STUDIA MATHEMATICA 273 (2) (2023)

Quasidiagonal weighted shifts on directed trees

by

Laurent W. Marcoux (Waterloo, Ont.) and Artur Płaneta (Kraków)

Abstract. We investigate quasidiagonality of weighted shifts on directed trees. We concentrate mainly on a subclass of weighted shifts operators called adjacency operators. In particular, we provide equivalent conditions for quasidiagonality of adjacency operators in terms of the structure of directed trees.

1. Introduction

1.1. Let $\mathscr{H}$ be a complex, separable Hilbert space, and denote by $\mathcal{B}(\mathscr{H})$ the set of bounded linear operators acting on $\mathscr{H}$. Let $\kappa := \dim \mathscr{H} \in \mathbb{N} \cup \{\aleph_0\}$ denote the dimension of $\mathscr{H}$. An operator $D \in \mathcal{B}(\mathscr{H})$ is said to be *diagonal relative to an orthonormal basis* $\{e_n\}_n$ for $\mathscr{H}$ if there exists a bounded sequence $(d_n)_n$ of complex numbers such that $De_n = d_n e_n$ for all n. In particular, each e_n is an eigenvector of D. We say that D is *diagonalisable* if there exists an orthonormal basis for $\mathscr{H}$ relative to which D is diagonal. Equivalently, having fixed the orthonormal basis $\{e_n\}_n$ for $\mathscr{H}$, $D \in \mathcal{B}(\mathscr{H})$ is diagonalisable if there exists a unitary operator $U \in \mathcal{B}(\mathscr{H})$ such that U^*DU is diagonal relative to $\{e_n\}_n$. A standard result from linear algebra – the spectral theorem for normal matrices – asserts that every normal operator acting on a finite-dimensional, complex Hilbert space is diagonalisable. When $\mathscr{H}$ is infinite-dimensional, this no longer holds, as normal operators may not have any eigenvalues. For example, if μ denotes Lebesgue measure on the interval $[0, 1]$, then the multiplication operator M_x acting on $\mathscr{H} = L^2([0, 1], d\mu)$ via $[(M_x)f](x) = xf(x)$ a.e.-μ is self-adjoint but has no eigenvalues. The infinite-dimensional version of the spectral theorem for normal operators guarantees that any normal operator is the norm-limit of diagonalisable normal operators, but as we shall soon see, one can do better.

2020 *Mathematics Subject Classification*: Primary 47A66; Secondary 47A65.
Key words and phrases: quasidiagonal operators, weighted shifts on directed trees, adjacency operators.
Received 30 August 2022; revised 2 May 2023.
Published online 27 November 2023.

DOI: 10.4064/sm220830-7-8

Recall that two operators $A, B \in \mathcal{B}(\mathscr{H})$ are said to be *unitarily equivalent* if there exists a unitary operator $U \in \mathcal{B}(\mathscr{H})$ such that $B = U^*AU$, and that they are *similar* if there exists an invertible operator $R \in \mathcal{B}(\mathscr{H})$ such that $B = R^{-1}AR$. Both unitary equivalence and similarity define equivalence relations on $\mathcal{B}(\mathscr{H})$. Let us denote by $\mathcal{K}(\mathscr{H}) \subseteq \mathcal{B}(\mathscr{H})$ the closed, two-sided ideal of compact operators. The *Calkin algebra* is the C^*-algebra $\mathcal{B}(\mathscr{H})/\mathcal{K}(\mathscr{H})$, and we denote by π the canonical quotient map from $\mathcal{B}(\mathscr{H})$ to $\mathcal{B}(\mathscr{H})/\mathcal{K}(\mathscr{H})$.

An operator $B \in \mathcal{B}(\mathscr{H})$ is said to be *block-diagonal* if there exists a sequence $(B_n)_n$ of operators acting on finite-dimensional complex Hilbert spaces $\mathscr{H}_n$, $n \geq 1$, such that B is unitarily equivalent to the direct sum $\bigoplus_n B_n$ acting on $\mathscr{H} = \bigoplus_n \mathscr{H}_n$. Clearly every diagonalisable operator $D \in \mathcal{B}(\mathscr{H})$ is block-diagonal. Letting P_n denote the orthogonal projection of $\mathscr{H}$ onto $\bigoplus_{k=1}^n \mathscr{H}_k$, we see that if B is block-diagonal, then $P_nB - BP_n = 0$ for all $n \geq 1$.

Block-diagonal operators are specific examples of quasidiagonal operators defined as follows.

Definition 1.1. An operator $T \in \mathcal{B}(\mathscr{H})$ acting on an infinite-dimensional, separable Hilbert space $\mathscr{H}$ is said to be *quasidiagonal* if there exists an increasing sequence $(P_n)_n$ of finite-rank projections converging strongly to the identity operator I such that

$$\lim_n \|P_nT - TP_n\| = 0.$$

The notion of quasidiagonality was first introduced by Halmos [11] in connection with the (then open) problem of deciding whether or not every normal operator can be expressed as a *compact* perturbation of a diagonalisable operator. This question admits an affirmative answer. Indeed, the Weyl–von Neumann–Berg/Sikonia Theorem [5] (see [9, Corollary II.4.2]) asserts that if $N \in \mathcal{B}(\mathscr{H})$ is normal and $\varepsilon > 0$, then there exists a compact operator K of norm less than ε such that $N - K$ is diagonalisable. (That hermitian operators acting on a separable Hilbert space are compact perturbations of diagonalisable operators was first shown by Weyl [21].)

Halmos showed that the set QD of quasidiagonal operators is closed in norm (and therefore contains all normal operators), and that QD coincides with the set of compact perturbations of block-diagonal operators. In fact, given a quasidiagonal operator T and $\varepsilon > 0$, one can find a compact operator K with $\|K\| < \varepsilon$ and a block-diagonal operator B such that $T = B + K$. As such, the set QD of all quasidiagonal operators is the norm-closure of the set BD of block-diagonal operators.

The concept of quasidiagonality was later extended to *sets* of operators (see e.g. [19]). A set $\mathfrak{S} \subseteq \mathcal{B}(\mathscr{H})$ is said to be *quasidiagonal* if for every $\varepsilon > 0$ and for all finite subsets $\mathfrak{F} \subseteq \mathfrak{S}$ and $\mathcal{X} \subseteq \mathscr{H}$, there exists a finite-rank orthogonal projection P such that $\|PT - TP\| < \varepsilon$ for all $T \in \mathfrak{F}$, and $\|(I - P)x\| < \varepsilon$ for all $x \in \mathcal{X}$. Note that $T \in \mathsf{QD}$ if and only if $\{T\}$ is quasidiagonal as a set. Furthermore, the quasidiagonality of a set $\mathfrak{S} \subseteq \mathcal{B}(\mathscr{H})$ is equivalent to that of the C^*-algebra $C^*(\mathfrak{S})$ generated by that set.

The set QD of quasidiagonal operators is stable under compact perturbations, Hilbert space adjoints, unitary equivalence and direct sums. Moreover, by a result of Luecke [14, Theorem 4], an operator Q is quasidiagonal if and only if $Q \oplus 0 \in \mathcal{B}(\mathscr{H} \oplus \mathscr{H})$ is quasidiagonal. Quasidiagonality, however, behaves very poorly under similarity. Indeed, if $Q \in \mathcal{B}(\mathscr{H})$ and $S^{-1}QS \in \mathsf{QD}$ for every invertible operator S, then Q has the property that its image $\pi(Q)$ in the Calkin algebra $\mathcal{B}(\mathscr{H})/\mathcal{K}(\mathscr{H})$ satisfies a polynomial equation of degree at most 2 [12]. In recent years, the notion of quasidiagonality for C^*-algebras has also been shown to occupy a central role in the classification program for simple, nuclear C^*-algebras (we draw the reader's attention to the survey [22]), although a full accounting of the developments there would take us rather far afield.

Quasidiagonality is a special case of a more general notion of *biquasitriangularity*. More specifically, an operator $T \in \mathcal{B}(\mathscr{H})$ is said to be *quasitriangular* if there exists an increasing sequence $(P_n)_n$ of finite-rank projections tending strongly to the identity operator such that $\lim_n \|(I - P_n)TP_n\| = 0$. We say that T is *biquasitriangular* if both T and T^* are quasitriangular. It is not hard to see that every quasidiagonal operator Q is biquasitriangular, and in fact, when Q is quasidiagonal, the sequence $(P_n)_n$ of finite-rank projections implementing the quasitriangularity of Q can be chosen to coincide with the sequence $(R_n)_n$ of finite-rank projections implementing the quasitriangularity of Q^*. Herrero [12] – extending a result of Luecke [14] – has nevertheless shown that the set of quasidiagonal operators is nowhere dense in the set of biquasitriangular operators.

There are a large number of equivalent formulations of biquasitriangularity for Hilbert space operators; Theorem 6.15 of [13] lists no fewer than eighteen equivalent such formulations. One of the most useful of these is a deep result of Apostol, Foiaş and Voiculescu [3]. Before describing that result, we recall some definitions. An operator $T \in \mathcal{B}(\mathscr{H})$ is said to be *semi-Fredholm* if $\operatorname{ran} T$ is closed, and at least one of $\dim\ker T$ and $\dim\ker T^*$ is finite. When this is the case, we define the *semi-Fredholm index* of T to be

$$\operatorname{ind} T := \dim\ker T - \dim\ker T^* \in \mathbb{Z} \cup \{-\infty, \infty\},$$

where it is understood that $\infty - n := \infty$ while $n - \infty := -\infty$ whenever $n \in \mathbb{Z}$. If T is semi-Fredholm and $\operatorname{ind} T \in \mathbb{Z}$, we say that T is *Fredholm*. It is well-known that T is semi-Fredholm if and only if its image $\pi(T)$ in the *Calkin* *algebra* is either left-invertible (corresponding to the case where $\dim\ker T < \infty$) or right-invertible (corresponding to the case where $\dim\ker T^* < \infty$), while T is Fredholm if and only if its image in the Calkin algebra is invertible. The *semi-Fredholm domain* $\varrho_{\mathrm{sF}}(T)$ of T is the set of all $\alpha \in \mathbb{C}$ for which $T - \alpha I$ is semi-Fredholm. Clearly $\varrho_{\mathrm{sF}}(T) = \varrho_{\mathrm{sF}}(T^*)$.

The result of Apostol, Foiaş and Voiculescu referred to above states that an operator T is quasitriangular if and only if $\operatorname{ind}(T - \alpha I) \geq 0$ for all $\alpha \in \varrho_{\mathrm{sF}}(T)$. If T is biquasitriangular, then both T and T^* are quasitriangular, and hence $\operatorname{ind}(T - \alpha I) \geq 0$ and $\operatorname{ind}(T - \alpha I)^* = -\operatorname{ind}(T - \alpha I) \geq 0$ for all $\alpha \in \varrho_{\mathrm{sF}}(T)$. That is, $\operatorname{ind}(T - \alpha I) = 0$ for all $\alpha \in \varrho_{\mathrm{sF}}(T)$. In particular, if Q is quasidiagonal, then $\operatorname{ind}(Q - \alpha I) = 0$ for all $\alpha \in \varrho_{\mathrm{sF}}(Q)$. Since Q is quasidiagonal if and only if $C^*(Q) \subseteq \mathsf{QD}$, we see that in order to prove that Q is *not* quasidiagonal, it suffices to find an operator $T \in C^*(Q)$ and $\alpha \in \varrho_{\mathrm{sF}}(T)$ such that $\operatorname{ind}(T - \alpha I) \neq 0$. That (non-zero) semi-Fredholm index should serve as an obstruction to being quasidiagonal was observed by a number of authors, including Apostol, Foiaş, Voiculescu and Zsidó [4, 1, 2, 3].

A specific example of a non-quasidiagonal operator which will be of paramount interest to us is the following. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for our Hilbert space $\mathscr{H}$. The unique operator $S \in \mathcal{B}(\mathscr{H})$ satisfying $Se_n = e_{n+1}$ for all $n \geq 1$ is referred to as the *unilateral shift operator* (with respect to $\{e_n\}_n$). It is readily verified that S is an isometry, and hence its range is closed and its kernel is trivial. Furthermore, its range has codimension 1 in $\mathscr{H}$, and thus $\operatorname{ind} S = 0 - 1 = -1$. By the Apostol–Foiaş–Voiculescu Theorem, $S \notin \mathsf{QD}$. Combining this with the result of Luecke mentioned above, $0 \oplus S \notin \mathsf{QD}$. Hence, if $T \in \mathcal{B}(\mathscr{H})$ and if there exists $X \in C^*(T)$ such that X is unitarily equivalent to $0 \oplus S$, then X – and consequently T – is not quasidiagonal.

1.2. In this paper we seek to characterise those directed trees whose associated so-called *adjacency operators* are quasidiagonal. Our work may be seen as an extension of the work of Smucker [18], who provided a characterisation of those bilateral weighted shifts acting on $\ell^2(\mathbb{N})$ which are quasidiagonal, and of Narayan [17], who extended the result of Smucker by determining necessary and sufficient conditions for the quasidiagonality of a finite direct sum of bilateral weighted shifts.

Recall that $W \in \mathcal{B}(\mathscr{H})$ is called a *bilateral weighted shift* if there exists an orthonormal basis $\{e_n\}_{\in\mathbb{Z}}$ for $\mathscr{H}$ and a bounded sequence $(w_n)_{n\in\mathbb{Z}}$ of scalars (called *weights*) such that $We_n = w_n e_{n+1}$ for all n. We say that $V \in \mathcal{B}(\mathscr{H})$ is a *unilateral forward weighted shift* if there exists an orthonormal basis $\{f_n\}_{n=1}^{\infty}$ for $\mathscr{H}$ and a bounded sequence $(v_n)_{n=1}^{\infty}$ such that $Vf_n = v_n f_{n+1}$, and that V is a *unilateral backward weighted shift* if V^* is a unilateral forward weighted shift. In each case, up to unitary equivalence, we may assume that all of the weights are non-negative real numbers. Smucker [18] defined a bilateral weighted shift W to be *block-balanced* if, given $\varepsilon > 0$ and $n \in \mathbb{N}$, there exist integers p and q such that $p + n < 0 < q$ and

$$\|(w_p, w_{p+1}, \ldots, w_{p+n}) - (w_q, w_{q+1}, \ldots, w_{q+n})\| < \varepsilon.$$

His classification of quasidiagonal weighted shifts then reads as follows:

THEOREM 1.2 (Smucker). *Let $W \in \mathcal{B}(\mathscr{H})$ be a bilateral weighted shift with weights $(w_n)_{n\in\mathbb{Z}}$. Then W is quasidiagonal if and only if one of the following two conditions holds: either*

(i) $\liminf_{n\geq 0} |w_n| = \liminf_{n\leq 0} |w_n| = 0$, *or*
(ii) *W is block-balanced.*

1.3. The bilateral weighted shift operator V obtained from W by changing the weight w_0 to 0 – a perturbation of rank at most 1 of W – is easily seen to be the direct sum of a backward unilateral weighted shift V_1 and a forward unilateral weighted shift V_2. Since QD is invariant under compact perturbations, $W \in \mathsf{QD}$ if and only if $V \in \mathsf{QD}$, and in this case, condition (i) above is easily seen to be the condition that both V_1 and V_2 are quasidiagonal.

Modern proofs of the sufficiency of the conditions in Smucker's result rely on an approximation technique due to Berg [6], subsequently referred to as *Berg's technique.* The fact that the above conditions (i) and (ii) are necessary for W to belong to QD is typically demonstrated as follows: one supposes that neither condition (i) nor (ii) above is met, and then one exhibits an element $Z \in C^*(W)$ and $\alpha \in \varrho_{\mathrm{sF}}(Z)$ for which $Z - \alpha I$ has non-zero semi-Fredholm index. As we have previously noted, this would imply that W would not be in QD.

Our goal is to examine the analogue of Smucker's theorem for a generalisation of weighted shifts known as *weighted shifts on directed trees.* More specifically, we shall focus on a particular subclass of weighted shifts on directed trees referred to as *adjacency operators* (see, e.g., [16, 15]). The basic strategy we shall employ to determine whether a given adjacency operator is quasidiagonal will be modelled along the lines of that described in the above paragraph, but – in the case of the proof of the sufficiency of our conditions – will rely on a generalisation of Berg's technique due to Berg and Davidson [7].

1.4. Let $\mathscr{H}$ be a complex Hilbert space. If $f, g \in \mathscr{H}$, then $f \otimes g$ denotes the rank 1 operator defined by $(f \otimes g)h = \langle h, g\rangle f$, $h \in \mathscr{H}$. For a non-empty set V, we denote by $\ell^2(V)$ the complex Hilbert space of all functions $f\colon V \to \mathbb{C}$ such that $\sum_{v\in V} |f(v)|^2 < \infty$ with the inner product given by $\langle f, g\rangle = \sum_{v\in V} f(v)\overline{g(v)}$. The set $\{e_u \colon u \in V\}$ is an orthonormal basis of $\ell^2(V)$, where

$$e_u(v) = \begin{cases} 1 & \text{if } u = v, \\ 0 & \text{if } u \neq v. \end{cases}$$

Let $\mathscr{T} = (V, E)$ be a directed tree (V and E stand for the sets of vertices and directed edges of $\mathscr{T}$, respectively). Set $\mathsf{Chi}(u) = \{v \in V \colon (u, v) \in E\}$ for $u \in V$. We refer to these as the *children* of u. If $u \in V$ is such that there exists a unique vertex $v \in V$ such that $(v, u) \in E$, we refer to v as the *parent* of u and we denote v by $\mathsf{par}(u)$. This induces a relation in V, denoted by par, which assigns to a vertex $u \in V$ (which admits a parent) its parent $\mathsf{par}(u)$. For $k \in \mathbb{N}$, par^k denotes the k-fold composition of the relation par; par^0 denotes the identity map on V. Following [15], we write $\mathsf{par}^k(u)$ only if u is in the domain of par^k. Given $u \in V$ and $n \in \mathbb{N}_0$, we set $\mathsf{Chi}^{\langle n\rangle}(u) = \{v \in V\colon \mathsf{par}^n(v) = u\}$, $\mathsf{Des}^n(u) = \bigcup_{j=0}^n \mathsf{Chi}^{\langle j\rangle}(u)$, and $\mathsf{Des}(u) = \bigcup_{j=0}^\infty \mathsf{Chi}^{\langle j\rangle}(u)$. Thus $\mathsf{Des}(u)$ describes all *descendants* of u. A vertex $u \in V$ is called a *root* of $\mathscr{T}$ if u has no parent. A root is unique (provided it exists); we denote it by root. The directed tree $\mathscr{T}$ is *rooted* if the root exists. The *height* of $\mathscr{T}$ is defined as $\sup\{n \in \mathbb{N}_0 \colon \exists_{u\in V}\ \mathsf{Chi}^{\langle n\rangle}(u) \neq \varnothing\} \in \mathbb{N}_0 \cup \{\infty\}$. In turn, $\mathscr{T}$ is *M-ary*, where $M \in \mathbb{N}_0$, if $M = \sup\{\#\mathsf{Chi}(v) \colon v \in V\}$. We will call the set $V_{\text{van}} = \{v \in V\colon\ \mathsf{Chi}^{\langle N\rangle}(v) = \varnothing \text{ for some } N \in \mathbb{N}\}$ the *vanishing subset* of $\mathscr{T}$. Finally, the tree $\mathscr{T}$ is *vanishing* if $V = V_{\text{van}}$.

Suppose $\mathscr{T}$ is rooted. We set $V^\circ = V \setminus \{\mathsf{root}\}$. If $v \in V$, then $|v|$ denotes the unique $k \in \mathbb{N}_0$ such that $\mathsf{par}^k(v) = \mathsf{root}$. A subgraph $\mathscr{S}$ of $\mathscr{T}$ which is a directed tree itself is called a *subtree* of $\mathscr{T}$. A *path* in $\mathscr{T}$ is a directed subtree $\mathscr{P} = (V_{\mathscr{P}}, E_{\mathscr{P}})$ of $\mathscr{T}$ which satisfies the following two conditions: (i) $\mathsf{root} \in \mathscr{P}$, (ii) for every $v \in V_{\mathscr{P}}$, $\operatorname{card}(\mathsf{Chi}(v) \cap V_{\mathscr{P}}) = 1$.

Weighted shifts on directed trees are defined as follows. Let $\mathscr{T} = (V, E)$ be a directed tree and let $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ} \subseteq \mathbb{C}$ be such that

$$\sup_{u\in V} \sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 < \infty.$$

Then the formula

$$(S_{\boldsymbol{\lambda}} f)(v) = \begin{cases} \lambda_v \cdot f(\mathsf{par}(v)) & \text{if } v \in V^\circ, \\ 0 & \text{if } v = \mathsf{root}, \end{cases} \qquad f \in \ell^2(V),$$

defines a bounded operator $S_{\boldsymbol{\lambda}}$ on $\ell^2(V)$ (see [15, Proposition 3.1.8]), which is called the *weighted shift on $\mathscr{T}$ with weights $\boldsymbol{\lambda}$*. It is not hard to see that $\mathscr{T} := (\mathbb{N}, \{(n, n+1)\colon n \in \mathbb{N}\})$ is an example of a rooted directed tree (in fact it is a path), and that the above notion of a weighted shift on the directed tree $\mathscr{T}$ coincides with the usual notion of a weighted shift on $\ell^2(\mathbb{N})$. The reader is referred to [15] for the foundations of the theory of weighted shifts on directed trees. In case $\lambda_v = 1$ for all $v \in V^\circ$, we use the symbol S_V instead of $S_{\boldsymbol\lambda}$. In particular, S_V is a bounded operator on $\ell^2(V)$ if and only if $\sup_{u\in V} \#\mathsf{Chi}(u) < \infty$. Moreover,

$$S_V e_u = \sum_{v\in\mathsf{Chi}(u)} e_v \quad \text{for every } u \in V. \tag{1}$$

Let $N \in \mathbb{N}_0$. Let $\mathcal{G}^N$ be a set such that $\mathcal{G}^N \cap [G]$ contains exactly one element for every finite directed tree G of height N, where $[G]$ denotes the class of all finite directed trees $G' \cong G$, where $\cong$ denotes the fact that two trees are isomorphic. (In other words, $\mathcal{G}^N$ is a set of representatives, one from each equivalence class of finite directed trees of height N.) Let $\mathcal{G}_N = \bigcup_{j=0}^N \mathcal{G}^j$. Let $\mathscr{T} = (V,E)$ be a directed graph. From now on, we shall adopt the following convention. Whenever $V' \subseteq V$, where $V' \neq \varnothing$, is supposed to be a directed tree, we consider the induced (directed) subgraph $\mathscr{T}[V'] = (V', (V'\times V')\cap E)$. For $W \subseteq V$, define

$$\mathcal{G}^N(W) = \{G \in \mathcal{G}^N : G \cong \mathsf{Des}^N(v) \text{ for some } v \in W\}$$

and

$$\mathcal{G}^N_{\mathrm{ess}}(W) = \bigcap_{W'\subseteq W,\, \#(W\setminus W')<\infty} \mathcal{G}^N(W').$$

If $\mathcal{W} \subset \mathcal{G}^N$, then

$$\mathsf{Ver}(\mathcal{W}) := \{v \in V : \exists_{G\in\mathcal{W}}\ \mathsf{Des}^N(v) \cong G\}.$$

In particular, if $G \in \mathcal{G}^N$, then we write $\mathsf{Ver}(G)$ for $\mathsf{Ver}(\{G\})$. In turn, $P^{\mathcal{W}}$ and P^G will denote the projections from $\ell^2(V)$ onto $\ell^2(\mathsf{Ver}(\mathcal{W}))$ and $\ell^2(\mathsf{Ver}(G))$, respectively. The next proposition describing properties of $\mathcal{G}^N(W)$ and $\mathcal{G}^N_{\mathrm{ess}}(W)$ will be used later to prove our main results.

Proposition 1.3. *Let $M \in \mathbb{N}$, $N \in \mathbb{N}_0$ and let $\mathscr{T} = (V,E)$ be an M-ary directed tree. If $W \subseteq V$, then*

(i) *for every $G \in \mathcal{G}^N$, $G \in \mathcal{G}^N(W)$ if and only if $\mathsf{Ver}(G)\cap W \neq \varnothing$,*
(ii) *for every $G \in \mathcal{G}^N$, $G \in \mathcal{G}^N_{\mathrm{ess}}(W)$ if and only if $\mathsf{Ver}(G)\cap W$ is infinite,*
(iii) *$\mathcal{G}^N(W)$ is finite,*
(iv) *$W \cap \mathsf{Ver}(\mathcal{G}^N \setminus \mathcal{G}^N_{\mathrm{ess}}(W))$ is finite,*
(v) *for every $n \in \mathbb{N}$, $\mathcal{G}^N_{\mathrm{ess}}(W_1 \cup \cdots \cup W_n) = \mathcal{G}^N_{\mathrm{ess}}(W_1) \cup \cdots \cup \mathcal{G}^N_{\mathrm{ess}}(W_n)$, $W_1,\ldots,W_n \subseteq V$.*

Proof. (i) and (ii) follow easily from the definitions.

(iii) is a consequence of the fact that there are only finitely many k-ary directed trees in $\mathcal{G}^N$, where $0 \le k \le M$, and (iii) combined with (ii) and the equality

$$W \cap \mathsf{Ver}(\mathcal{G}^N \setminus \mathcal{G}^N_{\mathrm{ess}}(W)) = \bigcup \{\mathsf{Ver}(G) \cap W : G \in \mathcal{G}^N(V) \text{ and } \#(\mathsf{Ver}(G) \cap W) < \infty\}$$

gives (iv).

Finally, (v) is a consequence of (ii). ∎

We close this section with a simple example illustrating our notation.

Example 1.4. Let $\mathscr{T} = (V, E)$ be a directed tree, where

$$V = \{(n, m) \in \mathbb{Z} \times \mathbb{N}_0 : m \le |n|\}$$

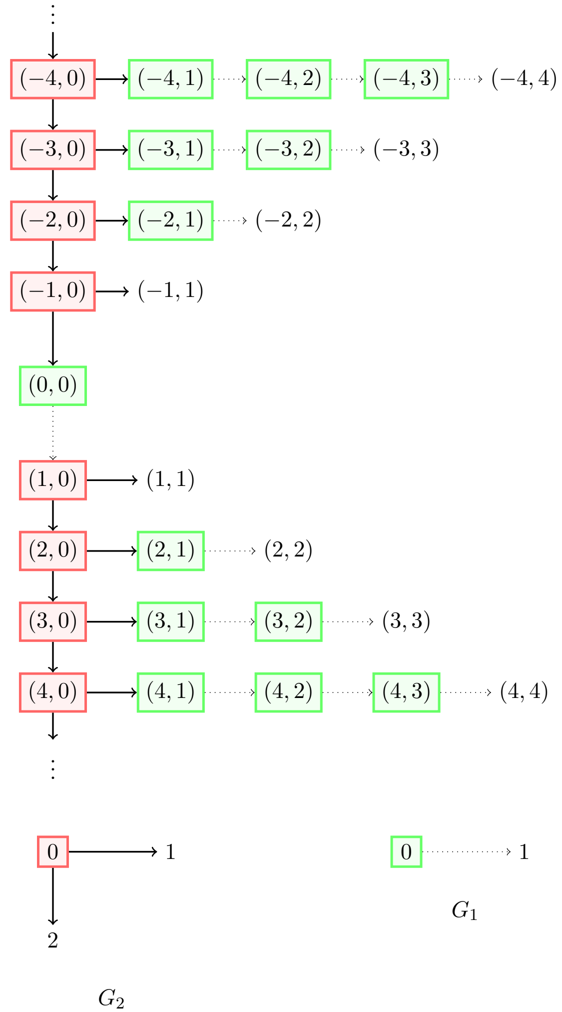


Fig. 1. A directed rooted tree and its subtrees of height 1

and $((n,m),(k,l)) \in E$ if and only if

- $k - n = 1$ and $m = l = 0$, or
- $n = k$ and $l - m = 1$.

Note that $\mathcal{G}^0 = \{G'_0\}$ and $\mathcal{G}^1 = \{G'_n : n \in \mathbb{N}\}$, where $G'_n \cong G_n = (V_n, E_n)$, $V_n = \{0, \ldots, n\}$, and

$$E_n = \begin{cases} \varnothing & \text{if } n = 0, \\ \{0\} \times \{1, \ldots, n\} & \text{if } n \in \mathbb{N}. \end{cases}$$

Then $\mathsf{Ver}(G'_0) = V$, $\mathsf{Ver}(G'_1) = \{(0,0)\} \cup \{(n,m) \in \mathbb{Z} \times \mathbb{N} : 1 \le m \le |n| - 1\}$, $\mathsf{Ver}(G'_2) = (\mathbb{Z} \setminus \{0\}) \times \{0\}$, and $\mathsf{Ver}(G'_n) = \varnothing$ for $n \ge 3$. Hence

$$\mathcal{G}^1(\mathbb{Z} \times \{0\}) = \{G'_1, G'_2\} \quad \text{and} \quad \mathcal{G}^1_{\text{ess}}(\mathbb{Z} \times \{0\}) = \{G'_2\}.$$

2. Quasidiagonality

2.1. In this section, we establish a minor variant of the Berg–Davidson technique for tridiagonal operators. We then demonstrate that the adjacency operators of interest below admit the required tridiagonal form, allowing us to apply the technique to prove the sufficiency of the conditions we use to characterise their quasidiagonality.

The definition of quasidiagonality of an operator $T \in \mathcal{B}(\mathscr{H})$ requires finding an increasing sequence $(P_n)_n$ of finite-rank projections converging strongly to the identity operator such that $\lim_n \|P_n T - T P_n\| = 0$.

Lemma 2.1 below allows us to reduce proving that a given operator is quasidiagonal to producing a single finite-rank projection satisfying two specific conditions. This will be employed throughout the remainder of the paper. The description of the spaces $\mathscr{L}_n$, $n \ge 1$, required in the statement of the lemma will depend upon whether the tree in question is rooted, rootless and vanishing, or a direct sum of a rooted and rootless tree. We shall address each of these cases separately. The idea, however, will always be that described by Lemma 2.1.

Lemma 2.1. *Let $\mathscr{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathscr{H})$. Let $(\mathscr{L}_n)_n$ be an increasing sequence of finite-dimensional subspaces of $\mathscr{H}$ whose union is dense in $\mathscr{H}$. Let $\mu > 0$ be a constant.*

Suppose that for all $N \in \mathbb{N}$, there exist $\kappa \ge N$ and a finite-rank projection $P \in \mathcal{B}(\mathscr{H})$ satisfying

(i) $\mathscr{L}_N \subseteq \operatorname{ran} P \subseteq \mathscr{L}_\kappa$,
(ii) $\|PT - TP\| \le \mu/N$.

Then $T \in \mathsf{QD}$.

Proof. Let $P_1 = 0$ and $\kappa_1 = 1$. By hypothesis, we may choose $\kappa_2 > \kappa_1$ and a projection P_2 such that $\mathscr{L}_{\kappa_1} \subseteq \operatorname{ran} P_2 \subseteq \mathscr{L}_{\kappa_2}$ and $\|P_2 T - T P_2\| < \mu/\kappa_1$.

More generally, having chosen $\kappa_1 < \cdots < \kappa_m$ and projections $P_1, \dots, P_m$ such that $\mathscr{L}_{\kappa_{j-1}} \subseteq \operatorname{ran} P_j \subseteq \mathscr{L}_{\kappa_j}$, and $\|P_j T - T P_j\| < \mu/\kappa_{j-1}$, $2 \le j \le m$, we may use the hypotheses to find an integer $\kappa_{m+1} > \kappa_m$ and a projection P_{m+1} such that $\mathscr{L}_{\kappa_m} \subseteq \operatorname{ran} P_{m+1} \subseteq \mathscr{L}_{\kappa_{m+1}}$ and $\|P_{m+1}T - TP_{m+1}\| < \mu/\kappa_m$.

Clearly $\operatorname{ran} P_m \subseteq \mathscr{L}_{\kappa_m} \subseteq \operatorname{ran} P_{m+1}$ implies that the sequence $(P_m)_m$ is increasing, and that each P_m is of finite rank. Combining this range inclusion with the fact that $\bigcup_m \mathscr{L}_m$ is dense in $\mathscr{H}$ implies that the sequence $(P_m)_m$ converges strongly to the identity operator.

Finally, since μ is fixed and $\lim_m \kappa_m = \infty$, we see that $\lim_m \|P_m T - TP_m\| = 0$, and thus $T \in \mathsf{QD}$. ∎

In any study of quasidiagonality of single operators, a technique originally developed by I. D. Berg (and known as *Berg's technique*) for weighted shifts [6], and later generalised by I. D. Berg and K. R. Davidson [7, Lemma 3.2], is indispensable. We shall need a minor modification of the latter result, which originally applied to the direct sum of two tridiagonal operators.

Proposition 2.2 (The Berg–Davidson technique revisited). *Suppose that $N \ge 1$ is an integer and that $T = [T_{i,j}]$ is tridiagonal with respect to the subspace decomposition $\mathscr{H} := \bigoplus_{k=0}^{2N+2} \mathscr{L}_k$. Suppose furthermore that $\mathscr{L}_k \simeq \mathscr{L}_{k+N+1}$ for all $1 \le k \le N$, and that $T_{i,j} = T_{i+N+1,j+N+1}$ for all $2 \le i+j \le 2N$. Let*

- *Q_0 be the orthogonal projection on $\mathscr{L}_0$;*
- *Q_{2N+2} be the orthgonal projection onto $\mathscr{L}_{2N+2}$;*
- *Q_k denote the orthogonal projection which acts on $\mathscr{L}_k \oplus \mathscr{L}_{k+N+1}$ via the operator matrix*

$$Q_k = \begin{bmatrix} c_k^2 I & c_k s_k I \\ c_k s_k I & s_k^2 I \end{bmatrix},$$

where $c_k = \cos\big(\frac{k\pi}{2N}\big)$ and $s_k = \sin\big(\frac{k\pi}{2N}\big)$ for each $1 \le k \le N$.

If $Q = Q_0 \oplus (\bigoplus_{k=1}^N Q_k) \oplus Q_{2N+2}$, then $\|QT - TQ\| \le \pi\|T\|/N$.

Proof. Since $T = [T_{i,j}]$ is tridiagonal with respect to the decomposition $\mathscr{H} = \bigoplus_{k=0}^{2N+2} \mathscr{L}_k$, we have $T_{i,j} = 0$ if $|i-j| \ge 2$.

If $N = 1$, then direct calculations show that

$$QT - TQ = \begin{bmatrix} 0 & T_{0,1} & 0 & 0 & 0 \\ -T_{1,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & T_{2,3} & 0 \\ 0 & 0 & -T_{3,2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with respect to the decomposition $\mathscr{H} = \mathscr{L}_0 \oplus \mathscr{L}_1 \oplus \mathscr{L}_2 \oplus \mathscr{L}_3 \oplus \mathscr{L}_4$. Hence,

$$\|QT - TQ\| \leq \|T\|$$

since every column has at most one non-zero term.

Now, let $N \geq 2$. Writing T relative to the decomposition

$$\mathscr{H} = (\mathscr{L}_0 \oplus \mathscr{L}_{N+1} \oplus \mathscr{L}_{2N+2}) \oplus \bigoplus_{k=1}^{N} (\mathscr{L}_k \oplus \mathscr{L}_{k+N+1}),$$

we obtain

$$T = \begin{bmatrix} X_{0,0} & X_{0,1} & 0 & \cdots & & & & X_{0,N} \\ X_{1,0} & X_{1,1} & X_{1,2} & 0 & & & & 0 \\ & X_{2,1} & X_{2,2} & X_{2,3} & & & & 0 \\ & 0 & X_{3,2} & X_{3,3} & X_{3,4} & \cdots & & 0 \\ \vdots & & & & & \ddots & & \vdots \\ & 0 & & & & X_{N-1,N-2} & X_{N-1,N-1} & X_{N-1,N} \\ X_{N,0} & 0 & & & & & X_{N,N-1} & X_{N,N} \end{bmatrix},$$

where

- $X_{0,0} = T_{0,0} \oplus T_{N+1,N+1} \oplus T_{2N+2,2N+2}$;
- $X_{0,1} = \begin{bmatrix} T_{0,1} & 0 \\ 0 & T_{N+1,N+2} \\ 0 & 0 \end{bmatrix}$;
- $X_{0,N} = \begin{bmatrix} 0 & 0 \\ T_{N+1,N} & 0 \\ 0 & T_{2N+2,2N+1} \end{bmatrix}$;
- $X_{1,0} = \begin{bmatrix} T_{1,0} & 0 & 0 \\ 0 & T_{N+2,N+1} & 0 \end{bmatrix}$;
- $X_{N,0} = \begin{bmatrix} 0 & T_{N,N+1} & 0 \\ 0 & 0 & T_{2N+1,2N+2} \end{bmatrix}$;
- $X_{i,j} = \begin{bmatrix} T_{i,j} & 0 \\ 0 & T_{i+N+1,j+N+1} \end{bmatrix} = \begin{bmatrix} T_{i,j} & 0 \\ 0 & T_{i,j} \end{bmatrix}$ if $|i-j| \leq 1$;
- $X_{i,j} = 0$ for all other i, j.

It follows that if $(i,j) \notin \{(0,0),(0,1),(0,N),(1,0),(N,0)\}$, then $Q_k X_{i,j} = X_{i,j} Q_k$ for all $1 \le k \le N$.

Relative to this decomposition of $\mathscr{H}$, and defining $Q^\circ := Q_0 \oplus 0 \oplus Q_{2N+2}$, we may write

$$Q = Q^\circ \oplus Q_1 \oplus \cdots \oplus Q_N.$$

We compute the entries of $[Q,T]$:

(i) The $(0,0)$ entry of $[Q,T]$ is $Q^\circ X_{0,0} - X_{0,0} Q^\circ = 0$.

(ii) The $(0,1)$ entry of $[Q,T]$ is

$$Q^\circ X_{0,1} - X_{0,1} Q_1 = \begin{bmatrix} T_{0,1} & 0 \\ 0 & T_{N+1,N+2} \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} - Q_1 \right).$$

Since $\left\| \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] - Q_1 \right\| < \frac{\pi}{2N}$, the norm of this entry is at most $\frac{\pi\|T\|}{2N}$.

(iii) The $(0,N)$ entry of $[Q,T]$ is

$$Q^\circ X_{0,N} - X_{0,N} Q_N = 0.$$

(iv) The $(1,0)$ entry of $[Q,T]$ is

$$Q_1 X_{1,0} - X_{1,0} Q^\circ = \left(Q_1 - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right) X_{1,0}.$$

Since $\left\| \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] - Q_1 \right\| < \frac{\pi}{2N}$, the norm of this entry is at most $\frac{\pi\|T\|}{2N}$.

(v) The $(N,0)$ entry of $[Q,T]$ is

$$Q_N X_{N,0} - X_{N,0} Q^\circ = 0.$$

(vi) If $2 \le i+j \le 2N$ and $|i-j| \le 1$, then the (i,j) entry of $[Q,T]$ is

$$Q_i X_{i,j} - X_{i,j} Q_j = (Q_i - \widetilde{Q}_j) X_{i,j},$$

where $\widetilde{Q}_j$ is the orthogonal projection which acts on $\mathscr{L}_i \oplus \mathscr{L}_{i+N+1}$ via the operator matrix

$$\widetilde{Q}_j = \begin{bmatrix} c_j^2 I & c_j s_j I \\ c_j s_j I & s_j^2 I \end{bmatrix}.$$

Since $|i-j| \le 1$, we have $\|Q_i - \widetilde{Q}_j\| \le \frac{\pi}{2N}$. Hence, the norm of this entry is at most $\frac{\pi\|T\|}{2N}$.

(vii) All other entries of $[Q,T]$ are zero.

In other words, $[Q,T]$ is of the form

$$\begin{bmatrix} 0 & Y_{0,1} & 0 & \cdots & & & 0 \\ Y_{1,0} & 0 & Y_{1,2} & 0 & & & 0 \\ 0 & Y_{2,1} & 0 & Y_{2,3} & & & 0 \\ & 0 & Y_{3,2} & 0 & Y_{3,4} & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & & & Y_{N-1,N-2} & 0 & Y_{N-1,N} \\ 0 & 0 & & & 0 & Y_{N,N-1} & 0 \end{bmatrix},$$

where each entry has norm at most $\frac{\pi\|T\|}{2N}$.

A simple estimate shows that

$$\|[Q,T]\| \leq \frac{\pi\|T\|}{N}.$$

Indeed, in general, if a Hilbert space $\mathscr{M}$ is a direct sum $\bigoplus_{i=1}^{m} \mathscr{M}_i$ of closed subspaces and if, relative to this decomposition, the operator matrix $Z=[Z_{ij}]$ of an operator $Z \in \mathcal{B}(\mathscr{M})$ has the property that there exists at most one non-zero entry in each row and in each column, then it is readily verified that the norm of Z is $\max(\|Z_{ij}\| : 1 \leq i,j \leq n)$. In our case, $[Q,T]$ may be expressed as the sum of two such operators (one whose non-zero entries live only on the first subdiagonal, and one whose non-zero entries live only on the first superdiagonal). Hence,

$$\begin{aligned} \|[Q,T]\| &\leq \max(\|Y_{i,i-1}\| : 1 \leq i \leq N) + \max(\|Y_{i-1,i}\| : 1 \leq i \leq N) \\ &\leq \frac{\pi\|T\|}{2N} + \frac{\pi\|T\|}{2N} = \frac{\pi\|T\|}{N}. \ \blacksquare \end{aligned}$$

REMARK 2.3. Keep in mind that the projection Q is at least as big as Q_0 (and hence $\operatorname{ran} Q \supseteq \mathscr{L}_0$), and if $\dim \mathscr{L}_k < \infty$ for all k except for $k = N+1$, then the rank of Q is finite.

2.2. A canonical (tridiagonal) form for shifts on directed trees. Here we outline a common strategy to determine sufficient conditions which guarantee that a directed tree gives rise to a quasidiagonal (unweighted) shift S_V. For each tree we shall consider, we shall identify an increasing sequence of finite-dimensional subspaces whose union is dense in $\ell^2(V)$ (this sequence will depend upon the structure of the tree), and we shall then apply the above generalisation of the Berg–Davidson technique to produce a finite-rank projection satisfying the conditions of Lemma 2.1.

In order to be able to do so, we require our operator S_V to admit a tridiagonal form. We now provide a general tridiagonal form which will apply to each case (i.e. the rooted case, the rootless, vanishing case and the double-ray case) studied below. We shall refer to these as our "canonical form" for the given shifts.

Note that depending upon the nature of the tree, the dimensions of these spaces will differ. We shall describe that in greater detail in each individual case. With the description below, the proofs in each case will reduce to identifying the spaces $\mathscr{H}_j$, $0 \le j \le N$, $\mathscr{K}$, and $\mathscr{K}_j$, $0 \le j \le N+1$, that we shall need, and showing that our choices in each instance result in the tridiagonal form described below.

(I) Let $u \in V$ and $N \in \mathbb{N}$ be arbitrary. Define $\mathscr{H}_{-1} := \ell^2(V) \ominus \ell^2(\mathsf{Des}(u)) = \ell^2(V \setminus \mathsf{Des}(u))$. For $0 \le j \le N$, set $\mathscr{H}_j := \ell^2(\mathsf{Chi}^{\langle j \rangle}(u))$. Finally, set $\mathscr{K} := \ell^2(V) \ominus \bigoplus_{j=-1}^{N} \mathscr{H}_j = \ell^2(\bigcup_{j=N+1}^{\infty} \mathsf{Chi}^{\langle j \rangle}(u))$.

Relative to the decomposition $\ell^2(V) = \mathscr{H}_{-1} \oplus \mathscr{H}_0 \oplus \cdots \oplus \mathscr{H}_N \oplus \mathscr{K}$, the operator matrix for S_V has the form

$$\begin{bmatrix} A_{-1,-1} & & & & & \\ A_{0,-1} & 0 & & & & \\ & A_{1,0} & 0 & & & \\ & & \ddots & \ddots & & \\ & & & A_{N,N-1} & 0 & \\ & & & & A_{N+1,N} & A_{N+1,N+1} \end{bmatrix}.$$

Indeed, take $w \in V$. If $w \in V \setminus \mathsf{Des}(u)$, then $\mathsf{Chi}(w) \subseteq (V \setminus \mathsf{Des}(u)) \cup \{u\}$. Then, by (1), $S_V(\mathscr{H}_{-1}) \subseteq \mathscr{H}_{-1} \oplus \mathscr{H}_0$. In turn, if $w \in \mathsf{Chi}^{\langle j \rangle}(u)$ for some $0 \le j \le N-1$, then $\mathsf{Chi}(w) \subseteq \mathsf{Chi}^{\langle j+1 \rangle}(u)$ and $S_V(\mathscr{H}_j) \subseteq \mathscr{H}_{j+1}$. Finally, for $w \in \bigcup_{j=N}^{\infty} \mathsf{Chi}^{\langle j \rangle}(u)$, $\mathsf{Chi}(w) \subseteq \bigcup_{j=N+1}^{\infty} \mathsf{Chi}^{\langle j \rangle}(u)$. Thus $S_V(\mathscr{H}_N \oplus \mathscr{K}) \subseteq \mathscr{K}$.

(II) Next, suppose that $v \in \mathsf{Des}(u) \setminus \mathsf{Des}^{N+1}(u)$, so that $\ell^2(v) \subseteq \mathscr{K}$. Applying the same analysis as above, we define $\mathscr{K}_{-1} := \mathscr{K} \ominus \ell^2(\mathsf{Des}(v))$, and for $0 \le j \le N$, we set $\mathscr{K}_j := \ell^2(\mathsf{Chi}^{\langle j \rangle}(v))$. Finally, we set $\mathscr{K}_{N+1} := \mathscr{K} \ominus \bigoplus_{j=-1}^{N} \mathscr{K}_j$.

Relative to the decomposition $\mathscr{K} = \mathscr{K}_{-1} \oplus \mathscr{K}_0 \oplus \cdots \oplus \mathscr{K}_{N+1}$, the operator matrix for $A_{N+1,N+1}$ has the form

$$\begin{bmatrix} B_{-1,-1} & & & & & \\ B_{0,-1} & 0 & & & & \\ & B_{1,0} & 0 & & & \\ & & \ddots & \ddots & & \\ & & & B_{N,N-1} & 0 & \\ & & & & B_{N+1,N} & B_{N+1,N+1} \end{bmatrix}.$$

Observe that $S_V(\mathscr{H}_N) \subseteq \ell^2(\mathsf{Chi}^{\langle N+1\rangle}(u)) \subseteq \mathscr{K}_{-1}$. From this it follows that the operator matrix $[T_{i,j}]$ for S_V relative to the decomposition $\ell^2(V) = \mathscr{H}_{-1} \oplus \mathscr{H}_0 \oplus \cdots \oplus \mathscr{H}_N \oplus \mathscr{K}_{-1} \oplus \mathscr{K}_0 \oplus \mathscr{K}_1 \oplus \cdots \oplus \mathscr{K}_{N+1}$ is tridiagonal, and the only non-zero entries appear either

- on the first subdiagonal, or
- at the $A_{-1,-1}, B_{-1,-1}$ and $B_{N+1,N+1}$ entries.

(iii) If $\mathsf{Des}^N(u) \cong \mathsf{Des}^N(v)$, we may further assume that (possibly after a unitary conjugation) $A_{i,j} = B_{i,j}$ for all $1 \le i + j \le 2N - 1$. If V is an M-ary directed tree, then $\|S_V\| \le M$ and thus $\|A_{i,j}\| \le M$ for all entries.

We can then apply our modified Berg–Davidson technique to produce a projection P satisfying

- $\mathscr{H}_{-1} \oplus \mathscr{K}_{N+1} \subseteq \operatorname{ran} P \subseteq (\bigoplus_{j=-1}^{N} \mathscr{H}_j) \oplus (\bigoplus_{j=0}^{N+1} \mathscr{K}_j)$;
- $\|PS_V - S_V P\| \le \frac{\pi M}{N+1}$.

Of course, $Q := I - P$ is also an orthogonal projection with $\|QS_V - S_V Q\| \le \frac{\pi M}{N+1}$, and we note that

$$\mathscr{K}_{-1} \subseteq \operatorname{ran} Q \subseteq \Big(\bigoplus_{j=0}^{N} \mathscr{H}_j\Big) \oplus \Big(\bigoplus_{j=-1}^{N} \mathscr{K}_j\Big).$$

In particular, the range of Q contains $\mathscr{K}_{-1}$ and is orthogonal to $\mathscr{H}_{-1} \oplus \mathscr{K}_{N+1}$.

Depending upon the structure of the tree we are considering, we shall sometimes need the projection P, and sometimes Q.

(iv) Note that if $v \in V_{\mathrm{van}}$ and $\mathscr{T}$ is M-ary, then $\dim \mathscr{K}_{N+1} < \infty$.

3. Rooted trees

3.1. We begin our study of quasidiagonality of (unweighted) shifts with rooted directed trees containing only one path. Let $M \in \mathbb{N}$ and let $\mathscr{T} = (V, E)$ be an M-ary directed tree. In [20, Lemma 4.4], it was proven that P^G, where G is a finite directed tree, belongs to the von Neumann algebra generated by S_V. In fact, it can be shown that P^G belongs to the C^*-algebra generated by S_V. This fact will be used in all cases considered in this paper.

Lemma 3.1. *Let $M \in \mathbb{N}$ and let $\mathscr{T} = (V, E)$ be an M-ary directed tree. If $\mathcal{W} \subset \mathcal{G}^N$ for some $N \in \mathbb{N}_0$, then $P^{\mathcal{W}} \in C^*(S_V)$.*

Proof. Without loss of generality, we can assume that $\mathcal{W} = \{G\}$, where $G \in \mathcal{G}^N$. Indeed, by Proposition 1.3(iii), $P^{\mathcal{W}}$ is a finite sum of projections P^G, where $G \in \mathcal{W} \cap \mathcal{G}^N(V)$.

The proof for G is by induction on N. If $N = 0$, then $P^G = I \in C^*(S_V)$.

Now, take $G \in \mathcal{G}^N$, where $N \geq 1$. Without loss of generality, we can assume that G is k-ary for some $1 \leq k \leq M$. Otherwise, $\mathsf{Ver}(G) = \varnothing$ and $P^G = 0 \in C^*(S_V)$ since $\mathscr{T}$ is M-ary. Denote by $\mathcal{G}_{N-1,M}$ the set of all directed trees $H \in \mathcal{G}_{N-1}$ such that H is k-ary for some $0 \leq k \leq M$. For every $H \in \mathcal{G}_{N-1,M}$, let $n_G(H)$ stand for the number of all vertices v of G such that v is a child of the root of G and $\mathsf{Des}^{N-1}(v) \cong H$. Let

$$\tilde{P}^G = \sum_{H \in \mathcal{G}_{N-1,M}} (S_V^* P^H S_V - n_G(H) I)^2. \tag{2}$$

By (1) and [15, Proposition 3.4.1],

$$\begin{aligned} S_V^* P^H S_V e_v &= S_V^* P^H \Big(\sum_{u \in \mathsf{Chi}(v)} e_u \Big) = S_V^* \Big(\sum_{u \in \mathsf{Chi}(v),\, \mathsf{Des}^{N-1}(u) \cong H} e_u \Big) \\ &= n_{\mathsf{Des}^N(v)}(H) e_v. \end{aligned}$$

Then

$$\tilde{P}^G e_v = \Big(\sum_{H \in \mathcal{G}_{N-1,M}} (n_{\mathsf{Des}^N(v)}(H) - n_G(H))^2 \Big) e_v, \qquad v \in V. \tag{3}$$

Note that $n_G(H), n_{\mathsf{Des}^N(v)}(H) \in \{0, \ldots, M\}$ for every $H \in \mathcal{G}_{N-1,M}$ and $v \in V$. Moreover, $\mathcal{G}_{N-1,M}$ is finite. Thus $\tilde{P}^G$ is a diagonal operator with finite spectrum $\sigma(\tilde{P}^G) \subseteq \mathbb{N}_0$. Fix a complex polynomial q such that $q(0) = 1$ and $q(\sigma(\tilde{P}^G) \setminus \{0\}) = \{0\}$. Then, by (2) and the induction hypothesis, $P = q(\tilde{P}^G) \in C^*(S_V)$. By (3),

$$P e_v = \begin{cases} e_v & \text{if } \mathsf{Des}^N(v) \cong G, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $P^G = P \in C^*(S_V)$. ∎

LEMMA 3.2. *Let $M \in \mathbb{N}$ and let $\mathscr{T} = (V, E)$ be an M-ary rooted directed tree. Assume $\mathscr{T}$ contains exactly one path $\mathscr{P}$, and for every $N \in \mathbb{N}$,*

$$\mathcal{G}^N(V_{\mathscr{P}}) \cap \mathcal{G}^N(V \setminus V_{\mathscr{P}}) \neq \varnothing.$$

Then, for every $N \in \mathbb{N}$,

$$\mathcal{G}^N_{\mathrm{ess}}(V_{\mathscr{P}}) \cap \mathcal{G}^N_{\mathrm{ess}}(V \setminus V_{\mathscr{P}}) \neq \varnothing.$$

Proof. Suppose that

$$\mathcal{G}^N_{\mathrm{ess}}(V_{\mathscr{P}}) \cap \mathcal{G}^N_{\mathrm{ess}}(V \setminus V_{\mathscr{P}}) = \varnothing \quad \text{for some } N \in \mathbb{N}. \tag{4}$$

By Proposition 1.3(iii) and the definition of $\mathcal{G}^N_{\mathrm{ess}}(W)$, there exists $\tilde{u} \in V_{\mathscr{P}}$ such that

$$\mathcal{G}^N(\mathsf{Des}(\tilde{u}) \cap V_{\mathscr{P}}) = \mathcal{G}^N_{\mathrm{ess}}(V_{\mathscr{P}}), \tag{5}$$

$$\mathcal{G}^N(\mathsf{Des}(\tilde{u}) \cap (V \setminus V_{\mathscr{P}})) = \mathcal{G}^N_{\mathrm{ess}}(V \setminus V_{\mathscr{P}}). \tag{6}$$

Let $N_0 := \max\{|w|\colon w \in V \setminus \mathsf{Des}(\tilde{u})\} + 1 \in \mathbb{N}$. By the assumption, there exist $u \in V_{\mathscr{P}}$ and $v \in V \setminus V_{\mathscr{P}}$ such that

$$\mathsf{Des}^{N+N_0}(u) \cong \mathsf{Des}^{N+N_0}(v). \tag{7}$$

Since $\mathscr{P}$ is a path, we can find a unique $u' \in \mathsf{Chi}^{\langle N_0\rangle}(u) \cap V_{\mathscr{P}}$. Denote by v' the corresponding vertex in $\mathsf{Des}^{N+N_0}(v)$ via the graph isomorphism in (7). Then $\mathsf{Des}^{N}(u') \cong \mathsf{Des}^{N}(v') \cong G$ for some $G \in \mathcal{G}^N$. Note that $v' \in V \setminus V_{\mathscr{P}}$, for otherwise $v \in V_{\mathscr{P}}$. Moreover, $u', v' \in \mathsf{Des}(\tilde{u})$ since $|u'| \geq N_0$ and $|v'| \geq N_0$. Hence, by (5) and (6),

$$G \in \mathcal{G}^N(\mathsf{Des}(\tilde{u}) \cap V_{\mathscr{P}}) \cap \mathcal{G}^N\big(\mathsf{Des}(\tilde{u}) \cap (V \setminus V_{\mathscr{P}})\big) = \mathcal{G}^N_{\mathrm{ess}}(V_{\mathscr{P}}) \cap \mathcal{G}^N_{\mathrm{ess}}(V \setminus V_{\mathscr{P}}),$$

which is a contradiction. This completes the proof. ∎

3.2. Let $M \in \mathbb{N}$ and let $\mathscr{T} = (V, E)$ be an M-ary rooted directed tree admitting a unique path $\mathscr{P}$. Denote $V_{\mathscr{P}} = \{u_0 := \mathsf{root}, u_1, u_2, \ldots\}$. For $n \geq 1$, we define the spaces $\mathscr{L}_n := \ell^2(V) \ominus \ell^2(\mathsf{Des}(u_n))$. It is not hard to see that $(\mathscr{L}_n)_n$ is a strictly increasing sequence of finite-dimensional subspaces and – since $\bigcap_n \ell^2(\mathsf{Des}(u_n)) = \{0\}$ – the union $\bigcup_n \mathscr{L}_n$ is dense in $\mathscr{H}$. We adopt this notation in the proof below.

PROPOSITION 3.3. *Let $M \in \mathbb{N}$ and let $\mathscr{T} = (V, E)$ be an M-ary rooted directed tree. Suppose that $\mathscr{T}$ contains exactly one path $\mathscr{P} = (V_{\mathscr{P}}, E_{\mathscr{P}})$. Let $N \geq 1$ and suppose that*

$$\mathcal{G}^N_{\mathrm{ess}}(V_{\mathscr{P}}) \cap \mathcal{G}^N_{\mathrm{ess}}(V \setminus V_{\mathscr{P}}) \neq \varnothing.$$

Then $S_V \in \mathsf{QD}$.

Proof. Let $u_0 := \mathsf{root}$ and denote $V_{\mathscr{P}} = \{u_0, u_1, \ldots\}$. Define the spaces $\mathscr{L}_n$, $n \geq 1$, as in the paragraph preceding the statement of the theorem.

Let $G \in \mathcal{G}^N_{\mathrm{ess}}(V_{\mathscr{P}}) \cap \mathcal{G}^N_{\mathrm{ess}}(V \setminus V_{\mathscr{P}})$. It follows that there exist $\kappa_1 \geq N$, $\kappa_2 \geq \kappa_1 + 3N$ and $v \in \mathsf{Des}(u_{\kappa_2 - 1}) \setminus (\mathsf{Des}(u_{\kappa_2}) \cup V_{\mathscr{P}})$ such that

$$\mathsf{Des}^N(u_{\kappa_1}) \cong G \cong \mathsf{Des}^N(v).$$

Let

- $\mathscr{H}_{-1} := \ell^2(V) \ominus \ell^2(\mathsf{Des}(u_{\kappa_1}))$,
- $\mathscr{H}_j := \ell^2(\mathsf{Chi}^{\langle j\rangle}(u_{\kappa_1}))$, $0 \leq j \leq N$,
- $\mathscr{K} := \ell^2(V) \ominus \bigoplus_{j=-1}^{N} \mathscr{H}_j$.

Note that each $\mathscr{H}_j$, $0 \leq j \leq N$, is finite-dimensional, as also is $\mathscr{H}_{-1}$.

Observe that $v \in \mathsf{Des}(u_{\kappa_2 - 1})$ implies that $\ell^2(\mathsf{Des}(v)) \subseteq \mathscr{K}$. In particular, if we set

- $\mathscr{K}_{-1} := \mathscr{K} \ominus \ell^2(\mathsf{Des}(v))$,
- $\mathscr{K}_j = \ell^2(\mathsf{Chi}^{\langle j\rangle}(v))$, $0 \leq j \leq N$,
- $\mathscr{K}_{N+1} := \mathscr{K} \ominus \bigoplus_{j=-1}^{N} \mathscr{K}_j$,

then each $\mathscr{K}_j$, $0 \le j \le N$, is finite-dimensional, and furthermore $\mathscr{K}_{N+1} \subseteq \ell^2(\mathsf{Des}(v))$. But $v \in V \setminus V_{\mathscr{P}}$, implying that $\mathsf{Des}(v)$ is finite, and thus $\ell^2(\mathsf{Des}(v))$ is finite-dimensional. *A fortiori*, $\mathscr{K}_{N+1}$ is also finite-dimensional.

Moreover, $S_V(\mathscr{H}_{-1}) \subseteq \mathscr{H}_{-1} \oplus \mathscr{H}_0$ and $S_V(\mathscr{H}_N) \subseteq \ell^2(\mathsf{Chi}^{N+1}(u_{\kappa_1})) \subseteq \mathscr{K}_{-1}$, so that S_V is tridiagonal relative to $\ell^2(V) = (\bigoplus_{j=-1}^N \mathscr{H}_j) \oplus (\bigoplus_{k=-1}^{N+1} \mathscr{K}_k)$.

As described in Section 2.2, we can then apply our modified Berg–Davidson technique (Proposition 2.2) to produce a projection P satisfying

- $\mathscr{H}_{-1} \oplus \mathscr{K}_{N+1} \subseteq \operatorname{ran} P$,
- $\mathscr{K}_{-1}$ is orthogonal to $\operatorname{ran} P$,
- $\|PS_V - S_V P\| \le \frac{\pi M}{N+1}$.

Now we have

(i) $\mathscr{L}_N \subseteq \mathscr{L}_{\kappa_1} \subseteq \mathscr{H}_{-1}$, so that $\mathscr{L}_N \subseteq \operatorname{ran} P$;
(ii) $\operatorname{ran} P \subseteq \mathscr{H}_{-1} \oplus (\bigoplus_{j=0}^N (\mathscr{H}_j \oplus \mathscr{K}_j)) \oplus \mathscr{K}_{N+1} \subseteq \mathscr{L}_{\kappa_2}$, so that P is of finite rank.

By Lemma 2.1, we conclude that S_V is quasidiagonal. ∎

THEOREM 3.4. *Let $M \in \mathbb{N}$ and let $\mathscr{T} = (V, E)$ be an M-ary rooted directed tree that contains exactly one path $\mathscr{P}$. Then the following conditions are equivalent:*

(i) $S_V \in \mathsf{QD}$,
(ii) *for every $N \in \mathbb{N}$,*
$$\mathcal{G}^N(V_{\mathscr{P}}) \cap \mathcal{G}^N(V \setminus V_{\mathscr{P}}) \neq \varnothing,$$
(iii) *for every $N \in \mathbb{N}$,*
$$\mathcal{G}^N_{\mathrm{ess}}(V_{\mathscr{P}}) \cap \mathcal{G}^N_{\mathrm{ess}}(V \setminus V_{\mathscr{P}}) \neq \varnothing.$$

Proof. (i)⇒(ii). Suppose that $\mathcal{G}^N(V_{\mathscr{P}}) \cap \mathcal{G}^N(V \setminus V_{\mathscr{P}}) = \varnothing$ for some $N \in \mathbb{N}$. Let $P = P^{\mathcal{W}}$, where $\mathcal{W} = \mathcal{G}^N(V_{\mathscr{P}})$. By Lemma 3.1 and [18, Theorem 3], $PS_V \in C^*(S_V)$ is quasidiagonal. Since $PS_V = S_{V_{\mathscr{P}}} \oplus 0$ with respect to the decomposition $\ell^2(V) = \ell^2(V_{\mathscr{P}}) \oplus \ell^2(V \setminus V_{\mathscr{P}})$, $S_{V_{\mathscr{P}}}$ is quasidiagonal. On the other hand, $S_{V_{\mathscr{P}}}$ is unitarily equivalent to a unilateral shift, which is not quasidiagonal. Hence S_V is not quasidiagonal, which completes the proof.

(ii)⇒(iii) follows from Lemma 3.2, and (iii)⇒(i) is Proposition 3.3. ∎

3.3. The authors would like to thank the anonymous referee for the following observation. Suppose that $\mathscr{T} = (V, E)$ is a rooted directed tree. We do not need to assume that $\mathscr{T}$ is M-ary. We say that $\mathscr{T}$ is *locally finite* if every vertex in V has finitely many children. We let $V_{\prec}$ denote the set of all vertices in $\mathscr{T}$ having at least two children. We say that $\mathscr{T}$ has *finite branching index* if $\sup\{|u| : u \in V_{\prec}\}$ is finite. The results of Chavan and Trivedi [8, Theorem 5.1(v) and Proposition 2.1] show that a left-invertible weighted shift S_λ on a locally finite, rooted directed tree $\mathscr{T}$ with finite branching index satisfies $\ker S_\lambda^* < \infty$ and $\operatorname{ind} S_\lambda = -\dim\ker S_\lambda^* < 0$ (as the characteristic function of root lies in the kernel of S_λ^*). Thus S_λ fails to be biquasitriangular, and so fails to be quasidiagonal.

4. Rootless trees

4.1. Vanishing, rootless trees. In this section, we turn our attention to vanishing, rootless trees. For $v \in V$, let $\mathscr{T}(v)$ denote the subtree of $\mathscr{T}$ such that $V_{\mathscr{T}(v)} = \{\mathsf{par}^k(v)\colon k \in \mathbb{N}\} \cup \mathsf{Des}(v)$. We begin by proving a counterpart of Lemma 3.2 for vanishing trees.

LEMMA 4.1. *Let $M \in \mathbb{N}$ and let $\mathscr{T} = (V,E)$ be an M-ary vanishing directed tree. Assume that V_1 and V_2 are subsets of V such that*

$$\mathcal{G}^N(V_1) \cap \mathcal{G}^N(V_2) \neq \varnothing \quad \textit{for every } N \in \mathbb{N}.$$

Then

$$\mathcal{G}^N_{\mathrm{ess}}(V_1) \cap \mathcal{G}^N_{\mathrm{ess}}(V_2) \neq \varnothing \quad \textit{for every } N \in \mathbb{N}.$$

Proof. By the assumption, there are sequences $\{\tilde{u}_n\}_{n=1}^\infty \subseteq V_1$ and $\{\tilde{v}_n\}_{n=1}^\infty \subseteq V_2$ such that $\mathsf{Chi}^{\langle n\rangle}(\tilde{u}_n) \neq \varnothing$ and

$$\mathsf{Des}^n(\tilde{u}_n) \cong \mathsf{Des}^n(\tilde{v}_n) \quad \text{for every } n \in \mathbb{N}.$$

Since $\mathscr{T}$ is vanishing, the sets $\{\tilde{u}_n\colon n \in \mathbb{N}\}$ and $\{\tilde{v}_n\colon n \in \mathbb{N}\}$ are infinite. After choosing an appropriate subsequence, we may also assume that there is $G \in \mathcal{G}^N(V)$ such that $\{\tilde{u}_n\colon n \in \mathbb{N}\} \cup \{\tilde{v}_n\colon n \in \mathbb{N}\} \subset \mathsf{Ver}(G)$ since $\mathcal{G}^N(V)$ is a finite set. Hence $G \in \mathcal{G}^N_{\mathrm{ess}}(V_1) \cap \mathcal{G}^N_{\mathrm{ess}}(V_2)$ by Proposition 1.3, which completes the proof. ∎

4.2. Let $M \in \mathbb{N}$ and let $\mathscr{T} = (V,E)$ be an M-ary vanishing rootless directed tree. We choose (and fix) an arbitrary vertex $u_0 \in V$, and set $u_n := \mathsf{par}^n(u_0)$, $n \geq 1$. We then define $\mathscr{L}_n := \ell^2(\mathsf{Des}(u_n))$, $n \geq 1$, observing that $\mathscr{L}_n$ is finite-dimensional for all $n \geq 1$, and that $\bigcup_n \mathscr{L}_n$ is dense in $\ell^2(V)$ since $\mathscr{T}$ is vanishing and rootless.

LEMMA 4.2. *Let $M, N \in \mathbb{N}$ and let $\mathscr{T} = (V,E)$ be an M-ary vanishing rootless directed tree. Suppose that*

$$\mathcal{G}^N_{\mathrm{ess}}(V_{\mathscr{T}(u)}) \cap \mathcal{G}^N_{\mathrm{ess}}(V \setminus V_{\mathscr{T}(u)}) \neq \varnothing \quad \textit{for all } u \in V.$$

Then there exist $v \in V \setminus V_{\mathscr{T}(u_N)}$ and $\kappa \geq 4N$ such that

$$v \in \mathsf{Des}(u_{\kappa-2N}) \quad \textit{and} \quad \mathsf{Des}^N(u_\kappa) \simeq \mathsf{Des}^N(v).$$

Proof. Let $G \in \mathcal{G}^N_{\mathrm{ess}}(V_{\mathscr{T}(u_N)}) \cap \mathcal{G}^N_{\mathrm{ess}}(V \setminus V_{\mathscr{T}(u_N)})$. By the definition of $\mathcal{G}^N_{\mathrm{ess}}(V \setminus V_{\mathscr{T}(u_N)})$, there exists $v \in V \setminus V_{\mathscr{T}(u_N)}$ such that $\mathsf{Des}^N(v) \cong G$. By [15, Proposition 2.1.4.], $v, u_0 \in \mathsf{Des}(u_m)$ for some $m \geq 2N$. Note that $\mathsf{Des}(u_{m+2N})$ is finite. Thus, by Proposition 1.3(ii), we can find $\kappa \geq m + 2N$ such that $\mathsf{Des}^N(u_\kappa) \cong G$ and $v \in \mathsf{Des}(u_m) \subseteq \mathsf{Des}(u_{\kappa-2N})$. ∎

Proposition 4.3. *Let $M \in \mathbb{N}$ and let $\mathscr{T} = (V, E)$ be an M-ary vanishing rootless directed tree. Suppose that*

$$\mathcal{G}^N_{\mathrm{ess}}(V_{\mathscr{T}(u)}) \cap \mathcal{G}^N_{\mathrm{ess}}(V \setminus V_{\mathscr{T}(u)}) \neq \varnothing \quad \textit{for all } N \in \mathbb{N} \textit{ and } u \in V.$$

Then $S_V \in \mathsf{QD}$.

Proof. Let $N \geq 1$. Take v and κ as in Lemma 4.2. The condition that $v \notin V_{\mathscr{T}(u_N)}$ guarantees that no vertex in $\mathsf{Des}(v)$ lies in $\mathsf{Des}(u_N)$. Similarly, the condition that $v \in \mathsf{Des}(u_{\kappa-2N})$ ensures that no vertex in $\mathsf{Des}^N(u_\kappa)$ lies in $\mathsf{Des}^N(v) \cup \mathsf{Des}(u_N)$.

Set $\mathscr{H}_{-1} = \ell^2(V) \ominus \ell^2(\mathsf{Des}(u_\kappa))$, and for $0 \leq j \leq N$, define $\mathscr{H}_j := \ell^2(\mathsf{Chi}^{\langle j \rangle}(u_\kappa))$. Let $\mathscr{K} := \ell^2(V) \ominus \bigoplus_{j=-1}^N \mathscr{H}_j$.

In particular, $\mathscr{K}$ contains $\ell^2(\mathsf{Des}(u_{\kappa-N-1}))$, and in particular $\ell^2(\mathsf{Des}(v))$ is a *finite-dimensional* subspace of $\mathscr{K}$. We next set $\mathscr{K}_{-1} = \mathscr{K} \ominus \ell^2(\mathsf{Des}(v))$, $\mathscr{K}_j := \ell^2(\mathsf{Chi}^{\langle j \rangle}(v))$ for $0 \leq j \leq N$, and $\mathscr{K}_{N+1} := \mathscr{K} \ominus \bigoplus_{j=-1}^N \mathscr{K}_j$.

Observe that $S_V(\mathscr{H}_N) = S_V(\ell^2(\mathsf{Chi}^{\langle N \rangle}(u_\kappa))) \subseteq \ell^2(\mathsf{Chi}^{\langle N+1 \rangle}(u_\kappa)) \subseteq \mathscr{K}_{-1}$, so that S_V is tridiagonal relative to $\ell^2(V) = (\bigoplus_{j=-1}^N \mathscr{H}_j) \oplus (\bigoplus_{k=-1}^{N+1} \mathscr{K}_k)$.

Appealing once again to the method of Section 2.2, we can apply our generalised Berg–Davidson technique (Proposition 2.2) to produce a projection P satisfying

- $\mathscr{H}_{-1} \oplus \mathscr{K}_{N+1} \subseteq \operatorname{ran} P$,
- $\mathscr{K}_{-1}$ is orthogonal to $\operatorname{ran} P$,
- $\|PS_V - S_V P\| \leq \frac{\pi M}{N+1}$.

Consider $Q := I - P$. Then Q is a projection with

$$\mathscr{K}_{-1} \subseteq \operatorname{ran} Q \subseteq (\mathscr{H}_{-1} \oplus \mathscr{K}_{N+1})^\perp,$$

and

$$\|QS_V - S_V Q\| \leq \frac{\pi M}{N+1}.$$

Note that $\mathscr{H}_{-1}^\perp = \ell^2(\mathsf{Des}(u_\kappa))$ is finite-dimensional, and thus Q has finite rank. Furthermore, $v \in V \setminus V_{\mathscr{T}(u_N)}$ implies that $\mathscr{L}_N = \ell^2(\mathsf{Des}(u_N)) \subseteq \mathscr{K}_{-1}$. Hence

$$\mathscr{L}_N \subseteq \operatorname{ran} Q \subseteq (\mathscr{H}_{-1} \oplus \mathscr{K}_{N+1})^\perp \subseteq \mathscr{L}_\kappa.$$

We may therefore apply Lemma 2.1 (with Q instead of P) to conclude that S_V is quasidiagonal. ■

Theorem 4.4. *Let $M \in \mathbb{N}$ and let $\mathscr{T} = (V, E)$ be an M-ary vanishing rootless directed tree. Then the following conditions are equivalent:*

(i) *S_V is quasidiagonal,*
(ii) *$\mathcal{G}^N(V_{\mathscr{T}(u)}) \cap \mathcal{G}^N(V \setminus V_{\mathscr{T}(u)}) \neq \varnothing$ for all $N \in \mathbb{N}$ and $u \in V$,*
(iii) *there exists $u \in V$ such that*

$$\mathcal{G}^N(V_{\mathscr{T}(u)}) \cap \mathcal{G}^N(V \setminus V_{\mathscr{T}(u)}) \neq \varnothing \quad \textit{for all } N \in \mathbb{N},$$

(iv) *there exists* $u \in V$ *such that*

$$\mathcal{G}_{\mathrm{ess}}^N(V_{\mathscr{T}(u)}) \cap \mathcal{G}_{\mathrm{ess}}^N(V \setminus V_{\mathscr{T}(u)}) \neq \varnothing \quad \textit{for all } N \in \mathbb{N},$$

(v) $\mathcal{G}_{\mathrm{ess}}^N(V_{\mathscr{T}(u)}) \cap \mathcal{G}_{\mathrm{ess}}^N(V \setminus V_{\mathscr{T}(u)}) \neq \varnothing$ *for all* $N \in \mathbb{N}$ *and* $u \in V$.

Proof. (i)⇒(ii). Suppose we can choose $N \in \mathbb{N}$ and $u \in V$ such that

$$\mathcal{G}^N(V_{\mathscr{T}(u)}) \cap \mathcal{G}^N(V \setminus V_{\mathscr{T}(u)}) = \varnothing.$$

Let $P = P^{\mathcal{W}}$, where $\mathcal{W} = \mathcal{G}^N(V_{\mathscr{T}(u)})$. By Lemma 3.1, $PS_V \in C^*(S_V)$, and since S_V is quasidiagonal by hypothesis, so is every operator in $C^*(S_V)$. In particular, PS_V is quasidiagonal. Set $\Omega = \mathsf{Des}(\mathsf{par}^N(u)) \cap V_{\mathscr{T}(u)}$. Then, with respect to the decomposition $\ell^2(V) = \ell^2(\Omega) \oplus \ell^2(V_{\mathscr{T}(u)} \setminus \Omega) \oplus \ell^2(V \setminus V_{\mathscr{T}(u)})$, we may write

$$PS_V = \begin{bmatrix} PS_\Omega & R \\ 0 & S_{V_{\mathscr{T}(u)} \setminus \Omega} \end{bmatrix} \oplus 0,$$

where $R = e_{\mathsf{par}^N(u)} \otimes e_{\mathsf{par}^{N+1}(u)}$. Note that PS_Ω and R are finite-rank operators.

It follows that $\left[\begin{smallmatrix} 0 & 0 \\ 0 & S_{V_{\mathscr{T}(u)} \setminus \Omega} \end{smallmatrix}\right] \oplus 0 \simeq S_{V_{\mathscr{T}(u)} \setminus \Omega} \oplus 0$ is a finite-rank perturbation of PS_V. Since the set of quasidiagonal operators is invariant under compact (and hence under finite-rank) perturbations [13, Theorem 6.12], we find that $S_{V_{\mathscr{T}(u)} \setminus \Omega} \oplus 0$ is quasidiagonal. By [14, Theorem 4], $S_{V_{\mathscr{T}(u)} \setminus \Omega}$ is quasidiagonal. Since V is rootless, $S_{V_{\mathscr{T}(u)} \setminus \Omega}$ is unitarily equivalent to the adjoint of the unilateral shift, which is not quasidiagonal (see Section 1.1). To see this, observe that there is a bijection $\varphi : \mathbb{N} \to V_{\mathscr{T}(u)} \setminus \Omega$ given by $\varphi(k) := \mathsf{par}^{N+k}(u)$, and that $S_{V_{\mathscr{T}(u)} \setminus \Omega}(e_{\varphi(k)}) = e_{\varphi(k-1)}$ for all $k \geq 1$, while $S_{V_{\mathscr{T}(u)} \setminus \Omega}(e_{\varphi(1)}) = 0$. This corresponds precisely to the action of S^* on the orthonormal basis $\{e_n\}_n$ with respect to which the unilateral forward shift satisfies $Se_n = e_{n+1}$ for all $n \geq 1$. This contradiction completes the proof.

The implication (ii)⇒(iii) is obvious, and (iii)⇒(iv) follows from Lemma 4.1.

(iv)⇒(v). Let $v \in V$. Since $u, v \in \mathsf{Des}(w)$ for some $w \in V$, the symmetric difference $V_{\mathscr{T}(u)} \triangle V_{\mathscr{T}(v)}$ is finite. Then, by Proposition 1.3,

$$\mathcal{G}_{\mathrm{ess}}^N(V_{\mathscr{T}(u)}) = \mathcal{G}_{\mathrm{ess}}^N(V_{\mathscr{T}(v)}) \quad\text{and}\quad \mathcal{G}_{\mathrm{ess}}^N(V \setminus V_{\mathscr{T}(u)}) = \mathcal{G}_{\mathrm{ess}}^N(V \setminus V_{\mathscr{T}(v)}).$$

This combined with our assumption gives (v).

Finally, (v)⇒(i) is the content of Proposition 4.3. ∎

5. Trees with one double ray

5.1. The case of an unweighted shift acting on a rooted tree containing only one path may be viewed as a generalisation of the case of the usual unilateral forward shift S acting on $\ell^2(\mathbb{N})$ (with standard orthonormal basis $\{e_n\}_{n=1}^{\infty}$) via $Se_n = e_{n+1}$. Indeed, $\mathscr{T} = (\mathbb{N}, \{(n, n+1)\colon n \in \mathbb{N}\})$ is a rooted tree with only one path, and so it is interesting to see that while S is definitely not quasidiagonal (due to index considerations – see Section 1.1), nevertheless, we may find examples of rooted trees $\mathscr{T} = (V, E)$ where the corresponding shift operator S_V is quasidiagonal. In this analogy, the shift operators acting on the vanishing trees of the previous section correspond to generalisations of the backward shift. The last case considered in this paper is of shifts on directed trees with one double ray ([1]). These may be thought of as generalisations of bilateral shifts. Smucker [18] obtained a characterisation of those *weighted* shifts on $\ell^2(\mathbb{Z})$ which are quasidiagonal. After a rank-one perturbation, such a shift – say W – is a direct sum of a unilateral backward weighted shift and a unilateral forward weighted shift. Smucker's result asserts that either both summands are themselves quasidiagonal, or they are *block-balanced*, in the sense of Theorem 1.2. Theorem 5.9 may be thought of as a generalisation of Smucker's result.

5.2. A *double* (directed) *ray* is an infinite graph $\mathscr{R} = (V_{\mathscr{R}}, E_{\mathscr{R}})$ of the form $V_{\mathscr{R}} = \{x_n\colon n \in \mathbb{Z}\}$ and $E_{\mathscr{R}} = \{(x_n, x_{n+1})\colon n \in \mathbb{Z}\}$, where the x_n are assumed to be distinct and $\mathbb{Z}$ is the set of all integers.

LEMMA 5.1. *Let $\mathscr{T} = (V, E)$ be a directed tree. Then the following conditions are equivalent:*

(i) *$\mathscr{T}$ contains exactly one double ray,*
(ii) *$V \setminus V_{\mathrm{van}}$ is a double ray.*

Proof. (i)⇒(ii). Let $W_1 \subseteq V$ be a double ray. Then $W_1 \subseteq V \setminus V_{\mathrm{van}}$. Suppose that $u \in V \setminus (W_1 \cup V_{\mathrm{van}})$. By [15, Proposition 2.1.4], we can find $v \in W_1$ such that $u \in \mathsf{Des}(v)$. In particular, $\mathsf{par}^n(u)$ is well-defined for every $n \in \mathbb{N}$. Then, applying König's Infinity Lemma (see [10, Lemma 8.1.2]), we obtain a set $W_2 = \{u_n \in V\colon n \in \mathbb{Z}\}$ such that $u_0 = u$ and $\mathsf{par}(u_n) = u_{n-1}$ for every $n \in \mathbb{Z}$. This means that W_2 is a double ray different from W_1, which is a contradiction. Thus $V \setminus V_{\mathrm{van}} = W_1$ and $V \setminus V_{\mathrm{van}}$ is a double ray.

(ii)⇒(i). Assume that $W_1, W_2 \subseteq V$ are different double rays. Then $W_1 \cup W_2 \subseteq V \setminus V_{\mathrm{van}}$. Hence, $V \setminus V_{\mathrm{van}}$ is not a double ray, a contradiction. ∎

Assume that $\mathscr{T} = (V, E)$ is an M-ary directed tree with one double ray $V' \subseteq V$. By Lemma 5.1, $V \setminus V'$ is the vanishing subset of V. Let $V' = \{u_n\colon n \in \mathbb{Z}\}$. Define

$$V_1' := \{u_n\colon n \geq 0\} \quad \text{and} \quad V_2' := \{u_n\colon n < 0\}.$$

According to Proposition 1.3(i)–(iii), $\mathcal{G}_{\mathrm{ess}}^N(V_j') \neq \varnothing$ for every $N \in \mathbb{N}_0$ and $j = 1, 2$. What is more, the sets $\mathcal{G}_{\mathrm{ess}}^N(V_j')$, $j = 1, 2$, do not depend on the

([1]) We adopt the term double ray from [10].

choice of a division of V' into two infinite subtrees V_1' and V_2'. That is, for any $m \in \mathbb{Z}$, we could just as well have defined $Z_1' := \{u_n\colon n \geq m\}$ and $Z_2' := \{u_n\colon n < m\}$, and we would find that $\mathcal{G}^N_{\mathrm{ess}}(Z_1') = \mathcal{G}^N_{\mathrm{ess}}(V_1')$ and $\mathcal{G}^N_{\mathrm{ess}}(Z_2') = \mathcal{G}^N_{\mathrm{ess}}(V_2')$.

We also define $V_1 := \mathsf{Des}(u_0)$ and $V_2 = V \setminus V_1$.

Let us formulate the following lemma which will be used several times in this section.

LEMMA 5.2. *Let $\mathscr{T} = (V, E)$ be an M-ary directed tree containing exactly one double ray, and suppose that for some $N \in \mathbb{N}$ and $j \in \{1, 2\}$,*

$$\mathcal{G}^N_{\mathrm{ess}}(V_j') \cap \mathcal{G}^N_{\mathrm{ess}}(V_{3-j} \setminus V_{3-j}') \neq \varnothing.$$

Then, for every $\kappa \geq 1$, there exist $\kappa_1, \kappa_2 \geq \kappa$ and $v \in V_{3-j} \setminus V_{3-j}'$ with $v \in \mathsf{Des}(u_{m-1}) \setminus \mathsf{Des}(u_m)$ such that

$$\mathsf{Des}^N(u_n) \cong \mathsf{Des}^N(v),$$

where $n = (-1)^{j+1}\kappa_j$ and $m = (-1)^j \kappa_{3-j}$.

Proof. Since $\mathscr{T}$ is M-ary and $V \setminus V'$ is the vanishing subset of $\mathscr{T}$, $W = \mathsf{Des}(u_{-\kappa+1}) \setminus \mathsf{Des}(u_\kappa)$ is finite. Let $G \in \mathcal{G}^N_{\mathrm{ess}}(V_j') \cap \mathcal{G}^N_{\mathrm{ess}}(V_{3-j} \setminus V_{3-j}')$. Then, by Proposition 1.3(ii), we may find $n \in \mathbb{Z}$ and $v \in V_{3-j} \setminus (V_{3-j}' \cup W)$ such that

$$u_n \in V_j' \setminus W \quad \text{and} \quad \mathsf{Des}^N(u_n) \cong \mathsf{Des}^N(v) \cong G. \tag{8}$$

Applying [15, Proposition 2.1.4.] for $\{u_0, v\}$, there exists $k \in \mathbb{Z}$ such that $v \in \mathsf{Des}(u_k)$. Then $m = 1 + \max\{l \in \mathbb{Z}\colon v \in \mathsf{Des}(u_l)\}$ is well-defined. Finally, define $\kappa_j = |n|$ and $\kappa_{3-j} = |m|$. By (8), $n = (-1)^{j+1}\kappa_j$ and $\kappa_j \geq \kappa$. The fact that $v \in V_{3-j} \setminus (V_{3-j}' \cup W)$ and $v \in \mathsf{Des}(u_{m-1}) \setminus \mathsf{Des}(u_m)$ imply that $|m| \geq \kappa$ and $m = (-1)^j \kappa_{3-j}$. ∎

For each $n \geq 1$, we define the spaces

$$\begin{aligned} \mathscr{L}_n^+ &:= \ell^2(\mathsf{Des}(u_0)) \ominus \ell^2(\mathsf{Des}(u_n)), \\ \mathscr{L}_n^- &:= \ell^2(\mathsf{Des}(u_{-n})) \ominus \ell^2(\mathsf{Des}(u_0)), \\ \mathscr{L}_n &:= \mathscr{L}_n^+ \oplus \mathscr{L}_n^-. \end{aligned}$$

Clearly $\mathscr{L}_n^+$ is orthogonal to $\mathscr{L}_m^-$ for all $m, n \geq 1$. Observe that $(\mathscr{L}_n^+)_n$ and $(\mathscr{L}_n^-)_n$ are increasing sequences of finite-dimensional subspaces, and $\bigcup_n \mathscr{L}_n$ is dense in $\ell^2(V)$.

We shall divide the proof of the main result of this section into several steps. The next result is an analogue of the case where the two summands of a weighted shift are block-balanced. In our case, there are no weights, but we may think of the “main diagonals” of the components V_1' and V_2' of our shift operator acting on our tree as having finite subgraphs of arbitrary height which are “block-balanced”, in the sense that we can find κ_1, κ_2 both arbitrarily large and positive such that $\mathsf{Des}^N(u_{-\kappa_2}) \cong \mathsf{Des}^N(u_{\kappa_1})$. Hence, the next proposition generalises Smucker's theorem in the case of the unweighted bilateral shift B, that is, the case where $\mathscr{H}$ admits an orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ relative to which $Be_n = e_{n+1}$ for all $n \in \mathbb{Z}$.

PROPOSITION 5.3. *Let $\mathscr{T} = (V, E)$ be an M-ary directed tree containing exactly one double ray, and suppose that for all $N \geq 1$,*

$$\mathcal{G}^N_{\mathrm{ess}}(V'_1) \cap \mathcal{G}^N_{\mathrm{ess}}(V'_2) \neq \varnothing.$$

Then S_V is quasidiagonal.

Proof. Fix $N \geq 1$. The assumption that $\mathcal{G}^N_{\mathrm{ess}}(V'_1) \cap \mathcal{G}^N_{\mathrm{ess}}(V'_2) \neq \varnothing$ implies that we can find $\kappa_1, \kappa_2 \geq 3N$ such that

$$\mathsf{Des}^N(u_{-\kappa_2}) \cong \mathsf{Des}^N(u_{\kappa_1}).$$

Let $\mathscr{H}_{-1} = \ell^2(V) \ominus \ell^2(\mathsf{Des}(u_{-\kappa_2}))$, and for $0 \leq j \leq N$, set $\mathscr{H}_j := \ell^2(\mathsf{Chi}^{\langle j\rangle}(u_{\kappa_2}))$. Let $\mathscr{K} := \ell^2(V) \ominus \bigoplus_{j=-1}^N \mathscr{H}_j$ and observe that $\ell^2(\mathsf{Des}^N(u_{\kappa_1})) \subseteq \mathscr{K}$. Define $\mathscr{K}_{-1} = \mathscr{K} \ominus \ell^2(\mathsf{Des}(u_{\kappa_1}))$, and for $0 \leq j \leq N$, we set $\mathscr{K}_j := \ell^2(\mathsf{Chi}^{\langle j\rangle}(u_{\kappa_1}))$. Finally, let $\mathscr{K}_{N+1} = \mathscr{K} \ominus \bigoplus_{j=-1}^N \mathscr{K}_j$, and note that $\mathscr{K}_{N+1}$ contains $\ell^2(\mathsf{Des}(u_{\kappa_1+N+1}))$.

Note that $\mathscr{L}^-_N, \mathscr{L}^+_N \subseteq \mathscr{K}_{-1}$.

Relative to $\ell^2(V) := (\bigoplus_{j=-1}^N \mathscr{H}_j) \oplus (\bigoplus_{j=-1}^{N+1} \mathscr{K}_j)$, we find that the operator matrix for S_V is of the canonical form described in Section 2.2.

As in Section 2.2, we note that we may apply the Berg–Davidson technique (Proposition 2.2) to produce a projection Q satisfying

- $\mathscr{K}_{-1} \subseteq \operatorname{ran} Q \subseteq (\mathscr{H}_{-1} \oplus \mathscr{K}_{N+1})^\perp$,
- $\|QS_V - S_V Q\| \leq \frac{\pi M}{N+1}$.

The fact that $\operatorname{ran} Q \subseteq (\mathscr{H}_{-1} \oplus \mathscr{K}_{N+1})^\perp$ implies that Q is of finite rank. But from the above,

$$\mathscr{L}_N := \mathscr{L}^-_N \oplus \mathscr{L}^+_N \subseteq \mathscr{K}_{-1} \subseteq \operatorname{ran} Q.$$

Applying Lemma 2.1, we see that S_V is quasidiagonal. ∎

Continuing our analogy with Smucker's result for weighted shift operators, the next two results are needed for the case where S_{V_1} is quasidiagonal and S_{V_2} is not.

LEMMA 5.4. *Let $\mathscr{T} = (V, E)$ be an M-ary directed tree containing exactly one double ray, and suppose that for all $N \geq 1$,*

$$\mathcal{G}^N_{\mathrm{ess}}(V'_2) \cap \mathcal{G}^N_{\mathrm{ess}}(V_1 \setminus V'_1) \neq \varnothing.$$

Then for all $\kappa_0 \geq 0$, there exist $\kappa_1, \kappa_2 \geq \max(\kappa_0, 3N)$ and a projection R such that

(i) $\ell^2(\mathsf{Des}(u_{\kappa_1})) \subseteq \operatorname{ran} R \subseteq \ell^2(\mathsf{Des}(u_{-\kappa_2}))$,
(ii) $\mathscr{L}_N \subseteq \operatorname{ran} R$,
(iii) $\|RS_V - S_V R\| \leq \frac{\pi M}{N+1}$.

Proof. Let $N \geq 1$ be fixed and $\kappa_0 \geq 1$. By Lemma 5.2, we may find $\kappa_1, \kappa_2 \geq \max(\kappa_0, 3N)$ and $v \in V_1 \setminus V_1'$ with $v \in \mathsf{Des}(u_{\kappa_1 - 1}) \setminus \mathsf{Des}(u_{\kappa_1})$ such that $\mathsf{Des}^N(u_{-\kappa_2}) \cong \mathsf{Des}^N(v)$.

Let

- $\mathscr{H}_{-1} := \ell^2(V) \ominus \ell^2(\mathsf{Des}(u_{-\kappa_2}))$,
- $\mathscr{H}_j := \ell^2(\mathsf{Chi}^{\langle j \rangle}(u_{-\kappa_2}))$, $0 \leq j \leq N$,
- $\mathscr{K} := \ell^2(V) \ominus \bigoplus_{j=-1}^N \mathscr{H}_j$.

Noting that $\ell^2(\mathsf{Des}(v)) \subseteq \mathscr{K}$, we set

- $\mathscr{K}_{-1} := \mathscr{K} \ominus \ell^2(\mathsf{Des}(v))$,
- $\mathscr{K}_j := \ell^2(\mathsf{Chi}^{\langle j \rangle}(v))$, $0 \leq j \leq N$,
- $\mathscr{K}_{N+1} := \mathscr{K} \ominus \bigoplus_{j=-1}^N \mathscr{K}_j$.

Applying the Berg–Davidson technique as in Section 2.2, we can find a projection R such that

- $\mathscr{K}_{-1} \subseteq \operatorname{ran} R \subseteq (\bigoplus_{j=0}^N \mathscr{H}_j) \oplus (\bigoplus_{j=-1}^N \mathscr{K}_j)$,
- $\|RS_V - S_V R\| \leq \frac{\pi M}{N+1}$.

Note that $\mathscr{H}_{-1} = (\ell^2(\mathsf{Des}(u_{-\kappa_2})))^\perp$, while $\ell^2(\mathsf{Des}(u_{\kappa_1})) \oplus \ell^2(\mathsf{Des}(u_{-N}) \setminus \mathsf{Des}(u_N)) \subseteq \mathscr{K}_{-1}$, so that

$$\ell^2(\mathsf{Des}(u_{\kappa_1})) \oplus \mathscr{L}_N \subseteq \operatorname{ran} R \subseteq \ell^2(\mathsf{Des}(u_{-\kappa_2})),$$

completing the proof. ∎

PROPOSITION 5.5. *Let $\mathscr{T} = (V, E)$ be an M-ary directed tree containing exactly one double ray, and suppose that for all $N \geq 1$,*

$$\mathcal{G}^N_{\mathrm{ess}}(V_2') \cap \mathcal{G}^N_{\mathrm{ess}}(V_1 \setminus V_1') \neq \varnothing.$$

If S_{V_1} is quasidiagonal, then so is S_V.

Proof. Fix $N \geq 1$ and $\kappa_0 \geq 3N$. By Lemma 5.4, there exists a projection R_1 such that

(i) $\ell^2(\mathsf{Des}(u_{\kappa_1})) \oplus \mathscr{L}_N \subseteq \operatorname{ran} R_1 \subseteq \ell^2(\mathsf{Des}(u_{-\kappa_2}))$,
(ii) $\|R_1 S_V - S_V R_1\| \leq \frac{\pi M}{N+1}$.

The statement that S_{V_1} is quasidiagonal implies that

$$\mathcal{G}^N_{\mathrm{ess}}(V_1') \cap \mathcal{G}^N_{\mathrm{ess}}(V_1 \setminus V_1') \neq \varnothing.$$

Thus, we can find $\kappa_4 \geq \kappa_3 + 2N \geq \max(\kappa_1, \kappa_2) + 4N$ and $v \in \mathsf{Des}(u_{\kappa_4 - 1}) \setminus \mathsf{Des}(u_{\kappa_4})$ such that

$$\mathsf{Des}^N(u_{\kappa_3}) \cong \mathsf{Des}^N(v).$$

Applying our canonical form with

- $\mathscr{H}_{-1} := \ell^2(V) \ominus \ell^2(\mathsf{Des}(u_{\kappa_3}))$,
- $\mathscr{H}_j := \ell^2(\mathsf{Chi}^{\langle j\rangle}(u_{\kappa_3}))$, $0 \le j \le N$,
- $\mathscr{K} := \ell^2(V) \ominus \bigoplus_{j=-1}^{N} \mathscr{H}_j$,

and

- $\mathscr{K}_{-1} := \mathscr{K} \ominus \ell^2(\mathsf{Des}(v))$,
- $\mathscr{K}_j := \ell^2(\mathsf{Chi}^{\langle j\rangle}(v))$, $0 \le j \le N$,
- $\mathscr{K}_{N+1} := \mathscr{K} \ominus \bigoplus_{j=-1}^{N} \mathscr{K}_j$,

we can find a projection R_2 satisfying

- $\mathscr{H}_{-1} \oplus \mathscr{K}_{N+1} \subseteq \operatorname{ran} R_2 \subseteq (\bigoplus_{j=-1}^{N} \mathscr{H}_j) \oplus (\bigoplus_{j=0}^{N+1} \mathscr{K}_j)$,
- $\|R_2 S_V - S_V R_2\| \le \frac{\pi M}{N+1}$.

In particular,

$$\mathscr{H}_{-1} = (\ell^2(\mathsf{Des}(u_{\kappa_3})))^\perp \subseteq \operatorname{ran} R_2, \qquad \ell^2(\mathsf{Des}(u_{\kappa_4})) \subseteq \mathscr{K}_{-1} \perp \operatorname{ran} R_2,$$

so that

$$\begin{aligned}\ell^2(\mathsf{Des}(u_{\kappa_1})) &\subseteq \operatorname{ran} R_1 \subseteq \ell^2(\mathsf{Des}(u_{-\kappa_2})),\\ \ell^2(\mathsf{Des}(u_{\kappa_3}))^\perp &\subseteq \operatorname{ran} R_2 \subseteq \ell^2(\mathsf{Des}(u_{\kappa_4}))^\perp.\end{aligned}$$

Since $\kappa_3 \ge \kappa_1 + 2N$, it follows that

(a) $P := R_2 R_1 = R_1 R_2$ is a projection,
(b) $\operatorname{ran} P = \operatorname{ran} R_2 \cap \operatorname{ran} R_1 \subseteq \ell^2(\mathsf{Des}(u_{-\kappa_2})) \ominus \ell^2(\mathsf{Des}(u_{\kappa_4}))$, which is finite-dimensional,
(c) $\operatorname{ran} P = \operatorname{ran} R_2 \cap \operatorname{ran} R_1 \supseteq \ell^2(\mathsf{Des}(u_{\kappa_3}))^\perp \cap \mathscr{L}_N = \mathscr{L}_N$.

Moreover,

$$\begin{aligned}\|PS_V - S_V P\| &= \|R_2 R_1 S_V - S_V R_2 R_1\|\\ &= \|R_2(R_1 S_V - S_V R_1) - (R_2 S_V - S_V R_2) R_1\|\\ &\le \|R_2\|\,\|R_1 S_V - S_V R_1\| + \|R_2 S_V - S_V R_2\|\,\|R_1\|\\ &< \frac{2\pi M}{N+1}.\end{aligned}$$

It now follows from Lemma 2.1 that S_V is quasidiagonal. ∎

Our final analogue of the bilateral weighted shift case is the case where S_{V_2} is quasidiagonal and S_{V_1} is not. We simplify the proof of the quasidiagonality of S_V by first proving that S_V° is quasidiagonal, where S_V° is a finite-rank perturbation of S_V.

LEMMA 5.6. *Let $\mathscr{T} = (V, E)$ be an M-ary directed tree containing exactly one double ray, and suppose that for all $N \ge 1$,*

$$\mathcal{G}_{\mathrm{ess}}^N(V_1') \cap \mathcal{G}_{\mathrm{ess}}^N(V_2 \setminus V_2') \ne \varnothing.$$

Let $S_V^\circ = S_V - e_{u_0} \otimes e_{u_{-1}}$. *Then for all* $\kappa_0 \geq 0$, *there exist* $\kappa_1, \kappa_2 \geq \max(\kappa_0, 2N)$ *and a finite-rank projection* R *such that*

(i) $\mathscr{L}_{\kappa_1}^+ \subseteq \operatorname{ran} R$,
(ii) $\mathscr{L}_{\kappa_2}^-$ *is orthogonal to* $\operatorname{ran} R$,
(iii) $\operatorname{ran} R \subseteq \mathscr{L}_\kappa$, *where* $\kappa = \max(\kappa_1 + N, \kappa_2 + 1)$,
(iv) $\|RS_V^\circ - S_V^\circ R\| \leq \frac{\pi M}{N+1}$.

Proof. Recall that $V_1 = \mathsf{Des}(u_0)$ and $V_2 = V \setminus V_1$. We now have $S_V^\circ \simeq S_{V_1}^\circ \oplus S_{V_2}^\circ$, where $S_{V_1}^\circ$ is a shift acting on a rooted tree with vertices V_1 and $\mathsf{root} = u_0$, and $S_{V_2}^\circ$ is a shift acting on a vanishing, rootless tree with vertices V_2.

Let $N \geq 1$ be fixed and $\kappa_0 \geq 1$. By Lemma 5.2, we may find $\kappa_1, \kappa_2 \geq \max(\kappa_0, 2N)$ and $v \in V_2 \setminus V_2'$ with $v \in \mathsf{Des}(u_{-\kappa_2 - 1}) \setminus \mathsf{Des}(u_{-\kappa_2})$ such that $\mathsf{Des}^N(u_{\kappa_1}) \cong \mathsf{Des}^N(v)$.

Let

- $\mathscr{H}_{-1} := \ell^2(V_1) \ominus \ell^2(\mathsf{Des}(u_{\kappa_1}))$,
- $\mathscr{H}_j := \ell^2(\mathsf{Chi}^{\langle j \rangle}(u_{\kappa_1}))$, $0 \leq j \leq N$,
- $\mathscr{H}_{N+1} := \ell^2(V_1) \ominus \bigoplus_{j=-1}^N \mathscr{H}_j$,

and

- $\mathscr{K}_{-1} := \ell^2(V_2) \ominus \ell^2(\mathsf{Des}(v))$,
- $\mathscr{K}_j := \ell^2(\mathsf{Chi}^{\langle j \rangle}(v))$, $0 \leq j \leq N$,
- $\mathscr{K}_{N+1} := \ell^2(V_2) \ominus \bigoplus_{j=-1}^N \mathscr{K}_j$.

This time, we may apply the generalised Berg–Davidson technique (Proposition 2.2) (although in this instance the original version of that result from [7] will suffice) to $S_V^\circ = S_{V_1}^\circ \oplus S_{V_2}^\circ$ to find a projection R such that

- $\mathscr{H}_{-1} \oplus \mathscr{K}_{N+1} \subseteq \operatorname{ran} R \subseteq (\oplus_{j=-1}^N \mathscr{H}_j) \oplus (\oplus_{j=0}^{N+1} \mathscr{K}_j)$;
- $\|RS_V^\circ - S_V^\circ R\| \leq \frac{\pi M}{N+1}$.

Now $\mathscr{L}_{\kappa_1}^+ = \mathscr{H}_{-1} \subseteq \operatorname{ran} R$. Since $\mathscr{L}_{\kappa_2}^- \subseteq \mathscr{K}_{-1}$, we also have $\operatorname{ran} R \subseteq (\mathscr{L}_{\kappa_2}^-)^\perp$.

Finally, note that $\mathscr{H}_j \subseteq \mathscr{L}_{\kappa_1+N}^+$ for all $-1 \leq j \leq N$, while $\mathscr{K}_j \subseteq \ell^2(\mathsf{Des}(v))) \subseteq \mathscr{L}_{\kappa_2+1}^-$, $0 \leq j \leq N+1$, implying that

$$\operatorname{ran} R \subseteq \mathscr{L}_\kappa,$$

where $\kappa = \max(\kappa_1 + N, \kappa_2 + 1)$. In particular, R has finite rank.

This completes the proof. ∎

Proposition 5.7. *Let* $\mathscr{T} = (V, E)$ *be an* M*-ary directed tree containing exactly one double ray, and suppose that for all* $N \geq 1$,

$$\mathcal{G}_{\mathrm{ess}}^N(V_1') \cap \mathcal{G}_{\mathrm{ess}}^N(V_2 \setminus V_2') \neq \varnothing.$$

Let $S_V^\circ = S_V - e_{u_0} \otimes e_{u_{-1}}$ *so that* $S_V^\circ \simeq S_{V_1}^\circ \oplus S_{V_2}^\circ$*, where* $S_{V_1}^\circ$ *is a shift acting on a rooted tree with vertices* V_1 *and* $\mathsf{root} = u_0$*, and* $S_{V_2}^\circ$ *is a shift acting on a vanishing, rootless tree with vertices* V_2*.*

If $S_{V_2}^\circ$ *is quasidiagonal, then so is* S_V*.*

Proof. By Theorem 4.4, the fact that $S_{V_2}^\circ$ is quasidiagonal implies that for all $N \geq 1$ and $u \in V_2$,

$$\mathcal{G}_{\text{ess}}^N(V_2 \cap V_{\mathscr{T}(u)}) \cap \mathcal{G}_{\text{ess}}^N(V_2 \setminus V_{\mathscr{T}(u)}) \neq \varnothing.$$

By Proposition 4.3 and its proof, we see that given $N \geq 1$, there exists a finite-rank projection R_2 (acting on $\ell^2(V_2)$) and an integer $\kappa_0 \geq N$ such that

$$\mathscr{L}_N^- \subseteq \operatorname{ran} R_2 \subseteq \mathscr{L}_{\kappa_0}^- \quad \text{and} \quad \|R_2 S_{V_2}^\circ - S_{V_2}^\circ R_2\| < \frac{\pi M}{N+1}.$$

We extend the domain of R_2 to all of $\ell^2(V)$ by setting $R_2|_{(\ell^2(V_2))^\perp} = 0$.

By Lemma 5.6, there exist $\kappa_1, \kappa_2 \geq \max(\kappa_0, 2N)$ and a finite-rank projection R_1 such that

- $\mathscr{L}_{\kappa_1}^+ \subseteq \operatorname{ran} R_1 \subseteq \mathscr{L}_\kappa$, where $\kappa = \max(\kappa_1 + N, \kappa_2 + 1)$,
- $\mathscr{L}_{\kappa_1}^+ \subseteq \operatorname{ran} R_1 \subseteq (\mathscr{L}_{\kappa_2}^-)^\perp$,
- $\|R_1 S_V^\circ - S_V^\circ R_1\| < \frac{\pi M}{N+1}$.

Since $\kappa_2 \geq \kappa_0$, it follows that the range of R_1 is orthogonal to that of R_2, so that $R := R_1 + R_2$ is a finite-rank projection. Moreover,

$$\mathscr{L}_N = \mathscr{L}_N^+ \oplus \mathscr{L}_N^- \subseteq \mathscr{L}_{\kappa_1}^+ \oplus \mathscr{L}_N^- \subseteq \operatorname{ran} R_1 \oplus \operatorname{ran}\, R_2 = \operatorname{ran} R.$$

Finally, a routine calculation shows that

$$\|R S_V^\circ - S_V^\circ R\| \leq \|R_1 S_V^\circ - S_V^\circ R_1\| + \|R_2 S_V^\circ - S_V^\circ R_2\| \leq \frac{2\pi M}{N+1}.$$

By Lemma 2.1, S_V° is quasidiagonal. But S_V° is a finite-rank perturbation of S_V, and thus S_V is also quasidiagonal. ∎

The proof of the next result is, unfortunately, significantly different from that of the previous results, as it involves first applying our standard technique to S_V, and then applying it once again to the result of the first application.

PROPOSITION 5.8. *Let* $\mathscr{T} = (V, E)$ *be an* M*-ary directed tree which contains exactly one double ray* $V' \subseteq V$*. Suppose that for every* $N \in \mathbb{N}$ *and* $j = 1, 2$,

$$\mathcal{G}_{\text{ess}}^N(V_j') \cap \mathcal{G}_{\text{ess}}^N(V_{3-j} \setminus V_{3-j}') \neq \varnothing.$$

Then S_V *is quasidiagonal.*

Proof. Let $N \geq 1$ be fixed.

First applying Lemma 5.2 for $j = 2$, we can find $\kappa_1, \kappa_2 \geq 4N$ and $v_1 \in \mathsf{Des}(u_{\kappa_1 - 1}) \setminus (\mathsf{Des}(u_{\kappa_1}) \cup V_1')$ such that

$$\mathsf{Des}^N(u_{-\kappa_2}) \cong \mathsf{Des}^N(v_1).$$

Let

- $\mathscr{H}_{-1} := \ell^2(V) \ominus \ell^2(\mathsf{Des}(u_{-\kappa_2}))$,
- $\mathscr{H}_j := \ell^2(\mathsf{Chi}^{\langle j \rangle}(u_{-\kappa_2}))$, $0 \le j \le N$,
- $\mathscr{K} := \ell^2(V) \ominus \bigoplus_{j=-1}^{N} \mathscr{H}_j$.

Note that $\ell^2(\mathsf{Des}(v_1)) \subseteq \mathscr{K}$. Define

- $\mathscr{K}_{-1} := \mathscr{K} \ominus \ell^2(\mathsf{Des}(v_1))$,
- $\mathscr{K}_j := \ell^2(\mathsf{Chi}^{\langle j \rangle}(v_1))$, $0 \le j \le N$,
- $\mathscr{K}_{N+1} := \mathscr{K} \ominus \bigoplus_{j=-1}^{N} \mathscr{K}_j$.

Note that $S_V(\mathscr{H}_N) \subseteq \ell^2(\mathsf{Chi}^{\langle N+1 \rangle}(u_{-\kappa_2})) \subseteq \mathscr{K}_{-1}$, so that S_V is tridiagonal with respect to the decomposition $\ell^2(V) = (\bigoplus_{j=-1}^{N} \mathscr{H}_j) \oplus (\bigoplus_{j=-1}^{N+1} \mathscr{K}_j)$, and it falls into the paradigm of Section 2.2.

As argued there, there exists a projection Q_1 satisfying

(i) $\mathscr{K}_{-1} \subseteq \operatorname{ran} Q_1 \subseteq (\bigoplus_{j=0}^{N} \mathscr{H}_j) \oplus (\bigoplus_{j=-1}^{N} \mathscr{K}_j)$;
(ii) $\|Q_1 S_V - S_V Q_1\| < \frac{\pi M}{N+1}$.

Applying Lemma 5.2 for $j = 1$ we may find $\kappa_3, \kappa_4 \ge \max(\kappa_1, \kappa_2) + 4N$ and $v_2 \in \mathsf{Des}(u_{-\kappa_4 - 1}) \setminus (\mathsf{Des}(u_{-\kappa_4}) \cup V_2')$ such that

$$\mathsf{Des}^N(v_2) \cong \mathsf{Des}^N(u_{\kappa_3}).$$

Observe that

- $\mathscr{L}_N \subseteq \mathscr{K}_{-1}$,
- $\ell^2(\mathsf{Des}(v_2)) \subseteq \operatorname{ran} Q_1^\perp$,
- $\ell^2(\mathsf{Des}(u_{\kappa_3})) \subseteq \operatorname{ran} Q_1$.

Let $W := Q_1^\perp S_V Q_1^\perp \oplus Q_1 S_V Q_1$, so that $\|W - S_V\| < \frac{\pi M}{N+1}$ by (ii) above. We define

- $\mathscr{M}_{-1} := \operatorname{ran} Q_1^\perp \ominus \ell^2(\mathsf{Des}(v_2))$,
- $\mathscr{M}_j := \ell^2(\mathsf{Chi}^{\langle j \rangle}(v_2))$, $0 \le j \le N$,
- $\mathscr{M}_{N+1} := \operatorname{ran} Q_1^\perp \ominus \bigoplus_{j=-1}^{N} \mathscr{M}_j$.

Relative to $\operatorname{ran} Q_1^\perp = \bigoplus_{j=-1}^{N+1} \mathscr{M}_j$, we find that $Q_1^\perp S_V Q_1^\perp$ is of the form

$$\begin{bmatrix} A_{-1,-1} & & & & & \\ A_{0,-1} & 0 & & & & \\ & A_{1,0} & 0 & & & \\ & & \ddots & \ddots & & \\ & & & A_{N,N-1} & 0 & \\ & & & & A_{N+1,N} & A_{N+1,N+1} \end{bmatrix}.$$

Next, we define

- $\mathscr{N}_{-1} := \operatorname{ran} Q_1 \ominus \ell^2(\mathsf{Des}(u_{\kappa 3}))$,
- $\mathscr{N}_j := \ell^2(\mathsf{Chi}^{\langle j \rangle}(u_{\kappa 3}))$, $0 \le j \le N$,
- $\mathscr{N}_{N+1} := \operatorname{ran} Q_1 \ominus \bigoplus_{j=-1}^{N} \mathscr{N}_j$.

Note that $\mathscr{L}_N \subseteq \mathscr{N}_{-1}$, and $\mathscr{N}_{-1}$ is finite-dimensional. Relative to $\operatorname{ran} Q_1 = \bigoplus_{j=-1}^{N+1} \mathscr{N}_j$, we find that $Q_1 S_V Q_1$ is of the form

$$\begin{bmatrix} B_{-1,-1} & & & & & & \\ B_{0,-1} & 0 & & & & & \\ & B_{1,0} & 0 & & & & \\ & & \ddots & \ddots & & & \\ & & & B_{N,N-1} & 0 & & \\ & & & & B_{N+1,N} & B_{N+1,N+1} \end{bmatrix}.$$

As always, the fact that $\mathsf{Des}^N(v_2) \cong \mathsf{Des}^N(u_{\kappa 3})$ implies that we may assume without loss of generality that $A_{i,j} = B_{i,j}$ for all $1 \le i + j \le 2N + 1$. Using our generalised Berg–Davidson technique (Proposition 2.2), we obtain a projection P such that

$$\mathscr{L}_N \subseteq \mathscr{N}_{-1} \subseteq \operatorname{ran} P \subseteq \Big(\bigoplus_{j=0}^{N} \mathscr{M}_j\Big) \oplus \Big(\bigoplus_{j=-1}^{N} \mathscr{N}_j\Big),$$

and

$$\|PW - WP\| \le \frac{\pi M}{N+1}.$$

Now $\dim \mathscr{M}_j = \dim \mathscr{N}_j < \infty$ for all $0 \le j \le N$, while $\dim \mathscr{N}_{-1} < \infty$ as noted above, so that P has finite rank.

Also,

$$\begin{aligned} \|PS_V - S_V P\| &\le \|P(S_V - W)\| + \|(S_V - W)P\| + \|PW - WP\| \\ &\le 2\|S_V - W\| + \frac{\pi M}{N+1} < \frac{3\pi M}{N+1}. \end{aligned}$$

We finally apply Lemma 2.1 to conclude that S_V is quasidiagonal. ∎

The next result is our main theorem for shifts acting on M-ary directed trees containing exactly one double ray.

Theorem 5.9. *Assume that an M-ary directed tree $\mathscr{T} = (V, E)$ contains exactly one double ray $V' \subseteq V$. Then the following conditions are equivalent:*

(i) *S_V is quasidiagonal,*
(ii) *for every $N \in \mathbb{N}$ and $j = 1, 2$,*

$$\mathcal{G}^N_{\mathrm{ess}}(V'_j) \cap \mathcal{G}^N_{\mathrm{ess}}(V \setminus V'_j) \ne \varnothing,$$

(iii) *one of the following holds:*

(a) S_{V_j} *is quasidiagonal for* $j = 1, 2$,

(b) *for some* $j \in \{1, 2\}$, S_{V_j} *is quasidiagonal and for all* $N \in \mathbb{N}$,

$$\mathcal{G}^N_{ess}(V'_{3-j}) \cap \mathcal{G}^N_{ess}(V_j \setminus V'_j) \neq \varnothing,$$

(c) *for every* $N \in \mathbb{N}$ *and* $j = 1, 2$,

$$\mathcal{G}^N_{\mathrm{ess}}(V'_{3-j}) \cap \mathcal{G}^N_{\mathrm{ess}}(V_j \setminus V'_j) \neq \varnothing,$$

(d) *for every* $N \in \mathbb{N}$,

$$\mathcal{G}^N_{\mathrm{ess}}(V'_1) \cap \mathcal{G}^N_{\mathrm{ess}}(V'_2) \neq \varnothing.$$

Proof. (i)⇒(ii). Suppose we can choose $N \in \mathbb{N}$ and $j \in \{1, 2\}$ such that

$$\mathcal{G}^N_{\mathrm{ess}}(V'_j) \cap \mathcal{G}^N_{\mathrm{ess}}(V \setminus V'_j) = \varnothing.$$

In particular,

$$(9) \qquad \mathcal{G}^N_{\mathrm{ess}}(V'_j) \cap \mathcal{G}^N_{\mathrm{ess}}(V_j \setminus V'_j) = \varnothing \quad \text{and} \quad \mathcal{G}^N_{\mathrm{ess}}(V'_j) \cap \mathcal{G}^N_{\mathrm{ess}}(V_{3-j}) = \varnothing.$$

By Proposition 1.3(iv), we can choose a vertex $u \in V'_1$ such that

$$(10) \qquad \mathsf{Des}(u) \cap \mathsf{Ver}(\mathcal{G}^N \setminus \mathcal{G}^N_{\mathrm{ess}}(V)) = \varnothing.$$

Moreover, applying [15, Proposition 2.1.4.], we can find $k \in \mathbb{N}$ such that

$$\mathsf{Ver}(\mathcal{G}^N \setminus \mathcal{G}^N_{\mathrm{ess}}(V)) \cup \{u_0\} \subset \mathsf{Des}(\mathsf{par}^k(u)).$$

Define

$$\Omega = \mathsf{Des}(\mathsf{par}^k(u)) \setminus \mathsf{Des}(u), \qquad \Omega_1 = \mathsf{Des}(u), \qquad \Omega_2 = V \setminus \mathsf{Des}(\mathsf{par}^k(u)).$$

Let $P_j = P^{\mathcal{W}_j}$, where $\mathcal{W}_j = \mathcal{G}^N_{\mathrm{ess}}(V'_j)$. Then, with respect to the decomposition $\ell^2(V) = \ell^2(\Omega_1) \oplus \ell^2(\Omega) \oplus \ell^2(\Omega_2)$, we may write

$$P_j S_V = \begin{bmatrix} P_j S_{\Omega_1} & R_1 & 0 \\ 0 & P_j S_\Omega & R_2 \\ 0 & 0 & P_j S_{\Omega_2} \end{bmatrix},$$

where $R_1 = (P_j e_u) \otimes e_{\mathsf{par}(u)}$ and $R_2 = (P_j e_{\mathsf{par}^k(u)}) \otimes e_{\mathsf{par}^{k+1}(u)}$. Taking into account (9) and (10), we get

$$P_j S_V = \begin{bmatrix} \delta_{1j} S_{\Omega_1 \cap V'_1} & R_1 & 0 \\ 0 & P_j S_\Omega & R_2 \\ 0 & 0 & \delta_{2j} S_{\Omega_2 \cap V'_2} \end{bmatrix}.$$

By Lemma 3.1 and [18, Theorem 3], $P_j S_V \in C^*(S_V)$ is quasidiagonal. Since $V \setminus V' = V_{\mathrm{van}}$, Ω is finite. Hence, by [14, Theorem 4], $S_{\Omega_j \cap V'_j}$ is also quasidiagonal. However, depending on j, $S_{\Omega_j \cap V'_j}$ is unitarily equivalent to the unilateral shift or to the adjoint of the unilateral shift, which is a contradiction.

(ii)⇒(iii). By Proposition 1.3, for every $j = 1, 2$,

$$\mathcal{G}^N_{\mathrm{ess}}(V'_j) \cap \mathcal{G}^N_{\mathrm{ess}}(V_j \setminus V'_j) \neq \varnothing \quad \text{for every } N \in \mathbb{N}, \tag{11}$$

or

$$\mathcal{G}^N_{\mathrm{ess}}(V'_j) \cap \mathcal{G}^N_{\mathrm{ess}}(V_{3-j} \setminus V'_{3-j}) \neq \varnothing \quad \text{for every } N \in \mathbb{N}, \tag{12}$$

or

$$\mathcal{G}^N_{\mathrm{ess}}(V'_j) \cap \mathcal{G}^N_{\mathrm{ess}}(V'_{3-j}) \neq \varnothing \quad \text{for every } N \in \mathbb{N}. \tag{13}$$

Condition (13) gives us (d). In turn, combining (11), (12), Theorem 3.4, and Theorem 4.4, we obtain (a), (b), or (c).

(iii)⇒(i). Indeed,

- (a)⇒(i): the fact that the direct sum of two quasidiagonal operators is quasidiagonal is standard (see [11, p. 902]);
- (b)⇒(i) follows from Propositions 5.5 and 5.7;
- (c)⇒(i) is the content of Proposition 5.8;
- (d)⇒(i) follows from Proposition 5.3. ∎

6. Examples

6.1. In this section we shall present two examples of quasidiagonal (unweighted) shifts on directed trees corresponding to (a) the rooted case, and (b) the double ray case. Before doing so, we describe an example of a quasidiagonal *weighted* shift acting on a rooted directed tree whose subgraphs of height N fail to satisfy the conditions of Theorem 3.4; more explicitly

$$\mathcal{G}^N_{\mathrm{ess}}(V_{\mathscr{P}}) \cap \mathcal{G}^N_{\mathrm{ess}}(V \setminus V_{\mathscr{P}}) = \varnothing \quad \text{for every } N \in \mathbb{N}.$$

Indeed, by that theorem, the unweighted shift corresponding to the same tree would not be quasidiagonal.

Example 6.1. Let $\mathscr{T} = (V, E)$ be a rooted directed tree, where

$$V = \{(n,m) \in \mathbb{N} \times \mathbb{N}_0 \colon m \le n\}$$

and $((n,m),(k,l)) \in E$ if and only if

- $k - n = 1$ and $m = l = 0$, or
- $n = k$ and $l - m = 1$.

For every $n \in \mathbb{N}$, let u_n denote the vertex $(n, 0)$. It is obvious that u_1 is the root of $\mathscr{T}$ and that $\mathscr{T}$ admits only one path $\mathscr{P} = (V_{\mathscr{P}}, E_{\mathscr{P}})$, where $V_{\mathscr{P}} = \{u_n \colon n \in \mathbb{N}\}$. Moreover,

$$\mathcal{G}^N_{\mathrm{ess}}(V_{\mathscr{P}}) \cap \mathcal{G}^N_{\mathrm{ess}}(V \setminus V_{\mathscr{P}}) = \varnothing \quad \text{for every } N \in \mathbb{N}.$$

Define $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$ by

$$\lambda_{(n,m)} = \begin{cases} 1/\sqrt{2} & \text{if } m \le 1 \text{ and } (n,m) \neq (1,0), \\ 1 & \text{if } m > 1. \end{cases}$$

(I) Let $N \in \mathbb{N}$ be arbitrary. Define $\mathscr{H}_{-1} := \ell^2(V) \ominus \ell^2(\mathsf{Des}(u_N))$. For $0 \leq j \leq N$, set $\mathscr{H}_j := \mathbb{C} S_{\boldsymbol{\lambda}}^j e_{u_N}$. Finally, set $\mathscr{K} := \ell^2(V) \ominus \bigoplus_{j=-1}^{N} \mathscr{H}_j$.

Relative to the decomposition $\ell^2(V) = \mathscr{H}_{-1} \oplus \mathscr{H}_0 \oplus \cdots \oplus \mathscr{H}_N \oplus \mathscr{K}$, the operator matrix for $S_{\boldsymbol{\lambda}}$ has the form

$$\begin{bmatrix} A_{-1,-1} & & & & & \\ A_{0,-1} & 0 & & & & \\ & A_{1,0} & 0 & & & \\ & & \ddots & \ddots & & \\ & & & A_{N,N-1} & 0 & \\ & & & & A_{N+1,N} & A_{N+1,N+1} \end{bmatrix}.$$

(II) Next, let $v_N = (2N+1, 1)$, so that $\ell^2(v_N) \subseteq \mathscr{K}$. We define $\mathscr{K}_{-1} := \mathscr{K} \ominus \ell^2(\mathsf{Des}(v_N))$, and for $0 \leq j \leq N$, we set $\mathscr{K}_j := \ell^2(\mathsf{Chi}^{\langle j \rangle}(v_N))$. Finally, we set $\mathscr{K}_{N+1} := \mathscr{K} \ominus \bigoplus_{j=-1}^{N} \mathscr{K}_j$.

Relative to the decomposition $\mathscr{K} = \mathscr{K}_{-1} \oplus \mathscr{K}_0 \oplus \cdots \oplus \mathscr{K}_{N+1}$, the operator matrix for $A_{N+1,N+1}$ has the form

$$\begin{bmatrix} B_{-1,-1} & & & & & \\ B_{0,-1} & 0 & & & & \\ & B_{1,0} & 0 & & & \\ & & \ddots & \ddots & & \\ & & & B_{N,N-1} & 0 & \\ & & & & B_{N+1,N} & B_{N+1,N+1} \end{bmatrix}.$$

Observe that $S_{\boldsymbol{\lambda}}(\mathscr{H}_N) \subseteq \ell^2(\mathsf{Chi}^{\langle N+1 \rangle}(u_N)) \subseteq \mathscr{K}_{-1}$. From this it follows that the operator matrix $[T_{i,j}]$ for $S_{\boldsymbol{\lambda}}$ relative to the decomposition $\ell^2(V) = \mathscr{H}_{-1} \oplus \mathscr{H}_0 \oplus \cdots \oplus \mathscr{H}_N \oplus \mathscr{K}_{-1} \oplus \mathscr{K}_0 \oplus \mathscr{K}_1 \oplus \cdots \oplus \mathscr{K}_{N+1}$ is tridiagonal, and the only non-zero entries appear either

- on the first subdiagonal, or
- at the $A_{-1,-1}, B_{-1,-1}$ and $B_{N+1,N+1}$ entries.

Moreover, $\mathscr{H}_j$ and $\mathscr{K}_j$ are one-dimensional Hilbert spaces for every $0 \leq j \leq N$.

(III) Let $1 \leq j \leq N$. Then, by [15, Lemma 6.1.1],

$$\begin{aligned} \|S_{\boldsymbol{\lambda}}^j e_{u_N}\|^2 &= \sum_{v \in \mathsf{Chi}^{\langle j \rangle}(u_N)} \Big| \prod_{i=0}^{j-1} \lambda_{\mathsf{par}^i(v)} \Big|^2 \\ &= \sum_{k=0}^{j} \Big| \prod_{i=0}^{j-1} \lambda_{\mathsf{par}^i((N+k, j-k))} \Big|^2 = \sum_{k=0}^{j-1} \frac{1}{2^{k+1}} + \frac{1}{2^j} = 1. \end{aligned}$$

Hence, since $\mathscr{H}_j$, $\mathscr{H}_{j-1}$, $\mathscr{K}_j$, and $\mathscr{K}_{j-1}$ are one-dimensional, we may further assume (after possibly applying a unitary conjugation) that $A_{j,j-1} = B_{j,j-1} = 1$, where $1 \leq j \leq N$.

Now, we can apply the Berg–Davidson technique (Proposition 2.2) to obtain a projection P satisfying

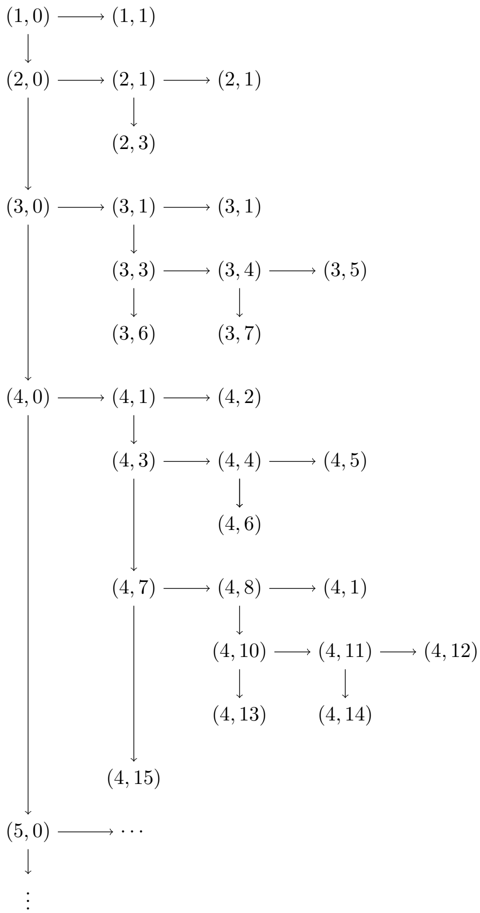
- $\mathscr{H}_{-1} \oplus \mathscr{K}_{N+1} \subseteq \operatorname{ran} P \subseteq (\bigoplus_{j=-1}^{N} \mathscr{H}_j) \oplus (\bigoplus_{j=0}^{N+1} \mathscr{K}_j)$,
- $\|PS_{\boldsymbol{\lambda}} - S_{\boldsymbol{\lambda}}P\| \leq \frac{\pi}{N+1}$.

Hence,

(i) $\mathscr{L}_N = \mathscr{H}_{-1} \subseteq \operatorname{ran} P$,
(ii) $\operatorname{ran} P \subseteq (\bigoplus_{j=-1}^{N} \mathscr{H}_j) \oplus (\bigoplus_{j=0}^{N+1} \mathscr{K}_j) \subseteq \mathscr{L}_{2N+2}$.

Applying Lemma 2.1 we conclude that $S_{\boldsymbol{\lambda}} \in \mathsf{QD}$.

Example 6.2. Consider the directed tree $\mathscr{T}$ described in Figure 2.



. Fig. 2. A directed rooted tree

The construction of this tree is as follows: we define inductively finite directed trees $\mathscr{T}_n$ for every $n \in \mathbb{N}$ and take the union $\bigcup_{n\in\mathbb{N}} \mathscr{T}_n$. Let $\mathscr{T}_1 = (V_1, E_1)$, where $V_1 = \{(1,0)\}$ and $E_1 = \varnothing$. Assume that $\mathscr{T}_n$ is defined for some $n \in \mathbb{N}$. Denote by V_n' the set $\{(n,\ell)\colon 1 \leq \ell \leq \#V_n\}$. Then

$$V_{n+1} = V_n \cup \{(n+1,0)\} \cup V_n'.$$

To define E_{n+1} one puts an edge between $(n,0)$ and $(n+1,0)$ and an edge between $(n,0)$ and $(n,1)$. One also "attaches" to the vertex $(n,1)$ a copy of $\mathscr{T}_n$ defined on V_n', which means that $\mathscr{T}_n \cong \mathscr{T}_n' = (V_n', E_n')$ for some $E_n' \subset V_n' \times V_n'$ and $(n,1)$ is the root of $\mathscr{T}_n'$. Hence,

$$E_{n+1} = E_n \cup \{((n,0),(n+1,0)), ((n,0),(n,1))\} \cup E_n'.$$

Set $\mathscr{T}_{n+1} = (V_{n+1}, E_{n+1})$. Thus we get a sequence of finite trees.

Define

$$\mathscr{T} = (V,E) = \Big(\bigcup_{n\in\mathbb{N}} V_n, \bigcup_{n\in\mathbb{N}} E_n\Big).$$

Note that $\mathscr{T}$ is a directed tree. Its root is $(1,0)$, and the unique path is $\mathscr{P} = \{(n,0)\colon n \in \mathbb{N}\}$ with edges $\{((n,0),(n+1,0))\colon n \in \mathbb{N}\}$.

It is relatively easy to see from the nature of our construction that for any $N \in \mathbb{N}$, the finite subtree $\mathsf{Des}^N(u_0)$ lies in

$$\mathcal{G}^N(V_{\mathscr{P}}) \cap \mathcal{G}^N(V \setminus V_{\mathscr{P}}) \neq \varnothing.$$

As an immediate consequence of Theorem 3.4, the unweighted shift S_V acting on this tree is quasidiagonal.

Example 6.3. Consider the directed tree $\mathscr{T}$ described in Figure 3.

This tree is constructed in the following way: Let $\mathscr{T}_0 = (V_0, E_0)$, where

$$V_0 = (\mathbb{N}_0 \times \{0\}) \cup \{(n,m) \in \mathbb{Z} \times \mathbb{N}_0\colon n < 0 \text{ and } 0 \leq m \leq -n\}$$

and $((n,m),(k,l)) \in E_0$ if and only if either

- $k - n = 1$ and $m = l = 0$, or
- $n = k$ and $l - m = 1$.

Define also $W_k = \{(n,m) \in \mathbb{Z} \times \mathbb{N}_0\colon -k \leq n < 0 \text{ and } 0 \leq m \leq -n\}$ for $k \in \mathbb{N}$. The sets W_n, $n \in \mathbb{N}$, will be considered as induced subtrees of $\mathscr{T}_0$. Denote by W_n' the set $\{(n^2,\ell)\colon \ell \in \mathbb{N}, 1 \leq \ell \leq \frac{1}{2}(n^2+3n)\}$ for $n \in \mathbb{N}$. Then

$$V = V_0 \cup \bigcup_{n\in\mathbb{N}} W_n'.$$

To define E one puts an edge between $(n^2,0)$ and $(n^2,1)$ and "attaches" to the vertex $(n^2,1)$ a copy of W_n defined on W_n' for every $n \in \mathbb{N}$. This means that $W_n \cong \mathscr{T}_n' = (W_n', E_n')$ for some $E_n' \subset W_n' \times W_n'$ and $(n^2,1)$ is the

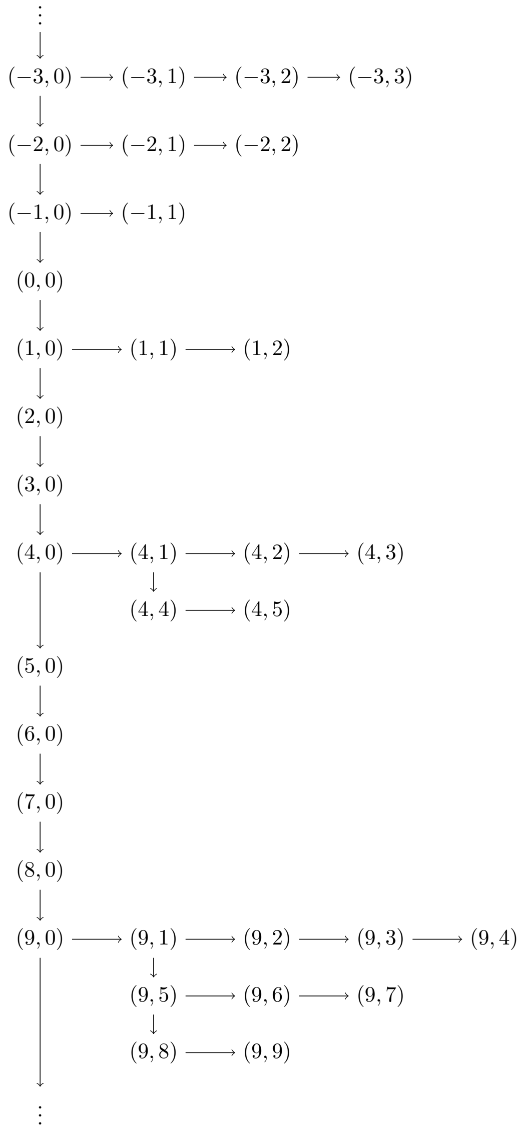


Fig. 3. A directed tree with one double ray

root of $\mathscr{T}_n'$. Hence,

$$E = E_0 \cup \bigcup_{n\in\mathbb{N}} \big(\{((n^2,0),(n^2,1))\} \cup E_n'\big).$$

Consider the infinite tree $\mathscr{T} = (V, E)$.

Letting $u_n = (n, 0)$ for $n \in \mathbb{Z}$, we see that $\mathscr{T}$ admits a unique double ray $\{u_n : n \in \mathbb{Z}\}$. In this case, $\mathcal{G}^N_{\mathrm{ess}}(V_1')$ contains arbitrarily long subtrees of the form $\{(m+1,0),\ldots,(m+N,0)\}$ with edges between $(m+i,0)$ and $(m+i+1,0)$ for $1 \le i \le N-1$.

Such subtrees clearly appear as subtrees in $\mathcal{G}^N_{\mathrm{ess}}(V_2 \setminus V_2')$, corresponding to the vertices $(-r,1),(-r,2),\ldots,(-r,N)$ for all $r \ge N$. Thus

$$\mathcal{G}^N_{\mathrm{ess}}(V_1') \cap \mathcal{G}^N_{\mathrm{ess}}(V_2 \setminus V_2') \neq \varnothing.$$

Meanwhile, by construction, for each $N \ge 1$, we have placed a copy of the subtree of V_2 corresponding to the vertices $\{(-m,\ell)\colon 1 \le m \le N, 0 \le \ell \le m\}$ starting at vertices $(N^2,1)$, from which we deduce that

$$\mathcal{G}^N_{\mathrm{ess}}(V_2') \cap \mathcal{G}^N_{\mathrm{ess}}(V_1 \setminus V_1') \neq \varnothing.$$

By Theorem 5.9, the corresponding unweighted shift S_V acting on this tree is quasidiagonal.

It is worth noting that if $V_1 = \mathsf{Des}(u_0)$, then the corresponding shift S_{V_1} (as defined in Proposition 5.7) is quasidiagonal by Proposition 3.3, since for each $N \ge 1$,

$$\mathcal{G}^N_{\mathrm{ess}}(V_1') \cap \mathcal{G}^N_{\mathrm{ess}}(V_1 \setminus V_1') \neq \varnothing.$$

(Indeed, there exist arbitrarily long subtrees of the form $\{(m+1,0),\ldots,(m+N,0)\}$ with edges between $(m+i,0)$ and $(m+i+1,0)$ for $1 \le i \le N-1$ which lie in the intersection of these two sets.)

Acknowledgements. The first author was supported in part by NSERC (Canada). The second author was supported by European Union under the European Social Fund – within the project: “Integrated Program of the University of Agriculture in Kraków”.

References

[1] C. Apostol, C. Foiaş, and D. Voiculescu, *Some results on non-quasitriangular operators II*, Rev. Roumaine Math. Pures Appl. 18 (1973), 159–181.

[2] C. Apostol, C. Foiaş, and D. Voiculescu, *Some results on non-quasitriangular operators III*, Rev. Roumaine Math. Pures Appl. 18 (1973), 309–324.

[3] C. Apostol, C. Foiaş, and D. Voiculescu, *Some results on non-quasitriangular operators IV*, Rev. Roumaine Math. Pures Appl. 18 (1973), 487–514.

[4] C. Apostol, C. Foiaş, and L. Zsidó, *Some results on non-quasitriangular operators*, Indiana Univ. Math. J. 22 (1973), 1151–1161.

[5] I. D. Berg, *An extension of the Weyl–von Neumann Theorem to normal operators*, Trans. Amer. Math. Soc. 160 (1971), 365–371.

[6] I. D. Berg, *On operators which almost commute with the shift*, J. Operator Theory 11 (1984), 365–377.

[7] I. D. Berg and K. R. Davidson, *Almost commuting matrices and a quantitative version of the Brown–Douglas–Fillmore theorem*, Acta Math. 166 (1991), 121–161.

[8] S. Chavan and S. Trivedi, *An analytic model for left-invertible weighted shifts on directed trees*, J. London Math. Soc. 94 (2016), 253–279.

[9] K. R. Davidson, *C^*-Algebras by Example*, Fields Inst. Monogr. 6, Amer. Math. Soc., Providence, RI, 1996.
[10] R. Diestel, *Graph Theory*, Grad. Texts in Math. 173, Springer, Berlin, 2017.
[11] P. R. Halmos, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. (N.S.) 76 (1970), 887–933.
[12] D. A. Herrero, *Quasidiagonality, similarity and approximation by nilpotent operators*, Indiana Univ. Math. J. 30 (1981), 199–233.
[13] D. A. Herrero, *Approximation of Hilbert Space Operators*, Vol. 1, 2nd ed., Research Notes in Math. 224, Pitman, London, 1990.
[14] G. Luecke, *A note on quasidiagonal and quasitriangular operators*, Pacific J . Math. 56 (1975), 179–185.
[15] Z. J. Jabłoński, I. B. Jung and J. Stochel, *Weighted shifts on directed trees*, Mem. Amer. Math. Soc. 216 (2012), no. 1017, viii+106 pp.
[16] M. Fujii, H. Sasaoka and Y. Watatani, *Adjacency operators of infinite directed graphs*, Math. Japon. 34 (1989), 727-–735.
[17] S. K. Narayan, *Quasidiagonality of direct sums of weighted shifts*, Trans. Amer. Math. Soc. 332 (1992), 757–774.
[18] R. A. Smucker, *Quasidiagonal weighted shifts*, Pacific J . Math. 98 (1982), 173–182.
[19] D. Voiculescu, *Around quasidiagonal operators*, Integral Equations Operator Theory 17 (1993), 137–149.
[20] L. Ward, *Shift operators on Hilbert spaces arising from trees*, Ph.D. thesis, Univ. of Florida, 2016, 43 pp.
[21] H. Weyl, *Über beschränkte quadratische Formen deren Differenz vollstetig ist*, Rend. Circ. Mat. Palermo 27 (1909), 373–392.
[22] W. Winter, *Structure of nuclear C^*-algebras: from quasidiagonality to classification and back again*, in: Proc. International Congress of Mathematicians—Rio de Janeiro 2018, Vol. III, Invited lectures, World Sci., Hackensack, NJ, 2018, 1801–1823.

Laurent W. Marcoux
Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1
E-mail: Laurent.Marcoux@uwaterloo.ca

Artur Płaneta
Katedra Zastosowań Matematyki
Uniwersytet Rolniczy w Krakowie
31-120 Kraków, Poland
E-mail: artur.planeta@urk.edu.pl