Hopf 2-Algebras: Homotopy Higher Symmetries in Physics

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Hank Chen is the sole author of Ch. 1 and 7, which were written under the supervision of Prof. Florian Girelli and not written for publication. Those including manuscripts written for publication are the following.

Chapter 2

This chapter is based on the article "Gauging the Gauge and Anomaly Resolution" for which Hank Chen and prof. Florian Girelli were the sole authors of this work. This research was conducted at the University of Waterloo by Hank Chen under the supervision of prof. Florian Girelli. Discussions with Dr. Justin Kulp (Simons Centre) and prof. Urs Schreiber (NYU Abu Dhabi) have also contributed to the preparation of this work.

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Chapter 3 and 4

This chapter is based on two articles,

1. "Integrability from categorification", for which Hank Chen and prof. Florian Girelli were the sole authors.
2. "Categorified Drinfel'd double and BF theory: 2-groups in 4D", for which Hank Chen and prof. Florian Girelli were the sole authors.

Both of these research efforts were conducted at the University of Waterloo by Hank Chen under the supervision of prof. Florian Girelli.

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Citations

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Chapter 5

This chapter is based on the article "Categorical quantum groups and braided monoidal 2-categories", for which Hank Chen was the primary investigator and prof. Florian Girelli played a supporting role. This research was conducted at the University of Waterloo by Hank Chen under the supervision of prof. Florian Girelli. Discussions with David Green (Ohio State U.), profs. Theo Johnson-Freyd (Dalhousie U.), Liang Kong (SUSTech), and Matt Hogancamp (Northeastern U.) have also contributed to the preparation of this work.

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Chapter 6

This chapter is based on the article "Drinfel'd double symmetry of the 4d Kitaev model", for which Hank Chen was the sole author. This research was conducted at the University of Waterloo by Hank Chen under the supervision of prof. Florian Girelli. Discussions with

profs. Theo Johnson-Freyd (Dalhousie U.), Liang Kong (SUSTech), Matthew Yu (Oxford U.), Ruochen Ma (U. Chicago) have also contributed to the preparation of this work.

Citations

H. Chen, *Drinfel'd double symmetry of the 4d Kitaev model.* J. High Energ. Phys. **2023**, 141 (2023).

Abstract

The theory of Hopf algebras and quantum groups have led to very rich and interesting developments in both mathematics and physics. In particular, they are known to play crucial roles in the interplay between 3d topological quantum field theories, categorical algebras, and the geometry of embedded links and tangles. Moreover, the semiclassical limits of quantum group Hopf algebras, in particular, are vital for the understanding of integrable systems in statistical mechanics and Poisson-Lie dualities in string theory. The goal of this PhD thesis is to study a higher-dimensional version of these correspondences, based on the very successful categorical ladder proposal: higher-dimensional physics and geometry is described by higher-categorical strutures. This is accomplished with the definition of a *higher homotopy Hopf algebra*, which can be understood as a quantization of the homotopy Lie bialgebra symmetries that have recently received attention in various fields of theoretical physics. These higher-homotopy symmetries are part of the study of the recently-popular categorical symmetries, which appear in the condensed matter literature, for instance, in relation to 1-form dipole symmetries in topologically ordered phases. However, here I will provide another physical motivation arising from the gauge theoretic perspective, which is natural in the context of the Green-Schwarz anomaly cancellation mechanism in quantum field theories. In particular, I use this perspective to prove various known structural theorems about Lie 2-bialgebras and their associated 2-graded classical R-matrices, as well as to provide a new definition and characterization of the so-called "quadratic 2-Casimir" elements. I will apply these higher homotopy symmetries to study the 4d 2-Chern-Simons topological quantum field theory, and to develop a notion of graded classical integrability for 2+1d bulk-boundary coupled systems. By following the philosophy of deformation quantization and the theory of A_∞-algbera, I then introduce the notion of a "Hopf 2-algebra" explicitly, and prove several of their structural theorems. I will in particular derive a novel definition of a universal quantum 2-R-matrix and the higher-Yang-Baxter equations they satisfy. The main result of this thesis is that the 2-representation 2-category of Hopf 2-algebras is cohesively braided monoidal iff it is equipped with a universal 2-R-matrix, and that (weak) Hopf 2-algebras admit (weak) Lie 2-bialgebras as semiclassical limits. Finally, an application of this quantization framework will be considered, in which I will explicitly compute the higher representation theory of Drinfel'd double Hopf 2-algebras of finite groups. The corresponding 2-group Dijkgraaf-Witten topological field theories are then constructed directly from these Hopf 2-algebras, and I show that they recover the known 2-categorical characterizations of 4d $\mathbb{Z}_2$ symmetry protected topological phases of matter.

Acknowledgments

I would like to thank all the people who have supported me throughout my PhD and made this thesis possible.

Dedication

This is dedicated to the one I love.

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Chapter 1

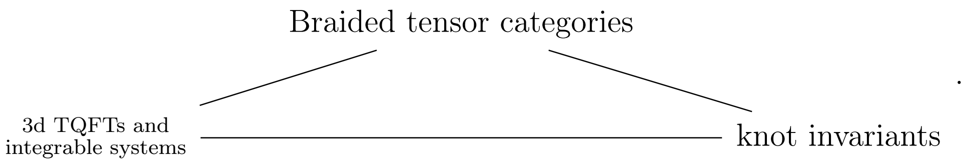
Introduction

The theory of quantum groups and Hopf algebras have received significant attention in various fields of mathematics and physics since the 1940's. They were first used by Hopf (hence the name "Hopf algebras") to describe structures of the cohomology rings of loop spaces, or more generally H-spaces [3], in algebraic topology. Somewhat independently, it was found that the $*$-algebra of operators in various statistical and quantum mechanical systems, such as integrable spin chains [4, 5] and 3d topological quantum field theories (TQFTs) [6] (eg. Chern-Simons/BF theory), also comes equipped with Hopf algebraic properties, in addition to extra functional analytic data attached. These "C^*-completed" Hopf algebras are also known as *quantum groups*, which are oftentimes infinite dimensional. In may cases, these quantum groups can be obtained from the corresponding "classical" Lie group/algebra objects in a systematic way through the so-called *Drinfel'd-Jimbo deformation* [7, 8]. These semiclassical limits of quantum groups are known as Poisson-Lie groups and Lie bialgebras [9, 10], which also turn out to play very important roles in integrability [11, 12] and T-duality in string theory [13, 14].

On the other hand, for topologists, it is known since the early 20th century that geometric and topological properties of spaces are best organized by algebraic gadgets known as *categories* [15]. These are abstract collection of objects and maps between them, usually endowed with additional structures such as a tensor product and a *braiding*. They in particular are understood to be crucial in describing the behaviour of embedded knots and links in 3-dimensional space — namely they capture invariants of *skein theory*. The tangle hypothesis [16] of Baez-Dolan is a vast generalization of this idea to higher-dimensions. It was also realized relatively recently that the structure of fusion categories — which are linear categories equipped with certain finitness conditions — can very generally be used to model gapped boundaries of topological phases in condensed matter physics [17, 18]. In this context, the celebrated *Levin-Wen model* [19, 20] describes an algorithm in which one can construct a concrete 2d lattice Hamiltonian from the data of a (spherical) fusion category. Such models can be understood as the lattice realization of the 2+1d *Turaev-Viro-Barrett-Westbury* topological quantum field theory (TQFT) [21, 22, 23, 24], the latter of which has very close relations to skein theory [25].

1.1 Hopf algebras at the centre of the 3d triangle

We have seen that there is a deep interplay of physics, categorical algebra and topology/geometry as captured by the following "3d triangle"



It turns out that the theory of Hopf algebras and quantum groups in fact play a central role in this story.

Hopf algebras and TQFTs/knot polynomials. It was known since the 1970's that, from certain statistical spin systems (ie. those that are integrable), an invariant of knots can in fact be computed from the partition function [26]. These were known to be the *Jones polynomial* invariants, which were found to be very closely related to 3d Chern-Simons quantum gravity [27]. From the quantum group Hopf algebra underlying these physical systems, the data of these geometric knot invariants can be systematically extracted out of their representations through the Kauffman bracket [28]. Indeed, the category of representations of a quantum group Hopf algebra turned out to have equipped precisely the structures required to describe embedded knots and links in 3-dimensions up to isotopy. These ideas have been generalized by Reshetikhin and Turaev [29, 30, 31] to define quantum invariants of 3-manifolds. This Reshetikhin-Turaev TQFT can be understood as a "modular refinement" of the Turaev-Viro TQFT, the former of which makes crucial use of the underlying braiding and ribbon data. These ideas have also been proposed to give rise to robust quantum computation [32, 33]. A crucial result by Witten [6] relates these Reshetikhin-Turaev quantum invariants back to the algebra of tangle operators (ie. Wilson lines) in $SU(2)_k$ Chern-Simons theory.

Hopf algebras and category theory. Somewhat independently of topology and physics, Hopf algebras were also known to play a significant role in the study of abstract category theory. Specifically, any fusion category (which are categories equipped with a monoidal product and satisfy certain finiteness conditions) equipped with a forgetful functor can in fact be realized up to equivalence as the representation category of a semisimple unital Hopf algebra [2].[1] This is known as the *Tannaka-Krein reconstruction* of monoidal categories [34, 35, 36, 37]. Inspired by this, the *Tannakian philosophy* is then the statement that structures on categories can be captured by modules/representations of Hopf-like algebras [38, 39].

Just very recently, in fact, the Tannakian philosophy has been concretely realized in the context of the Levin-Wen model [40]: string-net models can be realized as a gauge theory whose gauge algebra has Hopf-like properties.

[1]This statement does not require a fibre functor if we are content with Hopf algebras with weak units.

	Types of associative algebras	Category of modules
$k = 0$	algebra	category
$k = 1$	bialgebra	monoidal category (w/ forgetful)
...	Hopf algebra	tensor category (w/ forgetful)
$k = 2$	quasitriangular bialgebra	braided monoidal category (w/ forgetful)
...	quasitriangular Hopf algebra	braided tensor category (w/ forgetful)
...	Drinfel'd double	Drinfel'd centre

Table 1.1: A table summarizing Tannaka duality. The number k measures the level of "monoidality" of the category, with $k = 1$ denoting a monoidal product $\otimes$ and $k = 2$ denoting a braiding. Tensor categories have equipped duals and co/evaluation morphisms $X \otimes X^* \to 1, 1 \to X^* \otimes X$ that satisfy the snake equations [2].

Hopf algebras and vertex operator algebras. The above is not the full story. With compact quantum groups specifically, a correspondence of sorts was discovered by Kazhdan and Lusztik [41] between the representations of compact quantum groups — which labelled the tangle operators in 3d Chern-Simons theory [6] — and the *positive energy* representations of the Kac-Moody affine Lie algebra. The latter describes the algebra of operators in the 2d Wess-Zumino-Witten theory conformal field theory (CFT) [42], which lives at the boundary of the 3d Chern-Simons theory. Such an explicit correspondence between the operators of the bulk and boundary theories can be understood as one of the most mathematically well-understood instances of *holography* [43].

These facts have made Hopf algebras an extremely popular topic of research among both physicists and mathematicians, even until today.

1.2 Climbing the categorical ladder

The central theme of this PhD thesis is to motivate and understand a higher-dimensional version of the above story, focusing more on the physical and semiclassical side. This line of research has been very popular in the past few decades, following the "categorical ladder" proposal [44, 16, 45]. This is the proposal that higher-dimensional physics and geometry should be captured by higher-algebraic and higher-categorical structures [46, 47, 48]. The tangle hypothesis of Baez-Dolan [16] mentioned previously is part of this proposal.

This idea has been very successfully applied to many fields of theoretical physics, as a way to study emergent symmetry structures in field theory [49]. In fact, the development of the underlying mathematical theory of higher categories and categorical algebras is motivated in large part by the study of functorial TQFTs in the sense of Atiyah and Segal [50, 51, 52, 53, 54], which can be understood as a categorification of (framed) bordism invariants. The purported proof of the cobordism hypothesis by Lurie [55, 51], which sought to classify equivalence classes of (fully extended) functorial TQFTs in any dimension, sparked a series of developments in the field of categorical algebra that sought to pin down a notion of "higher-categories" [56, 57].

These higher categorical structures, specifically higher *fusion* categories [58, 59] and higher representations of finite groups [60, 61], have recently been successfully used as a way to record the renormalization group (RG) invariant properties of higher-dimensional phases of matter [62]. For a short and certainly non-exhaustive list of developments in this direction, see [48, 63, 64, 65, 66, 67, 46, 68]. The 4d topological sigma models associated to *finite* categorical groups have also been well-studied [69, 70, 71, 72, 73]. These can be understood as topological gauge theories whose structure groups form a special kind of finite category, called a **categorical group/2-group**,[2] and can be thought of as higher-dimensional generalizations of finite gauge theories.

One major success of the categorical ladder proposal is the recent work by Douglas and Reutter [65], in which the notion of "spherical fusion 2-categories" was defined, and a 4-dimensional analogue of the Turaev-Viro TQFT was constructed from its data. This led to many generalizations of known results and applications in physics to higher-dimensions, including a construction of 3d membrane-net Hamiltonians [74, 59], as well as exactly-solvable "fusion surface models" with 2-categorical symmetry [75]. Several very powerful classification and extension theorems for fusion 2-categories [48, 76, 63, 58, 77, 68, 78, 79] have been proven. These results have served to extend our understanding of higher-dimensional gapped topological phases.

The above results cement to an extent the connection between 4-dimensional TQFTs and higher-categorical algebras. In particular, a notion of *braided* monoidal 2-categories [80, 81] have been defined from the perspective of the 2-tangle hypothesis [82, 83]. Following similar ideas, knot polynomial invariants have seen a categorification in terms of bigraded complexes, called *knot homologies* [84, 85, 86]. Therefore, tentatively, there is a corresponding "4d triangle"

$$\begin{tikzcd}[column sep=small]
 & \text{Braided tensor 2-categories} \arrow[dl, dash] \arrow[dr, dash] & \\
\text{4d TQFTs} \arrow[rr, dash] & & \text{knot homologies}
\end{tikzcd}$$

that relates physics, algebra and geometry. Recent work as described above has cemented the edge on the top-left, namely that between higher categories and higher-dimensional topological field theories.

1.3 "Higher Hopf algebras" at the centre of the 4d triangle

What is the higher notion of Hopf algebras that sit in the centre of the 4d triangle? These algebraic gadgets, at the very least, should have braided monoidal 2-categories as their representations; see the categorical ladder diagram in 1.1.

One answer to this question came in the form of *Hopf monoidal categories* [87, 44], which can be understood as categories equipped with Hopf-like properties that hold *up to homotopy*. Representations of such Hopf monoidal categories have also been studied in [88], and they were found to have indeed the structure of a braided monoidal 2-category. A higher notion of the Tannaka-Krein duality has also appeared in [89, 90].

[2] The name "2-group" is ambiguous, as it can refer to a categorical group as well as a p-group where $p = 2$. We shall use the name "2-group" to refer strictly to the former throughout this thesis.

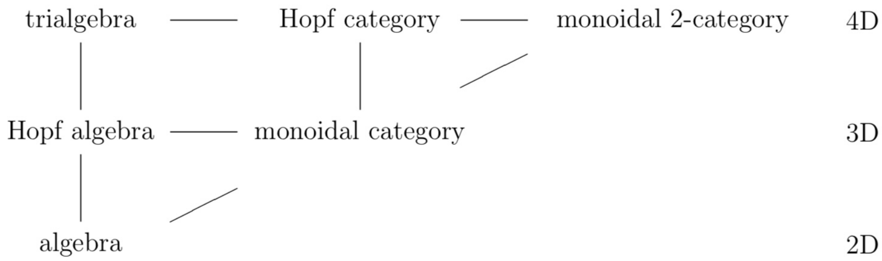


Figure 1.1: The categorical ladder as displayed in [1], which relates increasing categorical level with increasing dimensionality (the diagonal line). The horizontal axis represents the operation of taking modules/representations. The idea of a trialgebra is that their representations should form a Hopf monoidal category.

Alternatives to Hopf monoidal categories. In light of such rapid recent developments, however, several questions still remain open. One such question is the notion of a "quantum 2-group", which should be a categorical analogue of Drinfel'd-Jimbo deformed quantum groups. These should carry certain analytic data, and have a well-defined semiclassical limits; both are properties that Hopf monoidal categories lack. Many proposals for such objects have been proposed, such as the Hopf algebroids of Lu [91], quantum 2-groups of Majid [92], Hopf cat^1-algebras of Wagemann [93], and representations of trialgebras [1], to name a few.

However, it is not at all clear if the representations of these candidates for higher quantum groups have the right braided monoidal properties, or how any of them are related to higher-dimensional physics (aside from an application of trialgebras to 2+1d integrable spin systems [94]).

My PhD work is designed to precisely address this problem: I will provide first a motivation for the appearance of homotopy Lie algebras in higher gauge theory, then propose a notion of **homotopy Hopf algebra** which (i) serves as the quantum version of Lie 2-bialgebras [95]/Poisson-Lie 2-groups [96], (ii) whose representation 2-category is braided monoidal, and (iii) is closely related to many of the proposals for a "quantum 2-group" listed above.

1.4 A tale of two 2Vect's

An immediate issue one encounters is the *context* in which higher homotopy Hopf algebras should be defined. It is well-known that there are several inequivalent categorifications of the category of vector spaces Vect. Two of which of major interest in this thesis are the following.

1. Kapranov-Voevodsky (KV) 2-vector spaces $\mathsf{2Vect}^{KV}$ [97], which is a linear 2-category consisting of k-linear finite semisimple 1-categories (such as Vect), linear functors as 1-morphisms and natural transformations between these functors as 2-morphisms, and

2. Baez-Crans (BC) 2-vector spaces $\mathsf{2Vect}^{BC}$ [98], which is a linear 2-category consisting of k-linear 2-term cochain complexes of vector spaces, cochain maps as 1-morphisms and cochain homotopies between such chain maps as 2-morphisms.

The theory surrounding the KV 2-vector space has received much more attention in the literature (see eg. [48, 63, 64, 65, 58, 77]), and have seen successful applications to describe gapped topological phases in 4d [66, 67, 46, 68].

On the other hand, differential graded algebraic structures — such as L_∞-algebras [96, 99, 100, 101] and crossed-complexes of groups [102, 103, 104, 105, 106] — have also appeared in the literature as a way to model higher-dimensional physics, topology and geometry. These notions enjoy desirable properties, such as the fact that Lie 2-algebras serve as infinitesimal approximations of Lie 2-groups. The sort of gauge principles that are built out of the corresponding principal ∞-bundle [107, 108, 109] forms the basis of higher-gauge theories studied in the literature [110, 72, 111].

Hopf monoidal categories are, by definition, Hopf algebra objects in $\mathsf{2Vect}^{KV}$. They are linear semisimple categories $\mathcal{H}$ equipped with Hopf structure maps given by functors: for instance, the algebra map $\mu : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is understood as a functor between categories, whence associativity and unitality are witnessed by natural transformations [15]

$$\alpha : \mu \circ (\mu \times \mathrm{id}) \Rightarrow \mu \circ (\mathrm{id} \times \mu), \qquad \lambda^r : \mu \circ (\mathrm{id} \times \eta) \to \mathrm{id}, \qquad \lambda^l : \mu \circ (\eta \times \mathrm{id}) \Rightarrow \mathrm{id}\,.$$

These are known as **homotopy coherence data**, which must satisfy a complicated set of coherence conditions. Working with Hopf monoidal categories and their representations, however, is notoriously difficult [88].

On the other hand, the strict quantum 2-groups of Majid [92] can be viewed as Hopf algebra objects in the *strictification* of the bicategory of (linear semisimple) categories. By virtue of its definition, neither it nor its representation theory carry non-trivial coherence data, whence there are no coherence conditions to check and they are much easier to work with. Unfortunately, this is not at all a generic property, as it is known that monoidal 2-categories cannot be completely strictified [112]; contrast this with the case of monoidal 1-categories, which can always be strictified.[3] The same issue plagues the Hopf cat^1-algebras of Wagemann [93], which are Hopf algebra objects in the strict 2-category $\mathsf{2Vect}^{BC}$.

A major result of this PhD thesis is the resolution of this issue of the lack of coherence data in $\mathsf{2Vect}^{BC}$: I will develop a theory of Hopf A_∞-algebras which live, by the macroscopic principle [16], as Hopf algebra objects in a certain homotopy refinement of $\mathsf{2Vect}^{BC}$. This homotopy refinement $\mathsf{2Vect}^{hBC}$ exists, as I am able to explicitly write down all of the conditions that the coherence data must satisfy as cohomology descent equations. I will also show that these coherence conditions are very similar to those in $\mathsf{2Vect}^{KV}$. This framework of such Hopf A_∞-algebras appear directly from the fields in a 4d TQFT, similar to the factorization algebra approach of [101].

[3] This is known as the coherence theorem for monoidal categories by MacLane [15].

1.5 Outline

The outline of this thesis is as follows. I will begin with a pedagogical motivation for the appearance of higher homotopy Lie algebra symmetries from the perspective of gauge theory in §2, and show that we recover the known structures of (weak) 2-gauge theory [99]. This chapter is based on my paper [113]. I will then delve into the mathematical structures underlying Lie 2-bialgebras in §3, following the theory of (weak) Lie 2-bialgebras [95, 114, 115] known in the literature. Several major applications of such higher semiclassical symmetries to physics will be considered in §4, including 4d field theories and 2+1d integrable lattice systems. These chapters are based on my papers [115, 116].

The main portion of the thesis is §5, in which I develop the theory of (weak) Hopf 2-algebras as a Hopf algebra object in a homotopy refinement of the 2-category $\mathsf{2Vect}^{BC}$. I prove several key duality and factorizability properties à la Majid [117, 118], and define in a universal manner a higher notion of the quantum R-matrix. I will prove that they admit Lie 2-bialgebras as semiclassical limits, and their representations are cohesively braided monoidal in the sense of [81, 78]. This chapter is based on my paper [119].

Finally, as a proof of concept, I will apply the above framework to study 4d gapped topological phases with $\mathbb{Z}_2$ symmetry. Specifically, I will use the representation theory of Hopf 2-algebras to recover the Drinfel'd centre 2-categories that are used in the literature [76, 78, 120] to describe the 4d toric code and its spin version. This result unites the 2-categorical description of such phases with the 2-group gauge theoretic description of [71, 46]. This chapter is based on my paper [121].

Chapter 2

A procedure of "gauging the gauge"

In this Chapter, we introduce a procedure developed in [113], dubbed "gauging the gauge." This is a new pedagogical perspective on symmetries in which one can see the appearance of higher homotopy structures in gauge theory. We shall see that this "gauging"/localization of a global shift symmetry in 1-gauge theory gives rise precisely to a 2-gauge symmetry structure captured by Lie 2-algebras. Though such higher gauge structures have previously been studied [111, 96, 95, 99], this perspective provides a way to motivate the structure of Lie higher-algebras from physics and geometry. Moreover, we describe how the structures of a weak Lie 2-algebra [114, 98] manifests when the (1-)Bianchi identity is relaxed, and point out how the classifying *Postnikov class* [122, 69] contributes to the 2-curvature.

2.1 Gauging the 0-gauge

Let us begin by reviewing in a pedestrian way the notion of gauging a global symmetry. This is standard material, for which one can find many introductions (eg. [123]).

Let X denote a d-dimensional smooth manifold admitting an action by a Lie group G. Consider a (smooth) function ϕ on X transforming under a representation $\pi : G \to \mathrm{GL}(V)$ of the group G for some vector space V, that is $\phi \in C^\infty(X) \otimes V$ lies in the algebra of V-valued smooth functions on X.

Note that π is an homomorphism, and the field ϕ transforms as

$$\phi(x) \to \pi(g)\phi(x), \qquad g \in G.$$

If $g \in G$ is constant over X, then the derivative $d\phi$ transforms covariantly,

$$d\phi \to d(\pi(g)\phi) = \pi(g)d\phi,$$

and G encodes a (global) *0-gauge symmetry.*

We can promote g to be a G-valued function of X itself, such that we still have the transformation law

$$\phi(x) \to \pi(g(x))\phi(x) \equiv g(x) \cdot \phi(x) = \phi'.$$

In this case we are dealing with a principal bundle with fiber G and base X. Indeed, the Leibniz rule for the exterior derivative d dictates that[1]

$$d\phi \to g(d + g^{-1}dg) \cdot \phi.$$

As such it is not $d\phi$ that transforms covariantly, but the covariant derivative $\nabla\phi \equiv (d+g^{-1}dg)\phi$. Indeed, we can introduce the connection $A = g^{-1}dg \in \Omega^1(X) \otimes \mathfrak{g}$, to compensate for the lack of covariance,

$$gA\phi = d\phi' - gd\phi \to A = g^{-1}dg. \tag{2.1.1}$$

Notice that this connection has a natural invariance symmetry under the left translation for all $h \in G$ constant (ie. $dh = 0$).

$$(hg)^{-1}d(hg) = g^{-1}dg. \tag{2.1.2}$$

This is the well-known fact that this is a left-invariant form.

Given the covariant derivative $\nabla = d + g^{-1}dg$, its associated curvature

$$\mathrm{cur}\,\nabla = [\nabla, \nabla] = d(g^{-1}dg) + (g^{-1}dg) \wedge (g^{-1}dg) = 0$$

vanishes, where we have used the identity $d(1) = d(g^{-1}g) = (dg^{-1})g + g^{-1}dg = 0$. This means that the connection $A = g^{-1}dg$ is *flat*.

The 0-form symmetry and 1-gauge transformations. The connection 1-form in an arbitrary gauge, $A \in \Omega^1(X) \otimes \mathfrak{g}$ and the associated curvature 2-form $\mathrm{cur}\,A = F = d_A A = dA + \frac{1}{2}[A \wedge A]$ transform as

$$A \to A^g = g^{-1}Ag + g^{-1}dg, \qquad F \to F^g = g^{-1}Fg. \tag{2.1.3}$$

Expressing $g = \exp\lambda \approx 1 + \lambda$ in terms of the infinitesimal gauge parameter $\lambda \in \Omega^0(X) \otimes \mathfrak{g}$, we achieve the (infinitesimal) (1-)gauge transformation laws

$$\begin{aligned} A &\to A^\lambda = A + [A, \lambda] + d\lambda \equiv A + d_A\lambda, \\ F &\to F^\lambda = F + [F, \lambda]. \end{aligned}$$

They endow the bundle $P \to X$ with a *0-form gauge symmetry* parameterized by λ.

The *Bianchi identity* reads $d_A F = dF + [A \wedge F] = 0$, which holds in general for any principal G-bundle with connection A. Since F transforms covariantly, $d_A F$ also transforms covariantly

$$d_A F \to d_{A^\lambda} F^\lambda = d_A F + [d_A F, \lambda].$$

It is possible (and consistent) to achieve a *1-curvature anomaly* $F = \sigma \neq 0$, as long as $\sigma \in$

[1]Note that for notational simplicity we will not indicate π anymore. The representation π of G induces a representation $d\pi$ of its Lie algebra $\mathrm{Lie}\,G = \mathfrak{g}$. We will also omit $d\pi$ in this case.

$\Omega^2(X) \otimes \mathfrak{g}$ satisfies $d_A\sigma = 0$, and transforms covariantly $\sigma \to g^{-1}\sigma g$.

Global 1-form symmetry. What we have recalled here is that, by gauging the global symmetry understood as a "0-gauge" symmetry, we obtain an ordinary 1-gauge bundle $P \to X$ that is flat. However, one may notice that the curvature 2-form F has a hidden symmetry in the presence of a non-trivial center $Z(\mathfrak{g})$. This symmetry is given by

$$A \to A + \alpha, \tag{2.1.4}$$

where α is a *closed* 1-form valued in the center $Z(\mathfrak{g})$ of the Lie algebra $\mathfrak{G}$, that is $\alpha \in \Omega^1_0(X) \otimes Z(\mathfrak{g})$. As such the above gauge structure in fact manifests a "1-form symmetry" parameterized by α, on top of the pre-existing 1-gauge 0-form symmetry parameterized by λ. This 1-form symmetry is affecting the connection A but not its curvature.

2.2 Gauging the 1-gauge

In the 1-gauge case, we have highlighted two different types of invariance, one specified by a left multiplication, in (2.1.2), the other one by a 1-form shift in (2.1.4). It is natural to ask what happens when we gauge each symmetry, ie. we make them non-constant. For the former, making h non-constant amounts to just another gauge transformation, so there is nothing new to be gained. The latter is more interesting, as it leads to some new structures.

Relaxing the condition that α in (2.1.4) is constant and valued in the center $Z(\mathfrak{g})$ will be called "gauging the 1-form gauge." So, we allow ourselves to take $\alpha = a$ to be a generic 1-form $a \in \Omega^1(X) \otimes \mathfrak{g}$ that has non-trivial coordinate dependence on X, similar to the gauging procedure for the global/0-gauge symmetry.

2.2.1 Shifting the connection

Typically, one may *a priori* take a gauge bundle $P \to X$ with the non-trivial curvature $F = \sigma \neq 0$, then study the associated gauge theory. Alternatively, we may perform a *particular* 1-form shift such that $F \to F'$ is transformed to a non-trivial value.

Indeed, under a generic 1-form shift.

$$A \to A' = A + a,$$

we see that the curvature transforms accordingly as

$$F \to F' = d_{A'}A' = F + d_A a + \frac{1}{2}[a \wedge a] = F + d_A a + \frac{1}{2}[a \wedge a]. \tag{2.2.1}$$

In the gauge where $A = 0$, we just have

$$F' = da + \frac{1}{2}[a \wedge a],$$

which is the curvature of a considered as a G-connection. As such we may shift the curvature to any value from zero, which serves as the central key fact for anomaly resolution discussed later. Usually, the "gauging" story ends here, and we deal with an arbitrary curvature associated to the connection in a particular 1-form gauge $A = a$.

However, the above also shows that, by considering the 1-form shift as a higher-form gauge symmetry, the (1-)curvature quantity F is a *gauge datum*, the notion of curvature is gauge dependent. We have then a pair of gauge structures, one encoded in g which in a sense encodes the arbitrariness of the frame we deal with, and one encoded in a, which encodes the arbitrariness of the curvature.

One can realize that the transformation (2.2.1) can be seen as lack of covariance of the curvature 2-form under the arbitrary shift, analogous to the one of the derivative of the field ϕ under $\pi(g)$. To amend for the lack of covariance, we introduced a non-zero connection $A = gdg^{-1}$ in (2.1.1).

Hence in a similar manner, to amend for the lack of covariance of the curvature under the arbitrary shift, we introduce a *2-form* field $\Sigma \in \Omega^2(X) \otimes \mathfrak{g}$ such that, in the gauge where $A = 0$

$$\Sigma \equiv (F' - F) = F' = da + \frac{1}{2}[a \wedge a]. \tag{2.2.2}$$

If we define the curvature of Σ, as the 2-curvature,

$$K = d_A \Sigma,$$

then we see that by the Bianchi identity

$$d_A \Sigma = d_A F = 0,$$

so that this 2-connection is flat. Indeed as we shall see later, this 2-connection $\Sigma = da + \frac{1}{2}[a \wedge a]$ is a "pure 2-gauge", analogous to the flat pure 1-gauge $A = g^{-1}dg$ obtained from gauging the 0-gauge.

The construction so far is restrictive, in a sense since we focus on a 2-connection with value in the same Lie algebra $\mathfrak{g}$. It seems natural to make it valued in some other Lie algebra $\mathfrak{h}$, together with a map $t : \mathfrak{h} \to \mathfrak{g}$ (a homomorphism of Lie algebras), which plays in a sense the same role as the representation π when we dealt with a regular 1-gauge. The most natural notion to use is that of a Lie 2-algebra [98]. There are different notions of it. The first we are interested in is the notion of *strict* Lie 2-algebra, which can be equivalently viewed as a Lie algebra crossed-module [124]. The crossed-module formulation is most convenient to discuss the notion of 2-gauge theory. We shall also see how the notion of a weak Lie 2-algebra can be relevant in this setting.

2.2.2 Lie 2-algebras and Lie 2-groups

We first define the notion of Lie algebra crossed-modules, and introduce the fields relevant to building a 2-gauge theory. We will then seek to develop all the structures of a principal 2-bundle (see eg. [107]) from field-theoretic considerations.

Definition 2.2.1. A **Lie algebra crossed-module** $\mathfrak{G}$ is the data of a pair of Lie algebras $(\mathfrak{h}, [-,-]^{(-1)})$, $(\mathfrak{G}, [-,-]_0)$, a Lie algebra action $\rhd : \mathfrak{g} \to \operatorname{Der}\mathfrak{h}$ and a Lie algebra homomorphism $t : \mathfrak{h} \to \mathfrak{G}$ (called the *t-map*), satisfying the equivariance and the Peiffer identity

$$t(X \rhd Y) = [X, tY]_0, \qquad [Y, Y']^{(-1)} = (tY) \rhd Y', \tag{2.2.3}$$

as well as the 2-Jacobi identities

$$\begin{aligned} [X,[X',X'']_0]_0 + [X',[X'',X]_0]_0 + [X'',[X,X']_0]_0 &= 0, \\ X \rhd (X' \rhd Y) - X' \rhd (X \rhd Y) - [X,X']_0 \rhd Y &= 0, \end{aligned} \tag{2.2.4}$$

$\forall X, X', X'' \in \mathfrak{g}$, and $\forall Y, Y' \in \mathfrak{h}$.

The $\mathfrak{G}$-equivariance of t can be summarized by the following diagram

$$\begin{tikzcd} \mathfrak{h} \arrow[r, "t"] & \mathfrak{g} \arrow[dl, "\rhd"'] \arrow[d] \\ \operatorname{Der}\mathfrak{h} \arrow[r, "t"] & \operatorname{Inn}\mathfrak{g} \end{tikzcd} . \tag{2.2.5}$$

We shall denote a Lie algebra crossed-module by $\mathfrak{G} = (\mathfrak{h} \xrightarrow{t} \mathfrak{g}, \rhd, [-,-]_0)$ [115]. It is well-known that Lie algebra crossed-modules are equivalent to L_2-algebras, strict 2-term L_∞-algebra [95, 93].

Definition 2.2.2. A L_2**-algebra** is a graded space $\mathfrak{G} \cong \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ equipped with n-ary operations $\mu_n \in \operatorname{Hom}^{2-n}(\mathfrak{G}^{n\wedge}, \mathfrak{G})$ given by

$$n=1: \quad \mu_1 : \mathfrak{g}_{-1} \to \mathfrak{g}_0, \qquad n=2: \quad \mu_2 = [-,-] : (\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}) \otimes (\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}) \to (\mathfrak{g}_0 \oplus \mathfrak{g}_{-1})$$

such that the following *Koszul conditions* are satisfied,

$$\begin{aligned} &[X,X'] = -[X',X], \quad [X,Y] = -[Y,X], \quad \mu_1[X,Y] = [X,\mu_1 Y], \quad [\mu_1 Y, Y'] = [Y, \mu_1 Y'], \\ &[[X,X'],X''] + [[X'',X],X'] + [[X',X''],X] = 0, \qquad [[X,X'],Y] + [[X,Y],X'] + [X,[X',Y]] = 0, \end{aligned}$$

where $X, X', X'' \in \mathfrak{g}_0$, $Y, Y' \in \mathfrak{g}_{-1}$.

It is convenient to write the graded bracket $\mu_2 = [-,-] : \mathfrak{G}_i \otimes \mathfrak{G}_j \to \mathfrak{G}_{i+j}$ with $-2 \leqslant i+j \leqslant 0$, in terms of the degree $i, j \mod 2$ of $\mathfrak{G} \cong \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$, such that

$$\mu_2(Y+X, Y'+X') = [X,X'] + \big([X,Y'] + [Y,X']\big), \qquad X, X' \in \mathfrak{g}_0, \; Y, Y' \in \mathfrak{g}_{-1}. \tag{2.2.6}$$

In the following, we shall define μ_1 on the full space $\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ by $\mu_1(Y+X) = \mu_1 Y$.

Definition 2.2.3. A Lie algebra crossed-module map $\phi = (\phi, \phi_0) : \mathfrak{G} \to \mathfrak{G}'$ is a L_2-algebra homorphismsm consisting of a tuple of Lie algebra maps $\phi_{-1} : \mathfrak{h} \to \mathfrak{h}'$ and $\phi_0 : \mathfrak{G} \to \mathfrak{G}'$ such that

$$\phi_0 \circ t = t' \circ \phi_{-1}$$

and

$$\phi_{-1}(X \rhd Y) = (\phi_0(X)) \rhd' (\phi_{-1}(Y)), \qquad \forall\ X \in \mathfrak{G},\ Y \in \mathfrak{h}.$$

Given a Lie algebra crossed-module $\mathfrak{G} = (\mathfrak{h} \xrightarrow{t} \mathfrak{g}, \rhd, [-,-]_0)$, we simply identify $\mathfrak{g}_{-1} = \mathfrak{h}, \mathfrak{G} = \mathfrak{G}$ and $t = \mu_1$. Then, one reassembles the graded bracket μ_2 from the bracket $[-,-]_0$ on $\mathfrak{G}$ as well as the Lie algebra action $\rhd$ such that

$$\mu_2(Y + X, Y' + X') = [X, X']_0 + \big(X \rhd Y' - X' \rhd Y\big), \qquad X, X' \in \mathfrak{G},\ Y, Y' \in \mathfrak{h}.$$

It is then simple to check that the Lie algebra crossed-module conditions imply precisely the Koszul conditions; in particular, the Peiffer identity implies the Koszul identity

$$[\mu_1 Y, Y'] = [tY, Y'] = [Y, Y']^{(-1)} = -[Y', Y]^{(-1)} = -[tY', Y] = -[\mu_1 Y', Y] = [Y, \mu_1 Y']$$

as required. Conversely, given a strict L_2-algebra, one may recover a Lie algebra crossed-module with the above procedure, provided one *defines* the bracket $[-,-]^{(-1)}$ on $\mathfrak{h}$ by

$$[Y, Y']^{(-1)} \equiv [\mu_1 Y, Y'], \tag{2.2.7}$$

whence the Koszul conditions guarantee that this bracket is skew-symmetric and satisfies the Jacobi identity.

Due to this result, we will use "**strict Lie 2-algebras**" in the following to refer to both a Lie algebra crossed-module and a strict 2-term L_∞-algebra.

Lie theorem for Lie 2-groups. It is known that there is a one-to-one correspondence between (strict) Lie 2-algebras and connected, simply connected (strict) 2-groups [109, 125, 96], where the latter of which also admits a group crossed-module description.

Definition 2.2.4. A **Lie 2-group** $G = G_{-1} \xrightarrow{\mathbf{t}} G_0$ is the data of a pair of Lie groups G_{-1}, G_0, a smooth Lie group automorphism $\rhd : G_0 \times G_{-1} \to G_{-1}$ and a smooth group homomorphism $\mathbf{t} : G_{-1} \to G_0$ such that the following conditions

$$\mathbf{t}(x \rhd y) = x\mathbf{t}(y)x^{-1}, \qquad (\mathbf{t}y) \rhd y' = yy'y^{-1} \tag{2.2.8}$$

are satisfied for each $x \in G_0$ and $y, y' \in G_{-1}$.

It is easy to see that the t-map for the Lie algebra crossed-module is the tangent pushforward (ie. the derivative) of the smooth map $\mathbf{t}$ in the corresponding Lie 2-group G.

2.2.3 Connections and curvatures for Lie 2-algebras

Let us consider now the relevant connections: the 1-form connection A is valued in $\mathfrak{g}$, while the 2-form connection Σ is valued in $\mathfrak{h}$. As we will see in §2.2.3, t is a Lie algebra homomorphism that allows us to connect fields valued in $\mathfrak{h}$ to ones valued in $\mathfrak{g}$. This action $\triangleright$ can be viewed in a sense as the gauge transformations induced by $\mathfrak{g}$ on the fields/2-gauge parameters with value in $\mathfrak{h}$. This will be discussed in §2.3.

The covariant derivative we will use is still d_A, ie. it is defined in terms of the 1-connection A. We will therefore use the action to define the covariant derivative of a form with value in $\mathfrak{h}$. Taking an arbitrary $\mathfrak{h}$-valued n-form $S \in \Omega^n(X) \otimes \mathfrak{h}$, we introduce the wedge product $\wedge^{\triangleright}$ between a 1-form and and n-form,

$$\wedge^{\triangleright} : (\Omega^1(X) \otimes \mathfrak{g}) \otimes (\Omega^n(X) \otimes \mathfrak{h}) \xrightarrow{\wedge} \Omega^{n+1}(X) \otimes (\mathfrak{g} \otimes \mathfrak{h}) \xrightarrow{\triangleright} \Omega^{n+1}(X) \otimes \mathfrak{h}.$$

This allows to define the covariant derivative of $S \in \Omega^n(X) \otimes \mathfrak{h}$,

$$d_A S \equiv dS + A \wedge^{\triangleright} S.$$

Putting together the differential $d_A - = d - +A \wedge^{\triangleright} -$ on $\Omega^n(X) \otimes \mathfrak{h}$ with the t-map, and using the $\mathfrak{g}$-equivariance[2] implies that the covariant derivative d_A on $\mathfrak{h}$-valued forms is mapped under t to the covariant differential d_A on $\mathfrak{g}$-valued forms. This can be expressed compactly as

$$t d_A = d_A t. \tag{2.2.9}$$

Given the general 2-Lie algebra framework, we explore the different notions of curvature that appear. First we have the notion of fake flatness which relates the 2-connection to the 1-curvature up to the t-map. We then express the properties of the 2-curvature and highlight it also satisfies a type of Bianchi identity. Finally, we discuss how the one kind of violation of the 1-Bianchi identity can be recast in terms of a 2-gauge theory based on a *weak* 2-Lie algebra.

Fake-curvature

When using the crossed-module formalism, the relation between the 2-connection and the curvature we introduced in (2.2.2) can be rewritten as

$$t(\Sigma) = F' = da + a \wedge a,$$

with $\Sigma = dL + \frac{1}{2}[L \wedge L]$, provided that $t(L) = a$. In fact (2.2.2) can be readily obtained if $\mathfrak{h} = \mathfrak{g}$ and the t map is the identity. Hence the construction in (2.2.2) can be seen as an example of a 2-gauge theory based on the identity crossed-module.

[2] We have $t(A \wedge^{\triangleright} S) = [A \wedge t(S)]$.

The relation (2.2.2) can also be interpreted as a generalized notion of curvature

$$\mathcal{F} = F' - t(\Sigma),$$

which is known as *fake-curvature*. The condition in which it is constrained to be zero,

$$\mathcal{F} = F' - t(\Sigma) = 0, \tag{2.2.10}$$

is known as the *fake-flatness condition*. A naïve notion of "2-parallel transport" serves as a geometric motivation for imposing (2.2.10) [106], but we need not assume it at the infinitesimal level based on a Lie algebra crossed-module/*strict* Lie 2-algebra. We will see nevertheless that such condition can also appear when we consider 1- or 2-gauge transformations in §2.3.

Remark 2.2.1. As mentioned previously, we note that (2.2.10) can be interpreted as sourcing the curvature with $t(\Sigma)$, allowing us to break away from a flat 1-connection by sourcing it with a higher-gauge field. Further, it is possible to define a notion of higher-parallel transport *without* fake-flatness $\mathcal{F} \neq 0$, which would move us into the realm of *adjusted 2-parallel transport* [99]. We shall not consider this here.

2-curvature and 2-Bianchi identity

The 2-curvature is defined as the tensor $K = d_A\Sigma \in \Omega^3(X) \otimes \mathfrak{h}$. When the 2-connection is pure 2-gauge $\Sigma = dL + \frac{1}{2}[L \wedge L]$, we have as expected $K = 0$,

$$d_A\Sigma = d^2L + \frac{1}{2}d[L \wedge L] + t(L) \rhd (dL + \frac{1}{2}[L \wedge L]) = 0 \tag{2.2.11}$$

where for simplicity we picked the 1-gauge where $A = t(L)$ and we used that $d^2 = 0$, the Peiffer identity and the Jacobi identity for $\mathfrak{h}$.

One may insert a *2-curvature anomaly* $\kappa \neq 0$, such that $K = \kappa$, in which the principal 2-bundle under consideration is no longer trivial. We will study this in §2.3.2. As we are going to show, K (and hence κ) must be valued in $\ker t$ on-shell of the fake-flatness condition $\mathcal{F} = 0$. Indeed, for any 2-connection, as a consequence of the fake-flatness condition and the 1-Bianchi identity, the 2-curvature must be valued in $\ker t \subset \mathfrak{h}$.

$$t(K) = t(d_A\Sigma) = d_A t(\Sigma) = d_A F = 0. \tag{2.2.12}$$

As a consequence of the Bianchi identity, we have that $d_A K \in \ker t$.

On the other hand, by the graded Leibniz rule, the 2-curvature K satisfies

$$d_A K = d_A(d_A\Sigma) = F \wedge^{\rhd} \Sigma = t(\Sigma) \wedge^{\rhd} \Sigma = [\Sigma \wedge \Sigma]|_{\ker t},$$

where we used the Peiffer condition. Note that since $d_A K$ is valued in $\ker t$, we should project the commutator $[\Sigma \wedge \Sigma]$ to $\ker t$. However, since Σ is a 2-form and $[-,-] = (t\cdot) \rhd -$ is skew-symmetric, this term vanishes and hence we achieve the *2-Bianchi identity*

$$d_A K = 0. \tag{2.2.13}$$

1-Bianchi anomaly and weak 2-Lie algebras

Now suppose we relax the 1-Bianchi identity, such that it no longer holds. Then K needs not be valued in $\ker t$.

$$\begin{aligned} tK = d_A F = dF + [A, F] &= d^2 A + \frac{1}{2} d[A \wedge A] + [A \wedge dA] + \frac{1}{2}[A \wedge [A \wedge A]] \\ &= d^2 A + \frac{1}{2}[A \wedge [A \wedge A]] \neq 0, \end{aligned}$$

where we used that $d[A \wedge A] = [dA \wedge A] - [A \wedge dA] = -2[A \wedge dA]$. There are two different ways to do this, one is to let $d^2 A \neq 0$ (globally), in which case we have a monopole. The other way is if the second term is non-vanishing, which occurs when we let go of the Jacobi identity on $\mathfrak{g}$. In this case, $\mathfrak{g}$ is strictly speaking no longer a Lie algebra; however, we shall see that the following structure we shall derive can also be applied to the case where $\mathfrak{G}$ is a Lie algebra, but $t = 0$ must be identically zero.

Remark 2.2.2. The two ways in which the 1-Bianchi identity is violated are distinct. The violation of the Jacobi identity $[A \wedge [A \wedge A]]$ is of an algebraic nature, and hence introduces non-trivial modifications to our Lie 2-algebra structure; we shall focus on this case in the following. On the other hand, the monopole case $d^2 A \neq 0$ is of differential geometric nature, which indicates a non-trivial topology of the 1-gauge theory.

By relinquishing the Jacobi identity, we may write this term as a contribution to K by lifting it along t up to $\mathfrak{h}$. In other words, we introduce a skew-trilinear map — called appropriately the **Jacobiator** — satisfying

$$\mu : \mathfrak{g}^{\wedge 3} \to \mathfrak{h}, \qquad \frac{1}{3!} t\mu(A, A, A) = [A \wedge [A \wedge A]], \tag{2.2.14}$$

such that the **modified 2-flatness** reads

$$K = d_A \Sigma - \frac{1}{3!}\mu(A, A, A) = 0. \tag{2.2.15}$$

Since the term $\mu(A, A, A)$ arises due to the failure of the 1-Bianchi identity, we call it the *1-Bianchi anomaly.* This structure is captured algebraically by the following.

Definition 2.2.5. A **weak** Lie 2-algebra [96], or equivalently a **semistrict** [126] Lie 2-algebra, is a graded space $\mathfrak{G} \cong \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ equipped with n-ary operations $\mu_n \in \mathrm{Hom}^{2-n}(\mathfrak{G}^{n\wedge}, \mathfrak{G})$ where μ_1, μ_2 are given as in **Definition 2.2.2**, but with a non-trivial *homotopy map* $\mu = \mu_3 : \mathfrak{g}_0^{\wedge 3} \to \mathfrak{g}_{-1}$. The Koszul conditions now read

$$[X, X'] = -[X', X], \quad [X, Y] = -[Y, X], \quad \mu_1[X, Y] = [X, \mu_1 Y], \quad [\mu_1 Y, Y'] = [Y, \mu_1 Y'], \tag{2.2.16}$$

$$[X,[X',X'']] + [X',[X'',X]] + [X'',[X,X']] = t\mu(X,X',X''), \tag{2.2.17}$$
$$X \rhd (X' \rhd Y) - X' \rhd (X \rhd Y) - [X,X'] \rhd Y = \mu(X,X',t(Y)) \tag{2.2.18}$$

for each $X, X', X'' \in \mathfrak{g} = \mathfrak{g}_0$ and $Y \in \mathfrak{h} = \mathfrak{g}_{-1}$. Moreover, μ must satisfy the *3-cocycle condition*

$$x \rhd \mu(x_1,x_2,x_3) = \mu([x,x_1],x_2,x_3) + \mu(x_1,[x,x_2],x_3) + \mu(x_1,x_2,[x,x_3]). \tag{2.2.19}$$

Indeed, (2.2.14) is equivalent to the second line of these conditions. Note μ may only appear for non-Abelian $\mathfrak{g}$, and we note that the 1-gauge transformations need to be carefully analyzed in this case as $\mu(A,A,A)$ will not be a tensor.

Given the above structure, we can compute using the 3-cocycle condition (2.2.19) that

$$\begin{aligned} d_A\mu(A,A,A) &= d(\mu(A,A,A)) + A \wedge^{\rhd} \mu(A,A,A) && \mathfrak{g}\text{-equivariance and Leibniz rule} \\ &= \big(3\mu(dA,A,A)) + \frac{3}{2}\mu([A,A],A,A)\big) && \text{Trilinearity of } \mu \\ &= 3\mu(F,A,A), \end{aligned}$$

where $\circlearrowright$ denotes a summation over cyclic permutations. The factor of $\frac{3}{2}$ appears in the second line due to the fact that $\mu([A,A],A,A)$ is symmetric under an exchange of the first argument $[A,A]$ and the last two arguments A,A. This gives rise to the **modified 2-Bianchi identity**

$$d_AK = F \wedge^{\rhd} \Sigma - \frac{1}{2}\mu(F,A,A) = 0,$$

which has also appeared in the context of the gauge theory based on a weak Lie 2-algebra [99].

Remark 2.2.3. Notice that if the weak Lie 2-algebra is skeletal, namely $t = 0$, there is no violation to the Jacobi identity in the component $\mathfrak{g}$. An example is the skeletal model **string Lie 2-algebra** $\mathfrak{string}_k(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$ [99, 122], where $k \in \mathbb{Z}$ is called the *level*. The Lie 2-algebra structure is given by $t = 0, \rhd = 0$, and the Jacobiator is $\mu = k\omega$, where ω is the fundamental 3-cocycle

$$\omega = \langle -,[-,-]\rangle \in Z^3(\mathfrak{g},\mathbb{R}).$$

This is one of the most commonly-seen weak Lie 2-algebras in the physics literature. The bundle gerbe associated to the string Lie 2-algebra describes the *string structure* appearing in string theory [43, 100].

2.3 Gauge transformations

In this section, we review the different transformations we can perform and the inherited compatibility conditions.

2.3.1 1- and 2-gauge transformations

1-gauge transformations. In order to preserve the fake flatness condition, we derive the transformations of Σ and then K, from the transformation of the curvature 2-form (2.1.3).

$$\begin{aligned} F &\to F^\lambda = F + [F,\lambda] \Rightarrow t(\Sigma) \to t(\Sigma) + [t(\Sigma),\lambda] = t(\Sigma) - t(\lambda \rhd \Sigma) \\ \Sigma &\to \Sigma - \lambda \rhd \Sigma \\ K = d_A\Sigma &\to K - \lambda \rhd K, \end{aligned} \tag{2.3.1}$$

where $\lambda \in \Omega^0(X) \otimes \mathfrak{g}$.

Now suppose the underlying Lie 2-algebra is weak, with $\mu \neq 0$. We shall see that, provided Σ acquires an additional term [99]

$$\Sigma \to \Sigma^\lambda = \Sigma - \lambda \rhd \Sigma - \frac{1}{2}\mu(\lambda, A, A) \tag{2.3.2}$$

under 1-gauge transformation, then we preserve the covariance of the 2-curvature under the 1-gauge transformations,

$$K \to K^\lambda = K - \lambda \rhd K + \mu(\lambda, A, \mathcal{F}).$$

Indeed, working with the modified 2-curvature (2.2.15), we have from the definition (2.2.18),

$$-A \wedge^\rhd (\lambda \rhd \Sigma) + [A,\lambda] \wedge^\rhd \Sigma = -\mu(A,\lambda,t\Sigma) - \lambda \rhd (A \wedge^\rhd \Sigma) = \mu(\lambda, A, t\Sigma) - \lambda \rhd (A \wedge^\rhd \Sigma). \tag{2.3.3}$$

On the other hand, we have by the $\mathfrak{g}$-equivariance of μ, (2.2.19), that

$$\begin{aligned} \mu(d_A\lambda, A, A) &= \mu(d\lambda, A, A) - \frac{1}{2}(\mu([\lambda, A], A, A) - \mu([A,\lambda], A, A)) \\ &= d(\mu(\lambda, A, A)) + 2\mu(\lambda, A, dA) + \frac{1}{2}(\frac{2}{3}\lambda \rhd \mu(A,A,A) + 2\mu(\lambda, A, [A \wedge A]) \\ &\quad - A \wedge^\rhd \mu(\lambda, A, A)) \\ &= 2\mu(\lambda, A, F) + \frac{1}{3}\lambda \rhd \mu(A,A,A) - d_A\mu(\lambda, A, A). \end{aligned}$$

There are three such terms, hence we have

$$\frac{1}{3!}\mu(A,A,A) \to \frac{1}{3!}\mu(A,A,A) + \mu(\lambda, A, F) + \frac{1}{3!}\lambda \rhd \mu(A,A,A) - \frac{1}{2}d_A\mu(\lambda, A, A) + o(\lambda^2)$$

modulo terms of higher order in λ. These terms precisely cancel the $d_A\mu(\lambda, A, A)$ term in the 1-gauge transformation of K, as desired.

2-gauge transformations. The shift of the 1-connection parameterized by L such that $a = t(L)$ is interpreted as *the 2-gauge transformation*. Indeed, the 2-connection Σ was introduced such that the 1-form shift $A \to A' = A + t(L)$ in the 1-connection was interpreted as a (2-)*gauge* *symmetry.*

Given the 2-form connection Σ undergoes a corresponding 2-gauge transformation,

$$\Sigma \to \Sigma' = \Sigma + d_A L + \frac{1}{2}[L \wedge L], \tag{2.3.4}$$

parameterized by a 1-form $L \in \Omega^1(X) \otimes \mathfrak{h}$, we see that the fake-curvature $\mathcal{F} = F - t\Sigma$ is kept invariant, as desired. The 2-curvature is covariant under this 1-form shift transformation since, with $A' = A + t(L)$,

$$\begin{aligned}
K \to K' &= d_{A'}\Sigma' = d_A\Sigma + t(L) \wedge^{\rhd} \Sigma + d_{A+t(L)}(d_A L + \frac{1}{2}[L \wedge L]) \\
&= K + [L \wedge \Sigma] + F \wedge^{\rhd} L + \frac{1}{2} d_A[L \wedge L] + t(L) \wedge^{\rhd} d_A L + \frac{1}{2} t(L) \wedge^{\rhd} [L \wedge L] \\
&= K - t\Sigma \wedge^{\rhd} L + F \wedge^{\rhd} L + \frac{1}{2} d_A[L \wedge L] + [L \wedge d_A L] + \frac{1}{4}[L \wedge [L \wedge L]] \\
&= K + \mathcal{F} \wedge^{\rhd} L \sim K
\end{aligned} \tag{2.3.5}$$

where we used extensively the Peiffer conditions, and the Jacobi identity for the cubic term in L. Note K is invariant on-shell of the fake-flatness condition $\mathcal{F} = 0$.

Now let us consider how the modified 2-curvature K (2.2.15) transforms in the weak case $\mu \neq 0$. We seek to pick out terms in the computation of (2.3.5) that implicitly uses the 2-Jacobi identities. All such terms occur in the quantity

$$d_{A+t(L)}(d_A L + \frac{1}{2}[L \wedge L]),$$

which can be organized into three parts:

$$o(L): \ d_A d_A L, \qquad o(L^2): \ d_{tL} d_A L + \frac{1}{2} d_A [L \wedge L], \qquad o(L^3): \ \frac{1}{2} tL \wedge^{\rhd} [L \wedge L].$$

Consider first the term linear in L, which gives

$$d_A d_A L = (dA) \wedge^{\rhd} L + A \wedge^{\rhd} (A \wedge^{\rhd} L) = F \wedge^{\rhd} L + \frac{1}{2}\mu(A, A, tL)$$

by using (2.2.18). The additional μ-term here is compensated precisely by the linear $o(L)$-terms in the 2-gauge transformation of $\mu(A, A, A)$:

$$\frac{1}{3!}\mu(A, A, A) \to \frac{1}{3!}\mu(A, A, A) + \frac{1}{2}\mu(A, A, tL) + o(L^2).$$

Next we look at the terms quadratic in L. This gives

$$d_{tL} d_A L + \frac{1}{2} d_A [L \wedge L] = \frac{1}{2} A \wedge^{\rhd} [L \wedge L] + [L \wedge (A \wedge^{\rhd} L)] = \mu(A, tL, tL)$$

via (2.2.18), which is compensated precisely by the $o(L^2)$-terms in the transformation

$$\frac{1}{3!}\mu(A,A,A) \to \frac{1}{3!}\mu(A,A,A) + \frac{1}{2}\mu(A,A,tL) + \frac{3!}{3!}\mu(A,tL,tL) + o(L^3).$$

Finally, the cubic term is

$$tL \wedge^{\triangleright} [L \wedge L] = tL \wedge^{\triangleright} [L \wedge L] = [L \wedge [L \wedge L]] = \frac{1}{3!}\mu(tL,tL,tL),$$

which is compensated by the $o(L^3)$-term in the transformation

$$\frac{1}{3!}\mu(A,A,A) \to \frac{1}{3!}\mu(A,A,A) + \frac{1}{2}\mu(A,A,tL) + \mu(A,tL,tL) + \frac{1}{3!}\mu(tL,tL,tL).$$

As such, we see that the modified 2-curvature (2.2.15) follows also the 2-gauge transform law (2.3.5).

Compatibility between 1- and 2-gauge transformations. The shift has to be compatible with the 1-gauge transformation, so that the new curvature transforms covariantly,

$$A \to A' = A + a \to A' + d_{A'}\lambda \Rightarrow a = t(L) \to a + [a,\lambda] = t(L) + [t(L),\lambda] \tag{2.3.6}$$
$$L \to L - \lambda \triangleright L \tag{2.3.7}$$

where we used the Peiffer conditions, as always. It is interesting to note that 1-gauge $(\lambda, 0)$ and 2-gauge $(0, L)$ transformations do *not* commute. Through straightforward computations in the strict case $\mu = 0$ [69, 110, 127], we see that

$$[(\lambda,0),(0,L)] = (0, \lambda \triangleright L), \tag{2.3.8}$$

so 2-gauge transformations in general form a semidirect product [72, 110]

$$\mathrm{Gau}_2 = (\Omega^1(X) \otimes \mathfrak{h}) \rtimes (\Omega^0(X) \otimes \mathfrak{g})$$

defined by (2.3.8).

It is possible to perform the same kinematical analysis for the weak case, where $\mu \neq 0$. However, here the commutator between 2-gauge transformations read [99]

$$[(\lambda,L),(\lambda',L')] = ([\lambda,\lambda'], \lambda \triangleright L' - \lambda' \triangleright L) + (0, \mu(A,\lambda,\lambda')) + \mu(\mathcal{F},\lambda,\lambda'). \tag{2.3.9}$$

This is a major issue, because the additional term $\mu(\mathcal{F},\lambda,\lambda')$ is *not* a gauge transformation — the 2-gauge algebra Gau_2 fails to close unless the fake curvature condition $\mathcal{F} = 0$ is always satisfied! This is one of the motivations for the theory of adjusted parallel transport in [99]. Of course, when $\mu = 0$, we have a set of compatible gauge transformations, even if possibly $\mathcal{F} \neq 0$.

Generally, we also have a "higher gauge transformation" on the 2-gauge parameter $L \to$

$L + d_A\ell$, where $\ell \in \Omega^0(X) \otimes \mathfrak{h}$. If we take the two 2-gauge parameters $L, L' = L + d_A\ell$, and define

$$\Sigma' = \Sigma + d_A L + \frac{1}{2}[L \wedge L], \qquad \Sigma'' = \Sigma + d_A L' + \frac{1}{2}[L' \wedge L']$$
$$A' = A + tL, \qquad A'' = A + tL' = A + t(L + d_A\ell),$$

then we have

$$\begin{aligned} \Sigma'' - \Sigma' &= F \wedge^{\rhd} \ell + [L, d_A\ell] + \frac{1}{2}[d_A\ell, d_A\ell], \\ F'' - F' &= [F, t(\ell)] + [tL, t(d_A\ell)] + \frac{1}{2}[t(d_A\ell), t(d_A\ell)]. \end{aligned}$$

By the Peiffer conditions, we see that the two 2-gauge transformations $L, L' = L + d_A\ell$ act identically on the fake-curvature $\mathcal{F} = F - t\Sigma$ [69, 128]. The computation (2.3.5) then implies that the 2-curvature K is invariant on-shell of fake-flatness $\mathcal{F} = 0$ under both L, L'. Because of this, the study of such higher gauge transformation is not necessary in the context of higher-BF theories [110].

2.3.2 2-curvature anomaly and the first descendant

Recall from (2.3.5) that the 2-curvature K is covariant under a 2-gauge transformation. To introduce a 2-curvature anomaly κ into the theory, we require the anomaly equation of motion (EOM) $K = \kappa$ to transform covariantly, identically to how K transforms. On-shell of fake-flatness $\mathcal{F} = 0$, then, $\kappa = \kappa(A, \Sigma)$ must be 2-gauge invariant. Now since under a 2-gauge transformation, Σ shifts by an arbitrary element in $\mathfrak{h}$ and hence κ must be a constant as a function of $\mathfrak{h}$. On the other hand, *shift invariance* $\kappa(A) = \kappa(A + tL)$ implies that it can still have A-dependence through $\operatorname{coker} t = \mathfrak{g}/\operatorname{im} t$.

Here, we will study this particularly nice form of the 2-curvature anomaly $\kappa(A)$. We shall see that the covariance of the 2-curvature anomaly EOM $K = \kappa(A)$ will require a *twist* in the gauge transformations.

Twisting gauge transformations. Given the 1-form connection A transforms in the usual manner, we shall demonstrate here that the 1-gauge transformation of the 2-form connection Σ must be twisted by an additional term

$$\Sigma \to \Sigma^\lambda = \Sigma - \lambda \rhd \Sigma + \zeta_A(\lambda). \tag{2.3.10}$$

This additional contribution is required such that the 2-curvature anomaly equation $K = \kappa$ transforms appropriately.

Proposition 2.3.1.

1. *The quantity $\bar{K} = K - \kappa(A)$ transforms covariantly under 2-gauge transformations*
$$\bar{K} \to \bar{K}^L = \bar{K} + \mathcal{F} \wedge^{\rhd} L$$
iff the 2-form ζ is $\ker t$-valued and only a function of $\operatorname{coker} t$.

2. *the quantity $\bar{K}$ transforms covariantly under a 1-gauge transformation*
$$\bar{K} \to \bar{K}^\lambda = \bar{K} - \lambda \rhd \bar{K}$$
iff ζ_A satisfies the following descent equation
$$d_{A^\lambda}\zeta_A(\lambda) = \kappa(A^\lambda) - (\kappa(A) - \lambda \rhd \kappa(A)). \tag{2.3.11}$$

We call solutions ζ_A to (2.3.11) the ***first descendants*** *of the 2-curvature anomaly $\kappa(A)$ (cf. [69]).*

Proof. We prove the second statement first. Indeed, we first have the following computation
$$\begin{aligned} K^\lambda &= d_{A^\lambda}\Sigma^\lambda = d_{A^\lambda}(\Sigma - \lambda \rhd \Sigma) + d_{A^\lambda}\zeta_A(\lambda) \\ &= d_A\Sigma - \lambda \rhd (d_A\Sigma) + d_{A^\lambda}\zeta_A(\lambda) \end{aligned} \tag{2.3.12}$$
using (2.3.1). On the other hand, the 2-curvature anomaly transforms as $\kappa(A) \to \kappa(A^\lambda)$, hence from (2.3.12) we have
$$\begin{aligned} K - \kappa(A) &\to K^\lambda - \kappa(A^\lambda) \\ &= d_A\Sigma - \lambda \rhd (d_A\Sigma) + d_{A^\lambda}\zeta_A(\lambda) - \kappa(A^\lambda) \\ &= (K - \kappa(A)) - \lambda \rhd (K - \kappa(A)) \\ &\quad + d_{A^\lambda}\zeta_A(\lambda) - \kappa(A^\lambda) + \kappa(A) - \lambda \rhd \kappa(A). \end{aligned}$$

The last line is precisely the descent equation (2.3.11). Note moreover that $\zeta_A(\lambda)$ is valued in $\ker t$ iff it does not conflict with the covariance of the fake-curvature,
$$t(\Sigma) \to t(\Sigma^\lambda) = t(\Sigma) - t(\lambda \rhd \Sigma) + \underbrace{t(\zeta_A(\lambda))}_{=0}.$$

Now we consider a 2-gauge shift symmetry. Note the covariance of the transformation $\bar{K} \to \bar{K}^L$ (2.3.5) implies that $\bar{K}^L - \bar{K}$ is in fact independent of κ, and hence both κ and ζ cannot transform under L. By hypothesis, $\kappa(A)$ is shift invariant, hence we acquire the following terms from applying a 2-gauge transformation to the descent equation (2.3.11):
$$tL \wedge^{\rhd} \zeta_A(\lambda) = -t\zeta_A(\lambda) \wedge^{\rhd} L = 0,$$

$$
\begin{aligned}
[tL \wedge \lambda] \wedge^{\rhd} \zeta_A(\lambda) &= -t(\lambda \rhd L) \wedge^{\rhd} \zeta_A(\lambda) = (t\zeta_A(\lambda)) \wedge^{\rhd} (\lambda \rhd L) = 0, \\
\zeta_{A^L}(\lambda) &= \zeta_A(\lambda) + \zeta_{tL}(\lambda).
\end{aligned} \tag{2.3.13}
$$

where we have used the Peiffer identity and the fact that $\zeta_A(\lambda)$ is $\ker t$-valued. Note the last term remains ζ_A iff ζ depends on A only through $\operatorname{coker} t$, which would imply that (2.3.11) is invariant under 2-gauge transformations. This ensures that first descendants do not transform under L, as desired.

□

If $\kappa = 0$, then the first descendant $\zeta(A, \lambda)$ can be chosen to vanish, in which case we reproduce the covariance of K (2.3.1). Conversely, $\zeta_A(\lambda)$ *necessarily* occurs in the presence of a non-trivial $\kappa(A)$.

The descent equation (2.3.11) guarantees the 1-gauge covariance of the equation of motion $K = \kappa$, and provides a differential equation which allows to express κ in terms of ζ. As such, one may conversely view ζ as a particular twist in the 1-gauge transformation of Σ, which "inserts" the 2-curvature anomaly κ.

For readers familiar with the theory of Lie 2-algebras, this sort of 2-curvature anomaly $\kappa(A)$ is in fact precisely given by the cohomological cllassification of $\mathfrak{G}$. This class $[\kappa] \in H^3(\operatorname{coker} t, \ker t)$ is called the **Postnikov class** of $\mathfrak{G}$. We shall explain this in more detail in the Appendix.

2.4 2BF theory

The simplest action to consider is an action constructed from Lagrange multipliers enforcing the fake-flatness and 2-flatness constraints as equations of motion (EOMs). As such, this action is topological. By analogy to the BF case, we would call this action the *2BF action* [110, 127]. We shall see how the 2BF theory gives us a glimpse into the general symmetry structure of Lie 2-bialgebras and Drinfel'd 2-doubles, which we shall describe in detail in Chapter 3.

2.4.1 Action and EOMs

Let X be a manifold of dimension d and let us fix a Lie algebra crossed-module $\mathfrak{G} = \mathfrak{h} \xrightarrow{t} \mathfrak{g}$. Let $\mathfrak{G}^*[1]$ denote the dual space of linear functionals on $\mathfrak{G}$, and similarly let $\mathfrak{g}^*, \mathfrak{h}^*$ denote respectively the dual space of $\mathfrak{g}$ and $\mathfrak{h}$. We denote by $\langle -, - \rangle$ the duality pairing for them.

We begin by introducing Lagrange multipliers $B \in \Omega^{d-2} \otimes \mathfrak{g}^*$, $C \in \Omega^{d-3} \otimes \mathfrak{h}^*$ which implements the aforementioned flatness conditions. The 2-BF action in the absence of 2-curvature anomalies is

$$
S_{\mathrm{2BF}}[A, \Sigma] = \int_X \langle B \wedge \mathcal{F}(A, \Sigma) \rangle + \langle C \wedge \mathcal{G}(A, \Sigma) \rangle, \tag{2.4.1}
$$

where $\mathcal{F}(A, \Sigma) = F - t(\Sigma)$ and $\mathcal{G}(A, \Sigma) = K = d_A \Sigma$. For $d < 3$, the 2-BF theory reduces to a BF theory, since the dual field C does not exist.

The first half of the EOMs are

$$\delta B \Rightarrow \mathcal{F} = F - t(\Sigma) = 0, \qquad \delta C \Rightarrow \mathcal{G} = d_A\Sigma = 0,$$

which implement precisely the fake curvature and 2-flatness conditions, respectively. On the other hand, we also have the option to vary A and Σ. These must be done more carefully: we first introduce a map $\Delta : \mathfrak{h} \wedge \mathfrak{h}^* \to \mathfrak{g}^*$ dual to the crossed-module action:

$$\langle C \wedge (A \wedge^{\rhd} \Sigma)\rangle = -\langle \Delta(C \wedge \Sigma) \wedge A\rangle.$$

Second, we define the map $t^* : \mathfrak{g}^* \to \mathfrak{h}^*$ dual (with respect to the pairings $\langle -,-\rangle$) to the crossed-module map $t : \mathfrak{h} \to \mathfrak{g}$, and write

$$\langle B \wedge t(\Sigma)\rangle = \langle t^*(B) \wedge \Sigma\rangle.$$

We also introduce the dual of the action and adjoint representation,

$$\langle y, x \rhd y'\rangle = -\langle x \rhd^* y, y'\rangle, \qquad \langle x', [x, x'']\rangle = -\langle [x, x']^*, x''\rangle,$$

for all $y \in \mathfrak{h}, y' \in \mathfrak{h}^*, x \in \mathfrak{g}, x' \in \mathfrak{g}^*$.

These yield

$$\delta A \Rightarrow dB + [A \wedge B]^* - \Delta(C \wedge \Sigma) = 0, \qquad \delta\Sigma \Rightarrow t^*B + dC + A \wedge^{\rhd^*} C = 0.$$

If we define the quantities

$$\tilde{F} \equiv d_A C = dC + A \wedge^{\rhd^*} C, \qquad \tilde{K} \equiv d_A B = dB + [A \wedge B]^*,$$

we see that these sets of EOMs read

$$\tilde{F} = t^*(B), \qquad \tilde{K} = \Delta(C \wedge \Sigma), \tag{2.4.2}$$

the first of which looks like a fake-flatness condition for the dual fields. This suggests that B, C should be treated as a 2-connection as well, valued in a Lie algebra crossed-module of the form $t^* : \mathfrak{g}^* \to \mathfrak{h}^*$.

Remark 2.4.1. Indeed, dualizing the $t : \mathfrak{h} \to \mathfrak{g}$ gives $t^* : \mathfrak{g}^* \to \mathfrak{h}^*$, hence the dual Lie 2-algebra $\mathfrak{G}^*[1]$ comes with a shift [1] in the grading of the underlying vector spaces. This is a small subtlety in the mathematical notation that we shall keep in order to be consistent with the mathematical literature [95, 96, 114].

In the case of the specific action (2.4.1), the dual Lie 2-algebra is Abelian, equivalent to a 2-vector space [109]. More general cases can be studied by including certain coupling terms in the 2BF action; see [115].

2.4.2 Symmetries of the action

It was shown in [110] (see also [127]) that the 2BF action (2.4.1) is preserved under the operations

$$\lambda : \begin{cases} \mathcal{F} \to \mathcal{F}^\lambda = \mathcal{F} + [\mathcal{F}, \lambda] \\ \mathcal{G} \to \mathcal{G}^\lambda = \mathcal{G} + \lambda \rhd \mathcal{G} \end{cases}, \qquad L : \begin{cases} \mathcal{F} \to \mathcal{F}^L = \mathcal{F} \\ \mathcal{G} \to \mathcal{G}^L = \mathcal{G} + \mathcal{F} \wedge^\rhd L \end{cases}, \tag{2.4.3}$$

$$\lambda : \begin{cases} B \to B^\lambda = B + [\lambda, B]^* \\ C \to C^\lambda = C + \lambda \rhd^* C \end{cases}, \qquad L : \begin{cases} B \to B^L = B + \Delta(C \wedge L) \\ C \to C^L = C \end{cases}, \tag{2.4.4}$$

where we recognize the transformations of $\mathcal{F}$ and $\mathcal{G}$ we obtained in §2.3. Notice $\mathcal{G}^L$ is invariant only *on-shell* of the fake curvature condition $\mathcal{F} = 0$, which we had assumed in (2.3.5).

Algebraically, this implies that the 2-gauge algebra $\mathrm{Gau}_2 = (\Omega^1(X) \otimes \mathfrak{h}) \rtimes (\Omega^0(X) \otimes \mathfrak{g})$ acts naturally on the dual fields B, C. In other words, the original Lie 2-algebra $\mathfrak{G}$ has a natural action on the dual Lie 2-algebra $\mathfrak{G}^*[1]$ induced by the data $\rhd^*, \Delta$ emergent form the dual EOMs (2.4.2). These actions define a *strict coadjoint representation* [95] of the Lie 2-algebra $\mathfrak{G}$ on its dual $\mathfrak{G}^*[1]$.

A bit more structure can be inferred here, in fact. Generally, suppose the dual Lie 2-algebra $\mathfrak{G}^*[1]$ is non-Abelian and defines its own 2-gauge sector, then the corresponding gauge parameters $(\tilde{\lambda}, \tilde{L}) \in \mathfrak{G}^*[1]$ also acts on the dual fields (C, B) as

$$\tilde{\lambda} : \begin{cases} C \to C^{\tilde{\lambda}} = C + d_C \tilde{\lambda} \\ B \to B^{\tilde{\lambda}} = B + \tilde{\lambda} \rhd^* B \end{cases}, \qquad \tilde{L} : \begin{cases} C \to C^{\tilde{L}} = C + t^T \tilde{L} \\ B \to B^{\tilde{L}} = B + d_C \tilde{L} + \frac{1}{2}[\tilde{L} \wedge \tilde{L}]_* \end{cases}.$$

If there is a non-trivial back-action of $\mathfrak{G}^*[1]$ on $\mathfrak{G}$, then (A, Σ) would transform under $(\tilde{\lambda}, \tilde{L})$ as well, analogous to how (C, B) transforms under (λ, L) in (2.4.4). If certain compatibility conditions are satisfied between the mutual action of $\mathfrak{G}, \mathfrak{G}^*[1]$ between each other, then we obtain the structure of a *2-Manin triple*

$$\mathfrak{D} = \mathfrak{G} \;{}_{\mathrm{ad}^*}\!\bowtie_{\mathfrak{ad}^*} \mathfrak{G}^*[1],$$

which serves as a model for a "Drinfel'd 2-double" [95, 115] — a categorified, higher homotopy notion of the classical Drinfel'd double $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$ for a Lie algebra $\mathfrak{g}$ [9]. For a more detailed study and analysis, see Chapter 3.

In this Chapter, we have introduced a procedure of "gauging"/localizing the higher-form symmetry present in gauge theories. We showed that all the known 1- and 2-gauge transformations in a 2-gauge theory can be obtained from this perspective by imposing the condition that certain physical quantities transform appropriately and covariantly. We also demonstrated how certain well-known properties — such as flatness of the curvature and the Bianchi identity — can be relaxed *up to homotopy* by introducing the concept of weak Lie 2-algebras.

At the end, we described the simplest topological 2-gauge theory exhibiting Lie 2-group symmetry, which is the 2BF theory (2.4.1). This example provided us a firsthand glance into the fact that physical higher-gauge field theories typically exhibit a more intricate symmetry structure, namely that of **Lie 2-bialgebras** and Drinfel'd 2-doubles, than the usual Lie 2-algebra gauge symmetry. A deep dive in the mathematical formulation of such symmetry structures will be the main point of the following Chapter.

Chapter 3

Structure of Lie 2-bialgebras

We now dive into the full mathematical description of the Lie 2-bialgebra symmetry emergent in the previous Chapter. We shall describe the known notion of Lie 2-algebra 2-cocycles, the 2-graded classical r-matrix, as well as the Drinfel'd 2-double following recent mathematical literature [95, 114]. By leveraging these objects, I had developed a notion of graded Poisson structure suitable for differential grade (dg) manifolds, for which Poisson-Lie 2-groups [96] are examples. This perspective makes manifest the correspondence between (quasi) Poisson-Lie 2-groups and (quasi) Lie 2-bialgebras. This is based in part on my works [115, 116].

The classical r-matrix are known to play key roles in many areas of physics and mathematics, such as deformation quantization [9], 2+1d classical integrable systems [10] and 3d topological quantum field theories (TQFTs) [129, 130, 131]. As such, a homotopy categorification of usual classical r-matrix — namely the notion of a classical 2-r-matrix — are of particular interest, as they are expected to play key roles in 4d TQFTs. Motivated by this, I will give a characterization of the *quadratic 2-Casimirs* following my work [115], which controls the form of the classical 2-Yang-Bater equations [95].

3.1 Lie 2-bialgebras

Recall that Lie 2-algebras $\mathfrak{G} \cong \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ are synonymous with Lie algebra crossed-modules $t : \mathfrak{h} \to \mathfrak{G}$ in which $\mathfrak{G} = \mathfrak{g}_0$ has degree-0 and $\mathfrak{h} = \mathfrak{g}_{-1}$ has degree-(-1). Similarly, Lie 2-groups $\mathbb{G} = G_{-1} \rtimes G_0 \rightrightarrows G_0$ are synonymous with Lie group crossed-modules $t : H \to G$ with $H = G_{-1}$ and $G = G_0$.

Let us begin by introducing the following linear maps

$$\delta_{-1} : \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}^{2\otimes}, \qquad \delta_0 : \mathfrak{g}_0 \to (\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}) \oplus (\mathfrak{g}_{-1} \otimes \mathfrak{g}_0).$$

on the Lie 2-algebra $\mathfrak{G}$.

Definition 3.1.1. The tuple $\delta = \delta_{-1} + \delta_0$ is called **Lie 2-algebra 2-cocycle** — denoted

$\delta \in Z^1(\mathfrak{G}, \mathfrak{G} \wedge \mathfrak{G})$ — iff the following conditions are satisfied [95, 125]

$$\begin{aligned}
\delta_0 t &= (t \otimes 1 + 1 \otimes t)\delta_{-1}, \qquad (\textbf{ID1} \text{ in } \textbf{Theorem 2.15} \text{ of } [96]),\\
0 &= (t \otimes 1 - 1 \otimes t)\delta_0, \qquad (\textbf{ID2} \text{ in } \textbf{Theorem 2.15} \text{ of } [96]),\\
\delta_0([X, X']) &= (X \rhd \otimes 1 + 1 \otimes \mathrm{ad}_X)\delta_0(X')\\
&\quad - (X' \rhd \otimes 1 + 1 \otimes \mathrm{ad}_{X'})\delta_0(X), \qquad ({}_2\,\text{ad-invariance})\\
\delta_{-1}(X \rhd Y) &= (X \rhd \otimes 1 + 1 \otimes X \rhd)\delta_{-1}(Y)\\
&\quad + \delta_0(X)(\rhd Y \otimes 1 + 1 \otimes \rhd Y), \qquad (\textbf{ID3} \text{ in } \textbf{Theorem 2.15} \text{ of } [96]),
\end{aligned} \tag{3.1.1}$$

where $X, X' \in \mathfrak{G} = \mathfrak{g}_0$ and $Y \in \mathfrak{h} = \mathfrak{g}_{-1}$.

We can now define the notion of a Lie 2-bialgebra [95, 96, 115].

Definition 3.1.2. The tuple $(\mathfrak{G}; \delta)$ is a **(strict) Lie 2-bialgebra** iff the Lie 2-algebra 2-cocycle δ satisfies furthermore the following *2-cobracket conditions*

$$\begin{aligned}
0 &= \sum_{\text{cycl.}} ((\delta_{-1} + \delta_0) \otimes 1) \circ \delta_0 = (\delta_{-1} \otimes 1) \circ \delta_0 - (1 \otimes \delta_0) \circ \delta_0\\
&\quad - (\tau \otimes 1) \circ (1 \otimes \delta_0) \circ \delta_0\\
0 &= \sum_{\text{cycl.}} (\delta_{-1} \otimes 1) \circ \delta_{-1} = (\delta_{-1} \otimes 1) \circ \delta_{-1} - (1 \otimes \delta_{-1}) \circ \delta_{-1}\\
&\quad - (\tau \otimes 1) \circ (1 \otimes \delta_{-1}) \circ \delta_{-1},
\end{aligned} \tag{3.1.2}$$

where $\tau : \mathfrak{G} \otimes \mathfrak{G} \to \mathfrak{G} \otimes \mathfrak{G}$ swaps the tensor factors.

These conditions are equivalent to δ defining a Lie 2-algebra structure $[-,-]_*$ given by

$$\langle [f, f']_*, Y \rangle = \langle f \otimes f', \delta_{-1}(Y) \rangle, \qquad \langle f \rhd^* g, X \rangle = \langle f \otimes g, \delta_0(X) \rangle$$

for each $X \in \mathfrak{g}_0, Y \in \mathfrak{g}_{-1}$ and each element $f, f' \in \mathfrak{g}_{-1}^*, g \in \mathfrak{g}_0^*$ in the dual graded space $\mathfrak{G}^*[1]$, which we recall is equipped with a dual differential $t^T : \mathfrak{g}_0^* \to \mathfrak{g}_{-1}$. Indeed, the dual analogue of the conditions (2.2.3) read

$$t^T(f \rhd^* g) = [f, t^T g]_*, \qquad t^T g \rhd^* g' = [g, g']_*, \qquad \forall\ f \in \mathfrak{g}_{-1}^*,\ g, g' \in \mathfrak{g}_0^*,$$

which when written in terms of the 2-cochains (δ_{-1}, δ_0) are equivalent[1] to the first two lines of (3.1.1). The 2-Jacobi identities then follow from conditions (3.1.2).

Note the shift in the grading [1] upon dualizing the graded Lie algebra. The above definition is a direct generalization of the notion of a Lie 1-bialgebra $(\mathfrak{G}; \delta)$ [9] to the differential graded context. The following will describe results that also have lower-dimensional analogues.

[1]To see this, we first note $(t^T)^T = t$ and evaluate, for instance, $\delta_0 t$ to yield $(f \wedge g)(\delta_0 t Y) = (f \rhd^* g)(t(Y)) = (t^T(f \rhd^* g))(Y)$, while $(f \wedge g)((t \otimes 1 + 1 \otimes t)\delta_{-1}(Y)) = (f \wedge t^T g)(\delta_{-1} Y) = [f, t^T g]_*(Y)$.

Weak Lie 2-bialgebras. Now recall the notion of a *weak* Lie 2-algebra as given in **Definition 2.2.5**, in which the Jacobi identities are relaxed up to homotopy given by a skew-trilinear homotopy map $\mu : \mathfrak{g}_0^{3\wedge} \to \mathfrak{g}_{-1}$. The same idea can be applied to give the notion of a weak Lie 2-bialgebra [114], by relaxing the 2-cobracket conditions given in (3.1.2). To explain this, we first define the notation

$$D_a = \sum_{i=1}^{n} (-1)^i (1 \otimes \cdots \otimes \underbrace{a}_{i\text{-th position}} \otimes \cdots \otimes 1)$$

for the extension of a linear operator $a : V \to V$ to tensor products $V^{n\otimes}$.

Definition 3.1.3. A **weak Lie 2-bialgebra** is a tuple $(\mathfrak{G}; \delta, \eta)$ consisting of a weak Lie 2-algebra $\mathfrak{G}$, a Lie 2-algebra 2-cocycle δ and a **cohomotopy map** $\eta : \mathfrak{g}_0 \to \mathfrak{g}_{-1}^{3\wedge}$ satisfying the weak 2-cobracket conditions

$$\eta \circ t = \sum_{\text{cycl.}} (\delta_{-1} \otimes 1) \circ \delta_{-1}, \tag{3.1.3}$$

$$D_t \circ \eta = \sum_{\text{cycl.}} ((\delta_{-1} + \delta_0) \otimes 1) \circ \delta_0 \tag{3.1.4}$$

as well as the 3-cycle condition

$$D_{\delta_{-1}} \circ \eta = (1 \otimes \eta) \circ \delta_0. \tag{3.1.5}$$

In accordance with [96], we call $(\mathfrak{G}; \delta, \eta)$ in which $\mathfrak{G}$ is *strict* $\mu = 0$ a **quasi Lie 2-bialgebra**.

The strict notion of Lie 2-bialgebras is clearly obtained by taking $\eta, \mu = 0$.

As the name suggests, the cohomotopy map η dualizes to the homotopy map $\mu^* : (\mathfrak{g}_{-1}^*)^{3\wedge} \to \mathfrak{g}_0^*$ of the dual Lie 2-algebra $\mathfrak{G}^*[1]$,

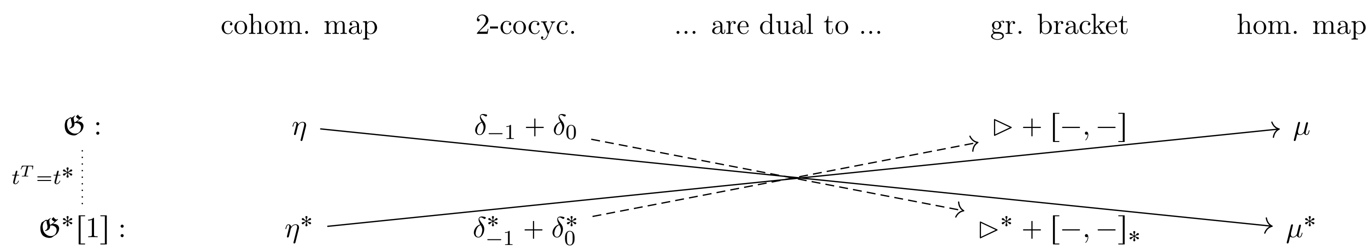
$$\langle Y_1 \otimes Y_2 \otimes Y_3, \eta(X) \rangle = \langle \mu^*(Y_1, Y_2, Y_3), X \rangle.$$

Indeed, the weak 2-cobracket conditions 3.1.4 imply (2.2.18), and the 3-cycle condition (3.1.5) implies (2.2.19), hence $\mathfrak{G}^*[1]$ forms a weak Lie 2-algebra by **Definition 2.2.5**.

Therefore, analogous to Lie bialgebras [9], we once again have the following self-duality property [95].

Proposition 3.1.1. *$(\mathfrak{G}; \delta)$ is a (weak) Lie 2-bialgebra iff $(\mathfrak{G}^*[1]; \delta^*)$ is a (weak) Lie 2-bialgebra.*

We can organize the situation like so:



3.1.1 Classical Drinfel'd 2-double

Recall the self-duality property of Lie bialgebra structures is key in forming the classical Drinfel'd double $\mathfrak{d} = \mathfrak{G} \bowtie \mathfrak{G}^*$ [132, 9]. Let us now turn to an analogous structure $\mathfrak{D}$ which we call a *classical Drinfel'd 2-double.* We shall focus on the strict case here.

Adjoint and coadjoint representation for Lie 2-algebras. Towards a description of $\mathfrak{D}$, we need to understand the adjoint representation of a Lie 2-algebra $\mathfrak{G}$. We denote this action of $\mathfrak{G}$ on itself by ${}_2\,\mathrm{ad}$, and it consists of the following graded components [95]

$$ {}_2\mathrm{ad} = (\mathrm{ad}_0, \mathrm{ad}_{-1}) : \mathfrak{g} \to \mathrm{End}\,\mathfrak{g}, \qquad \begin{cases} \mathrm{ad}_0 : \mathfrak{g}_0 \to \mathrm{End}(\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}) \\ \mathrm{ad}_{-1} : \mathfrak{g}_{-1} \to \mathrm{Hom}(\mathfrak{g}_0, \mathfrak{g}_{-1}) \end{cases}, \tag{3.1.6} $$

where

$$ \mathrm{ad}_0(X) = (\mathrm{ad}_X \equiv [X, -], \chi_X \equiv X \rhd -), \qquad \mathrm{ad}_{-1}(Y) = - \rhd Y $$

for each $X \in \mathfrak{g}_0, Y \in \mathfrak{g}_{-1}$. They satisfy the following key identities

$$ \mathrm{ad}_X\, t = t\chi_X, \qquad \mathrm{ad}_{-1}(Y)t = -\,\mathrm{ad}_Y, \qquad t\,\mathrm{ad}_{-1}(Y) = -\,\mathrm{ad}_{tY} \tag{3.1.7} $$

for each $X \in \mathfrak{g}_0, Y \in \mathfrak{g}_{-1}$, which come from the equivariance and the Peiffer identity conditions (2.2.3). We shall denote the adjoint representation of the dual Lie 2-algebra $\mathfrak{G}^*[1]$ by ${}_2\mathfrak{ad}$.

By dualizing the adjoint representations ${}_2\,\mathrm{ad}, {}_2\,\mathfrak{ad}$ (3.1.6) with respect to the canonical evaluation pairing, we define

$$ \begin{aligned} (\mathrm{ad}_0^*, \mathrm{ad}_{-1}^*) : \mathfrak{g} \to \mathrm{End}\,\mathfrak{g}^*[1], &\qquad (\mathfrak{ad}_0^*, \mathfrak{ad}_{-1}^*) : \mathfrak{g}^*[1] \to \mathrm{End}\,\mathfrak{g}, \\ \mathrm{ad}_0^* = (\mathrm{ad}^*, \chi^*) : \mathfrak{g}_0 \to \mathrm{End}(\mathfrak{g}_0^* \oplus \mathfrak{g}_{-1}^*), &\qquad \mathfrak{ad}_0^* = (\mathfrak{ad}^*, \eta^*) : \mathfrak{g}_{-1}^* \to \mathrm{End}(\mathfrak{g}_{-1} \oplus \mathfrak{g}_0), \\ \mathrm{ad}_{-1}^* \equiv \Delta : \mathfrak{g}_{-1} \to \mathrm{Hom}(\mathfrak{g}_{-1}^*, \mathfrak{g}_0^*), &\qquad \mathfrak{ad}_{-1}^* \equiv \tilde{\Delta} : \mathfrak{g}_0^* \to \mathrm{Hom}(\mathfrak{g}_0, \mathfrak{g}_{-1}). \end{aligned} \tag{3.1.8} $$

Explicitly for each $X, X' \in \mathfrak{g}_0, Y \in \mathfrak{g}_{-1}$ and $g, g' \in \mathfrak{g}_0^*, f \in \mathfrak{g}_{-1}^*$, they are defined in graded components by

$$ \begin{aligned} (\mathrm{ad}_X^*\, g)(X') = -g([X, X']), &\qquad f'(\mathfrak{ad}_f^* Y) = -[f, f']_*(Y), \\ (\chi_X^* f)(Y) = -f(X \rhd Y), &\qquad g(\eta_f^* X) = -(f \rhd^* g)(X), \\ (\Delta_Y(f))(X) = -f(X \rhd Y), &\qquad f(\tilde{\Delta}_g(X)) = -(f \rhd^* g)(X). \end{aligned} $$

It is clear that the canonical evaluation pairing

$$ \langle\langle g + f, X + Y \rangle\rangle = f(Y) + g(X) \tag{3.1.9} $$

is *by definition* invariant under the coadjoint representations (3.1.8). The equivariance of t

identities $t\chi = \operatorname{ad} t, t^T\eta = \mathfrak{ad} t^T$ then lead to

$$\chi^*_X t^T = t^T \operatorname{ad}^*_X, \qquad \eta^*_f t = t\mathfrak{ad}^*_f, \tag{3.1.10}$$
$$\Delta_Y \circ t^T = \operatorname{ad}^*_{tY}, \qquad \tilde{\Delta}_g \circ t = \mathfrak{ad}^*_{t^*g}. \tag{3.1.11}$$

If $\operatorname{ad}^*, \mathfrak{ad}^*$ satisfy (3.1.10), (3.1.11), then (3.1.8) define **strict coadjoint representations** of $\mathfrak{g}$ and $\mathfrak{g}^*[1]$ on each other.

We are now ready to define the classical Drinfel'd 2-double.

Definition 3.1.4. Let $(\mathfrak{G};\delta)$ denote a Lie 2-bialgebra. The **classical Drinfel'd 2-double $\mathfrak{D}$** of $\mathfrak{G}$ is given by the underlying differential graded (dg) vector space

$$\mathfrak{G}\oplus\mathfrak{G}^* \cong \underbrace{(\mathfrak{g}_{-1}\oplus\mathfrak{g}_0^*)}_{\deg=-1} \xrightarrow{t+t^T} \underbrace{(\mathfrak{g}_0\oplus\mathfrak{g}_{-1}^*)}_{\deg=0},$$

and the following Lie 2-algebra bracket $[\![-,-]\!]$ defined by

$$[\![X+Y, X'+Y']\!] = [X+Y, X'+Y'], \qquad [\![g+f, g'+f']\!] = [g+f, g'+f']_*$$
$$[\![X+Y, f'+g']\!] = {}_2\operatorname{ad}^*_{Y+X}(g+f) - {}_2\,\mathfrak{ad}^*_{g+f}(X+Y)$$

where $X\in\mathfrak{g}_0, Y\in\mathfrak{g}_{-1}$ and $f\in\mathfrak{g}^*_{-1}, g\in\mathfrak{g}^*_0$. We denote the classical Drinfel'd 2-double by $\mathfrak{D} = \mathfrak{G}\bar{\bowtie}\mathfrak{G}^*$.

The central characterization theorem of [95] is the following.

Theorem 3.1.1. *The tuple $(\mathfrak{G};\delta),(\mathfrak{G}^*;\delta^*)$ of mutually dual Lie 2-bialgebras form a classical 2-double $\mathfrak{D} = \mathfrak{G}\bowtie\mathfrak{G}^*$ iff the following compatibility conditions*

$$\begin{aligned}
\eta^*_f[X,X'] &= [\eta^*_f X, X'] + [X, \eta^*_f X'] - \eta^*_{\chi^*_X f}X' + \eta^*_{\chi^*_{X'}f}X, \\
\chi^*_X[f,f']_* &= [\chi^*_X f, f']_* + [f, \chi^*_X f]_* - \chi^*_{\eta^*_f X}f' + \chi_{\eta^*_{f'}X}f, && (3.1.12)\\
\mathfrak{ad}^*_f(X\rhd Y) &= X\rhd(\mathfrak{ad}^*_f Y) + (\eta^*_f X)\rhd Y - \mathfrak{ad}_{\chi^*_X f}Y + \tilde{\Delta}_{\Delta_Y(f)}(X), \\
\operatorname{ad}^*_X(f\rhd^* g) &= f\rhd^*(\operatorname{ad}^*_X g) + (\chi^*_X f)\rhd^* g - \operatorname{ad}_{\eta^*_f X}g + \Delta_{\tilde{\Delta}_g(X)}(f), && (3.1.13)\\
\tilde{\Delta}_g([X,X']) &= X\rhd\tilde{\Delta}_g(X') + X'\rhd\tilde{\Delta}_g(X) - \tilde{\Delta}_{\operatorname{ad}^*_X g}(X') + \tilde{\Delta}_{\operatorname{ad}^*_{X'}g}(X), \\
\Delta_Y([f,f']_*) &= f\rhd^*\Delta_Y(f') + f'\rhd^*\Delta_Y(f) - \Delta_{\mathfrak{ad}^*_f Y}(f') + \Delta_{\mathfrak{ad}^*_{f'}Y}(f) && (3.1.14)
\end{aligned}$$

are satisfied for each $X,X'\in\mathfrak{g}_0, f,f'\in\mathfrak{g}^*_{-1}, Y\in\mathfrak{g}_{-1}, g\in\mathfrak{g}^*_0$.

It is clear that each of piece of $\mathfrak{G},\mathfrak{G}^*$ in $\mathfrak{D}$ are *coisotropic* with respect to the canonical bilinear form (3.1.9),

$$\langle\langle\mathfrak{G},\mathfrak{G}\rangle\rangle = 0, \qquad \langle\langle\mathfrak{G}^*,\mathfrak{G}^*\rangle\rangle = 0,$$

hence the tuple $(\mathfrak{G},\mathfrak{G}^*,\langle\langle -,-\rangle\rangle)$ forms a **2-Manin triple**, called the standard 2-Manin triple.

The following result is also proven in [95].

Theorem 3.1.2. *All homomorphisms of 2-Manin triples (cf.* **Definition 2.2.3***) are isomorphisms.*

Hence 2-Manin triples are all isomorphic to the standard one, aka. a classical 2-double. I have provided a new proof of these known results from gauge theoretic considerations, but they are too lengthy to reproduce here. The interested reader is referred to [115] for details.

3.2 Poisson-Lie 2-groups

Recall the Lie theorem is well-known [98, 105] to generalize to Lie 2-groups, that there is a one-to-one correspondence between connected, simply-connected Lie 2-groups and Lie 2-algebras. An analogous statement for Lie 2-bialgebras would then involve structures of a so-called *Poisson-Lie 2-groups*. The following definition is due to [96].

Definition 3.2.1. A **Poisson-Lie 2-group** $(\mathbb{G}, \Pi)$ is a Lie 2-group $\mathbb{G} = G_{-1} \rtimes G_0 \rightrightarrows G_0$ equipped with a bivector field $\Pi \in \mathfrak{X}^2(\mathbb{G})$ that is multiplicative with respect to both the group multiplication and groupoid multiplication of $\mathbb{G}$.

We now wish to describe the structures of a Poisson-Lie 2-group fully. In order to do so, we formalize the definition of a Poisson 2-algebra for *differential graded (dg) manifolds* in general.

3.2.1 Poisson structure on dg manifolds

Let $M = M_{-1} \xrightarrow{\mathbf{t}} M_0$ denote a differential graded (dg) manifold consisting of only two terms, which is the data of a pair of manifolds M_{-1}, M_0 and a smooth map $\mathbf{t} : M_{-1} \to M_0$.

Definition 3.2.2. The smooth functions $C^\infty(M)$ on M make up a differential graded commutative algebra (dgca) $C^\infty(M_0) \xrightarrow{\mathbf{t}^*} C^\infty(M_{-1})$ given in terms of the graded sum $C^\infty(M_{-1}) \oplus C^\infty(M_0)$, and the pullback $\mathbf{t}^* : C^\infty(M_0) \to C^\infty(M_{-1})$.

Note the reversal of the degrees due to the pullback — $F_0 \in C^\infty(M_0)$ has degree-(-1) while $F_{-1} \in C^\infty(M_{-1})$ has degree-0. It will also be convenient to extend $\mathbf{t}^*$ to all of $C^\infty(M) \cong C^\infty(M_0) \oplus C^\infty(M_{-1})$ by

$$\mathbf{t}^* F = \mathbf{t}^*(F_0 \oplus F_{-1}) = \mathbf{t}^* F_{-1}, \qquad \forall\ F \in C^\infty(M).$$

The section $\Gamma(M, TM)$ of vector fields inherits the graded structure from $TM \cong TM_{-1} \times TM_0$. Hence, to build bivectors on M, we begin by first forming the following 3-term chain complex

$$\begin{aligned}\Gamma(M, TM \otimes TM) \quad = \quad & \Gamma(M, TM_{-1} \otimes TM_{-1}) \xrightarrow{D_t} \\ & \big(\Gamma(M, TM_{-1} \otimes TM_0)) \oplus \Gamma(M, TM_0 \otimes TM_{-1})\big) \xrightarrow{D_t} \Gamma(M, TM_0 \otimes TM_0),\end{aligned}$$

where $D_t = t \otimes \mathrm{id} \pm \mathrm{id} \otimes t$ and $t = T\mathbf{t}$ is the tangent pushforward of the anchor map $\mathbf{t} : M_{-1} \to M_0$. In accordance with the grading, we assign the degree -2, -1, 0 to the terms of the complex $\Gamma(M, TM^{2\otimes})$ from the left to right, and the sign in D_t depends on this grading.

We shall define the space of bivector fields $\mathfrak{X}^2(M)$ as a subcomplex of $\Gamma(M, TM^{2\otimes})$.

Definition 3.2.3. The **graded bivector fields** $\mathfrak{X}^2(M)$ on M consist of sections $\Pi \in \Gamma(M, TM^{2\otimes})$ such that the following conditions

$$\mathbf{t}^*\Pi^0 = D_t^+\Pi^{-1}, \qquad D_t^-\Pi^0 = 0 \tag{3.2.1}$$

are satisfied, where Π^{-1} has degree-(-2) and Π^0 has degree-(-1) in $\Gamma(M, TM^{2\otimes})$. Due to the second condition, we can introduce a component $\bar{\Pi}^0$ in degree-0 by

$$\bar{\Pi}^0 = (1 \otimes t)\Pi^0 = (t \otimes 1)\Pi^0.$$

One can compute that, for any smooth submersion $\phi : X \to Y$ and any vector $\xi \in \Gamma(X, TX)$, we have

$$\xi(\phi^* F) = (\phi_* \xi)(F), \qquad F \in C^\infty(Y),$$

and therefore

$$D_t^+\Pi^{-1} = \Pi^{-1} \circ (\mathbf{t}^* \otimes 1 + 1 \otimes \mathbf{t}^*). \tag{3.2.2}$$

This will be important in the following.

We use the subspace of *skew*-symmetric bivector fields $\mathfrak{X}^2_{\mathrm{sk}}(M) \subset \Gamma(M, TM \wedge TM)$ to define the following structure on $C^\infty(M)$. Let $\Pi = \Pi^{-1} + \Pi^0 \in \mathfrak{X}^2_{\mathrm{sk}}(M)$, we define

$$\{F, F'\} = \Pi(F \otimes F'), \qquad F, F' \in C^\infty(M), \tag{3.2.3}$$

which can be more explicitly written in the decomposed form

$$\begin{aligned}
\{F, F'\}_0 &= \{F_0, F_0'\}_0 = \bar{\Pi}^0(F_0 \otimes F_0'), \\
\{F, F'\}_{-1} &= \{F_{-1}, F_0'\}_{-1} + \{F_0, F_{-1}'\}_{-1} = \Pi^0(F_{-1} \otimes F_0' + F_0 \otimes F_{-1}'), \\
\{F, F'\}_{-2} &= \{F_{-1}, F_{-1}'\}_{-2} = \Pi^{-1}(F_{-1} \otimes F_{-1}'),
\end{aligned}$$

by leveraging the decomposition $F = F_{-1} \oplus F_0$ of functions on M. We now prove that $(C^\infty(M), \{-,-\})$ is in fact a Lie 2-algebra.

Lemma 3.2.1. *Let $\Pi = \Pi^{-1} + \Pi^0 \in \mathfrak{X}^2_{sk}(M)$ denote a* **Poisson bivector** *on M, namely a bivector field satisfying*

$$\sum_{cycl.} \Pi(\Pi \otimes 1) = 0. \tag{3.2.4}$$

Then the graded space $C^\infty(M) = C^\infty(M_0) \xrightarrow{\mathbf{t}^*} C^\infty(M_{-1})$ equipped with the bracket (3.2.3) a *strict Lie 2-algebra. We call* $(C^\infty(M), \{-,-\})$ *the* **Poisson 2-algebra** *of the graded Poisson manifold* (M, Π)*.*

Proof. The proof consists in showing that the different properties given in Definition 2.2.2 are satisfied. The skew-symmetry property is automatic. By a direct computation, the first condition in (3.2.1) implies

$$\begin{aligned} \mathbf{t}^*\{F, F'\}_{-1} &= (\mathbf{t}^*\Pi^0)(F_0 \otimes F'_{-1} + F_{-1} \otimes F'_0) \\ &= (D_t^+\Pi^{-1})(F_0 \otimes F'_{-1} + F_{-1} \otimes F'_0) \\ &= \Pi^{-1}(\mathbf{t}^*F_0 \otimes F'_{-1} + F_{-1} \otimes \mathbf{t}^*F'_0) \\ &= \{F, \mathbf{t}^*F'\}_{-2} + \{\mathbf{t}^*F, F'\}_{-2}, \end{aligned}$$

where we have also used (3.2.2).

On the other hand, $\{-,-\}_0$ is determined by $\{-,-\}_{-1}$, as $\bar{\Pi}^0$ is induced by Π^0 through D_t^+ from (3.2.1). We thus have

$$\begin{aligned} \{F, F'\}_0 &= \bar{\Pi}^0(F_0 \otimes F'_0) = \frac{1}{2}(D_t^+\Pi^0)(F_0 \otimes F'_0) \\ &= \frac{1}{2}\Pi^0(\mathbf{t}^*F_0 \otimes F'_0 + F_0 \otimes \mathbf{t}^*F'_0) \\ &= \frac{1}{2}(\{\mathbf{t}^*F, F'\}_{-1} + \{F, \mathbf{t}^*F'\}_{-1}) = \{\mathbf{t}^*F, F'\}_{-1}. \end{aligned}$$

From the Lie 2-algebraic perspective [95], the right-hand side of this computation should be taken as the *definition* of $\{-,-\}_0$.

Now it suffices to check the 2-Jacobi identities,

$$\begin{aligned} \{\{F, F'\}_{-2}, F''\}_{-1} + \{\{F', F''\}_{-1}, F\}_{-1} + \{\{F'', F\}_{-1}, F'\}_{-1} &= 0 \\ \{\{F, F'\}_{-2}, F''\}_{-2} + \{\{F', F''\}_{-2}, F\}_{-2} + \{\{F'', F\}_{-2}, F\}_{-2} &= 0. \end{aligned}$$

These are nothing but (3.2.4).

□

3.2.2 Poisson-Lie 2-groups as a Poisson dg manifold

The central example of a graded Poisson manifold (M, Π) is a **(strict) Poisson-Lie 2-group** $(\mathbb{G}, \Pi)$, where the graded Poisson bivector field $\Pi = \Pi^{-1} + \Pi^0 \in \mathfrak{X}^2_{\mathrm{sk}}(M)$ is given by

$$\Pi_y^{-1} = (L_y)_*(\hat{\delta}_{-1})_y, \qquad \Pi_x^0 = (L_x)_*(\hat{\delta}_0)_x, \qquad \bar{\Pi}_x^0 = \frac{1}{2}(L_x)_*(D_t^+\hat{\delta}_0)_x,$$

where $\hat{\delta}$ integrates the Lie 2-algebra 2-cocycle $\delta = \delta_{-1} + \delta_0$ on $\mathfrak{G}$, and L_* is the pushforward of the *group* left-multiplication on $\mathbb{G} = G_{-1} \rtimes G_0 \rightrightarrows G_0$. The conditions (3.2.1) are nothing but (3.2.5), and (3.2.4) follow from the 2-cobracket conditions (3.1.2). The rest of the 2-cocycle conditions, namely the third and fourth equations in (3.1.1), in fact implies the *multiplicativity* of the bivector Π with respect to the group and groupoid multiplications in $\mathbb{G}$ [96].

Theorem 3.2.1. *There is a one-to-one correspondence between connected, simply-connected Poisson-Lie 2-groups and Lie 2-bialgebras.*

To describe this correspondence, we write down how the Lie 2-algebra 2-cocycle δ can be integrated. Given $\mathfrak{G} = \operatorname{Lie}\mathbb{G}$, we can define the following maps $\mathbb{G}$,

$$\hat{\delta}_{-1} : G_{-1} \to \mathfrak{g}_{-1}^{2\wedge}, \qquad \hat{\delta}_0 : G_0 \to \mathfrak{g}_0 \wedge \mathfrak{g}_{-1},$$

given by

$$\delta_{-1}(Y) = -\frac{d}{ds}|_{s=0}(\hat{\delta}_{-1})_{\exp sY}, \qquad \delta_0(X) = -\frac{d}{ds}|_{s=0}(\hat{\delta}_0)_{\exp sX}.$$

For each $x \in G_0,\ y \in G_{-1}$ and $\omega_1, \omega_2 \in \mathfrak{G}^* = \mathfrak{g}_0^*$, the first two conditions in (3.1.1) imply the following

$$(\hat{\delta}_0)_{\mathbf{t}y} = D_t(\hat{\delta}_{-1})_y, \qquad \iota_{\omega_1}\iota_{t^T\omega_2}(\hat{\delta}_0)_x = -\iota_{\omega_2}\iota_{t^T\omega_1}(\hat{\delta}_0)_x, \tag{3.2.5}$$

which had also appeared in [96]. Here, $t^T : \mathfrak{g}_0^* \to \mathfrak{g}_{-1}^*$ denotes the dual of t, and $D_t^\pm = 1 \otimes t \pm t \otimes 1$ is the extension of the t-map t to the tensor product three-term complex $\mathfrak{G}^{2\otimes}$ [95].

The inner product ι in (3.2.5) is given by the evaluation pairing $\langle -, - \rangle : \mathfrak{G}^*[1] \otimes \mathfrak{G} \to k$ such that $\langle g + f, Y + X \rangle = f(Y) + g(X)$ for each $g \in \mathfrak{g}_0^*, f \in \mathfrak{g}_{-1}^*, X \in \mathfrak{g}_0, Y \in \mathfrak{g}_{-1}$. The dual t^T of the t-map is taken with respect to this pairing,

$$\langle t^T g, Y \rangle = \langle g, tY \rangle, \qquad \forall g \in \mathfrak{g}_0^*,\ Y \in \mathfrak{g}_{-1}.$$

A quadratic 2-Casimir [115] can also be used to induce such an invariant bilinear pairing; we shall explain this in more detail in §3.3.

2-graded Poisson maps. Let M, M' denote two 2-graded spaces, with t-maps $\mathbf{t}, \mathbf{t}'$, respectively. A smooth 2-graded map $\mathcal{J} = (\mathcal{J}_{-1}, \mathcal{J}_0) : M \to M'$ consists of smooth maps $\mathcal{J}_{-1,0} : M_{-1,0} \to M'_{-1,0}$ as its components, such that we have $\mathbf{t}'\mathcal{J}_{-1} = \mathcal{J}_0\mathbf{t}$. These maps pullback onto maps $\mathcal{J}_{-1,0}^* : C^\infty(M'_{-1,0}) \to C^\infty(M_{-1,0})$ on functions satisfying $\mathbf{t}^*\mathcal{J}_0^* = \mathcal{J}_{-1}^*\mathbf{t}'^*$, such that $\mathcal{J}^* = (\mathcal{J}_0^*, \mathcal{J}_{-1}^*) : C^\infty(M') \to C^\infty(M)$ defines the 2-graded map on the function algebra of M'.

When $M = \mathbb{G}, M' = \mathbb{G}'$ are two 2-groups, then $\mathcal{J}$ must be a **2-group homomorphism** [96]: the components $\mathcal{J}_0, \mathcal{J}_{-1}$ are group homomorphisms such that $\mathcal{J}_{-1}(x \rhd y) = (\mathcal{J}_0 x) \rhd' (\mathcal{J}_{-1} y)$ for each $x \in G_0, y \in G_{-1}$, in addition to the condition $\mathcal{J}_0\mathbf{t} = \mathbf{t}'\mathcal{J}_{-1}$. These imply that $\mathcal{J} = (\mathcal{J}_{-1}, \mathcal{J}_0)$ preserves the Peiffer identities (2.2.8) on $\mathbb{G}, \mathbb{G}'$. If we let $\jmath = (\jmath_{-1}, \jmath_0)$ denote the derivative of $\mathcal{J}$, then $\jmath$ preserves the Peiffer identities (2.2.3) on the Lie 2-algebras $\mathfrak{G}, \mathfrak{G}'$:

$$t'\jmath_{-1} = \jmath_0 t, \qquad \jmath_0[X, X'] = [\jmath_0 X, \jmath_0 X']', \qquad \jmath_{-1}(X \rhd Y) = (\jmath_0 X) \rhd' (\jmath_{-1} Y)$$

for each $X, X' \in \mathfrak{g}_0, Y \in \mathfrak{g}_{-1}$, where t, t' denote respectively the crossed-module maps on $\mathfrak{G}, \mathfrak{G}'$. We call such maps **Lie 2-algebra homomorphisms** [124].

Suppose $(\mathbb{G},\Pi)$ and $(\mathbb{G}',\Pi')$ are two Poisson-Lie 2-groups. The condition for $\mathcal{J}$ to be a Poisson map is that its pullback $\mathcal{J}^*: C^\infty(\mathbb{G}')\to C^\infty(\mathbb{G})$ commutes with the bivectors, anmely

$$\begin{aligned}(\mathcal{J}_0^*\Pi'^0)(F_0\otimes F'_{-1}+F_{-1}\otimes F'_0) &= \Pi^0(\mathcal{J}_0^*F_0\otimes\mathcal{J}_{-1}^*F'_{-1}+\mathcal{J}_{-1}^*F_{-1}\otimes\mathcal{J}_0^*F'_0),\\ (\mathcal{J}_{-1}^*\Pi'^{-1})(F_{-1}\otimes F'_{-1}) &= \Pi^{-1}(\mathcal{J}_{-1}^*F_{-1}\otimes\mathcal{J}_{-1}^*F'_{-1})\end{aligned}$$

for each $F,F'\in C^\infty(\mathbb{G}')$. If we let $\{-,-\},\{-,-\}'$ denote respectively the L_2-Poisson brackets induced on $C^\infty(\mathbb{G}),C^\infty(\mathbb{G}')$ via (3.2.3), then $\mathcal{J}^*$ is required to preserve them

$$\mathcal{J}^*\{-,-\}'=\{\mathcal{J}^*\cdot,\mathcal{J}^*\cdot\}.$$

This must hold for each graded component, hence they are nothing but the conditions for $\mathcal{J}^*$ to be a L_2-algebra homomorphism between Poisson 2-algebras. In other words, we have

Definition 3.2.4. Let $(\mathbb{G},\Pi),(\mathbb{G}',\Pi')$ denote two Poisson-Lie 2-groups. A 2-graded map $\mathcal{J}: G\to G'$ is a **2-graded Poisson map** iff $\mathcal{J}$ is a 2-group homomorphism such that its pullback $\mathcal{J}^*=(\mathcal{J}_0^*,\mathcal{J}_{-1}^*)$ is a Poisson 2-algebra homomorphism.

In particular, a Poisson-Lie 2-group $(\mathbb{G},\Pi)$ is precisely such that the group and groupoid multiplications are 2-graded Poisson maps [96].

Quasi Poisson-Lie 2-groups. As we have established, a Poisson bivector Π endows the graded functions $C^\infty(M)$ on a dg manifold M a structure of a (strict) Lie 2-algebra. A homotopy weakening of this structure is hence available, in which we introduce a homotopy map $\hat{\mu}: C^\infty(M_{-1})^{3\wedge}\to C^\infty(M_0)$ satisfying

$$\begin{aligned}&\sum_{\text{cycl.}}\{-,-\}_{-2}\circ(\{-,-\}_{-2}\otimes 1)=\mathbf{t}^*\hat{\mu},\\ &\sum_{\text{cycl.}}\{-,-\}_{-1}\circ((\{-,-\}_{-2}+\{-,-\}_{-1})\otimes 1)=\hat{\mu}\circ D_{\mathbf{t}^*}\end{aligned}\tag{3.2.6}$$

and the 3-cocycle condition equivalent to (2.2.19). Based on the duality (3.2.3) between the graded Poisson bracket $\{-,-\}$ and the bivector field Π, we see that $\hat{\mu}$ can equivalently be written

$$\hat{\mu}(F_1,F_2,F_3)(x)=\hat{\eta}_x(F_1\otimes F_2\otimes F_3),\qquad F_{1,2,3}\in C^\infty(M),\ x\in M_0$$

in terms of a trivector field $\hat{\eta}\in\Gamma(M_0,TM_{-1}^{3\wedge})$.

If we take $M=\mathbb{G}$ as a (strict) Lie 2-group, we have in fact rediscovered the notion of a **quasi Poisson-Lie 2-group** $(\mathbb{G},\Pi,\hat{\eta})$ above. Moreover, recalling the notion of a quasi Lie 2-bialgebra from **Definition 3.1.3**, the following is the main result in [96].

Theorem 3.2.2. *There is a one-to-one correspondence between connected, simply-connected quasi Poisson-Lie 2-groups and quasi Lie 2-biaglebras.*

The correspondence is given by integrating the cohomotopy map η to the trivector field $\hat{\eta}$ satisfying

$$\eta(X) = \frac{d}{ds}|_{s=0}\hat{\eta}_{e^{sX}}, \qquad X \in \mathfrak{g}_0.$$

3.3 The 2-graded classical r-matrix

Recall a (strict) Lie 2-bialgebra can be classified in terms of a Lie algebra 2-cocycle (δ_{-1}, δ_0) [95, 114], which induces a dual Lie 2-algebra $\mathfrak{D}^*[1]$. Moreover, the natural coadjoint representations (3.1.8) gives rise to the Drinfel'd 2-double $\mathfrak{D} = \mathfrak{G} \bowtie \mathfrak{G}^*[1]$.

Similar to the 1-algebra case, we begin by considering a 2-cocycle (δ_{-1}, δ_0) that is a "2-coboundary". We in particular focus on the form of the 2-coboundary generated by certain elements $r_0 \in \mathfrak{G}_0 \wedge \mathfrak{G}_{-1}$ and $r_{-1} \in \mathfrak{g}_{-1}^{2\wedge}$. These elements r_{-1}, r_0 form a **triangular 2-graded classical r-matrix** [95]

$$R = r_0 - D_t r_{-1} = r_0 - (t \otimes 1 + 1 \otimes t) r_{-1} \in \mathfrak{g}_0 \wedge \mathfrak{g}_{-1},$$

whence the 2-coboundary they form is given by

$$\delta_0(X) = [X \otimes 1 + 1 \otimes X, R], \qquad \delta_{-1}(Y) = [Y \otimes 1 + 1 \otimes Y, R]. \tag{3.3.1}$$

Here we are using the graded Lie bracket $[-,-] = l_2$ of the Lie 2-algebra.

More generally, suppose we are given $r_0 \in (\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}) \oplus (\mathfrak{g}_{-1} \otimes \mathfrak{g}_0), r_{-1} \in \mathfrak{g}_{-1}^{2\otimes}$ (namely not necessarily skew-symmetric elements). It was proven that [95]

Theorem 3.3.1. *The 2-cochain (δ_{-1}, δ_0) (3.3.4) makes $(\mathfrak{G}; \delta_{-1}, \delta_0)$ into a Lie 2-bialgebra iff for all $W \in \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$,*

- *$[W^{2\otimes}, R + \sigma(R)] = 0$ where σ is an exchange of tensor factors, and*
- *the* **2-graded classical Yang-Baxter equations** *[95] are satisfied:*
  1. *$D_t r_0 = 0$,*
  2. *$[W^{3\otimes}, [R_{12}, R_{13}] + [R_{13}, R_{23}] + [R_{12}, R_{23}]] = 0$*

where

$$W^{3\otimes} = W \otimes 1 \otimes 1 + 1 \otimes W \otimes 1 + 1 \otimes 1 \otimes W.$$

We call solutions $R \in (\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}) \oplus (\mathfrak{g}_{-1} \otimes \mathfrak{g}_0)$ to the above criteria a **2-graded classical r-matrix**.

In other words, the 2-graded classical Yang-Baxter equation implies the 2-cocycle condition [95] for 2-cochains (δ_{-1}, δ_0) defined in (3.3.1). If we write out the components

$$r_0 = \sum a \otimes b + \bar{a} \otimes \bar{b}, \qquad r_{-1} = \sum c \otimes d$$

for some $a, \bar{b} \in \mathfrak{g}_0$ and $\bar{a}, b, c, d \in \mathfrak{g}_{-1}$, then we have

$$R = \sum \underbrace{(a \otimes b - tc \otimes d)}_{\in \mathfrak{g}_0 \otimes \mathfrak{g}_{-1}} + \underbrace{(\bar{a} \otimes \bar{b} - c \otimes td)}_{\in \mathfrak{g}_{-1} \otimes \mathfrak{g}_0} \equiv \sum \rho + \bar{\rho}. \tag{3.3.2}$$

By decomposing into skew-symmetric and symmetric parts $R = R^\wedge + R^\odot$, we have $\bar{\rho} = -\sigma\rho$ in $R^\wedge$ while $\bar{\rho} = \sigma\rho$ in $R^\odot$ in terms of the components defined in (3.3.2), where σ permutes the tensor factors. In other words, we have

$$\begin{aligned} R^\wedge &= \sum \rho - \sigma\rho = \sum a \wedge b - tc \wedge d - c \wedge td = \sum a \wedge b - D_t(c \wedge d), \\ R^\odot &= \sum \rho + \sigma\rho = \sum a \odot b - tc \odot d - c \odot d = \sum a \odot b - D_t(c \odot d), \end{aligned}$$

If we write, using the graded Schouten bracket $[\![-,-]\!]$ [96, 95],

$$\Omega = -[\![R^\odot, R^\odot]\!] = -[R^\odot_{12}, R^\odot_{13}] + [R^\odot_{13}, R^\odot_{23}] + [R^\odot_{12}, R^\odot_{23}],$$

then the skew-symmetric part $R^\wedge$ satisfies the **modified 2-graded classical Yang-Baxter equation**

$$[\![R^\wedge, R^\wedge]\!] = \Omega. \tag{3.3.3}$$

This is an equivalent way of writing the second point in **Theorem 3.3.1**.

As in the 1-algebra case, the symmetric component $R^\odot \in \mathfrak{g}_0 \odot \mathfrak{g}_{-1}$ of R governs the form of (3.3.3), while the skew-symmetric component $R^\wedge \in \mathfrak{g}_0 \wedge \mathfrak{g}_{-1}$ contributes to the 2-coboundary (3.3.1). Recalling $D_t = \pm t \otimes 1 + 1 \otimes t$, the 2-coboundary (3.3.1) is given explicitly by

$$\begin{aligned} \delta_0(X) &= \sum [X, a] \wedge b + a \wedge (X \rhd b) \\ &\quad - \sum [X, tc] \wedge d + tc \wedge (X \rhd d) + (c \leftrightarrow d), \\ \delta_{-1}(Y) &= \sum c \wedge (td \rhd Y) + (c \leftrightarrow d) - \sum (a \rhd Y) \wedge b, \end{aligned} \tag{3.3.4}$$

where $c \leftrightarrow d$ indicates a swap of the elements c, d from the previous term.

One particular solution for the decomposition $R = R^\wedge + R^\odot$ is if the two quantities r_0, r_{-1} can themselves be decomposed into skew-symmetric and symmetric parts:

$$\begin{aligned} r_0 &= r_0^\wedge + r_0^\odot, \qquad & r_0^\wedge &\in \mathfrak{g}_0 \wedge \mathfrak{g}_{-1}, \qquad & r_{-1}^\wedge &\in \mathfrak{g}_{-1}^{2\wedge}, \\ r_{-1} &= r_{-1}^\wedge + r_{-1}^\odot, \qquad & r_0^\odot &\in \mathfrak{g}_0 \odot \mathfrak{g}_{-1}, \qquad & r_{-1}^\odot &\in \mathfrak{g}_{-1}^{2\odot}. \end{aligned}$$

The 2-graded r-matrix then reads

$$\begin{aligned} R^\wedge &= r_0^\wedge - D_t r_{-1}^\wedge = \sum a \wedge b - D_t(c \wedge d), \\ R^\odot &= r_0^\odot - D_t r_{-1}^\odot = \sum a \odot b - D_t(c \odot d). \end{aligned} \tag{3.3.5}$$

We stress that this may *not* be the most general form of the decomposition $R = R^\wedge + R^\odot$!

Due to the first condition in **Theorem 3.3.1**, we see that the symmetric contribution $R^{\odot}$ must be ${}_2\,\mathrm{ad}$-invariant, where ${}_2\,\mathrm{ad}$ is the strict adjoint representation (3.1.6) of $\mathfrak{G}$ on itself. We shall call $R^{\odot}$ a **quadratic 2-Casimir** of the Lie 2-algebra $\mathfrak{G}$.

3.3.1 Quadratic 2-Casimirs

Recall (3.3.3) constrains the symmetric piece $R^{\odot}$ of the classical 2-r-matrix to be invariant under the adjoint representation ${}_2\,\mathrm{ad}$, and to satisfy the condition $D_t^- R^{\odot} = 0$, thereby making it into a quadratic 2-Casimir. The following is from my paper [133], which completes the characterization of these objects $R^{\odot}$.

First, we note that $D_{{}_2\,\mathrm{ad}} = {}_2\,\mathrm{ad} \otimes 1 + 1 \otimes_2 \mathrm{ad}$ is the derivation on the tensor product $\mathfrak{G}^{2\otimes}$ associated to the strict adjoint representation ${}_2\,\mathrm{ad}$. We also recall that $\mathfrak{G}^{2\otimes}$ is a three-term graded complex, in which the differentials are given by $D_t = 1 \otimes t \pm t \otimes 1$ with the sign dependent on the degree.

Let $Y \otimes X + X' \otimes Y'$ denote an arbitrary element in $\mathfrak{g}_{-1} \otimes \mathfrak{g}_0 \oplus \mathfrak{g}_0 \otimes \mathfrak{g}_{-1}$, and let $X'' + Y'' \in \mathfrak{G}$, then

$$\begin{aligned} D_{{}_2\,\mathrm{ad}_{X''+Y''}}(Y \otimes X) &= \underbrace{(X'' \rhd Y) \otimes X + Y \otimes [X'', X]}_{\mathfrak{g}_{-1} \otimes \mathfrak{g}_0} + \underbrace{Y \otimes (X \rhd Y'')}_{\mathfrak{g}_{-1}^{2\otimes}}, \\ D_{{}_2\,\mathrm{ad}_{X''+Y''}}(X' \otimes Y') &= \underbrace{[X'', X'] \otimes Y' + X' \otimes (X'' \rhd Y')}_{\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}} + \underbrace{(X' \rhd Y'') \otimes Y'}_{\mathfrak{g}_{-1}^{2\otimes}}. \end{aligned}$$

Now if we take the symmetric tensor $Y \odot X = Y \otimes X + X \otimes Y$ and sum the above contributions, then the ${}_2\,\mathrm{ad}$-invariance condition $D_{{}_2\,\mathrm{ad}_{X''+Y''}}(Y \odot X) = 0$ gives rise to the following equations

$$[X'', X] \odot Y + X \odot (X'' \rhd Y) = 0, \qquad (X \rhd Y'') \odot Y = 0$$

for all $X'' + Y'' \in \mathfrak{G}$. The space of solutions is the subspace

$$\Theta_{\rhd} = \{X \odot Y \in \mathfrak{g}_0 \odot \mathfrak{g}_{-1} \mid \mathrm{ad}\, X \odot Y + X \odot \chi Y = 0,\ \chi_X = 0\},$$

where we recall $\chi = \rhd$ is the crossed-module action. Now the condition $D_t R = D_t r_0 = 0$ in **Theorem 3.3.1** constrains $R^{\odot}$ to lie in $\ker D(t)_0$, whence we assemble the elements

$$a \odot b \in 2\mathrm{Cas}_{\mathfrak{G}}[0] \equiv \Theta_{\rhd} \cap \ker D_t$$

as the quadratic 2-Casimirs of $\mathfrak{G}$.

On the other hand, for $Y \odot Y' \in \mathfrak{g}_{-1}^{2\odot}$ we have

$$\begin{aligned} D_{{}_2\,\mathrm{ad}_{X''+Y''}}(D(t)_{-1}(Y \odot Y')) &= (X'' \rhd Y) \odot tY' + Y \odot [X'', tY'] + Y \odot (tY' \rhd Y'') \\ &\quad + (X'' \rhd Y') \odot tY + Y' \odot [X'', tY] + Y' \odot (tY \rhd Y'') \\ &= (X'' \rhd Y) \odot tY' + Y \odot t(X'' \rhd Y') + Y \odot [Y', Y''] \end{aligned}$$

$$\begin{aligned} & +(X'' \rhd Y') \odot tY + Y' \odot t(X'' \rhd Y) + Y' \odot [Y, Y''] \\ = \; & D(t)_{-1}((X'' \rhd Y) \odot Y' + (X'' \rhd Y') \odot Y) \\ & -([Y'', Y'] \odot Y + Y' \odot [Y'', Y]), \end{aligned}$$

where we have used the conditions (2.2.3). Note $D(t)_{-1} = t \otimes 1 + 1 \otimes t$ on $\mathfrak{g}_{-1}^{2\odot}$, we define the subspaces

$$\begin{aligned} \Gamma_t &= \{Y \odot Y' \in \mathfrak{g}_{-1}^{2\odot} \mid \chi Y \odot Y' + Y \odot \chi Y' \in \ker D_t\}, \\ \mathrm{Cas}_{\mathfrak{g}_{-1}} &= \{Y \odot Y' \in \mathfrak{g}_{-1}^{2\odot} \mid \mathrm{ad}\, Y \odot Y' + Y \odot \mathrm{ad}\, Y' = 0\}, \end{aligned}$$

we see that the space of solutions is given by the intersection

$$c \odot d \in 2\mathrm{Cas}_{\mathfrak{G}}[-1] \equiv \Gamma_t \cap \mathrm{Cas}_{\mathfrak{g}_{-1}}.$$

Recall the adjoint action ad on $\mathfrak{g}_{-1}$ is defined via the Peiffer identity. If each term in $D_{2\,\mathrm{ad}_{X''+Y''}}(R^{\odot}) = 0$ vanishes, then we obtain the following characterization of quadratic 2-Casimirs:

$$R^{\odot} = \sum a \odot b + D(t)_{-1}(c \odot d) \in 2\mathrm{Cas}_{\mathfrak{G}}[0] \oplus D(t)_{-1}(2\mathrm{Cas}_{\mathfrak{G}}[-1]) \equiv 2\mathrm{Cas}_{\mathfrak{G}},$$

provided the decomposition (3.3.5) holds.

3.3.2 2-Casimirs of the Drinfel'd 2-double

We now use our above characterization of 2-Casimirs to classify the pairings that can be used to construct classical 2-doubles. To begin, let $(\mathfrak{G}, \mathfrak{G}^*[1])$ denote a matched pair of Lie 2-bialgebras, and we denote by $2\mathrm{Cas}_{\mathfrak{D}} \subset (\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}^*) \odot (\mathfrak{g}_{-1} \oplus \mathfrak{g}_0^*)$ the quadratic 2-Casimirs of the classical 2-double $\mathfrak{D} = \mathfrak{G} \bowtie \mathfrak{G}^*[1]$. Here, the adjoint and coadjoint actions (3.1.6), (3.1.8) of $\mathfrak{G}$ and $\mathfrak{G}^*[1]$ on each other both participate in the definition of the operator $D_{2\,\mathrm{ad}}$.

After a lengthy calculation, it can be explicitly shown that quadratic 2-Casimirs of $\mathfrak{d}$ satisfy the following invariance properties:

$$\begin{aligned} (\mathrm{ad}_{X'} X + \chi^*_{X'} f) \odot (Y + g) + (X + f) \odot (X' \rhd Y + \mathrm{ad}^*_{X'} g) &= 0, \\ (\eta^*_{f'} X + \mathfrak{ad}_{f'} f) \odot (Y + g) + (X + f) \odot (\mathfrak{ad}^*_{f'} Y + f' \rhd^* g) &= 0, \\ (X + f) \odot (\tilde{\Delta}_g(X') + \Delta_Y(f')) + (X' + f') \odot (\tilde{\Delta}_g(X) + \Delta_Y(f)) &= 0 \end{aligned} \tag{3.3.6}$$

for each $X, X' \in \mathfrak{g}_0, Y \in \mathfrak{g}_{-1}, f, f' \in \mathfrak{g}_{-1}^*, g \in \mathfrak{g}_0^*$. By expanding each row of (3.3.6) out, we see that these invariance properties encompass those of both the canonical evaluation pairing $\langle\langle -, - \rangle\rangle$ of (3.1.9), as well as the grading-inhomogeneous alternative pairing given by

$$\langle\langle (Y + g) + (X + f), (Y' + g') + (X' + f') \rangle\rangle' = (\langle Y, X' \rangle + \langle f, g' \rangle) + (\langle X, Y' \rangle + \langle g, f' \rangle). \tag{3.3.7}$$

For instance, expanding out the first equation yields

$$\begin{aligned} 0 &= (\mathrm{ad}_{X'} X \odot g + X \odot \mathrm{ad}^*_{X'} g) + (\chi^*_{X'} f \odot Y + f \odot X' \rhd Y) \\ &\quad + (\mathrm{ad}_{X'} X \odot Y + X \odot X' \rhd Y) + (\chi^*_{X'} f \odot g + f \odot \mathrm{ad}^*_{X'} g) \\ &= (g, \mathrm{ad}_{X'} X) + (\mathrm{ad}^*_{X'} g, X) + (\chi^*_{X'} f, Y) + (f, \chi_{X'} Y) \\ &\quad + \langle \mathrm{ad}_{X'} X, Y \rangle + \langle X, \chi_{X'} Y \rangle + \langle \chi^*_{X'} f, g \rangle + \langle f, \mathrm{ad}^*_{X'} g \rangle, \end{aligned}$$

where $(-,-)$ and $\langle -,- \rangle$ are the components of the pairings (3.1.9), (3.3.7), respectively. Similar computations can be carried out for the other two equations. Moreover, the condition that $D_{t+t^T} \, 2\mathrm{Cas}_{\mathfrak{d}} = 0$ implies that T is symmetric.

In other words, we have the following result.

Proposition 3.3.1. *Quadratic 2-Casimirs* $2\mathrm{Cas}_{\mathfrak{D}}$ *of the 2-Manin triple* $\mathfrak{D}$ *induces only the grading-odd pairings* (3.1.9), (3.3.7).

Note that (3.3.6) follows directly from the $D_{2\,\mathrm{ad}}$-invariance of $R^{\odot}$ itself as an element of $\mathfrak{g}_0 \odot \mathfrak{g}_{-1}$. Assumptions about its particular form, such as (3.3.5), are not necessary. In other words, **Proposition 3.3.1** is a general result that applies to any Drinfel'd 2-double as defined here and in the literature [95].

Chapter 4

Applications of Lie 2-bialgebra symmetries

Let us now turn to some applications of the semiclassical Lie 2-bialgebra symmetries that we have described in the previous chapter. The first application is the 4-dimensional analogue of the topological Chern-Simons theory defined using Lie 2-algebras and higher-gauge theory, following existing literature. The second application is a higher-dimensional notion of Lax integrability that I have derived in [116]. In particular, I shall describe an application of this general 2-Lax framework to study the Heisenberg spin rectangle.

The 4d 2-Chern-Simons theory is a TQFT which is expected to exhibit properties analogous to the usual 3d Chern-Simons theory, such as hosting (extended) topological operators that may give rise to novel 4d tangle invariants. Moreover, these theories are important to understand for quantum gravity [134, 44]. I have initiated a project which investigates the holographic principle for 2-Chern-Simons theories and the 3d integrable field theory that lies on its boundary [135].

4.1 2-Chern-Simons theory

In this section, we shall first give an overview of 2-Chern-Simons theory and its higher-gauge structures following [111]. I have also studied this theory in my paper [115] under the name "monster[1] BF theory". Recalling the notion of 2-BF theory was studied in §2.4, I have proven [115]:

Proposition 4.1.1. *Let $\mathfrak{D} = \mathfrak{G}^*[1] \bowtie \mathfrak{G}$ denote the classical Drinfel'd 2-double of $\mathfrak{G}$ with an Abelian dual $\mathfrak{G}^*[1]$. Then the 2-Chern-Simons theory on $\mathfrak{D}$ is equivalent to the 2-BF theory on $\mathfrak{G}$.*

An analogue of this statement in 3d is a well-known result [129] which identifies 3d gravity as a Chern-Simons theory on the Poincaré algebra $\mathbb{R}^{1,2} \rtimes \mathfrak{so}(1,2)$.

[1] No relation to the monster finite group.

4.1.1 2-Chern-Simons theory as a homotopy Maurer-Cartan theory

Let us begin with an exposition of homotopy Maurer-Cartan theory from the Batalin–Vilkovisky (BV) and derived superfield formulation, following [136] and [111]. This gives a general setting in which higher homotopy generalizations of higher Chern-Simons-like Poisson AKSZ models can be realized, of which "2-Chern-Simons theory" is an example.

Let $\mathfrak{G} \cong \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ denote a weak Lie 2-algebra equipped with n-nary skew-symmetric brackets μ_n, as in **Definition 2.2.5**. Tensoring with the de Rham complex $\Omega^*(X)$ over a space X gives rise to a Lie 2-algebra $\mathcal{L}$, with the graded components

$$\mathcal{L}_n = \bigoplus_{i+j=n} \Omega^i(X) \otimes \mathfrak{g}_j,$$

together with the differential $\ell_1 = d - \mu_1$ and $\ell_n = \mu_n \otimes \wedge^n$ for all $n \leqslant 3$.

Definition 4.1.1. An element $\mathcal{A} \in \mathcal{L}_1$ of degree 1 living in the space

$$\mathcal{A} = (A, B) \in \Omega^1(X) \otimes \mathfrak{g}_0 \oplus \Omega^2(X) \otimes \mathfrak{g}_{-1},$$

is a **Maurer-Cartan element** iff its curvature

$$\sum_{n=1}^{3} \frac{1}{n!} \ell_n(\mathcal{A}, \ldots, \mathcal{A}) = 0$$

vanishes.

We can compute the curvature explicitly as

$$\begin{aligned} \ell_1(\mathcal{A}) + \frac{1}{2}\ell_2(\mathcal{A}, \mathcal{A}) + \frac{1}{3!}\ell_3(\mathcal{A}, \mathcal{A}, \mathcal{A}) &= dA - \mu_1(B) + \frac{1}{2}\mu_2(A \wedge A) \\ &\quad + dB + \mu_2(A \wedge B) + \frac{1}{3!}\mu_3(A \wedge A \wedge A). \end{aligned}$$

Organizing this quantity by degree, we see that we obtain two equations

$$dA + \frac{1}{2}\mu_2(A \wedge A) - \mu_1(B) = 0, \qquad dB + \mu_2(A \wedge B) + \frac{1}{3!}\mu_3(A \wedge A \wedge A) = 0$$

that are identical, under the identification $\mu_2 = ([-, -], \rhd)$, to the fake-flatness (2.2.10) and (modified) 2-flatness (2.2.15) conditions that we have already found in Chapter 1.

We now define the action whose variational principle is associated to the zero-curvature condition; in other words, the minimal locus of the action consists of Maurer-Cartan elements. This is accomplished with an inner product $(-, -)$ on the Lie 2-algebra $\mathcal{L}_*$ of *degree* -3,

$$S_{2\text{MC}}[\mathcal{A}] = \sum_{m=1}^{3} \frac{1}{(m+1)!} (\mathcal{A}, \ell_m(\mathcal{A}, \ldots, \mathcal{A})).$$

This requirement arises from the fact that we must pair a degree-1 Maurer-Cartan element

$\mathcal{A} \in \mathcal{L}_1$ with its degree-2 curvature $F \in \mathcal{L}_2$, and end up with a real number in $\mathbb{R}$ at degree-0.

Now if X were 4-dimensional, this pairing must produce a 4-form on X. This implies that the invariant pairing on $\mathfrak{G}$ consistent with $(-,-)$ must have degree 1 [111, 115],

$$\langle -,- \rangle : (\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}) \oplus (\mathfrak{g}_{-1} \otimes \mathfrak{g}_0) \to \mathbb{R}.$$

This explains why dg Lie algebras are required to define a 4d analogue of Chern-Simons theory. The **2-Maurer-Cartan action** [136] is then

$$S_{2\mathrm{MC}}[\mathcal{A}] = \int_X \langle B, dA + \frac{1}{2}\mu_2(A \wedge A) - \frac{1}{2}\mu_1(B)\rangle + \frac{1}{4!}\langle A, \mu_3(A \wedge A \wedge A)\rangle,$$

where we have used the invariance property

$$\begin{aligned}
&\langle X_1, \mu_2(X_2, Y_1)\rangle = \langle \mu_2(X_1, X_2), Y_1\rangle, \\
&\langle Y_1, tY_2\rangle = \langle tY_1, Y_2\rangle, \qquad\qquad X_1, X_2 \in \mathfrak{g}_0,\ \ Y_1, Y_2 \in \mathfrak{g}_{-1}
\end{aligned} \tag{4.1.1}$$

of the pairing. The equations of motion implement precisely the fake- and modified 2-flatness conditions (2.2.10), (2.2.15).

When $\mathfrak{G}$ is strict such that $\mu_3 = 0$, we recover the **2-Chern-Simons action**

$$S_{2\mathrm{CS}}[A, B] = \int_X \langle B, F_A - \frac{1}{2}\mu_1(B)\rangle \tag{4.1.2}$$

as formulated in [111], which is also called the "4d BF theory" [134, 137, 138] in some literature. Indeed, a variation of the action

$$\begin{aligned}
\delta_B S_{2\mathrm{CS}}[A, B] = 0 &\implies \mathcal{F} = F_A - \mu_1(B) = 0, \\
\delta_A S_{2\mathrm{CS}}[A, B] = 0 &\implies K = dB + \mu_2(A \wedge B) = 0
\end{aligned}$$

imposes precisely the Maurer-Cartan condition for $\mathcal{A} \in \mathcal{L}_1$.

4.1.2 Gauge symmetries of the 2-Chern-Simons theory

Let us now introduce the gauge symmetries of the action (4.1.2) through the derived superfield formulation. Let $\mathcal{A} \in \mathcal{L}_1$ denote a Maurer-Cartan element, then a (finite) derived gauge transformation is given by

$$\mathcal{A} \to \mathcal{A}^U =_2 \mathrm{Ad}_U^{-1} \mathcal{A} + U^{-1}\ell_1 U, \tag{4.1.3}$$

where U is a degree-0 derived gauge parameter. In general, U is a polyform on X valued in $\mathbb{G}$.[2] But in order to understand its "2-adjoint action" ${}_2\mathrm{Ad}_U$, we need it to inherit the compatible group and groupoid structures of $\mathbb{G}$.

[2]We denote by $G = G_0$ and $H = G_{-1}$ the components of the Lie 2-group $\mathbb{G}$ as given in **Definition 2.2.4**, such that $\mathrm{Lie}\,\mathbb{G} = \mathfrak{G}$.

To do this, we shall parameterize $U = (g, e^{\alpha L})$ in terms of a real parameter $\alpha \in \mathbb{R}^{\dim \mathfrak{h}}$, where $g \in C^\infty(X) \otimes G$ is a G-valued function and $L \in \Omega^1(X) \otimes \mathfrak{h}$ is a $\mathfrak{h}$-valued 1-form. This is called the *derived 2-group* formalism [111]. Recalling $\ell_1 = d - \mu_1$, (4.1.3) can then be computed explicitly by (we without loss of generality absorb α into L)

$$\begin{aligned}\mathcal{A} = (A, B) &\to \mathcal{A}^{(g,L)} \\ &= (\mathrm{Ad}_g^{-1} A + g^{-1}dg - \mu_1(L), g^{-1} \rhd B + dL + \mu_2(A^{(g,L)} \wedge L) + \frac{1}{2}[L \wedge L]),\end{aligned}$$

which is precisely the form of the 2-gauge transformations that was found in §2.3 through ad hoc means. The same computations then implies the covariance of the higher curvature quantities

$$\mathcal{F}^U = \mathrm{Ad}_g^{-1} \mathcal{F}, \qquad K^U = g^{-1} \rhd K + \mu_2(\mathcal{F} \wedge L).$$

A lengthy computation shows that the gauge variation of the 2-Chern-Simons action (4.1.2) is a total boundary term [111],

$$S_{2\mathrm{CS}}[A^{(g,L)}, B^{(g,L)}] = S_{2\mathrm{CS}}[A, B] + \int_X d\Gamma,$$

where

$$\Gamma = 2(\langle g^{-1}F'g, L\rangle + \langle g^{-1}A'g + g^{-1}dg, L \wedge L\rangle) + L_{CS}(L).$$

We notice the appearance of a 3d Chern-Simons term

$$L_{CS}(L) = \langle \mu_1 L, dL\rangle + \frac{2}{3}\langle \mu_1 L, [L, L]\rangle.$$

In other words, the gauge non-invariance of 2-Chern-Simons theory is completely holographic, in contrast to the 3d Chern-Simons theory whose finite gauge variation contains the well-known bulk Wess-Zumino term, in addition to a total boundary term.

4.1.3 The underlying Lie 2-bialgebra symmetry of 2-Chern-Simons theory

Now let us examine the Lie 2-bialgebra underlying the 2-Chern-Simons theory. We begin by noting that the key ingredient in the construction of the Lagrangian is a bilinear form $\langle -, - \rangle$ on $\mathfrak{G}$ which is

1. non-degenerate, meaning $\dim \mathfrak{g}_0 = \dim \mathfrak{g}_{-1}$, and

2. invariant in the sense of (4.1.1).

Lie 2-algebras $\mathfrak{G}$ equipped with such a pairing is called *balanced* in [111].

Now as we have proven in §3.3, such pairings are in one-to-one correspondence with quadratic 2-Casimir elements $R^\odot$ of $\mathfrak{G}$. Correspondingly, the L_2-bracket μ_2 on $\mathfrak{G}$ dualizes to a 2-cobracket

$\delta : \mathfrak{G} \to \mathfrak{G} \wedge \mathfrak{G}$ given in graded components by

$$\langle Y, [X, X'] \rangle = \langle \delta_{-1}(Y), X \otimes X' \rangle, \qquad \langle X, X' \rhd Y \rangle = \langle \delta_0(X), X' \otimes Y \rangle,$$

where $X, X' \in \mathfrak{g}_0, Y \in \mathfrak{g}_{-1}$ and $\mu_2 = ([-,-], \rhd)$. This endows $\mathfrak{G}$ with a Lie 2-bialgebra structure δ, which is naturally encoded in the construction of 2-Chern-Simons action.

We are now ready to prove **Proposition 4.1.1**, following my paper [115]. We begin with the standard Manin triple (recall $t = \mu_1, t^T = \mu_1^T$)

$$\mathfrak{D} = \mathfrak{G} \bar{\bowtie} \mathfrak{G}^* = (\mathfrak{g}_{-1} \bowtie \mathfrak{g}_0^*) \xrightarrow{T = t + t^T} (\mathfrak{g}_0 \bowtie \mathfrak{g}_{-1}^*)$$

of the Lie 2-bialgebra $(\mathfrak{G}; \delta)$. An element $\boldsymbol{\mathcal{A}} \in \mathcal{L}_1$ of degree-1 in the dgla $\mathcal{L}_\bullet = \Omega^\bullet(X) \otimes \mathfrak{D}$ is given by $(\mathbf{A}, \mathbf{B}) = (A + C, \Sigma + B)$, where the fields are

$$A \in \Omega^1(X) \otimes \mathfrak{g}_0, \qquad C \in \Omega^1(X) \otimes \mathfrak{g}_{-1}^*, \qquad \Sigma \in \Omega^1(X) \otimes \mathfrak{g}_{-1}, \qquad B \in \Omega^2(X) \otimes \mathfrak{g}_0^*.$$

With these and the 2-coadjoint representations (3.1.8), we can decompose the curvature quantities

$$\begin{aligned}
\mathbf{F} &= (dA + \frac{1}{2}[A \wedge A] + \eta_C^*(\wedge A)) \\
&\quad + \ (dC + \frac{1}{2}[C \wedge C]_* + \chi_A^*(\wedge C)) \equiv \bar{F} + \bar{F}^*, \\
\boldsymbol{K} &= (d\Sigma + A \wedge^{\rhd} \Sigma + \tilde{\Delta}_B(\wedge A) - \mathfrak{ad}_C^*(\wedge \Sigma)) \\
&\quad + \ (dB + C \wedge^{\rhd^*} B + \Delta_\Sigma(\wedge C) - \mathrm{ad}_A^*(\wedge B)) \equiv \bar{K} + \bar{K}^*,
\end{aligned}$$

into $\mathfrak{G}$- and $\mathfrak{G}^*[1]$-valued sectors.

With the canonical evaluation pairing (3.1.9) on $\mathfrak{D}$, which we recall is coisotropic, the 2-Chern-Simons action on $\mathfrak{D}$, also called the "**monster 2-BF theory**" in [115], is given by

$$\begin{aligned}
S_{2\mathrm{CS}}[\mathbf{A}, \mathbf{B}] &= \int_X \langle \mathbf{B}, \mathbf{F} - \frac{1}{2} T\mathbf{B} \rangle \\
&= \int_X \langle B, \bar{F} \rangle + \langle \bar{F}^*, \Sigma \rangle - \frac{1}{2}\left[\langle B, t\Sigma \rangle + \langle \Sigma, t^T B \rangle \right].
\end{aligned}$$

Note by the symmetry condition $\langle t^T -, - \rangle = \langle -, t - \rangle$, the two final terms are equivalent. On the other hand, the first two terms read

$$\begin{aligned}
\langle B \wedge \bar{F} \rangle &= \langle B \wedge F \rangle + \langle B \wedge \eta_C^*(\wedge A) \rangle, \\
\langle \bar{F}^* \wedge \Sigma \rangle &= \langle dC + \frac{1}{2}[C \wedge C]_* + \chi_A^*(\wedge C) \wedge \Sigma \rangle.
\end{aligned}$$

An integration by parts (neglecting the boundary term $d\langle C \wedge \Sigma \rangle$) yields

$$\langle \bar{F}^* \wedge \Sigma \rangle = -\langle C \wedge K \rangle + \frac{1}{2}\langle [C \wedge C]_* \wedge \Sigma \rangle$$

in terms of the 2-curvature $K = d\Sigma + A \wedge^{\rhd} \Sigma$ on $\mathfrak{g}_{-1}$. Thus we see that the 2-Chern-Simons theory can be written as

$$S_{\mathbf{BF}}[\mathbf{A},\mathbf{B}] = \frac{1}{2}\int_X \langle B, F - t\Sigma\rangle - \langle C, K\rangle \\ + \frac{1}{2}\int_X \langle \frac{1}{2}[C \wedge C]_*, \Sigma\rangle - \langle C \wedge^{\rhd^*} B, A\rangle.$$

Now if $\mathfrak{G}^*[1]$ were Abelian, then $[-,-]_*, \rhd^* = 0$ whence the final two terms drop. The remaining action is precisely the 2-BF action (2.4.1).

4.2 Higher-dimensional integrability and the 2-Lax pair

The goal in this section is to lay out the general theory of 2-graded integrable systems following my work [116]. We defined an appropriate notion of a "2-graded Lax equation" as a categorification of the usual Lax equation. Once this is achieved, we specialize to the dual Lie 2-algebra $\mathfrak{G}^*[1]$ and construct a 2-graded Lax pair on it, in analogy with the 1-algebra case as reviewed in eg. [10]. We then work to prove that it does in fact satisfy the 2-graded Lax equations.

We begin with a dg manifold $M = M_{-1} \xrightarrow{\mathbf{t}} M_0$ equipped with a Poisson bivector $\Pi = \Pi^{-1} + \Pi^0$ satisfying (3.2.1) and (3.2.4). We let $(C^\infty(M), \mathbf{t}^*, \{-,-\})$ denote the Poisson 2-algebra via **Lemma 3.2.1**.

4.2.1 2-Lax pair

Consider smooth functions from the 2-graded space $M = M_{-1} \oplus M_0$ into a Lie 2-algebra $\mathfrak{g}$. We treat such functions as elements in the tensor product $C^\infty(M) \otimes \mathfrak{G}$, which is a 3-term complex (cf. [95])

$$\underbrace{C^\infty(M_0) \otimes \mathfrak{g}_{-1}}_{\text{deg-}(-2)} \xrightarrow{D} \underbrace{(C^\infty(M_{-1}) \otimes \mathfrak{g}_{-1}) \oplus (C^\infty(M_0) \otimes \mathfrak{g}_0)}_{\text{deg-}(-1)} \xrightarrow{D} \underbrace{C^\infty(M_{-1}) \otimes \mathfrak{g}_0}_{\text{deg-}0} \tag{4.2.1}$$

with the differentials $D = 1 \otimes t \pm \mathbf{t}^* \otimes 1$. The graded Lie bracket $[-,-]$ on $\mathfrak{G}$, together with the graded Poisson bracket $\{-,-\}$ on $C^\infty(M)$, as in **Proposition 3.2.1**, endow this complex with two Lie 2-algebra structures.

Let $H \in C^\infty(M)$ a Hamiltonian function on $M = M_{-1} \xrightarrow{\mathbf{t}} M_0$, which admits a graded decomposition $H = H_{-1} + H_0 \in C^\infty(M_{-1}) \oplus C^\infty(M_0)$.

Definition 4.2.1. A tuple of elements $(L, P) \in C^\infty(M) \otimes \mathfrak{G}$ is a **2-Lax pair**, of the Hamiltonian system $(M, \{-,-\}, H)$ iff it satisfies the **2-Lax equation**

$$\dot{L} = \{H, L\} = [P, L], \tag{4.2.2}$$

where $\{-,-\}, [-,-]$ are the graded Poisson/Lie brackets on the complex (4.2.1).

There is a subtlety associated to the meaning of "$\dot{L}$" in (4.2.2), as the Hamiltonian $H = H_{-1}+H_0$ here is itself graded. As such, the dynamics it generates is also graded, in the sense that there are essentially two Hamiltonians evolving under a single "time" parameter.

We note the functions $L, P : M \to \mathfrak{G}$ themselves need not be a 2-vector space homomorphisms. Indeed, such maps must only have components concentrated in degree-0 and degree-(-2) in (4.2.1) [125, 98].

4.2.2 Conserved quantities

Recall that in the 1-algebra case, the trace polynomials f_k of the Lax function L are constants of motion. We wish now to investigate the analogous notion of "2-graded integrability" afforded by the 2-Lax equations (4.2.2). Toward this, we must first explain how to construct trace polynomials in the 2-graded context and hence the relevant concept of 2-representation in our context.

Lie 2-algebra 2-representations. Let $V = V_{-1} \xrightarrow{\partial} V_0$ denote a 2-term complex of vector spaces.

Definition 4.2.2. The space of endomorphisms $\mathfrak{gl}(V) : \mathrm{End}_{-1}(V) \xrightarrow{\delta} \mathrm{End}_0(V)$ of V is a 2-graded space

$$\mathrm{End}_{-1}(V) = \mathrm{Hom}(V_0, V_{-1}), \qquad \mathrm{End}_0(V) = \{M + N \in \mathrm{End}(V_{-1}) \oplus \mathrm{End}(V_0) \mid N\partial = \partial M\}, \tag{4.2.3}$$

equipped with the following (strict) Lie 2-algebra structure [95, 125]

$$\begin{aligned} \delta : \mathrm{End}_{-1}(V) \to \mathrm{End}_0(V), \qquad & \delta(A) = A\partial + \partial A, \\ [M + N, M' + N']_C = [M, M'] + [N, N'], \qquad & (M + N) \rhd_C A = MA - AN, \\ [A, A']_C = A\partial A' - A'\partial A, \qquad & \end{aligned}$$

for each $M + N \in \mathrm{End}_0(V)$, $A \in \mathrm{End}_{-1}(V)$.

Definition 4.2.3. A (strict) **2-representation** $\rho : \mathfrak{G} \to \mathfrak{gl}(V)$ is a Lie 2-algebra homomorphism such that the following square

$$\begin{tikzcd} \mathfrak{g}_{-1} \arrow[r, "t"] \arrow[d, "\rho_1"] & \mathfrak{g}_0 \arrow[d, "\rho_0"] \\ \mathrm{End}_{-1}(V) \arrow[r, "\delta"] & \mathrm{End}_0(V) \end{tikzcd} \tag{4.2.4}$$

commutes. More explicitly, we have $\rho = (\rho_0, \rho_1)$ with $\rho_0(X) = (\rho_0^0(X), \rho_0^1(X)) \in \mathrm{End}_0(V)$ and $\rho_1(Y) \in \mathrm{End}_{-1}(V)$ for each $X \in \mathfrak{g}_0, Y \in \mathfrak{g}_{-1}$, such that the following conditions

$$\begin{aligned} &\rho_0^0(tY) = \partial\rho_1(Y), \qquad \rho_0^1(tY) = \rho_1(Y)\partial, \\ &\rho_1(X \rhd Y) = (\rho_0 X) \rhd_C \rho_1 Y = \rho_0^1(X)\rho_1(Y) - \rho_1(Y)\rho_0^0(X) \end{aligned} \tag{4.2.5}$$

are satisfied.

Furthermore, $\rho_0 = \rho_0^1 + \rho_0^0$ represents $\mathfrak{g}_0$ on respectively V_{-1} and V_0, with ∂ as the intertwiner. Elementary examples of 2-representations include the adjoint/coadjoint representations of $\mathfrak{G}$; see [95, 115] or Chapter 3.1.1.

Any 2-representation ρ as defined above gives rise to a *genuine* representation ρ^{gen} on the direct sum $V_{-1} \oplus V_0$, which takes the form of a block matrix

$$\rho^{gen}(L) = \begin{pmatrix} \rho_0^1(L_0 + tL_{-1}) & \rho_1(L_{-1}) \\ 0 & \rho_0^0(L_0) \end{pmatrix} \in \mathfrak{gl}(V_{-1} \oplus V_0), \quad L_0 \in \mathfrak{g}_0, \quad L_{-1} \in \mathfrak{g}_{-1}, \tag{4.2.6}$$

where L_{-1}, L_0 denotes the graded components of L that take values in $\mathfrak{g}_{-1}, \mathfrak{g}_0 \subset \mathfrak{G}$, respectively. This representation was shown to satisfy $\rho^{gen}([L,P]) = [\rho^{gen}(L), \rho^{gen}(P)]_C$ in [125], where $[-,-]_C$ is the matrix commutator on $\mathfrak{gl}(V_{-1} \oplus V_0)$.

Example 4.2.1. *The most relevant 2-representation for our current paper is the 2-coadjoint representation ${}_2\mathrm{Ad}^*$ of the 2-group G on its dual Lie 2-algebra $V = \mathfrak{G}^*[1]$. We shall now prove that ${}_2\mathrm{Ad}^* : G \to \mathrm{End}(\mathfrak{G}^*[1])$ is indeed a 2-representation.*

We define ${}_2\mathrm{Ad}^*$ by dualizing the adjoint representation ${}_2\mathrm{Ad} = (\mathrm{Ad}_0, \Upsilon)$ of G on $\mathfrak{g}$ defined in (3.1.6). Hence, ${}_2\mathrm{Ad}^*$ has the graded components

$$\mathrm{Ad}_0^* = (\mathcal{X}^*, \mathrm{Ad}^*) : \mathfrak{g}_0 \to \mathrm{End}(\mathfrak{g}_0^* \oplus \mathfrak{g}_{-1}^*), \qquad \Upsilon^* : \mathfrak{g}_{-1} \to \mathrm{Hom}(\mathfrak{g}_{-1}^*, \mathfrak{g}_0^*)$$

satisfying for each $x \in G_0, y \in G_{-1}$ and $X \in \mathfrak{g}_0, f \in \mathfrak{g}_{-1}, g \in \mathfrak{g}_0^*, f \in \mathfrak{g}_{-1}^*$ the invariance conditions

$$\begin{aligned} \langle \mathrm{Ad}_x^* g + \mathcal{X}_x^* f, Y + X' \rangle &= \langle g + f, \mathcal{X}_{x^{-1}} Y + \mathrm{Ad}_{x^{-1}} X' \rangle, \\ \langle \Upsilon_y^*(f), X \rangle &= \langle f, \Upsilon_{y^{-1}}(X) \rangle \end{aligned}$$

with respect to the natural pairing form $\langle -,- \rangle$ between $\mathfrak{G}^*[1]$ and $\mathfrak{G}$. Moreover, we see that ${}_2\mathrm{Ad}^*$ satisfies the following key identities

$$t^T \mathrm{Ad}_x^* = \mathcal{X}_x^* t^T, \qquad t^T \Upsilon_y^* = \mathrm{Ad}_{y^{-1}}^*, \qquad \Upsilon_y^* t^T = \mathrm{Ad}_{\mathrm{t}y^{-1}}^*,$$

where $t^T : \mathfrak{g}_0^* \to \mathfrak{g}_{-1}^*$ is the dual t-map on $\mathfrak{G}^*[1]$. The first identity implies $(\mathcal{X}^*, \mathrm{Ad}^*) \in \mathrm{End}(\mathfrak{G}^*[1])_0$, while the rest imply precisely the commutativity condition (4.2.4). Indeed, one explicitly computes for each $Y \in \mathfrak{g}_{-1}, X \in \mathfrak{g}_0$ that

$$\begin{aligned} \langle Y, t^T(\Upsilon_y^*(f)) \rangle &= \langle Y, \mathrm{Ad}_{y^{-1}}^* f \rangle = \langle \mathrm{Ad}_y Y, f \rangle = \langle \mathcal{X}_{\mathrm{t}y^{-1}} Y, f \rangle = \langle Y, \mathcal{X}_{\mathrm{t}y}^* f \rangle, \\ \langle X, \Upsilon_y^*(t^T g) \rangle &= \langle \Upsilon_{y^{-1}} X, t^T g \rangle = \langle t(\Upsilon_{y^{-1}} X), g \rangle = \langle \mathrm{Ad}_{\mathrm{t}y^{-1}} X, g \rangle = \langle X, \mathrm{Ad}_{\mathrm{t}y}^* g \rangle. \end{aligned}$$

Definition 4.2.4. A function $H \in C^\infty(\mathfrak{g}^*[1])$ is ${}_2\mathrm{Ad}^*$-invariant if

$$H_0 \circ \mathrm{Ad}_x^* = H_0, \qquad H_{-1} \circ \mathcal{X}_x^* = H_{-1}, \qquad H_0 \circ \Upsilon_y^* = H_{-1}, \tag{4.2.7}$$

for each $x \in G_0, y \in G_{-1}$, and where $H = H_{-1} + H_0 \in C^\infty(\mathfrak{g}^*[1]) \cong C^\infty(\mathfrak{g}_0^*) \oplus C^\infty(\mathfrak{g}_{-1}^*)$.

This notion of invariance will be useful later.

Constants of (graded) motion. We are now ready to characterize the notion of conserved quantities inherited from the construction of 2-representation built out on 2-vector spaces of the Baez-Crans type.

Theorem 4.2.1. *Let $\chi_V : \mathfrak{gl}(V) \to \mathbb{R}$ denote a class function; namely any linear map that is invariant under the L_2-bracket $[-,-]_C$ on $\mathfrak{gl}(V)$. The 2-Lax equation (4.2.2) implies that the polynomials*

$$\mathcal{F}_k = \chi_V \rho(L)^k$$

are constants of motion for any k and 2-representation ρ.

Proof. The proof runs in exact analogy with the 1-algebra case [10]. From (4.2.15) and the cyclicity of χ, we have

$$\dot{\mathcal{F}}_k = \sum_{i=0}^{k-1} \chi_V(\rho(L)^i \rho(\dot{L}) \rho(L)^{k-i-1}) = k\, \chi_V(\rho(L)^{k-1} \rho([L,P])).$$

By the fact that ρ is a homomorphism of Lie 2-algebras, we have $\rho([L,P]) = [\rho(L), \rho(P)]_C$ and hence

$$\begin{aligned} \chi_V(\rho(L)^{k-1}\rho([L,P])) &= \chi_V(\rho(L)^{k-1}[\rho(L), \rho(P)]_C) \\ &= \chi_V([\rho(L)^k, \rho(P)]_C) = 0, \end{aligned}$$

again from the invariance of χ_V. □

Note that the conservation of these trace polynomials is independent of the choice of the 2-representation ρ. However, what exactly is being conserved does depend on the representation — it is the eigenvalues of the matrix representation $\rho(L)$. The conservation of these eigenvalues can be understood as the notion of "2-graded integrability" that the 2-Lax pair in **Definition 4.2.2** affords.

By making use of the genuine representation ρ^{gen} given in (4.2.6), a straightforward example of a class function χ_V is given by merely the trace form on $\mathfrak{gl}(V_{-1} \oplus V_0)$. As such, the above result states that the trace polynomials

$$\mathcal{F}_k = \operatorname{tr}_V(\rho^{gen}(L)^k)$$

are conserved for any $k \in \mathbb{Z}_{\geqslant 0}$. By a fundamental result in linear algebra, the eigenvalues of a block-triangular matrix consist of the combined eigenvalues of its diagonal blocks:

$$\text{Eigen}\, \rho^{gen}(L) = \text{Eigen}\, \rho_0^1(L_0 + tL_{-1}) \coprod \text{Eigen}\, \rho_0^0(L_0).$$

These are example of the conserved quantities associated to the 2-Lax equation (4.2.2) that one can always compute, using the genuine representation (4.2.6).

4.2.3 2-Kirillov-Kostant Poisson structure on $C^\infty(\mathfrak{G}^*[1])$

We first generalize the standard Kirillov-Kostant Poisson structure to the Lie 2-algebra context. This shall serve as the appropriate setting for constructing a canonical 2-Lax pair on the dual space $\mathfrak{G}^*[1]$ of a given Lie 2-bialgebra $(\mathfrak{G}; dR^\wedge)$.

Proposition 4.2.1. *Let $\mathfrak{G}$ denote a Lie 2-bialgebra with the graded L_2-bracket $[-,-]$. The graded algebra of functions $C^\infty(\mathfrak{G}^*[1])$, equipped with the Poisson bracket $\{-,-\}^*$*

$$\{\phi,\phi'\}^*(g+f) = \langle g+f, [d_{g+f}\phi, d_{g+f}\phi']\rangle, \qquad \phi,\phi' \in C^\infty(\mathfrak{G}^*[1]), \tag{4.2.8}$$

where $g+f \in \mathfrak{G}^*[1]$, is a Poisson 2-algebra. We call this a **2-Kirillov-Kostant (2KK) Poisson structure** *on $C^\infty(\mathfrak{G}^*[1])$.*

Proof. It will be convenient to provide the explicit correspondence between the graded components of $\{-,-\}$ and $[-,-]$. For this, it is useful to recall that $\mathfrak{g}$ is dual to $\mathfrak{g}^*[1]$, so 1-forms on $\mathfrak{G}^*[1]$ are elements in $\mathfrak{G}$. In particular, $d\phi$ is valued in $\mathfrak{G}$ for $C^\infty(\mathfrak{G}^*[1]) \ni \phi = (\phi_{-1}+\phi_0) \in C^\infty(\mathfrak{g}_0^*) \oplus C^\infty(\mathfrak{g}_{-1}^*)$ and

$$d_g\phi_{-1} \in \mathfrak{G}_0, \qquad d_f\phi_0 \in \mathfrak{g}_{-1}.$$

With this in mind, we identify the components of the graded bracket $\{-,-\}$.

$$\begin{aligned}
\{\phi,\phi'\}^*_{-1}(g+f) &= \langle g, [d_f\phi_0, d_g\phi'_{-1}]_{-1} + [d_g\phi_{-1}, d_f\phi'_0]_{-1}\rangle,\\
\{\phi_0,\phi'_0\}^*_0(f) &= \langle f, [d_f\phi_0, d_f\phi'_0]^{(-1)}\rangle\\
\{\phi,\phi'\}^*_{-2}(g) &= \langle g, [d_g\phi_{-1}, d_g\phi'_{-1}]_0\rangle.
\end{aligned} \tag{4.2.9}$$

Now we must show that this graded Poisson bracket $\{-,-\}^*$ is a L_2-bracket, satisfying (3.2.1) and (3.2.4). To do so, first we note that we can decompose $\mathfrak{g}^*_{-1}$ as $\mathfrak{g}^*_{-1} \cong \operatorname{im} t^T \oplus \operatorname{coker} t^T$, hence every $f \in \mathfrak{g}^*_{-1}$ can be written as

$$f = t^T g' + f' \in \operatorname{im} t^T \oplus \operatorname{coker} t^T. \tag{4.2.10}$$

Next, using the rank-nullity theorem, we have that $\operatorname{coker} t^T \cong \ker t$ by duality, and hence

$$t(d_f\phi_0) = t(d_{t^Tg'}\phi_0 + d_{f'}\phi_0) = t(d_{t^Tg'}\phi_0) \equiv ((t^T)^* d\phi_0)_{g'} \tag{4.2.11}$$

for any $\phi_0 \in C^\infty(\mathfrak{g}^*_{-1})$; note the last equality is the *definition* of the pullback $(t^T)^*d\phi_0$. We can now directly compute

$$\begin{aligned}
((t^T)^*\{\phi_{-1},\phi'_0\}^*_{-1})(g) &= \langle t^T g, [(d_g\phi_{-1}), d_{t^Tg}\phi'_0]\rangle = \langle g, t[d_g\phi_{-1}, d_{t^Tg}\phi'_0]\rangle\\
&= \langle g, [d_g\phi_{-1}, t(d_{t^Tg}\phi'_0)]\rangle = \langle g, [d_g\phi_{-1}, ((t^T)^*d\phi')_g]\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \{\phi_{-1}, (t^T)^*\phi_0'\}^*_{-2}(g), \\
\{\phi_0, \phi_0'\}^*_0(f) &= \langle f, [d_f\phi_0, d_f\phi_0']\rangle = \langle f, [t(d_{t^Tg'}\phi_0)), d_f\phi_0']\rangle \\
&= \langle f, [((t^T)^* d\phi_0)_{g'}, d_f\phi_0']\rangle \\
&= \{(t^T)^*\phi_0, \phi_0'\}^*_{-1}(f),
\end{aligned} \tag{4.2.12}
$$

where we have used the equivariance and the Peiffer identity (2.2.3) in $\mathfrak{g}$. Similarly, the 2-Jacobi identities (3.2.4) follow from that (2.2.4) of $[-,-]$.

□

We now construct an alternative 2-KK Poisson structure on $\mathfrak{G}^*[1]$ by explicitly making use of the classical 2-r-matrix. We first define a map $\varphi = (\varphi_{-1}, \varphi_0) : \mathfrak{g} \to \mathfrak{g}$ of 2-graded vector spaces, then use it to define an alternative L_2-bracket $[-,-]_R$ on $\mathfrak{G}$. Let us fix the bases $\{T_i\}_i, \{S_a\}_a$ of $\mathfrak{g}_0, \mathfrak{g}_{-1}$ respectively.

Proposition 4.2.2. *The map $\varphi = (\varphi_{-1}, \varphi_0) : \mathfrak{g} \to \mathfrak{g}$ defined by*

$$
\begin{aligned}
\varphi_{-1} : \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}, \qquad & Y \mapsto (R^\wedge)^{ia}\langle Y, T_i\rangle S_a, \\
\varphi_0 : \mathfrak{g}_0 \to \mathfrak{g}_0, \qquad & X \mapsto (R^\wedge)^{ai}\langle X, S_a\rangle T_i,
\end{aligned}
$$

is a 2-vector space homomorphism if and only if $D_t^- R^\wedge = 0$.

Proof. Clearly, φ is linear, hence it remains to show that $t\varphi_{-1} = \varphi_0 t$. By definition, this requires

$$(R^\wedge)^{ia} T_i \wedge t(S_a) = (R^\wedge)^{ai} t(S_a) \wedge T_i$$

for each basis elements $T_i \in \mathfrak{g}_0, S_a \in \mathfrak{g}_{-1}$. In other words, the combination $(R^\wedge)^{ia} t_a^j$ is skew-symmetric; this is precisely the condition $D_t^- R^\wedge = 0$ in (3.3.3) [95].

□

Proposition 4.2.3. *Let $R \in \mathfrak{g}_0 \otimes \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1} \otimes \mathfrak{g}_0$ denote a solution to the modified 2-CYBE (3.3.3). The bracket defined by*

$$[Y + X, Y' + X']_R = [\varphi(Y + X), Y' + X'] + [Y + X, \varphi(Y' + X')],$$

is a Lie 2-algebra bracket which satisfies

$$[g + f, g' + f'] = \langle -, [Y + X, Y' + X']_R\rangle \tag{4.2.13}$$

where $f^{(\prime)} = \langle -, X^{(\prime)}\rangle \in \mathfrak{G}_0^*$ and $g^{(\prime)} = \langle -, Y^{(\prime)}\rangle \in \mathfrak{g}_{-1}^*$.

Proof. Recall [95] that the skew-symmetric piece $R^\wedge$ of a solution R to the modified 2-CYBE (3.3.3) defines the cobracket $dR^\wedge(Y + X) = \delta(Y + X) = \delta_{-1}(Y) + \delta_0(X)$ given by

$$\delta_{-1}(Y) = [Y \otimes 1 + 1 \otimes Y, R^\wedge], \qquad \delta_0(X) = [X \otimes 1 + 1 \otimes X, R^\wedge],$$

and the symmetric piece $R^{\odot} = \langle -, - \rangle$ defines a ${}_2\,$ad-invariant pairing. These facts allow us to compute directly (cf. [10]) that, for each basis element $Z_i = S_i + T_i \in \mathfrak{g}$,

$$\begin{aligned}
[h, h'](Z_i) &= \langle h \otimes h', \delta(Z_i) \rangle = \langle h \otimes h', [Z_i \otimes 1 + 1 \otimes Z_i, R^\wedge] \rangle \\
&= (R^\wedge)^{jk} \langle h \otimes h', [Z_i, Z_j] \otimes Z_k + Z_j \otimes [Z_i, Z_k] \rangle \\
&= (R^\wedge)^{jk} \left(R^{\odot}(Z, [Z_i, Z_j]) R^{\odot}(Z', Z_k) + R^{\odot}(Z, Z_j) R^{\odot}(Z', [Z_i, Z_k]) \right), \\
&= -(R^\wedge)^{jk} \left(R^{\odot}([Z, Z_j], Z_i) R^{\odot}(Z', Z_k) + R^{\odot}(Z, Z_j) R^{\odot}([Z', Z_k], Z_i) \right) \\
&= R^{\odot}([Z, \varphi(Z')], Z_i) + R^{\odot}([\varphi(Z), Z'], Z_i) \\
&= R^{\odot}([Z, Z']_R, Z_i) = \langle Z_i, [Z, Z']_R \rangle,
\end{aligned}$$

where we abbreviated the graded elements $h = g + f, h' = g' + f' \in \mathfrak{G}^*[1]$ and used that $Z^{(')} \equiv \langle h^{(')}, - \rangle \in \mathfrak{G}$. This proves (4.2.13).

Now let us establish that $[-, -]_R$ is a genuine L_2-bracket on $\mathfrak{g}$. Since $[-, -]$ by hypothesis is equivariant and satisfies the Peiffer identity with respect to t, the fact that φ is a 2-vector space homomorphism implies the same for $[-, -]_R$. It thus suffices to check the 2-Jacobi identities for $[-, -]_R$, but this directly follows from (4.2.13) (cf. [139]),

$$\langle Z_0, \circlearrowright [[Z, Z']_R, Z'']_R \rangle = \ (\circlearrowright [[h, h'], h''])(Z_0) = 0 \qquad \forall Z_0 \in \mathfrak{G}.$$

□

Lemma 4.2.1. *The Poisson bracket $\{-, -\}^*_R$, defined by the following formula*

$$\{\phi, \phi'\}^*_R(g + f) = \langle g + f, [d_{g+f}\phi, d_{g+f}\phi']_R \rangle, \tag{4.2.14}$$

where $\phi, \phi' \in C^\infty(\mathfrak{G}^*[1])$, $g + f \in \mathfrak{G}^*[1]$, is a 2KK Poisson structure.

Proof. This follows from the fact that $[-, -]_R$ is a L_2-bracket, hence the proof of **Proposition 4.2.1** applies. □

4.2.4 2-Lax pair on $\mathfrak{G}^*[1]$

Fix a ${}_2\,\mathrm{Ad}^*$-invariant Hamiltonian $H \in C^\infty(\mathfrak{G}^*[1])$ (as defined in Definition 4.2.4). We are now ready to finally canonically construct a 2-Lax pair (L, P) on $(\mathfrak{G}^*[1], \{-, -\}^*_R, H)$ in this section according to (4.2.2), based on the 2-KK Poisson structure $\{-, -\}^*_R$ (4.2.14) as well as the underlying classical 2-r-matrix. We will take

$$\begin{aligned}
L_0 &\in C^\infty(\mathfrak{g}^*_{-1}) \otimes \mathfrak{g}_0, \qquad & L_{-1} &\in C^\infty(\mathfrak{g}^*_0) \otimes \mathfrak{g}_{-1}, \\
P_{-1} &\in C^\infty(\mathfrak{g}^*_{-1}) \otimes \mathfrak{g}_{-1}, \qquad & P_0 &\in C^\infty(\mathfrak{g}^*_0) \otimes \mathfrak{g}_0,
\end{aligned}$$

hence L has degree-(-1) and P has degree-0 and -2 in the complex (4.2.1).

Fix bases $\{T_i\}_i, \{S_a\}_a$ of $\mathfrak{g}_0, \mathfrak{g}_{-1}$, and suppose the classical 2-r-matrix R on $\mathfrak{G}$ is invertible. We make use of a basic linear algebra fact [140] that the inverse of an off-diagonal block matrix, such as R where the off-diagonal pieces are given by R_1, R_2, is the off diagonal matrix with blocks R_2^{-1} and R_1^{-1}, and hence the inverse of the symmetric piece $(R_1^{\odot})_{ai}$, for instance, has matrix elements $((R_2^{\odot})^{-1})^{ai}$. Put

$$\begin{aligned} L_0 : f &\mapsto (R_2^{\odot})^{ai} f(S_a) T_i, \qquad & L_{-1} : g &\mapsto (R_1^{\odot})^{ia} g(T_i) S_a, \\ P_{-1} : f &\mapsto \varphi_{-1}(d_f H_0), \qquad & P_0 : g &\mapsto \varphi_0(d_g H_{-1}), \end{aligned} \tag{4.2.15}$$

and we wish to show that $(L, P) : \mathfrak{g}^*[1] \to \mathfrak{g}$ is indeed a 2-Lax pair as in **Definition 4.2.2**.

Theorem 4.2.2. *Let $H \in C^\infty(\mathfrak{G}^*[1])$ denote a ${}_2\operatorname{Ad}^*$-invariant Hamiltonian. Then (L, P) given in (4.2.15) is a 2-Lax pair of the 2-graded Hamiltonian system $(\mathfrak{G}^*[1], \{-,-\}_R^*, H)$ for which the Lax potential L satisfies*

$$\mathbf{t}^* L = tL, \qquad \{L, L\}_R^* = [L \otimes 1 + 1 \otimes L, R^\wedge], \tag{4.2.16}$$

where $\mathbf{t}^* = (t^T)^*$ is the pullback of $t^T : \mathfrak{g}_0^* \to \mathfrak{g}_{-1}^*$ and we have extended the L_2-bracket $[-,-]$ to $\mathfrak{g}^{2\otimes}$.

Proof. First we compute the coefficients

$$(d_f L_{-1})^i = R_2^{\odot bi} S_b, \qquad (d_g L_0)^a = R_1^{\odot ja} T_j.$$

We note also that the ${}_2\operatorname{Ad}^*$-invariance of H (4.2.7) implies, in particular, that

$$[Y + X, d_{g+f} H] = 0, \forall Y \in \mathfrak{g}_{-1}, \forall X \in \mathfrak{g}_0,$$

(we emphasize we use the bracket $[-,-]$ and not $[-,-]_R$).

Then from the 2-KK Poisson structure (4.2.14) we have

$$\begin{aligned} \{H, L\}_R^*(g+f) &= \langle g+f, [d_{g+f}H, d_{g+f}L^{i,a}]_R \rangle (T_i \oplus S_a) \\ &= \langle g+f, [d_{g+f}H, d_f L_{-1}^i]_R \rangle T_i + \langle g+f, [d_{g+f}H, d_g L_0^a]_R \rangle S_a \\ &= \langle g+f, [\varphi(d_{g+f}H), d_f L_{-1}^i] + [d_{g+f}H, \varphi(d_f L_{-1}^i)] \rangle T_i \quad \text{invariance of the Hamiltonian} \\ &\quad + \langle g+f, [\varphi(d_{g+f}H), d_g L_0^a] + [d_{g+f}H, \varphi(d_g L_0^a)] \rangle S_a \\ &= \langle g+f, [\varphi(d_{g+f}H), R_2^{\odot bi} S_b] \rangle T_i + \langle g+f, [\varphi(d_{g+f}H), R_1^{\odot ja} T_j] \rangle S_a \\ &= -R_2^{\odot bi} \langle f, S_b \rangle [\varphi(d_{g+f}H), T_i] - R_1^{\odot ja} \langle g, T_i \rangle [\varphi(d_{g+f}H), S_a] \\ &= [L(g+f), P(g+f)], \end{aligned}$$

where we have used the ${}_2\operatorname{ad}$-invariance of the 2-Casimir $R^{\odot}$. This proves the first statement.

To prove the second statement, we first note that we have the following expressions

$$\varphi_{-1}(S_a) = (R_1^{\odot})_{ai} (R_2^{\wedge})^{ic} S_c, \qquad \varphi_0(T_i) = (R_2^{\odot})_{ib} (R_1^{\wedge})^{bj} T_j$$

for φ. Hence by a direct computation,

$$\begin{aligned}
\{L,L\}_R^*(g+f) &= \{L^{a,i},L^{b,j}\}^*(g+f)(S_a+T_i)\otimes(S_b+T_j)\\
&= \langle g+f,[d_{g+f}L^{a,i},d_{g+f}L^{b,j}]_R\rangle(S_a+T_i)\otimes(S_b+T_j)\\
&= (R_2^{\odot ai'}+R_1^{\odot ia'})(R_2^{\odot bj'}+R_1^{\odot jb'})\langle g+f,[\varphi_0T_{i'}+\varphi_{-1}S_{a'},T_{j'}+S_{b'}]\\
&\quad+\ [T_{i'}+S_{a'},\varphi_0T_{j'}+\varphi_{-1}S_{b'}]\rangle(S_a+T_i)\otimes(S_b+T_j)\\
&= (R_1^{\wedge al}+R_2^{\wedge ic})(R_2^{\odot bj'}+R_1^{\odot jb'})\langle g+f,[T_l+S_c,T_{j'}+S_{b'}]\rangle((S_a+T_i)\otimes(S_b+T_j))\\
&\quad+(R_2^{\odot ai'}+R_1^{\odot ib'})(R_1^{\wedge bm}+R_2^{\wedge jd})\langle g+f,[T_{i'}+S_{a'},T_m+S_d]\rangle((S_a+T_i)\otimes(S_b+T_j))\\
&= -(R_1^{\wedge al}+R_2^{\wedge ic})(R_2^{\odot bj'}+R_1^{\odot jb'})\langle g+f,T_{j'}+S_{b'}\rangle((S_a+T_i)\otimes[T_l+S_c,S_b+T_j])\\
&\quad-(R_2^{\odot ai'}+R_1^{\odot ib'})(R_1^{\wedge bm}+R_2^{\wedge jd})\langle g+f,T_{i'}+S_{a'}\rangle([S_a+T_i,T_m+S_d]\otimes(S_b+T_j))\\
&= -(R_1^{\wedge al}+R_2^{\wedge ic})((S_a+T_i)\otimes[T_l+S_c,L(g+f)])\\
&\quad-(R_1^{\wedge bm}+R_2^{\wedge jd})([L(g+f),T_m+S_d]\otimes(S_b+T_j))\\
&= -[r,L\otimes 1+1\otimes L](g+f).
\end{aligned}\tag{4.2.17}$$

Finally, for each $g\in\mathfrak{g}_0^*$, we have

$$\begin{aligned}
\mathbf{t}^*L(g) &= L(t^Tg)=(R_2^{\odot})^{ai}(t^Tg)(S_a)T_i\\
&= (R_2^{\odot})^{ai}g(tS_a)(T_i)=(R_1^{\odot})^{ia}g(T_i)tS_a=tL(g)
\end{aligned}$$

as desired, where we have used the definition of the adjoint t^T as well as the symmetry of $R^{\odot}$. □

The special properties that the 2-Lax potential L satisfies in this case allows us to prove the following.

Corollary 4.2.1. *The 2-Lax pair (4.2.15) induces an ordinary Lax pair $(\underline{L},P_0):\mathfrak{g}_0^*\to\mathfrak{g}_0$ on the Hamiltonian system[3] $(\mathfrak{g}_0^*,\{-,-\}_0^*,H_{-1})$.*

Proof. Recall that we have extended the t-map to act on all of $\mathfrak{G}$, such that $t(Y+X)=tY$ for each $X\in\mathfrak{g}_0,Y\in\mathfrak{g}_{-1}$ [96]. Similarly, we shall extend the pullback map $(t^T)^*$ to act on all of $C^\infty(\mathfrak{G}^*[1])$ such that $(t^T)^*(F_0\oplus F_{-1})=(t^T)^*F_0$ for $F_0\in C^\infty(\mathfrak{g}_{-1}^*)$ and $F_{-1}\in C^\infty(\mathfrak{g}_0^*)$.

First, let us apply the t-map on $\mathfrak{G}$ to (4.2.2). This gives

$$t\dot{L}=t([L,P])=t[L_0,P_{-1}]+t[L_{-1},P_0]=[L_0,tP_{-1}]+[tL_{-1},P_0],$$

where we have used the equivariance of t. Considering this as an equation on the graded 3-term complex (4.2.1), we see that the term $[L_0,tP_{-1}]$ has total degree-(-1), while all the other terms have total degree-0. Therefore, we have

$$[L_0,tP_{-1}]=0,\qquad (\dot{tL_{-1}})=[tL_{-1},P_0].$$

[3] Note the degree convention from §3.2 is indeed such that $H_{-1}\in C^\infty(\mathfrak{g}_0^*)$.

We now apply $\mathbf{t}^* \equiv (t^T)^*$ to (4.2.2) and go through the same computation. We have

$$\mathbf{t}^*\dot{L} = \mathbf{t}^*(\{H, L\}^*_R) = \mathbf{t}^*\{H_0, L_{-1}\}^*_R + \mathbf{t}^*\{H_{-1}, L_0\}^*_R = \{\mathbf{t}^*H_0, L_{-1}\}^*_{R,0} + \{H_{-1}, \mathbf{t}^*L_0\}^*_{R,0},$$

where we have used the equivariance of $\mathbf{t}^*$ with respect to the 2-KK Poisson structure (4.2.14) (see also **Lemma 3.2.1**). We once again look at this equation within the graded complex (4.2.1), and see that $\{\mathbf{t}^*H_0, L_{-1}\}^*_0$ has total degree-(-1) while the other terms have degree-0,

$$\{\mathbf{t}^*H_0, L_{-1}\}^*_{R,0} = 0, \qquad (\dot{\mathbf{t}^*L_0}) = \{H_{-1}, \mathbf{t}^*L_0\}^*_{R,0}.$$

Theorem 4.2.2 allows us to define $\underline{L} = \mathbf{t}^*L_0 = tL_{-1}$. Hence we have

$$\dot{\underline{L}} = \{H_{-1}, \underline{L}\}^*_{R,0} = [\underline{L}, P_0],$$

completing the proof. □

Conversely, it is known [95] that a Lie bialgebra $\mathfrak{G}$ canonically gives rise to a Lie 2-bialgebra $\mathrm{id}_\mathfrak{g}$ given by $\mathrm{id}_\mathfrak{g} = \mathfrak{g} \xrightarrow{t=1} \mathfrak{g}$, where the dual t-map is the identity $t^T = \mathrm{id}$. Moreover, it is also known [115] that the 2-graded classical r-matrix $R = R_1 + R_2$ on $\mathrm{id}_\mathfrak{g}$ consist of two copies of the classical r-matrix $r = R_1 = R_2$ for $\mathfrak{g}$. Hence we immediately have the following.

Proposition 4.2.4. *Let $\mathfrak{g}$ be a Lie bialgebra. If $(\hat{L}, \hat{P}) : \mathfrak{g}^* \to \mathfrak{g}$ is a Lax pair on $\mathfrak{g}^*$, then the following graded functions $L = \hat{L} \oplus \hat{L}, P = \hat{P} \oplus \hat{P}$ consisting of two copies of the original Lax pair, is a 2-Lax pair on $\mathrm{id}_{\mathfrak{g}^*} = \mathrm{id}^*_\mathfrak{g}[1] : \mathfrak{g}^* \xrightarrow{\mathrm{id}} \mathfrak{g}^*$.*

These two above results show that our definition of the 2-Lax pair (4.2.2) is indeed a generalization (a categorification) of the usual Lax pair.

To conclude this Chapter, we briefly mention that a notion of a "2-Kac-Moody algebra" $\widehat{\Sigma_s\mathfrak{G}}$ was defined in my paper [116], which is a centrally-extended infinite-dimensional Lie 2-algebra living on a 2d surface Σ, such that the 2-Lax equations (4.2.2) valued in $\widehat{\Sigma_s\mathfrak{G}}$ can be written equivalently as a zero 2-curvature condition for a 2-connection A, B on $\Sigma \times \mathbb{R}$. I also showed that the topological-holomorphic 3d integrable field theory as derived in my paper [135] hosts higher-form currents that satisfy these 2-Lax equations, and is therefore an example of a physical theory that enjoys the above notion of 2-graded integrability.

Chapter 5

Hopf 2-Algebras: Quantization of Lie 2-bialgebras

In the previous Chapter, we have seen major applications of the semiclassical Lie 2-bialgebra symmetry to physical systems. They serve as motivation for the study of a *quantization* of these symmetry structures 'a la [101],

$$\begin{tikzcd} \text{Lie(/Poisson) algebra} \arrow[r, "\text{categorify}"] \arrow[d, "\text{quantize}"] & \text{(Poisson) } L_\infty\text{-algebras} \arrow[d, dashed, "\text{quantize}"] \\ \text{(Hopf) algebras} \arrow[r, "\text{categorify}"] & \text{(Hopf) } A_\infty\text{-algebras} \end{tikzcd} , \tag{5.0.1}$$

based on the ideas of deformation quantization (cf. [117, 7, 141]). Such Hopf A_n-algebras can be understood as the framed E_n-operads that fit into factorization algebras of Costello-Gwilliam [101].

In this Chapter, we give a proposal for the algebraic structure that captures such quantized, 2-term Hopf A_∞-objects, which we call **2-Hopf algebras** and **categorical quantum groups**. This Chapter is based on the work [119] and is the centrepiece of this PhD thesis.

There are two main theorems that we shall prove in this Chapter. The first one concerns the 2-representation theory of 2-Hopf algebras.

Theorem 5.0.1. *Let $(\mathcal{G}, \mathcal{T}; \delta, \Delta_1)$ denote a weak 2-bialgebra, and let $\mathcal{R} \in \mathcal{G}^{2\otimes}$. The weak 2-representation 2-category $2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$ is braided monoidal (à la Gurski [81, 78]) <u>iff</u> $\mathcal{R}$ is a **universal 2-R-matrix** of $\mathcal{G}$ satisfying the 2-Yang-Baxter equations (2YBE).*

The second one concerns the semiclassical limit of 2-Hopf algebras.

Theorem 5.0.2. *Let $\mathcal{L}$: $\mathsf{wk2Alg} \to \mathsf{wkLie2alg}$ denote the Lie-ification functor [93] taking (weak) 2-algebras to (weak) Lie 2-algebras.*

1. *$\mathcal{L}$ takes weak 2-biaglebras to a weak Lie 2-bialgebras, and*

2. *$\mathcal{L}$ takes solutions to the 2YBE to a solutions of the classical 2YBE (2CYBE).*

All the ingredients of these theorems will be defined explicitly in the following sections.

5.1 Strict 2-bialgebras

Quantum groups are Hopf algebras, hence we expect to define quantum 2-groups as "2-Hopf algebras". Different notions of 2-Hopf algebra have already been previously proposed in [92] and [93].

We begin with the following definition, and build up to the definition of an associative 2-algebra in [93].

Definition 5.1.1. Let $\mathcal{G}_0, \mathcal{G}_{-1}$ denote a pair of associative algebras. We say that $\mathcal{G}_{-1}$ is a $\mathcal{G}_0$**-bimodule** if we have a left and a right action[1] $\cdot_l, \cdot_r$ of $\mathcal{G}_0$ on $\mathcal{G}_{-1}$ which commute.

$$(x'x) \cdot y = x' \cdot (x \cdot y), \qquad (x \cdot y) \cdot x' = x \cdot (y \cdot x'), \quad y \cdot (xx') = (y \cdot x) \cdot x' \tag{5.1.1}$$

for all $y \in \mathcal{G}_{-1}$ and $x', x \in \mathcal{G}_0$.

Equivalently we can demand that the following diagrams are commutative. We note μ_i the multiplication in $\mathcal{G}_i$, $i = -1, 0$.

If we introduce a homomorphism t between $\mathcal{G}_{-1}$ and $\mathcal{G}_0$, subject to some conditions, then $\mathcal{G}_{-1}$ and $\mathcal{G}_0$ can be used to define a crossed module of algebras.

Definition 5.1.2. A crossed-module of (finite dimensional) associative algebras, $\mathcal{G}_0$, $\mathcal{G}_{-1}$, or an **associative 2-algebra**, is given by an algebra homomorphism $t : \mathcal{G}_{-1} \to \mathcal{G}_0$ where

1. $\mathcal{G}_{-1}$ is a $\mathcal{G}_0$-bimodule,
2. t is **two-sided** $\mathcal{G}_0$**-equivariant**,
$$t(x \cdot y) = xt(y), \qquad t(y \cdot x) = t(y)x \tag{5.1.2}$$
for all $y \in \mathcal{G}_{-1}, x \in \mathcal{G}_0$, and
3. the **Peiffer identity** is satisfied,
$$t(y) \cdot y' = yy' = y \cdot t(y'), \tag{5.1.3}$$
where $y, y' \in \mathcal{G}_{-1}$.

We call the latter two the *Peiffer conditions*. We denote an associative 2-algebra simply by $\mathcal{G}$, or by $(\mathcal{G}, \cdot)$ to emphasize the bimodule structure. Let k denote the ground ring of the 2-vector space underlying $\mathcal{G}$. We call $\mathcal{G}$ **unital** if there exists a unit map $\eta = (\eta_{-1}, \eta_0) : k \to \mathcal{G}$ such that

$$\eta_{-1}y = y\eta_{-1} = y, \quad \eta_0 x = x\eta_0 = x, \quad \eta_0 \cdot y = y \cdot \eta_0 = y, \tag{5.1.4}$$

for all $y \in \mathcal{G}_{-1}, x \in \mathcal{G}_0$. Moreover, t should respect the units such that $t(\eta_{-1}) = \eta_0$.

[1]We will often omit the subscript when there is no ambiguity.

Note that one may consider $\mathcal{G}_{-1}$ first as a vector space and *define* its product with the Peiffer identity. This notion is how one may show the bijective correspondence between Lie algebra crossed-modules and 2-term L_∞-algebras [95, 96]. However, in the skeletal case, since the Peiffer identity is empty, which forces the product on $\mathcal{G}_{-1}$ to be trivial.

Remark 5.1.1. If $t \neq 0$ were non-trivial then the Peiffer conditions, together with bimodularity, imply that

$$x \cdot (yy') = (x \cdot y)y', \qquad y(x \cdot y') = (y \cdot x)y', \qquad (yy') \cdot x = y(y' \cdot x)$$

for each $x \in \mathcal{G}_0, y, y' \in \mathcal{G}_{-1}$. This puts strong constraints on the algebra action $\cdot$, which is not necessarily imposed in the skeletal $t = 0$ case.

Classification of associative 2-algebras

A 2-algebra homomorphism $f = (f_{-1}, f_0) : \mathcal{G} \to \mathcal{G}'$ is a graded pair of algebra homomorphisms that respect the underlying bimodule structure, such that

1. $f_0 : \mathcal{G}_0 \to \mathcal{G}_0'$ and $f_1 : \mathcal{G}_{-1} \to \mathcal{G}_{-1}'$ are algebra homomorphisms,
2. $f_{-1}(x \cdot y) = (f_0 x) \cdot' (f_{-1} y)$ and $f_{-1}(y \cdot x) = (f_{-1} y) \cdot' (f_0 x)$ for each $x \in \mathcal{G}_0, y \in \mathcal{G}_{-1}$, and
3. $f_0 t = t' f_{-1}$.

We say that two 2-algebras are elementary equivalent, or quasi-isomorphic, if there exists an invertible 2-algebra homomorphism between them.

Theorem 5.1.1. (Gerstenhaber, attr. Wagemann [93]). *Associative 2-algebras are classified up to quasi-isomorphism by a degree-3* **Hochschild cohomology** *class* $\mathcal{T} \in HH^3(\mathcal{N}, V)$*, where* $\mathcal{N} = \operatorname{coker} t$ *and* $V = \ker t$.

See [93] for a definition of Hochschild cohomology of an algebra. The Peiffer identity implies that $V \subset Z(\mathcal{G}_{-1})$ is in the *nucleus* of $\mathcal{G}_{-1}$; it is in fact a square-free ideal [93]. Note the nucleus is *not* the same as the centre, which have commutative (but non-trivial) multiplication.

5.1.1 Associative 2-bialgebras

We seek a dual notion of an associative 2-algebra **Definition 5.1.2**. However, we must keep track of the degree-shift in our duality structure. This is a consequence of how "dualization" is defined in homological algebra [95, 115, 96, 114].

Coassociative 2-coalgebra. Let us consider a pair of vector spaces, $\mathcal{G}_0, \mathcal{G}_{-1}$ with the map $t : \mathcal{G}_{-1} \to \mathcal{G}_0$. In direct analogy with the 2-cocycle $\delta = \delta_{-1} + \delta_0$ that were introduced to define a classical Lie 2-bialgebra [95, 115], we introduce the *coproduct* maps

$$\Delta_{-1} : \mathcal{G}_{-1} \to \mathcal{G}_{-1} \otimes \mathcal{G}_{-1}, \qquad \Delta_0 : \mathcal{G}_0 \to (\mathcal{G}_{-1} \otimes \mathcal{G}_0) \oplus (\mathcal{G}_0 \otimes \mathcal{G}_{-1}). \tag{5.1.5}$$

Note that Δ_0 comes in two components, $\Delta_0 = \Delta_0^l + \Delta_0^r$ (we used the graded sum) with

$$\Delta_0^l : \mathcal{G}_0 \to \mathcal{G}_{-1} \otimes \mathcal{G}_0, \qquad \Delta_0^r : \mathcal{G}_0 \to \mathcal{G}_0 \otimes \mathcal{G}_{-1}.$$

In the following, we shall use extensively the conventional Sweedler notation

$$\Delta(y,x) \equiv \Delta_{-1}(y) + \Delta_0(x) = y_{(1)} \otimes y_{(2)} + (x^l_{(1)} \otimes x^l_{(2)} + x^r_{(1)} \otimes x^r_{(2)}) \tag{5.1.6}$$

where $x^l_{(1)}, x^r_{(2)} \in \mathcal{G}_0$ and $y_{(1)}, y_{(2)}, x^l_{(2)}, x^r_{(1)} \in \mathcal{G}_{-1}$.

Now let

$$\Delta_0' : \mathcal{G}_0 \to \mathcal{G}_0 \otimes \mathcal{G}_0$$

denote a coproduct in degree-0, such that Δ_{-1}, Δ_0' are subject to the following **coassociativity** conditions

$$(\mathrm{id} \otimes \Delta_{-1}) \circ \Delta_{-1} = (\Delta_{-1} \otimes \mathrm{id}) \circ \Delta_{-1}, \qquad (\mathrm{id} \otimes \Delta_0') \circ \Delta_0' = (\Delta_0' \otimes \mathrm{id}) \circ \Delta_0'. \tag{5.1.7}$$

$(\mathcal{G}_{-1}, \Delta_{-1})$ and $(\mathcal{G}_0, \Delta_0')$ are *coassociative coalgebras* if (5.1.7) is satisfied [117]. In the following, we shall use the Sweedler notation

$$\Delta_0'(x) = \bar{x}_{(1)} \otimes \bar{x}_{(2)} \in \mathcal{G}_0 \otimes \mathcal{G}_0. \tag{5.1.8}$$

Definition 5.1.3. Let $(\mathcal{G}_{-1}, \Delta_{-1})$ and $(\mathcal{G}_0, \Delta_0')$ denote a pair of coassociative coalgebras with the coactions Δ_0^l and Δ_0^r. We say that $\mathcal{G}_0$ forms a $\mathcal{G}_{-1}$**-cobimodule** if the following **cobimodularity conditions**

$$\begin{aligned}
(\Delta_{-1} \otimes \mathrm{id}) \circ \Delta_0^l &= (\mathrm{id} \otimes \Delta_0^l) \circ \Delta_0^l, \\
(\mathrm{id} \otimes \Delta_{-1}) \circ \Delta_0^r &= (\Delta_0^r \otimes \mathrm{id}) \circ \Delta_0^r, \\
(\mathrm{id} \otimes \Delta_0^r) \circ \Delta_0^l &= (\Delta_0^l \otimes \mathrm{id}) \circ \Delta_0^r
\end{aligned} \tag{5.1.9}$$

are satisfied.

Definition 5.1.4. A **coassociative 2-coalgebra** $(\mathcal{G}, \Delta)$ is a coalgebra homomorphism $t : \mathcal{G}_{-1} \to \mathcal{G}_0$ such that

1. $\mathcal{G}_0$ is a $\mathcal{G}_{-1}$-cobimodule,
2. t is **coequivariant**

$$D_t^+ \circ \Delta_{-1} = \Delta_0 \circ t, \tag{5.1.10}$$

where we have introduced a convenient tensor notation for the induced t-map

$$D_t^{\pm} := t \otimes 1 \pm 1 \otimes t$$

in terms of the graded sum.

3. the **coPeiffer identity**

$$(t\otimes\mathrm{id})\circ\Delta_0^l=\Delta_0'=(\mathrm{id}\otimes t)\circ\Delta_0^r, \tag{5.1.11}$$

which in particular means that we must necessarily have

$$D_t^-\Delta_0=(t\otimes\mathrm{id})\circ\Delta_0^l-(\mathrm{id}\otimes t)\circ\Delta_0^r=0.$$

We call $(\mathcal{G},\Delta)$ **counital** if there is a counit map $\epsilon=(\epsilon_{-1},\epsilon_0):\mathcal{G}\to k$ such that

$$\begin{aligned} \mathrm{id}&=(\mathrm{id}\otimes\epsilon_{-1})\circ\Delta_{-1}, &\qquad \mathrm{id}&=(\epsilon_{-1}\otimes\mathrm{id})\circ\Delta_{-1},\\ \mathrm{id}&=(\epsilon_{-1}\otimes\mathrm{id})\circ\Delta_0^l, &\qquad \mathrm{id}&=(\mathrm{id}\otimes\epsilon_{-1})\circ\Delta_0^r. \end{aligned} \tag{5.1.12}$$

Moreover, ϵ should respect the t-map such that $\epsilon_0=\epsilon_{-1}\circ t$.

Note again that in **Definition 5.1.4**, the coequivariance and coPeiffer identity are treated as constraints between two coalgebras and the coalgebra homomorphism t between them. With these constraints, we can deduce

$$\mathrm{id}=(\epsilon_0\otimes\mathrm{id})\circ\Delta_0'=(\mathrm{id}\otimes\epsilon_0)\circ\Delta_0' \tag{5.1.13}$$

from (5.1.8) and (5.1.12). In the skeletal $t=0$ case, the coproducts $\Delta_{-1},\Delta_0,\Delta_0'$ and the counits ϵ_{-1},ϵ_0 are independent, and this condition is separate from (5.1.12).

Remark 5.1.2. Similar to the 2-algebra case, if $t\neq 0$ were not trivial, then we could have the following conditions

$$\begin{aligned} (\mathrm{id}\otimes\Delta_0')\circ\Delta_0^l&=(\Delta_0^l\otimes\mathrm{id})\circ\Delta_0',\\ (\Delta_0'\otimes\mathrm{id})\circ\Delta_0^r&=(\mathrm{id}\otimes\Delta_0^r)\circ\Delta_0',\\ (\mathrm{id}\otimes\Delta_0^l)\circ\Delta_0'&=(\Delta_0^r\otimes\mathrm{id})\circ\Delta_0' \end{aligned} \tag{5.1.14}$$

between the coproducts Δ_0 and Δ_0'. By making use of the Sweedler notation (5.1.6), (5.1.8), these conditions translate to

$$\begin{cases} ty_{(1)}=(ty)^r_{(1)}\\ ty_{(2)}=(ty)^l_{(2)}\end{cases},\qquad \begin{cases}\bar{x}_{(1)}=tx^l_{(1)}=x^r_{(1)}\\ \bar{x}_{(2)}=x^l_{(2)}=tx^r_{(2)}\end{cases}. \tag{5.1.15}$$

When combined, they give $\bar{ty}_{(1)}=ty_{(1)},\bar{ty}_{(2)}=ty_{(2)}$ which will become important later. In the skeletal case, the constraints involving t drop and we would only have $\bar{x}_{(1)}=x^r_{(1)}$, $\bar{x}_{(2)}=x^l_{(2)}$.

2-bialgebra. Using the Sweedler notations (5.1.6), (5.1.8), we state the condition that the coproduct map Δ given in (5.1.5) preserves the algebra/bimodule structure:

$$\Delta_{-1}(x\cdot y)=\bar{x}_{(1)}\cdot y_{(1)}\otimes\bar{x}_{(2)}\cdot y_{(2)},\qquad \Delta_{-1}(y\cdot x)=y_{(1)}\cdot\bar{x}_{(1)}\otimes y_{(2)}\cdot\bar{x}_{(2)},$$

$$\Delta_0^l(xx') = x^l_{(1)}x'^l_{(1)} \otimes x^l_{(2)}x'^l_{(2)}, \qquad \Delta_0^r(xx') = x^r_{(1)}x'^r_{(1)} \otimes x^r_{(2)}x'^r_{(2)}. \tag{5.1.16}$$

We call these conditions the **2-bialgebra axioms**.

The bialgebra axioms in each degree,

$$\Delta_{-1}(yy') = y_{(1)}y'_{(1)} \otimes y_{(2)}y'_{(2)}, \qquad \Delta_0'(xx') = \bar{x}_{(1)}\bar{x}'_{(1)} \otimes \bar{x}_{(2)}\bar{x}'_{(2)},$$

follow directly from (5.1.16) and the coequivariance and coPeiffer identities (5.1.10), (5.1.11); see *Remark 5.1.2*.

Definition 5.1.5. The tuple $(\mathcal{G}, \cdot, \Delta)$ is an **associative 2-bialgebra** iff $(\mathcal{G}, \cdot)$ is an associative 2-algebra and $(\mathcal{G}, \Delta)$ is a coassociative 2-coalgebra that are mutually compatible, in the sense that the coproduct map Δ satisfies (5.1.7)-(5.1.11) and (5.1.16).

We call $(\mathcal{G}, \cdot, \eta, \Delta, \epsilon)$ **unital** if $(\mathcal{G}, \cdot, \eta)$ and $(\mathcal{G}, \Delta, \epsilon)$ are respectively unital and counital.

5.2 Strict quantum 2-doubles and the universal 2-R-matrix

In this section, we construct our main example of a strict 2-bialgebra given by the strict quantum 2-doubles which can be seen a categorification of the standard quantum double [117], and the quantization of a classical 2-double [96, 115] of Lie 2-algebras.

The goal for studying (2-)quantum doubles is that, for the ordinary 1-bialgebra H, the *skew-pairing* involved in the construction of the quantum double $D(H,H)$ of Majid [118] provides a characterization of R-matrices on H. Moreover, this construction is *universal* in the sense that any R-matrix on H can be derived this way from $D(H,H)$. We wish to directly categorify Majid's construction, and derive a universal characterization of *2-R-matrices* from our construction of a quantum 2-double.

Our strategy will be as follows. Firstly, we consider a pair of dual associative 2-bialgebras. They are dual in the sense that the coalgebra sector is given by the algebra sector of its dual counterpart. We then define a notion of a canonical *coadjoint action* of a 2-bialgebra on its dual. By requesting that the mutually-dual 2-bialgebras act on each other by such coadjoint actions, we are then able to form the quantum 2-double as a 2-bialgebra. We will then also prove a key factorization theorem for quantum 2-doubles.

5.2.1 Matched pair of 2-(bi)algebras

Dually paired 2-bialgebras Let $(\mathcal{G}, \cdot, \Delta)$ denote a (finite dimensional) 2-bialgebra, and let $\mathcal{G}^*$ denote its linear dual, defined with respect to the following duality evaluation/pairing map[2]

$$\langle (g,f), (y,x)\rangle = \langle f,y\rangle_{-1} + \langle g,x\rangle_0 \tag{5.2.1}$$

[2]We shall drop the subscripts on the pairing forms $\langle -,-\rangle$ when no confusion arises.

for each $x \in \mathcal{G}_0, y \in \mathcal{G}_{-1}, f \in \mathcal{G}_{-1}^*, g \in \mathcal{G}_0^*$. Note that the grading is flipped by dualizing the t-map: $\langle t^* \cdot, - \rangle = \langle -, t- \rangle$, whence $t^* : \mathcal{G}_0^* \to \mathcal{G}_{-1}^*$ and $\mathcal{G}^*$ is skeletal whenever $\mathcal{G}$ is. In the following, we shall denote this pairing also by an evaluation ev.

So far, $\mathcal{G}^*$ merely forms a 2-vector space. By leveraging the duality (5.2.1), we can induce algebraic structures on $\mathcal{G}^*$ according to the *co*algebraic structures (5.1.5), (5.1.8) on $\mathcal{G}$ as follows:

$$\begin{aligned} \langle f \otimes f', \Delta_{-1}(y) \rangle &= \langle ff', y \rangle, & \langle g \otimes g', \Delta_0'(x) \rangle &= \langle gg', x \rangle, \\ \langle f \otimes g, \Delta_0^l(x) \rangle &= \langle f \cdot^* g, x \rangle, & \langle g \otimes f, \Delta_0^r(x) \rangle &= \langle g \cdot^* f, x \rangle, \\ \langle \Delta_0^{*\prime} f, y \otimes y' \rangle &= \langle f, yy' \rangle, & \langle \Delta_{-1}^* g, x \otimes x' \rangle &= \langle g, xx' \rangle, \\ \langle \Delta_0^{*r} f, x \otimes y \rangle &= \langle f, x \cdot_l y \rangle, & \langle \Delta_0^{*l} f, y \otimes x \rangle &= \langle f, y \cdot_r x \rangle. \end{aligned}$$

The conditions (5.1.10), (5.1.11), (5.1.7), (5.1.9), then ensure that $(\mathcal{G}^*, \cdot^*)$ forms an associative 2-algebra. More is true, in fact, which we now prove in the following.

Proposition 5.2.1. *Let $\mathcal{G}, \mathcal{G}^*$ be dually paired as in (5.2.1), then $(\mathcal{G}, \cdot, \Delta)$ is an (unital) associative 2-bialgebra iff $(\mathcal{G}^*, \cdot^*, \Delta^*)$ is an (unital) associative 2-bialgebra.*

Proof. This is a straightforward computation using the pairing (5.2.1). In particular, the equivariance and Peiffer identity of t^*, as well as the fact that $\mathcal{G}_{-1}^*$ forms a $\mathcal{G}_0^*$-bimodule, follow directly from dualizing (5.1.10), (5.1.11), (5.1.7), (5.1.9).

What is non-trivial is (5.1.16). Define Δ_0^* by dualizing the bimodule structure $\cdot$ of $\mathcal{G}$, then we have

$$\begin{aligned} \langle (\Delta_0^*)^l(ff'), x \otimes y \rangle &= \langle f \otimes f', \Delta_{-1}(x \cdot y) \rangle, & \langle (\Delta_0^*)^r(ff'), y \otimes x \rangle &= \langle f \otimes f', \Delta_{-1}(y \cdot x) \rangle, \\ \langle (\Delta_{-1}^*)(f \cdot^* g), x \otimes x' \rangle &= \langle f \otimes g, \Delta_0^l(xx') \rangle, & \langle (\Delta_{-1}^*)(g \cdot^* f), x \otimes x' \rangle &= \langle f \otimes g, \Delta_0^r(xx') \rangle. \end{aligned}$$

We now compute using analogues of (5.1.16) for Δ^*, that

$$\begin{aligned} \langle f_{(1)}^l f_{(1)}'^l \otimes f_{(2)}^l f_{(2)}'^l, x \otimes y \rangle &= \langle (f_{(1)}^l \otimes f_{(1)}'^l) \otimes (f_{(2)}^l \otimes f_{(2)}'^l), (\bar{x}_{(1)} \otimes \bar{x}_{(2)}) \otimes (y_{(1)} \otimes y_{(2)}) \rangle \\ &= \langle (\Delta_0^*)^l(f) \otimes (\Delta_0^*)^l(f'), (\bar{x}_{(1)} \otimes y_{(1)}) \otimes (\bar{x}_{(2)} \otimes y_{(2)}) \rangle \\ &= \langle f \otimes f', (\bar{x}_{(1)} \cdot y_{(1)}) \otimes (\bar{x}_{(2)} \cdot y_{(2)}) \rangle, \\ \langle f_{(1)}^r f_{(1)}'^r \otimes f_{(2)}^r f_{(2)}'^r, y \otimes x \rangle &= \langle f \otimes f', (y_{(1)} \cdot \bar{x}_{(1)}) \otimes (y_{(2)} \cdot \bar{x}_{(2)}) \rangle, \end{aligned}$$

and similarly

$$\begin{aligned} \langle \bar{f}_{(1)} \cdot^* g_{(1)} \otimes \bar{f}_{(2)} \cdot g_{(2)}, x \otimes x' \rangle &= \langle (\bar{f}_{(1)} \otimes g_{(1)}) \otimes (\bar{f}_{(2)} \otimes g_{(2)}), (x_{(1)}^l \otimes x_{(2)}^l) \otimes (x_{(1)}'^l \otimes x_{(2)}'^l) \rangle \\ &= \langle \Delta_0'^*(f) \otimes \Delta_{-1}^*(g), (x_{(1)}^l \otimes x_{(1)}'^l) \otimes (x_{(2)}^l \otimes x_{(2)}'^l) \rangle, \\ &= \langle f \otimes g, x_{(1)}^l x_{(1)}'^l \otimes x_{(2)}^l x_{(2)}'^l \rangle, \\ \langle g_{(1)} \cdot^* \bar{f}_{(1)} \otimes g_{(2)} \cdot^* \bar{f}_{(2)}, x \otimes x' \rangle &= \langle g \otimes f, x_{(1)}^r x_{(1)}'^r \otimes x_{(2)}^r x_{(2)}'^r \rangle, \end{aligned}$$

hence Δ also satisfies (5.1.16). This proves that $(\mathcal{G}^*, \cdot^*, \Delta^*)$ is an associative 2-bialgebra iff $(\mathcal{G}, \cdot, \Delta)$ also is.

Now consider the units and counits. Given

$$\begin{aligned} \langle g, \eta x\rangle &= \langle (\eta^*\otimes \mathrm{id})\circ \Delta_{-1}^*(g), x\rangle, & \langle g, x\eta\rangle &= \langle (\mathrm{id}\otimes \eta^*)\circ \Delta_{-1}^*(g), x\rangle, \\ \langle f, \eta\cdot y\rangle &= \langle (\eta^*\otimes \mathrm{id})\circ (\Delta_0^*)^l(f), y\rangle, & \langle f, y\cdot\eta\rangle &= \langle (\mathrm{id}\otimes \eta^*)\circ (\Delta_0^*)^r(f), y\rangle, \end{aligned}$$

we see that η is a unit for $(\mathcal{G},\cdot)$ (ie. these quantities all vanish) iff η^* is a counit for $(\mathcal{G}^*,\Delta^*)$. Similarly, ϵ is a counit for $(\mathcal{G},\Delta)$ iff ϵ^* is a unit for $(\mathcal{G}^*,\cdot^*)$. The converse direction is identical. $\square$

Coadjoint action.

Definition 5.2.1. The canonical **coadjoint action** of $\mathcal{G}$ on $\mathcal{G}^*$ is specified in terms of three components, $\bar{\rhd} = ((\rhd_0,\rhd_{-1}),\Upsilon)$ given by

$$\begin{aligned} \rhd_0 : \mathcal{G}_0 &\to \operatorname{End}\mathcal{G}_0^*, & \langle g, xx'\rangle &= -\langle x\rhd_0 g, x'\rangle, \\ \rhd_{-1} : \mathcal{G}_0 &\to \operatorname{End}\mathcal{G}_{-1}^*, & \langle f, x\cdot y\rangle &= -\langle x\rhd_{-1} f, y\rangle, \\ \Upsilon : \mathcal{G}_{-1} &\to \operatorname{Hom}(\mathcal{G}_{-1}^*,\mathcal{G}_0^*), & \langle f, y\cdot x\rangle &= -\langle \Upsilon_y f, x\rangle. \end{aligned} \tag{5.2.2}$$

As we will see when discussing 2-representations in §5.5, the coadjoint action can also be interpreted as a 2-representation.

Analogously, we have the coadjoint back-action $\bar{\lhd} = ((\lhd_0,\lhd_{-1}),\tilde{\Upsilon})$ of $\mathcal{G}^*$ on $\mathcal{G}$, which we write from the right[3]. The "bar" notation is used to distinguish $\bar{\rhd}$ from the group action $\rhd$ in the case where $\mathcal{G} = kG$ is defined through a 2-group G.

Matched pair. Given the pair of strict 2-bialgebras $(\mathcal{G},\mathcal{G}^*)$, we allow them to act upon each other by coadjoint actions $\bar{\rhd}$ and $\bar{\lhd}$. In analogy with [117], we impose the following monstrous set of twelve compatibility conditions

$$\begin{aligned} x\rhd_{-1}(ff') &= (tx^l_{(1)}\rhd_0 f^l_{(1)})\cdot^*((x^l_{(2)}\lhd_{-1} f^l_{(2)})\rhd_{-1} f') + (x^r_{(1)}\rhd_0 f^l_{(1)})\cdot^*(\Upsilon_{x^r_{(2)}\lhd_0 f^l_{(2)}} f') \\ &\quad + (\Upsilon_{x^l_{(1)}} f^r_{(1)})\cdot^*(\Upsilon_{x^l_{(2)}\tilde{\Upsilon}_{f^r_{(2)}}} f') + (x^r_{(1)}\rhd_{-1} f^r_{(1)})\cdot^*(\Upsilon_{x^r_{(2)}\lhd_0(t^* f^r_{(2)})} f'), \\ \Upsilon_y(ff') &= ((ty_{(1)})\rhd_0 f^l_{(1)})\cdot^*(\Upsilon_{y_{(2)}\lhd_0 f^l_{(2)}} f') + (\Upsilon_{y_{(1)}} f^r_{(1)})\cdot^*(\Upsilon_{y_{(2)}\lhd_0(t^* f^r_{(2)})} f'), \\ x\rhd_0(f\cdot^* g) &= (tx^l_{(1)}\rhd_0 f^l_{(1)})\cdot^*((x^l_{(2)}\lhd_{-1} f^l_{(2)})\rhd_0 g) + (x^r_{(1)}\rhd_0 f^l_{(1)})\cdot^*(t(x^r_{(2)}\lhd_0 f^l_{(2)})\rhd_0 g) \\ &\quad + (\Upsilon_{x^l_{(1)}} f^r_{(1)})\cdot^*(t(x^l_{(2)}\tilde{\Upsilon}_{f^r_{(2)}})\rhd_0 g) + (x^r_{(1)}\rhd_{-1} f^r_{(1)})\cdot^*(t(x^r_{(2)}\lhd_0(t^* f^r_{(2)}))\rhd_0 g), \\ ty\rhd_0(f\cdot^* g) &= (ty_{(1)}\rhd_0 f^l_{(1)})\cdot^*(t(y_{(2)}\lhd_0 f^l_{(2)})\rhd_0 g) + (\Upsilon_{y_{(1)}} f^r_{(1)})\cdot^*(t(y_{(2)}\lhd_0(t^* f^r_{(2)}))\rhd_0 g), \\ x\rhd_0(g\cdot^* f) &= (tx^l_{(1)}\rhd_0 g_{(1)})\cdot^*(\Upsilon_{x^l_{(2)}\tilde{\Upsilon}_{g_{(2)}}} f) + (x^r_{(1)}\rhd_0 g_{(1)})\cdot^*(\Upsilon_{x^r_{(2)}\lhd_0(t^* g_{(2)})} f) \\ ty\rhd_0(g\cdot^* f) &= (ty_{(1)}\rhd_0 g_{(1)})\cdot^*(\Upsilon_{y_{(2)}\lhd_0(t^* g_{(2)})} f), \\ (xx')\lhd_{-1} f &= (x\tilde{\Upsilon}_{tx'^l_{(1)}\rhd_0 f^l_{(1)}})\cdot(x'^l_{(2)}\lhd_{-1} f^l_{(2)}) + (x\tilde{\Upsilon}_{x'^r_{(1)}\rhd_0 f^l_{(1)}})\cdot(x'^r_{(2)}\lhd_0 f^l_{(2)}) \\ &\quad + (x\tilde{\Upsilon}_{\Upsilon_{x'^l_{(1)}} f^r_{(1)}})\cdot(x'^l_{(2)}\tilde{\Upsilon}_{f^r_{(2)}}) + (x\lhd_{-1}(x'^r_{(1)}\rhd_{-1} f^r_{(1)}))\cdot(x'^r_{(2)}\lhd_0(t^* f^r_{(2)})), \end{aligned}$$

[3]This means that we have, for instance, $\langle g\cdot^* f, x\rangle = -\langle g, x\lhd_{-1} f\rangle$ and $\langle f\cdot^* g, x\rangle = -\langle f, x\tilde{\Upsilon}_g\rangle$.

$$
\begin{aligned}
(xx')\tilde{\Upsilon}_g &= (x\tilde{\Upsilon}_{tx'^l_{(1)}\rhd_0 g_{(1)}})\cdot(x'^l_{(2)}\tilde{\Upsilon}_{g_{(2)}}) + (x\tilde{\Upsilon}_{x'^r_{(1)}\rhd_0 g_{(1)}})\cdot(x'^r_{(2)}\lhd_0(t^*g_{(2)})),\\
(y\cdot x)\lhd_0 f &= (y\lhd_0 t^*(tx^l_{(1)}\rhd_0 f^l_{(1)}))\cdot(x^l_{(2)}\lhd_{-1} f^l_{(2)}) + (y\lhd_0 t^*(x^r_{(1)}\rhd_0 f^l_{(1)}))\cdot(x^r_{(2)}\lhd_0 f^l_{(2)})\\
&\quad + (y\lhd_0 t^*(\Upsilon_{x^l_{(1)}} f^r_{(1)}))\cdot(x^l_{(2)}\tilde{\Upsilon}_{f^r_{(2)}}) + (y\lhd_0(x^r_{(1)}\rhd_{-1} f^r_{(1)}))\cdot(x^r_{(2)}\lhd_0(t^*f^r_{(2)})),\\
(y\cdot x)\lhd_0 t^*g &= (y\lhd_0 t^*(tx^l_{(1)}\rhd_0 g_{(1)}))\cdot(x^l_{(2)}\tilde{\Upsilon}_{g_{(2)}}) + (y\lhd_0 t^*(x^r_{(1)}\rhd_0 g_{(1)}))\cdot(x^r_{(2)}\lhd_0 t{*}^{g}_{(2)}),\\
(x\cdot y)\lhd_0 f &= (x\tilde{\Upsilon}_{ty_{(1)}\rhd_0 f^l_{(1)}})\cdot(y_{(2)}\lhd_0 f^l_{(2)}) + (x\tilde{\Upsilon}_{\Upsilon_{y_{(1)}} f^r_{(1)}})\cdot(y_{(2)}\lhd_0(t^*f^r_{(2)})),\\
(x\cdot y)\lhd_0 t^*g &= (x\tilde{\Upsilon}_{ty_{(1)}\rhd_0 g_{(1)}})\cdot(y_{(2)}\lhd_0 t^*g_{(2)}),
\end{aligned}
$$

where we have made use of the Sweedler notation (5.1.6).

We define a shorthand notation where $z=(y,x)\in\mathcal{G}$, $h=(g,f)\in\mathcal{G}^*$, such that the following

$$
\begin{aligned}
z\bar{\rhd}(h\cdot^* h') &= (z_{(1)}\bar{\rhd}h_{(1)})\cdot^*((z_{(2)}\bar{\lhd}h_{(2)})\bar{\rhd}h'), &(5.2.3)\\
(z\cdot z')\bar{\lhd}h &= (z\bar{\lhd}(z'_{(1)}\bar{\rhd}h_{(1)}))\cdot(z'_{(2)}\bar{\lhd}h_{(2)}) &(5.2.4)
\end{aligned}
$$

encode respectively the first six and last six of the above conditions. We also have the cross relations

$$
z_{(1)}\bar{\lhd}h_{(1)}\otimes z_{(2)}\bar{\rhd}h_{(2)} = z_{(2)}\bar{\lhd}h_{(2)}\otimes z_{(1)}\bar{\rhd}h_{(1)}, \tag{5.2.5}
$$

as well as the unity axioms against the unit η and counit ϵ,

$$
z\bar{\rhd}\eta=\epsilon(z),\qquad \eta\bar{\lhd}h=\epsilon(h). \tag{5.2.6}
$$

Definition 5.2.2. We call a tuple $(\mathcal{G},\mathcal{G}^*)$ of (finite dimensional) 2-bialgebras satisfying (5.2.3)-(5.2.6) a **matched pair**.

Remark 5.2.1. Note that in the skeletal case $t,t^*=0$, the crossed relations (5.2.3), (5.2.4) reduce to just two non-trivial equations. These are given by

$$
\begin{aligned}
x\rhd_{-1}(ff') &= (x^r_{(1)}\rhd_0 f^l_{(1)})\cdot^*(\Upsilon_{x^r_{(2)}\lhd_0 f^l_{(2)}} f') + (\Upsilon_{x^l_{(1)}} f^r_{(1)})\cdot^*(\Upsilon_{x^l_{(2)}\tilde{\Upsilon}_{f^r_{(2)}}} f')\\
&\equiv (x_{(1)}\bar{\rhd}f_{(1)})\cdot^*((x_{(2)}\bar{\lhd}f_{(2)})\bar{\rhd}f'),\\
(xx')\lhd_{-1} f &= (x\tilde{\Upsilon}_{x'^r_{(1)}\rhd_0 f^l_{(1)}})\cdot(x'^r_{(2)}\lhd_0 f^l_{(2)}) + (x\tilde{\Upsilon}_{\Upsilon_{x'^l_{(1)}} f^r_{(1)}})\cdot(x'^l_{(2)}\tilde{\Upsilon}_{f^r_{(2)}})\\
&\equiv (x\bar{\lhd}(x'_{(1)}\bar{\rhd}f_{(1)}))\cdot(x_{(2)}\bar{\lhd}f_{(2)}),
\end{aligned} \tag{5.2.7}
$$

where we have used a convenient notation for brevity. One may notice that these are precisely the usual crossed relations for a quantum double group (cf. [117]) of a semidirect product 2-bialgebra $\mathcal{G}_{-1}\rtimes\mathcal{G}_0$, where $\mathcal{G}_{-1}$ is nuclear.

5.2.2 Construction of the strict quantum 2-double

We now begin our construction of the general quantum 2-double given a matched pair $(\mathcal{G},\mathcal{G}^*)$. We shall explicitly construct its 2-bialgebra structure such that its self-duality is manifest.

2-algebra structure. We consider $D(\mathcal{G})$ defined in terms of the graded components given by

$$D(\mathcal{G})_0 \cong \mathcal{G}_0 \otimes \mathcal{G}_{-1}^* \ni (x, f), \qquad D(\mathcal{G})_{-1} \cong \mathcal{G}_{-1} \otimes \mathcal{G}_0^* \ni (y, g),$$

for which we have a "right-moving" semidirect product $\overrightarrow{\ltimes} = (\cdot, \bar{\rhd})$, giving rise to $D(\mathcal{G})_{-1} \overrightarrow{\rtimes} D(\mathcal{G})_0$. Similarly, we also have a "left-moving" semidirect product $\overleftarrow{\ltimes} = (\cdot^*, \bar{\lhd})$ giving rise to $D(\mathcal{G})_{-1} \overleftarrow{\ltimes} D(\mathcal{G})_0$. The combined t-map $T = t \otimes t^*$ is equivariant with respect to these semidirect products

$$t^*(x \rhd_0 g) = x \rhd_{-1} t^* g, \qquad t(y \lhd_0 f) = (ty) \lhd_{-1} f, \tag{5.2.8}$$

since the coadjoint action is 2-representation, while the commutativity $\rhd \circ t = (-t^*, t^* -) \circ \Upsilon$ implies

$$\begin{aligned} (ty) \rhd_0 g &= \Upsilon_y(t^* g), & y \lhd_0 (t^* g) &= (ty)\tilde{\Upsilon}_g, \\ (ty) \rhd_{-1} f &= t^*(\Upsilon_y f), & x \lhd_{-1} (t^* g) &= t(x\tilde{\Upsilon}_g). \end{aligned} \tag{5.2.9}$$

These are in fact generalizations of the Peiffer identity.

Proposition 5.2.2. *If $\bar{\rhd}, \bar{\lhd}$ are given by the coadjoint representations (see (5.5.4)), then (5.2.9) reproduces the Peiffer identity.*

Proof. This is a direct computation. By the equality in the second row of (5.2.9), we have

$$\langle f, y \cdot ty' \rangle = -\langle t^* \Upsilon_y f, y' \rangle = -\langle (ty) \rhd_{-1} f, y' \rangle = \langle f, ty \cdot y' \rangle,$$

giving $ty \cdot y' = y \cdot ty'$. Now by the fact that t is an algebra homomorphism, we have

$$\begin{aligned} \langle (ty) \rhd_0 g, ty' \rangle &= -\langle g, (ty)(ty') \rangle = -\langle g, t(yy') \rangle, \\ \langle \Upsilon_y(t^* g), ty' \rangle &= -\langle t^* g, y \cdot ty' \rangle = -\langle g, t(y \cdot ty') \rangle, \end{aligned}$$

for which the first row of (5.2.9) states $yy' = y \cdot ty'$. Altogether yields

$$yy' = y \cdot t(y') = t(y) \cdot y'$$

for any $y, y' \in \mathcal{G}_{-1}$, which is precisely the Peiffer identity on $\mathcal{G}$. Similarly, if $\bar{\lhd}$ is the coadjoint representation then (5.2.9) reproduces the Peiffer identity on $\mathcal{G}^*$. □

In other words, the Peiffer identity in $D(\mathcal{G})$ is *by definition* given as in (5.2.9). The multiplication between the sectors $\mathcal{G}_{-1}, \mathcal{G}_{-1}^*$ is given by $yg = \Upsilon_y(t^* g)$ and $gy = (ty)\tilde{\Upsilon}_g$.

Now that we have defined the product of the graded components and the t-map associated to $D(\mathcal{G})$, we can identify the bimodule structure.

We combine the right-moving $\overrightarrow{\ltimes} = (\cdot, \bar{\rhd})$ and left-moving $\overleftarrow{\ltimes} = (\cdot^*, \bar{\lhd})$ multiplications on $D(\mathcal{G})$ to form $\hat{\cdot} = \overrightarrow{\ltimes} + \overleftarrow{\ltimes}$,

$$(z, h)\hat{\cdot}(z', h') = (z \cdot z' + z \bar{\lhd} h' + z' \bar{\lhd} h, h \cdot h' + z \bar{\rhd} h' + z' \bar{\rhd} h), \quad z, z' \in \mathcal{G},\ h, h' \in \mathcal{G}^*. \tag{5.2.10}$$

Since $\hat{\cdot}$ is a combination of the internal 2-algebra structures of $\mathcal{G}, \mathcal{G}^*$ and the 2-representations $\bar{\rhd}, \bar{\lhd}$, we have respectively the Peiffer conditions and associativity for $\mathcal{G}, \mathcal{G}^*$, as well as the 2-representation properties (5.2.8), (5.2.9) and the matched pair conditions (5.2.3), (5.2.4), (5.2.6). These imply that the map $\hat{\cdot}$

(i) is associative,

(ii) makes $D(\mathcal{G})_{-1}$ into a $D(\mathcal{G})_0$-bimodule,

(iii) satisfies the Peiffer conditions under $T = t \otimes t^*$.

Hence $(D(\mathcal{G}), \hat{\cdot}\,)$ is a 2-algebra.

2-coalgebra structure. We intend now to construct the coproduct $\Delta_D : D(\mathcal{G}) \to D(\mathcal{G})^{2\otimes}$. We have to build the components

$$\begin{aligned} \Delta_{D-1} \;&:\; D(\mathcal{G})_{-1} \to D(\mathcal{G})_{-1} \otimes D(\mathcal{G})_{-1} = (\mathcal{G}_{-1} \otimes \mathcal{G}_0^*) \otimes (\mathcal{G}_{-1} \otimes \mathcal{G}_0^*) \\ \Delta_{D0} \;&:\; D(\mathcal{G})_0 \to (D(\mathcal{G})_{-1} \otimes D(\mathcal{G})_0) \oplus (D(\mathcal{G})_0 \otimes D(\mathcal{G})_{-1}) \end{aligned}$$

We can directly infer some of the components Δ_{D-1} from the coproducts $\Delta_{-1}, \Delta_{-1}^*$ of $\mathcal{G}, \mathcal{G}^*$. Explicitly, it is defined as

$$\Delta_{D}{}^{d}_{-1} \;=\; \Delta_{-1} \otimes \Delta_{-1}^*.$$

This coproduct by construction encodes the separate coproducts $\Delta = \Delta_D|_{\mathcal{G}}, \Delta^* = \Delta_D|_{\mathcal{G}^*}$ by restriction and it is consistent with the products of each 2-algebras. These components are diagonal in a sense and we need to introduce some off diagonal contributions,

$$\xi_{-1} : \mathcal{G}_{-1} \to \mathcal{G}_0^* \otimes \mathcal{G}_{-1}, \quad \zeta_{-1} : \mathcal{G}_0^* \to \mathcal{G}_{-1} \otimes \mathcal{G}_0^*.$$

such that

$$\Delta_{D-1} = (\Delta_D)^d_{-1} + \xi_{-1} \otimes \zeta_{-1}. \tag{5.2.11}$$

ξ_{-1} and ζ_{-1} can be interpreted as coactions and are defined as dualized components of the coadjoint actions. Taking as usual $(x, f) \in D(\mathcal{G})_0 \cong \mathcal{G}_0 \otimes \mathcal{G}_{-1}^*$ and $(y, g) \in D(\mathcal{G})_{-1} \cong \mathcal{G}_{-1} \otimes \mathcal{G}_0^*$ we have

$$\langle \xi_{-1}(y), x \otimes f \rangle := \langle y, x \rhd_{-1} f \rangle, \qquad \langle \zeta_{-1}(g), f \otimes x \rangle := \langle g, x \lhd_{-1} f \rangle \tag{5.2.12}$$

These coactions are 2-algebra maps by (5.2.5), and hence Δ_{D-1} satisfies (5.1.16) on $D(\mathcal{G})$.

In a similar way, Δ_{D0} is also made of several components. We use the components $\Delta_0 : \mathcal{G}_0 \to (\mathcal{G}_0 \otimes \mathcal{G}_{-1}) \oplus (\mathcal{G}_{-1} \otimes \mathcal{G}_0)$ and $\Delta_0^* : \mathcal{G}_{-1}^* \to (\mathcal{G}_0^* \otimes \mathcal{G}_{-1}^*) \oplus (\mathcal{G}_{-1}^* \otimes \mathcal{G}_0^*)$ of $\mathcal{G}$ and $\mathcal{G}^*$ to define the "diagonal" contribution,

$$(\Delta_D^l)_0^d \quad := \quad \Delta_0^l \otimes \Delta_0^{*l}, \qquad (\Delta_D^r)_0 := \Delta_0^r \otimes \Delta_0^{*r}.$$

Once again, by restriction, one recovers the separate coproducts $\Delta_0^{r,l}$ and $\Delta_0^{*r,l}$ on respectively $\mathcal{G}$ and $\mathcal{G}^*$.

We also have to recover the mixed terms.

$$\xi_0^l : \mathcal{G}_0 \to \mathcal{G}_0^* \otimes \mathcal{G}_0, \quad \xi_0^r : \mathcal{G}_0 \to \mathcal{G}_{-1}^* \otimes \mathcal{G}_{-1}$$
$$\zeta_0^l : \mathcal{G}_{-1}^* \to \mathcal{G}_{-1} \otimes \mathcal{G}_{-1}^*, \quad \zeta_0^r : \mathcal{G}_0 \to \mathcal{G}_0 \otimes \mathcal{G}_0^*,$$

such that

$$(\Delta_D^{r,l})_0 := (\Delta_D^l)_0^d + \xi_0^{r,l} \otimes \zeta_0^{r,l}. \tag{5.2.13}$$

These mixed terms are again obtained by dualizing the components of the coadjoint actions

$$\langle \xi_0^l(x), x' \otimes g \rangle := \langle x, x' \rhd_0 g \rangle, \qquad \langle \xi_0^r(x), y \otimes f \rangle := \langle x, \Upsilon_y f \rangle,$$
$$\langle \zeta_0^l(f), f' \otimes y \rangle := \langle f, y \lhd_{-1} f' \rangle, \qquad \langle \zeta_0^r(f), g \otimes x \rangle := \langle f, x \tilde{\Upsilon}_g \rangle. \tag{5.2.14}$$

Once again, these coactions are 2-algebra maps by (5.2.5), and hence Δ_D satisfies (5.1.16) on $D(\mathcal{G})$.

We now need to show that it also satisfies (5.1.10), (5.1.11). We do this by leveraging the self-duality $D(\mathcal{G}) \cong D(\mathcal{G})^*$ under the natural non-degenerate self-pairing via (5.2.1) (cf. [95]),

$$\langle (z, h), (z', h') \rangle = \langle f, y' \rangle + \langle g, x' \rangle + \langle f', y \rangle + \langle g', x \rangle. \tag{5.2.15}$$

By **Proposition 5.2.1**, $(D(\mathcal{G}), \cdot)$ is an associative 2-algebra iff $(D(\mathcal{G})^* \cong D(\mathcal{G}), \Delta_D)$ is a coassociative 2-coalgebra, which implies (5.1.10)-(5.1.11) for Δ_D.

Definition 5.2.3. We call the 2-bialgebra

$$\mathcal{G} \bar{\bowtie} \mathcal{G}^* := D(\mathcal{G}) = (D(\mathcal{G})_{-1} \xrightarrow{T} D(\mathcal{G})_0, \hat{\cdot}, \Delta_D)$$

built out of the the matched pair of strict 2-bialgebras $(\mathcal{G}, \mathcal{G}^*)$ with the product, coproduct, and counit given respectively in (5.2.10), (5.2.11) and (5.2.13), (5.2.14), the **strict quantum 2-double** of $\mathcal{G}$.

5.2.3 Factorizability of 2-bialgebras

Conversely, we can determine when a strict 2-bialgebra is actually a strict quantum 2-double, which is given by a *factorizability/splitting* condition. In fact, we prove that any 2-bialgebra that factorizes appropriately into 2-bialgebras will automatically determine a quantum 2-double.

Theorem 5.2.1. *Suppose a (unital) 2-bialgebra ($\mathcal{K} = \mathcal{K}_{-1} \xrightarrow{T} \mathcal{K}_0, \hat{\cdot}$) factorizes into two (unital) sub-2-bialgebras $\mathcal{G}, \mathcal{H}$, meaning that there is a span of inclusions,*

$$\mathcal{G} \overset{\iota}{\hookrightarrow} \mathcal{K} \overset{\jmath}{\hookleftarrow} \mathcal{H}, \tag{5.2.16}$$

such that $\hat{\cdot} \circ (\iota \otimes \jmath)$ is an isomorphism of 2-vector spaces and such that the 2-sub-bialgebras $\mathcal{G}, \mathcal{H}$ are dually paired, with their t-maps satisfying $\langle t_{\mathcal{G}} -, - \rangle = \langle -, t_{\mathcal{H}} - \rangle$. Then $(\mathcal{G}, \mathcal{H})$ is a matched pair and $\mathcal{K} \cong \mathcal{G} \bar{\bowtie} \mathcal{H}$.

Proof. Let $\mathcal{K} = \mathcal{K}_{-1} \xrightarrow{T} \mathcal{K}_0$ be a 2-bialgebra factorizing into two 2-subbialgebras $\mathcal{G}, \mathcal{H}$, with typical elements $w \in \mathcal{K}_0$ and $e \in \mathcal{K}_{-1}$. Its 2-algebra structure $\hat{\cdot}$ contains a multiplication ww' in $\mathcal{K}_0$ and a $\mathcal{K}_0$-bimodule structure $w \hat{\cdot} e, e \hat{\cdot} w$ on $\mathcal{K}_{-1}$, which are both associative. Since (5.2.16) is a span of 2-vector spaces, we have

$$T \circ (\iota_{-1} \otimes \jmath_{-1}) = (\iota_0 \circ t_{\mathcal{G}}) \otimes (\jmath_0 \circ t_{\mathcal{H}}) = (\iota_0 \otimes \jmath_0) \circ (t_{\mathcal{G}} \otimes t_{\mathcal{H}}),$$

where t_G, t_H are the t-maps in $\mathcal{G}, \mathcal{H}$ respectively, and ι_{-1}, ι_0 are the graded components of the inclusion ι; similarly for $\jmath$.

We now separate the bimodule structure $\hat{\cdot}$ into components according to the span (5.2.16),

$$\hat{\triangleright} \equiv \hat{\cdot}|_{\operatorname{im}(\iota_0 \otimes \jmath_{-1})}, \qquad \hat{\Upsilon} \equiv \hat{\cdot}|_{\operatorname{im}(\iota_{-1} \otimes \jmath_0)},$$

then for $e = \iota_{-1}(y), e' = \jmath_{-1}(g)$ where $y \in G_{-1}, g \in H_{-1}$ we have

$$(T\iota_{-1}(y)) \hat{\triangleright} \jmath_{-1}(g) = \iota_0(t_{\mathcal{G}} y) \hat{\triangleright} \jmath_{-1}(g).$$

By the Peiffer identity in $\mathcal{K}$, this should read as a left-multiplication of y on g. We lift this action along $t_{\mathcal{H}}$ to create a map $\hat{\Upsilon}_y : \mathcal{H}_0 \to \mathcal{H}_{-1}$, for which $\hat{\Upsilon}_y(t_{\mathcal{H}} g)$ denotes the left-multiplicaion of y by g. Similarly we have the lift $\hat{\Upsilon}_g : \mathcal{G}_0 \to \mathcal{G}_{-1}$ of the right-multiplication of g on y.

Provided we identify $\hat{\Upsilon}_{y \otimes g} = \Upsilon_y \otimes \tilde{\Upsilon}_g$, the Peiffer conditions in $\mathcal{K}$ are then equivalent to the 2-representation properties (5.2.8), (5.2.9). In particular, the multiplication $y \cdot g = \Upsilon_y(t^* g) = (ty)\tilde{\Upsilon}_g$ is given by the generalized Peiffer identity as shown in **Proposition 5.2.2**.

Now we prove that (5.2.16) is in fact a span of 2-algebras. Due to the linear isomorphsm $\hat{\cdot} \circ (\iota \otimes \jmath)$, there exists a tuple of well-defined linear maps $\Psi = (\Psi_0, \Psi_{-1}; \bar{\Psi}) : \mathcal{G} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{G}$, called the **braided transposition**, such that

$$\begin{aligned}
\iota_0(x) \cdot \jmath_0(f) &= \cdot \circ (\jmath_0 \otimes \iota_0) \circ \Psi_0(x \otimes f), \\
\iota_0(x) \hat{\triangleright} \jmath_{-1}(g) &= \hat{\Upsilon} \circ (\jmath_{-1} \otimes \iota_0) \circ \Psi^r_{-1}(x \otimes g), \\
\iota_{-1}(y) \hat{\Upsilon} \jmath_0(f) &= \hat{\triangleleft} \circ (\jmath_0 \otimes \iota_{-1}) \circ \Psi^l_{-1}(y \otimes f), \\
\iota_{-1}(y) \cdot \jmath_{-1}(g) &= \cdot \circ (\jmath_{-1} \otimes \iota_{-1}) \circ \bar{\Psi}(y \otimes g),
\end{aligned}$$

where $\Psi_{-1} = \Psi^l_{-1} + \Psi^r_{-1}$ and $x \in \mathcal{G}_0, y \in \mathcal{G}_{-1}, f \in \mathcal{H}_0, g \in \mathcal{H}_{-1}$. Due to Peiffer conditions on $\mathcal{K}$, these braiding maps are not independent and must satisfy

$$(t_{\mathcal{H}}\otimes 1)\circ\Psi^r_{-1}=\Psi_0\circ(1\otimes t_{\mathcal{H}}),\qquad (1\otimes t_{\mathcal{G}})\circ\Psi^l_{-1}=\Psi_0\circ(t_{\mathcal{G}}\otimes 1),$$
$$\Psi^r_{-1}\circ(t_{\mathcal{G}}\otimes 1)=\bar{\Psi}=\Psi^l_{-1}\circ(1\otimes t_{\mathcal{H}}).$$

By collecting all of the graded components of Ψ in accordance with the shorthand notation $z=(y,x)\in\mathcal{G}, h=(g,f)\in\mathcal{H}$, the definition of Ψ can be concisely written as

$$\iota(z)\,\hat{\cdot}\,\jmath(h)=\hat{\cdot}\circ(\jmath\otimes\iota)\circ\Psi(z\otimes h),\tag{5.2.17}$$

and the relations between its components is summarized as

$$T'\circ\Psi_{-1}=\Psi_0\circ T,\qquad \bar{\Psi}=\Psi_{-1}\circ T,\tag{5.2.18}$$

where $T'=t_{\mathcal{H}}\otimes t_{\mathcal{G}}$ is the t-map of the 2-bialgebra $\mathcal{K}'\cong\mathcal{H}\otimes\mathcal{G}$ with $\mathcal{G},\mathcal{H}$ swapped in the span (5.2.16). (5.2.18) then implies in particular that $\Psi:\mathcal{K}\to\mathcal{K}'$ is a 2-vector space homomorphism.

We now proceed formally as in the 1-bialgebra case [117, 118]. The associativity in $\mathcal{K}$ is

$$\begin{aligned}(\iota(z)\hat{\cdot}\iota(z'))\hat{\cdot}\jmath(h) &= \iota(z)\hat{\cdot}(\iota(z')\hat{\cdot}\jmath(h)),\\ (\iota(z)\hat{\cdot}\jmath(h))\hat{\cdot}\jmath(h') &= \iota(z)\hat{\cdot}(\jmath(h)\hat{\cdot}\jmath(h')),\end{aligned}$$

which yields the **2-braiding relations**

$$\begin{aligned}\Psi\circ(\hat{\cdot}\otimes\mathrm{id}) &= (\mathrm{id}\otimes\hat{\cdot})\circ\Psi_{12}\circ\Psi_{23},\\ \Psi\circ(\mathrm{id}\otimes\hat{\cdot}) &= (\hat{\cdot}\otimes\mathrm{id})\circ\Psi_{23}\circ\Psi_{12}.\end{aligned}\tag{5.2.19}$$

This then allows us to define the actions

$$\begin{aligned}\bar{\rhd} &= (\mathrm{id}\otimes\epsilon)\circ\Psi:\mathcal{G}\otimes\mathcal{H}\to\mathcal{H},\\ \bar{\lhd} &= (\epsilon\otimes\mathrm{id})\circ\Psi:\mathcal{G}\otimes\mathcal{H}\to\mathcal{G},\end{aligned}$$

where ϵ denotes the counit map. Applying $\mathrm{id}\otimes\epsilon$ and $\epsilon\otimes\mathrm{id}$ respectively to the first and second equation of (5.2.19) implies that $\bar{\rhd},\bar{\lhd}$ respect the semidirect product structures $\mathcal{G}_{-1}\rtimes\mathcal{G}_0,\mathcal{H}_{-1}\rtimes\mathcal{H}_0$, respectively. Together with our above result, (5.2.16) is in fact a span of 2-algebras.

We now prove that (5.2.16) is actually a span of 2-bialgebras, which proves the theorem. Applying $\epsilon\otimes\mathrm{id}$ and $\mathrm{id}\otimes\epsilon$ respectively to the first and second (5.2.19) yields

$$(z\cdot z')\bar{\lhd}h=\hat{\cdot}(z\bar{\lhd}\Psi(z'\otimes h)),\qquad z\bar{\rhd}(h\cdot h')=\hat{\cdot}(\Psi(z\otimes h)\hat{\rhd}h').\tag{5.2.20}$$

We now take the coproduct $\Delta_K:\mathcal{K}\to\mathcal{K}^{2\otimes}$ on $\mathcal{K}$, given in components and Sweedler notation (see (5.1.5), (5.1.6)) by

$$(\Delta_\mathcal{K})_{-1}(e) = e_{(1)} \otimes e_{(2)}, \qquad (\Delta_\mathcal{K})_0(w) = w^l_{(1)} \otimes w^l_{(2)} + w^r_{(1)} \otimes w^r_{(2)};$$

note $w^l_{(1)}, w^r_{(2)} \in \mathcal{K}_{-1}$. With the span (5.2.16), we can write $w = \imath_0(x)\jmath_0(f), e = \imath_{-1}(y)\jmath_{-1}(g)$ for some appropriate elements x, f, y, g such that

$$\begin{aligned}
(\Delta_\mathcal{K})_{-1}(y,g) &= (y_{(1)} \otimes g_{(1)}) \otimes (y_{(2)} \otimes g_{(2)}), \\
(\Delta^l_\mathcal{K})_0(x,f) &= (x^l_{(1)} \otimes f^l_{(1)}) \otimes (x^l_{(2)} \otimes f^l_{(2)}), \\
(\Delta^r_\mathcal{K})_0(x,f) &= (x^r_{(1)} \otimes f^r_{(1)}) \otimes (x^r_{(2)} \otimes f^r_{(2)}).
\end{aligned}$$

This then allows us to define coproducts on $\mathcal{G}, \mathcal{H}$ by

$$\begin{aligned}
(\Delta_\mathcal{G})_{-1}(y) &= y_{(1)} \otimes y_{(2)}, & (\Delta_\mathcal{G})_0(x) &= x^l_{(1)} \otimes x^l_{(2)} + x^r_{(1)} \otimes x^r_{(2)}, \\
(\Delta_\mathcal{H})_{-1}(g) &= g_{(1)} \otimes g_{(2)}, & (\Delta_\mathcal{H})_0(f) &= f^l_{(1)} \otimes f^l_{(2)} + f^r_{(1)} + f^r_{(2)},
\end{aligned}$$

whence $\Delta_\mathcal{K} = \Delta_{\mathcal{G}\otimes\mathcal{H}}$, which implies that $\hat{\cdot} \circ (\imath \otimes \jmath)$ and $\hat{\cdot} \circ (\jmath \otimes \imath)$ by construction respects the coproducts.

As such, Ψ is a 2-coalgebra map. In particular, we have

$$\Delta_\mathcal{K} \circ \Psi = (\Psi \otimes \Psi) \circ \Delta_{\mathcal{K}'}, \qquad (\epsilon \otimes \epsilon) \circ \Psi = \epsilon \otimes \epsilon \tag{5.2.21}$$

where $\mathcal{K}'$ is the 2-bialgebra with $\mathcal{G}, \mathcal{H}$ swapped in the span (5.2.16). An application of $\epsilon \otimes \mathrm{id} \otimes \epsilon \otimes \mathrm{id}$ and $\mathrm{id} \otimes \epsilon \otimes \mathrm{id} \otimes \epsilon$ to (5.2.21) gives

$$\Delta_\mathcal{G} \circ \bar{\triangleleft} = (\bar{\triangleleft} \otimes \bar{\triangleleft}) \circ \Delta_\mathcal{K}, \qquad \Delta_\mathcal{H} \circ \bar{\triangleright} = (\bar{\triangleright} \otimes \bar{\triangleright}) \circ \Delta_\mathcal{K},$$

which ensures that $\bar{\triangleright}, \bar{\triangleleft}$ are 2-coalgebra maps.

Now applying $\epsilon \otimes \mathrm{id} \otimes \mathrm{id} \otimes \epsilon$ and $\mathrm{id} \otimes \epsilon \otimes \epsilon \otimes \mathrm{id}$ to (5.2.21) yields

$$\begin{aligned}
z_{(1)} \bar{\triangleleft} h_{(1)} \otimes z_{(2)} \bar{\triangleright} h_{(2)} &= \tau \circ \Psi(z \otimes h), \\
z_{(1)} \bar{\triangleright} h_{(1)} \otimes z_{(2)} \bar{\triangleleft} h_{(2)} &= \Psi(z \otimes h).
\end{aligned}$$

Using the second equation, together with (5.2.20), gives (5.2.6) and

$$\begin{aligned}
z \bar{\triangleright} (h \hat{\cdot}|_\mathcal{H} h') &= \hat{\cdot}((z_{(1)} \bar{\triangleright} h_{(1)} \otimes z_{(2)} \bar{\triangleleft} h_{(2)}) \hat{\triangleright} h') = (z_{(1)} \bar{\triangleright} h_{(1)}) \hat{\cdot}|_\mathcal{H} ((z_{(2)} \hat{\triangleleft} h_{(2)}) \bar{\triangleright} h'), \\
(z \hat{\cdot}|_\mathcal{G} z') \bar{\triangleleft} h &= \hat{\cdot}(z \bar{\triangleleft} (z'_{(1)} \bar{\triangleright} h_{(1)} \otimes z'_{(2)} \bar{\triangleleft} h_{(2)})) = (z \bar{\triangleleft} (z'_{(1)} \bar{\triangleright} h_{(1)})) \hat{\cdot}|_\mathcal{G} (z'_{(2)} \hat{\triangleleft} h_{(2)}),
\end{aligned}$$

which are precisely the mathced pair conditions (5.2.3), (5.2.4) for $\hat{\cdot}|_\mathcal{G} = \cdot, \hat{\cdot}|_\mathcal{H} = \cdot^*$. On the other hand, using the first equation gives (5.2.5). Thus (5.2.16) is a span of 2-bialgebras and so $K \cong \mathcal{G} \bar{\bowtie} \mathcal{H}$. $\square$

Note that the span (5.2.16) factorizes the 2-algebra structure on $\mathcal{K}$ into the right- $\overrightarrow{\times}$ =

$(\hat{\cdot}|_{\mathcal{G}}, \bar{\triangleleft})$ and left-moving $\overleftarrow{\times} = (\hat{\cdot}|_{\mathcal{H}}^{\text{opp}}, \bar{\triangleright}^{\text{opp}})$ 2-algebra structures. In other words, in order to identify $\mathcal{K}$ with a quantum 2-double, we must have [118]

$$\mathcal{K} \cong \mathcal{G} \bar{\bowtie} \mathcal{H} \cong D(\mathcal{G}, \mathcal{H}^{\text{opp}}), \tag{5.2.22}$$

where $\mathcal{H}^{\text{opp}}$ denotes the opposite 2-algebra. This is because, as can be seen in (5.2.14), the back-action $\bar{\triangleleft}$ is written from right to left.

5.2.4 Characterization of quantum 2-R-matrices

As we have mentioned in the beginning of this section, we wish to leverage the quantum 2-double construction we have given above in order to provide a notion of a quantum R-matrix on a 2-bialgebra $\mathcal{G}$. More precisely, we shall use the skew-pairing on $\mathcal{G}$ used in forming the quantum 2-double $D(\mathcal{G}, \mathcal{G}) = \mathcal{G} \bar{\bowtie} \mathcal{G}^{\text{opp}}$ in order to provide a definition of the 2-R-matrix on $\mathcal{G}$. We shall show in §5.6.2 that such a characterization is *universal*, in the sense that our definition of a 2-R-matrix gives rise to a *braiding* on the 2-representations of $\mathcal{G}$.

Review of the 1-bialgebra case. We first recall the explicit construction of the universal R-matrix for the ordinary 1-bialgebra H. It was noted by Majid (see eg. [118, 117]) that, in forming the quantum double $D(H, H) = H \bowtie H^{\text{opp}}$ as a bicrossed product, the (non-degenerate) skew-pairing which dualizes H with itself satisfies

$$\langle xx', g \rangle_{\text{sk}} = \langle x \otimes x', \Delta(g) \rangle_{\text{sk}} = \langle x, g_{(1)} \rangle_{\text{sk}} \langle x', g_{(2)} \rangle_{\text{sk}},$$
$$\langle x, gg' \rangle_{\text{sk}} = \langle \Delta(x), g' \otimes g \rangle_{\text{sk}} = \langle x_{(1)}, g' \rangle_{\text{sk}} \langle x_{(2)}, g \rangle_{\text{sk}},$$

where $x, x' \in H$ and $g, g' \in H^{\text{opp}} \cong H$. If we define this skew-pairing as a functional $\langle -, - \rangle_{\text{sk}} = R^* : H^{2\otimes} \to k$, then we see that the above conditions translate to

$$R^* \circ (\mu \otimes \text{id}) = R^*_{13} R^*_{23}, \qquad R^* \circ (\text{id} \otimes \mu) = R^*_{13} R^*_{12},$$

which is nothing but the defining properties of a *dual* R-matrix on H. Indeed, together with the property

$$g_{(1)}\, x_{(1)}\, R^*(x_{(2)}, g_{(2)}) = R^*(x_{(1)}, g_{(1)})\, x_{(2)}\, g_{(2)}, \tag{5.2.23}$$

we obtain the (dual) Yang-Baxter equations [118, 117].

In other words, the duality pairing $\langle -, - \rangle_{\text{sk}}$ on the bicrossed product quantum double $D(H, H) = H \bowtie H^{\text{opp}}$ gives rise to a R-matrix R on H, and conversely any R-matrix gives rise to such a duality bilinear form. Moreover, this pairing is non-degenerate iff the corresponding R-matrix is quasitriangular (ie. R is invertible).

(Dual) 2-R-matrix. We now follow an analogous treatment to characterize dual 2-R-matrices of a quasitriangular 2-bialgebra $\mathcal{G}$. Take the quantum 2-double $D(\mathcal{G}, \mathcal{G})$, whose underlying duality pairing (5.2.15) is given by a non-degenerate self-duality skew-pairing $\langle -,-\rangle_{\text{sk}} : \mathcal{G}\otimes\mathcal{G}\to k$. Explicitly, this pairing satisfies

$$\begin{aligned} \langle x\cdot_l y, f\rangle_{\text{sk}} &= \langle x\otimes y, \Delta_0^l(f)\rangle_{\text{sk}}, \qquad \langle y\cdot_r x, f\rangle_{\text{sk}} = \langle y\otimes x, \Delta_0^r(f)\rangle_{\text{sk}},\\ \langle x, f\cdot_l g\rangle_{\text{sk}} &= \langle \Delta_0^r(x), g\otimes f\rangle_{\text{sk}}, \qquad \langle x, g\cdot f\rangle_{\text{sk}} = \langle \Delta_0^l(x), f\otimes g\rangle_{\text{sk}}, \end{aligned} \tag{5.2.24}$$

and also in addition to the fact that it should respect the t-map $T = t\otimes t$ on $D(\mathcal{G},\mathcal{G})$,

$$\langle ty, g\rangle_{\text{sk}} = \langle y, tg\rangle_{\text{sk}},$$

where $x, f, f' \in \mathcal{G}_0$ and $y, g\in\mathcal{G}_{-1}$. Writing the skew-pairing in terms of a functional $\mathcal{R}^* : \mathcal{G}^{2\otimes}\to k$ by

$$\mathcal{R}_l^*(y,f) = \langle y, f\rangle_{\text{sk}}, \qquad \mathcal{R}_r^*(x,g) = \langle x, g\rangle_{\text{sk}},$$

we can rewrite (5.2.24) as

$$\begin{aligned} \mathcal{R}_l^*\circ(\cdot_l\otimes\text{id}) &= (\mathcal{R}_r^*)_{13}(\mathcal{R}_l^*)_{23}, \qquad \mathcal{R}_l^*\circ(\cdot_r\otimes\text{id}) = (\mathcal{R}_l^*)_{13}(\mathcal{R}_r^*)_{23},\\ \mathcal{R}_r^*\circ(\text{id}\otimes\cdot_l) &= (\mathcal{R}_l^*)_{13}(\mathcal{R}_r^*)_{12}, \qquad \mathcal{R}_r^*\circ(\text{id}\otimes\cdot_r) = (\mathcal{R}_r^*)_{13}(\mathcal{R}_l^*)_{12}, \end{aligned}$$

where $\cdot_l,\cdot_r$ denotes respectively the left and right $\mathcal{G}_0$-actions on $\mathcal{G}_{-1}$. We also have the compatibility conditions with the t-map:

$$\mathcal{R}_r^*\circ(\text{id}\otimes t) = \mathcal{R}_l^*\circ(t\otimes\text{id}) \in \mathcal{G}_{-1}^{2\otimes}.$$

By dualizing the above functional $\mathcal{R}^*$, we are able to characterize the 2-R-matrix $\mathcal{R}$ on $\mathcal{G}$.

Definition 5.2.4. A **2-R-matrix** associated to a 2-bialgebra $(\mathcal{G},\cdot,\Delta)$ is an element $\mathcal{R}\in\mathcal{G}\otimes\mathcal{G}$ consisting of the graded components

$$\mathcal{R}^l \in \mathcal{G}_{-1}\otimes\mathcal{G}_0, \qquad \mathcal{R}^r\in\mathcal{G}_0\otimes\mathcal{G}_{-1}.$$

such that the following identities are satisfied:

1. the **compatibility with the coproduct**

$$\begin{aligned} (\Delta_0^l\otimes\text{id})\mathcal{R}^r &= \mathcal{R}_{13}^l\cdot_l\mathcal{R}_{23}^r, \qquad (\Delta_0^r\otimes\text{id})\mathcal{R}^r = \mathcal{R}_{13}^r\cdot_r\mathcal{R}_{23}^l,\\ (\text{id}\otimes\Delta_0^l)\mathcal{R}^l &= \mathcal{R}_{13}^l\cdot_r\mathcal{R}_{12}^r, \qquad (\text{id}\otimes\Delta_0^r)\mathcal{R}^l = \mathcal{R}_{13}^r\cdot_l\mathcal{R}_{12}^l, \end{aligned} \tag{5.2.25}$$

2. the **coproduct permutation identity**

$$\mathcal{R}^r\Delta_0^r(x) = (\sigma\circ\Delta_0^l(x))\mathcal{R}^r, \qquad \mathcal{R}^l\Delta_0^l(x) = (\sigma\circ\Delta_0^r(x))\mathcal{R}^l \tag{5.2.26}$$

for each $x\in\mathcal{G}_0$, where $\sigma:\mathcal{G}\otimes\mathcal{G}\to\mathcal{G}\otimes\mathcal{G}$ is the permutation of tensor factors, and

3. the **equivariance condition**

$$D_t^- \mathcal{R} = 0 \iff (t \otimes \mathrm{id})\mathcal{R}^l = (\mathrm{id} \otimes t)\mathcal{R}^r \in \mathcal{G}_0^{2\otimes}. \tag{5.2.27}$$

We call the tuple $(\mathcal{G}, \cdot, \Delta, \mathcal{R})$ a **quasitriangular 2-bialgebra** if $\mathcal{R}^l, \mathcal{R}^r$ are both invertible.

We now derive the categorified notion of the Yang-Baxter equations.

Proposition 5.2.3. *The 2-R-matrix of a quasitriangular 2-bialgebra $(\mathcal{G}, \cdot, \Delta, \mathcal{R})$ satisfies the **2-Yang-Baxter equations***

$$\begin{aligned} &\mathcal{R}_{23}^r(\mathcal{R}_{13}^r \cdot_l \mathcal{R}_{12}^l) = (\mathcal{R}_{12}^l \cdot_r \mathcal{R}_{13}^r)\mathcal{R}_{23}^r, \qquad (\mathcal{R}_{23}^l \cdot_l \mathcal{R}_{13}^r)\mathcal{R}_{12}^r = \mathcal{R}_{12}^r(\mathcal{R}_{13}^r \cdot_r \mathcal{R}_{23}^l), \\ &\mathcal{R}_{23}^l(\mathcal{R}_{13}^l \cdot_r \mathcal{R}_{12}^r) = (\mathcal{R}_{12}^r \cdot_l \mathcal{R}_{13}^l)\mathcal{R}_{23}^l, \qquad (\mathcal{R}_{23}^r \cdot_r \mathcal{R}_{13}^l)\mathcal{R}_{12}^l = \mathcal{R}_{12}^l(\mathcal{R}_{13}^l \cdot_l \mathcal{R}_{23}^r). \end{aligned} \tag{5.2.28}$$

Proof. Recall that $\mathcal{R}$ is quasitriangular iff $\mathcal{R}^l, \mathcal{R}^r$ are square and invertible. This pairs $\mathcal{G}$ with itself and hence $\dim \mathcal{G}_0 = \dim \mathcal{G}_{-1}$. We calculate $(\mathrm{id} \otimes \sigma \circ \Delta_0^l)\mathcal{R}^l$ and $(\sigma \circ \Delta_0^l \otimes \mathrm{id})\mathcal{R}^r$, as well as $(\mathrm{id} \otimes \sigma \circ \Delta_0^r)\mathcal{R}^l$ and $(\sigma \circ \Delta_0^r \otimes \mathrm{id})\mathcal{R}^r$ in two ways. First using (5.2.25), we have

$$\begin{aligned} (\mathrm{id} \otimes \sigma \circ \Delta_0^l)\mathcal{R}^l &= (\mathrm{id} \otimes \sigma)\mathcal{R}_{13}^l \cdot_r \mathcal{R}_{12}^r = \mathcal{R}_{12}^l \cdot_r \mathcal{R}_{13}^r, \\ (\sigma \circ \Delta_0^l \otimes \mathrm{id})\mathcal{R}^r &= (\sigma \otimes \mathrm{id})\mathcal{R}_{13}^l \cdot_l \mathcal{R}_{23}^r = \mathcal{R}_{23}^l \cdot_l \mathcal{R}_{13}^r, \\ (\mathrm{id} \otimes \sigma \circ \Delta_0^r)\mathcal{R}^l &= (\mathrm{id} \otimes \sigma)\mathcal{R}_{13}^r \cdot_l \mathcal{R}_{12}^l = \mathcal{R}_{12}^r \cdot_l \mathcal{R}_{13}^l, \\ (\sigma \circ \Delta_0^r \otimes \mathrm{id})\mathcal{R}^r &= (\sigma \otimes \mathrm{id})\mathcal{R}_{13}^r \cdot_r \mathcal{R}_{23}^l = \mathcal{R}_{23}^r \cdot_r \mathcal{R}_{13}^l. \end{aligned}$$

On the other hand from (5.2.26), we have that,

$$\begin{aligned} (\mathrm{id} \otimes \sigma \circ \Delta_0^l)\mathcal{R}^l &= \mathcal{R}_{23}^r((\mathrm{id} \otimes \Delta_0^r)\mathcal{R}^l)\mathcal{R}^{r\,-1}_{23} = \mathcal{R}_{23}^r(\mathcal{R}_{13}^r \cdot_l \mathcal{R}_{12}^l)\mathcal{R}^{r\,-1}_{23} \\ (\sigma \circ \Delta_0^l \otimes \mathrm{id})\mathcal{R}^r &= \mathcal{R}_{12}^r((\Delta_0^r \otimes \mathrm{id})\mathcal{R}^r)\mathcal{R}^{r\,-1}_{12} = \mathcal{R}_{12}^r(\mathcal{R}_{13}^r \cdot_r \mathcal{R}_{23}^l)\mathcal{R}^{r\,-1}_{12}, \\ (\mathrm{id} \otimes \sigma \circ \Delta_0^r)\mathcal{R}^l &= \mathcal{R}_{23}^l((\mathrm{id} \otimes \Delta_0^l)\mathcal{R}^l)\mathcal{R}^{l\,-1}_{23} = \mathcal{R}_{23}^l(\mathcal{R}_{13}^l \cdot_r \mathcal{R}_{12}^r)\mathcal{R}^{l\,-1}_{23} \\ (\sigma \circ \Delta_0^r \otimes \mathrm{id})\mathcal{R}^r &= \mathcal{R}_{12}^l((\Delta_0^l \otimes \mathrm{id})\mathcal{R}^r)\mathcal{R}^{l\,-1}_{12} = \mathcal{R}_{12}^l(\mathcal{R}_{13}^l \cdot_l \mathcal{R}_{23}^r)\mathcal{R}^{l\,-1}_{12} \end{aligned}$$

Putting each equation with its above counterpart leads to (5.2.28). □

Remark 5.2.2. It is easy to see that, when $\mathcal{G} = D(\mathcal{H})$ is itself the quantum 2-double of a 2-bialgebra $\mathcal{H}$, then the skew-pairing required in forming the "2-quantum quadruple" $D(\mathcal{G}, \mathcal{G}) = D(D(\mathcal{H}), D(\mathcal{H}))$ splits into two copies the self-pairing form (5.2.15),

$$\langle [(y,x),(g,f)],[(y',x'),(g',f')] \rangle_{\mathrm{sk}} = \langle (g,f),(y',x') \rangle + \langle (g',f'),(y,x) \rangle.$$

Since (5.2.15) is non-degenerate, then so is $\langle -,- \rangle_{\mathrm{sk}}$ and the corresponding universal 2-R-matrix $\mathcal{R} \in D(\mathcal{H}, \mathcal{H})$ on $D(\mathcal{H})$ is automatically quasitriangular.

The (dual) 2-R-matrix from factorizability. Due to the factorizability result **Theorem 5.2.1**, we could have begun our characterization with a general associative 2-bialgebra $\mathcal{K}$ which factorizes into two copies of $\mathcal{G}$, instead of the quantum 2-double $D(\mathcal{G},\mathcal{G})$. This introduces the braided transposition $\Psi:\mathcal{G}\otimes\mathcal{G}\to\mathcal{G}\otimes\mathcal{G}$ given in (5.2.17) into the definition of the dual 2-R-matrix:

$$\mathcal{R}_l^* = \mathrm{ev}_l\circ\Psi_{-1}^l,\qquad \mathcal{R}_r^* = \mathrm{ev}_r\circ\Psi_{-1}^r$$

where $\mathrm{ev}=\mathrm{ev}_l+\mathrm{ev}_r$ is precisely the skew-pairing $\langle -,-\rangle_{\mathrm{sk}}$ that we have introduced previously.

Dualizing this construction then gives

$$\mathcal{R}^l = \Psi_{-1}^l\circ\mathrm{coev}_l,\qquad \mathcal{R}^r = \Psi_{-1}^r\circ\mathrm{coev}_r, \tag{5.2.29}$$

where $\mathrm{coev}=\mathrm{coev}_l+\mathrm{coev}_r:k\to\mathcal{G}\otimes\mathcal{G}$ is the coevaluation. In other words, we are able to reconstruct the 2-R-matrix from the braided transposition Ψ on the quantum 2-double $\mathcal{K}\cong D(\mathcal{G},\mathcal{G})$. Indeed, (5.2.18) gives the equivariance (5.2.27), and the relation (5.2.19) implies (5.2.25).

As mentioned in the proof of **Proposition 5.2.3**, having a *quasitriangular* structure on $\mathcal{G}$ implies that $\mathcal{G}$ is self-dual. This explains why only Ψ_{-1} appears in the reconstruction of the 2-R-matrix: the degree-0 component Ψ_0 dualizes to that in degree-(-2) $\bar{\Psi}^*$ for the dual $\mathcal{K}^*\cong\mathcal{K}$, which has the same t-map $T=t\otimes t$. As $\bar{\Psi}^*$ is determined by $(\Psi_{-1}^*)^{l,r}=\Psi_{-1}^{r,l}$ per (5.2.18), the component Ψ_0 is also completely determined by Ψ_{-1}.

Since **Theorem 5.2.1** implies that $\mathcal{K}\cong D(\mathcal{G},\mathcal{G})$, this particular construction is isomorphic to the one we have given above directly from $D(\mathcal{G},\mathcal{G})$. The characterization of the 2-R-matrix, **Definition 5.2.4**, thus does not depend on whether we induce R from the skew-pairing on $D(\mathcal{G},\mathcal{G})$ or the braiding trasposition Ψ on $\mathcal{K}$.

5.3 Weak 2-bialgebras

We now begin our endeavour to weaken the associativity conditions in the above quantum 2-double construction. The idea of *non-associative* 2-algebra has not been developed nearly as much as the associative ones, but we shall take inspiration from their Lie 2-algebra counterparts.

We provide the notion of a weak 2-algebra by generalizing **Definition 5.1.2**.

Definition 5.3.1. A **weak 2-algebra** $(\mathcal{G},\mathcal{T})$ is a map $t:\mathcal{G}_{-1}\to\mathcal{G}_0$ between a pair of *not necessarily associative* algebras, together with an **invertible homotopy map** $\mathcal{T}:\mathcal{G}_0^{3\otimes}\to\mathcal{G}_{-1}$ such that we have the conditions (5.1.2), (5.1.3), as well as

1. the **weak 1-associativity**,

$$(xx')x''-x(x'x'')=t\mathcal{T}(x,x',x''),\qquad (yy')y''-y(y'y'')=\mathcal{T}(ty,ty',ty'')$$

and the **weak bimodularity**,

$$x\cdot(x'\cdot y)-(xx')\cdot y=\mathcal{T}(x,x',ty)\qquad (x\cdot y)\cdot x'-x\cdot(y\cdot x')=\mathcal{T}(x,ty,x'),$$
$$(y\cdot x)\cdot x'-y\cdot(xx')=\mathcal{T}(ty,x,x'),$$

for each $x, x', x'' \in \mathcal{G}_0$ and $y, y', y'' \in \mathcal{G}_{-1}$,

2. the **Hochschild 3-cocycle condition**,

$$x_1 \cdot \mathcal{T}(x_2, x_3, x_4) + \mathcal{T}(x_1, x_2, x_3) \cdot x_4 = \mathcal{T}(x_1x_2, x_3, x_4) - \mathcal{T}(x_1, x_2x_3, x_4) + \mathcal{T}(x_1, x_2, x_3x_4)$$

for each $x_1, \ldots, x_4 \in \mathcal{G}_0$.

We call $(\mathcal{G}, \mathcal{T})$ a unital weak 2-algebra if we have a unit map $\eta : k \to \mathcal{G}$ that satisfies the usual conditions (5.1.4), and such that $\mathcal{T}$ is **normalized** — namely it vanishes whenever any of its arguments are 0 or η_0.

We note here that this structure is precisely the definition of a 2-term homotopy A_∞**-algebra** [142], together with the Peiffer identity constraint (5.1.3). The correspondence between the n-nary product $m_n \in \mathrm{Hom}^{n-2}(\mathcal{G}^{n\otimes}, \mathcal{G})$ and the weak 2-algebra structure is given by

$$m_1(-) = t(-), \qquad m_2(-,-) = (--, - \cdot -), \qquad m_3(-,-,-) = \mathcal{T}(-,-,-),$$

with $m_n = 0$ trivial for $n \geqslant 4$. Nevertheless, we shall see that the Peiffer identity on $\mathcal{G}$ shall play a very important role.

Similar to *Remark 5.1.1*, the Peiffer identity implies the further constraints

$$(x \cdot y)y' - x \cdot (yy') = \mathcal{T}(x, ty, ty'), \qquad (y \cdot x)y' - y(x \cdot y') = \mathcal{T}(ty, x, ty'),$$
$$y(y' \cdot x) - (yy') \cdot x = \mathcal{T}(ty, ty', x)$$

for $t \neq 0$, where $x \in \mathcal{G}_0, y, y' \in \mathcal{G}_{-1}$.

Weak 2-algebra homomorphisms

We define a map between weak 2-algebras $(\mathcal{G}, \mathcal{T}) \to (\mathcal{G}', \mathcal{T}')$ as a cochain map $F = (F_1, F_0, F_{-1}) : \mathcal{G} \to \mathcal{G}'$:

$$F_1 : \mathcal{G}_0^{2\otimes} \to \mathcal{G}'_{-1}, \qquad F_0 : \mathcal{G}_0 \to \mathcal{G}'_0, \qquad F_{-1} : \mathcal{G}_{-1} \to \mathcal{G}'_{-1},$$

such that $t' \circ F_{-1} = F_0 \circ t$ and the following conditions are satisfied,

$$\begin{aligned} t'F_1(x, x') &= F_0(xx') - F_0(x)F_0(x'), \\ F_1(x, ty) &= F_{-1}(x \cdot y) - F_0(x) \cdot' F_{-1}(y), \\ F_1(ty, x) &= F_{-1}(y \cdot x) - F_{-1}(y) \cdot' F_0(x) \\ \mathcal{T}'(F_0(x), F_0(x'), F_0(x'')) &= F_0(x) \cdot' F_1(x', x'') - F_1(xx', x'') \\ &\quad + F_1(x, x'x'') - F_1(x, x') \cdot' F_0(x'') \\ &\quad + F_{-1}(\mathcal{T}(x, x', x'')). \end{aligned} \tag{5.3.1}$$

In other words, F_1 contributes as an "obstruction" for the other components (F_0, F_{-1}) to define a strict 2-algebra homomorphism, but only up to homotopy in the sense that F_1 by definition (see the last equation of (5.3.1)) gives an explicit trivialization of the Hochschild cohomology class $[\mathcal{T}' \circ F_0] - [F_{-1} \circ \mathcal{T}] = 0$.

It can then be deduced that quasi-isomorphism classes of weak 2-algebras — where $\mathcal{G} \sim \mathcal{G}'$ are said to be quasi-isomorphic iff there exists a weakly inertible cochain map (5.3.1) between them — is still labeled by Hochschild cohomology classes $\mathcal{T} \in HH^3(\mathcal{N}, V)$, where $\mathcal{N} = \operatorname{coker} t$ and $V = \ker t$. In particular, $(\mathcal{G}, \mathcal{T})$ is always quasi-isomorphic to its skeleton $(\mathcal{N} \xrightarrow{0} V, [\mathcal{T}])$, which is in fact associative.

In summary, the difference between the strict and weak case is that there are distinguished associator chain homotopies

$$\mathcal{T}(x, x', x'') : (xx')x'' \to x(x'x''), \tag{5.3.2}$$

given by the homotopy map $\mathcal{T}$ witnessing associativity.

5.3.1 Weak 2-coalgebras

We begin by defining the notion of a weak 2-coalgebra. Recall that the weakening in **Definition 5.3.1** concerns only the associativity of the 2-algebra structure. Correspondingly, the weakening of a 2-coalgebra should only concern the coassociativity.

For brevity of notation later, we first rewrite the equations (5.1.7), (5.1.9) in a more concise way. Consider coassociativity (5.1.7); we naturally extend Δ_{-1} to act on tensor products (with alternating sign) such that

$$\Delta_{-1} \circ \Delta_{-1} \equiv (\mathrm{id} \otimes \Delta_{-1}) \circ \Delta_{-1} - (\Delta_{-1} \otimes \mathrm{id}) \circ \Delta_{-1}.$$

Secondly, we recombine $\Delta_0 = \Delta_0^l + \Delta_0^r$ and extend it as well to tensor products, such that

$$\begin{aligned}(\Delta_{-1} + \Delta_0) \circ \Delta_0 \quad &\equiv \quad \left[(\Delta_{-1} \otimes \mathrm{id}) \circ \Delta_0^l - (\mathrm{id} \otimes \Delta_0^l) \circ \Delta_0^l\right] \\ &\quad + \left[(\mathrm{id} \otimes \Delta_{-1}) \circ \Delta_0^r - (\Delta_0^r \otimes \mathrm{id}) \circ \Delta_0^r\right]\end{aligned}$$

encodes two expressions in (5.1.9). We extend the t-map to the triple tensor product,

$$D_t = \mathrm{id} \otimes \mathrm{id} \otimes t - \mathrm{id} \otimes t \otimes \mathrm{id} + t \otimes \mathrm{id} \otimes \mathrm{id},$$

such that the equation

$$D_t \circ \Delta_0 \circ \Delta_0 = \Delta_0 \circ D_t \circ \Delta_0$$

encodes all three equations in (5.1.14). For convenience, we define also the map

$$D_t[2] \equiv t \otimes t \otimes \mathrm{id} - t \otimes \mathrm{id} \otimes t + \mathrm{id} \otimes t \otimes t,$$

which is an extension of two applications of t to the 3-fold tensor product.

Definition 5.3.2. Let $\Delta_1 : \mathcal{G}_0 \to \mathcal{G}_{-1}^{3\otimes}$ denote an *invertible* trilinear map. Together with the maps (Δ_{-1}, Δ_0) defined as in (5.1.5), we say that the tuple $(\mathcal{G}, \Delta = (\Delta_{-1}, \Delta_0, \Delta_1))$ is a **weak 2-coalgebra** iff coequivariance (5.1.10), coPeiffer identity (5.1.11), **weak coassociativity**

$$\begin{aligned} \Delta_{-1} \circ \Delta_{-1} &= \Delta_1 \circ t, \\ (\Delta_{-1} + \Delta_0) \circ \Delta_0 &= D_t \circ \Delta_1, \end{aligned} \tag{5.3.3}$$

and **2-coassociativity**

$$\Delta_1 \circ \Delta_0 = \Delta_{-1} \circ \Delta_1 \tag{5.3.4}$$

are satisfied. In which case we call Δ_1 the **coassociator** of $\mathcal{G}$.

We call $(\mathcal{G}, \Delta)$ **counital** if it is equipped with a counit $\epsilon : k \to \mathcal{G}$ satisfying the usual conditions, and $\epsilon \circ \Delta_1 = 0$.

Notice that, provided the coequivariance and the coPeiffer identity are satisfied, applying one more t-map to (5.3.3) yields

$$\Delta_0' \circ \Delta_0 - \Delta_0 \circ \Delta_0' = D_t[2] \circ \Delta_1, \tag{5.3.5}$$

which is a monoidal weakening of the condition (5.1.14). Similarly, applying the t-map yet once more gives a map $\Phi \equiv (t \otimes t \otimes t)\Delta_1 : \mathcal{G}_0 \to \mathcal{G}_0^{3\otimes}$ that lands only in $\mathcal{G}_0$. We write this element multiplicatively such that

$$(\Delta_0' \otimes \mathrm{id}) \circ \Delta_0' = \Phi \circ (\mathrm{id} \otimes \Delta_0') \circ \Delta_0'. \tag{5.3.6}$$

Recall that, in the skeletal case where $t = 0$, the coproducts $\Delta_{-1}, \Delta_0, \Delta_0'$ are independent and hence (5.3.6) should also be imposed independently from (5.3.3).

5.3.2 Weak 2-bialgebras

Suppose now $(\mathcal{G}, \mathcal{T})$ is a weak 2-algebra equipped with the tuple $\Delta = (\Delta_{-1}, \Delta_0, \Delta_1)$ of linear maps. Recall the Sweedler notation (5.1.8) for $\Delta_0' : \mathcal{G}_0 \to \mathcal{G}_0^{2\otimes}$. We use it to state the condition that the coassociator Δ_1 preserves the algebra structure on $\mathcal{G}$,

$$\begin{aligned} (\Delta_{-1} \circ \mathcal{T})(x, x', x'') &= \mathcal{T}(\bar{x}_{(1)}, \bar{x}'_{(1)}, \bar{x}''_{(1)}) \otimes \mathcal{T}(\bar{x}_{(2)}, \bar{x}'_{(2)}, \bar{x}''_{(2)}), \\ \Delta_1(xx') &= x_{(1)}x'_{(1)} \otimes x_{(2)}x'_{(2)} \otimes x_{(3)}x'_{(3)}, \end{aligned} \tag{5.3.7}$$

for $x, x', x'' \in \mathcal{G}_0$. Note that $\bar{x}_{(1)}, \bar{x}_{(2)} \in \mathcal{G}_0$ are not to be confused with the elements $x_{(1)}^{l,r}$ in (5.1.6).

Definition 5.3.3. The tuple $(\mathcal{G}, \mathcal{T}, \Delta)$ is a (unital) **weak 2-bialgebra** iff $(\mathcal{G}, \mathcal{T})$ is a weak 2-algebra and $(\mathcal{G}, \Delta)$ is a (counital) weak 2-coalgebra. Equivalently, $(\mathcal{G}, \mathcal{T}, \Delta)$ is a weak 2-bialgebra iff the tuple $\Delta = (\Delta_1, \Delta_0, \Delta_{-1})$ satisfies (5.1.11), (5.1.10), (5.3.3)-(5.3.5), (5.1.16) and (5.3.7).

A weak 2-bialgebra $(\mathcal{G}, \mathcal{T}, \Delta)$ is called **quasi-2-bialgebra** if $\mathcal{T} = 0$.

Similar to what we have done for the strict case, we suppose $\mathcal{G}$ is dually paired with its dual 2-algebra through (5.2.1). The coassociator Δ_1 on $\mathcal{G}$ induces a linear map $\mathcal{T}^* : \mathcal{G}_{-1}^* \to \mathcal{G}_0^*$ by

$$\langle f \otimes f' \otimes f'', \Delta_1(x)\rangle = \langle \mathcal{T}^*(f, f', f''), x\rangle.$$

Similarly, the Hochschild 3-cocycle $\mathcal{T}$ on $\mathcal{G}$ induces a linear map $\Delta_1^* : \mathcal{G}_{-1}^* \to (\mathcal{G}_0^*)^{3\otimes}$. We form the tuple $\Delta^* = (\Delta_1^*, \Delta_0^*, \Delta_{-1}^*)$.

Proposition 5.3.1. *Let $\mathcal{G}, \mathcal{G}^*$ be dually paired, then $(\mathcal{G}, \mathcal{T}, \Delta)$ is a (unital) weak 2-bialgebra iff $(\mathcal{G}^*, \mathcal{T}^*, \Delta^*)$ is a (unital) weak 2-bialgebra.*

Proof. This follows directly from the definitions.

□

Given $(\mathcal{G}, \mathcal{G}^*)$ are dually paired 2-bialgebras, we see that a quasi-2-bialgebra $(\mathcal{G}, \mathcal{T} = 0, \Delta)$ encode the same data as a weak but coassociative 2-bialgebra $(\mathcal{G}^*, \mathcal{T}^*, \Delta^*)$, in which $\Delta_1^* = 0$.

5.4 Weak (skeletal) quantum 2-double

Let $\mathcal{G}, \mathcal{G}^*$ be dually paired (weak) 2-bialgebras. To form its weak quantum 2-double, we require them to act on each other *weakly*. This means, in particular, that the coadjoint actions $\bar{\rhd}, \bar{\lhd}$ now come with the additional components

$$\rhd_1 : \mathcal{G}_0^{2\otimes} \to \mathrm{Hom}(\mathcal{G}_{-1}^*, \mathcal{G}_0^*), \qquad \lhd_1 : (\mathcal{G}_{-1}^*)^{2\otimes} \to \mathrm{Hom}(\mathcal{G}_0, \mathcal{G}_{-1}).$$

This will be justified further in §5.5.1 where we show that the coadjoint action can be interpreted weak representation. More specifically, just like the product and actions in (5.2.2) contribute to defining dually some (crossed) relations, the cocycle $\mathcal{T}$ should also contribute dually to the adjoint action. This is what $\rhd_1$ and $\lhd_1$ stand for, as we will see in (5.4.1).

To construct non-skeletal weak quantum 2-doubles, one must explicitly keep track of how $\mathcal{T}, \mathcal{T}^*, \rhd_1, \lhd_1$ appear in the crossed-relations (5.2.3), (5.2.4), (5.2.5). For clarity and brevity, we will restrict for now to the skeletal case when defining the quantum double.

5.4.1 Matched pair of skeletal weak 2-bialgebras

Though the situation is drastically simplified in the skeletal case $t = 0$, it is now important for us to keep track of the associators. We shall do this by using the notation of (5.3.2).

The non-trivial crossed relations (5.2.7), in particular, are attached with the components $\rhd_1, \lhd_1$ of the coadjoint actions,

$$\begin{aligned} (x)\lhd_1^{f,f'} &: x \rhd_{-1} (ff') \xrightarrow{\sim} (x_{(1)}\bar{\rhd} f_{(1)}) \cdot^* ((x_{(2)}\bar{\lhd} f_{(2)})\bar{\rhd} f'), \\ \rhd_1^{x,x'}(f) &: (xx') \lhd_{-1} f \xrightarrow{\sim} (x\bar{\lhd}(x'_{(1)}\bar{\rhd} f_{(1)})) \cdot (x_{(2)}\bar{\lhd} f_{(2)}), \end{aligned}$$

where we have made use of the shorthand notation defined in *Remark 5.2.1*. These come together to allow us to define a Hochschild 3-cochain on the quantum 2-double $D(\mathcal{G})$,

$$\mathcal{T}_D : D(\mathcal{G})_0^{3\otimes} \to D(\mathcal{G})_{-1}, \qquad \mathcal{T}_D(w, w', w'') = \begin{cases} \mathcal{T}(x, x', x'') \\ \rhd_1^{x,x'}(f'') \\ (x)\lhd_1^{f',f''} \\ \mathcal{T}^*(f, f', f'') \end{cases}, \tag{5.4.1}$$

where $w = (x, f) \in D(\mathcal{G})_0$ is a degree-0 element, with $x \in \mathcal{G}_0$ and $f \in \mathcal{G}_{-1}^*$.

Definition 5.4.1. The pair $(\mathcal{G}, \mathcal{G}^*)$ of mutually paired weak skeletal 2-bialgebras forms a **(skeletal) matched pair** iff, in addition to the compatibility conditions (5.2.3)-(5.2.6), the 3-cochain $\mathcal{T}_D$ defined in (5.4.1) is a Hochschild 3-cocycle on $D(\mathcal{G}) \cong \mathcal{G} \otimes \mathcal{G}^*$.

For arguments contained solely in $\mathcal{G}_0$ or $\mathcal{G}_{-1}^*$, this condition merely states the 3-cocycle conditions for $\mathcal{T}, \mathcal{T}^*$, respectively. The other ones mix non-trivially the different components of the 3-cocycle $\mathcal{T}_D$,

$$\begin{aligned} x_1 \rhd_0 (\rhd_1^{x_2,x_3}(f)) - \mathcal{T}(x_1, x_2, x_3) \lhd_0 f &= \rhd_1^{x_1x_2,x_3}(f) - \rhd_1^{x_1,x_2x_3}(f) + \mathcal{T}(x_1, x_2, x_3 \lhd_{-1} f), \\ x_1 \cdot (x_2) \lhd_1^{f_1,f_2} - \rhd_1^{x_1,x_2}(f_1) \cdot^* f_2 &= (x_1x_2) \lhd_1^{f_1,f_2} - (x_1) \lhd_1^{x_2 \rhd_{-1} f_1, f_2} + \rhd_1^{x_1,x_2}(f_1f_2), \\ x \rhd_0 \mathcal{T}^*(f_1, f_2, f_3) - ((x)\lhd_1^{f_1,f_2}) \lhd_0 f_3 &= \mathcal{T}^*(x \rhd_{-1} f_1, f_2, f_3) - (x) \lhd_1^{f_1f_2,f_3} + (x) \lhd_1^{f_1,f_2f_3} \end{aligned} \tag{5.4.2}$$

Then, we construct $D(\mathcal{G})$ as a 2-bialgebra as in §5.2.

Since we are in the skeletal case, it is easy to see from (5.3.1) that the quantum 2-double is *weakly* self-dual $D(\mathcal{G}) \sim D(\mathcal{G})^*$, where we recall $\sim$ denotes equivalence of 2-algebras under the classification result **Theorem 5.1.1**. This means that the associated Hochschild 3-cocycles $\mathcal{T}_D, \mathcal{T}_D^*$ are cohomologous, where

$$\mathcal{T}_D^* : D(\mathcal{G})_0^{3\otimes} \to D(\mathcal{G})_{-1}, \qquad \mathcal{T}_D^*(w, w', w'') = \begin{cases} \mathring{\mathcal{T}}(f, f', f'') \\ \mathring{\rhd}_1^{f,f'}(x'') \\ (f)\mathring{\lhd}_1^{x',x''} \\ \mathring{\mathcal{T}}^*(x, x', x'') \end{cases},$$

denotes the dual of the 3-cocycle $\mathcal{T}_D$. The "dual" version of (5.4.2) reads

$$\begin{aligned} f_1 \lhd_0 (\mathring{\rhd}_1^{f_2,f_3}(x)) - \mathring{\mathcal{T}}(f_1, f_2, f_3) \rhd_0 x &= \mathring{\rhd}_1^{f_1f_2,f_3}(x) - \mathring{\rhd}_1^{f_1,f_2f_3}(x) + \mathring{\mathcal{T}}(f_1, f_2, f_3 \lhd_{-1} x), \\ f_1 \cdot^* (f_2)\mathring{\lhd}_1^{x_1,x_2} - \mathring{\rhd}_1^{f_1,f_2}(x_1) \cdot x_2 &= (f_1f_2)\mathring{\lhd}_1^{x_1,x_2} - (f_1)\mathring{\lhd}_1^{f_2 \lhd_{-1} x_1, x_2} + \mathring{\rhd}_1^{f_1,f_2}(x_1x_2), \\ f \lhd_0 \mathring{\mathcal{T}}^*(x_1, x_2, x_3) - ((f)\mathring{\lhd}_1^{x_1,x_2}) \rhd_0 x_3 &= \mathring{\mathcal{T}}^*(f \rhd_{-1} x_1, x_2, x_3) - (f)\mathring{\lhd}_1^{x_1x_2,x_3} + (f)\mathring{\lhd}_1^{x_1,x_2x_3} \end{aligned} \tag{5.4.3}$$

It is important to note that the components $\rhd_1, \lhd_1$ do *not* form Hochschild 3-cocycles by themselves, and similarly for the components $\mathring{\rhd}_1, \mathring{\lhd}_1$.

5.4.2 Factorizability of weak 2-bialgebras

We now prove the analogue of **Theorem 5.2.1**.

Theorem 5.4.1. *Suppose $(\mathcal{K}, \hat{\cdot}, \mathcal{T}_K)$ is a weak 2-bialgebra that weakly factors into two skeletal weak sub-2-bialgebras $\mathcal{G}, \mathcal{H}$, namely the inclusions in the span (5.2.16) are weak homomorphisms as defined in (5.3.1), then $\mathcal{K} \sim D(\mathcal{G})$ are equivalent as 2-bialgebras.*

Recall two weak 2-bialgebras are equivalent when there exists an invertible weak 2-homomorphism (5.3.1) between them.

Proof. The fact that $\mathcal{K}$ factors into skeletal 2-subalgebras means that it must also be skeletal itself. This allows us to leverage the proof of **Theorem 5.2.1** to reconstruct the underlying 2-bialgebra structure of $\mathcal{K} \cong D(\mathcal{G})$ as a quantum 2-double.

The subtlety here is that we must now keep track of the 3-cocycle $\mathcal{T}_K : \mathcal{K}_0^{3\otimes} \to \mathcal{K}_{-1}$ in $\mathcal{K}$ when we, in particular, invoke associativity in the form

$$\begin{aligned} \mathcal{T}_K(\iota_0(x), \iota_0(x'), \jmath_0(f)) &\equiv \rhd_1^{x,x'}(f) : (\iota_0(x)\hat{\cdot}\iota_0(x'))\hat{\cdot}\jmath_0(f) \xrightarrow{\sim} \iota_0(x)\hat{\cdot}(\iota_0(x')\hat{\cdot}\jmath_0(f)), \\ \mathcal{T}_K(\iota_0(x), \jmath_0(f), \jmath_0(f')) &\equiv (x)\lhd_1^{f,f'} : (\iota_0(x)\hat{\cdot}\jmath_0(f))\hat{\cdot}\jmath_0(f') \xrightarrow{\sim} \iota_0(x)\hat{\cdot}(\jmath_0(f)\hat{\cdot}\jmath_0(f')). \end{aligned}$$

Now in the skeletal case, the braiding map $\Psi = (\Psi_0, \Psi_{-1}; \bar{\Psi}) : \mathcal{G} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{G}$ is still defined as in (5.2.17). However, the components $\rhd_1, \lhd_1$ now give rise to associators

$$\begin{aligned} \rhd_1 &: \Psi \circ (\hat{\cdot} \otimes \mathrm{id}) \overset{\sim}{\Longrightarrow} (\mathrm{id} \otimes \hat{\cdot}) \circ \Psi_{12} \circ \Psi_{23}, \\ \lhd_1 &: \Psi \circ (\mathrm{id} \otimes \hat{\cdot}) \overset{\sim}{\Longrightarrow} (\hat{\cdot} \otimes \mathrm{id}) \circ \Psi_{23} \circ \Psi_{12} \end{aligned} \tag{5.4.4}$$

that implement the braiding relations (5.2.19). These braiding associators satisfy a set of algebraic conditions following from the 3-cocycle condition (5.4.2) for $\mathcal{T}_K$.

With the components $\rhd_1, \lhd_1$ as defined above, we now wish to reconstruct the Hochschild 3-cocycles $\mathcal{T}_G, \mathcal{T}_H$ of $\mathcal{G}, \mathcal{H}$ from $\mathcal{T}_K$. Note this cannot be achieved by just restricting $\mathcal{T}_K$ via the span (5.2.16), as this does not have the desired codomains. For instance, the restriction $\mathcal{T}_K|_{\mathrm{im}\,\iota = \mathcal{G}} : G_0^{3\otimes} \to \mathcal{K}_{-1} \cong \mathcal{G}_{-1} \otimes \mathcal{H}_{-1}$ in general lands in the tensor product, for which only the $\mathcal{G}_{-1}$-valued component gives the desired 3-cocycle $\mathcal{T}_G$ on $\mathcal{G}$. Nevertheless, with $\mathcal{T}_G, \mathcal{T}_H$ defined in this way, having the span (5.2.16) means that the 3-cocycle condition for $\mathcal{T}_K$ implies $(\mathcal{G}, \mathcal{T}_G), (\mathcal{H}, \mathcal{T}_H)$ form a matched pair of weak 2-bialgebras, as in (5.4.2).

The "undesirable" piece $\tilde{\mathcal{T}}_G$, namely the component of $\mathcal{T}_K|_\mathcal{G}$ valued in $\mathcal{H}_{-1}$, is a Hochschild coboundary. This follows from the definition of the inclusion $\iota = (\iota_{-1}, \iota_0, \iota_1) : \mathcal{G} \hookrightarrow \mathcal{K}$ as a weak homomorphism. Indeed, by projecting the last of (5.3.1) for ι_1 to $\mathcal{H}$, the first term $\iota_{-1}(\mathcal{T}_\mathcal{G}(x, x', x''))|_\mathcal{H} = 0$ vanishes whence

$$\begin{aligned} \tilde{\mathcal{T}}_G &\equiv \mathcal{T}_K(\iota_0(x), \iota_0(x'), \iota_0(x''))|_\mathcal{H} \\ &= \iota_0(x)\hat{\cdot}\iota_1(x', x'')|_\mathcal{H} - \iota_1(xx', x'')|_\mathcal{H} + \iota_1(x, x'x'')|_\mathcal{H} - \iota_1(x, x')|_\mathcal{H}\hat{\cdot}\iota_0(x'') \\ &= d_{HH}[\iota_1|_\mathcal{H}](x, x', x''), \end{aligned}$$

where d_{HH} is the Hochschild differential [93]. Similar arguments show that $\tilde{\mathcal{T}}_H = d_{HH}[\jmath_1|_{\mathcal{G}}]$ is a Hochschild coboundary as well. This establishes the weak equivalence $\mathcal{K} \sim D(\mathcal{G})$.

The same argument as above, but dualized, is applied to reconstruct $(\Delta_G)_1$ and $(\Delta_H)_1$ from the coassociator $(\Delta_K)_1$. The coassociator conditions (5.3.3)-(5.3.5), as well as (5.3.7), for them follow from those for $(\Delta_K)_1$. □

Note the coadjoint actions $\triangleright, \triangleleft$ only define genuine algebra representations when $\mathcal{T}, \mathcal{T}^* = 0$ (as in **Theorem 5.2.1**), or when $t, t^* = 0$. Without skeletality, the braiding transposition Ψ is no longer of the form given in (5.2.17). Terms like $\triangleright_1^{t\cdot,\cdot}, \triangleleft_1^{t^*\cdot,\cdot}$ must now appear. This, of course, would modify (5.2.19) in a complicated and intricate manner.

Remark 5.4.1. If the components $\iota_1, \jmath_1$ are not required as part of the data for the inclusions $\iota, \jmath$ in the span (5.2.16), then $\mathcal{K} \not\sim D(\mathcal{G})$ in general. In particular, without the component ι_1 trivializing $\tilde{\mathcal{T}}_G$ by (5.3.1), its (possibly non-trivial) Hochschild class $[\tilde{\mathcal{T}}_G] \in HH^3(\mathcal{K}_0, \mathcal{K}_{-1})$ is in fact an *extra* piece of data in $\mathcal{K}$ that is not in $D(\mathcal{G})$, despite them sharing the same 2-bialgebra structure. Such a factorizable weak 2-bialgebra is still weakly self-dual $\mathcal{K} \sim \mathcal{K}^*$.

In the following, we shall shift gears a bit and study the 2-representation theory of quasi-triangular 2-bialgebras.

5.5 The monoidal 2-category of 2-representations

With the above algebraic machinery in place, we are now ready to discuss the 2-representations of a strict or weak 2-bialgebra $\mathcal{G}$. In the following, we shall follow the Baez-Crans definition of a 2-vector space and the monoidal 2-category $\mathsf{2Vect}^{BC}$ they form [98, 143].

Definition 5.5.1. A 2-vector space is a 2-term cochain complex of vector spaces; equivalently, a 2-vector space is a *nuclear* 2-algebra [93], or an Abelian Lie 2-algebra [95, 96].

2-vector spaces of this type form a 2-category $\mathsf{2Vect}^{BC}$ in which the 1-morphisms are cochain maps and 2-morphisms are cochain homotopies. Concretely, let $V = V_{-1} \xrightarrow{\partial} V_0, W = W_{-1} \xrightarrow{\partial'} W_0$ denote two 2-vector spaces. A cochain map $f : V \to W$ is a collection linear maps $f_{0,-1} : V_{0,-1} \to W_{0,-1}$ such that

$$\partial' f_{-1} = f_0 \partial.$$

Given two such cochain maps f, g, a cochain homotopy $q : f \Rightarrow g$ is a linear map $q : V_0 \to W_{-1}$ such that

$$\partial q = f_0 - g_0, \qquad q\partial = f_{-1} - g_{-1}.$$

We shall refine these notions to fit the definition of a 2-representation of $\mathcal{G}$ in the following.

5.5.1 Weak 2-representations

Recall that a representation of an ordinary algebra A on the vector space V is an algebra homomorphism $A \to \mathrm{End}(V)$. Morally, a 2-representation should therefore be a 2-algebra homomorphism between a 2-algebra $\mathcal{G}$ and a "categorified" notion of the endomorphism algebra $\mathrm{End}(V)$. Correspondingly, a weak 2-representation should be a weak 2-homomorphism as in (5.3.1) into a "weak endomorphism 2-algebra".

Endomorphism 2-algebra on a 2-vector space

In the strict case, the endomorphisms of a 2-vector space are naturally given in the setting of $\mathsf{2Vect}^{BC}$ — namely $\mathrm{End}(V) = \mathrm{End}_{\mathsf{2Vect}^{BC}}(V)$, which forms an associative 2-algebra $\mathrm{End}(V) = \mathrm{End}(V)_{-1} \xrightarrow{\delta} \mathrm{End}(V)_0$ of linear transformations on a 2-term cochain complex V [125],

$$\begin{aligned} \mathrm{End}(V)_0 &= \{(M,N) \in \mathrm{End}(V_{-1}) \times \mathrm{End}(V_0) \mid \partial M = N\partial\}, \\ \mathrm{End}(V)_{-1} &= \{A \in \mathrm{Hom}(V_0, V_{-1}) \mid (A\partial, \partial A) \in \mathrm{End}(V_{-1}) \times \mathrm{End}(V_0)\}, \end{aligned}$$

equipped with the 2-algebra structure (take $A \in \mathrm{End}(V)_{-1},\ (M,N) \in \mathrm{End}(V)_0$)

$$\delta : A \mapsto (A\partial, \partial A), \qquad (M,N)\cdot A = MA, \qquad A \cdot (M,N) = AN.$$

The associativity of matrix multiplication implies that $\mathrm{End}(V)_{-1}$ is clearly a $\mathrm{End}(V)_0$-bimodule, Moreover, we have the Peiffer conditions (note $A, A' \in \mathrm{End}(V)_{-1}$)

$$\begin{aligned} \delta((M,N)\cdot A) &= (MA\partial, \partial MA) = (MA\partial, N\partial A) = (M,N)\delta(A), \\ \delta(A\cdot(M,N)) &= (AN\partial, \partial AN) = (A\partial M, \partial AN) = \delta(A)(M,N), \\ A * A' &\equiv \delta(A)\cdot A' = A\partial A' = A \cdot \delta(A'), \end{aligned}$$

and hence $\mathrm{End}(V)$ is an associative 2-algebra. Note that none of the matrices here are required to be invertible.

As weak 2-algebras are no longer associative, the above presentation of $\mathrm{End}(V)$ in terms of matrices is no longer sufficient: we require a weaker version of $\mathrm{End}(V)$. Such a notion of the weak endomorphism 2-algebra $\mathfrak{End}(V)$ would still have the same graded structure $\delta : \mathfrak{End}(V)_{-1} \to \mathfrak{End}(V)_0$ as in the strict case above, but its algebra structure should have its associativity controlled by a Hochschild 3-cocycle $\mathfrak{T}$, in accordance with **Definition 5.3.1**.

To begin, we extend the idea of [144] to weak 2-algebras. In essence, we leverage the observation in the strict case that an algebra 2-homomorphism $\mathcal{G} \to \mathfrak{End}(V)$ is equivalent to a $\mathcal{G}$-bimodule structure on V. We are going to provide a weak generalization of such a $\mathcal{G}$-bimodule structure in **Definition** 5.5.2.

Let $\mathsf{2Alg}$ denote the category of weak 2-algebras $(\mathcal{G}, \mathcal{T})$, which contains the full subcategory $\mathsf{2Alg}_{\mathrm{ass}}$ of *strict* 2-algebras. A Baez-Crans 2-vector space $V \in \mathsf{2Vect}^{BC} \subset \mathsf{2Alg}_{\mathrm{ass}} \subset \mathsf{2Alg}$ fits as a strict 2-algebra with trivial multiplication. We consider $\mathcal{G}$ a a weak 2-algebra (as defined in **Definition 5.3.1**). We then equip the direct sum $\mathcal{G} \oplus V$ with a semidirect product structure,

$$(z+u)\cdot(z'+u') \quad = \quad yy' + x \cdot_l y' + y \cdot_r x' + xx'$$

$$+ x \rhd w' + x \rhd v' + y \succ w' + y \succ v'$$
$$+ w \lhd x' + v \lhd x' + w \prec y' + v \prec y',$$

where we have used the shorthand notation $z = (y, x) \in \mathcal{G}_{-1} \times \mathcal{G}_0 = \mathcal{G}$, $u = (w, v) \in V_{-1} \times V_0 = V$ and where

$$\begin{aligned} \cdot_l : \mathcal{G}_0 \otimes \mathcal{G}_{-1} \to \mathcal{G}_{-1}, \qquad & \cdot_r : \mathcal{G}_{-1} \otimes \mathcal{G}_0 \to \mathcal{G}_{-1}, \\ \rhd : \mathcal{G}_0 \otimes V \to V, \qquad & \lhd : V \otimes \mathcal{G}_0 \to V, \\ \succ : \mathcal{G}_{-1} \otimes V \to V, \qquad & \prec : V \otimes \mathcal{G}_{-1} \to V \end{aligned}$$

are all bilinear maps.

Definition 5.5.2. We say that V is a **$\mathcal{G}$-bimodule** if $(\mathcal{G} \oplus V, \cdot) \in \mathsf{2Alg}$ is a weak 2-algebra. In other words,

(i) $(\mathcal{G} \oplus V)_{-1} \equiv \mathcal{G}_{-1} \oplus V_{-1}$ is a weak $(\mathcal{G} \oplus V)_0 := \mathcal{G}_0 \oplus V_0$-bimodule,

(ii) the map $t \oplus \partial : \mathcal{G}_{-1} \oplus V_{-1} \to \mathcal{G}_0 \oplus V_0$ is equivariant with respect to $\cdot$ and satisfies the Peiffer identity[4],

(iii) there exists a well-defined trilinear invertible map $(\mathcal{G}_0 \oplus V_0)^{3\otimes} \to \mathcal{G}_{-1} \oplus V_{-1}$ that satisfies the Hochschild 3-cocycle condition.

An equivalent characterization of weak $\mathcal{G}$-modules can be obtained as follows. By the macroscopic principle [16], there exists a k-linear 2-category $\mathsf{2Vect}^{hBC}$ of *homotopy* Baez-Crans 2-vector spaces, whose algebra objects in which are precisely two-term A_∞-algebras. For each $V \in \mathsf{2Vect}^{hBC}$, we call $\mathfrak{End}(V) = \mathrm{End}_{\mathsf{2Vect}^{hBC}}(V)$ the **weak endomorphism 2-algebra** on V, and denote by $\mathfrak{T} : \mathfrak{End}(V)_0^{3\otimes} \to \mathfrak{End}(V)_{-1}$ its Hochschild 3-cocycle. It is easy to see that a weak $\mathcal{G}$-module structures on V as given in **Definition 5.5.2** are in one-to-one correspondence with A_∞-algebra maps $\mathcal{G} \to \mathfrak{End}(V)$. This motivates our following theory of weak 2-representations.

Remark 5.5.1. We emphasize here that the 2-category $\mathsf{2Vect}^{BC}$ of Baez-Crans 2-vector spaces is completely strict [98], and hence its algebra objects (ie. associative 2-algebras/algebra crossed-modules) and its endomorphism categories $\mathrm{End}(V) = \mathrm{End}_{\mathsf{2Vect}^{BC}}(V)$ do not carry non-trivial homotopy data. Weak 2-algebras/2-term A_∞-algebras are therefore do not live in $\mathsf{2Vect}^{BC}$, but instead in its homotopy refinement $\mathsf{2Vect}^{hBC}$. The difference between the setting $\mathsf{2Vect}^{hBC}$ and the Kapranov-Voevodsky setting $\mathsf{2Vect}^{KV}$ is currently under investigation by the author; however, I have proven in [119] (see also §5.7) that 2-representation theory based on $\mathsf{2Vect}^{hBC}$ and those [59, 64, 65, 145] based on Vect^{KV} share the same homotopy coherences.

[4]The Peiffer identity states $y \succ w = (ty) \rhd w = y \succ (\partial w)$, and similarly $w \prec y = (\partial w) \prec y = w \lhd (ty)$. If we write $y \succ v = \Upsilon_y v$, then we reproduce precisely the 2-representation properties (5.2.9).

Weak 2-representations, weak 2-intertwiners and modifications

Definition 5.5.3. A **weak 2-representation** $(\varrho, \rho) : \mathcal{G} \to \mathfrak{End}(V)$ of $\mathcal{G}$ on V is a homomorphism between weak 2-algebras as in (5.3.1). In other words, $\rho = (\rho_0, \rho_1)$ is a chain map

$$\begin{tikzcd} \mathcal{G}_{-1} \arrow[r, "t"] \arrow[d, "\rho_1"] & \mathcal{G}_0 \arrow[d, "\rho_0"] \\ \mathfrak{End}(V)_{-1} \arrow[r, "\delta"] & \mathfrak{End}(V)_0 \end{tikzcd} \tag{5.5.1}$$

which preserves the 2-algebra structures *up to homotopy*,

$$\begin{aligned} \delta\varrho(x, x') &= \rho_0(xx') - \rho_0(x)\rho_0(x'), \\ \varrho(x, ty) &= \rho_1(x \cdot y) - \rho_0(x) \cdot \rho_1(y), \\ \varrho(ty, x) &= \rho_1(y \cdot x) - \rho_1(y) \cdot \rho_0(x), \end{aligned} \tag{5.5.2}$$

and for which the Hochschild 3-cocycles $\mathcal{T}, \mathfrak{T}$ of respectively $\mathcal{G}$ and $\mathfrak{End}(V)$ satisfy the following compatibility conditions

$$\begin{aligned} \rho_1(\mathcal{T}(x, x', x'')) = {} & \rho_0(x) \cdot \varrho(x', x'') - \varrho(xx', x'') \\ & + \varrho(x, x'x'') - \varrho(x, x') \cdot \rho_0(x'') \\ & + \mathfrak{T}(\rho_0(x), \rho_0(x'), \rho_0(x'')), \end{aligned} \tag{5.5.3}$$

where $x, x', x'' \in \mathcal{G}_0$ and $y \in \mathcal{G}_{-1}$. We require ϱ to be invertible.

We call ρ a **strict 2-representation** if $\varrho = 0$ identically.

As $\mathcal{T}, \mathfrak{T}$ are normalized, ϱ by definition vanishes if any of its arguments are 0 or the unit $\eta_0 \in \mathcal{G}_0$.

Remark 5.5.2. Due to the classification **Theorem 5.1.1** of 2-algebras [93], a non-trivial 2-algebra $\mathcal{G}$ with $\mathcal{T} \neq 0$ cannot admit a strict 2-representation. Conversely, however, 2-representations of a strict 2-algebra can still be weak, as (5.5.3) only states that the *cohomology class* of $\mathfrak{T}$ is trivial, not that it is trivial as a 3-cocycle. However, if we further restrict to the case where V is a strict $\mathcal{G}$-bimodule (ie. the trilinear map in **Definition 5.5.2** vanishes), then $\mathfrak{T} = 0$ and $\mathfrak{End}(V)$ is isomorphic to $\mathrm{End}(V)$.

Example: weak coadjoint representation. A very natural example of a 2-representation is achieved by dualizing, using (5.2.1), the 2-representation $\mathcal{G} \to \mathrm{End}(\mathcal{G})$ given by the weak 2-algebra structure of $\mathcal{G}$ on itself.

This gives rise to the *coadjoint representation* (cf. [95, 115]) $\bar{\rhd} = (\rhd_1, (\rhd_0, \rhd_{-1}), \Upsilon) : \mathcal{G} \to \mathfrak{End}(\mathcal{G}^*)$ of $\mathcal{G}$ on its dual $\mathcal{G}^*$, given explicitly by

$$\begin{aligned} \rhd_0 : \mathcal{G}_0 \to \mathfrak{End}(\mathcal{G}_0^*), \qquad & \langle g, xx' \rangle = -\langle x \rhd_0 g, x' \rangle, \\ \rhd_{-1} : \mathcal{G}_0 \to \mathfrak{End}(\mathcal{G}_{-1}^*), \qquad & \langle f, x \cdot y \rangle = -\langle x \rhd_{-1} f, y \rangle, \\ \Upsilon : \mathcal{G}_{-1} \to \mathrm{Hom}(\mathcal{G}_{-1}^*, \mathcal{G}_0^*), \qquad & \langle f, y \cdot x \rangle = -\langle \Upsilon_y f, x \rangle \end{aligned} \tag{5.5.4}$$

and

$$\rhd_1 : \mathcal{G}_0^{2\otimes} \to \mathfrak{End}(\mathcal{G}^*)_{-1} = \operatorname{Hom}(\mathcal{G}_{-1}^*, \mathcal{G}_0^*), \qquad \langle f, \mathcal{T}(x, x', x'')\rangle = +\langle \rhd_1^{x,x'}(f), x''\rangle. \tag{5.5.5}$$

Notice a plus sign occurs here, in contrast with the rest of the components defined in (5.5.4). This is because we have dualized two elements in $\mathcal{G}$, instead of one.

Analogously, we have the coadjoint back-action $\bar{\lhd} = ((\lhd_0, \lhd_{-1}), \tilde{\Upsilon})$ of $\mathcal{G}^*$ on $\mathcal{G}$, which we write from the right[5].

Due to (5.5.2), the components of a weak 2-representation are not genuine algebra representations in general, but only up to homotopy. We have in general that

$$(xx') \rhd_0 g = x \rhd_0 (x' \rhd_0 g) + \rhd_1^{x,x'}(t^* g), \qquad (xx') \rhd_{-1} f = x \rhd_{-1} (x' \rhd_{-1} f) + t^* \rhd_1^{x,x'}(f),$$

where t^* is the dual t-map on $\mathcal{G}^*$, and

$$\Upsilon_{x\cdot y} f = x \rhd_0 (\Upsilon_y f) + \rhd_1^{x,ty}(f), \qquad \Upsilon_{y\cdot x} f = \Upsilon_y (x \rhd_{-1} f) + \rhd_1^{ty,x}(f).$$

Of course, these components reduce to genuine strict algebra representations if $\rhd_1 = 0$ or $t = 0$, which simplifies the situation considerably.

1- and 2-morphisms on the weak 2-representation 2-category. With **Definition 5.5.3** in hand, we are now ready to define the morphisms on the weak 2-representations. Let $\rho = (\varrho, \rho_0, \rho_1)$ and $\rho' = (\varrho', \rho_0', \rho_1')$ denote two weak 2-representations on $V, W \in 2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$, respectively.

Definition 5.5.4. A **weak 2-intertwiner** $i = (I, i_1, i_0) : V \to W$ consist of a 2-vector space homomorphism $(i_1, i_0) : V \to W$ together with a collection of invertible cochain homotopies $I_{x,i} : V_0 \to W_{-1}$ satisfying

$$\partial I_{x,i} = i_0 \circ \rho_0^0(x) - \rho_0'^0(x) \circ i_0, \qquad I_{x,i}\partial = i_1 \circ \rho_0^1(x) - \rho_0'^1(x) \circ i_1$$

for each $x \in \mathcal{G}_0$, as well as

$$I_{ty,i} = i_1 \circ \rho_1(y) - \rho_1'(y) \circ i_0$$

for each $y \in \mathcal{G}_{-1}$. Moreover, $I_{\bullet,i}$ trivializes $\varrho - \varrho'$ as a Hochschild 2-cocycle, in the sense that for each $x, x' \in \mathcal{G}_0$,

$$\mathrm{id}_i \otimes \varrho(x, x') - \varrho'(x, x') \otimes \mathrm{id}_i = \mathrm{id}_{\rho_0(x)} \otimes I_{x',i} - I_{xx',i} + I_{x,i} \otimes \mathrm{id}_{\rho_0(x')}, \tag{5.5.6}$$

where $\mathrm{id}_i : i \Rightarrow i$ denotes the identity cochain homotopy on the intertwiner i.

[5] This means that we have, for instance, $\langle g \cdot^* f, x\rangle = -\langle g, x \lhd_{-1} f\rangle$ and $\langle f \cdot^* g, x\rangle = -\langle f, x\tilde{\Upsilon}_g\rangle$.

In other words, a weak 2-intertwiner $i : V \to W$ is such that the following diagrams

$$
\begin{tikzcd}
V_{-1} \arrow[rr, "\partial"] \arrow[dr, "i_1"] \arrow[dd, "\rho_0^1"'] & & V_0 \arrow[dr, "i_0"] \arrow[dd, bend left, "\rho_0^0"' near start] & \\
& W_{-1} \arrow[rr, bend left, "\partial'"] \arrow[dd, bend left, "\rho_0'^1"' near start] & & W_0 \arrow[dd, "\rho_0'^0"] \\
V_{-1} \arrow[rr, bend left, "\partial" near start] \arrow[dr, "i_1"] & & V_0 \arrow[dr, "i_0"] & \\
& W_{-1} \arrow[rr, "\partial'"] & & W_0
\end{tikzcd}
,\qquad
\begin{tikzcd}
V_0 \arrow[r, "\rho_1"] \arrow[d, "i_0"] & V_{-1} \arrow[d, "i_1"] \\
W_0 \arrow[r, "\rho_1'"] & W_{-1}
\end{tikzcd}
\tag{5.5.7}
$$

commute *up to a natural invertible 2-morphism* given by $I_{\bullet,i}$. By definition, we have $I_{0,i} = I_{\eta_0,i} = 0$ where η_0 is the unit of $\mathcal{G}_0$.

Now let $i, i' : \rho \to \rho'$ denote two weak 2-intertwiners, we have the following.

Definition 5.5.5. A **modification** $\mu : i \Rightarrow i'$ between two weak 2-intertwiners is a $\mathcal{G}$-equivariant cochain homotopy

$$
\begin{tikzcd}
V_{-1} \arrow[r, "\partial"] \arrow[d, "i_1 - i_1'"'] & V_0 \arrow[d, "i_0 - i_0'"] \arrow[dl, "\mu"'] \\
W_{-1} \arrow[r, "\partial'"] & W_0
\end{tikzcd}
,
\tag{5.5.8}
$$

where μ intertwines between $\rho_1(y), \rho_1'(y)$ for each $y \in \mathcal{G}_{-1}$, as cochain homotopies. Moreover, μ trivializes $I_{\cdot,i} - I_{\cdot,i'}$ as a Hochschild 1-cocycle, in the sense that

$$
I_{x,i} - I_{x,i'} = \mathrm{id}_{\rho_0(x)} \otimes \mu - \mu \tag{5.5.9}
$$

for all $x \in \mathcal{G}_0$, as a relation between cochain homotopies.

We shall denote by $2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$ the 2-category of weak 2-representations of the weak 2-bialgebra $(\mathcal{G}, \mathcal{T})$, consisting of weak 2-representation (V, ρ) objects, weak 2-intertwiners i as 1-morphisms and modifications μ as 2-morphisms. We devote the remainder of this section to proving that $2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$ forms a monoidal 2-category.

5.5.2 Monoidal structure on the 2-representations

Recall that vector space cochain complexes come equipped with natural notions of direct sum $\oplus$, as well as tensor product $\otimes$, which satisfy the distributive law

$$
V \otimes (W \oplus U) = (V \otimes W) \oplus (V \otimes U),
$$

where V, W, U are vector space cochains. For 2-vector spaces (or equivalently two-term cochain complexes of vector spaces [98]), the direct sum is given simply by

$$
V \oplus W = V_{-1} \oplus W_{-1} \xrightarrow{\partial \oplus \partial'} V_0 \oplus W_0,
$$

while the tensor product is given by the following 3-term complex (cf. [95])

$$V \otimes W = \underbrace{V_{-1} \otimes W_{-1}}_{\deg=-2} \xrightarrow{D^+} \underbrace{V_{-1} \otimes W_0 \oplus V_0 \otimes W_{-1}}_{\deg=-1} \xrightarrow{D^-} \underbrace{V_0 \otimes W_0}_{\deg=0}, \tag{5.5.10}$$

where $D^\pm = \pm 1 \otimes \partial' + \partial \otimes 1$ is the tensor extension of the differentials $\partial : V_{-1} \to V$ and $\partial' : W_{-1} \to W_0$.

We endow the direct sum and tensor product structure on 2-representations of $\mathcal{G}$ in the same way as above. Note the direct double $\mathcal{G}^{2\oplus}$ and the tensor square $\mathcal{G}^{2\otimes}$ of a strict 2-algebra $\mathcal{G}$ also have the same structure.

Direct sums. For the direct sum 2-representation, this is simply accomplished by extending **Definition 5.5.3** to a direct sum of 2-algebra homomorphisms

$$(\varrho, \rho) \oplus (\varrho', \rho') = (\varrho \oplus \varrho', \rho \oplus \rho') : \mathcal{G} \oplus \mathcal{G} \to \mathfrak{End}(V) \oplus \mathfrak{End}(W).$$

In particular, the direct sum $V \oplus W$ of 2-representations of $\mathcal{G}$ is given by the components

$$(\rho \oplus \rho')_0^0 = \rho_0^0 \oplus \rho'^0_0, \qquad (\rho \oplus \rho')_0^1 = \rho_0^1 \oplus \rho'^1_0, \qquad (\rho \oplus \rho')_1 = \rho_1 \oplus \rho'_1$$

such that the square (5.5.1) commutes,

$$(\rho \oplus \rho')_0 \circ (t \oplus t) = (\delta \oplus \delta') \circ (\rho \oplus \rho')_1,$$

where δ, δ' are the differentials of the two 2-algebras $\mathfrak{End}(V), \mathfrak{End}(W)$, respectively. The zero 2-representation under direct sum is of course the trivial complex $0 \to 0$.

Tensor product

As in the 1-bialgebra case, the tensor product of 2-representations is accomplished by precomposing with the coproduct. However, the graded components of the coproduct $\Delta = \Delta_{-1} + \Delta_0$ in (5.1.5), as well as Δ_0' in (5.1.8), allows us to define the tensor product between 2-representations $V \otimes W$

$$\rho_{V\otimes W}(x) = \big((\rho_V)_0 \otimes (\rho_W)_0\big) \circ \Delta_0'(x), \quad x \in \mathcal{G}_0, \tag{5.5.11}$$

as well as its weak component (cf. **Definition 5.5.3**)

$$\varrho_{V\otimes W}(x, x') = \varrho_V(\bar{x}_{(1)}, \bar{x}'_{(1)}) \otimes \varrho_W(\bar{x}_{(2)}, \bar{x}'_{(2)}), \qquad x, x' \in \mathcal{G}_0.$$

We also have the tensor product between a 2-intertwiner $i : V \to U$ and a 2-representation

$$\begin{aligned} \rho_{i\otimes W}(x) &= \big((\rho_U)_1 \circ i \otimes (\rho_W)_0\big) \circ \Delta_0^l(x) + (-1)^{\deg}\big(i \circ (\rho_V)_0 \otimes (\rho_W)_1\big) \circ \Delta_0^r(x), \\ \rho_{W\otimes i}(x) &= \big((\rho_W)_0 \otimes (\rho_U)_1 \circ i\big) \circ \Delta_0^r(x) + (-1)^{\deg}\big((\rho_W)_1 \otimes i \circ (\rho_V)_0\big) \circ \Delta_0^l(x) \end{aligned} \tag{5.5.12}$$

for each $x \in \mathcal{G}_0$, where the sign depends on the degree of the components in (5.5.10). Lastly, the tensor product between 2-intertwiners $i : V \to U, j : W \to T$ is given by just

$$\rho_{i\otimes j}(y) = \big((\rho_U)_1 \circ i \otimes (\rho_T)_1 \circ j + (-1)^{\deg} i \circ (\rho_V)_1 \otimes j \circ (\rho_W)_1\big) \circ \Delta_{-1}(y) \tag{5.5.13}$$

for each $y \in \mathcal{G}_{-1}$. This defines the invertible natural 2-morphism $I_{i\otimes j,\bullet}$ (cf. **Definition 5.5.4**).

The fact that (5.5.11), (5.5.12), (5.5.13) define genuine 2-representations (up to the homotopy ϱ; cf. **Definition 5.5.3** and (5.5.2)), for instance

$$\delta\varrho_{V\otimes W}(x, x') = \rho_{V\otimes W}(xx') - \rho_{V\otimes W}(x)\rho_{V\otimes W}(x'),$$

requires the 2-bialgebra axioms (5.1.16).

Tensor unit. Now if $\mathcal{G}$ is a unital 2-bialgebra, then there is a tensor unit, denoted by $I \in 2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$ given by the ground field complex $k \xrightarrow{1} k$, and a unit 2-intertwiner given by the identity $\mathrm{id}_I : 1 \to 1$, such that $\mathcal{G}$ acts on them through multiplication of the counit ϵ,

$$\rho_I(x) = \epsilon_0(x), \qquad \rho_{\mathrm{id}_I}(y) = \epsilon_{-1}(y).$$

From (5.5.2), the corresponding component $\varrho = \mathrm{id}$ for the tensor unit I is clearly the identity 2-morphism. In according with (5.5.11), (5.5.12), (5.5.13), the condition (5.1.12) then implies that the left- and right-unitor morphisms in $2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$ are all 1- and 2-isomorphisms. For instance, (5.1.13) implies

$$\rho_{V\otimes 1} = \rho_V = \rho_{1\otimes V},$$

whence $V \otimes 1, 1 \otimes V$ and V coincides as 2-representations.

Due to this, all coherence diagrams in $2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$ concerning the unitors, such as the homotopy triangle and the zig-zag axioms [79, 81], are trivially satisfied, and hence we will not directly prove them. The conditions (5.1.12), (5.1.13) can of course be easily relaxed to give non-trivial coherent unitors, but we shall not consider this here.

Naturality and Gray property of the tensor product

Recall the space $\mathfrak{End}(V)_{-1}$ is modelled by cochain homotopies, which can be interpreted as "endomorphisms" on $\mathfrak{End}(V)_0$. Using this perspective, we will prove the following key results.

Lemma 5.5.1. *Let $i : V \to U$ denote a 2-intertwiner. We have the following diagrams*

$$\begin{tikzcd} V\otimes W \arrow[r, "\rho_{V\otimes W}"] \arrow[d, "i"'] \arrow[dr, phantom, "\overset{\rho_{i\otimes W}}{\Longrightarrow}"] & V\otimes W \arrow[d, "i"] \\ U\otimes W \arrow[r, "\rho_{U\otimes W}"] & U\otimes W \end{tikzcd}, \qquad \begin{tikzcd} W\otimes V \arrow[r, "\rho_{W\otimes V}"] \arrow[d, "i"'] \arrow[dr, phantom, "\overset{\rho_{W\otimes i}}{\Longrightarrow}"] & W\otimes V \arrow[d, "i"] \\ W\otimes U \arrow[r, "\rho_{W\otimes U}"] & W\otimes U \end{tikzcd}$$

in $2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$.

Proof. Let us focus first on the left diagram. The goal is to show that $\rho_{i\otimes W}$ defines a cochain homotopy which fits into the following diagram

$$\begin{tikzcd} V_{-1}\otimes W_{-1} \arrow[r, "D^+"] \arrow[d] & V_{-1}\otimes W_0 \oplus V_0\otimes W_{-1} \arrow[r, "D^-"] \arrow[d] \arrow[dl, "\rho_{i\otimes W}"'] & V_0\otimes W_0 \arrow[d] \arrow[dl, "\rho_{i\otimes W}"'] \\ U_{-1}\otimes W_{-1} \arrow[r, "D^+"'] & U_{-1}\otimes W_0 \oplus U_0\otimes W_{-1} \arrow[r, "D^-"'] & U_0\otimes W_0 \end{tikzcd},$$

where the horizontal maps are the differentials given in (5.5.10), and the vertical maps are various components of $\rho_{V\otimes W}\circ i - i\circ\rho_{V\otimes W}$.

The key is the commutation relation (5.5.1), which allows us to write

$$\delta(\rho_1(y)) = (\rho_1(y)\partial, \partial\rho_1(y)) = (\rho_0^1(ty), \rho_0^0(ty))$$

for each $y\in\mathcal{G}_{-1}$, as well as the definition (5.1.8) of Δ_0'. Directly computing, we have for the rightmost triangle

$$\begin{aligned} D^-\rho_{i\otimes W} &= \partial_U(\rho_U)_1(x^l_{(1)})\circ i\otimes(\rho_W)_0^0(x^l_{(2)}) - (-1)^{\deg} i\circ(\rho_V)_0(x^r_{(1)})\otimes\partial_W(\rho_W)_0^0(x^r_{(2)}) \\ &= (\rho_U)_0^0(tx^l_{(1)})\circ i\otimes(\rho_W)_0^0(x^l_{(2)}) - i\circ(\rho_V)_0^0(x^r_{(1)})\otimes(\rho_W)_0^0(tx^r_{(2)}) \\ &= \rho_{U\otimes W}\circ i - i\circ\rho_{V\otimes W} \end{aligned}$$

as maps on $V_0\otimes W_0$ (with deg = 0), and similarly we have for the leftmost triangle

$$\begin{aligned} \rho_{i\otimes W}D^+ &= (\rho_U)_1(x^l_{(1)})\partial_U\circ i\otimes(\rho_W)_0^1(x^l_{(2)}) + (-1)^{\deg} i\circ(\rho_V)_0(x^r_{(1)})\otimes(\rho_W)_0^0(x^r_{(2)})\partial_W \\ &= (\rho_U)_0^1(tx^l_{(1)})\circ i\otimes(\rho_W)_0^1(x^l_{(2)}) - i\circ(\rho_V)_0^1(x^r_{(1)})\otimes(\rho_W)_0^1(tx^r_{(2)}) \\ &= \rho_{U\otimes W}\circ i - i\circ\rho_{V\otimes W} \end{aligned}$$

as maps on $V_{-1}\otimes W_{-1}$ (with deg = -1).

Now consider the middle section. We need to compute

$$\begin{aligned} D^+\rho_{i\otimes W} &= (\rho_U)_0^1(tx^l_{(1)})\circ i\otimes(\rho_W)_0^0(x^l_{(2)}) - i\circ(\rho_V)_0^0(x^r_{(1)})\otimes(\rho_W)_0^1(tx^r_{(2)}), \\ \rho_{i\otimes W}D^- &= (\rho_U)_0^0(tx^l_{(1)})\circ i\otimes(\rho_W)_0^1(x^l_{(2)}) - i\circ(\rho_V)_0^1(x^r_{(1)})\otimes(\rho_W)_0^0(tx^r_{(2)}), \end{aligned}$$

and sum them to find

$$\begin{aligned} D^+\rho_{i\otimes W} + \rho_{i\otimes W}D^- &= \left[(\rho_U)_0^1(tx^l_{(1)})\otimes(\rho_W)_0^0(x^l_{(2)}) + (\rho_U)_0^0(tx^l_{(1)})\otimes(\rho_W)_0^1(x^l_{(2)})\right]\circ i \\ &\quad - i\circ\left[(\rho_V)_0^0(x^r_{(1)})\otimes(\rho_W)_0^1(tx^r_{(2)}) + (\rho_V)_0^1(x^r_{(1)})\otimes(\rho_W)_0^0(tx^r_{(2)})\right] \\ &= \rho_{U\otimes W}\circ i - i\circ\rho_{V\otimes W} \end{aligned}$$

as maps on $V_{-1}\otimes W_0\oplus V_0\otimes W_{-1}$. The other diagram is treated identically. □

We now show that (5.5.13) is in fact *not* independently defined.

Lemma 5.5.2. *If $j : W \to T$ is another 2-intertwiner, then "$i \otimes j$" decomposes as two 2-isomorphic expressions*

$$i \otimes j = i \otimes T \circ V \otimes j \cong U \otimes j \circ i \otimes W. \tag{5.5.14}$$

The homotopy $I_{i\otimes j,\bullet} = I_{i\otimes \mathrm{id}_W} * I_{\mathrm{id}_V \otimes j}$ also decomposes accordingly.

Proof. What we need to show is that $\rho_{i\otimes j} = (\rho_{i\otimes T} * \rho_{V\otimes j}) \circ t = (\rho_{U\otimes j} * \rho_{i\otimes W}) \circ t$ as 2-morphisms. Recall cochain homotopies $q : f \Rightarrow g$, $p : g \Rightarrow h$ in $\mathsf{2Vect}^{hBC}$ compose by $p * q = p\partial_U q : f \Rightarrow h$, where U is the source 2-vector space of the cochain map g. Indeed, we have

$$\begin{aligned}\partial_W(p * q) &= (\partial_W p) \circ (\partial_U q) = (g_0 - h_0) \circ (f_0 - g_0),\\ (p * q)\partial_V &= (p\partial_U) \circ (q\partial_V) = (g_{-1} - h_{-1}) \circ (f_{-1} - g_{-1})\end{aligned}$$

as desired, where W is the target of h and V is the source of f. Notice this is exactly how elements in $\mathfrak{End}(V)_{-1}$ compose, $A * A' = A\delta A$.

The goal is to prove that $D^\pm \rho_{i\otimes j}(y)$ in fact decomposes as described above for each $y \in \mathcal{G}_{-1}$. This follows from the coequivariance condition (5.1.10). By direct computation, precomposing (5.5.12) yields (here we neglect the 2-vector space subscripts for brevity)

$$\begin{aligned}\rho_{i\otimes W} \circ t &= (\rho_1 i \otimes (\rho_0 t) + (-1)^{\deg} i(\rho_0 t) \otimes \rho_1) \circ \Delta_{-1}\\ &= (\rho_1 i \otimes \partial\rho_1 + (-1)^{\deg} i(\partial\rho_1) \otimes \rho_1) \circ \Delta_{-1},\\ \rho_{U\otimes j} \circ t &= ((\rho_0 t) \otimes \rho_1 j + (-1)^{\deg} \rho_1 \otimes i(\rho_0 t)) \circ \Delta_{-1}\\ &= ((\rho_1 \partial) \otimes \rho_1 j + (-1)^{\deg} \rho_1 \otimes j(\rho_1 \partial)) \circ \Delta_{-1},\end{aligned}$$

where we have used (5.5.1) to commute the t-map past the 2-representations to the differential ∂. Using the Sweeder notation (5.1.6) for Δ_{-1}, we compute their graded composition to be

$$\begin{aligned}(\rho_{U\otimes j})(ty) * (\rho_{i\otimes W})(ty) &= (\rho_1(y_{(1)})\partial)\rho_1(y_{(1)})i \otimes \rho_1(y_{(2)})j(\partial\rho_1(y_{(2)}))\\ &\quad + (-1)^{\deg}\rho_1(y_{(1)})(i\partial\rho_1(y_{(1)})) \otimes j(\rho_1(y_{(2)})\partial)\rho_1(y_{(2)})\\ &= (\rho_1(y_{(1)}) * \rho_1(y_{(1)}))i \otimes j(\rho_1(y_{(2)}) * \rho_1(y_{(2)}))\\ &\quad + (-1)^{\deg}(\rho_1(y_{(1)}) * \rho_1(y_{(1)}))i \otimes j(\rho_1(y_{(2)}) * \rho_1(y_{(2)}))\\ &= (\rho_1 i \otimes \rho_1 j + (-1)^{\deg} i\rho_1 \otimes j\rho_1) \circ \Delta_{-1}(y) = \rho_{i\otimes j}(y)\end{aligned}$$

as desired, where we have noted the property $i_{-1}(\rho_V)_1 = (\rho_U)_1 i_0$ of the 2-intertwiners i, j to permute them past the ρ's. This proves that the 2-algebra homomorphisms $\rho_{i\otimes j} = \rho_{i\otimes T} * \rho_{V\otimes j}$ coincide. A similar argument shows that the 2-algebra homomorphisms $\rho_{i\otimes j} = \rho_{U\otimes j} * \rho_{i\otimes W}$ also coincide.

This is not sufficient to imply that $i \otimes T \circ V \otimes j = U \otimes j \circ i \otimes W$, however. Indeed, the weak component ϱ of the two decomposed 2-representations in general may differ. After some computations, one can show that we have

$$\begin{aligned}\varrho_{(i\otimes T)\circ(V\otimes j)}\circ\Delta_0(x) &= \varrho(tx^l_{(1)},x^r_{(1)})\otimes\varrho(x^l_{(2)},tx^r_{(2)})+(-1)^{\deg}\varrho(x^r_{(1)},tx^l_{(1)})\otimes\varrho(tx^r_{(2)},x^l_{(1)}),\\ \varrho_{(U\otimes j)\circ(i\otimes W)}\circ\Delta_0(x) &= \varrho(x^r_{(1)},tx^l_{(1)})\otimes\varrho(tx^r_{(2)},x^l_{(1)})+(-1)^{\deg}\varrho(tx^l_{(1)},x^r_{(1)})\otimes\varrho(x^l_{(2)},tx^r_{(2)}).\end{aligned} \tag{5.5.15}$$

The difference $\varrho_{(i\otimes T)\circ(V\otimes j)}*\varrho^{-1}_{(U\otimes j)\circ(i\otimes W)}$ between these 2-morphisms is what gives rise to the 2-isomorphism $i\otimes T\circ V\otimes j\cong U\otimes j\circ i\otimes W$. □

The fact that the tensor product of 1-morphisms decompose into two 2-isomorphic "mixed" tensor products is a signature property of Gray-enriched categories [146, 88]. We call the property that "structures on the 1-morphisms are determined by the mixed structure, together with appropriate coherence 2-isomorphisms" the **Gray property**.

These lemmas are important, as its proof techniques will be used repeatedly in what follows.

5.5.3 Monoidal associators

In this section, we shall focus on the *associator* morphisms attached to the 2-representations in $2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$, as they play a direct role in the main theorem. Recall from §5.5.2 that the tensor product on $2\mathrm{Rep}(\mathcal{G})$ is given by the coproduct Δ. The associator morphisms a are therefore given by the coasscociator $\Delta_1:\mathcal{G}_0\to\mathcal{G}_{-1}^{3\otimes}$ attached to the coproduct in $\mathcal{G}$, and *not* the Hochschild 3-cocycle $\mathcal{T}$.

However, the data $\Delta_1,\mathcal{T}$ are dual to each other by **Proposition 5.3.1**, hence if $\mathcal{G}$ is self-dual (like the weak (skeletal) quantum 2-double as we constructed in §5.4), they in fact constitute the same data. As such we shall denote the weak 2-representation 2-category by $2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$. We shall neglect the tensor product notation $\otimes$ in the following.

We begin by constructing the associator 2-morphism $a_{ijk}:(i\otimes j)\otimes k\Rightarrow i\otimes(j\otimes k)$ on the triple $i:V\to V',j:W\to W',k:U\to U'$ of 2-intertwiners. By (5.3.3), we see that the following quantity

$$a_{ijk}=((\rho_{V'})_1\circ i\otimes(\rho_{W'})_1\circ j\otimes(\rho_{U'})_1\circ k+(-1)^{\deg}i\circ(\rho_V)_1\otimes j\circ(\rho_W)_1\otimes k\circ(\rho_U)_1)\circ(\Delta_1\circ t) \tag{5.5.16}$$

defines a cochain homotopy that fits into the following equation $\rho_{(ij)k}-\rho_{i(jk)}=a_{ijk}$, which induces a 2-morphism (also denoted by a_{ijk}) between the 2-intertwiners

$$a_{ijk}:(ij)k\Rightarrow i(jk).$$

Secondly, (5.3.3) implies that the following quantities based on $D_t\Delta_1$,

$$\begin{aligned} a_{Vjk} &= ((\rho_V)_0\otimes(\rho_{W'})_1\circ j\otimes(\rho_{U'})_1\circ k+(-1)^{\deg}(\rho_V)_0\otimes j\circ(\rho_W)_1\otimes k\circ(\rho_U)_1)\\ &\quad\circ(t\otimes 1\otimes 1)\Delta_1,\\ a_{iWk} &= ((\rho_{V'})_1\circ i\otimes(\rho_W)_0\otimes(\rho_{U'})_1\circ k+(-1)^{\deg}i\circ(\rho_V)_1\otimes(\rho_W)_0\otimes k\circ(\rho_U)_1)\end{aligned}$$

$$
\begin{aligned}
& \circ(1\otimes t\otimes 1)\Delta_1, \\
a_{ijU} \quad = \quad & ((\rho_{V'})_1\circ i\otimes(\rho_{W'})_1\circ j\otimes(\rho_U)_0+(-1)^{\deg}i\circ(\rho_V)_1\otimes j\circ(\rho_W)_1\otimes(\rho_U)_0) \\
& \circ(1\otimes 1\otimes t)\Delta_1,
\end{aligned} \tag{5.5.17}
$$

give rise to the associators for the following tensor products,

$$
a_{Vjk}:(Vj)k\Rightarrow V(jk),\qquad a_{iWk}:(iW)k\Rightarrow i(WK),\qquad a_{ijU}:(ij)U\Rightarrow i(jU)
$$

for the mixed tensor products defined by (5.5.12). Thirdly, (5.3.4) implies that the following quantities based in $D_t[2]\Delta_1$,

$$
\begin{aligned}
a_{VWk} \quad = \quad & ((\rho_V)_0\otimes(\rho_W)_0\otimes(\rho_{U'})_1\circ k+(-1)^{\deg}(\rho_V)_0\otimes(\rho_W)_0\otimes k\circ(\rho_U)_1) \\
& \circ(t\otimes t\otimes 1)\Delta_1, \\
a_{iWU} \quad = \quad & ((\rho_{V'})_1\circ i\otimes(\rho_W)_0\otimes(\rho_U)_0+(-1)^{\deg}i\circ(\rho_V)_1\otimes(\rho_W)_0\otimes(\rho_U)_0) \\
& \circ(1\otimes t\otimes t)\Delta_1, \\
a_{VjU} \quad = \quad & ((\rho_V)_0\otimes(\rho_{W'})_1\circ j\otimes(\rho_U)_0+(-1)^{\deg}(\rho_V)_0\otimes j\circ(\rho_W)_1\otimes(\rho_U)_0) \\
& \circ(t\otimes 1\otimes t)\Delta_1,
\end{aligned} \tag{5.5.18}
$$

serve as the associators

$$
a_{VWk}:(VW)k\Rightarrow V(Wk),\qquad a_{VjU}:(Vj)U\Rightarrow V(jU),\qquad a_{iWU}:(iW)\Rightarrow i(WU),
$$

Notice that these quantities we have defined so far are all cochain homotopies/2-mophisms in $2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$, due to the appearance of ρ_1 in their tensor products.

Lastly, (5.3.5) allows us to define the associator *1-morphism*,

$$
a_{VWU}=((\rho_V)_0\otimes(\rho_W)_0\otimes(\rho_U)_0)(\Phi), \tag{5.5.19}
$$

with $\Phi\equiv(t\otimes t\otimes t)\Delta_1:\mathcal{G}_0\to\mathcal{G}_0^{3\otimes}$, which induces an invertible 1-morphism

$$
a_{VWU}:(VW)U\to V(WU)
$$

that intertwines between $\rho_{(V\otimes W)\otimes U}$ and $\rho_{V\otimes(W\otimes U)}$.

The adjoint associator 2-morphism $a^\dagger$ is implemented by minus the corresponding cochain homotopy. For (5.5.19), however, the adjoint morphism $a^\dagger_{VWU}$ is given by the inverse Φ^{-1}.

The pentagon relation and naturality of the associator. We now prove the following.

Lemma 5.5.3. *Suppose the 3-cocycle $\mathfrak{T}=0$ is trivial for the moment. The pentagon relation for the associators a arising from* (5.5.16)*,* (5.5.17)*,* (5.5.18)*,* (5.5.19) *follows from the 2-coassociativity condition* (5.3.5) *for Δ_1.*

Proof. Consider first (5.5.16). We *precompose* (5.3.5) with t and reconstruct the associators corresponding to each term according to the definition,

$$\begin{gathered}
(\mathrm{id}\otimes(\Delta_1\circ t))\circ\Delta_{-1}\rightsquigarrow \mathrm{id}_i\otimes a_{jkl}, \qquad ((\Delta_1\circ t)\otimes\mathrm{id})\circ\Delta_{-1}\rightsquigarrow a_{ijk}\otimes\mathrm{id}_l,\\
(1\otimes\Delta_{-1}\otimes 1)\circ\Delta_1\circ t\rightsquigarrow a_{i(jk)l}, \qquad -(\Delta_{-1}\otimes 1\otimes 1)\circ\Delta_1\circ t\rightsquigarrow a^\dagger_{(ij)kl},\\
-(1\otimes 1\otimes\Delta_{-1})\circ\Delta_1\circ t\rightsquigarrow a^\dagger_{ij(kl)},
\end{gathered}$$

where $\mathrm{id}_i : i \Rightarrow i$ denotes the identity modification on the 2-intertwiner i. Now note that, by coequivariance (5.1.10) $D_t\circ\Delta_{-1} = \Delta_0\circ t$, we have

$$(\mathrm{id}\otimes(\Delta_1\circ t))\circ\Delta_{-1} = (\mathrm{id}\otimes\Delta_1)\circ\Delta_0^l\circ t, \qquad ((\Delta_1\circ t)\otimes\mathrm{id})\circ\Delta_{-1} = (\Delta_1\otimes\mathrm{id})\circ\Delta_0^r\circ t,$$

whence the pentagon relation

$$\begin{tikzcd}
 & & ((ij)k)l \arrow[drr, "a_{ijk}\,\mathrm{id}_l"] & & \\
(ij)(kl) \arrow[urr, "a^\dagger_{(ij)kl}"] & & & & (i(jk))l \arrow[dl, "a_{i(jk)l}"] \\
 & i(j(kl)) \arrow[ul, "a^\dagger_{ij(kl)}"] & & i((jk)l) \arrow[ll, "\mathrm{id}_i\, a_{jkl}"'] &
\end{tikzcd} \tag{5.5.20}$$

is equivalently expressed as

$$\begin{aligned}
0 &= (1\otimes\Delta_{-1}\otimes 1)\circ\Delta_1\circ t - (\Delta_{-1}\otimes 1\otimes 1)\circ\Delta_1\circ t - (1\otimes 1\otimes\Delta_{-1})\circ\Delta_1\circ t\\
&\quad +(\Delta_1\otimes 1)\circ\Delta_0^r\circ t + (1\otimes\Delta_1)\circ\Delta_0^l\circ t\\
&= [-\Delta_{-1}\circ\Delta_1 + \Delta_1\circ\Delta_0]\circ t,
\end{aligned}$$

which is nothing but the 2-coassociativity (5.3.5) precomposed with t. Now by the coPeiffer identity $\Delta_0' = D_t\Delta_0$ (5.1.8), the same argument shows that the pentagon relations for the rest of the associator 2-morphisms (5.5.17), (5.5.18) are equivalent to applying the t-map $D_t, D_t[2]$ to (5.3.5).

Similarly, under the complete t-map $D_t[3] = t\otimes t\otimes t$, the 2-coasscociativity condition (5.3.5) becomes

$$\Delta_0'\circ\Phi = \Phi\circ\Delta_0', \tag{5.5.21}$$

which by (5.5.11) implies the pentagon relation for the associator 1-morphism (5.5.19). □

We shall show in **Theorem 5.7.1** that $\mathfrak{T}$ gives rise to the *pentagonator* 2-morphism π in $2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$, which witnesses the pentagon relations (5.5.20) up to chain homotopy.

Recall from **Proposition 5.3.1** that, for a self-dual weak 2-bialgebra, (5.3.5) follows from the 3-cocycle condition for the Hochschild 3-cocycle $\mathcal{T}$. Thus the entirety of the 2-bialgebra (or 2-Hopf algebra) structure plays a central role, precisely as one would expect in Tannakian duality [38, 1].

Lemma 5.5.4. *The associator 2-morphism* (5.5.18) *fits into diagrams of the form*

$$\begin{tikzcd}
(VW)U \arrow[rr, "a_{VWU}"] \arrow[d, "k"'] & \arrow[d, phantom, "\xRightarrow{a_{VWk}}"] & V(WU) \arrow[d, "k"] \\
(VW)U' \arrow[rr, "a_{VWU'}"'] & & V(WU')
\end{tikzcd} \tag{5.5.22}$$

together with the associator morphism (5.5.19)*. Moreover, the associator 2-morphisms* (5.5.16)*,* (5.5.17) *are completely determined by* (5.5.18)*,* (5.5.19)*.*

Proof. The first statement follows directly from the definitions, and by using the same argument as in the proofs of **Lemma 5.5.1**, and also later in **Lemma 5.6.2**. Similarly, by adapting the proof of **Lemma 5.5.2**, we see that (5.5.16), (5.5.17) admit the following decompositions

$$a_{ijk} = (a_{V'W'k} \cdot a_{ijU}) \circ t = \dots \text{etc.}, \qquad D_\delta a_{ijU} = a_{V'jU} \cdot a_{iWU} = \dots \text{etc.},$$

where D_δ is the tensor triple of the t-map δ on $\mathfrak{End}(V)$, and "etc." means permutations of the subscripts. This proves the second statement. □

This naturaliy property shall become very important later in §5.6.2.

Remark 5.5.3. Suppose the endomorphism Φ in (5.5.19) is *inner*, in the sense that it is given by conjugation with an element — also denoted Φ — of $\mathcal{G}_0^{3\otimes}$, then the coassociativity condition becomes

$$(\mathrm{id} \otimes \Delta_0') \circ \Delta_0' = \Phi((\Delta_0' \otimes \mathrm{id}) \circ \Delta_0')\Phi^{-1},$$

and the 2-coassociativity condition (5.5.21) becomes

$$((\mathrm{id} \otimes \mathrm{id} \otimes \Delta_0')\Phi)((\Delta_0' \otimes \mathrm{id} \otimes \mathrm{id})\Phi) = (\Phi \otimes \eta_0)((\mathrm{id} \otimes \Delta_0' \otimes \mathrm{id})\Phi)(\eta_0 \otimes \Phi),$$

where η_0 is the unit of $\mathcal{G}_0$. In other words, $(\mathcal{G}_0, \Delta_0', \Phi)$ in fact forms a *quasi-bialgebra* [147] of Drinfel'd.

We have established $2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$ as a monoidal 2-category. We now turn to the braiding structure in the following.

5.6 The braided monoidal 2-category of 2-representations

We now turn to the braiding structure on the weak 2-representations afforded by the 2-R-matrix $\mathcal{R}$. We shall first examine some of the basic properties of the braiding map in §5.6.1. We will then study how such braiding maps interact with the weakened monoidal structures of the 2-representations in §5.6.2.

Let $(\mathcal{G},\cdot,\Delta,\mathcal{R})$ denote a strict quasitriangular 2-bialgebra as defined in §5.2.4. Recall that a 2-R-matrix $\mathcal{R}=\mathcal{R}^l+\mathcal{R}^r$ on the 2-bialgebra $\mathcal{G}$ consist of the following components

$$\mathcal{R}^l=\mathcal{R}^l_{(1)}\otimes\mathcal{R}^l_{(2)}\in\mathcal{G}_{-1}\otimes\mathcal{G}_0,\qquad \mathcal{R}^r=\mathcal{R}^r_{(1)}\otimes\mathcal{R}^r_{(2)}\in\mathcal{G}_0\otimes\mathcal{G}_{-1}$$

for which (5.2.25), (5.2.26), (5.2.27) are satisfied. The equivariance condition, (5.2.27), unambiguously defines an element

$$R=\mathcal{R}^r_{(1)}\otimes t\mathcal{R}^r_{(2)}(\equiv R^r)=t\mathcal{R}^l_{(1)}\otimes\mathcal{R}^l_{(2)}(\equiv R^l)\in\mathcal{G}_0\otimes\mathcal{G}_0,\tag{5.6.1}$$

where $t:\mathcal{G}_{-1}\to\mathcal{G}_0$ is the t-map on $\mathcal{G}$. Notice by applying the t-map (at every leg in $\mathcal{G}_{-1}$) to (5.2.28), we obtain two identical expressions that are equivalent to the usual 1-Yang-Baxter equations

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}$$

for the degree-0 R-matrix (5.6.1).

5.6.1 The braiding maps and their naturality

We shall use these components to define the braiding b on $2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$. Take two 2-representations V,W of $\mathcal{G}$; we define the braiding map between V,W by

$$b_{VW}:V\otimes W\to W\otimes V,\quad b_{VW}=\mathrm{flip}\circ\rho_0(R)\tag{5.6.2}$$

where $\rho_0=(\rho_V)_0\otimes(\rho_W)_0$ on $V\otimes W$, and $R\in\mathcal{G}_0\otimes\mathcal{G}_0$ is given in (5.6.1). By (5.5.11), the braiding between the tensor product 2-representations are then given by

$$b_{V(W\otimes U)}=\mathrm{flip}\circ\rho_0((1\otimes\Delta_0')R),\qquad b_{(V\otimes W)U}=\mathrm{flip}\circ\rho_0((\Delta_0'\otimes 1)R).$$

If $W=V$ are the same 2-representations of $\mathcal{G}$, then we have the *self-braiding* map $b_V=b_{VV}$. On the other hand, we define the *mixed braiding* map between a 1-morphism $i:V\to U$ and an object W by

$$\begin{aligned}b_{iW}&=\mathrm{flip}\circ\left[i\circ\rho_{10}(\mathcal{R}^l)+(-1)^{\deg}\rho_{01}(\mathcal{R}^r)\circ i\right],\\ b_{Wi}&=\mathrm{flip}\circ\left[i\circ\rho_{01}(\mathcal{R}^r)+(-1)^{\deg}\rho_{10}(\mathcal{R}^l)\circ i\right],\end{aligned}\tag{5.6.3}$$

where we have used the shorthand $\rho_{10}=(\rho_V)_1\otimes(\rho_W)_0$ and $\rho_{01}=(\rho_U)_0\otimes(\rho_W)_1$. The sign $(-1)^{\deg}$ depends on the degree of the complex $V\otimes W$; more explicitly, b_{iW} gives two maps

$$\begin{aligned}b^1_{iW}&:V_0\otimes W_0\to(W_{-1}\otimes U_0)\oplus(W_0\otimes U_{-1}),\\ b^2_{iW}&:(V_{-1}\otimes W_0)\oplus(V_0\otimes W_{-1})\to W_{-1}\otimes U_{-1}\end{aligned}$$

on the tensor product $V\otimes W$, the latter of which carries a non-trivial sign $(-1)^{\deg}=-1$;

similarly for b_{Wi}. Now in the spirit of **Lemma 5.5.2**, we shall *define* the braiding maps b_{ij} between two 1-morphisms i,j by the decomposition formula[6]

$$b_{ij} = b_{jU}\cdot b_{Wi} \cong b_{Ti}\cdot b_{jV}, \qquad \begin{cases} i: V\to U \\ j: W\to T \end{cases}. \tag{5.6.4}$$

Let $i: V\to V', j: U\to U'$ denote any 2-intertwiner. The above definition (5.6.3), together with (5.5.12) then allows us to form

$$\begin{aligned} b_{(i\otimes W)j} &= \mathrm{flip}_{(V'\otimes U')\otimes W}\circ\left[(i\otimes j)\rho_{101}((\Delta_0^l\otimes 1)\mathcal{R}^r) + (-1)^{\deg}\rho_{011}((\Delta_0^r\otimes 1)\mathcal{R}^r)\circ(i\otimes j)\right], \\ b_{i(W\otimes j)} &= \mathrm{flip}_{W\otimes(V'\otimes U')}\circ\left[(i\otimes j)\rho_{101}((\mathrm{id}\otimes\Delta_0^r)\mathcal{R}^l) + (-1)^{\deg}\rho_{110}((1\otimes\Delta_0^l)\mathcal{R}^l)\circ(i\otimes j)\right]. \end{aligned}$$

By applying strict 2-representations to (5.2.25), we obtain the following strict *higher hexagon relations*,

$$b_{(i\otimes W)j} = \mathrm{id}_i\otimes b_{Wj} * b_{Wi}\otimes\mathrm{id}_j, \qquad b_{i(W\otimes j)} = \mathrm{id}_i\otimes b_{jW} * b_{iW}\otimes\mathrm{id}_j, \tag{5.6.5}$$

in which the associator isomorphisms a have been suppressed. We will reinstate them later in §5.6.2.

With the definitions (5.6.2), (5.6.3) in hand, we now need to prove some very important lemmas.

Lemma 5.6.1. *The maps b_{VW} and b_{iW}, b_{Wi} are respectively 2-intertwiners and modifications in* $2\mathrm{Rep}(\mathcal{G})$ *for all 2-representation V,W and 2-intertwiner i iff (5.2.26) is satisfied.*

Proof. Note for each 2-representation ρ, the flip map, $\mathrm{flip}: V\otimes W\to W\otimes V$ is a 2-intertwiner between ρ and $\rho' = \rho\circ\sigma$. Moreover, we interpret the cochain homotopy defined by $(\rho_{V\otimes W})_0^1(x)$ for each $x\in\mathcal{G}_0$ as a modification between the action $(\rho_{V\otimes W})_0^0(x)$ and itself, treated as a 2-intertwiner; similarly for ρ'. Therefore, in order for the mixed braiding map b_{iW} to be a modification in $2\mathrm{Rep}(\mathcal{G})$, it must commute with the cochain homotopy $(\rho_{V\otimes W})_0^1(x)$ — namely

$$b_{iW} * (\rho_{V\otimes W})_0^1(x) = (\rho'_{W\otimes V})_0^1(x) * b_{iW},$$

where $*$ denotes the composition of cochain homotopies. With $\rho'_{W\otimes V} = (\rho_W\otimes\rho_V)\circ\sigma\circ\Delta$, this is satisfied by definition (5.6.2) of b_{iW} iff

$$\mathcal{R}^r\Delta_0^r(x) = \sigma(\Delta_0^l(x))\mathcal{R}^r, \qquad \mathcal{R}^l\Delta_0^l = \sigma(\Delta_0^r(x))\mathcal{R}^l, \tag{5.6.6}$$

which is precisely (5.2.26).

[6]Alternatively, provided there exists a well-defined R-matrix $R_{-1}\in\mathcal{G}_{-1}^{2\otimes}$ for the degree-(-1) coproduct Δ_{-1}, satisfying $(t\otimes 1 + 1\otimes t)R_{-1} = \mathcal{R}$, we can define

$$b_{ij} = \mathrm{flip}\circ\left[(i\otimes j)\circ\rho_{11}(R_{-1}) + (-1)^{\deg}\rho_{11}(R_{-1})\circ i\otimes j)\right]$$

such that (5.6.4) follows from the definition of R_{-1}.

Similarly, in order for the braiding map b_{VW} to be a 2-intertwiner, it must commute with the action $(\rho_{V\otimes W})_0^0(x)$ for each $x\in\mathcal{G}_0$:

$$b_{VW}\circ(\rho_{V\otimes W})_0^0(x)=(\rho'_{W\otimes V})_0^0(x)\circ b_{VW},$$

where $\circ$ denotes the composition of 2-intertwiners.

First if the 2-representation ρ were strict, then this translates to the algebraic condition

$$\sigma\Delta_0'(x)R=R\Delta_0'(x),$$

which in fact follows also from (5.2.26). To see this, we recall the definitions (5.6.1) of R and (5.1.8) of the coproduct Δ_0', and simply apply $t\otimes 1$ and $1\otimes t$ respectively to (5.2.26). The fact that t is an algebra homomorphism and that $(t\otimes 1)\circ\sigma=\sigma\circ(1\otimes t)$ proves the statement.

Second, if the 2-representation ρ were weak, then in general the component ϱ gives rise to a possibly non-trivial invertible natural 2-morphism

$$\varrho(\sigma\Delta_0'(x),R)-\varrho(R,\Delta_0'(x)).$$

We will not need this 2-morphism in the following so we shall suppose $I_{b_{VW},\bullet}=\mathrm{id}$.

∎

Notice this lemma implies that $(\mathcal{G}_0,\Delta_0',R)$ forms an ordinary quasitriangular 1-bialgebra. We can then leverage the well-known result in the literature [117, 39] that the Yang-Baxter equation for R implies the hexagon relation for the braiding structure b_{VW} at the level of the objects.

Next, we need to prove the naturality of b with respect to the 2-intertwiners $i:V\to U$. We shall do this via the same technique as **Lemma 5.5.1**.

Lemma 5.6.2. *Consider the intertwiners $i:V\to U$ and $j:U\to T$. The mixed braiding maps b_{iW},b_{Wi} fit into the following diagrams*

$$\begin{tikzcd}
V\otimes W \arrow[rr, "b_{VW}"] \arrow[d, "i"'] & {} \arrow[d, phantom, "\xRightarrow{b_{iW}}"] & W\otimes V \arrow[d, "i"] \\
U\otimes W \arrow[rr, "b_{UW}"'] & {} & W\otimes U
\end{tikzcd}, \qquad
\begin{tikzcd}
W\otimes V \arrow[rr, "b_{WV}"] \arrow[d, "i"'] & {} \arrow[d, phantom, "\xRightarrow{b_{Wi}}"] & V\otimes W \arrow[d, "i"] \\
W\otimes U \arrow[rr, "b_{WU}"'] & {} & U\otimes W
\end{tikzcd}$$

in $2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$. *Moreover,* $b_{jW}*b_{iW}=b_{j\circ i,W}$.

Proof. For brevity, we shall suppress the subscripts V,U,W on the 2-representations. Recall the two equivalent ways R^r,R^l to express R in (5.6.1). We can then write

$$b_{UW}\circ i=\mathrm{flip}\circ\rho_0(R^r)\circ i,\qquad i\circ b_{VW}=i\circ\mathrm{flip}\circ\rho_0(R^l).$$

Consider the left diagram. As 2-morphisms in $2\mathrm{Rep}(\mathcal{G})$ are given by cochain homotopies, we need to show that the definition (5.6.3) of the mixed braiding map $b_{iW} = b^1_{iW} + b^2_{iW}$ fits into the following diagram

$$\begin{tikzcd} V_{-1}\otimes W_{-1} \arrow[r, "D^+"] \arrow[d] & V_{-1}\otimes W_0 \oplus V_0\otimes W_{-1} \arrow[r, "D^-"] \arrow[d] \arrow[dl, "b^2_{iW}"'] & V_0\otimes W_0 \arrow[d] \arrow[dl, "b^1_{iW}"'] \\ W_{-1}\otimes U_{-1} \arrow[r, "D^+"'] & W_{-1}\otimes U_0 \oplus W_0\otimes U_{-1} \arrow[r, "D^-"'] & W_0\otimes U_0 \end{tikzcd}, \tag{5.6.7}$$

where the vertical arrows are the various graded components of $b_{UW}\circ i - i\circ b_{VW}$, and the horizontal arrows are the differentials on the three-term tensor product complex (5.5.10); for instance, the ones at the top row are given by $D^\pm = 1\otimes\partial_W \pm \partial_V\otimes 1$.

As in **Lemma 5.5.1**, the key towards this is the commutative square (5.5.1), which states that for each $y\in\mathcal{G}_{-1}$ we have

$$(\rho_1(y)\partial, \partial\rho_1(y)) = \delta(\rho_1)(y) = (\rho_0)(Ty) = (\rho_0^1(Ty), \rho_0^0(Ty)).$$

Let us examine first the commutative triangle on the ends of (5.6.7). First, for the right-most triangle, we compute in terms of the components $b^{1,2}_{iW}$ that

$$\begin{aligned} D^-b^1_{iW} &= (1\otimes\partial_V - \partial_W\otimes 1)\circ \mathrm{flip}\circ\rho(\mathcal{R}) \\ &= \mathrm{flip}\circ\left[\rho_0^0(\mathcal{R}^r_{(1)})\otimes\partial_W(\rho_1(\mathcal{R}^r_{(2)}))\circ i - i\circ i\partial_V(\rho_1(\mathcal{R}^l_{(1)}))\otimes\rho_0^0(\mathcal{R}^l_{(2)})\right] \\ &= \mathrm{flip}\circ\left[\rho_0^0(\mathcal{R}^r_{(1)}\otimes t\mathcal{R}^r_{(2)})\circ i + i\circ\rho_0^0(-t\mathcal{R}^l_{(1)}\otimes\mathcal{R}^l_{(2)})\right] \\ &= b_{UW}\circ i - i\circ b_{VW} \end{aligned}$$

as maps on $V_0\otimes W_0$. Similarly for the left-most triangle, we have

$$\begin{aligned} b^2_{iW}D^+ &= \mathrm{flip}\circ\rho(\mathcal{R})\circ(1\otimes\partial_W + \partial_V\otimes 1) \\ &= \mathrm{flip}\circ\left[\rho_0^1(\mathcal{R}^r_{(1)})\otimes(\rho_1(\mathcal{R}^r_{(2)}))\partial_W\circ i - i\circ(\rho_1(\mathcal{R}^l_{(1)}))\partial_V\otimes\rho_0^1(\mathcal{R}^l_{(2)})\right] \\ &= \mathrm{flip}\circ\left[\rho_0^1(\mathcal{R}^r_{(1)}\otimes t\mathcal{R}^r_{(2)})\circ i - i\circ\rho_0^1(t\mathcal{R}^l_{(1)}\otimes\mathcal{R}^l_{(2)})\right] \\ &= b_{UW}\circ i - i\circ b_{VW} \end{aligned}$$

as maps $V_{-1}\otimes W_{-1}$. Note the sign $(-1)^{\mathrm{deg}}$ in (5.6.3) is non-trivial here as $\mathcal{R}$ acts on the degree-(-1) part of the tensor product $V\otimes W$.

We now turn to the middle section of (5.6.7). We are required to compute the following,

$$\begin{aligned} D^+b^2_{iW} &= (1\otimes\partial_V + \partial_W\otimes 1)\circ\mathrm{flip}\circ\rho(\mathcal{R}) \\ &= \mathrm{flip}\circ\left[\rho_0^1(\mathcal{R}^r_{(1)})\otimes\partial_W(\rho_1(\mathcal{R}^r_{(2)}))\circ i - i\circ\partial_V(\rho_1(\mathcal{R}^l_{(1)}))\otimes\rho_0^1(\mathcal{R}^l_{(2)})\right] \\ &= \mathrm{flip}\circ\left[\rho_0^1(\mathcal{R}^r_{(1)})\otimes\rho_0^0(t\mathcal{R}^r_{(2)})\circ i - i\circ\rho_0^0(t\mathcal{R}^l_{(1)})\otimes\rho_0^1(\mathcal{R}^l_{(2)})\right], \\ b^1_{iW}D^- &= \mathrm{flip}\circ\rho(\mathcal{R})\circ(1\otimes\partial_W - \partial_V\otimes 1) \\ &= \mathrm{flip}\circ\left[\rho_0^0(\mathcal{R}^r_{(1)})\otimes\rho_1(\mathcal{R}^r_{(2)})\partial_W\circ i - i\circ\rho_1(\mathcal{R}^l_{(1)})\partial_V\otimes\rho_0^0(\mathcal{R}^l_{(2)})\right] \\ &= \mathrm{flip}\circ\left[\rho_0^0(\mathcal{R}^r_{(1)})\otimes\rho_0^1(t\mathcal{R}^r_{(2)})\circ i - i\circ\rho_0^1(t\mathcal{R}^)_{(1)}\otimes\rho_0^0(\mathcal{R}^l_{(2)})\right]. \end{aligned}$$

Summing these and rearranging terms gives, as maps on $V_{-1}\otimes W_0\oplus V_0\otimes W_{-1}$,

$$\begin{aligned}&\text{flip}\circ\left[\rho_0^0(\mathcal{R}^r_{(1)})\otimes\rho_0^1(t\mathcal{R}^r_{(2)})+\rho_0^1(\mathcal{R}^r_{(1)})\otimes\rho_0^0(t\mathcal{R}^r_{(2)})\right]\circ i\\ &-\text{flip}\circ i\circ\left[\rho_0^1(t\mathcal{R}^l_{(1)})\otimes\rho_0^0(\mathcal{R}^l_{(2)})+\rho_0^0(t\mathcal{R}^l_{(1)})\otimes\rho_0^1(\mathcal{R}^l_{(2)})\right]\\ &=b_{UW}\circ i-i\circ b_{VW}.\end{aligned}\tag{5.6.8}$$

The diagram on the right is treated identically, and this establishes the first statement. The second statement directly follows from the fact that $(j\circ i)\circ\rho_V=j\circ\rho_U\circ i=\rho_X\circ(j\circ i)$ for composable 2-intertwiners i,j. $\square$

In particular, since **Lemma 5.6.1** proves that b_{VW} is a 1-morphism, we can *iterate* the braiding maps and define $b_{b_{VW}U}$ as a 2-morphism. **Lemma 5.6.2** then implies that this is a 2-morphism

$$\begin{tikzcd}(V\otimes W)\otimes U \arrow[r, "b_{(VW)U}"] \arrow[d, "b_{VW}"] \arrow[dr, phantom, "\overset{b_{b_{VW}U}}{\Rrightarrow}"] & U\otimes(V\otimes W) \arrow[d, "b_{VW}"] \\ (W\otimes V)\otimes U \arrow[r, "b_{(WV)U}"] & U\otimes(W\otimes V)\end{tikzcd}\tag{5.6.9}$$

on three 2-representations V,W,U, and similarly for $b_{Vb_{WU}}$. This will be important later in §5.6.4.

Recall the "higher-hexagon relations" (5.6.5) following directly from the identities (5.2.25). We shall prove this in the weakened context in §5.6.2.

5.6.2 Braided 2-quasi-bialgebras; the modified hexagon relations

We now wish to keep track of the interplay between the fusion associators a and the braiding maps b — or, algebraically, the coassociator and the 2-R-matrix — on $2\text{Rep}^{\mathcal{T}}(\mathcal{G})$. We shall do this by revisiting the universal characterization of 2-R-matrices in §5.2.4. In other words, we are prompted to study the *weak* quantum 2-double $D(\mathcal{G},\mathcal{G})$ and its braided transposition Ψ.

Fix the weak 2-bialgebra $\mathcal{G}$. Despite the skeletal construction in §5.4, we are able to form $D(\mathcal{G},\mathcal{G})$ here *without* assuming skeletality, since we know exactly how $\mathcal{G}$ acts on itself by weak 2-representations — in the canonical way according to **Definition 5.3.1**. This fact also allows us to identify $\mathcal{T}_D$ as merely several copies of the 3-cocycle $\mathcal{T}$ on $\mathcal{G}$, and in particular the components $\rhd_1=\lhd_1=\mathcal{T}$ are equal.

To proceed, we recall two facts we have learned previously.

- The condition (5.2.25) in the strict case follows from dualizing the braiding relation (5.2.19) (see (5.2.29)).
- From (5.4.4), the braiding relation (5.2.19) for the braided transposition Ψ is modified by $\mathcal{T}$ whenever we invoke the associativity in $\mathcal{K}\cong D(\mathcal{G},\mathcal{G})$.

Combining these means that (5.2.25) is modified by the *dual* of $\mathcal{T}$ — ie. the coassociator Δ_1 — in the weakened case. More explicitly, we have

$$\begin{aligned} D_t\Delta_1(x)_{231}\cdot(1\otimes\Delta_0)\mathcal{R}\cdot D_t\Delta_1(x)_{123} &= \mathcal{R}_{13}\cdot D_t\Delta_1(x)_{213}\cdot\mathcal{R}_{12}, \\ D_t\Delta_1(x)^{-1}_{312}\cdot(\Delta_0\otimes 1)\mathcal{R}\cdot D_t\Delta_1(x)^{-1}_{213} &= \mathcal{R}_{13}\cdot D_t\Delta_1(x)^{-1}_{132}\cdot\mathcal{R}_{23}, \end{aligned} \tag{5.6.10}$$

for each $x\in\mathcal{G}_0$. This bears a striking resemblance to the defining relations of a *braided quasi-bialgebra* [147]; indeed, applying the double-t-map $D_t[2]$ to (5.6.10) yields, by definition (5.6.1), (5.5.19),

$$\Phi_{231}(x)(1\otimes\Delta_0')R\Phi_{123}(x) = R_{13}\Phi_{213}(x)R_{12},\qquad \Phi_{312}^{-1}(x)(\Delta_0'\otimes 1)R\Phi^{-1}_{213}(x) = R_{13}\Phi^{-1}_{132}(x)R_{23}, \tag{5.6.11}$$

which is precisely a braided quasi-bialgebra structure on $(\mathcal{G}_0,\Delta_0',R,\Phi)$; see *Remark 5.5.3*. This motivates the following definition.

Definition 5.6.1. A **braided 2-quasi-bialgebra**[7] $(\mathcal{G},\Delta=(\Delta_1,\Delta_0,\Delta_{-1}),\mathcal{T},\mathcal{R})$ is a weak 2-bialgebra equipped with a universal 2-R-matrix $\mathcal{R}$ and a coassociator $\Delta_1:\mathcal{G}_0\to\mathcal{G}_{-1}^{3\otimes}$ such that (5.6.10), (5.6.11), (5.2.26) and (5.2.27) hold.

Similar to (5.6.5), by applying *strict* 2-representations $\rho=(\rho_1,\rho_0)$ to (5.6.10), we obtain:

Lemma 5.6.3. *For each $X\in 2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$, we have the decompositions (the* **hexagon relations***)*

$$\begin{cases} b_{(VW)X} = a_{XVW}\circ b_{VX}\circ a^\dagger_{VXW}\circ b_{WX}\circ a_{VWX} \\ b_{V(WX)} = a^\dagger_{WXV}\circ b_{VX}\circ a_{WVX}\circ b_{VW}\circ a^\dagger_{VWX}\end{cases} \iff (5.6.11), \tag{5.6.12}$$

$$\begin{cases} b_{(Vj)X} = a_{XVj}*\mathrm{id}_{b_{VX}}*a^\dagger_{VXj}*b_{jX}*a_{VjX} \\ b_{V(jX)} = a^\dagger_{jXV}*\mathrm{id}_{b_{VX}}*a_{jVX}*b_{Vj}*a^\dagger_{VjX}\end{cases} \iff \textit{apply } D_t^+ \textit{ to } (5.6.10),$$

$$\begin{cases} b_{(iW)k} = a_{kiW}*b_{ik}*a^\dagger_{ikW}*b_{Wk}*a_{iWk} \\ b_{i(Wk)} = a^\dagger_{Wki}*b_{ik}*a_{Wik}*b_{iW}*a^\dagger_{iWk}\end{cases} \iff (5.6.10),$$

as 1-/2-morphisms, and similarly for all the other possible braiding maps on tensor products.

The decomposition formula for b_{ijk} follows from these, as well as the fact that b_{ij},a_{ijk} are all determined by the mixed braiding/associators.

The 2-morphism $b_{(iW)X}$, for instance, can be expressed in terms of the following composition diagram

$$\begin{tikzcd} (VW)X \arrow[r, "a_{VWX}"] \arrow[d, "i"] & V(WX) \arrow[r, "b_{WX}"] \arrow[d, "i"] & V(XW) \arrow[r, "a^\dagger_{VXW}"] \arrow[d, "i"] & (VX)W \arrow[r, "b_{VX}"] \arrow[d, "i"] & (XV)W \arrow[r, "a_{XVW}"] \arrow[d, "i"] & X(VW) \arrow[d, "i"] \\ (UW)X \arrow[r, "a_{UWX}"] & V(WX) \arrow[r, "b_{WX}"] & U(XW) \arrow[r, "a^\dagger_{UXW}"] & (UX)W \arrow[r, "b_{UX}"] & (XU)W \arrow[r, "a_{UVW}"] & X(UW) \end{tikzcd}, \tag{5.6.13}$$

[7]Note that a *quasi* 2-bialgebra, as opposed to a 2-quasi-bialgebra here, refers to a weak 2-bialgebra with trivial 3-cocycle $\mathcal{T}=0$ but non-trivial coassociator Δ_1.

which has also appeared in [78]. This establishes most of the structural properties of $2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$ as a braided 2-category, and the final ingredient to introduce is the *hexagonator*.

5.6.3 The braiding hexagonator

We obtained the decomposition **Lemma 5.6.3** by applying a strict 2-representation to (5.6.10). However, as we have noted previously in *Remark 5.5.2*, 2-representations of a weak 2-bialgebra $(\mathcal{G},\mathcal{T})$ *cannot* be strict, even when $\mathcal{G}$ is skeletal. As such, we must take into account the additional component $\varrho:\mathcal{G}_0^{2\otimes}\to\mathfrak{End}(V)_{-1}$ when deriving the decompositions above (in particular (5.6.12)).

For the rest of the paper, it suffices to consider the case $t=0$ or $t=\eta_0$, the constant map to the unit $\eta_0\in\mathcal{G}_0$. Since ϱ is normalized and the second and third equations in (5.5.2) involve *pre*-composing ϱ with t, the only non-trivial relation is

$$\rho_0(xx')-\rho_0(x)\rho_0(x')=\delta\varrho(x,x'),\qquad x,x'\in\mathcal{G}_0,$$

where we recall that $\delta:\mathfrak{End}(V)_{-1}\to\mathfrak{End}(V)_0$ is the t-map on the weak endomorphism 2-algebra. Therefore, in order to obtain the decomposition of the form (5.6.12) from (5.6.11), we must keep track of the terms involving ϱ that appear. For instance, we have

$$\rho_0^{3\otimes}(R_{13}\Phi_{213})-\rho_0^{3\otimes}(R_{13})\rho_0^{3\otimes}(\Phi_{213})=(\delta\varrho)^{3\otimes}(R_{13},\Phi_{213}),$$

in which we notice that the second term on the left-hand side is the composition $b_{VU}\circ a_{WVU}$.

More explicitly, translating (5.6.11) to (5.6.12) comes at a price given by a cochain homotopy

$$\begin{aligned}\Omega_{V|WU}(x)\quad&=\quad(\varrho_V\otimes\varrho_W\otimes\varrho_U)(\Phi_{231}(x),(1\otimes\Delta_0')R\Phi_{123}(x))\\&\quad-(\varrho_V\otimes\varrho_W\otimes\varrho_U)(R_{13},\Phi_{213}(x)R_{12})\\&\quad+(\varrho_V\otimes\varrho_W\otimes\varrho_U)((1\otimes\Delta_0')R,\Phi_{123}(x))\\&\quad-(\varrho_V\otimes\varrho_W\otimes\varrho_U)(\Phi_{213}(x),R_{12})\end{aligned}\tag{5.6.14}$$

between the two sides of (5.6.11) for each $x\in\mathcal{G}_0$, and similarly its adjoint $\Omega^\dagger_{V|WU}$. We thus have the following diagrams

$$\begin{tikzcd}(V_{-1}\otimes W_{-1})\otimes U_{-1} \arrow[r, "\partial^{3\otimes}"] \arrow[d] & (V_0\otimes W_0)\otimes U_0 \arrow[d] \arrow[dl, "\Omega_{V|WU}"'] \\ W_{-1}\otimes(U_{-1}\otimes V_{-1}) \arrow[r, "\partial^{3\otimes}"'] & W_0\otimes(U_0\otimes V_0)\end{tikzcd},$$

$$
\begin{tikzcd}
V_{-1}\otimes(W_{-1}\otimes U_{-1}) \arrow[r, "\partial^{3\otimes}"] \arrow[d] & V_0\otimes(W_0\otimes U_0) \arrow[d] \arrow[dl, "\Omega^\dagger_{V|WU}"'] \\
(U_{-1}\otimes V_{-1})\otimes W_{-1} \arrow[r, "\partial^{3\otimes}"'] & (U_0\otimes V_0)\otimes W_0
\end{tikzcd},
$$

where the vertical arrows denote the decomposition (5.6.12). These diagrams cast $\Omega,\Omega^\dagger$ as the **hexagonator** 2-morphisms in $2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$:

$$
\begin{tikzcd}
 & V(WU) \arrow[r, "b_{V(WU)}"] & (WU)V \arrow[dr, "a_{WUV}"] & \\
(VW)U \arrow[ur, "a_{VWU}"] \arrow[dr, "b_{VW}"'] \arrow[rrr, phantom, "\xRightarrow{\Omega_{V|WU}}"] & & & W(UV) \\
 & (WV)U \arrow[r, "a_{WVU}"'] & W(VU) \arrow[ur, "b_{VU}"'] &
\end{tikzcd},
$$

$$
\begin{tikzcd}
 & (VW)U \arrow[r, "b_{(VW)U}"] & U(VW) \arrow[dr, "a^\dagger_{UVW}"] & \\
V(WU) \arrow[ur, "a^\dagger_{VWU}"] \arrow[dr, "b_{WU}"'] \arrow[rrr, phantom, "\xRightarrow{\Omega^\dagger_{V|WU}}"] & & & (UV)W \\
 & V(UW) \arrow[r, "a^\dagger_{VUW}"'] & (VU)W \arrow[ur, "b_{VU}"'] &
\end{tikzcd}.
$$

In other words, the quantities $\Omega_{V|WU},\Omega^\dagger_{V|WU}$ by definition is an invertible modification implementing the two sides of the decomposition (5.6.12).

Now by the diagram (5.6.13), the 2-intertwiners $i:V\to U$ and their associated mixed braiding maps b_{iW} preserve these hexagon relations. This leads to the naturality of the hexagonator

$\Omega_{V|WU}$ with respect to 2-intertwiners such that we have (cf. diagram (2.2) in [78])

$$\begin{tikzcd}
 & V(WX) \arrow[rrr] \arrow[dr, "i"] & & & (WX)V \arrow[dl, "i"'] \arrow[ddr] & \\
 & & U(WX) \arrow[r, "b_{U(WX)}"] & (WX)U \arrow[dr, "a_{WXU}"] & & \\
(VW)X \arrow[uur, "a_{VWX}"] \arrow[r, "i"] \arrow[ddr, "b_{VW}"'] & (UW)X \arrow[ur, "a_{UWX}"] \arrow[dr, "b_{UW}"'] \arrow[rrr, phantom, "\xRightarrow[\Omega_{V|WX}]{\Omega_{U|WX}}"] & & & W(XU) & W(XV) \arrow[l, "i"'] \\
 & & (WU)X \arrow[r, "a_{WUX}"] & W(UX) \arrow[ur, "b_{UX}"'] & & \\
 & (WV)X \arrow[ur, "i"] \arrow[rrr, "a_{WVX}"] & & & W(VX) \arrow[ul, "i"] \arrow[uur, "b_{VX}"'] &
\arrow[from=1-2, to=2-5, phantom, "\xRightarrow{b_{i(WX)}}"]
\arrow[from=3-1, to=5-3, phantom, "\Downarrow b_{iW}"]
\arrow[from=3-6, to=5-4, phantom, "\Downarrow b_{iX}"]
\end{tikzcd},$$

and similarly for the adjoint diagrams with $\Omega^\dagger$. The tensor product VX of 2-representations is equipped with the tensor product $\Omega_{VX|WU}$ hexagonator, which are by construction natural and invertible.

Remark 5.6.1. Notice we did not define any associators for the 2-morphisms μ in $2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$. This is because 2-morphisms in a 2-category the tensor product $\mu\otimes\nu = \mu * \nu$ given by composition is *strictly* associative; indeed, such an associator $a_{\mu\nu\lambda} : (\mu\nu)\lambda \Rrightarrow \mu(\nu\lambda)$ would have to be a 3-morphism.

By the same token, the hexagon relations involving the mixed braiding maps (ie. the decompositions in **Lemma 5.6.3** aside from (5.6.12)), as well as the pentagon relations for the associator 2-morphisms (5.5.16), (5.5.17), (5.5.18), must hold strictly on-the-nose. However, the fact that a_{VWU} is a 1-morphism implies we can have a 2-morphism π, called the **pentagonator**, that implements its pentagon relation. We will show in **Theorem 5.7.1** that π is given by the Hochschild 3-cocycle $\mathfrak{T}$ attached to the weak endomorphism 2-algebra $\mathfrak{End}(V)$.

5.6.4 Proof of the main theorem

We are finally ready to state and prove the main theorem. As earlier, we will often omit the tensor products to lighten the notations.

Theorem 5.6.1. *The 2-representation 2-category* $2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$ *of a weak quasitriangular 2-bialgebra* $\mathcal{G}$ *is a braided monoidal 2-category with trivial left-/right-equivalences* $l : 1V \xrightarrow{\sim} V$, $r : V1 \xrightarrow{\sim} V$.

We will prove this by using algebraic and diagrammatic manipulations that we have outlined throughout the paper, and reproduce all the coherence relations defining a braided monoidal 2-category in [81]. On the way, we shall also construct quantities that has also appeared in [78].

Recall first that, from §5.5.2, we have trivial left- and right-unitors $l : 1V \to V, r : V1 \to V$, and hence all coherence relations involving them (ie. diagrams (2.5), (2.7)-(2.9) of [78]) are vacuously satisfied.

Braiding on the associator; the first axiom in [81]. Let $V, W, U \in 2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$ be four 2-representations. Consider the mixed braiding 2-morphism $b_{a_{VWU}X}$, which by **Lemma 5.6.2** fits into a diagram of the form

$$\begin{tikzcd} ((VW)U)X \arrow[rr, "b_{((VW)U)X}"] \arrow[d, "a_{VWU}"'] & \arrow[d, phantom, "\overset{b_{a_{VWU}X}}{\Longrightarrow}"] & X((VW)U) \arrow[d, "a_{VWU}"] \\ (V(WU))X \arrow[rr, "b_{(V(WU))X}"'] & & X(V(WU)) \end{tikzcd} .$$

Lemma 5.6.3 states that we can in fact decompose the top and bottom 1-morphisms in this diagram, provided we keep in mind the hexagonator $\Omega, \Omega^\dagger$ (5.6.14) that appears in doing so. We thus obtain a formula of the form

$$\begin{aligned} b_{((VW)U)X} &\xRightarrow{\Omega^\dagger_{(VW)|UX}} a_{X(VW)U} \circ b_{(VW)X} \circ a^\dagger_{(VW)XU} \circ b_{UX} \circ a_{(VW)UX} \\ &\xRightarrow{\Omega^\dagger_{V|WX}} a_{X(VW)U} \circ \Big[a_{XVW} \circ b_{VX} \circ a^\dagger_{VXW} \circ b_{WX} \circ a_{VWX} \Big] \\ &\qquad \circ a^\dagger_{(VW)XU} \circ b_{UX} \circ a_{(VW)UX}, \end{aligned} \tag{5.6.15}$$

and similarly for the bottom 1-morphism $b_{(V(WU))X}$,

$$\begin{aligned} b_{(V(WU))X} &\xRightarrow{\Omega^\dagger_{V|(WU)X}} a_{XV(WU)} \circ b_{VX} \circ a^\dagger_{VX(WU)} \circ b_{(WU)X} \circ a_{V(WU)X} \\ &\xRightarrow{\Omega^\dagger_{W|UX}} a_{XV(WU)} \circ b_{VX} \circ a^\dagger_{VX(WU)} \\ &\qquad \circ \Big[a_{XWU} \circ b_{WX} \circ a^\dagger_{WXU} \circ b_{UX} \circ a_{WUX} \Big] \circ a_{V(WU)X}. \end{aligned} \tag{5.6.16}$$

Now notice that there are three identical braiding maps that appear in both of these formulas, b_{VX}, b_{WX}, b_{UX}, but they act on objects that differ by an associator: we have $b_{UX} : (VW)(UX) \to (VW)(XU)$ from (5.6.15) and $b_{UX} : V(W(UX)) \to V(W(XU))$ from (5.6.16), for instance. Such a square is precisely given by the diagram (5.5.22),

$$\begin{tikzcd}
(VW)(UX) \arrow[r, "a_{VW(UX)}"] \arrow[d, "b_{UX}"'] \arrow[dr, phantom, "\xRightarrow{a_{VWb_{UX}}}"] & V(W(UX)) \arrow[d, "b_{UX}"] \\
(VW)(XU) \arrow[r, "a_{VW(XU)}"] & V(W(XU))
\end{tikzcd},$$

and similarly for the other braiding maps that occur in both (5.6.16), (5.6.15). Putting this all together, we achieve the following diagrammatic expression for $b_{a_{VWU}X}$ (labelling only the 2-morphisms for clarity):

$$\begin{tikzcd}
((VW)U)X \arrow[rrrrr] \arrow[d] & & & \arrow[d, phantom, "\Omega^\dagger_{(VW)|UX} \ \Searrow"] & & X((VW)U) \arrow[d] \\
(VW)(UX) \arrow[r] \arrow[dd] \arrow[rdd, phantom, "\xRightarrow{a_{VWb_{UX}}}"] & (VW)(XU) \arrow[r] \arrow[dd] & ((VW)X)U \arrow[rrr] \arrow[d] & & \arrow[d, phantom, "\Omega^\dagger_{V|WX} \Downarrow"] & (X(VW))U \arrow[d] \\
& & (V(WX))U \arrow[r] \arrow[d] \arrow[rd, phantom, "\xRightarrow{a_{Vb_{WX}U}}"] & (V(XW))U \arrow[r] \arrow[d] & ((VX)W)U \arrow[r] \arrow[dd] \arrow[rdd, phantom, "\xRightarrow{a_{b_{VX}WU}}"] & ((XV)W)U \arrow[dd] \\
V(W(UX)) \arrow[r] \arrow[d] & V(W(XU)) \arrow[r] & V((WX)U) \arrow[r] & V((XW)U) \arrow[d] & & \\
V((WU)X) \arrow[rrr] \arrow[d] \arrow[rr, phantom, "\Omega^\dagger_{W|UX} \Uparrow"] & & \arrow[d, phantom, "\Nwarrow \ \Omega^\dagger_{V|(WU)X}"] & V(X(WU)) \arrow[r] & (VX)(WU) \arrow[r] & (XV)(WU) \arrow[d] \\
(V(UW))X \arrow[rrrrr] & & & & & X(V(UW))
\end{tikzcd}$$

This is precisely the third axiom in [81]; cf. diagram (2.6) in [78].

Naturality of hexagonator $\Omega_{VX|WU}$; the third axiom of [81]. The strategy is to apply the same naturality procedure as above to expand the defining diagram for $\Omega_{VX|WU}$,

$$\begin{tikzcd}
& (VX)(WU) \arrow[r, "b_{(VX)(WU)}"] & (WU)(VX) \arrow[dr, "a_{WU(VX)}"] & \\
((VX)W)U \arrow[ur, "a_{(VX)WU}"] \arrow[dr, "b_{(VX)W}"'] \arrow[rrr, phantom, "\xRightarrow{\Omega_{VX|WU}}"] & & & W(U(VX)) \\
& (W(VX))U \arrow[r, "a_{W(VX)U}"'] & W((VX)U) \arrow[ur, "b_{(VX)U}"'] &
\end{tikzcd}.$$

For this, we wish to leverage a result that we will prove in §5.7: a weak 2-representation $(V,\rho)\in 2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$ has equipped the associator and pentagonator data

$$\alpha_{x_1x_2|V} = \varrho(\rho_0(x_1),\rho_0(x_2))(V), \qquad \pi_{x_1x_2x_3|V} = \mathfrak{T}(\rho_0(x_1),\rho_0(x_2),\rho_0(x_3))(V) \tag{5.6.17}$$

given by $\rho = (\varrho,\rho_0,\rho_1)$ as well as the Hochschild 3-cocycle $\mathfrak{T}$ on $\mathfrak{End}(V)$.

With this, we can begin by rewriting each of the associator and braiding maps appearing here using (5.6.3) and the pentagonator π (5.6.17) introduced above. We obtain precisely diagram (2.4) in [78] as the $\Omega_{VX|WU}$. The third axiom of [81] then follows.

Iterating the braiding map; the fourth axiom of [81]. Now consider the iterated braiding 2-morphism $b_{Vb_{UW}}$ (5.6.9). By the same logic as above, we can use the decomposition (5.6.12) once again on the top and bottom braiding morphisms that appear in the diagram,

$$b_{V(UW)} \xRightarrow{\Omega_{V|UW}} a^\dagger_{UWV}\circ b_{VW}\circ a_{UVW}\circ b_{VU}\circ a^\dagger_{VUW},$$
$$b_{V(WU)} \xRightarrow{\Omega_{V|WU}} a^\dagger_{WUV}\circ b_{VU}\circ a_{WVU}\circ b_{VW}\circ a^\dagger_{VWU}.$$

We can thus form the composition

$$b_{\Omega_{V|WU}} \equiv \Omega^{-1}_{V|WU}\cdot b_{Vb_{UW}}\cdot \Omega_{V|UW}, \tag{5.6.18}$$

which fits into a diagram that "pastes" two hexagon diagrams together,

$$\begin{tikzcd}
W(VU) \arrow[d, "b_{VU}"'] & (WV)U \arrow[l, "a_{WVU}"'] \arrow[d, phantom, "\xRightarrow{\Omega_{V|WU}}"] & (VW)U \arrow[l, "b_{VW}"'] \arrow[r, "a_{VWU}"] & V(WU) \arrow[d, bend right, "b_{V(WU)}"'] \arrow[d, phantom, "\xLeftarrow{b_{Vb_{UW}}}"'] & V(UW) \arrow[l, "b_{UW}"'] \arrow[dl, bend left, "b_{V(UW)}"] \arrow[d, phantom, "\xLeftarrow{\Omega_{V|UW}}"'] & (VU)W \arrow[l, "a_{VWU}"'] \arrow[d, "b_{VU}"] \arrow[lllll, bend right=25, "b_{(VU)W}"'] \\
W(UV) & (WU)V \arrow[l, "a_{WUV}"'] & (UW)V \arrow[l, "b_{UW}"'] \arrow[r, "a_{UWV}"] & U(WV) & U(VW) \arrow[l, "b_{VW}"'] & (UV)W \arrow[l, "a_{UVW}"'] \arrow[lllll, bend left=25, "b_{(UV)W}"]
\end{tikzcd}$$

Note that, by construction (5.6.18), the 2-morphisms $b_{\Omega_{V|\bullet\bullet}}$ are natural and invertible. Moreover, its definition is precisely (2.10) in [78], and hence the fourth axiom of [81] follows.

Cohomology descent equations; the second axiom of [81]. Let us now focus on (5.5.3). Recall that it states, for $x_1, x_2, x_3 \in \mathcal{G}_0$, that

$$\begin{aligned}\rho_1(\mathcal{T}(x_1,x_2,x_3)) - \mathfrak{T}(\rho_0(x_1),\rho_0(x_2),\rho_0(x_3)) \quad &= \quad \rho_0(x_1)\cdot\varrho(x_2,x_3) - \varrho(x_1x_2,x_3) \\ &\quad + \varrho(x_1,x_2x_3) - \varrho(x_1,x_2)\cdot\rho_0(x_3),\end{aligned}$$

where $\mathfrak{T}$ is the Hochschild 3-cocycle on the weak endomorphism 2-algebra $\mathfrak{End}(V)$ of a particularly chosen weak 2-vector space $V \in \mathsf{2Vect}^{hBC}$. We shall now specialize $x_1,\dots,x_3$ to the elements in $\mathcal{G}_0$ of (5.6.11), and let the equation act on V.

By some computations, we see that the right-hand side translates to the composition of 2-morphisms

$$\mathrm{id}_{\mathrm{id}_W}\,\Omega_{V|UX} * \Omega_{V|W(UX)} * \Omega^{-1}_{V|(WU)X} * (\Omega_{V|WU}\,\mathrm{id}_{\mathrm{id}_X})^{-1},$$

while on the term $\rho\circ\mathcal{T}$ on the left dualizes to terms of the form $(\rho_V\otimes\cdots\otimes\rho_X)(\Delta_1\circ R - \mathcal{R}\circ D_t^+\Delta_1)$, which translates to

$$a^\dagger_{Wb_{VU}X} * a_{b_{VW}UX} * \; a^\dagger_{WUb_{VX}} * b_{Va_{WUX}}.$$

Now recall from **Theorem 5.7.1** that $\mathfrak{T}\circ\rho_0^{3\otimes}$ in fact defines the pentagonator π on $2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$. The left-hand side then acquires also the contribution

$$\pi_{WVUX} * \pi_{WUVX} * \pi^\dagger_{VWUX} * \pi^\dagger_{WUXV},$$

where $\pi_{WUXV}(x) = \mathfrak{T}((\rho_W)_0(x),(\rho_U)_0(x),(\rho_X)_0(x))(V)$; see (5.6.17).

Altogether, this gives rise to the equation

$$\begin{aligned}\pi_{WVUX} * \pi_{WUVX} * \mathrm{id}_{\mathrm{id}_W}\,\Omega_{V|UX} * \Omega_{V|W(UX)} * a^\dagger_{b_{VW}UX} * b^\dagger_{Va_{WUX}} =\\ \pi_{VWUX} * \pi_{WUXV} * \Omega_{V|WU}\,\mathrm{id}_{\mathrm{id}_X} * \Omega_{V|(WU)X} * a^\dagger_{Wb_{VU}X} * a^\dagger_{WUb_{VX}}\end{aligned}$$

for $V,W,U,X \in 2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$, which is precisely the second axiom in [81] (or equivalently axiom (2.1) in [78]). In the group-theoretical case, this axiom was also captured in a cohomological manner in (3.2) of [78].

In summary, we find that $2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$ has the following ingredients:

objects	1-morphisms	2-morphisms
2-representations $(V, b_{V\bullet}, \Omega_{V\vert\bullet\bullet})$	2-intertwiners $(i, b_{i\bullet})$	equivariant cochain homotopies μ

This establishes **Theorem 5.6.1**.

5.7 Coherences of 2-representations

We first recall briefly some key aspects of a module 2-category [59, 64]. To be more concrete, let $\mathcal{C}$ denote a semisimple (monoidal) 2-category. A **$\mathcal{C}$-module 2-category** is a k-linear semisimple 2-category $\mathcal{D}$ with a $\mathcal{C}$ action 2-functor $\triangleright : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$ and a set of adjoint natural equivalences (the *associators*)

$$\alpha_{XY|A} : (X \otimes Y) \triangleright A \to X \triangleright (Y \triangleright A)$$

for each $X, Y \in \mathcal{C}$ and $A \in \mathcal{D}$, satisfying the module pentagon relations up to a possibly non-trivial *module pentagonator* 2-morphism $\pi_{XYZ|A}$. These pentagonators must satisfy *on the nose* an additional coherence condition, called the *associahedron condition*. The explicit expressions of these conditions can be found in [59, 64].

Consider a 2-bialgebra $\mathcal{G}$ as a connected 2-category

$$B\mathcal{G} = \mathcal{G}_{-1} \otimes \mathcal{G}_0 \rightrightarrows \mathcal{G}_0 \rightrightarrows \mathrm{pt},$$

which is a 2-category with a single object pt, 1-morphisms $\mathcal{G}_0$ and each 2-Hom space over $\mathcal{G}_0$ is a copy of $\mathcal{G}_{-1}$. Evaluating an action 2-functor $\triangleright : \mathcal{G} \times \mathsf{2Vect}^{hBC} \to \mathsf{2Vect}^{hBC}$ on the object V gives precisely a weak 2-representation $\rho : \mathcal{G} \to \mathrm{End}_{\mathsf{2Vect}^{hBC}}(V) = \mathfrak{End}(V)$ of $\mathcal{G}$ on V, as we have defined in the main text.

Theorem 5.7.1. *Weak 2-representations are $\mathcal{G}$-module categories over $\mathsf{2Vect}^{hBC}$:*

$$2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G}) = \mathrm{Mod}_{\mathsf{2Vect}^{hBC}}(\mathcal{G}).$$

Proof. The k-linearity is immediate. As foretold in (5.6.17), we reconstruct the module associator α and pentagonator π of the $\mathcal{G}$-module category $V \in \mathsf{2Vect}^{hBC}$ by taking

$$\alpha_{x_1x_2|V} = \varrho(\rho_0(x_1), \rho_0(x_2))(V), \qquad \pi_{x_1x_2x_3|V} = \mathfrak{T}(\rho_0(x_1), \rho_0(x_2), \rho_0(x_3))(V),$$

where $\rho = (\varrho, \rho_0, \rho_1) : \mathcal{G} \to \mathfrak{End}(V)$ is a weak 2-representation and $\mathfrak{T}$ is the Hochschild 3-cocycle on $\mathfrak{End}(V)$. We now proceed level by level.

Objects. We identify the action 2-functor $\triangleright$ as the weak 2-representation ρ such that $x \triangleright V = \rho_0(x)V$ for each $x \in \mathcal{G}_0$. An arrow $x \triangleright V \to x' \triangleright V$ is therefore expressed as $\rho_1(y)V$, where $y \in \mathcal{G}_{-1}$ is interpreted as a 2-morphism $x \overset{y}{\Rightarrow} x'$ between $x, x' = x + ty$ [93, 98], or simply by $\rho_1(y)$. What we need to prove is the pentagon relation between α, π, as well as the associahedron condition for π. The pentagon relation can be written as

$$\begin{tikzcd}
((x_1x_2)x_3)\rhd V \arrow[r, "\rho_1(\mathcal{T}(x_1,x_2,x_3))"] \arrow[dd, "\varrho(x_1x_2,x_3)"'] & (x_1(x_2x_3))\rhd V \arrow[r, "\varrho(x_1,x_2x_3)"] \arrow[d, phantom, "\overset{\pi_{x_1x_2x_3|V}}{\Rrightarrow}"] & x_1\rhd((x_2x_3)\rhd V) \arrow[dd, "\rho_0(x_1)\varrho(x_2,x_3)"] \\
& {} & \\
(x_1x_2)\rhd(x_3\rhd V) \arrow[rr, "\varrho(x_1,x_2))\rho_0(x_3)"] & & x_1\rhd(x_2\rhd(x_3\rhd V)))
\end{tikzcd}$$

Rewriting π in terms of the 3-cocycle $\mathfrak{T}$, we have

$$\begin{aligned}
\mathfrak{T}(\rho_0(x_1),\rho_0(x_2),\rho_0(x_3)) \quad &= \quad -\varrho(x_1x_2,x_3)-\varrho(x_1,x_2)\rho_0(x_3) \\
&\qquad + \rho_1(\mathcal{T}(x_1,x_2,x_3))+\varrho(x_1,x_2x_3)+\rho_0(x_1)\varrho(x_2,x_3),
\end{aligned}$$

which is nothing but the last equation of (5.3.1). It is then easy to see that the associahedron condition follows from the Hochschild 3-cocycle condition for $\mathfrak{T}$.

2-intertwiners. Recall the notion of weak 2-intertwiners that we have given in **Definition 5.5.4**. By treating V as a $\mathcal{G}$-module 2-category and taking $\rhd,\rhd'$ as the action 2-functors corresponding to the 2-representations ρ,ρ', we equivalently characterize the cochain homotopy I as a collection of invertible natural transformations $I_{\bullet,i}: i(\bullet\rhd V)\Rightarrow\bullet\rhd' i(V)$, such that the following pentagon relation

$$\begin{tikzcd}
i(\rho_0(xx')V) \arrow[rr, "I_{xx',i}"] \arrow[d, "i\circ\varrho(x,x')"'] & & \rho_0'(xx')\circ i(V) \\
i(\rho_0(x)\rho_0(x')V) \arrow[r, "I_{x,i}\otimes\mathrm{id}_{\rho_0(x')}"] & \rho_0'(x)\circ i(\rho_0(x')V) \arrow[r, "\rho_0'(x)\otimes I_{x',i}"] & \rho_0'(x)\rho_0'(x')\circ i(V) \arrow[u, "\varrho'(x,x')\circ i"']
\end{tikzcd}$$

follows directly from (5.5.6) This recovers precisely the notion of a $\mathcal{G}$-module functor [64]. Notice no pentagonator appears here, as this is a relation on the 2-morphisms in $2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$ and hence a pentagonator for it would have to be a 3-morphism.

Modifications. Now let us consider the notion of modifications in $2\mathrm{Rep}^{\mathcal{T}}(\mathcal{G})$ we have defined in **Definition 5.5.5**. The condition (5.5.9) is equivalent to the composition of 2-morphisms $(\mathrm{id}_{\rho_0(x)}\,\mu)*I_{x,i}=I_{x,i'}*\mu$, which is exactly a module natural transformation [64]. □

Following this, I have proven the following in [119].[8]

Proposition 5.7.1. *Let $G=G_{-1}\xrightarrow{1}G_0$ denote a split (ie. trivial Postnikov invariant τ) skeletal finite 2-group, and let $\mathcal{G}=kG$ denote its 2-group algebra. The 2-categories $2\mathrm{Rep}^{\mathcal{T}}(kG)=2\mathrm{Fun}((kG,*),2\mathsf{Vect}^{hBC})$ and $2\mathrm{Rep}_G=2\mathrm{Fun}((G,*),2\mathsf{Vect}^{KV})$ have the same homotopy theory.*

[8]The proof is a routine but lengthy check, so I will not reproduce it here.

What this means is that the coherence relations between the homotopy data — namely the associators, pentagonators, 2-intertwiners, and modifications attached respectively to the two 2-categories — coincide. The latter of which $2\mathrm{Rep}_G$ constitute the 2-representation theory of finite 2-groups as studied in the literature [59, 64, 65] over $\mathsf{2Vect}^{KV}$.

This result suggests that the 2-category $\mathsf{2Vect}^{hBC}$ that I have developed in this Chapter serves as a homotopy refinement of $\mathsf{2Vect}^{BC}$ that can capture, or "mimic", the homotopy coherence data in $\mathsf{2Vect}^{KV}$. This partially resolve the issues plaguing $\mathsf{2Vect}^{BC}$ raised in §1.4. An explicit comparison between $\mathsf{2Vect}^{hBC}$ and $\mathsf{2Vect}^{KV}$ is currently underway with a collaborator.

5.8 Classical limits of 2-bialgebras and 2-R-matrices

Motivated by the diagram (5.0.1), we shall prove in this section that the notion of quantum 2-doubles/2-bialgebras we have introduced in the main text reproduce the known notion of 2-Manin triples/Lie 2-bialgebras [95, 115, 96, 114] in the classical limit.

Classical limit and the Lie-ification functor. Given an (associative) algebra $A \in \mathsf{Alg}_{\mathrm{ass}}$, it is well-known [117, 93] that there is a *Lie-ification functor* $\mathscr{L} : \mathsf{Alg}_{\mathrm{ass}} \to \mathsf{LieAlg}$ that assigns A to its "classical" Lie algebra $\mathfrak{g}(A)$. The Lie bracket is given by the commutator $[X, X'] = XX' - X'X$, where $X \in \mathfrak{g}(A)$ is the image of an element $x \in A$ under $\mathscr{L}$. The associativity of A implies the Jacobi identity of $[-,-]$; note A only needs to be left-symmetric (not necessarily associative) in order for $\mathfrak{g}(A)$ to enjoy the Jacobi identity [95].

There is a left-adjoint to the Lie-ification functor given by the universal envelope $U : \mathfrak{g} \mapsto U(\mathfrak{g})$, which can be understood as a "quantization" map [118]. There is an analogous result for associative 2-algebras [93].

Lemma 5.8.1. *The Lie-ification functor $\mathscr{L} : \mathsf{2Alg}_{\mathrm{ass}} \to \mathsf{Lie2Alg}$ lifts to associative 2-algebras (see* **Definition 5.1.2***), where $\mathfrak{G}(\mathcal{G}) = \mathscr{L}(\mathcal{G}_{-1}) \xrightarrow{t} \mathscr{L}(\mathcal{G}_0)$ is a Lie 2-algebra with*

$$X \rhd Y = X \cdot Y - Y \cdot X, \qquad X = \mathscr{L}(x),\ Y = \mathscr{L}(y),$$

where $x \in \mathcal{G}_0, y \in \mathcal{G}_{-1}$. Moreover, the universal envelop functor U also lifts to Lie 2-algebras $U(\mathfrak{G}) = U(\mathfrak{g}_{-1}) \xrightarrow{t} U(\mathfrak{g}_0)$, such that U is left-adjoint to $\mathscr{L}$.

In the following, we shall write $[-,-] : \mathfrak{G}^{2\wedge} \to \mathfrak{G}$ as the binary L_2-bracket on $\mathfrak{G}$.

Note Lie-ification $\mathscr{L}$ is a functor. This means that, in particular, it sends a 2-algebra representation $\rho : \mathcal{G} \to \mathrm{End}(V)$ on 2-vector space V to a Lie 2-algebra representation $\mathscr{L}(\rho) : \mathfrak{G}(\mathcal{G}) \to \mathfrak{gl}(V)$ as defined in [95, 125].

5.8.1 Lie 2-bialgebras and the 2-classical double

We now extend the above lemma to associative quantum 2-doubles. Let $(\mathcal{G}, \cdot, \Delta)$ denote a strict 2-bialgebra as defined in **Definition 5.2.1**, and let $(\mathcal{G}^*, \cdot^*, \Delta^*)$ denote its dually-paired

2-algebra. We put $\mathfrak{G} = \mathscr{L}(\mathcal{G})$ and $\mathfrak{G}^*[1] = \mathscr{L}(\mathcal{G}^*)$ as the corresponding Lie-ification of these 2-bialgebras.

The Lie-ification procedure can be understood loosely as an "expansion", or linearization, $x \approx 1 + X$ near the identity. Indeed, we have

$$xx' - x'x \approx (1+X)(1+X') - (1+X')(1+X) \approx [X, X']$$

modulo terms of higher order. We make use of this notion on the coproduct (5.1.5), and also perform a skew-symmetrization, in order to define a Lie 2-algebra 2-cochain $\mathscr{L}(\Delta) = \delta = \delta_{-1} + \delta_0$ on $\mathfrak{G}$,

$$\begin{aligned} \delta_{-1}(Y) &= Y_{(1)} \wedge 1 + 1 \wedge Y_{(2)}, \\ \delta_0(X) &= \left[X^l_{(1)} - X^r_{(2)}\right] \wedge 1 + 1 \wedge \left[X^l_{(2)} - X^r_{(1)}\right] \\ &\equiv X_{(1)} \wedge 1 + 1 \wedge X_{(2)}, \end{aligned} \tag{5.8.1}$$

where we have made use of the Sweedler notation (5.1.6), and the conventional notation $\wedge$ to denote skew-symmetric tenor products. Note the skew-symmetrization $\mathcal{G}_{-1} \wedge \mathcal{G}_0$ lands as a subspace in $\mathcal{G}_{-1} \otimes \mathcal{G}_0 \oplus \mathcal{G}_0 \otimes \mathcal{G}_{-1}$.

In degree-0, we have of course also the coproduct Δ_0' defined in (5.1.8). It gives rise to a Lie algebra cochain on $\mathscr{L}(\mathcal{G}_0) = \mathfrak{g}_0$ by

$$\delta_0'(X) = \bar{X}_{(1)} \wedge 1 + 1 \wedge \bar{X}_{(2)} = tX_{(1)} \wedge 1 + 1 \wedge X_{(2)},$$

where $X_{(1)}, X_{(2)}$ have been given in (5.8.1).

Proposition 5.8.1. *The Lie-ification functor $\mathscr{L}$ sends a strict 2-bialgebra $(\mathcal{G}, \Delta)$ to a Lie 2-bialgebra $(\mathfrak{G}; \delta)$.*

Proof. Recall $(\mathfrak{G}; \delta)$ is a Lie 2-bialgebra iff δ is a Lie 2-algebra 2-cocycle [95]. Therefore it suffices to show that the 2-cochain defined in (5.8.1) is a 2-cocycle. This shall follow from the fact that $(\mathcal{G}, \cdot, \Delta)$ is a 2-bialgebra — namely the coproduct map Δ (5.1.5) satisfies (5.1.10), (5.1.11) and (5.1.16).

First note that (5.1.10) and (5.1.11) for the coproduct Δ translates directly to the conditions

$$(t \otimes 1 + 1 \otimes t)\delta_{-1} = \delta_0 \circ t, \qquad (t \otimes 1 - 1 \otimes t)\delta_0 = 0$$

for the 2-cochain $\delta = \delta_{-1} + \delta_0$. Now by a direct computation using (5.8.1), the condition (5.1.16) implies

$$\begin{aligned} \delta_0[X, X'] &= \delta_0(XX') - \delta_0(X'X) \\ &= X_{(1)}X'_{(1)} \wedge 1 + 1 \wedge X_{(2)}X'_{(2)} \\ &\quad - \left(X'_{(1)}X_{(1)} \wedge 1 + 1 \wedge X'_{(2)}X_{(2)}\right) \\ &= [X_{(1)}, X'_{(1)}] \wedge 1 + 1 \wedge [X_{(2)}, X'_{(2)}] \end{aligned}$$

$$
\begin{aligned}
&= \quad tX_{(1)} \rhd X'_{(1)} \wedge 1 + 1 \wedge [X_{(2)}, X'_{(2)}] \\
&= \quad (\bar{X}_{(1)} \rhd \otimes 1 + 1 \otimes \mathrm{ad}_{X_{(2)}})\delta_0(X') - (\bar{X}'_{(1)} \rhd \otimes 1 + 1 \otimes \mathrm{ad}_{X'_{(2)}})\delta_0(X),
\end{aligned}
$$

where we have used the the Peiffer identity and the fact that $\bar{X}_{(1)} = tX_{(1)}$ inherited from the constraints (5.1.15), and

$$
\begin{aligned}
\delta_{-1}(X \rhd Y) \quad &= \quad \delta_{-1}(X \cdot Y) - \delta_{-1}(Y \cdot X) \\
&= \quad \bar{X}_{(1)} \cdot Y_{(1)} \wedge 1 + 1 \wedge \bar{X}_{(2)} \cdot Y_{(2)} \\
&\quad\quad - \big(Y_{(1)} \cdot \bar{X}_{(1)} \wedge 1 - 1 \wedge Y_{(2)} \cdot \bar{X}_{(1)}\big) \\
&= \quad (\bar{X}_{(1)} \rhd Y_{(1)}) \wedge 1 + 1 \wedge (\bar{X}_{(2)} \rhd Y_{(2)}) \\
&= \quad [X_{(1)}, Y_{(1)}] \wedge 1 + 1 \wedge (X_{(2)} \rhd Y_{(2)}) \\
&= \quad (\mathrm{ad}_{X_{(1)}} \otimes 1 + 1 \otimes X_{(2)} \rhd)\delta_{-1}(Y) - (\mathrm{ad}_{Y_{(1)}} \otimes 1 - 1 \otimes \Upsilon_{Y_{(2)}})\delta_0(X),
\end{aligned}
$$

where $\bar{X}_{(2)} = X_{(2)}$. These are precisely the Lie 2-algebra 2-cocycle conditions for δ [95, 115]. $\square$

Now the characterization result in [95] states that $(\mathfrak{G}, \mathfrak{G}^*[1])$ form a matched pair of Lie 2-bialgebras iff δ is a Lie 2-algebra 2-cobracket on $\mathfrak{G}$, namely δ satisfies the 2-cobracket identities. For the 2-cocycle $\delta = \mathscr{L}(\Delta)$ defined in (5.8.1), this is guaranteed precisely by coassociativity (5.1.10), (5.1.11). We have therefore the immediate corollary:

Corollary 5.8.1. *Suppose $(\mathcal{G}, \mathcal{G}^*)$ form a matched pair of strict 2-bialgebras. The Lie-ification functor $\mathscr{L}$ sends a quantum 2-double $D(\mathcal{G}) = \mathcal{G}\bar{\bowtie}\mathcal{G}^*$ to a classical 2-double $\mathfrak{d} = \mathfrak{G} \bowtie \mathfrak{G}^*[1]$.*

In other words, our construction of the quantum 2-double $D(\mathcal{G})$ admits the classical 2-double as a classical limit, which directly categorifies an analogous statement between the general quantum double construction of Majid [118] and the classical Drinfel'd double [9].

5.8.2 The classical 2-r-matrix

Let us now turn to the classical limit of the 2-R-martrix as defined in §5.2.4. Prior to that, we first describe one of the key properties of the duality pairing on a quantum 2-double, namely its *invariance*. This is expressed by, for instance, (5.5.4) in the case of the coadjoint representation. For the sew-pairing $\langle -, - \rangle_{\mathrm{sk}}$ forming the quantum 2-double $D(\mathcal{G}, \mathcal{G}) = \mathcal{G}\bar{\bowtie}\mathcal{G}^{\mathrm{opp}}$, however, $\mathcal{G}$ acts on $\mathcal{G}^{\mathrm{opp}}$ via its underlying (opposite) 2-algebra structure, which means that the skew-pairing satisfies the invariance property

$$
\langle xx', g \rangle_{\mathrm{sk}} = -\langle x', g \cdot x \rangle_{\mathrm{sk}}, \qquad \langle x \cdot y, f \rangle_{\mathrm{sk}} = -\langle y, fx \rangle_{\mathrm{sk}}, \qquad \langle ff', y \rangle_{\mathrm{sk}} = -\langle f', f \cdot y \rangle_{\mathrm{sk}}.
$$

Given the adjoint action $\bar{\rhd} = (\Upsilon, (\rhd_0, \rhd_{-1}))$ of $\mathcal{G}$ on $\mathcal{G}^{\mathrm{opp}}$,

$$
x \rhd_0 g = g \cdot x, \qquad x \rhd_{-1} f = fx, \qquad \Upsilon_y f = f \cdot y,
$$

this invariance property translates to the following conditions on the 2-R-matrix $\mathcal{R}^{l,r}$,

$$(x\cdot\otimes 1+1\otimes x\rhd_0)\mathcal{R}^l=0,\qquad (x\cdot\otimes 1)\mathcal{R}^r+(1\otimes x\rhd_{-1})\mathcal{R}^l=0,\qquad (f\cdot\otimes 1+1\otimes f\rhd_0)\mathcal{R}^r=0.$$

Consider the first and last conditions with $x=f\in\mathcal{G}_0$. They can be rewritten equivalently as the conditions

$$(x\cdot\otimes 1)\mathcal{R}^l+(1\otimes x\rhd_0)\mathcal{R}^r=0,\qquad (x\cdot\otimes 1)\mathcal{R}^r+(1\otimes x\rhd_0)\mathcal{R}^l=0,$$

which together with the second condition may be compactly expressed as, using the graded sum,

$$(x\bar{\rhd}\otimes 1+1\otimes x\bar{\rhd})(\mathcal{R}+\sigma(\mathcal{R}))=0,\qquad \forall\ x\in\mathcal{G}_0,\tag{5.8.2}$$

where σ is a permutation of the $\mathcal{G}_0,\mathcal{G}_{-1}$ components.

Let us now finally recover the universal classical 2-r-matrix. This is once again accomplished by taking the Lie-ification functor on the universal quantum 2-R-matrix, $\mathfrak{r}=\mathscr{L}(\mathcal{R})\in\mathfrak{G}\otimes\mathfrak{G}$, whence

$$\mathfrak{g}_{-1}\otimes\mathfrak{g}_0\ni\mathfrak{r}^r=\mathscr{L}(\mathcal{R}^r),\qquad \mathfrak{g}_0\otimes\mathfrak{g}_{-1}\ni\mathfrak{r}^l=\mathscr{L}(\mathcal{R}^l).\tag{5.8.3}$$

The equivariance condition (5.2.27) clearly implies

$$D_t^-\mathfrak{r}=0,\tag{5.8.4}$$

while applying the Lie-ification functor $\mathscr{L}$ to (5.8.2) gives

$$[X\otimes 1+1\otimes X,\mathfrak{r}+\sigma(\mathfrak{r})]=0,\qquad X=\mathscr{L}(x)\in\mathfrak{g}_0.$$

Here, we have used the fact that the adjoint action ρ of $\mathcal{G}$ on itself gives rise to the adjoint representation (using the graded Lie bracket) $\mathscr{L}(\bar{\rhd})=[-,-]$ of $\mathfrak{G}$ on itself [95].

Finally, we consider the 2-Yang-Baxter equations (5.2.28). We sum each equation in (5.2.28) in the total graded complex $\mathcal{G}^{3\otimes}$, and rearragnge them to the form

$$\begin{aligned}0\quad=\quad&\big(\mathcal{R}^r_{23}(\mathcal{R}^r_{13}\cdot_l\mathcal{R}^l_{12})-(\mathcal{R}^l_{12}\cdot_r\mathcal{R}^r_{13})\mathcal{R}^r_{23}\big)\\&+\ \big(\mathcal{R}^l_{23}\cdot_l\mathcal{R}^r_{13})\mathcal{R}^r_{12}-\mathcal{R}^r_{12}(\mathcal{R}^r_{13}\cdot_r\mathcal{R}^l_{23})\big)\\&+\ \big(\mathcal{R}^l_{23}(\mathcal{R}^l_{13}\cdot_r\mathcal{R}^r_{12})-(\mathcal{R}^r_{12}\cdot_l\mathcal{R}^l_{13})\mathcal{R}^l_{23}\big)\\&+\ \big((\mathcal{R}^r_{23}\cdot_r\mathcal{R}^l_{13})\mathcal{R}^l_{12}-\mathcal{R}^l_{12}(\mathcal{R}^l_{13}\cdot_l\mathcal{R}^r_{23})\big)\,.\end{aligned}\tag{5.8.5}$$

Applying the Lie-ification functor $\mathscr{L}$ to this equation yields

$$\begin{aligned}0\quad=\quad&\big([\mathfrak{r}^r_{13},\mathfrak{r}^l_{12}]+[\mathfrak{r}^r_{23},\mathfrak{r}^r_{13}]+[\mathfrak{r}^r_{23},\mathfrak{r}^l_{12}]\big)\,|_{rrl}+\big([\mathfrak{r}^l_{23},\mathfrak{r}^r_{13}]+[\mathfrak{r}^l_{23},\mathfrak{r}^r_{12}]+[\mathfrak{r}^r_{13},\mathfrak{r}^r_{12}]\big)\,|_{lrr}\\&+\ \big([\mathfrak{r}^l_{13},\mathfrak{r}^r_{12}]+[\mathfrak{r}^l_{23},\mathfrak{r}^l_{13}]+[\mathfrak{r}^l_{23},\mathfrak{r}^r_{12}]\big)\,|_{llr}+\big([\mathfrak{r}^r_{23},\mathfrak{r}^l_{13}]+[\mathfrak{r}^r_{23},\mathfrak{r}^l_{12}]+[\mathfrak{r}^l_{13},\mathfrak{r}^l_{12}]\big)\,|_{rll},\end{aligned}$$

where the subscripts indicate where each term came from in (5.8.5).

Consider the two places in which $\mathfrak{r}_{23}^l\mathfrak{r}_{12}^r$ occurs in the above. These terms take the form respectively in Sweedler notation

$$\begin{aligned}\mathfrak{r}_{23}^l\mathfrak{r}_{12}^r|_{lrr} &= \mathfrak{r}_{(1)}^r\eta_0 \otimes \mathfrak{r}_{(1)}^l\mathfrak{r}_{(2)}^r \otimes \mathfrak{r}_{(2)}^l \cdot \eta_{-1},\\ \mathfrak{r}_{23}^l\mathfrak{r}_{12}^r|_{llr} &= \mathfrak{r}_{(1)}^r \cdot \eta_{-1} \otimes \mathfrak{r}_{(1)}^l\mathfrak{r}_{(2)}^r \otimes \mathfrak{r}_{(2)}^l\eta_0,\end{aligned}$$

where η_0, η_{-1} are the units in $\mathcal{G}_0, \mathcal{G}_{-1}$. By using the Peiffer identity and the equivariance condition (5.8.4)

$$(t\mathfrak{r}_{(1)}^l) \otimes \mathfrak{r}_{(2)}^l = (t\otimes 1)\mathfrak{r}^l = (1\otimes t)\mathfrak{r}^r = \mathfrak{r}_{(1)}^r \otimes (t\mathfrak{r}_{(2)}^r),$$

we can compute that

$$\begin{aligned}\mathfrak{r}_{23}^l\mathfrak{r}_{12}^r|_{llr} &= \mathfrak{r}_{(1)}^r \cdot \eta_{-1} \otimes \mathfrak{r}_{(1)}^l \cdot (t\mathfrak{r}_{(2)}^r) \otimes \mathfrak{r}_{(2)}^l\eta_0\\ &= (t\mathfrak{r}_{(1)}^l) \cdot \eta_{-1} \otimes \mathfrak{r}_{(1)}^l \cdot \mathfrak{r}_{(2)}^l \otimes \mathfrak{r}_{(2)}^l\eta_0\\ &= \mathfrak{r}_{(1)}^l\eta_{-1} \otimes \mathfrak{r}_{(1)}^l \cdot_r \mathfrak{r}_{(2)}^l \otimes \mathfrak{r}_{(2)}^l\eta_0\\ &= \mathfrak{r}_{23}^l \cdot_r \mathfrak{r}_{12}^l\end{aligned} \qquad \begin{aligned}\mathfrak{r}_{23}^l\mathfrak{r}_{12}^r|_{lrr} &= \mathfrak{r}_{(1)}^r\eta_0 \otimes (t\mathfrak{r}_{(1)}^l) \cdot \mathfrak{r}_{(2)}^r \otimes \mathfrak{r}_{(2)}^l \cdot \eta_{-1}\\ &= \mathfrak{r}_{(1)}^r\eta_0 \otimes \mathfrak{r}_{(1)}^r \cdot \mathfrak{r}_{(2)}^r \otimes (t\mathfrak{r}_{(2)}^r) \cdot \eta_{-1}\\ &= \mathfrak{r}_{(1)}^r\eta_0 \otimes \mathfrak{r}_{(1)}^r \cdot_l \mathfrak{r}_{(2)}^r \otimes \mathfrak{r}_{(2)}^r\eta_{-1}\\ &= \mathfrak{r}_{23}^r \cdot_l \mathfrak{r}_{12}^r\end{aligned},$$

As such, we have

$$[\mathfrak{r}_{23}^l, \mathfrak{r}_{12}^r] = [\mathfrak{r}_{23}^l, \mathfrak{r}_{12}^l] = [\mathfrak{r}_{23}^r, \mathfrak{r}_{12}^r],$$

and hence collecting all terms from the above gives

$$\begin{aligned}[\mathfrak{r}_{12}, \mathfrak{r}_{13}] &= [\mathfrak{r}_{12}^r, \mathfrak{r}_{13}^r] + [\mathfrak{r}_{12}^r, \mathfrak{r}_{13}^l] + [\mathfrak{r}_{12}^l, \mathfrak{r}_{13}^r] + [\mathfrak{r}_{12}^l, \mathfrak{r}_{13}^l]\\ [\mathfrak{r}_{13}, \mathfrak{r}_{23}] &= [\mathfrak{r}_{13}^r, \mathfrak{r}_{23}^r] + [\mathfrak{r}_{13}^r, \mathfrak{r}_{23}^l] + [\mathfrak{r}_{13}^l, \mathfrak{r}_{23}^r] + [\mathfrak{r}_{13}^l, \mathfrak{r}_{23}^l]\\ [\mathfrak{r}_{12}, \mathfrak{r}_{23}] &= [\mathfrak{r}_{12}^r, \mathfrak{r}_{23}^r] + [\mathfrak{r}_{12}^r, \mathfrak{r}_{23}^l] + [\mathfrak{r}_{12}^l, \mathfrak{r}_{23}^r] + [\mathfrak{r}_{12}^l, \mathfrak{r}_{23}^l]\end{aligned}$$

This is precisely the 2-graded classical Yang-Baxter equation of [95]

$$[\![\mathfrak{r}, \mathfrak{r}]\!] = [\mathfrak{r}_{12}, \mathfrak{r}_{13}] + [\mathfrak{r}_{13}, \mathfrak{r}_{23}] + [\mathfrak{r}_{12}, \mathfrak{r}_{23}] = 0$$

for the expansion $\mathfrak{r} = \mathscr{L}(\mathcal{R}) = \mathfrak{r}^r + \mathfrak{r}^l$.

Theorem 5.8.1. *The Lie-ification functor sends the universal quantum 2-R-matrix to a 2-graded classical r-matrix.*

In other words, the "quantization" of the classical 2-r-matrix and the associated Lie 2-bialgebra $\mathfrak{G}$ yields a universal 2-R-matrix with the associated quasitriangular 2-bialgebra $\mathcal{G}$.

5.8.3 Weak Lie 2-bialgebras

We now prove the weak analogues of the classical limit for 2-bialgebras.

Lemma 5.8.2. *The Lie-ification functor $\mathscr{L}$: $\mathsf{Alg} \to \mathsf{LieAlg}$ extends to weak 2-algebras, assigning $(\mathcal{G}, \mathcal{T})$ to a weak Lie 2-algebra $(\mathfrak{G}(\mathcal{G}), \mu_3)$ where the homotopy map μ_3 is the total skew-symmetrization of $\mathcal{T}$.*

Proof. We construct the Lie 2-algebra structure as in **Lemma 5.8.1**. Let $U_3 = \mathscr{L}\circ\mathcal{T}\circ\mathscr{L}$ denote the induced trilinear map on $\mathscr{L}(\mathcal{G})$. We apply $\mathscr{L}$ to the Jacobiator $J(X,X',X'') = [X,[X',X'']] + [X',[X'',X'] + [X'',[X,X']]$,

$$\begin{aligned}
J(X,X',X'') &= X(X'X'') - X(X''X') - (X'X'')X + (X''X')X \\
&\quad +X'(X''X) - X'(XX'') - (X''X)X' + (XX'')X' \\
&\quad +X''(XX') - X''(X'X) - (XX')X'' + (X'X)X'' \\
&= tU_3(X,X',X'') - tU_3(X,X'',X') + tU_3(X',X'',X) \\
&\quad - tU_3(X',X,X'') + tU_3(X'',X',X) - tU_3(X'',X,X') \\
&= t(U_3(X,X',X'') - U_3(X,X'',X') + U_3(X',X'',X) \\
&\quad - U_3(X',X,X'') + U_3(X'',X,X')) - U_3(X'',X',X),
\end{aligned}$$

where we have used the weak 1-associativity condition for $\mathcal{G}$. Similarly, for $J(X,X',Y) = X\rhd(X'\rhd Y) - X'\rhd(X\rhd Y) - [X,X']\rhd Y$ we have

$$\begin{aligned}
J(X,X',Y) &= t(U_3(X,X',tY) - U_3(X,tY,X') + U_3(X',tY,X) \\
&\quad - U_3(X',X,tY) + U_3(tY,X,X') - U_3(tY,X',X),
\end{aligned}$$

hence if we define the total skew-symmetrization

$$\begin{aligned}
\mu_3(X,X',X'') &\equiv U_3(X,X',X'') - U_3(X,X'',X') + U_3(X',X'',X) \\
&\quad -U_3(X',X,X'') + U_3(X'',X,X') - U_3(X'',X',X),
\end{aligned}$$

then weak 1-associativity implies the 2-Jacobi identity on $\mathscr{L}(\mathcal{G})$.

Using the Peiffer conditions on this fact, we see that the weak bimodularity condition also implies the 2-Jacobi identity, with two tY's inserted in U_3 instead. Similar computations show that the Hochschild 3-cocycle condition for $\mathcal{T}$ implies the Lie 3-cocycle condition for μ_3.

Finally, let $F:(\mathcal{G},\mathcal{T})\to(\mathcal{G}',\mathcal{T}')$ denote a weak 2-algebra homomorphism as defined in (5.3.1). By applying the Lie-ification functor and appropriately skew-symmetrizing $\mathcal{T},\mathcal{T}'$ and the 2-algebra structure, we recover precisely the definition of a weak 2-algebra map $\mathscr{L}(F):(\mathfrak{G},\mu)\to(\mathfrak{G}',\mu')$ [122]. Thus $\mathscr{L}$ is functorial. □

Similar to the Lie 2-algebra 2-cocycle (5.8.1) defined from the coproduct Δ, we form the classical limit of the coassociator Δ_1 by totally skew-symmetrizing and linearizing it, such that we have the Lie cochain

$$\eta(X) = X_{(1)}\wedge 1\wedge 1 - 1\wedge X_{(2)}\wedge 1 + 1\wedge 1\wedge X_{(3)},\qquad X\in\mathfrak{g}_0 = \mathscr{L}(\mathcal{G}_0). \tag{5.8.6}$$

It is not hard to see by, for instance, dualizing the computations in the proof of **Lemma 5.8.2**, that the conditions (5.3.3), (5.3.5) reduce to

$$\begin{aligned} \delta_{-1} \circ \delta_{-1} &= \eta \circ t, \qquad \text{cf. (42) in [96]} \\ (\delta_{-1} + \delta_0) \circ \delta_0 &= D_t \circ \eta, \qquad \text{cf. (43) in [96]}, \\ \eta \circ \delta_0 &= \delta_{-1} \circ \eta, \qquad \text{cf. (44) in [96]}. \end{aligned}$$

Let $(\mathcal{G}, \mathcal{T}, \Delta_1)$ be a weak 2-bialgebra as given in **Definition 5.3.3**. The conditions (5.3.7) translate to

$$\begin{aligned} \delta_{-1}(\mu_3(X, X', X'')) &= \mu_3(\bar{X}_{(1)}, \bar{X}'_{(1)}, \bar{X}''_{(1)}) \wedge \mu_3(\bar{X}_{(2)}, \bar{X}'_{(2)}, \bar{X}''_{(2)}), \\ \delta_1([X, X']) &= [X_{(1)}, X'_{(1)}] \wedge 1 \wedge 1 - 1 \wedge [X_{(2)}, X'_{(2)}] \wedge 1 + 1 \wedge 1 \wedge [X_{(3)}, X'_{(3)}], \end{aligned}$$

which are precisely the conditions for a *weak-Lie 2-bialgebra* $(\mathfrak{G}, \mu_3, \delta)$ [114], expressed explicitly. In other words, we have the weak version of **Proposition 5.8.1**:

Proposition 5.8.2. *The Lie-ification functor takes a weak 2-bialgebra* $(\mathcal{G}, \mathcal{T}, \Delta)$ *to a weak Lie 2-bialgebra* $(\mathfrak{G}, \mu, \delta)$*, with the 2-cocycle data given as in* (5.8.1)*,* (5.8.6)*.*

Note that this is a general result, which does not require any skeletality assumptions on $\mathcal{G}$. When $\mathcal{T} = 0$ and hence $\mu_3 = 0$, we recover the conditions for a quasi-Lie 2-bialgebra studied also in [96].

Chapter 6

The 4d Kitaev model

In this Chapter, I will apply the quantization framework established in §5 to study 4d $\mathbb{Z}_2$-gauge theories. I proved that this in fact recovers the known 2-categorical constructions of the charges in the 3+1d toric code, as well as the spin-$\mathbb{Z}_2$ gauge theory [76].

Theorem 6.0.1. *We have the following braided equivalences*

$$\begin{aligned} 2\mathrm{Rep}_{wk}(D^{\omega_b}(B\mathbb{Z}_2)) &\simeq \mathscr{R} \simeq Z_1(2\mathsf{Vect}^{KV}[\mathbb{Z}_2]), \\ 2\mathrm{Rep}_{wk}(D^{\omega_f}(B\mathbb{Z}_2)) &\simeq \mathscr{S} \simeq Z_1(\Sigma\mathsf{sVect}), \end{aligned}$$

where Z_1 is the Drinfel'd centre and Σ is the condensation completion functor[1] *defined in [148].*

This result makes concrete the equivalence between the 2-categorical [46] and field-theoretical [71] descriptions of 4d gapped $\mathbb{Z}_2$-symmetric topological phases, and provides an explicit machinery to produce 2-categories from a given 4d (finite) 2-group gauge theory action. This chapter is based on my work [121].

6.0.1 Skeletal 2-double of a finite cyclic Abelian group

We first begin with a quick but explicit description of the Drinfel'd double 2-bialgebra (see §5.4) that we are interested in. Let N be a finite cyclic Abelian group, and we take $M = \widehat{N}$ to be the Pontrjagyn dual of N. This makes the group algebra kM naturally as a kN-bimodule canonically through the dual left- and right- actions

$$(x \cdot y)(x') = y(x^{-1}x'), \qquad (y \cdot x)(x') = y(x'x), \tag{6.0.1}$$

where $x, x' \in N$ and $y \in \widehat{N} = M$. We denote the Pontrjagyn duality isomorphism by $p : x \mapsto p(x) = \hat{x}$ (recall both N, M are cyclic Abelian groups). We thus take $k\mathbb{G} = kBM[1] \xrightarrow{0} kN$ as our desired 2-algebra.

To make $k\mathbb{G}$ into a 2-bialgebra, we equip it with the grouplike graded coproduct Δ defined

[1] Note $\Sigma\mathsf{Vect} \simeq 2\mathsf{Vect}^{KV}$ [148].

by

$$\Delta_{-1}(y) = y\otimes y, \qquad \Delta_0'(x) = x\otimes x,$$
$$\Delta_0(x) = p(x)\otimes x + x\otimes p(x), \tag{6.0.2}$$

where $x\in N, y\in M$. By definition, Δ is coassociative and admits the usual antipode $S_0^0(x) = x^{-1}, S_0^1(y) = y^{-1}$, together with the unit/counit

$$\begin{cases}\eta_0 = 1\in N \\ \eta_{-1} = 1\in M\end{cases}, \qquad \begin{cases}\epsilon_0(x) = \delta_{x\eta_0}\in k \\ \epsilon_{-1}(y) = \delta_{y\eta_{-1}}\in k\end{cases}.$$

Moreover, this coproduct can be very easily shown to satisfy the 2-bialgebra axioms,

$$\Delta_{-1}(x\cdot y) = x\cdot y\otimes x\cdot y, \qquad \Delta_{-1}(y\cdot x) = y\cdot x\otimes y\cdot x,$$
$$\Delta_0(xx') = p(x)p(x')\otimes xx' + xx'\otimes p(x)p(x'),$$

where we have used the fact that p is a group homomorphism $p(xx') = p(x)p(x')$. This defines $(k\mathbb{G},\cdot,\Delta,S)$ as a unital Hopf 2-algebra (see Appendix A of [119]).

Moreover, it is easy to check that the grouplike coproduct Δ_0 (6.0.2) dualizes to the kN-bimodule structure (6.0.1) on $kM\cong k\widehat{N}$, as required by self-duality

$$k\mathbb{G}\equiv D(BM) = kBM\bar{\bowtie}kN, \qquad \begin{cases} BM = M\xrightarrow{0} * \\ N_* = *\xrightarrow{0} N\end{cases}.$$

We call this Hopf 2-algebra $(D(BM),\cdot,\Delta,S)$ the **(Drinfel'd) 2-double of** M.

Recall the factorizability property **Theorem 5.4.1** means that $D(BM)$ fits into a cospan of 2-bialgebras

$$kBM\hookrightarrow D(BM)\hookleftarrow kN.$$

However, since $HH^3(*,kM)=0, HH^3(kN,*)=0$, these 2-bialgebra injections cannot extend to an equivalence if we wish for $D(BM)$ to carry a non-trivial Hochschild class $\mathcal{T}\in HH^3(kN,kM)$. Due to a result

$$HH^*(kN,kM)\cong kM\otimes_k H^*(kN,k)\cong H^3(N,\widehat{N})\otimes k$$

of [149], we shall take $\mathcal{T}$ as coming from a Postnikov class $\tau\in H^3(N,\hat{N})$ of the 2-group $\mathbb{D} = \hat{N}\xrightarrow{1} N$. Since this is a bijection, we shall abuse notation and denote $\mathcal{T}$ as τ in the following.

6.1 2-BF theory on the Drinfel'd 2-double $D(BM)$

In this section, we specialize the above Drinfel'd double 2-bialgebra to the case $N=\mathbb{Z}_2, M = \widehat{\mathbb{Z}_2}\cong\mathbb{Z}_2$, and construct (3+1)D 2-BF theory based on $D(BM)$. We study its extended $\mathbb{Z}_2$-

charged excitations by studying $2\mathrm{Rep}_{\mathrm{wk}}(D(B\mathbb{Z}_2))$, and seek to prove the main result **Theorem 6.0.1**.

Along the way, we shall make concrete the connection8 between our 2-BF theory and the higher-gauge topological nonlinear σ-models (NLSMs) that have already appeared in the literature [71, 46]. We shall take the ground field $k=\mathbb{C}$ throughout the following.

Recall the Drinfel'd double 2-bialgebra $D(B\mathbb{Z}_2)$ has the structure of a skeletal 2-algebra

$$D(B\mathbb{Z}_2)=k\widehat{\mathbb{Z}_2}\xrightarrow{0}k\mathbb{Z}_2,$$

whose Hochschild class is determined by the choice of a Postnikov class

$$\tau\in H^3(\mathbb{Z}_2,\widehat{\mathbb{Z}_2})\cong\mathbb{Z}_2$$

of the underlying 2-group. Let $x\in k\mathbb{Z}_2$ and $y\in k\widehat{\mathbb{Z}_2}$ be understood as the non-trivial generators.

Let $k\mathbb{Z}_2=kN$ in degree-0 act on $k\widehat{\mathbb{Z}_2}=kM$ on the left by (6.0.1) as group algebras. There are two such algebra automorphisms: the trivial or the sign representation. We denote the Drinfel'd double 2-bialgebra by $D(B\mathbb{Z}_2)^{\mathrm{trv}}$ in the former case, while by $D(B\mathbb{Z}_2)^{\mathrm{sgn}}$ in the latter case. This then induces a non-trivial grouplike component $\Delta_0(x)=\hat{x}\otimes x+x\otimes\hat{x}$ of the coproduct Δ on $D(B\mathbb{Z}_2)$ (recall $\hat{x}=p(x)$ where p is the Pontrjagyn duality).

Now consider the discrete combined $D(B\mathbb{Z}_2)$-connection $(\mathbf{A},\mathbf{\Sigma})=(A+\Sigma,C+B)$ on a 4-manifold X [115]. These connection forms are given by cochains

$$A\in C^1(X,\mathbb{Z}_2),\qquad B\in C^2(X,\widehat{\mathbb{Z}_2}),$$

with the components $\Sigma=0,C=0$ trivial. Depending on the automorphism $\mathrm{Aut}(k\mathbb{Z}_2)$ encoded in the Drinfel'd double 2-bialgebra $D(B\mathbb{Z}_2)$, the 1- and 2-curvatures of the field theory are given by

$$F=\begin{cases}dA & ;\text{in } D(B\mathbb{Z}_2)^{\mathrm{triv}}\\ dA+\frac{1}{2}A\cup A & ;\text{in } D(B\mathbb{Z}_2)^{\mathrm{sgn}}\end{cases},\qquad d_AB=\begin{cases}dB & ;\text{in } D(B\mathbb{Z}_2)^{\mathrm{triv}}\\ dB+A\cup B & ;\text{in } D(B\mathbb{Z}_2)^{\mathrm{sgn}}\end{cases},$$

where the cup products are implemented through the automorphism $\mathrm{Aut}(k\mathbb{Z}_2)$ or its dual. The corresponding *monster 2-BF theory* [115] is given by the topological action

$$S[A,B]=\frac{1}{2}\int_X\langle B\cup F\rangle+\langle\tau(A)\cup A\rangle,\tag{6.1.1}$$

where we recall that $\tau\in H^3(\mathbb{Z}_2,\widehat{\mathbb{Z}_2})$ is the underlying Postnikov class of $D(B\mathbb{Z}_2)$.

Note that the discrete 1-form gauge fields must be flat, $F=dA=0$, and terms like $A^2=0$ mod 2 vanish, hence the classical equations of motion (EOMs) are given by

$$F=dA=0,\qquad d_AB=\tau(A).\tag{6.1.2}$$

These EOMs, together with the coefficient of $\frac{1}{2}$ in front of the topological action (6.1.1), tell us that the cochains A, B are $\mathbb{Z}_2$-valued. We will introduce in the following a non-trivial cohomological term that "mimics" $\frac{1}{2}A^2$. However, it is important to note that these cohomological terms constitute *twists* on the Drinfel'd double 2-bialgebra and are not dynamical; they do not alter the EOM (6.1.2).

We define the partition function corresponding to (6.1.1) on a 4-manifold X as a formal path integral

$$Z_{\mathrm{Kit}}(X) = \int dAdBe^{i2\pi S[A,B]}, \tag{6.1.3}$$

which should be appropriately normalized such that $Z_{\mathrm{Kit}}(S^4) = 1$ [71]. We call Z_{Kit} the **4d Kitaev model**. It should be understood as a collective of two such theories[2],

$$\text{(Invisible) toric code}: \; Z^0_{\mathrm{Kit}}, \qquad \text{Spin-Kiatev}: \; Z^s_{\mathrm{Kit}},$$

arising respectively from $D(B\mathbb{Z}_2)^{\mathrm{trv}}$ and $D(B\mathbb{Z}_2)^{\mathrm{sgn}}$. We shall refer to either of these Drinfel'd double 2-bialgebras collectively as $D(B\mathbb{Z}_2)$ in the following. The central idea is then that Z_{Kit} has a Drinfel'd double 2-bialgebra symmetry.

Z_{Kit} as a topological nonlinear σ-model

There had been proposals to construct (3+1)D topological phases with a higher-gauge field theory [104]. Specifically, [71] constructs a topological non-linear σ-model (NLSM) which corresponds to a higher-Dijkgraaf-Witten theory based on a 2-group, and claims that all (3+1)D topological phases can be described this way.

The NLSM they construct is characterized by the following data: (i) a (skeletal) 2-group $\mathbb{G} = \mathbb{Z}_2 \to G_b$, where G_b is a finite group labeling "stringlike bosonic charges", and $\mathbb{Z}_2$ is either fermion parity $\mathbb{Z}_2^f$ or a magnetic π-flux $\mathbb{Z}_2^m$, (ii) the first Postnikov class $\tau \in H^3(G_b, \mathbb{Z}_2^f)$ of $\mathbb{G}$ and (iii) a Dijkgraaf-Witten class $\omega \in H^4(\mathbb{G}, \mathbb{R}/\mathbb{Z})$ [104, 71]. We write the Hoáng data [103] of $\mathbb{G}$ as $(G_b, \mathbb{Z}_2^f, \tau)$.

Our construction of the Kitaev model (6.1.3) fits nicely into this framework, with the 2-group $\widehat{\mathbb{Z}_2} \xrightarrow{0} \mathbb{Z}_2$ given by the Hoáng data

$$(G_b = \mathbb{Z}_2, \mathbb{Z}_2^f \cong \widehat{\mathbb{Z}_2}, \tau).$$

To construct the Dijkgraaf-Witten cocycle, we begin with the group cohomology ring $H^*(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2[u]$ with a generator $u \in H^1(\mathbb{Z}_2, \mathbb{Z}_2)$ in degree-1 [150]. Considering $\mathbb{Z}_2$ as a trivial $\mathbb{Z}_2$-module, the sign representation $\mathrm{sgn} \in \mathrm{Aut}(k\mathbb{Z}_2) \cong \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2) \cong H^1(\mathbb{Z}_2, \mathbb{Z}_2)$ then serves as a representative of the generator u.

Now consider $D(B\mathbb{Z}_2)^{\mathrm{sgn}}$. The cup product for the term $\frac{1}{2}A \cup A$ in the curvature F is

[2] There is a slight misnomer here, where Z^0_{Kit} should really be called the "invisible" toric code, as it fails to satisfy the principle of *remote detecatbility* [79, 78, 76]; see *Remark 6.2.2* later.

implemented by the sign representation $\mathrm{sgn} = u \in H^1(\mathbb{Z}_2, \mathbb{Z}_2)$, from which

$$\frac{1}{2}A \cup A = \bar{e}(A), \qquad \bar{e} = \frac{1}{2}u \cup u \in H^2(\mathbb{Z}_2, \mathbb{Z}_2). \tag{6.1.4}$$

The factor of $1/2$ is very important as, without it, $u \cup u = 0 \mod 2$ is trivial in $\mathbb{Z}_2$-cohomology [150]. Dualizing the value of $\bar{e}$ to a class in $H^2(\mathbb{Z}_2, \widehat{\mathbb{Z}_2})$, it lifts the action $\triangleright$ of $k\mathbb{Z}_2$ on $k\widehat{\mathbb{Z}_2}$ to a central extension.

The term $\langle B \cup \bar{e}(A)\rangle$ that appears in (6.1.1) gives precisely the Dijkgraaf-Witten cocycle $\omega \in Z^4(G, \mathbb{R}/\mathbb{Z})$. Indeed, going on-shell of the EOM (6.1.2) reduces the spin-Kitaev partition function to

$$Z^s_{\text{Kit}}(X) \sim \sum_{\substack{dA=0\\ dB=\tau}} e^{i\pi\int_X \langle B\cup\bar{e}(A)\rangle}.$$

This gives exactly the NLSM constructed in [71] with $\omega(B, A) = B \cup \bar{e}(A)$, provided the **anomaly-free condition**

$$\tau \cup \bar{e} = 0 \tag{6.1.5}$$

is satisfied. This condition ensures that that the Dijkgraaf-Witten integrand $\omega(A, B) = \langle B \cup \bar{e}(A)\rangle$ is a cocycle $d\omega(A, B) = 0$ in light of the EOM $dB = \tau(A)$.

Classification of 4d topological phases with a single pointlike $\mathbb{Z}_2$-charge

The above describes the construction of a 4d Dijkgraaf-Witten topological field theory. As we have mentioned, these were proposed to describe [71, 104, 72, 151], in a very general sense, 4d gapped topological phases. Another approach towards this follows the program of "higher categorical symmetries" [46, 152, 47, 79, 48, 64, 66, 18]. In particular, the 4d toric code has been extensively studied in the literature [67, 68, 76] from this perspective, so we understand its corresponding braided fusion 2-category quite well.

By hypothesis, gapped topological phases are characterized by non-degenerate[3] braided fusion 2-categories, based on the physical principle of **remote detectability** [152, 47, 79, 48, 18]. In particular, those with a single pointlike $\mathbb{Z}_2$-charge have been classified in [76, 63]. These phases are

1. the 4d toric code $\mathscr{R} \simeq Z_1(2\mathsf{Vect}^{KV}[\mathbb{Z}_2])$,
2. the 4d spin-$\mathbb{Z}_2$ gauge theory $\mathscr{S} \simeq Z_1(\Sigma\mathsf{sVect})$,
3. the w_2w_3 gravitational anomaly $\mathscr{T}$,

where Z_1 denotes the Drinfel'd centre and Σ denotes the condensation completion functor [148]. Here, $\mathsf{Vect}[\mathbb{Z}_2]$ denotes the category of $\mathbb{Z}_2$-graded vector spaces, and sVect is the category of supervector spaces.

In this paper, we shall mostly focus on the gapped phases $\mathscr{R}, \mathscr{S}$, and leave the study of the gravitational anomaly $\mathscr{T}$ to a later work; the reason for this shall be given at the end of

[3]Namely the sylleptic/E_2-centre Z_2 is trivial.

§6.3. We will find explicit realizations of these phases as 2-representation 2-categories of certain versions of the quantum 2-double $D(B\mathbb{Z}_2)$. To do so, we study the excitations in the associated NLSM (6.1.1).

6.1.1 Anomaly-freeness of the 4d spin-Kitaev model

Recall from the above that the 4d Kitaev model Z_{Kit} is well-defined provided the non-trivial Postnikov class τ and extension class $\bar{e}$ of the underlying 2-group satisfies the anomaly free condition (6.1.5).

Let us here study, from the point of view of the 2-representation 2-category $2\mathrm{Rep}_{\mathrm{wk}}(D(B\mathbb{Z}_2))$, why the anomaly-free condition (6.1.5) is necessary. Recall that the self-duality of $D(B\mathbb{Z}_2)$ as a Drinfel'd double 2-bialgebra means that the Postnikov class τ dualizes to a coassociator $\Delta_1 : \mathbb{Z}_2 \to \widehat{\mathbb{Z}_2}^{3\otimes}$ defining the associator 1-morphism a_{VWU} for the objects $V, W, U \in 2\mathrm{Rep}_{\mathrm{wk}}(D(B\mathbb{Z}_2))$.

The key point is that, in general, the pentagon relation for a follows from the condition (5.3.4), which in turn follows from the 3-cocycle condition for τ. This notion generalizes to the case where $D(B\mathbb{Z}_2)$ is twisted by $\bar{e} \in H^2(\mathbb{Z}_2, \widehat{\mathbb{Z}_2})$; that is, the product in $D(B\mathbb{Z}_2)_0 = \mathbb{Z}_2$ is modified such that

$$x \times x' = \bar{e}(x, x')(xx'), \qquad x, x' \in \mathbb{Z}_2.$$

We shall denote the corresponding 2-representation 2-category by

$$2\mathrm{Rep}_{\mathrm{wk}}(D^{\bar{e}}(B\mathbb{Z})) = 2\mathrm{Rep}^{\tau}_{m}(D(B\mathbb{Z}_2)^{\mathrm{sgn}}).$$

This notation shall be explained later in §6.3. For now, we prove the following.

Lemma 6.1.1. *The anomaly-free condition* (6.1.5) *implies that the associator a of* $2\mathrm{Rep}^{\tau}_{m}(D(B\mathbb{Z}_2)^{\mathrm{sgn}})$ *satisfies the pentagon relations.*

Proof. In order to see the anomaly-free condition (6.1.5) manifest on the 2-representations, we begin with the observation that the component Δ_0 of the coproduct on $D(B\mathbb{Z}_2)$ satisfies

$$\Delta_0(x^2) = \bar{e}(x, x)\hat{x}^2 \otimes x^2 = \bar{e}(x, x) \otimes 1, \tag{6.1.6}$$

by (5.1.16). This means that Δ_0 is an algebra map on $k\mathbb{Z}_4$, not $k\mathbb{Z}_2$.

Because of (6.1.6), evaluating the condition (5.3.4) on $1 = x^2 \in k\mathbb{Z}_2$ gives

$$1^{4\otimes} = \Delta_{-1} \circ \Delta_1(1) = (\Delta_1 \otimes 1) \circ \Delta_0(x^2) = \bar{e}(x, x) \otimes \Delta_1(1),$$

which violates the pentagon relation unless the right-hand side is also trivial $1^{4\otimes}$. Pairing this equation with arbitrary $x_1, \ldots, x_3 \in k\mathbb{Z}_2$ gives

$$1 = \langle \bar{e}(x, x) \otimes \Delta_1(1), 1 \otimes x_1 \otimes x_2 \otimes x_3 \rangle = \langle \bar{e}(x, x) \otimes 1, 1 \otimes \tau(x_1, x_2, x_3) \rangle = \bar{e}(x, x)\tau(x_1, x_2, x_3).$$

This is nothing but $\bar{e} \cup \tau = 0$. □

Notice on the other hand that if $\tau = 0$ is trivial, then so is Δ_1 and the coassociativity condition simply implies the group cocycle condition for $\bar{e}$.

Weakening the anomaly-free condition. There is a way to weaken the anomaly-free condition, by imposing (6.1.5) only in *cohomology* $\tau \cup \bar{e} = 0 \in H^5(\mathbb{Z}_2, k^\times)$ [104, 153]. This means that the 4d Kitaev model gains an additional term $\nu(A)$ that trivializes the coboundary of the Dijkgraaf-Witten 4-cocycle,

$$d(\omega_b - \nu) = 0.$$

Algebraically, this 4-cocycle $\nu \in H^4(\mathbb{Z}_2, k^\times)$ is known to play the role of a "pentagonator" 2-morphism in the underlying 2-group [46, 71], implementing the pentagon relation **Lemma 5.5.3**.

This group 4-cocycle ν is intimately related to the Hochschild 3-cocycle $\mathfrak{T}$ attached to the *weak* endomorphism 2-algebra $\mathfrak{End}(V)$ (see **Definition 5.5.3**). Indeed, **Theorem 5.7.1** states that the module pentagonator π [64] is given by

$$\pi_{x_1x_2x_3|V} = \mathfrak{T}(\rho_0(x_1), \rho_0(x_2), \rho_0(x_3))(V), \qquad x_1, \dots, x_3 \in k\mathbb{Z}_2.$$

Given V is irreducible with an associated label $x_4 \in \mathbb{Z}_2$, then $\pi_{x_1x_2x_3|x_4} = \nu(x_1, x_2, x_3, x_4)$ defines a group 4-cocycle whose cocycle condition arises from the associahedron condition [64, 119]. Moreover, the fact that the Hochschild cohomology of

$$\rho_1 \circ \tau - \mathfrak{T} \circ \rho_0^{3\otimes}$$

is trivial (coming from (5.5.3)) translates to precisely the equation $d(\omega_b - \nu) = 0$. This relationship between τ and ν is intimately related to the conjecture [46] that the 4d ν-twisted gauge theory on G coincides with the 4d untwisted 2-gauge theory on $\mathbb{G} = (G, k^\times, \tau)$. We will not be proving this conjecture in this thesis, however.

6.2 Excitations in the (invisible) toric code $Z^0_{\mathbf{Kit}}$

Excitations are inserted into the theory Z_{Kit} with 2-representations ρ of $D(B\mathbb{Z}_2)$. Since $D(B\mathbb{Z}_2)$ is skeletal, it suffices to study 2-representations of the underlying 2-group. Let us first focus on the trivial case $D(B\mathbb{Z}_2)^{\text{trv}}$.

Recall that a 2-representation $\rho : D(B\mathbb{Z}_2)^{\text{trv}} \to \mathfrak{End}(V)$ on a 2-vector space $V = V_{-1} \xrightarrow{\partial} V_0$ consists of the following data:

1. a pair of $\mathbb{Z}_2$-representations

$$\rho_0 = \rho_0^1 \oplus \rho_0^0 : \mathbb{Z}_2 \to \mathfrak{End}(V_0) \oplus \mathfrak{End}(V_{-1}),$$

such that ∂ is an intertwiner between ρ_0^0 and ρ_0^1, and

2. a map $\rho_1 : \widehat{\mathbb{Z}_2} \to \mathrm{Hom}(V_0, V_{-1})$ such that $\rho_1(1) = 0$ on the identity $1 \in \widehat{\mathbb{Z}_2}$.

Since the t-map for $D(B\mathbb{Z}_2)$ is trivial, ρ must satisfy $\delta\rho_1 = (\rho_1 \circ \partial, \partial \circ \rho_1) = \rho_0 t = 0$, which means either $\rho_1 = 0$ or $\partial = 0$. For 1-dimensional irreducible representations (irreps) $V_0, V_{-1} \cong k$ over the ground field k, the value of ρ_1 on the non-trivial generator $y \in \widehat{\mathbb{Z}_2}$ is either 0 or a scalar multiplication. We write simply $\rho_1 = 0$ in the former case, while in the latter case we shall normalize the scalar $\rho_1(y)$ to $1 \in k^\times$, and denote this map by $\rho_1 = \hat{1}$.

Remark 6.2.1. Though ρ_1 need *not* be an intertwiner, we require it to preserve the identity $\rho_0^{0,1}(1) = \rho_0^{0,1}(x^2) = \mathrm{id}$ in the sense that

$$\rho_1(y) \circ \mathrm{id}_{V_0} = \mathrm{id}_{V_{-1}} \circ \rho_1(y), \qquad x \in \mathbb{Z}_2,\ y \in \widehat{\mathbb{Z}_2}.$$

This condition is vacuous here, but it shall become non-trivial later when we *twist* $D(BM)$. Strictly speaking, ρ_1 can be trivial as well if $\partial = 0$, but this distinction makes no difference for $D(B\mathbb{Z}_2)^{\mathrm{trv}}$.

Now given ρ_0^0, ρ_0^1 are irreducible, Schur's lemma implies that ∂ is either trivial or an isomorphism. Hence given $\partial \neq 0$, then ρ_0^0, ρ_0^1 are either both the trivial representation 1, or both the sign representation sgn. We therefore have four inequivalent irreducible 2-representations

Electric	$\mathbf{1} = (1 \oplus 1, \partial = 1, \rho_1 = 0)$	$\mathbf{c} = (1 \oplus \mathrm{sgn}, \partial = 0, \rho_1 = \hat{1})$
Magnetic	$\mathbf{1}^* = (\mathrm{sgn} \oplus \mathrm{sgn}, \partial = 1, \rho_1 = 0)$	$\mathbf{c}^* = (\mathrm{sgn} \oplus 1, \partial = 0, \rho_1 = \hat{1})$

Table 6.1: The list of the irreducible 2-representations of the Drinfel'd 2-double $D(B\mathbb{Z}_2)$.

which constitute the simple[4] objects in $2\mathrm{Rep}_{\mathrm{wk}}(D(B\mathbb{Z}_2))$. We call the first row the **electric** sector and the second row the **magnetic** sector; this partition will be clear in the following. Note that $\mathbf{c}$ is *not* equivalent to $\mathbf{c}^*$, because the map ∂ remembers its domain and codomain.

6.2.1 Fusion structure

We now investigate the monoidal structure of the 2-category $2\mathrm{Rep}(D(B\mathbb{Z}_2)^{\mathrm{trv}})$. Since the coproduct Δ on $D(B\mathbb{Z}_2)$ is grouplike, the tensor product of 2-representations ρ, ρ' is just the usual *graded* tensor product $\rho \otimes \rho'$. Graded here means (5.5.10), ie. equipped with the differential ∂; we demonstrate this through computations below.

Let us examine the 2-representations as listed in (6.1). In the electric sector, we use the Morita equivalence $\mathrm{sgn}^{2\otimes} \simeq 1^{2\otimes} \cong 1$ to have

$$\mathbf{c} \otimes \mathbf{c} = (1 \oplus \mathrm{sgn}) \otimes (1 \oplus \mathrm{sgn}) \cong 1 \oplus \mathrm{sgn} \oplus \mathrm{sgn}^{2\otimes} \oplus \mathrm{sgn} \simeq \mathbf{c} \oplus \mathbf{c}, \tag{6.2.1}$$

[4]Given any arbitrary 2-representation $\rho \in 2\mathrm{Rep}_{\mathrm{wk}}(D(B\mathbb{Z}_2))$, each graded component of the vector space complex $V = V_{-1} \xrightarrow{\partial} V_0$ carries a $\mathbb{Z}_2$-representation, which decomposes individually into direct sums of irreducible representations $1, \mathrm{sgn}$. As ∂ must be a $\mathbb{Z}_2$-intertwiner, it also decomposes accordingly as a direct sum on each irreducible summand, whence V is a direct sum of the objects listed in (6.1).

which tells us that $\mathbf{c}$ is a **Cheshire string** [76]; similarly, we compute

$$\mathbf{c}^* \otimes \mathbf{c}^* \cong \mathrm{sgn}^{2\otimes} \oplus \mathrm{sgn} \oplus 1 \oplus \mathrm{sgn} \simeq \mathbf{c} \oplus \mathbf{c}.$$

Note that the order of the direct sums matter, as we have are keeping track of the (trivial) differential $\partial = 0$. Indeed, we have on the other hand,

$$\mathbf{c} \otimes \mathbf{c}^* = (1 \oplus \mathrm{sgn}) \otimes (\mathrm{sgn} \oplus 1) \cong \mathrm{sgn} \oplus 1 \oplus \mathrm{sgn} \oplus 1 \simeq \mathbf{c}^* \oplus \mathbf{c}^* \simeq \mathbf{c}^* \otimes \mathbf{c}, \tag{6.2.2}$$

which is distinct from the above fusion rules.

Consider the mixed fusion $\mathbf{1}^* \otimes \mathbf{c}$. Here, we need to keep track of the non-trivial maps ∂,

$$\mathbf{1}^* \otimes \mathbf{c} \quad = \quad (\mathrm{sgn} \oplus \mathrm{sgn}) \otimes (1 \oplus \mathrm{sgn})$$

$$\begin{tikzcd}[column sep=small] \cong & \mathrm{sgn} \arrow[rrrr, bend left, "\partial=1"] & \oplus & \mathrm{sgn}^{2\otimes}(\simeq 1) \arrow[rrrr, bend right, "\partial=1"'] & \oplus & \mathrm{sgn} & \oplus & \mathrm{sgn}^{2\otimes}(\simeq 1) \end{tikzcd} .$$

Since these maps ∂ are intertwiners (in fact the identity), its domain and codomain are the same. We keep only one copy, so that

$$\mathbf{1}^* \otimes \mathbf{c} \simeq \mathrm{sgn} \oplus 1 = \mathbf{c}^*. \tag{6.2.3}$$

Through similar computations, we have

$$\mathbf{1} \otimes \mathbf{1} \cong \mathbf{1}, \qquad \mathbf{1} \otimes \mathbf{c} \simeq \mathbf{c}, \qquad \mathbf{1}^* \otimes \mathbf{1}^* \simeq \mathbf{1},$$

hence $\mathbf{1}, \mathbf{1}^*$ are the vacuum lines; in particular, $\mathbf{1}$ is the indecomposable identity object in $2\mathrm{Rep}(D(B\mathbb{Z}_2)^{\mathrm{trv}})$.

2-intertwiners; the 1-morphisms

Recall from **Definition 5.5.4** that the 1-morphisms in $2\mathrm{Rep}(D(B\mathbb{Z}_2)^{\mathrm{trv}})$ are given by $\mathbb{Z}_2$-equivariant *cochain maps*. Form the list (6.1), we clearly have identity self-2-intertwiners, such as $i[00] = \mathrm{id} : \mathbf{1} \to \mathbf{1}$ and $i[11] = \mathrm{id} : \mathbf{c} \to \mathbf{c}$. As the source and target are the same graded $\mathbb{Z}_2$-representations for self-2-intertwiners in particular, we can find two more. These are given by a swap of grading together with a certain twist,

$$i'[00] : (w, v) \mapsto (v, w), \qquad i'[11] : (w, v) \mapsto (\pm 1) \cdot (v, w), \tag{6.2.4}$$

where $(v, w) \in V_{-1} \oplus V_0$ denotes elements in $\mathbf{1}$ or $\mathbf{c}$. Clearly, the identity $i[00], i[11]$ admit trivial actions by ρ_1, in contrast to the grading swaps $i'[00], i'[11]$. Hence from the grouplike coproduct Δ_{-1} (6.0.2) we deduce the following fusion rules

$$i[00]\otimes i[00]=i'[00]\otimes i'[00]=i[00],\qquad i[00]\otimes i'[00]=i'[00]\otimes i[00]=i'[00],\tag{6.2.5}$$

and similarly for $i[11],i'[11]$. The same analysis applies to the dual sector $i^*[00]\in\mathfrak{End}(\mathbf{1}^*),i^*[11]\in\mathfrak{End}(\mathbf{c}^*)$.

Now consider a map $i[01]:\mathbf{1}\to\mathbf{c}$; in the absence of the homotopy I, the commutative diagrams (5.5.7) respectively enforce that

$$i[01]_0\circ 1=0\circ i[01]_1,\qquad i[01]_1\circ 0=\hat{1}(y)\circ i[01]_0,$$

where $\hat{1}(y)=\rho_1(y)\in\mathrm{Hom}(V_0,V_1)$ is a non-trivial scalar multiplication. These equations admit a non-trivial solution $i[01]_0=0,i[01]_1=1$, hence there is a non-trivial 2-intertwiner

$$i[01]=1\oplus 0:\mathbf{1}\to\mathbf{c};$$

similar arguments show that we also have a non-trivial 2-intertwiner

$$i[10]=0\oplus 1:\mathbf{c}\to\mathbf{1}.$$

These are the only possible 2-intertwiners between $\mathbf{1}$ and $\mathbf{c}$. Again, the same analysis applies to the dual sector. Since $i[01]$ and $i[10]$ have different domain and codomain, we must employ the decomposition (5.5.14) in order to find the tensor product between them [146]. However, since the coproduct $\Delta_0=0$ is trivial in $D(B\mathbb{Z}_2)^{\mathrm{trv}}$, we find their tensor product

$$i[01]\otimes i[10]=i[10]\otimes i[01]\simeq 1=i[00]$$

to be trivial as well. We shall see later in §6.3.1 that this will be different once we introduce twists on $D(B\mathbb{Z}_2)$.

Let us now come finally to the 2-intertwiners that map between dual sectors. First, consider maps such as $\mathbf{1}\to\mathbf{1}^*$ or $\mathbf{c}\to\mathbf{c}^*$. Any such maps must intertwine between different $\mathbb{Z}_2$-representations in both degrees, and the only such map is 0. Next, consider a map $\bar{i}[01]:\mathbf{1}\to\mathbf{c}^*$. The commutative diagrams (5.5.7) enforce

$$\bar{i}[01]_0\circ 0=1\circ\bar{i}[01]_1,\qquad \bar{i}[01]_1\circ 0=\hat{1}(1)\circ\bar{i}[01]_0.$$

The first equation says $\bar{i}[01]_1=0$, while the second equation says $\bar{i}[01]_0=0$, hence $\bar{i}[01]=0$. Similarly, any 2-intertwiner $\bar{i}[10]:\mathbf{c}^*\to\mathbf{1}$ must be trivial $\bar{i}[10]=0$.

The above paragraph proves that $2\mathrm{Rep}(D(B\mathbb{Z}_2)^{\mathrm{trv}})$ has two connected components made separately of the electric and magnetic objects in (6.1), which have no (invertible) 1-morphisms between them. We denote the **identity component** of $2\mathrm{Rep}(D(B\mathbb{Z}_2)^{\mathrm{trv}})$, namely the connected component of the fusion identity $\mathbf{1}$, by Γ, which consist of nothing but the electric sector. Relabeling $i[00],i[11]=\imath$ and $i'[00],i'[11]=\mathfrak{e}$, we achieve the following structure for

Γ from (6.2.5),

$$\begin{tikzcd} \mathbf{1} \arrow[loop left, "{\mathfrak{1},\mathfrak{e}}"] \arrow[r, bend left, "{i[01]}"] & \mathbf{c} \arrow[l, bend left, "{i[10]}"] \arrow[loop right, "{\mathfrak{1},\mathfrak{e}}"] \end{tikzcd}, \tag{6.2.6}$$

which shall become crucial in the following.

Cochain homotopies; the 2-morphisms

Recall from **Definition 5.5.4** that the 2-morphisms in $2\mathrm{Rep}_{\mathrm{wk}}(B\mathbb{Z}_2)$ are given by $\widehat{\mathbb{Z}_2}$-equivariant cochain homotopies. Of course, the monoidal structure of the 1-morphisms (eg. (6.2.5)) induce a monoidal structure on the modifications $\mu \otimes \mu' : i \otimes j \Rightarrow i' \otimes j'$, which by using the (so-far trivial) interchanger (5.5.15) can be expressed in terms of the composition $(\mu \otimes \mathrm{id}_j) \circ (\mathrm{id}_i \otimes \mu') \simeq \mu \circ \mu'$.

By inspection of the connected component Γ (6.2.6), one can argue that the only modifications possible in Γ are self-modifications $\mu : i \Rightarrow i$. To see this, we first note that there is only one unique 1-morphism $i[01]$ (or $i[10]$) between the simple objects $\mathbf{1}$ and $\mathbf{c}$, hence we only have the trivial identity cochain homotopy $\mathrm{id} : i[01] \Rightarrow i[01]$. On the other hand, there are two 1-endomorphisms on $\mathbf{1}$ (or equivalently $\mathbf{c}$), denoted by $\mathfrak{1}, \mathfrak{e}$. Each of these of course comes with its own trivial identity cochain homotopy, denoted by

$$1 : \mathfrak{1} \Rightarrow \mathfrak{1} \qquad \mu : \mathfrak{e} \Rightarrow \mathfrak{e}. \tag{6.2.7}$$

Here, we note that $\mu \simeq -1 \cdot \mathrm{id}$ carries a global sign due to a grading swap in $\mathfrak{e}$ (6.2.4).

It then remains to check that there are no non-trivial cochain homotopies between $\mathfrak{1}$ and $\mathfrak{e}$. Let $\bar{\mu} : \mathfrak{1} \Rightarrow \mathfrak{e}$ denote such a cochain homotopy. In order for $\bar{\mu}$ to denote a genuine 2-morphism in $2\mathrm{Rep}_{\mathrm{wk}}(D(B\mathbb{Z}_2))$, it must by definition (5.5.8) intertwine ρ_1. However, $\widehat{\mathbb{Z}_2}$ "acts" trivially on $\mathfrak{1}$, while non-trivially on $\mathfrak{e}$,

$$\rho_1 = \mathrm{id} : \mathfrak{1} \Rightarrow \mathfrak{1}, \qquad \rho_1 = \mathrm{sgn} \cdot \mathrm{id} : \mathfrak{e} \Rightarrow \mathfrak{e},$$

and hence $\bar{\mu} = 0$ must be trivial. This demonstartes that the only modofications in $2\mathrm{Rep}_{\mathrm{wk}}(D(B\mathbb{Z}_2))$ that exist are the self-modifications $\mu : i \Rightarrow i$, as desired.

We of course have the trivial 1- and 2-morphisms given by 0. More importantly, we note that the non-trivial 1- and 2-morphisms that we have identified above are not unique. In particular, we have made the choice to normalize all of the 2-intertwiners and the self-modifications, whereas any scalar multiple of them would also be valid. Further, we are also able to take direct sums of the 2-intertwiners that we have identified above; basically, §6.2.1 lists a minimal set of generators for the Hom-spaces of $2\mathrm{Rep}_{\mathrm{wk}}(D(B\mathbb{Z}_2))$.

Loop 1-category: the 1-endomorphism space of the tensor unit. Recall that, for any Abelian group N, $\rho_1 : \hat{N} \to \mathfrak{End}(V)_{-1}$ defines a "$\hat{N}$-action" by cochain homotopies on the endomorphisms $\mathfrak{End}(V)$ of a 2-representation $V \in 2\mathrm{Rep}_{\mathrm{wk}}(D(BN))$. Furthermore, modifications $\mu : i \Rightarrow i'$ between $i, i' \in \mathfrak{End}(V)$ by definition (5.5.8) must necessarily intertwine this $\hat{N}$-action.

On the tensor unit $V = \mathbf{1}$, in particular, the space $\mathfrak{End}(\mathbf{1}) \cong k \xrightarrow{1} k$ furnishes a 1-dimensional irreducible $\widehat{N}$-module, for which the intertwining modifications between the different $\widehat{N}$-module structures are either the identity or trivial. This allows us to conclude that

$$\Omega\, 2\mathrm{Rep}_{\mathrm{wk}}(D(BN)) = \mathrm{End}_{2\mathrm{Rep}_{\mathrm{wk}}(D(BN))}(\mathbf{1}) \simeq \mathrm{Rep}(\widehat{N}). \tag{6.2.8}$$

For $N = \mathbb{Z}_2$, we recover the result that $\mathfrak{End}(\mathbf{1}) \simeq \mathrm{Rep}(\mathbb{Z}_2) \simeq \mathsf{Vect}[\mathbb{Z}_2]$ has two distinct objects, $\mathfrak{1}, \mathfrak{c}$, with no non-trivial modifications between them.

Notice that in the usual theory of higher representations [46, 78, 63, 77], where $2\mathsf{Vect}^{KV}$ is 2-enriched in Vect [97, 154], the above statement follows immediately from definition. However, in the context of $2\mathsf{Vect}^{hBC}$ (which is *not* 2-enriched), we have to prove it by direct computation.

Proposition 6.2.1. *There is an equivalence between* $Z_1(\Sigma\mathsf{Vect}[\mathbb{Z}_2])$ *and* $2\mathrm{Rep}_{wk}(D(B\mathbb{Z}_2)^{\mathrm{trv}})$ *which is not compatible with the monoidal structure.*

Proof. We use the description of the braided fusion 2-category $\mathscr{R} \simeq Z_1(\Sigma\mathsf{Vect}[\mathbb{Z}_2])$ (with trivial associator class) describing the (3+1)D toric code given in [76]. This category has two identical components; the identity component $\Sigma\mathsf{Vect}[\mathbb{Z}_2]$ has two simple objects, given by the trivial $\mathbb{Z}_2$-algebra $I = \mathbb{C}$ and the Cheshire string $c = \mathbb{C}[x]/\langle x^2 - 1\rangle$, where $\mathbb{Z}_2$ acts non-trivially on x. Monoidally, the two components of $\mathscr{R}$ follow a fusion rule that is graded by $\mathbb{Z}_2$ [78],

$$I^2 \simeq m^2 \simeq I, \qquad c^2 \simeq m'^2 \simeq c \oplus c, \qquad c \otimes m = m \otimes c \simeq m', \qquad c \otimes m' \simeq m' \otimes c \simeq m' \oplus m',$$

where m, m' denotes the simple objects in the non-trivially graded copy of $\Sigma\mathsf{Vect}[\mathbb{Z}_2]$.

To show the desired equivalence, we need to exhibit a 2-functor $\mathfrak{F} : \mathscr{R} \to 2\mathrm{Rep}_{\mathrm{wk}}(D(B\mathbb{Z}_2))$ which is essentially surjective and fully faithful. This means that

1. $\mathfrak{F}$ is *essentially surjective*, namely a surjection on the equivalence classes of objects, and
2. $\mathfrak{F}$ is *fully faithful*, namely it is an equivalence of Hom-categories.

We begin by taking

$$\mathfrak{F}(I) = \mathbf{1}, \qquad \mathfrak{F}(c) = \mathbf{c}, \qquad \mathfrak{F}(m) = \mathbf{1}^*, \qquad \mathfrak{F}(m') = \mathbf{c}^*,$$

which is a bijection on the simple objects. Hence $\mathfrak{F}$ is essentially surjective, and furthermore preserves the identity. Now to check that $\mathscr{F}$ is fully-faithful, we must consider the Hom-categories. Since $Z_1(\Sigma\mathsf{Vect}[\mathbb{Z}_2]) \simeq \Sigma 2\mathsf{Vect}[\mathbb{Z}_2] \oplus \Sigma 2\mathsf{Vect}[\mathbb{Z}_2]$ [78], it suffices to show full and faithfulness on the corresponding identity components $\mathfrak{F} : \Sigma\mathsf{Vect}[\mathbb{Z}_2] \to \Gamma$.

To begin, we note that $\Sigma\mathsf{Vect}[\mathbb{Z}_2]$ is well-known to have the following form [65],

$$\begin{tikzcd} I \arrow[loop left, "{\mathsf{Vect}[\mathbb{Z}_2]}"] \arrow[rr, bend left, "\mathsf{Vect}"] && c \arrow[ll, bend left, "\mathsf{Vect}"] \arrow[loop right, "{\mathsf{Vect}[\mathbb{Z}_2]}"] \end{tikzcd},$$

with each of the Hom-categories labeled. We let $v_1 \cong k$ (resp. $v_2 \cong k$) denote respectively the simple object in the linear Hom-category $\mathsf{Vect} = \mathrm{Hom}_{\Sigma\mathsf{Vect}[\mathbb{Z}_2]}(I, c)$ (resp. $\mathsf{Vect} = \mathrm{Hom}_{\Sigma\mathsf{Vect}[\mathbb{Z}_2]}(I, c)$), which can be understood as the 1-dimensional vector space over k. Similarly, we let $\mathring{1}, e$ denote the two simple objects of the linear Hom-category $\mathsf{Vect}[\mathbb{Z}_2] = \mathrm{End}_{\Sigma\mathsf{Vect}[\mathbb{Z}_2]}(I) \simeq \mathrm{End}_{\Sigma\mathsf{Vect}[\mathbb{Z}_2]}(c)$; the direct sum $\mathring{1} \oplus e$ corresponds to a 1-dimensional super- (ie. $\mathbb{Z}_2$-graded) vector space.

By comparing with (6.2.6), we define the following component functors of the 2-functor $\mathfrak{F}$ by

$$\mathbb{1} = \begin{cases} i[00] = (\mathfrak{F})_{I\to I}(\mathring{1}) \\ i[11] = (\mathfrak{F})_{c\to c}(\mathring{1}) \end{cases}, \qquad \mathfrak{e} = \begin{cases} i'[00] = (\mathfrak{F})_{I\to I}(e) \\ i'[11] = (\mathfrak{F})_{c\to c}(e) \end{cases}$$
$$i[01] = (\mathfrak{F})_{I\to c}(v_1), \qquad i[10] = (\mathfrak{F})_{c\to I}(v_2),$$

which we note are all unit-preserving and essentially surjective. It then suffices to check that these component functors are fully faithful. By leveraging the linearity of the Hom-categories under consideration, this is equivalent to checking that each of the component functors send (additive) generating 2-morphisms to generating 2-morphisms.

This is indeed the case. Let $j_1 \in \mathrm{End}_{\mathsf{Vect}}(v_1) \cong k$ denote the non-trivial generating 2-morphism over the indecomposable 1-morphism $v_1 \in \mathsf{Vect} = \mathrm{Hom}_{\Sigma\mathsf{Vect}[\mathbb{Z}_2]}(I, c)$. Then by construction $\mathfrak{F}_{I\to c}(j_1 : v_1 \Rightarrow v_1) = \mathrm{id}_{i[01]} : i[01] \Rightarrow i[01]$ is the identity self-modification on $i[01] = \mathfrak{F}_{Ic}(v_1)$, which is the generating object in the Hom-category $\mathrm{Hom}_{2\mathrm{Rep}_{\mathrm{wk}}(D(B\mathbb{Z}_2))}(I, c)$ as required.

Similarly, as $\mathfrak{F}_{I\to I}, \mathfrak{F}_{c\to c}$ are additive, they send the generating 2-morphism $j_{\mathbb{Z}_2} : \mathring{1} \oplus e \Rightarrow \mathring{1} \oplus e$ over the indecomposable $\mathring{1} \oplus e \in \mathsf{Vect}[\mathbb{Z}_2]$ to the (graded) generating chain homotopy $1 + \mu$ (6.2.7) over $\mathbb{1} \oplus \mathfrak{e} = \mathfrak{F}_{I\to I}(\mathring{1} \oplus e)$. This shows that each component functor $\mathfrak{F}_{X\to Y}$ are equivalences of the corresponding Hom-categories, and hence $\mathfrak{F} : \Sigma\mathsf{Vect}[\mathbb{Z}_2] \to \Gamma$ is an equivalence of 2-categories.

We now wish to lift $\mathfrak{F}$ to a *monoidal* 2-functor, which requires the fusion rules to be preserved (up to coherence). The computations (6.2.1), (6.2.2), (6.2.3) show that $\mathfrak{F} : Z_1(\Sigma\mathsf{Vect}[\mathbb{Z}_2]) \to 2\mathrm{Rep}_{\mathrm{wk}}(D(B\mathbb{Z}_2)^{\mathrm{trv}})$ preserves the fusion rules of the simples, and is indeed monoidally essentially surjective. Next is to check that each component functor $\mathfrak{F}_{X\to Y}$, $X, Y \in \Sigma\mathsf{Vect}[\mathbb{Z}_2]$ is monoidal on the Hom-categories.

From the fusion rules (6.2.5) for the 1-morphisms, we see that $\mathfrak{F}_{X\to X}$ with $X = I, c$ are indeed monoidal, but the issue is that $\mathfrak{F}_{I\to c}$ (or $\mathfrak{F}_{c\to I}$) is *not*: $\mathfrak{F}_{I\to c}(v_1) \otimes \mathfrak{F}_{I\to c}(v_1) = i[01] \otimes i[10] \simeq \mathbb{1}$ is trivial in $\Gamma \subset 2\mathrm{Rep}_{\mathrm{wk}}(D(B\mathbb{Z}_2)^{\mathrm{trv}})$, while $v_1 \otimes v_1 \not\simeq \mathring{1}$ is not in $\Sigma\mathsf{Vect}[\mathbb{Z}_2]$. This prevents $\mathfrak{F}$ from being a monoidal equivalence. $\square$

The problem is in fact even worse — we will show in the following that $2\mathrm{Rep}_{\mathrm{wk}}(D(B\mathbb{Z}_2)^{\mathrm{trv}})$ does not even define a gapped topological order. We elaborate in §6.3 on how this problem can be amended by *twisting* the 2-algebra structure of $D(B\mathbb{Z}_2)$.

6.2.2 The braiding data

Let us for now turn to the *braiding* structure. From the perspective of $\mathscr{R}$, it is understood [76] in particular that there is the self-braiding

$$\beta : m \otimes m \to m \otimes m$$

on the magnetic m line, which can either be trivial $\mathring{1}$ or the electric $\mathbb{Z}_2$-particle e. An argument was given in [76] that states $\beta = \mathring{1}$ is in fact trivial. We will prove that this is also the case in $2\mathrm{Rep}_{\mathrm{wk}}(D(B\mathbb{Z}_2)^{\mathrm{trv}})$, but there is a major problem.

Theorem 6.2.1. *All braiding maps on* $2\mathrm{Rep}(D(B\mathbb{Z}_2)^{trv})$ *are trivial.*

Proof. Recall from (5.6.2), (5.6.3) that the braiding structure of $2\mathrm{Rep}(D(B\mathbb{Z}_2)^{\mathrm{trv}})$ is induced by a 2-R-matrix $(\mathcal{R}, R)$ on $D(B\mathbb{Z}_2)^{\mathrm{trv}}$. Since $D(B\mathbb{Z}_2)$ is a Drinfel'd double, we can make use of the braided transpositions $\Psi, \bar{\Psi}$ to characterize $(\mathcal{R}, R)$ using (5.2.29),

$$R = \bar{\Psi} \circ \mathrm{coev}, \tag{6.2.9}$$

$$\mathcal{R}^+ = \Psi^l_{-1} \circ \mathrm{coev}_l, \qquad \mathcal{R}^- = \Psi^r_{-1} \circ \mathrm{coev}_r \tag{6.2.10}$$

Here, coev is the coevaluation dual to the canonical pairing form on $\mathbb{Z}_2$ and $\mathrm{coev}_{l,r}$ is the coevaluation dual to the Pontrjagyn pairing. This method is based on the general quantum double construction of Majid [118, 117].

First, in degree-0, the braided transposition $\bar{\Psi} : k\mathbb{Z}_2 \otimes k\mathbb{Z}_2 \to k\mathbb{Z}_2 \otimes k\mathbb{Z}_2$ satisfies

$$xx' = \cdot \circ \bar{\Psi}(x' \otimes x), \qquad x, x' \in \mathbb{Z}_2.$$

Now since $\mathbb{Z}_2$ is Abelian, $\bar{\Psi}$ is simply the identity and hence (6.2.9) states that $R = \mathrm{id}$ is in fact the identity matrix. The braiding maps $b_{V,W} = 1$ are thus all trivial. Now in degree-(-1), the braided transpositions

$$\begin{aligned} \Psi^l_{-1} : \widehat{\mathbb{Z}_2} \otimes \mathbb{Z}_2 \to \mathbb{Z}_2 \otimes \widehat{\mathbb{Z}_2}, \qquad & \Psi^r_{-1} : \mathbb{Z}_2 \otimes \widehat{\mathbb{Z}_2} \to \widehat{\mathbb{Z}_2} \otimes \mathbb{Z}_2 \\ y \cdot f = \cdot \Psi^l_{-1}(y \otimes f), \qquad & x \cdot g = \cdot \circ \Psi^r_{-1}(x \otimes g) \end{aligned} \tag{6.2.11}$$

for $D(B\mathbb{Z}_2)^{\mathrm{trv}}$ is given by Pontrjagyn duality

$$\Psi^l_{-1}(y \otimes f) = \hat{y} \otimes \hat{f}, \qquad \Psi^r_{-1}(x \otimes g) = \hat{x} \otimes \hat{g},$$

whence (6.2.10) states that $\mathcal{R}^{\pm} = p \circ \mathrm{coev} = \mathrm{id}$ is the identity matrix. The mixed braiding maps $b_{i,W}, b_{W,i}$ are thus all trivial.

$\square$

The fact that all the braiding maps are trivial on $2\mathrm{Rep}_{\mathrm{wk}}(D(B\mathbb{Z}_2)^{\mathrm{trv}})$ can also be seen from the corresponding topological NLSM Z^0_{Kit}, which has no terms in its action that encode any non-trivial statistics of the charges in the theory [71, 46]. Of course, we already know from **Proposition 6.2.1** that $2\mathrm{Rep}_{\mathrm{wk}}(D(B\mathbb{Z}_2)^{\mathrm{trv}})$ not (braided) monoidally equivalent to the toric code $\mathscr{R}$, and hence calling Z^0_{Kit} the "4d toric code" is incorrect.

Remark 6.2.2. $2\mathrm{Rep}_{\mathrm{wk}}(D(B\mathbb{Z}_2)^{\mathrm{trv}})$ is "too trivial" to even describe a gapped topological phase, since it violates the principle of **remote detectability** [79, 78, 76]. This principle states that all non-trivial excitations can be detected by braiding, and it is part of the definition of a topological order (such as the toric code $\mathscr{R} \simeq \Sigma\mathsf{Vect}[\mathbb{Z}_2]$). In this simple $\mathbb{Z}_2$-charged case, this principle is encoded by the presence of the term $\langle B \cup \bar{e}(A)\rangle$ in the Dijkgraaf-Witten 4-cocycle ω [71, 76], which is only present for Z^s_{Kit}. Nevertheless, the above computations lay the foundation for our results in the following.

6.3 Excitations in the spin-Kitaev model Z^s_{Kit}

We now turn to the spin-Kitaev model Z^s_{Kit} given by the Drinfel'd double 2-bialgebra $D(B\mathbb{Z}_2)^{\mathrm{sgn}}$. Its 2-representations have the same ingredients as those of $D(B\mathbb{Z}_2)^{\mathrm{trv}}$, and hence the 2-category $2\mathrm{Rep}(D(B\mathbb{Z}_2)^{\mathrm{sgn}})$ also has four objects, similar to those in (6.1).

The difference here is that $D(B\mathbb{Z}_2)^{\mathrm{sgn}}_0 = \mathbb{Z}_2$ now acts non-trivially on $D(B\mathbb{Z}_2)^{\mathrm{sgn}}_{-1} = \widehat{\mathbb{Z}_2}$. This action was obtained by dualizing the non-trivial action $u \in \mathrm{Aut}(k\mathbb{Z}_2)$, which induces via (6.1.4) the class $\bar{e} = \frac{1}{2}u^2 \in H^2(\mathbb{Z}_2, \mathbb{Z}_2)$ determining the non-trivial central extension of $\mathbb{Z}_2$ by itself. This extension is $\mathbb{Z}_4$, which we interpret as a "semidirect product" $\mathbb{Z}_2 \rtimes \mathbb{Z}_2$ where the central element $x^2 \in \mathbb{Z}_2$ acts by -1.

As such, the component $\rho_0^0(x)^2$ "acts" non-trivially on the degree-(-1) component of the graded 2-representation spaces. In other words, provided ρ_0^0 is non-trivial, the component ρ_0 of the 2-representation ρ furnishes a representation of $\mathbb{Z}_2 \rtimes \mathbb{Z}_2$, satisfying

$$\rho_0(x^2)(w, v) = (\bar{e}(x, x) \cdot (\rho_0^1(x^2))w, \rho_0^0(x^2)v) = (-w, v), \qquad x \in \mathbb{Z}_2,$$

where $(w, v) \in V \cong V_{-1} \oplus V_0$. We denote such representations by $\rho_0 = \rho_0^1 \oplus_\pm \rho_0^0 = (\bar{e} \cdot \rho_0^1, \rho_0^0)$. From (6.1), we thus see that the magnetic vacuum line $\mathbf{1}^*$ and the Cheshire string $\mathbf{c}$ carry a $\mathbb{Z}_4$-representation, while the electric vacuum line $\mathbf{1}$ and the magnetic Cheshire $\mathbf{c}^*$ carry a $\mathbb{Z}_2 \times \mathbb{Z}_2$-representation.

Now recall from *Remark 6.2.1* that ρ should preserve the identity, which was a vacuous condition as $\rho_0^{0,1}(x^2) = 1$ are both trivial for $D(B\mathbb{Z}_2)^{\mathrm{trv}}$. However, due to the non-trivial sign coming from $\rho_1(\bar{e}(x, x)) = -1$ in the current case, this becomes a non-trivial relation that one must impose,

$$-1 \cdot \rho_1(y) = \rho_1(\bar{e}(x, x) \cdot y)\rho_0^1(x^2) = \rho_0^0(x^2)\rho_1(y) = \rho_1(y), \qquad y \in \widehat{\mathbb{Z}_2}.$$

The component ρ_1 is thus no longer required in general to preserve the identity. As $V_0, V_{-1} \cong k$

are both 1-dimensional vector spaces over the ground field k, we have

$$\rho_1(y^{-1})\rho_1(y) = \rho_1(y)^2 = \rho_1(y)^2(\rho_1(y^2))^{-1} \equiv \bar{c}(y,y) = -1 \tag{6.3.1}$$

by considering $\rho_1(y) \in k^\times$ as an invertible element. This defines a 2-cocycle $\bar{c} \in H^2(\widehat{\mathbb{Z}_2}, k^\times)$ at degree-(-1) carried by 2-representations that have $\rho_1 \neq 0$. In other words, the Cheshire strings $\mathbf{c}, \mathbf{c}^*$ are capable of carrying a minus sign due to $\bar{c}$, while the vacuum lines $\mathbf{1}, \mathbf{1}^*$ do not. We thus have two versions of the 2-category $2\mathrm{Rep}^\tau_{f,m}(D(B\mathbb{Z})^{\mathrm{sgn}})$, corresponding to the versions of $D(B\mathbb{Z}_2)$ that either carry the projective sign $\bar{c}$ or do not.

Twisted Drinfel'd 2-doubles. These 2-cocycles $\bar{c}, \bar{e}$ can alternatively be interpreted as "twists" in the 2-algebra structure of the Drinfel'd double 2-bialgebra. Moreover, they can also be interpreted as contributions to the 4-cocycles $H^4(D(B\mathbb{Z}_2), k^\times)$ of the (2-group underlying the) Drinfel'd double 2-bialgebra $D(B\mathbb{Z}_2)$. This is a categorification of the 3-cocycle twists of an ordinary 1-Drinfel'd double/3d tube algebra [155]; indeed, twists of 2-group(oid) algebras by 4-cocycles have also appeared in the construction of the 4d tube algebra [156].

More precisely, the degree-4 cohomology of $D(B\mathbb{Z}_2)$ was computed in [104] to take the form

$$H^4(D(B\mathbb{Z}_2), k^\times) \cong H^4(B^2\widehat{\mathbb{Z}_2}, k^\times) \oplus H^2(B\mathbb{Z}_2, \widehat{\mathbb{Z}_2}) \oplus H^4(\mathbb{Z}_2, k^\times).$$

The 2-cocycle $\bar{e}$ fits into the second term, while the double suspension map $\widehat{\mathbb{Z}_2} \to B^2\widehat{\mathbb{Z}_2}$ sends $\bar{c} \mapsto \bar{c}[1]$ into the first term [46, 63, 71]. This allows us to identify two different 2-group 4-cocycles

$$\omega_f = \bar{c}[1] + \bar{e}, \quad \omega_b = \bar{e} \quad \in H^4(\mathbb{G}, k^\times) \tag{6.3.2}$$

corresponding to twists of the Drinfel'd double 2-bialgebra $D(B\mathbb{Z}_2)$, where the notation "[1]" signifies a degree-shift under the double suspension map. These are the 4-cocycles that had appeared in **Theorem 6.0.1**.

In analogy with the 3-dimensional case [155], we shall denote the twisted Drinfel'd double 2-bialgebras by $D^\omega(B\mathbb{Z}_2)$, where $\omega \in H^4(D(B\mathbb{Z}_2), k^\times)$. We take, now with proper naming,

$$\textbf{Spin-Kitaev:} \quad 2\mathrm{Rep}^\tau_f(D(B\mathbb{Z}_2)^{\mathrm{sgn}}) = 2\mathrm{Rep}_{\mathrm{wk}}(D^{\omega_f}(B\mathbb{Z}_2)),$$
$$\textbf{Toric code:} \quad 2\mathrm{Rep}^\tau_m(D(B\mathbb{Z}_2)^{\mathrm{sgn}}) = 2\mathrm{Rep}_{\mathrm{wk}}(D^{\omega_b}(B\mathbb{Z}_2)),$$

in which the first version is called **fermionic** (f-subscript) while the second version is **bosonic** (m-subscript). This notation is suggestive, as it corresponds to whether the degree-(-1) $\widehat{\mathbb{Z}_2}$ of the Dijkgraaf-Witten NLSM associated to $D(B\mathbb{Z}_2)^{\mathrm{sgn}}$ is fermion parity $\mathbb{Z}_2^f$ or a bosonic π-flux $\mathbb{Z}_2^m$ [71, 46].

Strictly speaking, the monster 2-BF theory (6.1.1) associated to $2\mathrm{Rep}_f(D(B\mathbb{Z}_2)^{\mathrm{sgn}})$ should include a term $\bar{c}(B,B)$ given by the data of the 2-cocycle $\bar{c}$, whence the partition function

(6.1.3) reads

$$Z^s_{\text{Kit}}(X) \sim \sum_{\substack{dA=0\\ dB=\tau}} e^{i2\pi\int_X \langle B\cup\bar{e}(A)\rangle + \bar{c}(B,B)}. \tag{6.3.3}$$

Note that this term $\bar{c}(B,B)$, being cohomological, does not alter the EOM[5] for the fields (A,B). The theory Z^s_{Kit} has also appeared as part of the NLSM construction in [71], provided we identify

$$\bar{e}(A) = \frac{1}{2}\operatorname{Sq}^1 A, \qquad \bar{c}(B,B) = \frac{1}{2}\operatorname{Sq}^2 B \tag{6.3.4}$$

in terms of the $\mathbb{Z}_2$-cohomology operation $\operatorname{Sq}^i : H^j(X,\mathbb{Z}_2) \to H^{j+i}(X,\mathbb{Z}_2)$ called the **Steenrod square** [157].

Remark 6.3.1. In the spin-Kitaev model Z^s_{Kit}, the coefficient of 1/2 that appeared in front of the term $\operatorname{Sq}^2 B$ means that the point-like particle in the NLSM is a fermion [71]. If this coefficient is 1/4, then such a term $\frac{1}{4}\operatorname{Sq}^2 B = \mathfrak{p}_2(B)$ gives a cohomology operation called the *Pontrjagyn square* $\mathfrak{p}_2 : H^2(X,\mathbb{Z}_2) \to H^4(X,\mathbb{Z}_4)$ [104]. The point particle would then be a *semion* [71] in this case.

6.3.1 Fusion structure in the twisted case

Due to the presence of 2-cocycles $\bar{e}$ and $\bar{c}$ in $2\mathrm{Rep}_f(D(B\mathbb{Z}_2)^{\text{sgn}})$, the corresponding coproduct component Δ_0' governing the tensor product of 2-representations now satisfies a modified version of the condition (5.1.16),

$$\Delta_0'(x^2) = (\bar{e}(x,x)\cdot\bar{e}(x,x))\otimes x^2 = \bar{c}(y,y)1\otimes 1, \tag{6.3.5}$$

where we have noted $y = \bar{e}(x,x)$ and the twisted monoidal structure $y\cdot y = \bar{c}(y,y)\cdot 1$ for generators $x\in\mathbb{Z}_2, y\in\widehat{\mathbb{Z}_2}$. The presence of the sign $\bar{c}(y,y) = -1$ allows us to lift or trivialize certain $\mathbb{Z}_4$-representations. We demonstrate this with explicit computations.

Forming the tensor product, we see that the fusion rules in $2\mathrm{Rep}_f(D(B\mathbb{Z}_2)^{\text{sgn}})$ must be different than that in $2\mathrm{Rep}(D(B\mathbb{Z}_2)^{\text{trv}})$. To see this more explicitly, we perform a monoidal computation while keeping track of the data $\rho_1 : V_0 = \text{sgn} \to V_{-1} = 1$,

$$\begin{aligned}
\mathbf{c}\otimes\mathbf{c} &= (\rho_{\mathbf{c}}\otimes\rho_{\mathbf{c}})\circ\Delta_0' \\
&= (\bar{e}\cdot 1\otimes\bar{e}\cdot 1 \xleftarrow{\rho_1} \text{sgn}\otimes\text{sgn}(\simeq 1)) \oplus (\bar{e}\cdot 1\otimes\text{sgn} \underset{1\otimes\rho_1}{\overset{\rho_1\otimes 1}{\rightleftarrows}} \text{sgn}\otimes\bar{e}\cdot 1) \\
&\simeq (1\xleftarrow{\hat{1}} 1)\oplus(\bar{e}\cdot 1\otimes\text{sgn} \underset{1\otimes\rho_1}{\overset{\rho_1\otimes 1}{\rightleftarrows}} \text{sgn}\otimes\bar{e}\cdot 1),
\end{aligned}$$

where we we have used the fact that $(\bar{e}\cdot 1)^{2\otimes}\simeq 1$ and $\rho_1\otimes\rho_1\simeq\hat{1}$.

The first term is simply the trivial representation 1, while we use $\rho_1(y)^2 = \bar{c}(y,y) = -1$ in the second term to lift "sgn" to a sign representation of the subgroup $\mathbb{Z}_2\subset\mathbb{Z}_4$. However,

[5]Indeed, a 2-gauge theory with $F = B$ as an equation of motion would host instead a trivial 2-group $\mathbb{Z}\xrightarrow{1}\mathbb{Z}_2$ [115].

together with the factor $\bar{e}(x,x) \neq 1$, this allows to degenerate $\bar{e} \cdot 1 \otimes \mathrm{sgn} \simeq 1$ to the trivial representation; this is the effect of the condition (6.3.5). As such, we have

$$\mathbf{c} \otimes \mathbf{c} \simeq (1 \xleftarrow{\hat{1}} 1) \oplus (1 \xleftarrow{\hat{1}} 1) \simeq 1 \oplus 1 = \mathbf{1}, \tag{6.3.6}$$

which is indeed distinct from (6.2.1). The magnetic Cheshire $\mathbf{c}^*$, on the other hand, does not carry $\bar{e}$, so it furnishes a $k\mathbb{Z}_2 \times k\mathbb{Z}_2$-representation. However, it does carry the 2-cocycle $\bar{c}$, which lifts the sign representation of $\mathbb{Z}_2$ to the trivial one. Hence we deduce that we have $\mathbf{c}^* \otimes \mathbf{c}^* \simeq \mathbf{1}$ as well.

On the other hand, the above argument can be applied to compute the fusion

$$\mathbf{c} \otimes \mathbf{c}^* \simeq (1 \xleftarrow{\hat{1}} \mathrm{sgn}) \oplus (1 \xleftarrow{\hat{1}} \mathrm{sgn}) \simeq \mathrm{sgn} \oplus \mathrm{sgn} \simeq \mathbf{1}^*, \tag{6.3.7}$$

where a non-trivial sign representation is left over due to the lack of a 2-cocycle $\bar{e}$ carried by the magnetic Cheshire line $\mathbf{c}^*$. Similarly, we have $\mathbf{c}^* \otimes \mathbf{c} \simeq \mathbf{1}^*$.

The above computations for (6.3.6), (6.3.7) rely crucially on $\bar{c} \neq 0$. Therefore, if $\bar{c} = 0$ were trivial, then the Cheshire strings $\mathbf{c}, \mathbf{c}^* \in 2\mathrm{Rep}_m(D(B\mathbb{Z}_2)^{\mathrm{sgn}})$ revert to having the same fusion rules (6.2.1), (6.2.2) as those in $2\mathrm{Rep}(D(B\mathbb{Z}_2)^{\mathrm{trv}})$. This observation corroborates with [76].

Fusion rules for the 2-intertwiners $i[01], i[10]$. Now in contrast to the previous case of the invisible toric code, the coproduct component Δ_0 is non-trivial for the Drinfel'd double 2-bialgebra $D(B\mathbb{Z}_2)^{\mathrm{sgn}}$. This induces a tensor product between the 2-representations (6.1) and the 2-intertwiners on them. To be concrete and for brevity, we shall concentrate on the connected component $\Gamma = \mathrm{End}_{2\mathrm{Rep}_{\mathrm{wk}}(D(B\mathbb{Z}_2)^{\mathrm{sgn}}}(\mathbf{1})$ in the following.

The fusion rules for the self-2-intertwiners $i[00] = i[11] = \mathbb{1}, i[00]' = i[11]' = \mathfrak{c}$ remain the same as (6.3.6), hence we shall focus on the fusion rules between $i[01], i[10]$. For convenience, we relabel these 2-intertwiners as $v_{\mathbf{1}}, v_{\mathbf{c}}$ by their domains, and the goal is to directly compute the tensor product $v_{\mathbf{1}} \otimes v_{\mathbf{c}} = v_{\mathbf{c}} \otimes v_{\mathbf{1}}$ through the definition. Given the Gray-property we have noted in **Lemma 5.5.2**, the following two decompositions of $i \otimes j$

$$v_{\mathbf{1}} \otimes \mathbf{1} \circ \mathbf{1} \otimes v_{\mathbf{c}}, \qquad v_{\mathbf{c}} \otimes \mathbf{c} \circ \mathbf{c} \otimes v_{\mathbf{1}}$$

differ up to an invertible modification. This 2-isomorphism was computed in [119] to be given by the weak component $\varrho = \rho_1 \circ \bar{e}$, which in this case is determined by the 2-cocycle $\bar{e} \in H^2(\mathbb{Z}_2, \widehat{\mathbb{Z}_2})$ (see (6.3.10) later).

This fact is verified after a bit of a lengthy computation. We find that, for each non-trivial $x \in \mathbb{Z}_2$ (recall the counit ϵ defines the trivial 2-representation $\rho = 1$),

$$\begin{aligned} \rho_{v_{\mathbf{1}} \otimes \mathbf{1}} \cdot \rho_{\mathbf{1} \otimes v_{\mathbf{c}}}(x) &= \epsilon_{-1} \otimes \mathrm{id} \cong \rho_{\mathbb{1}}, \\ \rho_{v_{\mathbf{c}} \otimes \mathbf{c}} \cdot \rho_{\mathbf{c} \otimes v_{\mathbf{1}}}(x) &= (\epsilon_{-1} \otimes \rho_0(x)) \cdot (\epsilon_{-1} \otimes \rho_0(x)) = (\epsilon_{-1} \otimes \rho_0(x)^2). \end{aligned}$$

Upon using the extension class $\bar{e}$, the latter indeed becomes $\rho_1(\bar{e}(x,x)) \otimes \rho_0(x^2) = \rho_1(y) \otimes \mathrm{id} \cong \rho_{\mathfrak{c}}$, where $y \in \widehat{\mathbb{Z}_2}$ is the non-trivial generator. These contribute as (graded) summands into the tensor product, whence

$$v_{\mathbf{1}} \otimes v_{\mathbf{c}} (= v_{\mathbf{c}} \otimes v_{\mathbf{1}}) \simeq \mathbb{1} \oplus \mathfrak{e}. \tag{6.3.8}$$

This is required for the following.

Theorem 6.3.1. *There are monoidal equivalences*

$$\mathfrak{F}_m : Z_1(\Sigma\mathsf{Vect}[\mathbb{Z}_2]) \simeq 2\mathrm{Rep}_{wk}(D^{\omega_b}(B\mathbb{Z}_2)), \qquad \mathfrak{F}_f : Z_1(\Sigma\mathsf{sVect}) \simeq 2\mathrm{Rep}_{wk}(D^{\omega_f}(B\mathbb{Z}_2))$$

of fusion 2-categories.

Proof. Recall from proof of **Proposition 6.2.1** that the obstruction from lifting the equivalence $\mathfrak{F} : \Sigma\mathsf{Vect}[\mathbb{Z}_2] \to \Gamma \subset 2\mathrm{Rep}_{\mathrm{wk}}(D(B\mathbb{Z}_2)^{\mathrm{trv}})$ to a monoidal one is the component functor $\mathfrak{F}_{Ic}$ (or equivalently $\mathfrak{F}_{cI}$), where $I, c \in \Sigma\mathsf{Vect}[\mathbb{Z}_2]$ are respectively the tensor unit and the (electric) Cheshire string in $\mathscr{R}$.

Let $\Gamma_m \subset 2\mathrm{Rep}_{\mathrm{wk}}(D^{\omega_b}(B\mathbb{Z}_2))$ denote the identity component. By adapting $\mathfrak{F}$ to the twisted case $\mathfrak{F}_m : \Sigma\mathsf{Vect}[\mathbb{Z}_2] \to \Gamma_m \subset 2\mathrm{Rep}_{\mathrm{wk}}(D^{\omega_b}(B\mathbb{Z}_2))$, we see that the fusion rule (6.3.8) makes the component functors of $\mathfrak{F}_m$ monoidal,

$$\begin{aligned} (\mathfrak{F}_m)_{Ic\to cI}(v_1 v_2) &= (\mathfrak{F}_m)_{Ic\to Ic}(\mathring{1} \oplus e) = \mathbb{1} \oplus \mathfrak{e} \\ &\simeq v_{\mathbf{1}} v_{\mathbf{c}} = (\mathfrak{F}_m)_{I\to c}(v_1)(\mathfrak{F}_m)_{c\to I}(v_2), \end{aligned}$$

and identically for $(\mathfrak{F}_m)_{cI\to Ic}(v_2 v_1) \simeq (\mathfrak{F}_m)_{c\to I}(v_2)(\mathfrak{F}_m)_{I\to c}(v_1)$ (note the work [63] did not distinguish between v_1, v_2, so the fusion rule there is $v^2 \simeq \mathring{1} + e$). Therefore, $\mathfrak{F}_m : \Sigma\mathsf{Vect}[\mathbb{Z}_2] \to \Gamma_m$ is a monoidal equivalence. Since $\mathfrak{F}_m$ and its component functors preserve all units, it extends to a monoidal equivalence $\mathfrak{F}_m : Z_1(\Sigma\mathsf{Vect}[\mathbb{Z}_2]) \simeq \mathscr{R} \to 2\mathrm{Rep}_{\mathrm{wk}}(D^{\omega_b}(B\mathbb{Z}_2))$, as desired.

Now consider the fermionic case. We use the description of the braided fusion 2-category $\mathscr{S}$ describing the spin-$\mathbb{Z}_2$ gauge theory given in [76]. The 2-category $\mathscr{S}$ is very similar to $\mathscr{R}$: it has two identical components, with the endormophism category on the identity given by $\Omega\mathscr{S} = \mathrm{sVect} \simeq \mathsf{Vect}[\mathbb{Z}_2]$. The fusion rules of the two components are once again graded by $\mathbb{Z}_2$. The caveat, however, is that each component are monoidally equivalent to $\Sigma\mathsf{sVect}$ instead.

In the identity component $\Sigma\mathsf{sVect}$, the Cheshire string $c \in \mathrm{sVect}$ is the superalgebra $\mathrm{Cl}(1)$, ie. the Clifford algebra with one odd generator. It satisfies the well-known fusion rule $c \otimes c \simeq 1$ in the ambient category sVect. The rest of the fusion rules are then determined by the $\mathbb{Z}_2$-grading,

$$c^2 \simeq m'^2 \simeq 1, \qquad c \otimes m' \simeq m' \otimes c \simeq m, \qquad m \otimes c \simeq c \otimes m \simeq m'$$

Let Γ_f denote the identity component of $2\mathrm{Rep}_{\mathrm{wk}}(D^{\omega_f}(\mathbb{Z}_2))$. The 2-functor $\mathfrak{F}_f : \Sigma\mathsf{sVect} \to \Gamma_f$, defined in the same way as in **Proposition 6.2.1** and the above, the computations (6.3.6), (6.3.7) show that $\mathfrak{F}_f$ is monoidally essentially surjective.

Consider $\Omega\Gamma_f = \mathrm{End}_{2\mathrm{Rep}_{\mathrm{wk}}(D^{\omega_f}(\mathbb{Z}_2))}(\mathbf{1})$, whose unit is $\mathbb{1}$. Though $\Gamma_f \not\simeq \Gamma_m$ as monoidal 2-categories, we do have $\mathrm{sVect} \simeq \Omega\Gamma_f \simeq \Omega\Gamma_m \simeq \mathsf{Vect}[\mathbb{Z}_2]$ (only monoidally) [76], hence $\mathfrak{F}_f$ is monoidally fully faithful by the same argument as the above for Γ_m. Therefore, $\mathfrak{F}_f$ extends to a monoidal equivalence $\mathfrak{F}_f : \mathscr{S} \xrightarrow{\sim} 2\mathrm{Rep}_f(D(B\mathbb{Z}_2)^{\mathrm{sgn}})$ as desired. □

6.3.2 Proof of the main theorem

Let us now look at the braiding data. We recall that the braiding in the 4d toric code $\mathscr{R} \simeq Z_1(\Sigma\mathsf{Vect}[\mathbb{Z}_2])$ is known [78] to be given by

$$\beta_{X,Y}(X \otimes Y) = Y \otimes \mathrm{sgn}_{|Y|}X, \qquad X, Y \in Z_1(\Sigma\mathsf{Vect}[\mathbb{Z}_2]) \tag{6.3.9}$$

where sgn is the sign representation and $|Y| \in \mathbb{Z}_2$ denotes the $\mathbb{Z}_2$-grading of the object Y; namely, given $Y = m, m'$ is magnetic, $\mathrm{sgn}_{|Y|}$ acts non-trivially on the electric sector. This then gives rise, by naturality, to a non-trivial full mixed braiding [76, 68]

$$\beta_{e,Y} \circ \beta_{Y,e} = -1 \cdot \mathrm{id}, \qquad Y = m, m^*$$

between the non-trivial 1-morphism $e \in \Omega\mathscr{R} \simeq \mathsf{Vect}[\mathbb{Z}_2]$ and the magnetic objects m, m^*, as required by remote detectability (see *Remark 6.2.2*).

The spin-$\mathbb{Z}_2$ gauge theory $\mathscr{S} \simeq Z_1(\Sigma\mathsf{sVect})$, on the other hand, has $\Omega\Sigma\mathsf{sVect} \simeq \mathsf{sVect}$, which has a non-trivial self-braiding $\beta_e = -1 \cdot \mathrm{id}$ for the odd object e (this is what distinguishes sVect from $\mathsf{Vect}[\mathbb{Z}_2]$). Moreover, since the Cheshire strings c, m' are now invertible, either of them be self-braided. Given that the mixed braiding maps behave the same way as in $\mathscr{R}$ (namely the only non-trivial mixed braiding maps are between e and the magnetic sector, with non-trivial full-braiding), then it is one of the main results in [76] that *only* the electric Cheshire c carries a non-trivial self-braiding $\beta_c = e$ — a non-trivial self-braiding in m, m' would in fact trivialize the anomaly of $\mathscr{S}$.

We are now in a position to prove the main theorem.

Theorem 6.3.2. *The 2-functors $\mathfrak{F}_{m,f}$ in* **Theorem 6.3.1** *are* braided *equivalences.*

Proof. The strategy is to simply compute all of the braiding structures in $2\mathrm{Rep}_{\mathrm{wk}}(D(B\mathbb{Z}_2)^{\mathrm{sgn}})$, and match them to the topological orders $\mathscr{R}, \mathscr{S}$. To do this, we lift the 2-functors $\mathfrak{F}_{m,f}$ of **Theorem 6.3.1** to braided ones. This requires:

1. For each pair of simple objects $X, Y \in \mathscr{R}$, say, the 1-morphisms $\mathfrak{F}_m(\beta_{X,Y})$ and $b_{\mathfrak{F}_m(X),\mathfrak{F}_m(Y)}$ are 2-isomorphic in $2\mathrm{Rep}_{\mathrm{wk}}(D^{\omega_b}(B\mathbb{Z}_2))$, and
2. For each object X and morphism $f : Y \to Y$ in $\mathscr{R}$, the component functors $(\mathfrak{F}_m)_{XY\to Y'X}$ and $(\mathfrak{F}_m)_{YX\to XY'}$ satisfy

$$(\mathfrak{F}_m)_{XY\to Y'X}(\beta_{X,f}) = b_{\mathfrak{F}_m(X),(\mathfrak{F}_m)_{Y\to Y'}(f)}, \qquad (\mathfrak{F}_m)_{YX\to XY'}(\beta_{f,X}) = b_{(\mathfrak{F}_m)_{Y\to Y'}(f),\mathfrak{F}_m(X)}.$$

Of course, the same conditions must be met for $\mathfrak{F}_f : \mathscr{S} \to 2\mathrm{Rep}_{\mathrm{wk}}(D^{\omega_f}(B\mathbb{Z}_2))$.

We shall follow the proof of **Theorem 6.2.1** in order to construct the 2-R-matrix on $D^\omega(B\mathbb{Z}_2)$, which leads to the braiding in $2\mathrm{Rep}_{\mathrm{wk}}(D^\omega(B\mathbb{Z}_2))$ through (5.6.2), (5.6.3). We will see how each of the non-trivial 2-cocycle twists $\bar{e} \in H^2(\mathbb{Z}_2, \mathbb{Z}_2)$ and $\bar{c} \in H^2(\widehat{\mathbb{Z}_2}, k^\times)$ manifest in the braiding data.

Recall the 2-R-matrix $(\mathcal{R}, R)$ is determined by the braided transposition Ψ by (6.2.10), (6.2.9). Due to the "semidirect product" structure $\widehat{\mathbb{Z}_2} \rtimes \mathbb{Z}_2$ induced by the 2-cocycle $\bar{e}$, the degree-0 $\mathbb{Z}_2$ acts non-trivially on the degree-(-1) $\widehat{\mathbb{Z}_2}$ by a sign -1. The defining relations (6.2.11) then implies that 2-R-matrix $\mathcal{R}$ is non-trivial:

$$\mathcal{R} = (-1)^x \cdot y \otimes x + x \otimes (-1)^x y, \qquad R = (-1)^x x \otimes x.$$

By (5.6.2), (5.6.3), the off-diagonal nature of these R-matrices witness non-trivial braiding between the electric and magnetic sectors. Indeed, R acts non-trivially on 2-representations $V, W \in 2\mathrm{Rep}_{\mathrm{wk}}(D^\omega(B\mathbb{Z}_2))$ that differ in *both* of their graded $\mathbb{Z}_2$-representations, which is only possible if V, W lie in distinct sectors by (6.1). The sign then indicates that this braiding is non-trivial, consistent with (6.3.9).

Lemma 6.3.1. *The 2-cocycle $\bar{e} \in H^2(\mathbb{Z}_2, \mathbb{Z}_2)$ leads to non-trivial full braiding maps between $\mathfrak{e}$ and objects W in the magnetic sector.*

Proof. Recall $\bar{e} \in H^2(\mathbb{Z}_2, \mathbb{Z}_2)$ determines the non-trivial central extension $\mathbb{Z}_4$ of $\mathbb{Z}_2$ by itself. Provided that the component ρ_0^0 is non-trivial, then $\rho_0 = (\bar{e} \cdot \rho_0^1, \rho_0)$ furnishes a $k\mathbb{Z}_4$-representation.

In addition, this 2-cocycle also dualizes to $\bar{e} \in H^2(\mathbb{Z}_2, \widehat{\mathbb{Z}_2})$, which "twists" the algebra structure in $D(B\mathbb{Z}_2)^{\mathrm{sgn}}$ in the sense that

$$x \cdot (x \cdot y) = \bar{e}(x, x) y \neq x^2 \cdot y = y,$$

where $x \in \mathbb{Z}_2$ and $y \in k\widehat{\mathbb{Z}_2}$. In the 2-representation 2-category $2\mathrm{Rep}_{\mathrm{wk}}(D(B\mathbb{Z}_2))$, this manifests as the presence of the 2-morphism

$$\varrho(x_1, x_2) = \rho_1(\bar{e}(x_1, x_2)) : \rho_0(x_1) \circ \rho_0(x_2) \Rightarrow \rho_0(x_1 x_2), \qquad x_1, x_2 \in k\mathbb{Z}_2 \tag{6.3.10}$$

mentioned in **Definition 5.5.4**. This demonstrates why we must use the weak 2-representation theory based on $2\mathsf{Vect}^{BC}_{\mathrm{wk}}$, as the strict version does contain the component ϱ, and hence cannot detect any twists in the 2-bialgebra $D(B\mathbb{Z}_2)$.

Recall (6.2.4) that $\mathfrak{e}$ swaps the grading of the 2-representation spaces, and hence $\bar{e}$ will occur only in the full mixed braiding $B_{W,\mathfrak{e}} = B_{\mathfrak{e},W} = b_{W,\mathfrak{e}} \cdot b_{\mathfrak{e},W}$ between $\mathfrak{e}$ and those 2-representations W that carry a non-trivial sign representation *in degree-(-1)* — namely the magnetic sector in (6.1). The other full mixed braiding maps being trivial. A simple computation then gives

$$B_{W,\mathfrak{e}} : \rho_0^0(\mathcal{R}^+_{(2)}) \rho_0^0(\mathcal{R}^-_{(1)}) \Rightarrow \rho_0^0(\mathcal{R}^+_{(2)} \mathcal{R}^-_{(1)}) = 1, \tag{6.3.11}$$

which is precisely the map $\varrho_1(x, x) = \rho_1(\bar{e}(x, x)) \simeq -1$ from (6.3.10). In other words, the

$\mathbb{Z}_2$-particle $\mathfrak{e}$ braids non-trivially with the magnetic sector $\mathbf{1}^*, \mathbf{c}^*$, as required.

$\square$

Lemma 6.3.2. *The 2-cocycle $\bar{c} \in H^2(\widehat{\mathbb{Z}_2}, k^\times)$ gives the non-trivial self-braiding $b_{\mathfrak{e}} = -1$. Moreover, the self-braiding $b_{\mathbf{c}}$ is non-trivial in $2\mathrm{Rep}_{wk}(D^{\omega_f}(B\mathbb{Z}_2))$, but $b_{\mathbf{c}^*}, b_{\mathbf{1}^*}$ are trivial.*

Proof. Consider the first statement. By naturality, the braiding maps $b_{i,j}$ on 1-morphisms i, j can be decomposed into mixed braiding maps,

$$b_{i,j} = b_{i,W} b_{V,j}, \qquad \begin{cases} i : V \to U \\ j : W \to T \end{cases}.$$

Taking $i = j = \mathfrak{e}$ and the identity endomorphism $\mathbb{1}^* : W \to W$ on a magnetic line, we see that

$$\begin{aligned} b_{\mathfrak{e}} &= b_{\mathfrak{e},\mathbb{1}^*} b_{\mathbb{1}^*,\mathfrak{e}} = (b_{\mathfrak{e},W} b_{W,\mathfrak{e}})(b_{W,\mathfrak{e}} b_{\mathfrak{e},W}) \\ &= B_{\mathfrak{e},W} B_{W,\mathfrak{e}} = (\rho_1(\bar{e}(x,x)))^2 = \bar{c}(y,y) \cdot \mathrm{id} = -1 \cdot \mathrm{id} \end{aligned}$$

from the definition of $\bar{c}$ in (6.3.1) and the fact that $B_{\mathfrak{e},W} = \bar{e}$ from the above lemma. Here, note the extension cocycle $\bar{e}$ satisfies $\bar{e}(x,x) = y$ for the non-trivial generators $x \in \mathbb{Z}_2, y \in \widehat{\mathbb{Z}_2}$. This is consistent with the observation that $\bar{c}$ implements the fermionic statistics of the $\mathbb{Z}_2$-charged particle in [46, 71, 76].

Consider the second statement. Since $\bar{e}$ also determines a central extension of $D(B\mathbb{Z}_2)_0 = \mathbb{Z}_2$ by itself, an analogous argument as the previous lemma shows that, provided the 2-representation ρ_0 has the non-trivial sign representation at degree-0 (ie. the Cheshire string $\mathbf{c}$ or the magnetic vacuum line $\mathbf{1}^*$), then the self-braiding

$$b_V : \rho_0^0(R_{(1)})\rho_0^0(R_{(2)}) \Rightarrow \rho_0^0(R_{(1)}R_{(2)}) = 1$$

can carry the non-trivial 1-morphism $\rho_0(\bar{e}(x,x)) \simeq \mathfrak{e}$. In particular, this establishes that $b_{\mathbf{c}^*} \simeq \mathbb{1}$ is trivial while $b_{\mathbf{c}} \simeq \mathfrak{e}$ is not.

But what about the magnetic vacuum $\mathbf{1}^*$? The above argument does not force $b_{\mathbf{1}^*}$ to be trivial, but the fusion rule (6.2.3) (in the form $\mathbf{c} \otimes \mathbf{c}^* \simeq \mathbf{1}^*$) and the ribbon equation

$$b_{V \otimes W} \cong (V \otimes b_{V,W} \otimes W) \circ (b_V \otimes b_W) \circ (V \otimes b_{W,V} \otimes W)$$

do. Since the magnetic Cheshire $\mathbf{c}^*$ is bosonic, the *full* braiding $B_{\mathbf{c}^*,\mathbf{c}} \simeq b_{\mathbf{c}} \simeq \mathfrak{e}$ must be non-trivial. Using this along with (6.2.3) and the previous result then gives

$$\begin{aligned} b_{\mathbf{1}^*} &= b_{\mathbf{c}^* \otimes \mathbf{c}} \\ &\cong b_{\mathbf{c}^*,\mathbf{c}} \circ (b_{\mathbf{c}^*} \otimes b_{\mathbf{c}}) \circ b_{\mathbf{c},\mathbf{c}} \\ &\simeq (b_{\mathbf{c}^*} \otimes b_{\mathbf{c}}) \circ B_{\mathbf{c}^*,\mathbf{c}} \\ &\simeq \mathbb{1} \otimes \mathfrak{e} \otimes \mathfrak{e} \simeq \mathbb{1}, \end{aligned}$$

hence the magnetic vacuum $\mathbf{1}^*$ must have trivial self-braiding $b_{\mathbf{1}^*} = 1$.

Of course, in the absence of $\bar{c}$, the braiding maps considered above are all trivial. $\square$

These lemmas demonstrate that the non-trivial braiding data in $\mathscr{R}$ (resp. $\mathscr{S}$) appear in $2\mathrm{Rep}_{\mathrm{wk}}(D^{\omega_b}(B\mathbb{Z}_2))$ (resp. $2\mathrm{Rep}_{\mathrm{wk}}(D^{\omega_f}(B\mathbb{Z}_2))$), and identifies them from the 2-cocycle twists $\bar{e}, \bar{c}$ present in $D^\omega(B\mathbb{Z}_2)$.

$\square$

To further drive home the point of the main result **Theorem 6.3.2**, we shall recover the 5-dimensional cobordism invariant associated to the spin $\mathbb{Z}_2$-gauge theory $\mathscr{S}$ from the spin-Kitaev model. Recall the expressions of $\bar{e}(A) = \frac{1}{2}\mathrm{Sq}^1 A$ and $\bar{c}(B,B) = \frac{1}{2}\mathrm{Sq}^2 B$ in terms of the Steenrod square. Starting from the partition function (6.3.3),

$$Z^s_{\mathrm{Kit}}(X) \sim \sum_{\substack{dA=0\\ dB=\tau}} e^{i2\pi\int_X B\cup\frac{1}{2}\mathrm{Sq}^1 A + \frac{1}{2}\mathrm{Sq}^2 B},$$

we deduce that, given W is a 5-dimensional manifold with boundary $X = \partial W$, the bulk partition function takes the form [71]

$$Z^s_{\mathrm{Kit}}(X) \sim \exp\left[i\pi\int_W \tau(A)\cup\mathrm{Sq}^1 A + \mathrm{Sq}^2\tau(A)\right]$$

on-shell of the EOM $dA = 0, dB = \tau(A)$.

By interpreting the on-shell gauge fields (A,B) (ie. satisfying $dA = 0, dB = \tau(A)$) as a classifying map $f = (A,B) : W \to BD(B\mathbb{Z}_2)$ [71, 104], we can introduce group cohomology classes

$$E \in H^3(\mathbb{Z}_2, \widehat{\mathbb{Z}_2}), \qquad M \in H^2(\mathbb{Z}_2, \mathbb{Z}_2)$$

such that $f^*E = \tau(A)$ and $f^*M = \frac{1}{2}\mathrm{Sq}^1 A = \bar{e}(A)$. Then, the spin-Kitaev partition function can be written as

$$Z^s_{\mathrm{Kit}}(X) \sim \sum_{f\in[W,B\mathcal{K}]} ([W], f^*\alpha),$$

where $[W] \in H_5(W, \mathbb{C}^\times)$ is the fundamental homology class and α is a degree-5 group cohomology class given by

$$\alpha = (-1)^{\mathrm{Sq}^2 E + E\cup M} \in H^5(\mathbb{Z}_2[3]\times\mathbb{Z}_2[2], \mathbb{C}^\times). \tag{6.3.12}$$

This is precisely the anomaly of the fermionic phase $\mathscr{S}$ [76].

To conclude this Chapter, I would like to emphasize that I have generalized the above computations and results in [121] to study the 4d $\mathbb{Z}_p$ toric code, where p is an arbitrary prime. The partition function reads

$$Z_{\mathrm{Kit}^p}(X) \sim \sum_{\substack{dA=0\\ dB=\tau(A)}} e^{i2\pi\int_X\langle B\cup\bar{e}_m(A)\rangle},$$

where the 2-cocycle $\bar{e}_m \in H^2(\mathbb{Z}_p, \widehat{\mathbb{Z}_p})$ evaluated on A can be written in terms of the $\mathbb{Z}_p$-Bockstein homomorphism $\beta : H^1(X, \mathbb{Z}_p) \to H^2(X, \mathbb{Z}_p)$. We state the main results here without proof.

Theorem 6.3.3. *There is a monoidal equivalence between* $2\mathrm{Rep}^{\bar{e}}_{wk}(D(B\mathbb{Z}_p))$ *and* $Z_1(2\mathsf{Vect}^{KV}(\mathbb{Z}_p))$. *Further, the electric* $\mathbb{Z}_p$*-flavoured bosons* $\mathfrak{e}_k$ *have non-trivial full braidings with any of the a-magnetic objects* W^a,

$$B_{\mathfrak{e}_k, W^a} = \zeta^m \cdot \mathrm{id}, \qquad \forall a, k = 1, \ldots, p-1,$$

where $\mathrm{id} : \mathfrak{e}_k \otimes W^a \Rightarrow \mathfrak{e}_k \otimes W^a$ *denotes the identity 2-morphism.*

Chapter 7

Outlook

This PhD thesis has motivated the appearance of homotopy Lie algebra symmetries in gauge theories, and outlined several applications of the structure of Lie 2-bialgebras and the 2-graded classical R-matrix. I showed in my paper [113] that this gauge-theoretic perspective of higher homotopy symmetries is in fact very natural for the *anomaly cancellation* mechanism [100, 158, 159, 160], and the "gauging the gauge" idea extends straightforwardly to Lie *3*-algebras and 3-gauge theories [161]. These points did not make it into the main text of the thesis due to length constraints, but the interested reader is encouraged to check [113].

I then developed the algebraic structure of Hopf 2-algebras to serve as the quantization of the theory of (weak) Lie 2-bialgebras. One key point to emphasize is that no where in §5 did I require the underlying algebras to be semisimple or finite-dimensional. Hence the theory of Hopf 2-algebras can be used to describe a notion of **compact categorical quantum groups**, namely a deformation quantization, in the style of Drinfel'd-Jimbo, of compact Lie 2-groups. Such structures have been proposed to have important applications in 4d quantum gravity [44, 134, 1, 162].

These ideas that I have developed throughout my PhD allows one to tackle many open questions that remain to be explored. I end this thesis with a short list of them.

Higher-ribbon structures and modular tensor 2-categories. The reader may noticed that I have conveniently left out the study of the anomalous version $\mathscr{T}$ of the fermionic order $\mathscr{S}$ in §6. This is because $\mathscr{T}$ is not a Drinfel'd centre [63, 77], and hence a description in terms of a 4d Dijkgraaf-Witten TFT will not be straightforward. However, it is closely related to the w_2w_3 gravitational anomaly [76, 163], and there had been field theories and lattice models that are proposed to describe this anomaly [70, 113, 164].

The order $\mathscr{T}$ is known to be distinct from $\mathscr{S}$ as fusion 2-supercategories [76, 63]. As mentioned in *Remark 3.4* of [63], this can be understood as the difference between the self-duality datum they host for the magnetic line m, which prompts a notion of *ribbon* Hopf 2-algebras and their 2-representations. Such objects should morally be a quasitriangular Hopf 2-algebra equipped with a central *ribbon element* $\nu \in \mathcal{A}$ satisfying appropriate homotopy ribbon equations and coherences. Ideally, I wish to develop this theory in such a way that the ribbon data can be read off directly from the underlying Hopf 2-algebra of the 4d TQFT, or the underlying 2-groupoid algebra of the membrane-net lattice model [156].

The modular data of (possibly *non*-finite semisimple) ribbon tensor 2-categories — such as $2\mathrm{Rep}_{\mathrm{wk}}(\mathcal{A})$ for an infinite-dimensional ribbon 2-Hopf algebra $\mathcal{A}$ — could be used to construct a 4d version of the Reshetikhin-Turaev TQFT. As these TQFTs are non-semisimple [165], they can produce novel invariants of 4-manifolds that see exotic smooth structure.

Higher character theory and state sums for 4d TQFTs. The goal here is to provide a machinery that produces state sum invariants directly from the given 4d TQFT action. Towards this, I have initiated work with prof. Clement Delcamp (IHÉS) to establish a notion of delta functions and orthogonality of categorical characters [166, 167, 168] for higher representations of groups/2-groups [64, 60] from the **tensor networks/matrix product operators** perspective. This would allow us to explore the 4d analogue of the known deep relationship between Turaev-Viro invariants and the tangle operators in 3d Chern-Simons theory [30, 27, 25], which makes heavy use of character theory.

Having such a result would settle a conjecture [134] concerning the equivalence between 4d 2-Chern-Simons theory and the Crane-Yetter-Broda TQFT. Moreover, a higher homotopical version of the *Peter-Weyl theorem*, which states that the space of L^2-functions on a compact quantum group decomposes into (infinitely many!) finite-dimensional unitary irreps, would open the door towards the study of non-semisimple tensor 2-categories. Such algebraic gadgets would be very useful for both mathematics and physics, such as the construction of novel 4-manifold invariants and the classification of *gapless* conformal defects [169].

The holographic duality in higher Chern-Simons theory. The well-known 3d Chern-Simons/2d Wess-Zumino-Witten holographic correspondence [43], we expect a higher-integrable CFT to be associated to the boundary of higher-Chern-Simons theory [111]. In an upcoming work with Joaquin Liniado (La Plata U.), I have studied the homotopy 2+1d current algebra that lives on the boundary of 4d 2-Chern-Simons theory [135], based on the holomorphic Chern-Simons localization of [170]. The associated 2-Lax connections allowed us to construct higher conserved currents that live on surfaces. These can be used to model conformal defects in higher-dimensional CFTs.

In contrast to the twice-holomorphic homotopy current algebra of [171], the 3d currents are holomorphic-topological, and hence should admit a quantization in terms of a holomorphic-topological vertex operator algebra (VOA). It would be interesting to relate this VOA to the "Raviolo VOA" of [172, 173]. Similarly, its representations should also admit a higher homotopy version of the Kazhdan-Lusztik correspondence as mentioned in §1 — namely that "positive energy" 2-representations of this homotopy current algebra should be in one-to-one correspondence with 2-representations of compact categorical quantum groups.

2+1d quantum integrability. I propose that compact categorical quantum groups serves as the foundation for a **2-dimensional quantum inverse scattering method**. Such a notion of higher quantum integrability should realize the exact solvability of 2+1d lattice models with general 2-categorical symmetries [75]. This proposal is inspired by the Bethe ansatz for quantum integrable spin chains [174, 175] in 1-dimension, which gave rise to coherent states which diagonalizes the transfer matrix of the lattice model. Having control over higher-dimensional quantum integrability would also lead to the development of tools that are suitable for studying quantum entanglement properties of novel 3d quantum codes.

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Appendices

Appendix A

Classification of Lie 2-algebras

In this section we examine the classification of Lie algebra crossed-modules by Lie algebra cohomology, following [124]. Recall that a given two Lie algebras $\mathfrak{h}, \mathfrak{g}$ over a *fixed* field k of characteristic zero, a Lie algebra crossed-module is a map $t : \mathfrak{h} \to \mathfrak{g}$ and an action $\rhd$ of $\mathfrak{g}$ on $\mathfrak{h}$ such that the following *Peiffer conditions*

$$t(X \rhd Y) = [X, tY]_{\mathfrak{g}}, \qquad tY \rhd Y' = [Y, Y']_{\mathfrak{h}} \tag{A.0.1}$$

are satisfied for each $Y, Y' \in \mathfrak{h}, X \in \mathfrak{g}$. Mathematically, it is equivalent to a *strict* Lie 2-algebra[1], where the homotopy map $\mu = 0$ introduced in the main text vanishes.

Consider the following four-term algebra complex built from the Lie algebra crossed-module,

$$0 \to V \hookrightarrow \mathfrak{h} \xrightarrow{t} \mathfrak{g} \to \mathfrak{n} \to 0, \tag{A.0.2}$$

where $V = \ker t$ and $\mathfrak{n} = \operatorname{coker} t$. Due to the Peiffer identity in (A.0.1), the Lie algebra $V \subset Z(\mathfrak{h})$ must lie in the centre of $\mathfrak{h}$, and hence is Abelian. It admits an action by $\mathfrak{n}$ induced by the crossed-module action $\rhd$.

Definition A.0.1. We say that two crossed-modules $t : \mathfrak{h} \to \mathfrak{g}, t' : \mathfrak{h}' \to \mathfrak{g}'$ with the respective actions $\rhd, \rhd'$ are *elementary equivalent* if

1. $\ker t = \ker t' = V$ and $\operatorname{coker} t = \operatorname{coker} t' = \mathfrak{n}$,

2. there exists Lie algebra homomorphisms $\phi : \mathfrak{h} \to \mathfrak{h}', \psi : \mathfrak{g} \to \mathfrak{g}'$ compatible with the actions $\rhd, \rhd'$ such that
$$\phi(X \rhd Y) = \psi(X) \rhd' \phi(Y)$$

[1] Namely a two-term differential graded L_∞-algebra.

for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$. Moreover, the diagram

$$\begin{tikzcd}
 & & \mathfrak{h} \arrow[r, "t"] \arrow[dd, "\phi"] & \mathfrak{g} \arrow[dd, "\psi"] \arrow[dr] & & \\
0 \arrow[r] & V \arrow[ur] \arrow[dr] & & & \mathfrak{n} \arrow[r] & 0 \\
 & & \mathfrak{h}' \arrow[r, "t'"] & \mathfrak{g}' \arrow[ur] & &
\end{tikzcd}$$

commutes.

Let us denote the set of elementary equivalence classes of Lie algebra crossed-modules by $\mathbf{XMod}(\mathfrak{n}, V)$.

A.1 Lie algebra cohomology

We first review some basic facts about Lie algebra cohomology, which is a very powerful and important tool for classification of L_∞-algebras. We once again follow the treatment of [124].

Let $\mathfrak{n}$ be a Lie algebra over the field k and let V be an Abelian $\mathfrak{n}$-module. Define its differential graded *Chevalley-Eilenberg complex*

$$(C^\bullet(\mathfrak{n}, V), d), \qquad C^p(\mathfrak{n}, V) = \begin{cases} \Lambda(\mathfrak{n}^p, V) & ; p > 0 \\ V & ; p = 0 \end{cases},$$

where $\Lambda(\mathfrak{n}^p, V)$ denotes the exterior algebra of alternating forms on p-copies of $\mathfrak{n}$ over V. The differential $d : C^p(\mathfrak{n}, V) \to C^{p+1}(\mathfrak{n}, V)$ is given explicitly by

$$\begin{aligned} dc(x_0, \ldots, x_p) &= \sum_{i<j} (-1)^{i+j} c([x_i, x_j], x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_p) \\ &\quad - \sum_{i=1}^{p} (-1)^i x_i \rhd c(x_0, \ldots, \hat{x}_i, \ldots, x_p) \end{aligned}$$

for each cochain $c \in C^p(\mathfrak{n}, V)$, where $\hat{\cdot}$ denotes an omitted element.

Lemma A.1.1. $d^2 = 0$.

Proof. Recall the *Cartan formula*

$$L_x = d\iota_x + \iota_x d, \qquad x \in \mathfrak{n}$$

where $\iota_x : C^{p+1}(\mathfrak{n}, V) \to C^p(\mathfrak{n}, V)$ is the interior evaluation

$$\iota_x : c \mapsto ((x_1, \ldots, x_p) \mapsto c(x, x_1, \ldots, x_p))$$

and $L_x : C^p(\mathfrak{n}, V) \to C^p(\mathfrak{n}, V)$ is the Lie evaluation

$$L_x : c \mapsto ((x_1, \ldots, x_p) \mapsto x \rhd c(x_1, \ldots, x_p) - \sum_i c(x_1, \ldots, [x, x_i], \ldots, x_p)),$$

which by construction commutes with d. Now let $v \in V = C^0(\mathfrak{n}, V)$ be a 0-form, then

$$\begin{aligned} d^2v(x_1, x_2) &= -dv([x_1, x_2]) + x_1 \rhd dv(x_2) - x_2 \rhd dv(x_1) \\ &= [x_2, x_1] \rhd v + x_1 \rhd (x_2 \rhd v) - x_2 \rhd (x_1 \rhd v) = 0, \end{aligned}$$

which vanishes by the $\mathfrak{n}$-module structure on V.

Now let $p > 0$ and assume the induction hypothesis: $d^2 = 0$ on $C^{p-1}(\mathfrak{n}, V)$. Consider $c \in C^p(\mathfrak{n}, V)$, then by the Cartan formula

$$\begin{aligned} d^2c(x_{-1}, x_0, x_1, \ldots, x_p) &= \iota_{x_{-1}}(d^2c)(x_0, x_1, \ldots, x_p) \\ &= (L_{x_{-1}} - d\iota_{x_{-1}})dc(x_0, x_1, \ldots, x_p) \\ &= (L_{x_{-1}}d - d(L_{x_{-1}} - d\iota_{x_{-1}})c(x_0, x_1, \ldots, x_p) \\ &= (L_{x_{-1}}d - dL_{x_{-1}} + d^2\iota_{x_{-1}})c(x_0, x_1, \ldots, x_p) = 0, \end{aligned}$$

where the first two terms cancel by the property $L_x d = dL_x$, and the last term vanishes due to the induction hypothesis (recall $\iota_{x_{-1}} c \in C^{p-1}(\mathfrak{n}, V)$). $\square$

This nilpotency allows us to define the *Lie algebra cohomology*

$$H^\bullet(\mathfrak{n}, V) = \ker d / \operatorname{im} d.$$

These groups are extremely useful, as they are isomorphic to the de Rham cohomology of the topological group G [176]. Moreover, they classify various algebraic structures; for instance,

1. **Degree** $p = 0$: the group $H^0(\mathfrak{n}, V) = V^{\mathfrak{n}} \subset V$ classifies the $\mathfrak{n}$-invariants: namely elements $v \in V$ annihilated by $\mathfrak{n}$ via the action $\rhd$. Indeed, the 0-cocycle condition merely states

$$dv(x) = x \rhd v = 0, \qquad v \in V = C^0(\mathfrak{n}, V),$$

which means that $v \in Z^0(\mathfrak{n}, V)$ is $\mathfrak{n}$-invariant.

2. **Degree** $p = 1$: the group $H^1(\mathfrak{n}, V)$ classifies algebra representations of $\mathfrak{n}$ on V (i.e. derivations $\mathrm{Der}_{\mathfrak{n}}(V)$) modulo inner representations. Indeed, the 1-cocycle condition reads

$$dc(x_1, x_2) = c([x_1, x_2]) - x_1 \rhd c(x_2) + x_2 \rhd c(x_1) = 0,$$

which implies that $c \in Z^1(\mathfrak{n}, V)$ is a linear representation of $\mathfrak{n}$ on V. The 1-coboundaries are inner derivations $c(x) = dv(x) = x \rhd v$ for some $v \in V = C^0(\mathfrak{n}, V)$. If $\mathfrak{n}$ acts trivially on V, then $H^1(\mathfrak{n}, V)$ is in fact isomorphic to the (dual of the) Abelianization $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$.

3. **Degree** $p = 2$: the group $H^2(\mathfrak{n}, V)$ classifies *central extensions* $\hat{\mathfrak{n}}$ of $\mathfrak{n}$ by V, which fits in the three-term exact sequence

$$0 \to V \to \hat{\mathfrak{n}} \to \mathfrak{n} \to 0.$$

To see this at a glance, a *set-theoretic* section $s : \mathfrak{n} \to \hat{\mathfrak{n}}$ sees an obstruction to being a *Lie algebra-theoretic* section given by

$$c(x_1, x_2) = s([x_1, x_2]) - [s(x_1), s(x_2)].$$

It can be shown, with the $\mathfrak{n}$-module structure of V and the Jacobi identity, that $c \in Z^2(\mathfrak{n}, V)$ is a 2-cocycle, and any two choices of such sections s yields 2-cocycles c, c' that differ by a 2-coboundary $c - c' = da$.

In general, the set $H^p(\mathfrak{n}, V)$ classifies $(p+1)$-term extensions of $\mathfrak{n}$ by V. Moreover, equivalence classes of such extensions can be equipped with an Abelian group structure such that $H^p(\mathfrak{n}, V)$ coincides with it not just as a set, but also as a group.

We shall show in detail next that, at degree 3, $H^3(\mathfrak{n}, V)$ classifies precisely the four-term complex (A.0.2) of a Lie algebra crossed-module.

A.2 Theorem of Gerstenhaber

Before constructing the 3-cocycle $c \in Z^3(\mathfrak{n}, V)$, we introduce the notion of addition in the set of crossed-modules. Given two crossed-modules $t : \mathfrak{h} \to \mathfrak{g}, t' : \mathfrak{h}' \to \mathfrak{g}'$ with the same kernel V and cokernel $\mathfrak{n}$, it can be shown that

$$(t \oplus t') : \mathfrak{h} \oplus \mathfrak{h}'/\overline{\Delta} \to \mathfrak{g} \oplus_{\mathfrak{n}} \mathfrak{g}'$$

is another crossed-module, called the *crossed-module sum* of t and t'. Here, $\overline{\Delta}$ is the kernel of the addition map $+ : V \oplus V \to V$, while $\mathfrak{g} \oplus_{\mathfrak{n}} \mathfrak{g}'$ is the fibre pullback; explicitly,

$$\overline{\Delta} = \{(v, -v) \mid v \in V\}, \qquad \mathfrak{g} \oplus_{\mathfrak{n}} \mathfrak{g}' = \{(X, X') \in \mathfrak{g} \oplus \mathfrak{g}' \mid pX = p'X'\}.$$

Note that as direct sums are commutative, we have $(t \oplus t') \cong (t' \oplus t)$.

This notion descends to elementary equivalence classes of crossed-modules, and endows the set $\mathbf{XMod}(\mathfrak{n}, V)$ the structure of an Abelian group. We shall show that this Abelian group is isomorphic precisely to $H^3(\mathfrak{n}, V)$. To begin, we construct a bilinear skew-symmetric map

$$f(x_1, x_2) = s_1([x_1, x_2]) - [s_1(x_1), s_1(x_2)], \qquad x_1, x_2 \in \mathfrak{n}$$

from a section $s_1 : \mathfrak{n} \to \mathfrak{g}$ of the map $p : \mathfrak{g} \to \operatorname{coker} t = \mathfrak{n}$ in (A.0.2). Though s_1 may not be a Lie algebra map, the projection p is, so $pf = 0$ and f is valued in $\ker p$. By the exactness

$\ker p = \operatorname{im} t$ of (A.0.2), there exists a bilinear skew-symmetric map $e : \mathfrak{n}^{\wedge 2} \to \mathfrak{h}$ such that $f = te$.

We now pick another section $s_2 : \operatorname{im} t \subset \mathfrak{g} \to \mathfrak{h}$ of the crossed-module map $t : \mathfrak{h} \to \mathfrak{g}$, whence $e = s_2 f$. Let $\circlearrowleft$ denote a summation over cyclic permutations of x_1, x_2, x_3, then by construction,

$$\begin{aligned}
tde(x_1,x_2,x_3) &= t\left[\circlearrowleft e([x_1,x_2],x_3) - \circlearrowleft s_1(x_1) \rhd e(x_2,x_3)\right] \\
&= \circlearrowleft f([x_1,x_2],x_3) - \circlearrowleft t(s(x_1) \rhd e(x_2,x_3)) \qquad \text{Peiffer conditions (A.0.1)} \\
&= \circlearrowleft f([x_1,x_2],x_3) - \circlearrowleft [s_1(x_1), \underbrace{te(x_2,x_3)}_{=f(x_2,x_3)}] \qquad \text{Definition of } f \\
&= \circlearrowleft ([s_1([x_1,x_2]), s_1(x_3)] - s_1([[x_1,x_2],x_3])) \\
&\quad - \circlearrowleft ([s_1(x_1),[s_1(x_2),s_1(x_3)]] - [s_1(x_1), s_1([x_2,x_3])]) \qquad \text{Jacobi identity} \\
&= \circlearrowleft ([s_1([x_1,x_2]), s_1(x_3)] - [s_1([x_2,x_3]), s(x_1)]) \qquad \text{Cyclicity of summation} \\
&= 0,
\end{aligned}$$

as such de is in fact valued in $\ker t$. Again by the exactness of the sequence (A.0.2) we may find a skewsymmetric trilinear map $c : \mathfrak{n}^{\wedge 3} \to V$ such that $ic = de$, where $i : V \hookrightarrow \mathfrak{h}$ is the inclusion. Picking yet another section $s_3 : \mathfrak{h} \to V$ yields $c = s_3 De$.

Now we must show that $dc = 0$. It may be tempting to say that, since $ic = de$, we have $idc = dic = d^2 e = 0$ by the nilpotency $d^2 = 0$. However, this does not immediately follow, as s_1 is not necessarily a section and hence $s_1(\cdot)\rhd$ is not necessarily a well-defined action. By explicit computation, terms involving the problematic operation $s_1(\cdot)\rhd$ in idc read

$$\begin{aligned}
&\sum_{i<j} (-1)^{i+j} s_1([x_i,x_j]) \rhd e(x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,x_4) \\
&- \sum_{i=1}^{4} (-1)^i s_1(x_i) \rhd \left[\sum_{j\neq i} (-1)^j s_1(x_j) \rhd e(x_1,\ldots,\hat{x}_j,\ldots,x_3)\right] \qquad \text{Rearrange terms} \\
= \quad & \sum_{i<j} (-1)^{i+j} \left(s_1([x_i,x_j]) - [s_1(x_i), s_1(x_j)]\right) \rhd e(x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,x_4) \qquad \text{Definition of } f \\
= \quad & \sum_{i<j} (-1)^{i+j} \underbrace{f(x_i,x_j)}_{=te(x_i,x_j)} \rhd e(x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,x_4) \qquad \text{Peiffer conditions} \\
= \quad & \sum_{i<j} (-1)^{i+j} [e(x_i,x_j), e(x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,x_4)] \qquad \text{Cyclicity of summation} \\
= \quad & 0,
\end{aligned}$$

hence we nevertheless have $dc = 0$. This allows us to conclude that $c \in Z^3(\mathfrak{n}, V)$.

We now wish to show that changing the choices of the sections $s_{1,2,3}$ adds to c a 3-coboundary. By linearity, we can write $s_1' = s_1 + \delta$ for some map $\delta : \mathfrak{n} \to \mathfrak{g}$. Defining a bilinear skew-symmetric map f' analogously, we see that

$$f'(x_1,x_2) = f(x_1,x_2) + [s_1(x_1), \delta(x_2)] + [\delta(x_1), s_1(x_2)] + [\delta(x_1), \delta(x_2)] - \delta([x_1,x_2]).$$

Notice the terms $[s_1(x_1), \delta(x_2)] + [\delta(x_1), s_1(x_2)] - \delta([x_1, x_2])$ constitute precisely the coboundary $d\delta(x_1, x_2)$ of a cochain $\delta : \mathfrak{n} \to \mathfrak{g}$, with $x_1, x_2 \in \mathfrak{n}$ lifted up to $\mathfrak{g}$ by the map s_1.

Now as f', f are valued in $\ker p = \operatorname{im} t$, we can find $\mathfrak{h}$-valued bilinear maps ϵ, ε such that $t\epsilon(x_1, x_2) = d\delta(x_1, x_2)$ and $t\varepsilon(x_1, x_2) = [\delta(x_1), \delta(x_2)]$. Further, we can also find a $\ker t = \operatorname{im} i$-valued bilinear map φ such that

$$e'(x_1, x_2) = e(x_1, x_2) + \epsilon(x_1, x_2) + \varepsilon(x_1, x_2) + i\varphi(x_1, x_2)$$

when lifted by s_2. Our goal now is to apply the differential d; however, the trouble here is that d and s_2 need not commute, as s_2 is not in general a section. Now by computation

$$\begin{aligned} tds_2\delta(x_1, x_2) &= t(s_1(x_1) \rhd s_2\delta(x_2) + s_1(x_2) \rhd s_2\delta(x_1) - s_2\delta([x_1, x_2])), \qquad \text{Peiffer conditions} \\ &= ts_2([s_1(x_1), \delta(x_2)] - [s_1(x_2), \delta(x_1)] - \delta([x_1, x_2])) \\ &= ts_2 d\delta(x_1, x_2), \end{aligned}$$

so $\Delta_1 = ds_2\epsilon - s_2 d\epsilon$ is valued in $\ker t$. Similarly, the difference $\Delta_2 = ds_2\varepsilon - s_2 d\varepsilon$ also lies in $\ker t$, which allows us to finally write

$$c'(x_1, x_2, x_3) = c(x_1, x_2, x_3) + d\epsilon(x_1, x_2, x_3) + d\varepsilon(x_1, x_2, x_3) + i\left(\Delta_1 + \Delta_2\right)(x_1, x_2, x_3) + di\varphi(x_1, x_2, x_3).$$

Using the injectivity of i, we have $di\varphi = i(d|_V\varphi)$, hence defining $\sigma = \epsilon + \varepsilon$ and $\Gamma = \Delta_1 + \Delta_2 + d|_V\varphi$ yields

$$c' = c + d\sigma + i\Gamma = c + d\sigma \mod \ker t,$$

whence lifting by s_3 up to V yields $c' = c + d\sigma$. This shows that the cohomology class of c does not depend on the choice of the section s_1.

Now suppose we have distinct sections s_2, s_2', defining $e = s_2 f$ and $e' = s_2' f$. It is clear that $t(e - e') = ts_2 f - ts_2' f = f - f = 0$, hence $e - e'$ is valued in $\ker t = \operatorname{im} i$. This means that s_3 lifts $d(e - e')$ to a coboundary $d\omega$ such that $c' = c + d\omega$, demonstrating that the cohomnology class of c does not depend on the choice of the section s_2 as well. Lastly, any two sections s_3, s_3' must coincide, at least on the image $\operatorname{im} i = \ker t$, hence the *cocycle itself* c does not depend on the choice of s_3.

Lemma A.2.1. *Let t, t' denote two elementary equivalent crossed-modules, then the 3-cocycles c, c' they define coincide $[c] = [c'] \in H^3(\mathfrak{n}, V)$ in cohomology.*

Proof. First, pick sections $s_{1,2,3}, s'_{1,2,3}$ in the respective crossed-modules t, t' and construct the 3-cocycles $c, c' \in C^3(\mathfrak{n}, V)$. Suppose an elementary equivalence (ϕ, ψ) between the two crossed-modules exists, then ψs_1 is a section of p'. The above shows that the 3-cocycle $\tilde{c}'$ constructed from the sections $(\psi s_1, s_2', s_3')$ differ from that c' constructed from (s_1', s_2', s_3') only by a coboundary. Our task is thus to show that $\tilde{c}'$ also coincides with c up to coboundary.

Toward this, we define $s_2'\psi f \equiv \tilde{e}'$ and compare this to $\phi e = \phi s_2 f$. First, we know that $t's_2' = 1$, hence $\tilde{e}' - \phi e$ is valued in $\ker t' = \operatorname{im} i'$, so we can find a map $v : \mathfrak{n}^{\wedge 2} \to V$ such that

$\tilde{e}' - \phi e = i'v$.

We now take the differential d of this equation. By definition of the elementary equivalence, we can rewrite contributions $\psi(x_i) \rhd \phi(e) = \phi(x_i \rhd e)$ in the differential, as such $d(\phi e) = \phi de$. Now $s_3\phi$ is a section of i', hence

$$\tilde{c}' - c = s_3 D\tilde{e}' - (s_3\phi)de = dv$$

is a coboundary. This proves the lemma. $\square$

The lemma allows us to put a well-defined map $b : \mathbf{XMod}(\mathfrak{n}, V) \to H^3(\mathfrak{n}, V)$.

Theorem A.2.1. (Gerstenhaber, attr. by MacLane). *b is an isomorphism of Abelian groups.*

The classifying data of a Lie algebra crossed-module $t : \mathfrak{h} \to \mathfrak{g}$ is exactly $(\mathfrak{n}, V, c)$ with $c \in H^3(\mathfrak{n}, V)$.

A.3 The Postnikov class

Let us now turn to the reason why we called an element in $H^3(\mathfrak{n}, V)$ a "Postnikov class" in the main text. Formally, a Lie 2-algebra integrates to a Lie 2-group $t : H \to G$ [95, 122], for which a "Gerstenhaber theorem" also holds: $t : H \to G$ is classified by its *Hoang data* (N, V, κ) [177, 102], where $N = \operatorname{coker} t, V = \ker t$ and $\kappa \in H^3(N, V)$ is a group cohomology class (as opposed to a Lie algebra cohomology class).

The name "Postnikov class" comes from topology. Given any "nice" space X (a finite CW complex), its fundamental group $\pi_1(X)$ in general acts on higher homotopy groups $\pi_{\geqslant 2}(X)$ via *monodromy*. The *homotopy 2-type* $\Pi_2(X) = (\pi_1(X), \pi_2(X), \operatorname{Ptn}(X))$ is modeled by the group crossed-module [102]

$$1 \to \ker \partial = \pi_2(X) \to \pi_2(X, Y) \xrightarrow{\partial} \pi_1(Y) \to \pi_1(X) = \operatorname{coker} \partial \to 1,$$

where $Y \subset X$ is a closed subspace and ∂ is the natural boundary map. Up to homotopy, it is classified by the *Postnikov class* $\operatorname{Ptn}(X) \in H^3(\pi_1(X), \pi_2(X))$, which determines how 2-cells are glued upon the 1-cells.

It is possible to construct the classifying space $B(N, V)$ satisfying the condition $\Pi_2 B(N, V) = (N, V, \kappa)$ [69, 126]. Such a space sits in the *Postnikov tower* fibration sequence

$$B^2V \to B(N, V) \to BN,$$

where $BN = K(N, 1)$ is the classifying Eilenberg-MacLane space of N and $B^2V = K(V, 2)$ is the second delooping of V, satisfying $\pi_2(B^2V) = V$ with other homotopy groups vanishing.

In other words, the Postnikov class determines how $B(N, V)$ is constructed from the base BN by gluing the second delooping space B^2V. The homotopy classification theorem states that gauge-equivalent discrete flat 2-connections $H^1(X,(N,V))$ are isomorphic to homotopy classes of classifying maps $X \to B(N,V)$ [177, 126]; this is how 2-gauge topological field theories are constructed [69, 71].

The Postnikov class as a 2-curvature anomaly. The role the Postnikov class plays in the 2-gauge theory is as a *2-curvature anomaly*. Indeed, as we have seen already in §2.3.2, a 3-form contribution $\kappa(A)$ to the 2-curvature

1. does not violate the Bianchi identity $d_A\mathcal{F} = d_A(F - t\Sigma) = 0$ iff κ is $\ker t$-valued, and

2. is invariant under 2-gauge shifts $A \mapsto A + tL$ iff κ only depends on $\operatorname{coker} t$.

These desirable properties, as well as the descent equation (2.3.11), allows $\kappa(A)$ to have a cohomological interpretation in terms of a *Lie algebra 3-cocycle* $\kappa \in Z^3(\operatorname{coker} t, \ker t)$.

Notice that the function κ is only required to be a Lie algebra 3-cocycle, and hence is not necessarily covariantly closed. This means that, in the presence of $\kappa(A)$, the 2-Bianchi identity (2.2.13) can in fact be violated, due to the 2-curvature anomaly EOM $K = \kappa(A)$ giving $d_AK = d_A\kappa(A) \neq 0$.

As we see from the Gerstenhaber theorem above, the Postnikov class classifies the crossed-module $\mathfrak{G}$ up to elementary equivalence [124, 122]; in fact, Lie 2-algebras are classified by the same data [178, 126]. Indeed, the astute reader may have noticed a close parallel between the Postnikov anomaly $\kappa(A)$ and the Bianchi anomaly $\mu(A,A,A)$. They both define an anomaly of the 2-flatness condition, and the resulting 2-curvature quantity K have identical gauge transformation properties.

For $t \neq 0$, the two structures are actually different. Indeed, the 1-Bianchi anomaly $\mu(A,A,A)$ is *not* invariant under the 1-form shift symmetry $A \to A + tL$, while κ by hypothesis is. This speaks to the fact that, unlike their strict counterparts, weak Lie 2-algebras and non-trivial Lie algebra crossed-modules are *not* equivalent when $t \neq 0$. Indeed, the component $\mathfrak{G}$ in a weak Lie 2-algebra is not a Lie algebra, as the 2-Jacobi identities (2.2.18) do not hold. The quantity $\frac{1}{2}\mu(\lambda,A,A)$ that appeared in (2.3.2), which seems to serve as the first descendant of $\mu(A,A,A)$, does *not* satisfy the descent equation (2.3.11).

When $\mathfrak{G}$ is skeletal, on the other hand, the Postnikov class κ plays precisely the same role as a homotopy map for the Lie 2-algebra $V \xrightarrow{0} \mathfrak{n}$. No violation of the Jacobi identities are present due to $t = 0$. Therefore, algebraically, there is no distinction between a weak *skeletal* Lie 2-algebra and a Lie algebra crossed-module with Postnikov class.

However, in terms of the geometry of the principal 2-bundle, the Lie algebra crossed-module formulation has the distinct advantage that the 2-gauge theory it defines is free of the problems plaguing that of a weak Lie 2-algebra, such as the lack of closure of gauge transformations (2.3.9). This is because of the first descendant $\zeta_A(\lambda)$ of $\kappa(A)$ is part of the data of the 2-gauge theory. The descent equation (2.3.11) ensures that the 2-gauge structure closes and is consistent [99], even in the presence of a non-trivial Postnikov class [69].

Appendix B

2-bundle homomorphisms

In this Chapter, we show that an elementary equivalence gives rise to a homomorphism between 2-gauge bundles. We also generalize this perspective to the weak case.

Let $\mathcal{P}, \mathcal{P}' \to X$ denote two 2-gauge bundles on X, equipped with connections (A, Σ) and (A', Σ'), respectively. Intuitively, from the gauge theory perspective, a *2-bundle homomorphism* $g : \mathcal{P} \to \mathcal{P}'$ should satisfy two properties: (1) it is a bundle map over X; namely the triangle

$$\begin{tikzcd} \mathcal{P} \arrow[rr, "g"] \arrow[dr] & & \mathcal{P}' \arrow[dl] \\ & X & \end{tikzcd}$$

commutes, and (2) preserves all gauge-invariant data.

From our computations in the main text, the gauge-invariant data consist precisely of the fake-flatness $\mathcal{F}$ (2.2.10) and the 2-curvature $\mathcal{G} = K$. As such homomorphisms ψ must satisfy

$$\mathcal{F} = g^*\mathcal{F}', \qquad \mathcal{G} = g^*\mathcal{G}'.$$

Let us write, locally, $g^* = f^* \otimes \Psi$ in terms of components, where f^* is the pullback of $f : X \to X$ on forms and $\Psi = (\phi, \psi)$ is a map on the Lie algebras

$$\phi : \mathfrak{h}' \to \mathfrak{h}, \qquad \psi : \mathfrak{g}' \to \mathfrak{g}.$$

The fake-flatness condition $\mathcal{F} = \psi^*\mathcal{F}'$ implies

$$F = (f^* \otimes \psi)F', \qquad t\Sigma = (f^* \otimes \psi)t'\Sigma' = t(f^* \otimes \phi)\Sigma'; \tag{B.0.1}$$

by linearity and $F = d_A A, F' = d_{A'}A'$, the first condition in (B.0.1) means that f^* commutes with the de Rham differential d, and that ψ is a Lie algebra homomorphism[1]. The second condition means $t\phi = \psi t'$ commutes with the crossed-module maps t, t'.

[1]This means that $A = \psi A'$ and $[A \wedge A] = \psi[A' \wedge A'] = [\psi A' \wedge \psi A']$.

Equivalence of 2-gauge bundles. The 2-curvature condition reads

$$\mathcal{G} = d_A\Sigma = (f^*\otimes\phi)d_{A'}\Sigma' = (f^*\otimes\phi)(d\Sigma' + A\wedge^{\rhd'}\Sigma'),$$

where $\rhd'$ is the crossed-module action in $\mathcal{P}'$. Using the second condition from (B.0.1), the first term reads

$$(f^*\otimes\phi)d\Sigma' = d\Sigma = d(f^*\otimes\phi)\Sigma',$$

while the second term reads

$$A\wedge^{\rhd}\Sigma = (f^*\otimes\phi)A'\wedge^{\rhd'}\Sigma'.$$

However, the condition $A = (f^*\otimes\psi)A'$ means that we must have

$$(f^*\otimes\phi)A'\wedge^{\rhd'}\Sigma' = ((f^*\otimes\psi)A')\wedge^{\rhd}(f^*\otimes\phi)\Sigma'.$$

This tells us that, not only does g_{-1} also has to be a Lie algebra homomorphism, but also the condition

$$\phi(X\rhd' Y) = (\psi X)\rhd(\phi Y),\qquad \forall\ X\in\mathfrak{g}', Y\in\mathfrak{h}'. \tag{B.0.2}$$

This is precisely the definition of an *elementary equivalence* of Lie algebra crossed-modules [124, 122].

As such, we may interpret elementary equivalence as an equivalence of the gauge-invariant data on the 2-gauge bundles $\mathcal{P},\mathcal{P}'$. The *Gerstenhaber Theorem A.2.1* then implies

Corollary B.0.1. *If the 2-gauge bundles $\mathcal{P},\mathcal{P}'$ exhibit distinct Postnikov classes $\kappa\neq\kappa'\in H^3(\mathfrak{n},V)$ as 2-curvature anomalies, then there does not exist an invertible 2-bundle homomorphism between them.*