_0, \varphi_{\partial S}\}$ such that $\langle \varphi_0, z\rangle_{0;\Gamma_0} + \langle \varphi_{\partial S}, z\rangle_{0;\partial S} = 0$ for all $z \in \mathcal{F}$. $\widehat{\mathcal{H}}_{1/2}(\widehat{\Gamma})$ is the subspace of $H_{1/2}(\Gamma_0) \times H_{1/2}(\partial S)$ consisting of all $f = \{f^+, f_{\partial S}\}$ such that $\langle f^+, z\rangle_{0;\Gamma_0} + \langle f_{\partial S}, z\rangle_{0;\partial S} = 0$ for all $z \in \mathcal{F}$.

We define the single layer potential of density $\varphi \in \overset{\circ}{\mathcal{H}}_{-1/2}(\widehat{\Gamma})$ by

$$(V\varphi)(x) = (V_0\varphi_0)(x) + (V_{\partial S}\varphi_{\partial S})(x), \quad x \in \mathbb{R}^2,$$

where $V_0\varphi_0$ and $V_{\partial S}\varphi_{\partial S}$ are the single layer potentials defined on Γ_0 and ∂S, respectively. Let $\left\{\widehat{Z}^{(i)}\right\}_{i=1}^{3}$ be an $L^2(\widehat{\Gamma})$-orthonormal basis for $\mathcal{F}(\widehat{\Gamma})$, where $\widehat{Z}^{(i)} = \left\{Z^{(i)}, z^{(i)}\right\}$ and $Z^{(i)} = \{z^{(i)}, 0\}$. The rigid displacements $z^{(i)}$ satisfy (5.29) with boundary data $F = Z^{(i)}$, $f_{\partial S} = z^{(i)}$. We introduce the modified single layer potential

$$(\mathcal{V}\varphi)(x) = (V\varphi)(x) - \left[\left\langle (V\varphi)_0\,, z^{(i)}\right\rangle_{0;\Gamma_0} + \left\langle (V\varphi)_{\partial S}\,, z^{(i)}\right\rangle_{0;\partial S}\right] z^{(i)}(x), \quad x \in \mathbb{R}^2,$$

where $(V\varphi)_0$ and $(V\varphi)_{\partial S}$ are the restrictions of $V\varphi$ to Γ_0 and ∂S. The corresponding boundary operator $\mathcal{V}_{\widehat{\Gamma}}$ is defined by $\mathcal{V}_{\widehat{\Gamma}}\varphi = \left\{\gamma_0^+ (\mathcal{V}\varphi)_+\,, \gamma_{\partial S}^+ (\mathcal{V}\varphi)_\Omega\right\}$, where $(\mathcal{V}\varphi)_\pm$ are the restrictions of $\mathcal{V}\varphi$ to $\Omega^\pm$. We also introduce a boundary operator $\widehat{\mathcal{V}}$ by writing $\widehat{\mathcal{V}}\varphi = \left\{\gamma_0^+ (\mathcal{V}\varphi)_+\,, 0, \gamma_{\partial S}^+ (\mathcal{V}\varphi)_\Omega\right\}$.

Theorem 33 *$\mathcal{V}_{\widehat{\Gamma}}$ is a homeomorphism from $\overset{\circ}{\mathcal{H}}_{-1/2}(\widehat{\Gamma})$ to $\widehat{\mathcal{H}}_{1/2}(\widehat{\Gamma})$.*

Proof. From the properties of the single layer potential (Section 4.2) it follows that $(\mathcal{V}\varphi)_\Omega \in H_1(\Omega)$, $(\mathcal{V}\varphi)_{S^-} \in H_{1,\omega}(S^-)$, and

$$\|(\mathcal{V}\varphi)_\Omega\|^2_{1;\Omega} \leqslant c b_\Omega((\mathcal{V}\varphi)_\Omega\,, (\mathcal{V}\varphi)_\Omega). \tag{5.32}$$

Here $(\mathcal{V}\varphi)_{S^-}$ is the restriction of $\mathcal{V}\varphi$ to S^-. The properties of the Poincaré-Steklov operator $\mathcal{T}^-$ are given in Section 4.1. For any $\varphi = \{\varphi_0, \varphi_{\partial S}\} \in \overset{\circ}{\mathcal{H}}_{-1/2}(\widehat{\Gamma})$, the jump formula for normal boundary stresses and couple stresses can be written as

$$\varphi_0 = \left(\widehat{\mathcal{N}}\widehat{\mathcal{V}}\varphi\right)_1, \qquad \varphi_{\partial S} = \left(\widehat{\mathcal{N}}\widehat{\mathcal{V}}\varphi\right)_3 - \mathcal{T}^-\left(\mathcal{V}_{\widehat{\Gamma}}\varphi\right)_2, \tag{5.33}$$

where $\left(\widehat{\mathcal{N}}\widehat{\mathcal{V}}\varphi\right)_i$ are the components of $\widehat{\mathcal{N}}\widehat{\mathcal{V}}\varphi$ and $\left(\mathcal{V}_{\widehat{\Gamma}}\varphi\right)_\alpha$ are the components of $\mathcal{V}_{\widehat{\Gamma}}\varphi$. From (5.33) it follows that

$$\begin{aligned} & b_\Omega((\mathcal{V}\varphi)_\Omega\,, (\mathcal{V}\varphi)_\Omega) + b_{S^-}((\mathcal{V}\varphi)_{S^-}\,, (\mathcal{V}\varphi)_{S^-}) \\ = \;& \left\langle \left(\mathcal{V}_{\widehat{\Gamma}}\varphi\right)_1, \varphi_0\right\rangle_{0;\Gamma_0} + \left\langle \left(\mathcal{V}_{\widehat{\Gamma}}\varphi\right)_2, \varphi_{\partial S}\right\rangle_{0;\partial S}. \end{aligned} \tag{5.34}$$

We claim that $\mathcal{V}_{\widehat{\Gamma}}$ is continuous. Let $\varphi = \{\varphi_0, \varphi_{\partial S}\} \in \overset{\circ}{\mathcal{H}}_{-1/2}(\widehat{\Gamma})$. By the trace theorem [56], $\left\|\mathcal{V}_{\widehat{\Gamma}}\varphi\right\|_{H_{1/2}(\Gamma_0)\times H_{1/2}(\partial S)} \leqslant c\,\|(\mathcal{V}\varphi)_\Omega\|_{1;\Omega}$. By (5.32) and (5.34),

$$\begin{aligned} \left\|\mathcal{V}_{\widehat{\Gamma}}\varphi\right\|^2_{H_{1/2}(\Gamma_0)\times H_{1/2}(\partial S)} &\leqslant c\left[\left\langle \left(\mathcal{V}_{\widehat{\Gamma}}\varphi\right)_1, \varphi_0\right\rangle_{0;\Gamma_0} + \left\langle \left(\mathcal{V}_{\widehat{\Gamma}}\varphi\right)_2, \varphi_{\partial S}\right\rangle_{0;\partial S}\right] \\ &\leqslant c\left\|\mathcal{V}_{\widehat{\Gamma}}\varphi\right\|_{H_{1/2}(\Gamma_0)\times H_{1/2}(\partial S)} \|\varphi\|_{\overset{\circ}{H}_{-1/2}(\Gamma_0)\times H_{-1/2}(\partial S)}; \end{aligned}$$

consequently, $\left\|\mathcal{V}_{\widehat{\Gamma}}\varphi\right\|_{H_{1/2}(\Gamma_0)\times H_{1/2}(\partial S)} \leqslant c\,\|\varphi\|_{\overset{\circ}{H}_{-1/2}(\Gamma_0)\times H_{-1/2}(\partial S)}$, which proves the continuity of $\mathcal{V}_{\widehat{\Gamma}}$.

If $\mathcal{V}_{\widehat{\Gamma}}\varphi = 0$, then $\widehat{\mathcal{V}}\varphi = 0$ also, and (5.33) gives that $\varphi = 0$; therefore, $\mathcal{V}_{\widehat{\Gamma}}$ is injective. By (5.33) and Theorem 32,

$$\|\varphi\|_{\overset{\circ}{H}_{-1/2}(\Gamma_0)\times H_{-1/2}(\partial S)} \leqslant c \left\|\mathcal{V}_{\widehat{\Gamma}}\varphi\right\|_{H_{1/2}(\Gamma_0)\times H_{1/2}(\partial S)},$$

which means that $\mathcal{V}_{\widehat{\Gamma}}^{-1}$ is continuous.

To complete the proof, it suffices to show that the range of $\mathcal{V}_{\widehat{\Gamma}}$ is $\widehat{\mathcal{H}}_{1/2}(\widehat{\Gamma})$. Let $f = \{f^+, f_{\partial S}\} \in \widehat{\mathcal{H}}_{1/2}(\widehat{\Gamma})$ and $F = \{f^+, 0\} \in H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$, and let $\widehat{F} = \{F, f_{\partial S}\} \in H_{1/2}(\widehat{\Gamma})$. We denote by $u_\Omega \in H_1(\Omega)$ the solution of (5.29) and by $u_{S^-} \in H_{1,\omega}(S^-)$ the solution of problem

$$b_{S^-}(u_{S^-}, v_{S^-}) = 0 \quad \forall v_{S^-} \in \overset{\circ}{H}_{1,\omega}(S^-), \quad \gamma_{S^-}^- u_{S^-} = f_{\partial S}.$$

Let $\widehat{\mathcal{T}}\widehat{\mathcal{V}} = \widehat{G} = \{\delta g, g^-, g_{\partial S}^+\}$, and let $\mathcal{T}^- f_{\partial S} = g_{\partial S}^-$. We take $\varphi_0 = \delta g$, $\varphi_{\partial S} = g_{\partial S}^+ - g_{\partial S}^-$ and $\varphi = \{\varphi_0, \varphi_{\partial S}\}$, and write $w_\Omega = u_\Omega - (\mathcal{V}\varphi)_\Omega \in H_1(\Omega)$ and $w_{S^-} = u_{S^-} - (\mathcal{V}\varphi)_{S^-} \in H_{1,\omega}(S^-)$. Then

$$\begin{aligned}
\gamma_0 w_\Omega &= \{f^+ - \gamma_0^+(\mathcal{V}\varphi)_+, 0\},\\
\gamma_{\partial S}^+ w_\Omega &= f_{\partial S} - \gamma_{\partial S}^+(\mathcal{V}\varphi)_\Omega,\\
\gamma_{\partial S}^- w_{\partial S} &= f_{\partial S} - \gamma_{\partial S}^-(\mathcal{V}\varphi)_{S^-} = f_{\partial S} - \gamma_{\partial S}^+(\mathcal{V}\varphi)_\Omega.
\end{aligned}$$

From the jump formulae and the definition of φ it follows that

$$\begin{aligned}
&b_\Omega(w_\Omega, w_\Omega) + b_{S^-}(w_{S^-}, w_{S^-})\\
=\ &\left\langle g_{\partial S}^+ - \left(\widehat{\mathcal{N}}\widehat{\mathcal{V}}\varphi\right)_3, f_{\partial S} - \gamma_{\partial S}^+(\mathcal{V}\varphi)_\Omega\right\rangle_{0;\partial S}\\
&- \left\langle g_{\partial S}^- - \mathcal{T}^-\left(\mathcal{V}_{\widehat{\Gamma}}\varphi\right)_2, f_{\partial S} - \gamma_{\partial S}^-(\mathcal{V}\varphi)_{S^-}\right\rangle_{0;\partial S}\\
=\ &\left\langle g_{\partial S}^+ - g_{\partial S}^- - \varphi_{\partial S}, f_{\partial S} - \gamma_{\partial S}^+(\mathcal{V}\varphi)_\Omega\right\rangle_{0;\partial S} = 0;
\end{aligned}$$

hence, $\widehat{\Psi} = \{f^+ - \gamma_0^+(\mathcal{V}\varphi)_+, 0, f_{\partial S} - \gamma_{\partial S}^+(\mathcal{V}\varphi)_\Omega\} \in \mathcal{F}(\widehat{\Gamma})$. Since$\widehat{\Psi} \in \mathcal{H}_{1/2}(\widehat{\Gamma})$, we conclude that $\widehat{\Psi} = 0$, which completes the proof. ∎

Let $\mathcal{H}_{1/2}(\partial S)$ be the subspace of $H_{1/2}(\partial S)$ consisting of all f such that $\langle f, z\rangle_{0;\partial S} = 0$ for all $z \in \mathcal{F}$. $\mathcal{H}_{-1/2}(\partial S)$ is the subspace of $H_{-1/2}(\partial S)$ of all g such that $\langle g, z\rangle_{0;\partial S} = 0$ for all $z \in \mathcal{F}$.

We define the double layer potential of density $\psi = \{\psi_0, \psi_{\partial S}\} \in \overset{\circ}{H}_{1/2}(\Gamma_0) \times \mathcal{H}_{1/2}(\partial S)$ by

$$(W\psi)(x) = (W_0\psi_0)(x) + (W_{\partial S}\psi_{\partial S})(x), \qquad x \in \Omega, x \in S^-,$$

where $W_0\psi_0$ and $W_{\partial S}\psi_{\partial S}$ are the double layer potentials defined on Γ_0 and ∂S, respectively. We introduce the modified double layer potential

$$\begin{aligned}(\mathcal{W}\psi)(x) &= (W\psi)(x) - \left[\left\langle (W\psi)_0^+, z^{(i)}\right\rangle_{0;\Gamma_0} + \left\langle (W\psi)_{\partial S}^+, z^{(i)}\right\rangle_{0;\partial S}\right] z^{(i)}(x), \\ x &\in \Omega, \ \ x \in S^-,\end{aligned}$$

where $(W\psi)_0^+$ and $(W\psi)_{\partial S}^+$ are the limiting values of $W\psi$ on Γ_0 and ∂S from within Ω^+ and S^+. We also define the limiting values $\mathcal{W}^\pm$ of the modified double layer potential on Γ from within $\Omega^\pm$, by writing $\mathcal{W}^\pm\psi = \gamma^\pm\pi^\pm\mathcal{W}\psi$. The corresponding boundary operator $\widehat{\mathcal{W}}\psi = \left\{\pi_0\left(\mathcal{W}^+\psi\right), \pi_0(\mathcal{W}^+\psi - \mathcal{W}^-\psi), \gamma_{\partial S}^+\left(\mathcal{W}\psi\right)_\Omega\right\} = \left\{\gamma_0^+\pi^+\left(\mathcal{W}\psi\right), -\psi_0, \gamma_{\partial S}^+\left(\mathcal{W}\psi\right)_\Omega\right\}.$

Let $\widehat{\mathcal{G}} = \widehat{\mathcal{N}}\widehat{\mathcal{W}}$. From the jump formula for the normal boundary stresses and couple stresses of the double layer potential it follows that the first component of $\widehat{\mathcal{G}}\psi$ is zero for any $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0) \times \mathcal{H}_{1/2}(\partial S)$; therefore, we can write $\widehat{\mathcal{G}}\psi = \{0, \left(\widehat{\mathcal{G}}\psi\right)^-, \left(\widehat{\mathcal{G}}\psi\right)_{\partial S}\}$ for all $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0) \times \mathcal{H}_{1/2}(\partial S)$. We also define boundary operator $\mathcal{G}_{\widehat{\Gamma}}\psi = \{\left(\widehat{\mathcal{G}}\psi\right)^-, \left(\widehat{\mathcal{G}}\psi\right)_{\partial S}\}$ from $\overset{\circ}{H}_{1/2}(\Gamma_0) \times \mathcal{H}_{1/2}(\partial S)$ to $H_{-1/2}(\Gamma_0) \times$

$\mathcal{H}_{-1/2}(\partial S)$.

Theorem 34 *$\mathcal{G}_{\widehat{\Gamma}}$ is a homeomorphism from $\overset{\circ}{H}_{1/2}(\Gamma_0)\times\mathcal{H}_{1/2}(\partial S)$ to $H_{-1/2}(\Gamma_0)\times$ $\mathcal{H}_{-1/2}(\partial S)$.*

Proof. From the properties of the double layer potential (Section 4.2) it follows that $(\mathcal{W}\psi)_\Omega \in H_1(\Omega)$, $(\mathcal{W}\psi)_{S^-} \in H_{1,\omega}(S^-)$, and

$$\|(\mathcal{W}\psi)_\Omega\|^2_{1;\Omega} \leqslant c b_\Omega((\mathcal{W}\psi)_\Omega, (\mathcal{W}\psi)_\Omega). \tag{5.35}$$

Here $(\mathcal{W}\psi)_{S^-}$ is the restriction of $\mathcal{W}\psi$ to S^-. For any $\psi = \{\psi_0, \psi_{\partial S}\} \in \overset{\circ}{H}_{1/2}(\Gamma_0)\times\mathcal{H}_{1/2}(\partial S)$, the jump formula for double layer potential can be written as

$$\psi_0 = -\left(\widehat{\mathcal{W}}\psi\right)_2, \quad \psi_{\partial S} = -\left(\left(\widehat{\mathcal{W}}\psi\right)_3 - \gamma^-_{\partial S}(\mathcal{W}\psi)_{S^-}\right), \tag{5.36}$$

where $\left(\widehat{\mathcal{W}}\psi\right)_i$ are the components of $\widehat{\mathcal{W}}\psi$. From (5.36) it follows that

$$\begin{aligned} & b_\Omega((\mathcal{W}\psi)_\Omega, (\mathcal{W}\psi)_\Omega) + b_{S^-}((\mathcal{W}\psi)_{S^-}, (\mathcal{W}\psi)_{S^-}) \qquad (5.37)\\ = & -\left\langle (\mathcal{G}_{\widehat{\Gamma}}\psi)_1, \psi_0\right\rangle_{0;\Gamma_0} - \left\langle (\mathcal{G}_{\widehat{\Gamma}}\psi)_2, \psi_{\partial S}\right\rangle_{0;\partial S}. \end{aligned}$$

We claim that $\mathcal{G}_{\widehat{\Gamma}}$ is continuous. Let $\psi = \{\psi_0, \psi_{\partial S}\} \in \overset{\circ}{H}_{1/2}(\Gamma_0)\times\mathcal{H}_{1/2}(\partial S)$. By Theorem 32, (5.35) and (5.37),

$$\begin{aligned} \left\|\mathcal{G}_{\widehat{\Gamma}}\psi\right\|^2_{H_{-1/2}(\Gamma_0)\times H_{-1/2}(\partial S)} &\leqslant c\left\|\widehat{\mathcal{W}}\psi\right\|^2_{H_{-1/2}(\widehat{\Gamma})} \leqslant c\,\|(\mathcal{W}\psi)_\Omega\|^2_{1;\Omega}\\ &\leqslant c\left\{b_\Omega((\mathcal{W}\psi)_\Omega, (\mathcal{W}\psi)_\Omega) + b_{S^-}((\mathcal{W}\psi)_{S^-}, (\mathcal{W}\psi)_{S^-})\right\}\\ &\leqslant c\left\{\left|\left\langle (\mathcal{G}_{\widehat{\Gamma}}\psi)_1, \psi_0\right\rangle_{0;\Gamma_0}\right| + \left|\left\langle (\mathcal{G}_{\widehat{\Gamma}}\psi)_2, \psi_{\partial S}\right\rangle_{0;\partial S}\right|\right\}\\ &\leqslant c\left\|\mathcal{G}_{\widehat{\Gamma}}\psi\right\|_{H_{-1/2}(\Gamma_0)\times H_{1/2}(\partial S)}\|\psi\|_{\overset{\circ}{H}_{1/2}(\Gamma_0)\times H_{1/2}(\partial S)}; \end{aligned}$$

consequently, $\left\|\mathcal{G}_{\widehat{\Gamma}}\psi\right\|_{H_{-1/2}(\Gamma_0)\times H_{1/2}(\partial S)} \leqslant c\,\|\psi\|_{\overset{\circ}{H}_{1/2}(\Gamma_0)\times H_{1/2}(\partial S)}$, which proves the continuity of $\mathcal{G}_{\widehat{\Gamma}}$.

If $\mathcal{G}_{\widehat{\Gamma}}\psi = 0$, then from the properties of $\widehat{\mathcal{N}}$, $\widehat{\mathcal{W}}\psi = 0$ also. Noting that $\mathcal{T}^{-}\gamma_{\partial S}^{-}\left(\mathcal{W}\psi\right)_{S^-} = \left(\mathcal{G}_{\widehat{\Gamma}}\psi\right)_2 = 0$, from (5.36) we obtain that $\psi = 0$; therefore, $\mathcal{G}_{\widehat{\Gamma}}$ is injective. By (5.36) and Theorem 32,

$$\|\psi\|_{\overset{\circ}{H}_{1/2}(\Gamma_0)\times H_{1/2}(\partial S)} \leqslant c\left\|\mathcal{G}_{\widehat{\Gamma}}\psi\right\|_{H_{-1/2}(\Gamma_0)\times H_{1/2}(\partial S)},$$

which means that $\mathcal{G}_{\widehat{\Gamma}}^{-1}$ is continuous.

To complete the proof, it suffices to verify that the range of $\mathcal{G}_{\widehat{\Gamma}}$ is dense in $H_{-1/2}(\Gamma_0) \times \mathcal{H}_{-1/2}(\partial S)$. If the range of $\mathcal{G}_{\widehat{\Gamma}}$ is not dense in $H_{-1/2}(\Gamma_0) \times \mathcal{H}_{-1/2}(\partial S)$, then there is a nonzero ψ in the dual $\overset{\circ}{H}_{1/2}(\Gamma_0)\times\mathcal{H}_{1/2}(\partial S)$ such that $\left\langle\psi_0, \left(\mathcal{G}_{\widehat{\Gamma}}\beta\right)_1\right\rangle_{0;\Gamma_0} + \left\langle\psi_{\partial S}, \left(\mathcal{G}_{\widehat{\Gamma}}\beta\right)_2\right\rangle_{0;\partial S} = 0$ for all $\beta \in \overset{\circ}{H}_{1/2}(\Gamma_0) \times \mathcal{H}_{1/2}(\partial S)$. We take $\beta = \psi$ and obtain $\left\langle\psi_0, \left(\mathcal{G}_{\widehat{\Gamma}}\psi\right)_1\right\rangle_{0;\Gamma_0} + \left\langle\psi_{\partial S}, \left(\mathcal{G}_{\widehat{\Gamma}}\psi\right)_2\right\rangle_{0;\partial S} = \left[\psi, \widehat{\mathcal{G}}\psi\right]_{0;\widehat{\Gamma}} = 0$, which means that $\widehat{\mathcal{W}}\psi \in \mathcal{F}(\widehat{\Gamma})$; hence, $\psi_0 = -\left(\pi_0\mathcal{W}^+\psi - \pi_0\mathcal{W}^-\psi\right) = 0$, $\psi_{\partial S} = -\left(\gamma_{\partial S}^{+}\left(\mathcal{W}\psi\right)_\Omega - \gamma_{\partial S}^{-}\left(\mathcal{W}\psi\right)_{S^-}\right) = 0$. This contradiction completes the proof. ■

We represent the solution of (5.29) in the form

$$u = (\mathcal{V}\varphi)_\Omega + W_0\psi + z, \tag{5.38}$$

where $\varphi \in \overset{\circ}{\mathcal{H}}_{-1/2}(\widehat{\Gamma})$, $W_0\psi$ is the double layer potential of density $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0)$,

$$z = \left[\left\langle f^+ + \gamma_0^+(W_0\delta f)_+, z^{(i)}\right\rangle_{0;\Gamma} + \left\langle f_{\partial S} + \gamma_{\partial S}^+(W_0\delta f)_\Omega, z^{(i)}\right\rangle_{0;\partial S}\right] z^{(i)}, \tag{5.39}$$

and $(W_0\delta f)_+$ and $(W_0\delta f)_\Omega$ are the restrictions of $W_0\delta f$ to Ω^+ and Ω. This representation yields the system of boundary equations

$$\widehat{\mathcal{V}}\varphi + \left\{\gamma_0^+(W_0\psi)_+, -\psi, \gamma_{\partial S}^+(W_0\psi)_\Omega\right\} = \widehat{F} - \{z, 0, z\}. \tag{5.40}$$

Theorem 35 *For any $\widehat{F} \in H_{1/2}(\widehat{\Gamma})$, system (5.40) has a unique solution $\{\varphi, \psi\} \in \mathring{\mathcal{H}}_{-1/2}(\widehat{\Gamma}) \times \mathring{H}_{1/2}(\Gamma_0)$, which satisfies the estimate*

$$\| \{\varphi, \psi\} \|_{\mathring{H}_{-1/2}(\Gamma_0)\times H_{-1/2}(\partial S)\times \mathring{H}_{1/2}(\Gamma_0)} \leqslant c \| \widehat{F} \|_{H_{1/2}(\widehat{\Gamma})} .$$

In this case, u defined by (5.38) is a solution of (5.29).

Proof. We take $\psi = -\delta f$ and reduce (5.40) to the system

$$\mathcal{V}_{\widehat{\Gamma}}\varphi = \{f^+, f_{\partial S}\} + \left\{\gamma_0^+(W_0\delta f)_+, \gamma_{\partial S}^+(W_0\delta f)_\Omega\right\} - \{z, z\}. \tag{5.41}$$

By (5.39), the right-hand side in (5.41) belongs to $\widehat{\mathcal{H}}_{1/2}(\widehat{\Gamma})$. The assertion now follows from Theorem 33. ■

We represent the solution of (5.30) in the form

$$u = \mathcal{V}_{\Gamma_0}\varphi + (\mathcal{W}\psi)_\Omega + z, \tag{5.42}$$

where $\mathcal{V}_{\Gamma_0}\varphi$ is the modified single layer potential of density$\varphi \in \mathring{\mathcal{H}}_{-1/2}(\Gamma_0)$, φ and $\psi \in \mathring{H}_{1/2}(\Gamma_0) \times \mathcal{H}_{1/2}(\partial S)$ are unknown densities, and $z \in \mathcal{F}$ is arbitrary. This representation yields the system of boundary equations

$$\widehat{\mathcal{N}}\left\{\gamma_0^+(\mathcal{V}_{\Gamma_0}\varphi), 0, \gamma_{\partial S}^+(\mathcal{V}_{\Gamma_0}\varphi)_\Omega\right\} + \widehat{\mathcal{G}}\psi = \widehat{G}. \tag{5.43}$$

Theorem 36 *For any $\widehat{G} \in H_{-1/2}(\widehat{\Gamma})$, system (5.43) has a unique solution $\{\varphi, \psi\} \in \mathring{\mathcal{H}}_{-1/2}(\Gamma_0) \times \mathring{H}_{1/2}(\Gamma_0) \times \mathcal{H}_{1/2}(\partial S)$, which satisfies the estimate*

$$\| \{\varphi, \psi\} \|_{\mathring{H}_{-1/2}(\Gamma_0)\times \mathring{H}_{1/2}(\Gamma_0)\times H_{1/2}(\partial S)} \leqslant c \| \widehat{G} \|_{H_{-1/2}(\widehat{\Gamma})} .$$

In this case, u defined by (5.42) is a solution of (5.30)

Proof. From the jump formula for normal boundary stresses and couple stresses of the single layer potential (Theorem 1) the first component of

$$\widehat{\mathcal{N}}\left\{\gamma_0^+(\mathcal{V}_{\Gamma_0}\varphi), 0, \gamma_{\partial S}^+(\mathcal{V}_{\Gamma_0}\varphi)_\Omega\right\}$$

is equal to φ. Comparing the first components on the both sides of (5.43) we see that $\varphi = \delta g$. The assertion now follows from Theorems 32, 33 and 34. ■

Remark 37 *In this chapter we have assumed that* Γ *and* ∂S *are* C^2*-curves. It can be shown that all the above results remain valid for piecewise-smooth* $C^{0,1}$*-curves that consist of finitely many* C^2*-arcs [82].*

5.6 Summary

In this chapter we have formulated Dirichlet and Neumann boundary value problems for a domain weakened by a crack in Sobolev spaces and showed that these problems are well-posed and depend continuously (in a suitable Sobolev-type norm) on the data. This result is important for practical purposes, since it validates further applications of numerical procedures. We have also shown that the corresponding weak solutions can be represented in terms of modified integral potentials with unknown distributional densities, which facilitate the construction of appropriate boundary element methods for finding these distributional densities and solving the problem numerically.

Chapter 6

Example: Stress Distribution Around a Crack in a Human Bone

In Chapter 5 we have performed the rigorous analysis of Neumann boundary value problem for a domain weakened by a crack in Cosserat continuum and constructed the corresponding solution in the form of modified integral potentials with unknown distributional densities. Unfortunately, it is impossible to find these densities analytically, consequently, we have to find a numerical technique which will allow us to obtain a numerical approximation of the solution. One of the most effective approaches to achieve this goal is to use the boundary element method. This method has been developed by Brebbia [83] and has become very popular among researches in different areas including fracture mechanics (see, for example, [84] for references on applications of the boundary element method in science and engineering).

The boundary element method has originated from works on classical integral equations and finite elements and incorporates advantages of both techniques. On one hand, it allows to reduce the dimension of a problem by one and defines domains extending to infinity with a high degree of accuracy exactly as the boundary integral equation method. On the other hand, the boundary element method does not require the differentiation of shape functions, which is the major requirement of the finite element method when we have to find stresses, but allows us to differentiate the matrix of fundamental solutions instead, which makes the calculation of stresses easier and more accurate.

In this chapter we use the boundary element method to find the solution for an infinite domain weakened by a crack in Cosserat continuum, when stresses and couple stresses are prescribed along both sides of the crack (Neumann boundary value problem), and discuss its convergence. To illustrated the effectiveness of the method for applications we consider a crack in a human bone which is modelled under assumptions of plane micropolar elasticity. We find the numerical solution for stresses around the crack and show that the solution may be reduced to the classical one if we set all micropolar elastic constants equal to zero. We come to the conclusion that there is up to a 26 percent difference in quantitative characteristics of the stress around a crack in the micropolar case in comparison with the model when microstructure is ignored (classical case

[85]).

6.1 Boundary element method

Consider problem (5.16). In Theorem 30, we have shown that the solution to problem (5.16) can be represented in the form (5.26), i.e. $u = (\mathcal{V}\varphi)_\Omega + \mathcal{W}\psi + z$, and the corresponding boundary integral equations are uniquely solvable with respect to distributional densities φ and ψ. As we stated above, these densities cannot be found analytically. To approximate them numerically we use the boundary element method [86] which makes use of the following classical result.

Lemma 38 *(Somigliana formula) Using classical techniques, we can prove that if $u \in H_{1,\omega}(\Omega)$ is a solution of $Lu = 0$ in Ω, then*

$$\int_{\Gamma_0} \left[D(x,y)\delta\left(T(\partial_y)u(y)\right) - P(x,y)\delta u(y)\right]\, ds(y) = \frac{1}{2}\delta u(x), \quad x \in \Gamma_0, \tag{6.1}$$

where $\delta\left(T(\partial_y)u(y)\right)$ denotes the jump of $T(\partial_y)u(y)$ on the crack.

From Theorem 31, the density of the modified single layer potential is $\varphi = \delta\left(T(\partial_y)u(y)\right) = \delta g$. We need to find the density of the modified double layer potential $\psi = -\delta u$. To achieve this goal we divide Γ_0 into n elements $\Gamma_0^{(k)}$, each of which possesses one node $\xi^{(k)}$ located in the middle of the element. The values of δg and δu are constant throughout the element and correspond to the values at the node $\delta g(\xi^{(k)})$ and $\delta u(\xi^{(k)})$. Then (6.1) becomes

$$\sum_{k=1}^{n} \int_{\Gamma_0^{(k)}} \left[D(x,y)\delta g(\xi^{(k)}) - P(x,y)\delta u(\xi^{(k)})\right]\, ds(y) = \frac{1}{2}\delta u(x), \quad x \in \Gamma_0.$$

Placing x sequentially at all nodes, we obtain the linear algebraic system of equations

$$\begin{aligned} & \sum_{k=1}^{n} \left(\int_{\Gamma_0^{(k)}} D(\xi^{(i)}, y)\ ds(y) \right) \delta g(\xi^{(k)}) \\ & - \sum_{k=1}^{n} \left(\int_{\Gamma_0^{(k)}} P(\xi^{(i)}, y)\ ds(y) \right) \delta u(\xi^{(k)}) \\ = \ & \frac{1}{2} \delta u(\xi^{(i)}), \qquad i, k = \overline{1, n} \end{aligned} \tag{6.2}$$

with respect to $\delta u(\xi^{(i)})$.

We note that $\int_{\Gamma_0^{(k)}} D(\xi^{(i)}, y)\ ds(y)$ is defined for any i and k [44].

Solving (6.2) we construct the approximation to ψ. If we introduce the shape function $\Phi_k(x)$ by

$$\Phi_k(x) = \begin{cases} 1, & x \in \Gamma_0^{(k)} \\ 0, & x \in \Gamma_0 \backslash \Gamma_0^{(k)} \end{cases}$$

then the approximated densities are $\varphi^{(n)}(x) = \sum_{k=1}^{n} \Phi_k(x) \delta g(\xi^{(k)})$ and $\psi^{(n)}(x) = -\sum_{k=1}^{n} \Phi_k(x) \delta u(\xi^{(k)})$ and the approximate solution is $u^{(n)} = (\mathcal{V}\varphi^{(n)})_\Omega + \mathcal{W}\psi^{(n)} + z$, where z is arbitrary. Now we have to prove that approximate numerical solution $u^{(n)}$ will converge to exact analytical solution u when $n \to \infty$.

Theorem 39 *$u^{(n)} \to u$ as $n \to \infty$.*

Proof. Since we consider Neumann problem, rigid displacement terms are not determined. Consequently, it is enough to show that $V\varphi^{(n)} \to V\varphi$ and

$W\psi^{(n)} \to W\psi$ as $n \to \infty$. Consider $V\varphi^{(n)}$. For $x \in \Omega$

$$\begin{aligned}
& \left| V\varphi(x) - V\varphi^{(n)}(x) \right| \\
\leqslant \; & \sum_{i=1}^{3} \left| \int_{\Gamma_0} D^{(i)}(x,y)\varphi(y)\, ds(y) - \sum_{k=1}^{n} \left(\int_{\Gamma_0^{(k)}} D^{(i)}(x,y)\, ds(y) \right) \varphi(\xi^{(k)}) \right| \\
= \; & \sum_{i=1}^{3} \left| \sum_{k=1}^{n} \int_{\Gamma_0^{(k)}} \left[D^{(i)}(x,y)\varphi(y) - D^{(i)}(x,y)\varphi(\xi^{(k)}) \right] \, ds(y) \right| \\
= \; & \sum_{i=1}^{3} \left| \sum_{k=1}^{n} \int_{\Gamma_0^{(k)}} D^{(i)}(x,y) \left[\varphi(y) - \varphi(\xi^{(k)}) \right] \, ds(y) \right| \\
\leqslant \; & \sum_{i=1}^{3} \sum_{k=1}^{n} \parallel D^{(i)}(x,\cdot) \parallel_{1,\omega;\Omega} \left| \varphi(y) - \varphi(\xi^{(k)}) \right| h_k
\end{aligned}$$

where h_k is the length of the kth element assuming the elements are all equal $h_k = h = \frac{L}{n}$, where L is the length of Γ_0, and $\left| \varphi(y) - \varphi(\xi^{(k)}) \right| \leqslant \sum_{i=1}^{3} \left| \varphi_i(y) - \varphi_i(\xi^{(k)}) \right| = \sum_{i=1}^{3} \sum_{\alpha=1}^{2} \left| \partial_\alpha \varphi_i(\xi^{(k)}) \right| h + O(h^2)$. Denote by $M_1 = \max_{\alpha=1,2;i=\overline{1,3};k=\overline{1,n}} \left| \partial_\alpha \varphi_i(\xi^{(k)}) \right|$. Since $\parallel D^{(i)}(x,\cdot) \parallel_{1,\omega;\Omega}$ are uniformly bounded [40], that is, there exists $M_2 > 0$ such that $\parallel D^{(i)}(x,\cdot) \parallel_{1,\omega;\Omega} \leqslant M_2$ for any $x \in \Omega$ we may write

$$\left| V\varphi(x) - V\varphi^{(n)}(x) \right| \leqslant \frac{18 M_1 M_2 L^2}{n} \to 0 \text{ as } n \to \infty.$$

Repeating the proof for $W\psi^{(n)}$, we conclude that $u^{(n)} \to u$. ■

6.2 Example

As an example, we consider a longitudinal crack inside a bone in the case when constant normal stretching pressure of magnitude p is applied on both sides of the crack. If we consider a typical transversal cross-section of the bone and assume that this cross-section is small enough then the deformation of each cross-section under the prescribed load will be the same throughout the length of the bone and will develop in the plane of the cross-section. Consequently, such deformations may be considered under assumptions of plane micropolar elasticity. Such a model is not an idealization that lies far from reality, as it may seem first, but, as shown, for example, in [65-67], can describe actual cracks in bones very closely, since orthopedic biomechanics usually deals with cracks of a very small size.

We model a crack as an open arc of the circle given by equations: $x_1 = a\cos\theta$, $x_2 = a\sin\theta$, $\theta \in (0, \pi/6)$ (Figure 6.1). By changing the radius a of the circle we will change the length of the crack. We are interested in how the normal traction distributes at a distance from the crack tip along the line: $x_1 = a$, $x_2 < 0$. Clearly, this problem can be considered as Neumann problem described above.

Elastic constants for a human bone have been measured in [63] and take the following values: $\alpha = 4000\ MPa$, $\gamma = 193.6\ N$, $\varepsilon = 3047\ N$, $\lambda = 5332\ GPa$, $\mu = 4000\ MPa$. In our example we construct solutions for cracks of lengths equal to 0.26 mm, 0.52 mm, 0.75 mm, and 10 mm to show good agreement of our results with the results presented in the experimental study by Lakes and Nakamura [75] performed on human bone cracks of same lengths. We also assume that normal stretching pressure p to take a value of 2 MPa.

Let the distance from the tip of the crack be $\rho = |x_2|$. The numerical solution for boundary tractions and moments is found to be accurate to exact solution to five decimal place for $n = 52$ elements of Γ_0 (see Table 6.1).

Distance ρ (mm)		0.1	0.5	0.7	1
	T_n *(MPa)*	0.983256	0.108643	0.082351	0.057282
$n = 4$	T_s *(MPa)*	1.175489	0.538926	0.213505	0.756437
	M_3 *(N/m)*	162.5682	91.24553	79.45362	70.64523
	T_n *(MPa)*	0.569361	0.068735	0.052678	0.032536
$n = 10$	T_s *(MPa)*	0.634936	0.264282	0.091475	0.045343
	M_3 *(N/m)*	84.24634	49.86301	44.26856	36.09357
	T_n *(MPa)*	0.427549	0.061862	0.039754	0.021830
$n = 30$	T_s *(MPa)*	0.506874	0.184756	0.075982	0.033547
	M_3 *(N/m)*	69.24764	34.25447	30.25879	24.62579
	T_n *(MPa)*	0.398462	0.053918	0.030028	0.015244
$n = 50$	T_s *(MPa)*	0.454906	0.108063	0.061039	0.026688
	M_3 *(N/m)*	61.11549	29.69127	22.59611	15.80458
	T_n *(MPa)*	0.398456	0.053910	0.030021	0.015239
$n = 52$	T_s *(MPa)*	0.454899	0.108054	0.061033	0.026678
	M_3 *(N/m)*	61.11542	29.69119	22.59604	15.80451

Table 6.1: Approximate solution for crack of 0.52 mm length: T_n – normal traction, T_s – tangential traction, M_3 – moment about x_3-axis

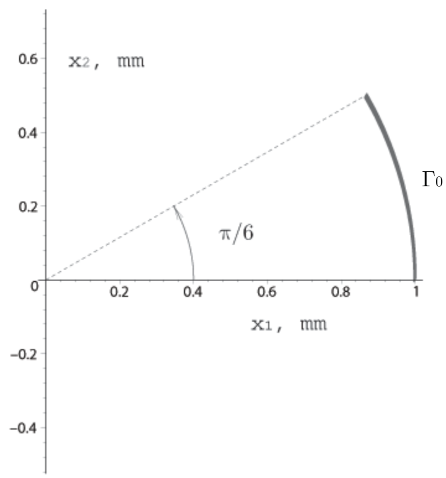

Figure 6.1: Crack in x_1x_2-plane. Length: 0.52 mm ($a = 1\ mm$)

Let us now compare the results for the normal traction in the micropolar case with the results of the classical theory for cracks of different lengths. This comparison is given in Tables 6.2-6.5 and Figures 6.2-6.5. In Figure 6.2-6.5 there is a graphical representation of the distribution for the normal traction at a distance from the lower crack tip for the cracks of lengths equal to 0.26 mm, 0.52 mm, 0.75 mm and 10 mm correspondingly. The bold curve characterizes the stress distribution in the micropolar case while the classical case is plotted by the normal curve. Tables 6.2-6.5 reflect values of the normal traction at representative points at a distance from the lower crack tip in both micropolar and classical cases. The distance between the first point, in which we measure the normal traction, and the tip of the crack is equal to the one tenth of the radius of the circle denoted by a. It can be observed that the normal traction is significantly higher in the vicinity of the crack tip in the micropolar case in comparison with the case when microstructure is ignored (classical theory). Depending on the crack length there can be up to a 26 % difference in the vicinity of the crack tip between the two cases. The negative sign of the difference in the value of the traction at representative points indicates that the value of the traction obtained under assumptions of classical elasticity is lower than that of the traction given by micropolar elasticity. As we move away from the tip of the crack the traction in the micropolar case decays faster than in the classical case so that approximately at a distance of one length of the crack the values of the normal traction become equal to each other. Further, the traction in the micropolar case becomes significantly lower than in the classical case and it may be observed that at a distance of approximately three crack lengths from the tip of the crack the effect of the crack on stresses is negligible according to Saint-Venant's principle.

Let $l_t = \sqrt{\frac{\gamma}{\mu}}$ and $l_b = \sqrt{\frac{\gamma+\varepsilon}{4\mu}}$ be the corresponding characteristic lengths of the micropolar material for torsion and bending [75]. If we take the values for elastic constants γ, ε, μ for a human bone as measured in [75], we obtain that l_t and l_b are equal to 0.22 mm and 0.45 mm in our case, respectively. Lakes and Nakamura [75] have shown experimentally that the difference in tractions between the micropolar case and the classical case is the largest when the crack length is comparable to the characteristic lengths.

As it may be seen from Tables 6.2-6.5 and Figures 6.2-6.5, the normal traction in the vicinity of the crack tip is significantly higher in comparison with that in the classical case for crack lengths of 0.26 mm, 0,.52 mm, 0.75 mm correspondingly. In the case when the length of the crack is equal to 0.75 mm we can observe that the normal traction in the vicinity of the crack tip is 26.8% higher under assumptions of Cosserat elasticity in comparison with the classical case. If the linear size of the crack is 10 mm, which is much longer than the corresponding characteristic lengths (see Table 6.5 and Figure 6.5) the tractions in the vicinity of the crack tip differ only by 13%. The fact that this difference is significantly lower than in the case of the shorter cracks may be explained by effect of the size of the crack opening, which is of the order of 10^{-5} for this crack in comparison with that of 10^{-6} in the other cases.

When it comes to the consideration of stresses at a distance from the crack tip, we can conclude that there is almost no difference in values of the tractions at the distance of one crack length from the tip and further for the crack which length is 10 mm for both micropolar and classical theories (Table 6.5 and Figure 6.5). However, for the crack lengths comparable to the characteristic lengths, this difference is still drastic and may be up to 19.8 % as in the case when the length of the crack is 0.26 mm (Table 6.2). At a distance from the crack tip stresses decay faster in the micropolar case.

The numbers presented in Tables 6.2-6.5 are in good agreement with experiments performed by Lakes and Nakamura [75]. However, they cannot be compared directly because in [75] a crack is considered as a squashed ellipse according to provisions of fracture mechanics. In our study, we represent a crack

Point (mm)	ρ (mm)	Micropolar T_n (MPa)	Classical T_n (MPa)	Difference (%)
(0.5,-0.05)	0.05	0.408419	0.331982	-18.7
(0.5,-0.15)	0.15	0.114308	0.111620	-2.35
(0.5,-0.25)	0.25	0.049057	0.058788	19.8
(0.5,-0.3)	0.3	0.034377	0.045874	33.4
(0.5,-0.35)	0.35	0.024762	0.036911	49

Table 6.2: Approximate solution for normal traction. Length of the crack: 0.26 mm ($a = 0.5$ mm)

by a piece of a plane curve so the shape of the crack is likely to have an effect on stress distribution in the vicinity and at a distance of the crack tip. Investigations relating to the effect of a crack shape on stress distribution is very challenging, complicated and, therefore, deserve to be performed in a separate work lying beyond the scope of this thesis.

Remark 40 *Recall that the traction on the boundary, hence, the corresponding displacements on the boundary depend on the outward normal to the boundary. When we move along one side of the crack, the normal is continuous at every point, consequently, the corresponding first derivatives of displacements on the crack sides are also continuous. However, when we change the sides of the crack, the normal has a jump, which generates a jump in the first derivatives of displacements. Therefore, the edges of the open crack have sharp corners.*

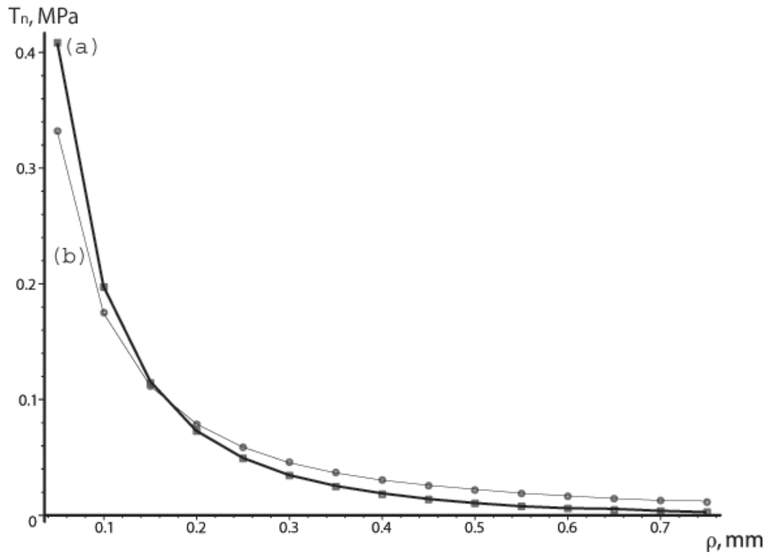

Figure 6.2: Normal traction on the edge of the crack. Length: 0.26 mm.

(a) – micropolar, (b) – classical

Point (mm)	ρ (mm)	Micropolar T_n (MPa)	Classical T_n (MPa)	Difference (%)
(1,-0.1)	0.1	0.398456	0.303602	-23.8
(1,-0.5)	0.5	0.053910	0.053702	-0.4
(1,-0.7)	0.7	0.030021	0.033719	12.3
(1,-0.8)	0.8	0.023434	0.027777	18.5
(1,-1)	1	0.015239	0.019865	30.3

Table 6.3: Approximate solution for normal traction. Length of the crack: 0.52 mm ($a = 1$ mm)

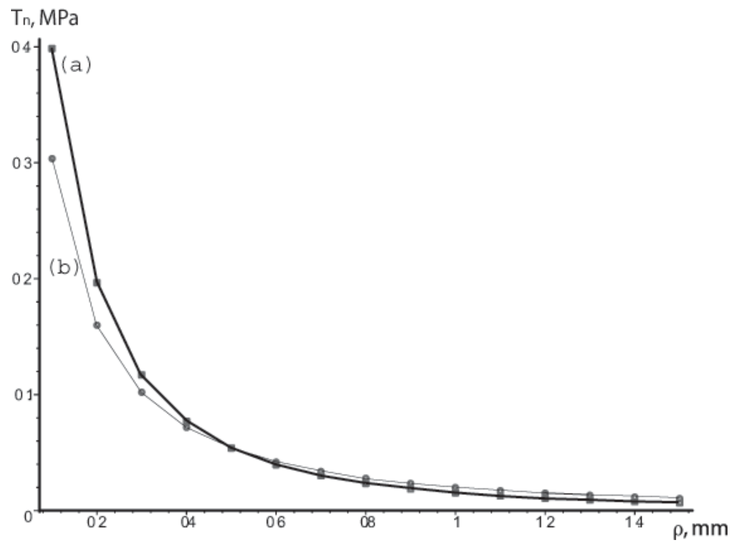

Figure 6.3: Normal traction on the edge of the crack. Length: 0.52 mm.

(a) – micropolar, (b) – classical

Point (mm)	ρ (mm)	Micropolar T_n (MPa)	Classical T_n (MPa)	Difference (%)
(1.42,-0.142)	0.142	0.394972	0.289178	-26.8
(1.42,-0.71)	0.71	0.056083	0.051118	-8.8
(1.42,-0.994)	0.994	0.032047	0.032097	0.1
(1.42,-1.136)	1.136	0.025338	0.026442	4.35
(1.42,-1.42)	1.42	0.016889	0.018912	12

Table 6.4: Approximate solution for normal traction. Length of the crack: 0.75 mm ($a = 1.42$ mm)

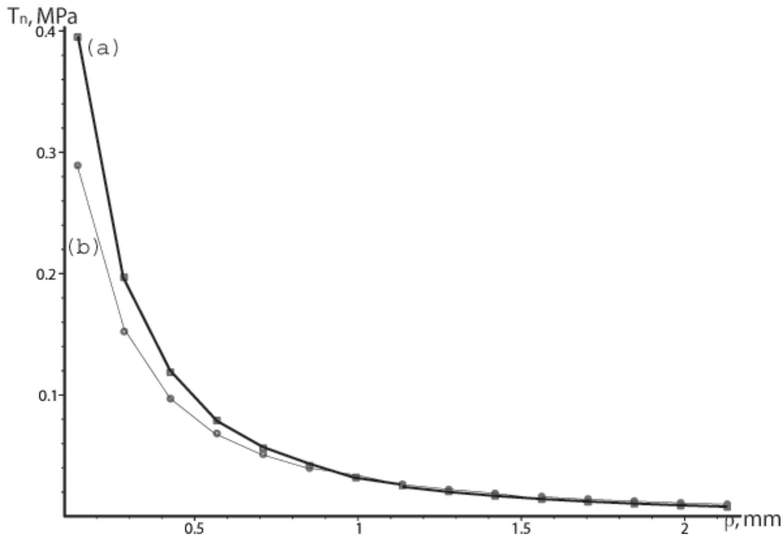

Figure 6.4: Normal traction on the edge of the crack. Length: 0.75 mm.

(a) – micropolar, (b) – classical

Point (mm)	ρ (mm)	Micropolar T_n (MPa)	Classical T_n (MPa)	Difference (%)
(20,-2)	2	0.208185	0.180806	-13
(20,-10)	10	0.032258	0.031698	-1.7
(20,-14)	14	0.019928	0.019907	-0.1
(20,-16)	16	0.016324	0.016405	0.5
(20,-20)	20	0.011579	0.011744	1.4

Table 6.5: Approximate solution for normal traction. Length of the crack: 10 mm ($a = 20\ \ mm$)

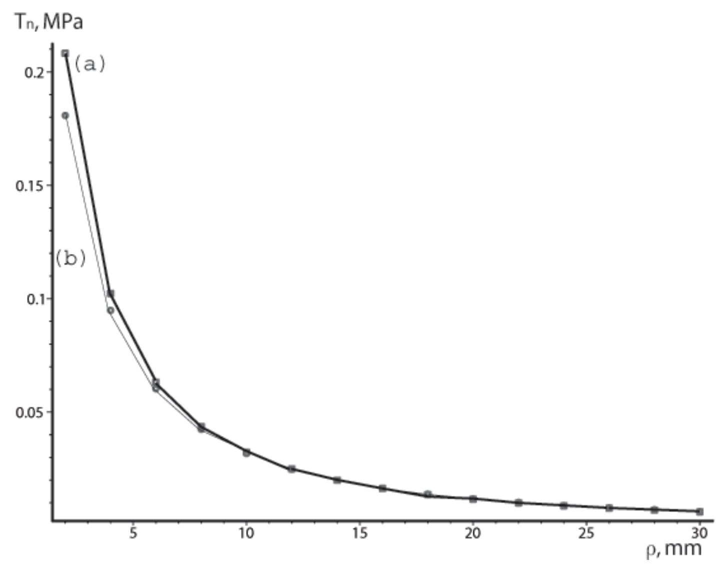

Figure 6.5: Normal traction on the edge of the crack. Length: 10 mm.

(a) – micropolar, (b) – classical

6.3 Summary

In this chapter we have illustrated the method introduced in Chapter 5 using the example of a crack in a human bone. We came to the conclusion that material microstructure does have a significant effect on the stress distribution around a crack in a human bone. The effect of material microstructure depends on the length of the crack and discovers the strongest influence in the vicinity of the crack tip. Results obtained in this chapter using the boundary element method are consistent with those obtained experimentally in [4-5] and by the finite element method in [75].

Chapter 7

Conclusions and Recommendations for Future Work

7.1 Conclusions

The present work has been devoted to investigation of the boundary value problems of plane Cosserat elasticity. In spite of the fact that the corresponding three-dimensional, plane and anti-plane problems have been carefully investigated for domains with smooth boundaries, the rigorous treatment of boundary value problems for domains with irregular boundaries remained practically untouched until today. At the same time investigations of boundary value problems for domains with irregular boundaries play a significant role in the analysis since they allow us to construct solutions for stresses and displacements in the vicinity of cracks and notches, which is very useful for applications in various fields of engineering science.

This dissertation is confined to the consideration of the statical problems of plane micropolar elasticity with emphasis on the crack problem. As a result of this work the following results have been obtained:

1. The interior and exterior Dirichlet and Neumann boundary value problems of plane micropolar elasticity have been formulated in Sobolev spaces, shown to be well-posed and rigorously solved by means of the boundary integral equation method. The uniqueness and existence theorems have been established and the exact analytical solutions have been obtained in the form of the corresponding integral potentials with distributional densities. Similar results have also been obtained for an infinite domain weakened by a crack.

2. As an example intended to demonstrate an important application of the proposed theory, the problem of a crack in a human bone has been considered under assumptions of micropolar elasticity. An efficient numerical scheme, based on the boundary element method has been developed, using which we have constructed the approximate solution allowing us to make the following important conclusions valuable for applications:

a. The normal traction in the vicinity of the crack tip is higher under assumptions of the Cosserat elasticity than in the classical case. The results can differ by up to 26 percent. This points to strong evidence that the material microstructure does have a significant effect especially in the case of domains with reduced boundary smoothness.

b. The difference between results of two theories (micropolar and classical) vary when we consider cracks of different sizes. For smaller cracks the difference in the value of stresses is greater but for larger cracks it is almost negligible. The fact that there can be up to 26 percent difference in normal traction between the micropolar and classical case and that this difference depends on the size of a crack is consistent with the results obtained by Savin [87] and Mindlin [15] for stress-concentration around holes in micropolar media, by Weitsman [88] and Hartranft [89] for stress concentration around inclusions in micropolar media and by Potapenko [46] for values of the warping function in the problem of torsion of cylindrical beams with significant microstructure. In addition, our results confirm the data relating to the stress distribution around a crack in Cosserat continuum obtained experimentally by Lakes and Nakamura [75].

7.2 Future Work

In this dissertation the main boundary value problems of plane micropolar elasticity for domains with irregular boundaries or domains containing cracks have been rigorously solved by means of the boundary integral equation method in the weak (Sobolev space) setting. In order to demonstrate the importance of the work for applications in mechanics we have also illustrated the effectiveness of the theory on the example of a crack in a human bone. We have shown that the unknown density, the most important part of the analytical solution in the form of an integral potential can be successfully approximated using boundary element method since it cannot be found analytically.

The methodology, analytical and numerical technique introduced in this work can be extended for the solution of a wide class of problems of micropolar and classical elasticity dealing with structures, three-dimensional and two-dimensional boundary value problems.

First of all, as a direct continuation of the work presented in this dissertation, the boundary element method can be applied for the derivation of a numerical solution of interior and exterior boundary value problems of plane micropolar elasticity. In spite of the fact that we have obtained an analytical solution to both interior and exterior Dirichlet and Neumann problems in the form of integral potentials in this work, we found it necessary to postpone the direct derivation of the unknown density for future work. The reason is, that in this dissertation we wanted to make an emphasis more on applications of the technique to the crack problem.

Second, we can formulate mixed boundary value problems of plane micropolar elasticity, i.e. when we assume that we impose Dirichlet conditions for two displacements and Neumann condition for microrotation. This particular problem then can be extended to the consideration of the crack problem which is reduced to the problem of displacement discontinuity and has a number of engineering applications in geomechanics.

As a third step, it would be possible and at the same time important, to formulate boundary value problems of anti-plane micropolar elasticity in Sobolev spaces. It has been shown in [45] that the problem of torsion of cylindrical beams with significant microstructure, introduced as a generalization of the Saint-Venant assumptions for the case of a classical beam [90], may be reduced to the interior Neumann boundary value problem of anti-plane micropolar elasticity. The major boundary value problems of anti-plane Cosserat elasticity for domains with twice differentiable boundary have been formulated and solved by means of boundary integral equation method in [45]. However, as we can see from the current work, material microstructure has the strongest effect on overall body's deformation when the domain has some irregularities. After formulating boundary value problems of anti-plane micropolar elasticity in Sobolev spaces we can extend the result from [45] to the problem of torsion of micropolar beams of more complicated cross-sections (for example, rectangular or square cross-section).

Another direction of the future work could be an incorporation of thermoelastic components into the model. Kupradze [38] has formulated fundamental boundary value problems of three-dimensional thermoelasticity and shown that they can be solved in a rigorous manner using the boundary integral equation method. Same technique has been applied to the investigation of two-dimensional problems of micropolar thermoelasticity: plane in [91] and anti-plane in [48]. This theory can be used to formulate the problem of thermoelastic deformations in a weak setting which will allow us to consider domains of more general form. In addition, the method can be applied for analysis of thermoelastic deformations of plates and shells.

In any case, this is the objective of the future work, as a result of which, we will definitely obtain a deeper understanding of the effects of material microstructure on deformations, stress concentration factors and elastic behavior of granular materials that will be very helpful for applications in the area of structural mechanics and modern day advanced composite materials.

In general, the approach presented in this thesis, is very new and not traditional for classical mechanics. At the same time it is very elegant, effective and has some certain advantages over the classical technique which has been in use in mechanics and engineering so far. It allows to find analytical solutions for a whole class of problems which could be worked out using only approximate numerical methods, such as finite element method, before. The boundary element method employed in this thesis for the consideration of a crack problem in plane micropolar elasticity allows to obtain numerical solutions for a various class of problems based on the exact analytical one, which is certainly a huge advantage over the finite element method.

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