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Five Hilbert Space Problems in Operator Algebras

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Abstract

A number of questions which have been solved over the years in the theory of single operators acting on Hilbert space have interesting analogues when recast in the setting of elements of C^*-algebras. We list five of these, as well as a number of "sub-problems" arising from them.

Keywords Nilpotents · Biquasitriangular · Algebraic elements · Commutators · Specht's theorem · Similarity orbits

Mathematics Subject Classification Primary 46C15; Secondary 47-02

Introduction

Those of us who grew up (at least in the mathematical sense) meandering through the green pleasant fields of single operator theory recall the years between 1960 and 1990 as a period of particularly intense activity, accompanied by exciting and important discoveries – including (but definitely not limited to) Lomonosov's Theorem [30], Voiculescu's non-commutative Weyl-von Neumann Theorem [50], Brown-Douglas-Fillmore (BDF) Theory [7], and the classification of the norm-closures of similarity-invariant sets by Apostol et al. [2]. Granted, Voiculescu's non-commutative Weyl-von Neumann Theorem is a result about C^*-algebras, while the BDF machinery was

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inspired in part by techniques from homology theory – but the former result is a crucial tool in the theory of approximation of Hilbert space operators, while the latter was borne of the desire to classify unitary orbits of normal elements of the Calkin algebra, a very "*single operator theoretic*" concept. That Banach algebra (and C^*-algebra) techniques should be used to resolve questions in operator theory is a concept so fundamental as to merit appearing as the title of the well-known monograph of Douglas [15].

In 1970, Paul Halmos published his famous *Ten problems in Hilbert space* [22], which inspired and helped focus a great deal of the research done in single operator theory for the subsequent twenty years (see also [23]). We modestly propose a list of five questions (plus a host of sub-questions) of our own, framing questions which appeared in Halmos' paper and elsewhere in the context of elements of C^*-algebras. We do not pretend that the questions we list below will shape the future of research in this area, though we do believe that they are non-trivial and interesting. We hope that at least some of the questions we ask will prove of interest to readers of the younger generation who may have more time on their hands than we have, and that they will lead to the development of techniques and solutions which will broaden our understanding of the internal structure of C^*-algebras. Of course, we particularly hope that the questions will appeal to those with an interest in single operator theory, and conversely, that they may generate a greater interest in single operator theory amongst those specialising in operator algebras. In the best of worlds, they would lead to even more questions and interplay between these two areas.

The choice of questions listed below is strictly personal: *we have chosen questions to which we would like to know the answer*.

Throughout this paper, $\mathcal{H}$ will denote an infinite-dimensional, complex, separable Hilbert space, $\mathcal{B}(\mathcal{H})$ will denote the set of bounded linear operators acting on $\mathcal{H}$, and $\mathcal{K}(\mathcal{H})$ will denote the closed, two-sided ideal of compact operators in $\mathcal{B}(\mathcal{H})$.

Most of the questions we pose may be asked in full generality in Banach algebras. Since this is not our area of specialty, we shall restrict our attention to (general) C^*-algebras, though in many cases a more useful starting point might be unital, simple C^*-algebras. An abundance of projections in the C^*-algebra $\mathbb{A}$ under investigation may often be useful as well (e.g. one might assume that $\mathbb{A}$ has real rank zero) and even supposing that $\mathbb{A}$ has stable rank one may be reasonable in some cases. In many, if not most instances, UHF C^*-algebras may serve as a prime candidate in which to first attempt a solution, though it may also be fruitful to consider the full panoply of "the extraordinary league of well-appreciated" C^*-algebras – AF C^*-algebras, Cuntz algebras, Bunce-Deddens algebras, irrational rotation algebras, group C^*-algebras, graph C^*-algebras, purely infinite C^*-algebras, etc. – a description of most of which may be found in the monograph of Davidson [9].

It might help to consider these problems in the setting of von Neumann algebras (e.g. a type II_1 or II_∞ factor), where the extra structure often present in these (e.g. the ubiquitousness of projections, polar decomposition, etc.) may provide extra tools. In short - there is no shortage of starting points for these journeys.

1 Limits of Nilpotents

The seventh of Halmos' problems from his 1970 paper [22] serves as the inspiration for our first question.

Recall that an element q of a Banach algebra $\mathcal{A}$ is said to be **quasinilpotent** if its spectral radius $\mathrm{spr}(q) = 0$, or equivalently, if its spectrum satisfies $\sigma(q) = \{0\}$. We say that $m \in \mathcal{A}$ is **nilpotent of index** $k \geq 1$ if $m^k = 0 \neq m^{k-1}$. Clearly every nilpotent element of $\mathcal{A}$ is quasinilpotent, though the converse typically fails. For example, the Volterra operator $Vf(x) := \int_x^1 f(t)dt$ is quasinilpotent but not nilpotent in the algebra of bounded linear operators acting on the Banach space $\mathcal{C}([0,1], \mathbb{C})$ of continuous, complex-valued functions on $[0, 1]$, equipped with the uniform norm. We denote by $\mathrm{NIL}(\mathcal{A})$ the set of nilpotent elements of $\mathcal{A}$.

Problem 7 of Halmos' paper asked: *is every quasinilpotent operator in $\mathcal{B}(\mathcal{H})$ a norm-limit of nilpotent operators?*

Even at the time of asking this question, Halmos acknowledged that it was in some sense "the wrong question" to ask.[1] Indeed, Kakutani [41, p. 282] had already exhibited an example of a non-quasinilpotent operator which is a limit of nilpotents (an operator now known as a *Kakutani shift*). As such, the "better" question to ask was: *which operators are limits of nilpotent operators?*

The question itself is interesting, but it is the scope and depth of the machinery that was developed to answer it that elevates it to one of extreme importance. It is beyond the reach of this article to give a complete history of the contributions which eventually led to the complete solution to this problem, and we shall constrain ourselves to a minimal set of salient observations.

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be **triangular** if there exists an orthonormal basis $\{e_n\}_{n=1}^\infty$ for $\mathcal{H}$ relative to which the operator-matrix $[T] = [t_{ij}]$ for T is upper-triangular; i.e. $t_{ij} := \langle Te_j, e_i\rangle = 0$ for all $i > j \geq 1$. We say that T is **quasitriangular** if $T = T_0 + K$, where $T_0 \in \mathcal{B}(\mathcal{H})$ is triangular and K is compact – we write $T \in$ (QT) – and that T is **biquasitriangular** if both T and T^* are quasitriangular, in which case we write $T \in$ (BQT). The orthonormal bases which "*quasitriangularise*" T and T^* need not be the same!

Let us recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be **semi-Fredholm** if its range is closed, and at least one of $\mathrm{NUL}\, T$ (the nullity of T) and $\mathrm{NUL}\, T^*$ (the nullity of T^*) is finite, in which case we define the **semi-Fredholm index**

$$\mathrm{IND}\, T := \mathrm{NUL}\, T - \mathrm{NUL}\, T^* \in \mathbb{Z} \cup \{-\infty, \infty\}.$$

(It is understood that $m - \infty := -\infty$ and $\infty - m := \infty$ for all $m \in \mathbb{N}$.) This generalises the notion of a **Fredholm operator**, which may be defined as a semi-Fredholm operator with finite index. Whereas an operator $R \in \mathcal{B}(\mathcal{H})$ is Fredholm if and only if its image $\pi(R)$ in the Calkin algebra under the canonical quotient map $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is invertible, it can be shown that an operator $T \in \mathcal{B}(\mathcal{H})$ is semi-Fredholm if and only if $\pi(T)$ is either left- or right-invertible (or both). The

[1] Halmos explains that his motivation in asking the question in this form was his desire to express all of his questions in such a way as to require a "yes" or "no" answer.

semi-Fredholm domain of an operator $T \in \mathcal{B}(\mathcal{H})$ is the set $\rho_{sF}(T) := \{\alpha \in \mathbb{C} : T - \alpha I$ is semi-Fredholm$\}$. We refer the reader to the monograph of Caradus et al. [8] for more information regarding semi-Fredholm operators.

It was Douglas and Pearcy [16] who first observed that the semi-Fredholm index provided an obstruction to membership in (QT); indeed, they observed that the unilateral forward shift S could not be a limit of triangular operators because $\text{IND}\, S := \text{NUL}\, S - \text{NUL}\, S^* = -1$. *A fortiori*, they observed that if there exists $\lambda \in \mathbb{C}$ such that $\text{IND}\,(T - \lambda I) < 0$, then $T \notin$ (QT). Shortly thereafter, Pearcy conjectured that index was the only obstruction to membership in (QT). He would eventually be proven correct by Apostol et al. [3] (see also [17]).

The relevance of quasitriangularity to Halmos' seventh problem is that every nilpotent operator is quasitriangular, and that the set (QT) of quasitriangular operators is norm-closed. Furthermore, the set $\text{NIL}(\mathcal{B}(\mathcal{H}))$ of nilpotent operators in $\mathcal{B}(\mathcal{H})$ is self-adjoint, and as a consequence we see that if $T \in \overline{\text{NIL}(\mathcal{B}(\mathcal{H}))}$, then $\text{IND}(T - \lambda) = 0$ whenever $T - \lambda I$ is semi-Fredholm.

Two other necessary conditions for membership in $\overline{\text{NIL}(\mathcal{B}(\mathcal{H}))}$ arise from the upper-semicontinuity of the spectrum and the fact that the invertible elements form an open set in any unital Banach algebra $\mathcal{A}$; if $t \in \overline{\text{NIL}(\mathcal{A})}$ and $\mathcal{K} \lhd \mathcal{A}$ is any closed, two-sided ideal of $\mathcal{A}$, then $\sigma_{\mathcal{A}/\mathcal{K}}(\pi_{\mathcal{K}}(t))$ is connected and contains the origin, where $\pi_{\mathcal{K}} : \mathcal{A} \to \mathcal{A}/\mathcal{K}$ denotes the canonical quotient map.

Six years after the question was originally raised by Halmos, as part of a seemingly infinite series of papers by an uncountable number of authors, Apostol, Foiaş and Voiculescu [4] proved that the above necessary conditions were also sufficient:

Theorem (Apostol, Foiaş and Voiculescu) *An operator $T \in \mathcal{B}(\mathcal{H})$ is a limit of nilpotent operators if and only if*

(i) *the spectrum $\sigma(T)$ of T is connected and contains the origin;*
(ii) *the essential spectrum $\sigma_e(T) := \sigma_{\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})}(\pi_{\mathcal{K}(\mathcal{H})}(T))$ of T is connected and contains the origin; and*
(iii) $\text{IND}\,(T - \lambda I) = 0$ *for all $\lambda \in \mathbb{C}$ for which $T - \lambda I$ is semi-Fredholm; equivalently, T is biquasitriangular.*

One of the key steps along the way to this general solution was earlier provided by D.A. Herrero [24], who proved that if $N \in \mathcal{B}(\mathcal{H})$ is a normal operator, then $N \in \overline{\text{NIL}(\mathcal{B}(\mathcal{H}))}$ if and only if $\sigma(N)$ is connected and contains the origin.

Of course, Halmos' question makes sense in any Banach algebra, and more specifically in any C^*-algebra, which leads us to ask the following questions. (Since the only quasinilpotent element of a commutative C^*-algebra is 0, the questions below are only interesting when $\mathbb{A}$ is non-commutative.)

Problem 1 Let $\mathbb{A}$ be a C^*-algebra. Characterise the norm-closure of the set $\text{NIL}(\mathbb{A})$ of nilpotent elements of $\mathbb{A}$.

It is to be expected that the solution to the problem will depend upon the nature of the C^*-algebra in question, meaning that it is in effect a collection of questions. As always, specific examples worth investigation might include UHF C^*-algebras, simple AF C^*-algebras, the Cuntz algebras $\mathcal{O}_n$, etc..

Naturally, Halmos' original question also makes sense in this context.

Problem 1.1 Let $\mathbb{A}$ be a C^*-algebra, and $q \in \mathbb{A}$ be quasinilpotent. Is q a limit of nilpotent elements of $\mathbb{A}$?

Phrased differently: does there exist a C^*-algebra $\mathbb{A}$ and a quasinilpotent element $q \in \mathbb{A}$ that is not a limit of nilpotent elements of $\mathbb{A}$? If so, can we characterise the C^*-algebras for which this can happen?

In the hope that Herrero's characterisation of normal operators lying in the closure of $\text{NIL}(\mathcal{B}(\mathcal{H}))$ has an extension to the C^*-algebra setting which proves as useful as the original result did in the Hilbert space setting, we are led to ask the following.

Problem 1.2 Let $\mathbb{A}$ be a C^*-algebra. Characterise the normal elements of $\mathbb{A}$ which lie in the closure of $\text{NIL}(\mathbb{A})$.

The presence of a (faithful) tracial state τ acting on the C^*-algebra makes life more interesting, and potentially more difficult. For example, if $N \in \mathcal{B}(\mathcal{H})$ is an hermitian operator whose spectrum is $[0, 1]$, then $N \in \overline{\text{NIL}(\mathcal{B}(\mathcal{H}))}$ by Herrero's result. If $\mathbb{A}$ is a UHF C^*-algebra with tracial state τ, for example, and $h = h^* \in \mathbb{A}$ satisfies $\sigma(h) = [0, 1]$, then – by the faithfulness of the trace – $\tau(h) > 0$, and so (by the continuity of τ) we find that $h \notin \overline{\text{NIL}(\mathbb{A})}$. It gets worse. If $h = h^*$, $\sigma(h) = [-1, 1]$ and even if $\tau(h) = 0$, then $\sigma(h^2) = [0, 1]$, and so $h^2 \notin \overline{\text{NIL}(\mathbb{A})}$, from which we again conclude that $h \notin \overline{\text{NIL}(\mathbb{A})}$. More generally, if $p \in \mathbb{C}[z]$ is a polynomials satisfying $p(0) = 0$ and if $n \in \mathbb{A}$ is normal, then $\tau(p(n)) \neq 0$ implies that $p(n) \notin \overline{\text{NIL}(\mathbb{A})}$, whence $n \notin \overline{\text{NIL}(\mathbb{A})}$.

By Gelfand Theory, the trace τ on $\mathbb{A}$ corresponds to a positive, regular Borel measure μ_τ on $C^*(n) \simeq \mathcal{C}(\sigma(n))$, in that for all $f \in \mathcal{C}(\sigma(n))$,

$$\tau(f(n)) = \int_{\sigma(n)} f(z) d\mu_\tau.$$

In effect, the question may be measure-theoretic in that we are looking for measures on $\sigma(n)$ which annihilate the polynomials which vanish at zero. This may not be the only property of the measure we require, but it does suggest that perhaps (an analogue of) a certain theorem of F. Riesz and M. Riesz [26] may have a role to play here. The interested reader should also see the paper of Skoufranis [45, Question 5.9].

Our next question is the restriction of the previous question to a specific C^*-algebra, but may prove to be the litmus test for any conjectures for more general C^*-algebras equipped with one or more tracial states.

Problem 1.3 Let $\mathbb{A}$ be a simple, unital AF C^*-algebra with a unique tracial state. Characterise the normal elements of $\mathbb{A}$ which lie in closure of $\text{NIL}(\mathbb{A})$.

Paul Skoufranis [46, Theorem 3.5] has shown that there exists a non-simple, AF-embeddable C^*-algebra $\mathbb{A}$ such that $\overline{\text{NIL}(\mathbb{A})}$ contains a positive element. Of course, such a C^*-algebra fails to admit a faithful tracial state.

At the other end of the spectrum (metaphorically speaking) are C^*-algebras which do not admit a trace. Again, we refer the reader to the same very interesting paper of

Skoufranis [46] who proved (Theorem 2.6 of that paper) that if $\mathbb{A}$ is unital, simple and purely infinite, then a normal element $n \in \mathbb{A}$ is a limit of nilpotent elements of $\mathbb{A}$ if and only if $\sigma(n)$ is connected and contains the origin, and $\lambda 1_{\mathbb{A}} - n$ lies in the connected component of the identity in the set of invertible elements of $\mathbb{A}$ for all $\lambda \in \mathbb{C}\backslash\sigma(n)$. The latter condition may be viewed as the C^*-analogue of the index obstruction observed in the $\mathcal{B}(\mathcal{H})$ setting.

There may prove to be more "room" to work in the setting of von Neumann algebras. Again, we refer to Skoufranis [45] for more information here. It would be interesting to know the answers to the following subquestions.

Problem 1.4 (a) Let $\mathfrak{M}$ be a type II_1 factor. Characterise the set $\overline{\mathrm{NIL}(\mathfrak{M})}$, as well as the set of normal elements of that set.

(b) Let $\mathfrak{N}$ be a type II_∞ factor. Characterise the set $\overline{\mathrm{NIL}(\mathfrak{M})}$, as well as the set of normal elements of that set.

2 Algebraic Elements of C^*-algebras

As mentioned above, the notion of biquasitriangularity plays a pivotal role in the characterisation of the closure of the set of nilpotent operators in $\mathcal{B}(\mathcal{H})$. The characterisation of biquasitriangularity in terms of index provides a simple and effective way of determining whether or not a given operator lies in the set (BQT). There is, however, a third characterisation, due to Voiculescu [49], which is easily expressed in a more general setting. We shall say that an element a of a Banach algebra $\mathcal{A}$ is **algebraic** if there exists a non-zero polynomial p such that $p(a) = 0$. We denote by $\mathrm{ALG}(\mathcal{A})$ the set of algebraic elements of $\mathcal{A}$.

Theorem (Voiculescu) *Let $\mathcal{H}$ be an infinite-dimensional, complex, separable Hilbert space. An operator $T \in \mathcal{B}(\mathcal{H})$ is biquasitriangular if and only if T is a limit of algebraic operators.*

This invites the question:

Problem 2 Let $\mathbb{A}$ be a unital C^*-algebra. Characterise the closure of the set $\mathrm{ALG}(\mathbb{A})$ of algebraic elements of $\mathbb{A}$.

Let $T \in \mathcal{B}(\mathcal{H})$ be arbitrary, and let $N \in \mathcal{B}(\mathcal{H})$ be a normal operator such that $\sigma(N) = \sigma_e(N) = \sigma(T)$. It is not hard to verify that the semi-Fredholm domain of $N \oplus T$ is exactly the common resolvent $\rho(T) = \rho(N)$ of T and N, and as a consequence, $(N \oplus T) - \lambda I$ is invertible (and thus has index zero) for all $\lambda \in \rho_{sF}(N \oplus T)$. In other words, $N \oplus T \in$ (BQT). By Voiculescu's Theorem above, $N \oplus T$ is a limit of algebraic operators.

Problem 2.1 Let $\mathbb{A}$ be a simple, unital C^*-algebra. Suppose that $t, n \in \mathbb{A}$, n is normal and $\sigma(n) \supseteq \sigma(t)$. Is $t \oplus n$ a limit of algebraic elements in $\mathbb{M}_2(\mathbb{A})$ (or, given $k \geq 3$ and a normal element $m \in \mathbb{M}_{k-1}(\mathbb{A})$, is $t \oplus m$ a limit of algebraic elements in $\mathbb{M}_k(\mathbb{A})$)?

(In the case where $\mathbb{A}$ is not simple, one may have to modify the above question to ensure that the spectrum of n should agree with the spectrum of its image in any quotient of $\mathbb{A}$ modulo a maximal ideal.)

3 Commutators in C^*-algebras

If $\mathcal{A}$ is any algebra and $a, b \in \mathcal{A}$, we shall refer to $[a, b] := ab - ba$ as the **commutator** of a and b. We denote the set of all commutators of elements of $\mathcal{A}$ by $\mathfrak{C}(\mathcal{A}) := \{[a, b] : a, b \in \mathcal{A}\}$.

Let $n \in \mathbb{N}$. It is a standard Theorem of Shoda [43] (see also [1]) that

$$\mathfrak{C}(\mathbb{M}_n(\mathbb{C})) = \{T \in \mathbb{M}_n(\mathbb{C}) : \text{Tr}(T) = 0\}.$$

Of course, $\mathbb{M}_n(\mathbb{C}) \simeq \mathcal{B}(\mathbb{C}^n)$. Let $\mathcal{H}$ be an infinite-dimensional, separable Hilbert space. We recall the following remarkable extension of Shoda's result by Brown and Pearcy [6].

Theorem (Brown and Pearcy) *If $\mathcal{H}$ is infinite-dimensional, separable Hilbert space. Then*

$$\mathfrak{C}(\mathcal{B}(\mathcal{H})) := \{T \in \mathcal{B}(\mathcal{H}) : T \notin \{\alpha I + K : 0 \neq \alpha \in \mathbb{C}, K \in \mathcal{K}(\mathcal{H})\}\}.$$

Once again, the necessity of this condition is relatively simple to demonstrate. A standard result in Banach algebra theory states that if $\mathcal{A}$ is a unital Banach algebra and $x, y \in \mathcal{A}$, then $\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$. Using this, we see that the equation $xy - yx = \alpha 1$ implies that $\sigma(xy) = \alpha + \sigma(yx)$, which contradicts the compactness of the spectrum of $\sigma(xy)$ if $\alpha \neq 0$.

In particular, if $T \in \mathfrak{C}(\mathcal{B}(\mathcal{H}))$ and $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the quotient map, then $\pi(T) \neq \alpha\pi(I)$ for any $\alpha \neq 0$, which implies that $T \notin \{\alpha I + K : 0 \neq \alpha \in \mathbb{C}, K \in \mathcal{K}(\mathcal{H})\}$. All of the work of Brown and Pearcy therefore goes into proving the sufficiency of this condition.

Problem 3 Let $\mathbb{A}$ be a C^*-algebra. Characterise $\mathfrak{C}(\mathbb{A}) := \{[a, b] : a, b \in \mathbb{A}\}$.

We stake no claim to authorship of this question, as it has been examined by a number of people in various specific C^*-algebras and algebras of (Hilbert and Banach space) operators. For example, Dykema and Skripka have investigated commutators in type II_1 factors; Dykema, Figiel, Weiss and Wodzicki have done a detailed study of commutators in operator ideals [18]; and Dykema and Krishnaswamy-Usha [19] have studied nilpotent compact operators as commutators of compact operators. Recently, Dosev [10] and Dosev and Johnson [11] have classified commutators in $\mathcal{B}(\ell_1)$ and in $\mathcal{B}(\ell_\infty)$ respectively (see also the work of Dosev et al. [12]).

Again, there is a dichotomy of strategies depending upon whether or not the algebra $\mathbb{A}$ admits a faithful tracial state τ. If it does, then clearly $\mathfrak{C}(\mathbb{A}) \in \ker\, \tau$. This is the case, therefore, when $\mathbb{A}$ is a UHF C^*-algebra. In that case, $\ker\, \tau$ has co-dimension 1, and it is not hard to show that every element of trace zero in $\mathbb{A}$ is a *limit* of commutators. (Indeed, if $z \in \ker\, \tau$, then $z = \lim_n a_n$, where $a_n \in \mathbb{A}_n \simeq \mathbb{M}_{k(n)}(\mathbb{C})$ for some $k(n) \in \mathbb{N}$. The continuity of the trace implies that $\alpha_n := \tau(z_n)$ converges to 0 as n tends to infinity, and thus $a_n - \alpha_n 1 \in \ker\, \tau$ converges to z.)

Furthemore, we have shown that every element of $\mathbb{A}$ is a *sum* of at most two commutators [32]. Despite this, the question of characterising the class $\mathfrak{C}(\mathbb{A})$ has eluded us. (That $\mathfrak{C}(\mathbb{A})$ consists of sums of two commutators in $\mathbb{A}$ happens in quite a few (simple) C^*-algebras. We direct the reader to [32, 33] for further details.) Also, P.W. Ng has shown that every trace zero element in the reduced C^*-algebra $C_r^*(\mathbb{F}_\infty)$ of the free group in infinitely many generators is a sum of three commutators [37], and that every trace zero element of the Jiang-Su algebra is a sum of at most five commutators [36].

The question of whether or not an operator in a C^*-algebra is a commutator may be easier to solve for certain specific subclasses of operators. Consider, for example, the following question.

Problem 3.1 Let $\mathbb{A}$ be a C^*-algebra. Which normal (or nilpotent) operators in $\mathbb{A}$ are commutators?

The next question stems from the seminal work of Fack [20] and Thomsen [48] on spans of commutators in C^*-algebras. Fack showed that if $\mathbb{A}$ is a simple, unital AF C^*-algebra with unique trace (or if $\mathbb{A}$ is a simple, unital, infinite C^*-algebra), then every trace zero element is a sum of no more than 14 commutators in $\mathbb{A}$, while Thomsen extended Fack's first theorem to certain unital, simple, homogeneous C^*-algebras (with a bound on the number of commutators required which depends upon the covering dimension of the spaces involved). As a consequence of their results, the linear span of the commutators in the C^*-algebras they consider is closed.

Problem 3.2 Characterise those C^*-algebras $\mathbb{A}$ for which the set $[\mathbb{A}, \mathbb{A}] := \operatorname{span} \mathfrak{C}(\mathbb{A})$ is closed.

If $T \in \mathbb{M}_n(\mathbb{C})$ has trace zero (equivalently, if $T \in \mathfrak{C}(\mathbb{M}_n(\mathbb{C}))$), then there exists an orthonormal basis $\{e_k\}_{k=1}^n$ relative to which the diagonal of the matrix of T is zero; i.e. $\langle Te_k, e_k\rangle = 0$, $1 \le k \le n$.

Problem 3.3 Let $a \in \mathfrak{C}(\mathbb{A})$. Does there exist a masa $\mathfrak{M}$ (which depends upon a) in $\mathbb{A}$ and a conditional expectation φ of $\mathbb{A}$ onto $\mathfrak{M}$ such that $\varphi(a) = 0$? If $\mathbb{A}$ admits a unique tracial state τ and $b \in \mathbb{A}$ satisfies $\tau(b) = 0$, do such a masa (again, dependent upon b) and conditional expectation exist? (In this last question, we do not assume that b is itself a commutator.)

There is an strong connection between commutators, nilpotents, idempotents and projections in C^*-algebras [33]. For example, using the fact that every trace-zero element of certain simple, unital C^*-algebras $\mathbb{A}$ of real rank zero is a sum of two commutators in the algebra, we were able to show [32] that every every element of $\mathbb{A}$ is a linear combination of at most 113 projections in $\mathbb{A}$. This estimate has virtually no chance of being sharp. Can we do better?

In his paper [5], Blackadar introduced several notions of comparability of projections in simple, unital C^*-algebras possessing a trace. To state our next question, we require the notion he refers to as the "*Fundamental Comparability Question 2*": a simple, unital C^*-algebra $\mathbb{A}$ is said to satisfy FCQ2 if for all projections $p, q \in \mathbb{A}$, we have that $p \prec q$ if $\tau(p) < \tau(q)$ for all traces τ on $\mathbb{A}$. As usual, $p \prec q$ means that there exists a partial isometry $v \in \mathbb{A}$ such that $p = v^*v$, and $q - vv^* > 0$.

Problem 3.4 Let $\mathbb{A}$ be a simple, unital C^*-algebra $\mathbb{A}$ of real rank zero, satisfying Blackadar's FCQ2 and admitting a unique tracial state. Find the minimum positive integer n such that every element of $\mathbb{A}$ may be written as a linear combination of at most n projections in $\mathbb{A}$.

We digress temporarily to mention that Paskiewicz [38] has shown that every self-adjoint element of $\mathcal{B}(\mathcal{H})$ is a linear combination of at most five projections, from which it trivially follows that every $T \in \mathcal{B}(\mathcal{H})$ is a linear combination of at most ten. It would be interesting to know what the minimum number of projections required for an arbitrary $T \in \mathcal{B}(\mathcal{H})$ setting really is.

4 Specht's Theorem in C^*-algebras

Let $n \geq 1$. The two most important equivalence relations on $\mathcal{B}(\mathbb{C}^n) \simeq \mathbb{M}_n(\mathbb{C})$ are *unitary equivalence* and *similarity*. Of course, two operators $A, B \in \mathcal{B}(\mathbb{C}^n)$ are *unitarily equivalent* (and we write $A \simeq B$), if there exists a unitary operator $U \in \mathcal{B}(\mathbb{C}^n)$ such that $B = U^*AU$, while A and B are *similar* (and we write $A \sim B$) if there exists an invertible operator $S \in \mathcal{B}(\mathbb{C}^n)$ such that $B = S^{-1}AS$.

A standard and elementary result from linear algebra shows that two matrices $A, B \in \mathcal{B}(\mathbb{C}^n)$ are similar if and only if they admit the same Jordan canonical form. Less well known, but equally important, is the following result of Specht [47]:

Theorem (Specht) *Two operators $A, B \in \mathcal{B}(\mathbb{C}^n)$ are unitarily equivalent if and only if*

$$\text{Tr}(w(A, A^*)) = \text{Tr}(w(B, B^*))$$

whenever $w(x, y)$ is a word in two non-commuting variables.

We shall refer to the "only if" half of this theorem as "*Specht's trace condition*". It is interesting to note that Pearcy [39] has shown that one need only consider words of length at most $2n^2$.

Observe that the group of unitary operators in $\mathcal{B}(\mathbb{C}^n)$ is compact, which is easily seen to imply that the **unitary orbit** $\mathcal{U}(T) := \{U^*TU : U \in \mathbb{M}_n(\mathbb{C}) \text{ unitary}\}$ of an element $T \in \mathbb{M}_n(\mathbb{C})$ is closed. If $\mathbb{A}$ is a C^*-algebra which admits a tracial state τ, and $w(x, y)$ is a word in two non-commuting variables, then a routine calculation shows that

$$\tau(w(a, a^*)) = \tau(w(b, b^*))$$

whenever b belongs to the *closure* of $\mathcal{U}(a) := \{u^*au : u \in \mathbb{A} \text{ unitary}\}$. In fact, the notion that $b \in \overline{\mathcal{U}(a)}$ defines an equivalence relation on $\mathbb{A}$ referred to as *approximate unitary equivalence* of a and b, and is denoted by writing $a \simeq_a b$.

In light of this, if one were to try to extend Specht's Theorem to more general C^*-algebras $\mathbb{A}$ (which admit a tracial state), then the best one could hope for is a result along the lines of: $a, b \in \mathbb{A}$ are *approximately* unitarily equivalent if and only if $\tau(w(a, a^*)) = \tau(w(b, b^*))$ for all words $w(x, y)$ in two non-commuting variables.

A natural first candidate to consider in extending Specht's Theorem to the setting of C^*-algebras is a UHF C^*-algebra. Since any such algebra $\mathbb{A} = \overline{\cup_n \mathbb{A}_n}$ is the norm-closure of an increasing union of algebras $\mathbb{A}_n$, each *-isomorphic to a full matrix algebra, it is clear that Specht's Theorem hold for pairs a, b in the dense subset of $\cup_n \mathbb{A}_n$ of $\mathbb{A}$.

Modulo a technical assumption (required to enable us to use a powerful result due to Schafhauser [42]), Y. Zhang and the present author [35] obtained a positive answer in the universal UHF C^*-algebra $\mathcal{Q}$. (The **universal** UHF C^*-algebra $\mathcal{Q}$ is the one whose supernatural number is divisible by any positive integer.)

Theorem (Marcoux and Zhang) *Let $\mathcal{Q}$ be the universal UHF C^*-algebra with tracial state τ, and let $a, b \in \mathcal{Q}$. Suppose furthermore that $C^*(a)$ satisfies the universal coefficient theorem* (UCT). *Then a and b are approximately unitarily equivalent if and only if*

$$\tau(w(a, a^*)) = \tau(w(b, b^*))$$

whenever $w(x, y)$ is a word in two non-commuting variables.

Perhaps surprisingly, there is a K-theoretic obstruction which prevents this generalisation of Specht's Theorem from holding in all UHF C^*-algebras. In fact, one can find a, b in the CAR algebra $\mathbb{M}_{2^\infty}$ such that $C^*(a)$ and $C^*(b)$ both satisfy the UCT, a and b satisfy Specht's trace condition, and yet a and b are *not* approximately unitarily equivalent in $\mathbb{M}_{2^\infty}$.

Hence we are left with the following question:

Problem 4 Determine necessary and sufficient conditions on a C^*-algebra $\mathbb{A}$ admitting a tracial state τ so that if $a, b \in \mathbb{A}$ satisfy $\tau(w(a, a^*)) = \tau(w(b, b^*))$ for all words $w(x, y)$ in two non-commuting variables, then a and b are approximately unitarily equivalent.

If $\mathbb{A}$ admits multiple tracial states, we require that Specht's trace condition should hold for each of them.

In the case where $\mathbb{A}$ is an arbitrary UHF C^*-algebra and $a \in \mathbb{A}$ is normal, the hypothesis that $C^*(a)$ satisfies the UCT is automatic, and the K-theoretic obstruction vanishes. Indeed, if $b \in \mathbb{A}$ and a, b satisfies Specht's trace condition, then $a \simeq_a b$.

Problem 4.1 Determine necessary and sufficient conditions on a C^*-algebra $\mathbb{A}$ admitting a tracial state τ so that if $m, n \in \mathbb{A}$ are *normal* and satisfy $\tau(w(m, m^*)) = \tau(w(n, n^*))$ for all words $w(x, y)$ in two non-commuting variables, then m and n are approximately unitarily equivalent.

In the paper [34] of Mastnak, Radjavi and the present author, it was shown that two matrices A and $B \in \mathbb{M}_n(\mathbb{C})$ are unitarily equivalent if and only if

$$\mathrm{Tr}(|p(A, A^*)|) = \mathrm{Tr}(|p(B, B^*)|)$$

for all *polynomials* p in two non-commuting variables. (That absolute values of words $w(x, y)$ in two non-commuting variables is not sufficient is easily seen by taking $A = I_n$ and B any other unitary operator.)

More generally, given a C^*-algebra $\mathbb{A}$ and $a, b \in \mathbb{A}$, we shall say that a and b satisfy the *approximate absolute value condition* (AAVC) if $|p(a, a^*)|$ is approximately unitarily equivalent to $|p(b, b^*)|$ for all polynomials p in two non-commuting variables. Proposition 4.3 of [35] asserts that if $\mathbb{A}$ admits a tracial state and $a, b \in \mathbb{A}$ satisfy the AAVC, then a and b satisfy Specht's tracial condition. In this sense, we may view the AAVC as an extension of Specht's tracial condition to C^*-algebras which do not necessarily admit a tracial state. This allows us to reformulate Problem 3 above to a much larger class of C^*-algebras:

Problem 4.2 Determine necessary and sufficient conditions on a C^*-algebra $\mathbb{A}$ such that if $a, b \in \mathbb{A}$ satisfy the AAVC, then a and b are approximately unitarily equivalent.

It is worth noting that two operators $A, B \in \mathcal{B}(\mathcal{H})$ satisfy the AAVC if and only if they are approximately unitarily equivalent [35, Theorem 4.7].

Problem 4.3 Determine necessary and sufficient conditions on a C^*-algebra $\mathbb{A}$ so that if $a, b \in \mathbb{A}$ satisfy the AAVC *and* a is normal, then a and b are approximately unitarily equivalent.

We conclude this section by mentioning that the question of extending Specht's Theorem to the von Neumann algebra setting has been considered before; for the sake of brevity, we cite only the papers of Pearcy [39] and of Pearcy and Ringrose [40].

5 Closures of Intermediate Similarity Orbits in C*-algebras

The monumental task of describing the closure of the similarity orbit $\mathcal{S}(T) := \{R^{-1}TR : R \in \mathcal{B}(\mathcal{H}) \text{ invertible}\}$ of (virtually) every Hilbert space operator T was undertaken by (amongst others) Apostol et al. [2], and in that volume they also explore the characterisation of other sets of operators invariant under similarity conjugation.

The closure of similarity-invariant subsets of $\mathcal{B}(\mathcal{H})$ is typically described in terms of spectral conditions (spectrum, essential spectrum, dimensions of Riesz subspaces, index), and in the infinite-dimensional setting, is usually far more tractable than a characterisation of the sets themselves. We are led to the following question.

Problem 5 Let $\mathbb{A}$ be a C^*-algebra and $a \in \mathbb{A}$. Characterise $\overline{\mathcal{S}(a)}$. More generally, if $\Omega \subseteq \mathbb{A}$ is a set which is invariant under conjugation by similarities, characterise $\overline{\Omega}$.

As in the $\mathcal{B}(\mathcal{H})$ setting, this lends itself to a multitude of subproblems.

Problem 5.1 Let $\mathbb{A}$ be a unital C^*-algebra. Characterise the norm-closure of each of the following sets:

(a) $K_\sigma := \{a \in \mathbb{A} : \sigma(a) = K\}$, where $K \subseteq \mathbb{C}$ is a fixed, non-empty compact set.
(b) $\mathfrak{C}(\mathbb{A}) := \{[a, b] := ab - ba : a, b \in \mathbb{A}\}$;
(c) $\mathfrak{C}_{\mathfrak{E}}(\mathbb{A}) := \{[e, f] := ef - fe : e, f \in \mathbb{A}, e = e^2, f = f^2\}$.

(d) $\text{NEG}(\mathbb{A}) := \{a \in \mathbb{A} : a \sim -a\}$.
(e) $\text{SIMNOR}(\mathbb{A}) := \{a \in \mathbb{A} : a \sim n$ for some normal element $n \in \mathbb{A}\}$.

The perspicacious reader (hopefully you) will have noticed that there is non-trivial intersection between this general problem and some of the problems we have mentioned earlier. In particular, by choosing $K_\sigma = \{0\}$ in part (a), we are looking for a characterisation of the norm-closure of the quasinilpotent elements of $\mathbb{A}$. The set $\mathfrak{C}_{\mathfrak{C}}(\mathbb{A})$ actually lies in $\text{NEG}(\mathbb{A})$, and one question is whether or not their closures are equal.

Our interest in $\text{SIMNOR}(\mathbb{A})$ stems from the fact that when $\mathbb{A} = \mathcal{B}(\mathcal{H})$, it is known [25] that the norm-closures of $\text{SIMNOR}(\mathcal{B}(\mathcal{H}))$ and $\text{ALG}(\mathcal{B}(\mathcal{H}))$ coincide. Is this a general phenomenon?

In connection with the question of determining the closure of the block-diagonal nilpotent operators (as first raised by L.R. Williams [51]), Herrero introduced a new orbit which he referred to as the $(\mathcal{U}+\mathcal{K})$-*orbit* of T, defined as:

$$(\mathcal{U}+\mathcal{K})(T) := \{R^{-1}TR : R = U + K \text{ invertible } U \in \mathcal{U}(\mathcal{H}),\ K \in \mathcal{K}(\mathcal{H})\}.$$

This provides us with a host of corresponding problems in the C^*-algebra setting.

Problem 5.2 Let $\mathbb{A}$ be a unital C^*-algebra and $\mathcal{J} \lhd \mathbb{A}$ be a closed, two-sided ideal of $\mathbb{A}$. Given $a \in \mathbb{A}$, characterise $\overline{(\mathcal{U}+\mathcal{J})(a)}$, where

$$(\mathcal{U}+\mathcal{J})(a) := \{r^{-1}ar : r \in \mathbb{A} \text{ invertible and } r = u + j,\ u \in \mathcal{U}(\mathbb{A}),\ j \in \mathcal{J}\}.$$

When $\mathcal{J} = \{0\}$, we are asking for a characterisation of the closure of the unitary orbit of a. At the other extreme, when $\mathcal{J} = \mathbb{A}$, we are asking for a characterisation of the closure of the similarity orbit of a. The $(\mathcal{U}+\mathcal{K})(T)$-orbit of $T \in \mathcal{B}(\mathcal{H})$ has been characterised in a small number of cases:

- T is normal [21];
- T is the unilateral shift [21];
- T is "*shift-like*", meaning that it has the same spectral picture as the shift [31];
- T is an "*essentially normal model*" [13, 14, 27–29, 52–54].

Each case mentioned above consists of essentially normal operators. The decision to restrict to this class results from the desire to take advantage of the fact that conjugating by an invertible operator of the form $R = U + K$ in $\mathcal{B}(\mathcal{H})$ translates to conjugating by the unitary $\pi(U)$ in the Calkin algebra, and unitary equivalence in the Calkin algebra is best understood for normal elements there.

This suggests that one might study the question:

Problem 5.3 Let $\mathbb{A}$ be a unital C^*-algebra and $n \in \mathbb{A}$ be normal. Characterise $\overline{(\mathcal{U}+\mathcal{J})(n)}$.

The work of Skoufranis [44] investigating the norm-closures of unitary and similarity orbits of normal operators in purely infinite C^*-algebras is relevant here. For example, Skoufranis characterises which normal operators sitting in a unital, simple, purely infinite C^*-algebra $\mathbb{A}$ lie in the norm-closure of the similarity orbit of a second normal operator in $\mathbb{A}$. Can one extend such a result to the setting of the more general orbits described above?

While the number of operators $T \in \mathcal{B}(\mathcal{H})$ for which $\overline{(\mathcal{U}+\mathcal{K})(T)}$ has been classified is rather small, the evidence nonetheless supports the conjecture that

$$\overline{(\mathcal{U}+\mathcal{K})(T)} = \overline{\mathcal{S}(T)} \cap \overline{\mathcal{U}(T)+\mathcal{K}(\mathcal{H})}.$$

We finish with the following problem.

Problem 5.4 Let $\mathbb{A}$ be a unital C^*-algebra. If $\mathcal{J} \lhd \mathbb{A}$ is a closed, two-sided ideal of $\mathbb{A}$ and $a \in \mathbb{A}$, is

$$\overline{(\mathcal{U}+\mathcal{J})(a)} = \overline{\mathcal{S}(a)} \cap \overline{\mathcal{U}(a)+\mathcal{J}}?$$

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References

1. Albert, A.A., Muckenhoupt, B.: On matrices of trace zero. Michigan Math. J. **3**, 1–3 (1957)
2. Apostol, C., Fialkow, L.A., Herrero, D.A., Voiculescu, D.: Approximation of Hilbert space operators II, volume 102 of Research Notes in Math. Pitman Advanced Publishing Program, Boston (1984)
3. Apostol, C., Foiaş, C., Voiculescu, D.: Some results on nonquasitriangular operators. IV. Rev. Roumaine Math. Pures Appl. **18**, 487–514 (1973)
4. Apostol, C., Foiaş, C., Voiculescu, D.: On the norm-closure of the nilpotents II. Rev. Roumaine Math. Pures Appl. **19**, 549–557 (1974)
5. Blackadar, B.E.: Comparison theory for simple C^*-algebras. In Operator algebras and applications, volume 1 of London Math. Soc. Lecture Note Ser., pp. 21–54. Cambridge University Press, Cambridge(1988)
6. Brown, A., Pearcy, C.: Structure of commutators of operators. Ann. Math. **82**, 112–127 (1965)
7. Brown, L.G., Douglas, R.G., Fillmore, P.A.: Unitary equivalence modulo the compact operators and extensions of C^*-algebras. In: Proceedings of a Conference on Operator Theory, Halifax, Nova Scotia, volume 23 of Lecture Notes in Math., pp. 58–128. Springer, Berlin (1973)
8. Caradus, S.R., Pfaffenberger, W.E., Yood, B.: Calkin algebras and algebras of operators on Banach spaces, volume 9 of Lecture Notes Pure Appl. Math. Marcel Dekker, Inc., New York (1974)
9. Davidson, K.R.: C^*-algebras by example, volume 6 of Fields Institute Monographs. Amer. Math. Soc., Providence, RI (1996)
10. Dosev, D.T.: Commutators on ℓ_1. J. Funct. Anal. **256**, 3490–3509 (2009)
11. Dosev, D.T., Johnson, W.B.: Commutators on ℓ_∞. Bull. London Math. Soc. **42**, 155–169 (2010)
12. Dosev, D.T., Johnson, W.B., Schechtman, G.: Commutators on l_p, $1 \le p < \infty$. J. Am. Math. Soc. **26**, 101–127 (2013)
13. Dostál, M.: Closures of $(\mathcal{U}+\mathcal{K})$-orbits of essentially normal models. PhD thesis, University of Alberta (1998)
14. Dostál, M.: $(\mathcal{U}+\mathcal{K})$-orbits, a block tridiagonal decomposition technique and a model with multiply connected spectrum. J. Oper. Th. **46**, 491–516 (2001)
15. Douglas, R.G.: Banach Algebra Techniques in Operator Theory. Academic Press, New York (1972)
16. Douglas, R.G., Pearcy, C.M.: A note on quasitriangular operators. Duke Math. J. **37**, 177–188 (1970)
17. Douglas, R.G., Pearcy, C.M.: Invariant subspaces of non-quasitriangular operators. Proc. Conf. Operator Theory **345**, 13–57 (1973)
18. Dykema, K., Figiel, T., Weiss, G., Wodzicki, M.: Commutator structure of operator ideals. Adv. Math. **185**, 1–79 (2002)

19. Dykema, K., Krishnaswamy-Usha, M.: Nilpotent elements of operator ideals as single commutators. Proc. Am. Math. Soc. **146**, 3031–3037 (2018)
20. Fack, Th.: Finite sums of commutators in C^*-algebras. Ann. Inst. Fourier, Grenoble **32**, 129–137 (1982)
21. Guinand, P.S., Marcoux, L.W.: Between the unitary and similarity orbits of normal operators. Pac. J. Math. **159**, 299–335 (1993)
22. Halmos, P.R.: Ten problems in Hilbert space. Bull. Am. Math. J. **76**, 887–933 (1970)
23. Halmos, P.R.: Ten year in Hilbert space. Integral Equ. Oper. Theory **2**, 529–564 (1979)
24. Herrero, D.A.: Normal limits of nilpotent operators. Indiana Univ. Math. J. **23**, 1097–1108 (1973/74)
25. Herrero, D.A.: Approximation of Hilbert space operators I, volume 224 of Pitman Research Notes in Math. Longman Scientific and Technical, Harlow, New York, second edition (1989)
26. Hoffman, K.: Banach spaces of analytic functions. Reprint of the 1962 original. Dover Publications, Inc., New York (1988)
27. Ji, Y., Jiang, C., Wang, Z.: The $(\mathcal{U}+\mathcal{K})$-orbit of essentially normal operators and compact perturbation of strongly irreducible operators. In: Functional Analysis in China, volume 356 of Mathematics and its applications, pages 307–314, Dordrecht-Boston, 1996. Kluwer Academic Publishers
28. Ji, Y., Jiang, C., Wang, Z.: The $(\mathcal{U}+\mathcal{K})$-orbit of essentially normal operators and compact perturbations of strongly irreducible operators. Chinese Ann. Math. Ser. B **21**, 237–248 (2001)
29. Ji, Y., Li, J.: The quasiapproximate $(\mathcal{U}+\mathcal{K})$-invariants of essentially normal operators. Integ. Equ. Oper. Th. **50**, 255–278 (2004)
30. Lomonosov, V.J.: Invariant subspaces for operators commuting with compact operators (Russian). Funkcional. Anal. i. Priložen **7**, 55–56 (1973)
31. Marcoux, L.W.: The closure of the $(\mathcal{U}+\mathcal{K})$-orbit of shift-like operators. Indiana Univ. Math. J. **41**, 1211–1223 (1992)
32. Marcoux, L.W.: Sums of small numbers of commutators. J. Oper. Th. **56**, 111–142 (2006)
33. Marcoux, L.W.: Projections, commutators and lie ideals in C^*-algebras. Math. Proc. R. Irish Acad. **110A**, 31–55 (2010)
34. Marcoux, L.W., Mastnak, M., Radjavi, H.: An approximate, multivariable version of Specht's theorem. Linear Mutlilinear Alg. **55**, 159–173 (2007)
35. Marcoux, L.W., Zhang, Y.: On Specht's Theorem in UHF C^*-algebras. J. Funct. Anal. **280**, 28 (2021)
36. Ng, P.W.: Commutators in the Jiang-Su algebra. Int. J. Math. **23**, 29 (2012)
37. Ng, P.W.: Commutators in $C^*_r(\mathbb{F}_\infty)$. Houston J. Math. **40**, 421–446 (2014)
38. Paszkiewicz, A.: Any selfadjoint operator is a finite linear combination of projectors. Bull. Acad. Polon. Sci. Sér. Sci. Math. **7–8**, 337–345 (1981)
39. Pearcy, C.: A complete set of unitary invariants for operators generating finite W^*-algebras of Type I. Pac. J. Math. **12**, 1405–1416 (1962)
40. Pearcy, C., Ringrose, J.R.: Trace-preserving isomorphisms in finite operator algebras. Am. J. Math. **90**, 444–455 (1968)
41. Rickart, C.E.: General theory of Banach algebras. Robert Krieger Publishing Co. Inc., Huntington, N.Y. (1974)
42. Schafhauser, C.: Subalgebras of simple AF-algebras. Ann. Math. **192**, 309–352 (2020)
43. Shoda, K.: Einige Sätze über Matrizen. Jpn. J. Math. **13**, 361–365 (1936)
44. Skoufranis, P.: Closed unitary and similarity orbits of normal operators in purely infinite C^*-algebras. J. Funct. Anal. **265**, 474–506 (2013)
45. Skoufranis, P.: Normal limits of nilpotent operators in von Neumann algebras. Integr. Equ. Oper. Theory **77**, 407–439 (2013)
46. Skoufranis, P.: Normal limits of nilpotent operators in C^*-algebras. J. Oper. Th. **72**, 135–158 (2014)
47. Specht, W.: Zur Theorie der Matrizen. II. Jahresber. Dtsch. Math-Ver. **50**, 19–23 (1940)
48. Thomsen, K.: Finite sums and products of commutators in inductive limit C^*-algebras. Ann. Inst. Fourier, Grenoble **43**, 225–249 (1993)
49. Voiculescu, D.: Norm-limits of algebraic operators. Rev. Roumaine Math. Pures Appl. **19**, 371–378 (1974)
50. Voiculescu, D.: A non-commutative Weyl-von Neumann theorem. Rev. Roumaine Math. Pures Appl. **21**, 97–113 (1976)
51. Williams, L.R.: On quasisimilarity of operators on Hilbert space. PhD thesis, University of Michigan (1976)

52. Zhai, F.H.: The closures of $(\mathcal{U}+\mathcal{K})$-orbits of class essentially normal operators. Houston J. Math. **30**, 1177–1194 (2004)
53. Zhai, F.H., Guo, X.Z.: The closures of $(\mathcal{U}+\mathcal{K})$-orbits of essentially normal triangular operator models. Sci China Math **53**, 1045–1066 (2010)
54. Zhai, F.H., Zhao, J.: A new characterization of the closure of the $(\mathcal{U}+\mathcal{K})$-orbit of certain essentially normal operators. Oper. Matrices **2**, 455–463 (2008)