An Improved Convex Optimization Model for Two-Dimensional Facility Layout

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The facility layout design problem is a fundamental optimization problem encountered in many manufacturing and service organizations that was originally formulated in 1963 by Armour & Buffa. This thesis derives a convex programming model, IBIMODEL, that is designed to improve upon the ModCoAR model of Anjos & Vannelli for the facility layout problem with unequal areas. The purpose of IBIMODEL is to find 'good' initial locations for the departments that a second model then uses to produce a detailed solution to the facility layout problem. The proposed model has four ideas behind it: unlike ModCoAR, it does not improve the objective function as the departments start overlapping, it takes into account the aspect ratio requirements, it introduces a systematic approach to making parameter choices, and it uses a new second stage recently proposed by Luo, Anjos & Vannelli to obtain the actual facility layouts. In this way, IBIMODEL efficiently generates a reasonably diverse set of superior solutions that allow the second stage to provide a wide variety of layouts with relatively low aspect ratios and total cost.

The proposed methodology was implemented and numerical results are presented on well-known large layout problems from the literature. To demonstrate the potential of the combination of IBIMODEL with Luo, Anjos & Vannelli's model, our results are compared with the best layouts found to date for these well-known large facility layout problems. The results support the conclusion that the propose a methodology consistently produces competitive, and often improved, layouts for large instances when compared with other approaches in the literature.

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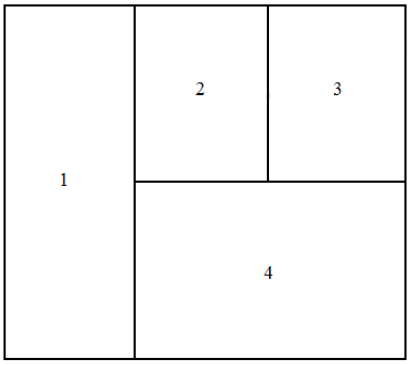
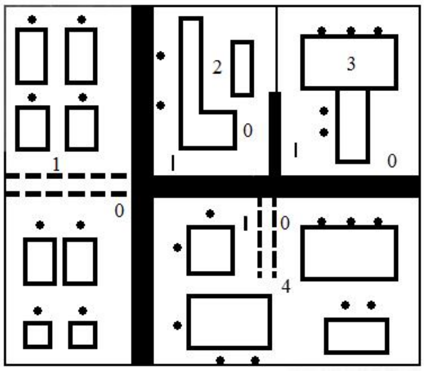
1 Introduction

The unequal-area facility layout problem (FLP) is concerned with finding the most efficient arrangement of m non-overlapping indivisible departments with unequal area requirements within a facility. The objective of the FLP is to minimize the total expected cost of flows inside the facility, where the cost incurred for each pair of departments is taken as the rectilinear distance between the centroids of the departments times the projected flow between them. The projected flow may reflect transportation costs, the construction of a material-handling system, the costs of laying communication wiring, or even adjacency preferences among departments. The problem contains two sets of constraints: department/floor area requirements and department location requirements (such as departments not overlapping, lying within the facility, and in some cases being fixed to a location, or being forbidden from specific regions). We assume that the facility and the departments are all rectangular. If the height and width of the departments can vary, then finding their optimal rectangular shapes is also part of the problem. The ratios height/width and width/height, called aspect ratios, also pose a challenge since square-like shaped departments with low aspect ratios are most practical in real-world applications, but this makes the problem harder. A solution to the FLP is a block layout that specifies the relative location and the dimensions of each department as illustrated in Figure 1(a). Once a block layout has been achieved, a detailed layout can be designed (Figure 1(b)) which specifies department locations, aisle structures and input/output (I/O) point locations (Kim & Kim 1999, Lee, Rohb & Jeong 2005, Takuya & Takashi 2006, Bock & Hoberg 2007). In this thesis, we will focus exclusively on block layout.

Facility layout design plays a significant role on manufacturing systems' effective utilization (Sule 1994, Tompkins, White, Bozer, Frazelle, Tanchoco & Trevino 1996), since a good solution for the facility layout problem contributes to the overall efficiency of operations. As stated in Chiang & Chiang (1998), a poor layout can lead to the accumulation of work-in-



(a) Block Layout



(b) Detailed Layout

Figure 1: Facility layout examples
(Figures taken from Meller & Gau (1996))

process inventory, overloading of material handling systems, inefficient set-ups, and longer queues. Furthermore, a facility layout represents a costly, long-term investment, since future large modification expenditures to the manufacturing system, if required, may not be easy to carry out. Modifying the layout of facilities is not only time-consuming but also disrupts worker activities and the flow of materials (Sule 1994). Other design environments where FLPs occur include; hospital, school and airport layouts (Singh & Sharma 2006), process plant layout (Patsiatzis & Papageorgiou 2002), piping design problems (Castillo, Westerlund, Emet & Westerlund 2005), VLSI macrocell design (Luo, Anjos & Vannelli n.d.), newspaper layout design (Yasdi 2000) and cutting/packing in various industries (Jakobs 1996, Liu & Teng 1999). All of these problems are very difficult, and are known to be NP-hard (Sahni & Gonzalez 1976), in particular because of the large number of possible physical layouts, and the existence of many locally (but not globally) optimal layouts.

The primary contribution of this thesis is the IBIMODEL that provides a block layout. IBIMODEL is a mathematical programming model that is designed to improve upon Anjos & Vannelli's (2006) ModCoAR model for the purpose of finding 'good' initial points for the detailed layout model of the second stage in a two-stage solution process. In Section 2, background on the facility layout problem is given. In Section 3, the IBIMODEL is motivated and derived, and Luo et al.'s (n.d.) second stage that takes these 'good' initial points from IBIMODEL and generates the final facility layout, is described. Computational results demonstrating the potential of the IBIMODEL model when used in conjunction with Luo et al.'s (n.d.) second stage model are presented in Section 4. Finally, possible directions for future research are discussed in Section 5.

2 Background

In the 1970s and 1980s, two major approaches were taken for modeling the FLP: the quadratic assignment problem (QAP) approach and the graph-theoretical approach. In the 1990s, two other approaches evolved based on mixed-integer linear programming and non-linear programming. Different solution methodologies have also been developed over the years to solve the different models, including exact, heuristic and metaheuristic algorithms.

2.1 Modeling Approaches

2.1.1 Quadratic Assignment Problem Model

The QAP approach was introduced by Koopmans & Beckman (1957) to model the problem of locating interacting plants of equal areas. The QAP is a special case of the facility layout problem because it assumes that all departments have equal shapes and all possible locations are fixed and known a priori. The QAP formulation assigns every department to one location and at most one department to a particular location. The cost of placing a department at a particular location is dependent on the location of the other departments' interacting with it. Such dependency leads to a quadratic objective that inspires the problem's name. Many researchers have addressed the importance of the QAP and its relevance to the equal area FLP (see e.g. Bazaraa 1975, Kusiak & Heragu 1987, Francis, McGinnis & White 1992).

Typically departments have unequal areas, and the QAP formulation for the unequal-area FLP is less appealing than for the equal-area FLP. Solution techniques for the QAP are based on variations of the branch-and-bound approaches initially proposed by Gilmore (1962) and Lawler (1963). Armour & Buffa (1963) applied a pair-wise exchange method to solve the unequal-area FLP, while others like Smith & Tate (1992) approached the problem with heuristic procedures. (Bazaraa 1975, Hassan, Hogg & Smith 1986, Kusiak & Heragu 1987) modeled the unequal-area facility layout problem as a special QAP by breaking the departments into small squares with equal areas, assigning a large artificial flow between the squares arising from the same department to ensure that they are not placed far apart, and solving the resulting QAP. However Bozer & Meller (1997) show that since these artificial flows are much larger than the real flows, the cost of 'travel' within departments will dominate the cost of travel between departments and negatively affect the quality of the solutions obtained. For instance, regardless of the flows among departments that were supplied by the analyst, each department is predisposed to assume a certain shape. In other words, high artificial flows implicitly add department shape constraints (Bozer & Meller 1997). Due to the discrete representation of the floor area, the layout frequently contains irregularly shaped departments that are not feasible in practice. There have been attempts made to place restrictions on the department shapes (Hassan et al. 1986, Kusiak & Heragu 1987). We refer the reader to Anstreicher (2003) for recent advances in QAP-solutions, description of new algorithms and computational structures used, and to Loiola, Maia de Abreu, Boaventura-Netto, Hahn & Querido (2007) for a thorough up-to-date review of the QAP.

The QAP has drawn a lot of attention in the past fifty years due to its complexity. Garey & Johnson (1979) showed that the QAP is NP-hard and NP-complete. To date, globally optimal solutions to general cases of the problem can only be obtained for problems with up to 36 departments. The 36-department problem Ste36a (Steinberg 1961) was solved to optimality by Brixius & Anstreicher (2001) and the Nugent 30 problem (Nugent, Vollman & Ruml 1968), one of the most well-known and challenging instances, was solved to optimality by Anstreicher & Brixius (2001).

2.1.2 Graph Theory Model

Graph-theoretical approaches assume that the preferences of any pair of departments to be placed adjacently is known. The area and shape of the departments are at first ignored, and each department is represented by a node in a graph. Satisfied department adjacency relationships are represented by an arc connecting the two adjacent departments in the graph (Hassan & Hogg 1987). The objective is to maximize the weight on the adjacencies (arcs) between department pairs (nodes). One of the most formative contributions to solving FLP with graph theory is Foulds & Robinson's (1978) model. Their method develops a family of interconnected planar graphs in a way that planarity is not violated and the need to constantly test for planarity is removed. Subsequent to Foulds & Robinson's (1978) contribution, most of the research has focused upon improving the selection and placement of vertices into such a graph. Chan & Milner (1982) developed a methodology that modified the means of vertex placement, Eades, Foulds & Griffin (1982) developed a wheel expansion algorithm based upon the deltahedron methodology of Foulds & Robinson (1978), and Al-Hakim (1991) developed two edge-elimination and edge-replacement algorithms for the purpose of improving the solution quality. Boswell (1992) developed TESSA, a heuristic that focuses on adding faces instead of single vertices at each iteration. (A face is a closed path in a graph that has no intersecting edges and no edge inside its enclosed region.) More thorough reviews of graph-theoretic approaches and heuristics can be found in Hassan & Hogg (1987) and Foulds (1991).

Unlike for the QAP approach, even unequal-area problems of small size (such as 10-department problems) cannot be solved to guaranteed optimality with graph-theoretic approaches. It is a time-consuming process that often requires trial and error. Moreover, the constructed layout is not guaranteed to satisfy the prescribed adjacencies (Heragu 1997).

2.1.3 Mixed-Integer Linear Programming Model

Recently, attention has shifted to formulating FLPs as mixed-integer linear programming (MILP) models, as first introduced by Montreuil (1990). The model uses a distance-based objective but is not based on the traditional discrete (QAP) framework because it utilizes a continuous representation of the layout. Meller & Gau (1996) separate the layout problem into two sub-problems: the block layout problem and the detailed layout problem. The block layout problem specifies sizes, shapes and relative locations of the departments within the facility, while the detailed layout problem addresses specific details such as exact department locations, aisle structures and input/output (I/O) points.

A block layout model was developed by Heragu & Kusiak (1991) where the department lengths, widths, and orientations are specified a priori. Many researchers have attempted to find 'good' feasible solutions for this model by heuristically fixing a subset of the binary variables and then optimizing the resulting linear program, see for example Montreuil, Ratliff & Goetschalckx (1987) and Langevin, Montreuil & Riopel (1996). Gau & Meller (1999) took Montreuil's (1990) model, redefined its binary variables and tightened the department area constraints. Meller, Narayanan & Vance (1999) also improved on Montreuil's (1990) model by approximating the non-linear area requirement for each department by an empirically derived relationship constraints (where area restrictions become more faithful as departments become more non-square) and strengthening the model with seven classes of valid inequalities to derive an augmented model formulation. Sherali, Fraticelli & Meller (2003) built upon Meller et al.'s (1999) model by further reducing problem symmetry, evaluating the performance of several classes of valid inequalities, exploring the construction of partial convex hull representations for the disjunctive constraints, and investigating judicious branching variable selection priority schemes. Sherali et al. (2003) achieved impressive results by solving a 9-department problem to global optimality. Castillo & Westerlund (2005) dealt with the nonconvex and hyperbolic area restrictions by devising a linear model that guarantees that, at optimality, the final area of each department is within an $\varepsilon\%$ error of the required area. Then Castillo et al. (2005) improved on Castillo & Westerlund's (2005) model by introducing a decision variable transformation and symmetric convex lower bound to gain an exact representation of the area restrictions, which together with a simple constraint avoids symmetric layout solutions. Others, such as Lacksonen (1994), proposed a two-step algorithm for the block layout problem that assumes departments can have varying areas and then modified their previous model in Lacksonen (1997) to deal with unequal areas and rearrangement costs.

Kim & Kim (1999) took the detailed layout problem and placed the input and output points of each department so that the total (shortest path) distance of material flows between the I/O points would be minimized. A branch and bound algorithm is developed to solve Kim & Kim's (1999) model, and find locally optimal solutions in a relatively short time even for large (20 and 30-department) problems. However, simultaneous solution of the block problem and the I/O points layout was not attempted in a single problem until Barbosa-Povoa, Mateus & Novais (2000). Then, in Barbosa-Povoa, Mateus & Novais (2001), different production sections and space, safety and operability restrictions are also addressed.

The mixed-integer linear programming approach is a promising technique that can solve larger problems than the QAP and graph theory models (Meller & Gau 1996). Currently only FLPs with up to 9 departments (Sherali et al. 2003) have been solved to global optimality.

2.1.4 Non-Linear Programming Model

Non-linear programming (NLP) models are typically nonconvex and, as a consequence they are not guaranteed to find global optimal solutions (Bazaraa, Sherali & Shetty 1993), but only local optima. Drezner (1980) uses a dispersion-concentration concept to formulate a nonconvex mathematical problem that he solves in two stages using the Lagrangian Differential Gradient Method. van Camp, Carter & Vannelli (1991) develop a three-stage penalty-based algorithm, called the nonlinear optimization layout technique (NLT). Then, Anjos & Vannelli (2002) proposed a convex optimization model, the CoAR model, based on an attractor-repeller concept. This model is solved with a nonlinear programming heuristic. Castillo & Sim (2004) base their model on Anjos & Vannelli's (2002) model, but change the objective function to reflect a spring-embedded (SE) concept. The SE concept is based on a system of particles with given areas that are mutually connected by strings of different strengths, and the system is allowed to oscillate until it stabilizes at a minimum-energy arrangement. The convexified Attractor-Repeller (CoAR) model is improved in Anjos & Vannelli (2006) by introducing generalized target distances that aim to account for the flows between departments. This thesis falls into this area of research, and therefore the models mentioned above are presented in greater detail in Section 3.

2.2 Solution Methodologies

2.2.1 Exact Algorithms

The branch-and-bound method was developed by Gilmore (1962). It is a popular method to solve QAP problems, as in White (1993) and Brixius & Anstreicher (2001), since the QAP model only involves binary variables. More recently (Lacksonen 1997, Sherali et al. 2003) have applied branch-and-bound to graph theory and MILP model formulations. Brixius & Anstreicher (2001) have solved problems with up to 32 departments (with the Gilmore-Lawler bound) to global optimality.

The cutting-plane method is another exact algorithm. It was developed by Bazaraa & Sherali (1982), and like the branch-and-bound algorithm, the cutting plane algorithm has a high time and storage complexity. The largest FLP solved by a cutting-plane algorithm was an 8-department layout problem (Kusiak & Heragu 1987).

2.2.2 Heuristic Algorithms

Heuristic algorithms can be classified as construction-type and improvement-type algorithms that are either adjacency-based or distance-based. Construction-type algorithms build a solution from scratch and are considered the simplest and oldest heuristic approaches (but provide less satisfactory solutions). Improvement-type algorithms take a feasible solution as an initial solution and try to improve it by performing interchanges of single assignments. The difference between adjacency-based and distance-based algorithms lies in the objective function. The objective function for an adjacency-based algorithm is defined as:

$$\max \sum_i \sum_j (r_{ij}) x_{ij}$$

where r_{ij} is the department closeness rating between department i to j and x_{ij} is 1 if department i is adjacent to department j, and 0 otherwise. Adjacency-based algorithms are usually used for graph-based models based on the principle that material handling costs are reduced if two departments are adjacent to one another (Meller & Gau 1996). On the other hand, the objective function for a distance-based algorithm is defined as:

$$\min \frac{1}{2} \sum_{i=1, i \neq k}^{n} \sum_{j=1, j \neq l}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} C_{ik} D_{jl} X_{ij} X_{kl}$$

where C_{ik} and D_{jl} stand for the material handling cost between departments i and k, and the distance between locations j and l, respectively. The variable X_{ij} is equal to 1 if department i is assigned to location j, or 0 otherwise. In the distance-based algorithm, the distance increases the total cost of traveling (Meller & Gau 1996).

Montreuil et al.'s (1987) MATCH is a construction-type adjacency-based algorithm that utilizes a discrete representation and uses integer programming to solve a matching problem. The algorithm attempts to maximize department adjacency scores that are based on adjacent segments. In an iterative approach MATCH generates departments that are rectangular in shape.

An example of a construction-type algorithm that is distance-based is SHAPE (Hassan et al. 1986). The algorithm uses department flows and user-defined critical flows to determine the department selection sequence as it begins placing departments at the center of the layout. Subsequent departments are placed on each of the layout's four slides, depending on their effect on the objective function value. Since the department shapes are controlled by the objective function, department shapes at the edges of the layout deteriorate.

LOGIC (Tam 1992*a*, Tam 1992*b*) is a distance-based improvement-type algorithm that uses a collection of rectangular partitions as the layout, and a slicing-tree to specify if departments on opposite sides of a branch are to the left, to the right, above, or below one another. The algorithm determines the layout by partitioning rectangular areas and placing the departments into the area specified by the branch operators (left, right, above or below). To avoid narrow department shapes, Tam introduces a penalty function to control department shapes. Tam (1992*a*) uses simulated annealing and Tam (1992*b*) uses a genetic algorithm to solve the resulting sub-problems.

Improvement-type algorithms that are distance-based include the commercial software CRAFT (Armour & Buffa 1963) and its improved algorithm, MULTIPLE (Bozer, Meller & Erlebacher 1994). Both CRAFT and MULTIPLE use pairwise interchanges, and are steepest-descent algorithms that converge to the first local minimum encountered. The performance of these algorithms is greatly affected by the given initial solution, and the department shapes obtained within the layout often have long thin edges, and hence could serve no practical purpose.

2.2.3 Metaheuristic Algorithms

Various metaheuristics have been employed to tackle the FLP and studies have demonstrated the efficacy and promise of such approaches (Tate & Smith 1995, Wu, Huang, Lau, Wong & Young 2002).

Simulated annealing (SA) is based on the analogy of annealing solids and was used for solving optimization problems by Burkard & Rendl (1984) for solving the QAP. The underlying dynamics of SA allow it to efficiently generate a reasonably diverse set of superior solutions, which is a critical factor for effective decision-making in layout design (Al-Hakim 2000). Promising results have been obtained by simulated annealing (Souliah 1995, Hopper & Turton 2001, Balakrishnan, Cheng & Wong 2003), and Singh & Sharma (2006) provide a survey of SA-based FLP papers.

Genetic algorithms (GA) have become fairly popular in the past decade. A genetic algorithm uses fixed-length strings of zeros and ones to encode layouts. In a parallel process, the GA searches for the global optimum from a small set of feasible solutions, generating the new solutions in some randomized fashion. Kochhar, Foster & Heragu (1998) and Balakrishnan et al. (2003) are some of the papers that use GA to solve FLP and Singh & Sharma (2006) provide a list of recent papers on GA-based FLP.

Tabu search (TS) is another metaheuristic that is actively researched and continues to evolve (Sporin-Kapov 1994, Helm & Hadley 2000), as are naive evolution (Hopper & Turton 2001) and neural networks (Tsuchiya, Bharitkar & Takefuji 1996).

2.3 Facility Layout Software Packages

Facility layout research has led to a variety of facility layout software packages. According to Sly (1995) and Heragu (1997), the leading packages are:

- *SPIRAL*, distributed by Marc Goetschalckx (Goetschalckx 1992), uses graph-theory-based algorithms to obtain block layouts.
- *VisFactory (or e-Factory)*, by CIMTECHNOLOGIE, includes multiple modules, namely, FactoryCAD, FactoryFLOW, FactoryPLAN/OPT, and FactoryVIEW, that incorporate a SPIRAL algorithm with some CRAFT-like improvement routines.

- *LayOPT*, by Production Modeling Corp., uses MULTIPLE.

- *Factory Modeler*, by Systems Espace Temps Inc., implements a MIP-based approach.

- *eM-Workplace*, by Tecnomatix Technologies Ltd., is a workplace/layout design and material flow simulation.

- *FACOPT* by (Balakrishnan et al. 2003) uses simulated annealing and a genetic algorithm.

Recently the systematic layout planning (SLP) approach has been applied in software packages to provide more practical layouts (Ferrari, Pareschi, Persona & Regattieri 2003). It consists of data collection, block layout planning, intra-block design and plant realization. The LRP (logistic and re-layout program) is a software product that uses SLP. It was developed in 1999 and focuses on continuous dual analysis of the flow of materials and activity relationships (Ferrari et al. 2003). Main features of LRP include human interactivity, modularity (with regards to data input and solution optimization) and lastly, compatibility and integration with CAD software.

2.4 Choice of Distance Measure

Choosing the criteria to be used to identify the 'best' solution from among several alternative solutions in the unequal-area FLP problem is not trivial. Most often the objective is to minimize the weighted distance traveled, with the weights representing the material flow, and therefore the Euclidean distance is appropriate to calculate the cost when it comes to overhead material handling systems, determining wiring between office units or in advanced flexible manufacturing systems. However, in office layouts that contain walls and corridors, in process plant layout or in piping design problems that incorporate piping, connection and other material handling costs, the rectilinear distance is more realistic. Cheng & Kuh (1984) and Weis & Mlynski (1987) have shown through computational experiments, and Blanks (1985) has shown probabilistically, that the Euclidean and rectilinear distances converge as the number of departments goes to infinity. In this thesis, we use the Euclidean distance in IBIMODEL, and the rectilinear distance in the second model.

3 Proposed Mathematical Model

3.1 Previous Nonlinear-Programming-Based Methods

In order to put this thesis's contribution into its technical context, some of the methods mentioned in Section 2 are reviewed in some detail, and related to the proposed new approach. The reviewed methods are the DISpersion-CONcentration (DISCON) method (Drezner 1980), the Non-linear optimization Layout Technique (NLT) (van Camp et al. 1991), the Attractor-Repeller (AR) (Anjos & Vannelli 2002), the Spring-Embedded (SE) (Castillo & Sim 2004), and MODified COnvexified Attractor-Repeller (ModCoAR)- Bilinear Penalty Layout (BPL) methods (Anjos & Vannelli 2006).

Throughout this thesis we label the departments $i = 1, ..., N$, where N is the total number number of departments, and each of these departments is approximated by a circle of radius r_i. The position of each department i is expressed by the coordinates of its center and is denoted by (x_i,y_i). It is assumed that the nonnegative costs c_{ij} per unit distance between departments i and j are given and are symmetric, i.e. $c_{ij}=c_{ji}$.

3.1.1 The DISCON Method

Drezner's (1980) DISCON method assumes that departments have circular shapes and arranges them so that they do not overlap and the Euclidean distance among them is at a local minimum. Furthermore, the model does not require that all departments be placed within a facility or that the layout area satisfy any dimensional requirements.

DISCON can be represented by the optimization problem:

(DISCON)

$$\begin{aligned} \text{Minimize} \quad & \sum_{1 \le i < j \le N} c_{ij} d_{ij} \\ \text{s.t.} \quad & d_{ij} \ge r_i + r_j, \qquad \text{for all } 1 \le i < j \le N, \end{aligned}$$

where r_{ij} is the radius of circle i, and $d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$ is the Euclidean distance between the centers of the circles. The objective function minimizes the total material handling cost inside the facility and the constraints require that the circles corresponding to each pair of departments do not overlap.

Conceptually, Drezner (1980) solves DISCON in two stages using the Lagrangian Differential Gradient Method, a penalty method for which the penalty factors are changed continuously. The first stage is the dispersion phase, where all the departments are dispersed far from the center of the layout area. Initially circles are placed very close to the origin where the minimum of the objective function is attained, but constraints are strongly violated. The constraint violations generate repulsion forces which in turn disperse the circles from the origin and expand the whole system. Drezner (1980) uses the following system of ordinary differential equations with free parameter t for the first stage of the method:

$$\begin{aligned} \frac{dx_i}{dt} &= 2\sum_{j=1}^{n} \frac{(u_{ij}-c_{ij})(x_i-x_j)}{d_{ij}} && \text{for } i = 1, \ldots, n \\ \frac{dy_i}{dt} &= 2\sum_{j=1}^{n} \frac{(u_{ij}-c_{ij})(y_i-y_j)}{d_{ij}} && \text{for } i = 1, \ldots, n \\ \frac{du_{ij}}{dt} &:= \begin{cases} \lambda^2(r_i + r_j - d_{ij}) & \text{if } d_{ij} \le r_i + r_j \quad \text{or} \quad u_j(t) > 0, \\ 0 & \text{otherwise,} \end{cases} && \text{for } 1 \le i < j \le n \end{aligned}$$

where the variable $u_{ij} \ge 0$ is the dual variable associated with the constraint and is proportional to the force of repulsion. If the solution to the above system converges over time and if x^0 and y^0 reach a steady value, then (x^0,y^0) satisfies the Karush-Kuhn-Tucker optimality conditions for the problem with u^0 as the Lagrange multiplier vector, and DISCON has been solved.

The first stage provides a good starting point for the second stage, since the circles are widely dispersed and are not touching. In the second stage, the concentration phase, all the departments are brought back together to converge to a local minimum. The second stage formulation introduces a new vector, v_{ij}, that is associated with u_{ij}, yielding the following system of ordinary differential equations:

$$
\begin{aligned}
&\tfrac{dx_i}{dt} = 2\sum_{j=1}^{n} \tfrac{(u_{ij}-c_{ij})(x_i-x_j)}{d_{ij}} && \text{for } i = 1, \ldots, n \\
&\tfrac{dy_i}{dt} = 2\sum_{j=1}^{n} \tfrac{(u_{ij}-c_{ij})(y_i-y_j)}{d_{ij}} && \text{for } i = 1, \ldots, n \\
&\tfrac{dv_{ij}}{dt} := \begin{cases} \lambda^2(r_i + r_j - d_{ij}) & \text{if } v_{ij} + \lambda^2(r_i + r_j - d_{ij}) > 0, \\ -v_{ij} & \text{otherwise,} \end{cases} && \text{for } 1 \le i < j \le n \\
&u_{ij} = v_{ij} + \tfrac{dv_{ij}}{dt} && \text{for } 1 \le i < j \le n
\end{aligned}
$$

where x, u and $v \geq 0$. The DISCON second stage local minimum provides the final solution for the problem, as illustrated in Figure 2 for a 30-department problem.

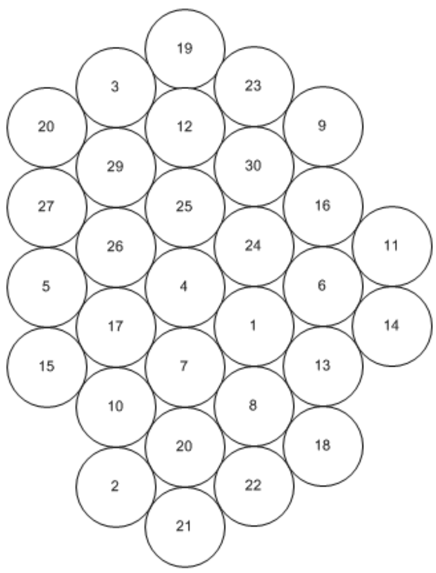


Figure 2: Typical layout for Drezner's DISCON model
(Figure taken from Drezner (1980))

3.1.2 The NLT Method

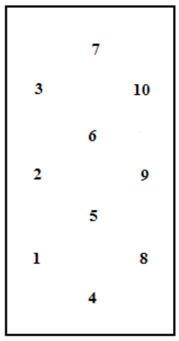
The NLT method (van Camp et al. 1991) uses a three-stage penalty-based algorithm to solve the facility layout problem. The first two stages are solved by a relaxation of the NLT model that approximates each department by a circle whose radius is proportional to the area of the department, while the third stage determines the final layout solution by solving the following optimization problem:

(NLT)

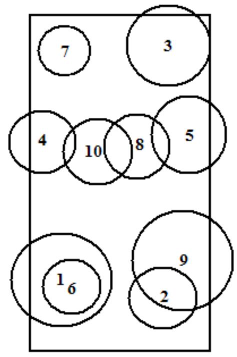
$$
\begin{array}{rlll}
\text{Minimize} & \sum_{1\le i<j\le N} c_{ij}d_{ij} & & \\
\text{s.t.} & |x_i - x_j| - \frac{1}{2}(w_i + w_j) \ge 0 & \text{if} & |y_i - y_j| - \frac{1}{2}(h_i + h_j) < 0 \\
 & |y_i - y_j| - \frac{1}{2}(h_i + h_j) \ge 0 & \text{if} & |x_i - x_j| - \frac{1}{2}(w_i + w_j) < 0 \\
 & \frac{1}{2}w_F - (x_i + \frac{1}{2}w_i) \ge 0 & \text{for} & i = 1, ..., N \\
 & (x_i - \frac{1}{2}w_i) + \frac{1}{2}w_F \ge 0 & \text{for} & i = 1, ..., N \\
 & \frac{1}{2}h_F - (y_i + \frac{1}{2}h_i) \ge 0 & \text{for} & i = 1, ..., N \\
 & (y_i - \frac{1}{2}h_i) + \frac{1}{2}h_F \ge 0 & \text{for} & i = 1, ..., N \\
 & \min(w_i, h_i) - l_i^{min} \ge 0 & \text{for} & i = 1, ..., N \\
 & l_i^{max} - \min(w_i, h_i) \ge 0 & \text{for} & i = 1, ..., N \\
 & \min(w_F, h_F) - l_F^{min} \ge 0 & & \\
 & l_F^{max} - \min(w_F, h_F) \ge 0 & &
\end{array}
$$

where (x_i,y_i) and d_{ij} are as previously defined; w_i, h_i are the width and height of department i; l_i^{min},l_i^{max} are the minimum and maximum allowable lengths for the sides of department i; w_F, h_F are the width and height of the facility; and l_F^{min},l_F^{max} are the minimum and maximum allowable lengths for the facility.

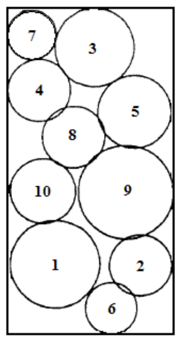
In the first stage, the model first evenly distributes the centers of the departments inside the facility. The algorithm calculates the number of evenly spaced rows and columns to place



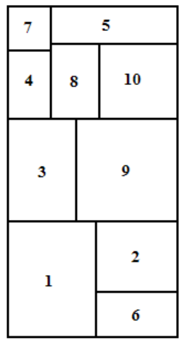
(a) Starting layout for Stage One



(b) Stage One solution



(c) Stage Two solution



(d) Stage Three solution

Figure 3: Layouts from van Camp et al. (1991) three-stage NLT model
(Figures taken from van Camp et al. (1991))

the departments in the plant layout. This 'ideal' even spread, shown in Figure 3 (a), serves as the starting point for the first stage model. Next, a set of constraints are enforced on the centers of the departments to ensure that the moments about the x- and y-axis of the computed solution correspond to an evenly spread solution. van Camp et al. (1991) define the mth moment about the axes of the ideal layout to be

$$M_m(k) = \sum_{i=1}^{N} x_i^{(m-k)} y_i^k, \qquad k = 0, \ldots, m$$

and require $\overline{M}_m(k)$, the mth moment of the actual layout being generated, to satisfy

$$\overline{M}_m(k) = M_m(k), \qquad \text{for } k = 0, \ldots, m \text{ and } m = 1, \ldots, N.$$

In particular, this constraint will ensure that the sum of the department centers to the right (or above) the origin will equal the sum of department centers to the left (or below) the origin. Therefore the model for the first stage is:

$$\text{Minimize} \sum_{1 \leq i < j \leq N} c_{ij} d_{ij}^2$$

$$\text{s.t. } \overline{M}_m(k) = M_m(k), \qquad \text{for } k = 0, \ldots, m \text{ and } m = 1, \ldots, N.$$

where the distance measure, d_{ij}, is squared to avoid discontinuities in the gradient of the Euclidean distance. (The gradient is undefined when $d_{ij} = 0$ which is when the departments completely overlap.)

First, Stage One is taken with the two lowest orders of moments being constrained and is run until the penalty cost (proportional to the square of a measure of the degree to which a constraint is violated) is less than one percent of the total cost. This solution is then used as the initial point to re-solve the Stage One model with the three lowest orders of moments being constrained. The resulting solution is once again considered as the initial point for Stage One, and it is re-solved with the four lowest orders of moments being constrained, and so on. The Stage One model is thus re-solved with appropriate initial points until all of the moments have been constrained. van Camp et al. (1991) illustrated their solution for the first stage in Figure 3(b).

In the second stage, the objective is to reduce the overlap among departments. The Stage Two model is:

$$
\begin{aligned}
\text{Minimize} \quad & \textstyle\sum_{1 \le i < j \le N} c_{ij} d_{ij} \\
\text{s.t.} \quad & d_{ij} - (r_i + r_j) \ge 0 \\
& \tfrac{1}{2} w_F - (x_i + r_i) \ge 0 && \text{for } i = 1, \ldots, N \text{ and } j = 1, \ldots, N \\
& \tfrac{1}{2} h_F - (y_i + r_i) \ge 0 && \text{for } i = 1, \ldots, N \text{ and } j = 1, \ldots, N \\
& (x_i - r_i) + \tfrac{1}{2} w_F \ge 0 && \text{for } i = 1, \ldots, N \\
& (y_i - r_i) + \tfrac{1}{2} h_F \ge 0 && \text{for } i = 1, \ldots, N \\
& \min(w_F, h_F) - l_F^{min} \ge 0 \\
& l_F^{max} - \min(w_F, h_F) \ge 0
\end{aligned}
$$

where r_i is the radius of the circle representing department i. This model ensures that no two departments overlap, all departments remain inside the facility, and the dimensions of the facility as a whole remain within acceptable limits. The solution of this problem, shown in Figure 3 (c), is found by solving the Stage Two model until the penalty cost (using a penalty in the objective function for every constraint that is violated) is less than one percent of the total function value.

Lastly in the third stage, the NLT model is solved using the solution from Stage Two as an initial solution. After each run, departmental interchanges are attempted to try to improve the layout. Figure 3 (d) is an example of a successful layout that the NLT model produced.

3.1.3 The AR Method

Anjos & Vannelli (2002) derived the Attractor-Repeller (AR) model to improve on van Camp et al.'s (1991) NLT method. The AR model replaces the first two stages of the NLT method, and finds a good initial point for NLT method's third stage.

The AR model is derived from nature's concept of attractor and repeller particles, as shown in Figure 4. An attractor element attracts particles according to set strength and range parameters. Each attractor can be set to attract any combination of particles. A repeller

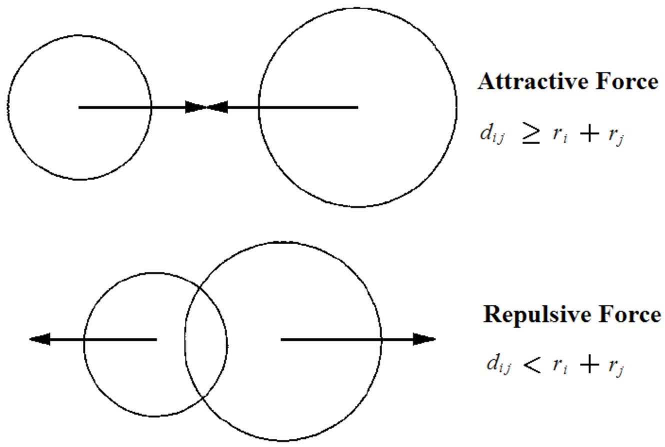


Figure 4: Repulsive and attractive forces on particles
(Figure taken from Anjos & Vannelli (2002))

element is identical to an attractor but has a negative strength parameter which makes it push particles away from each other. Anjos & Vannelli (2002) define the attractor component to be $\sum_{1\leq i<j\leq N} c_{ij}d_{ij}^2$ in the objective function. When the objective function is minimized, this attractor component will try to minimize distances by moving the circles closer together until they are on top of each other, at which point $d_{ij}^2 = 0$. Then, to prevent overlap, the AR model employs the concept of *target distances*. The target distance between any two circles i and j is denoted by $\sqrt{t_{ij}}$, where $t_{ij} = \alpha(r_i + r_j)^2$ and $\alpha > 0$. The parameter t_{ij} is the target value for D_{ij}, which is the square of the distance between each pair of circles: $D_{ij} := d_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2$. The α parameter provides flexibility as to how tightly the user wishes to enforce the constraint of circles non-overlapping. Hence, choosing $\alpha < 1$ allows a priori some overlap of the areas of the respective circles, while α=1 means that there should be no overlap and the circles should intersect at exactly one point on their boundaries. The target distance is enforced by penalizing the overlap with respect to the target distance by introducing the term $K \cdot f(\frac{D_{ij}}{t_{ij}})$ in the objective function, where $f(z) = \frac{1}{z} - 1,\ \ z > 0$ and $K = 10 \cdot (\sum_{i,j} c_{ij})$. Therefore if the target distance is reached right on, then $z = 1$ which makes the $f(z)$ term disappear. However, if the distance is a lot smaller then the target distance, then $f(z) \longrightarrow \infty$ since z becomes really small. Hence, Anjos & Vannelli (2002) define the repeller component to be $\sum_{1\leq i<j\leq N} Kf\left(\frac{D_{ij}}{t_{ij}}\right)$ in the objective function, which prevents circles from overlapping. The AR model is formulated as:

(AR)

$$
\begin{array}{rll}
\text{Minimize} & \sum_{1\leq i<j\leq N} c_{ij}D_{ij} + K\cdot f\left(\frac{D_{ij}}{t_{ij}}\right) & \\
\text{s.t.} & \frac{1}{2}w_f \geq x_i + r_i & \text{for } \ i=1,...,N \\
& \frac{1}{2}w_f \geq r_i - x_i & \text{for } \ i=1,...,N \\
& \frac{1}{2}h_f \geq y_i + r_i & \text{for } \ i=1,...,N \\
& \frac{1}{2}h_f \geq r_i - y_i & \text{for } \ i=1,...,N \\
& w_F^{max} \geq w_F \geq w_F^{min} & \\
& h_F^{max} \geq h_F \geq h_F^{min}. &
\end{array}
$$

The first four constraints require that all the circles be entirely contained within the facility, and the remaining two constraints bound the width and height of the facility. However, the AR model is not convex, and therefore the optimal solution is not necessarily unique. Anjos & Vannelli (2002) also modify the AR model's objective function to yield a convex problem, the convexified AR model, CoAR. The CoAR model formulation is:

(CoAR)

$$
\begin{array}{rll}
\text{Minimize} & \sum_{1\leq i<j\leq N} f_{ij}\left(x_i, x_j, y_i, y_j\right) & \\
\text{s.t.} & \frac{1}{2}w_f \geq x_i + r_i & \text{for } \ i=1,...,N \\
& \frac{1}{2}w_f \geq r_i - x_i & \text{for } \ i=1,...,N \\
& \frac{1}{2}h_f \geq y_i + r_i & \text{for } \ i=1,...,N \\
& \frac{1}{2}h_f \geq r_i - y_i & \text{for } \ i=1,...,N \\
& w_F^{max} \geq w_F \geq w_F^{min} & \\
& h_F^{max} \geq h_F \geq h_F^{min}. &
\end{array}
$$

where

$$
f_{ij}(x_i, x_j, y_i, y_j) := \begin{cases} c_{ij}D_{ij} + t_{ij}/D_{ij} - 1 & \text{if } D_{ij} > \sqrt{t_{ij}/c_{ij}} \\ 2\sqrt{c_{ij}t_{ij}} - 1 & \text{if } 0 \leq D_{ij} \leq \sqrt{t_{ij}/c_{ij}} \end{cases}
$$

and $c_{ij} > 0$, $t_{ij} > 0$ and $D_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2$. This way f_{ij} attains its minimum value whenever the positions of circles i and j satisfy $D_{ij} \leq \sqrt{t_{ij}/c_{ij}}$. Since whenever $D_{ij} < \sqrt{t_{ij}/c_{ij}}$ there will be overlap among circles, it is therefore desirable to have $D_{ij} \approx \sqrt{t_{ij}/c_{ij}}$. Another way of looking at this is that if $D_{ij} \approx T_{ij}$ then D_{ij} is proportional to both t_{ij} and $1/c_{ij}$ and D_{ij} is close to the geometric average of these two quantities. Note that when the c_{ij} is large, $1/c_{ij}$ is small, and therefore it is desirable for the circles to be near each other (and vice versa when the cost is small).

However, Anjos & Vannelli (2002) do not use the CoAR model because it requires a carefully designed line search procedure and the non-convex AR model with the scaling by K performs well in practice, in spite of lacking convexity.

3.1.4 The SE Method

Castillo & Sim (2004) introduced the SE Method which is based on relating the arrangement of circles within a facility to a dynamically balanced spring system. The strength of the spring between each pair of circles is represented by c_{ij}, and the force created in the spring is represented by the Euclidean distance between the centers of the circles.

The SE model is based on the AR model and introduces a new objective function. As in the AR model, the first four constraints require that all the circles be entirely contained within the facility, and the remaining two constraints bound the width and height of the facility. The SE model is formulated as:

(SE)

$$
\begin{aligned}
\text{Minimize} \quad & \textstyle\sum_{1\le i<j\le N} c_{ij} d_{ij}^2 + \max\{0, K_{ij}(r_i + r_j - d_{ij})\} & \\
\text{s.t.} \quad & \tfrac{1}{2} w_F \ge x_i + r_i & \text{for } i = 1, ..., N \\
& \tfrac{1}{2} w_F \ge r_i - x_i & \text{for } i = 1, ..., N \\
& \tfrac{1}{2} h_F \ge y_i + r_i & \text{for } i = 1, ..., N \\
& \tfrac{1}{2} h_F \ge r_i - y_i & \text{for } i = 1, ..., N \\
& w_F^{max} \ge w_F \ge w_F^{min} & \\
& h_F^{max} \ge h_F \ge h_F^{min} &
\end{aligned}
$$

where $K_{ij} > 0$, $1 \le i < j \le N$.

As in previously described methods, the $\sum_{1\le i<j\le N} c_{ij} d_{ij}^2$ component seeks to make the distances between departments as small as possible by attracting all pairs of circles to each other, and the penalty term representing the total energy of the springs is introduced to enforce non-overlapping. The penalty term assumes a non-negative value proportional to the magnitude of the area overlap of the circles and results in a repulsive force. If there is no overlap, the total energy function is not penalized, and only an attractive force remains. In summary, the objective function of the SE model represents the total energy function and minimizes the degree of imbalance inside a facility.

3.1.5 The ModCoAR-BPL Method

Anjos & Vannelli (2006) builds upon the convexified CoAR model (Anjos & Vannelli 2002) and the second stage of the NLT method (van Camp et al. 1991).

Anjos & Vannelli (2006) define a generalized target distance as:

$$T_{ij} := \sqrt{\frac{t_{ij}}{c_{ij} + \epsilon}}, \qquad 1 \le i < j \le N,$$

where $\epsilon > 0$ is a small enough number such that if $c_{ij} = 0$ then T_{ij} will not be undefined, but if $D_{ij} \approx T_{ij}$ then $D_{ij} \approx \sqrt{t_{ij}/c_{ij}}$. As with the target distance of the CoAR model, both the relative size of the departments and the connection cost between them are taken into account. However, the generalized target distance is defined even when $c_{ij} = 0$. Anjos & Vannelli (2006) also sacrifice the convexity of the model for computational practicality, by adding the term $-\ln(D_{ij}/T_{ij})$ to the objective function so that an appropriate algorithm will stop at a solution that is on or near the flat portion of the objective function but is farthest from the origin, i.e., where $D_{ij} \approx T_{ij}$. Hence the final model is formulated as:

(ModCoAR)

$$\begin{array}{lll}
\text{Minimize} & \sum_{1\le i<j\le N} F_{ij}(x_i, x_j, y_i, y_j) - K \, \ln\left(\frac{D_{ij}}{T_{ij}}\right) & \\
\text{s.t.} & \frac{1}{2} w_F \ge x_i + r_i & \\
& \frac{1}{2} w_F \ge r_i - x_i & \\
& \frac{1}{2} h_F \ge y_i + r_i & \\
& \frac{1}{2} h_F \ge r_i - y_i & \\
& w_F^{max} \ge w_F \ge w_F^{min} & \\
& h_F^{max} \ge h_F \ge h_F^{min} & \\
\text{where} & F_{ij}(x_i, x_j, y_i, y_j) := \begin{cases} c_{ij} D_{ij} + t_{ij}/D_{ij} - 1 & \text{if } D_{ij} > T_{ij} \\ 2\sqrt{c_{ij} t_{ij}} - 1 & \text{if } 0 \le D_{ij} \le T_{ij} \end{cases} &
\end{array}$$

where the constant K is a large penalty factor (typically $10 \sum c_{ij}$).

Next, Anjos & Vannelli (2006) consider the final stage of the method and reformulate the disjunctive nonoverlap constraints within the NLT model (van Camp et al. 1991) that are the most difficult to handle. By introducing two new complementary variables, X_{ij} and Y_{ij}, that satisfy $X_{ij} \ge \frac{1}{2}(w_i + w_j) - |x_i - x_j|$, $X_{ij} \ge 0$ and $Y_{ij} \ge \frac{1}{2}(h_i + h_j) - |y_i - y_j|$, $Y_{ij} \ge 0$, the nonoverlap constraints are equivalent to $X_{ij} Y_{ij} = 0$. Because of these last constraints, this model is a mathematical program with complementarity constraints (MPCC). To enforce the restriction that at any feasible point $X_{ij} = 0$ or $Y_{ij} = 0$ must hold, Anjos & Vannelli (2006) penalize the constraints in the objective to obtain the bilinear penalty layout model (BPL). Hence the formulation of the BPL model is as follows:

(BPL)

$$
\begin{array}{rll}
\text{Minimize} & \sum_{1\leq i<j\leq N} c_{ij}d_{ij} + KX_{ij}Y_{ij} & \\
\text{s.t.} & X_{ij} \geq \frac{1}{2}(w_i + w_j) - |x_i - x_j| & \text{for } 1 \leq i < j \leq N \\
& Y_{ij} \geq \frac{1}{2}(h_i + h_j) - |y_i - y_j| & \text{for } 1 \leq i < j \leq N \\
& X_{ij} \geq 0,\ \ Y_{ij} \geq 0,\ \text{ and } X_{ij}Y_{ij} = 0 & \text{for } 1 \leq i < j \leq N \\
& \frac{1}{2}w_F - (x_i + \frac{1}{2}w_i) \geq 0 & \text{for } i = 1, ..., N \\
& (x_i - \frac{1}{2}w_i) + \frac{1}{2}w_F \geq 0 & \text{for } i = 1, ..., N \\
& \frac{1}{2}h_F - (y_i + \frac{1}{2}h_i) \geq 0 & \text{for } i = 1, ..., N \\
& (y_i - \frac{1}{2}h_i) + \frac{1}{2}h_F \geq 0 & \text{for } i = 1, ..., N \\
& w_i h_i = a_i & \text{for } i = 1, ..., N \\
& \min(w_i, h_i) - l_i^{min} \geq 0 & \text{for } i = 1, ..., N \\
& l_i^{max} - \min(w_i, h_i) \geq 0 & \text{for } i = 1, ..., N \\
& \min(w_F, h_F) - l_F^{min} \geq 0 & \\
& l_F^{max} - \min(w_F, h_F) \geq 0 &
\end{array}
$$

where $K = \sum_{1\leq i<j\leq N} c_{ij}$.

As shown in Figure 5, Anjos & Vannelli (2006) solved the well-known 20-department problem of Armour & Buffa (1963) using their ModCOAR and BPL combination and obtained layouts that are up to 20% better that the results reported in Smith & Tate (1992).

3.2 The IBIMODEL

The contribution of this thesis is a model that is built upon Anjos & Vannelli's (2006) ModCoAR model. The first main idea behind the proposed model is that it does not improve

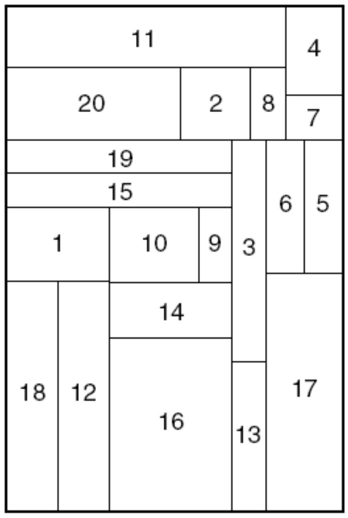


Figure 5: ModCoAR and BPL layout for Armour & Buffa's (1963) 20-department problem (Figure taken from Anjos & Vannelli (2006))

the objective function as the circles start overlapping and the distance between the circle centers becomes less than $r_i + r_j$. A second idea is to try to include some information about aspect ratios. Thirdly, a systematic approach to making parameter choices is introduced. Lastly, a new final stage recently proposed by Luo et al. (n.d.) is used to obtain the actual facility layouts. The net result is a methodology that consistently produces competitive, and often better layouts for large FLPs, when compared with other approaches in the literature.

3.2.1 Improved Objective Function

The means for not rewarding the objective function as the circles start overlapping is based on $r_i + r_j$ (the actual sum of radii), $\sqrt{\frac{t_{ij}}{c_{ij}+\epsilon}}$ (the generalized target distance from Anjos & Vannelli (2006)) and τ_{ij} (a new target distance). In IBIMODEL, t_{ij} is defined slightly differently than in the previously mentioned models. The α is dropped from t_{ij} and thus $t_{ij} := (r_i + r_j)^2$ is the actual sum of radii squared. Furthermore, we set a new parameter v_{ij} and the new target distance τ_{ij} as follows:

$$\text{if } t_{ij} > \sqrt{\tfrac{t_{ij}}{c_{ij}+\epsilon}} \quad \text{then } v_{ij} = c_{ij}t_{ij} \text{ and } \tau_{ij} = t_{ij}$$
$$\text{else } v_{ij} = 2\sqrt{t_{ij}c_{ij}+\epsilon} - 1 \text{ and } \tau_{ij} = \sqrt{\tfrac{t_{ij}}{c_{ij}+\epsilon}}.$$

The idea is that if the generalized target distance is less than the actual sum of radii squared, then the function is truncated at a higher level than the previous value in ModCoAR (so that overlapping is not rewarded) by setting the new target distance, τ_{ij}, to t_{ij} and setting the cost to $c_{ij}t_{ij}$. On the other hand, if the generalized target distance is greater than or equal to the actual sum of radii squared, then this indicates that t_{ij} is in the flat part of the convexified function and therefore the cost can be set to the value at the flat part of the function, $v_{ij} = 2\sqrt{t_{ij}c_{ij}+\epsilon} - 1$, and the new target distance can be set to the generalized target distance. This concept leads to the IBIMODEL below:

(IBIMODEL)

$$\begin{aligned}
\text{Minimize} \quad & \textstyle\sum_{i<j} F_{ij}(x_i, x_j, y_i, y_j) - K \log\left(\frac{D_{ij}}{t_{ij}}\right) & \\
\text{s.t.} \quad & \tfrac{1}{2}w_F \geq x_i + r_i & \text{for } i = 1, ..., N \\
& \tfrac{1}{2}w_F \geq r_i - x_i & \text{for } i = 1, ..., N \\
& \tfrac{1}{2}h_F \geq y_i + r_i & \text{for } i = 1, ..., N \\
& \tfrac{1}{2}h_F \geq r_i - y_i & \text{for } i = 1, ..., N
\end{aligned}$$

$$\text{where} \quad \tau_{ij} = \begin{cases} t_{ij} & \text{if } t_{ij} \geq \sqrt{\frac{t_{ij}}{c_{ij}+\epsilon}} \\ \sqrt{\frac{t_{ij}}{c_{ij}+\epsilon}} & \text{otherwise,} \end{cases}$$

$$v_{ij} = \begin{cases} c_{ij}t_{ij} & \text{if } t_{ij} \geq \sqrt{\frac{t_{ij}}{c_{ij}+\epsilon}} \\ 2\sqrt{t_{ij}c_{ij}+\epsilon} - 1 & \text{otherwise,} \end{cases}$$

$$F_{ij}(x_i, x_j, y_i, y_j) = \begin{cases} c_{ij}D_{ij} + \frac{\alpha\, t_{ij}}{D_{ij}} - 1 & \text{if } \tau_{ij} \leq D_{ij} \\ v_{ij} & \text{if } \tau_{ij} > D_{ij} \end{cases}$$

where D_{ij} and ϵ are both as previously defined in Section 3.1.5.

3.2.2 Aspect Ratios

One way to obtain very low costs in the FLP is by aligning the departments in a stack of long, narrow departments where the centroids are very close to one another. Therefore, another goal of the FLP is to try to give the rectangular-shaped departments dimensions that are not too far off from being square. The concept of aspect ratio measures how far off a department's shape is from being square. The aspect ratio of department i is defined as $\beta_i := \max \frac{h_i}{w_i}, \frac{w_i}{h_i}$, where h_i is the height and w_i is the width of department i. As the aspect ratio becomes smaller (approaching 1), the problem becomes more constrained, the total cost increases, and feasible solutions become harder to find. With the exception of the last stage of the NLT method and of BPL, all of the above models do not have any control over the aspect ratios.

IBIMODEL tries to include some information about the desired width and height for each department. The aspect ratios can be thought of as a bound on a department's width and height. For example, if department i has area a_i and a restriction that its minimum length should be larger then μ_i, so that $h_i \geq \mu_i$, $w_i \geq \mu_i$ must hold (as in van Camp et al.'s (1991) example), then (provided $\mu_i > 0$) one can show that

$\frac{h_i}{w_i} = \frac{h_i^2}{w_i h_i} \geq \frac{\mu_i^2}{a_i}$ and $\frac{w_i}{h_i} = \frac{w_i^2}{h_i w_i} \geq \frac{\mu_i^2}{a_i}$, as well as $\frac{h_i}{w_i} = \frac{h_i w_i}{w_i^2} \leq \frac{a_i}{\mu_i^2}$ and $\frac{w_i}{h_i} = \frac{w_i h_i}{h_i^2} \leq \frac{a_i}{\mu_i^2}$.

Given that the aspect ratio is $\beta_i = \max\{\frac{h_i}{w_i}, \frac{w_i}{h_i}\}$, the implied bound on the aspect ratio is $\beta_i \leq \frac{a_i}{\mu_i^2}$. Therefore, by altering the minimum side length of departments, one can control the upper bound of the aspect ratio. It is important to note that this upper bound will be different for each department i as the area a_i for each department differs.

Since the aspect ratio is closely related to the minimum side length of the department, the idea of controlling the aspect ratio via the minimum side length of departments provides a link back to the notion of having the departments represented by circles. (To date only circles have been considered as initial estimators for departments, likely because of their symmetry.) In the proposed first stage of the facility layout design, IBIMODEL, the radii of circles that represent departments are considered to be a given parameter, even though one only knows the desired area of the departments. If one simply takes the radius to be $r_i = \sqrt{a_i/\pi}$, the results tend to yield costs that are relatively low but also departments with relatively large aspect ratios, i.e., with a large difference between the length and width. Intuitively, one reason for the large aspect ratios is that it is much harder for the larger departments than for smaller departments to shape themselves into square-like rectangles when the circle sizes only depend on the desired department areas. Large departments usually have many more neighbours than smaller departments, and rely more heavily on these neighbours to move/jump around to allow the large department the flexibility it needs to form itself into a square-like shape. Therefore if larger departments have relatively larger circles to represent their areas, the circles may reserve enough area inside the facility for the second stage model to be able to form departments with lower aspect ratios. For this reason in IBIMODEL, the radii are calculated to be $r_i = \sqrt{\frac{a_i}{\pi}} * \log_2(1 + \frac{a_i}{\varphi^2})$ where φ is a parameter for controlling the desired smallest length or width for each department in the layout. By construction, the log scaling factor is greater than or equal to 1, and as φ decreases, the log factor increases and IBIMODEL aims for lower aspect ratios, while if φ increases then the log factor decreases and allows IBIMODEL to aim for higher aspect ratios.

Since the areas of the circles that represent the departments are increased, the floor dimensions also have to be adjusted in order to allow the circles to fit with a nice spread (and reduced overlap). Therefore, the facility dimensions are also adjusted by a factor $\chi = \max \log_2(1 + \frac{a_i}{\varphi^2})$. This way the circles will fit nicely and have the potential of leading to small aspect ratios for the department in the layout obtained using the second stage.

For example, consider a 10-department example with a facility of dimensions 20×30, depart-

Area a_i	Radius $\sqrt{a_i/\pi}$	Adjustment $\log_2(1+\frac{a_i}{2^2})$	Adjusted Radius $\sqrt{a_i/\pi} * \log_2(1+\frac{a_i}{2^2})$
9	1.69257	1.7004	5.8870
18	2.39365	2.4594	5.8870
24	2.76395	2.8074	7.7594
27	2.93161	2.9542	8.6606
41	3.61257	3.4542	12.6146
42	3.65636	3.5236	12.8834
45	3.78469	3.6147	13.6806
60	4.37019	4.0000	17.4808
64	4.51351	4.0875	18.4488
70	4.72034	4.2094	19.8700

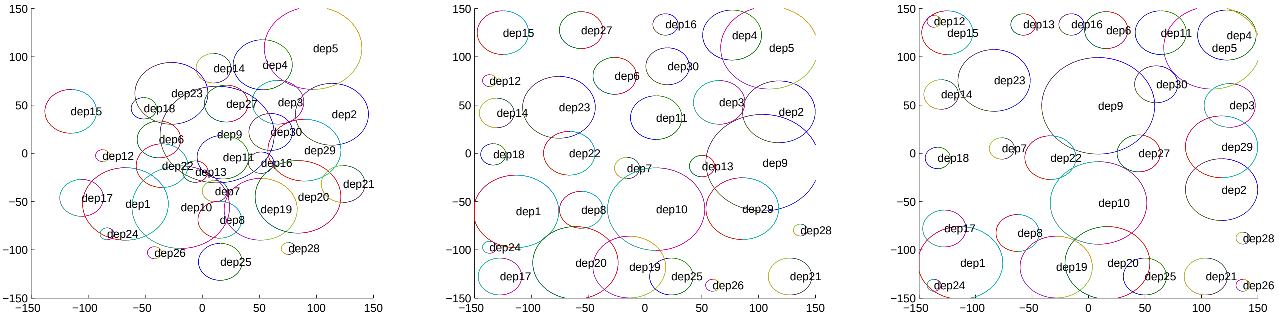
Table 1: Radius calculation for example.

ments with areas 9, 18, 24, 27, 41, 42, 45, 60, 64, and 70, and a restriction that the minimum height and width of a department is 2. The adjusted radii calculated with $\varphi = 2$ are given in last column of Table 1. Since the circles have been increased, the facility layout has to be increased as well. Calculating $\chi = \max \log_2(1+\frac{a_i}{2^2})$ for this problem gives $\chi = 4.3038$ and hence the facility length and width are increased by a factor of χ to 86×129.

It is shown in Section 4 that, together with the final stage of Luo et al. (n.d.), IBIMODEL does provide a wide variety of layouts with relatively low aspect ratios and costs. Since real-world problems are customarily simplified before they are modeled using the FLP, even if the model does solve the problem to optimality, the original problem being solved is much too simplified compared to its real-world setting. With IBIMODEL and the second stage of Luo et al. (n.d.), it is possible to get a wide variety of layouts with competitive costs. In this way, a layout that most closely meets all the problems requirements (even the ones that were not accounted for in the FLP model) can be selected from the range of layouts obtained.

3.2.3 Parameter Selection of α and K

IBIMODEL has two main parameters that need to be adjusted experimentally: α and K. The parameter α controls the amount of overlap amongst circles, by penalizing overlaps in the objective function. K is the dispersion parameter, so if K is too small, all the circles will be placed on top of each other around the center of the layout, as illustrated in Figure 6 (a). If K is too large, then all the circles will be pushed to the edges of the layout as illustrated in Figure 6 (c). Finding a good balance between these two extremes will result in the most promising layouts after the second stage, as in Figure 6 (b).



(a) K is too small
Max Aspect Ratio*: 10.27
Cost*: 24303

(b) K is good
Max Aspect Ratio*: 3.75
Cost*: 25000

(c) K is too big
Max Aspect Ratio*: 16.77
Cost*: 27257

Figure 6: Layouts for Tam's 30-department problem with different K parameter values in IBIMODEL

* Calculated after the 2nd stage has computed the layout.

Since the ranges of α and K within which good layouts can be found vary from one instance to another, one must first experiment with extreme values for each parameter to get a good sense of which values will yield the most promising results. This is done by fixing α first to a value around 3 (since this was found to be always within α's range) and varying K, to find a value of K for which all the circles are approaching the center of the layout and are practically on top of each other, and another value of K that starts pushing all the circles to the layout edges. Afterwards K is fixed to a value that provided one of the smallest costs in the previous experiments (when α was fixed and K was varied) and α is varied to extreme values that provide solutions where all circles start overlapping or are pushed to the edges of the layout. Once the ranges of the parameters have been determined, one can either solve the IBIMODEL for all possible combinations of α and K on a grid within these extreme valued ranges (to find the best possible layout) or randomly sample within the same ranges.

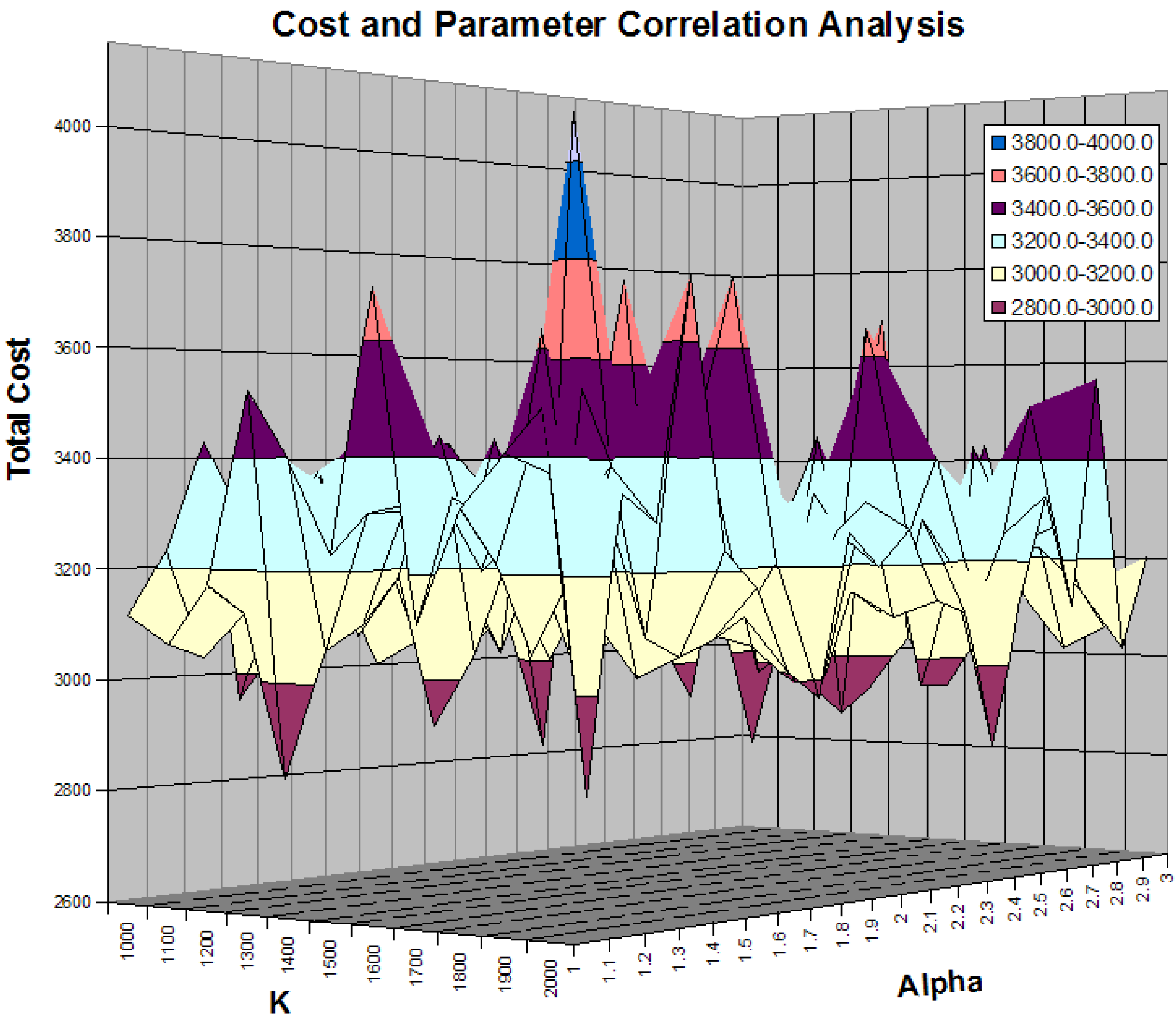


Figure 7: Cost fluctuations on the 20-department problem

In order to experimentally determine if random sampling was a reasonable way of choosing values for α and K, we conducted a test using Armour & Buffa's (1963) 20-department problem. First, a regular grid coverage on the parameters was used to solve the problem

231 times (contained in Appendix I and II), with 21 different α values between 1 and 3 in increments of .1, and with 11 different K values between 1000 and 10000 in increments of 100. The result of IBIMODEL was used as input to the second stage described in Section 3.4. Figure 7 illustrates clearly that there is no definite correlation between parameter values and final layout costs. As observed, there is not one region in the graph that has consistently low layout costs, and actually the costs fluctuate significantly with even small changes in the parameters. Next, the 20-department problem was solved using increasing sets of 20, 30, 40 and 50 randomly generated values for α and K. When the results of the grid coverage and random sampling are compared in Tables 2 and 3, one can observe that the difference between the results is not large. As illustrated in Table 3, up to the 3rd quartile, the best layouts for the sample sets are at most 3.2% worse than those from the grid coverage.

Data Set	Min	1st quartile	2nd quartile	3rd quartile	Max
20 Samples	2902.1	3074.8	3181.1	3266.0	3642.9
30 Samples	2902.1	3101.7	3181.1	3318.9	3642.9
40 Samples	2902.1	3119.3	3237.0	3351.5	3642.9
50 Samples	2847.7	3096.4	3181.1	3323.2	3642.9
Grid Coverage	2811.0	3084.7	3208.1	3321.1	4100.4

Table 2: Cost statistics for the 20, 30, 40, 50 sample and grid coverage data sets

Data Set	Min	1st Quartile	2nd Quartile	3rd Quartile	Max
20 Samples	3.2%	0.3%	0.8%	1.7%	11.2%
30 Samples	3.2%	0.6%	0.8%	0.1%	11.2%
40 Samples	3.2%	1.1%	0.9%	0.9%	11.2%
50 Samples	1.3%	0.4%	0.8%	0.1%	11.2%

Table 3: Percentage difference between the 20, 30, 40, 50 sample data set costs and the grid coverage data set costs

Hence, based upon this experiment, it can be concluded that randomly sampling parameter values may yield approximately the same quality of layouts as tediously large grid samplings would, allowing the method to be more efficient.

3.3 First Stage Model Variations

Facility layout problems sometimes require departments not to be narrower than a certain value. For example, van Camp et al. (1991) has a 10-department problem with the restriction that the departments cannot be narrower then 5 meters in either dimension. This restriction can be accounted for in IBIMODEL by changing the φ parameter. As explained in Section 3.2.1, the log function helps in adjusting the respective circle sizes for each department to allow them to have enough space in order to be able to shape themselves closer to a square shape.

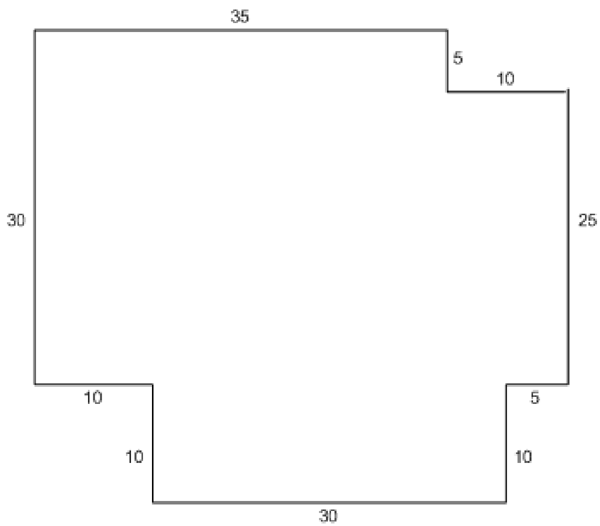


Figure 8: Floor plan for the 30-department layout
(Figure taken from Tam (1992a))

Often facility layout problems do not allow the entire floor plan to be used for placing departments. There may be locations on the floor plan that are already occupied by existing facilities such as elevator shafts, utilities and columns, etc. For example, Tam (1992*a*) and Tam (1992*b*) consider a 30-department problem with 3 occupied spaces at the corner, as shown in Figure 8. With IBIMODEL, two different approaches can be used to solve this 30-department problem. The first approach considers the 3 occupied spaces as additional 'departments', hence solving the problem at the first stage as a 33-department problem.

These additional 3 departments have associated flows of zero, and fixed positions at the spots where they are required to be. The second approach initially ignores these 3 occupied spaces and solves the first stage as a 30-department problem, and then adds these 3 extra departments into the problem at the second stage. The results in Section 4 suggest that the latter approach yields layouts with lower costs.

3.4 Second Stage Model

The second stage model that we used was developed by Luo et al. (n.d.). Given the fixed-outline of the facility and the locations of circles (from the first stage), semidefinite programming is used to provide the precise location and rectangular dimensions of the departments while minimizing the layout costs.

3.4.1 Semidefinite Programming

Definition 1 (*Positive Semidefiniteness*) *A matrix $X \in S^n$, where S^n denotes the set of $n \times n$ real symmetric matrices, is said to be positive semidefinite (psd) if for all $y \in R^n$:*

$$y^T X y = \sum_{i,j} X_{i,j} y_i y_j \geq 0.$$

Definition 2 (*Semidefinite Programming*) *In $\mathcal{S}_n$, a semidefinite programming (SDP) problem has the form:*

$$\begin{aligned} &\mathit{min} \quad C \bullet X \\ &s.t. \\ &\qquad A_i \bullet X = b_i, \ \mathit{for} \ i = 1, 2, ...m, \\ &\qquad X \succeq 0. \end{aligned}$$

where $A \bullet X = \sum_{i,j} A_{i,j} B_{i,j} = \text{trace}(B^T A)$ and X is psd.

SDP is a generalization of linear programming and is a convex optimization problem that can be solved effectively using interior-point algorithms (Wolkowicz, Vandenberghe & Saigal 2000). Based on the relative position of departments obtained from the first stage, we formulate the FLP as a convex optimization problem using SDP. In this section, the second stage model is formulated by applying semidefinite optimization techniques to the area and aspect ratio constraints.

3.4.2 SDP Model

One property of psd matrices is that all principal minors are non-negative. Therefore for $a_i > 0$, if we relax the area constraint $w_i h_i = a_i$ to $w_i h_i \geq a_i$, we can express the relaxed constraint as a PSD constraint:

$$\begin{pmatrix} w_i & \sqrt{a_i} \\ \sqrt{a_i} & h_i \end{pmatrix} \succeq 0.$$

Assuming that the aspect ratio of department i must be bounded above by a given value $\beta_i^* > 0$, then letting $w_i^{low} = h_i^{low} = \sqrt{a_i/\beta_i^*}$ where $a_i = w_i h_i$, we have:

$$\begin{aligned} & w_i \geq w_i^{low} > 0, \\ & w_i^2 \geq a_i/\beta_i^*, \\ & \beta_i^* w_i^2 \geq a_i, \\ & \beta_i^* \geq h_i/w_i. \end{aligned}$$

Similarly, as $h_i \geq h_i^{low} > 0$ we can conclude that $\beta_i^* \geq w_i/h_i$.

With $\beta_i^* \geq h_i/w_i$ and $w_i h_i = a_i$, Luo et al. (n.d.) obtain $\beta_i^* \geq h_i^2/a_i$ and $a_i\beta_i^* \geq h_i^2$.

Thus, $\beta_i^* \geq h_i/w_i$ is equivalent to the following psd constraint:

$$\begin{pmatrix} \beta_i^* & h_i \\ h_i & a_i \end{pmatrix} \succeq 0,$$

and $\beta_i^* \geq w_i/h_i$ is equivalent to

$$\begin{pmatrix} \beta_i^* & w_i \\ w_i & a_i \end{pmatrix} \succeq 0.$$

Combining all these constraints yields the second stage model for the FLP:

$$\begin{aligned}
&\min_{(x_i,y_i),w_i,h_i} \quad \sum_{1\leq i<j\leq n} c_{ij}\left(|x_i - x_j| + |y_i - y_j|\right) \\
&\text{s.t.} \\
&\quad x_i + \frac{1}{2}w_i \leq \frac{1}{2}w_F \quad \forall\, i \\
&\quad y_i + \frac{1}{2}h_i \leq \frac{1}{2}h_F \quad \forall\, i \\
&\quad \frac{1}{2}w_i - x_i \leq \frac{1}{2}w_F \quad \forall\, i, \\
&\quad \frac{1}{2}h_i - y_i \leq \frac{1}{2}h_F \quad \forall\, i, \\
&\quad w_i^{low} \leq w_i \leq w_i^{up} \quad \forall\, i, \\
&\quad h_i^{low} \leq h_i \leq h_i^{up} \quad \forall\, i, \\
&\quad \begin{pmatrix} w_i & \sqrt{a_i} \\ \sqrt{a_i} & h_i \end{pmatrix} \succeq 0 \quad \forall\, i, \\
&\quad \begin{pmatrix} \beta_i^* & h_i \\ h_i & a_i \end{pmatrix} \succeq 0, \quad \forall\, 1 \leq i < j \leq n, \\
&\quad \begin{pmatrix} \beta_i^* & w_i \\ w_i & a_i \end{pmatrix} \succeq 0, \quad \forall\, 1 \leq i < j \leq n
\end{aligned}$$

where w_F and h_F are the fixed width and height of the facility. Note that the non-overlap constraints are absent from this formulation. They are enforced using additional linear constraints in the manner that we now describe.

3.4.3 The Relative Position Matrix

Recall from the NLT model that the overlap-free constraints for each pair of departments can be expressed as:

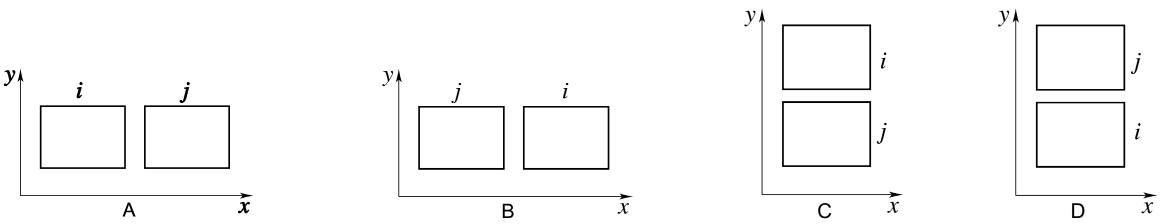
$$\begin{aligned} \tfrac{1}{2}(w_i + w_j) &\leq |x_i - x_j| \quad \text{if } |y_i - y_j| < \tfrac{1}{2}(h_i + h_j) \\ \tfrac{1}{2}(h_i + h_j) &\leq |y_i - y_j| \quad \text{if } |x_i - x_j| < \tfrac{1}{2}(w_i + w_j). \end{aligned}$$

Since the relative positions of the departments are already determined in the first stage, one can determine which of these two conditions applies, and therefore also eliminate the symbols of absolute values, thus linearizing the non-overlap constraints.

The relative positions of the departments are stored in a Relative Position Matrix (RPM). A RPM is an $N \times N$ non-negative, symmetric matrix with zeros on the principal diagonal, where N is the number of departments in the problem. Since the matrix is symmetric, the information in the upper triangular matrix is sufficient. The RPM matrix is populated in the following manner:

- "1-" is used to represent that department i is horizontally separated from department j. Furthermore, "11" means that department i is to the left of department j (Figure 9(a)), and "12" means that department i is to the right of department j (Figure 9(b)).
- "2-" is used to represent that department i is vertically separated from department j. Furthermore, "21" means that department i is above department j (Figure 9(c)), and "22" means that department i is below department j (Figure 9(d)).

If the relative position of two departments is diagonal (Figure 10), the following rule is applied



(a) "11" (b) "12" (c) "21" (d) "22"

Figure 9: Horizontal and vertical relative positioning amongst modules

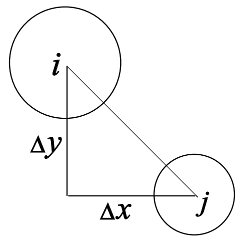
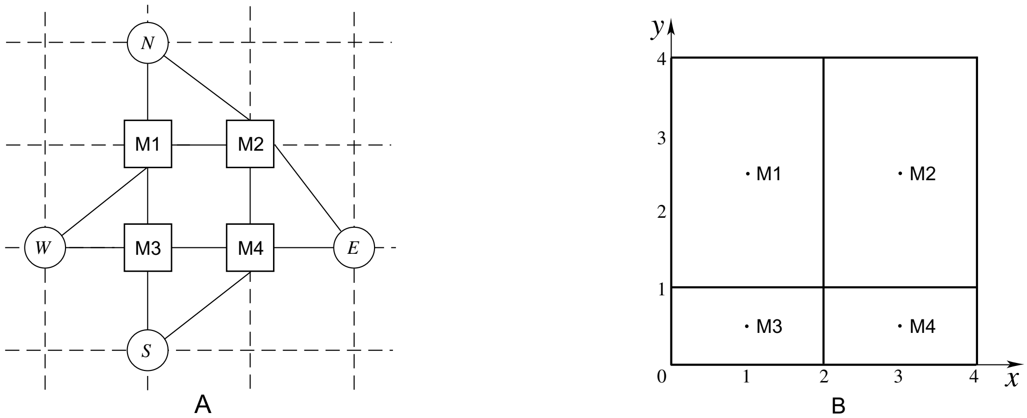


Figure 10: Diagonal relative positioning amongst modules

to determine how the two modules are separated. If $\Delta y \geq \Delta x$, then only the vertical relative positioning is considered, and if $\Delta x > \Delta y$, then only the horizontal relative positioning is considered.



(a) The relational position graph (b) The final layout

Figure 11: A 4-department example from Luo et al. (to be published)

We use a 4-department example to show how a RPM can be generated. A relation position graph obtained from the first stage for the four-department case is shown in Figure 11(a). Based upon the labeling system explained, the following RPM matrix is obtained for this example:

$$\begin{pmatrix} 0 & 11 & 21 & 0 \\ 0 & 0 & 0 & 21 \\ 0 & 0 & 0 & 11 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where the following relative position relations are given by the entries of the RPM:

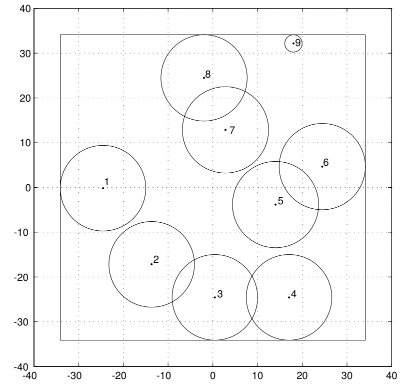
- "11" in entry (1,2): Department 1 is on the left of department 2;
- "11" in entry (3,4): Department 3 is on the left of department 4;
- "21" in entry (1,3): Department 1 is above department 3;
- "21" in entry (2,4): Department 2 is above department 4.

The final layout for this example, based on this RPM, is illustrated in Figure 11(b).

3.4.4 The Voronoi Diagram

It only remains to present how a planar graph that reflects the relative positions of the departments is obtained from the solution of IBIMODEL. Luo et al. (n.d.) converts the output of IBIMODEL into a planar graph using the Delaunay Triangulation (DT) and its dual construction, the Voronoi diagram (VD). As illustrated in Figure 13(a), we use the positions of the centers of the circles obtained from IBIMODEL to partition the plane into the resulting Voronoi diagram. The idea is that each cell in the VD contains exactly one department center and every point in a given cell is closer to its generating department center than to any other.

The DT for a set of points P in the plane is the triangulation $DT(P)$ such that no point in P is inside the circumcircle of any triangle in $DT(P)$. Delaunay triangulations maximize the minimum angle of all the angles of the triangles in the triangulation, and hence avoid sliver-like triangles. In Figure 13(b), the dashed circles represent the solution of the first stage, the fine lines represent the corresponding DT and the black bold lines represent the cells of the corresponding VD. The edges of the DT represent the relative positions of the 9 departments. These relationships are then included in the RPM matrix.

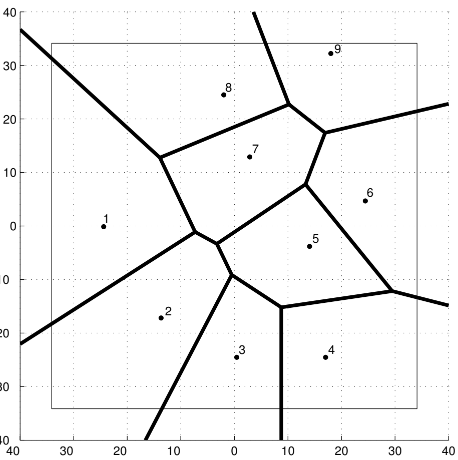


(a) Layout after the first stage.

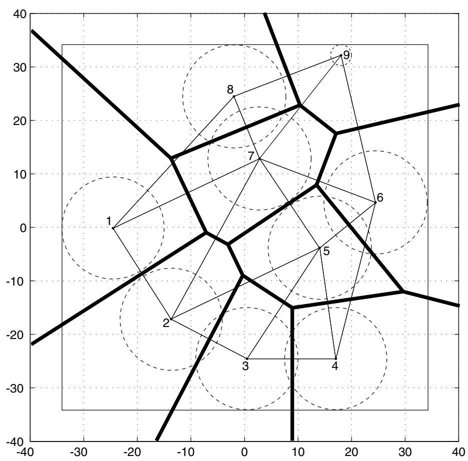
$$\begin{pmatrix} 0 & 21 & 11 & 11 & 11 & 11 & 11 & 22 & 11 \\ 0 & 0 & 11 & 11 & 11 & 11 & 22 & 22 & 22 \\ 0 & 0 & 0 & 11 & 22 & 22 & 22 & 22 & 22 \\ 0 & 0 & 0 & 0 & 22 & 22 & 22 & 22 & 22 \\ 0 & 0 & 0 & 0 & 0 & 11 & 22 & 22 & 22 \\ 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 22 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 22 & 22 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) RPM

Figure 12: 9-department example from Luo et al.
(Figure taken from Luo et al.)



(a) Voronoi diagram on top of the graph



(b) Delaunay triangulations added to graph

Figure 13: Relative position graph with the Voronoi diagram and Delaunay triangulation for 9-department example

(Figure taken from Luo et al.)

4 Computational Results

4.1 Computational Details

IBIMODEL was tested using the non-linear optimization solver KNITRO 5.0 accessed via the modeling language AMPL on a Microsoft Windows Server 2003 with a 2.84 GHz processor. The radii of the approximating circles was set to $r_i = \sqrt{\frac{a_i}{\pi}} * \log_2(1 + \frac{a_i}{\varphi^2})$ where $\varphi = 2$ or the desired minimum length for the instance (if any is specified) and the generalized target distance was set with $\epsilon = 0.1$. When solving IBIMODEL, KNITRO requires initial starting points for the centers of the circles. Since it is not clear a priori what the 'best' starting configuration is, following Anjos & Vannelli (2006), the centers of the N circles were initially placed at regular intervals around a large circle of radius $r = (w_F + h_F) * \chi$. The χ variable is the adjusting ratio that corresponds to the ratios that the circle sizes were enlarged by, and hence $\chi = \max(\log_2(1 + \frac{a_i}{\varphi^2}))$. Therefore, the initial centers (x_i, y_i) of the departmental circles can be set to $x_i = r \cos \theta_i * \chi$ and $y_i = r \sin \theta_i * \chi$, where $\theta_i = 2\pi(i - 1)$.

The IBIMODEL has $2N$ variables and $3\binom{n}{2} + N + 3$ parameters with only the objective function being non-linear.

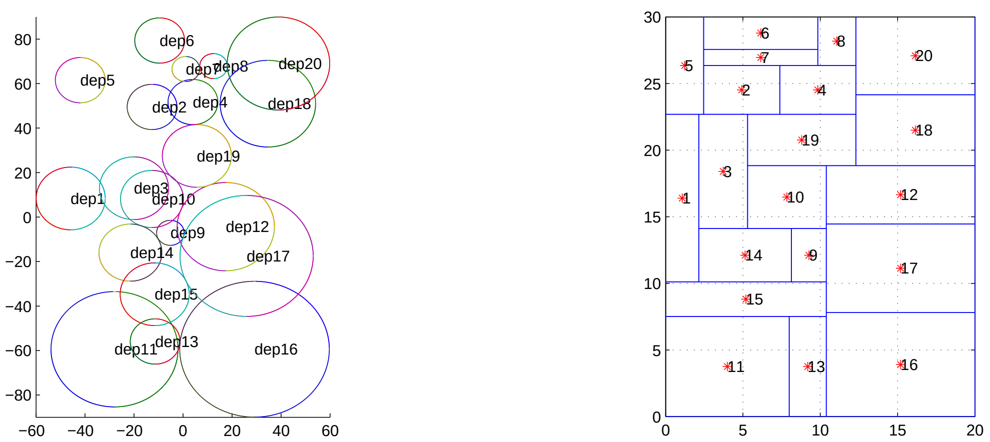
The AMPL code of IBIMODEL can be found in Appendix III.

4.2 20-Department Test Problem

Arguably the best known large benchmark instance in FLP research is Armour & Buffa's (1963) 20-department problem. This instance uses a symmetrical flow matrix and rectilinear distances. As proposed originally, it does not have any requirements on the minimum side length or the maximum allowable aspect ratio. Armour & Buffa (1963) approached this problem by requiring all departments to be made up of contiguous rectangular building blocks, and then applied departmental adjacent pairwise exchanges. Tong (1991) approached this problem by assuming rectangular departmental shapes placed in bays. Tate & Smith (1995) applied genetic optimization with adaptive penalty functions to improve the solution. Kim & Kim (1998) used a simulated annealing approach with probabilistically based aspect ratios. Enea, Galante & Panascia (2005) also used a genetic algorithm that employed concepts of evolutionary hybrid algorithms to obtain a better local optima. Anjos & Vannelli (2006) were able to find the best costs in the literature using the two-stage non-linear framework described earlier.

4.2.1 Computational Results on Armour & Buffa's 20-Department Problem

Reported below in Table 4 are the results that were obtained from applying IBIMODEL and the SDP-based second stage to the Armour & Buffa (1963) 20-department problem. (The corrected cost matrix from Scriabin & Vergin (1975) and Huntley & Brown (1991) was used.)



(a) Layout of department circles using IBIMODEL (b) Final layout using Luo et al. (n.d.)

Figure 14: Best layout achieved for the 20-department problem

As explained in Section 3.2.3, the α and K parameters were varied randomly and also

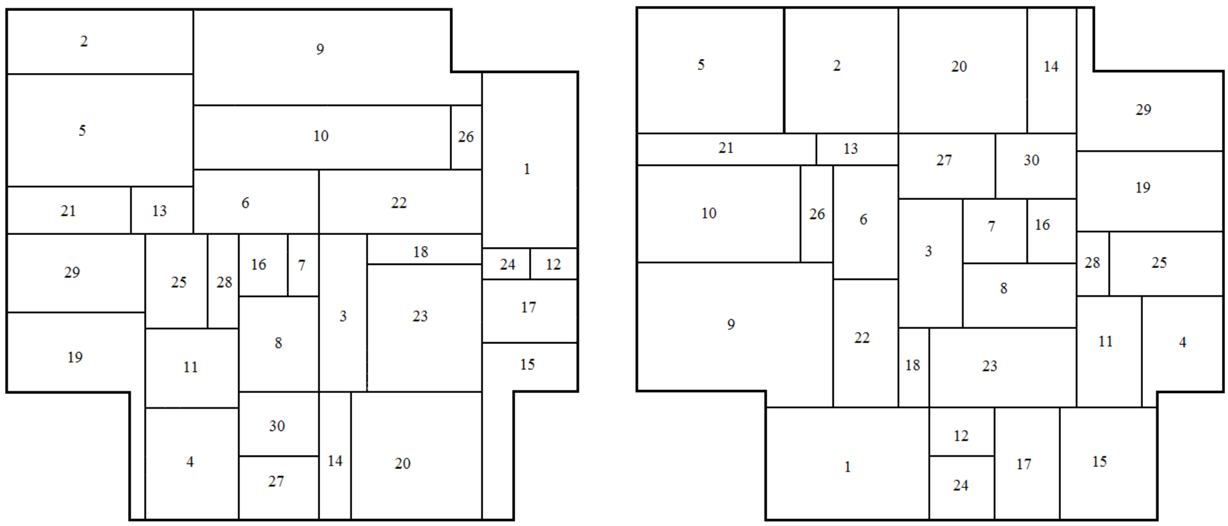
β_i^*	Cost of best layout in TS	Cost of best layout in EGP	Cost of best layout in AV	Cost of best layout by IBIMODEL
8	5255.0†	4793.5†	4591.3†	3014.2
7	5255.0	4793.5	4591.3†	2979.3
6	5524.7†	5397.6†	4591.3†	**2708.0**
5	5524.7	5397.6	4591.3	3009
4	5743.1	5370.9	4786.4	2960.5
3	5832.6	5594.3	5140.1	-*
2	6171.1	6023.2	5224.7	-*

Table 4: Comparison of the algorithms for the Armour and Buffa problem
(*No feasible layout found, †Cost of layout for that specific aspect ratio is the best for a lower aspect ratio)
(TS is Tate & Smith (1995), AV is Anjos & Vannelli (2006), EGP is Enea, Galante & Panascia (2005))

systematically to obtain different layouts. The combination of our first stage algorithm and Luo et al.'s (n.d.) second stage model found the layout in Figure 14, with a cost of 2708. The layout in Figure 14 shows department 7 as having the largest aspect ratio (of 6). Parameter combination α=1.04 and K=1690 was used to solve the first stage model in 0.769 seconds and the second stage model in 16.6 seconds. In comparison, the genetic algorithm in Tate & Smith (1995) with a (lower) bound of 7 on the aspect ratio found a best layout with a cost of 5255.0 (over 10 runs of the algorithm) and the best layout reported in Anjos & Vannelli (2006) was 4591.3 (with an aspect ratio of 5 however). Thus the new layout obtained using IBIMODEL as the first stage algorithm found improved layouts with aspect ratios as low as 4 and improved by 41% on the previous best cost found for an aspect ratio bound of 7.

4.3 30-Department Test Problem

Another large problem that is well known in the literature is the 30-department Nugent et al. (1968) QAP problem. Tam (1992*a*) and Tam (1992*b*) transformed this problem into a FLP and included three additional pre-positioned departments, as illustrated in Figure 8. The three pre-positioned departments represent areas that are occupied by existing facilities such as elevator shafts or columns. Tam (1992*a*) uses a genetic algorithm and Tam (1992*b*) uses a simulated annealing algorithm with a slicing structure/tree method that contains information on partitioning the floor. Tam (1992a) and Tam (1992b) aim for relatively low aspect ratios (all below 2.5) and minimal deadspace. Deadspace refers to overlap, both between two departments and between a department and the occupied areas. Their algorithm uses a penalizing component in the objective function to compensate for larger aspect ratios and existing deadspace (the latter gauges the degree of shape distortion of departments due to overlap of occupied areas). Hence, Tam's (1992a) and Tam's (1992b) layout solutions provide some awkwardly shaped facilities that are not always practical in the real-world (see departments 15 and 19 in Figure 15(a) and department 29 in Figure 15(b)).



(a) Tam's (1992a) minimum cost layout (b) Tam's (1992b) minimum cost layout

Figure 15: Tam's 30-department minimum cost layouts

Other papers have attempted to solve this hard problem, but due to its size, the aspect ratio requirements, and the three pre-positioned departments, each one of these approaches made its unique simplifications. The unfortunate result is that it is not possible to compare our costs directly with theirs. Kim & Kim (1998) solved this 30-department problem using a simulated annealing algorithm in which a solution is encoded as a matrix that has information about relative locations of the facilities on the floor. To compare their results with Tam (1992a) and Tam (1992b), they converted Tam's costs to minimum total transportation distances by subtracting the penalties from the objective function to obtain an estimate that their model beat. Lee & Lee (2002) developed a shape-based block layout approach that uses a hybrid genetic algorithm (which adds the strength of simulated annealing and tabu search algorithms to the genetic algorithm). Even though they claim to be basing their test data on Nugent et al. (1968), Tam (1992*a*) and Tam (1992*b*), their areas and aspect ratios are different from those of the original problem, and the pre-positioned departments are completely ignored. Balakrishnan et al. (2003) developed the FACOPT software package that uses two algorithms, simulated annealing and a genetic algorithm, to solve the 30-department problem. However, FACOPT only beats Tam's (1992*a*) and Tam's (1992*b*) results with its genetic algorithm-based model and there is no indication in the paper about whether the pre-positioned departments were considered and what aspect ratios were obtained in these layouts.

4.3.1 Computational Results on Tam's 30-Department Problem

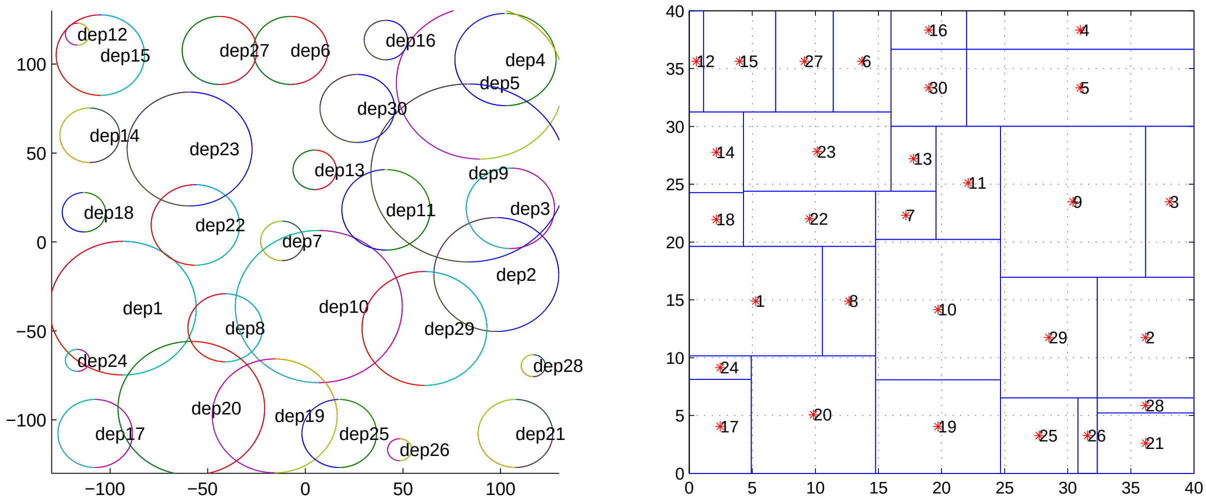
Table 5 illustrates the lowest costs that were obtained by Tam (1992*a*), Tam (1992*b*), Lee & Lee (2002) and Balakrishnan et al. (2003) for the 30-department problem with occupied spaces. Also, all of these models contain a penalty component in their objective function, hence these models minimize a penalty-added objective function value for feasible solutions (including penalties incurred due to unsatisfied shape constraints). Tam (1992*a*) and Tam (1992*b*) have aspect ratio requirements that vary from 1 to 2.5, Balakrishnan et al. (2003) use an upper bound of 3 as the aspect ratio requirement, and Lee & Lee (2002) do not mention what aspect ratio bound was used.

Kim & Kim (1998) were the first to use a model that did not contain a penalty component in its objective function. However, it did not consider the occupied areas. Their objective is to find the best feasible solution with a maximum aspect ratio of 2. Kim & Kim (1998) also published Tam's (1992) unpenalized layout by subtracting the penalites from the objective

	Cost of best layout in Tam (1992*a*)	Cost of best layout in Tam (1992*b*)	Cost of best layout in Lee & Lee (2002)†	Cost of best layout in Balakrishnan et al. (2003)
Layout Cost	47422.3	47483.7	51373	42638.6

Table 5: Comparison of the best results in the literature for the Tam's 30-department problem with penalty costs

†Layout dimension is not considered and aspect ratios are not mentioned in publication



(a) Layout of departments using IBIMODEL (b) Final layout using Luo et al.

Figure 16: Best floor plan for 30-department problem (with no occupied area)

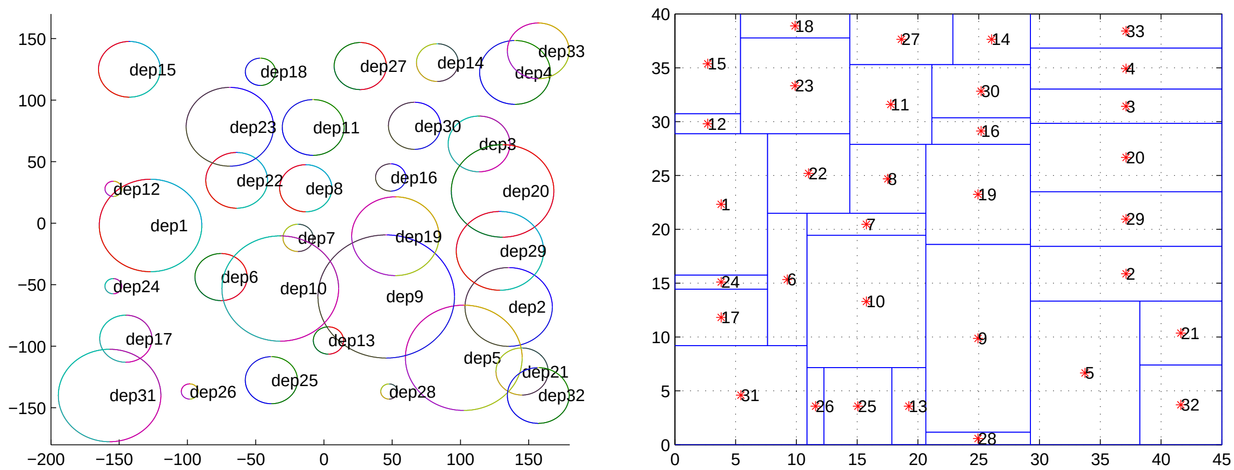
function value as done in Meller & Bozer (1996). Table 6 compares the results of the unpenalized Tam (1992*b*), Kim & Kim (1998) and our algorithm's costs for the problem without the pre-occupied areas. As shown, our model provides very consistent costs as the aspect ratios increase or decrease. Also, for this specific problem, our model was not able to achieve better results then Kim & Kim (1998), however we obtained costs that were only 8.6% higher. Figure 16 illustrates the facility layout obtained by IBIMODEL and Luo et al.'s (n.d.) model that has a cost of 23420 and a maximum aspect ratio of 7.67 (which took 2.91 seconds for the first stage and 255.2 seconds for the second stage to solve).

Table 7 on the other hand shows results that were obtained with our model for the 30-

β_i^*	Cost of best layout in Tam (1992*b*)*	Cost of best layout in Kim & Kim (1998)†	Cost of best layout found by our algorithm
10	23416.5	21560.6	24098
9	23416.5	21560.6	23924
8	23416.5	21560.6	**23420**
7	23416.5	21560.6	23974
6	23416.5	21560.6	23770
5	23416.5	21560.6	24916
4	23416.5	21560.6	25000

Table 6: Comparison of the algorithms for Tam's 30-department problem without occupied areas

*Minimum cost reported in Meller & Bozer (1996) was estimated by subtracting the penalties from the objective value give in Tam (1992*b*), † Facilities have an upper limit of 2.0 for their aspect ratios.



(a) Layout of departments using IBIMODEL

(b) Final layout using Luo et al.

Figure 17: Best floor plan for 30-department problem (with circles 31-33 representing occupied areas)

β_i^*	Cost of best layout found by our algorithm
10	24017
9	24477
8	23994
7	24200

Table 7: Results of our algorithm for Tam's 30-department problem with occupied areas and no penalty costs

department problem with pre-occupied areas. Figure 17 illustrates the solution obtained for the 30-department problem that uses 3 extra circles to estimate the preoccupied areas. This layout has a cost of 23994 and a largest aspect ratio of 7.41 (which took 4.1 seconds for the first stage and 236 seconds for the second stage to solve). Once again, it can be noted that the layout costs are very consistent for varying aspect ratios, which is a fairly important feature of our algorithm.

5 Conclusions and Future Research

IBIMODEL is a promising method that efficiently generates a reasonably diverse set of 'good' initial locations for departments in the FLP. The proposed model does not improve the objective function as the circles start overlapping and the distance between the circle centers becomes less than $r_i + r_j$. IBIMODEL also provides a systematic approach to setting the required parameters and has the ability to account for aspect ratio requirements. When IBIMODEL is used together with Luo et al.'s (n.d.) second stage, the net result is a methodology that consistently produces competitive, and often better layouts for large FLPs when compared with other approaches in the literature.

Nonetheless, future research should include investigating how to control the aspect ratios better. Potentially, using ellipsoids instead of circles to approximate the initial positions of departments could provide better results (this has not been attempted in the literature to date). Ellipsoids would likely provide more realistic estimations of department positions, since departments in real-world applications are not square-shaped.

Further work also includes adjusting the φ (the parameter that can control what the desired smallest length or width should be in each department's layout) and potentially personalizing it for each department (hence turning it into φ_i).

Finally, different combinations of first stage and second stage models from past papers should be tested, to see which combination provide the best overall layouts.

References

Al-Hakim, L. (1991), ‘Two graph-theoretic procedures for an improved solution to the facilities layout problem’, *International Journal of Production Research* **29**, 1701–1718.

Al-Hakim, L. (2000), ‘A note on ’a genetic algorithm approach to multiple criteria facility layout design”, *International Journal of Production Research* **38**(4), 985–989.

Anjos, M. F. & Vannelli, A. (2002), ‘An attractor-repeller approach to floorplanning’, *Mathematic Methods of Operations Research* **56**(1), 3–27.

Anjos, M. F. & Vannelli, A. (2006), ‘A new mathematical-programming framework for facility-layout design’, *INFORMS Journal on Computing* **18**(1), 111–118.

Anstreicher, K. (2003), ‘Recent advances in the solution of quadratic assignment problems’, *Mathematical Programming* **97**(1–2), 27–42.

Anstreicher, K. & Brixius, N. (2001), ‘A new bound for the quadratic assignment problem based on convex quadratic programming’, *Mathematical Programming* **89**(3), 341–357.

Armour, G. & Buffa, E. (1963), ‘A heuristic algorithm and simulation approach to relative location of facilities’, *Management Science* **9**, 294–309.

Balakrishnan, J., Cheng, C.-H. & Wong, K.-F. (2003), ‘FACOPT: a User friendly FACility layout OPTimization system’, *Computers and Operations Research* **30**, 1625–1641.

Barbosa-Povoa, A., Mateus, R. & Novais, A. (2000), ‘Optimal facility layout - a generic approach to process plants’, *Proceedings of the Manufacturing and Production & Control and Logistics* .

Barbosa-Povoa, A., Mateus, R. & Novais, A. (2001), ‘Optimal two-dimensional layout of industrial facilities’, *International Journal of Production Research* **39**(12), 2567–2593.

Bazaraa, M. (1975), ‘Computerized layout design: A branch and bound approach’, *AIIE Transactions* **7**(4), 432–438.

Bazaraa, M., Sherali, H. & Shetty, C. (1993), *Nonlinear programming: theory and algorithms*, Wiley, New York, NY.

Bazaraa, M. & Sherali, M. (1982), ‘On the use of exact and heuristic cutting plane methods for the quadratic assignment problem’, *Journal of the Operational Research Society* **33**(1), 991–1003.

Blanks, J. (1985), ‘Near-optimal quadratic-based placement for a class of ic layout problem’, *IEEE Circuits and Devices magazine* **1**(5), 31–39.

Bock, S. & Hoberg, K. (2007), ‘Detailed layout planning for irregularly-shaped machines with transportation path design’, *European Journal of Operational Research* **177**, 693–718.

Boswell, S. (1992), 'TESSA - a new greedy heuristic for facility layout planning', *International Journal of Production Research* **30**, 1957–1968.

Bozer, Y. & Meller, R. (1997), 'A reexamination of the distance-based facility layout problem', *IIE Transactions* **29**(7), 549–560.

Bozer, Y., Meller, R. & Erlebacher, S. (1994), 'An improvement-type layout algorithm for single and multiple floor facilities', *Management Science* **40**(7), 918–932.

Brixius, N. & Anstreicher, K. (2001), 'Solving quadratic assignment problems using convex quadratic programming relaxations', *Optimization Methods and Software* **16**, 49–68.

Burkard, R. & Rendl, F. (1984), 'A thermodynamically motivated simulation procedure for combinatorial optimization problems', *European Journal of Operational Research* **17**, 169–174.

Castillo, I. & Sim, T. (2004), 'A spring-embedding appraoch for the facility layout problem', *Journal of the Operational Research Society* **55**, 73–81.

Castillo, I., Westerlund, J., Emet, S. & Westerlund, T. (2005), 'Optimization of block layout design problems with unequal areas: A comparison of MILP and MINLP optimixation methods', *Computers and Chemical Engineering* **30**(1), 54–69.

Castillo, I. & Westerlund, T. (2005), 'An ε-accurate model for optimal unequal-area block layout design', *Computers and Operations Research* **32**(3), 429–447.

Chan, H. & Milner, D. (1982), 'Direct clustering algorithm for group formation in cellular manufacturing', *Journal of Manufacturing Systems* **1**, 65–75.

Cheng, C. & Kuh, E. (1984), 'Module placement based on resistive network optimization', *IEEE Transactions on Computer Aided Design* **3**(3), 218–225.

Chiang, W. & Chiang, C. (1998), 'Intelligent local search strategies for solving facility layout problems with the quadratic assignment problem formulation', *European Journal of Operational Research* **106**, 457–488.

Drezner, Z. (1980), 'DISCON: A new method for the layout problem', *Operations Research* **28**(6), 1375–1384.

Eades, P., Foulds, L. & Griffin, J. (1982), 'An efficient heuristic for identifying a maximum weight planar subgraph', *In Lecture Notes in Mathematics No. 952* .

Enea, M., Galante, G. & Panascia, E. (2005), 'The facility layout problem approached using a fuzzy model and a genetic search', *Journal of Intelligent Manufacturing* **16**, 303–316.

Ferrari, E., Pareschi, A., Persona, A. & Regattieri, A. (2003), 'Plant layout computerised design: Logistic and relayout program (lrp)', *The International Journal of Advanced Manufacturing Technology* **21**, 917–922.

Foulds, L. (1991), *Graph Theory Applications*, Springer-Verlag, New York.

Foulds, L. & Robinson, F. (1978), 'Graph theoretic heuristics for the plant layout problem', *International Journal of Production Research* **16**, 27–37.

Francis, R., McGinnis, L. & White, J. (1992), *Facility Layout and Location: An Analytical Approach*, Prentice Hall, Englewood Cliffs, New Jersey.

Garey, M. & Johnson, D. (1979), *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W.H. Freeman and Co., New York.

Gau, K. & Meller, R. (1999), 'An iterative facility layout algorithm', *International Journal of Production Research* **37**(16), 3739–3758.

Gilmore, P. (1962), 'Optimal and suboptimal algorithms for the quadratic assignment problem', *Journal of the Society for Industrial and Applied Mathematics* **10**, 305–313.

Goetschalckx, M. (1992), 'SPIRAL: An efficient and interactive adjacency graph heuristic for rapid prototyping of facilities design', *European Journal of Operational Research* (Special Issue on Facilities Design), 304–321.

Hassan, M. & Hogg, G. (1987), 'A review of graph theory applications to the facilities layout problem', *Omega* **15**(4), 291–300.

Hassan, M., Hogg, G. & Smith, D. (1986), 'SHAPE: A construction algorithm for area placement evaluation', *International Journal of Production Research* **24**(5), 1283–1295.

Helm, S. & Hadley, S. (2000), 'Tabu search based heuristics for multi floor facility layout', *International Journal of Production Research* **38**(2), 365–383.

Heragu, S. (1997), *Facilities Design*, PWS Publishing Company, Boston, MA.

Heragu, S. & Kusiak, A. (1991), 'Efficient models for the facility layout problem', *European Journal of Operational Research* **53**(1), 1–13.

Hopper, E. & Turton, B. (2001), 'An empirical investigation of meta-heuristic and heuristic algorithms for a 2d packing problem', *European Journal of Operational Research* **128**(1), 34–37.

Huntley, C. & Brown, D. (1991), 'A parallel heuristic for quadratic assignment problems', *Computers Ops Res* **18**(3), 275–289.

Jakobs, S. (1996), 'On genetic algorithms for packing of polygons', *European Journal of Operational Research* **88**, 165–181.

Kim, J. & Kim, Y. (1998), 'A space partitioning method for facility layout problems with shape constraints', *IIE Transactions* **30**(10), 947–957.

Kim, J. & Kim, Y. (1999), 'Branch and bound algorithm for locating input and output points of departments on the block layout', *Journal of the Operational Research Society* **50**(5), 517–525.

Kochhar, J., Foster, B. & Heragu, S. (1998), 'Hope: a genetic algorithm for the unequal area facility layout problem', *Computers and Operations Research* **25**, 583–594.

Koopmans, T. & Beckman, M. (1957), 'Assignment problems and the location of economic activities', *Econometrica* **25**, 53–76.

Kusiak, A. & Heragu, S. (1987), 'The facility layout problem', *European Journal of Operational Research* **29**, 229–251.

Lacksonen, T. (1994), 'Static and dynamic facility layout problems with varying areas', *Journal of the Operational Research Society* **35**(4), 1095–1106.

Lacksonen, T. (1997), 'Preprocessing for static and dynamic facility layout problems', *International Journal of Production Research* **35**(4), 1095–1106.

Langevin, A., Montreuil, B. & Riopel, D. (1996), 'Spine layout design', *International Journal of Production Research* **32**(2), 429–442.

Lawler, E. (1963), 'The quadratic assignment problem', *Management Science* **9**, 586–599.

Lee, K., Rohb, M. & Jeong, H. (2005), 'An improved genetic algorithm for multi-floor facility layout problems having inner structure walls and passages', *Computers & Operations Research* **32**, 879–899.

Lee, Y. & Lee, M. (2002), 'A shape-based block layout approach to facility layout problems using hybrid genetic algorithm', *Computers and Industrial Engineering* **42**, 237–248.

Liu, D. & Teng, H. (1999), 'An improved BL-algorithm for genetic algorithm of the orthogonal packing of rectangles', *European Journal of Operational Research* **112**, 158–166.

Loiola, E., Maia de Abreu, N., Boaventura-Netto, P., Hahn, P. & Querido, T. (2007), 'A survey for the quadratic assingment problem', *European Journal of Operational Research* **176**, 657–690.

Luo, C., Anjos, M. & Vannelli, A. (n.d.), A nonlinear optimization methodology for VLSI fixed-outline floorplanning.

Meller, R. & Bozer, Y. (1996), 'A new simulated annealing algorithm for the facility layout problem', *International Journal of Production Research* **34**(6), 1675–1692.

Meller, R. & Gau, K. (1996), 'The facility layout problem: recent and emerging trends and perspectives', *Journal of Manufacturing Systems* **15**, 351–366.

Meller, R., Narayanan, V. & Vance, P. (1999), 'Optimal facility layout design', *Operations Research Letters* **23**, 117–127.

Montreuil, B. (1990), 'A modeling framework for integrating layout design and flow network design', *Proceedings for the material handling research colloquium,* pp. 43–58.

Montreuil, B., Ratliff, H. & Goetschalckx, M. (1987), 'Matching based interactive facility layout', *IEE Transactions* **19**(3), 271–279.

Nugent, C., Vollman, T. & Ruml, J. (1968), 'An experimental comparison of techniques for the assignment of facilities to locations', *Operations Research* **16**, 150–173.

Patsiatzis, D. & Papageorgiou, L. (2002), 'Optimal multi-floor process plant layout', *Computers and Chemical Engineering* **26**(4), 575–583.

Sahni, S. & Gonzalez, T. (1976), 'P-complete approximation problems', *Journal of the Association for Computing Machinery* **23**(1), 555–565.

Scriabin, M. & Vergin, R. (1975), 'Comparison of computer algorithms and visual based methods for plant layout', *Management Science* **22**(2), 172–181.

Sherali, H., Fraticelli, B. & Meller, R. (2003), 'Enhanced model formulation for optimal facility layout', *Operations Research* **51**(4), 629–644.

Singh, S. & Sharma, R. (2006), 'A review of different approaches to the facility layout problems', *The International Journal of Advanced Manufacturing Technology* **30**(5–6), 425–433.

Sly, D. (1995), 'Computerized facilities design and management: An overview', *IIE Solutions* **27**(8), 43–47.

Smith, A. & Tate, D. (1992), *Using Genetic Algorithms for Equal Area Facilities Layout Design*, Vol. 4, American Society of Mechanical Engineers Press, New York.

Souliah, A. (1995), 'Simulated annealing for manufacturing systems layout design', *European Journal of Operational Research* **82**(3), 592–614.

Sporin-Kapov, J. (1994), 'Extensions of the tabu-search adaptation to the quadratic assignment problem', *Computers and Operations Research* **21**(8), 855–865.

Steinberg, L. (1961), 'The backboard wiring problem: A placement algorithm', *SIAM Review* **3**, 37–50.

Sule, D. (1994), *Manufacturing Facilities: Lacation, Planning and Design*, 2 edn, PWS Publishing, Boston.

Takuya, Y. & Takashi, I. (2006), 'Detailed layout design methodology using mixed integer programming and simulated annealing', *Journal of Japan Industrial Management Association* **57**(1), 39–50.

Tam, K. (1992*a*), 'Genetic algorithms, function optimization, and facility layout design', *European Journal of Operational Research* **63**, 322–346.

Tam, K. (1992*b*), 'A simulated annealing algorithm for allocating space to manufacturing cells', *International Journal of Production Research* **30**(1), 63–87.

Tate, D. & Smith, A. (1995), 'Unequal-area facility layout by genetic search', *IIE Transactions* **27**, 465–472.

Tompkins, J., White, J., Bozer, Y., Frazelle, E., Tanchoco, J. & Trevino, J. (1996), *Facilities Planning*, 2 edn, Wiley, New York.

Tong, X. (1991), SECOT: a sequential construction technique for facility design, Doctoral dissertation, University of Pittsburgh.

Tsuchiya, K., Bharitkar, S. & Takefuji, Y. (1996), 'A neural network approach to facility layout problems', *European Journal of Operational Research* **89**(3), 556–563.

van Camp, D., Carter, M. & Vannelli, A. (1991), 'A nonlinear optimization approach for solving facility layout problems', *European Journal of Operational Research* **57**, 174–189.

Weis, B. & Mlynski, D. (1987), 'A new relative placement procedure based on msst and linear programming', *Proceedings of the International Symposium on Circuits and Systems* **2**, 564–567.

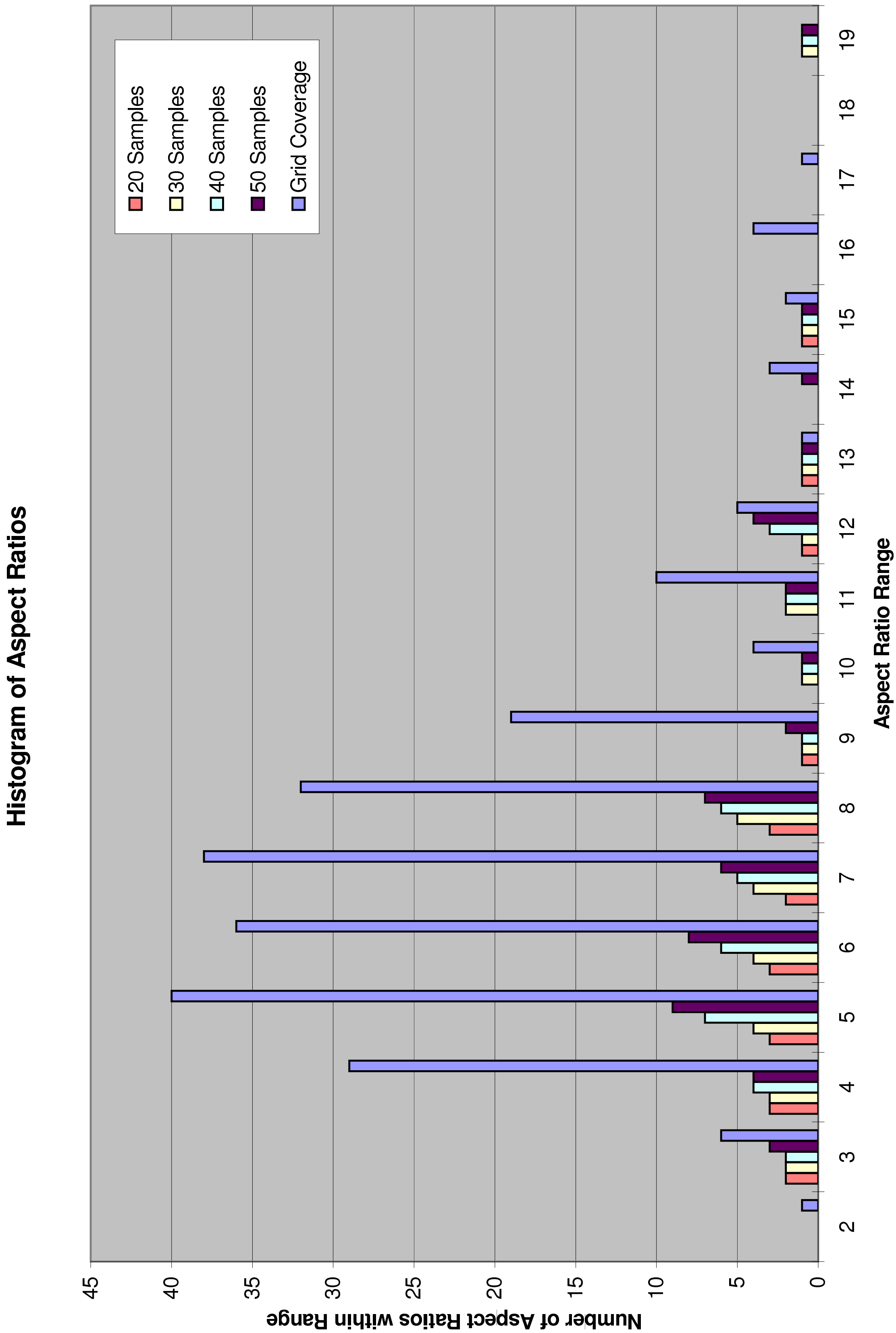
White, D. (1993), 'A convex form of the quadratic assignment problem', *European Journal of Operational Research* **65**(3), 407–416.

Wolkowicz, H., Vandenberghe, L. & Saigal, R. (2000), *Handbook of Semidefinite Programming: Theory, Algorithms, and Applications*, Kluwer Academic Publishers, Dordrecht.

Wu, Y., Huang, W., Lau, S., Wong, C. & Young, G. (2002), 'An effective quasi-human based heuristic for solving the rectangle packing problem', *European Journal of Operational Research* **141**, 341–358.

Yasdi, R. (2000), 'A literature survey on applications of neural networks for human-computer interacion', *Neural Computing and Applications* **9**, 245–258.

Appendix I



Histogram of Aspect Ratios
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40 Samples
50 Samples
Grid Coverage
Number of Aspect Ratios within Range
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40
45
Aspect Ratio Range
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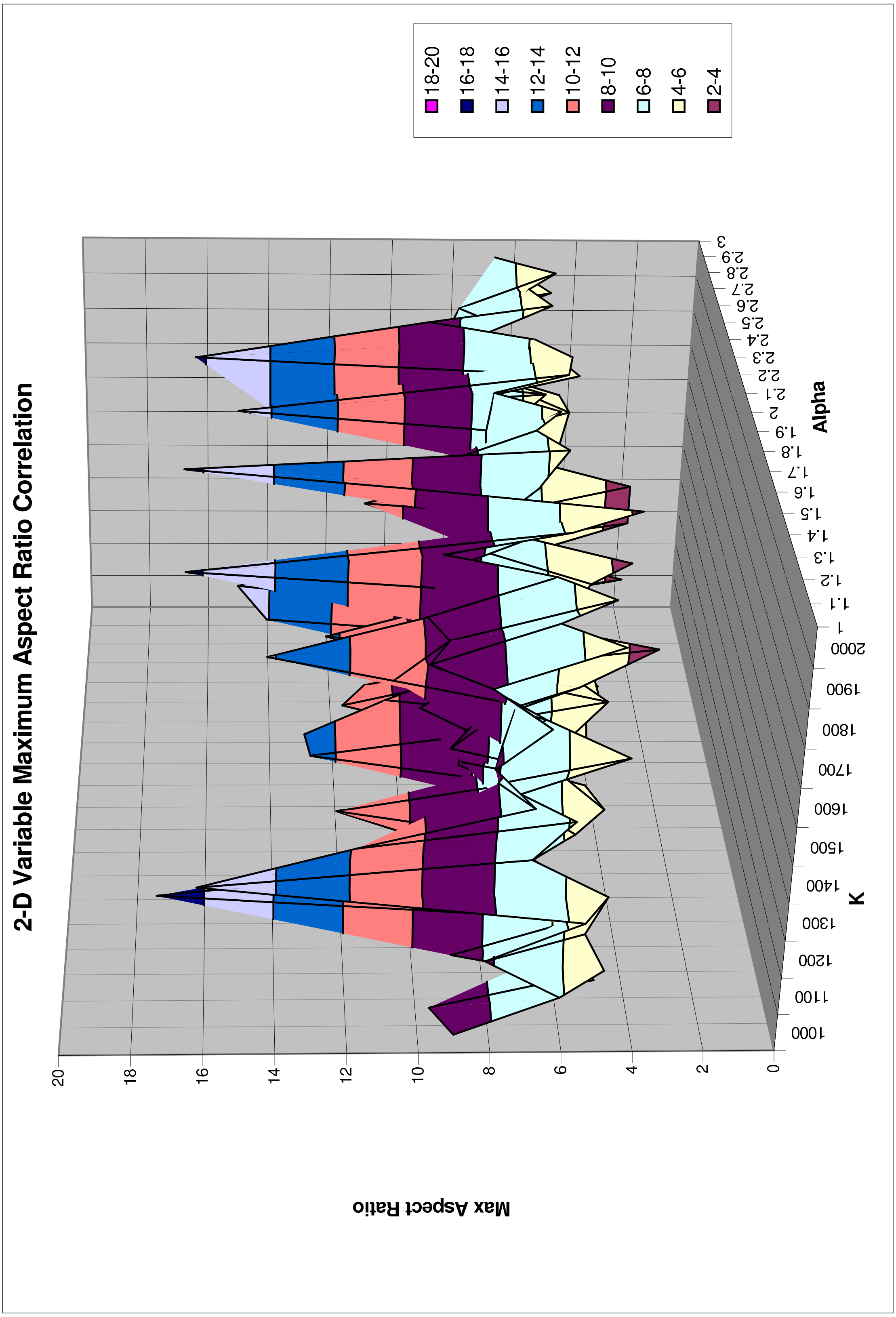
Appendix II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Appendix III - AMPL Code used for IBIMODEL

IBIMODEL was implemented in AMPL modeling language for mathematical programming. The AMPL code is shown below:

set DEP ordered;
param $w_f^{min} > 0$;
param $w_f^{max} \geq w_f^{min}$;
param $h_f^{min} > 0$;
param $h_f^{max} \geq h_f^{min}$;
param a {DEP} > 0;
param r {i in DEP} ≥ 0, := $\sqrt{\frac{\text{a[i]}}{\pi}} * \log_2(1 + \text{a[i]}/\varphi^2)$;
param c {DEP, DEP} ≥ 0;
param $\alpha > 0$;
param $\epsilon \geq 0$;
param t {i in DEP, j in DEP} := $(\text{r[i]+r[j]})^2$;
param τ {i in DEP, j in DEP} := if t[i,j] > $\sqrt{\frac{\text{t[i,j]}}{\text{c[i,j]}+\epsilon}}$ then t[i,j]
else $\sqrt{\frac{\text{t[i,j]}}{\text{c[i,j]}+\epsilon}}$;

param v {i in DEP, j in DEP} := if t[i,j] > $\sqrt{\frac{\text{t[i,j]}}{\text{c[i,j]}+\epsilon}}$ then c[i,j]*t[i,j]
else $\sqrt{\text{t[i,j]*c[i,j]} + \epsilon - 1}$;

param D {i in DEP, j in DEP} := $(\text{x[i]-x[j]})^2 + (\text{y[i]-y[j]})^2$;
param k ≥ 0;

var x {DEP} ;
var y {DEP} ;
var $w_f \geq w_f^{min}, \leq w_f^{max}$;
var $h_f \geq h_f^{min}, \leq h_f^{max}$;

minimize Cost: $\sum_{i<j} \text{c[i,j]*D[i,j]} + \frac{\alpha t\text{[i,j]}}{\text{D[i,j]}} - 1$

$+ \max(0, \min(1, \lceil \tau\text{[i,j]-D[i,j]} \rceil)) * (\text{v[i,j]-c[i,j]D[i,j]-} \frac{\alpha t\text{[i,j]}}{\text{D[i,j]}} + 1) \text{ - k log} \left(\frac{D[i,j]}{t[i,j]} \right)$;

s.t. Const1 {i in DEP} : x[i] + r[i] - $\frac{1}{2} w_f$ <= 0;
s.t. Const2 {i in DEP} : r[i] - x[i] - $\frac{1}{2} w_f$ <= 0;
s.t. Const3 {i in DEP} : y[i] + r[i] - $\frac{1}{2} h_f$ <= 0 ;
s.t. Const4 {i in DEP} : r[i] - y[i] - $\frac{1}{2} h_f$ <= 0 ;