Exponential Stability of Maxwell's Equations

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

The control of electromagnetic systems, which is governed by Maxwell's equations, is of interest for various reasons such as controlling plasma in a nuclear fusion reactor or magnetically trapping antiparticles for observation. Proving that a controlled system is exponentially stable is of particular interest, as exponential stability infers that the system will converge asymptotically to a steady state solution. A tool called the multiplier method is considered, which allows for exponential stability to be demonstrated by defining an auxiliary functional and proving it is bounded by the system energy and the energy's time derivative in a particular way. If an appropriate bound is shown, exponential stability is not only guaranteed, but the exponential function which bounds the L2 norm of the system variables will be fully determined in terms of the systems parameters. Currently, there is ongoing work into generalizing the multiplier method approach for a class of problems known as Port-Hamiltonian systems. This thesis aims to contribute to this work by formulating Maxwell's equations as a Port-Hamiltonian system, and using this formulation as a basis for determining how to choose the auxiliary functional needed in the multiplier method.

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3.1 Let $\mathbf{f}, \mathbf{g}, \mathbf{h} \in \mathbb{R}^3$ be sufficiently differentiable. Let $\times$ denote the cross product and $\cdot$ denote the dot product. The table forms a list of relevant vector calculus and algebra identities. More details on these identities can be found in [19, Chapter 1 and 2] 47

List of Abbreviations

C_0-semigroups strongly continuous semigroups 5

ODE ordinary differential equations 1

PDE partial differential equations 1

PHS port-Hamiltonian systems 1

List of Symbols

Spatial Variables

t	Time
$\mathbf{x}$	Spatial Variables
Ω	Spatial Domain
Γ	Boundary of Ω

Mathematical Spaces

$\mathbb{R}^n$	Real valued vectors of degree n
$\mathbb{R}$	Real numbers
$\mathbb{R}^{n\times k}$	Real valued matrices with dimension $n \times k$
$L^2(\Omega)$	Square integrable functions over Ω
$H^1(\Omega)$	Square integrable functions with square integrable derivative over Ω

Mathematical Notation

^{T}	Transpose of a vector or matrix
$\mathbf{f}$	Boldface indicates a vector quantity
$\mathbf{f} \cdot \mathbf{g}$	Dot product, $\mathbf{f} \cdot \mathbf{g} = \sum_i f_i g_j$
$\mathbf{f} \times \mathbf{g}$	Cross product
$\lvert\mathbf{f}\rvert$	Euclidean norm, $\lvert\mathbf{f}\rvert = \sqrt{\mathbf{f} \cdot \mathbf{f}}$
$< \mathbf{f}, \mathbf{g} >$	L^2 inner product, $< \mathbf{f}, \mathbf{g} >= \int_\Omega \mathbf{f} \cdot \mathbf{g} \, d\Omega$
$\lvert\lvert\mathbf{f}\rvert\rvert$	L^2 norm, $\lvert\lvert\mathbf{f}\rvert\rvert = \sqrt{< \mathbf{f}, \mathbf{f} >}$
$\mathcal{O}$	Zero operator, $\mathcal{O}(\mathbf{f}) = \mathbf{0}$
$\mathcal{I}$	Identity operator, $\mathcal{I}(\mathbf{f}) = \mathbf{f}$

Differential Operators

$\dot{f}$	Total derivative in time, $\frac{d}{dt}f$
∂_x	Partial derivative
∇	Gradient
$\nabla\cdot$	Divergence
$\nabla\times$	Curl

Port-Hamiltonian Systems

$\mathcal{H}$	Hamiltonian or total energy
h	Hamiltonian density
$\mathbf{u}$	Input
$\mathbf{y}$	Output
e_∂	Boundary effort
f_∂	Boundary flow

Electromagnetic Variables

ε_0	Vacuum permittivity
χ_e	Electric susceptibility
ε	Permittivity, $\varepsilon = \varepsilon_0(1 + \chi_e)$
μ_0	Vacuum permeability
χ_e	Magnetic susceptibility
μ	Permeability, $\mu = (\mu_0^{-1} - \chi_m)^{-1}$
$\mathbf{E}$	Electric field
$\mathbf{P}$	Electric polarization
$\mathbf{D}$	Electric displacement, $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$
$\mathbf{B}$	Magnetic field
$\mathbf{M}$	Magnetisation vector
$\mathbf{H}$	Magnetic displacement, $\mathbf{H} = \mu_0^{-1}\mathbf{B} - \mathbf{M}$
$\mathbf{J}$	Electric current density
ρ	Electric charge density
η	Electrical resistivity

Chapter 1

Introduction

Control of electromagnetic systems is a topic that has been of interest in recent years, with applications such as the control of plasma inside a nuclear fusion reactor [35, 37]. An important aspect of control that is worth considering is stabilization. Stabilization is the process of stabilizing a system via a control input in order to have the system decay to its steady state. One approach to stability analysis is the multiplier method [14], which this thesis aims to apply to electromagnetic systems.

This thesis is structured as follows. In Chapter 2 the necessary requisite mathematics needed to prove well-posedness is discussed. The tools used to analyse partial differential equations (PDE) is the formalism of abstract PDE systems and the Lumer-Phillips theorem [10, 22]. Chapter 2 also reviews the topic of port-Hamiltonian systems (PHS). Port-Hamiltonian systems are a particular subclass of differential equations which nice properties such as passivity (a notion that relates to the rate of change of the systems total energy). Port-Hamiltonian systems are broken into two categories: finite-dimensional or ordinary differential equations (ODE) systems and infinite-dimensional or PDE systems. The study of Maxwell's equations as a port-Hamiltonian system is an active area of research of growing interest [7, 27, 31, 35]. Finally Chapter 2 will also discuss stability, and in particular exponential stability. Exponential stability provides an exponential bound on the decay of solutions to the steady state. The main tool this thesis will use to prove exponential stability is the multiplier method approach which was outlined in [14, 36]. Recently, [21] has demonstrated a methodology for applying the multiplier method to determine the exponential stability of port-Hamiltonian systems.

In Chapter 3 necessary background into electromagnetism is provided. Mostly adapting material from [3, 6, 18], the start of this chapter will introduce Maxwell's equations, formulate an equivalent problem in terms of free charge and current, and discuss physically relevant boundary conditions. The second half of this chapter is dedicated to setting up a system that is of interest to study and in proving well-posedness of the boundary value control problem.

Finally, Chapter 4 will outline the main result of the thesis. Several papers over the last 20 years [11, 16, 25, 26, 28, 32] have discussed the stabilization of Maxwell's equations for various circumstances, but the main result this chapter will follow is [14]. This paper considered Maxwell's equations in a region with zero internal charge or current, and was able to apply the multiplier method to demonstrate exponential stability. Section 4.1 will go over the results outlined in [14] in detail, while also expanding the results to allow for arbitrary positive constants ε and μ, which the paper set to be equal to one. Section 4.2 will expand on the results to demonstrate that a controlled system with controller $\mathbf{u} = k\mathbf{y}$ is exponentially stable for $0 \leq k < 1$.

Chapter 2

Mathematical Background

Before establishing a physical model for an electromagnetic system, some mathematical concepts need to be established which will be used throughout this thesis.

It is import to prove the well-posedness of the model to ensure a unique solution exists. To this end, section 2.1 and section 2.2 will present some background on the well-posedness of PDEs through the use of semigroup theory including the Lumer-Phillips theorem.

Motivation for some work done later in the thesis will be obtained by using the port-Hamiltonian structure of the model, so the concept of port-Hamiltonian systems needs to be established. To introduce the topic, finite-dimensional PHS will be discussed in section 2.3. After the concept is motivated, section 2.4 will discuss infinite-dimensional PHS in a single spatial dimension. The topic of higher spatial dimensions will be withheld until the end of Chapter 3 so that Maxwell's equations can be introduced first.

The last topic this chapter will cover is the multiplier method approach for stability. Section 2.5 will introduce the notion of stability and discuss the multiplier method approach which will be used later in the thesis.

2.1 Well-Posedness and Strongly Continuous Semigroups

For a problem to be considered well-posed in mathematics, it must satisfy the following three conditions [12]:

1. A solution to the problem exists,
2. this solution is unique, and
3. the solution depends continuously on the initial data.

The first two conditions make sense since the analysis of a problem most often requires a solution to the problem to exist and for this solution to be unique. In our case it does not make sense to discuss the exponential decay of the systems solution if the solution does not exist. The last condition requires that we can bound any two solutions if their initial conditions are close. Closeness is always considered with regards to some norm that applies to the function spaces being considered. For this thesis, the L^2 norm is the natural choice since it will be shown later that the system energy is just a weighted L^2 norm. The L^2 norm over a region (denoted $||\cdot||$) is defined by the L^2 inner product (denoted $<\cdot,\cdot>$). Let $\mathbf{f}, \mathbf{g} \in \mathbb{R}^n$ and let our region of interest be denoted as Ω, then the inner product and norm over the space $L^2(\Omega)$ are defined as

$$< \mathbf{f}, \mathbf{g} >= \int_{\Omega} \mathbf{f} \cdot \mathbf{g} \; d\Omega, \tag{2.1}$$

$$||\mathbf{f}|| = \sqrt{< \mathbf{f}, \mathbf{f} >}, \tag{2.2}$$

where $\mathbf{f} \cdot \mathbf{g} = \sum_{k=1}^{n} f_k g_k$ is the standard dot product.

Remark 2.1.1. *It should be noted that the well-posedness of a problem depends on the choice of what Hilbert space defined the state space for the problem. A problem may not be well-posed on a given Hilbert space and may require a subspace to be taken. From the discussion in [22, Examples 2.32 and 2.33], choosing a suitable state space for PDEs is not a trivial choice and restrictions on the state space often need to be made to ensure well-posedness. Often, these restrictions can be motivated by analysis of the systems energy.*

There are several established methods of proving well-posedness for PDEs which could be used, but for this thesis the notion of strongly continuous semigroups (C_0-semigroups) will be used to prove Well-Posedness. Definitions, theorems, and proofs will follow from Chapters 5-6 of [10]. Since our target goal is to analyze a PDE system, we should first set up an abstract PDE system to consider. Let $z(t) \in \mathbb{R}^n$ we are interested in proving the well posedness of the differential equation

$$\dot{z}(t) = Az(t) \quad z(0) = z_0 \tag{2.3}$$

where $\dot{z}$ refers to the total time derivative $\frac{d}{dt}z$, and A is a differential operator.

Next the definition of a C_0-semigroup is provided as follows.

Definition 2.1.2 (C_0-Semigroup)**.** *Let $\mathcal{Z}$ be a Hilbert Space. $(T(t))_{t\geq 0}$ is called a* $\mathbf{C_0}$-***semigroup*** *if the following holds:*

1. *For all $t \geq 0$, $T(t)$ is a bounded linear operator on $\mathcal{Z}$,*

2. *$T(0) = \mathcal{I}$, the identity operator,*

3. *$T(t+s) = T(t)T(s)$ for all $t, s \geq 0$,*

4. *for $z_0 \in \mathcal{Z}$, we have that $||T(t)z_0 - z_0||_{\mathcal{Z}}$ converges to zero as t goes to zero.*

At the moment it is not entirely clear how the mathematical structure of C_0-semigroups will help in the analysis of the well-posedness of PDEs. The next definition will serve as a bridge between semigroup theory and PDEs.

Definition 2.1.3 (Infinitesimal Generator)**.** *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup of the Hilbert space $\mathcal{Z}$. If the limit*

$$\lim_{t\to 0} \frac{T(t)z_0 - z_0}{t}$$

exists, then we say z_0 is in the domain of A ($z_0 \in \mathcal{D}(A)$) and we define

$$Az_0 = \lim_{t \to 0} \frac{T(t)z_0 - z_0}{t}. \tag{2.4}$$

*We call A the **infinitesimal generator** of the C_0-semigroup $(T(t))_{t \geq 0}$.*

With the concept of an infinitesimal generator defined, we can finally establish the connection between C_0-semigroups and PDEs. It will be shown that if the differential operator A in the abstract system (2.3) generates a C_0-semigroup, then for initial data $z_0 \in \mathcal{D}(A) \subseteq \mathcal{Z}$ a classical solution exists. If z_0 is instead only in the set $\mathbf{Z}$, then we say a mild solution exists. We define a classical solution and a mild solution as follows.

Definition 2.1.4 (Classic solution)**.** *A differentiable function $z : \mathbb{R} \to \mathcal{Z}$ is called a **classical solution** of (2.3) if for all $t \geq 0$ we have that $z(t) \in \mathcal{D}(A)$ and (2.3) is satisfied.*

Definition 2.1.5 (Mild solution)**.** *A continuous function $z(t) : \mathbb{R} \to \mathcal{Z}$ is called a **mild solution** of (2.3) if $\int\limits_0^t z(s)ds \in \mathcal{D}(A)$, $z(0) = z_0$ and*

$$z(t) - z(0) = A \int\limits_0^t z(s)ds, \quad \textit{for all } t \geq 0 \tag{2.5}$$

The following theorem will be needed to prove the existence of a solution when A generates a C_0-semigroup.

Theorem 2.1.6. *Let $(T(t))_{t \geq 0}$ be a C_0-semigroup on a Hilbert space $\mathcal{Z}$ with an infinitesimal generator $A : \mathcal{D}(A) \subseteq \mathcal{Z} \to \mathcal{Z}$. Then the following results hold:*

1. *for $z_0 \in \mathcal{D}(A)$ and $t \geq 0$, $T(t)z_0 \in \mathcal{D}(A)$*
2. *$\frac{d}{dt}(T(t)z_0) = AT(t)z_0 = T(t)Az_0$ for $z_0 \in \mathcal{D}(A)$ and $t \geq 0$,*
3. *for all $z_0 \in \mathcal{Z}$, $\lim\limits_{s \to \infty} \left(\frac{1}{s} \int_0^s T(t)z_0 dt \right) = z_0$.*

Proof. For 1 and 2, let $s \geq 0$ and consider $\dfrac{T(t+s)z_0 - T(t)z_0}{s}$. Since $T(t+s) = T(t)T(s) = T(s)T(t)$, this expression can be rewritten as

$$\frac{T(t+s)z_0 - T(t)z_0}{s} = T(t)\left(\frac{T(s)z_0 - z_0}{s}\right) = \frac{T(s)T(t)z_0 - T(t)z_0}{s}.$$

Taking the limit as $s \to 0$, this expression becomes

$$\lim_{s\to 0}\frac{T(t+s)z_0 - T(t)z_0}{s} = T(t)\lim_{s\to 0}\left(\frac{T(s)z_0 - z_0}{s}\right) = \lim_{s\to 0}\frac{T(s)T(t)z_0 - T(t)z_0}{s}.$$

First we analyse the middle term. Not that since A is the infinitesimal generator of $(T(t))_{t\geq 0}$, by definition the limit $\lim_{s\to 0}\left(\frac{T(s)z_0 - z_0}{s}\right)$ exists and is equal to Az_0.

Since the limit exists, we know that $T(t)Az_0$ also exists.

Since the limit exists, $\lim_{s\to 0}\frac{T(s)T(t)z_0 - T(t)z_0}{s} = AT(t)z_0$ and $T(t)z_0 \in \mathcal{D}(A)$ by definition

Lastly, $\lim_{s\to 0}\frac{T(t+s)z_0 - T(t)z_0}{s}$ is just the definition of the derivative of $T(t)z_0$ with respect to t. Therefore, we conclude

$$\frac{d}{dt}(T(t)z_0) = AT(t)z_0 = T(t)Az_0.$$

For 3, let $z_0 \in \mathcal{Z}$ and $\alpha > 0$. By the definition of a C_0-semigroup, we can choose $\tau > 0$ such that $||T(t)z_0 - z_0||_{\mathcal{Z}} \leq \alpha$ for all $t \in [0, \tau]$.

$$\begin{aligned}
||\frac{1}{s}\int_0^s T(t)z_0 dt - z_0||_{\mathcal{Z}} &= ||\frac{1}{s}\int_0^s T(t)z_0 - z_0 dt||_{\mathcal{Z}} \\
&\leq \frac{1}{t}\int_0^s ||T(t)z_0 - z_0||_{\mathcal{Z}} dt \\
&\leq \alpha \quad \text{for all } t \in [0, \tau].
\end{aligned}$$

This is the definition of convergence, so we can conclude that limit exists and $\frac{1}{s}\int_0^s T(t)z_0 dt \to z_0$.

$\square$

The fact that a classical solution exists is a consequence of this theorem, as stated in the following corollary.

Corollary 2.1.7. *Consider the abstract PDE*

$$\dot{z}(t) = Az(t) \quad z(0) = z_0,$$

where $z_0 \in \mathcal{D}(A) \subseteq \mathcal{Z}$. If A generates a C_0-semigroup $(T(t))_{t\geq 0}$, then $z(t) = T(t)z_0$ is the classical solution to the problem, and the problem is well-posed.

If we relax the requirement for the initial condition such that $z_0 \in \mathcal{Z}$, then $z = T(t)z_0$ is the unique mild solution. If the classical solution exists, then the mild solution is equal to the classical solution.

Proof. Proof for the classical solution.

By theorem 2.1.6,

$$\dot{z}(t) = \frac{d}{dt}(T(t)z_0) = AT(t)z_0 = Az(t).$$

Moreover, since $T(0) = \mathcal{I}$ we have that $z(0) = T(0)z_0 = z_0$. Therefore, this is a solution to the PDE.

Now to prove the solution is unique, suppose for contradiction there is another solution $y(t) \neq z(t)$. Define $w(s) = T(t-s)y(s)$.

Let $s, h \in \mathbb{R}$ such that $s, s+h \in [0,t]$, then

$$\frac{w(s+h)+w(s)}{h} = \frac{T(t-s-h)y(s) - T(t-s)y(s)}{h} + T(t-s-h)\frac{y(s+h)-y(s)}{h}. \quad (2.6)$$

Since y is a classical solution, $y \in \mathcal{D}(A)$. This means

$$\begin{aligned}
&\lim_{h \to 0} \frac{T(t-s-h)y(s) - T(t-s)y(s)}{h} \\
&= -T(t-s) \lim_{h \to 0} \frac{T(h)y(s) - y(s)}{h} = -T(t-s)Ay(s).
\end{aligned} \tag{2.7}$$

Now we consider the term $T(t-s-h)\dfrac{y(s+h)-y(s)}{h}$. Taking the limit as h goes to zero,

$$\begin{aligned}
\lim_{h \to 0} T(t-s-h)\frac{y(s+h)-y(s)}{h} &= \lim_{h \to 0} T(t-s-h) \lim_{h \to 0} \frac{y(s+h)-y(s)}{h} \\
&= T(t-s)\dot{y}(s),
\end{aligned}$$

where separating the limit like this is valid because $T(t)$ is continuous. Since y is a classical solution, $\dot{y}(s) = Ay(s)$ so we have

$$\lim_{h \to 0} T(t-s-h)\frac{y(s+h)-y(s)}{h} = T(t-s)Ay(s) \tag{2.8}$$

Taking the limit as $h \to 0$ of (2.1) and substituting (2.1) and (2.1) in, we obtain

$$\dot{w}(s) = \lim_{h \to 0} \frac{w(s+h)+w(s)}{h} = -T(t-s)Ay(s) + T(t-s)Ay(s) = 0. \tag{2.9}$$

Since $\dot{w}(s) = 0$, it follows that $w(s)$ is constant for all s. Therefore

$$y(t) = T(0)w(t) = w(t) = w(0) = T(t)y(0) = T(t)z_0 = z(t),$$

We have proven that $y(t) = z(t)$, so the solution is unique.

Lastly, it is easy to confirm that the solutions depend continuously on the initial data. Let $w = T(t)w_0$ and $y = T(t)y_0$ be the classical solutions to $\dot{z}(t) = Az(t)$ with initial conditions $z(0) = w_0$ and $z(0) = y_0$ respectively. Let $\alpha > 0$ be given. The goal is to show that for some $\delta > 0$ if $||w_0 - y_0|| \leq \delta$ then $||w(t) - y(t)|| \leq \alpha$.

Since $(T(t))_{t\geq 0}$ is a C_0-semigroup, $T(t)$ is bounded. For an operator to be bounded, it means that for all $z \in \mathcal{D}(A)$ then there exists a $C > 0$ such that $||T(t)z|| \leq C||z||$. Let $\delta = \dfrac{\alpha}{C}$. Then

$$\begin{aligned}
||w(t) - y(t)|| &= ||T(t)w_0 - T(t)y_0|| \\
&= ||T(t)(w_0 - y_0)|| \\
&\leq ||T(t)||_{op}||w_0 - y_0|| \\
&\leq C\delta \\
&= \alpha.
\end{aligned}$$

Therefore if A generates a C_0-semigroup $(T(t))_{t\geq 0}$, the PDE described by (2.3) has a classical solution $z = T(t)z_0$ which is unique and continuously depends on the initial data. Hence, the problem is well-posed.

Proof for the mild solution.

Let $z_0 \in \mathcal{Z}$. Let $s \leq t$ and consider the following integral

$$\frac{T(s) - \mathcal{I}}{s}\int_0^t T(u)z_0du = \frac{1}{s}\int_0^t T(s+u)z_0du - \frac{1}{s}\int_0^t T(u)z_0du. \tag{2.10}$$

Using the substitution $v = s + u$ on the second integral,

$$\frac{T(s) - \mathcal{I}}{s}\int_0^t T(u)z_0du = \frac{1}{s}\int_s^{s+t} T(v)z_0dv - \frac{1}{s}\int_0^t T(u)z_0du. \tag{2.11}$$

The two integrals on the right hand side of (2.1) can be broken apart as

$$\frac{1}{s}\int_{s}^{s+t} T(v)z_0 dv = \frac{1}{s}\int_{s}^{t} T(v)z_0 dv + \frac{1}{s}\int_{t}^{s+t} T(v)z_0 dv, \tag{2.12}$$

and

$$\frac{1}{s}\int_{0}^{t} T(u)z_0 du = \frac{1}{s}\int_{0}^{s} T(u)z_0 du + \frac{1}{s}\int_{s}^{t} T(u)z_0 du. \tag{2.13}$$

Substituting (2.1)-(2.1) into (2.1), the integrals from s to t cancel resulting in the expression

$$\frac{T(s)-\mathcal{I}}{s}\int_{0}^{t} T(u)z_0 du = \frac{1}{s}\int_{t}^{s+t} T(v)z_0 dv - \frac{1}{s}\int_{0}^{s} T(u)z_0 du. \tag{2.14}$$

Preforming the substitution $v = t + u$ on the second integral, we finally arrive at

$$\begin{aligned}\frac{T(s)-\mathcal{I}}{s}\int_{0}^{t} T(u)z_0 du &= \frac{1}{s}\int_{0}^{s} T(u+t)z_0 du - \frac{1}{s}\int_{0}^{s} T(u)z_0 du \\ &= \frac{1}{s}\int_{0}^{s} T(u)(T(t)-\mathcal{I})z_0 du. \end{aligned} \tag{2.15}$$

From Theorem 2.1.6, $\lim_{s\to 0} \frac{1}{s}\int_{0}^{s} T(u)(T(t)-\mathcal{I})z_0 = T(t)z_0 - z_0$. Since this limit exists, we have that

$$\lim_{s\to 0}\frac{T(s)-\mathcal{I}}{s}\int_{0}^{t} T(u)z_0 du = T(t)z_0 - z_0.$$

Thus, since the limit exists by Definition 2.1.3

$$\int_0^t T(u)z_0 du \in \mathcal{D}(A),$$

and the limit is equal to $A\int_0^t T(u)z_0 du$.

For $z(t) = T(t)z_0$, $\int_0^t z(s)ds \in \mathcal{D}(A)$, $z(0) = z_0$, and

$$z(t) - z(0) = A\int_0^t z(s)ds.$$

Therefore, $z(t) = T(t)z_0$ is the mild solution.

To prove uniqueness, assume for contradiction that there exists a mild solution $y(t)$ such that $y(t) \neq z(t)$. Let $w(t) = y(t) - z(t)$. Then $w(0) = y(0) - z(0) = z_0 - z_0 = 0$. Moreover since both $y(t)$ and $z(t)$ are mild solutions,

$$w(t) = y(t) - z(t) = y(0) - z(0) + A\int_0^t (y(s) - z(s))ds = A\int_0^t w(t)ds.$$

In addition,

$$\lim_{h\to 0}\frac{T(h)\int_0^t w(s)ds - \int_0^t w(s)ds}{h} = \lim_{h\to 0}\frac{T(h)\int_0^t y(s)ds - \int_0^t y(s)}{h} - \lim_{h\to 0}\frac{T(h)\int_0^t y(s)ds - \int_0^t y(s)}{h}.$$

Since both $y(t)$ and $z(t)$ are mild solutions, both of the limits on the right exist. We can conclude then that $\int_0^t w(s)ds \in \mathcal{D}(A)$. Hence, $w(t)$ is itself a mild solution to the abstract problem $\dot{w}(t) = Aw(t)$ with $w(0) = 0$. We conclude by the existence proof that the mild solution to this problem is $w(t) = T(t)w(0) = 0$. So $y(t) = z(t)$ which contradicts the assumption that the mild solutions are distinct.

Continuous dependence on initial conditions follows from the same proof as the in the classical solution proof.

□

2.2 Lumer-Phillips Theorem

The previous section demonstrated that for an abstract PDE $\dot{z} = Az(t)$ if A is an infinitesimal generator of a C_0-semigroup then the problem is well-posed. The remaining issue is how do we prove A generates a C_0-semigroup? This section will answer this question by using the Lumer-Phillips theorem to conclude a class of problems do in fact generate a C_0-semigroup. This section will follow the definitions and theorem statements outlined in chapter two of [22]. The Lumer-Phillips theorem provides a condition for A to generate a contraction semigroup.

Definition 2.2.1 (Contraction semigroup)**.** *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup with infinitesimal generator A. $(T(t))_{t\geq 0}$ is called a contraction semigroup if the operator norm of $T(t)$ given by*

$$||T(t)||_{op} = \sup_{z\in\mathcal{D}(A), z\neq 0} \frac{||T(t)z||}{||z||},$$

satisfies $||T(t)||_{op} \leq 1$ for all $t \geq 0$.

Theorem 2.2.2 (Lumer-Phillips)**.** *Let $A : \mathcal{D}(A) \subseteq \mathcal{Z} \to \mathcal{Z}$ be a closed, densely defined operator on a Hilbert space $\mathcal{Z}$. The operator A generates a contraction semigroup on $\mathcal{Z}$ if and only if for all real $\omega > 0$,*

1. *$||(\omega\mathcal{I} - A)z|| \geq \omega||z||$ for all $z \in \mathcal{D}(A)$,*

2. *$||(\omega\mathcal{I} - A^*)z|| \geq \omega||z||$ for all $z \in \mathcal{D}(A^*)$,*

where A^* refers to the adjoint operator of A.

Since this theorem requires the operator to be closed and densely defined, what it means to be closed and dense both need to be defined. Consider the following definitions.

Definition 2.2.3 (Closed)**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces and let $A : \mathcal{D}(A) \subseteq \mathcal{X} \to \mathcal{Y}$ be a linear operator. A is called **closed** if for any sequence $\{z_k\}$ satisfying $z_k \in \mathcal{D}(A)$, $\lim\limits_{k\to\infty} z_k = z$, and $\lim\limits_{k\to\infty} Az_k = y$, we have that $z \in \mathcal{D}(A)$ and $Az = y$.*

Definition 2.2.4 (Dense)**.** *Let $\mathcal{Z}$ be a Hilbert space. The subspace $\mathcal{W} \subseteq \mathcal{Z}$ is called **dense** in $\mathcal{Z}$ if for every $z \in \mathcal{Z}$ and $\alpha > 0$ there is a $w \in \mathcal{W}$ such that $||w - z|| < \alpha$*

Moreover, a proper definition of what an adjoint operator is will be needed in order to use the Lumer-Phillips theorem. The definition of an adjoint operator is taken from [23] as follows.

Definition 2.2.5 (Adjoint operator)**.** *Let A be a linear operator on a Hilbert space $\mathcal{Z}$. The **adjoint** of $A : \mathcal{D}(A) \subseteq \mathcal{Z} \to \mathcal{Z}$, denoted A^*, is given by the operator that satisfies*

$$< Az, w >_{\mathcal{Z}} = < z, A^*w >_{\mathcal{Z}} \quad \forall z, w \in \mathcal{Z} \tag{2.16}$$

with the domain

$$\mathcal{D}(A^*) = \{w \in \mathcal{Z} :< Az, w >_{\mathcal{Z}} = < z, v >_{\mathcal{Z}} \text{ For some } v \in \mathcal{Z} \text{ and all } z \in \mathcal{Z}\}. \tag{2.17}$$

*Moreover, we call A **Self-adjoint** if $A = A^*$ and $\mathcal{D}(A) = \mathcal{D}(A^*)$.*

Proof of the Lumer-Phillips theorem follows from the more general Hille-Yosida theorem. For a statement of the Hille-Yosida theorem and proof of both the Hille-Yosida theorem and the Lumer-Phillips theorem, see [10] chapter 6.1.

The following corollary serves as a sufficient condition to satisfy the Lumer-Phillips theorem without having to consider a constant ω.

Corollary 2.2.6. *Let $A : \mathcal{D}(A) \subseteq \mathcal{Z} \to \mathcal{Z}$ be a closed, densely defined operator on a Hilbert space $\mathcal{Z}$. The operator A generates a contraction semigroup on $\mathcal{Z}$ if*

- *$Re < Az, z > \leq 0$ for all $z \in \mathcal{D}(A)$*

- *$Re < A^*z, z > \leq 0$ for all $z \in \mathcal{D}(A^*)$*

Where Re refers to the real component of the inner product.

Proof. Suppose the conditions the corollary hold. First we prove $||(\omega\mathcal{I} - A)z|| \geq \omega||z||$. Let $z \in \mathcal{D}(A)$, we have that

$$\begin{aligned}||(\omega\mathcal{I} - A)z||^2 &= \omega^2||z||^2 + ||Az||^2 - < \omega z, Az > - < Az, \omega z > \\ &= \omega^2||z||^2 + ||Az||^2 - \omega < z, Az > - \omega < Az, z >\end{aligned}$$

Inner products satisfy conjugate symmetry, so $< a, b >= \overline{< b, a >}$ where the overbar notation refers to the complex conjugate. Moreover, for any complex or real number a, $a + \bar{a} = 2Re(a)$. Thus we can conclude

$$||(\omega\mathcal{I} - A)z||^2 = \omega^2||z||^2 + ||Az||^2 - 2\omega Re < Az, z > \geq \omega^2||z||^2$$

since $||Az||^2 \geq 0$ and by assumption $Re < Az, z > \leq 0$. Taking the root of both sides, this is equivalent to $||(\omega\mathcal{I} - A)z|| \geq \omega||z||$.

The proof for $||(\omega\mathcal{I} - A^*)z|| \geq \omega||z||$ is the exact same except replace A with A^* and let $z \in \mathcal{D}(A^*)$. □

2.3 Finite-Dimensional Port-Hamiltonian Systems

A port-Hamiltonian system (PHS) is a particular subclass of systems of differential equations that have a dissipative structure. We refer to systems of ODEs that obey a port-Hamiltonian structure as finite-dimensional port-Hamiltonian systems. This is to distinguish these systems from PDE systems, which are referred to as infinite-dimensional PHS. In this chapter, the idea of PHS will be introduced through the easier finite-dimensional case first in order to build intuition and help motivate the infinite-dimensional case.

Port-Hamiltonian systems get their name from the Hamiltonian, which in the case of physical systems which do not have explicit time dependence is equal to the total energy of the system [34]. A PHS is a differential equation which is formulated in terms of the system energy, which is useful for showing things such as energy decay. This section will follow [5] chapter 2 and [10] chapters 2 and 4.

There are many ways to formulate a PHS, but for our uses the following definition will suffice.

Definition 2.3.1 (Finite dimensional Port-Hamiltonian system). *Let $z \in \mathcal{Z} \subseteq \mathbb{R}^n$, the state space of a dynamical system. Moreover, let $\mathcal{H} : \mathcal{Z} \to \mathbb{R}$ be the systems Hamiltonian.*

A finite dimensional PHS is given as

$$\begin{aligned} \dot{\boldsymbol{z}}(t) &= [\mathcal{J}(\boldsymbol{z}) - \mathcal{R}(\boldsymbol{z})]\frac{\partial \mathcal{H}}{\partial \boldsymbol{z}} + G(z)\boldsymbol{u}(t) \\ \boldsymbol{y} &= G(\boldsymbol{z})^T \frac{\partial \mathcal{H}}{\partial \boldsymbol{z}} \end{aligned} \tag{2.18}$$

which satisfies the conditions that $\mathcal{J}(\boldsymbol{z}) = -\mathcal{J}(\boldsymbol{z})^T$ is skew-adjoint and $\mathcal{R}(\boldsymbol{z}) \geq 0$ is positive semi-definite. Here $\boldsymbol{u}(t) \in \mathbb{R}^k$ is a control (input) for the system, and $\boldsymbol{y}(t) \in \mathbb{R}^k$ is an observer (output). $\mathcal{R}(\boldsymbol{z})$ represents some resistive structure inherent to the system.

The conditions on $\mathcal{J}(\mathbf{z})$, and $\mathcal{R}(\mathbf{z})$ were made to ensure the system is passive. A passive system is a system defined as follows[22, Definition 7.1].

Definition 2.3.2 (Passivity). *Consider a dynamical system*

$$\dot{\boldsymbol{z}}(t) = A\boldsymbol{z}(t) + B\boldsymbol{u}(t), \tag{2.19}$$

$$\boldsymbol{y}(t) = C\boldsymbol{z}(t) + D\boldsymbol{u}(t), \tag{2.20}$$

where $\boldsymbol{z}(t) \in \mathcal{Z}$, $\boldsymbol{u}(t) \in \mathcal{U}$, $\boldsymbol{y} \in \mathcal{Y}$. Here $\mathcal{Z}, \mathcal{U}, \mathcal{Y}$ are all Hilbert spaces, B,C, and D are all bounded operators on their respective spaces and A generates a C_0-semigroup.

*This system is called **passive** if for some positive semi-definite self-adjoint operator P we have*

$$\frac{1}{2}\frac{d}{dt} < \boldsymbol{z}(t), P\boldsymbol{z}(t) > \quad \leq \quad < \boldsymbol{u}(t), \boldsymbol{y}(t) > .$$

For finite-dimensional systems with quadratic Hamiltonian, this condition is equivalent to $\dot{\mathcal{H}}(t) \leq \mathbf{u} \cdot \mathbf{y}$.

Passivity is a desirable property because it allows us to make conclusions about the stability of a closed loop system with imperfect knowledge of the component systems. A system being passive allows us to apply the passivity theorem [22, Corollary 7.8] to relate the external stability of either the control system or the open loop system to the stability of the closed loop system.

Definition 2.3.3 (External stability). *A system is externally stable if for every input $u(t) \in L^2(0, \infty; \mathcal{U})$ and zero initial condition, the output is also in $L^2(0, \infty; \mathcal{Y})$.*

Theorem 2.3.4 (Passivity Theorem). *If both plant and controller are passive, and at least one of the two systems is externally stable, then the closed loop is externally stable and also passive.*

This theorem says that if the uncontrolled system and the system for the control law are both passive and we demonstrate the external stability of either of these systems, then the controlled system is stable.

Theorem 2.3.5 (Energy Balance). *The PHS (2.18) is passive.*

Proof.

$$\begin{aligned}
\dot{\mathcal{H}}(\mathbf{z}) &= \frac{\partial \mathcal{H}}{\partial \mathbf{z}}^T \dot{\mathbf{z}} \\
&= \frac{\partial \mathcal{H}}{\partial \mathbf{z}}^T \left([\mathcal{J}(\mathbf{z}) - \mathcal{R}(\mathbf{z})] \frac{\partial \mathcal{H}}{\partial \mathbf{z}} + G(z)\mathbf{u}(t) \right) \\
&= \frac{\partial \mathcal{H}}{\partial \mathbf{z}}^T \mathcal{J} \frac{\partial \mathcal{H}}{\partial \mathbf{z}} - \frac{\partial \mathcal{H}}{\partial \mathbf{z}}^T \mathcal{R} \frac{\partial \mathcal{H}}{\partial \mathbf{z}} + \frac{\partial \mathcal{H}}{\partial \mathbf{z}}^T G\mathbf{u}.
\end{aligned}$$

Since $\mathcal{J} = -\mathcal{J}^T$ we have $\frac{\partial \mathcal{H}}{\partial \mathbf{z}}^T \mathcal{J} \frac{\partial \mathcal{H}}{\partial \mathbf{z}} = -\mathcal{J} |\frac{\partial \mathcal{H}}{\partial \mathbf{z}}|^2 \leq 0$. Moreover, since $\mathcal{R}$ is positive semi-definite, $-\frac{\partial \mathcal{H}}{\partial \mathbf{z}}^T \mathcal{R} \frac{\partial \mathcal{H}}{\partial \mathbf{z}} \leq 0$.

Using these inequalities, the equality above can be rewritten as

$$\begin{aligned}\dot{\mathcal{H}}(\mathbf{z}) &\leq \frac{\partial \mathcal{H}}{\partial \mathbf{z}}^T G u \\ &= \left(G^T \frac{\partial \mathcal{H}}{\partial \mathbf{z}}\right)^T \mathbf{u} \\ &= \mathbf{y}^T \mathbf{u} \\ &= \mathbf{u}^T \mathbf{y}.\end{aligned}$$

□

A consequence of this theorem gives a direct relation between the energy decay and the input-output of the system. To better see how systems can fit into a PHS framework, consider the following example.

Example 2.3.6. *Consider the model of a cart attached to a spring and a damper on a track (Figure 2.1). Assume the spring has a spring constant k, the damper has a damping constant c, and that the cart has an external force applied $F_{applied} = \alpha u(x)$. For simplicity, it is assumed that friction is negligible and can be ignored.*

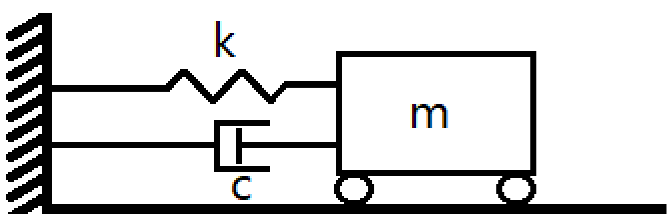

Figure 2.1: Cart attached to a spring and damper

The spring force is modeled by Hooke's law $F_s = -kx$, and the damper is modeled by the damping force $F_d = -c\dot{x}$.

From Newton's 3rd law, the dynamics of the system are described by

$$m\ddot{x} = F_{net} = -kx - c\dot{x} + \alpha u(x). \tag{2.21}$$

In this case, there is no explicit time dependence in the total energy, so the Hamiltonian and energy coincide and are given by the spring potential and kinetic energy of the cart

$$\mathcal{H} = \frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2. \tag{2.22}$$

To convert this problem into a PHS, we need to reduce the order from a second-order ODE to a first-order system of ODEs. Let $z_1 = x$*,* $z_2 = \dot{x}$ *so that the system can be rewritten as*

$$\dot{\boldsymbol{z}} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \frac{1}{m}\begin{bmatrix} mz_2 \\ -kz_1 - cz_2 + \alpha u \end{bmatrix}.$$

Computing the derivatives of $\mathcal{H}$ *w.r.t* z_1 *and* z_2 *we have* $\frac{\partial \mathcal{H}}{\partial \boldsymbol{z}} = \begin{bmatrix} kz_1 \\ mz_2 \end{bmatrix}$*. Therefore the system can be rewritten in the form*

$$\dot{\boldsymbol{z}} = \left(\frac{1}{m}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \frac{c}{m^2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \frac{\partial \mathcal{H}}{\partial \boldsymbol{z}} + \frac{\alpha}{m}\begin{bmatrix} 0 \\ 1 \end{bmatrix} u(x).$$

Clearly, $\mathcal{J} = \frac{1}{m}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *is skew-ajoint and* $\mathcal{R} = \frac{c}{m^2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ *is positive semi-definite.*

Pairing the system with an observer $y = \frac{\alpha}{m}[0\ 1]\frac{\partial \mathcal{H}}{\partial \boldsymbol{z}} = \alpha z_2 = \alpha \dot{x}$*, it has been demonstrated that this mass cart system can be modelled in a port-Hamiltonian framework.*

2.4 Infinite-Dimensional Port-Hamiltonian Systems in One Spatial Dimension

In the prior section the topic of PHS for ODEs was discussed. Our goal is to find an analogous setup for PDEs which obey the nice properties of finite-dimensional PHS. In particular, we want a system which is formulated in terms of the Hamiltonian and is passive. This section will follow [10] chapters 7 and 11, as well as [20].

Before attempting to set up our abstract PHS in finite dimensions, some intricacies that come as a result of working with PDEs need to be discussed. First and foremost, PDEs have a spatial domain that needs to be considered. While ODEs only need an initial condition, PDE theory tells us that we need to also mention boundary conditions if we have any hope for our problem to be well-posed. Moreover, unlike ODE systems which can only have a control acting directly on the system, PDEs permit the use of boundary control and observation. With this in mind, our definition of an infinite-dimensional PHS will be formulated as a boundary control problem.

Another difference that a PDE system will have comes from the form of the Hamiltonian. For ODEs, the Hamiltonian is a function which maps the state space to a real value. Since the Hamiltonian corresponds to the total energy for most physical systems, PDEs will have a Hamiltonian of the form

$$\mathcal{H}(t) = \int\limits_{\Omega} h(\mathbf{z}, x)\ d\Omega, \tag{2.23}$$

where Ω is the domain of the space being considered, and $h(\mathbf{z}, x)$ is the Hamiltonian density. A particular subclass of problems that are of interest are problems with a quadratic Hamiltonian density. In this case, the Hamiltonian is given by

$$\mathcal{H}(t) = \frac{1}{2}\int\limits_{\Omega} \mathbf{z}^T h(x)\mathbf{z}\ d\Omega, \tag{2.24}$$

for some weight matrix h. Since we want the PHS to be passive, we will need to look at the total time derivative of the Hamiltonian. Taking the time derivative of (2.23) results in

$$\dot{\mathcal{H}}(t) = \int\limits_{\Omega} \frac{\partial h}{\partial \mathbf{z}}^T \partial_t \mathbf{z}\ d\Omega.$$

This would suggest that the analogous system should be reformulated in terms of $\dfrac{\partial h}{\partial \mathbf{z}}$.

We are interested in first order partial differential equations of the form

$$\partial_t \mathbf{z} = (P_1 \partial_x + P_0)(h\mathbf{z}). \tag{2.25}$$

To define a infinite-dimensional PHS that has dynamics which are structured as (2.25), we need to discuss a suitable boundary control and observer. To accomplish this, we first define the boundary flow and boundary effort of the system, since formulating the control and observer in terms of these quantities will be convenient.

Definition 2.4.1 (Boundary Flow and Effort)**.** *Recall $\Omega = [a, b]$. Consider the PDE system defined by (2.25). The boundary effort e_∂ and boundary flow f_∂ are defined by*

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = R_0 \begin{bmatrix} (h\boldsymbol{z})(b) \\ (h\boldsymbol{z})(a) \end{bmatrix}, \tag{2.26}$$

where

$$R_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} P_1 & -P_1 \\ \mathcal{I} & \mathcal{I} \end{bmatrix}. \tag{2.27}$$

Now we can define a first-order infinite-dimensional PHS.

Definition 2.4.2 (First-order linear PHS)**.** *Let $\mathcal{Z}$ be the state space of a dynamic system over a region $\Omega = [a, b] \in \mathbb{R}$ equipped with the inner product*

$$< \boldsymbol{f}, \boldsymbol{g} >_{\mathcal{Z}} = \frac{1}{2} \int_a^b g^T h f dx. \tag{2.28}$$

Let $P_0, P_1 \in \mathbb{R}^{nxn}$ be $n \times n$ square matrices such that P_0 is skew-adjoint (i.e $P_0^T = -P_0$) and P_1 is self-adjoint (i.e $P_1^T = P_1$). Let $h(x)$ be self-adjoint and bounded such that for some $m, M > 0$ we have $m\mathcal{I} \leq h(x) \leq M\mathcal{I}$ for all x. Finally let W_B and W_C be matrices such that W_B and $\begin{bmatrix} W_B \\ W_C \end{bmatrix}$ have full rank. The system which satisfies these conditions and is given by

$$\partial_t \boldsymbol{z} = (P_1 \partial_x + P_0)(h\boldsymbol{z}), \tag{2.29}$$

$$\boldsymbol{u} = W_B \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}, \tag{2.30}$$

$$\boldsymbol{y} = W_C \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}, \tag{2.31}$$

is called a linear first-order infinite-dimensional PHS. The associated Hamiltonian is the quadratic Hamiltonian

$$\mathcal{H}(t) = \frac{1}{2}\int_a^b \boldsymbol{z}^T h \boldsymbol{z} dx.$$

With the conditions set in this definition, well-posedness of the boundary control problem is guaranteed via Lumer-Phillips theorem (see [10, Theorem 11.3.5] for details).

This definition can be extended to allow for a resistive structure similar to the ODE case. From [20], we can instead write (2.29) as $\partial_t \mathbf{z} = (\mathcal{J} - \mathcal{R})(h\mathbf{z})$, where $\mathcal{J} = P_1\partial_x + P_0$ and $\mathcal{R} \geq 0$ is positive semi-definite.

Now all that remains is to prove that the system as defined is passive. First a couple of technical results are proven.

Lemma 2.4.3. *Let* $R_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} P_1 & -P_1 \\ \mathcal{I} & \mathcal{I} \end{bmatrix}$ *and define* $\Sigma = \begin{bmatrix} \mathcal{O} & \mathcal{I} \\ \mathcal{I} & \mathcal{O} \end{bmatrix}$,

Then $R_0^T \Sigma R_0 = \begin{bmatrix} P_1 & \mathcal{O} \\ \mathcal{O} & -P_1. \end{bmatrix}$

Proof. Doing simple block matrix multiplication and using the fact that $P_1^T = P_1$,

$$\frac{1}{2}\begin{bmatrix} P_1 & \mathcal{I} \\ -P_1 & \mathcal{I} \end{bmatrix}\begin{bmatrix} \mathcal{O} & \mathcal{I} \\ \mathcal{I} & \mathcal{O} \end{bmatrix}\begin{bmatrix} P_1 & -P_1 \\ \mathcal{I} & \mathcal{I} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \mathcal{I} & P_1 \\ \mathcal{I} & -P_1 \end{bmatrix}\begin{bmatrix} P_1 & -P_1 \\ \mathcal{I} & \mathcal{I} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2P_1 & \mathcal{O} \\ \mathcal{O} & -2P_1 \end{bmatrix}$$

□

Theorem 2.4.4. *Assume the system* $\partial_t \boldsymbol{z} = (P_1\partial_x + P_0 - \mathcal{R})(h\boldsymbol{z})$ *which obeys all of the conditions outlined in Definition 2.4.2. Moreover assume that* $\boldsymbol{y}, \boldsymbol{u} \in \mathbb{R}^n$ *so that the system has full measurements. Then the system is passive with an energy balance*

$$\dot{\mathcal{H}(t)} \leq \frac{1}{2}\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{y} \end{bmatrix}^T P_{W_B,W_c}\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{y} \end{bmatrix}, \tag{2.32}$$

where

$$P_{W_B,W_c} = \left(\begin{bmatrix} W_B \\ W_C \end{bmatrix} \Sigma \begin{bmatrix} W_B \\ W_C \end{bmatrix}^T \right)^{-1} \tag{2.33}$$

Proof. Taking a time derivative of $\mathcal{H}(t)$ and using product rule

$$\begin{aligned}
\dot{\mathcal{H}}(t) &= \frac{1}{2}\int_a^b \partial_t \mathbf{z}^T h\mathbf{z} + \mathbf{z}h\partial_t \mathbf{z} dx \\
&= \frac{1}{2}\int_a^b ((P_1\partial_x + P_0 - \mathcal{R})(h\mathbf{z}))^T(h\mathbf{z}) + (h\mathbf{z}))(h\mathbf{z}))^T(P_1\partial_x + P_0 - \mathcal{R})(h\mathbf{z})dx \\
&= \frac{1}{2}\int_a^b (\partial_x(h\mathbf{z}))^T P_1^T(h\mathbf{z}) + (h\mathbf{z})P_1\partial_x(h\mathbf{z}) + (h\mathbf{z})^T(P_0^T + P_0)(h\mathbf{z}) - (h\mathbf{z})^T(\mathcal{R}+\mathcal{R}^T)(h\mathbf{z})dx.
\end{aligned}$$

Using the fact that $P_0^T = -P_0$, $P_1^T = P_1$, $\mathcal{R}^T = \mathcal{R}$,

$$\begin{aligned}
\dot{\mathcal{H}}(t) &= \frac{1}{2}\int_a^b (\partial_x(h\mathbf{z}))^T P_1(h\mathbf{z}) + (h\mathbf{z})P_1\partial_x(h\mathbf{z}) - (h\mathbf{z})^T\mathcal{R}(h\mathbf{z})dx \\
&= \frac{1}{2}\int_a^b \partial_x((h\mathbf{z})^T P_1(h\mathbf{z})) - (h\mathbf{z})^T\mathcal{R}(h\mathbf{z})dx \\
&\le \frac{1}{2}\left[(h\mathbf{z})^T P_1(h\mathbf{z})\right]_{x=a}^b .
\end{aligned}$$

The last step used the fundamental theorem of calculus and the fact that $\mathcal{R}$ is positive definite. Notice that

$$
\begin{aligned}
\left[(h\mathbf{z})^T P_1 (h\mathbf{z})\right]_{x=a}^{b} &= ((h\mathbf{z})(b))^T P_1((h\mathbf{z})(b)) - ((h\mathbf{z})(b))^T P_1((h\mathbf{z})(b)) \\
&= \begin{bmatrix} (h\mathbf{z})(b) \\ (h\mathbf{z})(a) \end{bmatrix}^T \begin{bmatrix} P_1 & \mathcal{O} \\ \mathcal{O} & -P_1. \end{bmatrix} \begin{bmatrix} (h\mathbf{z})(b) \\ (h\mathbf{z})(a) \end{bmatrix} \\
&= \begin{bmatrix} (h\mathbf{z})(b) \\ (h\mathbf{z})(a) \end{bmatrix}^T R_0^T \Sigma R_0 \begin{bmatrix} (h\mathbf{z})(b) \\ (h\mathbf{z})(a) \end{bmatrix} \\
&= \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}^T \Sigma \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}.
\end{aligned}
$$

Since we are assuming both $\mathbf{u}$ and $\mathbf{y}$ are n-dimensional, $\begin{bmatrix} W_B \\ W_C \end{bmatrix}$ is a square matrix with full rank. Linear algebra tells us this matrix is invertible, so we can write

$$
\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} W_B \\ W_C \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{u} \\ \mathbf{y} \end{bmatrix}.
$$

Moreover, it is easy to verify using block multiplication that $\Sigma = \Sigma^{-1}$. Putting this all together,

$$
\begin{aligned}
\frac{1}{2}\left[(h\mathbf{z})^T P_1 (h\mathbf{z})\right]_{x=a}^{b} &= \begin{bmatrix} \mathbf{u} \\ \mathbf{y} \end{bmatrix}^T \begin{bmatrix} W_B \\ W_C \end{bmatrix}^{-T} \Sigma^{-1} \begin{bmatrix} W_B \\ W_C \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{u} \\ \mathbf{y} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{u} \\ \mathbf{y} \end{bmatrix}^T \left(\begin{bmatrix} W_B \\ W_C \end{bmatrix}^T \Sigma \begin{bmatrix} W_B \\ W_C \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{u} \\ \mathbf{y} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{u} \\ \mathbf{y} \end{bmatrix}^T P_{W_B, W_c} \begin{bmatrix} \mathbf{u} \\ \mathbf{y} \end{bmatrix},
\end{aligned}
$$

completing the proof. □

As desired, we have formulated a passive system which interacts via the boundary and is formulated in terms of the Hamiltonian.

To demonstrate how in practice this works, consider the following example.

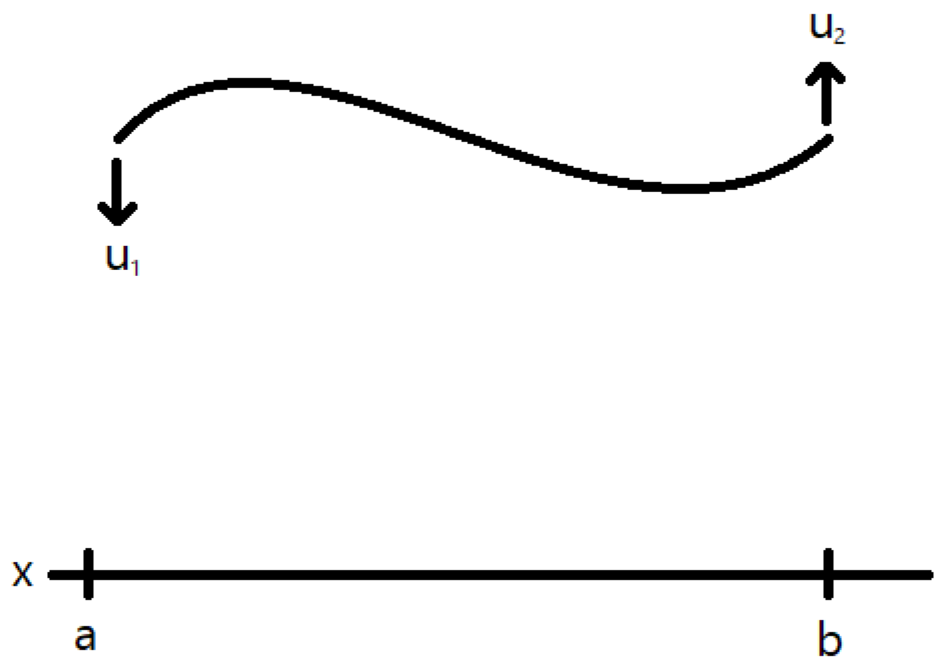

Figure 2.2: 1D vibrating string with both ends free

Example 2.4.5 ([10], Example 11.3.6)**.** *Consider the problem of a 1D vibrating string on the domain $[a, b] \in \mathbb{R}$. We assume that both sides of the string are free to move up and down (Figure 2.2).*

The model for this system is given by

$$\partial_{tt} w(x,t) = \frac{1}{d(x)} \partial_x (T(x) \partial_x w(x,t)), \tag{2.34}$$

where $d(x)$ is the mass density of the string and $T(x)$ is Young's modulus, both properties of the material the string is made of. Both d and T are taken to be strictly positive and bounded quantities. Physically $\partial_t w$ represents the velocity of the string, and $T(x)\partial_x w$ represents the force acting on the string. Assume a classical solution $w \in C^2([a, b] \times \mathbb{R})$ exists.

The Hamiltonian for this system is given as

$$\mathcal{H}(t) = \frac{1}{2}\int_a^b d(x)(\partial_t w)^2 + T(x)(\partial_x w)^2 dx. \tag{2.35}$$

Suppose both ends of the string are controlled via an external force. This is modelled by the boundary condition

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} T(a)\partial_x w(a,t) \\ T(b)\partial_x w(b,t) \end{bmatrix}. \tag{2.36}$$

Moreover, the physically measurable quantity in this problem is the velocity of the strings at both ends. Thus we desire an observer of the form

$$\begin{bmatrix} y_1(t) \\ 2_2(t) \end{bmatrix} = \begin{bmatrix} \partial_t w(a,t) \\ \partial_t w(b,t) \end{bmatrix}. \tag{2.37}$$

First, we aim to recast our dynamics into a port-Hamiltonian structure. Notice the Hamiltonian can be rewritten as

$$\mathcal{H}(t) = \frac{1}{2}\int_a^b \begin{bmatrix} d(x)\partial_t w \\ \partial_x w \end{bmatrix}^T \begin{bmatrix} \frac{1}{d(x)} & 0 \\ 0 & T(x) \end{bmatrix} \begin{bmatrix} d(x)\partial_t w \\ \partial_x w \end{bmatrix} dx. \tag{2.38}$$

This gives us a quadratic Hamiltonian with density $h(x) = \begin{bmatrix} \frac{1}{d(x)} & 0 \\ 0 & T(x) \end{bmatrix}$ *and state space variable* $\boldsymbol{z} = \begin{bmatrix} d(x)\partial_t w \\ \partial_x w \end{bmatrix}$.

Taking a time derivative of $\boldsymbol{z}$ *while making use of the fact that since we are assuming* $w \in C^2$ *we can interchange the order of second derivatives to obtain* $\partial_{xt} w = \partial_{tx} = \partial_x(\frac{1}{d(x)} z_1)$,

$$\partial_t \boldsymbol{z} = \begin{bmatrix} d(x)\partial_{tt} w \\ \partial_t x w \end{bmatrix} = \begin{bmatrix} \partial_x(T(x) z_2) \\ \partial_x(\frac{1}{d(x)} z_1). \end{bmatrix}$$

Notice that $h\boldsymbol{z} = \begin{bmatrix} \frac{1}{d(x)} z_1 \\ T(x) z_2 \end{bmatrix}$, *so by using a rotation matrix we can write*

$$\partial_t \boldsymbol{z} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \partial_x (h\boldsymbol{z}).$$

This is the form of (2.25) *with* $P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $P_0 = \mathcal{O}$. *What is left is to recast the boundary conditions in port-Hamiltonian form. First we determine the boundary flow and effort.*

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = R_0 \begin{bmatrix} (h\mathbf{z})(\mathbf{b}) \\ (h\mathbf{z})(\mathbf{a}) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} (h\mathbf{z})(\mathbf{b}) \\ (h\mathbf{z})(\mathbf{a}) \end{bmatrix}. \tag{2.39}$$

When worked out, this matrix multiplication results in

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} T(b)z_2(b,t) - T(a)z_2(a,t) \\ \frac{1}{d(b)} z_1(b,t) - \frac{1}{d(a)} z_1(a,t) \\ \frac{1}{d(b)} z_1(b,t) + \frac{1}{d(a)} z_1(a,t) \\ T(b)z_2(b,t) + T(a)z_2(a,t) \end{bmatrix}.$$

Setting $W_B = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$, *we have that*

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = W_B \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}.$$

Moreover we set $W_C = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$, *which results in*

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = W_C \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}.$$

Clearly W_B *has rank 2, and* $\begin{bmatrix} W_B \\ W_C \end{bmatrix}$ *has rank 4, so the requirements for this to be a PHS are met.*

2.5 Stability

Exponential stability is defined as follows [10, Definition 8.8.1].

Definition 2.5.1 (Exponential Stability)**.** *Let $\boldsymbol{z}(\boldsymbol{x}, t)$ be the state space variables for a dynamical system defined over a region Ω. We say that the system is **exponentially stable** with regards to some norm $||\cdot||$ if there exists constants $M, q > 0$ such that for all $t \geq 0$*

$$||\boldsymbol{z}(\boldsymbol{x}, t)|| \leq Me^{-qt}||\boldsymbol{z}(\boldsymbol{x}, 0)||. \tag{2.40}$$

One of the more common approaches to proving a system's stability comes from Lyapunov theory. In Lyapunov theory, we demonstrate exponential stability by defining a suitable Lyapunov functional $V(t)$.

Coming up with a suitable Lyapunov function is not always trivial. Often, the system energy $\mathcal{H}(t)$ can be used as a Lyapunov function but this is not always the case. In cases where the system energy is not enough to prove exponential stability, one approach is modify the energy with additional terms to define a new Lyapunov functional which will demonstrate exponential stability. For this we introduce the multiplier method approach based on [36, Chapter 7.6].

Example 2.5.2. *Consider the initial and boundary value problem for the 1D wave equation on an open domain $(a, b) \in \mathbb{R}$*

$$\begin{aligned}
\partial_{tt}w(x,t) &= \partial_{xx}w(x,t) & (x,t) &\in (a,b)\times[0,\infty), && (2.41)\\
w(a,t) &= 0 & t &\in [0,\infty), && (2.42)\\
\partial_x w(b,t) + \beta^2\partial_t w(b,t) &= 0 & t &\in [0,\infty), && (2.43)\\
w(x,0) = w_0(x), \partial_t w(x,0) &= q_0(x) & x &\in (a,b). && (2.44)
\end{aligned}$$

Assume that a unique classical solution $w \in C^2((a, b))$ exists. The energy for this system is

$$\mathcal{H}(t) = \int_a^b (\partial_x w)^2 + (\partial_t w)^2 dx. \tag{2.45}$$

We can set this up as a system by taking $z_1 = \partial_x w$ and $z_2 = \partial_t w$. When written like this, $\mathcal{H} =< \boldsymbol{z}, \boldsymbol{z} >$. Taking the time derivative of the energy

$$\begin{aligned}\dot{\mathcal{H}} &= 2\int_a^b \partial_x w \partial_{xt} w + \partial_t w \partial_{tt} w dx \\ &= 2\int_a^b \partial_x w \partial_{tx} w + \partial_t w \partial_{tt} w dx,\end{aligned}$$

where the order of the partial derivatives could be swapped since $w \in C^2$. Doing integration by parts on the first term,

$$\dot{\mathcal{H}} = 2\ \partial_x w(x,t)\partial_t w(x,t)\big|_{x=a}^{b} + \int_a^b \partial_t w \partial_{xx} w - \partial_t w \partial_{tt} w dx.$$

The integrand is zero since $\partial_{tt} w = \partial_{xx} w$. Substituting the boundary conditions into the first term leaves us with

$$\dot{\mathcal{H}} = -2\beta(b)^2(\partial_t w(b,t))^2. \tag{2.46}$$

From (2.46) it is clear that the energy is decreasing. This implies that $\partial_t w(x,t)$ and $\partial_x w(x,t)$ must be decreasing. This is not enough information to conclude that the system is stable. To gain more information about the system, we use the multiplier method.

Let $m = x$ be our multiplier, and define

$$w(t) = \int_a^b 2m(\partial_t w)(\partial_x w) dx, \tag{2.47}$$

First we take a time derivative of $w(t)$ and simplify,

$$
\begin{aligned}
\dot{w}(t) &= 2\int_a^b m(\partial_{tt}w)(\partial_x w) + m(\partial_t w)(\partial_{xt}w)dx \\
&= 2\int_a^b m(\partial_{xx}w)(\partial_x w) + m(\partial_t w)(\partial_{xt}w)dx. \qquad (2.48)
\end{aligned}
$$

Notice

$$
\partial_x(m(\partial_x w)^2) = (\partial_x)^2 + 2m(\partial_{xx}w)(\partial_x w), \qquad (2.49)
$$
$$
\partial_x(m(\partial_t w)^2) = (\partial_t)^2 + 2m(\partial_{xt}w)(\partial_t w), \qquad (2.50)
$$

Substituting (2.49) *and* (2.50) *into* (2.48) *and applying the boundary conditions,*

$$
\begin{aligned}
\dot{w}(t) &= \int_a^b \partial_x(m(\partial_x w)^2 + m(\partial_t w)^2) - (\partial_x w)^2 - (\partial_t w)^2 dx \\
&= \left[m(\partial_x w)^2 + m(\partial_t w)^2\right]_{x=a}^{b} - \mathcal{H}(t) \\
&= b(\partial_x w(b,t))^2 + b(\partial_t w(b,t))^2 - \mathcal{H}(t) \\
&= b(1+\beta^4)(\partial_t w(b,t))^2 - \mathcal{H}(t) \\
&= -b\frac{1+\beta^4}{\beta^2}\dot{\mathcal{H}}(t) - \mathcal{H}(t).
\end{aligned}
$$

Moreover,

$$
|w(t)| = 2\left|\int_a^b m(\partial_t w)(\partial_x w)dx\right| \leq 2b\int_a^b |\partial_t w||\partial_x|dx \leq 2b\int_a^b (\partial_t 2)^2 + (\partial_x w)^2 dx = 2b\mathcal{H}(t). \qquad (2.51)
$$

Here an argument from [21] is applied. Define for $\alpha > 0$,

$$V_\alpha = \mathcal{H} + \alpha w. \tag{2.52}$$

Then

$$\mathcal{H}(t) - \alpha|w(t)| \leq V_\alpha \leq \mathcal{H}(t) + \alpha|w(t)|.$$

Using (2.51), $(1-2b\alpha)\mathcal{H}(t) \leq \mathcal{H}(t) - \alpha|w(t)|$ *and* $\mathcal{H}(t) + \alpha|w(t)| \leq (1+2b\alpha)\mathcal{H}(t)$. *Thus,*

$$(1-2b\alpha)\mathcal{H}(t) \leq V_\alpha(t) \leq (1+2b\alpha)\mathcal{H}(t) \tag{2.53}$$

For $\alpha \leq \frac{1}{2b}$ *we are guaranteed that* $V_\alpha \geq 0$. *Taking a time derivative of* V_α,

$$\dot{V}_\alpha(t) = \dot{\mathcal{H}}(t) + \dot{w}(t) \leq \left(1 - \alpha b \frac{1+\beta^4}{\beta^2}\right)\dot{\mathcal{H}}(t) - \alpha\mathcal{H}(t).$$

For $\alpha \geq \frac{\beta^2}{b(1+\beta^4)}$,

$$\dot{V}_\alpha(t) \leq -\alpha\mathcal{H}(t).$$

Finally, (2.53) *implies* $-\alpha\mathcal{H}(t) \leq -\frac{1}{(1+2b\alpha)}V_\alpha(t)$, *so*

$$\dot{V}_\alpha(t) \leq -\frac{1}{(1+2b\alpha)}V_\alpha(t).$$

A direct consequence of this is $V_\alpha(t) = V_\alpha(0)e^{-qt}$ *where* $q = \frac{1}{(1+2b\alpha)}$.

We conclude exponential stability by observing that from (2.53)

$$\begin{aligned} V_\alpha(0) &\leq (1+2b\alpha)\mathcal{H}(0), \\ (1-2b\alpha)\mathcal{H}(t) &\leq V_\alpha(t), \end{aligned}$$

which means for $M = \frac{1+2b\alpha}{1-2b\alpha}$, $q = \frac{1}{(1+2b\alpha)}$ *and* $\alpha \in [\frac{\beta^2}{b(1+\beta^4)}, \frac{1}{2b}]$,

$$\mathcal{H}(t) \leq Me^{-qt}\mathcal{H}(0).$$

Remark 2.5.3. *Note that* $\frac{\beta^2}{(1+\beta^4)} \leq \frac{1}{2}$ *for all* β *This can be seen by considering the extrema of the function* $f(x) = \frac{x^2}{1+x^4}$.

$$f'(x) = \frac{2x}{1+x^4} - \frac{x^2(4x^3)}{(1+x^4)^2} = \frac{2x(1-x^4)}{(1+x^4)^2}$$

Clearly, $f'(x) = 0$ *for* $x = 0$ *and* $|x| = 1$. *The minimum is* $f(0) = 0$, *and the maximum(s) are* $f(1) = f(-1) = \frac{1^2}{1+1^4} = \frac{1}{2}$. *Therefore* $0 \leq f(x) \leq \frac{1}{2}$ *for all* x.

The consequence of this is is the set $[\frac{\beta^2}{b(1+\beta^4)}, \frac{1}{2b}]$ *is guaranteed to be non-empty, so there is always an* α *such that the system is exponentially stable.*

Chapter 3

Maxwell's Equations

Electromagnetism is modelled by a set of four equations known as Maxwell's equations. These equations relate the electric field, magnetic fields, electric charge density, and the charge electric current density. Maxwell's equations are a fundamental set of governing equations for electromagnetism, so they are usually considered to be the starting set of equations when dealing with classical electrodynamics. In free space, Maxwell's equations are given as

$$\nabla \cdot \mathbf{E} = \varepsilon_0 \rho_{tot} \tag{3.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3.2}$$

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E} \tag{3.3}$$

$$\varepsilon_0 \mu_0 \partial_t \mathbf{E} = \nabla \times \mathbf{B} - \mu_0 \mathbf{J}_{tot}. \tag{3.4}$$

Here $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, $\mathbf{J}_{tot}$ is the total current density and ρ_{tot} is the total charge density. This set of equations is a collection of previously deduced physical laws. In particular, (3.1) is Gauss' law, (3.3) is Faraday's law, and (3.4) is Ampére's law with a correction by Maxwell. The second equation (3.2) is an unnamed divergence free condition for the magnetic field.

This chapter will discuss Maxwell's equations in matter. Section 3.1 will discuss how the existence of matter causes there to be bound charge and current. Since bound charge and bound current cannot be directly interacted with, it is convenient to absorb terms related to the bound charge and current into the fields and reformulate in terms of just the free charge and current. Section 3.2 will go over a derivation for physical boundary conditions and discuss the impedance boundary condition that is commonly considered for electromagnetic systems. After the physical boundary conditions are established, sections 3.3-3.5 will set up a particular system we are interested in analysing, discussing the energy of the system, and converting the problem to an abstract system so well-posedness can be proven.

3.1 Maxwell's Equations in Matter

While Maxwell's equations in terms of the total charge and density are always valid, this is not always the most convenient way to consider the system. Bound charges and currents cannot be directly interfaced with, so if the goal is to control an electromagnetic system it is ideal to reformulate in terms of things that can be interacted with. The goal of this section will be to rewrite Maxwell's equations in terms of only free charge and current density. Most of this section will follow [6], with discussion on non-linear materials from [3].

In matter, an electric field will cause the electrons to align with the field. This alignment will cause the existence of a bound charge density which in turn will create an internal electric field called the polarization (denoted $\mathbf{P}$). Similarly a magnetic field will cause electrons to flow and create an internal bound current, causing an internal magnetisation (denoted $\mathbf{M}$). From here on, we will assume we are dealing with electromagnetism in a region Ω made of some material (interchangeably, the material will also be referred to as Ω).

The following lemma will state how the bound charge density and bound current density relates to the electric polarization and the magnetisation. Derivations of these quantities can be found in [6, Chapters 4.2.1, 6.2.1, and 7.3.5].

Lemma 3.1.1. *Let $\boldsymbol{P}$ be the electric polarization and $\boldsymbol{M}$ be the magnetization of a material Ω. The associated bound charge density is*

$$\rho_b = -\nabla \cdot \boldsymbol{P}. \tag{3.5}$$

Moreover, there are two types of bound current density. The first, denoted $\boldsymbol{J}_b$ comes directly from the magnetization and is given by the expression

$$\boldsymbol{J}_b = \nabla \times \boldsymbol{M}. \tag{3.6}$$

The second type of bound current density, which we will call the polarization current density, comes from the flow of bound charges and is given by

$$\boldsymbol{J}_p = \partial_t \boldsymbol{P} \tag{3.7}$$

Denote the free charge and current densities as ρ and $\mathbf{J}$ respectively. The total charge (resp. current) density can be written as the sum of the free and bound charge (resp. current) as $\rho_{tot} = \rho + \rho_b$ (resp. $\mathbf{J}_{tot} = \mathbf{J} + \mathbf{J}_b + \mathbf{J}_p$). Applying Lemma 3.1.1, we obtain

$$\rho = \rho_f - \nabla \cdot \mathbf{P} \tag{3.8}$$

$$\mathbf{J} = \mathbf{J}_f + \nabla \times \mathbf{M} + \partial_t \mathbf{P}. \tag{3.9}$$

Before proceeding to convert Maxwell's equations to being in terms of only the free charge and current density, we need to define the following:

Definition 3.1.2. *Consider a material Ω. Let $\boldsymbol{E}$ be the electric field, $\boldsymbol{B}$ be the magnetic field, $\boldsymbol{P}$ be the electric polarization inside Ω, and $\boldsymbol{M}$ be the magnetization inside Ω.*

*The **electric displacement** is*

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}. \tag{3.10}$$

*The **magnetic displacement** is*

$$\boldsymbol{H} = \mu_0^{-1} \boldsymbol{B} - \boldsymbol{M} \tag{3.11}$$

(See page x for a complete list of notation).

Now we are equipped to convert Maxwell's equations to a form where only the free charge and current density are explicitly included.

Theorem 3.1.3. *Maxwell's equations (3.1)-(3.4) inside a material Ω with an electric polarization $\boldsymbol{P}$ and magnetization $\boldsymbol{M}$ can be written as*

$$\nabla \cdot \boldsymbol{D} = \rho_f \tag{3.12}$$
$$\nabla \cdot \boldsymbol{B} = 0 \tag{3.13}$$
$$\partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E} \tag{3.14}$$
$$\partial_t \boldsymbol{D} = \nabla \times \boldsymbol{H} - \boldsymbol{J}_f. \tag{3.15}$$

Proof. First consider Gauss' law (3.1). Substituting in the expression for the total charge density (3.8),

$$\varepsilon_0 \nabla \cdot \mathbf{E} = \rho - \nabla \cdot \mathbf{P}.$$

Rearranging, we obtain $\nabla \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \rho$. The term inside the divergence is exactly what was defined to be $\mathbf{D}$, completing the first part of the proof.

Now consider Ampére's law (3.4). Substituting the expression for the total current density (3.9)

$$\varepsilon_0 \partial_t \mathbf{E} = \mu_0^{-1} \nabla \times \mathbf{B} - \mathbf{J} - \nabla \times \mathbf{M} - \partial_t \mathbf{P}.$$

Putting the derivative terms together, we arrive at the expression

$$\partial_t (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \nabla \times (\mu_0^{-1} \mathbf{B} - \mathbf{M}) - \mathbf{J}.$$

The term inside the time derivative is once again by definition $\mathbf{D}$, and the term inside the curl was defined to be $\mathbf{H}$, completing the proof. □

While this new formulation of Maxwell's equations is only in terms of the free charge and current density, the system has too few equations to solve for all of the unknown variables. In total (3.12)-(3.15) represent 16 unknowns (three for each component of $\mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H}, \mathbf{J}$ and one for ρ) to be solved in 8 equations (1 each for the first two equations, 3 each for the second two). In order to have a well-posed system we need some constitutive laws that relate some of the variables.

First, note that $\mathbf{J}$ and ρ must be related. Current represents the flow of charges, so the existence of a free current density implies that the charge distribution must change in time. This relationship is given by the continuity equation

$$\partial_t \rho = -\nabla \cdot \mathbf{J}. \tag{3.16}$$

The remaining constitutive laws relate the electric field (resp. magnetic displacement) to the polarization (resp. magnetization). Since the polarization is created in response to the electric field, it is expected that the polarization should be a function of the electric field. Similarly, the the magnetization should be related to the magnetic field. A power series relationship

$$\mathbf{P} = \chi_e^{(1)}\mathbf{E} + \chi_e^{(2)}|\mathbf{E}|^2 + \cdots , \tag{3.17}$$

$$\mathbf{M} = \chi_m^{(1)}\mathbf{H} + \chi_m^{(2)}|\mathbf{H}|^2 + \cdots , \tag{3.18}$$

is typically assumed, where in general $\chi^{(i)}$ are tensors of sufficient rank to ensure that $\mathbf{P}, \mathbf{M} \in \mathbb{R}^3$. While this general case is important in many applications such as non-linear optics [3], most applications can just assume a linear model [18]. The following simplifications are defined as below:

Definition 3.1.4. *Let Ω be a material. We say Ω is* ***linear*** *if the relationship between the polarization and electric field (resp. magnetization and magnetic displacement) are*

$$\boldsymbol{P} = \varepsilon_0 \chi_e \boldsymbol{E}, \tag{3.19}$$

$$\boldsymbol{M} = \chi_m \boldsymbol{H}. \tag{3.20}$$

The values χ_e and χ_m (which are known as electric and magnetic susceptibility respectively) are in general non-constant 3×3 matrices.

We further say that Ω is ***homogeneous*** *if χ_e and χ_m are constants, and* ***isotropic*** *if χ_e and χ_m are scalar quantities.*

Going forward, we will assume that the material Ω being considered is a linear, isotropic, and homogeneous material. This means we can write our constitutive laws as

$$\mathbf{D} = \varepsilon_0(1+\chi_e)\mathbf{E} = \varepsilon\mathbf{E}, \tag{3.21}$$

$$\mathbf{B} = \mu_0(1+\chi_m)\mathbf{H} = \mu\mathbf{H}. \tag{3.22}$$

The constants ε and μ will be assumed positive. While there is some interest in materials which have negative values for the permittivity and permeability such as the study of metamaterials (i.e materials created as a composite mesh of different substances in a lattice) [29], these quantities are positive for normal materials.

The remaining quantity for which we need a constitutive relation is the free current density. For many materials such as conductors, this relationship comes in the form of Ohm's law [6, Chapter 7.1.1],

$$\mathbf{J} = \eta^{-1}\mathbf{E}. \tag{3.23}$$

The quantity η is known as the electrical resistivity of the system, and in the case of homogeneous and isotropic materials is constant.

Remark 3.1.5. *The typical notation in the literature for (3.23) is $\boldsymbol{J} = \sigma\boldsymbol{E}$, where σ is the conductivity. The conductivity is exactly equal to the inverse of the resistivity. Since σ is also commonly used to denote the surface current density, the choice to use η has been made to avoid confusion.*

Remark 3.1.6. *Ohm's law as stated holds well for conductors where the velocity of the flowing electrons is sufficiently small. In the case where the velocity is large, the relation is*

$$\boldsymbol{J} = \eta^{-1}(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}). \tag{3.24}$$

This type of constitutive law is necessary in the study of plasma [35]), but for our sake we will assume that (3.23) holds.

These constitutive laws have brought the number of unknown functions down to 6 unknown quantities. Making the choice of using $\mathbf{D}$ and $\mathbf{B}$ as the representative vectors describing the electric and magnetic fields, these quantities are the three components each for $\mathbf{D}, \mathbf{B}$. Substituting in the constitutive relationships for $\mathbf{D}$ and $\mathbf{B}$ we have the following system

$$\nabla \cdot \mathbf{D} = \rho, \tag{3.25}$$
$$\nabla \cdot \mathbf{B} = 0, \tag{3.26}$$
$$\partial_t \mathbf{D} = \mu^{-1} \nabla \times \mathbf{B} - \mathbf{J}, \tag{3.27}$$
$$\partial_t \mathbf{B} = -\varepsilon^{-1} \nabla \times \mathbf{D}, \tag{3.28}$$
$$\partial_t \rho = -\nabla \cdot \mathbf{J}, \tag{3.29}$$

with $\mathbf{J} = \eta^{-1}\varepsilon^{-1}\mathbf{D}$. This system has now gone from under defined to over defined. As it turns out, the first two equations are not necessary to understand the dynamics of the system. Using the interchange of derivatives it can be shown that if the first two equations hold for the initial conditions, then (3.27)-(3.29) imply they hold for all time.

Proposition 3.1.7. *Consider the system described in* (3.25)- (3.29)*. Let* $(\boldsymbol{D}(\boldsymbol{x}, 0), \boldsymbol{B}(\boldsymbol{x}, 0)) = (\boldsymbol{D}_0(\boldsymbol{x}), \boldsymbol{B}_0(\boldsymbol{x}))$ *be the initial state of the system and let* $\rho_0(\boldsymbol{x}) = \rho(\boldsymbol{x}, t)$ *be the initial free charge density. If* $\nabla \cdot \boldsymbol{D}_0 = \rho_0$ *and* $\nabla \cdot \boldsymbol{B}_0 = 0$*, then* $\nabla \cdot \boldsymbol{D} = \rho$ *and* $\nabla \cdot \boldsymbol{B} = 0$ *for all times.*

Proof. Assume initially $\nabla \cdot \mathbf{D}_0 = \rho_0$ holds. From Faraday's law

$$\partial_t \mathbf{D} = \mu^{-1} \nabla \times \mathbf{B} - \mathbf{J}.$$

Assume continuous mixed partials. Taking the divergence of both sides

$$\begin{aligned} \partial_t \nabla \cdot \mathbf{D} &= \mu^{-1} \nabla \cdot (\nabla \times \mathbf{B}) - \nabla \cdot \mathbf{J} \\ &= 0 + \partial_t \rho \\ &= \partial_t \rho, \end{aligned}$$

where the fact that the divergence of a curl is always zero was used. Integrating both sides with respect to t from 0 to t,

$$\nabla \cdot \mathbf{D} - \nabla \cdot \mathbf{D}_0 = \rho - \rho_0.$$

If $\nabla \cdot \mathbf{D}_0 = \rho_0$, then $\nabla \cdot \mathbf{D} = \rho$ as required.

Now consider Amphére's law

$$\partial_t \mathbf{B} = -\varepsilon^{-1}\nabla \times \mathbf{D}.$$

Assume continuous mixed partials and take the divergence of both sides

$$\partial_t \nabla \cdot \mathbf{B} = -\varepsilon^{-1}\nabla \cdot (\nabla \times \mathbf{D}) = 0.$$

Integrate with respect to t from 0 to t

$$\nabla \cdot \mathbf{B} - \nabla \cdot \mathbf{B}_0 = 0.$$

If $\nabla \cdot \mathbf{B}_0 = 0$, then $\nabla \cdot \mathbf{B} = 0$ as required. □

Proposition 3.1.7 demonstrates that if the initial conditions satisfy (3.25) and (3.26), the divergence equations will be satisfied for all time. Therefore we can absorb the first two equations into the set of permissible initial conditions and consider only (3.27) − (3.29) when discussing the dynamics of the system.

3.2 Boundary Conditions

To derive physically relevant boundary conditions, the fact that Maxwell's equations should be valid locally both just inside and just outside of Ω will be used. The method outlined in [6] will be followed for this derivation, but done more rigorously by dealing with limits of regions. These conditions hold for general materials so the derivation will be done using (3.12)-(3.15) instead of using the equations for linear media.

Theorem 3.2.1. *Consider Maxwell's equations (3.12)- (3.15) for a region Ω. Then at the boundary of Ω, which we denote as Γ, the following equations must hold locally for each point on the boundary.*

$$\boldsymbol{n} \cdot (\boldsymbol{D} - \boldsymbol{D}^{ex}) = -\sigma, \tag{3.30}$$
$$\boldsymbol{n} \cdot (\boldsymbol{B} - \boldsymbol{B}^{ex}) = 0, \tag{3.31}$$
$$\boldsymbol{n} \times (\boldsymbol{E} - \boldsymbol{E}^{ex}) = \boldsymbol{0}, \tag{3.32}$$
$$\boldsymbol{n} \times (\boldsymbol{H} - \boldsymbol{H}^{ex}) = -\boldsymbol{K}, \tag{3.33}$$

where $\boldsymbol{E}^{ex}, \boldsymbol{D}^{ex}, \boldsymbol{B}^{ex}, \boldsymbol{H}^{ex}$ refer to external fields outside Ω, σ is the surface charge density, and $\boldsymbol{K}$ is the surface current density.

Proof. A proof of (3.30) and (3.33) will be provided, since the other two proofs are just easier variations of the same techniques. In order to prove these local conditions, we need to first convert the problem to its integral form. Let $\Sigma \subset \mathbb{R}^3$ be a closed region with a boundary $\partial\Sigma$. Integrating Gauss' law over this region results in

$$\int_{\Sigma} \nabla \cdot \mathbf{D}\, d\Omega = \int_{\Sigma} \rho\, d\Omega.$$

By definition, since ρ is the free charge density, the right side of the quality will just be the free enclosed charge Q_{enc}. Applying the Divergence Theorem to the left side to convert to an integral over the boundary will yield

$$\int_{\partial\Sigma} \mathbf{D} \cdot \mathbf{n}\, d\Gamma = Q_{enc}.$$

Let $\mathbf{p} = (p_1, p_2, p_3) \in \Gamma$ be a point on the surface of Ω. Without loss of generality take the coordinate system to be such that the z-axis is in the direction of the outward unit normal at $\mathbf{p}$. Let $\mathbf{D}_i$ denote the i-th component of $\mathbf{D}$ for $i = x, y, z$. Finally, we let our region Σ be defined for some $a, b > 0$ as the box

$$\Sigma = \left\{\mathbf{x} \in \mathbb{R}^3 : |x - p_1| \le a, |y - p_2| \le a, |z - p_3| \le b; a, b > 0\right\}$$

The box Σ can be broken up into six surfaces. These are the faces corresponding to $x = p_1 - a$, $x = p_1 + a$, $y = p_2 - a$, $y = p_2 + a$, $z = p_3 - b$, and $z = p_3 + b$. Labeling these regions as ∂_1 through $\partial\Sigma_6$, and denote D_i to refer to the i-th component of $\mathbf{D}$ for $i = x, y, z$,

$$
\begin{aligned}
\mathbf{D}\cdot\mathbf{n} &= -D_z(x,y,p_3-b) && \mathbf{x}\in\partial\Sigma_1,\\
\mathbf{D}\cdot\mathbf{n} &= D_z(x,y,p_3+b) && \mathbf{x}\in\partial\Sigma_2,\\
\mathbf{D}\cdot\mathbf{n} &= -D_z(x,p_2-a,z) && \mathbf{x}\in\partial\Sigma_3,\\
\mathbf{D}\cdot\mathbf{n} &= D_z(x,p_2+a,z) && \mathbf{x}\in\partial\Sigma_4,\\
\mathbf{D}\cdot\mathbf{n} &= -D_z(p_1-a,y,z) && \mathbf{x}\in\partial\Sigma_5,\\
\mathbf{D}\cdot\mathbf{n} &= D_z(p_1+a,y,z) && \mathbf{x}\in\partial\Sigma_6,
\end{aligned}
$$

where

$$
\begin{aligned}
\partial\Sigma_1 &= \{\mathbf{x}\in\mathbb{R} : z = p_3 - b, |x-p_1|\le a, |y-p_2|\le a\},\\
\partial\Sigma_2 &= \{\mathbf{x}\in\mathbb{R} : z = p_3 + b, |x-p_1|\le a, |y-p_2|\le a\},\\
\partial\Sigma_3 &= \{\mathbf{x}\in\mathbb{R} : y = p_2 - a, |x-p_1|\le a, |z-p_3|\le b\},\\
\partial\Sigma_4 &= \{\mathbf{x}\in\mathbb{R} : y = p_2 + a, |x-p_1|\le a, |z-p_3|\le b\},\\
\partial\Sigma_5 &= \{\mathbf{x}\in\mathbb{R} : x = p_1 - a, |y-p_2|\le a, |z-p_3|\le b\},\\
\partial\Sigma_6 &= \{\mathbf{x}\in\mathbb{R} : x = p_1 + a, |y-p_2|\le a, |z-p_3|\le b\}.
\end{aligned}
$$

Combining these results,

$$
\begin{aligned}
\int_{\partial\Sigma}\mathbf{D}\cdot\mathbf{n}\,d\Gamma = &\iint\limits_{\substack{|x-p_1|<a\\|y-p_1|<a}} D_z(x,y,p_3+b) - D_z(x,y,p_3-b)\;dxdy\\
&+ \iint\limits_{\substack{|x-p_1|<a\\|z-p_3|<b}} D_y(x,p_2+a,z) - D_z(x,p_2-a,z)\;dxdz\\
&+ \iint\limits_{\substack{|x-p_1|<a\\|z-p_3|<b}} D_x(p_1+a,y,z) - D_x(p_1-a,y,z)\;dydz.
\end{aligned}
$$

Taking the limit as $b \to 0$, the last two integrals vanish since their domains of integration go to zero. For the first integral, denote $D_z(x, y, p_3)$ to be the z-component of the field interior to Ω, and $D_z^{ex}(x, y, p_3)$ to be the z-component of the field exterior to Ω at $\mathbf{z} = p_3$. Let $g(x, y) = -(D_z(x, y, p_3) - D_z^{ex}(x, y, p_3))$. We can now write

$$\iint\limits_{\substack{|x-p_1|<a \\ |y-p_1|<a}} g(x, y)\ dxdy = Q_{enc}. \tag{3.34}$$

By the Average Value Theorem (see any introductory calculus textbook, for instance, [33, pg. 997])

$$g_{avg} = \frac{1}{a^2} \iint\limits_{\substack{|x-p_1|<a \\ |y-p_1|<a}} g(x, y)\ dxdy. \tag{3.35}$$

In the limit as $b \to 0$ the only enclosed charge is the surface charge $\sigma(x, y)$. Applying the Average Value Theorem,

$$Q_{enc} = \iint\limits_{\substack{|x-p_1|<a \\ |y-p_1|<a}} \sigma(x, y)\ dxdy = a^2 \sigma_{avg}. \tag{3.36}$$

Substituting (3.35) and (3.36) into (3.34),

$$a^2 g_{avg} = a^2 \sigma_{avg}.$$

Since $a > 0$, we can divide by a^2 on both sides. As $a \to 0$, the average value of $g(x, y)$ will approach $g(p_1, p_2)$ and similarly $\sigma_{avg} = \sigma(p_1, p_2)$. This means locally

$$g(p_1, p_2) = -(D_z(p_1, p_2, p_3) - D_z^{ex}(p_1, p_2, p_3)) = \sigma(p_1, p_2, p_3)$$

Since $\mathbf{p}$ was arbitrary, this relation must hold locally for all $\mathbf{p} \in \Gamma$. Note that since it was chosen that the coordinates ensured z was in the same direction as the outward normal at $\mathbf{p}$, this is equivalent to saying

$$\mathbf{n} \cdot (\mathbf{D} - \mathbf{D}^{ex}) = -\sigma.$$

Now to prove (3.33), we similarly aim to convert the equation into its integral form to proceed. Now let $S \subset \mathbb{R}^3$ be a surface enclosed by a loop ∂S. Taking the surface integral of (3.33) over S yields

$$\int_S (\nabla \times \mathbf{H}) \cdot \mathbf{n} \, d\Gamma = \int_S \partial_t \mathbf{D} \cdot \mathbf{n} \, d\Gamma + \int_S \mathbf{J} \cdot \mathbf{n} \, d\Gamma.$$

The free surface current density is defined as the quantity $\mathbf{J}$ such that the enclosed free current is equal to $I_{enc} = \int_S \mathbf{J} \cdot \mathbf{n} \, d\Gamma$. Applying Stokes' Theorem to the integral with the curl to convert to a line integral over ∂S we have

$$\int_{\partial S} \mathbf{H} \cdot d\ell = \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{n} \, d\Gamma + I_{enc}$$

As before, let $\mathbf{p} = (p_1, p_2, p_3)$ be some point on the surface of Ω. Without loss of generality, we assume the z-axis is in the direction of the outward unit normal, and the x-axis is in the direction of $\mathbf{K}$, the free surface current density at $\mathbf{p}$. Note that this choice in axis is okay since $\mathbf{K}$ must be tangent to the surface and thus orthogonal to the normal vector, allowing for an orthonormal coordinate system to be defined. Finally, let S be defined by the square

$$S = \{\mathbf{x} \in \mathbb{R}^3 : x = p_1, |y - p_2| \leq a, |z - p_3| \leq b; a, b > 0\}.$$

We will consider each part of the overall equality separately. First note that that in the limit as $b \to 0$

$$\frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{n} \, d\Gamma \to 0,$$

because the domain of integration will vanish. Next, we note that for the average current density, $I_{enc} = 2a\mathbf{K}_{avg} \cdot \hat{x}$ since $\mathbf{K}_{avg} \cdot \hat{x} = |K_{avg}|$. Finally,

$$\int\limits_{\partial S} \mathbf{H}\cdot d\ell = \int\limits_{p_2-b}^{p_3+b} H_z(p_1,p_2+a,z) - H_z(p_1,p_2-a,z)dz + \int\limits_{p_3-a}^{p_2+a} H_y(p_1,y,p_3-b) - H_y(p_1,y,p_3+b)dy.$$

In the limit as $b \to 0$, the first integral's domain of integration collapses, so the first term goes to zero. By the same argument as in the case of Gauss' law, for $f(y) = H_y(p_1, y, p_3, p_3) - H_y^{ex}(p_1, y, p_3)$,

$$\int\limits_{p_3-a}^{p_2+a} H_y(p_1,y,p_3) - H_y^{ex}(p_1,y,p_3)dy = 2af_{avg}(p_1,y,p_3).$$

Putting things together,

$$f_{avg}(p_1,y,p_3) = \mathbf{K}_{avg} \cdot \hat{x}.$$

Taking a to be small but nonzero, the average values approach the values evaluated at $\mathbf{p}$, so once again we have

$$H_y(p_1,p_2,p_3) - H_y^{ex}(p_1,p_2,p_3) = \mathbf{K} \cdot \hat{x}.$$

Since $\mathbf{p}$ is arbitrary, this must hold for all $\mathbf{p} \in \Gamma$. Note that $\hat{y} = \hat{z} \times \hat{x}$, so we can rewrite the above as

$$(\mathbf{H} - \mathbf{H}^{ex}) \cdot (\hat{z} \times \hat{x}) = \mathbf{K} \cdot \hat{x}.$$

Using the cyclical property of the scalar triple product and replacing $\hat{z}$ with $\mathbf{n}$,

$$-(\mathbf{n} \times (\mathbf{H} - \mathbf{H}^{ex})) \cdot \hat{x} = \mathbf{K} \cdot \hat{x}.$$

Since $\hat{x}$ is just in the direction of $\mathbf{K}$, this is just telling us that the component of $(\mathbf{n} \times (\mathbf{H} - \mathbf{H}^{ex}))$ that is parallel to $\mathbf{K}$ is equal to $\mathbf{K}$. Both components are already in this direction, so we can just drop the dot product with $\hat{x}$ to finally obtain

$$\mathbf{n} \times (\mathbf{H} - \mathbf{H}^{ex}) = -\mathbf{K}.$$

□

These are general continuity equations that any electromagnetic field must satisfy. Since these equations were derived using Maxwell's equations, from Proposition 3.1.7, if the initial conditions satisfy Gauss' law (3.1) and the divergence condition for $\mathbf{B}$ (3.2), then (3.30) and (3.31) are satisfied.

It is worth making note of some particular physical boundary conditions.

Definition 3.2.2 (perfect conductor)**.** *A **perfect conductor** is defined as a material which has zero resistivity.*

Since Ohm's law holds for perfect conductors, this is equivalent to saying a perfect conductor permits zero internal electric field, since $\mathbf{J} = \eta^{-1}\mathbf{E}$ is only finite if $\mathbf{E} = \mathbf{0}$.

A perfect conductor gives rise to prefect conducting boundary condition. Suppose Ω is connected to a perfect conductor at the boundary Γ, then there is no electric field external to Ω inside the perfect conducting region. From (3.32) we require

$$\mathbf{n} \times \mathbf{D} = \mathbf{0}, \tag{3.37}$$

where $\mathbf{E}$ has been swapped with $\mathbf{D}$ by dividing both sides by ε. Another important model for the boundary conditions is the impedance boundary condition. According to [18], in many substances such as metals the penetrative depth of the electromagnetic fields is low. If the penetrative depth is small in comparison to the thickness of a conductor, the fields in the medium will be a highly damped plane wave [8]. A consequence of this is that the tangential components of the electric and magnetic fields (and thus, the magnetic field $\mathbf{B}$ and the electric displacement $\mathbf{D}$) can be related. This relationship can be used in place of the boundary condition for the surface, and is given by

$$\mathbf{n} \times (\mathbf{n} \times \mathbf{D}) + \sqrt{\frac{\varepsilon}{\mu}}\mathbf{n} \times \mathbf{B} = \mathbf{0}. \tag{3.38}$$

This condition is a first-order approximation to the dynamics at the boundary and has been shown to be accurate numerically for short computational time [1]. This model assumes that at least locally, the problem can be considered as a plane wave entering a flat surface [8]. To account for regions where the surface is not sufficiently flat locally to make this approximation, this condition can be generalized as

$$\mathbf{n} \times (\mathbf{n} \times \mathbf{D}) + r\mathbf{n} \times \mathbf{B} = \mathbf{0}, \tag{3.39}$$

for a positive scalar function $r > 0$ to provide better modeling. The linear case is recovered when $r = \sqrt{\frac{\varepsilon}{\mu}}$. The scalar function r being used instead of the simple constant condition allows more freedom when modeling, which is useful for numerical computations. r is left as arbitrary in this thesis, but for a particular problem it could be chosen to match the desired boundary condition.

1.	$\nabla \cdot (\mathbf{f} \times \mathbf{g}) = \mathbf{g} \cdot \nabla \times \mathbf{f} - \mathbf{f} \cdot \nabla \times \mathbf{g}$
2.	$\mathbf{f} \times \nabla \times \mathbf{f} = \frac{1}{2}\nabla \lvert\mathbf{f}\rvert^2 - (\mathbf{f} \cdot \nabla)\mathbf{f}$
3.	$\nabla \cdot (\nabla \times \mathbf{f}) = 0$
4.	$\mathbf{f} \times \mathbf{f} = \mathbf{0}$
5.	$\mathbf{f} \times \mathbf{g} = -\mathbf{g} \times \mathbf{f}$
6.	$(\mathbf{f} \times \mathbf{g}) \cdot \mathbf{h} = (\mathbf{h} \times \mathbf{f}) \cdot \mathbf{g} = (\mathbf{g} \times \mathbf{h}) \cdot \mathbf{f}$
7.	$(\mathbf{f} \times \mathbf{g}) \times \mathbf{h} = (\mathbf{f} \cdot \mathbf{h})\mathbf{g} - (\mathbf{f} \cdot \mathbf{g})\mathbf{h}$

Table 3.1: Let $\mathbf{f}, \mathbf{g}, \mathbf{h} \in \mathbb{R}^3$ be sufficiently differentiable. Let $\times$ denote the cross product and $\cdot$ denote the dot product. The table forms a list of relevant vector calculus and algebra identities. More details on these identities can be found in [19, Chapter 1 and 2]

Proposition 3.2.3. *The quantity $\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{D})$ is the tangential component of $\boldsymbol{D}$ at the surface.*

Proof. Any vector $\mathbf{f} \in \mathbb{R}^3$ can be decomposed into its normal and tangential components to a unit normal vector $\mathbf{n}$. By definition, the normal component is the projection of $\mathbf{f}$ onto $\mathbf{n}$, which is given as $proj_{\mathbf{n}}(\mathbf{f}) = \dfrac{\mathbf{n} \cdot \mathbf{f}}{|\mathbf{n}|^2}\mathbf{n} = (\mathbf{n} \cdot \mathbf{f})\mathbf{n}$. The tangential component, given by the perpendicular, is solved for by subtracting the the projection from $\mathbf{f}$,

$$perp_{\mathbf{n}}(\mathbf{f}) = \mathbf{f} - (\mathbf{n} \cdot \mathbf{f})\mathbf{n}.$$

Noting that the vector triple product gives the relation yields (Table 3.1 identity 7)

$$\mathbf{n} \times (\mathbf{n} \times \mathbf{f}) = (\mathbf{n} \cdot \mathbf{f})\mathbf{n} - (\mathbf{n} \cdot \mathbf{n})\mathbf{f} = (\mathbf{n} \cdot \mathbf{f})\mathbf{n} - \mathbf{f},$$

we can conclude that the tangential component for any vector $\mathbf{f} \in \mathbb{R}^3$ is therefore $-\mathbf{n} \times (\mathbf{n} \times \mathbf{f})$. $\square$

For this thesis, we will only consider boundaries which are either perfectly conducting, or obey the impedance boundary condition.

3.3 Control System Formulation

The problem we are interested in discussing is adapted from the boundary value problem discussed in [38]. We consider a region $\Omega \subset \mathbb{R}^3$ with a C^1 boundary $\Gamma = \partial\Omega$. The boundary Γ is decomposed into two regions. The first region $\Gamma_0 \subset \Gamma$ corresponds to a perfect conducting boundary, and $\Gamma_1 = \Gamma \backslash \Gamma_0$ is the remainder of the boundary and obeys the impedance boundary condition. Assume Ω is a region which obeys Ohm's law.

Since Γ_0 is a perfect conducting region it must have the boundary condition $\mathbf{n} \times \mathbf{D} = 0$ on Γ_0. Moreover, the impedance boundary satisfies the condition that $\mathbf{n} \times (\mathbf{n} \times \mathbf{D}) = -r\mathbf{n} \times \mathbf{B}$ on Γ_1 where $r > 0$ is a scalar function such that both r, r^{-1} are bounded on Γ_1. We assume that a control is implemented only on the impedance boundary, which could be physically implemented by running a current along the surface. Define the control and measurement used in [38] on Γ_1

$$\mathbf{u} = \frac{1}{\sqrt{2}} \left(\mathbf{n} \times (\mathbf{n} \times \mathbf{D}) + r\mathbf{n} \times \mathbf{B} \right), \tag{3.40}$$

$$\mathbf{y} = \frac{1}{\sqrt{2}} \left(\mathbf{n} \times (\mathbf{n} \times \mathbf{D}) - r\mathbf{n} \times \mathbf{B} \right), \tag{3.41}$$

Here, $\mathbf{y}$ represents measurements of the components of the electromagnetic field tangential to the surface. Physically, recall that $\mathbf{n} \times (\mathbf{n} \times \mathbf{D})$ is the tangential component of $\mathbf{D}$ at the surface, and $\mathbf{n} \times \mathbf{B}$ appears in the derivation for the general boundary conditions in section 3.2. If there is no external field this condition is (after converting from $\mathbf{H}$ to $\mathbf{B}$),

$$\mu^{-1}\mathbf{n} \times \mathbf{B} = -\mathbf{K}.$$

This suggests that our control $\mathbf{u}$ can be implemented by running a surface current along the boundary of Ω. It will be shown in section 3.5 that this choice of control and observation leads to a well-posed system.

Our control system is

$$\begin{aligned}
\partial_t \mathbf{D} &= \mu^{-1}\nabla \times \mathbf{B} - \eta^{-1}\varepsilon^{-1}\mathbf{D} && \mathbf{x} \in \Omega, t \geq 0, && (3.42)\\
\partial_t \mathbf{B} &= -\varepsilon^{-1}\nabla \times \mathbf{D} && \mathbf{x} \in \Omega, t \geq 0, && (3.43)\\
\mathbf{0} &= \mathbf{n} \times \mathbf{D} && \mathbf{x} \in \Gamma_0, t \geq 0, && (3.44)\\
\mathbf{u} &= \frac{1}{\sqrt{2}}\left(\mathbf{n} \times (\mathbf{n} \times \mathbf{D}) + r\mathbf{n} \times \mathbf{B}\right) && \mathbf{x} \in \Gamma_1, t \geq 0, && (3.45)\\
(\mathbf{D}(\mathbf{x},0), \mathbf{B}(\mathbf{x},0)) &= (\mathbf{D}_0(\mathbf{x}), \mathbf{B}_0(\mathbf{x})) && \mathbf{x} \in \Omega. && (3.46)
\end{aligned}$$

We define the set of possible initial conditions to be

$$\mathcal{Z}_0 = \{(\mathbf{f}, \mathbf{g}) \in L^2(\Omega)^6 : \nabla \cdot \mathbf{f}, \nabla \cdot \mathbf{g} \text{ exist and } \nabla \cdot \mathbf{f} = \rho, \nabla \cdot \mathbf{g} = 0 \text{ in } \Omega\}, \tag{3.47}$$

which as discussed before ensures the first two of Maxwell's equations are satisfied. The functions must exist in $L^2(\Omega)$ to ensure that initial conditions are square-integrable. This is required since as will be demonstrated, the total energy for an electromagnetic system in linear media is a weighted L^2 norm. If the initial conditions were not square integrable, the total energy could be infinite, which is not physical.

3.4 Energy Balance

Since electromagnetism is a physical phenomena there is a physical energy associated with electromagnetic systems. For linear media, the Hamiltonian density is given as $h = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$ [6]. Since there is no explicit time dependence, the Hamiltonian density directly corresponds to the energy density for the system, so we can conclude that the total energy is

$$\mathcal{H}(t) = \frac{1}{2}\int\limits_{\Omega} \mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}\; d\Omega. \tag{3.48}$$

Since we are only considering linear media, we can write the total energy in terms of just $\mathbf{D}$ and $\mathbf{B}$. The result is

$$\mathcal{H}(t) = \frac{1}{2}\int\limits_{\Omega} \varepsilon^{-1}|\mathbf{D}|^2 + \mu^{-1}|\mathbf{B}|^2\; d\Omega. \tag{3.49}$$

Proving that $\mathcal{H}(t)$ is strictly non-increasing in time will be important for proving exponential stability. To this end, we first take a time derivative of the Hamiltonian and substitute in the dynamics of the system.

Theorem 3.4.1 (Energy Balance)**.** *Let $\mathcal{H}(t)$ be given as (3.49), where $\boldsymbol{D}$ and $\boldsymbol{B}$ are classical solutions to (3.42)-(3.46). Then*

$$\dot{\mathcal{H}}(t) = -\varepsilon^{-1}\mu^{-1}\int\limits_{\Gamma}(\boldsymbol{D}\times\boldsymbol{B})\cdot\boldsymbol{n}\; d\Gamma - \eta^{-1}\varepsilon^{-2}\int\limits_{\Omega}|\boldsymbol{D}|^2\; d\Omega. \tag{3.50}$$

Proof.

$$\begin{aligned}
\dot{\mathcal{H}}(t) &= \int\limits_{\Omega} \varepsilon^{-1}\mathbf{D}\cdot\partial_t\mathbf{D} + \mu^{-1}\mathbf{B}\cdot\partial_t\mathbf{B}\; d\Omega \\
&= \int\limits_{\Omega} \varepsilon^{-1}\mathbf{D}\cdot(\mu^{-1}\nabla\times\mathbf{B} - \eta^{-1}\varepsilon^{-1}\mathbf{D}) + \mu^{-1}\mathbf{B}\cdot(-\varepsilon^{-1}\nabla\times\mathbf{D})\; d\Omega \\
&= -\varepsilon^{-1}\mu^{-1}\int\limits_{\Omega}(\mathbf{B}\cdot\nabla\times\mathbf{D} - \mathbf{D}\cdot\nabla\times\mathbf{B})\; d\Omega - \eta^{-1}\varepsilon^{-2}\int\limits_{\Omega}|\mathbf{D}|^2\; d\Omega.
\end{aligned}$$

Using the vector calculus identity $\nabla\cdot(\mathbf{f}\times\mathbf{g}) = \mathbf{g}\cdot\nabla\times\mathbf{f} - \mathbf{f}\cdot\nabla\times\mathbf{g}$ (Table 3.1 identity 1),

$$\dot{\mathcal{H}}(t) = -\varepsilon^{-1}\mu^{-1}\int\limits_{\Omega}\nabla\cdot(\mathbf{D}\times\mathbf{B})\; d\Omega - \eta^{-1}\varepsilon^{-2}\int\limits_{\Omega}|\mathbf{D}|^2\; d\Omega.$$

Finally, applying the divergence theorem

$$\dot{\mathcal{H}}(t) = -\varepsilon^{-1}\mu^{-1}\int_{\Gamma}(\mathbf{D}\times\mathbf{B})\cdot\mathbf{n}\,d\Gamma - \eta^{-1}\varepsilon^{-2}\int_{\Omega}|\mathbf{D}|^2\,d\Omega.$$

□

The first term in (3.50),

$$-\varepsilon^{-1}\mu^{-1}\int_{\Gamma}(\mathbf{D}\times\mathbf{B})\cdot\mathbf{n}\,d\Gamma,$$

physically represents the energy flowing across the boundary. The quantity $\mathbf{S} = \varepsilon^{-1}\mu^{-1}\mathbf{D}\times\mathbf{B}$ is typically referred to as the Poynting vector, and is the energy flux density. The second term,

$$-\eta^{-1}\varepsilon^{-2}\int_{\Omega}|\mathbf{D}|^2\,d\Omega,$$

might seem confusing since it seems to violate the conservation of energy. This is however a consequence of our model not considering the full system. This term relates to the conversion of energy inside Ω into heat via the Joule effect, so if the thermal domain were considered the total energy of the system remains conserved.

Since we wish to use the impedance boundary condition, reformulating (3.50) to contain the boundary terms in the impedance condition will be of use. Consider the following corollary to Theorem 3.4.1

Corollary 3.4.2. *The expression for $\dot{\mathcal{H}}(t)$ in (3.50) can be equivalently rewritten as*

$$\dot{\mathcal{H}}(t) = \varepsilon^{-1}\mu^{-1}\int_{\Gamma}(\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{D}))\cdot(\boldsymbol{n}\times\boldsymbol{B})\,d\Gamma - \eta^{-1}\varepsilon^{-2}\int_{\Omega}|\boldsymbol{D}|^2\,d\Omega. \tag{3.51}$$

Proof. By Proposition 3.2.3, we can decompose $\mathbf{D}$ into its normal and tangential components as

$$\mathbf{D} = (\mathbf{n} \cdot \mathbf{D})\mathbf{n} - \mathbf{n} \times (\mathbf{n} \times \mathbf{D}).$$

Substituting this into $(\mathbf{D} \times \mathbf{B}) \cdot \mathbf{n}$ results in

$$\begin{aligned}(\mathbf{D} \times \mathbf{B}) \cdot \mathbf{n} &= (((\mathbf{n} \cdot \mathbf{D})\mathbf{n} - \mathbf{n} \times (\mathbf{n} \times \mathbf{D})) \times \mathbf{B}) \cdot \mathbf{n} \\ &= (\mathbf{n} \cdot \mathbf{D})(\mathbf{n} \times \mathbf{B}) \cdot \mathbf{n} - ((\mathbf{n} \times (\mathbf{n} \times \mathbf{D})) \times \mathbf{B}) \cdot \mathbf{n}.\end{aligned}$$

Using the cyclic property of the scalar triple product, $(\mathbf{n} \times \mathbf{B}) \cdot \mathbf{n} = (\mathbf{n} \times \mathbf{n}) \cdot \mathbf{B} = 0$ and

$$((\mathbf{n} \times (\mathbf{n} \times \mathbf{D})) \times \mathbf{B}) \cdot \mathbf{n} = ((\mathbf{n} \times (\mathbf{n} \times \mathbf{D})) \cdot (\mathbf{B} \times \mathbf{n}) = -((\mathbf{n} \times (\mathbf{n} \times \mathbf{D})) \cdot (\mathbf{n} \times \mathbf{B}).$$

Substituting this back into $(\mathbf{D} \times \mathbf{B}) \cdot \mathbf{n}$ obtains the desired result. □

This corollary directly includes both terms in the impedance boundary condition, which will be useful for proving that the energy is strictly non-increasing for systems which have an impedance boundary.

Theorem 3.4.3. *Let $\boldsymbol{D}$ and $\boldsymbol{B}$ be classical solutions to Maxwell's equations in a region Ω with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ satisfying*

1. *$\boldsymbol{n} \times \boldsymbol{D} = \boldsymbol{0}$ on Γ_0, and*
2. *$\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{D}) + r\boldsymbol{n} \times \boldsymbol{B} = \boldsymbol{0}$ for some positive scalar function $r > 0$ on Γ_1.*

Then the total energy $\mathcal{H}(t)$ is non-increasing.

Proof. From (3.50) we have

$$\dot{\mathcal{H}}(t) = -\varepsilon^{-1}\mu^{-1} \int\limits_{\Gamma} (\mathbf{D} \times \mathbf{B}) \cdot \mathbf{n} \, d\Gamma - \varepsilon^{-2}\eta^{-1} \int\limits_{\Omega} |\mathbf{D}|^2 \, d\Omega.$$

By the cyclic property of the triple scalar product, $(\mathbf{D} \times \mathbf{B}) \cdot \mathbf{n} = (\mathbf{n} \times \mathbf{D}) \cdot \mathbf{B} = 0$ on Γ_0. Thus,

$$\dot{\mathcal{H}}(t) = -\varepsilon^{-1}\mu^{-1}\int\limits_{\Gamma_1}(\mathbf{D}\times\mathbf{B})\cdot\mathbf{n}\; d\Gamma - \varepsilon^{-2}\eta^{-1}\int\limits_{\Omega}|\mathbf{D}|^2\; d\Omega.$$

From Corollary 3.4.2, this can be rewritten as

$$\dot{\mathcal{H}}(t) = \varepsilon^{-1}\mu^{-1}\int\limits_{\Gamma_1}(\mathbf{n}\times(\mathbf{n}\times\mathbf{D}))\cdot(\mathbf{n}\times\mathbf{B})\; d\Gamma - \varepsilon^{-2}\eta^{-1}\int\limits_{\Omega}|\mathbf{D}|^2\; d\Omega.$$

Substitute the boundary condition over Γ_1 in,

$$\dot{\mathcal{H}}(t) = -\varepsilon^{-1}\mu^{-1}\int\limits_{\Gamma_1} r|\mathbf{n}\times\mathbf{B}|^2\; d\Gamma - \varepsilon^{-2}\eta^{-1}\int\limits_{\Omega}|\mathbf{D}|^2\; d\Omega.$$

Since r, $|\mathbf{n}\times\mathbf{B}|^2$, and $|\mathbf{D}|^2$ are all non-negative, the integrals are all non-positive. Therefore $\dot{\mathcal{H}}(t)\le 0$ for all $t\ge 0$.

□

3.5 Abstract System Description

To analyse well-posedness results using the formalism of semigroups, Maxwell's equations need to be rewritten as an abstract semigroup.

Given the system (3.42)-(3.46) with $\mathbf{u}=0$, let $\mathbf{z}=\begin{bmatrix}\mathbf{D}\\ \mathbf{B}\end{bmatrix}\in\mathcal{Z}$ where $\mathcal{Z}$ is the Hilbert Space $H^1(\Omega)^6$ equipped with the inner product

$$<\begin{bmatrix}\mathbf{f}_1\\ \mathbf{f}_2\end{bmatrix},\begin{bmatrix}\mathbf{g}_1\\ \mathbf{g}_2\end{bmatrix}>_{\mathcal{Z}}=\int\limits_{\Omega}\begin{bmatrix}\mathbf{f}_1\\ \mathbf{f}_2\end{bmatrix}^T\begin{bmatrix}\varepsilon^{-1}\mathcal{I} & \mathcal{O}\\ \mathcal{O} & \mu^{-1}\mathcal{I}\end{bmatrix}\begin{bmatrix}\mathbf{g}_1\\ \mathbf{g}_2\end{bmatrix}\; d\Omega. \tag{3.52}$$

Recall $\mathcal{O}$ is the zero operator and $\mathcal{I}$ is the identity operator. The system can be written in an abstract form as

$$\partial_t \mathbf{z} = A\mathbf{z}, \tag{3.53}$$

$$A = \begin{bmatrix} -\eta^{-1}\varepsilon^{-1}\mathcal{I} & \mu^{-1}\nabla\times \\ -\varepsilon^{-1}\nabla\times & \mathcal{O} \end{bmatrix}, \tag{3.54}$$

$$\begin{aligned}\mathcal{D}(A) = \{(\mathbf{D}, \mathbf{B}) \in H^1(\Omega)^6 \cap \mathcal{Z}_0 : \mathbf{0} = \mathbf{n} \times \mathbf{D} \text{ for } \mathbf{x} \in \Gamma_0, \\ \mathbf{n} \times (\mathbf{n} \times \mathbf{D}) + r\mathbf{n} \times \mathbf{B} = \mathbf{0} \text{ for } \mathbf{x} \in \Gamma_1\},\end{aligned} \tag{3.55}$$

where $\mathcal{Z}_0$ was defined in (3.47). The domain of A was taken to be in $H^1(\Omega)$ to ensure that not only $\mathbf{D}$ and $\mathbf{B}$ are square-integrable, but their derivatives are also square-integrable. This is necessary to ensure that both $\mathcal{H}(t)$ and $\dot{\mathcal{H}}(t)$ are finite and so $A : \mathcal{D}(A) \subset L^2(\Omega) \to L^2(\Omega)$

We will find the adjoint of A, as well as show A is densely defined and closed in order to apply the Lumer-Phillips theorem.

Lemma 3.5.1. *Let A be given by (3.54) with the domain $\mathcal{D}(A)$ given by (3.55). Then the adjoint operator of A is*

$$A^* = \begin{bmatrix} -\eta^{-1}\varepsilon^{-1}\mathcal{I} & -\mu^{-1}\nabla\times \\ \varepsilon^{-1}\nabla\times & \mathcal{O} \end{bmatrix}, \tag{3.56}$$

with domain

$$\begin{aligned}\mathcal{D}(A^*) = \{(\boldsymbol{D}, \boldsymbol{B}) \in H^1(\Omega)^6 \cap \mathcal{Z}_0 : \boldsymbol{0} = \boldsymbol{n} \times \boldsymbol{D} \textit{ for } \boldsymbol{x} \in \Gamma_0, \\ \boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{D}) - r\boldsymbol{n} \times \boldsymbol{B} = \boldsymbol{0} \textit{ for } \boldsymbol{x} \in \Gamma_1\}.\end{aligned} \tag{3.57}$$

Proof. Let $\mathbf{z} \in \mathcal{D}(A)$ and let $\mathbf{w}$ be sufficiently differentiable. Recall from Definition 2.2.5, the adjoint of $A : \mathcal{D}(A) \subseteq \mathcal{Z} \to \mathcal{Z}$ is the operator A^* which satisfies

$$< A\mathbf{z}, \mathbf{w} >_{\mathcal{Z}} = < \mathbf{z}, A^*\mathbf{w} >_{\mathcal{Z}} \quad \forall \mathbf{z}, \in \mathcal{D}(A),$$

and its domain is the set of elements such that this relation holds. In practice, we find A^* and its domain by taking the inner product of $A\mathbf{z}$ and $\mathbf{w}$.

Note that $A\mathbf{z} = \begin{bmatrix} \mu^{-1}\nabla \times \mathbf{z}_2 - \eta^{-1}\varepsilon^{-1}\mathbf{z}_1 \\ \varepsilon^{-1}\nabla \times \mathbf{z}_1 \end{bmatrix}.$

$$
\begin{aligned}
< A\mathbf{z}, \mathbf{w} >_{\mathcal{Z}} &= \frac{1}{2} \int\limits_{\Omega} \begin{bmatrix} \mu^{-1} \nabla \times \mathbf{z}_2 - \eta^{-1} \varepsilon^{-1} \mathbf{z}_1 \\ -\varepsilon^{-1} \nabla \times \mathbf{z}_1 \end{bmatrix}^T \begin{bmatrix} \varepsilon^{-1} \mathcal{I} & \mathcal{O} \\ \mathcal{O} & \mu^{-1} \mathcal{I} \end{bmatrix} \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} \, d\Omega \\
&= \frac{1}{2} \int\limits_{\Omega} \varepsilon^{-1} \mathbf{w}_1 \cdot (\mu^{-1} \nabla \times \mathbf{z}_2 - \varepsilon^{-1} \eta^{-1} \mathbf{z}_1) + \mu^{-1} \mathbf{w}_2 \cdot (-\varepsilon^{-1} \nabla \times \mathbf{z}_1) \, d\Omega \\
&= \varepsilon^{-1} \mu^{-1} \frac{1}{2} \int\limits_{\Omega} (\mathbf{w}_1 \cdot \nabla \times \mathbf{z}_2 - \mathbf{w}_2 \cdot \nabla \times \mathbf{w}_1) - \varepsilon^{-2} \eta^{-1} \mathbf{z}_1 \cdot \mathbf{w}_1 \, d\Omega.
\end{aligned}
$$

Using the vector calculus identity $\nabla \cdot (\mathbf{f} \cdot \mathbf{g}) = \mathbf{g} \cdot \nabla \times \mathbf{f} - \mathbf{f} \cdot \nabla \times \mathbf{g}$,

$$
\begin{aligned}
< A\mathbf{z}, \mathbf{w} >_{\mathcal{Z}} = \varepsilon^{-1} \mu^{-1} & \int\limits_{\Omega} \nabla \cdot (\mathbf{z}_1 \times \mathbf{w}_2 - \mathbf{z}_2 \times \mathbf{w}_1) \, d\Omega \\
& + \varepsilon^{-1} \mu^{-1} \int\limits_{\Omega} (\mathbf{z}_2 \cdot \nabla \times \mathbf{w}_1 - \mathbf{z}_1 \cdot \nabla \times \mathbf{w}_2) - \varepsilon^{-2} \eta^{-1} \mathbf{z}_1 \cdot \mathbf{w}_1 \, d\Omega.
\end{aligned}
$$

The integrand of the second integral can be rewritten as

$$
\begin{aligned}
& \varepsilon^{-1} \mu^{-1} (\mathbf{z}_2 \cdot \nabla \times \mathbf{w}_1 - \mathbf{z}_1 \cdot \nabla \times \mathbf{w}_2) - \varepsilon^{-2} \eta^{-1} \mathbf{z}_1 \cdot \mathbf{w}_1 \\
& = \varepsilon^{-1} \mathbf{z}_1 (-\mu^{-1} \nabla \times \mathbf{w}_2 - \varepsilon^{-1} \eta^{-1} \mathbf{w}_1) + \mu^{-1} \mathbf{z}_2 \cdot (\varepsilon^{-1} \nabla \times \mathbf{w}_2) \\
& = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix}^T \begin{bmatrix} \varepsilon^{-1} \mathcal{I} & \mathcal{O} \\ \mathcal{O} & \mu^{-1} \mathcal{I} \end{bmatrix} \begin{bmatrix} -\mu^{-1} \nabla \times \mathbf{w}_2 - \varepsilon^{-1} \eta^{-1} \mathbf{w}_1 \\ \varepsilon^{-1} \nabla \times \mathbf{w}_2 \end{bmatrix}.
\end{aligned}
$$

Defining $A^* = \begin{bmatrix} -\eta^{-1} \varepsilon^{-1} \mathcal{I} & -\mu^{-1} \nabla \times \\ \varepsilon^{-1} \nabla \times & \mathcal{O} \end{bmatrix}$ the second integral is $< \mathbf{z}, A^* \mathbf{w} >_{\mathcal{Z}}$ for $\mathbf{w} \in H^1(\Omega)^6$. Now what remains to determine what domain will ensure $< A\mathbf{z}, \mathbf{w} >_{\mathcal{Z}} - < \mathbf{z}, A^* \mathbf{w} >_{\mathcal{Z}} = 0$.

$$
\begin{aligned}
< A\mathbf{z}, \mathbf{w} >_{\mathcal{Z}} - < \mathbf{z}, A^* \mathbf{w} >_{\mathcal{Z}} &= \varepsilon^{-1} \mu^{-1} \frac{1}{2} \int\limits_{\Omega} \nabla \cdot (\mathbf{z}_1 \times \mathbf{w}_2 - \mathbf{z}_2 \times \mathbf{w}_1) \, d\Omega \\
&= \varepsilon^{-1} \mu^{-1} \frac{1}{2} \int\limits_{\Gamma} (\mathbf{z}_1 \times \mathbf{w}_2) \cdot \mathbf{n} - (\mathbf{z}_2 \times \mathbf{w}_1) \cdot \mathbf{n} \, d\Gamma.
\end{aligned}
$$

First, note that since $\mathbf{z} \in \mathcal{D}(A)$, the cyclic property of the scalar triple product implies $(\mathbf{z}_1 \times \mathbf{w}_2) \cdot \mathbf{n} = (\mathbf{n} \times \mathbf{z}_1) \cdot \mathbf{w}_2 = 0$ on Γ_0. Moreover, using the fact that $\mathbf{z}$ satisfies the impedance boundary condition on Γ_1,

$$\begin{aligned}
(\mathbf{z}_1 \times \mathbf{w}_2) \cdot \mathbf{n} - (\mathbf{z}_2 \times \mathbf{w}_1) \cdot \mathbf{n} &= -(\mathbf{n} \times \mathbf{w}_2) \cdot \mathbf{z}_1 + (\mathbf{n} \times \mathbf{z}_2) \cdot \mathbf{w}_1 \\
&= -(\mathbf{n} \times \mathbf{w}_2) \cdot \mathbf{z}_1 + r^{-1}(\mathbf{n} \times (\mathbf{n} \times \mathbf{z}_1)) \cdot \mathbf{w}_1 \\
&= -(\mathbf{n} \times \mathbf{w}_2) \cdot \mathbf{z}_1 + r^{-1}(\mathbf{n} \times \mathbf{z}_1) \cdot (\mathbf{n} \times \mathbf{w}_1). \\
&= -r^{-1}\mathbf{z}_1 \cdot (r\mathbf{n} \times \mathbf{w}_2 - \mathbf{n} \times (\mathbf{n} \times \mathbf{w}_1)).
\end{aligned}$$

Thus,

$$\begin{aligned}
< A\mathbf{z}, \mathbf{w} >_{\mathcal{Z}} - < \mathbf{z}, A^*\mathbf{w} >_{\mathcal{Z}} = -\varepsilon^{-1}\mu^{-1}\frac{1}{2}\int\limits_{\Gamma_0} (\mathbf{z}_2 \times \mathbf{w}_1) \cdot \mathbf{n} \; d\Gamma \\
+ \varepsilon^{-1}\mu^{-1}\frac{1}{2}\int\limits_{\Gamma_1} -r^{-1}\mathbf{z}_1 \cdot (r\mathbf{n} \times \mathbf{w}_2 - \mathbf{n} \times (\mathbf{n} \times \mathbf{w}_1)) \; d\Gamma.
\end{aligned}$$

If we define

$$\begin{gathered}
\mathcal{D}(A^*) = \{(\mathbf{D}, \mathbf{B}) \in H^1(\Omega)^6 : \mathbf{0} = \mathbf{n} \times \mathbf{D} \text{ for } \mathbf{x} \in \Gamma_0, \\
\mathbf{n} \times (\mathbf{n} \times \mathbf{D}) - r\mathbf{n} \times \mathbf{B} = \mathbf{0} \text{ for } \mathbf{x} \in \Gamma_1\},
\end{gathered}$$

then $< A\mathbf{z}, \mathbf{w} >_{\mathcal{Z}} - < \mathbf{z}, A^*\mathbf{w} >_{\mathcal{Z}} = 0$ for all $\mathbf{z} \in \mathcal{D}(A)$, $\mathbf{w} \in \mathcal{D}(A^*)$. □

Now that the adjoint of A has been found, the next step is to prove A is a closed operator.

Lemma 3.5.2. *Let A be given by (3.54) with the domain $\mathcal{D}(A)$ given by (3.55). Then $A : \mathcal{D}(A) \subseteq L^2(\Omega)^6 \to L^2(\Omega)^6$ is closed in $L^2(\Omega)^6$.*

Proof. Let $\{\mathbf{z}_k\}$ be any sequence in the domain of A such that $\mathbf{z}_k \to \mathbf{z}$ and $A\mathbf{z}_k \to \mathbf{y}$. Here

$$\mathbf{z}_k = \begin{bmatrix} \mathbf{D}_k \\ \mathbf{B}_k \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} \mathbf{D} \\ \mathbf{B} \end{bmatrix}.$$

From [4, Chapter IX: Proposition 3] the image of curl, *curl* $H^1(\Omega)^3$, is closed in $L^2(\Omega)^3$. This means for any convergent series $\mathbf{f}_k \in H^1(\Omega)^3$ which converge to some $\mathbf{f}$, and $\nabla\times\mathbf{f}_k \to \mathbf{g}$, then $\mathbf{g} = \nabla \times \mathbf{f}$ and $\mathbf{f} \in H^1(\Omega)^3$.

First we prove that $A\mathbf{z} = \mathbf{y}$ using the the fact that the curl operator is closed.

$$\begin{aligned}
\mathbf{y} &= \lim_{k\to\infty} A\mathbf{z}_k \\
&= \lim_{k\to\infty} \begin{bmatrix} -\eta^{-1}\varepsilon^{-1}\mathbf{D}_k + \mu^{-1}\nabla \times \mathbf{B}_k \\ -\varepsilon^{-1}\nabla \times \mathbf{D}_k \end{bmatrix} \\
&= \begin{bmatrix} \lim_{k\to\infty}(-\eta^{-1}\varepsilon^{-1}\mathbf{D}_k + \mu^{-1}\nabla \times \mathbf{B}_k) \\ \lim_{k\to\infty}(-\varepsilon^{-1}\nabla \times \mathbf{D}_k) \end{bmatrix}
\end{aligned}$$

Since the curl operator is closed, $\lim_{k\to\infty} \nabla \times \mathbf{D}_k = \nabla \times \mathbf{D}$ and $D \in H^1(\Omega)^3$. Similarly, $\lim_{k\to\infty} \nabla \times \mathbf{B}_k = \nabla \times \mathbf{B}$ and $B \in H^1(\Omega)^3$.

Thus,

$$\mathbf{y} = \begin{bmatrix} -\eta^{-1}\varepsilon^{-1}\mathbf{D} + \mu^{-1}\nabla \times \mathbf{B} \\ (-\varepsilon^{-1}\nabla \times \mathbf{D}) \end{bmatrix} = A\mathbf{z},$$

and $(\mathbf{D}, \mathbf{B}) \in H^1(\Omega)^6$. It will be shown that $\mathbf{D}$ and $\mathbf{B}$ satisfy the boundary conditions. The cross product is a continuous operator, so it can be pulled outside a limit. For the first condition, starting with the fact that $\mathbf{n} \times \mathbf{D}_k = \mathbf{0}$ on Γ_0 and taking the limit as k goes to infinity of both sides,

$$\mathbf{0} = \lim_{k\to\infty}(\mathbf{n} \times \mathbf{D}_k) = \mathbf{n} \times \lim_{k\to\infty} \mathbf{D}_k = \mathbf{n} \times \mathbf{D}.$$

Similarly over Γ_1, starting with $\mathbf{n} \times (\mathbf{n} \times \mathbf{D}_k) + r\mathbf{n} \times \mathbf{B}_k = \mathbf{0}$, by the same reasoning the limit can be passed through all the terms proving that on Γ_1

$$\mathbf{n} \times (\mathbf{n} \times \mathbf{D}) + r\mathbf{n} \times \mathbf{B} = \mathbf{0}.$$

Therefore, $\mathbf{z} \in \mathcal{D}(A)$, proving A is closed. $\square$

Finally, A needs to be shown to be dense.

Lemma 3.5.3. *Let A be given by (3.54) with the domain $\mathcal{D}(A)$ given by (3.55). Then $A : \mathcal{D}(A) \subseteq H^1(\Omega)^6 \to L^2(\Omega)^6$ is dense in $L^2(\Omega)^6$.*

Proof. It is a well known fact that $C^1(\Omega)$ is dense in $L^2(\Omega)$ [22]. Since $C^1(\Omega) \subseteq H^1(\Omega)$, we directly get that $H^1(\Omega)$ is dense in $L^2(\Omega)$. The inclusion of boundary conditions does not affect the L^2 norm since functions in $L^2(\Omega)$ that differ by a finite measure are regarded as equivalent. Therefore we can conclude that $\mathcal{D}(A)$ is dense in $L^2(\Omega)^6$. $\square$

Now that we have shown A is close and densely defined we can prove well-posedness.

Theorem 3.5.4. *The operator A defined in (3.54) with domain $\mathcal{D}(A)$ defined in (3.55) generates a contraction semigroup.*

Proof. By Lemma 3.5.2, Lemma 3.5.3, A is closed and densely defined. What remains to be shown

$$
\begin{aligned}
< \mathbf{z}, A\mathbf{z} >_{\mathcal{Z}} \leq 0 \quad & \text{for all } \mathbf{z} \in \mathcal{D}(A), \\
< \mathbf{z}, A^*\mathbf{z} >_{\mathcal{Z}} \leq 0 \quad & \text{for all } \mathbf{z} \in \mathcal{D}(A^*).
\end{aligned}
$$

For the first condition, note that $< \mathbf{z}, A\mathbf{z} >_{\mathcal{Z}} = < \mathbf{z}, \partial_t \mathbf{z} >_{\mathcal{Z}} = 2\dot{\mathcal{H}}(t)$. In was already shown in Theorem 3.4.3 that $\dot{\mathcal{H}}(t) \leq 0$ for $\mathbf{z} \in \mathcal{D}(A)$. For the second condition, we work through the inner product to show the inequality holds.

$$
\begin{aligned}
< \mathbf{z}, A^*\mathbf{z} >_{\mathcal{Z}} &= \frac{1}{2} \int\limits_{\Omega} \varepsilon^{-1}\mathbf{D} \cdot (-\mu^{-1}\nabla \times \mathbf{B} - \varepsilon^{-1}\eta^{-1}\mathbf{D}) + \mu^{-1}\mathbf{B} \cdot (\varepsilon^{-1}\nabla \times \mathbf{D}) \, d\Omega \\
&= -\frac{1}{2} \int\limits_{\Omega} \varepsilon^{-1}\mu^{-1}(\mathbf{D} \cdot \nabla \times \mathbf{B} - \mathbf{B} \cdot \nabla \times \mathbf{D}) \, d\Omega - \varepsilon^{-2}\eta^{-1} \int\limits_{\Omega} |\mathbf{D}|^2 \, d\Omega \\
&= \frac{1}{2}\varepsilon^{-1}\mu^{-1} \int\limits_{\Omega} \nabla \cdot (\mathbf{D} \times \mathbf{B}) \, d\Omega - \varepsilon^{-2}\eta^{-1}\frac{1}{2} \int\limits_{\Omega} |\mathbf{D}|^2 \, d\Omega \\
&= \frac{1}{2}\varepsilon^{-1}\mu^{-1} \int\limits_{\Gamma} (\mathbf{D} \times \mathbf{B}) \cdot \mathbf{n} \, d\Gamma - \varepsilon^{-2}\eta^{-1}\frac{1}{2} \int\limits_{\Omega} |\mathbf{D}|^2 \, d\Omega.
\end{aligned}
$$

Since $\mathbf{z} \in \mathcal{D}(A^*)$, we have that $(\mathbf{D} \times \mathbf{B}) \cdot \mathbf{n} = (\mathbf{n} \times \mathbf{D}) \cdot \mathbf{B} = 0$ on Γ_0. From Corollary 3.4.2 we know we can rewrite

$$(\mathbf{D} \times \mathbf{B}) \cdot \mathbf{n} = -(\mathbf{n} \times (\mathbf{n} \times \mathbf{D})) \cdot (\mathbf{n} \times \mathbf{B}).$$

Combining these results back into the expression for the inner product,

$$\begin{aligned} < \mathbf{z}, A^*\mathbf{z} >_{\mathcal{Z}} &= -\frac{1}{2}\varepsilon^{-1}\mu^{-1} \int_{\Gamma_1} (\mathbf{n} \times (\mathbf{n} \times \mathbf{D})) \cdot (\mathbf{n} \times \mathbf{B}) \, d\Gamma - \varepsilon^{-2}\eta^{-1}\frac{1}{2} \int_{\Omega} |\mathbf{D}|^2 \, d\Omega \\ &= -\frac{1}{2}\varepsilon^{-1}\mu^{-1} \int_{\Gamma_1} r|\mathbf{n} \times \mathbf{B}|^2 \, d\Gamma - \varepsilon^{-2}\eta^{-1}\frac{1}{2} \int_{\Omega} |\mathbf{D}|^2 \, d\Omega. \end{aligned}$$

Both integrals are non-positive, so we can conclude that $< \mathbf{z}, A^*\mathbf{z} >_{\mathcal{Z}} \leq 0$ for all $\mathbf{z} \in \mathcal{D}(A^*)$. By Corollary 2.2.6, A generates a contraction semigroup. □

Since a contraction semigroup is a C_0-semigroup, by Corollary 2.1.7 a classical solution exists for all $(\mathbf{D}_0(\mathbf{x}), \mathbf{B}_0(\mathbf{x})) \in \mathcal{D}(A)$. This proves that the uncontrolled problem is well-posed.

Now that the basic problem has been shown to be well-posed, an argument proving that the controlled system is also well-posed needs to be made. Following [38], consider the following theorem.

Theorem 3.5.5. *Let*

$$R = r^{-1}\varepsilon^{-1}\mu^{-1}, G = 2\varepsilon^{-1}\eta^{-1}.$$

Let $\boldsymbol{u} \in \mathcal{U}, \boldsymbol{y} \in \mathcal{Y}, \boldsymbol{z} \in \mathcal{Z}$ where $\mathcal{Z}$ is the Hilbert space with inner product (3.52), $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces with the inner product

$$< \boldsymbol{f}, \boldsymbol{g} >_{\mathcal{U}} = < \boldsymbol{f}, \boldsymbol{g} >_{\mathcal{Y}} = \int_{\Gamma_1} R\boldsymbol{f} \cdot \boldsymbol{g} \, d\Gamma. \tag{3.58}$$

Then the solutions to the system (3.42)-(3.46) satisfy

$$\frac{d}{dt} < \boldsymbol{z}, \boldsymbol{z} >_{\mathcal{Z}} = < \boldsymbol{u}, \boldsymbol{u} >_{\mathcal{U}} - < \boldsymbol{y}, \boldsymbol{y} >_{\mathcal{Y}} - < G\boldsymbol{D}, \boldsymbol{D} >_{\mathcal{Z}} . \tag{3.59}$$

Proof. First note that

$$\frac{d}{dt} < \mathbf{z}, \mathbf{z} >_{\mathcal{Z}} = \frac{d}{dt} \int_{\Omega} \varepsilon^{-1} |\mathbf{D}|^2 + \mu^{-1} |\mathbf{B}|^2 \, d\Omega = 2\dot{\mathcal{H}}(t).$$

Now,

$$\begin{aligned}
< \mathbf{u}, \mathbf{u} >_{\mathcal{U}} &= \frac{1}{2} < r^{-1}\varepsilon^{-1}\mu^{-1}\mathbf{n} \times (\mathbf{n} \times \mathbf{D}) + \mathbf{n} \times \mathbf{B}, \mathbf{n} \times (\mathbf{n} \times \mathbf{D}) + r\mu^{-1}\mathbf{n} \times \mathbf{B} >_{L^2(\Gamma_1)} \\
&= \frac{1}{2} < r^{-1}\varepsilon^{-1}\mu^{-1}\mathbf{n} \times (\mathbf{n} \times \mathbf{D}), \mathbf{n} \times (\mathbf{n} \times \mathbf{D}) >_{\mathcal{U}} + \frac{1}{2} < \mathbf{n} \times \mathbf{B}, r\mathbf{n} \times \mathbf{B} >_{L^2(\Gamma_1)} \\
&+ \varepsilon^{-1}\mu^{-1} < \mathbf{n} \times (\mathbf{n} \times \mathbf{D}), \mathbf{n} \times \mathbf{B} >_{L^2(\Gamma_1)} .
\end{aligned}$$

Similarly,

$$\begin{aligned}
< \mathbf{y}, \mathbf{y} >_{\mathcal{Y}} &= \frac{1}{2} < r^{-1}\varepsilon^{-1}\mu^{-1}\mathbf{n} \times (\mathbf{n} \times \mathbf{D}) - \mathbf{n} \times \mathbf{B}, \mathbf{n} \times (\mathbf{n} \times \mathbf{D}) - r\mu^{-1}\mathbf{n} \times \mathbf{B} >_{L^2(\Gamma_1)} \\
&= \frac{1}{2} < r^{-1}\varepsilon^{-1}\mu^{-1}\mathbf{n} \times (\mathbf{n} \times \mathbf{D}), \mathbf{n} \times (\mathbf{n} \times \mathbf{D}) >_{\mathcal{Y}} + \frac{1}{2} < \mathbf{n} \times \mathbf{B}, r\mathbf{n} \times \mathbf{B} >_{L^2(\Gamma_1)} \\
&- \varepsilon^{-1}\mu^{-1} < \mathbf{n} \times (\mathbf{n} \times \mathbf{D}), \mathbf{n} \times \mathbf{B} >_{L^2(\Gamma_1)} .
\end{aligned}$$

By Corollary 3.4.2,

$$\begin{aligned}
2\dot{\mathcal{H}}(t) &= 2\varepsilon^{-1}\mu^{-1} \int_{\Gamma_1} (\mathbf{n} \times (\mathbf{n} \times \mathbf{D})) \cdot (\mathbf{n} \times \mathbf{B}) \, d\Gamma - 2\varepsilon^{-2}\eta^{-1} \int_{\Omega} |\mathbf{D}|^2 \, d\Omega \\
&= < \mathbf{u}, \mathbf{u} >_{\mathcal{U}} - < \mathbf{y}, \mathbf{y} >_{\mathcal{Y}} - < G\mathbf{D}, \mathbf{D} >_{\mathcal{Z}} .
\end{aligned}$$

□

To prove well-posedness, first we must establish what is meant by well-posedness in the sense of a control system. Following [9], we define a well-posed control system as follows.

Definition 3.5.6 (Well-posed control system)**.** *A control system is considered well-posed if the four maps from input and initial conditions to state and output are bounded for every* $t > 0$*.*

Theorem 3.5.7. *Consider the system* (3.42)-(3.46)*. Then for any $T > 0$ the following holds.*

1. $||\boldsymbol{z}(\boldsymbol{x},T)||^2_{\mathcal{Z}} \leq ||\boldsymbol{z}(\boldsymbol{x},0)||^2_{\mathcal{Z}}$,
2. $||\boldsymbol{z}(\boldsymbol{x},T)||^2_{\mathcal{Z}} \leq \int\limits_0^T ||\boldsymbol{u}||^2_{\mathcal{U}} dt$,
3. $||\boldsymbol{y}||^2_{\mathcal{Y}} \leq ||\boldsymbol{u}||^2_{\mathcal{U}}$,
4. $\int\limits_0^T ||\boldsymbol{y}||^2_{\mathcal{Y}} dt \leq ||\boldsymbol{z}(\boldsymbol{x},0)||^2_{\mathcal{Z}}$.

Proof. Proof of 1.

First, we know that $\dot{\mathcal{H}}(t) \leq 0$ from Theorem 3.4.3. Thus,

$$\frac{d}{dt}||\mathbf{z}(\mathbf{x},t)||^2_{\mathcal{Z}} \leq 0.$$

This implies $||\mathbf{z}||^2_{\mathcal{Z}}$ is non-increasing, so for any $T > 0$,

$$||\mathbf{z}(\mathbf{x},T)||^2_{\mathcal{Z}} \leq ||\mathbf{z}(\mathbf{x},0)||^2_{\mathcal{Z}}$$

.

Proof of 2.

Integrating (3.59) over t from 0 to T,

$$||\mathbf{z}(\mathbf{x},T)||^2_{\mathcal{Z}} - ||\mathbf{z}(\mathbf{x},0)||^2_{\mathcal{Z}} = \int\limits_0^T < \mathbf{u},\mathbf{u} >_{\mathcal{U}} dt - \int\limits_0^T < \mathbf{y},\mathbf{y} >_{\mathcal{Y}} dt - \int\limits_0^T < G\mathbf{D},\mathbf{D} >_{\mathcal{Z}} dt \quad (3.60)$$

Take $\mathbf{z}(\mathbf{x},0) = \mathbf{0}$. Then,

$$||\mathbf{z}(\mathbf{x},T)||^2_{\mathcal{Z}} \leq \int\limits_0^T < \mathbf{u},\mathbf{u} >_{\mathcal{U}} dt = \int\limits_0^T ||\mathbf{u}||^2_{\mathcal{U}} dt.$$

Proof of 3.

Take $\mathbf{z}(\mathbf{x},0)=\mathbf{0}$. From (3.60),

$$||\mathbf{z}(\mathbf{x},T)||^2_{\mathcal{Z}} \leq \int_0^T <\mathbf{u},\mathbf{u}>_{\mathcal{U}} dt - \int_0^T <\mathbf{y},\mathbf{y}>_{\mathcal{Y}} dt,$$

$$\int_0^T <\mathbf{y},\mathbf{y}>_{\mathcal{Y}} dt \leq \int_0^T <\mathbf{u},\mathbf{u}>_{\mathcal{U}} dt - ||\mathbf{z}(\mathbf{x},T)||^2_{\mathcal{Z}},$$

$$\int_0^T ||\mathbf{y}||^2_{\mathcal{Y}} dt \leq \int_0^T ||\mathbf{u}||_{\mathcal{U}} dt^2.$$

Since $T>0$ is arbitrary, this implies $||\mathbf{y}||^2_{\mathcal{Y}} \leq ||\mathbf{u}||_{\mathcal{U}} dt^2$.

Proof of 4.

Now take $\mathbf{u}=\mathbf{0}$.

$$||\mathbf{z}(\mathbf{x},T)||^2_{\mathcal{Z}} - ||\mathbf{z}(\mathbf{x},0)||^2_{\mathcal{Z}} \leq -\int_0^T <\mathbf{y},\mathbf{y}>_{\mathcal{Y}} dt.$$

Clearly,

$$\int_0^T <\mathbf{y},\mathbf{y}>_{\mathcal{Y}} dt \leq -||\mathbf{z}(\mathbf{x},T)||^2_{\mathcal{Z}} + ||\mathbf{z}(\mathbf{x},0)||^2_{\mathcal{Z}} \leq ||\mathbf{z}(\mathbf{x},0)||^2_{\mathcal{Z}},$$

so $\int_0^T ||\mathbf{y}||_{\mathcal{Y}} dt \leq ||\mathbf{z}(\mathbf{x},0)||^2_{\mathcal{Z}}$.

$\square$

Theorem 3.5.7 demonstrates that the maps from input and initial condition to state and output are all bounded, so we conlcude that the control problem is well-posed.

3.6 Infinite-Dimensional Port-Hamiltonian Systems in Higher Spatial Dimensions

Most of the literature present for infinite-dimensional port-Hamiltonian systems is for PDEs with a single spatial dimension. The problem of considering higher spatial dimensions is still an active field of research, so there is far less literature for higher-dimensional models. The problem of irrotational and isentropic fluids was considered in [20]. This paper did not consider a control problem so no discussion on a suitable control or observer is present.

Maxwell's equations were put in port-Hamiltonian form in a 2020 conference paper [27] and in a 2022 follow up paper [7]. In these papers, the general methodology was to take the general equations and put them in the form

$$\partial_t \mathbf{z} = (\mathcal{J} - \mathcal{R})(h\mathbf{z}),$$

where

$$\mathcal{J} = \begin{bmatrix} \mathcal{O} & \nabla\times \\ -\nabla\times & \mathcal{O} \end{bmatrix}, \mathcal{R} = \begin{bmatrix} \eta^{-1}\mathcal{I} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{bmatrix}, h\mathbf{z} = \begin{bmatrix} \varepsilon^{-1}\mathbf{D} \\ \mu^{-1}\mathbf{B} \end{bmatrix},$$

and then proceed to choose the boundary effort and flow in reverse. Instead of mathematically deducing what e_∂ and f_∂ were and then using them to to determine the passivity of the system, $\dot{\mathcal{H}}(t)$ was computed first and then the boundary effort and flow were chosen such that passivity was ensured. In this case they obtain the same result as Corollary 3.4.2, so they chose

$$f_\partial = \mathbf{n} \times \mathbf{B}, e_\partial = \mathbf{n} \times (\mathbf{n} \times \mathbf{D}).$$

The first paper [27] did not discuss well-posedness, and [7] mentioned well-posedness briefly but not including the controls.

Skrepek [31] in 2021 discussed PHS in higher spatial dimensions and some of the difficulty associated with it. In particular for Maxwell's equations he made note that some standard assumptions need to be relaxed to formulate Maxwell's equations in a manner approaching a PHS. Skrepek also provided a framework which derives the system (3.42)-(3.46) set up by [38] as a valid choice of system to use for a well-posed problem.

Some additional discussion is present in [5], where differential calculus which is beyond the scope of this thesis is used. This is again done for Maxwell's equations, and the system formulation agrees with [7, 27].

Chapter 4

Exponential Stability of Maxwell's Equations

The first and most relevant book to better motivate the work in this chapter was written by Komornik [14] in 1994. Komornik considered problems without internal current or charge and an impedance boundary modelled by

$$\partial_t \mathbf{D} = \nabla \times \mathbf{B} \qquad \Omega \times (0, \infty), \tag{4.1}$$
$$\partial_t \mathbf{B} = -\nabla \times \mathbf{D} \qquad \Omega \times (0, \infty), \tag{4.2}$$
$$0 = \nabla \cdot \mathbf{D} = \nabla \cdot \mathbf{B} \qquad \Omega \times (0, \infty), \tag{4.3}$$
$$0 = \mathbf{n} \times (\mathbf{n} \times \mathbf{D}) + \mathbf{n} \times \mathbf{B} \qquad \Gamma \times (0, \infty), \tag{4.4}$$
$$(\mathbf{D}(\mathbf{x}, 0), \mathbf{B}(\mathbf{x}, 0)) = (\mathbf{D}(\mathbf{x}), \mathbf{B}(\mathbf{x})) \qquad \Omega. \tag{4.5}$$

In this paper, Komornik set $\varepsilon = \mu = 1$ for his analysis. Komornik was able to use the multiplier method to show that the systems energy obeyed the following exponential decay law under the condition that Ω is a strictly star-shaped region.

Definition 4.0.1 (Star-shaped region)**.** *Let $\Omega \subset \mathbb{R}^n$ with a boundary Γ. We say Ω is a* ***star-shaped region*** *(Figure 4.1) if there exists some $\boldsymbol{x}_0 \in \Omega$ such that for all $\boldsymbol{x} \in \Omega$ the line*

$$s\boldsymbol{x} + (1 - s)\boldsymbol{x}_0 \quad t \in [0, 1],$$

is contained entirely inside Ω. This is equivalent to saying that for some $\boldsymbol{x}_0 \in \Omega$ and all $\boldsymbol{x} \in \Gamma$

$$\boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{x}_0) \geq 0,$$

where $\boldsymbol{n}$ is the outward unit normal to the surface at $\boldsymbol{x}$.

Moreover, we say a region is strictly star-shaped if for the first definition holds for all $\boldsymbol{x} \in \bar{\Omega}$, that is $\boldsymbol{x}$ is in the closure of Ω. This is equivalent to the condition that

$$\boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{x}_0) \geq \alpha > 0.$$

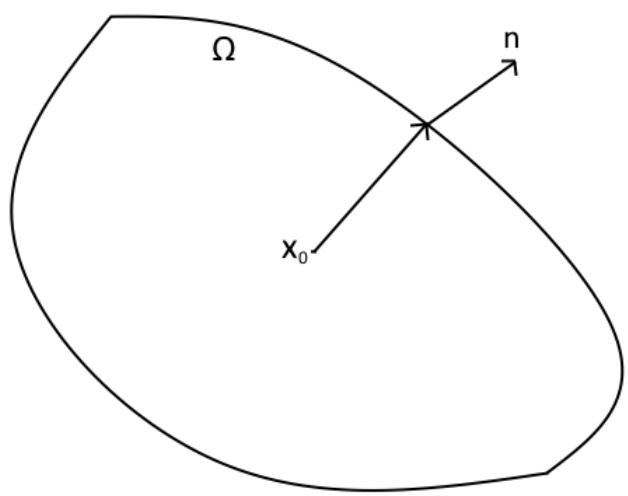

Figure 4.1: An example of a star-shaped region in two dimensions.

With the star-shaped assumption, he demonstrated that

$$\mathcal{E}(t) \leq \mathcal{E}(0)e^{1-t/(R+R_1)} \quad \forall \quad t \geq 0, \tag{4.6}$$

where $R = \sup_{\mathbf{x}\in\Omega} |\mathbf{x}|$, $R_1 = \max_{\mathbf{x}\in\Gamma} \dfrac{(\mathbf{x}\cdot\mathbf{n})^2 + |\mathbf{x}|^2}{2\mathbf{x}\cdot\mathbf{n}}$. Here $|\cdot|$ refers to the Euclidean norm, $|\mathbf{x}| = \sqrt{\mathbf{x}\cdot\mathbf{x}}$.

The condition that Ω is strictly star-shaped is not especially restrictive since it is a valid assumption for many regions including cylinders, spheres, or boxes.

This paper outlines much of the methodology that will be used later, so the proof outlined within will be redone in the case that ε and μ are both not equal to one will be presented in the next section.

Later, the work by Komornik was expanded by Nicaise and Pignotti [25] in 2003 by having less strict conditions. Specifically, they allowed for ε and μ to be functions of space and time, and they used a more general version of the impedance boundary condition known as the Silver-Müller condition

$$\mathbf{n}\times\mathbf{B} - \mathbf{n}\times\mathbf{g}(\mathbf{x}, \mathbf{n}\times\mathbf{D}) = \mathbf{0}\ \forall\ \mathbf{x}\in\Gamma, t\geq 0 \tag{4.7}$$

Here, $\mathbf{g}$ is some continuous function which satisfies some conditions. These conditions are not relevant to this thesis so they will not be specifically mentioned. Once again this paper made use of the multiplier method approach to demonstrate an energy decay.

In a subsequent 2005 paper [26], Nicaise and Pignotti considered the problem with spatially varying ε and μ but now instead of using the homogeneous Maxwell's equations it was no longer assumed that $\mathbf{J} = \mathbf{0}$. Instead of the impedance condition, they used the perfect conducting boundary condition. The method used in this paper involved splitting the solution into a homogeneous and inhomogeneous solution.

The above papers all only considered stability over domains where only one boundary condition was used across the entire boundary. Skrepek and Waurick [32] recently demonstrated semi-uniform stability for Maxwell's equations with a boundary broken into a perfect conducting region and a region obeying the impedance condition. Semi-uniform stability was described by them as a form of stability between exponential and asymptotic stability.

Some additional papers exploring the topic include Jochmann [11] who considered the homogeneous Maxwell's equations but allowed for the electric polarization to be non-linear, Pokojovy and Schnaubelt [28] who allowed for ε and μ to be rank-2 non-constant tensors (which serves as a model for regions that are heterogeneous and not isotropic), and Labrunie and Zaafanti [16] who discuss the stabilization of a cold magnetized plasma.

While the above deal directly with stabilization, there is a lot of literature that deal with the idea of controllability which are worth also mentioning. Controllability describes the ability to drive a system via the use of a control to an arbitrary state in finite time. For finite-dimensional systems stabilization and controllability are directly related. In infinite-dimensional systems, some aspects of this relationship can be extended [30, 2].

The controllability of Maxwell's equations on various regions has been considered. For the charge and current free Maxwell's equations, [13] demonstrated the boundary controllability over a sphere. A general star-shaped region was shown to be controllable in [17]. In another paper, [15] demonstrated boundary controllability controllability for a cube with perfect conducting boundaries by implementing a control on only a single face of the cube. Heterogeneous media were considered in [24].Also, Zhou [39] demonstrated internal conrollability (as opposed to the boundary control used in the other papers) by making use of the multiplier method.

4.1 Stability of Maxwell's Equations with an Impedance Boundary Condition and no Charge or Current

This section will go over the stability results presented by Komornik in [14]. Komornik formulated the problem in terms of $\mathbf{E}$ and $\mathbf{H}$, but this is equivalent to using $\mathbf{D}$ and $\mathbf{B}$ for a linear system. Moreover, Komornik set $\varepsilon = \mu = 1$ for his analysis. We will go through the same process Komornik did but in terms of $\mathbf{D}$ and $\mathbf{B}$ and with ε and μ arbitrary positive constants.

Let $\Omega \subset \mathbb{R}^3$ be an open domain with a C^1 boundary Γ. Let Ω have no free charge density ρ, or free current density $\mathbf{J}$. We assume Γ obeys the first-order impedance boundary condition (3.38); that is,

$$\partial_t \mathbf{D} = \mu^{-1}\nabla\times\mathbf{B},\ \partial_t\mathbf{B} = -\varepsilon^{-1}\nabla\times\mathbf{D} \qquad \mathbf{x}\in\Omega, t\geq 0, \tag{4.8}$$

$$\nabla\cdot\mathbf{D} = \nabla\cdot\mathbf{B} = 0 \qquad \mathbf{x}\in\Omega, t\geq 0, \tag{4.9}$$

$$\mathbf{n}\times(\mathbf{n}\times\mathbf{D}) = -\sqrt{\frac{\varepsilon}{\mu}}\mathbf{n}\times\mathbf{B} \qquad \mathbf{x}\in\Gamma, t\geq 0, \tag{4.10}$$

$$(\mathbf{D}(\mathbf{x},0),\mathbf{B}(\mathbf{x},0)) = (\mathbf{D}_0(\mathbf{x}),\mathbf{B}_0(\mathbf{x})) \qquad \mathbf{x}\in\Omega. \tag{4.11}$$

This system very close to the system (3.42)-(3.46). The main differences are that $\Gamma_0 = \emptyset$ is empty, the system is taken to be uncontrolled and the terms associated with the the charge and current density are gone. A consequence of having no internal current is there is no internal dissipation for the energy. Without free flowing electrons, there is no mechanism for the system to dissipate energy as heat.

Written in abstract form, this problem has an operator A_f with domain $\mathcal{D}(A_f)$ given by

$$A_f = \begin{bmatrix} \mathcal{O} & \mu^{-1}\nabla\times \\ -\varepsilon^{-1}\nabla\times & \mathcal{O}\end{bmatrix},$$

$$\mathcal{D}(A_f) = \{(\mathbf{D},\mathbf{B})\in H^1(\Omega)^6\cap\mathcal{Z}_f : \mathbf{n}\times(\mathbf{n}\times\mathbf{D}) + \sqrt{\frac{\varepsilon}{\mu}}\mathbf{n}\times\mathbf{B} = \mathbf{0} \text{ for } \mathbf{x}\in\Gamma\},$$

$$\mathcal{Z}_f = \{(\mathbf{f},\mathbf{g})\in L^2(\Omega)^6 : \nabla\cdot\mathbf{f}, \nabla\cdot\mathbf{g} \text{ exist and } \nabla\cdot\mathbf{f} = \nabla\cdot\mathbf{g} = 0 \text{ in } \Omega\}.$$

This problem is a simplification of the model explored in Section 3.5. We can apply the well-posedness results from the previous section here for this more specific case.

Take $(\mathbf{D}(\mathbf{x},0),\mathbf{B}(\mathbf{x},0))\in\mathcal{D}(A_f)$ to ensure a unique classical solution exists. Moreover, it is assumed that Ω is strictly star-shaped with respect to the origin such that for some $\alpha > 0$ we have $\mathbf{x}\cdot\mathbf{n}\geq\alpha$ on Γ.

Before stating the main theorem, we first state and prove several technical lemmas which will be needed.

Lemma 4.1.1. *For $\boldsymbol{x}\in\mathbb{R}^3$ and any differentiable function $\boldsymbol{f}\in\mathbb{R}^3$*

$$\boldsymbol{x}\cdot(\boldsymbol{f}\cdot\nabla)\boldsymbol{f} = \boldsymbol{f}\cdot\nabla(\boldsymbol{x}\cdot\boldsymbol{f}) - |\boldsymbol{f}|^2.$$

Proof. Use Einstein implied summation notation (i.e, for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^k$, $a_i b_i = \sum_{i=1}^{k} a_i b_i$) to ease the notation. Moreover, let ∂_i denote ∂_{x_i}, the partial derivative with respect to x_i.

$$\begin{aligned} \mathbf{f} \cdot \nabla(\mathbf{x} \cdot \mathbf{f}) - |\mathbf{f}|^2 &= f_i \partial_i (x_j f_j) - f_i f_i \\ &= f_i x_j \partial_i f_j + f_i f_j \partial_i x_j - f_i f_i. \end{aligned}$$

Note that $\partial_i x_j = \delta_{ij}$, the Kronecker delta, which is equal to 1 if $i = j$ and 0 otherwise.

$$\begin{aligned} \mathbf{f} \cdot \nabla(\mathbf{x} \cdot \mathbf{f}) - |\mathbf{f}|^2 &= f_i x_j \partial_i f_j + f_i f_j \delta_{ij} - f_i f_i \\ &= f_i x_j \partial_i f_j + f_i f_i - f_i f_i \\ &= x_j f_i \partial_i f_j \\ &= \mathbf{x} \cdot (\mathbf{f} \cdot \nabla)\mathbf{f}. \end{aligned}$$

□

Now recall that the total energy in given by

$$\mathcal{H}(t) = \frac{1}{2} \int_{\Omega} \varepsilon^{-1} |\mathbf{D}|^2 + \mu^{-1} |\mathbf{B}|^2 \, d\Omega. \tag{4.12}$$

Taking a derivative with respect to time,

$$\begin{aligned} \dot{\mathcal{H}}(t) &= \int_{\Omega} \varepsilon^{-1} \mathbf{D} \cdot \partial_t \mathbf{D} + \mu^{-1} \mathbf{B} \cdot \partial_t \mathbf{B} \, d\Omega \\ &= \varepsilon^{-1} \mu^{-1} \int_{\Omega} \mathbf{D} \cdot (\nabla \times \mathbf{B}) - \mathbf{B} \cdot (\nabla \times \mathbf{D}) \, d\Omega \end{aligned}$$

By the vector calculus identity $\nabla \cdot (\mathbf{f} \times \mathbf{g}) = \mathbf{g} \cdot (\nabla \times \mathbf{f}) - \mathbf{f} \cdot (\nabla \times \mathbf{g})$ and the divergence theorem,

$$\dot{\mathcal{H}}(t) = -\varepsilon^{-1}\mu^{-1}\int\limits_{\Gamma}(\mathbf{D}\times\mathbf{B})\cdot\mathbf{n}\;d\Gamma. \tag{4.13}$$

Lemma 4.1.2. *Let $\mathcal{H}(t)$ be given by (4.12), then*

$$\mathcal{H}(S)-\mathcal{H}(T) = \frac{1}{\sqrt{\varepsilon\mu}}\int\limits_{S}^{T}\int\limits_{\Gamma}\varepsilon^{-1}|\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{D})|^2\;d\Gamma dt = \frac{1}{\sqrt{\varepsilon\mu}}\int\limits_{S}^{T}\int\limits_{\Gamma}\mu^{-1}|\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{B})|^2\;d\Gamma dt.$$

Proof. It was shown in Corollary 3.4.2 that

$$(\mathbf{D}\times\mathbf{B})\cdot\mathbf{n} = (\mathbf{n}\times\mathbf{B})\cdot(\mathbf{n}\times(\mathbf{n}\times\mathbf{D})). \tag{4.14}$$

Applying the boundary condition, this term is exactly equal to $\sqrt{\frac{\mu}{\varepsilon}}|\mathbf{n}\times(\mathbf{n}\times\mathbf{D})|^2$ on the boundary. Substituting this into the expression for $\dot{\mathcal{H}}(t)$ and integrating with respect to t from S to T, we obtain

$$\mathcal{H}(T)-\mathcal{H}(S) = -\frac{1}{\sqrt{\varepsilon\mu}}\int\limits_{S}^{T}\int\limits_{\Gamma}\varepsilon^{-1}|\mathbf{n}\times(\mathbf{n}\times\mathbf{D})|^2\;d\Gamma dt.$$

To prove the second equality, we rewrite the boundary condition by decomposing $\mathbf{D}$ into its normal and tangential components by Proposition 3.2.3 and obtain

$$(\mathbf{D}\cdot\mathbf{n})\mathbf{n}-\mathbf{D} = \sqrt{\frac{\varepsilon}{\mu}}\mathbf{n}\times\mathbf{B}.$$

Applying $\mathbf{n}\times$ on the left of both sides of the equality, we obtain

$$\mathbf{n}\times\mathbf{D} = -\sqrt{\frac{\varepsilon}{\mu}}\mathbf{n}\times(\mathbf{n}\times\mathbf{B}). \tag{4.15}$$

Moreover, by the cyclic property of the scalar triple product, we can write (4.14) as

$$
\begin{aligned}
(\mathbf{D}\times\mathbf{B})\cdot\mathbf{n} &= (\mathbf{n}\times\mathbf{B})\cdot(\mathbf{n}\times(\mathbf{n}\times\mathbf{D}))\\
&= ((\mathbf{n}\times\mathbf{B})\times\mathbf{n})\cdot(\mathbf{n}\times\mathbf{D})\\
&= -(\mathbf{n}\times(\mathbf{n}\times\mathbf{B}))\cdot(\mathbf{n}\times\mathbf{D}).
\end{aligned}
$$

Substituting in $\mathbf{n}\times\mathbf{D} = -\sqrt{\frac{\varepsilon}{\mu}}\mathbf{n}\times(\mathbf{n}\times\mathbf{B})$, this term is equal to $\frac{\varepsilon}{\mu}|\mathbf{n}\times(\mathbf{n}\times\mathbf{B})|^2$ on the boundary. By the same argument as before, when we integrate $\dot{\mathcal{H}}(t)$ and integrating with respect to t from S to T we obtain

$$
\mathcal{H}(T)-\mathcal{H}(S) = -\frac{1}{\sqrt{\varepsilon\mu}}\int_S^T\int_\Gamma \mu^{-1}|\mathbf{n}\times(\mathbf{n}\times\mathbf{B})|^2\, d\Gamma dt.
$$

□

Another technical lemma we need is as follows.

Lemma 4.1.3. *Define $\boldsymbol{m}(\boldsymbol{x}) = \boldsymbol{x}$ and $w(t) = \int_\Omega(\boldsymbol{D}\times\boldsymbol{B})\cdot\boldsymbol{m}\, d\Omega$. Then*

$$
\begin{aligned}
&2\int_S^T \mathcal{H}(t)dt = 2\left[\int_\Omega(\boldsymbol{D}\times\boldsymbol{B})\cdot\boldsymbol{m}\, d\Omega\right]_{t=S}^T\\
&+\int_S^T\int_\Gamma(\boldsymbol{m}\cdot\boldsymbol{n})(\varepsilon^{-1}|\boldsymbol{D}|^2+\mu^{-1}|\boldsymbol{B}|^2-2\varepsilon^{-1}(\boldsymbol{m}\cdot\boldsymbol{D})(\boldsymbol{D}\cdot\boldsymbol{n})-2\mu^{-1}(\boldsymbol{m}\cdot\boldsymbol{B})(\boldsymbol{B}\cdot\boldsymbol{n})\, d\Gamma dt
\end{aligned}
$$

Proof.

$$
\begin{aligned}
\dot{w}(t) &= \int_\Omega(\mathbf{D}\times\partial_t\mathbf{B})\cdot\mathbf{m}+(\partial_t\mathbf{D}\times\mathbf{B})\cdot\mathbf{m}\, d\Omega\\
&= -\int_\Omega \varepsilon^{-1}(\mathbf{D}\times\nabla\times\mathbf{D})\cdot\mathbf{m}+\mu^{-1}(\mathbf{B}\times\nabla\times\mathbf{B})\cdot\mathbf{m}\, d\Omega.
\end{aligned}
$$

By the vector calculus identity $\mathbf{f}\times\nabla\times\mathbf{f} = \frac{1}{2}\nabla|\mathbf{f}|^2-(\mathbf{f}\cdot\nabla)\mathbf{f}$ and Lemma 4.1.1,

$$\mathbf{m}\cdot(\mathbf{D}\times\nabla\times\mathbf{D}) = \frac{1}{2}\mathbf{m}\cdot\nabla|\mathbf{D}|^2 - \mathbf{m}\cdot(\mathbf{D}\cdot\nabla)\mathbf{D} = \frac{1}{2}\mathbf{m}\cdot\nabla|\mathbf{D}|^2 - \mathbf{D}\cdot\nabla(\mathbf{m}\cdot\mathbf{D}) + |\mathbf{D}|^2,$$
$$\mathbf{m}\cdot(\mathbf{B}\times\nabla\times\mathbf{B}) = \frac{1}{2}\mathbf{m}\cdot\nabla|\mathbf{B}|^2 - \mathbf{m}\cdot(\mathbf{B}\cdot\nabla)\mathbf{B} = \frac{1}{2}\mathbf{m}\cdot\nabla|\mathbf{B}|^2 - \mathbf{B}\cdot\nabla(\mathbf{m}\cdot\mathbf{B}) + |\mathbf{B}|^2.$$

Substituting these into the expression for $\dot{w}(t)$

$$\begin{aligned}\dot{w}(t) = &- \int_{\Omega} \frac{1}{2}\mathbf{m}\cdot\nabla(\varepsilon^{-1}|\mathbf{D}|^2 + \mu^{-1}|\mathbf{B}|^2) - \varepsilon^{-1}\mathbf{D}\cdot\nabla(\mathbf{m}\cdot\mathbf{D}) - \mu^{-1}\mathbf{B}\cdot\nabla(\mathbf{m}\cdot\mathbf{B})\, d\Omega \\ &- \int_{\Omega} \varepsilon^{-1}|\mathbf{D}|^2 + \mu^{-1}|\mathbf{B}|^2\, d\Omega. \end{aligned} \tag{4.16}$$

From the vector calculus identity $\nabla\cdot(\phi\mathbf{f}) = \phi\nabla\cdot\mathbf{f} + \mathbf{f}\cdot\nabla\phi$ and the divergence theorem, we have a higher dimensional form of integration by parts

$$\int_{\Omega} \mathbf{f}\cdot\nabla\phi\, d\Omega = \int_{\Gamma} \phi\mathbf{f}\cdot\mathbf{n}\, d\Gamma - \int_{\Omega} \phi\nabla\cdot\mathbf{f}\, d\Omega.$$

Using the fact that we have $\nabla\cdot\mathbf{D} = \nabla\cdot\mathbf{B} = 0$ and $\nabla\cdot\mathbf{m} = \nabla\cdot\mathbf{x} = 3$,

$$\begin{aligned}\int_{\Omega} \mathbf{m}\cdot\nabla(\varepsilon^{-1}|\mathbf{D}|^2 + \mu^{-1}|\mathbf{B}|^2)\, d\Omega &= \int_{\Gamma} (\mathbf{m}\cdot\mathbf{n})(\varepsilon^{-1}|\mathbf{D}|^2 + \mu^{-1}|\mathbf{B}|^2)\, d\Gamma \\ &\quad - \int_{\Omega} (\nabla\cdot\mathbf{m})(\varepsilon^{-1}|\mathbf{D}|^2 + \mu^{-1}|\mathbf{B}|^2)\, d\Omega \\ &= \int_{\Gamma} (\mathbf{m}\cdot\mathbf{n})(\varepsilon^{-1}|\mathbf{D}|^2 + \mu^{-1}|\mathbf{B}|^2)\, d\Gamma \\ &\quad - 3\int_{\Omega} (\varepsilon^{-1}|\mathbf{D}|^2 + \mu^{-1}|\mathbf{B}|^2)\, d\Omega, \end{aligned} \tag{4.17}$$

$$\begin{aligned}\int_\Omega \varepsilon^{-1}\mathbf{D}\cdot\nabla(\mathbf{m}\cdot\mathbf{D}) &= \int_\Gamma \varepsilon^{-1}(\mathbf{m}\cdot\mathbf{D})(\mathbf{D}\cdot\mathbf{n})\ d\Gamma \\ &\quad - \int_\Omega \varepsilon^{-1}(\nabla\cdot\mathbf{D})(\mathbf{m}\cdot\mathbf{D})\ d\Omega \\ &= \int_\Gamma \varepsilon^{-1}(\mathbf{m}\cdot\mathbf{D})(\mathbf{D}\cdot\mathbf{n})\ d\Gamma, \end{aligned} \tag{4.18}$$

$$\begin{aligned}\int_\Omega \mu^{-1}\mathbf{B}\cdot\nabla(\mathbf{m}\cdot\mathbf{B}) &= \int_\Gamma \mu^{-1}(\mathbf{m}\cdot\mathbf{B})(\mathbf{B}\cdot\mathbf{n})\ d\Gamma \\ &\quad - \int_\Omega \mu^{-1}(\nabla\cdot\mathbf{B})(\mathbf{m}\cdot\mathbf{B})\ d\Omega \\ &= \int_\Gamma \mu^{-1}(\mathbf{m}\cdot\mathbf{B})(\mathbf{B}\cdot\mathbf{n})\ d\Gamma. \end{aligned} \tag{4.19}$$

Substituting (4.17)-(4.19) into (4.16),

$$\begin{aligned}\dot{w}(t) &= \frac{1}{2}\int_\Omega (\varepsilon^{-1}|\mathbf{D}|^2 + \mu^{-1}|\mathbf{B}|^2)\ d\Omega \\ &\quad - \frac{1}{2}\int_\Gamma (\mathbf{m}\cdot\mathbf{n})(\varepsilon^{-1}|\mathbf{D}|^2 + \mu^{-1}|\mathbf{B}|^2) - 2\varepsilon^{-1}(\mathbf{m}\cdot\mathbf{D})(\mathbf{D}\cdot\mathbf{n}) - 2\mu^{-1}(\mathbf{m}\cdot\mathbf{B})(\mathbf{B}\cdot\mathbf{n})\ d\Gamma.\end{aligned}$$

Moving the integral of the boundary to the other side, multiplying everything by 2, and noting that the integral over Ω is just the total energy,

$$\begin{aligned}2\mathcal{H}(t) &= 2\dot{w}(t) \\ &\quad + \int_\Gamma (\mathbf{m}\cdot\mathbf{n})(\varepsilon^{-1}|\mathbf{D}|^2 + \mu^{-1}|\mathbf{B}|^2) - 2\varepsilon^{-1}(\mathbf{m}\cdot\mathbf{D})(\mathbf{D}\cdot\mathbf{n}) - 2\mu^{-1}(\mathbf{m}\cdot\mathbf{B})(\mathbf{B}\cdot\mathbf{n})\ d\Gamma.\end{aligned}$$

Integrating with respect to t from S to T yields the desired result. □

Finally, we need to deal with the integrand of the boundary integral in the previous lemma.

Lemma 4.1.4. *Let* $R_1 = \max_{\boldsymbol{x}\in\Gamma} \dfrac{(\boldsymbol{m}\cdot\boldsymbol{n})^2 - |\boldsymbol{m}|^2}{\boldsymbol{m}\cdot\boldsymbol{n}}$. *Then*

$$\begin{aligned}
&(\boldsymbol{m}\cdot\boldsymbol{n})(\varepsilon^{-1}|\boldsymbol{D}|^2 + \mu^{-1}|\boldsymbol{B}|^2) - 2\varepsilon^{-1}(\boldsymbol{m}\cdot\boldsymbol{D})(\boldsymbol{D}\cdot\boldsymbol{n}) - 2\mu^{-1}(\boldsymbol{m}\cdot\boldsymbol{B})(\boldsymbol{B}\cdot\boldsymbol{n})\\
&\leq R_1(\varepsilon^{-1}|\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{D})|^2 + \mu^{-1}|\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{B})|^2).
\end{aligned}$$

Proof. Following Proposition 3.2.3, we decompose $\mathbf{D}$ and $\mathbf{B}$ into normal and tangential components

$$\begin{aligned}
\mathbf{D} &= (\mathbf{D}\cdot\mathbf{n})\mathbf{n} - \mathbf{n}\times(\mathbf{n}\times\mathbf{D}),\\
\mathbf{B} &= (\mathbf{B}\cdot\mathbf{n})\mathbf{n} - \mathbf{n}\times(\mathbf{n}\times\mathbf{B}).
\end{aligned}$$

From this we obtain

$$|\mathbf{D}|^2 = (\mathbf{D}\cdot\mathbf{n})^2 + |\mathbf{n}\times(\mathbf{n}\times\mathbf{D})|^2, \tag{4.20}$$
$$\mathbf{m}\cdot\mathbf{D} = (\mathbf{D}\cdot\mathbf{n})(\mathbf{m}\cdot\mathbf{n}) - \mathbf{m}\cdot(\mathbf{n}\times(\mathbf{n}\times\mathbf{D})), \tag{4.21}$$
$$|\mathbf{B}|^2 = (\mathbf{B}\cdot\mathbf{n})^2 + |\mathbf{n}\times(\mathbf{n}\times\mathbf{B})|^2, \tag{4.22}$$
$$\mathbf{m}\cdot\mathbf{B} = (\mathbf{B}\cdot\mathbf{n})(\mathbf{m}\cdot\mathbf{n}) - \mathbf{m}\cdot(\mathbf{n}\times(\mathbf{n}\times\mathbf{B})). \tag{4.23}$$

Using (4.20) and (4.21),

$$\begin{aligned}
&(\mathbf{m}\cdot\mathbf{n})|\mathbf{D}|^2 - 2(\mathbf{m}\cdot\mathbf{D})(\mathbf{D}\cdot\mathbf{n})\\
&= (\mathbf{m}\cdot\mathbf{n})((\mathbf{D}\cdot\mathbf{n})^2 + |\mathbf{n}\times(\mathbf{n}\times\mathbf{D})|^2) - 2(\mathbf{D}\cdot\mathbf{n})((\mathbf{D}\cdot\mathbf{n})(\mathbf{m}\cdot\mathbf{n}) - \mathbf{m}\cdot(\mathbf{n}\times(\mathbf{n}\times\mathbf{D})))\\
&= (\mathbf{m}\cdot\mathbf{n})(|\mathbf{n}\times(\mathbf{n}\times\mathbf{D})|^2 - (\mathbf{D}\cdot\mathbf{n})^2) + 2(\mathbf{D}\cdot\mathbf{n})(\mathbf{m}\cdot(\mathbf{n}\times(\mathbf{n}\times\mathbf{D}))).
\end{aligned}$$

By Young's inequality, for any $a, b, c \in \mathbb{R}$ with $c > 0$

$$2ab \leq ca^2 + \frac{1}{c}b^2$$

Taking $a = (\mathbf{D} \cdot \mathbf{n})$, $b = \mathbf{m} \cdot (\mathbf{n} \times (\mathbf{n} \times \mathbf{D}))$, and $c = \mathbf{m} \cdot \mathbf{n}$ (which is positive since we are assuming the region is strictly star-shaped)

$$2(\mathbf{D} \cdot \mathbf{n})(\mathbf{m} \cdot (\mathbf{n} \times (\mathbf{n} \times \mathbf{D}))) \le (\mathbf{m} \cdot \mathbf{n})(\mathbf{D} \cdot \mathbf{n})^2 + \frac{|\mathbf{m} \cdot (\mathbf{n} \times (\mathbf{n} \times \mathbf{D}))|^2}{\mathbf{m} \cdot \mathbf{n}}. \tag{4.24}$$

Similarly for $\mathbf{B}$,

$$2(\mathbf{B} \cdot \mathbf{n})(\mathbf{m} \cdot (\mathbf{n} \times (\mathbf{n} \times \mathbf{B}))) \le (\mathbf{m} \cdot \mathbf{n})(\mathbf{B} \cdot \mathbf{n})^2 + \frac{|\mathbf{m} \cdot (\mathbf{n} \times (\mathbf{n} \times \mathbf{B}))|^2}{\mathbf{m} \cdot \mathbf{n}}. \tag{4.25}$$

By (4.24)-(4.25),

$$\begin{aligned}
&(\mathbf{m} \cdot \mathbf{n})(\varepsilon^{-1}|\mathbf{D}|^2 + \mu^{-1}|\mathbf{B}|^2 - 2\varepsilon^{-1}(\mathbf{m} \cdot \mathbf{D})(\mathbf{D} \cdot \mathbf{n}) - 2\mu^{-1}(\mathbf{m} \cdot \mathbf{B})(\mathbf{B} \cdot \mathbf{n}) \\
&\le (\mathbf{m} \cdot \mathbf{n})(\varepsilon^{-1}|\mathbf{n} \times (\mathbf{n} \times \mathbf{D})|^2 + \mu^{-1}|\mathbf{n} \times (\mathbf{n} \times \mathbf{B})|^2) \\
&+ \frac{\varepsilon^{-1}|\mathbf{m} \cdot (\mathbf{n} \times (\mathbf{n} \times \mathbf{D}))|^2 + \mu^{-1}|\mathbf{m} \cdot (\mathbf{n} \times (\mathbf{n} \times \mathbf{B}))|^2}{\mathbf{m} \cdot \mathbf{n}}.
\end{aligned} \tag{4.26}$$

Applying the Cauchy-Schwarz inequality,

$$\varepsilon^{-1}|\mathbf{m} \cdot (\mathbf{n} \times (\mathbf{n} \times \mathbf{D}))|^2 + \mu^{-1}|\mathbf{m} \cdot (\mathbf{n} \times (\mathbf{n} \times \mathbf{B}))|^2 \le |\mathbf{m}|^2(\varepsilon^{-1}|\mathbf{n} \times (\mathbf{n} \times \mathbf{D})|^2 + \mu^{-1}|\mathbf{n} \times (\mathbf{n} \times \mathbf{B})|^2). \tag{4.27}$$

Using (4.27) as an upper bound in (4.26),

$$\begin{aligned}
&(\mathbf{m} \cdot \mathbf{n})(\varepsilon^{-1}|\mathbf{D}|^2 + \mu^{-1}|\mathbf{B}|^2 - 2\varepsilon^{-1}(\mathbf{m} \cdot \mathbf{D})(\mathbf{D} \cdot \mathbf{n}) - 2\mu^{-1}(\mathbf{m} \cdot \mathbf{B})(\mathbf{B} \cdot \mathbf{n}) \\
&\le \left((\mathbf{m} \cdot \mathbf{n}) + \frac{|\mathbf{m}|^2}{\mathbf{m} \cdot \mathbf{n}}\right)(\varepsilon^{-1}|\mathbf{n} \times (\mathbf{n} \times \mathbf{D})|^2 + \mu^{-1}|\mathbf{n} \times (\mathbf{n} \times \mathbf{B})|^2) \\
&= \left(\frac{(\mathbf{m} \cdot \mathbf{n})^2 + |\mathbf{m}|^2}{\mathbf{m} \cdot \mathbf{n}}\right)(\varepsilon^{-1}|\mathbf{n} \times (\mathbf{n} \times \mathbf{D})|^2 + \mu^{-1}|\mathbf{n} \times (\mathbf{n} \times \mathbf{B})|^2) \\
&\le R_1(\varepsilon^{-1}|\mathbf{n} \times (\mathbf{n} \times \mathbf{D})|^2 + \mu^{-1}|\mathbf{n} \times (\mathbf{n} \times \mathbf{B})|^2),
\end{aligned}$$

setting $R_1 = \max_{\mathbf{x} \in \Gamma} \dfrac{(\mathbf{m} \cdot \mathbf{n})^2 - |\mathbf{m}|^2}{\mathbf{m} \cdot \mathbf{n}}$. □

Now we are ready to state the main result.

Theorem 4.1.5. *Let Ω be an open, bounded and star-shaped region containing the origin in $\mathbb{R}^3$ with a C^1 boundary. Let $(\boldsymbol{D}_0, \boldsymbol{B}_0) \in \mathcal{D}(A_0)$ so classical solutions to (4.8)-(4.11) exist. Define*

$$R = \sup_{\boldsymbol{x} \in \Omega} |\boldsymbol{m}|, \tag{4.28}$$

$$R_1 = \max_{\boldsymbol{x} \in \Gamma} \frac{(\boldsymbol{m} \cdot \boldsymbol{n})^2 - |\boldsymbol{m}|^2}{\boldsymbol{m} \cdot \boldsymbol{n}}, \tag{4.29}$$

where $\boldsymbol{m} = \boldsymbol{x}$. Then the total energy decays exponentially as

$$\mathcal{H}(t) \leq \mathcal{H}(0) \exp\left(1 - \frac{t}{\sqrt{\varepsilon\mu}(R + R_1)}\right). \tag{4.30}$$

Proof. Using Lemma 4.1.2 and Lemma 4.1.4

$$\begin{aligned}
&\int_S^T \int_\Gamma (\mathbf{m} \cdot \mathbf{n})(\varepsilon^{-1}|\mathbf{D}|^2 + \mu^{-1}|\mathbf{B}|^2 - 2\varepsilon^{-1}(\mathbf{m} \cdot \mathbf{D})(\mathbf{D} \cdot \mathbf{n}) - 2\mu^{-1}(\mathbf{m} \cdot \mathbf{B})(\mathbf{B} \cdot \mathbf{n}) \; d\Gamma dt \\
&\leq R_1 \int_S^T \int_\Gamma \varepsilon^{-1}|\mathbf{n} \times (\mathbf{n} \times \mathbf{D})|^2 + \mu^{-1}|\mathbf{n} \times (\mathbf{n} \times \mathbf{B})|^2 \; d\Gamma dt \\
&= 2\sqrt{\varepsilon\mu} R_1 (\mathcal{H}(S) - \mathcal{H}(T)).
\end{aligned} \tag{4.31}$$

Moreover,

$$|2w(t)| = \left| 2 \int_\Omega (\mathbf{D} \times \mathbf{B}) \cdot \mathbf{m} \; d\Omega \right| \leq 2 \int_\Omega |(\mathbf{D} \times \mathbf{B}) \cdot \mathbf{m}| \; d\Omega.$$

We want to show this is bounded by the total energy. To accomplish this note that $|(\mathbf{D} \times \mathbf{B}) \cdot \mathbf{m}| \leq |\mathbf{m}||\mathbf{D}||\mathbf{B}|$ and

$$2|\mathbf{D}||\mathbf{B}| = \sqrt{\varepsilon\mu}\left(2\frac{|\mathbf{D}|}{\sqrt{\varepsilon}}\frac{|\mathbf{B}|}{\sqrt{\mu}}\right) \leq \sqrt{\varepsilon\mu}\left(\varepsilon^{-1}|\mathbf{D}|^2 + \mu^{-1}|\mathbf{B}|^2\right),$$

we have

$$|2w(t)| \leq \sqrt{\varepsilon\mu}\int_\Omega |\mathbf{m}|\left(\varepsilon^{-1}|\mathbf{D}|^2 + \mu^{-1}|\mathbf{B}|^2\right)\, d\Omega \leq \sqrt{\varepsilon\mu}R\int_\Omega \varepsilon^{-1}|\mathbf{D}|^2 + \mu^{-1}|\mathbf{B}|^2\, d\Omega = 2\sqrt{\varepsilon\mu}R\mathcal{H}(t),$$

for $R = \sup_{\mathbf{x}\in\Omega}|\mathbf{x}|$. From Lemma 4.1.3 we have

$$\begin{aligned}
&2\int_S^T \mathcal{H}(t)dt = [w(t)]_{t=S}^T \\
&+ \int_S^T\int_\Gamma (\mathbf{m}\cdot\mathbf{n})(\varepsilon^{-1}|\mathbf{D}|^2 + \mu^{-1}|\mathbf{B}|^2 - 2\varepsilon^{-1}(\mathbf{m}\cdot\mathbf{D})(\mathbf{D}\cdot\mathbf{n}) - 2\mu^{-1}(\mathbf{m}\cdot\mathbf{B})(\mathbf{B}\cdot\mathbf{n})\, d\Gamma dt.
\end{aligned}$$

Since $w(t) \leq |w(t)|$, we get $|w(T) - w(S)| \leq 2R(\mathcal{H}(S) + \mathcal{H}(T))$, so

$$\begin{aligned}
2\int_S^T \mathcal{H}(t)dt &\leq 2\sqrt{\varepsilon\mu}R(\mathcal{H}(S) + \mathcal{H}(T)) + 2\sqrt{\varepsilon\mu}R_1(\mathcal{H}(S) - \mathcal{H}(T)) \\
&= 2\sqrt{\varepsilon\mu}(R + R_1)\mathcal{H}(S) + 2\sqrt{\varepsilon\mu}(R - R_1)\mathcal{H}(T).
\end{aligned}$$

Using Young's inequality, it is easy to see that

$$(\mathbf{x}\cdot\mathbf{n})^2 + |\mathbf{x}|^2 \geq 2(\mathbf{x}\cdot\mathbf{n})|\mathbf{x}|.$$

Thus, if we take $|\mathbf{x}| = R$, the value of the supremum of $|\mathbf{x}|$ over Ω, then

$$R_1 = \max_{\mathbf{x}\in\Gamma}\frac{(\mathbf{x}\cdot\mathbf{n})^2 + |\mathbf{x}|^2}{\mathbf{x}\cdot\mathbf{n}} \geq \frac{2(\mathbf{x}\cdot\mathbf{n})R}{\mathbf{x}\cdot\mathbf{n}} = 2R \geq R.$$

Since $R_1 \geq R$, the coefficient in front of $\mathcal{H}(T)$ is negative. Since $\mathcal{H}(t)$ is strictly positive we can conclude

$$2\int_S^\infty \mathcal{H}(t)dt \leq 2\sqrt{\varepsilon\mu}(R+R_1)\mathcal{H}(S). \tag{4.32}$$

Let $\omega = \sqrt{\varepsilon\mu}(R+R_1)$, multiplying both sides by $\frac{1}{2}e^{s/\omega}$ the above inequality can be rewritten as

$$\frac{d}{ds}\left(e^{s/\omega}\int_s^\infty \mathcal{H}(t)dt\right) \leq 0.$$

Since the derivative is non-positive, the quantity is non-increasing in s. Thus,

$$e^{s/\omega}\int_s^\infty \mathcal{H}(t)dt \leq \int_0^\infty \mathcal{H}(t)dt. \tag{4.33}$$

From (4.32),

$$\int_0^\infty \mathcal{H}(t)dt \leq \omega\mathcal{H}(0). \tag{4.34}$$

Moreover, since $\mathcal{H}(t)$ is non-negative, truncating the integral on the left will result in a lower bound

$$e^{s/\omega}\int_s^{s+\omega} \mathcal{H}(t)dt \leq e^{s/\omega}\int_s^\infty \mathcal{H}(t)dt.$$

Finally note that $\mathcal{H}$ is decreasing and non-negative, so $\mathcal{H}(s+\omega) \leq \mathcal{H}(t)$ for all $t \in [s, s+\omega]$, which implies

$$\omega e^{s/\omega}\mathcal{H}(s+\omega) = e^{s/\omega}\int_s^{s+\omega} \mathcal{H}(s+\omega)dt \leq e^{s/\omega}\int_s^{s+\omega} \mathcal{H}(t)dt \leq e^{s/\omega}\int_s^{s+\omega} \mathcal{H}(t)dt. \tag{4.35}$$

Combining the upper bound (4.34) and lower bound (4.35)

$$\omega e^{s/\omega}\mathcal{H}(s+\omega) \leq \omega\mathcal{H}(0).$$

Set $t = s + \omega$ and move the exponential factor to the other side, leading to

$$\mathcal{H}(t) \leq e^{-(t-\omega)/\omega}\mathcal{H}(0) = e^{1-t/\omega}\mathcal{H}(0).$$

□

4.2 Stability Results for the Charge and Current Free Controlled Maxwell's Equations

In this section the results from Komornik [14] are extended for a controlled system. Refer back to the model in Chapter 3 for the full discussion of the model.

Let Ω be an open region in $\mathbb{R}^3$ with a C^1 boundary Γ. Let Ω have no free charge or current densities. Let $r > 0$ be a positive scalar function such that r and r^{-1} are both bounded away from zero on Ω. We consider the following controlled electromagnetic system with impedance boundary

$$\begin{aligned}
\partial_t \mathbf{D} = \mu^{-1}\nabla \times \mathbf{B},\ \partial_t \mathbf{B} = -\varepsilon^{-1}\nabla \times \mathbf{D} \qquad & \mathbf{x} \in \Omega, t \geq 0, && (4.36)\\
\nabla \cdot \mathbf{D} = \nabla \cdot \mathbf{B} = 0 \qquad & \mathbf{x} \in \Omega, t \geq 0, && (4.37)\\
\mathbf{u} = \frac{1}{\sqrt{2}}\left(\mathbf{n} \times (\mathbf{n} \times \mathbf{D}) + r\mathbf{n} \times \mathbf{B}\right) \qquad & \mathbf{x} \in \Gamma, t \geq 0, && (4.38)\\
(\mathbf{D}(\mathbf{x},0), \mathbf{B}(\mathbf{x},0)) = (\mathbf{D}_0(\mathbf{x}), \mathbf{B}_0(\mathbf{x})) \qquad & \mathbf{x} \in \Omega, && (4.39)
\end{aligned}$$

with a measurement

$$\mathbf{y} = \frac{1}{\sqrt{2}}\left(\mathbf{n} \times (\mathbf{n} \times \mathbf{D}) - r\mathbf{n} \times \mathbf{B}\right) \quad \mathbf{x} \in \Gamma, t \geq 0. \qquad (4.40)$$

Since $\Gamma_0 = \emptyset$ is empty and there is no internal current or charge, our domains of interest are now

$$
\begin{aligned}
A &= \begin{bmatrix} \mathcal{O} & \mu^{-1}\nabla\times \\ -\varepsilon^{-1}\nabla\times & \mathcal{O} \end{bmatrix}, \\
\mathcal{D}(A) &= \{(\mathbf{D},\mathbf{B}) \in H^1(\Omega)^6 \cap \mathcal{Z}_0 : \mathbf{n}\times(\mathbf{n}\times\mathbf{D}) + r\mathbf{n}\times\mathbf{B} = \mathbf{0} \text{ for } \mathbf{x}\in\Gamma\}, \\
\mathcal{Z}_0 &= \{(\mathbf{f},\mathbf{g}) \in L^2(\Omega)^6 : \nabla\cdot\mathbf{f}, \nabla\cdot\mathbf{g} \text{ exist and } \nabla\cdot\mathbf{f} = \nabla\cdot\mathbf{g} = 0 \text{ in } \Omega\}.
\end{aligned}
$$

As before, we assume that Ω is strictly star-shaped with respect to the origin. Moreover, we assume we can take $\mathbf{u}$ to be in the feedback form. for some value of $0 \le k < 1$,

$$
\mathbf{u} = k\mathbf{y}. \tag{4.41}
$$

Since in practice $\mathbf{u}$ is given by the surface current which we choose, it is assumed that we can choose this current to be a scalar multiple of the measurement $\mathbf{y}$. The boundary condition becomes

$$
0 = \sqrt{2}(\mathbf{u} - k\mathbf{y}) = (1-k)\mathbf{n}\times(\mathbf{n}\times\mathbf{D}) + r(1+k)\mathbf{n}\times\mathbf{B}.
$$

Moreover, the same trick of taking $\mathbf{n}\times$ from the left of both sides works and yields

$$
0 = (1-k)\mathbf{n}\times\mathbf{D} - (1+k)r\mathbf{n}\times(\mathbf{n}\times\mathbf{B}).
$$

As it turns out, only Lemma 4.1.2 needs to be modified for this new system, as the rest of the proof in Section 4.1 did not make use of the boundary condition.

Lemma 4.2.1. *Let $\mathcal{H}(t)$ be the total energy for a linear electromagnetic system. Let $\lambda_r = \inf_{\boldsymbol{x}\in\Gamma} r$ and $\lambda_{r^{-1}} = \inf_{\boldsymbol{x}\in\Gamma} r^{-1}$, Then*

$$
\mathcal{H}(S) - \mathcal{H}(T) \ge \mu^{-1}\frac{1-k}{1+k}\lambda_{r^{-1}} \int_S^T \int_\Gamma \varepsilon^{-1}|\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{D})|^2 \, d\Gamma dt, \text{ and}
$$

$$
\mathcal{H}(S) - \mathcal{H}(T) \ge \varepsilon^{-1}\frac{1+k}{1-k}\lambda_r \int_S^T \int_\Gamma \mu^{-1}|\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{B})|^2 \, d\Gamma dt
$$

Proof. As in the proof of Lemma 4.1.2,

$$\dot{\mathcal{H}}(t) = \varepsilon^{-1}\mu^{-1}\int\limits_{\Gamma}(\mathbf{n}\times\mathbf{B})\cdot(\mathbf{n}\times(\mathbf{n}\times\mathbf{D}))\; d\Gamma,$$

$$\dot{\mathcal{H}}(t) = -\varepsilon^{-1}\mu^{-1}\int\limits_{\Gamma}(\mathbf{n}\times\mathbf{D})\cdot(\mathbf{n}\times(\mathbf{n}\times\mathbf{B}))\; d\Gamma.$$

First applying $(1-k)\mathbf{n}\times(\mathbf{n}\times\mathbf{D}) + r(1+k)\mathbf{n}\times\mathbf{B} = 0$ on Γ, we have

$$\dot{\mathcal{H}}(t) = -\mu^{-1}\frac{1-k}{1+k}\int\limits_{\Gamma} r^{-1}\varepsilon^{-1}|\mathbf{n}\times(\mathbf{n}\times\mathbf{D})|^2\; d\Gamma.$$

Integrating with respect to time from S to T,

$$\mathcal{H}(T) - \mathcal{H}(S) = -\mu^{-1}\frac{1-k}{1+k}\int\limits_{S}^{T}\int\limits_{\Gamma} r^{-1}\varepsilon^{-1}|\mathbf{n}\times(\mathbf{n}\times\mathbf{D})|^2\; d\Gamma dt.$$

Since r^{-1} is bounded and positive, $\lambda_{r^{-1}} = \inf\limits_{\mathbf{x}\in\Gamma} r^{-1} > 0$, and the integrand is minimized by this value. Therefore,

$$\mathcal{H}(S) - \mathcal{H}(T) \geq \mu^{-1}\frac{1-k}{1+k}\lambda_{r^{-1}}\int\limits_{S}^{T}\int\limits_{\Gamma} \varepsilon^{-1}|\mathbf{n}\times(\mathbf{n}\times\mathbf{D})|^2\; d\Gamma dt.$$

Now starting with

$$\dot{\mathcal{H}}(t) = -\varepsilon^{-1}\mu^{-1}\int\limits_{\Gamma}(\mathbf{n}\times\mathbf{D})\cdot(\mathbf{n}\times(\mathbf{n}\times\mathbf{B})),$$

and using $0 = (1-k)\mathbf{n}\times\mathbf{D} - (1+k)r\mathbf{n}\times(\mathbf{n}\times\mathbf{B})$,

$$\dot{\mathcal{H}}(t) = -\varepsilon^{-1}\frac{1+k}{1-k}\int\limits_{\Gamma} r\mu^{-1}|\mathbf{n}\times(\mathbf{n}\times\mathbf{B})|^2\; d\Gamma.$$

Integrating with respect to time from S to T,

$$\mathcal{H}(T) - \mathcal{H}(S) = -\varepsilon^{-1}\frac{1+k}{1-k}\int_S^T \int_\Gamma r\mu^{-1}|\mathbf{n} \times (\mathbf{n} \times \mathbf{B})|^2 \, d\Gamma dt.$$

Since r is bounded and positive, $\lambda_r = \inf_{\mathbf{x}\in\Gamma} r > 0$, and the integrand is minimized by this value. Therefore,

$$\mathcal{H}(S) - \mathcal{H}(T) \geq \varepsilon^{-1}\frac{1+k}{1-k}\lambda_r \int_S^T \int_\Gamma \mu^{-1}|\mathbf{n} \times (\mathbf{n} \times \mathbf{B})|^2 \, d\Gamma dt.$$

□

Now to state the main result of this section.

Theorem 4.2.2. *Let Ω be an open and bounded, star-shaped region in $\mathbb{R}^3$ with a C^1 boundary. Consider solutions to (4.36)-(4.39). If $\boldsymbol{u} = k\boldsymbol{y}$ with $0 \leq k < 1$, then for all $(\boldsymbol{D}_0, \boldsymbol{B}_0) \in \mathcal{D}(A)$, the total energy decays exponentially as*

$$\mathcal{H}(t) \leq \mathcal{H}(0) \exp\left(1 - \frac{t}{\omega}\right) \qquad t \geq 0. \tag{4.42}$$

where

$$\omega = \sqrt{\varepsilon\mu}R + \frac{1}{2}R_1\left(\mu\frac{1-k}{1+k}\frac{1}{\lambda_{r^{-1}}} + \varepsilon\frac{1+k}{1-k}\frac{1}{\lambda_r}\right), \tag{4.43}$$

$$\lambda_r = \inf_{\boldsymbol{x}\in\Gamma} r, \tag{4.44}$$

$$\lambda_{r^{-1}} = \inf_{\boldsymbol{x}\in\Gamma} r^{-1}, \tag{4.45}$$

$$R = \sup_{\boldsymbol{x}\in\Omega} |\boldsymbol{x}| \tag{4.46}$$

$$R_1 = \max_{\boldsymbol{x}\in\Gamma} \frac{(\boldsymbol{x}\cdot\boldsymbol{n})^2 - |\boldsymbol{x}|^2}{\boldsymbol{x}\cdot\boldsymbol{n}}, \tag{4.47}$$

Proof. Using Lemma 4.1.2 and Lemma 4.1.4

$$\int_S^T \int_\Gamma (\mathbf{m}\cdot\mathbf{n})(\varepsilon^{-1}|\mathbf{D}|^2 + \mu^{-1}|\mathbf{B}|^2 - 2\varepsilon^{-1}(\mathbf{m}\cdot\mathbf{D})(\mathbf{D}\cdot\mathbf{n}) - 2\mu^{-1}(\mathbf{m}\cdot\mathbf{B})(\mathbf{B}\cdot\mathbf{n})\; d\Gamma dt$$
$$\leq R_1 \int_S^T \int_\Gamma \varepsilon^{-1}|\mathbf{n}\times(\mathbf{n}\times\mathbf{D})|^2 + \mu^{-1}|\mathbf{n}\times(\mathbf{n}\times\mathbf{B})|^2\; d\Gamma dt$$
$$\leq R_1 \left(\varepsilon^{-1}\frac{1-k}{1+k}\lambda_{r^{-1}} + \mu^{-1}\frac{1+k}{1-k}\lambda_r\right)(\mathcal{H}(S) - \mathcal{H}(T)).$$

Moreover, as was shown in Theorem 4.1.5,

$$|2w(t)| \leq 2\sqrt{\varepsilon\mu}R\mathcal{H}(t).$$

By Lemma 4.1.3

$$2\int_S^T \mathcal{H}(t)dt = [w(t)]_{t=S}^T$$
$$+ \int_S^T \int_\Gamma (\mathbf{m}\cdot\mathbf{n})(\varepsilon^{-1}|\mathbf{D}|^2 + \mu^{-1}|\mathbf{B}|^2 - 2\varepsilon^{-1}(\mathbf{m}\cdot\mathbf{D})(\mathbf{D}\cdot\mathbf{n}) - 2\mu^{-1}(\mathbf{m}\cdot\mathbf{B})(\mathbf{B}\cdot\mathbf{n})\; d\Gamma dt.$$

Since $w(t) \leq |w(t)|$, we get

$$2\int_S^T \mathcal{H}(t)dt \leq 2\sqrt{\varepsilon\mu}R(\mathcal{H}(S) + \mathcal{H}(T)) + R_1\left(\mu\frac{1-k}{1+k}\frac{1}{\lambda_{r^{-1}}} + \varepsilon\frac{1+k}{1-k}\frac{1}{\lambda_r}\right)(\mathcal{H}(S) - \mathcal{H}(T)).$$

Dividing by 2, taking $T \to \infty$ such that $\mathcal{H}(T) \to 0$, and setting $\omega = \sqrt{\varepsilon\mu}R + \frac{1}{2}R_1\left(\mu\frac{1-k}{1+k}\frac{1}{\lambda_{r-1}} + \varepsilon\frac{1+k}{1-k}\frac{1}{\lambda_r}\right)$.

$$\int_S^\infty \mathcal{H}(t)dt \le \omega\mathcal{H}(S).$$

The proof follows from the same argument shown in the proof of Theorem 4.1.5.

$\square$

Corollary 4.2.3. *Consider the special case where $k = 0$ (the system is uncontrolled) and $r = \sqrt{\frac{\varepsilon}{\mu}}$ (the case corresponding to the first-order impedance boundary). Then $\omega = \sqrt{\varepsilon\mu}(R + R_1)$*

Proof. In this case, r is a constant, so $\lambda_r = \sqrt{\frac{\varepsilon}{\mu}}$ and $\lambda_{r^{-1}} = \sqrt{\frac{\mu}{\varepsilon}}$. Thus,

$$\begin{aligned}
\omega &= \sqrt{\varepsilon\mu}R + \frac{1}{2}R_1\left(\mu\frac{1-k}{1+k}\frac{1}{\lambda_{r^{-1}}} + \varepsilon\frac{1+k}{1-k}\frac{1}{\lambda_r}\right) \\
&= \sqrt{\varepsilon\mu}R + \frac{1}{2}R_1\left(\mu\sqrt{\frac{\varepsilon}{\mu}} + \varepsilon\sqrt{\frac{\mu}{\varepsilon}}\right) \\
&= \sqrt{\varepsilon\mu}R + \frac{1}{2}R_1\left(2\sqrt{\varepsilon\mu}\right) \\
&= \sqrt{\varepsilon\mu}(R + R_1).
\end{aligned}$$

$\square$

This is exactly the same result found in Theorem 4.1.5. Corollary 4.2.3 demonstrates that this extension is consistent with the prior results by Komornik in [14].

Chapter 5

Conclusion and Future Work

In this thesis, the problem of demonstrating exponential stability of a controlled electromagnetic system by use of the multiplier method was considered. It was demonstrated in Chapter (3) that the system (3.42)-(3.46) the system was well-posed via use of the Lumer-Phillips theorem.

The main contribution of this thesis comes from Chapter 4. In this chapter, prior results considering an uncontrolled system with a first-order impedance boundary with no internal free charge or current was modified to allow for arbitrary permittivity ε and permeability μ. It was then demonstrated that exponential stability could obtained for a controlled system given by $\mathbf{u} = k\mathbf{y}$ for $0 \leq k < 1$. This result was also shown to be consistent with the earlier findings.

For the future, work needs to be done to determine a control and observer setup such that the system is both well-posed and passive. The system used in this thesis was well-posed but not passive. Ideally, the system could be proven to be passive in a port-Hamiltonian sense so that the passivity theorem could be applied.

Additionally, extending the work done to allow for free current and charge to be present is of interest, since many applications such as the control of plasma in reactors require the existence of free current. Some additional extensions to consider is loosening the requirement the region is homogeneous (i.e, allowing the system parameters to be spatially varying), and isotropic (i.e, allowing the system parameters to be matrices) to allow for a more broad result to be found.

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