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On *-similarity in C^*-algebras

by

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Abstract. Two subsets $\mathcal{X}$ and $\mathcal{Y}$ of a unital C^*-algebra $\mathcal{A}$ are said to be **-similar via* $s \in \mathcal{A}^{-1}$ if $\mathcal{Y} = s^{-1}\mathcal{X}s$ and $\mathcal{Y}^* = s^{-1}\mathcal{X}^*s$. We show that this relation imposes a certain structure on the sets $\mathcal{X}$ and $\mathcal{Y}$, and that under certain natural conditions (for example, if $\mathcal{X}$ is bounded), *-similar sets must be unitarily equivalent. As a consequence of our main results, we present a generalized version of a well-known theorem of W. Specht.

1. Introduction

1.1. Let $\mathcal{H}$ be a finite- or infinite-dimensional complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators acting on $\mathcal{H}$. A standard fact is that if $A, B \in \mathcal{B}(\mathcal{H})$ are *normal* and A is *similar* to B, i.e. there exists an invertible operator $S \in \mathcal{B}(\mathcal{H})$ such that $B = S^{-1}AS$, then B is unitarily equivalent to A, i.e. S may be taken to be unitary. (See Proposition 1.2 below.) We are interested in extensions of this fact to *sets* of (not necessarily normal) operators.

More precisely, let $\mathcal{X}$ and $\mathcal{Y}$ be two subsets of $\mathcal{B}(\mathcal{H})$ that are (simultaneously) similar, i.e. there exists an invertible operator $S \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{Y} = \{S^{-1}XS : X \in \mathcal{X}\}$.

Under what conditions can S be replaced by a unitary operator U? Clearly, when this is the case, the same operator U will also implement the unitary equivalence of the sets $\mathcal{X}^* = \{X^* : X \in \mathcal{X}\}$ and $\mathcal{Y}^* = \{Y^* : Y \in \mathcal{Y}\}$. In light of this, we investigate the case where the operator S which implements the similarity of $\mathcal{X}$ and $\mathcal{Y}$ coincides with that which implements the similarity of $\mathcal{X}^*$ and $\mathcal{Y}^*$. We refer to this concept as **-similarity* of $\mathcal{X}$ and $\mathcal{Y}$. We show that under certain additional hypotheses (e.g. when $\mathcal{X}$ and $\mathcal{Y}$ are

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bounded), *-similarity of $\mathcal{X}$ and $\mathcal{Y}$ is sufficient for unitary equivalence of $\mathcal{X}$ and $\mathcal{Y}$.

As a consequence of our results, we are able to generalize a theorem of Specht regarding unitary equivalence of two $n \times n$ complex matrices, namely that $A, B \in \mathbb{M}_n(\mathbb{C})$ are unitarily equivalent if and only if

$$\operatorname{tr}(w(A, A^*)) = \operatorname{tr}(w(B, B^*))$$

for all words w in two non-commuting variables. The generalization (Theorem 2.21) extends this to arbitrary families $\mathcal{X} = \{X_\lambda\}_\lambda$ and $\mathcal{Y} = \{Y_\lambda\}_\lambda$ in $\mathbb{M}_n(\mathbb{C})$, by requiring that

$$\operatorname{tr}(w(X_{\lambda_1}, \dots, X_{\lambda_m}, X^*_{\lambda_1}, \dots, X^*_{\lambda_m})) = \operatorname{tr}(w(Y_{\lambda_1}, \dots, Y_{\lambda_m}, Y^*_{\lambda_1}, \dots, Y^*_{\lambda_m}))$$

for all words w in $2m$ non-commuting variables.

We state our main results in the context of C^*-algebras.

The following result appears as an exercise in textbooks on operator theory (see, for example, [H82, Problem 192]).

1.2. PROPOSITION. *Let $\mathcal{A}$ be a unital C^*-algebra and $m, n \in \mathcal{A}$ be normal elements. Suppose that $s \in \mathcal{A}$ is invertible with polar decomposition $s = u|s|$. If $m = s^{-1}ns$, then $m = u^*nu$.*

1.3. In this paper we shall examine to what extent Proposition 1.2 can be generalized. Given a subset $\mathcal{X}$ of a unital C^*-algebra $\mathcal{A}$ and an element s in $\mathcal{A}^{-1}$, the invertible group of $\mathcal{A}$, we note that if $\mathcal{Y} := s^{-1}\mathcal{X}s$ is similar to $\mathcal{X}$, then obviously $\mathcal{Y}^*$ is similar to $\mathcal{X}^*$, as $\mathcal{Y}^* = s^*\mathcal{X}^*(s^*)^{-1}$.

We introduce a stronger form of similarity, which we call **-similarity*, that asks that $\mathcal{Y} = s^{-1}\mathcal{X}s$ and $\mathcal{Y}^* = s^{-1}\mathcal{X}^*s$, and we investigate the consequences of this relation on the structure of $\mathcal{X}$ and $\mathcal{Y}$.

2. *-similarity.

Throughout the remainder of this paper, $\mathcal{A}$ will denote a unital C^*-algebra. By $\mathcal{A}^{-1}$ we denote the set of invertible elements in $\mathcal{A}$. We begin with the following definition.

2.1. DEFINITION. Let $\emptyset \neq \mathcal{X}, \mathcal{Y} \subseteq \mathcal{A}$. We say that $\mathcal{X}$ and $\mathcal{Y}$ are **-similar via* $s \in \mathcal{A}^{-1}$ if $\mathcal{Y} = s^{-1}\mathcal{X}s := \{s^{-1}xs : x \in \mathcal{X}\}$ and

$$\mathcal{Y}^* = \{y^* : y \in \mathcal{Y}\} = s^{-1}\mathcal{X}^*s := \{s^{-1}x^*s : x \in \mathcal{X}\}.$$

It is clear that if $\mathcal{X}$ and $\mathcal{Y}$ are selfadjoint sets, then *-similarity of $\mathcal{X}$ and $\mathcal{Y}$ coincides with similarity.

2.2. NOTATION. When considering *-similar subsets $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{A}$, unless explicitly stated otherwise, we shall assume that the similarity which implements the *-similarity is denoted by s, and has polar decomposition $s = u|s|$. We shall also assume that $\mathcal{X} = \{x_\lambda : \lambda \in \Lambda\}$ is indexed by a non-empty set Λ and that $\mathcal{Y} = \{y_\lambda : \lambda \in \Lambda\}$, where $y_\lambda = s^{-1}x_\lambda s$ for all λ.

The condition $\mathcal{Y}^* = s^{-1}\mathcal{X}^*s$ implies that there exists a bijection $\theta : \Lambda \to \Lambda$ such that $y_\lambda^* = s^{-1}x_{\theta(\lambda)}^*s$ for all $\lambda \in \Lambda$.

We emphasize that if $y_\lambda = s^{-1}x_\lambda s$ for some $\lambda \in \Lambda$, then there is no reason why y_λ^* should equal $s^{-1}x_\lambda^*s$.

2.3. Definition. We say that two non-empty sets $\mathcal{X}$ and $\mathcal{Y} \subseteq \mathcal{A}$ are *strongly* **-similar via* $s \in \mathcal{A}^{-1}$ if they are *-similar via s, and if $y = s^{-1}xs$ implies that $y^* = s^{-1}x^*s$ for all $x \in \mathcal{X}$.

In other words, $\mathcal{X}$ and $\mathcal{Y}$ are strongly *-similar via s if the corresponding bijection $\theta : \Lambda \to \Lambda$ above is the identity map.

2.4. Proposition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be non-empty subsets of $\mathcal{A}$ and suppose that $\mathcal{X}$ is strongly *-similar to $\mathcal{Y}$ via $s \in \mathcal{A}^{-1}$. Then $\mathcal{X}$ and $\mathcal{Y}$ are unitarily equivalent.*

Proof. Write $\mathcal{X} = \{x_\lambda : \lambda \in \Lambda\}$, $\mathcal{Y} = \{y_\lambda : \lambda \in \Lambda\}$ and $s = u|s|$ as above. The hypothesis that $y_\lambda = s^{-1}x_\lambda s$ and $y_\lambda^* = s^{-1}x_\lambda^*s$ for each λ implies that

$$\operatorname{Re} y_\lambda = \frac{y_\lambda + y_\lambda^*}{2} = s^{-1}\frac{x_\lambda + x_\lambda^*}{2}s = s^{-1}\operatorname{Re} x_\lambda\, s,$$

and similarly

$$\operatorname{Im} y_\lambda = s^{-1}\operatorname{Im} x_\lambda\, s.$$

By Proposition 1.2, $\operatorname{Re} y_\lambda = u^*\operatorname{Re} x_\lambda\, u$ and $\operatorname{Im} y = u^*\operatorname{Im} x_\lambda\, u$, whence

$$y_\lambda = u^*x_\lambda u \quad \text{for all } \lambda \in \Lambda.$$

That is, $\mathcal{Y} = u^*\mathcal{X}u$. ∎

Again, it is worth noting that the unitary operator that implements the unitary equivalence of $\mathcal{X}$ and $\mathcal{Y}$ is the unitary u arising from the polar decomposition of $s = u|s|$.

2.5. We next establish an interesting and useful structure theorem for *-similar sets. For $x \in \mathcal{A}$ and $0 \le k \in \mathcal{A}^{-1}$, we define the (two-sided) *orbit* of x under k to be

$$\mathcal{O}(x) := \{k^{-m}xk^m : m \in \mathbb{Z}\}.$$

Clearly, any two elements of $\mathcal{O}(x)$ are similar to each other in $\mathcal{A}$.

2.6. Proposition. *Let $\mathcal{X}, \mathcal{Y}$ be non-empty subsets of $\mathcal{A}$ and suppose that there exists an invertible operator $s = u|s| \in \mathcal{B}(\mathcal{H})$ such that*

- *$\mathcal{Y} = s^{-1}\mathcal{X}s$, and*
- *$\mathcal{Y}^* = s^{-1}\mathcal{X}^*s$.*

Let $k := ss^*$. Then the orbits $\mathcal{O}(x)$ for $x \in \mathcal{X}$ are each contained in $\mathcal{X}$ and they partition $\mathcal{X}$, i.e.

$$\mathcal{X} = \dot{\bigcup_{x\in\mathcal{X}}} \mathcal{O}(x).$$

Furthermore, the sets $\mathcal{Q}(x) := s^{-1}\mathcal{O}(x)s$ *for* $x \in \mathcal{X}$ *partition* $\mathcal{Y}$*:*

$$\mathcal{Y} := \dot{\bigcup_{x\in\mathcal{X}}} \mathcal{Q}(x).$$

Finally, $\mathcal{Q}(x)^* = s^{-1}\mathcal{O}(x)^*s$ *for all* $x \in \mathcal{X}$*, so that each* $\mathcal{O}(x)$ *is* **-similar to* $\mathcal{Q}(x)$ *via* s*.*

Proof. Fix $x_0 \in \mathcal{X}$. Set $y_0 := s^{-1}x_0 s \in \mathcal{Y}$, and choose $x_1 \in \mathcal{X}$ such that $y_0^* = s^{-1}x_1^* s$. Equivalently, $sy_0 = x_0 s$ and $y_0 s^* = s^* x_1$. Let $k = ss^*$.

It easily follows that

$$kx_1 = s(s^*x_1) = s(y_0 s^*) = (sy_0)s^* = (x_0 s)s^* = x_0 k.$$

As s is invertible, so is k. Hence

$$x_1 = k^{-1}x_0 k.$$

By applying the same procedure, setting $y_1 := s^{-1}x_1 s \in \mathcal{Y}$, we can find $x_2 \in \mathcal{X}$ such that $y_1^* = s^{-1}x_2^* s$, and the above computation shows that

$$x_2 = k^{-1}x_1 k.$$

More generally, for each $m \geq 1$, having chosen $x_0, x_1, \ldots, x_m$ and $y_k = s^{-1}x_k s$, $0 \leq k \leq m$, we can find $x_{m+1} \in \mathcal{X}$ for which $y_m^* = s^{-1}x_{m+1}^* s$, and

$$x_{m+1} = k^{-1}x_m k.$$

This process is reversible as well. If we set $y_{-1}^* := s^{-1}x_0^* s$, then $y_{-1}^* \in \mathcal{Y}^*$, and so there exists $x_{-1} \in \mathcal{X}$ such that $y_{-1} = s^{-1}x_{-1}s$. Computing as before, we find that

$$x_0 = k^{-1}x_{-1}k,$$

and indeed for all $m \leq 0$, we have

$$x_m = k^{-1}x_{m-1}k.$$

We have shown that the orbit $\mathcal{O}(x_0)$ of each $x_0 \in \mathcal{X}$ lies in $\mathcal{X}$.

If $z \in \mathcal{X}$ and $\mathcal{O}(z) \cap \mathcal{O}(x_0) \neq \emptyset$, then it is routine to verify that $\mathcal{O}(z) = \mathcal{O}(x_0)$ and thus these orbits partition $\mathcal{X}$.

Moreover, given an orbit $\mathcal{O}(x_0) := \{x_m := k^{-m}x_0 k^m : m \in \mathbb{Z}\}$, set $y_m := s^{-1}x_m s$ for $m \in \mathbb{Z}$. Observe that $\mathcal{Q}(x_0) = \{y_m : m \in \mathbb{Z}\}$ and that

$$\begin{aligned} y_m^* &= s^* x_m^* (s^{-1})^* = s^*(ss^*)^m x_0^* (ss^*)^{-m}(s^{-1})^* \\ &= s^{-1}(ss^*)^{m+1}x_0^*(ss^*)^{-(m+1)}s = s^{-1}x_{m+1}^* s, \quad m \in \mathbb{Z}. \end{aligned}$$

From this it follows that $\mathcal{Q}(x)^* = (s^{-1}\mathcal{O}(x_0)s)^* = s^{-1}\mathcal{O}(x_0^*)s = s^{-1}\mathcal{O}(x_0)^*s$, so that $\mathcal{Q}(x_0)$ and $\mathcal{O}(x_0)$ are *-similar via s. ∎

For a selfadjoint operator $X \in \mathcal{B}(\mathcal{H})$ and a Borel set $\Omega \subseteq \mathbb{R}$, we write $E_\Omega(X)$ to denote the spectral projection for X corresponding to Ω.

2.7. Lemma. *Let $0 \leq K \in \mathcal{B}(\mathcal{H})$ be an invertible operator, $X \in \mathcal{B}(\mathcal{H})$ and suppose that the orbit $\mathcal{O}(X) := \{K^{-m} X K^m : m \in \mathbb{Z}\}$ of X under K is bounded. Then X commutes with K, and $\mathcal{O}(X) = \{X\}$.*

Proof. Once we show that X commutes with K, the fact that $\mathcal{O}(X) = \{X\}$ is obvious.

We shall prove the contrapositive of our claim. Suppose that X does not commute with K. Then there must exist $\alpha > 0$ and a spectral projection $E_{[0,\alpha]}(K)$ for K corresponding to the interval $[0, \alpha]$ which does not commute with X.

Write $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1 = E_{[0,\alpha]}(K)\mathcal{H}$, and $\mathcal{H}_2 = \mathcal{H}_1^{\perp}$. Relative to this decomposition, we may write

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \quad \text{and} \quad E_{[0,\alpha]}(K) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

The fact that X does not commute with $E_{[0,\alpha]}(K)$ means that either $X_2 \neq 0$ or $X_3 \neq 0$.

Suppose for example that X_2 is non-zero. For each $n \geq 1$, set $P_n := E_{[\alpha+1/n, \|K\|]}(K)\mathcal{H}$. Then $(P_n)_{n=1}^{\infty}$ is an increasing sequence of orthogonal projections converging strongly to $E_{(\alpha,\|K\|]}(K)$. Since $X_2 = X_2 E_{(\alpha,\|K\|]}(K)$, we see that $(X_2 P_n)_{n=1}^{\infty}$ converges strongly to X_2.

Since $X_2 \neq 0$, there must exist $n_0 \geq 1$ such that $X_2 P_{n_0} \neq 0$. Consider next the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus (\mathcal{H}_2 \ominus P_{n_0}\mathcal{H}_2) \oplus P_{n_0}\mathcal{H}_2$, and write

$$X = \begin{bmatrix} X_1 & X_{21} & X_{22} \\ X_{31} & X_{41} & X_{42} \\ X_{32} & X_{43} & X_{44} \end{bmatrix}.$$

Observe that relative to this decomposition, $K = K_1 \oplus K_2 \oplus K_3$, where $K_1 = K E_{[0,\alpha]}(K)$ has spectrum in $[0, \alpha]$, K_2 has spectrum in $[\alpha, \alpha + 1/n_0]$, and $K_3 = K P_{n_0}$ has spectrum in $[\alpha + 1/n_0, \|K\|]$.

Note that

$$K^{-m} X K^m = \begin{bmatrix} * & * & K_1^{-m} X_{22} K_3^m \\ * & * & * \\ * & * & * \end{bmatrix}.$$

(The starred entries are irrelevant here.) Now $X_{22} = X_2 P_{n_0} \neq 0$, and we are multiplying it on the left by the *inverse* of a positive operator bounded above by $\alpha^m I$, and multiplying it on the right by a positive operator bounded below by $(\alpha + 1/n_0)^m I$. It follows that

$$\lim_{m \to \infty} \|K_1^{-m} X_{22} K_3^m\| = \infty,$$

whence $\mathcal{O}(X)$ is not bounded in norm. This completes the proof of the contrapositive of our statement.

The proof for $X_3 \neq 0$ is similar. ∎

2.8. REMARK. We point out that there is a rather simpler proof of this result when there exist $m < n \in \mathbb{Z}$ such that

$$K^{-m} X K^m = K^{-n} X K^n.$$

Then $K^{n-m} X = X K^{n-m}$, so that X commutes with the positive operator K^{n-m}, so by the functional calculus, X commutes with K.

2.9. COROLLARY. *Let $0 \leq k \in \mathcal{A}^{-1}$, $x \in \mathcal{A}$ and suppose that the orbit $\mathcal{O}(x)$ of x under k is bounded. Then x commutes with k, and $\mathcal{O}(x) = \{x\}$.*

Proof. Let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a non-degenerate, isometric representation of $\mathcal{A}$, $X = \pi(x)$ and $K = \pi(k)$. The result now follows from Lemma 2.7. ∎

2.10. THEOREM. *Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{X}, \mathcal{Y}$ be non-empty subsets of $\mathcal{A}$. Suppose that there exists an invertible element $s = u|s| \in \mathcal{A}$ such that*

- $\mathcal{Y} = s^{-1} \mathcal{X} s$, *and*
- $\mathcal{Y}^* = s^{-1} \mathcal{X}^* s$.

If $\mathcal{X}$ is bounded, then $\mathcal{Y} = u^* \mathcal{X} u$. In particular, if $\mathcal{X}$ (and therefore $\mathcal{Y}$) is finite, then $\mathcal{Y} = u^* \mathcal{X} u$.

Proof. By Proposition 2.6, we may partition $\mathcal{X}$ as the disjoint union of distinct orbits $\mathcal{O}(x) = \{k^{-m} x k^m : m \in \mathbb{Z}\}$, where $k = ss^*$. Moreover, this partitions $\mathcal{Y}$ as a disjoint union of the corresponding sets $\mathcal{Q}(x) = s^{-1} \mathcal{O}(x) s$, with $\mathcal{Q}(x)^* = s^{-1} \mathcal{O}(x)^* s$.

By Lemma 2.7, the boundedness of $\mathcal{X}$ and therefore of each orbit $\mathcal{O}(x)$ implies that each $\mathcal{O}(x) = \{x\}$ is a singleton set.

In light of Proposition 2.6, we see that $\mathcal{X}$ and $\mathcal{Y}$ must be strongly *-similar, and thus by Proposition 2.4, $\mathcal{Y} = u^* \mathcal{X} u$. ∎

2.11. EXAMPLE. The hypothesis that $\mathcal{X}$ be bounded cannot, in general, be removed, *even when $\mathcal{X}$ is a semigroup.* For example, consider

$$\mathcal{X} = \left\{ \begin{bmatrix} 3^m & 0 \\ 0 & 3^{-m} \end{bmatrix}, \begin{bmatrix} 0 & 3^n \\ 3^{-n} & 0 \end{bmatrix} : m, n \in \mathbb{Z} \right\} \subseteq \mathbb{M}_2(\mathbb{C}) = \mathcal{B}(\mathbb{C}^2).$$

Set

$$S = \begin{bmatrix} 3^{-1/4} & 0 \\ 0 & 3^{1/4} \end{bmatrix}.$$

Then

$$\mathcal{Y} = S^{-1}\mathcal{X}S = \left\{ \begin{bmatrix} 3^m & 0 \\ 0 & 3^{-m} \end{bmatrix}, \begin{bmatrix} 0 & 3^{n+1/2} \\ 3^{-n-1/2} & 0 \end{bmatrix} : m, n \in \mathbb{Z} \right\}.$$

It is routine to verify that $\mathcal{X} = \mathcal{X}^*$ and $\mathcal{Y} = \mathcal{Y}^*$, so that $\mathcal{X}$ and $\mathcal{Y}$ are *-similar groups.

Nevertheless, that $Y := \left[\begin{smallmatrix} 0 & \sqrt{3} \\ 1/\sqrt{3} & 0 \end{smallmatrix}\right] \in \mathcal{Y}$, and $\|Y\| = \sqrt{3}$, while $X \in \mathcal{X}$ implies that $\|X\| = 3^k$ for some $0 \le k \in \mathbb{Z}$. Thus $\mathcal{Y}$ cannot be unitarily equivalent to $\mathcal{X}$.

In fact, if we set $\mathcal{X}_h = \mathbb{C}\mathcal{X}$ and $\mathcal{Y}_h = \mathbb{C}\mathcal{Y}$, then $\mathcal{X}_h = \mathcal{X}_h^*$ and $\mathcal{Y}_h = S^{-1}\mathcal{X}_h S = \mathcal{Y}_h^*$. (We refer to $\mathcal{X}_h$ and to $\mathcal{Y}_h$ as the *homogenizations* of $\mathcal{X}$ and of $\mathcal{Y}$ respectively.)

The matrix $Y := \left[\begin{smallmatrix} 0 & \sqrt{3} \\ 1/\sqrt{3} & 0 \end{smallmatrix}\right]$ is in $\mathcal{Y} \subseteq \mathcal{Y}_h$, and $\|Y\| = \sqrt{3}$ and $\det Y = -1$. If $\mathcal{Y}_h$ is to be unitarily equivalent to $\mathcal{X}_h$, then there must exist $X_0 \in \mathcal{X}$ with $\|X_0\| = \sqrt{3}$ and $\det X_0 = -1$. Note that if $X \in \mathcal{X}_h$, then either $X = \left[\begin{smallmatrix} \lambda 3^m & 0 \\ 0 & \lambda 3^{-m} \end{smallmatrix}\right]$ or $X = \left[\begin{smallmatrix} 0 & \lambda 3^m \\ \lambda 3^{-m} & 0 \end{smallmatrix}\right]$ for some $\lambda \in \mathbb{C}$ and some $m \in \mathbb{Z}$, in which case the condition $\det X = -1$ implies that $|\lambda| = 1$, from which we deduce that $\|X\| = 3^{|m|} \neq \sqrt{3}$.

Hence $\mathcal{X}_h$ and $\mathcal{Y}_h$ are not unitarily equivalent. However, we shall see later (in Example 2.17) that if $\mathcal{K} = \operatorname{span}\mathcal{X}$ and $\mathcal{L} = \operatorname{span}\mathcal{Y}$, then $\mathcal{K}$ and $\mathcal{L}$ *are* unitarily equivalent!

The hypothesis of Theorem 2.10 that $\mathcal{X}$ (and hence $\mathcal{Y}$) be bounded is not necessary when $\mathcal{X}$ consists of normal elements. The following may be viewed as a generalization of Proposition 1.2 above.

2.12. Theorem. *Let $\mathcal{A}$ be a unital C^*-algebra, $\mathcal{N} \subseteq \mathcal{A}$ be a non-empty collection of normal elements, and $\mathcal{M} \subseteq \mathcal{A}$. If $\mathcal{M}$ and $\mathcal{N}$ are *-similar via $s = u|s| \in \mathcal{A}^{-1}$, then they are unitarily equivalent via u. In particular, $\mathcal{M}$ consists of normal elements of $\mathcal{A}$ as well.*

Proof. Write $\mathcal{N} = \{n_\lambda : \lambda \in \Lambda\}$ and $\mathcal{M} = \{m_\lambda : \lambda \in \Lambda\}$, where $m_\lambda = s^{-1}n_\lambda s$ for all $\lambda \in \Lambda$.

By Proposition 2.6, letting $k = ss^*$, we may partition $\mathcal{N}$ as

$$\mathcal{N} = \dot{\bigcup_{n \in \mathcal{N}}} \mathcal{O}(n),$$

where $\mathcal{O}(n) = \{k^{-p}nk^p : p \in \mathbb{Z}\}$. Note that for each $p \in \mathbb{Z}$, $k^{-p}nk^p \in \mathcal{O}(n)$ is similar to n, and it is normal (as $\mathcal{O}(n) \subseteq \mathcal{N}$). By Proposition 1.2 above, it is unitarily equivalent to n, and so has the same norm. It follows that $\mathcal{O}(n)$ is bounded. By Corollary 2.9, $\mathcal{O}(n) = \{n\}$.

By Proposition 2.6, we also find that each $\mathcal{O}(n)$ is *-similar to $\mathcal{Q}(n) = \{s^{-1}ns\}$. It follows that $\mathcal{N}$ and $\mathcal{M}$ are strongly *-similar via s, and by Proposition 2.4 they are unitarily equivalent via u.

The last statement is obvious. ∎

We now wish to consider *-similarity of norm-closed linear subspaces of the unital C^*-algebra $\mathcal{A}$. We begin with the following remarkable result due to Gardner [G64].

2.13. THEOREM (Gardner). *Let $\mathcal{L}$ be a norm-closed linear subspace of $\mathcal{B}(\mathcal{H})$, and $R \in \mathcal{B}(\mathcal{H})$ be a positive, invertible operator. If $R^{-1}\mathcal{L}R \subseteq \mathcal{L}$, then $R^{-\alpha}\mathcal{L}R^{\alpha} \subseteq \mathcal{L}$ for all $\alpha \in \mathbb{R}$.*

That Gardner's theorem holds in a unital C^*-algebra $\mathcal{A}$ can be easily seen by simply representing this algebra faithfully on a Hilbert space. Thus we have

2.14. COROLLARY. *Let $\mathcal{L}$ be a norm-closed linear subspace of a unital C^*-algebra $\mathcal{A}$, and $r \in \mathcal{A}^{-1}$ be a positive, invertible element. If $r^{-1}\mathcal{L}r \subseteq \mathcal{L}$, then $r^{-\alpha}\mathcal{L}r^{\alpha} \subseteq \mathcal{L}$ for all $\alpha \in \mathbb{R}$.*

As an immediate consequence, one obtains the following. (We note that Gardner stated the result below for the case where $\mathcal{L}$ and $\mathcal{M}$ are C^*-subalgebras of $\mathcal{B}(\mathcal{H})$; his proof—essentially reproduced below—works just as well for any norm-closed, selfadjoint subspaces of a unital C^*-algebra $\mathcal{A}$.) Recall that an *operator system* is a selfadjoint, unital linear subspace of a unital C^*-algebra. These need not be norm-closed.

2.15. COROLLARY. *Let $\mathcal{L}$ and $\mathcal{M}$ be selfadjoint, norm-closed subspaces of a unital C^*-algebra $\mathcal{A}$ and suppose that $s \in \mathcal{A}^{-1}$ is an element for which*

$$\mathcal{M} = s^{-1}\mathcal{L}s.$$

If $s = u|s|$ denotes the polar decomposition of s, then

$$\mathcal{M} = u^*\mathcal{L}u.$$

In particular, similar norm-closed operator systems are unitarily equivalent.

Proof. Since $\mathcal{L}$ and $\mathcal{M}$ are selfadjoint, the relation $s\mathcal{M} = \mathcal{L}s$ implies that $\mathcal{M}s^* = s^*\mathcal{L}$, and therefore

$$|s|^2\mathcal{M} = (s^*s)\mathcal{M} = s^*(s\mathcal{M}) = s^*(\mathcal{L}s) = (s^*\mathcal{L})s = (\mathcal{M}s^*)s = \mathcal{M}|s|^2,$$

or equivalently,

$$|s|^2\mathcal{M}|s|^{-2} = \mathcal{M}.$$

By Gardner's theorem,

$$|s|\mathcal{M}|s|^{-1} = \mathcal{M},$$

and thus $\mathcal{L} = s\mathcal{M}s^{-1} = u|s|\mathcal{M}|s|^{-1}u^* = u\mathcal{M}u^*$, which is equivalent to our claim. ∎

Turning our attention to *-similarity, we have

2.16. THEOREM. *Let $\mathcal{L}$ and $\mathcal{M}$ be closed subspaces of a unital C^*-algebra $\mathcal{A}$ and suppose that $\mathcal{L}$ and $\mathcal{M}$ are *-similar via $s = u|s| \in \mathcal{A}^{-1}$. Then $\mathcal{L}$ and $\mathcal{M}$ are unitarily equivalent via u.*

Proof. By hypothesis, $\mathcal{M} = s^{-1}\mathcal{L}s$ and $\mathcal{M}^* = s^{-1}\mathcal{L}^*s$. Thus $s\mathcal{M} = \mathcal{L}s$ and $\mathcal{M}s^* = s^*\mathcal{L}$. The rest of the argument follows the proof of Corollary 2.15. ∎

2.17. EXAMPLE. Let $\mathcal{X}$, $\mathcal{Y}$, and S be as in Example 2.11. As we saw there, $\mathcal{X}$ and $\mathcal{Y}$ are *-similar subgroups of $\mathbb{M}_2(\mathbb{C})$, but $\mathcal{X}$ is not unitarily equivalent to $\mathcal{Y}$. If $\mathcal{L} = \operatorname{span} \mathcal{X}$ and $\mathcal{M} = \operatorname{span} \mathcal{Y}$, then it is routine to verify that $\mathcal{L}$ and $\mathcal{M}$ are *-similar, and thus by Theorem 2.16, $\mathcal{L}$ and $\mathcal{M}$ are unitarily equivalent.

Let us now consider a couple of applications of *-similarity.

2.18. PROPOSITION. *Let $\mathcal{A}$ be a unital C^*-subalgebra of the unital C^*-algebra $\mathcal{B}$, and denote by $\mathcal{A}_1$ the closed unit ball $\{a \in \mathcal{A} : \|a\| \le 1\}$. Let $s = u|s| \in \mathcal{B}^{-1}$. If $s^{-1}\mathcal{A}_1 s$ is selfadjoint, then it is unitarily equivalent to $\mathcal{A}_1$.*

Proof. Let $\mathcal{Y} := s^{-1}\mathcal{A}_1 s$. Note that $\mathcal{A}_1$ is obviously selfadjoint, while $\mathcal{Y}$ is selfadjoint by hypothesis. Thus $\mathcal{A}_1$ and $\mathcal{Y} \subseteq \mathcal{B}$ are *-similar via s. Since $\mathcal{A}_1$ is bounded, we may apply Theorem 2.10 to conclude that $\mathcal{Y} = u^*\mathcal{A}_1 u$. ∎

A theorem of Specht [S40] asserts that if $n \ge 1$ is an integer and $A, B \in \mathbb{M}_n(\mathbb{C})$, then A is unitarily equivalent to B if and only if

$$\operatorname{tr}(w(A, A^*)) = \operatorname{tr}(w(B, B^*))$$

for all words $w(x, y)$ in two non-commuting variables x and y.

In [MMR07], while studying *approximate* multi-variable versions of this theorem for semigroups of matrices, the first two authors—in collaboration with M. Mastnak—obtained the following result:

2.19. PROPOSITION ([MMR07, Corollary 3.10]). *Suppose that $\mathcal{A} = \{A_\alpha\}_{\alpha \in \Lambda}$ and $\mathcal{B} = \{B_\alpha\}_{\alpha \in \Lambda}$ are two families of invertible matrices in $\mathbb{M}_n(\mathbb{C})$ such that*

(a) *if $A \in \mathcal{A}$ then $A^{-1} \in \mathcal{A}$, and similarly if $B \in \mathcal{B}$ then $B^{-1} \in \mathcal{B}$; and*
(b) *the algebras generated by $\mathcal{A}$ and $\mathcal{B}$ are semisimple.*

If for every $m \ge 1$, every word w in m non-commuting variables, and every choice of $\{\lambda_1, \ldots, \lambda_m\} \subseteq \Lambda$ we have

$$|\operatorname{tr}(w(A_{\lambda_1}, \ldots, A_{\lambda_m})) - \operatorname{tr}(w(B_{\lambda_1}, \ldots, B_{\lambda_m}))| < 1,$$

then there exists $R \in \mathbb{M}_n(\mathbb{C})$ invertible such that

$$A_\alpha = R^{-1} B_\alpha R \quad \textit{for all } \alpha \in \Lambda.$$

Thanks to the following result of Freedman, Gupta and Guralnick [FGG97], we can extend the multi-variable version to more general sets of not necessarily invertible matrices, and also deduce that R above must be unitary, providing that we ask that the traces of words in the matrices *agree*. The result we quote now is actually a special case of [FGG97, Corollary 2.7].

2.20. Theorem (Freedman, Gupta and Guralnick). *Suppose that $\mathcal{X} = \{X_1, \ldots, X_m\}$ and $\mathcal{Y} = \{Y_1, \ldots, Y_m\}$ are non-empty subsets of $\mathbb{M}_n(\mathbb{C})$, and the algebras generated by $\mathcal{X}$ and by $\mathcal{Y}$ are semisimple. Suppose furthermore that for each word w in m non-commuting variables we have*

$$\mathrm{tr}(w(X_1, \ldots, X_m)) = \mathrm{tr}(w(Y_1, \ldots, Y_m)).$$

Then there exists $S \in \mathbb{M}_n(\mathbb{C})$ invertible such that $Y_j = S^{-1}X_jS$, $1 \le j \le m$.

2.21. Theorem (A generalized version of Specht's theorem). *Suppose that $\mathcal{X} := \{X_\lambda : \lambda \in \Lambda\}$ and $\mathcal{Y} := \{Y_\lambda : \lambda \in \Lambda\}$ are non-empty subsets of $\mathbb{M}_N(\mathbb{C})$. Suppose that for every $m \ge 1$, every word w in $2m$ variables, and every choice $\{\lambda_1, \ldots, \lambda_m\} \subseteq \Lambda$ we have*

$$\mathrm{tr}(w(X_{\lambda_1}, \ldots, X_{\lambda_m}, X^*_{\lambda_1}, \ldots, X^*_{\lambda_m})) = \mathrm{tr}(w(Y_{\lambda_1}, \ldots, Y_{\lambda_m}, Y^*_{\lambda_1}, \ldots, Y^*_{\lambda_m})).$$

Then there exists a unitary matrix $U \in \mathbb{M}_N(\mathbb{C})$ such that $Y_\lambda = U^*X_\lambda U$ for all $\lambda \in \Lambda$.

Proof. Let $F = \{\lambda_1, \ldots, \lambda_m\} \subseteq \Lambda$ be a finite set, and consider $\mathcal{X}_F := \{X_\lambda : \lambda \in F\}$ and $\mathcal{Y}_F := \{Y_\lambda : \lambda \in F\}$. The algebras generated by $\mathcal{X}_F \cup \mathcal{X}_F^*$ and $\mathcal{Y}_F \cup \mathcal{Y}_F^*$ are C^*-algebras, and are therefore semisimple.

By the result of Freedman, Gupta and Guralnick applied to the sets $\mathcal{X}_F \cup \mathcal{X}_F^*$ and $\mathcal{Y}_F \cup \mathcal{Y}_F^*$, there exists $S_F \in \mathbb{M}_n(\mathbb{C})$ such that $Y_\lambda = S_F^{-1}X_\lambda S_F$ and $Y_\lambda^* = S_F^{-1}X_\lambda^* S_F$, for $\lambda \in F$. This implies that $\mathcal{X}_F \cup \mathcal{X}_F^*$ is *-similar to $\mathcal{Y}_F \cup \mathcal{Y}_F^*$, and thus by Theorem 2.10, $Y_\lambda = U_F^* X_\lambda U_F$ for $\lambda \in F$, where $S_F = U_F|S_F|$ is the polar decomposition of S_F.

Let $\mathcal{F} = \{F \subseteq \Lambda : F \text{ is finite}\}$, partially ordered by inclusion. Then $(U_F)_{F\in\mathcal{F}}$ is a net of unitary operators in $\mathbb{M}_N(\mathbb{C})$. Since the set of unitary matrices in $\mathbb{M}_N(\mathbb{C})$ is compact, we can find a subnet $(U_{F_\gamma})_{\gamma\in\Gamma}$ which converges in norm to a unitary matrix U.

Let $\lambda \in \Lambda$. The argument of the previous paragraphs shows that $Y_\lambda = U_F^* X_\lambda U_F$ for all $F \in \mathcal{F}$ for which $\lambda \in F$. Since $(U_{F_\gamma})_{\gamma\in\Gamma}$ is a subnet of $(U_F)_{F\in\mathcal{F}}$, there exists $\gamma_0 \in \Gamma$ such that $\{\lambda\} \subseteq F_{\gamma_0}$. But then $\gamma \ge \gamma_0$ implies that $F_\gamma \supseteq \{\lambda\}$. Hence $Y_\lambda = U^*_{F_\gamma} X_\lambda U_{F_\gamma}$ for all $\gamma \ge \gamma_0$, and thus $Y_\lambda = U^* X_\lambda U$. ∎

2.22. We conclude with a question which we have been unable to answer so far.

Let $\mathcal{A}$ be a unital C^*-algebra and suppose that $K, L \subseteq \mathcal{A}$ are closed, convex sets in $\mathcal{A}$. If K is *-similar to L via $s \in \mathcal{A}^{-1}$, are K and L unitarily equivalent?

(Of course, if K or L is bounded, then the answer is yes, as an immediate consequence of Theorem 2.10.)

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