Topics in the Geometry of Special Riemannian Structures

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Pure Mathematics

Waterloo, Ontario, Canada, 2024

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

The thesis consists of two chapters. The first chapter is the paper named "Betti numbers of nearly G_2 and nearly Kähler 6-manifolds with Weyl curvature bounds" which is now in the journal Geometriae Dedicata. Here we use the Weitzenböck formulas to get information about the Betti numbers of compact nearly G_2 and compact nearly Kähler 6-manifolds. First, we establish estimates on two curvature-type self adjoint operators on particular spaces assuming bounds on the sectional curvature. Then using the Weitzenböck formulas on harmonic forms, we get results of the form: if certain lower bounds hold for these curvature operators then certain Betti numbers are zero. Finally, we combine both steps above to get sufficient conditions of vanishing of certain Betti numbers based on the bounds on the sectional curvature.

The second chapter is the paper written with my supervisor Spiro Karigiannis named "A special class of k-harmonic maps inducing calibrated fibrations", to appear in the journal Mathematical Research Letters. Here we consider two special classes of k-harmonic maps between Riemannian manifolds which are related to calibrated geometry, satisfying a first order fully nonlinear PDE. The first is a special type of weakly conformal map $u\colon (L^k, g) \to (M^n, h)$ where $k \leqslant n$ and α is a calibration k-form on M. Away from the critical set, the image is an α-calibrated submanifold of M. These were previously studied by Cheng–Karigiannis–Madnick when α was associated to a vector cross product, but we clarify that such a restriction is unnecessary. The second, which is new, is a special type of weakly horizontally conformal map $u\colon (M^n, h) \to (L^k, g)$ where $n \geqslant k$ and α is a calibration $(n-k)$-form on M. Away from the critical set, the fibres $u^{-1}\{u(x)\}$ are α-calibrated submanifolds of M. We also review some previously established analytic results for the first class; we exhibit some explicit noncompact examples of the second class, where (M, h) are the Bryant–Salamon manifolds with exceptional holonomy; we remark on the relevance of this new PDE to the Strominger–Yau–Zaslow conjecture for mirror symmetry in terms of special Lagrangian fibrations and to the G_2 version by Gukov–Yau–Zaslow in terms of coassociative fibrations; and we present several open questions for future study.

Acknowledgments

I would like to express my deepest gratitude to my supervisor, Professor Spiro Karigiannis, whose expertise, understanding, and patience added considerably to my graduate experience. Without his guidance and persistent help, this thesis would not have been possible.

Spiro Karigiannis provided me with the freedom to explore on my own, while also steering me in the right direction whenever I needed it. His insightful feedback, constant support, and encouragement were instrumental in the successful completion of this work. I am extremely fortunate to have had the opportunity to work under his supervision.

I would also like to extend my thanks to the faculty and staff of the Department of Pure Mathematics at the University of Waterloo for their invaluable assistance and for providing a stimulating environment for research.

Additionally, I must express my profound gratitude to my family for their unfailing support and continuous encouragement throughout my years of study and through the process of researching and writing this thesis. This accomplishment would not have been possible without them.

Finally, I would like to thank Uwe Semmelmann who communicated with my supervisor about some of the topics, and the reviewer for the detailed and useful feedback on an earlier version of the paper which is the first chapter. Secondly, Spiro Karigiannis and I thank Da Rong Cheng and Jesse Madnick for useful discussions in relation to the paper which makes up the second chapter.

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Preliminaries

Both chapters of this thesis are parts of Riemannian geometry, however they are only tangentially related to each other. They both involve studies of special structures on Riemannian manifolds. Hence, we keep motivations of each topic separate, and in this section we just introduce common notation as to not repeat it twice. However, each chapter will also have its own small notation section.

All manifolds are *oriented* Riemannian manifolds. For the first chapter we crucially need the assumption of compactness, however, not for the second chapter. As usual a superscript on a manifold such as M^n means $\dim M = n$.
We often use the Riemannian metric (via the musical isomorphism) to identify vector fields and 1-forms. By $\mathcal{T}^k$ we denote k-tensors, by $\mathcal{S}^k$ the symmetric k-tensors, by $\mathcal{S}^2_0$ the traceless symmetric 2-tensors, by Ω^k the k-forms, and $\star$ for the Hodge star operator.
We also define the wedge product without any constants, meaning that for $\alpha, \beta \in \Omega^1$ we set

$$\alpha \wedge \beta := \alpha \otimes \beta - \beta \otimes \alpha,$$

and extend to the higher order forms to preserve associativity.
The inner product on k-forms we define as follows. For $\alpha, \beta \in \Omega^k$:

$$\langle \alpha, \beta \rangle = \frac{1}{k!} \alpha_{i_1 \dots i_k} \beta_{i_1 \dots i_k},$$

in terms of a local orthonormal frame.
We write $\operatorname{div}\colon \mathcal{T}^m \to \mathcal{T}^{m-1}$ for the Riemannian *divergence*, given in terms of a local orthonormal frame by

$$(\operatorname{div} A)_{j_1 \cdots j_{m-1}} = \nabla_i A_{i j_1 \cdots j_{m-1}}.$$

Finally, for $\sigma \in \Omega^k$ and $h \in \mathcal{T}^2$, we define $h \diamond \sigma \in \Omega^k$ as:

$$(h \diamond \sigma)_{i_1 \cdots i_k} := h_{i_1 p} \sigma_{p i_2 \cdots i_k} + h_{i_2 p} \sigma_{i_1 p i_3 \cdots i_k} + \cdots + h_{i_k p} \sigma_{i_1 \cdots i_{k-1} p}.$$

Chapter 1

Betti numbers of nearly G_2 and nearly Kähler 6-manifolds with Weyl curvature bounds

1.1 Introduction

1.1.1 Motivation

There is a long history of using Bochner-Weitzenböck technique to conclude vanishing results of Betti numbers of compact Riemannian manifolds assuming curvature bounds. In this chapter we establish several results, particulary for compact nearly G_2 and compact nearly Kähler 6-manifolds. We show that certain bounds on the sectional curvature imply vanishing of the second or the third Betti numbers.
Nearly G_2 and nearly Kähler 6-manifolds are spin, positive Einstein manifolds, which by Myers's theorem implies that they have finite fundamental group and hence $b_1 = 0$. They are the only possible manifolds whose metric cones have Spin(7) and G_2 holonomy, respectively. These, in turn, are useful from the physics perspective as they provide local models for the simplest type of interesting singularities. Hence, studying the topology of compact nearly G_2 and compact nearly Kähler 6-manifolds might lead to new insights. See [36] and [37] for results relating Betti numbers and linear stability.

1.1.2 Organization of the chapter and main results

Following Bourguignon-Karcher [6], we consider two curvature-type operators $\hat{R} \in \mathcal{S}^2(\Omega^2)$, $\mathring{R} \in \mathcal{S}^2(\mathcal{S}^2)$ and the usual sectional curvature $\bar{R}$ coming from the Riemannian curvature. We prove the following theorems that give us bounds on these operators in terms of the bounds

on the sectional curvature.
Here is a summary of the main results. Throughout, $[a \pm b]$ means $[a-b, a+b]$.
First, we reprove the following result from [6].
Theorem 1.2.12 Assume $\delta \leqslant \bar{R} \leqslant \Delta$. Then the eigenvalues of $\hat{R}$ on Ω^2 lie in the following interval:

$$\left[-(\Delta+\delta) \pm \frac{4\lfloor \frac{n}{2} \rfloor - 1}{3} (\Delta - \delta) \right].$$

Then, for nearly G_2 or nearly Kähler 6-manifolds, we improve the previous result on certain subspaces: **Corollary 1.2.14** Assume $\delta \leqslant \bar{R} \leqslant \Delta$. Moreover let M be a nearly G_2 or a nearly Kähler 6-manifold. Then on Ω^2_{14} or Ω^2_8, respectively, the eigenvalues of $\hat{R}$ lie in the following interval:

$$\left[-(\Delta+\delta) \pm \frac{7}{3} (\Delta - \delta) \right].$$

Next, we again reprove a theorem from [6] for $\mathring{R}$ in the general setting:
Corollary 1.2.16 Assume $\delta \leqslant \bar{R} \leqslant \Delta$. Then all but one of the eigenvalues of $\mathring{R}$ on $\mathcal{S}^2$ lie in the following interval:

$$\left[\frac{1}{2} \Big((\Delta+\delta) \pm (n-1)(\Delta-\delta) \Big) \right],$$

and the other one lies in the interval:

$$[-(n-1)\Delta, -(n-1)\delta].$$

Following, we slightly improve the estimates for $\mathring{R}$ in the Einstein case:
Theorem 1.2.17 Suppose M is Einstein with Einstein constant k. Assume $\delta \leqslant \bar{R} \leqslant \Delta$. Then the eigenvalues of $\mathring{R}$ on $\mathcal{S}^2_0$ lie in the intersection of the following intervals:

$$[-k+n\delta, k-(n-2)\delta], [k-(n-2)\Delta, -k+n\Delta].$$

Next, for the nearly Kähler 6-manifolds, we can talk about eigenvalues of $\mathring{R}$ on $\mathcal{S}^2_{+0} \subseteq \mathcal{S}^2$ (see Remarks 1.5.30 and 1.5.49). Hence, we are able to get a better estimate in this case:
Theorem 1.2.19 Assume $\delta \leqslant \bar{R} \leqslant \Delta$. Assume we are in the setting of a nearly Kähler 6-manifold. Then the eigenvalues of $\mathring{R}$ on $\mathcal{S}^2_{+0}$ (see Remark 1.5.30 for definition) lie in the following interval:

$$\left[\frac{1}{2} \Big((\Delta+\delta) \pm 3(\Delta-\delta) \Big) \right] = [2\delta - \Delta, 2\Delta - \delta].$$

Finally, we will see that again, on a nearly Kähler 6-manifold, we have a specific relationship between $\hat{R}$ on Ω^2_8 and $\mathring{R}$ on $\mathcal{S}^2_{+0}$, see Remark 1.2.22. This allows us to get estimates for $\hat{R}$ on Ω^2_8 in terms of the ones for $\mathring{R}$ on $\mathcal{S}^2_{+0}$ and vice versa. That is we can combine

Corollary 1.2.14 and Theorem 1.2.19 to get the following two statements:

Theorem 1.2.15 Let M be a nearly Kähler 6-manifold. Let $\delta \leqslant \bar{R} \leqslant \Delta$. Then the eigenvalues of $\hat{R}$ on Ω^2_8 lie in the intersection of the following intervals:

$$[-4 + (\Delta + \delta) \pm 3(\Delta - \delta)] \, , \left[-(\Delta + \delta) \pm \frac{7}{3}(\Delta - \delta)\right].$$

Theorem 1.2.25 Let M be a nearly Kähler 6-manifold. Let $\delta \leqslant \bar{R} \leqslant \Delta$. Then the eigenvalues of $\mathring{R}$ on $\mathcal{S}^2_{+0}$ lie in the intersection of the following intervals:

$$\left[\frac{1}{2}\Big((\Delta + \delta) \pm 3(\Delta - \delta)\Big)\right], \left[2 + \frac{1}{2}\Big(-(\Delta + \delta) \pm \frac{7}{3}(\Delta - \delta)\Big)\right].$$

In the next section, where we introduce the Weitzenböck formulas (which relates the Laplacian Δ and the rough Laplacian (or Bochner Laplacian) $\nabla^*\nabla$ in terms of the Riemannian and Ricci curvatures) for 2-forms and 3-forms on nearly G_2 or nearly Kähler 6-manifolds, the results are not new and can be found in the literature. However, we aim to keep the chapter as self-contained as possible, so we include all the proofs, but we cite the results when appropriate.

The main idea is that for nearly Kähler and nearly G_2 manifolds, harmonic 2-forms and harmonic 3-forms are of a special algebraic type. In the case of 2-forms this means that we need to consider the map $\hat{R}$ (or $\hat{W}$, where W is the Weyl tensor) only on certain subspaces of Ω^2.

Moreover, when we apply the Weitzenböck formulas to harmonic forms to obtain sufficient conditions for certain Betti numbers to vanish in terms of lower bounds of $\hat{W}$ and $\mathring{W}$ (which is equivalent to some lower bounds on $\bar{R}$ and $\mathring{R}$), we get better estimates by considering the Weitzenböck formulas written in the intermediate forms. For example, consider (1.5.61):

$$\Delta\beta = \nabla^*\nabla\beta + 8\beta + \hat{W}\beta, \text{ for any } \beta \in \Omega^2.$$

Assuming $\beta = h \diamond \omega$ (see Section 1.5.2) is harmonic for some $h \in \mathcal{S}^2_{+0}$, we can rewrite this as:

$$0 = \nabla^*\nabla\beta + (8h + 2\mathring{W}h) \diamond \omega = (\nabla^*\nabla h - 2h) + (8h + 2\mathring{W}h) \diamond \omega,$$

which is Proposition 1.5.62. So, even though the last part is a well-known formula, we actually get better sufficient conditions for vanishing of b_2 in terms of the lower bound of $\mathring{W}$ by using the intermediate step above. Similar things happen in other cases as well.

We summarize the results we obtain in the folowing table:

Sufficient conditions for vanishing of Betti numbers		
Manifold type	$b_2 = 0$	$b_3 = 0$
Compact nearly G_2	$\mathcal{S}^2(\Omega^2_{14}) \ni \hat{W} \geqslant -\frac{5\tau_0^2}{8}$ (1.4.7)	$\mathcal{S}^2(\mathcal{S}^2_0) \ni \mathring{W} \geqslant -\frac{3\tau_0^2}{8}$ (1.4.14), or $\mathcal{S}^2(\Omega^2_{14}) \ni \hat{W} \geqslant -\frac{\tau_0^2}{4}$ (1.4.14)
Compact nearly Kähler of dim 6	$\mathcal{S}^2(\Omega^2_8) \ni \hat{W} \geqslant -8$ (1.5.63), or $\mathcal{S}^2(\mathcal{S}^2_{+0}) \ni \mathring{W} \geqslant -4$ (1.5.63)	$\mathcal{S}^2(\mathcal{S}^2_-) \ni \mathring{W} \geqslant -\frac{9}{2}$ (1.5.70), or $\mathcal{S}^2(\Omega^2_8) \ni \hat{W} \geqslant -3$ (1.5.70)

We also use the fact that there are no parallel non-zero 2-forms and no parallel non-zero traceless symmetric 2-tensors. This is true because the restricted holonomy is exactly $\mathrm{SO}(n)$. One can see this by observing that nearly G_2 and nearly Kähler manifolds admit a Killing spinor which implies that they are not locally reducible and nonsymmetric (the arguments can be found in [4]), hence the result follows by Berger's classification. As corollaries, we obtain sufficient conditions for vanishing of the Betti numbers from inequalities $\delta \leqslant \bar{R} \leqslant \Delta$, which we again summarize in the following table:

Sufficient conditions for vanishing of Betti numbers		
Manifold type	$b_2 = 0$	$b_3 = 0$
Compact nearly G_2	$-(\Delta+\delta) - \frac{7}{3}(\Delta-\delta) \geqslant -\frac{3\tau_0^2}{4}$ (1.4.8)	$\Delta \leqslant \frac{11\tau_0^2}{80}$ (1.4.15), or $\delta \geqslant \frac{\tau_0^2}{112}$ (1.4.15)
Compact nearly Kähler of dim 6	$-(\Delta+\delta) - \frac{7}{3}(\Delta-\delta) \geqslant -10$ (1.5.64), or $(\Delta+\delta) - 3(\Delta-\delta) \geqslant -6$ (1.5.64)	$\delta \geqslant \frac{1}{4}$ (1.5.71), or $\Delta \leqslant \frac{17}{8}$ (1.5.71)

Finally, for both nearly G_2 and nearly Kähler cases, we check our results on one example of a compact normal homogeneous manifold. The corollaries discussed above may not appear to be that useful for the known examples, as calculating the bounds for $\bar{R}$ is a harder process than just getting the bounds for $\hat{R}$ and $\mathring{R}$. However, our theorems are not limited to just these known examples, hence are interesting on their own.

Since the work of this chapter is mostly algebraic in nature, it is possible it can be adapted to other settings. For this, we would need an Einstein metric, a decomposition of forms which is preserved by $\hat{R}$ and $\mathring{R}$ and harmonic forms to be of special algebraic type.

1.1.3 Notation

Throughout this paper (M^n, g) is a compact connected Riemannian manifold.
We define the *Kulkarni–Nomizu* product as follows. For $s, t \in \mathcal{T}^2$ we define $s \owedge t \in \mathcal{T}^4$ to be:

$$(s \owedge t)_{ijkl} := s_{il}t_{jk} + s_{jk}t_{il} - s_{ik}t_{jl} - s_{jl}t_{ik}. \tag{1.1.1}$$

To simplify the interval notation, by $[a \pm b]$, we mean $[a - b, a + b]$, for $a, b \in \mathbb{R}, b > 0$.
Finally, whenever we refer to $\delta \leqslant \Delta$, these are any real numbers.

Remark 1.1.2. Let Riem be the Riemann curvature tensor. We will write R_{ijkl} for Riem_{ijkl}, R_{ij} for $\mathrm{Ric}_{ij} := R_{kijl}g^{kl}$ and $R := \mathrm{Ric}_{ij}\, g^{ij}$ for the scalar curvature when there is no confusion.
Also we define the traceless Ricci tensor:

$$\mathrm{Ric}^0 := \mathrm{Ric} - \frac{1}{n} Rg.$$

Then on a general Riemannian manifold of dimension $n \geqslant 3$ we have the following orthogonal decomposition of Riem (see [5]). Define

$$\begin{aligned} \text{traceless Ricci part: } E &:= \frac{1}{n-2} \mathrm{Ric}^0 \owedge g, \\ \text{scalar part: } S &:= \frac{R}{2n(n-1)} g \owedge g, \\ \text{Weyl part: } W &:= \mathrm{Riem} - E - S. \end{aligned}$$

Then we have:

$$\mathrm{Riem} = S + E + W.$$

Also, we say that (M^n, g) is *Einstein* with *Einstein constant* k if $\mathrm{Ric} = kg$. In this case the scalar curvature is $R = nk$ and $\mathrm{Ric}^0 = 0$, thus $E = 0$. So, $\mathrm{Riem} = S + W$, for an Einstein metric.
We also have, by construction, that $W_{kijl}g_{kl} = 0$.

1.2 Curvature estimates

Throughout this section, we let (M, g) be a Riemannian manifold. First, we define a notion of a curvature tensor A. Then following Bourguignon-Karcher [6] we introduce two self-adjoint operators $\hat{A}$ and $\mathring{A}$ and in Sections 1.2.1 and 1.2.2 we obtain multiple results for bounds of $\hat{W}$ and $\mathring{W}$ in terms of bounds on the sectional curvature $\hat{R}$, where W is the Weyl tensor. In particular, we strengthen some of the results from [6] in the nearly G_2 and nearly Kähler of dimension 6 settings.

Definition 1.2.1. We say an element $A \in \mathcal{T}^4$ is an *algebraic curvature tensor*, if the following properties hold:

- $A_{ijkl} = -A_{jikl} = -A_{ijlk} = A_{klij}$.
- $A_{ijkl} + A_{kijl} + A_{jkil} = 0$ (Bianchi identity).

Let $\mathcal{R}$ be the set of algebraic curvature tensors. Note that $\mathcal{R}$ is a module over C^∞. ▲

Remark 1.2.2. If $s, t \in \mathcal{S}^2$, then $s \owedge t \in \mathcal{R}$. This follows directly from the definition of $\owedge$ in (1.1.1). Hence, it follows from Remark 1.1.2 that W is also a curvature tensor.

Definition 1.2.3. Let $A \in \mathcal{R}$. Following Bourguignon-Karcher [6], we define

$$\hat{A} \in \mathcal{S}^2(\Omega^2) \text{ as } (\hat{A}\beta)_{ij} = A_{ijkl}\beta_{kl}, \text{ for } \beta \in \Omega^2.$$

$$\mathring{A} \in \mathcal{S}^2(\mathcal{S}^2) \text{ by } (\mathring{A}h)_{ij} = A_{kilj}h_{kl}, \text{ by } h \in \mathcal{S}^2,$$

$$\bar{A} \text{ by } \bar{A}(X \wedge Y) = \frac{A(X,Y,Y,X)}{\|X \wedge Y\|^2}, \text{ for linearly independent } X, Y \in \Gamma(TM).$$

In particular, in an orthonormal frame: $\bar{A}(e_i \wedge e_j) = A_{ijji}$, for $i \neq j$ (with no sum over indices). We also call $\bar{A}$ the *sectional curvature* of A, it is a smooth function on the space of 2-planes on M. ▲

For the sake of completeness, we show that indeed, $\hat{A} \in \mathcal{S}^2(\Omega^2)$ and $\mathring{A} \in \mathcal{S}^2(\mathcal{S}^2)$.
Let $\beta, \gamma \in \Omega^2$. Then:

$$(\hat{A}\beta)_{ij} = A_{ijkl}\beta_{kl} = -A_{jikl}\beta_{kl} = -(\hat{A}\beta)_{ji}.$$

$$\langle \hat{A}\beta, \gamma \rangle = \frac{1}{2}A_{ijkl}\beta_{kl}\gamma_{ij} = \frac{1}{2}\beta_{kl}A_{klij}\gamma_{ij} = \langle \beta, \hat{A}\gamma \rangle.$$

Now, let $h, s \in \mathcal{S}^2$. Then:

$$(\mathring{A}h)_{ij} = A_{kilj}h_{kl} = A_{ljki}h_{lk} = (\mathring{A}h)_{ji}.$$

$$\langle \mathring{A}h, s \rangle = A_{kilj}h_{kl}s_{ij} = h_{kl}A_{ljki}s_{ij} = h_{lk}A_{jlik}s_{ji} = \langle h, \mathring{A}s \rangle.$$

Remark 1.2.4. We can extend the map $\hat{A}$ to any k-form for $k > 2$ as follows: for $\beta \in \Omega^k$ we define $\hat{A}\beta \in \Omega^2 \bigotimes \Omega^{k-2}$ as

$$(\hat{A}\beta)_{i_1 \ldots i_{k-2}} := A_{i_1 i_2 ab}\beta_{abi_3 \ldots i_{k-2}},$$

that is, we just fix the last $k-2$ indices and think of β as a 2-form in the first two indices.
Also, note that $W_{kijl}g_{kl} = 0$ implies that for any $h \in \mathcal{S}^2$, we have $\mathring{W}h \in \mathcal{S}^2_0$.
From now on we will also use $R, \hat{R}, \mathring{R}, \bar{R}$ instead of Riem, $\widehat{\text{Riem}}$, etc., which should be clear from the context, and similarly for W.

Lemma 1.2.5. *The following identities hold:*

- $g \bar{\owedge} g = 2.$
- $g \hat{\owedge} g = -4 \operatorname{Id}.$
- $g \mathring{\owedge} g = 2 \operatorname{Id}$ *on* $\mathcal{S}^2_0$,

where by $g \bar{\owedge} g$ *we mean that we apply the* $^-$ *operator to* $g \owedge g \in \mathcal{R}$*. Similarly, for the* $g \hat{\owedge} g$ *and* $g \mathring{\owedge} g$*.*

Proof. For any $X, Y \in \Gamma(TM)$, we have:

$$\begin{aligned}(g \bar{\owedge} g)(X \wedge Y) &= \frac{(g \owedge g)(X, Y, Y, X)}{\|X \wedge Y\|^2} \\ &= \frac{2\|X\|^2\|Y\|^2 - 2\langle X, Y\rangle^2}{\|X \wedge Y\|^2} \\ &= 2.\end{aligned}$$

Next, let $\beta \in \Omega^2$ and $h \in \mathcal{S}^2_0$. Then in an orthonormal frame:

$$\begin{aligned}((g \hat{\owedge} g)\beta)_{ij} &= (g \owedge g)_{ijkl}\beta_{kl} \\ &= 2(g_{il}g_{jk} - g_{ik}g_{jl})\beta_{kl} \\ &= 2(\beta_{ji} - \beta_{ij}) \\ &= -4\beta_{ij},\end{aligned}$$

and

$$\begin{aligned}((g \mathring{\owedge} g)h)_{jl} &= (g \owedge g)_{ijkl}h_{ik} \\ &= 2(g_{il}g_{jk} - g_{ik}g_{jl})h_{ik} \\ &= 2(h_{lj} - \operatorname{tr}(h)g_{jl}) \\ &= 2h_{jl}.\end{aligned}$$

giving us the required results. □

In order to simplify the proofs of the following theorems we make the following definition:

Definition 1.2.6. Assume that:

$$\delta \leqslant \bar{R} \leqslant \Delta,$$

where δ, Δ are any real constants. This means that for all $X, Y \in \Gamma(TM)$ with $\|X \wedge Y\|^2 = 1$, we have $\delta \leqslant \bar{R}(X \wedge Y) \leqslant \Delta$.
Define

$$R_0 := R - \frac{\delta + \Delta}{4} g \owedge g.$$

Now, by Lemma 1.2.5, $\bar{R}_0 = \bar{R} - \frac{\delta+\Delta}{2}$, so that

$$|\bar{R}_0| \leqslant \frac{\Delta - \delta}{2}. \tag{1.2.7}$$

Note that $R_0 \in \mathcal{R}$, because both $R, g \owedge g \in \mathcal{R}$. ▲

Next, we note that in the Einstein case, $\hat{W}$ and $\hat{R}$ differ by a constant multiple of the identity. The same holds for $\mathring{W}$ and $\mathring{R}$ on $\mathcal{S}_0^2$ (the constant is not the same though).

Lemma 1.2.8. *Assume M is Einstein with Einstein constant k. Then*

$$\hat{W} = \hat{R} + \frac{2k}{n-1}\,\mathrm{Id},$$
$$\mathring{W} = \mathring{R} - \frac{k}{n-1}\,\mathrm{Id}, \quad \text{on } \mathcal{S}_0^2.$$

Proof. By Remark 1.1.2, $W = R - S$. Using Lemma 1.2.5, we have

$$\hat{S} = \frac{R}{2n(n-1)} g \hat{\owedge} g = -\frac{nk}{2n(n-1)} 4\,\mathrm{Id} = -\frac{2k}{n-1}\,\mathrm{Id}\,.$$

Similarly on $\mathcal{S}_0^2$ we have

$$\mathring{S} = \frac{R}{2n(n-1)} g \mathring{\owedge} g = \frac{nk}{2n(n-1)} 2\,\mathrm{Id} = \frac{k}{n-1}\,\mathrm{Id},$$

hence, the results follow. □

Finally, we have an observation about the a priori values of δ, Δ in the Einstein case.

Remark 1.2.9. Assume (M^n, g) is Einstein with Einstein constant k. Let $\delta \leqslant \bar{R} \leqslant \Delta$. Then:

$$\delta \leqslant \frac{k}{n-1} \leqslant \Delta.$$

Proof. We compute

$$nk = R = \sum_{i=1}^{n} R_{ii} = \sum_{i,j=1}^{n} R_{ijji} = \sum_{i \neq j} \bar{R}(e_i \wedge e_j) \leqslant n(n-1)\Delta,$$

as when $i = j$, $R_{ijji} = 0$. So, $k \leqslant (n-1)\Delta$. The other inequality is done similarly. □

1.2.1 Estimates for $\hat{R}$

In this section we investigate what sectional curvature bounds tell us about the bounds of $\hat{R}$. Since in the Einstein case, $\hat{R}$ and $\hat{W}$ differ by a constant multiple of the identity map, one can use the result above to get bounds for $\hat{W}$.
First, we prove a lemma which gives us bounds for R_0 in terms of bounds of $\bar{R}$. Note that one can similarly obtain bounds for R itself, but we do not need this.

Lemma 1.2.10. *Assume $\delta \leqslant \bar{R} \leqslant \Delta$. Let $X, Y, Z, W \in TM$ be unit length. Then $|R_0(X,Y,Z,W)| \leqslant \frac{2}{3}(\Delta - \delta)$.*

Proof. This result is Lemma 3.7 in [6], but we provide all the details.
Without loss of generality, assume $X \neq \pm W$ and $Y \neq \pm Z$. Otherwise, swap Z and W. If even after swapping, that is not achieved, it means, Z and W are multiples of each other, so $R_0(X,Y,Z,W) = 0$.
We claim that

$$\begin{aligned} 6R_0(X,Y,Z,W) =& R_0(X,Y+Z,Y+Z,W) - R_0(Y,X+Z,X+Z,W) \\ &- R_0(X,Y-Z,Y-Z,W) + R_0(Y,X-Z,X-Z,W). \end{aligned} \tag{1.2.11}$$

Expanding the RHS we get:

$$\begin{aligned} &R_0(X,Y,Y,W) + R_0(X,Z,Z,W) + R_0(X,Y,Z,W) + R_0(X,Z,Y,W) \\ &- R_0(Y,X,X,W) - R_0(Y,Z,Z,W) - R_0(Y,X,Z,W) - R_0(Y,Z,X,W) \\ &- R_0(X,Y,Y,W) - R_0(X,Z,Z,W) + R_0(X,Y,Z,W) + R_0(X,Z,Y,W) \\ &+ R_0(Y,X,X,W) + R_0(Y,Z,Z,W) - R_0(Y,X,Z,W) - R_0(Y,Z,X,W) \\ =&4R_0(X,Y,Z,W) - 2\Big(R_0(Z,X,Y,W) + R_0(Y,Z,X,W)\Big) \\ =&6R_0(X,Y,Z,W), \end{aligned}$$

as claimed. Now, consider one of the terms $R_0(X,Y+Z,Y+Z,W)$:

$$\begin{aligned} &R_0(X,Y+Z,Y+Z,W) \\ =&\frac{1}{4}\Big(R_0(X+W,Y+Z,Y+Z,X+W) - R_0(X-W,Y+Z,Y+Z,X-W)\Big) \\ =&\frac{\|Y+Z\|^2}{4}\Big(\|X+W\|^2 R_0(\frac{X+W}{\|X+W\|}, \frac{Y+Z}{\|Y+Z\|}, \frac{Y+Z}{\|Y+Z\|}, \frac{X+W}{\|X+W\|}) \\ &- \|X-W\|^2 R_0(\frac{X-W}{\|X-W\|}, \frac{Y+Z}{\|Y+Z\|}, \frac{Y+Z}{\|Y+Z\|}, \frac{X-W}{\|X-W\|})\Big). \end{aligned}$$

Now, note that for unit length vectors S, T we have:

$$|R_0(S,T,T,S)| = |\bar{R}_0(S,T)|(\|S\|^2\|T\|^2 - \langle S,T\rangle^2) \leqslant |\bar{R}_0(S \wedge T)|.$$

Thus:

$$
\begin{aligned}
|R_0(X, Y+Z, Y+Z, W)| &\leqslant \frac{\|Y+Z\|^2}{4}\Big(\|X+W\|^2|\bar{R}_0(\frac{X+W}{\|X+W\|} \wedge \frac{Y+Z}{\|Y+Z\|})| \\
&+ \|X-W\|^2|\bar{R}_0(\frac{X-W}{\|X-W\|} \wedge \frac{Y+Z}{\|Y+Z\|})|\Big) \\
&\leqslant \frac{\|Y+Z\|^2}{4}(\|X+W\|^2 + \|X-W\|^2)\frac{\Delta-\delta}{2} \text{ (by (1.2.7))} \\
&= \|Y+Z\|^2\frac{\Delta-\delta}{2}.
\end{aligned}
$$

Hence, applying the same inequalities for the other terms, equation (1.2.11) becomes:

$$
\begin{aligned}
6|R_0(X,Y,Z,W)| &\leqslant (\|Y+Z\|^2 + \|X+Z\|^2 + \|Y-Z\|^2 + \|X-Z\|^2)\frac{\Delta-\delta}{2} \\
&= 4(\Delta-\delta),
\end{aligned}
$$

which yields the desired result. □

We are ready to get to the main theorem of this section. The first part applies to any manifold, however on certain subspaces of manifolds with G_2 or $SU(3)$-structure, we can improve the result.

Theorem 1.2.12. *Assume $\delta \leqslant \bar{R} \leqslant \Delta$. Then the eigenvalues of $\hat{R}$ lie in the following interval:*

$$
\left[-(\Delta+\delta) \pm \frac{4\lfloor\frac{n}{2}\rfloor - 1}{3}(\Delta-\delta)\right].
$$

Proof. Assume $\hat{r}$ is an eigenvalue of $\hat{R}$ with $0 \neq \beta \in \Omega^2$ the corresponding unit eigenvector. Note that $\hat{R}_0 = \hat{R} + (\Delta+\delta)\,\mathrm{Id}$, by Remark 1.2.5 and Definition 1.2.6. So, β is also an eigenvector for $\hat{R}_0$ with the eigenvalue $\hat{r}_0 = \hat{r} + (\delta+\Delta)$.
Assume β is of rank $2p$, so there exists an orthonormal basis $\{e_1, \dots, e_n\}$ such that $\beta =$

$\sum_{i=1}^{p} \beta_i e_i \wedge e_{\bar{i}}$, where $\bar{i} = i + p$. Then we have

$$\begin{aligned}
\hat{r}_0 \beta_j &= (\hat{r}_0 \beta)_{j\bar{j}} \\
&= (\hat{R}_0 \beta)_{j\bar{j}} \\
&= \sum_{i=1}^{p} \beta_i (\hat{R}_0 (e_i \wedge e_{\bar{i}}))_{j\bar{j}} \\
&= \sum_{i=1}^{p} \beta_i (R_0)_{plj\bar{j}} (e_i \wedge e_{\bar{i}})_{pl} \\
&= \sum_{i=1}^{p} \beta_i (R_0)_{plj\bar{j}} (\delta_{ip}\delta_{\bar{i}l} - \delta_{il}\delta_{\bar{i}p}) \\
&= 2 \sum_{i=1}^{p} \beta_i (R_0)_{i\bar{i}j\bar{j}}.
\end{aligned}$$

Now, take $|\beta_j| \neq 0$ maximal to obtain from the above that

$$\begin{aligned}
|\hat{r}_0| &\leqslant 2 \sum_{i=1}^{p} \Big|\frac{\beta_i}{\beta_j}\Big| |(R_0)_{i\bar{i}j\bar{j}}| \\
&= 2 \sum_{i \neq j} \Big|\frac{\beta_i}{\beta_j}\Big| |(R_0)_{i\bar{i}j\bar{j}}| + 2 |(\bar{R}_0)(e_j \wedge e_{\bar{j}})| \qquad (1.2.13) \\
&\leqslant 2(p-1)\frac{2}{3}(\Delta - \delta) + 2\frac{\Delta - \delta}{2} \text{ (by Lemma 1.2.10 and (1.2.7))} \\
&= \frac{4p-1}{3}(\Delta - \delta) \\
&\leqslant \frac{4\lfloor \frac{n}{2} \rfloor - 1}{3}(\Delta - \delta).
\end{aligned}$$

Recalling that $\hat{r}_0 = \hat{r} + (\delta + \Delta)$, we get the required result. □

Adding onto the work of Bourguignon-Karcher [6], the previous theorem can be improved for nearly G_2 or nearly Kähler 6-manifolds on certain subspaces.

Corollary 1.2.14. *In the nearly G_2 case on Ω^2_{14} or in the nearly Kähler case on Ω^2_8 the eigenvalues of $\hat{R}$ lie in the following interval:*

$$\left[-(\Delta + \delta) \pm \frac{7}{3}(\Delta - \delta) \right].$$

See Sections 1.4.1 and 1.5.1 for the descriptions of these manifolds and subspaces. Note that just the presence of a G_2 or an $SU(3)$ structure is not enough, as we need $\hat{R}$ to preserve

those subspaces.
Note that the previous Theorem 1.2.12, only would have given us $\frac{11}{3}$ instead of $\frac{7}{3}$.

Proof. For both the G_2-structure case on Ω^2_{14} or for the $SU(3)$-structure case on Ω^2_8, if we assume β is of rank $2p = 2, 4, 6$, then there exist canonical forms $\beta = \sum_{i=1}^{p} \beta_i e_i \wedge e_{\bar{i}}$, where $\bar{i} = i + p$, such that $\sum_{i=1}^{k} \beta_i = 0$, for some orthonormal basis $\{e_1, \dots e_n\}$ (in the case of G_2-structures, see [9], and in the case of $SU(3)$-structure, this follows because $\Lambda^2_8 \cong \mathfrak{su}(3)$). Taking $|\beta_j| \neq 0$ maximal forces the other β_i's, of which there are at most two, to be of the same sign, meaning that $|\beta_j| = \sum_{i \neq j} |\beta_i|$. Thus, continuing from (1.2.13), we can improve the previous estimate to:

$$\begin{aligned}
|\hat{r}_0| &\leqslant 2 \sum_{i \neq j} |\frac{\beta_i}{\beta_j}||(R_0)_{i\bar{i}j\bar{j}}| + 2|(\bar{R}_0)(e_j \wedge e_{\bar{j}})| \\
&\leqslant 2\frac{2}{3}(\Delta - \delta) + 2\frac{\Delta - \delta}{2} \text{ (by Lemma 1.2.10 and (1.2.7))} \\
&= \frac{7}{3}(\Delta - \delta).
\end{aligned}$$

which is enough to conclude the result. □

We will see that in the nearly Kähler case, the operators $\hat{W}$ and $\mathring{W}$ are closely related on certain subspaces. See Remark 1.2.22. Hence, we summarize the estimates for $\hat{R}$ on Ω^2_8 in the following Corollary:

Corollary 1.2.15. *Let M be a nearly Kähler 6-manifold. Let $\delta \leqslant \bar{R} \leqslant \Delta$. Then the eigenvalues of $\hat{R}$ on Ω^2_8 lie in the intersection of the following intervals:*

$$[-4 + (\Delta + \delta) \pm 3(\Delta - \delta)], \left[-(\Delta + \delta) \pm \frac{7}{3}(\Delta - \delta)\right].$$

Proof. This follows from Remark 1.2.22. □

1.2.2 Estimates for $\mathring{R}$

Note that when M is Einstein, $\mathring{R}, \mathring{W}$ preserve $\mathcal{S}^2_0$. This is because $\mathring{W}h \in \mathcal{S}^2_0$ for any $h \in \mathcal{S}$, by the properties of the Weyl tensor, and since $\mathring{R}$ and $\mathring{W}$ differ by a constant on $\mathcal{S}^2_0$, we get the required observation.

First, we prove a therorem that gives us bounds for $\mathring{R}$ on $\mathcal{S}^2$ in terms of bounds of $\bar{R}$. Next, we assume that M is Einstein which allows us to improve the result on $\mathcal{S}^2_0$.

Theorem 1.2.16. *Assume $\delta \leqslant \bar{R} \leqslant \Delta$. Then all but one of the eigenvalues of $\mathring{R}$ on $\mathcal{S}^2$ lie in the following interval:*

$$\left[\frac{1}{2}\Big((\Delta+\delta) \pm (n-1)(\Delta-\delta)\Big)\right],$$

and the other one lies in the interval:

$$\left[-(n-1)\Delta, -(n-1)\delta\right].$$

Proof. On $\mathcal{S}^2_0$, $\mathring{R} = \mathring{R}_0 + \frac{\delta+\Delta}{4} g \mathbin{\mathring{\owedge}} g = \mathring{R}_0 + \frac{\delta+\Delta}{2}\,\mathrm{Id}$, by Lemma 1.2.5 and Definition 1.2.6.
Recall that by Definition 1.2.6 we have that $\mathring{R} = \mathring{R}_0 + \frac{\delta+\Delta}{4} g \mathbin{\mathring{\owedge}} g$.
First, we show that $|\mathring{R}_0| \leqslant \frac{n-1}{2}(\Delta-\delta)$: Let $0 \neq h \in \mathcal{S}^2_0$ be a unit eigenvector of $\mathring{R}_0$ with the eigenvalue $\mathring{r}_0$. Assume h is of rank p for some $1 \leqslant p \leqslant n$. Then there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ such that $h = \sum_{i=1}^{p} h_i e_i \otimes e_i$. Thus:

$$\begin{aligned}
\mathring{r}_0 h_j &= (\mathring{R}_0 h)_{jj} \\
&= (R_0)_{mjlj} h_{ml} \\
&= (R_0)_{mjlj} h_m \delta_{ml} \\
&= \sum_m (R_0)_{mjmj} h_m.
\end{aligned}$$

Take $|h_j| \neq 0$ maximal. We obtain from the above that:

$$\begin{aligned}
|\mathring{r}_0| &\leqslant \sum_m \left|\frac{h_m}{h_j}\right| |(R_0)_{mjmj}| \\
&\leqslant (p-1)|\bar{R}_0| \\
&\leqslant (n-1)\frac{\Delta-\delta}{2},
\end{aligned}$$

yielding the required result.
Next, we investigate the eigenvalues of $\mathring{R} - \mathring{R}_0 = \frac{\delta+\Delta}{4} g \mathbin{\mathring{\owedge}} g$. It is easy to check that $(g \mathbin{\mathring{\owedge}} g)g = 2(1-n)g$, and we know that $g \mathbin{\mathring{\owedge}} g = 2\,\mathrm{Id}$ on $\mathcal{S}^2_0$, by Lemma 1.2.5.
Hence, the result follows from the Weyl's inequality for eigenvalues applied to $\mathring{R} = \mathring{R}_0 + (\mathring{R} - \mathring{R}_0)$. □

Theorem 1.2.17. *Suppose M is Einstein with Einstein constant k. Assume $\delta \leqslant \bar{R} \leqslant \Delta$. Then the eigenvalues of $\mathring{R}$ on $\mathcal{S}^2_0$ lie in the intersection of the following intervals:*

$$[-k+n\delta, k-(n-2)\delta], [k-(n-2)\Delta, -k+n\Delta].$$

Proof. For simplicity, introduce $R' := R - \frac{\delta}{2} g \owedge g$. Then $\bar{R}' = \bar{R} - \delta$ and $\mathring{R}' = \mathring{R} - \delta\,\mathrm{Id}$, by Remark 1.2.5. Hence, $\bar{R}' \geqslant 0$. By $\mathring{R}'$ we will mean $^\circ$ applied to R', and similarly for $\bar{R}'$. Let $0 \neq h \in \mathring{S}^2_0$ be a unit eigenvector of $\mathring{R}$ with the eigenvalue $\mathring{r}$. Note that h is also an eigenvector of $\mathring{R}'$ with the eigenvalue $\mathring{r}' = \mathring{r} - \delta$. Assume h is of rank p for some $1 \leqslant p \leqslant n$. Then there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ such that $h = \sum_{i=1}^{p} h_i e_i \otimes e_i$. Thus:

$$
\begin{aligned}
\mathring{r}' h_j &= (\mathring{R}' h)_{jj} \\
&= (R')_{mjlj} h_{ml} \\
&= (R')_{mjlj} h_m \delta_{ml} \\
&= \sum_{m=1}^{p} (R')_{mjmj} h_m \\
&= -\sum_{m=1}^{p} (\bar{R}')_{mj} h_m.
\end{aligned}
$$

Take $|h_j| \neq 0$ maximal. By replacing h by $-h$, if necessary, assume that $h_j > 0$. Note that now for all m, $-1 \leqslant \frac{h_m}{h_j} \leqslant 1$. Then since $\bar{R}' \geqslant 0$, we have:

$$
\begin{aligned}
-\mathring{r}' &= \sum_{m=1}^{p} \frac{h_m}{h_j} \bar{R}'(e_m \wedge e_j) \\
&\leqslant \sum_{m=1}^{n} \bar{R}'(e_m \wedge e_j) \\
&= \sum_{m=1}^{n} (\bar{R} - \delta)(e_m \wedge e_j) \\
&= \sum_{m=1}^{n} \bar{R}(e_m \wedge e_j) - (n-1)\delta.
\end{aligned}
$$

Finally, note that $\sum_{m=1}^{n} (\bar{R})_{mj} = \sum_{m=1}^{n} R_{mjjm} = R_{jj} = k g_{jj} = k$ (where the j was fixed.) Hence,

$$
-(\mathring{r} - \delta) = -\mathring{r}' \leqslant k - (n-1)\delta,
$$

which gives the required

$$
\mathring{r} \geqslant -k + n\delta.
$$

However, (this was not present in the Bourguignon-Karcher paper) since $-1 \leqslant \frac{h_m}{h_j}$, we can also do the following:

$$
\begin{aligned}
-\mathring{r}' &= \sum_{m=1}^{p} \frac{h_m}{h_j} \bar{R}'(e_m \wedge e_j) \\
&\geqslant -\sum_{m=1}^{p} \bar{R}'(e_m \wedge e_j) \\
&\geqslant -\sum_{m=1}^{n} \bar{R}'(e_m \wedge e_j) \\
&= -(k-(n-1)\delta).
\end{aligned}
$$

Hence, we also get

$$-(\mathring{r}-\delta) = -\mathring{r}' \geqslant -k+(n-1)\delta,$$

which is just

$$\mathring{r} \leqslant k-(n-2)\delta.$$

Thus, we have

$$-k+n\delta \leqslant \mathring{r} \leqslant k-(n-2)\delta.$$

The other inequality

$$k-(n-2)\Delta \leqslant \mathring{r} \leqslant -k+n\Delta$$

is proven in the similar way by introducing $R'' := R - \frac{\Delta}{2} g \owedge g$, so $\bar{R}'' \leqslant 0$. ☐

Remark 1.2.18. In [6], the authors proved the estimate

$$-k+n\delta \leqslant \mathring{R} \leqslant -k+n\Delta, \qquad \text{on } \mathcal{S}^2_0.$$

which is weaker than Theorem 1.2.17.

Proposition 1.2.19. *Assume $\delta \leqslant \bar{R} \leqslant \Delta$. Assume we are in the setting of a nearly Kähler 6-manifold. Then by Remark 1.5.49, $\mathring{R}$ preserves $\mathcal{S}^2_{0+}$ and we claim that the eigenvalues of $\mathring{R}$ on $\mathcal{S}^2_{+0}$ lie in the following interval:*

$$\left[\frac{1}{2}\Big((\Delta+\delta) \pm 3(\Delta-\delta)\Big)\right] = [2\delta-\Delta, 2\Delta-\delta].$$

Proof. As in Corollary 1.2.14, we use the fact that for an element $h \in \mathcal{S}^2_{0+}$ (which by Proposition 1.5.31 is isomorphic to $\Lambda^2_8 \cong \mathfrak{su}(3)$) we can find a canonical form $h = \sum_{i=1}^6 h_i e_i \otimes e_i$ with $e_1, \ldots, e_6$ an orthonormal frame and $h_1 = h_2, h_3 = h_4, h_5 = h_6$ with $h_1 + h_2 + h_3 = 0$. We proceed in the same way as in the proof of Theorem 1.2.16. Let $0 \neq h \in \mathcal{S}^2_{0+}$ be a unit eigenvector of $\mathring{R}_0$ and let $\mathring{r}_0$ be the corresponding eigenvalue. Put h in the canonical form as above. By replacing h by $-h$ and by swapping h_i's if necessary, we can assume $|h_1| > 0$ is maximal, $h_1 = h_2 > 0$ and as before, $h_3 = h_4, h_5 = h_6$. This forces h_3, h_5 to be non-positive with

$$|h_3| + |h_5| = |h_4| + |h_6| = h_1. \tag{1.2.20}$$

As before, we get:

$$\mathring{r}_0 h_1 = \sum_m (R_0)_{m1m1} h_m.$$

Dividing through by $h_1 > 0$, and using (1.2.7) with (1.2.20) we get:

$$\begin{aligned}
|\mathring{r}_0| &\leqslant \sum_m \frac{h_m}{h_1} |(R_0)_{m1m1}| \\
&= |(R_0)_{2121}| + \frac{|h_3|}{h_1}|(R_0)_{3131}| + \frac{|h_4|}{h_1}|(R_0)_{4141}| + \frac{|h_5|}{h_1}|(R_0)_{5151}| + \frac{|h_6|}{h_1}|(R_0)_{6161}| \\
&\leqslant 3 \max |\bar{R}_0| \\
&\leqslant \frac{3(\Delta - \delta)}{2},
\end{aligned}$$

which along with the same details as in Theorem 1.2.16 concludes the proof. □

Remark 1.2.21. In the Einstein setting, Theorem 1.2.16 tells us that the eigenvalues of $\mathring{R}$ on $\mathcal{S}_0^2$ lie in

$$\left[\frac{1}{2}\big((\Delta + \delta) \pm (n-1)(\Delta - \delta)\big)\right],$$

which is a weaker result than the one in Theorem 1.2.17. It is immediately clear that on a nearly Kähler 6-manifold, on $\mathcal{S}_{0+}^2$, the interval from Theorem 1.2.19 is a better result than Theorem 1.2.16.
One can also show that Theorem 1.2.19 is also stronger than Theorem 1.2.17. For example, let us show that $2\delta - \Delta \geqslant -5 + 6\delta$. So, we need $5 \geqslant 4\delta + \Delta$. This is clearly true, as we can pick an orthonormal frame where $R_{1212} = \Delta$. Then we know that $R_{1i1i} \geqslant \delta$ for $i = 3, 4, 5$ and $\sum_{i=2}^5 R_{1i1i} = 5$, which is the Einstein constant for a nearly Kähler 6-manifold. Hence the result follows. All the other inequalities are similar.

Remark 1.2.22. In the proof of Theorem 1.5.62, we will show that on a nearly Kähler 6-manifold, for $\beta \in \Omega_8^2$, which must equal $h \diamond \omega$ for some unique $h \in \mathcal{S}_{0+}^2$, we have that $\hat{W}\beta = (2\mathring{W}h) \diamond \omega$. Hence, if $\beta = h \diamond \omega$ is an eigenvector of $\hat{W}$ with the eigenvalue λ, then h is an eigenvector $\mathring{W}$ with the eigenvalue $\frac{\lambda}{2}$. This clearly means that $\operatorname{range}(\hat{W}) = 2 \operatorname{range}(\mathring{W})$, where by range of a self-adjoint operator we mean the closed interval from the smallest eigenvalue to the largest one.

By Lemma 1.2.8 we have that $\hat{W} = \hat{R} + 2\,\mathrm{Id}, \mathring{W} = \mathring{R} - \mathrm{Id}$ on $\mathcal{S}^2_0$.
Hence, assume that $\delta \leqslant \bar{R} \leqslant \Delta$. Then by Corollary 1.2.14, the range of $\hat{W}$ on Ω^2_8 lies in

$$\left[2 - (\Delta + \delta) \pm \frac{7}{3}(\Delta - \delta)\right]. \tag{1.2.23}$$

Similarly, by Theorem 1.2.19, the range of $\mathring{W}$ on $\mathcal{S}^2_{0+}$ lies in

$$\left[-1 + \frac{1}{2}\Big((\Delta + \delta) \pm 3(\Delta - \delta)\Big)\right]. \tag{1.2.24}$$

However, since $\mathrm{range}(\hat{W}) = 2\,\mathrm{range}(\mathring{W})$, $\hat{W}$ on Ω^2_8 also lies in

$$\left[-2 + (\Delta + \delta) \pm 3(\Delta - \delta)\right],$$

which is clearly not the same interval as in (1.2.23). Since we cannot say that one of the intervals is always better than the other one, we will use them both by taking their intersection. Similarly, we can also obtain a second interval for $\mathring{W}$ on $\mathcal{S}^2_{0+}$.

We summarize the estimates in the case of a nearly Kähler 6-manifold for $\mathring{R}$ on $\mathcal{S}^2_0$.

Corollary 1.2.25. *Let M be a nearly Kähler 6-manifold. Let $\delta \leqslant \bar{R} \leqslant \Delta$. Then the eigenvalues of $\mathring{R}$ on $\mathcal{S}^2_{+0}$ lie in the intersection of the following intervals:*

$$\left[\frac{1}{2}\Big((\Delta + \delta) \pm 3(\Delta - \delta)\Big)\right], \left[2 + \frac{1}{2}\Big(-(\Delta + \delta) \pm \frac{7}{3}(\Delta - \delta)\Big)\right].$$

Proof. This follows from Remark 1.2.22. □

1.3 General Weitzenböck formulas

In this section we rederive the well-known general Weitzenböck formula and then simplify it in the case that the manifold is Einstein. More information can be found in [38], [33], [35].
Let (M^n, g) be a Riemannian manifold. For $\alpha \in \Omega^k$ and $T \in \mathcal{T}^k$ we have:

$$(d\alpha)_{i_1 \ldots i_{k+1}} = \sum_{j=1}^{k+1} (-1)^{j-1} \nabla_{i_j} \alpha_{i_1 \ldots \hat{i_j} \ldots i_{k+1}},$$

$$\begin{aligned}(d^*\alpha)_{i_1 \ldots i_{k-1}} &= -\nabla_p \alpha_{p i_1 \ldots i_{k-1}},\\ (\nabla^*\nabla\alpha)_{i_1 \ldots i_{k-1}} &= -\nabla_p \nabla_p \alpha_{i_1 \ldots i_{k-1}},\\ (\nabla^* T)_{i_2 \ldots i_k} &= -\nabla_p T_{p i_2 \ldots i_k}.\end{aligned}$$

Proposition 1.3.1. ***General Weitzenböck formula***
For $\alpha \in \Omega^k$ we have:

$$
\begin{aligned}
(\Delta\alpha)_{i_1\dots i_k} = (\nabla^*\nabla\alpha)_{i_1\dots i_k} &+ \sum_{j=1}^{k} \alpha_{i_1\dots u\dots i_k} R_{i_j u} \ \textit{(u is at j^{th} position)} \\
&+ \sum_{1\leqslant l<j\leqslant k} \alpha_{i_1\dots u\dots p\dots i_k} R_{i_j i_l p u} \ \textit{(u and p are at l^{th} and j^{th} positions respectively)}
\end{aligned}
$$

Proof. First we compute

$$
\begin{aligned}
(dd^*\alpha)_{i_1\dots i_k} &= \sum_{j=1}^{k} (-1)^{j-1} \nabla_{i_j} (d^*\alpha)_{i_1\dots \hat{i_j}\dots i_k} \\
&= \sum_{j=1}^{k} (-1)^{j-1} \nabla_{i_j} (-\nabla_p \alpha_{p i_1\dots \hat{i_j}\dots i_k}) \\
&= \sum_{j=1}^{k} (-1)^{j} \nabla_{i_j} \nabla_p \alpha_{p i_1\dots \hat{i_j}\dots i_k},
\end{aligned}
$$

and

$$
\begin{aligned}
(d^*d\alpha)_{i_1\dots i_k} &= -\nabla_p (d\alpha)_{p i_1\dots i_k} \\
&= -\nabla_p (\nabla_p \alpha_{i_1\dots i_k} + \sum_{j=1}^{k} (-1)^j \nabla_{i_j} \alpha_{p i_1\dots \hat{i_j}\dots i_k}) \\
&= (\nabla^*\nabla\alpha)_{i_1\dots i_k} - \sum_{j=1}^{k} (-1)^j \nabla_p \nabla_{i_j} \alpha_{p i_1\dots \hat{i_j}\dots i_k}.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
(\Delta\alpha)_{i_1\dots i_k} &= (dd^*\alpha)_{i_1\dots i_k} + (d^*d\alpha)_{i_1\dots i_k} \\
&= (\nabla^*\nabla\alpha)_{i_1\dots i_k} + \sum_{j=1}^{k} (-1)^j (\nabla_{i_j}\nabla_p - \nabla_p\nabla_{i_j}) \alpha_{p i_1\dots \hat{i_j}\dots i_k}.
\end{aligned}
$$

Apply the Ricci identity to the last term to get:

$$\sum_{j=1}^{k}(-1)^j(\nabla_{i_j}\nabla_p-\nabla_p\nabla_{i_j})\alpha_{pi_1\ldots\hat{i_j}\ldots i_k} = -\sum_{j=1}^{k}(\nabla_{i_j}\nabla_p-\nabla_p\nabla_{i_j})\alpha_{i_1\ldots p\ldots i_k} \text{ (where } p \text{ is at } j^{th} \text{ position)}$$

$$= \sum_{j=1}^{k}\sum_{l=1}^{k} R_{i_jpi_lu}\alpha_{i_1\ldots u\ldots p\ldots i_k} \text{ (} u \text{ and } p \text{ are at } l^{\text{th}} \text{ and } j^{\text{th}} \text{ positions respectively)}$$

$$= \sum_{j=1}^{k} R_{i_jppu}\alpha_{i_1\ldots u\ldots i_k} + \sum_{j=1}^{k}\sum_{j>l} R_{i_jpi_lu}\alpha_{i_1\ldots u\ldots p\ldots i_k} \text{ (first term is when } l=j\text{)}$$

$$+ \sum_{j=1}^{k}\sum_{j<l} R_{i_jpi_lu}\alpha_{i_1\ldots p\ldots u\ldots i_k}$$

$$= \sum_{j=1}^{k} R_{i_ju}\alpha_{i_1\ldots u\ldots i_k} + \sum_{l=1}^{k}\sum_{j>l} R_{i_jpi_lu}\alpha_{i_1\ldots u\ldots p\ldots i_k} + \sum_{j=1}^{k}\sum_{l>j} R_{i_lui_jp}\alpha_{i_1\ldots p\ldots u\ldots i_k}$$

$$= \sum_{j=1}^{k} R_{i_ju}\alpha_{i_1\ldots u\ldots i_k} + 2\sum_{j>l} R_{i_jpi_lu}\alpha_{i_1\ldots u\ldots p\ldots i_k}$$

Now let

$$L := 2\sum_{j>l} R_{i_jpi_lu}\alpha_{i_1\ldots u\ldots p\ldots i_k}.$$

By the first Bianchi identity, we have

$$\begin{aligned} L &= 2\sum_{j>l} R_{i_jpi_lu}\alpha_{i_1\ldots u\ldots p\ldots i_k} \\ &= -2\sum_{j>l}(R_{i_ji_lup}+R_{i_jupi_l})\alpha_{i_1\ldots u\ldots p\ldots i_k} \\ &= 2\sum_{j>l} R_{i_ji_lpu}\alpha_{i_1\ldots u\ldots p\ldots i_k} + 2\sum_{j>l} R_{i_jui_lp}\alpha_{i_1\ldots u\ldots p\ldots i_k} \\ &= 2\sum_{j>l} R_{i_ji_lpu}\alpha_{i_1\ldots u\ldots p\ldots i_k} - L. \end{aligned}$$

Thus:

$$L = \sum_{j>l} R_{i_ji_lpu}\alpha_{i_1\ldots u\ldots p\ldots i_k}.$$

Concluding, we get the required:

$$(\Delta\alpha)_{i_1\ldots i_k} = (\nabla^*\nabla\alpha)_{i_1\ldots i_k} + \sum_{j=1}^{k} R_{i_j u}\alpha_{i_1\ldots u\ldots i_k} + \sum_{j>l} R_{i_j i_l p u}\alpha_{i_1\ldots u\ldots p\ldots i_k}. \quad \square$$

From Proposition 1.3.1 we obtain the **Weitzenböck formula for 2-forms**:
Let $\beta \in \Omega^2$. Then:

$$(\Delta\beta)_{ab} = (\nabla^*\nabla\beta)_{ab} + R_{ap}\beta_{pb} + R_{bp}\beta_{ap} + R_{abpq}\beta_{pq}. \tag{1.3.2}$$

Corollary 1.3.3. *Assume M is Einstein with Einstein constant k. Then the Weitzenböck formula for 2-forms simplifies to:*

$$\Delta\beta = \nabla^*\nabla\beta + 2k\frac{n-2}{n-1}\beta + \hat{W}\beta,$$

where the $\hat{W}$ notation is defined in Section 1.2.3.

Proof. In the Einstein case we have $Ric = kg$. Hence, each of the Ricci terms in (1.3.2) is equal to $k\beta_{ab}$.
The last term is:

$$R_{abpq}\beta_{pq} = (\hat{R}\beta)_{ab} = ((\hat{W} - \frac{2k}{n-1}\,\mathrm{Id})\beta)_{ab} = (\hat{W}\beta)_{ab} - \frac{2k}{n-1}\beta_{ab},$$

by Lemma 1.2.8. Thus, putting everything together, we get:

$$\Delta\beta = \nabla^*\nabla\beta + 2(k\beta) + (\hat{W}\beta - \frac{2k}{n-1}\beta) = \nabla^*\nabla\beta + 2k\frac{n-2}{n-1}\beta + \hat{W}\beta,$$

as required. $\square$

From Proposition 1.3.1 we obtain the **Weitzenböck formula for 3-forms**:
Let $\beta \in \Omega^3$. Then:

$$(\Delta\beta)_{abc} = (\nabla^*\nabla\beta)_{abc} + R_{au}\beta_{ubc} + R_{bu}\beta_{auc} + R_{cu}\beta_{abu} + R_{abpu}\beta_{puc} + R_{acpu}\beta_{pbu} + R_{bcpu}\beta_{apu}. \tag{1.3.4}$$

Corollary 1.3.5. *Assume M is Einstein with Einstein constant k. Then the Weitzenböck formula for 3-forms simplifies to:*

$$(\Delta\beta)_{abc} = (\nabla^*\nabla\beta)_{abc} + 3k\frac{n-3}{n-1}\beta_{abc} + W_{abpu}\beta_{puc} + W_{acpu}\beta_{pbu} + W_{bcpu}\beta_{apu}.$$

Note that the last three terms can be written as $(\hat{W}\beta)_{abc} + (\hat{W}\beta)_{cab} + (\hat{W}\beta)_{bca}$, by using notation of Remark 1.2.4.

Proof. In the Einstein case we have $Ric = kg$. Hence, each of the Ricci terms in (1.3.4) is equal to $k\beta_{abc}$.
Now, consider the term $R_{abpu}\beta_{puc}$ of (1.3.4). This is $(\hat{R}\beta)_{abc}$, which is equal to $((\hat{W} - \frac{2k}{n-1}\,\mathrm{Id})\beta)_{abc} = W_{abpu}\beta_{puc} - \frac{2k}{n-1}\beta_{abc}$, by Lemma 1.2.8. Similarly, we can do the same for the other terms to get the required result:

$$\begin{aligned}(\Delta\beta)_{abc} &= (\nabla^*\nabla\beta)_{abc} + 3(k\beta_{abc}) + W_{abpu}\beta_{puc} + W_{acpu}\beta_{pbu} + W_{bcpu}\beta_{apu} - 3(\frac{2k}{n-1}\beta_{abc}) \\ &= (\nabla^*\nabla\beta)_{abc} + 3k\frac{n-3}{n-1}\beta_{abc} + W_{abpu}\beta_{puc} + W_{acpu}\beta_{pbu} + W_{bcpu}\beta_{apu}.\end{aligned}$$

□

1.4 Nearly G_2 manifolds

First, in Section 1.4.1 we recall some facts about G_2 structures and nearly G_2 manifolds. In Section 1.4.2 we observe some properties about the curvature of nearly G_2 manifolds. Finally, in Section 1.4.3 we simplify the Weitzenböck formulas for harmonic 2 and 3-forms and using the assumption of compactness of our manifolds, we get the necessary conditions of vanishing of b_2 and b_3 in terms of bounds on $\bar{R}$, $\mathring{R}$, and $\hat{R}$.

1.4.1 Preliminaries

Throughout this section M^7 is a manifold with a G_2 structure. That means that M admits a non-degenerate 3-form φ (see [24] for more details). Note that φ determines a metric g and orientation, hence also the Hodge-star $\star$. Then we also have that $\psi := \star\varphi$ is a non-degenerate 4-form. First, we list some results for manifolds with a G_2-structure.

Proposition 1.4.1. *We use the following identities from [24]:*

- $\varphi_{ijk}\varphi_{abk} = \delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja} - \psi_{ijab}$.
- $\varphi_{ijk}\psi_{abck} = \delta_{ia}\varphi_{jbc} + \delta_{ib}\varphi_{ajc} + \delta_{ic}\varphi_{abj} - \delta_{aj}\varphi_{ibc} - \delta_{bj}\varphi_{aic} - \delta_{cj}\varphi_{abi}$.
- $\varphi_{ijk}\psi_{abjk} = -4\varphi_{iab}$.
- $\psi_{ijkl}\psi_{abkl} = 4\delta_{ia}\delta_{jb} - 4\delta_{ib}\delta_{ja} - 2\psi_{ijab}$.
- $\psi_{ijkl}\psi_{ajkl} = 24\delta_{ia}$.

Remark 1.4.2. We have the following descriptions of the orthogonal decompositions of Ω^2 and Ω^3 into irreducible subspaces (see [24]):

$$\begin{aligned}\Omega^2 &= \Omega^2_7 \oplus \Omega^2_{14} \\ \Omega^3 &= \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27},\end{aligned}$$

where the subscripts denote the corresponding dimensions. In particular:

- $\Omega^2_7 = \{X \lrcorner \varphi : X \in \Gamma(TM)\} = \{\beta \in \Omega^2 : \star(\varphi \wedge \beta) = -2\beta\}$,
  or equivalently, $\beta \in \Omega^2_7$ iff $\beta_{ij}\psi_{ijkl} = -4\beta_{kl} \Leftrightarrow \beta_{ij} = X_k\varphi_{ijk}$ for $X_k = \frac{1}{6}\beta_{ij}\varphi_{ijk}$
- $\Omega^2_{14} = \{\beta \in \Omega^2 : \beta \wedge \psi = 0\} = \{\beta \in \Omega^2 : \star(\varphi \wedge \beta) = \beta\}$,
  or equivalently, $\beta \in \Omega^2_{14}$ iff $\beta_{ij}\psi_{ijkl} = 2\beta_{kl} \Leftrightarrow \beta_{ij}\varphi_{ijk} = 0$.
- $\Omega^3_1 = \{f\varphi : f \in C^\infty(M)\} \cong \mathbb{R}g$,
- $\Omega^3_7 = \{X \lrcorner \psi : X \in \Gamma(TM)\} \cong \Omega^2_7$,
- $\Omega^3_{27} \cong \mathcal{S}^2_0$,

where the isomorphisms are obtained using the $\diamond$ map with φ (although different notation is used in [24], instead of $\diamond$ there).
Similarly, we have isomorphisms between the irreducible subspaces of Ω^4 and $\mathcal{S}^2 \oplus \Omega^2_7$ via $\diamond$ with ψ. In particular, let $\gamma \in \Omega^3, \zeta \in \Omega^4$. Then $\gamma = A \diamond \varphi$ and $\zeta = B \diamond \psi$ for some unique $A = \frac{1}{7}(\operatorname{tr} A)g + A_0 + A_7, B = \frac{1}{7}(\operatorname{tr} B)g + B_0 + B_7 \in \mathcal{S}^2 \oplus \Omega^2_7$. Define

$$\hat{\gamma}_{ia} := \gamma_{ijk}\varphi_{ajk} \qquad \text{and} \qquad \hat{\zeta}_{ia} := \zeta_{ijkl}\psi_{ajkl}.$$

Then:

$$\begin{aligned}
\operatorname{tr} A &= \frac{1}{18}\operatorname{tr}(\hat{\gamma}),\\
(A_0)_{ia} &= \frac{1}{8}(\hat{\gamma}_{ia} + \hat{\gamma}_{ai}) - \frac{1}{28}\operatorname{tr}(\hat{\gamma})g_{ia},\\
(A_7)_{ia} &= \frac{1}{24}(\hat{\gamma}_{ia} - \hat{\gamma}_{ai}).
\end{aligned}$$

and we have similar formulas for B, but they will not be used here.

Definition 1.4.3. A manifold M with a G_2 structure φ has four independent torsion forms corresponding to a G_2 structure φ:

$$\tau_0 \in C^\infty(M), \qquad \tau_1 \in \Omega^1_7, \qquad \tau_2 \in \Omega^2_{14}, \qquad \tau_3 \in \Omega^3_{27},$$

defined by the equations:

$$\begin{aligned}
d\varphi &= \tau_0\psi + 3\tau_1 \wedge \varphi + \star\tau_3\\
d\psi &= 4\tau_1 \wedge \psi + \star\tau_2.
\end{aligned}$$

We say that M is **nearly** G_2 if $d\varphi = \tau_0\psi$ and $d\psi = 0$ for some $\tau_0 \neq 0$. It follows in this case that τ_0 must be constant. Note that the condition of being nearly G_2 is also equivalent to the fact that the only non-zero component of the torsion tensor is τ_0. These manifolds are positive Einstein and one might want to scale the metric so that $\tau_0 = 4$ (in this case we will also have $\operatorname{Ric} = 6g$), as we do for the nearly Kähler case. However, we keep it more general. ▲

Proposition 1.4.4. *For a nearly G_2 manifold we have the following formulas:*

$$\begin{aligned}\nabla_p\varphi_{ijk} =& \frac{\tau_0}{4}\psi_{pijk},\\ \nabla_p\psi_{ijkl} =& \frac{\tau_0}{4}(\delta_{lp}\varphi_{ijk}+\delta_{jp}\varphi_{ikl}-\delta_{kp}\varphi_{ijl}-\delta_{ip}\varphi_{ljk}),\\ \sum_p\nabla_p\nabla_p\varphi_{ijk} =& -\frac{\tau_0^2}{4}\varphi_{ijk}.\end{aligned}$$

Proof. The first two formulas are in [24]. The third formula is demonstrated in Proposition 2.4 of [1]. □

1.4.2 Curvature identities

On a nearly G_2 manifold we have: $Ric = \frac{3\tau_0^2}{8}g$ (also see [24]), so the Einstein constant $k = \frac{3\tau_0^2}{8}$ and $R = \frac{21\tau_0^2}{8}$. Applying the result from Lemma 1.2.8 we get:

$$\begin{aligned}\hat{W} &= \hat{R} + \frac{\tau_0^2}{8}\,\mathrm{Id},\\ \mathring{W} &= \mathring{R} - \frac{\tau_0^2}{16}\,\mathrm{Id},\ \text{ on } \mathcal{S}_0^2.\end{aligned} \tag{1.4.5}$$

Also, note that the Weyl tensor W_{ijkl} lies in Ω^2_{14} in the first two or the last two indices. This is because from Theorem 4.2 in [24], we have $R_{ijkl}\varphi_{klm} = -\frac{\tau_0^2}{8}\varphi_{ijm}$. By Remark 1.2.4, we can write this as $\hat{R}\varphi = -\frac{\tau_0^2}{8}\varphi$. By (1.4.5), we get $\hat{W}\varphi = \hat{R}\varphi + \frac{\tau_0^2}{8}\varphi = 0$. By Remark 1.4.2, this is equivalent to the fact that W lies in Ω^2_{14} in the first two indices. Because of its symmetries, the same holds in the last two indices.
Hence, we can also conclude that $\hat{W}$, $\hat{R}$ preserve the space Ω^2_{14}. Consider $\hat{W}$ first. The 2-form $(\hat{W}\beta)_{ab} = W_{abij}\beta_{ij}$ will always lie in Ω^2_{14}. So, vacuously, it preserves Ω^2_{14}. Next, since $\hat{R}$ and $\hat{W}$ differ by a constant multiple of the identity, $\hat{R}$ also preserves Ω^2_{14}. This fact means that we can consider $\hat{W}$ (and $\hat{R}$) as a self-adjoint operator only on Ω^2_{14} which will provide better estimates when we apply the Bochner-Weitzenböck techniques.

1.4.3 Weitzenböck formulas

In this section we establish sufficient conditions for the Betti numbers b_2 or b_3 to vanish, in terms of bounds on $\hat{W}$ and $\mathring{W}$ respectively. As a corollary, we can get those sufficient conditions in terms of bounds on $\hat{R}$.
The simplified Weitzenböck formulas obtained in this section can be found in the literature, but possibly in different forms (see [1]). As we mentioned in the introduction, we again reprove all the results in a simple, direct way.

We will use Theorems 3.8 and 3.9 from [12], which state the every harmonic 2-form lies in Ω^2_{14}, and every harmonic 3-form lies in Ω^3_{27}.

2-forms

We apply Corollary 1.3.3 to the nearly G_2 setting to get:

$$\Delta\beta = \nabla^*\nabla\beta + \frac{5\tau_0^2}{8}\beta + \hat{W}\beta, \text{ for any } \beta \in \Omega^2. \tag{1.4.6}$$

Theorem 1.4.7. *Let M be a compact nearly G_2 manifold. If $\mathcal{S}^2(\Omega^2_{14}) \ni \hat{W} \geqslant -\frac{5\tau_0^2}{8}$, then $b_2 = 0$.*

Proof. Let $\beta \in \Omega^2$ be harmonic. Then $\beta \in \Omega^2_{14}$. Substituting it in (1.4.6), and using the assumption that $\hat{W} \geqslant -\frac{5\tau_0^2}{8}$, we get by integration that $\nabla\beta = 0$. Hence $\beta = 0$, as there are no parallel non-zero 2-forms. □

Theorem 1.4.8. *Let M be a compact nearly G_2 manifold. Let $\delta \leqslant \bar{R} \leqslant \Delta$ with $-(\Delta + \delta) - \frac{7}{3}(\Delta - \delta) \geqslant -\frac{3\tau_0^2}{4}$. Then $b_2 = 0$.*

Proof. If $-(\Delta + \delta) - \frac{7}{3}(\Delta - \delta) \geqslant -\frac{3\tau_0^2}{4}$, then by Corollary 1.2.14 we have that $\hat{R} \geqslant -\frac{3\tau_0^2}{4}$. So, we use equation (1.4.5) to get that $\hat{W} \geqslant -\frac{5\tau_0^2}{8}$ and hence $b_2 = 0$ by Theorem 1.4.7. □

3-forms

We apply Corollary 1.3.5 to the nearly G_2 setting to get:

$$(\Delta\beta)_{abc} = (\nabla^*\nabla\beta)_{abc} + \frac{3}{4}\tau_0^2\beta_{abc} + W_{abpu}\beta_{puc} + W_{acpu}\beta_{pbu} + W_{bcpu}\beta_{apu}, \text{ for any } \beta \in \Omega^3. \tag{1.4.9}$$

The aim now is to simplify this formula for harmonic β. First, we do this more generally, we will just assume $\beta \in \Omega^3_1 \oplus \Omega^3_{27}$, which includes all the harmonic forms. Then we will see what we can get from the assumption of β being harmonic and then we use all these steps to get a simpler formula.
Recall definitions of Div and $\diamond$ from Section 1.1.3, and also for $h \in \mathcal{S}^2$, let $\tilde{h} \in \mathcal{T}^2$ be defined as

$$\tilde{h}_{kc} = (\nabla_i h_{jk})\varphi_{ijc}. \tag{1.4.10}$$

Proposition 1.4.11. *Let M be nearly G_2. Let $\beta \in \Omega^3_1 \oplus \Omega^3_{27}$, so that $\beta = h \diamond \varphi$ for some $h \in \mathcal{S}^2$. Then:*

$$\Delta(h \diamond \varphi) = (\nabla^*\nabla h + \tau_0^2 h + \frac{\tau_0}{2}\tilde{h}_{symm} - \frac{\tau_0}{12}((\operatorname{Div} h - \nabla \operatorname{tr} h) \lrcorner \varphi) + 2\mathring{W}h) \diamond \varphi.$$

Proof. First, consider the term $W_{abpu}\beta_{puc}$ from (1.4.9):

$$\begin{aligned} W_{abpu}\beta_{puc} &= W_{abpu}(h_{ps}\varphi_{suc} + h_{us}\varphi_{psc} + h_{cs}\varphi_{pus}) \\ &= 2W_{abpu}h_{ps}\varphi_{suc}, \end{aligned}$$

as the first two terms in the brackets are skew in p, u and the last term vanishes because $W \in \mathcal{S}^2(\Omega^2_{14})$. Hence,

$$W_{abpu}\beta_{puc} + W_{acpu}\beta_{pbu} + W_{bcpu}\beta_{apu} = 2h_{ps}(W_{abpu}\varphi_{suc} + W_{acpu}\varphi_{sbu} + W_{bcpu}\varphi_{asu}),$$

and define γ_{abc} to be equal to this expression. Since $\gamma \in \Omega^3, \gamma = A \diamond \varphi$ for some $A \in \mathcal{S}^2 \oplus \Omega^2_7$. We have:

$$\begin{aligned} \hat{\gamma}_{at} =& \gamma_{abc}\varphi_{tbc} \\ =& 2h_{ps}(W_{abpu}\varphi_{suc} + W_{acpu}\varphi_{sbu} + W_{bcpu}\varphi_{asu})\varphi_{tbc} \\ =& 4h_{ps}(W_{abpu}\varphi_{suc}\varphi_{tbc}) \\ & \text{(since } W \in \mathcal{S}^2(\Omega^2_{14}) \text{ and we use skew-symmetry in } b, c \text{ on the first two terms)} \\ =& 4h_{ps}W_{abpu}(\delta_{st}\delta_{ub} - \delta_{sb}\delta_{ut} - \psi_{sutb}) \text{ (by Proposition 1.4.1)} \\ =& -4(h_{ps}W_{aspt} + h_{ps}W_{abpu}\psi_{sutb}). \text{ (because the Ricci tensor of } W \text{ is zero, i.e. } W_{abbu} = 0) \end{aligned}$$

Now, note that:

$$\begin{aligned} W_{abpu}\psi_{sutb} =& -(W_{apub} + W_{aubp})\psi_{sutb} \\ =& W_{apub}\psi_{stub} - W_{aubp}\psi_{sutb} \\ =& 2W_{apst} - W_{abup}\psi_{sbtu} \\ & \text{(swap the indices } b, u \text{ and use that } W \in S^2(\Omega^2_{14}) \text{ with Remark 1.4.2)} \\ =& 2W_{apst} - W_{abpu}\psi_{sutb}. \end{aligned}$$

Hence, we have

$$W_{abpu}\psi_{sutb} = W_{apst},$$

and thus

$$\begin{aligned} \hat{\gamma}_{at} &= -4(h_{ps}W_{aspt} + h_{ps}W_{apst}) \\ &= 8(\mathring{W}h)_{at}. \end{aligned}$$

Next, we have that $\operatorname{tr}(\hat{\gamma}) = 0$ because $W_{ipqi} = 0$. Also, we see that $\hat{\gamma}$ is symmetric. Hence, by Remark 1.4.2,

$$A = A_0 = \frac{1}{4}\hat{\gamma} = 2\mathring{W}h.$$

Thus, the term with Weyl tensors in the Weitzenböck formula is equal to $\gamma = A \diamond \varphi = 2(\mathring{W}h) \diamond \varphi$.
Next, we compute $\nabla^*\nabla\beta = \nabla^*\nabla(h \diamond \phi)$ as follows:

$$\begin{aligned}
\nabla^*\nabla(h \diamond \phi)_{abc} =& -\nabla_s\nabla_s(h_{ap}\varphi_{pbc} + h_{bp}\varphi_{apc} + h_{cp}\varphi_{abp}) \\
=& -(\nabla_s(\nabla_s h_{ap})\varphi_{pbc} + \nabla_s(\nabla_s h_{bp})\varphi_{apc} + \nabla_s(\nabla_s h_{cp})\varphi_{abp})) \\
&- 2(\nabla_s(h_{ap})\nabla_s(\varphi_{pbc}) + \nabla_s(h_{bp})\nabla_s(\varphi_{apc}) + \nabla_s(h_{cp})\nabla_s(\varphi_{abp})) \\
&- (h_{ap}\nabla_s\nabla_s(\varphi_{pbc}) + h_{bp}\nabla_s\nabla_s(\varphi_{apc}) + h_{cp}\nabla_s\nabla_s(\varphi_{abp})) \\
=& ((\nabla^*\nabla h + \frac{\tau_0^2}{4}h) \diamond \varphi)_{abc} \text{ (by Proposition 1.4.4)} \\
&- \frac{\tau_0}{2}(\nabla_s(h_{ap})\psi_{spbc} + \nabla_s(h_{bp})\psi_{sapc} + \nabla_s(h_{cp})\psi_{sabp}).
\end{aligned}$$

Let $\sigma_{abc} := \nabla_s(h_{ap})\psi_{spbc} + \nabla_s(h_{bp})\psi_{sapc} + \nabla_s(h_{cp})\psi_{sabp}$. As $\sigma \in \Omega^3$, $\sigma = B \diamond \varphi$ for some unique $B \in \mathcal{S}^2 \oplus \Omega^2_7$. Then:

$$\begin{aligned}
\hat{\sigma}_{at} &= \sigma_{abc}\varphi_{tbc} \\
&= \nabla_s(h_{ap})\psi_{spbc}\varphi_{tbc} + 2\nabla_s(h_{bp})\psi_{sapc}\varphi_{tbc} \\
&= \nabla_s(h_{ap})(-4\varphi_{tsp}) + 2\nabla_s(h_{bp})(\delta_{ts}\varphi_{bap} + \delta_{ta}\varphi_{sbp} + \delta_{tp}\varphi_{sab} - \delta_{sb}\varphi_{tap} - \delta_{ab}\varphi_{stp} - \delta_{pb}\varphi_{sat}) \\
&\qquad \text{(by Proposition 1.4.1)} \\
&= -4\nabla_s(h_{pa})\varphi_{spt} - 2\nabla_s(h_{bt})\varphi_{sba} + 2\nabla_s(h_{sp})\varphi_{pat} + 2\nabla_s(h_{pa})\varphi_{spt} - 2\nabla_s(h_{pp})\varphi_{sat} \\
&= -2\tilde{h}_{at} - 2\tilde{h}_{ta} + 2(\operatorname{Div} h \lrcorner \varphi)_{at} - 2(\nabla \operatorname{tr} h \lrcorner \varphi)_{at} \\
&= -4(\tilde{h}_{symm})_{at} + 2((\operatorname{Div} h - \nabla \operatorname{tr} h) \lrcorner \varphi)_{at}.
\end{aligned}$$

Note that $\operatorname{tr}\hat{\sigma} = -4(\tilde{h}_{symm})_{aa} = -4\tilde{h}_{aa} = -4\nabla_i(h_{ja})\varphi_{ija} = 0$. Thus, by Remark 1.4.2 we have:

$$(B_0)_{ia} = \frac{1}{8}(\hat{\sigma}_{ia} + \hat{\sigma}_{ai}) = -(\tilde{h}_{symm})_{ia},$$

$$(B_7)_{ia} = \frac{1}{24}(\hat{\sigma}_{ia} - \hat{\sigma}_{ai}) = \frac{1}{6}((\operatorname{Div} h - \nabla \operatorname{tr} h) \lrcorner \varphi)_{ia}.$$

We conclude that:

$$\sigma = (-\tilde{h}_{symm} + \frac{1}{6}((\operatorname{Div} h - \nabla \operatorname{tr} h) \lrcorner \varphi)) \diamond \varphi.$$

Putting everything together we get:

$$\begin{aligned}
\Delta(h \diamond \varphi) =& (\nabla^*\nabla h + \frac{\tau_0^2}{4}h - \frac{\tau_0}{2}(-\tilde{h}_{symm} + \frac{1}{6}((\operatorname{Div} h - \nabla \operatorname{tr} h) \lrcorner \varphi)) + \frac{3}{4}\tau_0^2 h + 2\mathring{W}h) \diamond \varphi \\
=& (\nabla^*\nabla h + \tau_0^2 h + \frac{\tau_0}{2}\tilde{h}_{symm} - \frac{\tau_0}{12}((\operatorname{Div} h - \nabla \operatorname{tr} h) \lrcorner \varphi) + 2\mathring{W}h) \diamond \varphi,
\end{aligned}$$

as claimed. □

Proposition 1.4.12. *Let M be a compact nearly G_2 manifold. Let $\beta \in \Omega^3_{27} \supseteq \mathcal{H}^3$, so that $\beta = h \diamond \varphi$ for some $h \in \mathcal{S}^2_0$. Then β is harmonic if and only if:*

$$\tilde{h} = -\frac{3\tau_0}{4} h \in \mathcal{S}^2 \qquad \textit{and} \qquad \operatorname{Div} h = 0.$$

Proof. Since M is compact, β is harmonic if and only if $d^*\beta = 0$ and $d\beta = 0$.
First, we calculate the Ω^2_7 component of $\tilde{h}$. This is done by contracting it with φ. That is, by Remark 1.4.2, $\pi_7(\tilde{h}) = X \lrcorner \varphi$, where $X_k = \frac{1}{6}\tilde{h}_{ij}\varphi_{kij}$. So:

$$\begin{aligned}
X_k &= \frac{1}{6}\tilde{h}_{ij}\varphi_{kij} \\
&= \frac{1}{6}(\nabla_a h_{bi})\varphi_{abj}\varphi_{kij} \\
&= \frac{1}{6}(\nabla_a h_{bi})(\delta_{ak}\delta_{bi} - \delta_{ai}\delta_{bk} - \psi_{abki}) \\
&= \frac{1}{6}(\nabla_k h_{bb} - \nabla_i h_{ki}) \\
&= -\frac{1}{6}(\operatorname{Div} h)_k.
\end{aligned}$$

Thus:

$$\pi_7(\tilde{h}) = X \lrcorner \varphi \qquad \text{where} \qquad X = -\frac{1}{6}\operatorname{Div} h.$$

Now, consider the condition $d^*\beta = 0$:

$$\begin{aligned}
-(d^*\beta)_{kl} =& \nabla_j \beta_{jkl} \\
=& \nabla_j (h_{jp}\varphi_{pkl} + h_{kp}\varphi_{jpl} + h_{lp}\varphi_{jkp}) \\
=& \nabla_j(h_{jp})\varphi_{pkl} + h_{jp}\nabla_j(\varphi_{pkl}) + \nabla_j(h_{kp})\varphi_{jpl} \\
&+ h_{kp}\nabla_j(\varphi_{jpl}) + \nabla_j(h_{lp})\varphi_{jkp} + h_{lp}\nabla_j(\varphi_{jkp}) \\
=& (\operatorname{Div} h \lrcorner \varphi)_{kl} + h_{jp}\frac{\tau_0}{4}\psi_{jpkl} + \tilde{h}_{kl} + h_{kp}\frac{\tau_0}{4}\psi_{jjpl} - \tilde{h}_{lk} + h_{lp}\frac{\tau_0}{4}\psi_{jjkp} \\
=& (\operatorname{Div} h \lrcorner \varphi)_{kl} + 2(\tilde{h}_{skew})_{kl}.
\end{aligned}$$

Hence, using the formula for $\pi_7(\tilde{h})$, we get:

$$\begin{aligned}
-d^*\beta &= \operatorname{Div} h \lrcorner \varphi + 2(\pi_7(\tilde{h}) + \pi_{14}(\tilde{h})) \\
&= \operatorname{Div} h \lrcorner \varphi - \frac{1}{3}\operatorname{Div} h \lrcorner \varphi + 2\pi_{14}(\tilde{h}) \\
&= 2\pi_{14}(\tilde{h}) + \frac{2}{3}\operatorname{Div} h \lrcorner \varphi.
\end{aligned}$$

Thus, we have that:

$$d^*\beta = 0 \qquad \text{if and only if} \qquad \begin{cases} \pi_{14}(\tilde{h}) = 0, \\ \operatorname{Div} h = 0. \end{cases}$$

Next, we consider the second condition $d\beta = 0$. Since $d\beta \in \Omega^4$, by Remark 1.4.2, $d\beta = B \diamond \psi$ for some unique $B \in \mathcal{S}^2 \oplus \Omega^2_7$. Then $d\beta = 0$ iff $B = 0$ iff $\widehat{d\beta} = 0$. So:

$$\begin{aligned}
(\widehat{d\beta})_{ia} =& (d\beta)_{ijkl}\psi_{ajkl} \\
=& (\nabla_i\beta_{jkl} - \nabla_j\beta_{ikl} + \nabla_k\beta_{ijl} - \nabla_l\beta_{ijk})\psi_{ajkl} \\
=& (\nabla_i\beta_{jkl} - 3\nabla_j\beta_{ikl})\psi_{ajkl} \\
=& (\nabla_i(h_{jp}\varphi_{pkl} + h_{kp}\varphi_{jpl} + h_{lp}\varphi_{jkp}))\psi_{ajkl} \\
& - 3(\nabla_j(h_{ip}\varphi_{pkl} + h_{kp}\varphi_{ipl} + h_{lp}\varphi_{ikp}))\psi_{ajkl} \\
=& 3\nabla_i(h_{jp}\varphi_{pkl})\psi_{ajkl} - 3\nabla_j(h_{ip}\varphi_{pkl})\psi_{ajkl} - 6\nabla_j(h_{kp}\varphi_{ipl})\psi_{ajkl}.
\end{aligned}$$

We calculate each of these terms separately:

$$\begin{aligned}
3\nabla_i(h_{jp}\varphi_{pkl})\psi_{ajkl} &= 3\nabla_i(h_{jp})\varphi_{pkl}\psi_{ajkl} + 3h_{jp}\nabla_i(\varphi_{pkl})\psi_{ajkl} \\
&= -12\nabla_i(h_{jp})\varphi_{paj} + \frac{3\tau_0}{4}h_{jp}\psi_{ipkl}\psi_{ajkl} \\
&= \frac{3\tau_0}{4}h_{jp}(4\delta_{ia}\delta_{pj} - 4\delta_{ij}\delta_{pa} - 2\psi_{ipaj}) \text{ (by Proposition 1.4.1)} \\
&= 3\tau_0((\operatorname{tr} h)\delta_{ia} - h_{ia}) \\
&= -3\tau_0 h_{ia}.
\end{aligned}$$

Next:

$$\begin{aligned}
-3\nabla_j(h_{ip}\varphi_{pkl})\psi_{ajkl} &= -3\nabla_j(h_{ip})\varphi_{pkl}\psi_{ajkl} - 3h_{ip}\nabla_j(\varphi_{pkl})\psi_{ajkl} \\
&= 12\nabla_j(h_{ip})\varphi_{paj} - \frac{3\tau_0}{4}h_{ip}\psi_{jpkl}\psi_{ajkl} \\
&= 12\nabla_j(h_{ip})\varphi_{paj} + \frac{3\tau_0}{4}h_{ip}24\delta_{pa} \text{ (by Proposition 1.4.1)} \\
&= 12\tilde{h}_{ia} + 18\tau_0 h_{ia}.
\end{aligned}$$

Finally:

$$\begin{aligned}
-6\nabla_j(h_{kp}\varphi_{ipl})\psi_{ajkl} &= -6\nabla_j(h_{kp})\varphi_{ipl}\psi_{ajkl} - 6h_{kp}\frac{\tau_0}{4}\psi_{jipl}\psi_{ajkl} \\
&= -6\nabla_j(h_{kp})\varphi_{ipl}\psi_{ajkl} - \frac{3\tau_0}{2}h_{kp}\psi_{pijl}\psi_{akjl} \\
&= -6\nabla_j(h_{kp})(\delta_{ia}\varphi_{pjk} + \delta_{ij}\varphi_{apk} + \delta_{ik}\varphi_{ajp} - \delta_{ap}\varphi_{ijk} - \delta_{jp}\varphi_{aik} - \delta_{kp}\varphi_{aji}) \\
&\quad - \frac{3\tau_0}{2}h_{kp}(4\delta_{pa}\delta_{ik} - 4\delta_{pk}\delta_{ia} - 2\psi_{piak}) \text{ (by Proposition 1.4.1)} \\
&= -6(\nabla_j(h_{ip})\varphi_{ajp} - \nabla_j(h_{ka})\varphi_{ijk} - \nabla_j(h_{kj})\varphi_{aik} - \nabla_j(\operatorname{tr} h)\varphi_{aji}) \\
&\quad - 6\tau_0(h_{ia} - (\operatorname{tr} h)\delta_{ia}) \\
&= -6\tilde{h}_{ia} + 6\tilde{h}_{ai} - 6(\operatorname{Div} h \lrcorner \varphi)_{ia} - 6\tau_0 h_{ia}.
\end{aligned}$$

Combining all the results we get:

$$\begin{aligned}
(\widehat{d\beta})_{ia} &= 6(\tilde{h}_{ia} + \tilde{h}_{ai}) + 9\tau_0 h_{ia} - 6(\operatorname{Div} h \lrcorner \varphi)_{ia} \\
&= 12(\tilde{h}_{symm})_{ia} + 9\tau_0 h_{ia} - 6(\operatorname{Div} h \lrcorner \varphi)_{ia}.
\end{aligned}$$

Thus, we have that:

$$d\beta = 0 \qquad \text{if and only if} \qquad \begin{cases} \tilde{h}_{symm} = -\frac{3\tau_0}{4}h, \\ \operatorname{Div} h = 0. \end{cases}$$

Summarizing, β is harmonic if and only if $\tilde{h}_{symm} = -\frac{3\tau_0}{4}h, \pi_{14}(\tilde{h}) = 0, \operatorname{Div} h = 0$. But we know that $\operatorname{Div} h$ vanishes if and only if $\pi_7(\tilde{h})$ vanishes, so $\tilde{h}_{skew} = 0$ and $\tilde{h} = \tilde{h}_{symm}$. Hence we get the required result. □

Corollary 1.4.13. *Let M be a compact nearly G_2 manifold. Assume β is harmonic. Then $\beta \in \Omega^3_{27}$, so that $\beta = h \diamond \varphi$ for some $h \in \mathcal{S}^2_0$. Then:*

$$\nabla^*\nabla h + \frac{5\tau_0^2}{8}h + 2\mathring{W}h = 0.$$

Proof. By assumption, β is harmonic, so the left hand side of the Weitzenböck formula in Proposition 1.4.11 vanishes. Next, by Proposition 1.4.12, $\tilde{h}_{symm} = \tilde{h} = -\frac{3\tau_0}{4}h$ and $\operatorname{Div} h = 0$. Also, we know $\operatorname{tr} h = 0$. Substituting all these terms into Proposition 1.4.11, we get the required result. □

Theorem 1.4.14. *Let M be a compact nearly G_2 manifold. If $\mathcal{S}^2(\mathcal{S}^2_0) \ni \mathring{W} \geqslant -\frac{3\tau_0^2}{8}$ **or** $\mathcal{S}^2(\Omega^2_{14}) \ni \hat{W} \geqslant -\frac{\tau_0^2}{4}$, then $b_3 = 0$.*

Proof. For the first part, we use (1.4.9) and the proof of Proposition 1.4.11 to get that if $\beta = h \diamond \varphi \in \Omega^3_{27}$ is harmonic for some $h \in \mathcal{S}^2_0$, then:

$$\nabla^*\nabla\beta + (\frac{3\tau_0^2}{4}h + 2\mathring{W}h) \diamond \varphi = 0.$$

Hence, the result follows by integration and the fact that there are no parallel non-zero 3-forms. Note that using Corollary 1.4.13 in order to get a similar result would have been worse, as we would have been able to only conclude that if $\mathcal{S}^2(\mathcal{S}^2_0) \ni \mathring{W} \geqslant -\frac{5\tau_0^2}{16}$ then $b_3 = 0$. This is because $\nabla^*\nabla\beta = (\nabla^*\nabla h - \frac{\tau_0^2}{8}h) \diamond \varphi$, so we can see that even though the left hand side is obviously non-negative, we cannot conclude that from the right hand side.
The second part trivially follows from (1.4.9). □

Theorem 1.4.15. *Let M be a compact nearly G_2 manifold. Let $\delta \leqslant \bar{R} \leqslant \Delta$ with $\Delta \leqslant \frac{11\tau_0^2}{80}$* ***or*** *$\delta \geqslant \frac{\tau_0^2}{112}$. Then $b_3 = 0$.*

Proof. Recall that the Einstein constant $k = \frac{3\tau_0^2}{8}$. Then by Theorem 1.2.17, on $\mathcal{S}^2_0$, $\mathring{R} \geqslant -\frac{3\tau_0^2}{8} + 7\delta$ and $\mathring{R} \geqslant \frac{3\tau_0^2}{8} - 5\Delta$. Hence, by (1.4.5), $\mathring{W} \geqslant -\frac{7\tau_0^2}{16} + 7\delta$ and $\mathring{W} \geqslant \frac{5\tau_0^2}{16} - 5\Delta$.
In order for $b_3 = 0$, by Theorem 1.4.14 we want $\mathring{W} \geqslant -\frac{3\tau_0^2}{8}$. We have $-\frac{7\tau_0^2}{16} + 7\delta \geqslant -\frac{3\tau_0^2}{8}$ iff $\delta \geqslant \frac{\tau_0^2}{112}$; and $\frac{5\tau_0^2}{16} - 5\Delta \geqslant -\frac{3\tau_0^2}{8}$ iff $\Delta \leqslant \frac{11\tau_0^2}{80}$. Hence, the result follows from Theorem 1.4.14. Recall that, a priori, by Remark 1.2.9, we have that $\delta \leqslant \frac{\tau_0^2}{16} \leqslant \Delta$.
Also, note that we do not use Corollary 1.2.14 along with the statement that $\mathcal{S}^2(\Omega^2_{14}) \ni \hat{W} \geqslant -\frac{\tau_0^2}{4}$ implies that $b_3 = 0$. This is because the sufficient conditions in terms of the bounds on the sectional curvature we would have obtained imply that $\Delta \leqslant \frac{11\tau_0^2}{80}$ or $\delta \geqslant \frac{\tau_0^2}{112}$. □

1.5 Nearly Kähler 6-manifolds

First, in Sections 1.5.1 and 1.5.2 we establish some preliminaries for 6-manifolds with an $SU(3)$-structure. In particular, we give various descriptions of irreducible subspaces of Ω^2 using the $\diamond$ map. Next, in Section 1.5.3 we introduce nearly Kähler 6-manifolds and in Sections 1.5.4 and 1.5.5 we establish several identities involving curvature and harmonic 2 and 3-forms. Finally, in Section 1.5.6 we simplify the Weizenbock formulas for harmonic 2 and 3-forms and using the assumption of compactness of our manifolds, we get the necessary conditions of vanishing of b_2 and b_3 in terms of bounds on $\bar{R}$, $\mathring{R}$, and $\hat{R}$.

1.5.1 Preliminaries

For this section as well, most of the results can be found in [14], [34], [36], [31], [30] and other sources on nearly Kähler manifolds. Nevertheless, we include as many details as

possible.
First we consider a general $SU(3)$-structure on a Riemannian 6-manifold (M,g). This means that M^6 has an almost complex structure J compatible with the metric g, and a complex 3-form $\Omega = \psi^+ + i\psi^-$ satisfying

$$\psi^+ \wedge \psi^- = \frac{2}{3}\omega^3 = 4\,\mathrm{vol_M}\,.$$

Also in this case, g_C, φ and ψ defined as:

$$\begin{aligned} g_C &:= r^2 g + dr^2, \\ \varphi &:= -r^2 dr \wedge \omega + r^3\psi^+, \\ \psi &:= -r^3 dr \wedge \psi^- - r^4\frac{w^2}{2}. \end{aligned}$$

give a metric cone G_2-structure on $\mathbb{R}^+ \times M$.
Hence, in a local orthonormal frame we get:

$$\begin{aligned} \varphi_{0ij} &= -\omega_{ij} \qquad & \psi_{0ijk} &= -\psi^-_{ijk} \\ \varphi_{ijk} &= \psi^+_{ijk} \qquad & \psi_{ijkl} &= -(\star\omega)_{ijkl}. \end{aligned} \tag{1.5.1}$$

We also list the following identities that hold, without proof:

$$\omega_{ik}\omega_{il} = \delta_{kl}, \star\psi^+ = \psi^-, \star\psi^- = -\psi^+, \star\omega = \frac{1}{2}\omega^2. \tag{1.5.2}$$

Looking at the last identity in coordinates gives us:

$$(\star\omega)_{ijkl} = \omega_{ij}\omega_{kl} + \omega_{jk}\omega_{il} + \omega_{lj}\omega_{ik}. \tag{1.5.3}$$

Proposition 1.5.4. *The following identities hold:*

- $\psi^+_{ijk}\omega_{ak} = -\psi^-_{ija}$. (1.5.5)
- $\psi^-_{ijk}\omega_{ak} = \psi^+_{ija}$. (1.5.6)
- $\psi^+_{ijk}\omega_{jk} = 0 = \psi^-_{ijk}\omega_{jk}$. (1.5.7)
- $\psi^+_{ijk}\psi^+_{abk} = \delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja} - \omega_{ia}\omega_{jb} + \omega_{ib}\omega_{ja}$. (1.5.8)
- $\psi^+_{ijk}\psi^+_{ajk} = 4\delta_{ik}$. *(contraction of the previous one)* (1.5.9)
- $\psi^+_{ijk}\psi^-_{abk} = \delta_{ia}\omega_{jb} + \delta_{jb}\omega_{ia} - \delta_{ib}\omega_{ja} - \delta_{ja}\omega_{ib}$. (1.5.10)
- $\psi^+_{ijk}\psi^-_{ajk} = 4\omega_{ia}$. *(contraction of the previous one)* (1.5.11)
- $\psi^-_{ijk}\psi^-_{abk} = \delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja} - \omega_{ia}\omega_{jb} + \omega_{ib}\omega_{ja}$. *(same as $\psi^+_{ijk}\psi^+_{abk}$)* (1.5.12)

- $\psi^-_{ijk}\psi^-_{ajk} = 4\delta_{ik}$. *(contraction of the previous one)* (1.5.13)
- $\omega_{ik}(\star\omega)_{abck} = \delta_{ia}\omega_{bc} + \delta_{ib}\omega_{ca} + \delta_{ic}\omega_{ab}$. (1.5.14)
- $\omega_{ik}(\star\omega)_{abik} = 4\omega_{ab}$. *(contraction of the previous one)* (1.5.15)
- $$\begin{aligned}\psi^+_{ijk}(\star\omega)_{abck} = &-\delta_{ia}\psi^+_{jbc} - \delta_{ib}\psi^+_{ajc} - \delta_{ic}\psi^+_{abj} \\ &+\delta_{aj}\psi^+_{ibc} + \delta_{bj}\psi^+_{aic} + \delta_{cj}\psi^+_{abi} - \omega_{ij}\psi^-_{abc}\end{aligned} \tag{1.5.16}$$
- $\psi^+_{ijk}(\star\omega)_{abck} = -\psi^-_{ija}\omega_{bc} - \psi^-_{ijb}\omega_{ca} - \psi^-_{ijc}\omega_{ab}$. *(alternative expression)* (1.5.17)
- $\psi^+_{ijk}(\star\omega)_{abjk} = 2\psi^+_{iab}$. *(contraction of the previous one)* (1.5.18)
- $\psi^-_{ijk}(\star\omega)_{abck} = \psi^+_{ija}\omega_{bc} + \psi^+_{ijb}\omega_{ca} + \psi^+_{ijc}\omega_{ab}$. (1.5.19)
- $\psi^-_{ijk}(\star\omega)_{abjk} = 2\psi^-_{iab}$. *(contraction of the previous one)* (1.5.20)
- $(\star\omega)_{ijkl}(\star\omega)_{abkl} = 2\delta_{ia}\delta_{jb} - 2\delta_{ib}\delta_{ja} + 2\omega_{ij}\omega_{ab}$. (1.5.21)
- $(\star\omega)_{ijkl}(\star\omega)_{ajkl} = 12\delta_{ia}$. *(contraction of the previous one)* (1.5.22)

Proof. We repeatedly use (1.5.1) along with G_2-contraction identities. For (1.5.5):

$$\begin{aligned}\psi^+_{ijk}\omega_{ak} &= \sum_{k=1}^{6}\varphi_{ijk}(-\varphi_{0ak}) \\ &= -\sum_{k=0}^{6}\varphi_{ijk}\varphi_{0ak} \\ &= -(\delta_{i0}\delta_{ja} - \delta_{ia}\delta_{j0} - \psi_{ij0a}) \\ &= \psi_{0ija} \\ &= -\psi^-_{ija}.\end{aligned}$$

Contracting both sides of (1.5.5) with w_{au} we get:

$$\begin{aligned}\psi^+_{ijk}\omega_{ak}\omega_{au} &= -\psi^-_{ija}\omega_{au} \\ \psi^+_{ijk}\delta_{ku} &= \psi^-_{ija}\omega_{ua} \\ \psi^+_{iju} &= \psi^-_{ija}\omega_{ua},\end{aligned}$$

which gives us (1.5.6).
Contracting on (1.5.5) and (1.5.6) on j, a immediately gives (1.5.7).

Next, for (1.5.8):

$$\begin{aligned}
\psi^+_{ijk}\psi^+_{abk} &= \sum_{k=1}^{6} \varphi_{ijk}\varphi_{abk} \\
&= \sum_{p=0}^{6} \varphi_{ijk}\varphi_{abk} - \varphi_{ij0}\varphi_{ab0} \\
&= (\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja} - \psi_{ijab}) - \omega_{ij}\omega_{ab} \\
&= (\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}) + ((\star\omega)_{ijab} - \omega_{ij}\omega_{ab}). \\
&= \delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja} + \omega_{ja}\omega_{ib} + \omega_{bj}\omega_{ia} \text{ (by (1.5.3))} \\
&= \delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja} - \omega_{ia}\omega_{jb} + \omega_{ib}\omega_{ja}.
\end{aligned}$$

Contracting (1.5.8) on j, b gives us (1.5.9):

$$\begin{aligned}
\psi^+_{ijk}\psi^+_{ajk} &= 6\delta_{ia} - \delta_{ia} + \omega_{ij}\omega_{ja} \\
&= 6\delta_{ia} - \delta_{ia} - \delta_{ia} \\
&= 4\delta_{ia}.
\end{aligned}$$

Next, for (1.5.10):

$$\begin{aligned}
\psi^+_{ijk}\psi^-_{abk} &= \sum_{k=1}^{6} \varphi_{ijk}(-\psi_{0abk}) \\
&= -\sum_{k=0}^{6} \varphi_{ijk}\psi_{0abk} \\
&= -(\delta_{i0}\varphi_{jab} + \delta_{ia}\varphi_{0jb} + \delta_{ib}\varphi_{0aj} - \delta_{0j}\varphi_{iab} - \delta_{aj}\varphi_{0ib} - \delta_{bj}\varphi_{0ai}) \\
&= \delta_{ia}\omega_{jb} + \delta_{ib}\omega_{aj} - \delta_{aj}\omega_{ib} - \delta_{bj}\omega_{ai} \\
&= \delta_{ia}\omega_{jb} + \delta_{jb}\omega_{ia} - \delta_{ib}\omega_{ja} - \delta_{ja}\omega_{ib}.
\end{aligned}$$

Contracting (1.5.10) on j, b will give us (1.5.11):

$$\begin{aligned}
\psi^+_{ijk}\psi^-_{ajk} &= 6\omega_{ia} - \omega_{ia} - \omega_{ia} \\
&= 4\omega_{ia}.
\end{aligned}$$

Next, we show that $\psi^-_{ijk}\psi^-_{abk} = \psi^+_{ijk}\psi^+_{abk}$, which means that (1.5.12) and (1.5.8) are the same. Using (1.5.5), we have:

$$\begin{aligned}
\psi^-_{ijk}\psi^-_{abk} &= \psi^+_{ijs}\omega_{ks}\psi^+_{abt}\omega_{kt} \\
&= \psi^+_{ijs}\psi^+_{abt}\delta_{st} \\
&= \psi^+_{ijs}\psi^+_{abs}.
\end{aligned}$$

Thus, we also get (1.5.13):

$$\psi^-_{ijk}\psi^-_{ajk} = \psi^+_{ijk}\psi^+_{ajk} = 4\delta_{ia}.$$

Next, for (1.5.14):

$$\begin{aligned}
\omega_{ik}(\star\omega)_{abck} &= \sum_{k=1}^{6}(-\varphi_{0ik})(-\psi_{abck}) \\
&= \sum_{k=0}^{6}\varphi_{0ik}\psi_{abck} \\
&= \delta_{0a}\varphi_{ibc} + \delta_{0b}\varphi_{aic} + \delta_{0c}\varphi_{abi} - \delta_{ai}\varphi_{0bc} - \delta_{bi}\varphi_{a0c} - \delta_{ci}\varphi_{ab0} \\
&= \delta_{ia}\omega_{bc} + \delta_{ib}\omega_{ca} + \delta_{ic}\omega_{ab}.
\end{aligned}$$

Contracting (1.5.14) on i, c yields (1.5.15):

$$\begin{aligned}
\omega_{ik}(\star\omega)_{abik} &= \omega_{ba} + \omega_{ba} + 6\omega_{ab} \\
&= 4\omega_{ab}.
\end{aligned}$$

For the next identity, there are two ways of computing the desired contractions yielding two different expressions (1.5.16) and (1.5.17). First, we use the usual way:

$$\begin{aligned}
\psi^+_{ijk}(\star\omega)_{abck} &= \sum_{k=1}^{6}\varphi_{ijk}(-\psi_{abck}) \\
&= -\sum_{k=0}^{6}\varphi_{ijk}\psi_{abck} + \varphi_{ij0}\psi_{abc0} \\
&= -(\delta_{ia}\varphi_{jbc} + \delta_{ib}\varphi_{ajc} + \delta_{ic}\varphi_{abj} - \delta_{aj}\varphi_{ibc} - \delta_{bj}\varphi_{aic} - \delta_{cj}\varphi_{abi}) + (-\omega_{ij})\psi^-_{abc} \\
&= -\delta_{ia}\psi^+_{jbc} - \delta_{ib}\psi^+_{ajc} - \delta_{ic}\psi^+_{abj} + \delta_{aj}\psi^+_{ibc} + \delta_{bj}\psi^+_{aic} + \delta_{cj}\psi^+_{abi} - \omega_{ij}\psi^-_{abc}.
\end{aligned}$$

Second, we can also use the previous results to get:

$$\begin{aligned}
\psi^+_{ijk}(\star\omega)_{abck} &= \psi^-_{iju}\omega_{ku}(\star\omega)_{abck} \\
&= -\psi^-_{iju}\omega_{uk}(\star\omega)_{abck} \\
&= -\psi^-_{iju}(\delta_{au}\omega_{bc} + \delta_{bu}\omega_{ca} + \delta_{cu}\omega_{ab}) \\
&= -\psi^-_{ija}\omega_{bc} - \psi^-_{ijb}\omega_{ca} - \psi^-_{ijc}\omega_{ab}.
\end{aligned}$$

Note that both contractions of (1.5.16) and (1.5.17) on j, c yiled the same result (1.5.18):

$$\begin{aligned}
\psi^+_{ijk}(\star\omega)_{abjk} &= -\psi^+_{abi} + \psi^+_{iba} + \psi^+_{aib} + 6\psi^+_{abi} - \omega_{ij}\psi^-_{abj} \\
&= -\psi^+_{iab} - \psi^+_{iab} - \psi^+_{iab} + 6\psi^+_{iab} - \psi^+_{abi} \\
&= 2\psi^+_{iab},
\end{aligned}$$

and

$$\begin{aligned}\psi^+_{ijk}(\star\omega)_{abjk} &= -\psi^-_{ija}\omega_{bj} - \psi^-_{ijb}\omega_{ja}\\ &= \psi^-_{iaj}\omega_{bj} - \psi^-_{ibj}\omega_{aj}\\ &= \psi^+_{iab} - \psi^+_{iba}\\ &= 2\psi^+_{iab}.\end{aligned}$$

Since the second way of computing the contraction of ψ^+ and $\star\omega$ gave us a nicer expression, we use it again for (1.5.19):

$$\begin{aligned}\psi^-_{ijk}(\star\omega)_{abck} &= -\psi^+_{iju}\omega_{ku}(\star\omega)_{abck}\\ &= \psi^+_{iju}\omega_{uk}(\star\omega)_{abck}\\ &= \psi^+_{iju}(\delta_{au}\omega_{bc} + \delta_{bu}\omega_{ca} + \delta_{cu}\omega_{ab})\\ &= \psi^+_{ija}\omega_{bc} + \psi^+_{ijb}\omega_{ca} + \psi^+_{ijc}\omega_{ab}.\end{aligned}$$

Contracting (1.5.19) on j, c yields (1.5.20):

$$\begin{aligned}\psi^-_{ijk}(\star\omega)_{abjk} &= \psi^+_{ija}\omega_{bj} + \psi^+_{ijb}\omega_{ja}\\ &= -\psi^+_{iaj}\omega_{bj} + \psi^+_{ibj}\omega_{aj}\\ &= \psi^-_{iab} - \psi^-_{iba}\\ &= 2\psi^-_{iab}.\end{aligned}$$

Finally, we compute (1.5.21):

$$\begin{aligned}(\star\omega)_{ijkl}(\star\omega)_{abkl} &= \sum_{k,l=1}^{6}\psi_{ijkl}\psi_{abkl}\\ &= \sum_{k,l=0}^{6}\psi_{ijkl}\psi_{abkl} - \sum_{k=0}^{6}\psi_{ijk0}\psi_{abk0} - \sum_{l=0}^{6}\psi_{ij0l}\psi_{ab0l}\\ &= \sum_{k,l=0}^{6}\psi_{ijkl}\psi_{abkl} - 2\sum_{k=1}^{6}\psi^-_{ijk}\psi^-_{abk}\\ &=(4\delta_{ia}\delta_{jb} - 4\delta_{ib}\delta_{ja} - 2\psi_{ijab}) - 2(\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja} - \omega_{ia}\omega_{jb} + \omega_{ib}\omega_{ja})\\ &=2\delta_{ia}\delta_{jb} - 2\delta_{ib}\delta_{ja} + 2((\star\omega)_{ijab} + \omega_{ia}\omega_{jb} + \omega_{aj}\omega_{ib})\\ &=2\delta_{ia}\delta_{jb} - 2\delta_{ib}\delta_{ja} + 2((\omega_{ij}\omega_{ab} + \omega_{ja}\omega_{ib} + \omega_{bj}\omega_{ia}) + \omega_{ia}\omega_{jb} + \omega_{aj}\omega_{ib})\\ &=2\delta_{ia}\delta_{jb} - 2\delta_{ib}\delta_{ja} + 2\omega_{ij}\omega_{ab}.\end{aligned}$$

Contracting (1.5.21) on b, j gives us (1.5.22):

$$(\star\omega)_{ijkl}(\star\omega)_{ajkl} = 12\delta_{ia} - 2\delta_{ia} + 2\delta_{ia} = 12\delta_{ia}. \qquad \square$$

Remark 1.5.23. We have the following descriptions of the orthogonal decompositions of Ω^2 and Ω^3 into irreducible subspaces (see [14]):

$$\Omega^2 = \Omega^2_1 \oplus \Omega^2_6 \oplus \Omega^2_8,$$
$$\Omega^3 = \Omega^3_{1\oplus 1} \oplus \Omega^3_6 \oplus \Omega^3_{12},$$

where the indices denote the corresponding dimensions. In particular:

- $\Omega^2_1 = \{\beta \in \Omega^2 : \star(\beta \wedge \omega) = 2\beta\} = \mathbb{R}\omega$,
- $\Omega^2_6 = \{\beta \in \Omega^2 : \star(\beta \wedge \omega) = \beta\} = \{X \lrcorner \psi^+ : X \in \Gamma(TM)\}$,
- $\Omega^2_8 = \{\beta \in \Omega^2 : \star(\beta \wedge \omega) = -\beta\}$ is the space of primitive forms of type $(1,1)$, or equivalently, $\beta \in \Omega^2_8$ iff $\beta_{ij}\psi^+_{ijk} = 0$ and $\beta_{ij}w_{ij} = 0$,
- $\Omega^3_{1\oplus 1} = \mathbb{R}\psi^+ \bigoplus \mathbb{R}\psi^-$,
- $\Omega^3_6 = \{X \wedge \omega : X \in \Gamma(TM)\}$,
- Ω^3_{12} is the space of primitive forms of type $(1,2)+(2,1)$, or equivalently, $\Omega^3_{12} = \mathcal{S}^2_- \diamond \psi^+$,

where the $\mathcal{S}^2_-$ is defined in Section 1.5.2.

Remark 1.5.24. Consider the map $\mathcal{P} : \Omega^2 \to \Omega^2$ given by $\mathcal{P}(\beta) = \star(\beta \wedge \omega)$, for $\beta \in \Omega^2$. In a local orthonormal frame:

$$(\mathcal{P}\beta)_{ab} = \frac{1}{2}\beta_{ij}(\frac{\omega^2}{2})_{ijab} = \frac{1}{2}\beta_{ij}(\star\omega)_{ijab}.$$

We can extend the map $\mathcal{P}$ to all of $\mathcal{T}^2$ via the formula above. Then we have $\mathcal{S}^2 = \ker(\mathcal{P})$ and for $\beta \in \Omega^2$, Remark 1.5.23 says that:

$$\mathcal{P}(\beta) = 2\pi_1(\beta) + \pi_6(\beta) - \pi_8(\beta). \tag{1.5.25}$$

Proposition 1.5.26. *Let $\beta = \beta_0 + \lambda\omega + X \lrcorner \psi^+$, where $\beta_0 \in \Omega^2_8$. Then:*

- $\lambda = \frac{1}{6}\beta_{ij}\omega_{ij}$.
- $X_k = \frac{1}{4}\beta_{ij}\psi^+_{kij}$.

Proof. Recall that $(\beta_0)_{ij}\psi^+_{ijk} = 0, (\beta_0)_{ij}\omega_{ij} = 0$, and $\omega_{ij}\psi^+_{ijk} = 0$.
For the first identity, contract

$$\beta_{ij} = (\beta_0)_{ij} + \lambda\omega_{ij} + X_a\psi^+_{aij} \tag{1.5.27}$$

with ω_{ij} to get:

$$\beta_{ij}\omega_{ij} = \lambda\omega_{ij}\omega_{ij} = 6\lambda.$$

Similarly, contracting (1.5.27) with ψ^+_{kij} gives us:

$$\beta_{ij}\psi^+_{kij} = X_a\psi^+_{aij}\psi^+_{kij} = 4X_a\delta_{ak} = 4X_k.$$

as claimed. □

Lemma 1.5.28. *Let $\beta \in \Omega^2_8$. Then $\beta\omega = \omega\beta$, where by $\beta\omega \in \mathcal{T}^2$ we mean $(\beta\omega)_{ij} = \beta_{ik}\omega_{kl}$, and similarly for $\omega\beta$.*

Proof. Since, $\beta \in \Omega^2_8$, by Remark 1.5.24, we have that $\mathcal{P}\beta = -\beta$, which in a local orthonormal frame is $\frac{1}{2}\beta_{ij}(\star\omega)_{ijab} = -\beta_{ab}$. Also, recall that $\beta_{ij}\omega_{ij} = 0$. Using Proposition 1.5.4, we have:

$$\begin{aligned}
(\beta\omega)_{st} &= \beta_{su}\omega_{ut} \\
&= -\frac{1}{2}\beta_{ij}(\star\omega)_{ijsu}\omega_{ut} \\
&= \frac{1}{2}\beta_{ij}\omega_{tu}(\star\omega)_{ijsu} \\
&= \frac{1}{2}\beta_{ij}(\delta_{it}\omega_{js} + \delta_{jt}\omega_{si} + \delta_{st}\omega_{ij}) \\
&= \beta_{ij}\delta_{it}\omega_{js} + 0 \\
&= \beta_{tj}\omega_{js} \\
&= \omega_{sj}\beta_{jt} \\
&= (\omega\beta)_{st},
\end{aligned}$$

as claimed. □

1.5.2 The $\diamond$ operator

The results in this section are very similar to the ones in Remark 1.4.2. We describe the isomorphisms coming from the $\diamond$ map. This time, however, we give most of the details. Recall the definition of the $\diamond$ map: let $\sigma \in \Omega^k$. For $h \in \mathcal{T}^2$, we define:

$$(h \diamond \sigma)_{i_1\cdots i_k} := h_{i_1p}\sigma_{pi_2\cdots i_k} + h_{i_2p}\sigma_{i_1pi_3\cdots i_k} + \cdots + h_{i_kp}\sigma_{i_1\cdots i_{k-1}p}.$$

Definition 1.5.29. Let β be a 2, 3, or 4-form. Then we define $\hat{\beta} \in \mathcal{T}^2$ as follows:

$$\begin{aligned}
\text{for } \beta \in \Omega^2, &\quad \hat{\beta}_{ia} := \beta_{ik}\omega_{ak}, \\
\text{for } \beta \in \Omega^3, &\quad \hat{\beta}_{ia} := \beta_{ijk}\psi^+_{ajk}, \\
\text{for } \beta \in \Omega^4, &\quad \hat{\beta}_{ia} := \beta_{ijkl}(\star\omega)_{ajkl}.
\end{aligned}$$

▲

Remark 1.5.30. Let

$$\mathcal{S}^2_+ := \{h \in \mathcal{S}^2 | h\omega - \omega h = 0\}$$
$$\mathcal{S}^2_- := \{h \in \mathcal{S}^2 | h\omega + \omega h = 0\}$$

which are the spaces of symmetric 2-tensors which commute and anti-commute with ω (or equivalently with J), respectively.
Note that $\mathcal{S}^2_- \subset \mathcal{S}_0$. This can be easily seen by recalling that $\omega^2 = -\operatorname{Id}$, and so

$$\operatorname{tr}(h) = -\operatorname{tr}((h\omega)\omega) = -\operatorname{tr}((-\omega h)\omega) = \operatorname{tr}(\omega h\omega) = \operatorname{tr}(h)\omega^2 = -\operatorname{tr}(h).$$

Hence, we can further decompose

$$\mathcal{S}^2_+ = \mathbb{R}g \oplus \mathcal{S}^2_{+0}$$

where $\mathcal{S}^2_{+0}$ are the traceless elements of $\mathcal{S}^2_+$.
Concluding, we have the orthogonal decomposition:

$$\mathcal{S}^2 = \mathbb{R}g \oplus \mathcal{S}^2_{+0} \oplus \mathcal{S}^2_-.$$

It is easy to check that $\mathcal{S}^2_{+0}$ has dimension 8 and $\mathcal{S}^2_-$ has dimension 12.

2-forms

Proposition 1.5.31. *In the case of 2-forms, the $\cdot \diamond \omega$ map is an isomorphism of the following spaces:*

$$\begin{aligned} \mathbb{R}g &\cong \Omega^2_1, \\ \Omega^2_6 &\cong \Omega^2_6, \\ \mathcal{S}^2_{+0} &\cong \Omega^2_8. \end{aligned}$$

Proof. For the first two maps it will be clear that they are isomorphisms. For the last one, we just check that the image under the $\cdot \diamond \omega$ map lies in the required subspace. Then by the next Proposition 1.5.32, which shows that the map is invertible, we conclude that it is also an isomorphism. So, we have:

$$(g \diamond \omega)_{ij} = g_{ip}\omega_{pj} + g_{jp}\omega_{ip} = 2\omega_{ij}.$$

Next, take any $\beta = X \lrcorner \psi^+ \in \Omega^2_6$. By Proposition 1.5.4 and 1.5.26, we have:

$$\begin{aligned}(\beta \diamond \omega)_{ij} &= \beta_{ip}\omega_{pj} + \beta_{jp}\omega_{ip} \\ &= X_a\psi^+_{aip}\omega_{pj} + X_a\psi^+_{ajp}\omega_{ip} \\ &= -X_a\psi^+_{aip}\omega_{jp} + X_a\psi^+_{ajp}\omega_{ip} \\ &= X_a\psi^-_{aij} - X_a\psi^-_{aji} \\ &= 2X_a\psi^-_{aij} \\ &= 2X_a\psi^-_{ija} \\ &= -2X_a\omega_{ap}\psi^+_{ijp} \\ &= -(2J(X) \lrcorner \psi^+)_{ij},\end{aligned}$$

where we have used that

$$(J(X))_p = \langle J(X), e_p \rangle = \omega(X, e_p) = X_a\omega_{ap}.$$

Finally, take any $h \in S^2_{+0}$. Then:

$$\begin{aligned}(h \diamond \omega)_{ij}\omega_{ij} &= (h_{ip}\omega_{pj} + h_{jp}\omega_{ip})\omega_{ij} \\ &= h_{ip}\delta_{pi} + h_{jp}\delta_{pj} \\ &= 2\operatorname{tr} h \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}(h \diamond \omega)_{ij}\psi^+_{ijk} &= (h_{ip}\omega_{pj} + h_{jp}\omega_{ip})\psi^+_{ijk} \\ &= ((h\omega)_{ij} + (\omega h)_{ij})\psi^+_{ijk} \\ &= 2(h\omega)_{ij}\psi^+_{ijk} \text{ (since } h\omega = \omega h \text{ for } h \in \mathcal{S}^2_+) \\ &= 2h_{ip}\omega_{pj}\psi^+_{ijk} \\ &= 2h_{ip}\omega_{pj}\psi^+_{kij} \\ &= -2h_{ip}\psi^-_{kip} \\ &= 0,\end{aligned}$$

because $h \in \mathcal{S}^2$. This shows that $h \diamond \omega \in \Omega^2_8$. Hence, the result follows. □

Proposition 1.5.32. *Let $\beta \in \Omega^2$. Then $\beta = h \diamond \omega$ for some unique $h = \frac{1}{6}\operatorname{tr}(h)g + X \lrcorner \psi^+ + h_{+0}$, where $X \in \Gamma(TM)$, $h_{+0} \in \mathcal{S}^2_{+0}$. Then $h = \frac{1}{2}\hat{\beta}$, where $\hat{\beta}$ is as in Definition 1.5.29. This implies that*

$$\operatorname{tr}(h) = \frac{1}{2}\operatorname{tr}(\hat{\beta}), \qquad X_k = \frac{1}{8}\hat{\beta}_{ia}\psi^+_{kia}, \qquad h_{+0} = \frac{1}{2}\hat{\beta}_{symm} - \frac{1}{12}\operatorname{tr}(\hat{\beta})g.$$

Also, clearly $\beta = 0$ iff $h = 0$ iff $\hat{\beta} = 0$.

Proof. We compute $\hat{\beta}$ as follows:

$$\begin{aligned}
\hat{\beta}_{ia} &= \beta_{ik}\omega_{ak} \\
&= (h \diamond \omega)_{ik}\omega_{ak} \\
&= (h_{ip}\omega_{pk} + h_{kp}\omega_{ip})\omega_{ak} \\
&= h_{ip}\delta_{ap} + h_{kp}\omega_{ip}\omega_{ak} \\
&= h_{ia} + (\frac{1}{6}\operatorname{tr}(h)\delta_{kp} + X_u\psi^+_{ukp} + (h_{+0})_{kp})\omega_{ip}\omega_{ak} \\
&= h_{ia} + \frac{1}{6}\operatorname{tr}(h)\delta_{ia} + X_u\psi^+_{ukp}\omega_{ip}\omega_{ak} - \omega_{ip}(h_{+0})_{pk}\omega_{ka} \\
&= h_{ia} + \frac{1}{6}\operatorname{tr}(h)\delta_{ia} - X_u\psi^-_{uki}\omega_{ak} - (\omega h_{+0}\omega)_{ia} \\
&= h_{ia} + \frac{1}{6}\operatorname{tr}(h)\delta_{ia} + X_u\psi^-_{uik}\omega_{ak} - (h_{+0}\omega^2)_{ia} \\
&= h_{ia} + \frac{1}{6}\operatorname{tr}(h)\delta_{ia} + X_u\psi^+_{uia} + (h_{+0})_{ia} \\
&= 2h_{ia}.
\end{aligned}$$

as claimed.
So, we get $2\operatorname{tr}(h) = \operatorname{tr}(\hat{\beta})$ along with

$$\hat{\beta}_{symm} = 2(\frac{1}{6}\operatorname{tr}(h)g + h_{+0})$$

which means that

$$h_{+0} = \frac{1}{2}\hat{\beta}_{symm} - \frac{1}{6}\operatorname{tr}(h)g = \frac{1}{2}\hat{\beta}_{symm} - \frac{1}{12}\operatorname{tr}(\hat{\beta})g.$$

Finally, by Proposition 1.5.26, we get that

$$2X_k = \frac{1}{4}(\hat{\beta}_{skew})_{ia}\psi^+_{kia} = \frac{1}{4}\hat{\beta}_{ia}\psi^+_{kia}. \quad \square$$

4-forms

Proposition 1.5.33. *Let $\beta \in \Omega^4$. Then $\beta = h \diamond (\star\omega)$ for some unique $h \in \Omega^2_6 \oplus \mathcal{S}^2_+$.*

Proof. It is easy to check that:

$$\star(h \diamond \omega) = (\frac{1}{4}\operatorname{tr}(h)g - h^T) \diamond (\star\omega).$$

Now, since $\beta \in \Omega^4$, we have $\star\beta \in \Omega^2$. Then by Proposition 1.5.31, $\star\beta = h \diamond \omega$, for some unique $h \in \Omega^2_6 \oplus \mathcal{S}^2_+$.
Hence,

$$\beta = \star(\star\beta) = (\frac{1}{4}\operatorname{tr}(h)g - h^T) \diamond (\star\omega).$$

Note that the map $h \mapsto (\frac{1}{4}\operatorname{tr}(h)g - h^T)$ is an automorphism of $\Lambda^2_6 \oplus \mathcal{S}^2_+$. This is because it can be seen that under this map, Ω^2_6 is mapped to itself and for $h \in \mathcal{S}^2_+$ we have:

$$h \mapsto \frac{1}{4}\operatorname{tr}(h)g - h,$$

which is injective, as $\frac{1}{4}\operatorname{tr}(h)g - h = 0$ iff $h = cg$, for some $c \in \mathbb{R}$, but then $\frac{1}{4}6cg = cg$, hence $c = 0$. Also, since g, h commute with ω, $\frac{1}{4}\operatorname{tr}(h)g - h$ also commutes with ω, so is in $\mathcal{S}^2_+$. Thus, we get the required result. □

Proposition 1.5.34. *Let $\beta \in \Omega^4$, so $\beta = h \diamond (\star\omega)$ for some unique $h \in \Omega^2_6 \oplus \mathcal{S}^2_+$. Then*

$$\hat{\beta} = 8\operatorname{tr}(h)g + 12h_6 + 12h_{+0},$$

where $\hat{\beta}$ is as in Definition 1.5.29.

Proof. We compute $\hat{\beta}$:

$$\begin{aligned}
\hat{\beta}_{ia} &= \beta_{ijkl}(\star\omega)_{ajkl} \\
&= (h_{ip}(\star\omega)_{pjkl} + h_{jp}(\star\omega)_{ipkl} + h_{kp}(\star\omega)_{ijpl} + h_{lp}(\star\omega)_{ijkp})(\star\omega)_{ajkl} \\
&= h_{ip}(\star\omega)_{pjkl}(\star\omega)_{ajkl} + 3h_{jp}(\star\omega)_{ipkl}(\star\omega)_{ajkl} \\
&= h_{ip}12\delta_{pa} + 3h_{jp}(2\delta_{ia}\delta_{pj} - 2\delta_{ij}\delta_{pa} + 2\omega_{ip}\omega_{aj}) \\
&= 12h_{ia} + 6\operatorname{tr}(h)\delta_{ia} - 6h_{ia} + 6h_{jp}\omega_{ip}\omega_{aj}
\end{aligned}$$

In the proof of Proposition 1.5.32, we computed that for $h \in \Omega^2_6 \oplus \mathcal{S}^2_+$, $h_{kp}\omega_{ip}\omega_{ak} = h_{ia}$. Hence,

$$\begin{aligned}
\hat{\beta}_{ia} &= 6h_{ia} + 6\operatorname{tr}(h)\delta_{ia} + 6h_{ia} \\
&= 12h_{ia} + 6\operatorname{tr}(h)\delta_{ia} \\
&= 12(\frac{1}{6}\operatorname{tr}(h)g + h_6 + h_{+0})_{ia}) + 6\operatorname{tr}(h)\delta_{ia} \\
&= 8\operatorname{tr}(h)\delta_{ia} + 12(h_6)_{ia} + 12(h_{+0})_{ia}.
\end{aligned}$$

□

Corollary 1.5.35. *Let $\beta \in \Omega^4$, so $\beta = h \diamond (\star\omega)$ for some unique $h = \frac{1}{6}\operatorname{tr}(h)g + h_6 + h_{+0}$. Then:*

$$\begin{aligned}
\operatorname{tr}(h) =& \frac{1}{48}\operatorname{tr}(\hat{\beta}), \\
(h_{+0})_{ia} =& \frac{1}{12}(\hat{\beta}_{symm})_{ia} - \frac{1}{72}\operatorname{tr}(\hat{\beta})\delta_{ia} = \frac{1}{24}(\hat{\beta}_{ia} + \hat{\beta}_{ai}) - \frac{1}{72}\operatorname{tr}(\hat{\beta})\delta_{ia}, \\
(h_6)_{ia} =& \frac{1}{12}(\hat{\beta}_{skew})_{ia} = \frac{1}{24}(\hat{\beta}_{ia} - \hat{\beta}_{ai}).
\end{aligned}$$

Also, clearly $\beta = 0$ *iff* $h = 0$ *iff* $\hat{\beta} = 0$.

Proof. In Proposition 1.5.34 we proved that

$$\hat{\beta} = 8\,\mathrm{tr}(h)g + 12h_6 + 12h_{+0}. \tag{1.5.36}$$

Taking traces of both sides, we get

$$\mathrm{tr}(\hat{\beta}) = 48\,\mathrm{tr}(h).$$

Hence,

$$h_{+0} = \frac{1}{12}(\hat{\beta}_{symm} - 8\,\mathrm{tr}(h)g) = \frac{1}{12}\hat{\beta}_{symm} - \frac{2}{3}\cdot\frac{1}{48}\,\mathrm{tr}(\hat{\beta})g = \frac{1}{12}\hat{\beta}_{symm} - \frac{1}{72}\,\mathrm{tr}(\hat{\beta})g.$$

Taking skew-symmetric parts of (1.5.36) gives us the required

$$12h_6 = \hat{\beta}_{skew}. \quad \square$$

3-forms

Proposition 1.5.37. *In the case of the* 3*-forms, the* $\cdot \diamond \psi^+$ *map is an isomorphism of the following spaces:*

$$\begin{aligned} \mathbb{R}g \oplus \mathbb{R}\omega &\cong \Omega^3_{1\oplus 1}, \\ \Omega^2_6 &\cong \Omega^3_6, \\ \mathcal{S}^2_- &\cong \Omega^3_{12}. \end{aligned}$$

Proof. Computing $g \diamond \psi^+$ and $\omega \diamond \psi^+$ gives us:

$$\begin{aligned} g \diamond \psi^+ &= 3\psi^+. \\ (\omega \diamond \psi^+)_{ijk} &= \omega_{ip}\psi^+_{pjk} + \omega_{jp}\psi^+_{ipk} + \omega_{kp}\psi^+_{ijp} \\ &= \omega_{ip}\psi^+_{jkp} + \omega_{jp}\psi^+_{kip} + \omega_{kp}\psi^+_{ijp} \\ &= -\psi^-_{jki} - \psi^-_{kij} - \psi^-_{ijk} \\ &= -3\psi^-_{ijk}, \end{aligned}$$

which is enough to conclude that $\mathbb{R}g \oplus \mathbb{R}\omega \cong \Omega^3_{1\oplus 1}$.
Next, take any $X \lrcorner \psi^+$, with $X \in \Gamma(TM)$. Then:

$$\begin{aligned}((X \lrcorner \psi^+) \diamond \psi^+)_{ijk} =&(X \lrcorner \psi^+)_{ip}\psi^+_{pjk} + (X \lrcorner \psi^+)_{jp}\psi^+_{ipk} + (X \lrcorner \psi^+)_{kp}\psi^+_{ijp}\\ =&X_u\psi^+_{uip}\psi^+_{pjk} + X_u\psi^+_{ujp}\psi^+_{ipk} + X_u\psi^+_{ukp}\psi^+_{ijp}\\ =&X_u(\psi^+_{uip}\psi^+_{jkp} + \psi^+_{ujp}\psi^+_{kip} + \psi^+_{ukp}\psi^+_{ijp})\\ =&X_u(\delta_{uj}\delta_{ik} - \delta_{uk}\delta_{ij} - \omega_{uj}\omega_{ik} + \omega_{uk}\omega_{ij}\\ &+ \delta_{uk}\delta_{ji} - \delta_{ui}\delta_{jk} - \omega_{uk}\omega_{ji} + \omega_{ui}\omega_{jk}\\ &+ \delta_{ui}\delta_{kj} - \delta_{uj}\delta_{ki} - \omega_{ui}\omega_{kj} + \omega_{uj}\omega_{ki})\\ =&2X_u(\omega_{ui}\omega_{jk} + \omega_{uj}\omega_{ki} + \omega_{uk}\omega_{ij})\\ =&2(JX \wedge \omega)_{ijk},\end{aligned}$$

which again is enough to see that $\Omega^2_6 \cong \Omega^3_6$.
For the last isomorphism, we avoid the details, because this is how we defined Ω^3_{12} in Remark 1.5.23. ☐

Proposition 1.5.38. *Let $\beta \in \Omega^3$, so $\beta = h \diamond \psi^+$ for some unique $h \in \mathbb{R}g \oplus \mathbb{R}\omega \oplus \Omega^2_6 \oplus \mathcal{S}^2_-$. Then*

$$\hat{\beta} = 2\,\mathrm{tr}(h)g + 12\lambda\omega + 4h_6 + 4h_-,$$

where $\hat{\beta}$ is as in Definition 1.5.29, and λ is the coefficient of ω in h, meaning that the unique part of h in $\mathbb{R}\omega$ is $\lambda\omega$.

Proof. Let $h_6 = X \lrcorner \psi^+$, for some unique $X \in \Gamma(TM)$. Now, we just compute $\hat{\beta}$:

$$\begin{aligned}\hat{\beta}_{ia} &= \beta_{ijk}\psi^+_{ajk}\\ &= (h \diamond \psi^+)\psi^+_{ajk}\\ &= (h_{ip}\psi^+_{pjk} + h_{jp}\psi^+_{ipk} + h_{kp}\psi^+_{ijp})\psi^+_{ajk}\\ &= h_{ip}\psi^+_{pjk}\psi^+_{ajk} + 2h_{jp}\psi^+_{ipk}\psi^+_{ajk}\\ &= h_{ip}4\delta_{pa} + 2h_{jp}(\delta_{ia}\delta_{pj} - \delta_{ij}\delta_{pa} - \omega_{ia}\omega_{pj} + \omega_{ij}\omega_{pa})\\ &= 4h_{ia} + 2\,\mathrm{tr}(h)\delta_{ia} - 2h_{ia} + 2h_{jp}\omega_{jp}\omega_{ia} + 2h_{jp}\omega_{pa}\omega_{ij}.\end{aligned}$$

We compute the last two terms separately:

$$\begin{aligned}
2h_{jp}\omega_{pa}\omega_{ij} &= 2(\frac{1}{6}\operatorname{tr}(h)\delta_{jp} + \lambda\omega_{jp} + X_u\psi^+_{ujp} + (h_-)_{jp})\omega_{pa}\omega_{ij} \\
&= \frac{1}{3}\operatorname{tr}(h)\omega_{pa}\omega_{ip} + 2\lambda\omega_{jp}\omega_{pa}\omega_{ij} - 2X_u\psi^+_{ujp}\omega_{ap}\omega_{ij} + 2\omega_{ij}(h_-)_{jp}\omega_{pa} \\
&= -\frac{1}{3}\operatorname{tr}(h)\delta_{ia} - 2\lambda\delta_{aj}\omega_{ij} + 2X_u\psi^-_{uja}\omega_{ij} + 2(\omega h_-\omega)_{ia} \\
&= -\frac{1}{3}\operatorname{tr}(h)\delta_{ia} - 2\lambda\omega_{ia} - 2X_u\psi^-_{uaj}\omega_{ij} - 2(h_-\omega^2)_{ia} \\
&= -\frac{1}{3}\operatorname{tr}(h)\delta_{ia} - 2\lambda\omega_{ia} - 2X_u\psi^+_{uai} + 2(h_-)_{ia} \\
&= -\frac{1}{3}\operatorname{tr}(h)\delta_{ia} - 2\lambda\omega_{ia} + 2(h_6)_{ia} + 2(h_-)_{ia}.
\end{aligned}$$

Next,

$$\begin{aligned}
2h_{jp}\omega_{jp}\omega_{ia} &= 2(\frac{1}{6}\operatorname{tr}(h)\delta_{jp} + \lambda\omega_{jp} + X_u\psi^+_{ujp} + (h_-)_{jp})\omega_{jp}\omega_{ia} \\
&= 0 + 12\lambda\omega_{ia} + 0 + 0 \\
&= 12\lambda\omega_{ia}.
\end{aligned}$$

Hence, combining these parts we get:

$$\begin{aligned}
\hat{\beta}_{ia} &= 2h_{ia} + 2\operatorname{tr}(h)\delta_{ia} + (-\frac{1}{3}\operatorname{tr}(h)\delta_{ia} - 2\lambda\omega_{ia} + 2(h_6)_{ia} + 2(h_-)_{ia}) + 12\lambda\omega_{ia} \\
&= 2(\frac{1}{6}\operatorname{tr}(h)\delta_{ia} + \lambda\omega_{ia} + (h_6)_{ia} + (h_-)_{ia}) + \frac{5}{3}\operatorname{tr}(h)\delta_{ia} + 10\lambda\omega_{ia} + 2(h_6)_{ia} + 2(h_-)_{ia} \\
&= 2\operatorname{tr}(h)\delta_{ia} + 12\lambda\omega_{ia} + 4(h_6)_{ia} + 4(h_-)_{ia},
\end{aligned}$$

as claimed. □

Corollary 1.5.39. *Let $\beta \in \Omega^3$, so $\beta = h \diamond \psi^+$ for some unique $h = \frac{1}{6}\operatorname{tr}(h)g + \lambda\omega + X \lrcorner \psi^+ + h_-$, where $X \in \Gamma(TM)$. Then:*

$$\begin{aligned}
\operatorname{tr}(h) =& \frac{1}{12}\operatorname{tr}(\hat{\beta}), \\
(h_-)_{ia} =& \frac{1}{4}(\hat{\beta}_{symm})_{ia} - \frac{1}{24}\operatorname{tr}(\hat{\beta})\delta_{ia} = \frac{1}{8}(\hat{\beta}_{ia} + \hat{\beta}_{ai}) - \frac{1}{24}\operatorname{tr}(\hat{\beta})\delta_{ia}, \\
\lambda =& \frac{1}{72}\hat{\beta}_{ia}\omega_{ia}, \\
X_k =& \frac{1}{16}\hat{\beta}_{ia}\psi^+_{kia}.
\end{aligned}$$

Also, clearly $\beta = 0$ iff $h = 0$ iff $\hat{\beta} = 0$.

Proof. In Proposition 1.5.38 we proved that:

$$\hat{\beta} = 2\,\mathrm{tr}(h)g + 12\lambda\omega + 4h_6 + 4h_-. \tag{1.5.40}$$

Taking traces of both sides yields

$$\mathrm{tr}(\hat{\beta}) = 12\,\mathrm{tr}(h).$$

Next, taking symmetric parts of both sides of (1.5.40) gives us:

$$\hat{\beta}_{symm} = 2\,\mathrm{tr}(h)g + 4h_-.$$

Thus,

$$h_- = \frac{1}{4}(\hat{\beta}_{symm} - 2\,\mathrm{tr}(h)g) = \frac{1}{4}\hat{\beta}_{symm} - \frac{1}{24}\,\mathrm{tr}(\hat{\beta})g.$$

On the other hand, comparing skew-symmetric parts of both sides of (1.5.40) gives us:

$$\hat{\beta}_{skew} = 12\lambda\omega + 4h_6$$

We recall Proposition 1.5.26 to get

$$12\lambda = \frac{1}{6}(\hat{\beta}_{skew})_{ia}\omega_{ia} = \frac{1}{6}\hat{\beta}_{ia}\omega_{ia}$$

and

$$4X_k = \frac{1}{4}(\hat{\beta}_{skew})_{ia}\psi^+_{kia} = \frac{1}{4}\hat{\beta}_{ia}\psi^+_{kia},$$

which concludes the proof. □

1.5.3 Nearly Kähler 6-manifolds

Let (M^6, g, J, Ω) be a compact connected 6-manifold with an $SU(3)$-structure. We say it is **nearly Kähler** if:

$$\nabla_X\omega = -X \lrcorner \psi^+ \text{ and } \nabla_X\psi^+ = X \wedge \omega. \tag{1.5.41}$$

In dimension 6 it is equivalent to $(\nabla_X J)(X) = 0$, for all $X \in \Gamma(TM)$, but $\nabla J \neq 0$. Also, by [34] it is also equivalent to $d\omega = 3\nabla\omega$ or that $d\omega = -3\psi^+$ and $d\psi^- = 4\frac{\omega^2}{2}$. Moreover, one can check that in this case the conical G_2 structure on $M \times \mathbb{R}$ is torsion-free. Finally, it is a fact that all nearly Kähler manifolds in dimension 6 are positive Einstein. With our choice of normalization, the Einstein constant is 5.
In a local orthonormal frame we can write (1.5.41) as:

$$\nabla_i\omega_{jk} = -\psi^+_{ijk} \text{ and } \nabla_i\psi^+_{jkl} = \delta_{ij}\omega_{kl} + \delta_{ik}\omega_{lj} + \delta_{il}\omega_{jk} \tag{1.5.42}$$

Note that contracting the second identity on i, j gives us

$$\nabla_i\psi^+_{ikl} = 6\omega_{kl} + \omega_{lk} + \omega_{lk} = 4\omega_{kl}. \tag{1.5.43}$$

1.5.4 Curvature identities

On a nearly Kähler manifold we have the Einstein constant $k = 5$. Applying the result from Lemma 1.2.8 we get:

$$\begin{aligned} \hat{W} &= \hat{R} + 2\,\mathrm{Id}, \\ \mathring{W} &= \mathring{R} - \mathrm{Id}, \text{ on } \mathcal{S}_0^2. \end{aligned} \tag{1.5.44}$$

Proposition 1.5.45. *The following identities hold:*

- $R_{pqiu}\psi^+_{liu} = -2\psi^+_{pql}$. (1.5.46)
- $R_{pqiu}\psi^-_{viu} = -2\psi^-_{pqv}$. (1.5.47)
- $R_{pqju}\omega_{ju} = -2\omega_{pq}$. (1.5.48)

Proof. For the first identity (1.5.46), we show that computing contraction of ψ^- and $\nabla\nabla\omega$ yields the required result. Explicitly,

$$\begin{aligned} \nabla_p\nabla_q\omega_{ij} &= \nabla_p(-\psi^+_{qij}) \\ &= -(\delta_{pq}\omega_{ij} + \delta_{pi}\omega_{jq} + \delta_{pj}\omega_{qi}) \text{ (by (1.5.42))} \end{aligned}$$

Now, we use the Ricci identity to get:

$$\begin{aligned} -R_{pqiu}\omega_{uj} - R_{pqju}\omega_{iu} &= (\nabla_p\nabla_q - \nabla_q\nabla_p)\omega_{ij} \\ &= -(\delta_{pi}\omega_{jq} + \delta_{pj}\omega_{qi}) + (\delta_{qi}\omega_{jp} + \delta_{qj}\omega_{pi}). \end{aligned}$$

Contracting both sides with ψ^-_{ijl} and using skew-symmetry of both sides in i, j we get:

$$\begin{aligned} -2R_{pqiu}\omega_{uj}\psi^-_{ijl} &= -2\delta_{pi}\omega_{jq}\psi^-_{ijl} + 2\delta_{qi}\omega_{jp}\psi^-_{ijl} \\ 2R_{pqiu}\psi^-_{ilj}\omega_{uj} &= -2\psi^-_{pjl}\omega_{jq} + 2\psi^-_{qjl}\omega_{jp} \\ 2R_{pqiu}\psi^+_{ilu} &= -2\psi^-_{plj}\omega_{qj} + 2\psi^-_{qlj}\omega_{pj} \\ -2R_{pqiu}\psi^+_{liu} &= -2\psi^+_{plq} + 2\psi^+_{qlp} \\ &= 4\psi^+_{pql}, \end{aligned}$$

which yields (1.5.46).
For (1.5.47), contract (1.5.46) with ω_{vl} to get:

$$\begin{aligned} R_{pqiu}\psi^+_{liu}\omega_{vl} &= -2\psi^+_{pql}\omega_{vl} \\ R_{pqiu}\psi^+_{iul}\omega_{vl} &= -2(-\psi^-_{pqv}) \\ R_{pqiu}(-\psi^-_{iuv}) &= -2(-\psi^-_{pqv}) \\ R_{pqiu}\psi^-_{viu} &= -2\psi^-_{pqv}, \end{aligned}$$

as desired.
Finally, as for the first identity, we first compute $\nabla\nabla\psi^+$ and then contract it with ψ^-. Explicitly,

$$\begin{aligned}\nabla_p\nabla_q\psi^+_{jkl} &= \nabla_p(\delta_{qj}\omega_{kl} + \delta_{qk}\omega_{lj} + \delta_{ql}\omega_{jk}) \\ &= -(\delta_{qj}\psi^+_{pkl} + \delta_{qk}\psi^+_{plj} + \delta_{ql}\psi^+_{pjk}) \text{ (by (1.5.42))}.\end{aligned}$$

Now we use the Ricci identity to get:

$$\begin{aligned}-R_{pqju}\psi^+_{ukl} - R_{pqku}\psi^+_{jul} - R_{pqlu}\psi^+_{jku} =& (\nabla_p\nabla_q - \nabla_q\nabla_p)\psi^+_{jkl} \\ =& -(\delta_{qj}\psi^+_{pkl} + \delta_{qk}\psi^+_{plj} + \delta_{ql}\psi^+_{pjk}) \\ &+ (\delta_{pj}\psi^+_{qkl} + \delta_{pk}\psi^+_{qlj} + \delta_{pl}\psi^+_{qjk}).\end{aligned}$$

Contracting both sides with ψ^-_{jkl} and using the skew-symmetry in j, k, l we get (1.5.48):

$$\begin{aligned}-3R_{pqju}\psi^+_{ukl}\psi^-_{jkl} &= -3\delta_{qj}\psi^+_{pkl}\psi^-_{jkl} + 3\delta_{pj}\psi^+_{qkl}\psi^-_{jkl} \\ -3R_{pqju}(4\omega_{uj}) &= -3\psi^+_{pkl}\psi^-_{qkl} + 3\psi^+_{qkl}\psi^-_{pkl} \\ 12R_{pqju}\omega_{ju} &= -12\omega_{pq} + 12\omega_{qp} \\ R_{pqju}\omega_{ju} &= -2\omega_{pq}.\end{aligned}$$

□

Remark 1.5.49. Proposition 1.5.45 says that $\hat{R} = -2\,\mathrm{Id}$ on $\psi^+_{ijk}, \psi^+_{ijk}, \omega_{ij}$. Recall that by (1.5.44), we have $\hat{W} = \hat{R} + 2\,\mathrm{Id}$. Hence, $\hat{W}\psi^+, \hat{W}\psi^-, \hat{W}\omega$ are all equal to 0, which is exactly what is needed in order for W to be in Ω^2_8 (in the first two or the last two indices), by Remark 1.5.23. Hence, we have that $(W\beta)_{ab} = W_{abij}\beta_{ij}$ will always lie in Ω^2_8. Thus, since $\hat{R}$ and $\hat{W}$ differ by a constant, we can conclude that both $\hat{W}$ and $\hat{R}$ preserve Ω^2_8.
We claim that $\mathring{W}$ preserves both $\mathcal{S}^2_-$ and $\mathcal{S}^2_{+0}$. For the first subspace, let $h \in \mathcal{S}^2_-$. Then we compute:

$$\begin{aligned}((\mathring{W}h)\omega)_{ab} &= (\mathring{W}h)_{au}\omega_{ub} = W_{kalu}h_{kl}\omega_{ub} \\ &= -(W_{klua} + W_{kual})h_{kl}\omega_{ub} \\ &= -W_{kual}h_{kl}\omega_{ub} \\ &= -W_{ubal}h_{kl}\omega_{ku} \text{ (because } W \in \Omega^2_8 \text{ in the first (last) two indices)} \\ &= -W_{ubal}\omega_{kl}h_{ku} \text{ (because } h \in \mathcal{S}^2_-) \\ &= W_{ublk}\omega_{al}h_{ku} \\ &= -W_{klub}h_{ku}\omega_{al} \\ &= -(\mathring{W}h)_{lb}\omega_{al} \\ &= -(\omega(\mathring{W}h))_{ab},\end{aligned}$$

as claimed. The other case is similar, along with recalling that W is Ricci-traceless. Finally, since $\mathring{W}$ and $\mathring{R}$ differ by a constant on $\mathcal{S}^2_0$, $\mathring{R}$ also preserves that splitting.
These fact mean that we can consider $\hat{W}$ ($\mathring{W}$ resp.) as a self-adjoint operator only on Ω^2_8 ($\mathcal{S}^2_-$ and $\mathcal{S}^2_{+0}$ resp.) which will provide better estimates when we apply the Bochner-Weitzenböck techniques.

1.5.5 Harmonic forms

In this section we derive some useful properties about the harmonic forms. We will use the fact that harmonic 2-forms lie in Ω^2_8 and harmonic 3-forms lie in Ω^3_{12}. See [14, Theorem 3.8].

Definition 1.5.50. For $h \in \mathcal{S}^2$, let $\tilde{h} \in \mathcal{T}^2$ be defined as

$$\tilde{h}_{kc} := (\nabla_i h_{jk})\psi^+_{ijc}. \quad \blacktriangle$$

Proposition 1.5.51. *Let $h \in \mathcal{S}^2_-$. Then:*

- $(\nabla_a h_{ki})\omega_{ak} = -(\operatorname{Div} h)_k \omega_{ki}$.
- $(\nabla_u h_{ik})\psi^-_{uib}\omega_{ka} = \tilde{h}_{ab} + 4(h\omega)_{ab}$.

Proof. Since, $h \in \mathcal{S}^2_-$, we have

$$h_{ik}\omega_{ka} + \omega_{ik}h_{ka} = 0.$$

Differentiate it to get:

$$\begin{aligned} 0 &= (\nabla_u h_{ik})\omega_{ka} + h_{ik}(\nabla_u \omega_{ka}) + (\nabla_u \omega_{ik})h_{ka} + \omega_{ik}(\nabla_u h_{ka}) \\ &= (\nabla_u h_{ik})\omega_{ka} - h_{ik}\psi^+_{uka} - \psi^+_{uik}h_{ka} + \omega_{ik}(\nabla_u h_{ka}). \end{aligned} \tag{1.5.52}$$

Contract (1.5.52) on a, u to get:

$$\begin{aligned} 0 &= (\nabla_a h_{ik})\omega_{ka} + \omega_{ik}(\nabla_a h_{ka}) \\ &= -(\nabla_a h_{ki})\omega_{ak} + \omega_{ik}(\operatorname{Div} h)_k \end{aligned}$$

which gives the desired

$$(\nabla_a h_{ki})\omega_{ak} = -(\operatorname{Div} h)_k \omega_{ki}.$$

For the second identity, contract both sides of (1.5.52) with ψ^-_{uib} to get:

$$0 = (\nabla_u h_{ik})\omega_{ka}\psi^-_{uib} - h_{ik}\psi^+_{uka}\psi^-_{uib} - \psi^+_{uik}h_{ka}\psi^-_{uib} + \omega_{ik}(\nabla_u h_{ka})\psi^-_{uib}$$

The first term is what we need to solve for. So let us simplify the others separately:

$$
\begin{aligned}
h_{ik}\psi^+_{uka}\psi^-_{uib} &= h_{ik}\psi^+_{kau}\psi^-_{ibu} \\
&= h_{ik}(\delta_{ki}\omega_{ab} + \delta_{ab}\omega_{ki} - \delta_{kb}\omega_{ai} - \delta_{ai}\omega_{kb}) \\
&= 0 + 0 - h_{ib}\omega_{ai} - h_{ak}\omega_{kb} \\
&= -(\omega h + h\omega)_{ab} \\
&= 0.
\end{aligned}
$$

Similarly, we have:

$$
\psi^+_{uik}h_{ka}\psi^-_{uib} = h_{ka}\psi^+_{kui}\psi^-_{bui} = h_{ka}4\omega_{kb} = 4(h\omega)_{ab},
$$

and

$$
\omega_{ik}(\nabla_u h_{ka})\psi^-_{uib} = (\nabla_u h_{ka})\psi^-_{ubi}\omega_{ki} = (\nabla_u h_{ka})\psi^+_{ubk} = -(\nabla_u h_{ka})\psi^+_{ukb} = -\tilde{h}_{ab}.
$$

Hence,

$$
(\nabla_u h_{ik})\omega_{ka}\psi^-_{uib} = \tilde{h}_{ab} + 4(h\omega)_{ab}. \quad \square
$$

Proposition 1.5.53. *Let $h \in \mathcal{S}^2$, so that $\tilde{h} \in \mathcal{T}^2 = \mathcal{S}^2 \oplus \Omega^2$. Then $\tilde{h}_{skew} \in \Omega^2_8$.*

Proof. By Remark 1.5.24, it is enough to show that $\mathcal{P}\tilde{h} = -\tilde{h}_{skew}$. So, we compute:

$$
\begin{aligned}
(\mathcal{P}\tilde{h})_{ij} &= \frac{1}{2}\tilde{h}_{ab}(\star\omega)_{ijab} \\
&= \frac{1}{2}(\nabla_u h_{va})\psi^+_{uvb}(\star\omega)_{ijab} \\
&= \frac{1}{2}(\nabla_u h_{va})(-\delta_{ui}\psi^+_{vja} - \delta_{uj}\psi^+_{iva} - \delta_{ua}\psi^+_{ijv} + \delta_{iv}\psi^+_{uja} + \delta_{jv}\psi^+_{iua} + \delta_{av}\psi^+_{iju} - \omega_{uv}\psi^-_{ija}) \\
&= \frac{1}{2}(0 + 0 - (\nabla_a h_{va})\psi^+_{ijv} + (\nabla_u h_{ia})\psi^+_{uja} + (\nabla_u h_{ja})\psi^+_{iua} + 0 - (\nabla_u h_{va})\omega_{uv}\psi^-_{ija}) \\
&= \frac{1}{2}(-((\operatorname{Div} h) \lrcorner \psi^+)_{ij} - (\nabla_u h_{ai})\psi^+_{uaj} + (\nabla_u h_{aj})\psi^+_{uai} + (\operatorname{Div} h)_k\omega_{ka}\psi^-_{ija}) \\
&\qquad \text{(by Proposition 1.5.51)} \\
&= \frac{1}{2}(-((\operatorname{Div} h) \lrcorner \psi^+)_{ij} - \tilde{h}_{ij} + \tilde{h}_{ji} + (\operatorname{Div} h)_k\psi^+_{kij}) \\
&= \frac{1}{2}(-((\operatorname{Div} h) \lrcorner \psi^+)_{ij} - 2(\tilde{h}_{skew})_{ij} + ((\operatorname{Div} h) \lrcorner \psi^+)_{ij}) \\
&= -(\tilde{h}_{skew})_{ij}.
\end{aligned}
$$

as claimed, concluding the proof. $\square$

Proposition 1.5.54. *Let M be compact nearly Kähler. Let $\beta \in \Omega^3_{12} \supseteq \mathcal{H}^3$. Hence, $\beta = h \diamond \psi^+$ for some unique $h \in \mathcal{S}^2_-$. Then β is harmonic iff $\operatorname{Div} h = 0, \tilde{h} = 2\omega h = -2h\omega \in \mathcal{S}^2$.*

Proof. Note that in fact, since h is symmetric, ω is skew, and that they anticommute, we have $\omega h \in \mathcal{S}^2$. So the last condition is equivalent to $\tilde{h}_{symm} = 2\omega h$ and $\tilde{h}_{skew} = 0$.
Since M is compact, β is harmonic if and only if $d^*\beta = 0$ and $d\beta = 0$. Let us look at each of these conditions separately. So, we have:

$$\begin{aligned}
0 =& -(d^*\beta)_{kl}\\
=&\nabla_j \beta_{jkl}\\
=&\nabla_j (h_{jp}\psi^+_{pkl} + h_{kp}\psi^+_{jpl} + h_{lp}\psi^+_{jkp})\\
=&(\nabla_j h_{jp})\psi^+_{pkl} + (\nabla_j h_{kp})\psi^+_{jpl} + (\nabla_j h_{lp})\psi^+_{jkp} + h_{jp}(\nabla_j \psi^+_{pkl}) + h_{kp}(\nabla_j \psi^+_{jpl}) + h_{lp}(\nabla_j \psi^+_{jkp})\\
=&(\operatorname{Div} h \lrcorner \psi^+)_{kl} + (\nabla_j h_{pk})\psi^+_{jpl} - (\nabla_j h_{pl})\psi^+_{jpk} + h_{jp}(\delta_{jp}\omega_{kl} + \delta_{jk}\omega_{lp} + \delta_{jl}\omega_{pk})\\
&+ 4h_{kp}\omega_{pl} + 4h_{lp}\omega_{kp} \text{ (by (1.5.43))}\\
=&(\operatorname{Div} h \lrcorner \psi^+)_{kl} + \tilde{h}_{kl} - \tilde{h}_{lk} + (0 + h_{kp}\omega_{lp} + h_{lp}\omega_{pk}) + 4(h_{kp}\omega_{pl} + h_{lp}\omega_{kp})\\
=&(\operatorname{Div} h \lrcorner \psi^+)_{kl} + 2(\tilde{h}_{skew})_{kl} + 3(h\omega + \omega h)_{kl}\\
=&(\operatorname{Div} h \lrcorner \psi^+)_{kl} + 2(\tilde{h}_{skew})_{kl},
\end{aligned}$$

where we have used that $h \in \mathcal{S}^2_- \subseteq \mathcal{S}^2_0$. Recall that by Proposition 1.5.53, $\tilde{h}_{skew} \in \Omega^2_8$, hence, looking at the types we get:

$$d^*\beta = 0 \qquad \text{if and only if} \qquad \begin{cases} \operatorname{Div} h = 0, \\ \tilde{h}_{skew} = 0. \end{cases}$$

Next, by Corollary 1.5.35, we know that $d\beta = 0$ iff $\widehat{d\beta} = 0$. We have:

$$\begin{aligned}
\widehat{d\beta}_{ia} &= (d\beta)_{ijkl}(\star\omega)_{ajkl}\\
&= (\nabla_i \beta_{jkl} - \nabla_j \beta_{ikl} + \nabla_k \beta_{ijl} - \nabla_l \beta_{ijk})(\star\omega)_{ajkl}\\
&= (\nabla_i \beta_{jkl})(\star\omega)_{ajkl} - 3(\nabla_j \beta_{ikl})(\star\omega)_{ajkl}. \qquad (1.5.55)
\end{aligned}$$

We will compute each term of (1.5.55) separately. First we have:

$$\begin{aligned}(\nabla_i \beta_{jkl})(\star\omega)_{ajkl} &= \nabla_i(h_{jp}\psi^+_{pkl} + h_{kp}\psi^+_{jpl} + h_{lp}\psi^+_{jkp})(\star\omega)_{ajkl}\\ &= 3\nabla_i(h_{jp}\psi^+_{pkl})(\star\omega)_{ajkl}\\ &= 3(\nabla_i h_{jp})\psi^+_{pkl}(\star\omega)_{ajkl} + 3h_{jp}(\nabla_i\psi^+_{pkl})(\star\omega)_{ajkl}\\ &= 3(\nabla_i h_{jp})2\psi^+_{paj} + 3h_{jp}(\delta_{ip}\omega_{kl} + \delta_{ik}\omega_{lp} + \delta_{il}\omega_{pk})(\star\omega)_{ajkl}\\ &= 0 + 3h_{ij}\omega_{kl}(\star\omega)_{ajkl} + 6h_{jp}\delta_{ik}\omega_{lp}(\star\omega)_{ajkl}\\ &= 12h_{ij}\omega_{aj} - 6h_{jp}\omega_{pl}(\star\omega)_{ajil}\\ &= 12h_{ij}\omega_{aj} - 6h_{jp}(\delta_{ap}\omega_{ji} + \delta_{jp}\omega_{ia} + \delta_{ip}\omega_{aj})\\ &= -12(h\omega)_{ia} - 6h_{ja}\omega_{ji} - 6\,\mathrm{tr}(h)\omega_{ia} - 6h_{ji}\omega_{aj}\\ &= -12(h\omega)_{ia} + 6(\omega h + h\omega)_{ia}\\ &= -12(h\omega)_{ia}.\end{aligned}$$

For the second term of (1.5.55), we have:

$$\begin{aligned}-3(\nabla_j \beta_{ikl})(\star\omega)_{ajkl} &= -3\nabla_j(h_{ip}\psi^+_{pkl} + h_{kp}\psi^+_{ipl} + h_{lp}\psi^+_{ikp})(\star\omega)_{ajkl}\\ &= -3\nabla_j(h_{ip}\psi^+_{pkl})(\star\omega)_{ajkl} - 6\nabla_j(h_{kp}\psi^+_{ipl})(\star\omega)_{ajkl}.\end{aligned} \tag{1.5.56}$$

Here again, we compute both terms of (1.5.56) separately. First we have:

$$\begin{aligned}-3\nabla_j(h_{ip}\psi^+_{pkl})(\star\omega)_{ajkl} &= -3(\nabla_j h_{ip})\psi^+_{pkl}(\star\omega)_{ajkl} - 3h_{ip}(\nabla_j\psi^+_{pkl})(\star\omega)_{ajkl}\\ &= -3(\nabla_j h_{ip})2\psi^+_{paj} - 3h_{ip}(\delta_{jp}\omega_{kl} + \delta_{jk}\omega_{lp} + \delta_{jl}\omega_{pk})(\star\omega)_{ajkl}\\ &= -6(\nabla_j h_{pi})\psi^+_{jpa} - 3h_{ip}\delta_{jp}\omega_{kl}(\star\omega)_{ajkl}\\ &= -6\tilde{h}_{ia} - 3h_{ij}4\omega_{aj}\\ &= -6\tilde{h}_{ia} + 12(h\omega)_{ia}.\end{aligned}$$

For the second term of (1.5.56) we use Proposition 1.5.51 to get:

$$\begin{aligned}-6\nabla_j(h_{kp}\psi^+_{ipl})(\star\omega)_{ajkl} &= -6(\nabla_j h_{kp})\psi^+_{ipl}(\star\omega)_{ajkl} - 6h_{kp}(\nabla_j\psi^+_{ipl})(\star\omega)_{ajkl}\\ &= 6(\nabla_j h_{kp})(\psi^-_{ipa}\omega_{jk} + \psi^-_{ipj}\omega_{ka} + \psi^-_{ipk}\omega_{aj}) - 6h_{kp}(\delta_{ji}\omega_{pl} + \delta_{jp}\omega_{li} + \delta_{jl}\omega_{ip})(\star\omega)_{ajkl}\\ &= -6(\mathrm{Div}\,h)_s\omega_{sp}\psi^-_{ipa} + 6(\nabla_j h_{kp})\psi^-_{ipj}\omega_{ka} + 0 - 6h_{kp}\omega_{pl}(\star\omega)_{aikl} - 6h_{kp}\omega_{li}(\star\omega)_{apkl} + 0\\ &= 6(\mathrm{Div}\,h)_s\psi^-_{iap}\omega_{sp} - 6(\nabla_j h_{pk})\psi^-_{jpi}\omega_{ka} - 6h_{kp}(\delta_{ap}\omega_{ik} + \delta_{ip}\omega_{ka} + \delta_{kp}\omega_{ai}) + 0\\ &= 6(\mathrm{Div}\,h)_s\psi^+_{ias} - 6(\tilde{h}_{ai} + 4(h\omega)_{ai}) - 6h_{ka}\omega_{ik} - 6h_{ki}\omega_{ka} + 0\\ &= 6(\mathrm{Div}\,h \lrcorner \psi^+)_{ia} - 6\tilde{h}_{ai} - 24(h\omega)_{ia} - 6(\omega h + h\omega)_{ia}\\ &= 6(\mathrm{Div}\,h \lrcorner \psi^+)_{ia} - 6\tilde{h}_{ai} - 24(h\omega)_{ia}.\end{aligned}$$

Thus, combining the last two results we simplify (1.5.56) to get:

$$\begin{aligned}-3(\nabla_j\beta_{ikl})(\star\omega)_{ajkl} &= (-6\tilde{h}_{ia} + 12(h\omega)_{ia}) + (6(\operatorname{Div} h \lrcorner \psi^+)_{ia} - 6\tilde{h}_{ai} - 24(h\omega)_{ia}) \\ &= 6(\operatorname{Div} h \lrcorner \psi^+)_{ia} - 12(\tilde{h}_{symm})_{ia} - 12(h\omega)_{ia}.\end{aligned}$$

And so, returning to (1.5.55), we have:

$$\begin{aligned}\widehat{d\beta}_{ia} &= -12(h\omega)_{ia} + 6(\operatorname{Div} h \lrcorner \psi^+)_{ia} - 12(\tilde{h}_{symm})_{ia} - 12(h\omega)_{ia} \\ &= 6(\operatorname{Div} h \lrcorner \psi^+)_{ia} - 12(\tilde{h}_{symm})_{ia} - 24(h\omega)_{ia},\end{aligned}$$

which implies:

$$d\beta = 0 \qquad \text{if and only if} \qquad \begin{cases}\operatorname{Div} h = 0, \\ \tilde{h}_{symm} = -2h\omega.\end{cases}$$

Hence, we get that β is harmonic iff $\operatorname{Div} h = 0$ and $\tilde{h} = -2h\omega = 2\omega h$. □

Proposition 1.5.57. *Let M be compact nearly Kähler. Let $\beta \in \Omega^2_8 \supseteq \mathcal{H}^2$. Hence, $\beta = h \diamond \omega$ for some unique $h \in \mathcal{S}^2_{+0}$. Then β is harmonic iff* $\operatorname{Div} h = 0, \tilde{h} = -3h\omega \in \Omega^2_8$.

Proof. First, note that

$$\beta_{ij} = (h \diamond \omega)_{ij} = h_{ip}\omega_{pj} + h_{jp}\omega_{ip} = (h\omega)_{ij} + (\omega h)_{ij} = 2(h\omega)_{ij}.$$

As in the proof of the previous theorem, β is harmonic if and only if $d^*\beta = 0$ and $d\beta = 0$. Looking at each of the conditions separately, we get:

$$\begin{aligned}0 = -(d^*\beta)_k = \nabla_p\beta_{pk} &= 2\nabla_p(h_{pu}\omega_{uk}) \\ &= 2(\operatorname{Div} h)_u\omega_{uk} + 2h_{pu}\nabla_p\omega_{uk} \\ &= 2(\operatorname{Div} h)_u\omega_{uk} - 2h_{pu}\psi^+_{puk} \\ &= 2(\operatorname{Div} h)_u\omega_{uk}.\end{aligned}$$

Since ω is non-degenerate, we get that:

$$d\beta = 0 \qquad \text{if and only if} \qquad \operatorname{Div} h = 0.$$

Next, by Corollary 1.5.39, we have that $d\beta = 0$ iff $\widehat{d\beta} = 0$. We have:

$$\begin{aligned}0 = \widehat{d\beta}_{ia} &= (d\beta)_{ijk}\psi^+_{ajk} \\ &= (\nabla_i\beta_{jk} - \nabla_j\beta_{ik} + \nabla_k\beta_{ij})\psi^+_{ajk} \\ &= (\nabla_i\beta_{jk})\psi^+_{ajk} - 2(\nabla_j\beta_{ik})\psi^+_{ajk}. \qquad (1.5.58)\end{aligned}$$

We will compute each term of (1.5.58) separately. First we have:

$$
\begin{aligned}
(\nabla_i \beta_{jk})\psi^+_{ajk} &= 2\nabla_i(h_{ju}\omega_{uk})\psi^+_{ajk} \\
&= 2(\nabla_i h_{ju})\omega_{uk}\psi^+_{ajk} + 2h_{ju}(\nabla_i \omega_{uk})\psi^+_{ajk} \\
&= 2(\nabla_i h_{ju})(-\psi^-_{uaj}) + 2h_{ju}(-\psi^+_{iuk})\psi^+_{ajk} \\
&= 0 - 2h_{ju}(\delta_{ia}\delta_{uj} - \delta_{ij}\delta_{ua} - \omega_{ia}\omega_{uj} + \omega_{ij}\omega_{ua}) \qquad \text{(by (1.5.8))} \\
&= -2\delta_{ia} \operatorname{tr} h + 2h_{ia} + 0 - 2(\omega h \omega)_{ia} \\
&= 2h_{ia} - 2(h\omega^2)_{ia} \qquad \text{(because } h \in \mathcal{S}^2_{+0}) \\
&= 4h_{ia}.
\end{aligned} \tag{1.5.59}
$$

For the second term of (1.5.58) we have:

$$
\begin{aligned}
-2(\nabla_j \beta_{ik})\psi^+_{ajk} &= -4\nabla_j(h_{iu}\omega_{uk})\psi^+_{ajk} \\
&= -4(\nabla_j h_{iu})(-\psi^-_{uaj}) - 4h_{iu}(-\psi^+_{juk})\psi^+_{ajk} \\
&= 4(\nabla_j h_{iu})\psi^-_{uaj} - 4h_{iu}(4\delta_{ua}) \qquad \text{(by (1.5.9))} \\
&= 4(\nabla_j h_{iu})\psi^-_{uaj} - 16h_{ia}.
\end{aligned} \tag{1.5.60}
$$

Conbining (1.5.59) and (1.5.60), we get that:

$$0 = \widehat{d\beta}_{ia} = 4(\nabla_j h_{iu})\psi^-_{uaj} - 12h_{ia}.$$

So, $d\beta = 0$ iff $\widehat{d\beta} = 0$ iff $(\nabla_j h_{iu})\psi^-_{uaj} = 3h_{ia}$ iff $(\nabla_j h_{iu})\psi^-_{uaj}\omega_{at} = 3h_{ia}\omega_{at}$.
Since $(\nabla_j h_{iu})\psi^-_{uaj}\omega_{at} = (\nabla_j h_{iu})\psi^+_{ujt} = -\tilde{h}_{it}$, we get that:

$$d\beta = 0 \qquad \text{if and only if} \qquad \tilde{h} = -3h\omega.$$

Hence, we conclude that β is harmonic iff $\operatorname{Div} h = 0$ and $\tilde{h} = -3h\omega$.
It is easy to see that $h\omega \in \Omega^2$ for $h \in \mathcal{S}^2_{+0}$, so by Proposition 1.5.53, in this case we indeed have $\tilde{h} = -3h\omega \in \Omega^2_8$. □

1.5.6 Weitzenböck formulas

The following formulas can be found in [36], however we include the proofs, and when deriving sufficient conditions for vanishing of b_2 and b_3, we use slightly different forms of these formulas.

2-forms

We apply Corollary 1.3.3 to the nearly Kähler setting to get:

$$\Delta\beta = \nabla^*\nabla\beta + 8\beta + \hat{W}\beta, \text{ for any } \beta \in \Omega^2. \tag{1.5.61}$$

Proposition 1.5.62. *Let $\beta = h \diamond \omega \in \Omega^2_8$ for some $h \in \mathcal{S}^2_{+0}$. Assume β is harmonic. Then:*

$$\nabla^*\nabla h + 6h + 2\mathring{W}h = 0.$$

Proof. Using (1.5.61), it is enough to show that $\nabla^*\nabla\beta = (\nabla^*\nabla h - 2h) \diamond \omega$ and $\hat{W}\beta = 2(\mathring{W}h) \diamond \omega$. So, we proceed with the first claim:

$$\begin{aligned}
(\nabla^*\nabla\beta)_{ab} &= -\nabla_s\nabla_s\beta_{ab}\\
&= -\nabla_s\nabla_s(h \diamond \omega)_{ab}\\
&= -\nabla_s\nabla_s(h_{ap}\omega_{pb} + h_{bp}\omega_{ap})\\
&= -((\nabla_s\nabla_s h) \diamond \omega)_{ab} - 4(\nabla_s h_{ap})(\nabla_s\omega_{pb})\\
&\quad - 2h_{ap}\nabla_s\nabla_s\omega_{pb} \text{ (because } h \in \mathcal{S}^2_{+0} \text{ and hence } h_{ap}\omega_{pb} = h_{bp}\omega_{ap})\\
&= -((\nabla_s\nabla_s h) \diamond \omega)_{ab} + 4(\nabla_s h_{ap})\psi^+_{spb} + 8h_{ap}\omega_{pb}\\
&\qquad\qquad \text{(by (1.5.42) and (1.5.43), } \nabla_s\nabla_s\omega_{pb} = -4\omega_{pb})\\
&= ((\nabla^*\nabla h) \diamond \omega)_{ab} + 4\tilde{h}_{ab} + 8(h\omega)_{ab}\\
&= ((\nabla^*\nabla h) \diamond \omega)_{ab} - 12(h\omega)_{ab} + 8(h\omega)_{ab} \text{ (by Proposition 1.5.57)}\\
&= ((\nabla^*\nabla h) \diamond \omega)_{ab} - 4(h\omega)_{ab}\\
&= ((\nabla^*\nabla h - 2h) \diamond \omega)_{ab} \text{ (becase } h \diamond \omega = 2h\omega \text{ for } h \in \mathcal{S}^2_{+0}).
\end{aligned}$$

For the second claim, we know that since $\beta \in \Omega^2_8$, then $\hat{W}\beta \in \Omega^2_8$. Hence by Proposition 1.5.32, $\hat{W}\beta = f \diamond \omega$, for some $f \in \mathcal{S}^2_{+0}$. The same proposition also tells us that $f = \frac{1}{2}(\hat{W}\beta)_{ik}\omega_{ak}$. Computing, we have:

$$\begin{aligned}
f_{ia} = \frac{1}{2}(\hat{W}\beta)_{ik}\omega_{ak} &= \frac{1}{2}W_{ikuv}\beta_{uv}\omega_{ak} = \frac{1}{2}W_{ikuv}(h \diamond \omega)_{uv}\omega_{ak} = W_{ikuv}h_{up}\omega_{pv}\omega_{ak}\\
&= -(W_{kuiv} + W_{uikv})h_{up}\omega_{pv}\omega_{ak} \qquad \text{(by the Bianchi identity)}\\
&= (W_{ivuk} + W_{uivk})\omega_{ka}h_{up}\omega_{vp}\\
&= W_{ivuk}\omega_{ka}h_{up}\omega_{vp} + W_{uivk}\omega_{ka}h_{up}\omega_{vp}\\
&= W_{ivak}\omega_{ku}h_{up}\omega_{vp} + W_{uiak}\omega_{kv}h_{up}\omega_{vp}\\
&\qquad \text{(by Lemma 1.5.28 and } W \in \Omega^2_8 \text{ in first (last) two indices)}\\
&= W_{ivak}h_{ku}\omega_{up}\omega_{vp} - W_{uiak}h_{up}\delta_{kp} \text{ (we use that } h \in \mathcal{S}^2_{+0} \text{ and that } \omega^2 = -\operatorname{Id})\\
&= W_{ivak}h_{ku}\delta_{uv} - W_{uiak}h_{uk}\\
&= W_{ivak}h_{kv} - W_{uiak}h_{uk}\\
&= W_{vika}h_{vk} + W_{uika}h_{uk}\\
&= 2(\mathring{W}h)_{ia},
\end{aligned}$$

as claimed. Hence, the proof is complete. □

Theorem 1.5.63. *Let M be a compact nearly Kähler 6-manifold. If $\mathcal{S}^2(\Omega^2_8) \ni \hat{W} \geqslant -8$, or equivalenty $\mathcal{S}^2(\mathcal{S}^2_{+0}) \ni \mathring{W} \geqslant -4$, then $b_2 = 0$.*

Proof. Let $\beta \in \Omega^2$ be harmonic. Then $\beta \in \Omega^2_8$, as mentioned in the start of Section 1.5.5. Substituting it in (1.5.61), and using the assumption that $\hat{W} \geqslant -8$, we get that $\beta = 0$, as there are no parallel non-zero 2-forms.
Using the fact that $\hat{W}\beta = 2(\mathring{W}h) \diamond \omega$, where $\beta = h \diamond \omega \in \Omega^2_8$, for $h \in \mathcal{S}^2_{+0}$ we get the other equivalent condition.
Note that using Proposition 1.5.62 in order to get a similar result would have been worse, as we would have been able to only conclude that if $\mathcal{S}^2(\mathcal{S}^2_{+0}) \ni \mathring{W} \geqslant -3$ then $b_2 = 0$. This is because $\nabla^*\nabla\beta = (\nabla^*\nabla h - 2h) \diamond \omega$, so we can see that even though the left hand side is obviously non-negative, we cannot conclude that from the right hand side. □

Theorem 1.5.64. *Let M be a compact nearly Kähler 6-manifold. Let $\delta \leqslant \bar{R} \leqslant \Delta$ with $-(\Delta + \delta) - \frac{7}{3}(\Delta - \delta) \geqslant -10$ **or** $(\Delta + \delta) - 3(\Delta - \delta) \geqslant -6$. Then $b_2 = 0$.*

Proof. If the conditions above hold, then by Corollary 1.2.15 we have that $\hat{R} \geqslant -10$. So, we use (1.5.44) to get that $\hat{W} \geqslant -8$ and hence $b_2 = 0$ by Theorem 1.5.63. □

3-forms

We apply Corollary 1.3.5 to the nearly Kähler setting to get:

$$(\Delta\beta)_{abc} = (\nabla^*\nabla\beta)_{abc} + 9\beta_{abc} + W_{abpu}\beta_{puc} + W_{acpu}\beta_{pbu} + W_{bcpu}\beta_{apu}, \text{ for any } \beta \in \Omega^3. \quad (1.5.65)$$

Proposition 1.5.66. *Let $\beta \in \Omega^3_{12}$, so $\beta = h \diamond \psi^+$ for some unique $h \in \mathcal{S}^2_-$. Assume β is harmonic. Then:*

$$\nabla^*\nabla h + 8h + 2\mathring{W}h = 0.$$

Proof. Substitute a harmonic β into (1.5.65) to get the vanishing of the left hand side. Now, the goal is to rewrite the RHS as $A \diamond \psi^+$ for some $A \in \mathbb{R}g \oplus \mathbb{R}\omega \oplus \mathcal{S}^2_- \oplus \Omega^2_6$. Then we can conclude that $A = 0$. By Proposition 1.5.54, since β is harmonic, $\operatorname{Div} h = 0$ and $\tilde{h} = 2\omega h$. Keeping this in mind, we will simplify each term of the RHS of (1.5.65) one by one. We start with $\nabla^*\nabla\beta$:

$$\begin{aligned}(\nabla^*\nabla\beta)_{abc} =& -\nabla_s\nabla_s(h_{ap}\psi^+_{pbc} + h_{bp}\psi^+_{apc} + h_{cp}\psi^+_{abp})\\ =&((\nabla^*\nabla h) \diamond \psi^+)_{abc} - 2((\nabla_s h_{ap})(\nabla_s\psi^+_{pbc}) + (\nabla_s h_{bp})(\nabla_s\psi^+_{apc}) + (\nabla_s h_{cp})(\nabla_s\psi^+_{abp}))\\ &- (h_{ap}(\nabla_s\nabla_s\psi^+_{pbc}) + h_{bp}(\nabla_s\nabla_s\psi^+_{apc}) + h_{cp}(\nabla_s\nabla_s\psi^+_{abp})).\end{aligned} \quad (1.5.67)$$

First, note that:

$$\begin{aligned}\nabla_s\nabla_s\psi^+_{ijk} &= \nabla_s(\delta_{si}\omega_{jk} + \delta_{sj}\omega_{ki} + \delta_{sk}\omega_{ij})\\ &= \delta_{si}(-\psi^+_{sjk}) + \delta_{sj}(-\psi^+_{ski}) + \delta_{sk}(-\psi^+_{sij})\\ &= -3\psi^+_{ijk}.\end{aligned}$$

Hence, the third term in (1.5.67) is equal to:

$$-(h_{ap}(\nabla_s\nabla_s\psi^+_{pbc})+h_{bp}(\nabla_s\nabla_s\psi^+_{apc})+h_{cp}(\nabla_s\nabla_s\psi^+_{abp})) = 3(h_{ap}\psi^+_{pbc}+h_{bp}\psi^+_{apc}+h_{cp}\psi^+_{abp}) = (3h\diamond\psi^+)_{abc}.$$

In order to calculate the second term of (1.5.67), we define the 3-form σ by:

$$\sigma_{abc} := (\nabla_s h_{ap})(\nabla_s\psi^+_{pbc}) + (\nabla_s h_{bp})(\nabla_s\psi^+_{apc}) + (\nabla_s h_{cp})(\nabla_s\psi^+_{abp}).$$

We claim that $\sigma = 2h\diamond\psi^+$. In order to get this, we first calculate $\hat{\sigma}$ and then use Corollary 1.5.39. So,

$$\begin{aligned}\hat{\sigma}_{at} =&\sigma_{abc}\psi^+_{tbc}\\ =&((\nabla_s h_{ap})(\nabla_s\psi^+_{pbc}) + (\nabla_s h_{bp})(\nabla_s\psi^+_{apc}) + (\nabla_s h_{cp})(\nabla_s\psi^+_{abp}))\psi^+_{tbc}\\ =&(\nabla_s h_{ap})(\nabla_s\psi^+_{pbc})\psi^+_{tbc} + 2(\nabla_s h_{bp})(\nabla_s\psi^+_{apc})\psi^+_{tbc}\\ =&(\nabla_s h_{ap})(\delta_{sp}\omega_{bc} + \delta_{sb}\omega_{cp} + \delta_{sc}\omega_{pb})\psi^+_{tbc} + 2(\nabla_s h_{bp})(\delta_{sa}\omega_{pc} + \delta_{sp}\omega_{ca} + \delta_{sc}\omega_{ap})\psi^+_{tbc}\\ =&0 + (\nabla_b h_{ap})\psi^+_{btc}\omega_{pc} + (\nabla_c h_{ap})\psi^+_{ctb}\omega_{pb} + 2(\nabla_a h_{bp})\psi^+_{tbc}\omega_{pc}\\ &+ 2(\nabla_p h_{bp})\psi^+_{btc}\omega_{ac} - 2(\nabla_c h_{bp})\psi^+_{cbt}\omega_{ap} \text{ (as Div } h = 0)\\ =&-(\nabla_b h_{ap})\psi^-_{btp} - (\nabla_c h_{ap})\psi^-_{ctp} - 2(\nabla_a h_{bp})\psi^-_{tbp} - 0 - 2\tilde{h}_{pt}\omega_{ap}\\ =&-2(\nabla_b h_{ap})\psi^-_{btp} - 0 - 2\tilde{h}_{pt}\omega_{ap}\\ =&2(\nabla_b h_{pa})\psi^-_{bpt} - 2(\omega\tilde{h})_{at}\\ =&-2(\nabla_b h_{pa})\psi^+_{bpu}\omega_{tu} - 2(\omega\tilde{h})_{at}\\ =&-2\tilde{h}_{au}\omega_{tu} - 2(\omega\tilde{h})_{at}\\ =&2(\tilde{h}\omega)_{at} - 2(\omega\tilde{h})_{at}\\ =&4(\omega h\omega)_{at} - 4(\omega^2 h)_{at} \text{ (because } \tilde{h} = 2\omega h)\\ =&-4(h\omega^2)_{at} + 4h_{at}\\ =&8h_{at}.\end{aligned}$$

Hence, $\hat{\sigma} = 8h \in \mathcal{S}^2_0$. Thus, by Proposition 1.5.39, $\sigma = \frac{1}{4}\hat{\sigma}\diamond\psi^+ = 2h\diamond\psi^+$, as claimed. Thus, returning to (1.5.67), we get:

$$\begin{aligned}\nabla^*\nabla\beta =&(\nabla^*\nabla h)\diamond\psi^+ - 2\sigma + 3h\diamond\psi^+\\ =&(\nabla^*\nabla h - 4h + 3h)\diamond\psi^+\\ =&(\nabla^*\nabla h - h)\diamond\psi^+. \qquad (1.5.68)\end{aligned}$$

Next, we proceed to the terms in (1.5.65) with the Weyl tensors. Recall that W is in Ω^2_8 with respect to the first two or the last two indices. Hence, $W_{abij}\psi^+_{abc} = 0$. So, we have:

$$\begin{aligned} W_{abpu}\beta_{puc} &= W_{abpu}(h_{ps}\psi^+_{suc} + h_{us}\psi^+_{psc} + h_{cs}\psi^+_{pus}) \\ &= 2h_{ps}W_{abpu}\psi^+_{suc}. \end{aligned}$$

Similarly, we have:

$$\begin{aligned} W_{acpu}\beta_{pbu} &= 2h_{ps}W_{acpu}\psi^+_{sbu}, \\ W_{bcpu}\beta_{apu} &= 2h_{ps}W_{bcpu}\psi^+_{asu}. \end{aligned}$$

Thus, the Weyl terms in (1.5.65) are equal to:

$$W_{abpu}\beta_{puc} + W_{acpu}\beta_{pbu} + W_{bcpu}\beta_{apu} = 2h_{ps}(W_{abpu}\psi^+_{suc} + W_{acpu}\psi^+_{sbu} + W_{bcpu}\psi^+_{asu}).$$

Now, we define the 3-form γ via

$$\gamma_{abc} \coloneqq h_{ps}(W_{abpu}\psi^+_{suc} + W_{acpu}\psi^+_{sbu} + W_{bcpu}\psi^+_{asu}).$$

We claim that $\gamma = (\mathring{W}h) \diamond \psi^+$. Again, to get this, we first need to calculate $\hat{\gamma}$. We have:

$$\begin{aligned} \hat{\gamma}_{at} &= \gamma_{abc}\psi^+_{tbc} \\ &= h_{ps}(W_{abpu}\psi^+_{suc} + W_{acpu}\psi^+_{sbu} + W_{bcpu}\psi^+_{asu})\psi^+_{tbc} \\ &= 2h_{ps}W_{abpu}\psi^+_{suc}\psi^+_{tbc} + 0 \\ &= 2h_{ps}W_{abpu}(\delta_{st}\delta_{ub} - \delta_{sb}\delta_{ut} + \omega_{ut}\omega_{sb} + \omega_{bu}\omega_{st}) \\ &= 0 - 2h_{pb}W_{abpt} + 2h_{ps}W_{abpu}\omega_{ut}\omega_{sb} + 2h_{ps}W_{abpu}\omega_{bu}\omega_{st}. \end{aligned} \tag{1.5.69}$$

We will simplify the last two terms of (1.5.69) separately. Recall Lemma 1.5.28 which implies that W and ω commute. Also, we have that h and ω anticommute. Hence, for the third term we have:

$$\begin{aligned} 2h_{ps}W_{abpu}\omega_{ut}\omega_{sb} &= 2W_{abpu}\omega_{ut}h_{ps}\omega_{sb} \\ &= -2\omega_{pu}W_{abut}\omega_{ps}h_{sb} \\ &= -2\delta_{us}W_{abut}h_{sb} \\ &= -2W_{abst}h_{sb} \\ &= 2W_{bast}h_{bs} \\ &= 2(\mathring{W}h)_{at}. \end{aligned}$$

For the fourth term of (1.5.69) we have:

$$\begin{aligned} 2h_{ps}W_{abpu}\omega_{bu}\omega_{st} &= -2W_{abpu}\omega_{ub}h_{ps}\omega_{st} \\ &= -2\omega_{pu}W_{abub}h_{ps}\omega_{st} \\ &= 0, \end{aligned}$$

because $W_{abub} = 0$. Thus, returning to (1.5.69), we get:

$$\begin{aligned} \hat{\gamma}_{at} &= -2h_{pb}W_{abpt} + 2(\mathring{W}h)_{at} + 0 \\ &= 2W_{abtp}h_{bp} + 2(\mathring{W}h)_{at} + 0 \\ &= 4(\mathring{W}h)_{at}. \end{aligned}$$

Hence, $\hat{\gamma} = 4\mathring{W}h \in \mathcal{S}^2_0$. Thus, by Proposition 1.5.39, $\gamma = \frac{1}{4}\hat{\gamma} \diamond \psi^+ = (\mathring{W}h) \diamond \psi^+$, as claimed. This finishes the proof of the proposition, as substituting all the results into (1.5.65), we get:

$$0 = ((\nabla^*\nabla h - h) + 9h + 2\gamma) \diamond \psi^+ = (\nabla^*\nabla h + 8h + 2\mathring{W}h) \diamond \psi^+. \quad \square$$

Theorem 1.5.70. *Let M be a compact nearly Kähler 6-manifold. If $\mathcal{S}^2(\mathcal{S}^2_-) \ni \mathring{W} \geqslant -\frac{9}{2}$* ***or*** *$\mathcal{S}^2(\Omega^2_8) \ni \hat{W} \geqslant -3$, then $b_3 = 0$.*

Proof. The first statement follows from the fact that using (1.5.68) we can rewrite Proposition 1.5.66 as: if $\beta = h \diamond \psi^+$ is harmonic for some $h \in \mathcal{S}^2_-$, then:

$$0 = \nabla^*\nabla\beta + (9h + 2\mathring{W}h) \diamond \psi^+.$$

Hence, assuming $\mathcal{S}^2(\mathcal{S}^2_-) \ni \mathring{W} \geqslant -\frac{9}{2}$ and using the fact that there are no nonzero parallel $h \in \mathcal{S}^2_0$, we get $b_3 = 0$.
Note that using Proposition 1.5.66 in order to get a similar result would have been worse, as we would have been able to only conclude that if $\mathcal{S}^2(\mathcal{S}^2_-) \ni \mathring{W} \geqslant -4$ then $b_3 = 0$. This is because $\nabla^*\nabla\beta = (\nabla^*\nabla h - h) \diamond \psi^+$, so we can see that even though the left hand side is obviosuly non-negative, we cannot conclude that from the right hand side.
Next, $\mathcal{S}^2(\Omega^2_8) \ni \hat{W} \geqslant -3$ implies $b_3 = 0$ because of (1.5.65).
Note that the condition $\mathcal{S}^2(\Omega^2_8) \ni \hat{W} \geqslant -3$ is weaker than the condition $\mathcal{S}^2(\mathcal{S}^2_-) \ni \mathring{W} \geqslant -\frac{9}{2}$. This is because in the proof of Proposition 1.5.66 we show that $W_{abpu}\beta_{puc} + W_{acpu}\beta_{pbu} + W_{bcpu}\beta_{apu} = (2(\mathring{W}h) \diamond \psi^+)_{abc}$, for $\beta = h \diamond \psi^+ \in \Omega^3_{12}$, where $h \in \mathcal{S}^2_-$. That means if we assume that $\mathcal{S}^2(\Omega^2_8) \ni \hat{W} \geqslant c$, then $\mathring{W} \geqslant \frac{3c}{2}$, where $c \in \mathbb{R}$, but not vice versa. $\square$

Theorem 1.5.71. *Let M be a compact nearly Kähler 6-manifold. Let $\delta \leqslant \bar{R} \leqslant \Delta$ with $\delta \geqslant \frac{1}{4}$* ***or*** *$\Delta \leqslant \frac{17}{8}$. Then $b_3 = 0$.*

Proof. Recall that the Einstein constant $k = 5$. Then by Theorem 1.2.17, on $\mathcal{S}^2_0$, $\mathring{R} \geqslant -5 + 6\delta$ and $\mathring{R} \geqslant 5 - 4\Delta$. Hence, by (1.5.44), $\mathring{W} \geqslant -6 + 6\delta$ and $\mathring{W} \geqslant 4 - 4\Delta$.
In order for $b_3 = 0$, by Theorem 1.5.70 we want $\mathring{W} \geqslant -\frac{9}{2}$. We have $-6 + 6\delta \geqslant -\frac{9}{2}$ iff $\delta \geqslant \frac{1}{4}$; and $4 - 4\Delta \geqslant -\frac{9}{2}$ iff $\Delta \leqslant \frac{17}{8}$. Hence, the result follows. Recall, that a priori, by Remark 1.2.9, we have that $\delta \leqslant 1 \leqslant \Delta$.
Also, note that we do not use Corollary 1.2.15 along with the statement that $\mathcal{S}^2(\Omega^2_8) \ni \hat{W} \geqslant -3$ implies that $b_3 = 0$. This is because the sufficient conditions in terms of the bounds on the sectional curvature we would have obtained imply that $\Delta \leqslant \frac{17}{8}$ or $\delta \geqslant \frac{1}{4}$. □

1.6 Examples

We only consider normal homogeneous spaces G/H (see [8].) Denote the Lie algebras of G and H by $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$.
Having a bi-invariant metric on G induces a metric on G/H which gives us a Riemannian submersion $\pi : G \to G/H$. The usual decomposition into vertical and horizontal subspaces corresponds to the decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. Hence, using the formula (3.30) and Corollary 3.19 from [8] gives us that for $X, Y, Z, W \in \mathfrak{m}$ we have:

$$\begin{aligned} R(X,Y,Z,W) =& \frac{1}{4}(\langle [X,W],[Y,Z]\rangle - \langle [X,Z],[Y,W]\rangle) + \frac{1}{4}(\langle [X,W]_{\mathfrak{h}},[Y,Z]_{\mathfrak{h}}\rangle \\ &- \langle [X,Z]_{\mathfrak{h}},[Y,W]_{\mathfrak{h}}\rangle) - \frac{1}{2}\langle [Z,W]_{\mathfrak{h}},[X,Y]_{\mathfrak{h}}\rangle. \end{aligned}$$

Letting $X = W, Y = Z$ yields:

$$R(X,Y,Y,X) = \frac{1}{4}\|[X,Y]_{\mathfrak{m}}\|^2 + \|[X,Y]_{\mathfrak{h}}\|^2. \tag{1.6.1}$$

The first formula will allow us to calculate sharp bounds for $\mathring{R}$, $\hat{R}$ and the second one bounds for $\bar{R}$, which we use to check the theorems.

Before going to specific examples, we briefly outline the process of how we get the bounds for $\hat{R}$ and $\mathring{R}$.
Consider $\hat{R}$ first. Note that this is a self-adjoint operator, hence it is bounded by the smallest and the largest eigenvalues. So, if we take any local orthonormal frame f_α of Ω^2, find all the entries of the matrix $\hat{R}_{\alpha\beta}$ corresponding to this linear operator, we can find its eigenvalues.
We already have that $e_i \wedge e_j$ for $i < j$ is an orthonormal frame for Ω^2. Let $f_\alpha = e_i \wedge e_j, f_\beta = e_u \wedge e_v$ be any two such basis elements. Then from the proof of Theorem 1.2.12, we have

$$\bar{R}_{\alpha\beta} = (\hat{R}f_\alpha, f_\beta) = \frac{1}{2}(\hat{R}f_\alpha)_{kl}(f_\beta)_{kl} = \frac{1}{2}(\hat{R}(e_i \wedge e_j))_{kl}(e_u \wedge e_v)_{kl} = (\hat{R}(e_i \wedge e_j))_{uv} = 2R_{ijuv}.$$

So, we use Maple to find all the values R_{ijuv} and thus the matrix $\hat{R}_{\alpha\beta}$. As mentioned before, its largest and smallest eigenvalues are the sharp bounds we are looking for.

Next, we want to also find the bounds for $\mathring{R}$ on $\mathcal{S}^2$. As before, it is enough to find the eigenvalues corresponding to this linear self-adjoint operator.
Let M be of dimension n. Then $\mathcal{S}^2$ has dimension $\frac{n(n+1)}{2}$.
Let $\{e_1, \ldots, e_n\}$ be an orthonormal frame, with the dual frame $\{e^1, \ldots, e^n\}$. Now, let:

$$f_{ij} := e_i \otimes e_j + e_j \otimes e_i, \text{ for } i < j,$$

$$f_{ii} := e_i \otimes e_i, \text{ for } i = 1, \ldots, n.$$

Note that these f_{ij}, f_{ii} form a frame for $\mathcal{S}^2$. Let them be denoted just as f_α. We will still specify if the f_α we take is one of f_{ij}, for $i < j$, or one of f_{ii}. So, now, we want to find the matrix representation of $\mathring{R}$ in terms of the basis of f_α's. Note that this frame is not orthonormal, but we do not need it to be, since the eigenvalues of the matrix will turn out to be all the same.
First, note that for $h \in \mathcal{S}^2$ we have:

$$h = \sum_{i,j=1}^{n} h_{ij} e_i \otimes e_j = \sum_{i<j} h_{ij}(e_i \otimes e_j + e_j \otimes e_i) + \sum_{i=1}^{n} h_{ii} e_i \otimes e_i = \sum_{i<j} h_{ij} f_{ij} + \sum_{i=1}^{n} h_{ii} f_{ii}.$$

That means that the f_β component of h, which we will denote by h^β is equal to h_{ij}, for $f_\beta = f_{ij}, i \leqslant j$. Next, we need to find how $\mathring{R}$ acts on these basis elements f_α. We claim that:

$$(\mathring{R} f_{ij})_{ab} = R_{iajb} + R_{jaib}, \text{ for } i < j, \tag{1.6.2}$$

$$(\mathring{R} f_{ii})_{ab} = R_{iaib}. \tag{1.6.3}$$

We calculate:

$$\begin{aligned}(\mathring{R} f_{ij})_{ab} &= \sum_{k,l} R_{kalb}(f_{ij})_{kl} \\ &= \sum_{k,l} R_{kalb}(e_i \otimes e_j + e_j \otimes e_i)_{kl} \\ &= \sum_{k,l} R_{kalb}(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) \\ &= R_{iajb} + R_{jaib}.\end{aligned}$$

Also,

$$(\mathring{R} f_{ii})_{ab} = (\mathring{R}(e_i \otimes e_i))_{ab} = R_{iaib}.$$

as claimed.

From (1.6.2) and (1.6.3) it is easy to get that the f_β component of $\mathring{R} f_\alpha$, which we denote by $\mathring{R}_{\alpha\beta}$ is equal to:

- if $f_\alpha = f_{ij}, f_\beta = f_{st}, i < j$ and $s < t$, then $\mathring{R}_{\alpha\beta} = R_{isjt} + R_{jsit}$,
- if $f_\alpha = f_{ij}, f_\beta = f_{ss}, i < j$, then $\mathring{R}_{\alpha\beta} = 2R_{isjs}$,
- if $f_\alpha = f_{ii}, f_\beta = f_{st}, s < t$, then $\mathring{R}_{\alpha\beta} = R_{isit}$,
- if $f_\alpha = f_{ii}, f_\beta = f_{ss}$, then $\mathring{R}_{\alpha\beta} = R_{isis}$.

Note that we actually need bounds for $\hat{R}$ or $\mathring{R}$ on specific subspaces of Ω^2 or $\mathcal{S}^2$ respectively, but this will be easy to get as we know what these operators do on the complements of the subspaces we are looking for.

Also, for both examples we identify $\mathfrak{su}(2)$ with $\mathbb{R}^3$ as follows: $\frac{a_1}{\sqrt{2}}I + \frac{a_2}{\sqrt{2}}J + \frac{a_3}{\sqrt{2}}K \longleftrightarrow (a_1, a_2, a_3)$, where $I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ is the standard basis for $\mathfrak{su}(2)$. This takes the inner product $\operatorname{tr}(a^* b)$ on $\mathfrak{su}(2)$ to the usual one in $\mathbb{R}^3$. For $a, b, c, d \in \mathfrak{su}(2)$, it is straightforward to verify that:

$$\begin{aligned} \langle [a,b], [c,d] \rangle &= 2(\langle a,c \rangle \langle b,d \rangle - \langle a,d \rangle \langle b,c \rangle), \\ |[a,b]|^2 &= 2|a|^2|b|^2 - 2\langle a,b \rangle^2. \end{aligned} \tag{1.6.4}$$

1.6.1 $\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{U}(1)\times\mathrm{SU}(2)}$

We describe some of the aspects of the nearly G_2 structure on this G/H. See [1] for more information. By $\mathrm{SU}(2)_d$ we denote the following embedding of SU(2) into $\mathrm{SU}(3) \times \mathrm{SU}(2)$:

$$\mathrm{SU}(2)_d = \{(\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, A), A \in \mathrm{SU}(2)\}.$$

Also, by U(1) we mean the following embedding into subgroup of $\mathrm{SU}(3) \times \{\mathrm{I}\} \subseteq \mathrm{SU}(3) \times \mathrm{SU}(2)$:

$$\mathrm{U}(1) = \{(\begin{pmatrix} e^{it} & 0 & 0 \\ 0 & e^{it} & 0 \\ 0 & 0 & e^{-2it} \end{pmatrix}, \mathrm{I}), t \in \mathbb{R}\}.$$

Then $\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{U}(1)\times\mathrm{SU}(2)}$ is a normal homogeneous space with the metric $B = -\frac{1}{24}(6\operatorname{tr}(uv)) + 4\operatorname{tr}(wz))$ (this is a multiple of the Killing form), for $(u,w), (v,z) \in \mathfrak{g} = \mathfrak{su}(3) \oplus \mathfrak{su}(2)$. With such a choice of a metric one obtains a nearly G_2 structure with $\tau_0 = -\frac{12}{\sqrt{5}}$ and hence the Einstein constant $k = \frac{54}{5}$ with $R = \frac{378}{5}$. Then we have the following orthogonal decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},$$

with

$$\mathfrak{h} = \mathfrak{u}(1) \oplus \mathfrak{su}(2)_d \text{ and } \mathfrak{m} = \mathfrak{su}(2)_o \oplus \mathfrak{m}'$$

where:

$$\mathfrak{u}(1) = \operatorname{span}\{(\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}, 0)\}, \mathfrak{su}(2)_d = \{(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, a), a \in \mathfrak{su}(2)\},$$

$$\mathfrak{su}(2)_o = \{(\begin{pmatrix} 2a & 0 \\ 0 & 0 \end{pmatrix}, -3a), a \in \mathfrak{su}(2)\}, \mathfrak{m}' = \{(\begin{pmatrix} 0 & z \\ -\bar{z}^T & 0 \end{pmatrix}, 0), z \in \mathbb{C}^2\}.$$

We define the following quantities:

$$\begin{aligned}
f_1(a) &:= (\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, a) \in \mathfrak{su}(2)_d \subseteq \mathfrak{h}, \text{ for } a \in \mathfrak{su}(2), \\
f_2(a) &:= (\begin{pmatrix} 2a & 0 \\ 0 & 0 \end{pmatrix}, -3a) \in \mathfrak{su}(2)_o \subseteq \mathfrak{m}, \text{ for } a \in \mathfrak{su}(2), \\
g_1(r) &:= (r \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}, 0) \in \mathfrak{u}(1) \subseteq \mathfrak{h}, \text{ for } r \in \mathbb{R}, \\
g_2(z) &:= (\begin{pmatrix} 0 & z \\ -\bar{z}^T & 0 \end{pmatrix}, 0) \in \mathfrak{m}' \subseteq \mathfrak{m}, \text{ for } z \in \mathbb{C}^2, \\
|z|^2 &:= |z_1|^2 + |z_2|^2 = \bar{z}^T z, \text{ for } z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2.
\end{aligned}$$

Note that all f_1, f_2, g_1, g_2 are linear. Next, we compute their norms with respect to the metric B, where the norm squared is denoted by $\|\cdot\|^2 = B(\cdot,\cdot)$. So:

$$\|f_1(a)\|^2 = -\frac{1}{24}(6\operatorname{tr}(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}^2) + 4\operatorname{tr}(a^2)) = -\frac{1}{24}(6\operatorname{tr}(a^2) + 4\operatorname{tr}(a^2)) = -\frac{5}{12}\operatorname{tr}(a^2) = \frac{5}{12}|a|^2.$$

$$\begin{aligned}
\|f_2(a)\|^2 &= -\frac{1}{24}(6\operatorname{tr}(\begin{pmatrix} 2a & 0 \\ 0 & 0 \end{pmatrix}^2) + 4\operatorname{tr}((-3a)^2)) = -\frac{1}{24}(24\operatorname{tr}(a^2) + 36\operatorname{tr}(a^2))\operatorname{tr}(a^2) \\
&= -\frac{5}{2} = \frac{5}{2}|a|^2. \qquad (1.6.5)
\end{aligned}$$

$$\|g_1(r)\|^2 = -\frac{1}{24}(6r^2\operatorname{tr}(\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}^2)) = -\frac{1}{24}(6r^2(-6)) = \frac{3}{2}r^2.$$

$$\begin{aligned}
\|g_2(z)\|^2 &= -\frac{1}{24}(6\operatorname{tr}(\begin{pmatrix} 0 & z \\ -\bar{z}^T & 0 \end{pmatrix}^2)) = -\frac{1}{4}\operatorname{tr}(\begin{pmatrix} -z\bar{z}^T & 0 \\ 0 & -\bar{z}^T z \end{pmatrix}) \\
&= \frac{1}{4}(\operatorname{tr}(z\bar{z}^T) + \bar{z}^T z) = \frac{1}{4}2\bar{z}^T z = \frac{1}{2}|z|^2.
\end{aligned}$$

Now, we want to find bounds on $\bar{R}$, so take $X = f_2(a) + g_2(z), Y = f_2(b) + g_2(w) \in \mathfrak{m}$ for some $a, b \in \mathfrak{su}(2)$ and $z, w \in \mathbb{C}^2$, with $\|X\|^2 = \|Y\|^2 = 1, B(X,Y) = 0$. So, we have:

$$
\begin{aligned}
1 &= \|X\|^2 = \|f_2(a)\|^2 + \|g_2(z)\|^2 = \frac{5}{2}|a|^2 + \frac{1}{2}|z|^2, \\
1 &= \|Y\|^2 = \|f_2(b)\|^2 + \|g_2(w)\|^2 = \frac{5}{2}|b|^2 + \frac{1}{2}|w|^2, \\
0 &= B(X,Y) = B(f_2(a) + g_2(z), f_2(b) + g_2(w)) \\
&= B((\begin{pmatrix} 2a & z \\ -\bar{z}^T & 0 \end{pmatrix}, -3a), (\begin{pmatrix} 2b & w \\ -\bar{w}^T & 0 \end{pmatrix}, -3b)) \\
&= -\frac{1}{24}(6\operatorname{tr}(\begin{pmatrix} 2a & z \\ -\bar{z}^T & 0 \end{pmatrix} \begin{pmatrix} 2b & w \\ -\bar{w}^T & 0 \end{pmatrix}) + 4\operatorname{tr}((-3a)(-3b))) \\
&= -\frac{1}{24}(6\operatorname{tr}(\begin{pmatrix} 4ab - z\bar{w}^T & 2aw \\ -2\bar{z}^T b & -\bar{z}^T w \end{pmatrix}) + 36\operatorname{tr}(ab)) \\
&= -\frac{1}{24}(24\operatorname{tr}(ab) - 6\operatorname{tr}(z\bar{w}^T) - 6\bar{z}^T w + 36\operatorname{tr}(ab)) \\
&= -\frac{1}{24}(60\operatorname{tr}(ab) - 6\bar{w}^T z - 6\bar{z}^T w),
\end{aligned}
$$

and thus:

$$\bar{w}^T z + \bar{z}^T w = 10\operatorname{tr}(ab).$$

Next, we need to calculate $[X,Y]$. We have:

$$
\begin{aligned}
[X,Y] &= [f_2(a) + g_2(z), f_2(b) + g_2(w)] \\
&= [f_2(a), f_2(b)] + [g_2(z), f_2(b)] + [f_2(a), g_2(w)] + [g_2(z), g_2(w)].
\end{aligned}
$$

We will calculate each term separately:

$$
\begin{aligned}
[f_2(a), f_2(b)] &= [(\begin{pmatrix} 2a & 0 \\ 0 & 0 \end{pmatrix}, -3a), (\begin{pmatrix} 2b & 0 \\ 0 & 0 \end{pmatrix}, -3b)] \\
&= (\begin{pmatrix} 4[a,b] & 0 \\ 0 & 0 \end{pmatrix}, 9[a,b]) \\
&= 6f_1([a,b]) - f_2([a,b]).
\end{aligned}
$$

Next:

$$\begin{aligned}
[g_2(z), f_2(b)] &= [(\begin{pmatrix} 0 & z \\ -\bar{z}^T & 0 \end{pmatrix}, 0), (\begin{pmatrix} 2b & 0 \\ 0 & 0 \end{pmatrix}, -3b)] \\
&= ([\begin{pmatrix} 0 & z \\ -\bar{z}^T & 0 \end{pmatrix}, \begin{pmatrix} 2b & 0 \\ 0 & 0 \end{pmatrix}], 0) \\
&= (\begin{pmatrix} 0 & 0 \\ -2\bar{z}^T b & 0 \end{pmatrix} - \begin{pmatrix} 0 & 2bz \\ 0 & 0 \end{pmatrix}, 0) \\
&= (\begin{pmatrix} 0 & -2bz \\ -2\bar{z}^T b & 0 \end{pmatrix}, 0) \\
&= -2g_2(bz).
\end{aligned}$$

Similarly:

$$[f_2(a), g_2(w)] = -[g_2(w), f_2(a)] = 2g_2(aw).$$

Finally:

$$\begin{aligned}
[g_2(z), g_2(w)] &= ([\begin{pmatrix} 0 & z \\ -\bar{z}^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & w \\ -\bar{w}^T & 0 \end{pmatrix}], 0) \\
&= (\begin{pmatrix} -z\bar{w}^T & 0 \\ 0 & -\bar{z}^T w \end{pmatrix} - \begin{pmatrix} -w\bar{z}^T & 0 \\ 0 & -\bar{w}^T z \end{pmatrix}, 0) \\
&= (\begin{pmatrix} -z\bar{w}^T + w\bar{z}^T & 0 \\ 0 & -\bar{z}^T w + \bar{w}^T z \end{pmatrix}, 0) \\
&= (\frac{-\bar{z}^T w + \bar{w}^T z}{-2i} \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}, 0) + (\begin{pmatrix} -z\bar{w}^T + w\bar{z}^T + \frac{-\bar{z}^T w + \bar{w}^T z}{2} \mathrm{I} & 0 \\ 0 & 0 \end{pmatrix}, 0) \\
&\quad (\text{Let } A := -z\bar{w}^T + w\bar{z}^T + \frac{-\bar{z}^T w + \bar{w}^T z}{2} \mathrm{I} \in \mathfrak{su}(2)) \\
&= g_1(\frac{-\bar{z}^T w + \bar{w}^T z}{-2i}) + (\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, 0) \\
&= g_1(\frac{-\bar{z}^T w + \bar{w}^T z}{-2i}) + \frac{3}{5} f_1(A) + \frac{1}{5} f_2(A).
\end{aligned}$$

Hence we conclude that:

$$[X, Y] = [X, Y]_{\mathfrak{m}} + [X, Y]_{\mathfrak{h}},$$

where

$$[X, Y]_{\mathfrak{m}} = f_2(-[a, b] + \frac{1}{5} A) + g_2(2(aw - bz)),$$

$$[X, Y]_{\mathfrak{h}} = f_1(6[a, b] + \frac{3}{5} A) + g_1(\frac{-\bar{z}^T w + \bar{w}^T z}{-2i}).$$

Applying the formula (1.6.1) for the sectional curvature, along with (1.6.5), we get:

$$\begin{aligned}
\bar{R}(X \wedge Y) =& \frac{1}{4}\|[X,Y]_{\mathfrak{m}}\|^2 + \|[X,Y]_{\mathfrak{h}}\|^2 \\
=& \frac{1}{4}(\|f_2(-[a,b] + \frac{1}{5}A)\|^2 + \|g_2(2(aw - bz))\|^2) + \|f_1(6[a,b] + \frac{3}{5}A)\|^2 \\
&+ \|g_1(\frac{-\bar{z}^T w + \bar{w}^T z}{-2i})\|^2 \\
=& \frac{1}{4}(\frac{5}{2}|-[a,b] + \frac{1}{5}A|^2 + \frac{1}{2}|2(aw - bz)|^2) + \frac{5}{12}|6[a,b] + \frac{3}{5}A|^2 \\
&+ \frac{3}{2}\Big(\frac{-\bar{z}^T w + \bar{w}^T z}{-2i}\Big)^2 \\
=& \frac{5}{8}(|[a,b]|^2 + \frac{1}{25}|A|^2 + \frac{2}{5}\operatorname{tr}([a,b]A)) + \frac{1}{2}|aw - bz|^2 \\
&+ \frac{5}{12}(36|[a,b]|^2 + \frac{9}{25}|A|^2 - \frac{36}{5}\operatorname{tr}([a,b]A)) - \frac{3}{8}(-\bar{z}^T w + \bar{w}^T z)^2 \\
=& \frac{125}{8}|[a,b]|^2 + \frac{7}{40}|A|^2 - \frac{11}{4}\operatorname{tr}([a,b]A) + \frac{1}{2}|aw - bz|^2 - \frac{3}{8}(-\bar{z}^T w + \bar{w}^T z)^2.
\end{aligned}$$

It is straightforward to check that for $a \in \mathfrak{su}(2), w \in \mathbb{C}^2$ we have:

$$|aw|^2 = \frac{1}{2}|a|^2|w|^2. \tag{1.6.6}$$

Polarizing, we also get:

$$\begin{gathered}
\langle aw, az\rangle = \frac{1}{2}|a|^2\langle w, z\rangle, \\
\langle az, bz\rangle + \langle bz, az\rangle = |z|^2\langle a, b\rangle, \\
\langle az, bw\rangle + \langle bw, az\rangle + \langle bz, aw\rangle + \langle aw, bz\rangle = \langle a, b\rangle(\langle z, w\rangle + \langle w, z\rangle).
\end{gathered}$$

For simplicity, define:

$$\begin{aligned}
\alpha &:= \langle z, w\rangle \in \mathbb{C}, \\
\sigma &:= -\langle aw, bz\rangle \in \mathbb{C}, \\
\varphi &:= -\langle az, bw\rangle \in \mathbb{C}.
\end{aligned}$$

Then some calculation yields that:

$$\begin{aligned}
\frac{125}{8}|[a,b]|^2 &= \frac{125}{4}|a|^2|b|^2 - \frac{125}{4}\langle a,b\rangle^2,\\
\frac{7}{40}|A|^2 &= \frac{7}{20}|z|^2|w|^2 - \frac{7}{80}\alpha^2 - \frac{7}{80}\bar{\alpha}^2 - \frac{7}{40}\alpha\bar{\alpha},\\
-\frac{11}{4}\operatorname{tr}([a,b]A) &= \frac{11}{4}(\sigma+\bar{\sigma}-\varphi-\bar{\varphi}),\\
\frac{1}{2}|aw-bz|^2 &= \frac{1}{4}|a|^2|w|^2 + \frac{1}{4}|b|^2|z|^2 - \sigma - \bar{\sigma},\\
-\frac{3}{8}(-\bar{z}^T w + \bar{w}^T z)^2 &= -\frac{3}{8}\alpha^2 - \frac{3}{8}\bar{\alpha}^2 + \frac{3}{4}\alpha\bar{\alpha}.
\end{aligned}$$

Recall that we assumed:

$$\begin{aligned}
1 &= \frac{5}{2}|a|^2 + \frac{1}{2}|z|^2,\\
1 &= \frac{5}{2}|b|^2 + \frac{1}{2}|w|^2,\\
\alpha + \bar{\alpha} &= -10\langle a,b\rangle.
\end{aligned}$$

Isolating $\langle a,b\rangle, |a|^2, |b|^2$ and substituting these resullts into expressions found earlier, we get that:

$$\begin{aligned}
\bar{R}(X\wedge Y) = 5 &- \frac{12}{5}|z|^2 - \frac{12}{5}|w|^2 + \frac{3}{2}|z|^2|w|^2 - \frac{31}{40}\alpha^2 - \frac{31}{40}\bar{\alpha}^2 - \frac{1}{20}\alpha\bar{\alpha}\\
&+ \frac{7}{4}\sigma + \frac{7}{4}\bar{\sigma} - \frac{11}{4}\varphi - \frac{11}{4}\bar{\varphi}.
\end{aligned}$$

Note that each $\sigma, \bar{\sigma}, \varphi, \bar{\varphi}$ in absolute value is $\leqslant \frac{1}{2}|a||b||z||w|$, by Cauchy-Schwarz and (1.6.6). Hence:

$$\begin{aligned}
\bar{R}(X\wedge Y) &\leqslant 5 - \frac{12}{5}|z|^2 - \frac{12}{5}|w|^2 + \frac{3}{2}|z|^2|w|^2 + \frac{3}{2}|\alpha|^2 + \frac{9}{2}|a||b||z||w|\\
&\leqslant 5 - \frac{12}{5}|z|^2 - \frac{12}{5}|w|^2 + 3|z|^2|w|^2 + \frac{9}{2}|a||b||z||w|.
\end{aligned}$$

One can check that on $1 = \frac{5}{2}|a|^2 + \frac{1}{2}|z|^2, 1 = \frac{5}{2}|b|^2 + \frac{1}{2}|w|^2$,

$$\bar{R}(X\wedge Y) \leqslant \frac{37}{5}.$$

Numerical evidence suggests that $\frac{1}{5} \leqslant \bar{R}(X\wedge Y)$, however the author was unable to verify this. Nevertheless, we have $0 \leqslant \bar{R}(X\wedge Y)$ and we can show that both values $\frac{1}{5}$ and $\frac{37}{5}$ can be achieved:

$$a = b = 0, z = \sqrt{2}\begin{pmatrix}1\\0\end{pmatrix}, w = \sqrt{2}\begin{pmatrix}i\\0\end{pmatrix}$$

gives orthonormal X, Y with $\bar{R}(X \wedge Y) = \frac{37}{5}$, and

$$a = -\frac{\sqrt{5}}{5}I, b = 0, z = 0, w = \sqrt{2}\begin{pmatrix}0\\ i\end{pmatrix}$$

gives orthonormal X, Y with $\bar{R}(X \wedge Y) = \frac{1}{5}$.

A computation on Maple reveals that eigenvalues of $\hat{R}$ on Ω^2 are $\left(-\frac{114}{5}\right)_1, \left(-\frac{66}{5}\right)_3$, $\left(-\frac{18}{5}\right)_7, \left(\frac{6}{5}\right)_{10}$ where by the subscript we denote its multiplicity. Note that this makes sense, because for $\beta \in \Omega^2_7$, from Remark 1.4.2, $\beta = X \lrcorner \varphi$ and hence a quick calculation gives that $\hat{R}\beta = X \lrcorner (\hat{R}\varphi) = X \lrcorner (-\frac{\tau_0^2}{8}\varphi) = -\frac{\tau_0^2}{8}\beta = -\frac{18}{5}\beta$, so we get seven eigenvalues $-\frac{18}{5}$. Hence, we conlude that on Ω^2_{14}, $-\frac{114}{5} \leqslant \hat{R} \leqslant \frac{6}{5}$.

Similarly, Maple shows that eigenvalues of $\mathring{R}$ on $\mathcal{S}^2$ are $\left(-\frac{54}{5}\right)_1, \left(-\frac{47}{5}\right)_1, \left(-\frac{23}{5}\right)_7, \left(\frac{13}{5}\right)_8, 5_5, \left(\frac{37}{5}\right)_6$. Again, this makes sense, as we know that $\mathring{R}g = -\frac{3\tau_0^2}{8}g = -\frac{54}{5}g$. Hence, on $\mathcal{S}^2_0$, $-\frac{47}{5} \leqslant \mathring{R} \leqslant \frac{37}{5}$.

So, we summarize and check the theorems. We have:

$$\begin{aligned}\frac{1}{5} &\leqslant \bar{R} \leqslant \frac{37}{5},\\ -\frac{114}{5} &\leqslant \hat{R} \leqslant \frac{6}{5} \text{ on } \Omega^2_{14},\\ -\frac{47}{5} &\leqslant \mathring{R} \leqslant \frac{37}{5} \text{ on } \mathcal{S}^2_0.\end{aligned}$$

Corollary 1.2.14 gives us that $-\frac{122}{5} \leqslant \hat{R} \leqslant \frac{46}{5}$ on Ω^2_{14} which is consistent.

Corollary 1.2.17 gives us that $-\frac{47}{5} \leqslant \mathring{R} \leqslant \frac{49}{5}$ on $\mathcal{S}^2_0$, which is also consistent with the first inequality being sharp.

For the main Theorems, we know in this case that $b_2 = 1$. So it must be false that $\hat{W} \geqslant -18$ on Ω^2_{14}, by Theorem 1.4.7. By (1.4.5), we have that $\hat{W} \geqslant -19.2$ on Ω^2_{14}, with the eigenvalue value -19.2 achieved. Hence, we get no contradiction.

As for Theorem 1.4.14 we cannot predict whether $\mathring{W} \geqslant -\frac{54}{5}$ on $\mathcal{S}^2_0$ or $\hat{W} \geqslant -\frac{36}{5}$ on Ω^2_{14} must hold or not, because $b_3 = 0$. However, these inequalities do not hold as we have $\mathring{W} \geqslant -\frac{56}{5}$ on $\mathcal{S}^2_0$ and $\hat{W} \geqslant -\frac{96}{5}$ on Ω^2_{14} with these lower bounds attained. This shows that, in general, these sufficient conditions are not necessary.

1.6.2 $\frac{\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)}{\mathrm{SU}(2)}$

First, we describe the nearly Kähler structure on this G/H.

The $\mathrm{SU}(2)$ in the denominator is embedded diagonally in the numerator, meaning it is:

$$\{(A, A, A) \in \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2) : A \in \mathrm{SU}(2)\}.$$

Recall that we decompose $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then $\mathfrak{m} = \{(a, b, -(a+b) : a, b \in \mathfrak{su}(2))\}$.
Equipping G with the metric $B((a,b,c),(u,v,w)) = \frac{1}{3}(\operatorname{tr}(a^*u) + \operatorname{tr}(b^*v) + \operatorname{tr}(c^*w))$, for $(a,b,c),(u,v,w) \in \mathfrak{g}$ makes G/H into a normal homogeneous space with scalar curvature 30.
The nearly Kähler structure is obtained from the almost complex structure, which is defined as follows:

$$J((a,b,c)) = \frac{2}{\sqrt{3}}(b,c,a) + \frac{1}{\sqrt{3}}(a,b,c),$$

for $(a,b,c) \in \mathfrak{m}$. It follows from $c = -a-b$ that $J^2 = -\mathrm{I}$.
Define the following quantities:

$$\begin{aligned} f(a) &:= (a,a,a) \in \mathfrak{h} \subseteq \mathfrak{g}, \text{ for } a \in \mathfrak{su}(2) \\ g(b,c) &:= (b,c,-(b+c)) \in \mathfrak{m} \subseteq \mathfrak{g}, \text{ for } b,c \in \mathfrak{su}(2). \end{aligned}$$

Then for $(a,b,c) \in \mathfrak{g}$, we have that

$$\begin{aligned} (a,b,c)_{\mathfrak{h}} &= f\left(\frac{a+b+c}{3}\right), \\ (a,b,c)_{\mathfrak{m}} &= g\left(\frac{2a-b-c}{3}, \frac{-a+2b-c}{3}\right). \end{aligned} \tag{1.6.7}$$

Note that also $|f(a)|^2 = |a|^2$, and $|g(b,c)|^2 = \frac{1}{3}(|b|^2 + |c|^2 + |b+c|^2)$.
We want to calculate the bounds on $\bar{R}$. Clearly from the formula (1.6.1) for $\bar{R}$, we see that $0 \leqslant \bar{R}$. We claim that $\bar{R} \leqslant \frac{9}{4}$. Take $X, Y \in \mathfrak{m}$ with $\|X\|^2 = 1 = \|Y\|^2, B(X,Y) = 0$. Let $X = g(b,c), Y = g(d,e)$, for $b,c,d,e \in \mathfrak{su}(2)$. Then:

$$\begin{aligned} [X,Y] &= [(b,c,-(b+c)),(d,e,-(d+e))] \\ &= ([b,d],[c,e],[b,d]+[c,e]+[b,e]+[c,d]). \end{aligned}$$

Let $A := [b,d], B := [c,e], C := [b,e]+[c,d]$ so that $[X,Y] = (A,B,A+B+C)$. By (1.6.7):

$$\begin{aligned} [X,Y]_{\mathfrak{h}} &= f\left(\frac{1}{3}(2A+2B+C)\right), \\ [X,Y]_{\mathfrak{m}} &= g\left(\frac{1}{3}(A-2B-C), \frac{1}{3}(-2A+B-C)\right). \end{aligned}$$

Hence, equation (1.6.1) gives:

$$\begin{aligned}
\bar{R}(X \wedge Y) =& \frac{1}{4}|[X,Y]_{\mathfrak{m}}|^2 + |[X,Y]_{\mathfrak{h}}|^2 \\
=& \frac{1}{4} \cdot \frac{1}{3}(|\frac{1}{3}(A-2B-C)|^2 + |\frac{1}{3}(-2A+B-C)|^2 + |\frac{1}{3}(-A-B-2C)|^2) \\
& + |\frac{1}{3}(2A+2B+C)|^2 \\
=& \frac{1}{108}|A-2B-C|^2 + \frac{1}{108}|-2A+B-C|^2 + \frac{1}{108}|-A-B-2C|^2 \\
& + \frac{1}{9}|2A+2B+C|^2 \\
=& \frac{1}{108}(|A|^2 + 4|B|^2 + |C|^2 - 4\langle A,B\rangle - 2\langle A,C\rangle + 4\langle B,C\rangle) \\
& + \frac{1}{108}(4|A|^2 + |B|^2 + |C|^2 - 4\langle A,B\rangle + 4\langle A,C\rangle - 2\langle B,C\rangle) \\
& + \frac{1}{108}(|A|^2 + |B|^2 + 4|C|^2 + 2\langle A,B\rangle + 4\langle A,C\rangle + 4\langle B,C\rangle) \\
& + \frac{1}{9}(4|A|^2 + 4|B|^2 + |C|^2 + 8\langle A,B\rangle + 4\langle A,C\rangle + 4\langle B,C\rangle) \\
=& \frac{1}{2}|A|^2 + \frac{1}{2}|B|^2 + \frac{1}{6}|C|^2 + \frac{5}{6}\langle A,B\rangle + \frac{1}{2}\langle A,C\rangle + \frac{1}{2}\langle B,C\rangle.
\end{aligned}$$

Using (1.6.4) we now get:

$$\begin{aligned}
|A|^2 &= |[b,d]|^2 = 2|b|^2|d|^2 - 2\langle b,d\rangle^2, \\
|B|^2 &= |[c,e]|^2 = 2|c|^2|e|^2 - 2\langle c,e\rangle^2, \\
|C|^2 &= |[b,e] + [c,d]|^2 = |[b,e]|^2 + |[c,d]|^2 + 2\langle [b,e],[c,d]\rangle \\
&= 2|b|^2|e|^2 - 2\langle b,e\rangle^2 + 2|c|^2|d|^2 - 2\langle c,d\rangle^2 + 4(\langle b,c\rangle\langle e,d\rangle - \langle b,d\rangle\langle c,e\rangle), \\
\langle A,B\rangle &= \langle [b,d],[c,e]\rangle = 2(\langle b,c\rangle\langle d,e\rangle - \langle b,e\rangle\langle c,d\rangle), \\
\langle A,C\rangle &= \langle [b,d],[b,e]+[c,d]\rangle = \langle [b,d],[b,e]\rangle + \langle [b,d],[c,d]\rangle \\
&= 2(|b|^2\langle d,e\rangle - \langle b,e\rangle\langle b,d\rangle + \langle b,c\rangle|d|^2 - \langle b,d\rangle\langle c,d\rangle), \\
\langle B,C\rangle &= \langle [c,e],[b,e]+[c,d]\rangle = \langle [c,e],[b,e]\rangle + \langle [c,e],[c,d]\rangle \\
&= 2(\langle c,b\rangle|e|^2 - \langle c,e\rangle\langle b,e\rangle + |c|^2\langle d,e\rangle - \langle c,d\rangle\langle c,e\rangle).
\end{aligned}$$

Substituting the above into $\bar{R}(X \wedge Y)$, we get:

$$\begin{aligned}\bar{R}(X \wedge Y) =& |b|^2|d|^2 - \langle b,d\rangle^2 + |c|^2|e|^2 - \langle c,e\rangle^2 \\ &+ \frac{1}{3}|b|^2|e|^2 - \frac{1}{3}\langle b,e\rangle^2 + \frac{1}{3}|c|^2|d|^2 - \frac{1}{3}\langle c,d\rangle^2 + \frac{2}{3}\langle b,c\rangle\langle e,d\rangle \\ &- \frac{2}{3}\langle b,d\rangle\langle c,e\rangle + \frac{5}{3}\langle b,c\rangle\langle d,e\rangle - \frac{5}{3}\langle b,e\rangle\langle c,d\rangle \\ &+ |b|^2\langle d,e\rangle - \langle b,e\rangle\langle b,d\rangle + |d|^2\langle b,c\rangle - \langle b,d\rangle\langle c,d\rangle + |e|^2\langle c,b\rangle \\ &- \langle c,e\rangle\langle b,e\rangle + |c|^2\langle d,e\rangle - \langle c,d\rangle\langle c,e\rangle.\end{aligned} \tag{1.6.8}$$

Recall that we assumed $\|X\|^2 = 1 = \|Y\|^2, B(X,Y) = 0$. Hence $1 = \|X\|^2\|Y\|^2 - B(X,Y)^2$. (Note this is just saying $\|X \wedge Y\|^2 = 1$. We could have assumed just this, however, the first assumption makes the argument easier.) We have:

$$\begin{aligned}\|X\|^2 &= \frac{1}{3}(|b|^2 + |c|^2 + |b+c|^2) = \frac{2}{3}(|b|^2 + |c|^2 + \langle b,c\rangle). \\ \|Y\|^2 &= \frac{1}{3}(|d|^2 + |e|^2 + |d+e|^2) = \frac{2}{3}(|d|^2 + |e|^2 + \langle d,e\rangle). \\ B(X,Y) &= \frac{1}{3}(\langle b,d\rangle + \langle c,e\rangle + \langle b+c, d+e\rangle) \\ &= \frac{1}{3}(2\langle b,d\rangle + 2\langle c,e\rangle + \langle b,e\rangle + \langle c,d\rangle).\end{aligned} \tag{1.6.9}$$

Hence,

$$\begin{aligned}1 =& \|X\|^2\|Y\|^2 - B(X,Y)^2 \\ =& \frac{4}{9}(|b|^2 + |c|^2 + \langle b,c\rangle)(|d|^2 + |e|^2 + \langle d,e\rangle) - \frac{1}{9}(2\langle b,d\rangle + 2\langle c,e\rangle + \langle b,e\rangle + \langle c,d\rangle)^2.\end{aligned}$$

or equivalently,

$$\begin{aligned}\frac{9}{4} =& |b|^2|d|^2 + |b|^2|e|^2 + |b|^2\langle d,e\rangle + |c|^2|d|^2 + |c|^2|e|^2 + |c|^2\langle d,e\rangle + |d|^2\langle b,c\rangle + |e|^2\langle b,c\rangle \\ &+ \langle b,c\rangle\langle d,e\rangle - \langle b,d\rangle^2 - \langle c,e\rangle^2 - \frac{1}{4}\langle b,e\rangle^2 - \frac{1}{4}\langle c,d\rangle^2 - 2\langle b,d\rangle\langle c,e\rangle - \langle b,d\rangle\langle b,e\rangle \\ &- \langle b,d\rangle\langle c,d\rangle - \langle c,e\rangle\langle b,e\rangle - \langle c,e\rangle\langle c,d\rangle - \frac{1}{2}\langle b,e\rangle\langle c,d\rangle,\end{aligned}$$

which can be rearranged to get:

$$\begin{aligned}&|b|^2|d|^2 + |b|^2\langle d,e\rangle + |c|^2|e|^2 + |c|^2\langle d,e\rangle + |d|^2\langle b,c\rangle + |e|^2\langle b,c\rangle - \langle b,d\rangle^2 - \langle c,e\rangle^2 \\ &-\langle b,d\rangle\langle b,e\rangle - \langle b,d\rangle\langle c,d\rangle - \langle c,e\rangle\langle b,e\rangle - \langle c,e\rangle\langle c,d\rangle \\ &= \frac{9}{4} - |b|^2|e|^2 - |c|^2|d|^2 - \langle b,c\rangle\langle d,e\rangle + \frac{1}{4}\langle b,e\rangle^2 + \frac{1}{4}\langle c,d\rangle^2 + 2\langle b,d\rangle\langle c,e\rangle + \frac{1}{2}\langle b,e\rangle\langle c,d\rangle.\end{aligned}$$

Substituting this into (1.6.8), we get:

$$\begin{aligned}\bar{R}(X,Y) =& \frac{9}{4} - \frac{2}{3}|b|^2|e|^2 - \frac{2}{3}|c|^2|d|^2 + \frac{4}{3}\langle b,c\rangle\langle d,e\rangle - \frac{1}{12}\langle b,e\rangle^2 - \frac{1}{12}\langle c,d\rangle^2 \\ &+ \frac{4}{3}\langle b,d\rangle\langle c,e\rangle - \frac{7}{6}\langle b,e\rangle\langle c,d\rangle \\ =& \frac{9}{4} - \frac{1}{12}\Bigg(8\Big(|b|^2|e|^2 + |c|^2|d|^2 - 2\langle b,c\rangle\langle d,e\rangle - 2\langle b,d\rangle\langle c,e\rangle + 2\langle b,e\rangle\langle c,d\rangle\Big)\Bigg) \\ &- \frac{1}{12}\Big(\langle b,e\rangle - \langle c,d\rangle)^2\Big).\end{aligned}$$

We claimed that $\bar{R} \leqslant \frac{9}{4}$, hence it is enough to show that

$$|b|^2|e|^2 + |c|^2|d|^2 - 2\langle b,c\rangle\langle d,e\rangle - 2\langle b,d\rangle\langle c,e\rangle + 2\langle b,e\rangle\langle c,d\rangle \geqslant 0.$$

We note that this expression is equal to:

$$\begin{aligned}&(|b|^2|e|^2 - \langle b,e\rangle^2 + |c|^2|d|^2 - \langle c,d\rangle^2 - 2\langle b,c\rangle\langle d,e\rangle + 2\langle b,d\rangle\langle c,e\rangle) \\ &+ (\langle b,e\rangle^2 + 2\langle b,e\rangle\langle c,d\rangle + \langle c,d\rangle^2) - 4\langle b,d\rangle\langle c,e\rangle \\ &= (\frac{1}{2}|[b,e]|^2 + \frac{1}{2}|[c,d]|^2 - \langle [b,e],[c,d]\rangle) + (\langle b,e\rangle + \langle c,d\rangle)^2 - 4\langle b,d\rangle\langle c,e\rangle \\ &= \frac{1}{2}|[b,e] - [c,d]|^2 + (\langle b,e\rangle + \langle c,d\rangle)^2 - 4\langle b,d\rangle\langle c,e\rangle. \end{aligned} \tag{1.6.10}$$

We assumed that $B(X,Y) = 0$, so from (1.6.9), we get that $\langle b,e\rangle + \langle c,d\rangle = -2(\langle b,d\rangle + \langle c,e\rangle)$. Thus, continuing with (1.6.10), we get:

$$\frac{1}{2}|[b,e] - [c,d]|^2 + 4(\langle b,d\rangle + \langle c,e\rangle)^2 - 4\langle b,d\rangle\langle c,e\rangle,$$

which is always non-negative because for any real x, y, we have $4(x+y)^2 - 4xy = 4(x^2 + xy + y^2) \geqslant 0$.
Finally, we need to show that the bounds $0 \leqslant \bar{R} \leqslant \frac{9}{4}$ are sharp. To do this, we take an explicit orthonormal basis for $\mathfrak{m}$:

$$e_1 = g(\frac{\sqrt{3}}{2}I, 0), \qquad e_2 = g(\frac{\sqrt{3}}{2}J, 0), \qquad e_3 = g(\frac{\sqrt{3}}{2}K, 0),$$

$$e_4 = g(\frac{1}{2}I, -I), \qquad e_5 = g(\frac{1}{2}J, -J), \qquad e_6 = g(\frac{1}{2}K, -K).$$

For this basis we also have: $Je_i = e_{i+3}, 1 \leqslant i \leqslant 3$.
Then one easily calculates that $\bar{R}(e_1 \wedge e_4) = 0, \bar{R}(e_1 \wedge e_2) = \frac{9}{4}$, as we claimed.
A computation on Maple reveals that eigenvalues of $\hat{R}$ on Ω^2 are $-7_3, -2_7, 1_5$, where by

the subscript we denote its multiplicity. Note that this makes sense, as we know from the discussion following the proof of Proposition 1.5.45 that $\hat{R}$ is $-2\,\mathrm{Id}$ on Ω^2_1 and Ω^2_6, so we get seven eigenvalues -2. Hence, we conlude that on Ω^2_8, $-7 \leqslant \hat{R} \leqslant 1$.
Similarly, Maple shows that eigenvalues of $\mathring{R}$ on $\mathcal{S}^2$ are $-5_1, -4_2, \left(-\frac{3}{2}\right)_3, 2_{10}, \left(\frac{5}{2}\right)_5$. Again, this makes sense, as we know that $\mathring{R}g = -5g$, because the Einstein constant is 5. Hence, on $\mathcal{S}^2_0$,$-4 \leqslant \mathring{R} \leqslant \frac{5}{2}$. Furthermore, using that $\hat{W}\beta = 2(\mathring{W}h) \diamond \omega$ for $\beta = h \diamond \omega \in \Omega^2_8$, where $h \in \mathcal{S}^2_{+-}$, it can be easily shown that in fact the eigenvalues $\left(-\frac{3}{2}\right)_3, \left(\frac{5}{2}\right)_5$ occur on $\mathcal{S}^2_{+0}$ and $-4_2, 2_{10}$ occur on $\mathcal{S}^2_-$. So, we summarize and check the theorems. We have:

$$
\begin{aligned}
0 \leqslant \bar{R} &\leqslant \frac{9}{4}, \\
-7 \leqslant \hat{R} &\leqslant 1 \text{ on } \Omega^2_8, \\
-4 \leqslant \mathring{R} &\leqslant \frac{5}{2} \text{ on } \mathcal{S}^2_0, \\
-\frac{3}{2} \leqslant \mathring{R} &\leqslant \frac{5}{2} \text{ on } \mathcal{S}^2_{+0}, \\
-4 \leqslant \mathring{R} &\leqslant 2 \text{ on } \mathcal{S}^2_-.
\end{aligned}
$$

Corollary 1.2.15 gives us that $-\frac{15}{2} \leqslant \hat{R} \leqslant 3$ on Ω^2_8 which is consistent.
Corollary 1.2.17 gives us that $-4 \leqslant \mathring{R} \leqslant 5$ on $\mathcal{S}^2_0$, which is consistent with the first inequality being sharp.
Corollary 1.2.25 gives us that $-\frac{7}{4} \leqslant \mathring{R} \leqslant \frac{7}{2}$ on $\mathcal{S}^2_{+0}$, which is also consistent.
For the main theorems, we know in this case that $b_3 = 2$. So it must be false that $\mathring{W} \geqslant -4$ on $\mathcal{S}^2_0$, by Theorem 1.5.70. By (1.5.44), we have that $-5 \leqslant \mathring{W}$ on $\mathcal{S}^2_0$, with the eigenvalue value -5 achieved. Hence, we get no contradiction. Similarly, $\hat{R}$ achieves -7, so $\hat{W}$ achieves -5, hence we indeed have that $\hat{W} \geqslant -3$ is false.
As for Theorem 1.5.63 we cannot predict whether $\hat{W} \geqslant -8$ on Ω^2_8 (or equivalenty $\mathcal{S}^2(\mathcal{S}^2_{+0}) \ni \mathring{W} \geqslant -4$) must hold or not, because $b_2 = 0$. However, we can actually deduce the vanishing of b_2, since by (1.5.44), we can get that $-5 \leqslant \hat{W}$ on Ω^2_8, so the assumption of the theorem is satisfied. Finally, note that it is even possible to deduce that $b_2 = 0$ from Theorem 1.5.64, since we get that $-(\Delta+\delta) - \frac{7}{3}(\Delta-\delta) \geqslant -10$ and $(\Delta+\delta) - 3(\Delta-\delta) \geqslant -6$ both hold.

Chapter 2

A special class of k-harmonic maps inducing calibrated fibrations

2.1 Introduction

The natural partial differential equations which arise in Riemannian geometry are usually second order. Some important examples are:

(i) an Einstein metric [$\mathrm{Ric}_g = \lambda g$, where Ric is the Ricci curvature]

(ii) a minimal submanifold [$H = 0$, where H is the mean curvature]

(iii) a Yang–Mills connection ∇ on a vector bundle [$(d^\nabla)^* F^\nabla = 0$, where F^∇ is the curvature]

(iv) a k-harmonic map $u\colon (M_1, g_1) \to (M_2, g_2)$ between Riemannian manifolds $\mathrm{div}(|du|^{k-2} du) = 0$

All of the above geometric objects are also *variational*. That is, the PDEs are Euler–Lagrange equations for some natural geometric functional or "energy", and hence such objects are *critical points* of these functionals, but may not in general be (local) minima.

A common feature is that when there is additional geometric structure present, one can identify a natural *special class of solutions* which:

- satisfy a (usually fully nonlinear) *first order* PDE, and
- are actually global minimizers of the functional within a particular class of variations.

With respect to the particular examples above, these special first order solutions are:

(i) a *special holonomy metric*: Calabi–Yau, hyperkähler, quaternionic-Kähler, G_2, or Spin(7). These are all Einstein, and most are Ricci-flat. [The condition of special holonomy is first order on the metric in each case, but there does not seem to be any unified way of describing these, and it is unknown if they are global minimizers of the Einstein–Hilbert functional within some particular class of variations.]

(ii) a *calibrated submanifold* of a special holonomy manifold. These are all minimal. The calibrated condition is a first order condition on the immersion. They are global minimizers of the volume functional in a given homology class.

(iii) an *instanton* on a vector bundle over a special holonomy manifold. These are all Yang–Mills. The instanton condition is a first order condition on the connection, being an algebraic condition on the curvature. In many cases, a characteristic class argument shows that they are global minimizers of the Yang–Mills energy.

Note that all the special first order solutions in (i), (ii), and (iii) described above are related to Riemannian manifolds with special holonomy. [This is not necessary. Classical self-dual and anti-self-dual instantons are special Yang–Mills connections on a Riemannian 4-manifold, with no special holonomy.]

In this paper, we discuss two classes of special first order solutions to (iv) above, called *Smith maps*. They are special types of k-harmonic maps $u\colon (M_1, g_1) \to (M_2, g_2)$ between pairs of Riemannian manifolds, which are intimately related to both *calibrated geometry* and *conformal geometry*:

- For $u\colon (L^k, g) \to (M^n, h)$, with $k \leqslant n$ and $\alpha \in \Omega^k(M)$ a closed calibration, we define a *Smith immersion*, which is a special type of weakly conformal k-harmonic map. If L^0 is the open subset on which $du \neq 0$, then $u\colon L^0 \to M$ is an immersion, whose image $u(L^0)$ is k-dimensional α-calibrated submanifold of (M, h). Moreover, the notion of Smith immersion is invariant under *conformal change* of the domain metric g. Conversely, if $u\colon (L^k, g) \to (M^n, h)$ is a weakly conformal k-harmonic map such that $u(L^0)$ is α-calibrated, then u is a Smith immersion. (Theorem 2.3.2.)
- For $u\colon (M^n, h) \to (L^k, g)$, with $n \geqslant k$ and $\alpha \in \Omega^{n-k}(M)$ a closed calibration, we define a *Smith submersion*, which is a special type of weakly horizontally conformal k-harmonic map. If M^0 is the open subset on which $du \neq 0$, then the fibres $u^{-1}\{u(x)\}$ of $u\colon M^0 \to L$ are $(n-k)$-dimensional α-calibrated submanifolds of (M, h). Moreover, the notion of Smith submersion is invariant under *horizontally conformal change* of the domain metric h. Conversely, if $u\colon (M^n, h) \to (L^k, g)$ is a weakly horizontally conformal k-harmonic map such that the fibres of $u|_{M^0}$ are α-calibrated, then u is a Smith submersion. (Theorem 2.4.10.)

The notion of Smith immersions was previously studied by Cheng–Karigiannis–Madnick in [10] and [11, Section 3.3], inspired by an unpublished preprint of Smith [40]. We review it here, and clarify that it extends from calibrations associated to vector cross products to any calibrations. (This was implicit in [11, Section 3.3].) The notion of Smith submersions is *new* in the present chapter.

In each case, we establish a fundamental pointwise inequality in Theorems 2.3.2 and 2.4.10, respectively, which itself is obtained by combining the fundamental inequality of calibrated geometry and the Hadamard inequality. We then use these pointwise inequalities, together with the assumption that $d\alpha = 0$, to prove the associated integral *energy inequalities* in Theorems 2.3.6 and 2.4.18, respectively, when the domain is compact. This immediately yields the k-harmonicity of such maps. We also give direct proofs of k-harmonicity by differentiating the Smith equations, which also explicitly show the importance of the $d\alpha = 0$ assumption.

The two constructions should also be viewed as special first order versions of the following particular classical results [44, (3.5) and (3.10)] from harmonic map theory:

- a Riemannian immersion $u\colon (L, g) \to (M, h)$ is harmonic $\iff$ the image is minimal,
- a Riemannian submersion $u\colon (M, h) \to (L, g)$ is harmonic $\iff$ the fibres are minimal.

In the final section, we briefly discuss the analytic results for Smith immersions which were established in [10], discuss several explicit examples of Smith submersions with non-compact domains, comment on the relevance of Smith submersions to the SYZ and GYZ "conjectures" involving special Lagrangian and coassociative fibrations, and collect several open questions for future study.

Conventions and notation.

All manifolds are *oriented* Riemannian manifolds, though not necessarily compact. As usual a superscript on a manifold such as M^n means $\dim M = n$. All maps between manifolds are smooth.

We often use the Riemannian metric (via the musical isomorphism) to identify vector fields and 1-forms, and more generally tensors of mixed type with covariant tensors. We use $\mathcal{T}^m$ for the space of smooth m-tensors (that is, smooth sections of the m^{th} tensor power of the cotangent bundle), we use Ω^p for the space of p-forms, $\star$ for the Hodge star operator, and vol for the Riemannian volume form. We use ∇ for the Levi-Civita connection. We write $\operatorname{div}\colon \mathcal{T}^m \to \mathcal{T}^{m-1}$ for the Riemannian *divergence*, given in terms of a local orthonormal frame by $(\operatorname{div} A)_{j_1\cdots j_{m-1}} = \nabla_i A_{ij_1\cdots j_{m-1}}$. (We sum over repeated indices.)

For us, a *calibration* α is a comass one differential form, not necessarily closed. (Some authors call this a semi-calibration or pre-calibration.) When α is also closed, we call it a *closed calibration.*

The following result is a version of *Hadamard's inequality* that we use frequently.

Proposition 2.1.1 (Hadamard's inequality)**.** *Let* $A\colon (V_1^{n_1}, g_1) \to (V_2^{n_2}, g_2)$ *be a linear map between real inner product spaces where* $n_k = \dim V_k$*. Define* $|A|^2 := \operatorname{tr}(A^*A)$ *(and similarly* *for other linear maps between real inner product spaces). Then* $|\Lambda^{n_1} A| \leqslant \frac{1}{(\sqrt{n_1})^{n_1}} |A|^{n_1}$ *with equality if and only if* $A^* g_2 = \lambda^2 g_1$ *with* $\lambda^2 = \frac{1}{n_1}|A|^2$.

Proof. A proof can be found, for example, in [10, Corollary 2.5 and Lemma 2.1]. □

2.2 Preliminaries

In this section we review some standard material on calibrations and p-harmonic maps.

2.2.1 Calibrations

The classical theory of calibrated geometry was initiated by Harvey–Lawson [20]. A good reference for beginners is the text of Joyce [23]. Let (M^n, h) be a Riemannian manifold.

Definition 2.2.1. Let $\alpha \in \Omega^k$ on (M^n, h). We say that α is a *calibration* if

$$\alpha(v_1 \wedge \cdots \wedge v_k) \leqslant |v_1 \wedge \cdots \wedge v_k| \quad \text{for all } v_1, \ldots, v_k \in T_x M \text{ and all } x \in M. \tag{2.2.2}$$

This is clearly equivalent to saying that

$$-1 \leqslant \alpha(e_1, \ldots, e_k) \leqslant 1 \quad \text{for all } \textit{orthonormal } e_1, \ldots, e_k \in T_x M \text{ and all } x \in M.$$

Let L^k be an oriented submanifold of M. We say L is *calibrated* with respect to α if $\alpha|_L = \mathsf{vol}_L$, where vol_L is the Riemannian volume form associated to the orientation and the induced metric $h|_L$. (That is, L is α-calibrated if equality in (2.2.2) is attained on each oriented tangent space $T_x L$ of L.) ▲

The classical *fundamental theorem of calibrated geometry* of Harvey–Lawson [20] says that if the calibration form α is *closed*, then a calibrated submanifold is locally volume minimizing in its homology class. In particular, if $d\alpha = 0$, then a calibrated submanifold is *minimal* (has vanishing mean curvature).

We collect here some results and definitions on calibrations which are needed later.

Lemma 2.2.3 (The first cousin principle)**.** *Let* $\alpha \in \Omega^k$ *be a calibration, and let* $L_x \in \Lambda^k(T_x M)$ *be an oriented* k*-dimensional subspace which is calibrated with respect to* α*. If* $e_1, \ldots e_{k-1}$ *are orthonormal in* L_x *and* $w \in L_x^\perp$*, then* $\alpha(e_1, \ldots, e_{k-1}, w) = 0$.

Proof. We can choose $e_k \in L_x$ so that $e_1, \ldots, e_k$ is an oriented orthonormal basis of L_x. Let $w_t = (\cos t)e_k + (\sin t)w$. Then $e_1, \ldots, e_{k-1}, w_t$ are orthonormal for all $t \in \mathbb{R}$. Thus we have that

$$f(t) := \alpha(e_1, \ldots, e_{k-1}, w_t) = (\cos t)\alpha(e_1, \ldots, e_{k-1}, e_k) + (\sin t)\alpha(e_1, \ldots, e_{k-1}, w)$$

satisfies $f(t) \leqslant 1$ for all $t \in \mathbb{R}$ with equality at $t = 0$. Thus $f'(0) = \alpha(e_1, \ldots, e_{k-1}, w) = 0$. ☐

Proposition 2.2.4. *If $\alpha \in \Omega^k$ is calibration, then $\star\alpha \in \Omega^{n-k}$ is also a calibration.*

Proof. Using the metric we can identify $\Lambda^k(T_xM)$ with $\Lambda^k(T_x^*M)$. Let $\Pi_x = e_1 \wedge \cdots \wedge e_k$, where $e_1, \ldots, e_k$ are orthonormal. Then using the fact that $\star$ is an isometry, we have

$$(\star\alpha)(\Pi_x) = g(\star\alpha, \Pi_x) = g(\star^2\alpha, \star\Pi_x) = \pm g(\alpha, \star\Pi_x) = \pm\alpha(\star\Pi_x) \in [-1, 1],$$

because α is a calibration. ☐

Definition 2.2.5. Let $\alpha \in \Omega^k$. Define $P_\alpha \colon \Gamma(\Lambda^{k-1}(TM)) \to \Gamma(TM)$ by

$$g(P_\alpha(v_1 \wedge \cdots \wedge v_{k-1}), v_k) = \alpha(v_1 \wedge \cdots \wedge v_k).$$

That is, P_α is the vector-valued $(k-1)$-form obtained by "raising an index" on α using the metric. ▲

Remark 2.2.6. For some calibrations α, the vector-valued form P_α is a *vector cross product*. This means that $|P_\alpha(v_1 \wedge \cdots \wedge v_{k-1})|^2 = |v_1 \wedge \cdots \wedge v_{k-1}|^2$. This holds, in particular, for the Kähler calibration of degree 2, and for the associative and Cayley calibrations. See [10, Section 2] for more details. One of the key points of our Section 2.3 below is the observation that the results of [10] continue to hold for all calibrations, not just for those for which P_α is a vector cross product.

Proposition 2.2.7. *Let $\alpha \in \Omega^k$. The adjoint $P_\alpha^\top \colon \Gamma(TM) \to \Gamma(\Lambda^{k-1}(TM))$ is given by*

$$P_\alpha^\top(v) = (-1)^{k-1} v \lrcorner \alpha.$$

(There is a metric identification here of $\Lambda^{k-1}(TM)$ and $\Lambda^{k-1}(T^*M)$.)

Proof. Let $v_1, \ldots, v_k \in \Gamma(TM)$. We compute

$$\begin{aligned} g(P_\alpha(v_1 \wedge \cdots \wedge v_{k-1}), v_k) &= \alpha(v_1 \wedge \cdots \wedge v_k) \\ &= g(v_1 \wedge \cdots \wedge v_k, \alpha) \\ &= (-1)^{k-1} g(v_k \wedge v_1 \wedge \cdots \wedge v_{k-1}, \alpha) \\ &= (-1)^{k-1} g(v_1 \wedge \cdots \wedge v_{k-1}, v_k \lrcorner \alpha), \end{aligned}$$

hence the result follows. ☐

2.2.2 Harmonic maps and p-harmonic maps

We briefly review some basic facts about harmonic maps and p-harmonic maps. For more details, the reader can consult Eells–Lemaire [13] or Baird–Gudmundsson [2].

If $u\colon (M_1^{n_1}, g_1) \to (M_2^{n_2}, g_2)$ is a smooth map between Riemannian manifolds, then its differential du is a smooth section of $T^*M_1 \otimes u^*TM_2$, and its value at $x \in M_1$ is the linear map $du_x\colon T_xM_1 \to T_{u(x)}M_2$. The bundle $T^*M_1 \otimes u^*TM_2$ has a natural fibre metric $g_1^{-1} \otimes u^*g_2$ which allows us to define the smooth function $|du|^2$ on M_1. One can also verify that

$$|du|^2 = \mathrm{tr}_{g_1}(u^*g_2). \tag{2.2.8}$$

A useful observation is that if $e_1, \ldots, e_{n_1}$ is a local orthonormal frame for (M_1, g_1), then

$$|du_x|^2 = \sum_{i=1}^{n_1} (u^*g_2)_x(e_i, e_i) = \sum_{i=1}^{n_1} g_2(du_x(e_i), du_x(e_i)). \tag{2.2.9}$$

Definition 2.2.10. Let $u\colon (M_1, g_1) \to (M_2, g_2)$ be a smooth map. Let $p \in [2, \infty)$. If M_1 is compact, then the *p-energy* of u is defined to be

$$E_p(u) := \frac{1}{(\sqrt{p})^p} \int_{M_1} |du|^p \mathsf{vol}_{M_1}.$$

Note that up to a constant factor (which is chosen for later convenience), the p-energy is the p^{th} power of the L^p norm of du. We say that a map u is *p-harmonic* if it is a critical point of the functional E_p. That is, a p-harmonic map is a solution to the Euler–Lagrange equation for the p-energy functional. This equation is

$$\mathrm{div}(|du|^{p-2} du) = 0 \in \Gamma(u^*TM_2), \tag{2.2.11}$$

and is called the *p-harmonic map equation*. When $p = 2$, this reduces to the classical elliptic harmonic map equation $\mathrm{div}(du) = 0$, and a 2-harmonic map is just called a harmonic map. But for $p > 2$ this equation is a *degenerate* elliptic equation.

More generally, the section of u^*TM_2 given by

$$\tau_p(u) := \mathrm{div}(|du|^{p-2} du) \tag{2.2.12}$$

is called the *p-tension* of u, so a map u is p-harmonic if and only if it has vanishing p-tension. In fact, the p-tension $\tau_p(u)$ is, up to a positive factor, the negative gradient of the p-energy functional with respect to the L^2 inner product.

Note that if M_1 is not compact we can still take equation (2.2.11) as the definition of p-harmonic. ▲

The p-energy and p-harmonic map equation are related to conformal geometry as follows. Let f be a positive function on M_1, so $\tilde{g}_1 = f^2 g_1$ is another metric on M_1 in the same conformal class as g_1. Then we have

$$|du|^2_{\tilde{g}_1,g_2} = f^{-2}|du|^2_{g_1,g_2} \qquad \text{and} \qquad \mathsf{vol}_{M_1,\tilde{g}_1} = f^{n_1}\mathsf{vol}_{M_1,g_1}.$$

It follows that

$$E_{p,\tilde{g}_1,g_2}(u) = \frac{1}{(\sqrt{p})^p}\int_{M_1} |du|^p_{\tilde{g}_1,g_2}\mathsf{vol}_{M_1,\tilde{g}_1} = \frac{1}{(\sqrt{p})^p}\int_{M_1} f^{n_1-p}|du|^p_{g_1,g_2}\mathsf{vol}_{M_1,g_1},$$

and thus the p-energy of a map $u\colon (M_1^{n_1}, g_1) \to (M_2^{n_2}, g_2)$ is *conformally invariant* (that is, depends only on the conformal class of g_1) if $p = n_1$. With a bit more effort, one can similarly compute that

$$\tau_{p,\tilde{g}_1,g_2}(u) = f^{-p}\tau_{p,g_1,g_2}(u) + f^{-p-1}|du|^{p-2}(n_1 - p)g_1(df, du),$$

which again shows that the notion of a p-harmonic map depends only on the conformal class of g_1 if $p = n_1$.

The case that has received the most attention classically is the conformal invariance of the 2-energy (also called the Dirichlet energy) from a 2-dimensional oriented Riemannian manifold (Σ^2, g) into another Riemannian manifold (M, h). Since this depends only on the conformal class of g on Σ^2, we see that the notion of a harmonic map from a *Riemann surface* Σ^2 into a Riemannian manifold is well-defined.

See Remarks 2.3.7 and Remarks 2.4.19 for the precise formulation of "conformal invariance" for Smith immersions and Smith submersions.

2.3 Smith immersions

The notion of a *Smith immersion* was studied by Cheng–Karigiannis–Madnick in [10] and [11, Section 3.3] where it was assumed that the calibration form α is induced from a vector cross product. In this section we introduce a slightly modified definition of Smith immersions which applies to *any* calibration α, not just those induced by vector cross products. In the vector cross product case, our new definition is equivalent to the earlier definition. Moreover, our more general definition still enjoys all the analytic properties established in [10, Sections 4 and 5]. See Section 2.5.1.

In this section, $u\colon (L^k, g) \to (M^n, h)$ is a smooth map between Riemannian manifolds, with $k \leqslant n$. Recall that $u\colon (L^k, g) \to (M^n, h)$ is an *immersion* if $\operatorname{rank}(du_x) = k$ for all $x \in L$.

2.3.1 Smith immersions and the energy inequality

Before we can define Smith immersions, we recall some facts about (weakly) conformal maps.

Definition 2.3.1. A smooth map $u\colon (L^k, g) \to (M^n, h)$ is called *(weakly) conformal* if

$$u^*h = \lambda^2 g$$

for some smooth function $\lambda \geqslant 0$ which is continuous (and smooth away from 0) on L. This function λ is called the dilation. It then follows from (2.2.8) that necessarily $\lambda^2 = \frac{1}{k}|du|^2$.

Let $L^0 \subseteq L$ be the open set where $|du| \neq 0$. From $u^*h = \frac{1}{k}|du|^2 g$, we deduce that $u|_{L^0}\colon L^0 \to M$ is an immersion. When $L^0 = L$, we say that u is a *conformal immersion*. An immersion $u\colon (L^k, g) \to (M^n, h)$ is called a *Riemannian immersion* if $u^*h = g$ on L, or equivalently if it is a conformal immersion with dilation $\lambda = 1$. ▲

Theorem 2.3.2. *Let $u\colon (L^k, g) \to (M^n, h)$ be a smooth map. Let $\alpha \in \Omega^k(M)$ be a calibration. Then*

$$u^*\alpha \leqslant \lambda^k \mathsf{vol}_L, \quad \textit{where } \lambda = \tfrac{1}{\sqrt{k}}|du|. \tag{2.3.3}$$

Moreover, equality holds if and only if:

- *$u^*h = \lambda^2 g$ (so u is a weakly conformal immersion), and*
- *the image $u(L^0)$ is calibrated with respect to α.*

Proof. We trivially have equality at points where du is zero. Let $x \in L^0$. Let $e_1, \ldots, e_k$ be an orthonormal frame for T_xL. Then we have

$$\begin{aligned}
(u^*\alpha)(e_1 \wedge \cdots \wedge e_k) &= \alpha((\Lambda^k du)(e_1 \wedge \cdots \wedge e_k)) \\
&\leqslant |(\Lambda^k du)(e_1 \wedge \cdots \wedge e_k)| && \text{(because } \alpha \text{ is a calibration)} \\
&= |\Lambda^k du|\,|e_1 \wedge \cdots \wedge e_k| \\
&\leqslant \lambda^k && \text{(by Proposition 2.1.1)},
\end{aligned}$$

which concludes the proof of (2.3.3).

Equality holds if and only if equality holds in the two inequalities of the above computation. If the second inequality above is an equality, then by Proposition 2.1.1 we have $u^*h = \lambda^2 g$, so u is weakly conformal. Let $x \in L^0$ and let $e_1, \ldots, e_k$ be an orthonormal frame for T_xL, so $\frac{1}{\lambda}du(e_1), \ldots, \frac{1}{\lambda}du(e_k)$ is an orthonormal frame for $du(T_xL) \subseteq T_{u(x)}M$. If the first inequality above is an equality, then we see that we must have $\alpha(\frac{1}{\lambda}du(e_1) \wedge \cdots \wedge \frac{1}{\lambda}du(e_k)) = 1$. That is, the image $u(L^0)$ is calibrated with respect to α. □

Definition 2.3.4. If equality holds in (2.3.3), we say that u is a **Smith immersion** with respect to α. That is, a Smith immersion with respect to α is a smooth map $u\colon (L^k, g) \to (M^n, h)$ such that

$$u^*\alpha = \frac{1}{(\sqrt{k})^k}|du|^k \mathsf{vol}_L, \qquad u^*h = \frac{1}{k}|du|^2 g, \tag{2.3.5}$$

at all points on L. [However, recall that the first equation automatically implies the second equation.] Note that, strictly speaking, a Smith immersion is only actually an immersion on the open subset $L^0 = \{x \in L : du_x \neq 0\}$ of L. ▲

Theorem 2.3.6 (Energy Inequality)**.** *Let $\alpha \in \Omega^k(M)$ be a* closed *calibration. Let $u\colon (L^k, g) \to (M^n, h)$ be a Smith immersion with respect to α. Suppose L is compact. Then u is k-harmonic in the sense that it is a critical point of E_k.*

Proof. For any smooth map $u\colon (L^k, g) \to (M^n, h)$, let $\lambda = \frac{1}{\sqrt{k}}|du|$. Using (2.3.3) we have

$$E_k(u) = \frac{1}{(\sqrt{k})^k}\int_L |du|^k \mathsf{vol}_L = \int_L \lambda^k \mathsf{vol}_L \geqslant \int_L u^*\alpha = [\alpha] \cdot u_*[L],$$

where we have used the fact that α is closed. Thus the k-energy of u is bounded from below by a topological quantity, as it depends only on the cohomology class $[\alpha]$ and the homotopy class of u. Moreover, by Theorem 2.3.2, equality holds if and only if u is a Smith immersion. This shows that such maps are local minimizers of E_k and thus are k-harmonic. □

We note that Theorem 2.3.6 still holds if L is noncompact. See Theorem 2.3.15.

Remark 2.3.7. Since a Smith immersion $u\colon (L^k, g) \to (M^n, h)$ with respect to $\alpha \in \Omega^k(M)$ is in particular a k-harmonic map (when $d\alpha = 0$), by the discussion at the end of Section 2.2.2, we expect that the notion of a Smith immersion should depend only on the conformal class $[g]$ of the metric on the domain L. Indeed, this is true even without the assumption that $d\alpha = 0$. To see this, suppose $\tilde{g} = f^2 g$ for some smooth positive function on L. From (2.2.8) we get

$$\widetilde{\lambda}^2 = \frac{1}{k}|du|^2_{\tilde{g},h} = f^{-2}\frac{1}{k}|du|^2_{g,h} = f^{-2}\lambda^2,$$

and clearly $\widetilde{\mathsf{vol}}_L = f^k \mathsf{vol}_L$. It follows that the Smith immersion equations $u^*\alpha = \lambda^k \mathsf{vol}_L$ and $u^*h = \lambda^2 g$ are *invariant under conformal scaling* of the domain metric g on L.

2.3.2 Direct proof that Smith immersions are k-harmonic

In Theorem 2.3.15 below we show directly that a Smith immersion satisfies the k-harmonic map equation, in the sense that $\tau_k(u) = 0$, without assuming L is compact. This argument appeared earlier in [10, Section 3.5] under the assumption that α induces a vector cross product P_α by raising an index. We provide a slightly modified argument here to show that this assumption was in fact unnecessary. First we need some preliminary results.

Proposition 2.3.8. *Let $u\colon (L^k, g) \to (M^n, h)$ be a Smith immersion with respect to the calibration form $\alpha \in \Omega^k$ on M. Then we have*

$$P_\alpha \circ \Lambda^{k-1}(du) \circ \star_L = \frac{(-1)^{k-1}}{(\sqrt{k})^{k-2}} |du|^{k-2} du. \tag{2.3.9}$$

Proof. The equation is trivially satisfied at points where du is zero. Let $x \in L^0$. Also, recall that we necessarily have $u^*h = \lambda^2 g$. Let $e_1, \ldots, e_k$ be an oriented orthonormal basis for T_xL. We compute

$$\begin{aligned}
h(P_\alpha(\Lambda^{k-1}du)(e_1 \wedge \cdots \wedge e_{k-1}), du(e_k)) &= \alpha((\Lambda^{k-1}du)(e_1 \wedge \cdots \wedge e_{k-1}), du(e_k)) \\
&= u^*\alpha(e_1 \wedge \cdots \wedge e_k) \\
&= \lambda^k \mathsf{vol}_L(e_1 \wedge \cdots \wedge e_k) \\
&= \lambda^k g(\star(e_1 \wedge \cdots \wedge e_{k-1}), e_k) \\
&= \lambda^{k-2} u^*h(\star(e_1 \wedge \cdots \wedge e_{k-1}), e_k) \\
&= \lambda^{k-2} h(du(\star(e_1 \wedge \cdots \wedge e_{k-1})), du(e_k)).
\end{aligned}$$

Denoting $A := P_\alpha(\Lambda^{k-1}du)\colon \Lambda^k(T_xL) \to T_{u(x)}M$, the above says

$$h(A(e_1 \wedge \cdots \wedge e_{k-1}), du(e_k)) = \lambda^{k-2} h(du(\star(e_1 \wedge \cdots \wedge e_{k-1})), du(e_k)). \tag{2.3.10}$$

Recall that $du(T_xL)$ is α-calibrated by Theorem 2.3.2. Suppose $w \in (\operatorname{im} du_x)^\perp$. Then we have

$$\begin{aligned}
h(A(e_1 \wedge \cdots \wedge e_{k-1}), w) &= h(P_\alpha(du(e_1) \wedge \cdots \wedge du(e_{k-1}), w) \\
&= \alpha(du(e_1), \ldots, du(e_{k-1}), w) = 0
\end{aligned}$$

by Lemma 2.2.3. Hence we have shown that $\operatorname{im} A \subseteq \operatorname{im} du_x$. It therefore follows from (2.3.10) and the fact that du_x is injective that

$$P_\alpha \circ (\Lambda^{k-1}du) = \lambda^{k-2} du \circ \star_L.$$

Using that $\star^2 = (-1)^{k-1}$ on 1-forms, we obtain the desired result. □

In the case where P_α is a *vector cross product*, it was shown in [10, Proposition 2.32] that (2.3.9) is *equivalent* to our Smith immersion equation (2.3.5). In fact this holds in general.

Proposition 2.3.11. *We have shown that if* $u\colon (L^k, g) \to (M^n, h, \alpha)$ *is a Smith immersion, then*

$$P_\alpha \circ \Lambda^{k-1}(du) \circ \star_L = (-1)^{k-1} \frac{|du|^{k-2}}{\sqrt{k}^{k-2}} du. \tag{2.3.12}$$

The converse also holds. That is, if (2.3.12) *holds, then* u *is a Smith immersion.*

Proof. Let $x \in L$. If $du_x = 0$, which satisfies (2.3.12) at x, then u is a Smith immersion at x. Now assume $du_x \neq 0$. Let $e_1, \ldots, e_k$ be a oriented orthonormal basis of T_xL. Let $i, j \in \{1, \ldots, k\}$. Then we have

$$\star_L e_i = (-1)^{i-1} e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_k.$$

Evaluating both sides of (2.3.12) on e_i and taking inner product with $du(e_j)$ we get

$$\begin{aligned}(-1)^{k-1}\lambda^{k-2} h(du(e_i), du(e_j)) &= (-1)^{i-1} h(P_\alpha(du(e_1) \wedge \cdots \wedge \widehat{du(e_i)} \wedge \cdots \wedge du(e_k)), du(e_j)) \\ &= (-1)^{i-1} u^*\alpha(e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_k \wedge e_j) \\ &= (-1)^{k-1} u^*\alpha(e_1 \wedge \cdots \wedge e_j \wedge \cdots \wedge e_k).\end{aligned}$$

We deduce that

$$\begin{cases} \lambda^{k-2} h(du(e_i), du(e_j)) \mathsf{vol}_L = u^*\alpha & \text{if } i = j, \\ h(du(e_i), du(e_j)) = 0 & \text{if } i \neq j. \end{cases}$$

Using the above we compute

$$\begin{aligned} u^*\alpha &= \frac{1}{k}\lambda^{k-2} \sum_i h(du(e_i), du(e_i)) \mathsf{vol}_L \\ &= \frac{1}{k}\lambda^{k-2} \sum_{i,j} h(du(e_i), du(e_j)) \mathsf{vol}_L \\ &= \frac{1}{k}\lambda^{k-2} |du|^2 \mathsf{vol}_L = \lambda^k \mathsf{vol}_L, \end{aligned}$$

and thus u is a Smith immersion in the sense of Definition 2.3.4. ☐

Lemma 2.3.13. *Let* $u\colon (L^k, g) \to (M^n, h)$ *be a Smith immersion with respect to the calibration* α. *Then* $u^*(\nabla_V \alpha) = 0$ *for any* $V \in \Gamma(TM)$.

Proof. The equation is trivially satisfied at points where du is zero. Let $x \in L^0$. Let $e_1, \ldots, e_k$ be an oriented orthonormal basis for $T_x L$. Then from the proof of Theorem 2.3.2, we have that $\frac{1}{\lambda} du(e_1), \ldots, \frac{1}{\lambda} du(e_k)$ is an oriented orthonormal basis for $du(T_x L) \subseteq T_{u(x)} M$, which is calibrated by α. Thus we have

$$\begin{aligned} \tfrac{1}{\lambda^k} u^*(\nabla_V \alpha)(e_1 \wedge \cdots \wedge e_k) &= u^*(\nabla_V \alpha)(\tfrac{1}{\lambda} e_1 \wedge \cdots \wedge \tfrac{1}{\lambda} e_k) \\ &= V\big(\alpha(\tfrac{1}{\lambda} du(e_1) \wedge \cdots \wedge \tfrac{1}{\lambda} du(e_k))\big) \\ &\quad - \sum_{j=1} \alpha(\tfrac{1}{\lambda} du(e_1) \wedge \cdots \wedge \nabla_V(\tfrac{1}{\lambda} du(e_j)) \wedge \cdots \wedge \tfrac{1}{\lambda} du(e_k)). \end{aligned} \tag{2.3.14}$$

The first term in (2.3.14) vanishes because α calibrates $du(T_x L)$. By skew-symmetry of α, the only component of $\nabla_V(\frac{1}{\lambda} du(e_j))$ in the span of $\frac{1}{\lambda} du(e_1), \ldots, \frac{1}{\lambda} du(e_k)$ which can contribute to

$$\alpha(\tfrac{1}{\lambda} du(e_1) \wedge \cdots \wedge \nabla_V(\tfrac{1}{\lambda} du(e_j)) \wedge \cdots \wedge \tfrac{1}{\lambda} du(e_k))$$

is the $\frac{1}{\lambda} du(e_j)$ component. But since $\frac{1}{\lambda} du(e_j)$ has constant (unit) length, the covariant derivative $\nabla_V(\frac{1}{\lambda} du(e_j))$ is orthogonal to $\frac{1}{\lambda} du(e_j)$. We deduce that

$$\alpha(\tfrac{1}{\lambda} du(e_1) \wedge \cdots \wedge \nabla_V(\tfrac{1}{\lambda} du(e_j)) \wedge \cdots \wedge \tfrac{1}{\lambda} du(e_k)) = \alpha(\tfrac{1}{\lambda} du(e_1) \wedge \cdots \wedge w \wedge \cdots \wedge \tfrac{1}{\lambda} du(e_k))$$

for some vector w orthogonal to the α-calibrated k-plane spanned by $\frac{1}{\lambda} du(e_1), \ldots, \frac{1}{\lambda} du(e_k)$. It then follows from Lemma 2.2.3 that each of the terms in the last line of (2.3.14) also vanish, so $u^*(\nabla_V \alpha) = 0$. □

The next result is exactly [10, Proposition 3.20], but with a harmless sign error corrected. We include it for completeness and comparison with Theorem 2.4.29 in the case of Smith submersions.

Theorem 2.3.15. *Let $u \colon (L^k, g) \to (M^n, h)$ be a Smith immersion with respect to the calibration form $\alpha \in \Omega^k$. If $d\alpha = 0$, then u is k-harmonic in the sense that $\tau_k(u) = 0$.*

Proof. We show that the k-tension $\tau_k(u)$ of equation (2.2.12) vanishes at any point $x \in L$. Let

$$B = P_\alpha \circ \Lambda^{k-1}(du) \circ \star_L \in \Gamma(T^* L \otimes u^* TM).$$

By Proposition 2.3.8, it suffices to show that $\operatorname{div}(B) = 0$, which is a smooth section of $u^* TM$. Let μ denote the Riemannian volume form on L, and identify 1-forms and vector fields using the musical isomorphism. Recall that $\star v = v \lrcorner \mu$ for any vector field v on L, so $(\star v)_{i_1 \cdots i_{k-1}} = v_j \mu_{j i_1 \cdots i_{k-1}}$. We also have $(P_\alpha)_{b_1 \cdots b_{k-1} a} = \alpha_{b_1 \cdots b_{k-1} a}$.

Take Riemannian normal coordinates $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a}$ centred at x and $u(x)$ respectively. At the point x, we compute

$$\begin{aligned}
\operatorname{div}(B)_a &= (\nabla_j B)_{aj} \\
&= \nabla_j (P_\alpha \circ \Lambda^{k-1}(du) \circ \star_L)_{aj} \\
&= \frac{1}{(k-1)!} \nabla_j (P_\alpha \circ \Lambda^{k-1}(du))_{i_1 \cdots i_{k-1} a} \mu_{j i_1 \cdots i_{k-1}} \\
&= \frac{1}{(k-1)!} \nabla_j \left(\frac{\partial u^{b_1}}{\partial x^{i_1}} \cdots \frac{\partial u^{b_{k-1}}}{\partial x^{i_{k-1}}} (P_\alpha)_{b_1 \cdots b_{k-1} a} \right) \mu_{j i_1 \cdots i_{k-1}} \\
&= \frac{1}{(k-1)!} \nabla_j \left(\frac{\partial u^{b_1}}{\partial x^{i_1}} \cdots \frac{\partial u^{b_{k-1}}}{\partial x^{i_{k-1}}} \alpha_{b_1 \cdots b_{k-1} a} \right) \mu_{j i_1 \cdots i_{k-1}} \\
&= \frac{1}{(k-1)!} \frac{\partial u^{b_1}}{\partial x^{i_1}} \cdots \frac{\partial u^{b_{k-1}}}{\partial x^{i_{k-1}}} (\nabla_j \alpha_{b_1 \cdots b_{k-1} a}) \mu_{j i_1 \cdots i_{k-1}} \\
&\quad + \frac{1}{(k-1)!} \sum_{\ell=1}^{k-1} \frac{\partial^2 u^{b_\ell}}{\partial x^j \partial x^{i_\ell}} \frac{\partial u^{b_1}}{\partial x^{i_1}} \cdots \widehat{\frac{\partial u^{b_\ell}}{\partial x^{i_\ell}}} \cdots \frac{\partial u^{b_{k-1}}}{\partial x^{i_{k-1}}} \alpha_{b_1 \cdots b_{k-1} a} \mu_{j i_1 \cdots i_{k-1}},
\end{aligned}$$

where the $\widehat{}$ as usual denotes omission. The second term vanishes by (skew)-symmetry in j, i_ℓ. For the first term, we have

$$\nabla_j \alpha = \nabla_{\frac{\partial}{\partial x^j}} \alpha = \frac{\partial u^{b_k}}{\partial x^j} \nabla_{\frac{\partial}{\partial y^{b_k}}} \alpha,$$

which we write as $\frac{\partial u^{b_k}}{\partial x^j} \nabla_{b_k} \alpha$. Thus we have

$$\operatorname{div}(B)_a = \frac{1}{(k-1)!} \frac{\partial u^{b_1}}{\partial x^{i_1}} \cdots \frac{\partial u^{b_{k-1}}}{\partial x^{i_{k-1}}} \frac{\partial u^{b_k}}{\partial x^j} (\nabla_{b_k} \alpha_{b_1 \cdots b_{k-1} a}) \mu_{j i_1 \cdots i_{k-1}}.$$

Relabelling j as i_k, we have

$$\operatorname{div}(B)_a = \frac{(-1)^{k-1}}{(k-1)!} \frac{\partial u^{b_1}}{\partial x^{i_1}} \cdots \frac{\partial u^{b_k}}{\partial x^{i_k}} (\nabla_{b_k} \alpha_{b_1 \cdots b_{k-1} a}) \mu_{i_1 \cdots i_k}.$$

By the skew-symmetry of μ, if we swap b_ℓ and b_m in the factor $(\nabla_{b_k} \alpha_{b_1 \cdots b_{k-1} a})$ above, the sign of the right hand side changes. We therefore can write

$$\operatorname{div}(B)_a = \frac{(-1)^{k-1}}{(k-1)!} \frac{\partial u^{b_1}}{\partial x^{i_1}} \cdots \frac{\partial u^{b_k}}{\partial x^{i_k}} \frac{1}{k} \sum_{\ell=1}^{k} (\nabla_{b_\ell} \alpha_{b_1 \cdots b_{\ell-1} a b_{\ell+1} \cdots b_k}) \mu_{i_1 \cdots i_k}$$

because for each ℓ when we swap a with b_k and then b_k with b_ℓ we introduce two minus signs which cancel. Finally we use the fact that α is closed to write

$$0 = (d\alpha)_{a b_1 \cdots b_k} = \nabla_a \alpha_{b_1 \cdots b_k} - \sum_{\ell=1}^{k} (\nabla_{b_\ell} \alpha_{b_1 \cdots b_{\ell-1} a b_{\ell+1} \cdots b_k}).$$

Combining these we obtain

$$\begin{aligned}\operatorname{div}(B)_a &= \frac{(-1)^{k-1}}{k!}\frac{\partial u^{b_1}}{\partial x^{i_1}}\cdots\frac{\partial u^{b_k}}{\partial x^{i_k}}\nabla_a\alpha_{b_1\cdots b_k}\mu_{i_1\cdots i_k}\\ &= (u^*\nabla_a\alpha)_{i_1\cdots i_k}\mu_{i_1\cdots i_k},\end{aligned}$$

which vanishes by Lemma 2.3.13, completing the proof. □

2.4 Smith submersions

We introduce a new class of maps $u\colon (M^n, h) \to (L^k, g)$ between Riemannian manifolds with $n \geqslant k$, where the *domain* is equipped with a calibration form α of degree $n-k$. These maps are a special class of k-harmonic maps satisfying a first order nonlinear differential equation, and have the property that when $d\alpha = 0$, the *smooth fibres* are α-calibrated submanifolds of M.

In this section, $u\colon (M^n, h) \to (L^k, g)$ is a *surjective* smooth map between Riemannian manifolds, with $n \geqslant k$. Recall that $u\colon (M^n, h) \to (L^k, g)$ is a *submersion* if $\operatorname{rank}(du_x) = k$ for all $x \in M$.

2.4.1 (Weakly) conformally horizontal submersions

In order to be able to define the submersion analogue of "weakly conformal", we need to first recall the horizontal/vertical splitting of TM associated to a submersion $u\colon M \to L$.

Definition 2.4.1. Let $u\colon (M^n, h) \to (L^k, g)$ be a smooth surjection. Let $M^0 \subseteq M$ be the open set where $|du| \neq 0$. Suppose that the restriction $u|_{M^0}\colon M^0 \to L$ is a submersion, so that $\operatorname{rank}(du_x) = k$ for all $x \in M^0$. Then the tangent bundle TM^0 of M^0 decomposes as

$$TM^0 = (\ker du) \oplus_\perp (\ker du)^\perp,$$

where $\ker du = VM^0$ is the *vertical* subbundle, which has rank $n-k$, and $(\ker du)^\perp = HM^0$ is the *horizontal* subbundle, which has rank k.

It follows that an m-tensor $\alpha \in \mathcal{T}^m$ on M^0 is a smooth section of

$$\bigoplus_{p+q=m} (\ker du)^{\otimes p} \otimes ((\ker du)^\perp)^{\otimes q},$$

with $p \leqslant n-k, q \leqslant k$. We denote by $\alpha^{(p,q)}$ the component of α which lies in

$$\mathcal{T}^{(p,q)} := \Gamma((\ker du)^{\otimes p} \otimes ((\ker du)^\perp)^{\otimes q})$$

and we say that $\alpha^{(p,q)}$ is of *type* (p, q).

It follows that the metric h on M^0 decomposes as $h = h^{2,0} + h^{0,2}$, where $h^{2,0}$ is the metric on the vertical subbundle $\ker du$, and $h^{0,2}$ is the metric on the horizontal subbundle $(\ker du)^{\perp}$. In particular, we have

$$\mathrm{tr}_h(h^{0,2}) = k. \tag{2.4.2}$$

Finally, we use $\Omega^{(p,q)}$ to denote the totally skew-symmetric elements of $\mathcal{T}^{(p,q)}$. ▲

Definition 2.4.3. A smooth surjection $u\colon (M^n, h) \to (L^k, g)$ is called *(weakly) horizontally conformal* if for every point $x \in M$, we either have $du_x = 0$, or if $du_x \neq 0$, then $\mathrm{rank}(du_x) = k$ is maximal and

$$u^* g = \lambda^2 h^{(0,2)}$$

for some smooth function $\lambda > 0$ on M^0. We can extend λ^2 by zero to obtain a continuous non-negative function on M. This function λ is called the dilation. It then follows from (2.2.8) that necessarily $\lambda^2 = \frac{1}{k}|du|^2$.

When $M^0 = M$, we say that u is a *horizontally conformal submersion*. A submersion $u\colon (M^n, h) \to (L^k, g)$ is called a *Riemannian submersion* if $u^* g = h^{(0,2)}$ on M, or equivalently if it is a horizontally conformal submersion with dilation $\lambda = 1$. ▲

Remark 2.4.4. Let $u\colon (M^n, h) \to (L^k, g)$ be weakly horizontally conformal. Restricted to the open subset M^0, the map $u|_{M^0}$ is a submersion, and thus by the implicit function theorem each fibre $M^0 \cap u^{-1}\{u(x)\}$ for $x \in M^0$ is a smooth $(n-k)$-dimensional submanifold of M^0.

Remark 2.4.5. Let $u\colon (M^n, h) \to (L^k, g)$ be a smooth surjection. Over M^0 we get a canonical orientation on the horizontal subbundle $(\ker du)^{\perp}$ from the class $[u^*\mathsf{vol}_L]$. Then the vertical subbundle $\ker du$ inherits a unique orientation such that $\mathsf{vol}_{\ker du} \wedge \mathsf{vol}_{(\ker du)^{\perp}} = \mathsf{vol}_M$.

If u is (weakly) horizontally conformal, then by Definition 2.4.3, we have that for any $x \in M^0$, the map

$$(du)_x\colon \big((\ker du_x)^{\perp}, \lambda^2(x) h_x^{(0,2)}\big) \cong \big(T_{u(x)}L, g_{u(x)}\big)$$

is an orientation preserving isometry.

For the remainder of this section, we assume that $u\colon (M^n, h) \to (L^k, g)$ is horizontally conformal. (Equivalently, it is weakly horizontally conformal and we work only on the open subset M^0 where it is horizontally conformal.) We collect several results that are needed to study Smith submersions.

Lemma 2.4.6. *Let $\beta \in \Omega^p(L)$. Then $u^*\beta$ is of type $(0,p)$.*

Proof. Let $v_1, \ldots, v_p \in \Gamma(TM)$. Then $(u^*\beta)(v_1, \ldots, v_p) = \beta(du(v_1), \ldots, du(v_p))$, so if at least one of the v_i lies in $\ker du$, then $(u^*\beta)(v_1, \ldots, v_p) = 0$. □

Lemma 2.4.7. *Let $\alpha \in \Omega^{(p,q)}(M)$. Then $\star\alpha \in \Omega^{(n-k-p,k-q)}(M)$. Moreover, for any form β, we have $(\star\beta)^{(n-k-p,k-q)} = \star(\beta^{(p,q)})$.*

Proof. This follows from the fact that $\mathsf{vol}_M \in \Omega^{(n-k,k)}(M)$. □

Lemma 2.4.8. *Let $\alpha \in \Omega^{(p,q)}(M)$. Then for any $v \in \Gamma(TM)$, the form $v^{(1,0)} \lrcorner \alpha$ is of type $(p-1,q)$ and the form $v^{(0,1)} \lrcorner \alpha$ is of type $(p,q-1)$.*

Proof. This is clear from definition of the interior product. □

Lemma 2.4.9. *Let $\alpha \in \Omega^{(0,k)}(M)$. Let P_α be as in Definition 2.2.5, and let $P_\alpha^\top$ be its adjoint map as in Proposition 2.2.7. Then we have*

$$P_\alpha P_\alpha^\top = |\alpha|^2 \pi^{(0,1)},$$

where $\pi^{(0,1)} \colon \Gamma(TM) \to \Gamma(TM^{(0,1)})$ is the orthogonal projection.

Proof. First, note that since α is of type $(0,k)$, and the metric h on TM is of type $(2,0)+(0,2)$, the map P_α takes values in the horizontal subbundle $TM^{(0,1)} = (\ker du)^\perp$. Consider any $v \in \Gamma(TM)$ and $w \in \Gamma(TM^{(0,1)})$. By Proposition 2.2.7 we have $P_\alpha^\top v = (-1)^{k-1} v \lrcorner \alpha$. Hence we have

$$\begin{aligned} g(P_\alpha P_\alpha^\top v, w) &= (-1)^{k-1} g(P(v \lrcorner \alpha), w) \\ &= (-1)^{k-1} \alpha((v \lrcorner \alpha) \wedge w) \\ &= g(\alpha, w \wedge (v \lrcorner \alpha)). \end{aligned}$$

Recall that $v \lrcorner (w \wedge \alpha) = (v \lrcorner w)\alpha - w \wedge (v \lrcorner \alpha)$, and thus $w \wedge (v \lrcorner \alpha) = g(v,w)\alpha$ because $w \wedge \alpha = 0$ since it is of type $(0,k+1)$. Hence, we get

$$g(P_\alpha P_\alpha^\top v, w) = g(v,w)|\alpha|^2,$$

and the result follows. □

2.4.2 Smith submersions and the energy inequality

We can now consider the notion of a Smith submersion.

Theorem 2.4.10. *Let $u \colon (M^n, h) \to (L^k, g)$ be a smooth surjection. Let $\alpha \in \Omega^{n-k}(M)$ be a calibration. Then*

$$\alpha \wedge u^* \mathsf{vol}_L \leqslant \lambda^k \mathsf{vol}_M, \quad \text{where } \lambda = \frac{|du|}{\sqrt{k}}. \tag{2.4.11}$$

Moreover, equality holds if and only if:

- $u^*g = \lambda^2 h^{(0,2)}$ *(so u is a weakly horizontally conformal submersion) and,*
- *the fibres of the restriction of u to M^0 are calibrated with respect to α.*

Proof. We trivially have equality at points where du is zero. Let $x \in M^0$. If du_x is not maximal rank, then $u^*\mathsf{vol}_L$ vanishes, while $\lambda > 0$, so the inequality (2.4.11) is satisfied and indeed is always a *strict* inequality at such points.

Now consider $x \in M^0$ such that du_x has maximal rank k. Let $e_1, \dots, e_k$ be an oriented orthonormal basis of $(\ker du_x)^\perp$ and $\tilde{e}_1, \dots, \tilde{e}_{n-k}$ be an oriented orthonormal basis of $\ker du_x$. With our choice of orientations from Remark 2.4.5 we have $\mathsf{vol}_M = \tilde{e}_1 \wedge \cdots \wedge \tilde{e}_{n-k} \wedge e_1 \wedge \cdots \wedge e_k$. Then we have

$$
\begin{aligned}
&(\alpha \wedge u^*\mathsf{vol}_L)(\tilde{e}_1 \wedge \cdots \wedge \tilde{e}_{n-k} \wedge e_1 \wedge \cdots \wedge e_k) \\
&\qquad = \alpha(\tilde{e}_1 \wedge \cdots \wedge \tilde{e}_{n-k}) u^*\mathsf{vol}_L(e_1 \wedge \cdots \wedge e_k) && \text{(by Lemma 2.4.6)} \\
&\qquad \leqslant 1 \cdot \mathsf{vol}_L((\Lambda^k du)(e_1 \wedge \cdots \wedge e_k)) && \text{(because } \alpha \text{ is a calibration)} \\
&\qquad = |(\Lambda^k du)(e_1 \wedge \cdots \wedge e_k)| \\
&\qquad = |\Lambda^k du|\, |(e_1 \wedge \cdots \wedge e_k)| \\
&\qquad \leqslant \lambda^k && \text{(by Proposition 2.1.1),}
\end{aligned}
$$

which concludes the proof of (2.4.11).

Equality holds if and only if equality holds in the two inequalities of the above computation. If the second inequality above is an equality, then by Proposition 2.1.1 we have $u^*g = \lambda^2 h^{(0,2)}$, so u is weakly horizontally conformal. Let $x \in M^0$ and let $\tilde{e}_1, \dots, \tilde{e}_{n-k}$ be an orthonormal frame for $\ker du_x$. If the first inequality above is an equality, then we see that we must have $\alpha(\tilde{e}_1 \wedge \cdots \wedge \tilde{e}_{n-k}) = 1$. That is, the smooth fibre $M^0 \cap u^{-1}\{u(x)\}$ is calibrated with respect to α. □

Definition 2.4.12. If equality holds in (2.4.11), we say that u is a **Smith submersion** with respect to α. That is, a Smith submersion with respect to α is a smooth map $u \colon (M^n, h) \to (L^k, g)$ such that

$$
\alpha \wedge u^*\mathsf{vol}_L = \frac{1}{(\sqrt{k})^k} |du|^k \mathsf{vol}_M, \qquad u^*g = \frac{1}{k}|du|^2 h^{(0,2)}, \tag{2.4.13}
$$

at all points on M. [However, recall that the first equation automatically implies the second equation.] Note that, strictly speaking, a Smith submersion is only actually a submersion on the open subset $M^0 = \{x \in M : du_x \neq 0\}$ of M. ▲

Before we prove the Smith submersion energy inequality in Theorem 2.4.18 below, which is analogous to Theorem 2.3.6 for Smith immersions, we first show that in the Smith submersion case we can rewrite the equation in a useful alternative form.

Lemma 2.4.14. *Let $u\colon (M^n, h) \to (L^k, g)$ be weakly horizontally conformal with dilation λ. Let $\alpha \in \Omega^{n-k}(M)$ be a calibration, so $\star\alpha \in \Omega^k(M)$ is also a calibration by Proposition 2.2.4. At at point x where $du_x \neq 0$, the following are equivalent:*

(i) $u^*\mathsf{vol}_L = \lambda^k(\star\alpha)^{(0,k)}$,

(ii) $(\ker du)^\perp$ *is calibrated with respect to* $\star\alpha$,

(iii) $\ker du$ *is calibrated with respect to* α.

Proof. (i) $\iff$ *(ii).* Let $e_1, \dots, e_k$ be an oriented orthonormal basis of $(\ker du_x)^\perp$. Then since

$$(\star\alpha)(e_1, \dots, e_k) = (\star\alpha)^{(0,k)}(e_1, \dots, e_k) \quad \text{and} \quad (u^*\mathsf{vol}_L)(e_1, \dots, e_k) = \lambda^k,$$

we have that $u^*\mathsf{vol}_L = \lambda^k(\star\alpha)^{(0,k)}$ if and only if $(\star\alpha)(e_1, \dots, e_k) = 1$ if and only if $(\ker du)^\perp$ is calibrated with respect to $\star\alpha$.

(ii) $\iff$ *(iii).* Let $\tilde{e}_1, \dots, \tilde{e}_{n-k}$ be an oriented orthonormal basis of $\ker du_x$. Note that

$$\mathsf{vol}_M = \tilde{e}_1 \wedge \cdots \wedge \tilde{e}_{n-k} \wedge e_1 \wedge \cdots \wedge e_k.$$

Thus we have

$$\begin{aligned}\alpha(\tilde{e}_1, \dots, \tilde{e}_{n-k}) &= h(\alpha, \tilde{e}_1 \wedge \cdots \wedge \tilde{e}_{n-k}) \\ &= h(\star\alpha, \star(\tilde{e}_1 \wedge \cdots \wedge \tilde{e}_{n-k})) = h(\star\alpha, e_1 \wedge \cdots \wedge e_k) = (\star\alpha)(e_1, \dots, e_k),\end{aligned}$$

and the result follows. □

Corollary 2.4.15. *Let $u\colon (M^n, h) \to (L^k, g)$ be a smooth surjection. Let $\lambda = \frac{|du|}{\sqrt{k}}$ and $\alpha \in \Omega^{n-k}(M)$ be a calibration. Then the following are equivalent:*

(i) $u^*\mathsf{vol}_L = \lambda^k(\star\alpha)^{(0,k)}$ *and* $u^*g = \lambda^2 h^{(0,2)}$,

(ii) $\alpha \wedge u^*\mathsf{vol}_L = \lambda^k \mathsf{vol}_M$.

Proof. Both equations are trivially satisfied at the points where du is zero. Let $x \in M^0$.

Suppose that *(i)* holds. By Lemma 2.4.14, we have that $(\ker du)^\perp$ is calibrated with respect to $\star\alpha$. Combining this with $u^*g = \lambda^2 h^{(0,2)}$ and using Theorem 2.4.10, we obtain $\alpha \wedge u^*\mathsf{vol}_L = \lambda^k \mathsf{vol}_M$.

Conversely, suppose *(ii)* holds. From Theorem 2.4.10 we know that u is horizontally conformal and α calibrates $\ker du$. Hence by Lemma 2.4.14 we also have $u^*\mathsf{vol}_L = \lambda^k(\star\alpha)^{(0,k)}$, so *(i)* holds. □

Remark 2.4.16. Corollary 2.4.15 establishes two equivalent formulations of Smith submersion. The original definition of Smith submersion in (2.4.13) is precisely *(ii)* of Corollary 2.4.15, since the first equation in (2.4.13) implies the second. However, in the alternative formulation *(i)* of Corollary 2.4.15, we need *both* equations. The first does not in general imply the second.

Moreover, the original definition in (2.4.13) arises as the case of equality in the general inequality of (2.4.11). Similarly, we can show that *if we assume the second equation in (i) of Corollary 2.4.15*, then we claim that we always have the inequality

$$u^*\mathsf{vol}_L \geqslant \lambda^k(\star\alpha)^{(0,k)}. \tag{2.4.17}$$

However, the inequality (2.4.17) need not hold in general if we do not assume $u^*g = \lambda^2 h^{(0,2)}$.

To see that (2.4.17) holds if $u^*g = \lambda^2 h^{(0,2)}$, note that both sides are sections of the oriented line bundle $\ker(du)^\perp$ whose space of sections is $\Omega^{(0,k)}$. Hence we can compare any two elements. Clearly the inequality holds on $M\backslash M^0$ as both sides are zero. Let $x \in M^0$. Let $e_1, \ldots, e_k$ be an oriented orthonormal basis of $(\ker du_x)^\perp$. Then since $u^*g = \lambda^2 h^{(0,2)}$ we have

$$u^*\mathsf{vol}_L(e_1 \wedge \cdots \wedge e_k) = \lambda^k,$$

and since $\star\alpha$ is also a calibration we have

$$\lambda^k(\star\alpha)^{(0,k)}(e_1 \wedge \cdots \wedge e_k) = \lambda^k(\star\alpha)(e_1 \wedge \cdots \wedge e_k) \leqslant \lambda^k.$$

Thus the inequality (2.4.17) holds if $u^*g = \lambda^2 h^{(0,2)}$.
Finally, as in the immersion case, there is another equivalent form of the Smith equation, which we prove in Propositions 2.4.21 and 2.4.25.

Theorem 2.4.18 (Energy Inequality)**.** *Let $\alpha \in \Omega^{n-k}(M)$ be a* closed *calibration. Let $u\colon (M^n, h) \to (L^k, g)$ be a Smith submersion with respect to α. Suppose M is compact. Then u is k-harmonic in the sense that it is a critical point of E_k.*

Proof. For any smooth map $u\colon (M^n, h) \to (L^k, g)$, let $\lambda = \frac{1}{\sqrt{k}}|du|$. Using (2.4.11) we have

$$E_k(u) = \frac{1}{(\sqrt{k})^k}\int_M |du|^k \mathsf{vol}_M = \int_M \lambda^k \mathsf{vol}_M \geqslant \int_M \alpha \wedge u^*\mathsf{vol}_L = ([\alpha] \cup u^*[\mathsf{vol}_L]) \cdot [M],$$

where we have used the fact that α is closed. Thus the k-energy of u is bounded from below by a topological quantity, as it depends only on the cohomology class $[\alpha]$ and the homotopy class of u. Moreover, by Theorem 2.4.10, equality holds if and only if u is a Smith submersion. This shows that such maps are local minimizers of E_k and thus are k-harmonic. □

We note that Theorem 2.4.18 still holds if M is noncompact. See Theorem 2.4.29.

Remark 2.4.19. Smith submersions also enjoy a sort of "conformal invariance", but it is slightly more complicated. (This is expected, because a Smith submersion $u\colon (M^n, h) \to (L^k, g)$ with respect to $\alpha \in \Omega^{n-k}(M)$ is in particular a k-harmonic map (when $d\alpha = 0$), so by the discussion at the end of Section 2.2.2, this notion would depend only on the conformal class $[h]$ of the metric on the domain M only in the particular special case $n = k$.

In general, if $n > k$, we have the following. Let $h = h^{(2,0)} + h^{(0,2)}$ be the decomposition of the metric h on M in terms of the horizontal/vertical splitting as in Definition 2.4.1. A *horizontally conformal scaling* of h is a new metric $\widetilde{h} = h^{(2,0)} + f^2 h^{(0,2)}$ for some smooth positive function on L. (That is, we only conformally scale the *horizontal* part of the metric h). Since du is zero on vertical vectors, from (2.2.9) we get

$$\widetilde{\lambda}^2 = \frac{1}{k}|du|^2_{\widetilde{h},g} = f^{-2}\frac{1}{k}|du|^2_{h,g} = f^{-2}\lambda^2,$$

and clearly $\widetilde{\mathsf{vol}}_M = f^k \mathsf{vol}_M$. It follows that the Smith submersion equations $\alpha \wedge u^* \mathsf{vol}_L = \lambda^k \mathsf{vol}_M$ and $u^* g = \lambda^2 h^{(0,2)}$ are *invariant under horizontally conformal scaling* of the domain metric h on M.

2.4.3 Direct proof that Smith submersions are k-harmonic

In Theorem 2.3.15 below we show directly that a Smith submersion satisfies the k-harmonic map equation, in the sense that $\tau_k(u) = 0$, without assuming M is compact. First we need some preliminary results.

Lemma 2.4.20. *Let $\alpha \in \Omega^{n-k}(M)$ be a calibration. Let $u\colon (M^n, h) \to (L^k, g)$ be a Smith submersion with respect to α. Then we have*

$$(\star\alpha)^{(1,k-1)} = 0, \qquad \text{and} \qquad \nabla\alpha = 0 \ \text{on} \ \ker du.$$

Proof. The first statement follows from Lemma 2.2.3, because by Corollary 2.4.15 and Lemma 2.4.14, the form $\star\alpha$ calibrates $(\ker du)^\perp$. For the second statement, since $\alpha \in \Omega^{n-k}(M)$ and $\ker du$ is $(n-k)$-dimensional, it is enough to show that

$$(\nabla_X \alpha)(\tilde{e}_1 \wedge \ldots \wedge \tilde{e}_{n-k}) = 0,$$

for any local orthonormal frame $\tilde{e}_1, \ldots, \tilde{e}_{n-k}$ of $\ker du$. Since by Lemma 2.4.14, α calibrates $\ker du$, we have that $\alpha(\tilde{e}_1 \wedge \cdots \wedge \tilde{e}_{n-k}) = 1$. Hence we have

$$\begin{aligned}(\nabla_X \alpha)(\tilde{e}_1 \wedge \cdots \wedge \tilde{e}_{n-k}) &= X\big(\alpha(\tilde{e}_1 \wedge \cdots \wedge \tilde{e}_{n-k})\big) - \sum_{j=1}^{n-k} \alpha(\tilde{e}_1 \wedge \cdots \wedge (\nabla_X \tilde{e}_j) \wedge \cdots \wedge \tilde{e}_{n-k}) \\ &= 0 - \sum_{j=1}^{n-k} \alpha(\tilde{e}_1 \wedge \cdots \wedge (\nabla_X \tilde{e}_j) \wedge \cdots \wedge \tilde{e}_{n-k}).\end{aligned}$$

Now for any fixed j, the term $\alpha(\tilde{e}_1 \wedge \cdots \wedge (\nabla_X \tilde{e}_j)^{(0,1)} \wedge \cdots \wedge \tilde{e}_{n-k})$ vanishes by the first statement. Next note that since the $\tilde{e}_j$ are of norm 1, the vector field $\nabla_X \tilde{e}_j$ is always orthogonal to $\tilde{e}_j$, and thus $\tilde{e}_1, \ldots, (\nabla_X \tilde{e}_j)^{(1,0)}, \ldots, \tilde{e}_{n-k}$ are linearly dependent for any j, so $\alpha(\tilde{e}_1 \wedge \cdots \wedge (\nabla_X \tilde{e}_j)^{(1,0)} \wedge \cdots \wedge \tilde{e}_{n-k})$ also vanishes, which concludes the proof. □

Proposition 2.4.21. *Let $u\colon (M^n, h) \to (L^k, g)$ be a Smith submersion with respect to the calibration form $\alpha \in \Omega^{n-k}$ on M. Then we have*

$$\star_L \Lambda^{k-1}(du)(\cdot \lrcorner \star\alpha) = \frac{(-1)^{k-1}}{(\sqrt{k})^{k-2}} |du|^{k-2} du. \tag{2.4.22}$$

Proof. The equation is trivially satisfied at points where du is zero. Let $x \in M^0$. Also, recall that we necessarily have $u^*g = \lambda^2 h^{(0,2)}$, and that from Corollary 2.4.15 we also have $u^*\mathsf{vol}_L = \lambda^k (\star\alpha)^{(0,k)}$.

For simplicity of notation, let $P_{(0,k)}$ denote $P_{(\star\alpha)^{(0,k)}}$. Note that $P_{(0,k)} \in \Gamma(\Lambda^{(0,k-1)}(TM) \otimes (TM)^{(0,1)})$. Using this, for any $v_1, \ldots, v_k \in T_x M$ we have

$$\begin{aligned} g(\star_L(du(v_1) \wedge \cdots \wedge du(v_{k-1})), du(v_k)) &= \mathsf{vol}_L(du(v_1) \wedge \cdots \wedge du(v_k)) \\ &= (u^*\mathsf{vol}_L)(v_1, \ldots, v_k) \\ &= \lambda^k (\star\alpha)^{(0,k)}(v_1, \ldots, v_k) \\ &= \lambda^k h(P_{(0,k)}(v_1, \ldots, v_{k-1}), v_k) \\ &= \lambda^k h^{(0,2)}(P_{(0,k)}(v_1, \ldots, v_{k-1}), v_k) \\ &= \lambda^{k-2} (u^*g)(P_{(0,k)}(v_1, \ldots, v_{k-1}), v_k) \\ &= \lambda^{k-2} g(du(P_{(0,k)}(v_1, \ldots, v_{k-1})), du(v_k)). \end{aligned}$$

Since du_x is surjective, we get

$$\star_L(du(v_1) \wedge \cdots \wedge du(v_{k-1})) = \lambda^{k-2} du(P_{(0,k)}(v_1, \ldots, v_{k-1}))$$

or equivalently

$$\star_L \Lambda^{k-1}(du) = \lambda^{k-2} du \circ P_{(0,k)} \quad \text{on } \Lambda^{k-1}(T_x M). \tag{2.4.23}$$

From the proof of Corollary 2.4.15, we had $|(\star\alpha)^{(0,k)}| = 1$. Combining this with Lemma 2.4.9 gives

$$P_{(0,k)} P_{(0,k)}^{\top} = |(\star\alpha)^{(0,k)}|^2 \, \pi^{(0,1)} = \pi^{(0,1)}. \tag{2.4.24}$$

Composing with $P_{(0,k)}^{\top}$ on the right of both sides of (2.4.23) and using (2.4.24) and Proposition 2.2.7, since $du \circ \pi^{(0,1)} = du$, we obtain

$$\star_L \Lambda^{k-1}(du)(\cdot \lrcorner (\star\alpha)^{(0,k)}) = \frac{(-1)^{k-1}}{(\sqrt{k})^{k-2}} |du|^{k-2} du.$$

Comparing the above with (2.4.22), we see that it remains to verify that

$$\Lambda^{k-1}(du)(\cdot \lrcorner (\star\alpha)^{(0,k)}) = \Lambda^{k-1}(du)(\cdot \lrcorner \star\alpha).$$

To see this, we take any $v \in T_x M$ and compute

$$\begin{aligned}
&\Lambda^{k-1}(du)(v \lrcorner (\star\alpha)^{(0,k)}) \\
&\qquad = \Lambda^{k-1}(du)((v^{(1,0)} + v^{(0,1)}) \lrcorner (\star\alpha)^{(0,k)}) \\
&\qquad = \Lambda^{k-1}(du)(v^{(0,1)} \lrcorner (\star\alpha)^{(0,k)}) && \text{(because } v^{(1,0)} \lrcorner (\star\alpha)^{(0,k)} = 0\text{)} \\
&\qquad = \Lambda^{k-1}(du)(v^{(0,1)} \lrcorner (\star\alpha)^{(0,k)} + v^{(1,0)} \lrcorner (\star\alpha)^{(1,k-1)}) && \text{(because } (\star\alpha)^{(1,k-1)} = 0 \text{ by Lemma 2.4.20)} \\
&\qquad = \Lambda^{k-1}(du)(v \lrcorner \star\alpha)^{(0,k-1)} && \text{(by Lemma 2.4.8)} \\
&\qquad = \Lambda^{k-1}(du)(v \lrcorner \star\alpha) && \text{(because } du \text{ is zero on vertical vectors),}
\end{aligned}$$

concluding the claim. □

Proposition 2.4.25. *We have shown that if $u\colon (M^n, h, \alpha) \to (L^k, g)$ is a Smith submersion, then*

$$\star_L \Lambda^{k-1}(du)(\cdot \lrcorner \star\alpha) = (-1)^{k-1} \frac{|du|^{k-2}}{\sqrt{k}^{k-2}} du. \tag{2.4.26}$$

The converse also holds. That is, if (2.4.26) holds, then u is a Smith submersion.

Proof. Let $x \in M$. If $du_x = 0$, which satisfies (2.4.26) at x, then u is a Smith submersion at x. Now assume $du_x \neq 0$. Let $e_1, \ldots, e_m$ be an oriented orthonormal bases of $(\ker(du)_x)^\perp$. Note that a priori we do *not* know that $m = k$. However, we have that $1 \leqslant m \leqslant k$. Let $i, j \in \{1, \ldots .m\}$.

We first observe that

$$\begin{aligned}
\Lambda^{k-1}(du)(e_i \lrcorner \star\alpha) &= \Lambda^{k-1}(du)(e_i \lrcorner \star\alpha)^{(0,k-1)} \\
&= \Lambda^{k-1}(du)(e_i^{(0,1)} \lrcorner \star\alpha)^{(0,k-1)} \quad \text{(because } e_i \text{ is already of type } (0,1)\text{)} \\
&= \Lambda^{k-1}(du)(e_i^{(0,1)} \lrcorner (\star\alpha)^{(0,k)}) \\
&= \Lambda^{k-1}(du)(e_i \lrcorner (\star\alpha)^{(0,k)}).
\end{aligned} \tag{2.4.27}$$

Evaluating both sides of (2.4.26) on e_i, using (2.4.27), and taking inner product with $du(e_j)$

we get

$$
\begin{aligned}
(-1)^{k-1}\lambda^{k-2}g(du(e_i),du(e_j)) &= g(\star_L\Lambda^{k-1}(du)(e_i\lrcorner(\star\alpha)^{(0,k)}),du(e_j))\\
&= u^*\mathsf{vol}_L((e_i\lrcorner(\star\alpha)^{(0,k)})\wedge e_j)\\
&= u^*\mathsf{vol}_L(e_i\lrcorner((\star\alpha)^{(0,k)}\wedge e_j)-(-1)^k(\star\alpha)^{(0,k)}e_i\lrcorner e_j)\\
&= u^*\mathsf{vol}_L(0+(-1)^{k-1}\delta_{ij}(\star\alpha)^{(0,k)}) \quad (\text{because } \Omega^{(0,k+1)}=0)\\
&= (-1)^{k-1}\delta_{ij}u^*\mathsf{vol}_L((\star\alpha)^{(0,k)})\\
&= (-1)^{k-1}\delta_{ij}u^*\mathsf{vol}_L(\star\alpha).
\end{aligned}
$$

Note that

$$u^*\mathsf{vol}_L(\star\alpha)\,\mathsf{vol}_M = h(u^*\mathsf{vol}_L,\star\alpha)\,\mathsf{vol}_M = u^*\mathsf{vol}_L\wedge\star^2\alpha = u^*\mathsf{vol}_L\wedge(-1)^{k(n-k)}\alpha = \alpha\wedge u^*\mathsf{vol}_L.$$

We deduce that

$$
\begin{cases}
\lambda^{k-2}g(du(e_i),du(e_j))\,\mathsf{vol}_M = \alpha\wedge u^*\mathsf{vol}_L & \text{if } i=j,\\
g(du(e_i),du(e_j)) = 0 & \text{if } i\neq j.
\end{cases}
$$

Using the above we compute

$$
\begin{aligned}
\alpha\wedge u^*\mathsf{vol}_L &= \frac{1}{m}\lambda^{k-2}\sum_i g(du(e_i),du(e_i))\,\mathsf{vol}_M\\
&= \frac{1}{m}\lambda^{k-2}\sum_{i,j} g(du(e_i),du(e_j))\,\mathsf{vol}_M\\
&= \frac{1}{m}\lambda^{k-2}|du|^2\mathsf{vol}_M\\
&\geqslant \frac{1}{k}\lambda^{k-2}|du|^2\mathsf{vol}_M\\
&= \lambda^k\mathsf{vol}_M.
\end{aligned}
$$

Combining with Theorem 2.4.10 we get the desired equality, and thus u is a Smith submersion in the sense of Definition 2.4.12. ☐

Proposition 2.4.28. *Let $P \in \Gamma(T^*M\otimes\Lambda^q(TM))$. Under the identification of vector fields with 1-forms using the metric, assume that P is totally skew-symmetric. Then $\operatorname{div}(\Lambda^q(du)(P)) = \Lambda^q(du)(\operatorname{div}(P))$.*

Proof. We trivially have equality at points where du is zero. Let $x \in M^0$. Take Riemannian normal coordinates $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a}$ centred at x and $u(x)$ respectively. For simplicity of notation, let

$$A := \Lambda^q(du)(P) \in \Gamma(T^*M\otimes\Lambda^q(TL)).$$

Expressing the components of A and P in terms of these normal coordinates at the point x, we compute

$$\begin{aligned}
A_j^{v_1\cdots v_q} &= \left(\Lambda^q(du)(P_j)\right)^{v_1\cdots v_q} \\
&= \frac{1}{q!}P_j^{t_1\cdots t_q}\left(\Lambda^q(du)\left(\frac{\partial}{\partial x^{t_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{t_q}}\right)\right)^{v_1\cdots v_q} \\
&= \frac{1}{q!}P_j^{t_1\cdots t_q}\left(\frac{\partial u^{s_1}}{\partial x^{t_1}}\frac{\partial}{\partial y^{s_1}}\wedge\cdots\wedge\frac{\partial u^{s_q}}{\partial x^{t_q}}\frac{\partial}{\partial y^{s_q}}\right)^{v_1\cdots v_q} \\
&= P_j^{t_1\cdots t_q}\frac{\partial u^{v_1}}{\partial x^{t_1}}\cdots\frac{\partial u^{v_q}}{\partial x^{t_q}} \qquad \text{(by skew-symmetry of } P \text{ in } t_1\cdots t_q).
\end{aligned}$$

From this we obtain

$$\begin{aligned}
(\operatorname{div} A)^{v_1\cdots v_q} &= (\nabla_j A)_j^{v_1\cdots v_q} \\
&= (\nabla_j P)_j^{t_1\cdots t_q}\frac{\partial u^{v_1}}{\partial x^{t_1}}\cdots\frac{\partial u^{v_q}}{\partial x^{t_q}} \\
&\quad + P_j^{t_1\cdots t_q}\sum_{\ell=1}^{q}\frac{\partial^2 u^{v_\ell}}{\partial x^j\partial x^{t_\ell}}\frac{\partial u^{v_1}}{\partial x^{t_1}}\cdots\widehat{\frac{\partial u^{v_\ell}}{\partial x^{t_\ell}}}\cdots\frac{\partial u^{v_q}}{\partial x^{t_q}},
\end{aligned}$$

where the second term above is zero by symmetry in j, t_ℓ of $\frac{\partial^2 u^{v_\ell}}{\partial x^j\partial x^{t_\ell}}$ and skew-symmetry of $P_j^{t_1\cdots t_q}$, by our assumption on P. But then the first term is just:

$$\Lambda^q(du)(\operatorname{div}(P))^{v_1\cdots v_q},$$

which completes the proof. □

Theorem 2.4.29. *Let $u\colon (M^n, h) \to (L^k, g)$ be a Smith submersion with respect to the calibration form $\alpha \in \Omega^{n-k}$. If $d\alpha = 0$, then u is k-harmonic in the sense that $\tau_k(u) = 0$.*

Proof. By equation (2.2.12) and Proposition 2.4.21, we need to show that

$$\operatorname{div}(\star_L\Lambda^{k-1}(du)(\cdot \lrcorner \star\alpha)) = 0.$$

However, $\star_L$ commutes with ∇. Moreover, the section $\cdot \lrcorner \star\alpha \in \Gamma(T^*M \otimes \Lambda^{k-1}(T^*M))$ is totally skew-symmetric. Hence, by Proposition 2.4.28, it is enough to show that

$$\operatorname{div}(\cdot \lrcorner \star\alpha) = 0.$$

But for any $\beta \in \Omega^q$ we have $\operatorname{div}(\cdot \lrcorner \beta) = -d^*\beta$, because

$$\operatorname{div}(\cdot \lrcorner \beta)_{s_1\cdots s_{q-1}} = \nabla_i((\cdot \lrcorner \beta)_i)_{s_1\cdots s_{q-1}} = \nabla_i\beta_{is_1\cdots s_{q-1}} = -(d^*\beta)_{s_1\cdots s_{q-1}}.$$

So if $d\alpha = 0$ then $\operatorname{div}(\cdot \lrcorner \star\alpha) = -d^* \star \alpha = 0$, which concludes the proof. □

2.5 Discussion

In this section we review analytic properties of Smith immersions, discuss examples of Smith immersions and Smith submersions, make some remarks on the relevance to the SYZ and GYZ conjectures of mirror symmetry involving calibrated fibrations, and present several questions for future study.

2.5.1 Analytic results for Smith immersions

Numerous analytic results for Smith immersions were proved in Cheng–Karigiannis–Madnick [10, Sections 4 and 5]. In that paper the authors assumed that the calibration form $\alpha \in \Omega^k(M)$ was associated to a vector cross product (VCP), but as we showed in Section 2.3, this assumption was not necessary. All the analytic results used the form (2.3.5) of the Smith immersion equation. In this section we informally review these analytic results. (Note that when $k = 2$ these analytic results concern J-holomorphic maps and are classical.) See [10] for precise statements.

Removable singularities. If u is a C^1_{loc} Smith immersion on a punctured open ball in $\mathbb{R}^k$ with finite k-energy, then u extends to a C^1 Smith immersion across the puncture.

Energy gap. There exists a "threshold energy" $\varepsilon_0 > 0$ such that every Smith immersion $u\colon S^k \to M$ with k-energy less than ε_0 is *constant*. (That is, any nontrivial solution has a *minimum* k-energy.) This is used to show that there are only a finite number of "bubbles".

Compactness modulo bubbling. Let $W \subseteq L$ be open, and let $\{W_m\}_{m\in\mathbb{N}}$ an increasing sequence of open sets exhausting W, and g_m a sequence of metrics on W_m such that $g_m \to g$ in C^∞_{loc} on W. Let $u_m\colon (W_m, [g_m]) \to (M, h)$ be a sequence of Smith immersions with *uniformly bounded k-energy*.

Then there exists a Smith immersion $u_\infty\colon (W, g|_W) \to (M, h)$ and a (possibly empty) finite subset $\mathcal{B} = \{x_1, \ldots, x_N\}$ of L such that (after passing to a subsequence) the following three properties hold:

(a) $u_m \to u_\infty$ in C^1_{loc} on $W \backslash \mathcal{B}$ uniformly on compact subsets of $W \backslash \mathcal{B}$,

(b) as Radon measures on L, we have $|du_m|^k \mathsf{vol}_L \to |du_\infty|^k \mathsf{vol}_L + \sum_{i=1}^N c_i \delta(x_i)$, where $\delta(x_i)$ is a Dirac measure at x_i, and each $c_i \geqslant \frac{1}{2}\varepsilon_0$, where ε_0 is the "threshold energy". This says that the energy density can concentrate at points, where a minimum amount of energy is lost.

(c) If the u_m have uniformly bounded p-energy for some $p \in (k, \infty]$, then $\mathcal{B} = \varnothing$. (There is no bubbling.)

(In practice we take $W = L$ or $L = S^k$ and $W = S^k \backslash \{p^-\}$, where p^- is the south pole. See [10, Remark 4.13] for details.)

This result can be applied to a sequence $u_m\colon L \to M$ of Smith immersions representing *the same homology class in* $H_k(M)$, as they have a uniform k-energy bound. For each x_i, by rescaling about x_i and using *conformal invariance*, and reapplying this result, we obtain a "bubbled off" Smith immersion $\tilde{u}_{\infty,i}\colon S^k \to M$. This process stops after a *finite number of iterations* due to the energy gap.

No energy loss. We have $\lim_{m\to\infty} E_k(u_m) = E_k(u_\infty) + \sum_i E_k(\tilde{u}_{\infty,i})$. This says that the limiting k-energy is the sum of the k-energy of u_∞ plus the k-energy of each of the bubble maps.

Zero neck length. We have $u_\infty(x_i) = \tilde{u}_{\infty,i}(p^-)$, where p^- is the south pole of S^k. This says that for $m >> 0$, then u_m is *homotopic* to the connect sum $u_\infty \#(\underset{i}{\#}\tilde{u}_{\infty,i})$.

It would of course be very interesting to establish analogous analytic results for Smith submersions. However, the *conformal invariance* of Smith immersions, as detailed in Remark 2.3.7, was used crucially to establish the above analytic results. By contrast, Remark 2.4.19 says that Smith submersions are only *horizontally conformally invariant*. But perhaps this is indeed the right notion that is needed in this context. The authors plan to investigate this question further.

2.5.2 Examples of Smith maps

In this section we discuss some examples of Smith maps.

Example 2.5.1. Let (M^n, h) be a Riemannian manifold equipped with a calibration form $\alpha \in \Omega^k(M)$. Let $\iota\colon L^k \to M^n$ be an immersion of an oriented manifold L^k into M, and equip L with the pullback metric $g = \iota^* h$, so that ι is a Riemannian immersion. Suppose that $\iota(L)$ is α-calibrated, which means that $\iota^*\alpha = \mathsf{vol}_L$. Then ι is a Smith immersion with dilation $\lambda = 1$. Thus, any α-calibrated submanifold gives rise to a Smith immersion, but the notion of Smith immersion is more general.

Indeed, if $f\colon (L, g) \to (L, g)$ is an orientation-preserving conformal diffeomorphism, then $u = \iota \circ f$ is also a Smith immersion, with the same image $u(L) = \iota(L)$, but u need not be a Riemannian immersion. ▲

There are several examples of Smith submersions where the domain (M^n, h) is noncompact, given by explicit *cohomogeneity one* special holonomy metrics on total spaces M^n of vector bundles over a base L^k, and equipped with a parallel calibration form $\alpha \in \Omega^k(M)$. These include the Bryant–Salamon examples [7] of G_2 and Spin(7) manifolds, and (very likely) also include the Stenzel examples [41] of Calabi–Yau metrics on T^*S^m. The Smith submersion is the projection map $u\colon M \to L$, and the fibres are $(n-k)$-dimensional submanifolds calibrated by $\star\alpha \in \Omega^{n-k}(M)$.

In these examples, we have $du \neq 0$ everywhere on M, so $M^0 = M$. (See the discussion in Section 2.5.3 for why we cannot expect this to happen if M is compact.) We now discuss these examples in detail.

Example 2.5.2. Consider the spinor bundle $M^7 = \slashed{S}(S^3)$ over the round S^3. There is a torsion-free G_2-structure φ on M^7, with dual 4-form $\psi = \star\varphi$, inducing a metric h which has holonomy G_2. The projection $u\colon (M^7, h) \to (S^3, g)$ is a submersion. We claim that the map u is a Smith submersion with respect to the calibration form $\alpha = \psi \in \Omega^4(M)$.

To see this, we use the notation of [25, Section 3.1]. We have local vertical vector fields $\zeta_0, \zeta_1, \zeta_2, \zeta_3$ and horizontal vector fields b_1, b_2, b_3. The function $r \geqslant 0$ is the distance from the zero section in the fibres of M. Then it is known that for $c_0, c_1 > 0, \kappa > 0$ we have a torsion-free G_2 structure defined by

$$\varphi = 3\kappa(c_0 + c_1 r^2)u^*\mathsf{vol}_{S^3} + 4c_1(b_1 \wedge \Omega_1 + b_2 \wedge \Omega_2 + b_3 \wedge \Omega_3), \tag{2.5.3}$$

where Ω_i are vertical 2-forms and such that the induced metric is

$$h = (3\kappa)^{\frac{2}{3}}(c_0 + c_1 r^2)^{\frac{2}{3}} u^* g_{S^3} + 4\left(\frac{c_1^3}{3\kappa}\right)^{\frac{1}{3}}(c_0 + c_1 r^2)^{-\frac{1}{3}}(\zeta_0^2 + \zeta_1^2 + \zeta_2^2 + \zeta_3^2).$$

Hence, we see that $h^{(0,2)} = (3\kappa)^{\frac{2}{3}}(c_0 + c_1 r^2)^{\frac{2}{3}} u^* g_{S^3}$ which gives $u^* g_{S^3} = \lambda^2 h^{(0,2)}$ for

$$\lambda = (3\kappa)^{-\frac{1}{3}}(c_0 + c_1 r^2)^{-\frac{1}{3}}.$$

By Corollary 2.4.15, it remains to verify that $u^*\mathsf{vol}_{S^3} = \lambda^3 \varphi^{(0,3)}$. But we immediately see from (2.5.3) that

$$\varphi^{(0,3)} = 3\kappa(c_0 + c_1 r^2)u^*\mathsf{vol}_{S^3} = \lambda^{-3} u^*\mathsf{vol}_{S^3},$$

which gives the desired equality.

Since the G_2-structure is torsion-free, in particular we have that $d\psi = 0$. Consequently, the map $u\colon M \to S^3$ is 3-harmonic and the fibres are calibrated by ψ. (That the fibres of this G_2-manifold are coassociative submanifolds is of course well-known.) ▲

Example 2.5.4. Consider the manifold $M^7 = \Lambda^2_-(T^* X^4)$ of anti-self dual 2-forms over X, where X^4 is either the round S^4 or the Fubini–Study $\mathbb{CP}^2$. There is a torsion-free G_2-structure φ on M^7, with dual 4-form $\psi = \star\varphi$, inducing a metric h which has holonomy G_2. The projection $u\colon (M^7, h) \to (X^4, g)$ is a submersion. We claim that the map u is a Smith submersion with respect to the calibration form $\alpha = \varphi \in \Omega^3(M)$.

To see this, we use the notation of [26, Section 4.1]. There exist positive functions w and v which depend only on the radial coordinate in the vertical fibres and satisfy certain differential equations such that we have a torsion-free G_2 structure given by

$$\varphi = v^3 \mathsf{vol}_{\mathcal{V}} + w^2 v d\theta,$$

where $\mathsf{vol}_{\mathcal{V}}$ is the volume form for the vertical part and θ is the canonical 2-form on $\Lambda^2_-(T^*X)$. The dual 4-form can be expressed as

$$\psi = \psi^{(0,4)} + \psi^{(2,2)} \qquad \text{where} \quad \psi^{(0,4)} = w^4\, u^*\mathsf{vol}_X, \tag{2.5.5}$$

and the metric h induced by φ is given by

$$h = w^2\, u^* g_X + v^2 g_{\mathcal{V}}.$$

Hence, we see that $h^{(0,2)} = \lambda^{-2} u^* g_X$ for $\lambda = w^{-1}$. By Corollary 2.4.15, it remains to verify that $u^*\mathsf{vol}_X = \lambda^4 \psi^{(0,4)}$. But this is immediate from (2.5.5).

Since the G_2-structure is torsion-free, in particular we have that $d\varphi = 0$. Consequently, the map $u\colon M \to X^4$ is 4-harmonic and the fibres are calibrated by φ. (That the fibres of this G_2-manifold are associative submanifolds is of course well-known.) ▲

Example 2.5.6. Consider the manifold $M^8 = \$_-(S^4)$ of negative chirality spinors over the round S^4. There is a torsion-free Spin(7)-structure Φ on M^8, inducing a metric h which has holonomy Spin(7). The projection $u\colon (M^8, h) \to (S^4, g)$ is a submersion. We claim that the map u is a Smith submersion with respect to the calibration form $\alpha = \Phi \in \Omega^4(M)$.

To see this, we use the notation of [26, Section 4.2]. There exist positive functions w and v which depend only on the radial coordinate in the vertical fibres and satisfy certain differential equations such that we have a torsion-free Spin(7) structure given by

$$\Phi = w^4\, u^*\mathsf{vol}_{S^4} + w^2 v^2 \beta + v^4 \mathsf{vol}_{\mathcal{V}}, \tag{2.5.7}$$

where $\mathsf{vol}_{\mathcal{V}}$ is the volume form on the vertical part and β is some $(2,2)$-form. The metric h induced by Φ is given by

$$h = w^2\, u^* g_{S^4} + v^2 g_{\mathcal{V}}.$$

Hence, we see that $h^{(0,2)} = \lambda^{-2} u^* g_{S^4}$ for $\lambda = w^{-1}$. By Corollary 2.4.15, it remains to verify that $u^*\mathsf{vol}_{S^4} = \lambda^4 \Phi^{(0,4)}$. But this is immediate from (2.5.7).

Since the Spin(7)-structure is torsion-free, in particular we have that $d\Phi = 0$. Consequently, the map $u\colon M \to S^4$ is 4-harmonic and the fibres are calibrated by Φ. (That the fibres of this Spin(7)-manifold are Cayley submanifolds is of course well-known.) ▲

Example 2.5.8. There is an explicit cohomogeneity one Calabi–Yau metric h on the total space of $M^{2m} = T^*(S^m)$, called the *Stenzel* metric. When $m = 2$ this is the classical Eguchi–Hanson metric, and when $m = 3$ it is the Candelas–de la Ossa *conifold metric*. (See the paper of Ionel–Min-Oo [21] for a concrete simple description of these metrics.) Being Calabi–Yau, this Riemannian manifold (M^{2m}, h) is equipped with a holomorphic complex volume form $\Upsilon \in \Omega^{(m,0)}(M)$ such that $\alpha = \mathrm{Re}(\Upsilon) \in \Omega^m(M)$ is a special Lagrangian calibration.

Let $u\colon M^{2m} \to S^m$ be the projection. The fibres of u are special Lagrangian submanifolds. It seems very likely that u is a Smith submersion, so that it is horizontally conformal and an m-harmonic map. The authors did not explicitly verify this. At least when $m = 4$, such a verification should be possible using the many useful explicit formulas in Papoulias [32]. ▲

It would be interesting to examine if other known calibrated fibrations can be described by Smith submersions. For example, Goldstein exhibits a special Lagrangian torus fibration on the Borcea–Voisin manifold in [16] and other special Lagrangian fibrations in noncompact Calabi–Yau manifolds with symmetry are discussed by Gross [17] and Goldstein [15].

Moreover, Karigiannis–Lotay [25] exhibit other coassociative fibrations on the Bryant–Salamon G_2-manifold $\Lambda^2_-(S^4)$, very different from the obvious one in Example 2.5.4, and Trinca [43] similarly exhibits a nontrivial Cayley fibration on the Bryant–Salamon Spin(7)-manifold $\not{S}_-(S^4)$, very different from the obvious one in Example 2.5.6. Attempting to verify if these fibrations can be described by a Smith submersion seems to be an interesting but difficult problem.

2.5.3 Calibrated fibrations and the SYZ and GYZ "conjectures"

In this section we briefly discuss the potential relevance of Smith submersions to the Strominger–Yau–Zaslow [42] "conjecture" in Calabi–Yau geometry, as well as to the analogous Gukov–Yau–Zaslow "conjecture" in G_2 geometry. The authors are certainly not experts on the mathematics involved here, and we know even less about the physics. Nevertheless, we feel it worthwhile to make a few remarks. We put "conjecture" in quotes in both cases, as these ideas are predominantly motivated by physics, and their precise mathematical formulations are constantly evolving. Our brief discussion here is far from exhaustive, and is only meant to pique the reader's interest for further inquiry.

Roughly speaking, Strominger–Yau–Zaslow argue in [42] that one should expect (at least for certain types of points near the boundary of the moduli space) that a compact Calabi–Yau complex 3-fold should admit a fibration over a real 3-dimensional base, necessarily with singular fibres. The generic (smooth) fibre should be a special Lagrangian torus. The mathematical inspiration comes from the deformation theory of McLean [29], which shows that a compact special Lagrangian 3-manifold L^3 in a Calabi–Yau 6-manifold locally smoothly deforms in a family of dimension $b^1(L^3)$. One then expects to construct the "mirror Calabi–Yau manifold" by dualizing smooth fibres and then somehow compactifying.

Similarly, Gukov–Yau–Zaslow explain in [18] that, again under certain conditions, a compact torsion-free G_2-manifold should admit a fibration over a 3-dimensional base, again with singular fibres. The generic (smooth) fibre should be a coassocative submanifold with is topologically either T^4 or $K3$. Again, this is inspired by McLean's result in [29] that a compact coassociative 4-manifold L^4 in a torsion-free G_2-manifold locally smoothly deforms in a family of dimension $b^2_+(L^4)$, modulo orientations.

A kew observation by Joyce [22], discussed also in [23, Chapter 9], is that special Lagrangian fibrations of compact Calabi–Yau manifolds should not be expected to be smooth generically. Rather, Joyce provides evidence that they should be *piecewise-smooth*, with the singularities of the map being related to topology change of the fibres. This suggests that the set of critical fibres should be relatively large. Indeed, Joyce argues that singular fibres should generically be of codimension one. It is reasonable to believe that analogous statements should hold for coassociative fibrations of compact torsion-free G_2-manifolds. (Baraglia [3] gives a rigorous intricate argument proving that such coassociative fibrations necessarily must have singular fibres.)

When the domain (M, h) of a Smith submersion is noncompact, there exist many explicit examples of calibrated fibrations, and at least some are definitely Smith submersions, as discussed in Section 2.5.2. However, if (M, h) is compact, then we expect that there must necessarily exist singular fibres. It would be interesting to see this directly by studying the PDE (2.4.13) satisfied by a Smith submersion.

More generally, it is crucially important to understand the size of the *critical set*

$$M^c = M \backslash M^0 = \{x \in M : du_x = 0\}$$

of a Smith submersion. Similarly, the critical set $L^c = L \backslash L^0 = \{x \in L : du_x = 0\}$ of a Smith immersion is still very mysterious. In the classical case, when (M, h) is an almost Kähler manifold equipped with the Kähler calibration form $\alpha = \omega \in \Omega^2(M)$, then a Smith immersion $u : (L^2, g) \to (M, h)$ with respect to ω is a J-holomorphic map. In this case, when L is compact it is known, by methods of *unique continuation*, that the critical set L^c is a finite set of points. (See McDuff–Salamon [28, Sections 2.3–2.4] for details.) It is an important open problem to see if such methods can in any way be effectively applied to general Smith immersions and Smith submersions. Of course, we certainly do not expect the critical sets to be of dimension zero in general.

2.5.4 Questions for future study

Many questions arise naturally from our study, which are somewhat speculative. Some of these are:

Deformation theory of Smith maps. What is the deformation theory of a Smith map (immersion or submersion)? From Example 2.5.1, any calibrated submanifold gives rise to a Smith immersion. The work of McLean [29] studies the deformation theory of (compact) calibrated submanifolds. Interestingly, there are two kinds of behaviours. Special Lagrangian and coassociative submanifolds deform smoothly, while complex, associative, and Cayley submanifolds in general have obstructed deformations. (The second class are essentially those calibrated submanifolds whose calibration forms are associated to vector cross products, except for higher dimensional complex submanifolds.)

However, at first glance, the Smith submersion equation does not seem to see the difference between those calibrations which have smooth deformation theories and those which are obstructed (respectively called branes and instantons by Leung–Lee [27]). Thus, it is important to reconcile the distinction in McLean's deformation theories with the existence theory of Smith submersions. For example, if the domain (M, h) is compact, so that the smooth fibres of a Smith submersion are compact calibrated submanifolds, and if α is an associative or Cayley calibration, then we should not in general expect existence of Smith submersions with respect to α, because associative and Cayley submanifolds are in general obstructed. (Of course, examples *do occur*, such as the obvious projections from a 7-torus or 8-torus with their standard G_2 or Spin(7)-structures.)

It would be interesting to see if the deformation theory of Smith immersions is "better behaved". Note that we aways have the freedom of precomposing by an orientation-preserving conformal diffeomorphism. Such deformations should be considered in some sense trivial. We are interested in deformations of Smith immersions which are transverse to such trivial deformations. For example, start with a (compact) associative or Cayley submanifold, and describe it by a Smith immersion. Can we always deform it (nontrivially) *as a Smith immersion*? This would give a class of calibrated submanifolds with a particular type of allowed singularities which nevertheless have smooth deformation spaces.

Stability. We have seen from the energy inequalities that Smith immersions and Smith submersions are global minimizers of the k-energy in a particular class of maps. Suppose that u is a k-harmonic map, which is *stable* in the sense that the second variation of the k-energy at u is nonnegative, so u is a local minimum of the k-energy. Under what additional assumptions on the geometry of the source and target could we ensure that such a stable k-harmonic map is necessarily a Smith map? The classical example of such a stability theorem is the demonstration by Siu–Yau [39] that a stable harmonic map from $S^2 = \mathbb{CP}^1$ into a compact Kähler manifold (M, h, ω) with positive holomorphic bisectional curvature is necessarily $\pm$-holomorphic. Generalizing such a result should involve finding analogues of "positive holomorphic bisectional curvature" in Riemannian manifolds with special holonomy.

Constructing Smith maps via flows. If a general stability theorem as described in the previous paragraph could be established, then one could use this to attempt to construct examples of Smith immersions or Smith submersions by running the k-harmonic map heat flow. This is the negative gradient flow of the k-energy. One would have to show that (under certain assumptions on the geometries of the source and target) that the flow exists for all time and converges to a k-harmonic map. Then one would hope to argue that the limit must in fact be a Smith map.

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