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Mean-Expectile Portfolio Selection

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Abstract We consider a mean-expectile portfolio selection problem in a continuous-time diffusion model. We exploit the close relationship between expectiles and the Omega performance measure to reformulate the problem as the maximization of the Omega measure, and show the equivalence between the two problems. After showing that the solution for the mean-expectile problem is not attainable but that the value function is finite, we modify the problem by introducing a bound on terminal wealth and obtain the solution by Lagrangian duality. The global expectile minimizing portfolio and efficient frontier with a terminal wealth bound are also discussed.

Keywords Expectiles · Portfolio Selection · Expectiles · Efficient Frontier · Performance Measures · Omega

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1 Introduction

Since Markowitz [1952] introduced the classical mean-variance model, a large literature has developed on portfolio selection models in which measures of risk and return are balanced. A significant vein in this literature has focused on incorporating risk measures other than variance into the portfolio selection framework. Among others, Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR, a.k.a. Expected Shortfall), are two alternatives that enjoy significant popularity in both academia and practice. For example, Alexander and Baptista [2002] consider a mean-VaR model for portfolio selection assuming normally distributed returns and relate the model to the classical mean-variance analysis. Campbell et al. [2001] consider portfolio selection by maximizing expected return subject to a constraint on VaR. Although VaR is popular, it has been widely criticized for some of its undesirable properties such as the lack of subadditivity, see Artzner et al. [1999]. Recognizing the shortcomings of VaR, the mean-CVaR model has attracted significant attention (see, e.g. Rockafellar and Uryasev [2000]). Other work has focused general classes of risk measures, such as distortion risk measures, and spectral risk measures; see Sereda et al. [2010], Adam et al. [2008].

Expectiles were introduced by Newey and Powell [1987], as the minimizers of a piecewise quadratic loss function. In recent years, there has been an increased interest in expectiles as an alternative risk measure, as they are the only coherent risk measures with the property of elicitability. Elicitability is a concept introduced by Osband [1985]. In practice, elicitability corresponds to the existence of a natural backtesting methodology making it possible to compare different statistical methods when estimating risk from historical data. Further details on elicitability and other properties of expectiles can be found in Emmer et al. [2015], which also presents comparisons between several widely used risk measures.

To our knowledge, only a small number of papers have investigated the mean-risk portfolio selection problem with expectiles. For example, Jakobsons [2016] used a scenario aggregation method for expectile optimization, and Cai and Weng [2016] studied the problem of optimal reinsurance design using the expectile as a risk measure. While both Cai and Weng [2016] and the present paper employ expectiles as the objective to optimize, there are substantial differences between these two papers. First, Cai and Weng [2016] seeks to determine an optimal partition on a given insurance risk (a given nonnegative random variable) into two nonnegative parts: the risk ceded to a reinsurer, and the risk retained by an insurer. In contrast, the present paper aims to develop an optimal portfolio strategy rebalancing over multiple risky assets. Second, the two papers apply distinct methodologies in deriving optimal solutions. The problem of Cai and Weng [2016] was formulated into determining an optimal function over a certain feasible set, and an optimal solution was explicitly constructed. In contrast, the present paper applies the martingale method and Lagrangian duality to determine an optimal solution to the portfolio selection problem.

Most of the above-mentioned literature concerning mean-risk analysis with different risk measures is in a discrete-time framework, typically on a finite sample space. There have been extensions of the classical mean-variance model from the discrete-time setting to a dynamic continuous-time framework; see Zhou and Li [2000]. Applications of other risk measures in the mean-risk portfolio selection problem under a dynamic continuous-time setting have been developed as well in the past decades, such as Jin et al. [2005], and He et al. [2015]. It is worth noting that the results in He et al. [2015] show a vertical line efficient frontier in the mean-risk plane, which differs from our results for a constrained version of the problem.

Our paper contributes to the literature by considering a mean-risk portfolio choice problem with the expectile risk measure in a dynamic continuous-time framework. Owing to the implicit definition of the expectile as the minimizer of a piecewise quadratic loss function, our problem lacks an explicit form for the objective function. We exploit its connection with the Omega performance measure (see Bellini et al. [2016]) to relate the expectile minimization problem to an Omega maximization problem. Based on this relationship, we show that the expectile minimization problem has a finite optimal value, but this value is not attainable (i.e. there does not exist a feasible portfolio that attains the finite infimum). Employing a modification from the literature, e.g. Bernard et al. [2019] and Chiu et al. [2012], we impose an additional constraint by introducing an upper bound on the terminal wealth. In this modified setting, we consider the global expectile minimizing portfolio and obtain a mean-expectile efficient frontier, resembling the one from the classical mean-variance model. It is worth stressing that our findings are based on a complete market model. The optimal mean-expectile portfolio strategies may be attainable in a general incomplete model.

The remainder of the paper is structured as follows. Section 2 presents the formulation of a portfolio selection problem with the expectile as objective function, introduces an optimization problem with the Omega measure, discusses the relationship between both problems, and shows that there is no optimal solution for the mean-expectile problem, even though the optimal value is finite. In Section 3, we modify the problem by imposing an upper bound on the terminal wealth and solve the problem using Lagrangian duality. Section 4 considers the global expectile minimizing portfolio. Section 5 presents the mean-expectile efficient frontier, and a numerical example. Section 6 summarizes and presents concluding remarks. The Appendix contains some technical proofs.

2 Model Formulation and Preliminary Analysis

2.1 Financial Market Model

We assume that an agent, with initial wealth $x_0 > 0$, invests capital in a risk-free bond B and p risky assets S with price processes as follows:

$$\begin{cases} dB_t = rB_t dt, \\ dS_t^{(i)} = S_t^{(i)} \left[\mu^{(i)} dt + \sum_{j=1}^{p} \sigma_{ij} dW_t^{(j)} \right], \quad i = 1, \cdots, p, \end{cases} \tag{1}$$

where $r > 0$ is the risk-free rate, $\mu^{(i)} > r$ is the growth rate of risky asset i, for $i = 1, \cdots, p$, and $\mu = (\mu^{(1)}, \cdots, \mu^{(p)})^\top$. $\sigma = \{\sigma_{ij}\}_{1 \leq i,j \leqslant p}$ is the corresponding volatility matrix, which is invertible with inverse σ^{-1}. $W \equiv \{W_t, t \geqslant 0\} := \{(W_t^{(1)}, \cdots, W_t^{(p)})^\top, t \geqslant 0\}$ is a standard $\mathbb{R}^p$-valued Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We use $\mathbb{F} := \{\mathcal{F}_t, t \geqslant 0\}$ to denote the $\mathbb{P}$-augmentation of the natural filtration generated by the Brownian motion W.

We consider a finite investment time horizon $[0, T]$ with $T > 0$. Let $\pi_t := (\pi_t^{(1)}, \cdots, \pi_t^{(p)})^\top$, where $\pi_t^{(i)}$ denotes the dollar amount of capital invested in the ith risky asset at time t, for $t \geqslant 0$ and $i = 1, \ldots, p$. With the trading strategy $\pi := \{\pi_t, 0 \leqslant t \leqslant T\}$, the portfolio value process, denoted by X_t^π, evolves according to the following stochastic differential equation (SDE):

$$dX_t^\pi = [rX_t^\pi + \pi_t^\top (\mu - r\mathbf{1})]dt + \pi_t^\top \sigma dW_t, \;\; t \geqslant 0, \tag{2}$$

where $\mathbf{1}$ denotes the p-dimensional column vector with each element equal to 1. It is natural to assume that the trading strategy π is $\mathbb{F}$-progressively measurable and satisfies $\int_0^T \|\pi_t\|^2 dt < \infty$ a.s., where $\|\pi_t\|^2 = \sum_{i=1}^p \left(\pi_t^i\right)^2$, so that a unique strong solution exists for the SDE (2).

Definition 1 A trading strategy $\pi := \{\pi_t, 0 \leqslant t \leqslant T\}$ is called admissible with initial wealth $x_0 > 0$ if it belongs to the following set:

$$\mathcal{A}(x_0) := \{\pi \in \mathcal{S} : \;\; X_0^\pi = x_0 \;\text{ and }\; X_t^\pi \geqslant 0, \text{a.s.}, \forall\, 0 \leqslant t \leqslant T\},$$

where $\mathcal{S}$ denotes the set of $\mathbb{F}$-progressively measurable processes π such that $\int_0^T \|\pi_t\|^2 dt < \infty$ a.s.

We consider the market price of risk, defined as

$$\zeta \equiv (\zeta_1, \ldots, \zeta_p)^\top := \sigma^{-1}(\mu - r\mathbf{1}),$$

and the state-price density process, given by

$$\xi_t := \exp\left\{-\left(r + \frac{\|\zeta\|^2}{2}\right)t - \zeta^\top W_t\right\}, \tag{3}$$

We further employ the notation:

$$\xi_{t,s} = \xi_t^{-1}\xi_s = \exp\left[-\left(r + \frac{\|\zeta\|^2}{2}\right)(s - t) - \zeta^\top (W_s - W_t)\right], \;\; t \leqslant s, \tag{4}$$

Note that $\xi_t = \xi_{0,t}$, and $\xi_{t,s}$ is independent of $\mathcal{F}_t$ under $\mathbb{P}$. Consequently, we can introduce an equivalent risk-neutral measure $\mathbb{Q}$ defined by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{rt}\xi_t$$

so that $W_t^{\mathbb{Q}} := W_t + \zeta t$ is a Brownian motion under $\mathbb{Q}$, and

$$\xi_t := \exp\left\{-\left(r - \frac{\|\zeta\|^2}{2}\right)t - \zeta^\top W_t^{\mathbb{Q}}\right\}. \tag{5}$$

2.2 Expectiles

The expectile $\mathcal{E}_Y(\alpha)$ of a loss random variable Y with $\mathbb{E}[Y^2] < \infty$ at a confidence level $\alpha \in (0,1)$ is defined as the unique minimizer of an asymmetric quadratic function:

$$\mathcal{E}_Y(\alpha) = \arg\min_{m\in\mathbb{R}} \left\{ \alpha\mathbb{E}\left[(Y-m)_+^2\right] + (1-\alpha)\mathbb{E}\left[(m-Y)_+^2\right]\right\}, \tag{6}$$

where $(x)_+ := \max(x,0)$. It is easy to show (see, e.g. Bellini et al. [2014]) that $\mathcal{E}_Y(\alpha)$ solves the above optimization problem (6) if and only if

$$\alpha\mathbb{E}\left[(Y-\mathcal{E}_Y(\alpha))_+\right] = (1-\alpha)\mathbb{E}\left[(\mathcal{E}_Y(\alpha)-Y)_+\right]. \tag{7}$$

It can be verified that there exists a unique solution $\mathcal{E}_Y(\alpha)$ to equation (7) (e.g., Newey and Powell [1987], and Cai and Weng [2016]). Further, a simple rearrangement of equation (7) using the equality $(x)_+ - (-x)_+ = x$ yields the following expression:

$$\mathcal{E}_Y(\alpha) = \mathbb{E}[Y] + \beta\,\mathbb{E}\left[(Y-\mathcal{E}_Y(\alpha))_+\right] \quad \text{with} \quad \beta = \frac{2\alpha-1}{1-\alpha} \quad \text{and} \quad 0<\alpha<1. \tag{8}$$

In particular, for $\alpha = 1/2$, $\beta = 0$, and thus $\mathcal{E}_Y(1/2) = \mathbb{E}[Y]$. For a random variable Y with $\mathbb{E}[|Y|] < \infty$, we adopt equation (7) or equivalently (8) as the definition of the expectile.

The following lemma summarizes some properties of expectiles that are useful in the sequel.

Lemma 1 *For a loss random variable Y with $\mathbb{E}[|Y|] < \infty$ and $\alpha \in (0,1)$, we have the following:*

(a) $\mathcal{E}_{Y+h}(\alpha) = \mathcal{E}_Y(\alpha) + h$, *for each* $h \in \mathbb{R}$,

(b) $\mathcal{E}_{-Y}(\alpha) = -\mathcal{E}_Y(1-\alpha)$,

(c) $\mathcal{E}_Y(\alpha)$ *is strictly increasing and continuous with respect to* α *for a given* Y *with a non-degenerate distribution under* $\mathbb{P}$,

(d) $\lim_{\alpha\to 0^+} \mathcal{E}_Y(\alpha) = \operatorname{ess\,inf}(Y)$ *and* $\lim_{\alpha\to 1^-} \mathcal{E}_Y(\alpha) = \operatorname{ess\,sup}(Y)$.

Proof See Bellini et al. [2014, Propositions 5 and 7]. □

Bellini et al. [2014] show that when $\alpha > 1/2$, the expectile is a coherent risk measure in the sense of Artzner et al. [1999]. Consequently, in the sequel we restrict ourselves to the case $\alpha \in (\frac{1}{2}, 1)$.

2.3 Relationship between Expectiles and the Omega Performance Measure

The Omega was introduced by Keating and Shadwick [2002], and has become a popular performance measure. For a random return R and a benchmark return level l, it is defined as follows:

$$\Omega_R(l) = \frac{\int_l^M [1-F_R(x)]\,dx}{\int_m^l F_R(x)dx} = \frac{\mathbb{E}\left[(R-l)_+\right]}{\mathbb{E}\left[(l-R)_+\right]}. \tag{9}$$

where F_R denotes the cumulative distribution function of R, and m and M are respectively the essential infimum and essential supremum of the return under the physical measure $\mathbb{P}$.

A simple connection between the Omega measure and expectile can be observed by comparing (7) and (9) as follows:

$$\Omega_Y\left(\mathcal{E}_Y(\alpha)\right)=\frac{\mathbb{E}\left[(Y-\mathcal{E}_Y(\alpha))_+\right]}{\mathbb{E}\left[(\mathcal{E}_Y(\alpha)-Y)_+\right]}=\frac{1-\alpha}{\alpha}, \tag{10}$$

which, as observed by Bellini et al. [2016], yields the following one-to-one relation:

$$\Omega_Y(l)=\frac{1-\mathcal{E}_Y^{-1}(l)}{\mathcal{E}_Y^{-1}(l)},\ \ l\in\mathbb{R}, \tag{11}$$

with $\mathcal{E}_Y^{-1}(\cdot)$ denoting the inverse function of $\mathcal{E}_Y(\cdot)$ which exists due to part (c) of Lemma 1 for Y with a non-degenerate distribution.

From (10), we see that the expectile is the value of the threshold that makes the ratio of the *expectation* of the amount by which the loss exceeds the threshold to the *expectation* of the amount by which it is below the threshold equal to $\frac{1-\alpha}{\alpha}$. An analogous property holds for VaR, which is the value of the threshold such that the ratio of the *probability* of exceeding the threshold to the *probability* of being below the threshold equals $\frac{1-\alpha}{\alpha}$; see Bellini and Di Bernardino [2017] for a comparison of the financial meanings of expectiles, VaR, and CVaR.

We summarize some useful properties of the Omega measure in the following lemma (see also Theorem 2 in Bellini et al. [2016]).

Lemma 2 *Denote* $m:=\operatorname{ess\,inf}(R)$ *and* $M:=\operatorname{ess\,sup}(R)$ *for a nondegenerate random variable* R*. The function* $\Omega_R:(m,M)\to(0,\infty)$ *is strictly positive, continuous and strictly decreasing with* $\lim\limits_{l\to m^+}\Omega_R(l)=\infty$, $\lim\limits_{l\to M^-}\Omega_R(l)=0$ *and* $\Omega_R\left(\mathbb{E}[R]\right)=1$.

Proof See Section 3 of Keating and Shadwick [2002].

2.4 The Mean-Expectile Optimization Problem

We consider a mean-risk portfolio choice problem using the expectile as the risk measure. An agent has initial wealth x_0 and undertakes dynamic trading strategies to minimize the risk of the portfolio measured by the expectile of the loss random variable at the final time T, given a prespecified expected wealth target at T. The loss random variable at T is defined as $L:=x_0e^{rT}-X_T^{\pi}$ where X_T^{π} is the wealth accumulated at T and x_0e^{rT} is the terminal wealth attained by allocating all the capital to the risk-free asset. The optimization problem is formulated as follows:

$$\begin{cases}\inf\limits_{\pi\in\mathcal{A}(x_0)} & \mathcal{E}_L(\alpha),\\ \text{subject to} & \mathbb{E}[X_T^{\pi}]=d,\\ & \mathbb{E}[\xi_TX_T^{\pi}]\leqslant x_0.\end{cases} \tag{12}$$

By the martingale approach (see Karatzas and Shreve [1998]), as well as parts (a) and (b) of Lemma 1, it is equivalent to study the following optimal terminal payoff problem:

$$\begin{cases}\sup\limits_{Z\in\mathcal{M}_+} & \mathcal{E}_Z(1-\alpha),\\ \text{subject to} & \mathbb{E}[Z]=d,\\ & \mathbb{E}[\xi_TZ]\leqslant x_0,\end{cases} \tag{13}$$

where $\mathcal{M}_+$ denotes the set of non-negative $\mathcal{F}_T$-measurable random variables. We denote the feasible set of the above problem by $C_1(d, x_0)$, i.e.,

$$C_1(d, x_0) := \{Z \in \mathcal{M}_+ \mid \mathbb{E}[Z] = d \text{ and } \mathbb{E}[\xi_T Z] \le x_0\}. \tag{14}$$

Remark 1 From the financial point of view, it is more meaningful to consider the optimization problem:

$$\begin{cases} \inf_{\pi \in \mathcal{A}(x_0)} & \mathcal{E}_L(\alpha), \\ \text{subject to} & \mathbb{E}[Z] \ge d, \\ & \mathbb{E}[\xi_T Z] \leqslant x_0, \end{cases} \tag{15}$$

in which an inequality constraint is applied to expected terminal wealth $\mathbb{E}[Z]$, instead of an equality as in (13). Considering the equality constraint in (13) simplifies the problem. In fact, assuming the existence of the solution to both problems (12) and (15), the strategy obtained from problem (12) (resp. problem (15)) corresponds to a strategy lying on an expectile minimizing frontier (resp. efficient frontier). Later on, we will show how to obtain the solution with an inequality constraint on the mean from the one with an equality constraint; see Section 5.

We impose the following assumptions throughout.

H1. The required expected wealth d satisfies $d > x_0 e^{rT}$.

H2. The confidence level α satisfies $\frac{1}{2} < \alpha < 1$.

Remark 2 Without assumption **H1**, investing all wealth in the risk-free asset would yield the required terminal wealth without any risk. As noted above, **H2** implies that the expectile is a coherent risk measure. In addition, it implies that $0 < \mathcal{E}_Z(1-\alpha) < \mathbb{E}[Z] = d$ for any $Z \in C_1(d, x_0)$ by equation (8) and **H1**.

2.5 The Mean-Omega Optimization Problem

As the expectile is defined implicitly, through a minimization problem, it is difficult to obtain a solution for the optimization problem (13) directly. However, given the close relationship between the Omega and expectiles, i.e., equations (10) and (11), we propose a family of Mean-Omega optimization problems indexed by $K \in (0, d)$ to connect to the problem (13) as follows:

$$g(K; x_0) = \sup_{Z \in C_1(d, x_0)} \Omega_Z(K), \tag{16}$$

where $C_1(d, x_0)$ is defined in (14). We confine the parameter K within $(0, d)$ because the equivalence between problems (13) and (16) only requires a subset of $(0, d)$ for K under assumption **H2**; see Proposition 3 below for details. While $\Omega_Z(K)$ is the Omega measure applied to the terminal portfolio value, it can indeed be interpreted as the Omega measure applied to the simple return: let $R = \frac{Z}{x_0} - 1$ to get

$$\mathcal{E}_R(l) = \frac{\mathbb{E}[(R-l)_+]}{\mathbb{E}[(l-R)_+]} = \frac{\mathbb{E}[(Z - x_0(1+l))_+]}{\mathbb{E}[(x_0(1+l) - R)_+]} = \frac{\mathbb{E}[(Z-K)_+]}{\mathbb{E}[(K-Z)_+]}, \quad \text{with } K = x_0(1+l).$$

Since $\mathbb{E}\left[(Z-K)_+\right] = \mathbb{E}\left[Z-K\right] + \mathbb{E}\left[(K-Z)_+\right] = d - K + \mathbb{E}\left[(K-Z)_+\right]$ for $Z \in C_1(d, x_0)$, the objective in (16) can be rewritten as

$$\frac{d-K}{\mathbb{E}[(K-Z)_+]} + 1.$$

Thus, in order to study properties of problem (16), we consider the following problem:

$$\tilde{g}(K; x_0) = \inf_{Z \in C_1(d,x_0)} \mathbb{E}\left[(K-Z)_+\right]. \tag{17}$$

$\tilde{g}$ is clearly increasing in K. The following proposition establishes that $\tilde{g}(\cdot; x_0)$ is Lipschitz continuous, with constant 1.

Proposition 1 *Assume **H1**, and suppose that $K_1, K_2 \in (0, d)$. Then $|\tilde{g}(K_1; x_0) - \tilde{g}(K_2; x_0)| \le |K_1 - K_2|$.*

Proof Without loss of generality, assume $K_1 > K_2$. Let $\varepsilon > 0$ and Z_i be such that $\mathbb{E}\left[(K_i - Z_i)_+\right] \le \tilde{g}(K_i; x_0) + \varepsilon$, $i = 1, 2$. Using the inequality $\mathbb{E}\left[(K_1 - Z)_+\right] - \mathbb{E}\left[(K_2 - Z)_+\right] \le K_1 - K_2$, we get

$$\tilde{g}(K_1; x_0) \le \mathbb{E}\left[(K_1 - Z_2)_+\right] \le \mathbb{E}\left[(K_2 - Z_2)_+\right] + (K_1 - K_2) \le \tilde{g}(K_2; x_0) + \varepsilon + (K_1 - K_2),$$

and the result follows by letting $\varepsilon \searrow 0$. □

The following proposition demonstrates some properties of both problems (16) and (17).

Proposition 2 *Assume **H1** and $0 < K < d$.*

(a) If $0 < K < x_0 e^{rT} < d$, then $\tilde{g}(K; x_0) = 0$ and $g(K; x_0) = \infty$, i.e. problem (16) is unbounded.
(b) If $0 < x_0 e^{rT} = K < d$, then there exists a sequence of $Z_n \in C_1(d, x_0)$ such that $\lim_{n\to\infty} \mathbb{E}\left[(K - Z_n)_+\right] = 0$ and $\lim_{n\to\infty} \Omega_{Z_n}(K) = \infty$.
(c) If $0 < x_0 e^{rT} < K < d$, then $\tilde{g}(K; x_0) > 0$, $g(K; x_0) < \infty$, and the optima for both problems (16) and (17) are not attained.

Proof We only present the proof of part (a) and relegate those of parts (b) and (c) to Appendix A. Consider Z defined as follows:

$$Z = K + \left(x_0 - K e^{-rT}\right) \frac{\xi_T^{\beta-1} \mathbf{1}_{\{\xi_T \le \delta\}}}{\mathbb{E}\left[\xi_T^{\beta} \mathbf{1}_{\{\xi_T \le \delta\}}\right]}, \tag{18}$$

for some $\beta > 1$, where δ is chosen so that $\mathbb{E}[Z] = d$. It is easy to see that $\mathbb{E}[\xi_T Z] = x_0$.

Since $Z \ge K$ a.s., $\mathbb{E}\left[(K - Z)_+\right] = 0$, thus $\tilde{g}(K; x_0) = 0$ and $g(K; x_0) = \infty$, so that problem (16) is unbounded.

It remains to justify the existence of both β and δ. From (3) and (5) we know that

$$\begin{aligned}\xi_T^{\beta-1} &= \exp\left\{-\left(r + \frac{||\zeta||^2}{2}\right)(\beta-1)T - (\beta-1)\zeta^\top W_T\right\} =: m_{\beta,1}(T)\Lambda_{\beta,1}(T),\\ &= \exp\left\{-\left(r - \frac{||\zeta||^2}{2}\right)(\beta-1)T - (\beta-1)\zeta^\top W_T^{\mathbb{Q}}\right\} =: m_{\beta,2}(T)\Lambda_{\beta,2}(T),\end{aligned}$$

where

$$\begin{cases} m_{\beta,1}(T) := \exp\left\{-r(\beta-1)T + \frac{||\zeta||^2}{2}(\beta-1)(\beta-2)T\right\}, \\ \Lambda_{\beta,1}(T) := \exp\left\{-\frac{||\zeta||^2}{2}(\beta-1)^2\cdot T - (\beta-1)\zeta^\top W_T\right\}, \\ m_{\beta,2}(T) := \exp\left\{-r(\beta-1)T + \frac{||\zeta||^2}{2}(\beta-1)\beta T\right\}, \\ \Lambda_{\beta,2}(T) := \exp\left\{-\frac{||\zeta||^2}{2}(\beta-1)^2\cdot T - (\beta-1)\zeta^\top W_T^{\mathbb{Q}}\right\}. \end{cases}$$

We introduce two equivalent measures defined by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \Lambda_{\beta,1}(T) \text{ and } \frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = \Lambda_{\beta,2}(T)$$

so that $W_T^{\tilde{\mathbb{P}}} = W_T + (\beta-1)\zeta\cdot T$ and $W_T^{\tilde{\mathbb{Q}}} = W_T^{\mathbb{Q}} + (\beta-1)\zeta\cdot T$ are Brownian motions under $\tilde{\mathbb{P}}$ and $\tilde{\mathbb{Q}}$ respectively, from which we obtain that

$$\frac{\mathbb{E}\left[\xi_T^{\beta-1}\mathbf{1}_{\{\xi_T\le\delta\}}\right]}{\mathbb{E}\left[\xi_T^{\beta}\mathbf{1}_{\{\xi_T\le\delta\}}\right]} = \frac{e^{rT}\mathbb{E}\left[\xi_T^{\beta-1}\mathbf{1}_{\{\xi_T\le\delta\}}\right]}{\mathbb{E}^{\mathbb{Q}}\left[\xi_T^{\beta-1}\mathbf{1}_{\{\xi_T\le\delta\}}\right]} = \frac{e^{rT}m_{\beta,1}(T)\cdot\tilde{\mathbb{P}}\left(\xi_T\le\delta\right)}{m_{\beta,2}(T)\cdot\tilde{\mathbb{Q}}\left(\xi_T\le\delta\right)} = \frac{e^{\left[r-||\zeta||^2(\beta-1)\right]T}\cdot\Phi(a)}{\Phi(a-||\zeta||\sqrt{T})},$$

where $a = \frac{\ln\delta + \left[r-(\beta-\frac{3}{2})||\zeta||^2\right]T}{||\zeta||\sqrt{T}}$.

We can verify that

$$\begin{cases} \frac{\mathbb{E}\left[\xi_T^{\beta-1}\mathbf{1}_{\{\xi_T\le\delta\}}\right]}{\mathbb{E}\left[\xi_T^{\beta}\mathbf{1}_{\{\xi_T\le\delta\}}\right]} \to \infty \text{ as } \delta\to 0, \\ \frac{\mathbb{E}\left[\xi_T^{\beta-1}\mathbf{1}_{\{\xi_T\le\delta\}}\right]}{\mathbb{E}\left[\xi_T^{\beta}\mathbf{1}_{\{\xi_T\le\delta\}}\right]} \to e^{\left[r-||\zeta||^2(\beta-1)\right]T} < e^{rT} \text{ as } \delta\to\infty. \end{cases}$$

Consequently, given β we can find δ such that $\frac{\mathbb{E}[\xi_T^{\beta-1}\mathbf{1}_{\{\xi_T\le\delta\}}]}{\mathbb{E}[\xi_T^{\beta}\mathbf{1}_{\{\xi_T\le\delta\}}]} = \frac{d-K}{x_0-Ke^{-rT}} > e^{rT}$, i.e. $\mathbb{E}[Z]=d$. □

2.6 Nonexistence of Optimal Solutions to the Mean-Expectile Problem

Denote the mean-expectile optimal payoff problem (13) and the mean-omega optimal payoff problem (16) by $P_1(\alpha)$ and $P_2(K)$ respectively. Consider the set of all optimal solutions to $P_1(\alpha)$ as α ranges over $(\frac{1}{2},1)$:

$$\Pi_{P_1} := \bigcup_{\alpha\in(\frac{1}{2},1)} \{Z^* \mid Z^* \text{ is optimal for } P_1(\alpha)\}. \tag{19}$$

In this section, we will show that $\Pi_{P_1}=\emptyset$, which motivates us to consider a modification of the problem in the next section.

We have already shown in Proposition 2 that the corresponding optimal solution set for the Mean-Omega problem is the empty set, i.e.

$$\Pi_{P_2} := \bigcup_{K\in(x_0e^{rT},d)} \{Z^* \mid Z^* \text{ is optimal to } P_2(K)\} = \emptyset. \tag{20}$$

Emptyness of Π_{P_1} then follows from the following proposition.

Proposition 3 $\Pi_{P_1} \subseteq \Pi_{P_2}$.

Proof Suppose $Z^* \in \Pi_{P_1}$, and let $\alpha \in (\frac{1}{2}, 1)$ be such that $\mathcal{E}_{Z^*}(1-\alpha) \geq \mathcal{E}_Z(1-\alpha)$ for any $Z \in C_1(d, x_0)$. Letting $K = \mathcal{E}_{Z^*}(1-\alpha)$, we obtain

$$\Omega_{Z^*}(K) = \Omega_{Z^*}\left(\mathcal{E}_{Z^*}(1-\alpha)\right) = \frac{\alpha}{1-\alpha} = \Omega_Z\left(\mathcal{E}_Z(1-\alpha)\right) \geq \Omega_Z\left(\mathcal{E}_{Z^*}(1-\alpha)\right) = \Omega_Z(K), \tag{21}$$

where the second and the third equalities follow from equation (10), and the inequality follows because $\Omega_Z(\cdot)$ is decreasing by Lemma 2.

It remains to prove that $K \in (x_0 e^{rT}, d)$. Firstly, $K < d$ by equation (8) and the fact that $\alpha \in (\frac{1}{2}, 1)$[1]. If $K \leq x_0 e^{rT}$, by Proposition 2, we can construct a feasible strategy (for $K < x_0 e^{rT}$) or a sequence of strategies (for $K = x_0 e^{rT}$) leading to $\Omega_{Z^*}(K) = \infty$, contradicting $\Omega_{Z^*}(K) = \frac{\alpha}{1-\alpha}$. □

3 Optimal Solutions with Bounded Terminal Wealth

In this section, we modify the portfolio choice problem by imposing an upper bound on the terminal wealth. This technique has been used in the literature, see Chiu et al. [2012] and more recently Bernard et al. [2019]. It may be thought of as a constructive method of producing (for large values of the bound) nearly optimal strategies. The modified problem is as follows:

$$\begin{cases} \inf\limits_{\pi \in \mathcal{A}(x_0)} & \mathcal{E}_L(\alpha), \\ \text{subject to} & \mathbb{E}[X_T^\pi] = d, \\ & \mathbb{E}[\xi_T X_T^\pi] \leqslant x_0, \\ & 0 \leq X_T^\pi \leq M, \ \text{ a.s.} \end{cases} \tag{22}$$

Using the fact that $\mathcal{E}_L(\alpha) = x_0 e^{-rT} - \mathcal{E}_{X_T^\pi}(1-\alpha)$, we will apply the martingale approach and consider the following problem:

$$\begin{cases} \sup\limits_{Z \in \mathcal{M}_+} & \mathcal{E}_Z(1-\alpha), \\ \text{subject to} & \mathbb{E}[Z] = d, \\ & \mathbb{E}[\xi_T Z] \leqslant x_0, \\ & 0 \leq Z \leq M, \ \text{ a.s.} \end{cases} \tag{23}$$

For a nonempty and nontrivial feasible set, we should have $M > d$.

We denote the feasible set of the above problem (23) by

$$C_2(d, x_0, M) = \left\{ Z \in \mathcal{M}_+ \mid \mathbb{E}[Z] = d, \ \mathbb{E}\left[\xi_T Z\right] \leq x_0 \text{ and } 0 \leq Z \leq M \text{ a.s.} \right\}. \tag{24}$$

We once again consider the corresponding Mean-Omega problem indexed by $K \in (0, d)$, now with a bound on the terminal wealth:

$$G(K; x_0) = \sup_{Z \in C_2(d, x_0, M)} \Omega_Z(K). \tag{25}$$

[1] $Z \equiv d$ is not feasible, since $\mathbb{E}[\xi_T] = e^{-rT} > x_0/d$; so $\mathbb{E}[(Y - \mathcal{E}_Y(\alpha))_+] > 0$.

The connection between the Mean-Omega problem and the Mean-Expectile problem will be described in Proposition 7 in the sequel.

We know that $\mathbb{E}\left[(Z-K)_+\right] = \mathbb{E}[Z-K] + \mathbb{E}\left[(K-Z)_+\right] = d - K + \mathbb{E}\left[(K-Z)_+\right]$ for any $Z \in C_2(d, x_0, M)$. Thus, we consider the following problem in order to study the properties of problem (25):

$$\tilde{G}(K; x_0) = \inf_{Z \in C_2(d, x_0, M)} \mathbb{E}\left[(K-Z)_+\right]. \tag{26}$$

From the proof of part (a) of Proposition 2, for K in the subset of $(0, x_0 e^{rT})$, some problems (25), indexed by K, are also unbounded when the upper bound M is large enough. Thus, hereafter we focus on the case where $x_0 e^{rT} \le K < d$.

Proposition 4 *Suppose $x_0 e^{rT} \le K < d$.*

(a) $\tilde{G}(K; x_0)$ is Lipschitz continuous and strictly increasing with respect to K.

(b) If $\tilde{G}(x_0 e^{rT}; x_0) > 0$, then $G(K; x_0)$ is Lipschitz continuous and strictly decreasing with respect to K.

Proof The Lipschitz continuity of $\tilde{G}(K; x_0)$ regarding K can be proved in the same way as in Proposition 1. Below we show the strict monotonicity of $\tilde{G}(K; x_0)$ with respect to K. We claim that $\mathbb{P}(Z \le K) > 0$ for any $Z \in C_2(d, x_0, M)$ since $x_0 e^{rT} \le K < d$. Indeed, if $\mathbb{P}(Z \le K) = 0$, then $\mathbb{P}(Z > K) = \mathbb{Q}(Z > K) = 1$ implies $\mathbb{E}[\xi_T Z] = \mathbb{E}^{\mathbb{Q}}[e^{-rT} Z] > e^{-rT} K \ge x_0$, which contradicts the budget constraint. Denote $\varepsilon = K_1 - K_2 > 0$ for $x_0 e^{rT} \le K_2 < K_1 < d$ and take $\tilde{Z} \in C_2(d, x_0, M)$ such that $\mathbb{E}[(K_1 - \tilde{Z})_+] \le \tilde{G}(K_1; x_0) + \frac{b}{2}\varepsilon$, where $b = \mathbb{P}(\tilde{Z} \le K_2) > 0$. Then, it follows that

$$\begin{aligned}
\tilde{G}(K_1; x_0) &\ge \mathbb{E}[(K_1 - \tilde{Z})_+] - \frac{b}{2}\varepsilon \\
&= \mathbb{E}[(K_2 - \tilde{Z})_+] + (K_1 - K_2)\mathbb{P}(\tilde{Z} \le K_2) + \mathbb{E}[(K_1 - \tilde{Z})_+ \mathbf{1}_{\{K_2 < \tilde{Z} \le K_1\}}] - \frac{b}{2}\varepsilon \\
&\ge \mathbb{E}[(K_2 - \tilde{Z})_+] + \varepsilon b - \frac{b}{2}\varepsilon \ge \tilde{G}(K_2; x_0) + \frac{b}{2}\varepsilon > \tilde{G}(K_2; x_0),
\end{aligned}$$

which implies the strict monotonicity of $\tilde{G}(K; x_0)$ with respect to K. Finally, (b) is a straightforward consequence of (a) due to the relationship between the objective functions of problems (25) and (26). □

3.1 Choosing the Upper Bound M

As advised by Chiu et al. [2012], M should be chosen sufficiently large to approximate well the original problem, which does not have the bound constraint, resulting in a "nearly-optimal" strategy. Let φ be the standard normal probability density function, Φ be the standard normal cumulative distribution function, and Φ^{-1} be its inverse. The following result is useful in selecting M (see Section 3.3).

Lemma 3 *Assume* **H1** *holds, i.e. $d > x_0 e^{rT}$. Then:*

$$\lim_{M \to \infty} \Phi^{-1}\left(\frac{d}{M}\right) - \Phi^{-1}\left(\frac{x_0 e^{rT}}{M}\right) = 0 \tag{27}$$

Proof By the concavity of Φ^{-1} near 0, for sufficiently large M,

$$0 \leq \Phi^{-1}\left(\frac{d}{M}\right) - \Phi^{-1}\left(\frac{x_0 e^{rT}}{M}\right) \leq \frac{1}{\varphi\left(\Phi^{-1}\left(\frac{x_0 e^{rT}}{M}\right)\right)}\left[\frac{d}{M} - \frac{x_0 e^{rT}}{M}\right]$$
$$= \sqrt{2\pi}\exp\left\{\frac{1}{2}\left(\Phi^{-1}\left(\frac{x_0 e^{rT}}{M}\right)\right)^2\right\} \cdot \frac{d - x_0 e^{rT}}{M} \to 0, \text{ as } M \to \infty.$$

The claim follows from the Squeeze Theorem. □

In particular, Lemma 3 ensures the existence of a constant $M_{\min} > d$ such that for all $M > M_{\min}$, we have

$$\Phi^{-1}\left(\frac{d}{M}\right) - \Phi^{-1}\left(\frac{x_0 e^{rT}}{M}\right) < ||\zeta||\sqrt{T}. \tag{28}$$

We require that M satisfies this inequality in order to obtain the solution for the modified problems (25) and (26); see Lemma 5 and Proposition 5 in the sequel.

H3. The upper bound M satisfies $M > d$ and (28).

3.2 Lagrangian Duality and Pointwise Optimization

We now focus on problem (26), which we will solve by a Lagrangian duality method in conjunction with a pointwise optimization procedure. This entails introducing the following optimization problems with multipliers β_1 and β_2:

$$\inf_{\substack{Z\in\mathcal{M}_+ \\ 0\leq Z\leq M \text{ a.s.}}} \mathbb{E}\left\{(K-Z)_+ + (\beta_2\xi_T - \beta_1)Z\right\}, \quad \beta_1 \in \mathbb{R}, \quad \beta_2 > 0, \tag{29}$$

where we recall from (25) that the parameter K is within $(0, d)$. We solve the above problem by resorting to a pointwise optimization procedure and consider the following problem for $y_1 > 0$ and $y_2 > 0$:

$$\inf_{0\leq x\leq M} \quad \{(K-x)_+ + (y_2 - y_1)x\}, \tag{30}$$

Given that **H3** holds, it is easy to verify that the solution to the pointwise optimization problem (30) is as follows:

$$x^*(y_1, y_2) = K\mathbf{1}_{\{y_1 < y_2 \leq y_1+1\}} + M\mathbf{1}_{\{y_2\leq y_1\}}. \tag{31}$$

Lemma 4

(a) $Z^*_{\beta_1,\beta_2} := x^*(\beta_1, \beta_2\xi_T)$ *solves problem* (29) *where* x^* *is given in* (31).

(b) If there exist two constants $\beta_1^* > 0$ *and* $\beta_2^* > 0$ *such that* $Z^* := x^*(\beta_1^*, \beta_2^*\xi_T) \in \mathcal{F}_T$ *satisfies* $\mathbb{E}[Z^*] = d$ *and* $\mathbb{E}[\xi_T Z^*] = x_0$. *Then* Z^* *solves both problems* (25) *and* (26).

Proof The proof resembles those of Lemmas 3.1 and 3.2 in Lin et al. [2017], and is thus omitted. □

3.3 Solutions to the Mean-Omega Problems

In this section, we investigate the solutions of problems (25) and (26). The following lemma will be employed later for determining the solutions.

Lemma 5 *Suppose $x_0e^{rT} \leq K < d$ and* **H3** *holds. There exists a unique solution pair $(\tilde{q}_1, \tilde{q}_2)$ satisfying $1 > \tilde{q}_2 > \tilde{q}_1 > 0$ to the following system:*

$$\begin{cases} p_1(\tilde{q}_1, \tilde{q}_2) := \tilde{q}_2 + \left(\frac{M}{K} - 1\right)\tilde{q}_1 - \frac{d}{K} = 0, \\ p_2(\tilde{q}_1, \tilde{q}_2) := \varPhi\left(\varPhi^{-1}(\tilde{q}_2) - ||\zeta||\sqrt{T}\right) + \left(\frac{M}{K} - 1\right)\varPhi\left(\varPhi^{-1}(\tilde{q}_1) - ||\zeta||\sqrt{T}\right) - \frac{x_0e^{rT}}{K} = 0. \end{cases} \tag{32}$$

Proof For each q_1, equation $p_1(q_1, q_2) = 0$ is equivalent to $q_2 = \frac{d}{K} - \left(\frac{M}{K} - 1\right)q_1$. Thus, the condition $1 > q_2 > q_1$ implies that $\frac{d-K}{M-K} < q_1 < \frac{d}{M}$. Write $q_2(q_1) := \frac{d}{K} - \left(\frac{M}{K} - 1\right)q_1$ to get $\frac{dq_2}{dq_1} = -\left(\frac{M}{K} - 1\right) < 0$ and

$$\begin{aligned} \frac{dp_2(q_1, q_2(q_1))}{dq_1} &= \frac{\varphi\left(\varPhi^{-1}(q_2) - ||\zeta||\sqrt{T}\right)}{\varphi\left(\varPhi^{-1}(q_2)\right)}\frac{dq_2}{dq_1} + \left(\frac{M}{K} - 1\right)\frac{\varphi\left(\varPhi^{-1}(q_1) - ||\zeta||\sqrt{T}\right)}{\varphi\left(\varPhi^{-1}(q_1)\right)} \\ &= e^{-\frac{1}{2}||\zeta||^2T}\left(e^{||\zeta||\sqrt{T}\varPhi^{-1}(q_2)} - e^{||\zeta||\sqrt{T}\varPhi^{-1}(q_1)}\right)\frac{dq_2}{dq_1} < 0, \end{aligned}$$

for $q_2 > q_1$. This implies that $p_3(q_1) := p_2(q_1, q_2(q_1))$ is decreasing in q_1.

Furthermore, as $q_1 \nearrow \frac{d}{M}$ we have

$$p_3(q_1) \to \frac{M}{K}\varPhi\left(\varPhi^{-1}(\frac{d}{M}) - ||\zeta||\sqrt{T}\right) - \frac{x_0e^{rT}}{K} < 0,$$

where the inequality follows from assumption **H3**.

As $q_1 \searrow \frac{d-K}{M-K}$, we obtain

$$\begin{aligned} p_3(q_1) \to & 1 + \left(\frac{M}{K} - 1\right)\varPhi\left(\varPhi^{-1}\left(\frac{d-K}{M-K}\right) - ||\zeta||\sqrt{T}\right) - \frac{x_0e^{rT}}{K} \\ & \geq 1 + \left(\frac{M}{K} - 1\right)\varPhi\left(\varPhi^{-1}\left(\frac{d-K}{M-K}\right)\right) - \frac{x_0e^{rT}}{K} = \frac{d - x_0e^{rT}}{K} > 0 \end{aligned}$$

where the inequality follows from $x_0e^{rT} < d$. Therefore, we conclude that there exists a unique solution $(\tilde{q}_1, \tilde{q}_2)$ to the system (32). □

Proposition 5 *Suppose that $x_0e^{rT} \leq K < d$ and* **H3** *holds. There exist two unique constants $\beta_1^* > 0$ and $\beta_2^* > 0$ such that $Z^* := x^*(\beta_1^*, \beta_2^*\xi_T)$ satisfies $\mathbb{E}[Z^*] = d$ and $\mathbb{E}[\xi_T Z^*] = x_0$, where x^* is given in (31).*

Proof From (31), we know that

$$Z^*_{\beta_1,\beta_2} := x^*(\beta_1, \beta_2\xi_T) = K\mathbf{1}_{\{\beta_1 < \beta_2\xi_T \leq \beta_1 + 1\}} + M\mathbf{1}_{\{\beta_2\xi_T \leq \beta_1\}}.$$

Thus,

$$\begin{cases} \mathbb{E}\left[Z^*_{\beta_1,\beta_2}\right] = K\mathbb{P}\left(\beta_1 < \beta_2\xi_T \le \beta_1+1\right) + M\mathbb{P}\left(\beta_2\xi_T \le \beta_1\right) \\ \qquad\qquad\quad = K\mathbb{P}\left(\beta_2\xi_T \le \beta_1+1\right) + (M-K)\mathbb{P}\left(\beta_2\xi_T \le \beta_1\right), \\ \mathbb{E}[\xi_T Z^*_{\beta_1,\beta_2}] = Ke^{-rT}\mathbb{Q}\left(\beta_1 < \beta_2\xi_T \le \beta_1+1\right) + Me^{-rT}\mathbb{Q}\left(\beta_2\xi_T \le \beta_1\right) \\ \qquad\qquad\quad = Ke^{-rT}\mathbb{Q}\left(\beta_2\xi_T \le \beta_1+1\right) + (M-K)e^{-rT}\mathbb{Q}\left(\beta_2\xi_T \le \beta_1\right) \\ \qquad\qquad\quad = Ke^{-rT}\Phi\left(\Phi^{-1}\left[\mathbb{P}\left(\beta_2\xi_T \le \beta_1+1\right)\right] - ||\zeta||\sqrt{T}\right) \\ \qquad\qquad\qquad + (M-K)e^{-rT}\Phi\left(\Phi^{-1}\left[\mathbb{P}\left(\beta_2\xi_T \le \beta_1\right)\right] - ||\zeta||\sqrt{T}\right), \end{cases}$$

where the last equality follows from the fact that $\mathbb{Q}(\xi_T \le a) = \Phi\left(\Phi^{-1}\left(\mathbb{P}(\xi_T \le a)\right) - ||\zeta||\sqrt{T}\right)$ for a positive constant a. Denote $\tilde{q}_1 := \mathbb{P}\left(\beta_2\xi_T \le \beta_1\right)$ and $\tilde{q}_2 := \mathbb{P}\left(\beta_2\xi_T \le \beta_1+1\right)$ to get that $1 > \tilde{q}_2 > \tilde{q}_1 > 0$. Then by Lemma 5, the claim follows. □

Let β_1^* and β_2^* be the two unique constants that satisfy both constraints $\mathbb{E}\left[x^*(\beta_1^*, \beta_2^*\xi_T)\right] = d$ and $\mathbb{E}[\xi_T x^*(\beta_1^*, \beta_2^*\xi_T)] = x_0$. We characterize the optimal value $G(K; x_0)$ of problem (25) and optimal value $\tilde{G}(K; x_0)$ of problem (26) in the following proposition.

Proposition 6 *Suppose $x_0e^{rT} \le K < d$ and* **H3** *holds. Then, $x^*(\beta_1^*, \beta_2^*\xi_T)$ solves problems* (25) *and* (26)*, where x^* is given in* (31)*. The optimal values $G(K; x_0)$ and $\tilde{G}(K; x_0)$ of the two problems are respectively given as follows:*

$$\begin{cases} G(K; x_0) = \left(\dfrac{M}{K} - 1\right) \dfrac{\mathbb{P}\left(\beta_2^*\xi_T \le \beta_1^*\right)}{1 - \mathbb{P}\left(\beta_2^*\xi_T \le \beta_1^* + 1\right)} = \dfrac{\frac{d}{K} - \mathbb{P}\left(\beta_2^*\xi_T \le \beta_1^* + 1\right)}{1 - \mathbb{P}\left(\beta_2^*\xi_T \le \beta_1^* + 1\right)}, \\ \tilde{G}(K; x_0) = K\left[1 - \mathbb{P}\left(\beta_2^*\xi_T \le \beta_1^* + 1\right)\right]. \end{cases} \tag{33}$$

Proof The claims follow immediately from Lemma 4 and Proposition 5. □

3.4 Optimal Solution to the Mean-Expectile Problem with a Bounded Wealth Constraint

As previously noted, we focus on the case $x_0e^{rT} \le K < d$. In the earlier sections, for ease of notation, we suppressed the dependence of $\tilde{q}_{1,2}$, $\beta_1^*, \beta_2^*, x^*$ and the optimal solution Z^* on K and M. In this section, we make the dependence explicit when it is necessary for clarity. No confusion should result from this abuse of notation. We can now proceed to investigate the optimal solution for problem (23).

Proposition 7 *Assume that there exists a K^* such that $x_0e^{rT} \le K^* < d$ and $G(K^*; x_0) = \frac{\alpha}{1-\alpha}$. Then $Z^*_{K^*} = x^*_{K^*}(\beta_1^*(K^*), \beta_2^*(K^*)\xi_T)$ is an optimal solution to problem* (23) *and K^* is the optimal objective value.*

Proof The proof is similar to that of the analogous result in Proposition 3, and is therefore omitted.

Remark 3 Given an upper bound M, Proposition 7 specifies how to recover the optimal solution to problem (23). Since we consider the case $x_0e^{rT} \le K < d$, from Proposition 6 we have that $\tilde{G}(x_0e^{rT}; x_0) > 0$ due to the facts that $\beta_1^*(x_0e^{rT}) > 0$ and $\beta_2^*(x_0e^{rT}) > 0$, as shown in Proposition 5. Thus according to Proposition 4, $G(K; x_0)$ is Lipschitz continuous and strictly decreasing with respect to K. It is clear that $G(K; x_0) \in (1, G(x_0e^{rT}; x_0)]$. By **H2**, $\frac{\alpha}{1-\alpha} > 1$, however, if $\frac{\alpha}{1-\alpha} > G(x_0e^{rT}; x_0)$, then obviously the method from Proposition 7 fails since $\frac{\alpha}{1-\alpha}$ is outside of the range of $G(K; x_0)$ (and we need to consider larger values of M). □

The following proposition implies that increasing M will increase the value of $G_M(x_0e^{rT};x_0) := G(x_0e^{rT};x_0)$, where G is defined in (25) and we have adjusted the notation in order to make its dependence on M explicit.

Proposition 8 *For $x_0e^{rT} \le K < d$, if $M_2 > M_1$ and both M_2 and M_1 satisfy* (28)*, then $G_{M_2}(K;x_0) > G_{M_1}(K;x_0)$.*

Proof It is obvious that $G_{M_2}(K;x_0) \ge G_{M_1}(K;x_0)$. It remains to rule out equality. Suppose $G_{M_2}(K;x_0) = G_{M_1}(K;x_0)$. By Proposition 6, we obtain:

$$\tilde{q}_2(M_2) := \mathbb{P}\left(\beta_2^*(K,M_2)\xi_T \le \beta_1^*(K,M_2)+1\right) = \mathbb{P}\left(\beta_2^*(K,M_1)\xi_T \le \beta_1^*(K,M_1)+1\right) =: \tilde{q}_2(M_1).$$

By Lemma 5 and Proposition 5, there should exist a unique solution $(\tilde{q}_1(M_1), \tilde{q}_1(M_2))$ to the following equations:

$$\begin{cases} \left(\frac{M_1}{K}-1\right)\tilde{q}_1(M_1) = \left(\frac{M_2}{K}-1\right)\tilde{q}_1(M_2), \\ \left(\frac{M_1}{K}-1\right)\Phi\left(\Phi^{-1}(\tilde{q}_1(M_1)) - ||\zeta||\sqrt{T}\right) = \left(\frac{M_2}{K}-1\right)\Phi\left(\Phi^{-1}(\tilde{q}_1(M_2)) - ||\zeta||\sqrt{T}\right). \end{cases} \tag{34}$$

Suppose $\left(\frac{M_1}{K}-1\right)\tilde{q}_1(M_1) = \left(\frac{M_2}{K}-1\right)\tilde{q}_1(M_2)$. Then $\tilde{q}_1(M_1) = \frac{M_2-K}{M_1-K}\tilde{q}_1(M_2) > \tilde{q}_1(M_2)$. Defining

$$f(q) := \left(\frac{M_1}{K}-1\right)\Phi\left(\Phi^{-1}\left(\frac{M_2-K}{M_1-K}\cdot q\right) - ||\zeta||\sqrt{T}\right) - \left(\frac{M_2}{K}-1\right)\Phi\left(\Phi^{-1}(q) - ||\zeta||\sqrt{T}\right),$$

we should have $f(\tilde{q}_1(M_2)) = 0$ by (34). However, as $q \nearrow 1$, $f(q) \to K^{-1}(M_1 - M_2) < 0$, and differentiating gives:

$$\begin{aligned} f'(q) &= \left(\frac{M_2}{K}-1\right)\left[\frac{\varphi\left(\Phi^{-1}\left(\frac{M_2-K}{M_1-K}\cdot q\right) - ||\zeta||\sqrt{T}\right)}{\varphi\left(\Phi^{-1}\left(\frac{M_2-K}{M_1-K}\cdot q\right)\right)} - \frac{\varphi\left(\Phi^{-1}(q) - ||\zeta||\sqrt{T}\right)}{\varphi\left(\Phi^{-1}(q)\right)}\right] \\ &= \left(\frac{M_2}{K}-1\right)e^{-\frac{1}{2}||\zeta||^2T}\left(e^{||\zeta||\sqrt{T}\Phi^{-1}\left(\frac{M_2-K}{M_1-K}\cdot q\right)} - e^{||\zeta||\sqrt{T}\Phi^{-1}(q)}\right) > 0. \end{aligned}$$

so $f(q) < 0$ for $q \in (0,1)$, and we have a contradiction. Thus $G_{M_2}(K;x_0) > G_{M_1}(K;x_0)$. □

The next Proposition shows that as $M \to \infty$, the optimal values of the problem with the upper bound on wealth tend to the optimal value of the problem with unbounded wealth.

Proposition 9 *For $x_0e^{rT} \le K < d$, $\lim_{M\to\infty} \tilde{G}_M(K;x_0) = \tilde{g}(K;x_0)$, where $\tilde{g}$ is defined in* (17).

Proof $\tilde{G}_M(K;x_0) \ge \tilde{g}(K;x_0) > 0$ is clear. Since $G_M(K;x_0) = \frac{d-K}{\tilde{G}_M(K;x_0)} + 1$, Proposition 8 implies that $\tilde{G}_M$ is strictly decreasing in M, hence $\kappa := \lim_{M\to\infty} \tilde{G}_M(K;x_0) \ge \tilde{g}(K;x_0)$ exists. For small enough $\varepsilon > 0$, Lemma 10 guarantees the existence of a bounded Z_ε such that $\kappa \le \mathbb{E}\left[(K - Z_\varepsilon)_+\right] \le \tilde{g}(K;x_0) + \varepsilon$, and the result follows by letting $\varepsilon \searrow 0$. □

By the virtue of both Proposition 2 and Proposition 9, we know that $\lim_{M\to\infty} \tilde{G}_M(x_0e^{rT};x_0) = \tilde{g}(x_0e^{rT};x_0) = 0$, and thus $\lim_{M\to\infty} G_M(x_0e^{rT};x_0) = g(x_0e^{rT};x_0) = \infty$. Therefore, if M tends to infinity, the optimal value of our modified problem approaches the optimal value of our original problem without a bound on the terminal wealth.

We arrive at following algorithm for producing approximate solutions:

1. Fix M and derive the optimal function x^* for the pointwise optimization problem (30) using equation (31);
2. For each $x_0e^{rT} \leq K < d$, search for the unique solution pair to both equations $\mathbb{E}[x_K^*(\beta_1^*(K), \beta_2^*(K)\xi_T)] = d$ and $\mathbb{E}[\xi_T x_K^*(\beta_1^*(K), \beta_2^*(K)\xi_T)] = x_0$. Then set $Z_K^* = x_K^*(\beta_1^*(K), \beta_2^*(K)\xi_T)$;
3. Invoke Proposition 7 to get $Z^* := Z_{K^*}^*$ by solving for K^* from $G(K^*; x_0) = \frac{\alpha}{1-\alpha}$. If K^* exists, then stop. If there is no K^* such that $G(K^*; x_0) = \frac{\alpha}{1-\alpha}$ or equivalently, $\frac{\alpha}{1-\alpha} > G(x_0e^{rT}; x_0)$, increase the upper bound M, and return to **Step 1.**

4 Global Expectile Minimizing Strategies with Terminal Wealth Bound Constraint

In this section we consider the following global expectile minimization problem:

$$\begin{cases} \inf\limits_{\pi\in\mathcal{A}(x_0)} & \mathcal{E}_L(\alpha), \\ \text{subject to} & \mathbb{E}[\xi_T X_T^\pi] \leqslant x_0, \\ & 0 \leq X_T^\pi \leq M, \ \text{ a.s.,} \end{cases} \tag{35}$$

which differs from problem (23) by the exclusion of the mean constraint $\mathbb{E}[X_T^\pi] = d$. A solution to problem (35) leads to the global minimum expectile without a constraint on the expected terminal wealth, which is an interesting problem in its own right. In addition, an analysis of the problem will shed some light on the mean-expectile efficient frontier, which we will study in Section 5.

We also follow the martingale approach and consider the following problem:

$$\begin{cases} \sup\limits_{Z\in\mathcal{F}_T} & \mathcal{E}_Z(1-\alpha), \\ \text{subject to} & \mathbb{E}[\xi_T Z] \leqslant x_0, \\ & 0 \leq Z \leq M, \ \text{ a.s.} \end{cases} \tag{36}$$

We denote the feasible set of the above problem by $C_3(x_0, M)$, i.e.,

$$C_3(x_0, M) := \{Z \in \mathcal{M}_+ \mid \mathbb{E}\left[\xi_T Z\right] \leq x_0 \text{ and } 0 \leq Z \leq M \text{ a.s.}\}\,. \tag{37}$$

Similarly to the previous sections, we consider an associated Omega maximization problem:

$$\sup_{Z\in C_3(x_0,M)} \Omega_Z(K). \tag{38}$$

The above Omega maximization problem has been solved by Bernard et al. [2019] (see also Proposition 12 below). Following the same proof idea as in Propositions 3 and 7, we need to find a K^* such that $\Omega_{Z_{K^*}^*}(K^*) = \frac{\alpha}{1-\alpha}$ where Z_K^* denotes the solution for problem (38) at K. This nonlinear optimization problem can be reduced to the following linearized optimization problem:

$$H(K; x_0) = \sup_{Z\in C_3(x_0,M)} \mathbb{E}\left[(Z-K)_+\right] - \frac{\alpha}{1-\alpha}\mathbb{E}\left[(K-Z)_+\right]. \tag{39}$$

Proposition 10 *Suppose there exists a K^* such that $x_0e^{rT} \leq K^* < d$ and $H(K^*; x_0) = 0$. Then $Z_{K^*}^*$ is the optimal solution to problem* (36) *and K^* is the optimal objective value for problem* (36)*, provided that* $\mathbb{E}\left[(K^* - Z_{K^*}^*)_+\right] > 0$.

Proof Since $H(K^*;x_0)=0$ and $\mathbb{E}\left[(K^*-Z^*_{K^*})_+\right]>0$, we obtain that $\Omega_{Z^*_{K^*}}(K^*)=\frac{\alpha}{1-\alpha}$. The rest of the proof is similar to the proofs of the analogous results in Propositions 3 and 7. □

Proposition 11 *$H(K;x_0)$ is Lipschitz continuous and strictly decreasing with respect to K. Furthermore, $H(x_0e^{rT};x_0)\geq 0$ and $H(M;x_0)<0$.*

Proof The Lipschitz continuity can be proved in the same way as in Proposition 1. The strict monotonicity can be proved in the same way as that of Proposition 4.

When $K=x_0e^{rT}$, investing all wealth in the risk-free asset (i.e. $Z=x_0e^{rT}$, or equivalently $\pi_t=0$ for all $0\leq t\leq T$) will achieve a zero objective value. Thus $H(x_0e^{rT};x_0)\geq 0$. When $K=M$, the objective value of $Z\in C_3(x_0,M)$ is $-\frac{\alpha}{1-\alpha}\mathbb{E}\left[(M-Z)_+\right]<0$. Therefore, $H(M;x_0)<0$. □

We can derive the solution to (39) using the pointwise optimization technique and Lagrangian duality method. A similar result is given in Bernard et al. [2019, Proposition 1].

Proposition 12 *The unique optimal solution to* (39) *is given by:*

$$Z^*_K=M\mathbf{1}_{\{\beta^*\xi_T\leq 1\}}+K\mathbf{1}_{\{1<\beta^*\xi_T\leq\frac{\alpha}{1-\alpha}\}},\tag{40}$$

where β^* is such that $\mathbb{E}\left[\xi_T Z^*_K\right]=x_0$. The value function $H(K;x_0)$ is given as

$$H(K;x_0)=(M-K)\,\mathbb{P}\left(\beta^*\xi_T\leq 1\right)-\frac{\alpha}{1-\alpha}K\mathbb{P}\left(\beta^*\xi_T\geq\frac{\alpha}{1-\alpha}\right).\tag{41}$$

Proof The existence of the given solution (40) can be proved in the same manner as for problems (25) and (26) in Section 3. Uniqueness can be proved in a manner similar to that used in Bernard et al. [2019]. □

By the above two propositions, there is a unique K^* such that $H(K^*;x_0)=0$ (since $H(K;x_0)$ is strictly decreasing with respect to K). Thus, we can obtain the unique global expectile minimizing portfolio. The corresponding mean, denoted by d_{gem} and uniquely determined by K^*, is given by

$$d_{gem}:=\mathbb{E}\left[Z^*_{K^*}\right]=\left(M-K^*\right)\mathbb{P}\left(\beta^*\xi_T\leq 1\right)+K^*\mathbb{P}\left(\beta^*\xi_T\leq\frac{\alpha}{1-\alpha}\right),\tag{42}$$

and the global minimum expectile is $\mathcal{E}_{L^*}(\alpha)=x_0e^{rT}-\mathcal{E}_{Z^*_{K^*}}(1-\alpha)=x_0e^{rT}-K^*$.

5 Efficient Frontier with a Bound on Terminal Wealth

In this section, we will construct the efficient portfolios and derive the efficient frontier of our mean-risk portfolio selection problem using the expectile risk measure with an upper bound on terminal wealth. We begin with the following definitions, which are adapted from the corresponding notions in the mean-variance analysis (see, e.g. Markowitz et al. [2000]).

Definition 2 The mean-risk portfolio selection problem using the expectile risk measure with bounded terminal wealth is formulated as the following multi-objective optimization problem:

$$\begin{cases} \inf\limits_{\pi\in\mathcal{A}(x_0)} & (J_1(\pi), J_2(\pi)) := (\mathcal{E}_L(\alpha), -\mathbb{E}[X_T^\pi]), \\ \text{subject to} & \mathbb{E}[\xi_T X_T^\pi] \leqslant x_0, \\ & 0 \le X_T^\pi \le M, \ \text{ a.s.,} \end{cases} \tag{43}$$

where $L := x_0e^{rT} - X_T^\pi$. A feasible portfolio π^* is called an "efficient portfolio" if there exists no feasible portfolio π such that

$$J_1(\pi) \le J_1(\pi^*), \ \ J_2(\pi) \le J_2(\pi^*),$$

with at least one of the inequalities being strict. In this case, we call $(J_1(\pi^*), -J_2(\pi^*)) \in \mathbb{R}^2$ an efficient point. The set of all efficient points is called the efficient frontier.

The efficient frontier can be generated by solving the following optimization problem:

$$\begin{cases} \inf\limits_{\pi\in\mathcal{A}(x_0)} & J_1(\pi) := \mathcal{E}_L(\alpha), \\ \text{subject to} & J_2(\pi) := -\mathbb{E}[X_T^\pi] \le -d, \\ & \mathbb{E}[\xi_T X_T^\pi] \leqslant x_0, \\ & 0 \le X_T^\pi \le M, \ \text{ a.s.} \end{cases} \tag{44}$$

for all $d \ge 0$. The set $(J(\pi^*), -J_2(\pi^*)) \in \mathbb{R}^2$ for all optimal π^* is the efficient frontier.

In the remainder of the paper, we write Z_d^* for the optimal solution Z^* for problem (23), in order to make its dependence on d explicit. Note that (23) has an equality constraint on the mean of Z, rather than an inequality constraint as in (44).

Proposition 13 *Let $d_2 > d_1 \ge d_{gem}$, where d_{gem} is given by (42). Then $\mathcal{E}_{Z^*_{d_1}}(1-\alpha) > \mathcal{E}_{Z^*_{d_2}}(1-\alpha)$. For $d_{gem} \ge d_3 > d_4 > x_0e^{rT}$, $\mathcal{E}_{Z^*_{d_3}}(1-\alpha) > \mathcal{E}_{Z^*_{d_4}}(1-\alpha)$. Furthermore, $\mathcal{E}_{Z^*_d}(1-\alpha)$ is a concave function with respect to d.*

Proof For $d_2 > d_1 \ge d_{gem}$, let $a := \frac{d_1-d_{gem}}{d_2-d_{gem}} \in [0,1)$, so that $d_1 = ad_2 + (1-a)d_{gem}$. Then $Z := aZ^*_{d_2} + (1-a)Z^*_{d_{gem}} \in C_2(d_1, x_0, M)$, and

$$\mathcal{E}_{Z^*_{d_1}}(1-\alpha) \ge \mathcal{E}_Z(1-\alpha) \ge a\mathcal{E}_{Z^*_{d_2}}(1-\alpha) + (1-a)\mathcal{E}_{Z^*_{d_{gem}}}(1-\alpha) > \mathcal{E}_{Z^*_{d_2}}(1-\alpha) \tag{45}$$

where the first inequality is due to the optimality of $Z^*_{d_1}$ to maximize $\mathcal{E}_Z(1-\alpha)$ among the class $C_2(d_1, x_0, M)$, the second equality is due to the concavity of $\mathcal{E}_Z(1-\alpha)$ with respect to Z for $\alpha > 0.5$, and the last inequality follows from the uniqueness of the global expectile minimizing portfolio. The proof for the case $d_{gem} \ge d_3 > d_4 > x_0e^{rT}$ is similar.

The concavity of $\mathcal{E}_{Z^*_d}(1-\alpha)$ also follows in the same manner. Indeed, for $d_5 > d_6 > x_0e^{rT}$ and $\gamma \in (0,1)$, set $\tilde{d} = \gamma d_5 + (1-\gamma)d_6$ and construct $\tilde{Z} := \gamma Z^*_{d_5} + (1-\gamma)Z^*_{d_6} \in C_2(\tilde{d}, x_0, M)$. Similarly to (45) we have

$$\mathcal{E}_{Z^*_{\tilde{d}}}(1-\alpha) \ge \mathcal{E}_{\tilde{Z}}(1-\alpha) \ge \gamma\mathcal{E}_{Z^*_{d_5}}(1-\alpha) + (1-\gamma)\mathcal{E}_{Z^*_{d_6}}.$$

□

Since $\mathcal{E}_{L^*}(\alpha) = x_0 e^{rT} - \mathcal{E}_{Z_d^*}(1-\alpha)$, we are now ready to summarize the final result on the efficient frontier.

Proposition 14 *The efficient portfolio for the mean-risk portfolio selection problem using the expectile risk measure with bounded terminal wealth, i.e. the optimal portfolio for problem (44), is determined by those solutions to problem (22) with $d \geq d_{gem}$, where d_{gem} is given by (42). The points $(\mathcal{E}_{L^*}(\alpha), d) \in \mathbb{R}^2$ for all $d \geq d_{gem}$ form the corresponding efficient frontier. Moreover, the minimum expectile $\mathcal{E}_{L^*}(\alpha)$ as a function of the expected terminal wealth d is convex.*

Proof The proof follows from the definition of the efficient frontier and Proposition 13. □

Example 1 We consider the parameter values given in Table 1.

x_0	T	r	α	μ	σ
100	5	0.03	0.75	0.07	0.3

Table 1: Parameter Setting for Numerical Illustration

We vary the choice of d over $(x_0 e^{rT}, x_0 e^{rT} + 20)$ for numerical illustration. We use $M = 500$ for our analysis. The frontier is shown in Figure 1. For the global expectile minimizing portfolio, we try two different approaches and the results from both approaches agree within accepted tolerance. The first approach, which is more accurate because the result is computed from an analytical formula, is to use the results in Section 4. The coordinate for the global expectile minimizing portfolio is $(\mathcal{E}_{L^*}(\alpha), d_{gem}) = (-1.5607, 125.7551)$, as illustrated in Figure 1.

The second approach is to solve the problem (23) for each d and find the minimum objective value by a numerical search. We need to solve problem (25) and find K^* such that $G(K^*; x_0) = \frac{\alpha}{1-\alpha}$ to recover the solution. In this approach, we pick two different K's that lead to two objective values above and below $\frac{\alpha}{1-\alpha}$ respectively, then use the bisection method to approach K^* such that $G(K^*; x_0) = \frac{\alpha}{1-\alpha}$, where we select the tolerance for root finding to be 1.0×10^{-10}. We repeat the procedure for each d to obtain the curve. We employ step sizes of 0.001 and 0.0001 for d and find the coordinates for the global expectile minimizing portfolio are $(\mathcal{E}_{L^*}(\alpha), d_{gem}) = (-1.5607, 125.7554)$ and $(\mathcal{E}_{L^*}(\alpha), d_{gem}) = (-1.5607, 125.7551)$ respectively. The values differ after four decimal places; choosing smaller step sizes should lead to convergence to the global expectile minimizing portfolio obtained from the first approach.

The numerical results agree with our analytical findings. When $d \in (x_0 e^{rT}, d_{gem})$, $\mathcal{E}_{L^*}(\alpha)$ decreases with d, whereas when $d \in (d_{gem}, x_0 e^{rT} + 20)$, $\mathcal{E}_{L^*}(\alpha)$ increases with d and the curve in this case is the efficient frontier. This observation is consistent with Proposition 13. The entire curve in Figure 1 is the expectile minimizing frontier. Recalling the curves for the variance minimizing frontier and the efficient frontier for the mean-variance problem, the shapes of both curves are similar to their counterparts which are obtained in Figure 1.

In addition, we carry out sensitivity analysis with respect to the upper bound M. The results are shown in Fig 2. Here, we consider three cases, $M = 500, 600$, and 700. When M gets large, the entire curve of the expectile minimizing frontier shifts to the left upper in the mean-expectile plane. This finding is also revealed

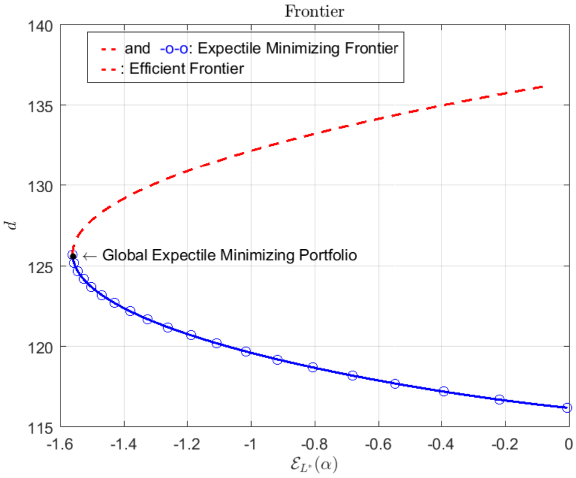

Fig. 1: Frontier: $\mathcal{E}_{L^*}(\alpha)$ versus d

in the global expectile minimizing portfolio. In other words, a larger upper bound M allows the investors to construct more efficient portfolios in that it generates more return but the same risk or smaller risk for the same return.

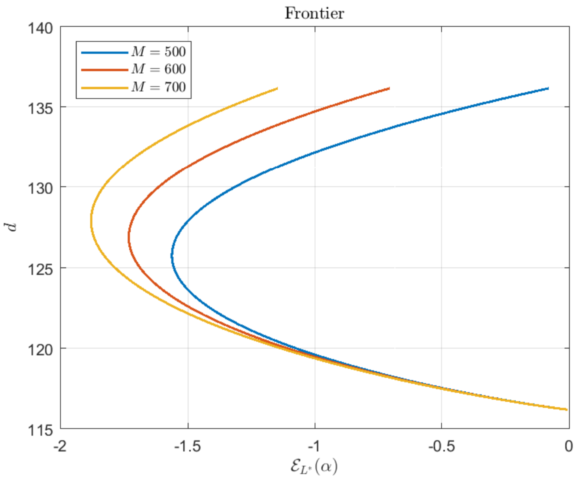

Fig. 2: Frontier: $\mathcal{E}_{L^*}(\alpha)$ versus d

6 Conclusion

In this paper, we consider a mean-risk portfolio selection problem for an expectile minimizing investor. Relying on the close relationship between the expectile and the Omega measure, we propose an alternative problem with the Omega measure as an objective and conclude that the original mean-expectile portfolio choice problem has no solution, i.e. the solution is not attainable. Following the literature, we impose an upper bound on terminal wealth and solve the modified problem by a Lagrangian approach and the pointwise optimization technique. We proved that the optimal value of the problem with an upper bound on the terminal wealth converges to that of the problem without such an upper bound as the imposed bound increases to infinity. Thus, the optimal solution obtained for the problem with an upper bound can be taken as an approximate solution to the mean-expectile problem with no upper bound on the terminal wealth. We also consider the global expectile minimizing portfolio and the mean-expectile efficient frontier.

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A Proof of part (b) and part (c) of Proposition 2

In this appendix, we provide the proof of parts (b) and (c) of Proposition 2. The proof essentially consists of a series of lemmas adapted from Section 5 of Jin et al. [2005], in which a similar result is shown.

Recall that $0 < x_0 e^{rT} \leq K < d$ is assumed for both parts (b) and (c). To proceed, we let $Y := Z - d$ and $y_0 := x_0 - de^{-rT} < 0$, so that the problem (16) can be equivalently cast as

$$\begin{cases} \inf_{Y \in \mathcal{F}_T} & \mathbb{E}\left[(K - d - Y)_+\right], \\ \text{subject to} & \mathbb{E}[Y] = 0, \\ & \mathbb{E}[\xi_T Y] \leqslant y_0 \\ & Y \geq -d \text{ a.s.} \end{cases} \tag{46}$$

where $Y \in \mathcal{F}_T$ means that Y is $\mathcal{F}_T$ measurable. Consider the optimization problem that relaxes the constraint on $\mathbb{E}[Y]$.

$$\begin{cases} \inf_{Y \in \mathcal{F}_T} & \mathbb{E}\left[(K - d - Y)_+\right], \\ \text{subject to} & \mathbb{E}[\xi_T Y] \leqslant y_0, \\ & Y \geq -d \text{ a.s.} \end{cases} \tag{47}$$

The following lemma is due to Cvitanić and Karatzas [1999].

Lemma 6 *Assuming $0 < x_0 \leq Ke^{-rT}$ or equivalently $y_0 \in (-de^{-rT}, Ke^{-rT} - de^{-rT}]$, an optimal solution to problem (47) is given by*

$$Y^* = Z^* - d = K\mathbf{1}_{\{\beta^* \xi_T \leq 1\}} - d, \tag{48}$$

where $\beta^* = \exp\left\{\|\zeta\|\sqrt{T}\Phi^{-1}\left(1 - \frac{y_0 e^{rT} + d}{K}\right) + \left(r - \frac{1}{2}\|\zeta\|^2\right)T\right\}$. The corresponding value function, denoted by $h(y_0)$, is

$$h(y_0) = K\Phi\left(\Phi^{-1}\left(1 - \frac{y_0 e^{rT} + d}{K}\right) - \|\zeta\|\sqrt{T}\right). \tag{49}$$

In Lemma 6, when $x_0 = Ke^{-rT}$, i.e. $y_0 e^{rT} + d = K$, we have $\beta^* = 0$, $Y^* = K - d$ and $h(y_0) = 0$, which means that the optimal solution to problem (47) is to invest only in the risk-free asset, and the optimal value is zero.

It is obvious that $h(y_0)$ is strictly decreasing with respect to $y_0 \in (-de^{-rT}, Ke^{-rT} - de^{-rT}]$.

Lemma 7 *For any sufficiently small $\varepsilon > 0$ and $y_0 \in (-de^{-rT}, Ke^{-rT} - de^{-rT}]$, there exists a feasible solution Y to problem (47) such that $h(y_0) \leq \mathbb{E}\left[(K - d - Y)_+\right] = h(y_0) + \frac{\varepsilon}{2}$ and $\mathbb{E}[\xi_T Y] = y_0$.*

Proof For any feasible solution Y of problem (47), $h(y_0) \leq \mathbb{E}\left[(K - d - Y)_+\right]$ by definition. Consider Y_ε defined as follows:

$$Y_\varepsilon = \left(K - d - \frac{\varepsilon}{2b\mathbb{E}\left[\xi_T \mathbf{1}_{\{\beta^* \xi_T \leq 1\}}\right]}\right)\mathbf{1}_{\{\beta^* \xi_T \leq 1\}} + \left(\frac{\varepsilon}{2b\mathbb{E}\left[\xi_T \mathbf{1}_{\{\beta^* \xi_T > 1\}}\right]} - d\right)\mathbf{1}_{\{\beta^* \xi_T > 1\}}, \tag{50}$$

where $b = \frac{1}{\mathbb{E}[\xi_T | \beta^* \xi_T \leq 1]} - \frac{1}{\mathbb{E}[\xi_T | \beta^* \xi_T > 1]} \geq 0$ and β^* is given in Lemma 6. For $\varepsilon > 0$ small enough, $Y_\varepsilon \geq -d$ a.s. It can be verified that $\mathbb{E}[\xi_T Y_\varepsilon] = y_0$ and

$$\begin{aligned} \mathbb{E}\left[(K - d - Y_\varepsilon)_+\right] &= \mathbb{E}\left[\left(K - \frac{\varepsilon}{2b\mathbb{E}\left[\xi_T \mathbf{1}_{\{\beta^* \xi_T > 1\}}\right]}\right)\mathbf{1}_{\{\beta^* \xi_T > 1\}}\right] + \mathbb{E}\left[\frac{\varepsilon}{2b\mathbb{E}\left[\xi_T \mathbf{1}_{\{\beta^* \xi_T \leq 1\}}\right]}\mathbf{1}_{\{\beta^* \xi_T \leq 1\}}\right] \\ &= h(y_0) + \frac{\varepsilon}{2b}\left(\frac{\mathbb{P}(\beta^* \xi_T \leq 1)}{\mathbb{E}\left[\xi_T \mathbf{1}_{\{\beta^* \xi_T \leq 1\}}\right]} - \frac{\mathbb{P}(\beta^* \xi_T > 1)}{\mathbb{E}\left[\xi_T \mathbf{1}_{\{\beta^* \xi_T > 1\}}\right]}\right) \\ &= h(y_0) + \frac{\varepsilon}{2b} b = h(y_0) + \frac{\varepsilon}{2}. \end{aligned}$$

Therefore, Y_ε constructed in (50) meets the requirement.

The following is Lemma 5.2 in Jin et al. [2005].

Lemma 8 *For any $\alpha > 0$, $\delta > 0$, and $0 < \beta < \alpha\delta$, there exists a bounded random variable $\tilde{Y} \geq 0$ such that $\mathbb{E}[\tilde{Y}] = \alpha$, $\mathbb{E}[\xi_T \tilde{Y}] = \beta$ and $\tilde{Y} = 0$ on the set $\{\xi_T \geq \delta\}$.*

Lemma 9 *For any sufficiently small $\varepsilon > 0$ and $y_0 \in (-de^{-rT}, Ke^{-rT} - de^{-rT}]$, given the feasible solution Y_ε in (50) to problem (47) such that $h(y_0) \leq \mathbb{E}\left[(K - d - Y_\varepsilon)_+\right] = h(y_0) + \frac{\varepsilon}{2}$ and $\mathbb{E}[\xi_T Y_\varepsilon] = y_0$, we have the following.*

(a) There exists a unique $\delta_0(a)$ for any $a \in (-de^{-rT}, y_0]$ such that

$$\mathbb{E}\left[\frac{a}{y_0}\xi_T Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \delta_0(a)\}}\right] = y_0.$$

(b) $\lim_{a \nearrow y_0} \delta_0(a) = 0$.

(c) There exists a $\delta_1(a)$ such that $0 < \delta_1(a) < \delta_0(a)$ and

$$\frac{\mathbb{E}\left[\frac{a}{y_0} Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \delta_1(a)\}}\right]}{\mathbb{E}\left[\xi_T \frac{a}{y_0} Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \delta_1(a)\}}\right] - y_0} > \frac{1}{\delta_1(a)},$$

(d) $\lim_{a \nearrow y_0} \delta_1(a) = 0$.

Proof (a) From (50), notice that $\mathbb{E}\left[\frac{a}{y_0}\xi_T Y_\varepsilon\right] = a$ and $Y_\varepsilon \leq 0$ a.s. for any sufficiently small $\varepsilon > 0$. Define $X_\beta := \frac{a}{y_0}\xi_T Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \beta\}}$ and $H(\beta) := \mathbb{E}(X_\beta) = \mathbb{E}\left[\frac{a}{y_0}\xi_T Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \beta\}}\right]$ for $\beta > 0$. We observe that X_β increases in β and tends to 0 and $\xi_T \frac{a}{y_0} Y_\varepsilon$ a.s. as β tends to ∞ and 0 respectively. Furthermore, $|X_{\beta'}| \leq \left|\xi_T \frac{a}{y_0} Y_\varepsilon\right|$ for all β, so the Dominated Convergence Theorem implies that $H(\beta)$ is continuous on $(0, \infty)$ with $\lim_{\beta \to \infty} H(\beta) = 0$ and $\lim_{\beta \to 0} H(\beta) = a < 0$. The existence of $\delta_0(a)$ follows. Uniqueness follows from the strict monotonicity of H, since for $\beta_1 > \beta_2 > 0$, we have

$$H(\beta_1) - H(\beta_2) = \mathbb{E}\left[\frac{a}{y_0}\xi_T Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \beta_1\}}\right] - \mathbb{E}\left[\frac{a}{y_0}\xi_T Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \beta_2\}}\right] = \mathbb{E}\left[\frac{a}{y_0}\xi_T (-Y_\varepsilon) \mathbf{1}_{\{\beta_2 \leq \xi_T < \beta_1\}}\right] > 0.$$

(b) It is clear that $\delta_0(y_0) = 0$. Continuity of $\delta_0(a)$ follows from the continuity and strict monotonicity of H.

(c) Define $G(\lambda) = \mathbb{E}\left[\frac{a}{y_0}(-\lambda Y_\varepsilon)\mathbf{1}_{\{\xi_T \geq \lambda\}}\right] - \left(y_0 - \mathbb{E}\left[\xi_T \frac{a}{y_0} Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \lambda\}}\right]\right)$ for $\lambda \in (0, \delta_0(a))$. The continuity of $G(\lambda)$ with respect to λ can be proved in the same way as in part (a).

Both random variables inside the corresponding expectations are integrable, with magnitudes bounded by $\left|\xi_T \frac{a}{y_0} Y_\varepsilon\right|$. Dominated Convergence then yields

$$\begin{aligned}\lim_{\lambda \nearrow \delta_0(a)} G(\lambda) &= \mathbb{E}\left[\frac{a}{y_0}(-\delta_0(a) Y_\varepsilon)\mathbf{1}_{\{\xi_T \geq \delta_0(a)\}}\right] - \left(y_0 - \mathbb{E}\left[\xi_T \frac{a}{y_0} Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \delta_0(a)\}}\right]\right) \\ &= \delta_0(a)\mathbb{E}\left[\frac{a}{y_0}(-Y_\varepsilon)\mathbf{1}_{\{\xi_T \geq \delta_0(a)\}}\right] > 0\end{aligned}$$

where the second equality follows from part (a). The continuity of G implies that there exists a $0 < \delta_1(a) < \delta_0(a)$ such that $G(\delta_1(a)) > 0$. Notice that for such a $\delta_1(a)$, we can obtain that $\delta_1(a)\mathbb{E}\left[\frac{a}{y_0}(-Y_\varepsilon)\mathbf{1}_{\{\xi_T \geq \delta_1(a)\}}\right] > 0$ and $y_0 - \mathbb{E}\left[\xi_T \frac{a}{y_0} Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \delta_1(a)\}}\right] > 0$, where the latter inequality follows from strict monotonicity of H from part (a). The result follows by rearranging $G(\delta_1(a)) > 0$.

(d) With $0 < \delta_1(a) < \delta_0(a)$ and $\lim_{a \nearrow y_0} \delta_0(a) = 0$, the claim follows by Squeeze Theorem.

Lemma 10 *For any sufficiently small $\varepsilon > 0$ and $y_0 \in (-de^{-rT}, Ke^{-rT} - de^{-rT}]$, there exists a feasible solution Y_ε^* to problem (46) such that $\mathbb{E}\left[(K - d - Y_\varepsilon^*)_+\right] < h(y_0) + \varepsilon$.*

Proof Using Lemma 8, we define

$$\bar{Y}_a = \frac{a}{y_0} Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \delta_1(a)\}} + \tilde{Y}_a \mathbf{1}_{\{\xi_T < \delta_1(a)\}} \tag{51}$$

where Y_ε is defined in (50) and $\tilde{Y}_a \geq 0$ a.s. is such that $\tilde{Y}_a = 0$ on the set $\{\xi_T \geq \delta_1(a)\}$ and

$$\begin{cases} \mathbb{E}\left[\tilde{Y}_a\right] = \mathbb{E}\left[\tilde{Y}_a \mathbf{1}_{\{\xi_T < \delta_1(a)\}}\right] = -\mathbb{E}\left[\frac{a}{y_0} Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \delta_1(a)\}}\right] > 0 \\ \mathbb{E}\left[\xi_T \tilde{Y}_a\right] = \mathbb{E}\left[\xi_T \tilde{Y}_a \mathbf{1}_{\{\xi_T < \delta_1(a)\}}\right] = y_0 - \mathbb{E}\left[\xi_T \frac{a}{y_0} Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \delta_1(a)\}}\right] > 0 \end{cases}$$

where $\delta_1(a) > 0$ and the two inequalities follow from proof of part (c) in Lemma 9. Consequently, $\mathbb{E}[Y_a] = 0$ and $\mathbb{E}[\xi_T Y_a] = y_0$. For $\bar{Y}_a$, we have:

$$\begin{aligned}\mathbb{E}\left[\left(K-d-\bar{Y}_a\right)_+\right] &= \mathbb{E}\left[\left(K-d-\frac{a}{y_0}Y_\varepsilon\right)_+ \mathbf{1}_{\{\xi_T \geq \delta_1(a)\}}\right] + \mathbb{E}\left[\left(K-d-\tilde{Y}_a\right)_+ \mathbf{1}_{\{\xi_T < \delta_1(a)\}}\right] \\ &= \mathbb{E}\left[\left(K-d-\frac{a}{y_0}Y_\varepsilon\right)_+ \mathbf{1}_{\{\xi_T \geq \delta_1(a)\}}\right].\end{aligned}$$

since $\tilde{Y}_a \geq 0$ a.s. Since $\left|\left(K-d-\frac{a}{y_0}Y_\varepsilon\right)_+ \mathbf{1}_{\{\xi_T \geq \delta_1(a)\}}\right| \leq \left|K-d-\frac{a}{y_0}Y_\varepsilon\right| \leq K+d+\frac{a}{y_0}|Y_\varepsilon|$ and $|Y_\varepsilon|$ is integrable from (50), the Dominated Convergence Theorem implies:

$$\lim_{a \nearrow y_0} \mathbb{E}\left[\left(K-d-\bar{Y}_a\right)_+\right] = \mathbb{E}\left[\left(K-d-Y_\varepsilon\right)_+\right] = h(y_0) + \frac{\varepsilon}{2}$$

where the second equality is due to the definition of Y_ε (50) in Proposition 7. Thus, we can find an $a < y_0$ such that $\mathbb{E}\left[\left(K-d-\bar{Y}_a\right)_+\right] < h(y_0) + \varepsilon$.

Lemma 11 *Given $y_0 < 0$, for any feasible solution Y of* (46), $\mathbb{E}\left[\left(K-d-Y\right)_+\right] > h(y_0)$.

Proof Note that for any Y feasible for problem (46), $\mathbb{E}[\xi_T Y_+] > 0$, since otherwise $Y_+ = 0$ a.s., and then $\mathbb{E}[Y] = 0$ implies $Y = 0$ a.s., and thus $\mathbb{E}[\xi_T Y] = 0 > y_0$, contradicting feasibility.

Let $Y_- = \max(-Y, 0)$. Then $b := \mathbb{E}[\xi_T(-(Y_-))] \leq y_0 - \mathbb{E}[\xi_T Y_+] < y_0$, and $-(Y_-) \geq -d$. Thus $\tilde{Y} := -(Y_-)$ is a feasible solution to (47), and we have:

$$\mathbb{E}\left[\left(K-d-Y\right)_+\right] \geq \mathbb{E}\left[\left(K-d-\tilde{Y}\right)_+\right] \geq h(b) > h(y_0).$$

using the fact that h is strictly decreasing by Lemma 6.

Lemmas 10 and 11 yield the claims in both part (b) and part (c) of Proposition 2.