Fair transition from defined benefit to target benefit

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Abstract

Target benefit (TB) plans that incorporate intergenerational risk-sharing have been demonstrated to be welfare improving over the long term. However, there has been little discussion of the short-term benefits for members in a Defined Benefit (DB) plan that is transitioning to TB. In this paper, we adopt a two-step approach designed to ensure the long-term sustainability of the new plan, without unduly sacrificing the benefit security of current retirees. We propose a cohort-based transition plan for reducing intergenerational inequity. Our study is based on simulations using an economic scenario generator, with some theoretical results under simplified settings.

1 Introduction

Over the past two decades the rising costs of traditional defined benefit (DB) pension plans have led to significant numbers of DB plan closures, and a growth in defined contribution (DC) plan membership. However, there is growing recognition that DC plans may not be fit for purpose, due to the lack of assurance that the resulting pension will be adequate, and the invidious burden placed on the individual, with respect to the management of investment and longevity risk (**?**).

DB and DC plans can be seen as two extremes on a spectrum of potential plan designs; traditional DB plans carry cost risk but no benefit risk, other than from insolvency. Traditional DC plans carry benefit risk but no cost risk. The fact that both DB and DC are unsatisfactory to many stakeholders leads us to consider non-corner solutions. Target Benefit (TB) plans (also known as Defined Ambition, or Intergenerational risk-sharing plans) are hybrid plans, designed to incorporate the best features of both DB and DC plans. The term 'Target Benefit' covers a wide range of plans, but all have the common feature that risk is shared across generations, which means that benefits in payment are adjustable. Contributions may be fixed or variable. If contributions are fixed, then the TB plan has some mechanism to hold back surplus in good times to support benefits in lean times (otherwise it would be a DC plan). If both contributions and benefits are variable, then some mechanism must be implemented to share surplus and deficits between workers and retirees.

Compared to a DB plan, the ability to adjust benefits allows for better risk-sharing across generations (for example, younger workers are not fully responsible to make up deficits caused by inadequate contributions paid by the previous generations). It also means that the default risk (particularly relevant for private sector plans) is much reduced, compared with traditional DB plans, and that intergenerational transfers are better controlled. In addition, TB plans can provide smoother and more predictable retirement income than conventional DC plans.

Various forms of TB plans have been proposed or implemented in various countries, including the UK (**?**, **?**), the Netherlands (**?**), the US (**?**) and Canada (**?**). For a general discussion of the TB plans see the published reports by the Canadian Institute of Actuaries and the Society of Actuaries, **?** and **?**. There has also been considerable theoretical research interest, including **?**, **?**, **?**, **?**, **?**, **?**, and **?**. Each of these research papers concluded that TB plans are preferred to both DB and DC plans, based on some objective function; the papers differ in their assumed benefit structure, objective functions and solution methods. However, all of these papers study optimality on a collective basis, which leads to a natural question on how to balance group optimality and individual optimality. Pioneered by **?**, a value-based approach is often used to evaluate the gains and losses for each generation. In particular, **?** and **?** compared the value transfer across generations between DB plans and risk-sharing pension plans. The value-based approach is clearly a zero-sum game, where some cohorts benefit and some suffer, which suggests that not every cohort favours the TB design.

However, our study demonstrates that pension reform is not a zero-sum game if the following factors are taken into consideration. Firstly, the default risk in the DB plan is often neglected in the literature, leading to significant over-estimation of the value of the DB plan for each member. This is not a trivial detail; in practice, DB plan failures have caused misery to millions of plan members who believed their pensions were 'gold standard'. Secondly, the value-based approach ignores individual cohort priorities; the risks that are most important to younger generations will differ from older generations. Lastly, none of the aforementioned papers have incorporated any transition procedure; the comparison is made by assuming all members will be moved into the new plan. In this paper, we propose a phased transition strategy from a DB to a TB plan that benefits all cohorts when default risk and time-varying preference are taken into account. We illustrate that a commonly adopted strategy in practice, letting the older workers and retirees stay in the (closed) DB plan while younger workers are transferred to the new plan, is a special case of our phased transition approach.

To assess the effectiveness of the phased transition quantitatively, we adopt a simple linear risk-sharing structure for the TB design, where both contributions and benefits are adjustable based on the plan's funding level. The optimal transition strategy is obtained by separating the objectives of pension reform into long-term and short-term goals. Solving for long-term optimality provides the risk-sharing parameters for future cohorts. Then, solving for short-term optimality, we devise parameters for a phased transition for older members at the time of transition. We illustrate the model and develop solutions numerically, using Monte Carlo simulation.

The remainder of the paper is structured as follows. Section 2 introduces our model. Section 3 discusses the choice of objective functions in pension reform. Section 4 provides numerical results. Section 5 concludes.

2 Model

Our study focuses on the transition from DB to TB, therefore, it is necessary to set up models for both the DB plan and TB plan, and highlight their structural differences. For both plans, members have a risk from pension plan deficits. However, the TB plan applies on-going risk-sharing between generations, while the DB plan tends to transfer deficit risk ahead. Ultimately, the DB plan may be forced to terminate, in which case the then current members will lose some or all of their accrued benefit, absorbing all that risk transferred forward from previous cohorts.

We describe the model for the population and economy in Sections 2.1 and 2.2. The dynamics of the pension assets are defined in Section 2.3, and the liability calculation is given in Section 2.6. We assume the liabilities for TB are defined-benefit-like, and so they use the same valuation formula as the DB benefits. The stylized DB and TB designs are described in Sections 2.4 and 2.5.

2.1 Overlapping Generations

We adopt the multi-period overlapping generations (OLG) model to project the performance of the TB plan. We assume homogeneity for all employees, such that they start their career at entry age x_e and retire at age x_r, with a maximum age of ω. In the numerical examples, in later sections, we set $x_e = 25$, $x_r = 65$, and $\omega = 115$. Let ${}_tp_x$ denote the probability that an individual aged x survives for t years, and let $l_x(t)$ denote the number of lives aged x at time t. For simplicity, we ignore longevity risk, and assume that the pension plan is large enough such that the mortality risk is fully diversifiable. We also assume that the number of new entries to the plan is the same each year. By unitizing the population, we have $l_x(t) = l_x = {}_xp_{x_e}$. Mortality is assumed to follow the Canadian Insured Payout Mortality 2014 (CIP2014, male), so the population consists of roughly 39 active workers and 19 retirees, representing an old-age dependency ratio of 50%.

We assume homogeneity in the population such that all employees earn one unit of currency in real terms. Therefore, the same cohorts are earning the same nominal amount at each time. The consumable income that active workers receive is their nominal income reduced by their assigned pension contributions. For retirees, the individuals receive no income other than their pensions. In addition, we assume there is no death benefit.

2.2 Economic Assumptions

Analyzing pension plans requires a long-run projection of financial risk factors and economic variables. Economic scenario generators (ESGs) are a popular choice in the application of actuarial risk management. Here we adopt the famous Wilkie ESG, (**?**) as developed and fitted in **?**. The following economic variables are generated:

1. Inflation Index I_t. Note that salaries are assumed to grow at the inflation rate.
2. Equity Index S_t.
3. Long-term bond rates R_t.
4. Short rates r_t.[1]

An outline of the model with parameter values, is given in the Appendix A.

[1] The short rate is not included in **?**; we adopt the original version of Wilkie's model (**?**).

It is important to point out that the simplicity of the structure of Wilkie's model inevitably has some disadvantages, as noted through the back-testing exercise in **?**. For example, the model tends to overestimate inflation and underestimate the total return on stocks. However, many ESGs are not fully disclosed, and even for models that are open access, there is a lack of maintenance to update the parameters. Some other ESGs that appear in the academic literature include: the Vector Autoregressive Model (VAR), as studied in **?** and **?**, the Generic ESG constructed in **?**, and the Exponential Regressive Conditional Heteroscedasticity (ERCH) model proposed in **?**.

2.3 Pension Assets

We denote by A_t the value of the pension assets at time t, which we assume to be invested in three securities: one-year Treasury Bills, long-term bonds, and an equity index fund (for example, tracking the S&P 500 index). Let b_t and c_t denote the actual benefit and actual contribution for each individual at time t. The aggregate amount of contribution received at t is $\sum_{x=x_e}^{x_r-1} l_x \times c_t$ and the aggregate amount of benefit paid at t is $\sum_{x=x_r}^{\omega-1} l_x \times b_t$.

Since the focus of the paper is on intergenerational risk-sharing, we exclude sponsor contributions and implicitly assume the pension is fully funded by reducing employee salaries.

The evolution of A_t is:

$$A_{t+1} = \left(A_t + \sum_{x=x_e}^{x_r-1} l_x \times c_t - \sum_{x=x_r}^{\omega-1} l_x \times b_t \right) \left(\pi_S \frac{S_{t+1}}{S_t} + \pi_B e^{R_t} + (1 - \pi_S - \pi_B) e^{r_t} \right)$$

where π_S and π_B are the weights of the portfolio invested in equity and long-term corporate bonds, which are assumed to be constant. The default investment strategy is an equity allocation of $\pi_S = 60\%$, a long term bond allocation of $\pi_B = 30\%$, and the remainder allocated to 1-year Treasuries. This meets the commonly applied "60/40" rule of thumb for pension investments. To model the dynamics of S_t, R_t, r_t and other economic variables, we use the Wilkie ESG (see Appendix A).

2.4 Contributions and Benefits – DB plan

As we are evaluating a transition from a DB plan to a TB plan, the details of the DB plan are relevant only insofar as we want to ensure that current retirees at transition are no worse off in the TB plan than they would be in the DB plan, and in order to compare default risk for retirees under the two plans.

We assume that existing DB plan has defined benefits that are fixed in real terms, so that, if the plan is not wound up, the actual benefit paid at t, denoted b_t, is based on a fixed real rate of benefit, b^*, rolled up for inflation. That is, $b_t = b^* \times I_t$ where I_t is the inflation index generated using the Wilkie ESG. This benefit is funded by contributions from active employees, which are set at a rate of c^* for all employees, in real terms. The actual contribution at t is $c_t = c^* \times I_t$.

For a given b^*, the value of c^* is determined through a specified actuarial pricing principle. In Section 4, we set the target real contribution rate to be $c^* = 15\%$, and determine b^* such that the median long term funding level of the DB plan is equal to 1.0, which gives a replacement rate of $b^* = 70\%$.

If the DB plan becomes severely underfunded, it will be wound up. We set a funding ratio of $A_t/L_t = 40\%$ as the wind-up threshold for our numerical illustrations, where L_t is the actuarial value of the pension liability at t (see Section 2.6).[2] After deducting wind-up expenses of $(e_p \times A_t)$, the remaining asset balance will be used to purchase annuities at market consistent prices. In our numerical illustrations, we set $e_p = 10\%$.

For retirees, the reduced benefit payable at $t + k$, given wind-up at t, for $k = 0, 1, \ldots$, is

$$b_{t+k} = \frac{(1-u) \times (1-e_p) \times A_t}{\sum_{x=x_r}^{\omega-1} l_x \times \sum_{y=x}^{\omega-1} {}_{y-x}p_x \times (1+i_t)^{-(y-x)}}$$

where $1-u$ is the proportion of assets allocated to the retirees, and i_t is the valuation interest rate for the plan at time t (see Section 2.6).

To better compare the structural difference between DB and TB designs, we exclude regulatory interventions such as Pension Benefit Guarantee payouts.

2.5 Contributions and Benefits – TB plan

The TB structure in our study has target benefits and contributions which are identical to the benefits and contributions of the DB plan. The risk sharing mechanism is based on **?**, **?** and **?**, where the risk-sharing formula uses a linear allocation of the surplus or deficit to the active and retired members. The model is similar to **?** but with separate risk-sharing parameters depending on whether the plan is in surplus or deficit. This more realistically considers management of funding levels based on a corridor of acceptable values, with adjustments only applying when the funding level moves outside the corridor, and with surplus management being a different issue than deficit management.

[2]It would be very difficult for a DB plan to recover from such a funding level as low as 40%; see, for example, **?**; $e_p = 10\%$ represents a 10% expense rate on default, or the market cost for annuitization.

We specify the surplus and deficit sharing as follows. The target values at t for the contributions and benefits are denoted c_t^* and b_t^*, respectively, and are assumed to be equal to the DB plan benefits and contributions; that is, $c_t^* = c^* \times I_t$ and $b_t^* = b^* \times I_t$.

The actual contributions and benefits at t are denoted c_t and b_t, and are adjusted from the target values to allow for sharing of any surplus or deficit. In particular, we have

$$c_t = c_t^* - \alpha^s \frac{(A_t - \psi^s \times L_t)^+}{\sum\limits_{x=x_e}^{x_r-1} l_x} + \alpha^d \frac{(\psi^d \times L_t - A_t)^+}{\sum\limits_{x=x_e}^{x_r-1} l_x}$$

$$b_t = b_t^* + \beta^s \frac{(A_t - \psi^s \times L_t)^+}{\sum\limits_{x=x_r}^{\omega-1} l_x} - \beta^d \frac{(\psi^d \times L_t - A_t)^+}{\sum\limits_{x=x_r}^{\omega-1} l_x},$$

where L_t is the value at t of the pension liabilities (see Section 2.6), ψ^s is the funding level at which surplus will start to be distributed to retirees and active employees, and ψ^d is the funding level at which a funding deficit will be recovered through contribution and benefit adjustments. These are assumed to be externally imposed, and may be based on regulatory constraints. The α and β parameters represent the percentage of surplus or loss that is distributed to active employees and retirees, respectively, with values that may differ if the plan is in surplus (α^s, β^s), or in deficit (α^d, β^d). The default TB design in this paper uses $\psi^s = \psi^d = 1$, and $\alpha^s \neq \alpha^d$ and $\beta^s \neq \beta^d$. Notice that if we assume additionally that $\alpha^s = \alpha^d$ and $\beta^s = \beta^d$, we recover the TB design of **?**.

We have not proposed any constraints on c_t and b_t, which means we are allowing unlimited surplus distribution for active workers, and unlimited deficit sharing for retirees. In principle, this could lead to negative income for active workers, whose income at t is $I_t - c_t$, or to negative income for retirees, whose income is b_t. In practice negative income did not arise in any of our simulations; if it had we would have imposed a lower bound of zero.

Although the linear risk-sharing structure is simple, it does have theoretical support. **?** and **?** formulate the risk-sharing plan as an optimal control problem in a continuous setting (albeit simpler than used here) and both derived optimal risk-sharing structures in linear form.

2.6 The Pension Liabilities

As our TB plan is very similar to the existing DB plan, we use identical liability valuation methods for both. We follow the traditional unit credit (TUC) approach, which means that only the accrued benefits at t are included in the liability valuation. Let L_t denote the liability value at time t, then

$$L_t = \mathbb{E}_t\left[\underbrace{\sum_{x=x_e}^{x_r-1} l_x}_{\text{Active Workers}} \left(\underbrace{\frac{x-x_e}{x_r-x_e}\, b_t^*}_{\text{Accrued Benefit}} \underbrace{\sum_{y=x_r}^{\omega-1} {}_{y-x}p_x\, v_{i_t}^{(y-x)}}_{\text{Actuarial value of annuity}} \right)\right] + \mathbb{E}_t\left[\underbrace{\sum_{x=x_r}^{\omega-1} l_x}_{\text{Retirees}}\, b_t^* \underbrace{\left(\sum_{y=x}^{\omega-1} {}_{y-x}p_x\, v_{i_t}^{(y-x)} \right)}_{\text{Actuarial value of annuity}} \right],$$

where $\mathbb{E}_t$ is the expectation at t, conditional on the filtration generated by the ESG. The interest rate, i_t ($v_{i_t} = \frac{1}{1+i_t}$), depends on the type of valuation; for going-concern valuations, i_t is often chosen to be the current long-term corporate bond rate; for wind-up or solvency valuations, i_t is usually close to the risk-free rate. In this paper, we set the interest rate to be the long-term bond rate plus 2% for going-concern liability (loosely representing a corporate bond yield), and the short rate plus 2% (capped at the long-term bond rate) for the wind-up liability. See **?** and **?** for a short discussion on the liability valuation techniques and discounting rate selection from a practitioner perspective.

3 Transition from DB to TB

The flexible nature of the Target Benefit plan allows a range of risk allocations, through the parameters α and β. **?** and **?** maximize the utility of the plan participant, while **?** focus on the optimal plan structure that improves the overall welfare of all participants, and **?** focus on the optimal pension allocation between a TB plan and an annuity for each individual. In addition, **?** minimize the benefit risk, while constraining the insolvency risk of the pension fund. All papers demonstrate the necessity of including TB plans in the retirement portfolio. However, those studies have not addressed the potential conflict between group optimality and individual cohort optimality. **?** demonstrate that an unconstrained optimal control solution will sacrifice the interest of nearby generations for the benefit of generations in the distant future. This unfairness is amplified as the time horizon increases. They also demonstrate that a constrained problem may mitigate the unfairness, but does not eliminate the problem; certain cohorts may still suffer from lack of ex-ante fairness in the TB plan design, in the sense that the optimization allocates significant risk to them, in order to build up (on average) sufficient funds to eliminate risk for future generations. The objective of this paper is to explore whether this ex-ante unfairness can be mitigated through design of the TB plan, and through allowing short-term adjustments to the TB plan for members in force or in retirement at the time of transition.

This objective suggests a two-stage procedure, with the first stage focusing on long-term optimality, and the second adjusting for the short-term transition effects.

3.1 Long-term optimality

For long-term optimality, we adopt an objective function similar to **?** and **?**, under which we minimize the squared deviation between the actual and the benchmark income, and solve for the risk-sharing parameters, α^s, α^d, β^s and β^d. The squared deviation objective function places value on predictability and smoothness of income. High side deviations are treated the same as low side deviations, which means that efficiency of funding is also captured, as high income values reflect the distribution of excess surplus. Note that, as we are not using a utility-based optimization, it is not necessary to assume that individuals consume all their income, but for convenience we will use the term 'target consumption' to mean the individual's preferred target income.

The income of active workers in our model is the salary (modelled by the inflation index) minus the contribution, that is, $I_t - c_t$. The income of retirees is b_t.

Let C^w and C^r denote the target consumption, in real terms, for active workers and retirees. The inflation adjusted target consumption at t is then $C_t^w = C^w \times I_t$ for workers, and $C_t^r = C^r \times I_t$ for retirees. If the target consumption matches the TB targets, then we have $C_t^w = I_t - c_t^*$ and $C_t^r = b_t^*$. However, it is also possible that the target consumption may differ from the targets within the pension plan, and we allow for the possibility of treating C^w and C^r as external inputs to the plan design.

The optimal TB plan parameters are set by minimizing the squared difference between the target consumption and the actual income, in real terms, over a horizon of T years, with respect to the parameter set $\theta = \{\alpha^d, \alpha^s, \beta^d, \beta^s\} \in \Theta$, where Θ is the parameter space.

Consider all generations born before time T (so that all are deceased by time $T + \omega$). The optimal TB plan is determined in the general case, without assuming population stationarity, as:[3]

$$\arg\inf_{\theta\in\Theta} \frac{1}{T}\mathbb{E}\left\{ \underbrace{\sum_{x=x_e-T}^{\omega-1} \sum_{t=\max(x_r-x,0)}^{\omega-1-x} l_{x+t}(t) \left(\frac{b_t - C_t^r}{I_t}\right)^2}_{\text{retirement income for person born before T}} + \underbrace{\sum_{x=x_e-T}^{x_r-1} \sum_{t=\max(x_e-x,0)}^{x_r-x-1} l_{x+t}(t) \left(\frac{I_t - c_t - C_t^w}{I_t}\right)^2}_{\text{contribution for person born before T}} \right\} \tag{1}$$

As $T \to \infty$, if the population structure is stationary, the optimal TB plan design is:

$$\arg\inf_{\theta\in\Theta} \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} \mathbb{E}\left\{ \left(\sum_{x=x_r}^{\omega-1} l_x\right) \left(\frac{b_t - C_t^r}{I_t}\right)^2 + \left(\sum_{x=x_e}^{x_r-1} l_x\right) \left(\frac{I_t - c_t - C_t^w}{I_t}\right)^2 \right\}. \tag{2}$$

[3] In **?**, the authors add a linear penalty terms (i.e. $\varrho_r \times (C_t^r - b_t)$). It is easy to show that in this case including the penalty is equivalent to changing the target consumption C_t^r.

The derivation of this formula is given in Appendix C.

? used a horizon of $T = 20$ years, with an additional penalty term at the terminal date. The penalty is the squared deviation between the actual asset level and the target, which represents a weighting function between current generations and future generations. We find that the choice of the penalty function has a significant impact on the results. We can avoid this subjectivity by solving for the limiting case, when $T \to \infty$. Numerical results in Section 4 demonstrate that for our examples, the optimal pension design does converge in time, with all economic variables and the asset level stabilized in the long run. The convergence speed depends on the initial states of the economic variables; usually a projection period of at least 100 years is required. To ensure convergence, we approximate the infinite time horizon study using standard Monte Carlo methods.

We note that the current workers and retirees are irrelevant in equation (2). This points to the importance of adding another stage in the optimization, targeting their interests to achieve a fair transition.

3.2 Relation between optimal α and β

It is interesting to explore the relationship between the risk-sharing parameters for workers and retirees. In this section we simplify the model by setting $\alpha^s = \alpha^d = \alpha$, $\beta^s = \beta^d = \beta$ and letting $C_t^w = I_t - c_t^*$ and $C_t^r = b_t^*$. This allows us to to derive analytic results that can be compared with the numerical results determined for the full model.

Proposition 1. *Let α and β denote the risk-sharing parameters for workers and retirees respectively. Then the optimal risk-sharing parameters, $\hat{\alpha}$ and $\hat{\beta}$ satisfy:*

$$\frac{\hat{\alpha}}{\Sigma^W} = \frac{\hat{\beta}}{\Sigma^R}, \tag{3}$$

where $\Sigma^W = \sum_{x=x_e}^{x_r-1} l_x$ is the number of workers in force at any time, and $\Sigma^R = \sum_{x=x_r}^{\omega-1} l_x$ is the number of retirees.

Proof: See Appendix B.

The proposition provides an intuitive result that for an optimal TB plan, the amount of risk borne by any group is proportional to the group population size.

Notice that $\alpha = \beta = 0$ is a trivial solution to the optimization problem, as the objective function is equal to its minimum value of 0 throughout. However, with no risk sharing the long-term funding level will go to either infinity or negative infinity, which is clearly infeasible. Therefore, we construct a constrained space for the risk-sharing parameters such that $\Theta = \{\alpha > 0, \beta > 0\}$.[4]

The graph of the objective function as a function of α and β is given in Figure 1, using the benchmark parameters outlined in Section 4. Note that the objective function does not decrease as α and β approach zero. This is because no matter how small the risk-sharing parameters are, as long as they are strictly above zero, the divergence of the asset (after inflation adjustment) eventually increases the value of the objective function.

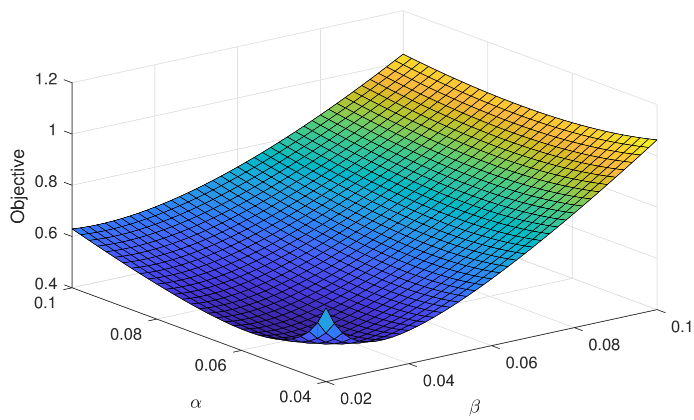


Figure 1: Objective function for different α and β.

3.3 The short-term problem: a fair transition criterion

The optimal plan design determined by Equation (2), in general, results in non-zero values for β^d and β^s. This means that, in the long term, intergenerational risk-sharing does benefit members. For future entrants, the advantage is measured over the total impact on income through the full working and retired lifetimes of the members. Compared to the DB plan, the TB plan offers more stable contributions, and better protection against default. However, for those members who have already retired at the time of transition it is too late to benefit from the reduced contribution volatility, and the enhanced protection from default will have little value to older members, as default is unlikely during their remaining lifetimes. Hence, retirees have little incentive to move to the TB plan; to force them to do so would not be permitted in many jurisdictions, and would be unfair in any jurisdiction. However, it is possible to structure the transition without disadvantaging any groups if transition is phased in for current retirees, to take into account the different impact of the new risk-sharing structure on different cohorts at transition.

[4] One may consider a smaller feasible set by incorporating regulatory constraints, as suggested by **?**.

In this section we consider how to ensure a fair transition from DB to TB, where we define fair to mean that no single cohort is worse-off, and at least some are better off, after the change.[5]

We interpret the "not worse off" criterion based on the downside risk associated with aggregate future retirement income after transition. In the DB plan, all downside risk comes from default on wind-up. In the TB plan, there is much less default risk, but there is ongoing downside risk from deficit sharing, based on the parameter β^d. In the long term, all retirees share the same value of β^d, derived from optimizing the objective function in equation (2). For the short-term, in order to ensure a fair transition for all cohorts, we derive different values of β^d for each retirement age group. Let β_x^d denote the deficit sharing parameter for retirees aged x at transition, where $x_r \leq x < \omega$. Our objective is to determine β_x^d such that the downside risk measure for each age group at transition, under the TB plan, is less than or equal to the downside risk measure under the DB plan.

Mathematically, let $\mathcal{E}(x; \beta_x^d)$ denote the downside risk measure for retirees aged x at the transition date, based on the downside risk-sharing parameter β_x^d; we omit β_x^d for simplicity when there is no confusion. Then we seek $\left\{\beta_x^d\right\}_{x_r \leq x < \omega}$ such that

$$\left\{\beta_x^d : \mathcal{E}^{\text{TB}}(x; \beta_x^d) \leq \mathcal{E}^{\text{DB}}(x), \text{ for all } x_r \leq x < \omega\right\}. \tag{4}$$

There may exist infinitely many transition strategies satisfying the inequality, so we consider the most sustainable one, which is the boundary case that solves the equations:

$$\beta_x^d = \min\left\{\beta_x' \middle| \mathcal{E}^{\text{DB}}(x) = \mathcal{E}^{\text{TB}}(x; \beta_x'), \beta_x' \geq 0\right\} \wedge \beta^d, \quad x_r \leq x < \omega \tag{5}$$

where retirees will never share more deficit than with future generations, that is, $\beta_x^d \leq \beta^d$, and, since we assume no default risk in the TB plan, $\mathcal{E}^{\text{DB}}(x) \geq \mathcal{E}^{\text{TB}}(x; \beta_x^d = 0)$ which ensures that the transition strategy is guaranteed to be beneficial to the retirees.

For the downside risk measure $\mathcal{E}(x)$, we use the expected remaining lifetime downside squared deviation:

$$\mathcal{E}(x) = \sum_{t=0}^{\omega-1-x} {}_tp_x \, \mathbb{E}\left[\left(\frac{(b_t - C_t^r)_-}{I_t}\right)^2\right]. \tag{6}$$

Additionally, in Appendix D, we derive analytic results using the expected remaining lifetime downside deviation:

$$\mathcal{E}(x) = \sum_{t=0}^{\omega-1-x} {}_tp_x \, \mathbb{E}\left[\left(\frac{(b_t - C_t^r)_-}{I_t}\right)\right]. \tag{7}$$

[5]This is consistent with the criteria reported by **?** from a small sample interview with younger generations in the U.S.

We note that there exist other reasonable choices for $\mathcal{E}(x)$. Examples include the lower quantiles of the retirement income, the conditional tail expectation of benefit loss, or a weighted average for expected surplus and loss. The common idea is to introduce a phased transition where the oldest cohorts are protected from risk through partial enrollment in the new plan, where the definition of 'protected' depends on the specific metric chosen. The metric used in this paper seems a reasonable choice, as retirees value downside stability, and this measure penalises large drops in income more severely than a series of smaller movements (**?**).

A different approach that is sometimes seen in practice is to introduce the new pension scheme to all cohorts younger than the retirement age, but leave all retirees in the DB plan. This is a special case of phased transition, which is equivalent to setting $\beta_x^d = 0$ for all $x_r \leq x < \omega$. We refer to this as the "cut-off" strategy. It clearly satisfies condition (4). In Section 4, we will compare the two strategies and highlight the sustainability improvement of the phased transition approach.

Note that our approach involves phasing in only for the retiree group at transition; we assume that younger members are automatically fully enrolled into the TB plan. We could extend the optimization to include workers who are near to retirement at transition, but in our numerical experiments it was not necessary to do so, as we show in Section 4.3.

4 Numerical Analysis

In this section, we develop numerical results to illustrate the optimal phased transition strategy described in the previous section. In Section 4.1 we discuss the choice of target contribution and target benefit level. Section 4.2 displays the optimal TB design in the long run. In Section 4.3 we demonstrate the effectiveness of phased transition.

4.1 The target benefit level (b^*)

Given the target contribution level $c^* = 15\%$, the target benefit $b^* = 70\%$ is determined such that the long-term median funding ratio (Asset/Liability) of the DB plan is equal to 1.0. We assume the same targets for the TB plan, making before-and-after comparisons more transparent. We note however that, given the different levels of risk involved, it may be appropriate to set different targets for the TB plan. For a sensitivity study, we include other benefit targets such that the long-run probability of full funding is (i) 90%, (ii) 95% and (iii) 99%. The corresponding b^* are (i) 0.54, (ii) 0.52, (iii) 0.47.

The optimization results depend on the choice of forecasting horizon T. We found that 100 years is sufficiently long for the optimal values to converge.

4.2 Long-term optimal TB plan design

Table 1 presents the optimal long-term TB plan design for a range of values of b^*, and where $C^r = b^*$, which means that the desired consumption matches the target benefit income in retirement. All other assumptions remain unchanged.

b^*	β^s	β^d	α^s	α^d
0.6984	0.0348	0.0225	0.0704	0.0455
0.5447	0.0537	0.0143	0.1086	0.0289
0.5165	0.0594	0.0131	0.1201	0.0264
0.4701	0.0667	0.0105	0.1348	0.0213

Table 1: Optimal TB plan Structure under long-term Objective; $C^r = b^*$

We notice monotone relationships between b^* and each optimal risk-sharing parameter. The overall surplus share $(\alpha^s + \beta^s)$ is between 10% and 20%, and the overall deficit share $(\alpha^d + \beta^d)$ is between 3% and 7%. Although we choose different α and β for surplus and deficit shares, Proposition 1 still roughly holds, that is

$$\frac{\alpha^d}{\beta^d} \approx \frac{\alpha^s}{\beta^s} \approx \frac{\Sigma^W}{\Sigma^R} \approx 2.$$

We emphasize that it is not necessary for C^r and b^* to be the same. Table 2 assumes the desired retirement income (replacement rate) is $C^r = 69.84\%$, but the target benefit b^* differs. Again, we see monotone relationships between b^* and each risk-sharing parameter. We also see that when b^* is smaller than C^r, the surplus sharing parameters, β^s, are driven upwards, to try to recover more income to meet the desired income of C^r, and the deficit sharing parameters, β^d fall to zero. The deficit risk then falls wholly on the active employees, who also receive a smaller share of the surplus.

b^*	β^s	β^d	α^s	α^d
0.6984	0.0348	0.0225	0.0704	0.0455
0.5447	0.0629	0.0000	0.0481	0.0517
0.5165	0.0669	0.0000	0.0426	0.0553
0.4701	0.0726	0.0000	0.0345	0.1034

Table 2: Optimal TB plan Structure under long-term Objective; $C^r = 0.6984$

More sensitivity tests for the optimal TB design over different c^* and b^* are given in Figure 2. We have also conducted sensitivity tests for the portfolio mix, with results shown in Figure 3. The

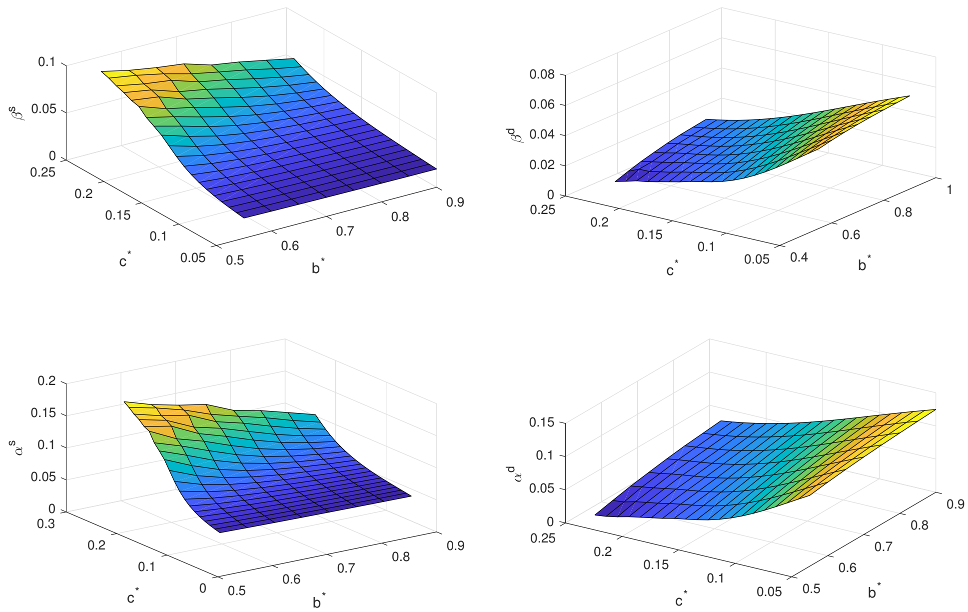


Figure 2: The optimal risk-sharing parameters for different $c^* = 1 - C^w$ and $b^* = C^r$.

main highlight observed in both graphs is that the optimal TB design is rather robust. Only under extreme scenarios, where the target contribution c^* is high and the target benefit b^* is low, does the optimal TB plan collapse to a DB-like design. In other words, when the contribution is far more than enough to cover the promised pension, the fund will be sustainable without any need to distribute its deficit, and $\beta^d = \alpha^d = 0$.

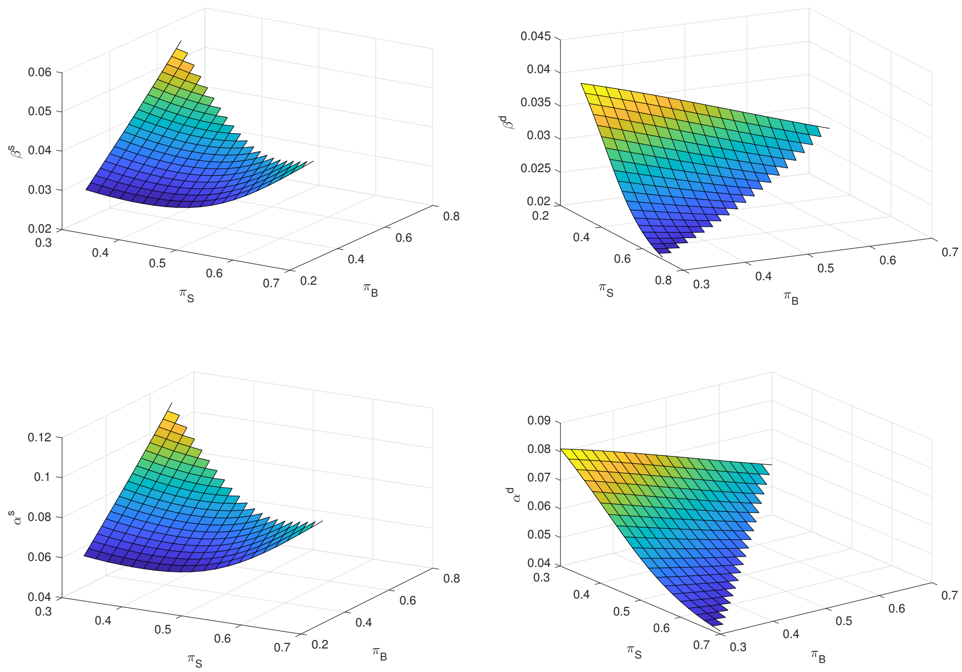


Figure 3: The optimal risk-sharing parameters for different investment strategies, $c^* = 1 - C^w = 0.15$, $b^* = C^r = 0.6984$

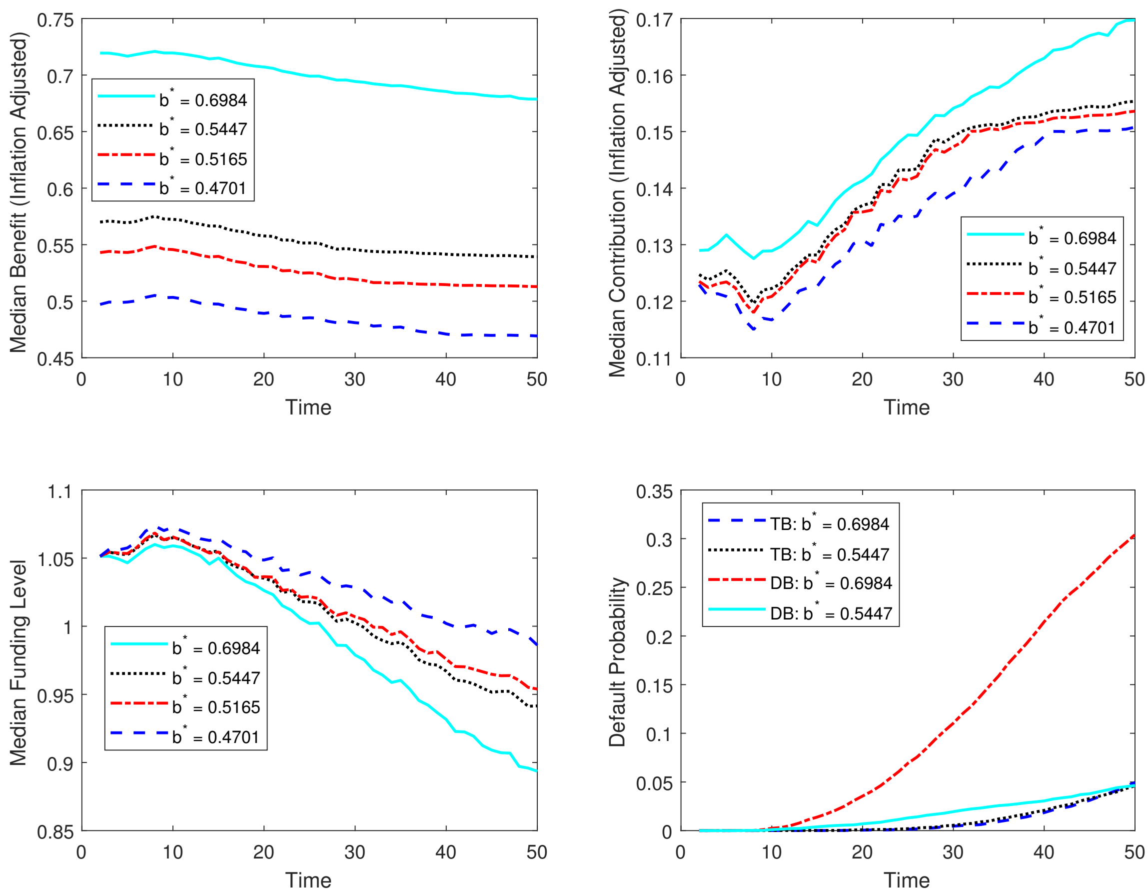


Figure 4: Sensitivity tests of median benefits, contributions, funding level and default risk for different b^*, when C^r is fixed at 0.6366.

The projected median benefit level, contribution rate, funding levels and default risk are shown in Figure 4. We have displayed simulation results for fifty years ahead, which is as long a horizon as most actuaries would consider relevant, but some values have not yet converged to their long-run levels. Consider, for example, $b^* = 0.6984$. For this case, the long-term median funding levels, benefit and contributions are 0.92, 0.68 and 0.2 respectively. At the first glance, it seems that the TB plan underperforms the DB plan, since the long term DB metrics are 1.0, 0.7, and 0.15 for funding, benefit and contributions, respectively. However, this is explained by the fact that the TB plan reduces the default risk by holding back surplus and reducing deficits; the impact is that the payouts are lower than under the DB plan, most of the time, but the default risk is significantly smaller, reducing the chance of catastrophic reduction in benefits.

As expected, the long-term median funding level is a decreasing function of b^*. Even though the long-term median funding level when $b^* = 0.6984$ falls below 90%, the probability of default ($A_t < 0.4L_t$) remains very small. In addition, although we did not display it here, we notice that in the long run ($T > 200$) the overall median consumption level (allowing for salary minus contributions for workers, and pension income for retirees) remains similar for different target benefit designs; if $Q_{0.5}$ denotes the median function, then the median total population income at t is

$$Q_{0.5}(b_t) \times \Sigma^R + Q_{0.5}(1 - c_t) \times \Sigma^W,$$

and this value is roughly the same for different b^*. The choice of b^* reflects the consumption balance for pre- and post-retirement periods.

We note that the median benefit level is not a consistent measure with respect to our objective function, which reflects the deviations between the target and actual income. We present the median values because they roughly match the behaviour of our objective, and the median function is more easily interpretable than the objective function. We note also that the sustainability advantage of the TB plan is not fully reflected in this analysis, as we have not allowed for the fact that the defined benefit is a contractual obligation, unlike the target benefit which is fully adjustable. Ways of allowing for this might include adjusting the default threshold to a lower value for the TB plan, or adjusting the liability valuation assumptions for a less conservative valuation of the TB benefits. Such questions are interesting, but beyond the scope of this paper, where we use the similarity of the benefits, thresholds and valuation assumptions to highlight the different risks and rewards from the TB and DB plans.

For the remaining analysis in this section, the benchmark long-term TB design is based on $C^r = b^* = 0.6984$ and $C^w = 1 - c^* = 1 - 0.15 = 0.85$.

4.3 Cohort-based transition

In this section we develop a fair, cohort-based transition from the DB plan to the TB plan. We measure pension sustainability for each cohort, using the expected probability of funding insolvency for that cohort, where insolvency is defined as having assets of less than 40% of liabilities. We denote the probability for the cohort age x at transition as DP_x, so that:

$$DP_x = \sum_{t=0}^{\omega - x} {}_t p_x \, \mathbb{E}\left[\mathbf{1}_{A_t \le 0.4 \times L_t} \,\middle|\, A_s > 0.4 \times L_s, \ \forall s < t \right]. \tag{8}$$

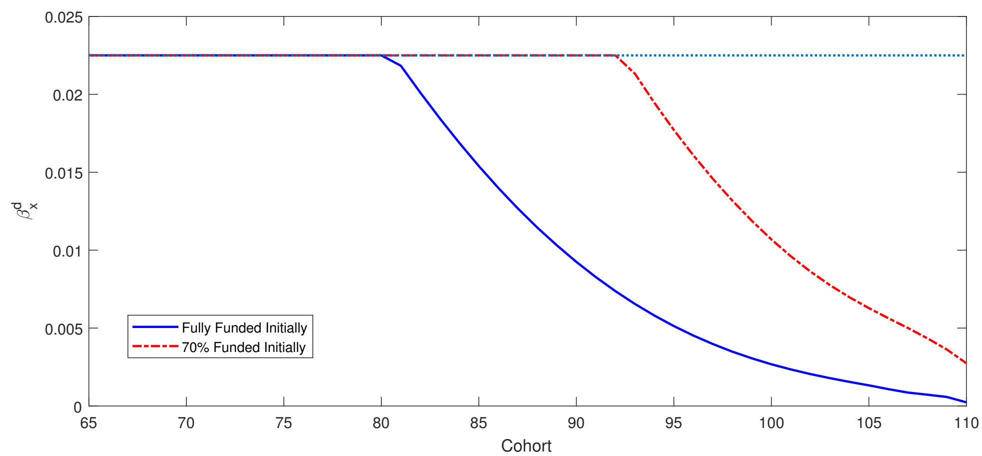
We emphasize that DP_x is a downside risk measure, based on a subjective ‘insolvency’ threshold, chosen to represent a practical lower bound for the funding level of the TB plan.

4.3.1 Comparison of TB and DB default risk, phased transition

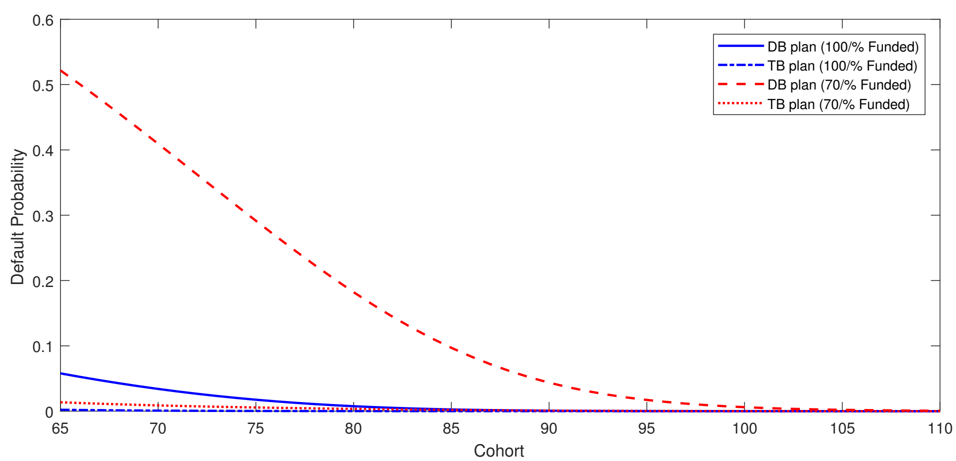
In Figure 5(a) the horizontal line is the optimal β^d obtained from the long-term optimization, which would determine deficit sharing for all future retirees, but which would not necessarily be suitable for those who have already retired at transition. For current retirees, different values of β_x^d are determined, where x is the age at transition, based on the downside risk measure criterion in equation (6). These values depend on the default risk of the DB plan, which in turn depends on the funding level at transition. The two curves shown indicate values for β_x^d for (i) a plan which is fully funded at transition and (ii) a plan which is 70% funded at transition. When β_x^d is equal to the long-term value, then lives aged x at transition should optimally transfer fully to the TB plan. Where β_x^d is close to zero, then lives aged x at transition should retain their DB plan benefits, including the possibility of default. We see that members of a highly under-funded DB plan will have more incentive to participate in the TB plan, as they would significantly reduce the risk of catastrophic default which is inherent in the DB plan. Based on our assumptions, for a 70% funded plan, full participation in the TB plan is optimal for all cohorts except the very oldest. When the DB plan is 100% funded at transition, partial participation in the TB plan is recommended for retirees aged 80 and above, with full transition indicated for all retirees below age 80.

In Figure 5(b) we show the lifetime plan default risk by age at transition. This demonstrates that a phased transition to the TB plan significantly enhances benefit security for all retirees, where the DB plan is only 70% funded, but as the default risk is low, provides little benefit for lives over age 80 where the plan is fully funded at transition. That explains why partial participation for these lives is optimal.

To demonstrate that the phased transition strategy is still applicable for other TB structures, in Figure 6 we show the same results for the the optimal TB structure derived in **?**, where $\alpha^s = \alpha^d = 0.06$, $\beta^s = \beta^d = 0.02$, $c^* = 0.14$ and $b^* = 0.66$. The graphs show very similar results to those for the TB plan of this paper, with phased transition for the very highest ages for 70% initial funding level, and for ages over 82 when the initial funding level is 100%.

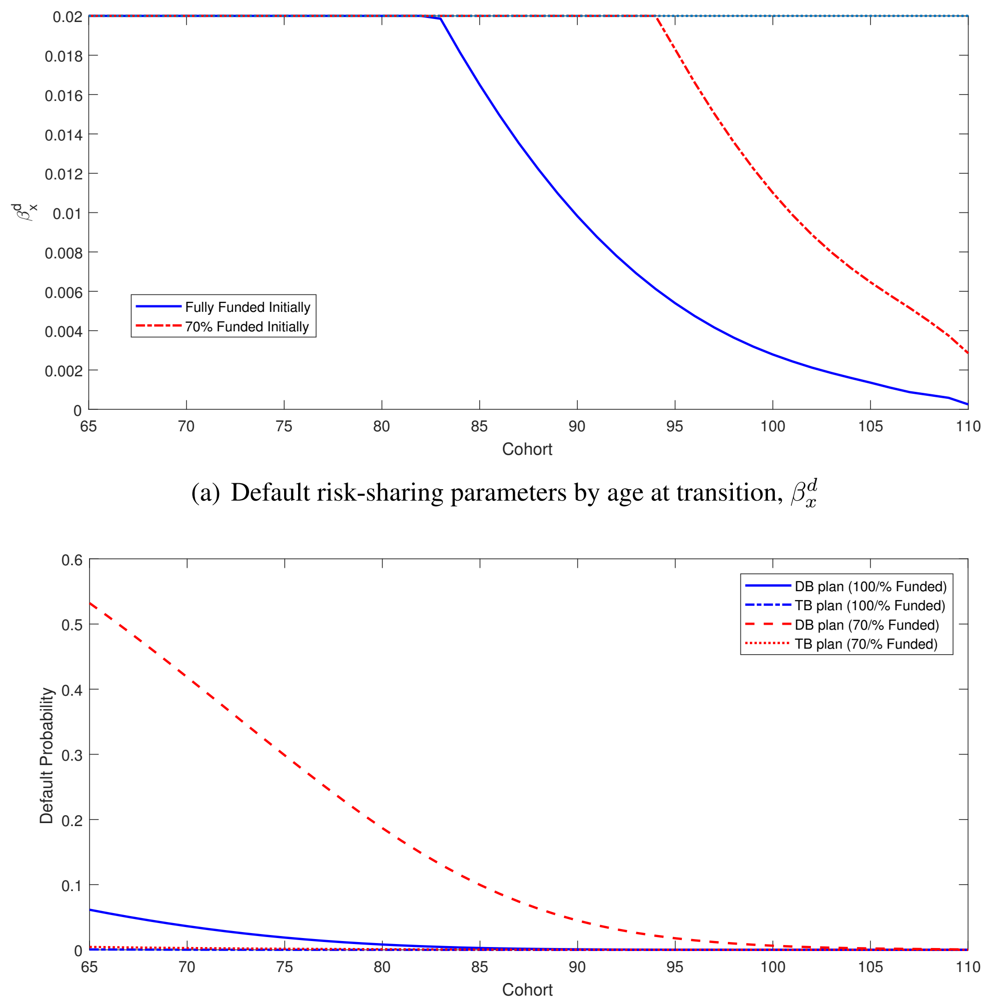


(a) Default risk-sharing parameters by age at transition, β_x^d



(b) Lifetime default probability by age at transition

Figure 5: Transition by cohort; risk sharing parameters (β_x^d) and lifetime default risks; $b^* = C^r = 0.6984$; 100% or 70% initial funding level.



(a) Default risk-sharing parameters by age at transition, β_x^d

(b) Lifetime default probability by age at transition

Figure 6: Transition by cohort for the model of **?**; risk sharing parameters (β_x^d) and lifetime default risks; 100% or 70% initial funding level.

4.3.2 Comparison to cut-off transition

A more straightforward way to differentiate the interests of different generations is to adopt a cut-off strategy. Current retirees retain their full DB benefits, while all active members move to the new TB plan. In this section we compare the efficiency of a phased transition illustrated in Figure 5, with the cut-off strategy, where the cut-off point is age 65, the retirement age.

Figure 7 presents the sustainability of the two transition methods. If the plan is fully funded at transition, there is little difference in default risk between the cut-off and phased transition.

However, when the plan is significantly under-funded at transition, excluding the current retirees from the risk-sharing design exposes the plan to significant default risk.

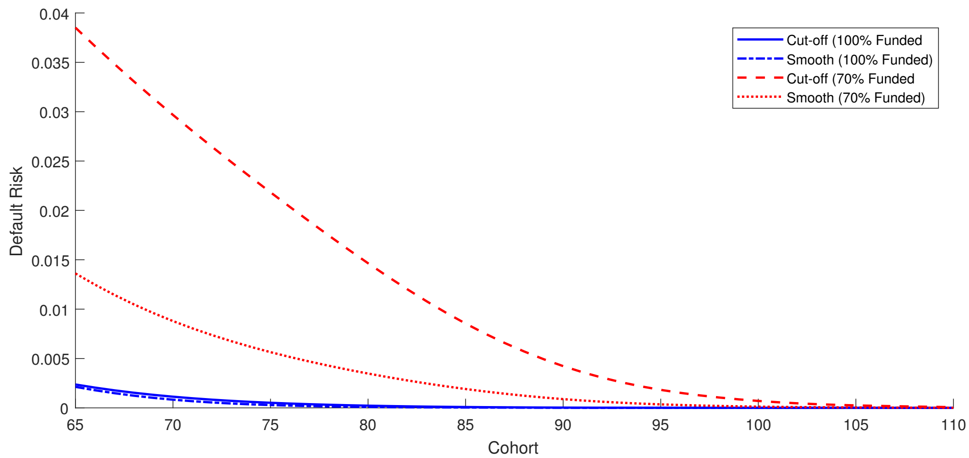


Figure 7: Cut-off strategy – the probability of lifetime future default for each cohort, $b^* = C^r = 0.6366$.

The cut-off strategy may be optimal if the target benefit level b^* is much smaller than the target consumption level C^r. Recall that for the cases when $b^* \leq 0.55$ where $C^r = 0.6984$ in Table 2, the optimal loss sharing for retirees is $\beta^d = 0$.

4.4 Alternative phased transition strategies

Here we demonstrate that the phased transition idea can be applied in alternative ways by setting other parameters to be cohort-based. In the previous analysis, the funding ratio thresholds for both profit and deficit shares are assumed to be 1.0, but in reality, alternative risk-sharing thresholds exist. For example, the inflation indexation for Dutch pension funds will be reduced if the funding level is below 130%. Here we first perform a sensitivity test for the optimal TB design with respect to different sets of funding thresholds, to check whether a cohort-based ψ^d is reasonable, see Figure 8 below.

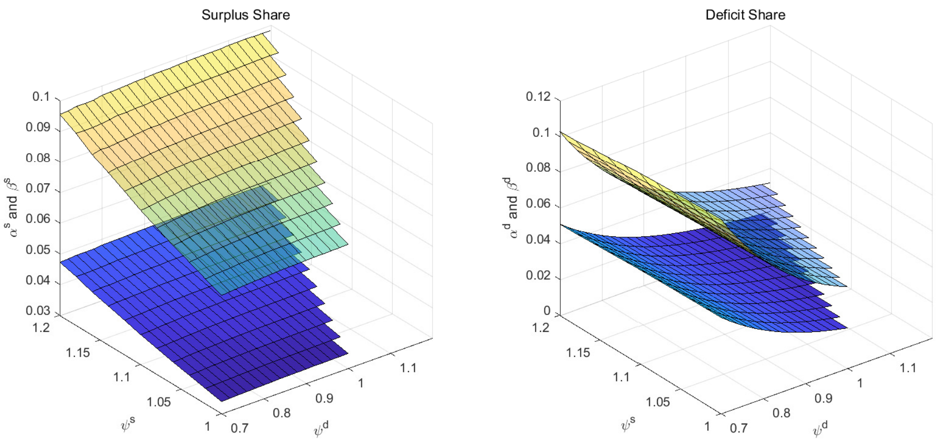


Figure 8: The optimal risk-sharing parameters for different sets of ψ. The upper layer represents α^s and α^d and the lower layer represents β^s and β^d.

Immediately we notice that the profit sharing parameters (α^s and β^s) are barely affected by the choice of ψ^d, and similarly that the deficit sharing parameters are not sensitive to the choice of ψ^s. The TB parameters are monotone with respect to the choice of funding threshold, such that increases in ψ^s reduce the overall distribution of surplus. Therefore, to maintain the same level of benefit and contribution stability, the optimal TB design will increase the profit sharing parameters, and similarly for the deficit case. The ratios between the risk-sharing parameters (α^s/β^s and α^d/β^d) are also roughly equal to the ratio between population sizes of retirees and active workers. The overall surplus sharing rate is kept between 10% and 15%, and the overall deficit sharing rate is between 3% and 15%.

The monotone behaviour shown in Figure 8 suggests that a cohort-based risk-sharing threshold ψ_x^d is also a monotone function of x, where x is the age of the retiree. For the same deficit sharing parameter β^d but with lower ψ_x^d, it would be intuitive to expect that a retiree aged x will be less involved in the deficit sharing. To ensure that the transition to TB will be beneficial to all retirees, we set a short-term transition objective using an age-based deficit threshold,

$$\psi_x^d = \max\left\{\psi_x' \middle| \mathcal{E}^{\mathrm{DB}}(x) = \mathcal{E}^{\mathrm{TB}}(x;\psi_x')\right\} \wedge \psi^d, \quad x_r \le x < \omega \tag{9}$$

where we set the upper bound of ψ_x^d as ψ^d, which is equal to 1.0 in our numerical illustrations; the constraint $\wedge\psi^d$ ensures that current retirees will not take a larger share of deficits than future generations. The objective function $\mathcal{E}$ is chosen to be the same as Equation (6). The transition strategy results are shown in Figure 9.

The interpretation of the transition strategy is similar to that for Figure 5, in that full participation

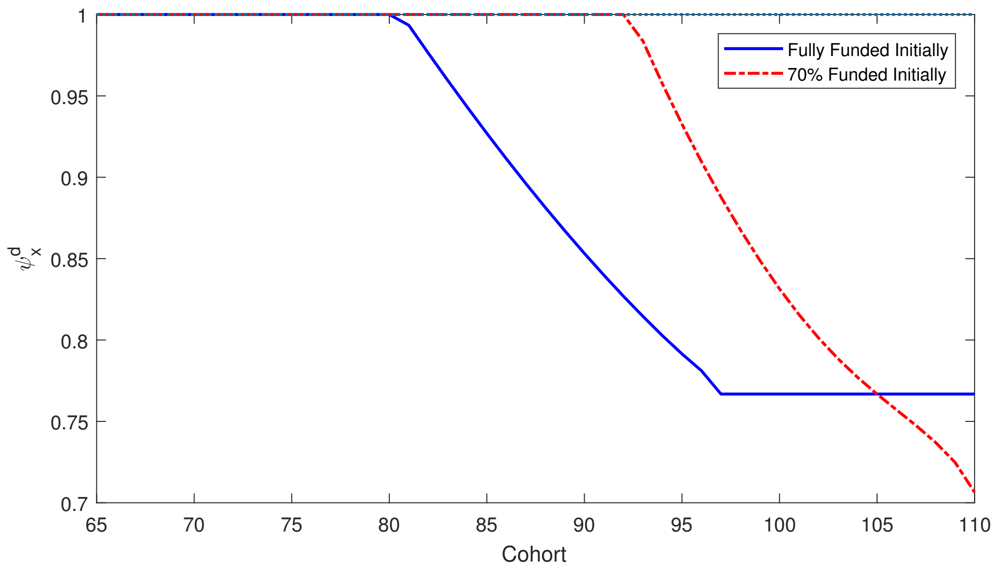


Figure 9: ψ_x^d under different initial funding levels, $b^* = C^r = 0.6984$. The horizontal dotted line is the value of β^d, representing the full participation in the TB plan.

in the TB plan is optimal for ages below 80, for initial funding of 70% and below 95 for initial funding of 100%. However, we also see that ψ_x^d behaves differently from β_x^d in that the oldest generations are more involved in the new TB plan. This is due to the fact that under the TB plan, the sustainability of the fund has been greatly improved, and the probability of extremely low funding levels can be neglected in the short run. For a TB plan that is 100% funded initially, there is virtually no benefit risk for the oldest generations with $\psi_x^d < 0.75$.

5 Conclusion

The actuarial literature on risk-sharing pension plans has generally assumed very simple risk-sharing structures and interest rate and investment returns. In this paper we have loosened those assumptions to create a model that, whilst still highly simplified, at least allows for some of the assumptions and inputs to better reflect the real world. In particular:

(1) We allow the risk-sharing parameters to differ depending on whether the plan is in surplus or deficit.

(2) We have incorporated different valuations for funding (going-concern) and wind-up, and allowed for additional expenses in the event of insolvency.

(3) We have used an ESG, allowing us to include dependent models of equity returns, inflation, and long and short-term interest rates.

(4) We have specifically addressed the issue of fairness to retiree cohorts at the transition date.

We find that the results are consistent with those from the simpler models, including for example **?**. The linear risk-sharing structure proposed has some desirable properties, including long-term convergence, as well as being very transparent for members.

Phasing in the deficit risk-sharing for older cohorts at transition can achieve a fair result, with respect to downside deviation measures.

The plan demographics, benefit structures and valuation methods in this paper are still highly simplified. Nevertheless, the results are sufficiently promising that it will be worthwhile to continue exploring the linear risk-sharing hybrid pension design in a more realistic setting.

6 Acknowledgements

This project was supported by the Global Risk Institute, through the National Pension Hub. Mary Hardy and David Saunders also acknowledge the support of the Natural Sciences and Engineering Research Council of Canada (NSERC), funding reference numbers RGPIN-03754-2018 (Hardy) and RGPIN-2017-04220 (Saunders). The authors wish to acknowledge the very helpful suggestions of two anonymous referees.

A Economic Scenario Generator

The Wilkie ESG is described in detail in **?**, **?**, and **?**. It is a cascade model with annual time steps. The cascade structure that we have used is illustrated in Figure 10.

The inflation model uses an AR(1) structure for the continuously compounded force of inflation. We have used this to generate the inflation index, I_t. The long interest rate series, $R(t)$, is used to model the yield to maturity on the long term bond part of the pension assets. The short rate, $r(t)$ is used to model the return on short term fixed interest investments. The interest rate processes, $R(t)$ and $r(t)$, are also used to determine the pension valuation interest rates through the Monte Carlo projections. The dividend growth process and dividend yield process are combined to create a stock price process, which is used to model the return on the stock part of the pension assets.

Updated parameters using US data, together with a discussion of the model fit, are given in **?**. We have used the following parameters from the US 1926-2014 data. For an interpretation of these parameters, see **?** or **?**.

Inflation: $\mu_q = 0.0307,\ a_q = 0.5731,\ \sigma_q = 0.0337$

Dividend yield: $\mu_y = 0.0309,\ a_y = 0.9368,\ \sigma_y = 0.1632,\ w_y = 0.0$

Dividend growth: $\mu_d = 0.0129,\ d_d = 0.38,\ b_d, = -0.6004,\ \sigma_d = 0.1581,\ y_d = 0.0$

Long bond rate: $\mu_c = 0.0238,\ d_c = 0.058,\ a_c = 0.9175,\ \sigma_c = 0.2832,\ c_{\min} = 0.005,\ y_c = 0.0$

Short rate: $\mu_l = 0.6516,\ a_l = 0.8966,\ \sigma_l = 0.3843$

We define the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which $Z(t) = [z_q(t), z_y(t), z_d(t), z_c(t), z_l(t)]$, and denote the filtration as $\mathbb{F} = \{\mathcal{F}_t = \sigma(Z(s) : 0 \leq s \leq t) | t \geq 0\}$. The expectation conditional on the filtration is denoted as $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_t]$.

B Proof of Proposition 1

$$
\begin{aligned}
A_{t+1} &= \left[A_t + \sum_{x=x_e}^{x_r-1} l_x \left(c_t^* - \alpha \frac{A_t - L_t}{\sum_{x=x_e}^{x_r-1} l_x} \right) - \sum_{x=x_r}^{\omega} l_x \left(b_t^* + \beta \frac{A_t - L_t}{\sum_{x=x_r}^{\omega} l_x} \right) \right] Z_{t+1} \\
&= \left[A_t(1 - \alpha - \beta) + \Sigma^W c_t^* - \Sigma^R b_t^* + (\alpha + \beta) L_t \right] Z_{t+1} \\
&= (1 - \alpha - \beta)^{t+1} A_0 \prod_{s=1}^{t+1} Z_s + \sum_{u=0}^{t} (1 - \alpha - \beta)^{t-u} \left(\Sigma^W c_u^* - \Sigma^R b_u^* + (\alpha + \beta) L_u \right) \prod_{s=u+1}^{t+1} Z_s.
\end{aligned}
$$

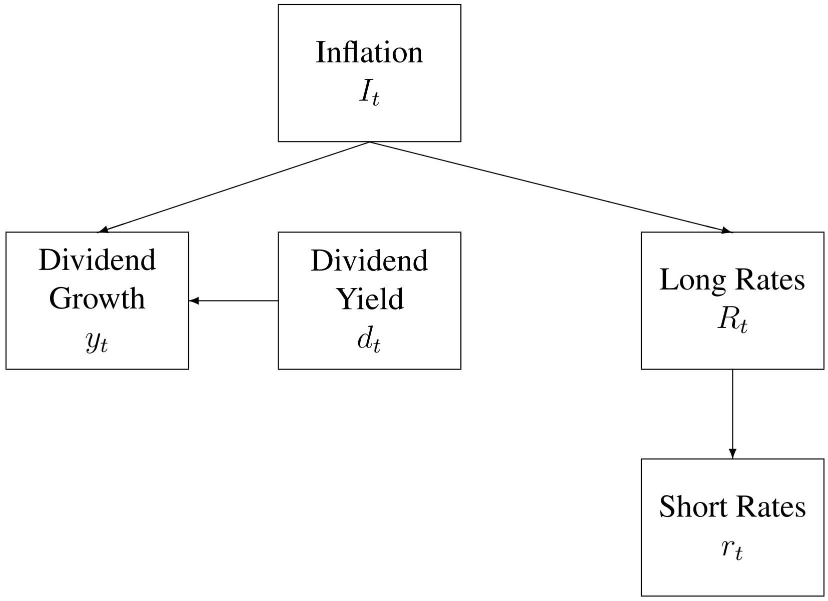


Figure 10: The Wilkie ESG, reduced form from **?**, with short rate extension from **?**.

Since $C_t^w = I_t - c_t^*$, the long-term objective function can be simplified as:

$$\frac{1}{T}\mathbb{E}\left[\sum_{t=0}^{T} \Sigma^W \left(\frac{I_t - c_t - C_t^w}{I_t}\right)^2 + \Sigma^R \left(\frac{b_t - C_t^r}{I_t}\right)^2\right]$$

$$= \frac{1}{T}\mathbb{E}\left[\sum_{t=0}^{T} \frac{1}{\Sigma^W}\left(\alpha\frac{A_t - L_t}{I_t}\right)^2 + \frac{1}{\Sigma^R}\left(\beta\frac{A_t - L_t}{I_t}\right)^2\right].$$

Notice that only A_t depends on the risk-sharing parameters α and β; L_t and I_t do not. Taking the derivative with respect to α we have

$$\frac{1}{T}\mathbb{E}\left[\sum_{t=0}^{T} 2\frac{1}{\Sigma^W}\left(\alpha\left(\frac{A_t - L_t}{I_t}\right)^2 + \alpha^2\frac{\partial A_t}{\partial \alpha}\frac{1}{I_t}\frac{A_t - L_t}{I_t}\right) + 2\frac{1}{\Sigma^R}\beta^2\left(\frac{A_t - L_t}{I_t}\right)\frac{\partial A_t}{\partial \alpha}\frac{1}{I_t}\right],$$

and since $\dfrac{\partial A_t}{\partial \alpha} = \dfrac{\partial A_t}{\partial \beta}$, for the derivative with respect to β, we have

$$\frac{1}{T}\mathbb{E}\left[\sum_{t=0}^{T} 2\frac{1}{\Sigma^W}\alpha^2\frac{\partial A_t}{\partial \alpha}\frac{1}{I_t}\frac{A_t - L_t}{I_t} + 2\frac{1}{\Sigma^R}\left(\beta\left(\frac{A_t - L_t}{I_t}\right)^2 + \beta^2\left(\frac{A_t - L_t}{I_t}\right)\frac{\partial A_t}{\partial \alpha}\frac{1}{I_t}\right)\right].$$

Setting both derivatives to zero gives:

$$\mathbb{E}\left[\sum_{t=0}^{T}\left(\frac{\alpha}{\Sigma^W} - \frac{\beta}{\Sigma^R}\right)\left(\frac{A_t - L_t}{I_t}\right)^2\right] = 0 \implies \frac{\alpha}{\Sigma^W} = \frac{\beta}{\Sigma^R}.$$

C Proof of Equation (2)

Here we simplify the first part of Equation (1), focusing the on the future benefit payment; the second part can be simplified similarly. Assume a stationary population such that $l_x(t) = l_x$:

$$\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left\{\sum_{x=x_e-T}^{\omega-1}\sum_{t=\max(x_r-x,0)}^{\omega-1-x} l_{x+t}(t)\left(\frac{b_t-C_t^r}{I_t}\right)^2\right\}$$

$$=\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left\{\left(\sum_{x=x_e-T}^{x_r}\sum_{t=x_r-x}^{\omega-1-x}+\sum_{x=x_r+1}^{\omega-1}\sum_{t=0}^{\omega-1-x}\right) l_{x+t}\left(\frac{b_t-C_t^r}{I_t}\right)^2\right\}$$

$$=\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left\{\sum_{x=x_e-T}^{x_r}\sum_{t=x_r-x}^{\omega-1-x} l_{x+t}\left(\frac{b_t-C_t^r}{I_t}\right)^2\right\}+\underbrace{\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left\{\sum_{x=x_r+1}^{\omega-1}\sum_{t=0}^{\omega-1-x} l_{x+t}\left(\frac{b_t-C_t^r}{I_t}\right)^2\right\}}_{=0}$$

$$=\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left\{\underbrace{\sum_{t=0}^{\omega-1-x_e+T}\sum_{x=x_r-t}^{\omega-1-t}}_{\text{interchange summation}} l_{x+t}\left(\frac{b_t-C_t^r}{I_t}\right)^2\right\}$$

$$=\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left\{\sum_{t=0}^{\omega-1-x_e+T}\sum_{z=x_r}^{\omega-1} l_z\left(\frac{b_t-C_t^r}{I_t}\right)^2\right\}$$

$$=\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left\{\sum_{t=0}^{T}\sum_{x=x_r}^{\omega-1} l_x\left(\frac{b_t-C_t^r}{I_t}\right)^2\right\}+\underbrace{\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left\{\sum_{t=T+1}^{T+1+\omega-2-x_e}\sum_{x=x_r}^{\omega-1} l_z\left(\frac{b_t-C_t^r}{I_t}\right)^2\right\}}_{=0}$$

$$=\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T}\mathbb{E}\left\{\left(\sum_{x=x_r}^{\omega-1} l_x\right)\left(\frac{b_t-C_t^r}{I_t}\right)^2\right\}$$

D Theoretical Analysis in a Simplified Setting

Equation (5) outlines the procedure in obtaining a phased transition from a DB plan to a TB plan that ensures some benefit for all cohorts. However, the complexity makes the problem analytically intractable. In this appendix, we simplify some of the assumptions, allowing us to develop theoretical support when the objective function is the expected remaining lifetime downside payment, as defined in equation (7), and repeated here for convenience:

$$\mathcal{E}(x) = \sum_{t=0}^{\omega-1-x} {}_t p_x \, \mathbb{E}\left[\left(\frac{(b_t - b_t^*)_-}{I_t}\right)\right].$$

We also use the following assumptions and notation:

- We assume the liability is a constant $L_t \equiv L \times I_t$. This is equivalent to an ideal situation where the discount rate used for liability calculation is robust with respect to the market condition, that is, $d_t \equiv \hat{d}$ is a constant.
- We denote the portfolio return from period $t-1$ to t by Z_t, and we assume Z_t and the inflation rate, $I_t/I_{t-1} - 1$ are Markov processes.
- $\alpha^s = \alpha^d = 0$, and $\beta^s = \beta^d = \beta$, that is, the contribution is fixed, and we have the same risk-sharing for surplus and loss. This is how a typical Canadian TB plan is structured (see, for example, **?**).
- We denote the real portfolio return index by $\mathcal{R}_t = \mathbb{E}\left[\frac{Z_t}{I_t/I_{t-1}}\right]$, with $\mathcal{R}_0 = 1$, and $\mathcal{R}_t > 0$ to have an overall positive expected return.

Proposition 2. *Consider a set of risk-sharing parameters $\boldsymbol{\beta}_x = \{\beta_j, x_r \leq j < \omega\}$ that satisfies condition (4) for a retiree aged x. Denote another set of risk-sharing parameters $\hat{\boldsymbol{\beta}}_x$ that is the same as $\boldsymbol{\beta}_x$, except $\hat{\beta}_x < \beta_x$. Then*

$$\left[\prod_{j=1}^{\omega-x} \frac{1}{\mathcal{R}_j} - \frac{\hat{\beta}_x}{\sum_x l_x} \sum_{j=1}^{\omega-x} {}_j p_x \prod_{i=j+1}^{\omega-x} \frac{1}{\mathcal{R}_i}\right] > 0 \implies \mathcal{E}^{\boldsymbol{\beta}_x}(x) - \mathcal{E}^{\hat{\boldsymbol{\beta}}_x}(x) \geq 0, \tag{10}$$

where $\mathcal{E}^{\boldsymbol{\beta}_x}(x)$ is the objective function (7) under the set of risk-sharing parameters $\boldsymbol{\beta}_x$.

Proof. We prove the proposition through induction. For any arbitrary age $x \in [x_r, \omega)$, let A_t be the asset process where the risk-sharing for retirees is based on $\boldsymbol{\beta}_x$ and $\hat{A}_t$ with risk-sharing strategy $\hat{\boldsymbol{\beta}}_x$. Then, it is easy to show that:

$$\hat{A}_t = A_t + \xi_t \times Z_t, \qquad t \geq 1,$$

where

$$\xi_1 = l_x \left(\beta_x - \hat{\beta}_x\right) \frac{A_0 - L \times I_0}{\sum_x l_x},$$

$$\xi_{t+1} = \xi_t Z_t \left(1 - \sum_j l_j \frac{\beta_j}{\sum_x l_x} + l_{x+t}(\beta_x - \hat{\beta}_x)\frac{1}{\sum_x l_x}\right) + l_{x+t}(\beta_x - \hat{\beta}_x)\frac{A_t - L \times I_t}{\sum_x l_x}.$$

To simplify the notation, we denote $\Sigma_l = \sum_x l_x$ and $\Sigma_B = \sum_j l_j \frac{\beta_j}{\sum_x l_x}$.

Next, denote by $\mathcal{E}^{\boldsymbol{\beta}_x,T}(x)$ the loss function:

$$\mathcal{E}^{\boldsymbol{\beta}_x,T}(x) = \sum_{t=0}^{T} {}_t p_x \mathbb{E}\left[\frac{(b_t(x) - b_t(x)^*)_-}{I_t}\right], \quad T < \omega - x.$$

Then for $T = 0$, we have

$$\mathcal{E}^{\boldsymbol{\beta}_x,0}(x) - \mathcal{E}^{\hat{\boldsymbol{\beta}}_x,0}(x) = {}_0 p_x \left(\beta_x - \hat{\beta}_x\right) \frac{(A_0 - L \times I_0)_-}{\Sigma_l I_0} \geq 0.$$

For $T = 1$, we have

$$\begin{aligned}
&\mathcal{E}^{\boldsymbol{\beta}_x,1}(x) - \mathcal{E}^{\hat{\boldsymbol{\beta}}_x,1}(x) \\
&\qquad = {}_0 p_x \left[\left(\beta_x - \hat{\beta}_x\right) \frac{(A_0 - L)_-}{\Sigma_l I_0}\right] + {}_1 p_x \mathbb{E}\left[\beta_x \left(\frac{(A_1 - L \times I_1)_-}{\Sigma_l I_1}\right) - \hat{\beta}_x \left(\frac{(\hat{A}_1 - L \times I_1)_-}{\Sigma_l I_1}\right)\right] \\
&\qquad \geq {}_0 p_x \frac{-\xi_1}{I_0 l_x} \mathbf{1}_{\xi_1 < 0} + {}_1 p_x \mathbb{E}\left\{\hat{\beta}_x \left[\left(\frac{(A_1 - L \times I_1)_-}{\Sigma_l I_1}\right) - \left(\frac{(A_1 + \xi_1 Z_1 - L \times I_1)_-}{\Sigma_l I_1}\right)\right] \mathbf{1}_{\xi_1 < 0}\right\} \\
&\qquad\qquad + {}_1 p_x \mathbb{E}\left[(\beta_x - \hat{\beta}_x)\left(\frac{(A_1 - L \times I_1)_-}{\Sigma_l I_1}\right)\right] \\
&\qquad \geq \mathbb{E}\left\{\left[{}_0 p_x \frac{-\xi_1}{I_0 l_x} + {}_1 p_x \hat{\beta}_x \frac{\xi_1 Z_1}{\Sigma_l I_1}\right] \mathbf{1}_{\xi_1 < 0}\right\} + {}_1 p_x \mathbb{E}\left[(\beta_x - \hat{\beta}_x)\left(\frac{(A_1 - L \times I_1)_-}{\Sigma_l I_1}\right)\right] \\
&\qquad \geq \mathbb{E}\left\{-\xi_1 \left[\frac{{}_0 p_x}{I_0 l_x} - {}_1 p_x \hat{\beta}_x \frac{Z_1}{\Sigma_l I_1}\right] \mathbf{1}_{\xi_1 < 0}\right\} + {}_1 p_x \mathbb{E}\left[(\beta_x - \hat{\beta}_x)\left(\frac{(A_1 - L \times I_1)_-}{\Sigma_l I_1}\right)\right] \\
&\qquad = \mathbb{E}\left[\frac{-\xi_1}{I_0} \mathbf{1}_{\xi_1 < 0}\right] \mathbb{E}\left[\frac{{}_0 p_x}{l_x} - {}_1 p_x \hat{\beta}_x \frac{Z_1}{\Sigma_l \frac{I_1}{I_0}}\right] + {}_1 p_x \mathbb{E}\left[(\beta_x - \hat{\beta}_x)\left(\frac{(A_1 - L \times I_1)_-}{\Sigma_l I_1}\right)\right],
\end{aligned}$$

which is greater than zero if

$$\frac{{}_0p_x}{I_0 l_x} - {}_1p_x \hat{\beta}_x \mathbb{E}\left[\frac{Z_1}{\Sigma_l I_1}\right] > 0.$$

Our conjecture is that, for any $t \in [1, \omega - x]$, if

$$\left[\prod_{j=1}^{t} \frac{1}{\mathcal{R}_j} - \frac{\hat{\beta}_x}{\Sigma_l} \sum_{j=1}^{t} {}_jp_x \prod_{i=j+1}^{t} \frac{1}{\mathcal{R}_i}\right] > 0. \tag{11}$$

Then

$$\begin{aligned}\mathcal{E}^{\boldsymbol{\beta}_x,t}(x) - \mathcal{E}^{\hat{\boldsymbol{\beta}}_x,t}(x) &\geq \mathbb{E}\left[-\xi_t \mathbf{1}_{\xi_t<0}\right] \times \mathcal{R}_t \times \left(\prod_{j=1}^{t} \frac{1}{\mathcal{R}_j} - \frac{\hat{\beta}_x}{\Sigma_l} \sum_{j=1}^{t} \prod_{i=j+1}^{t} \frac{{}_jp_x}{\mathcal{R}_i}\right) \\ &\quad + {}_tp_x \mathbb{E}\left[\left(\beta_x - \hat{\beta}_x\right) \frac{(A_t - L \times I_t)_-}{\Sigma_l I_t}\right] \\ &\geq 0.\end{aligned}$$

Clearly for $t = 1$, we have

$$\mathcal{E}^{\boldsymbol{\beta}_x,1}(x) - \mathcal{E}^{\hat{\boldsymbol{\beta}}_x,1}(x) \geq \mathbb{E}\left[-\xi_1 \mathbf{1}_{\xi_1<0}\right] \left(1 - {}_1p_x \frac{\hat{\beta}_x}{\Sigma_l} \times \mathcal{R}_t\right) + {}_1p_x \mathbb{E}\left[(\beta_x - \hat{\beta}_x)\left(\frac{(A_1 - L \times I_1)_-}{\Sigma_l I_1}\right)\right],$$

which satisfies the conjecture.

Now suppose the statement is true for $1, \cdots, t$, then for $t+1$, we have

$$\begin{aligned}&\mathcal{E}^{\boldsymbol{\beta}_x,t+1}(x) - \mathcal{E}^{\hat{\boldsymbol{\beta}}_x,t+1}(x) \\ &\quad \geq \mathbb{E}\left[-\xi_t \mathbf{1}_{\xi_t<0}\right] \times \mathcal{R}_t \times \mathbb{E}\left[\prod_{j=1}^{t} \frac{1}{\mathcal{R}_j} - \frac{\hat{\beta}_x}{\Sigma_l} \sum_{j=1}^{t} {}_jp_x \prod_{i=j+1}^{t} \frac{1}{\mathcal{R}_i}\right] \\ &\qquad + {}_tp_x \mathbb{E}\left[(\beta_x - \hat{\beta}_x)\left(\frac{(A_t - L \times I_t)_-}{\Sigma_l I_t}\right)\right] + {}_{t+1}p_x \mathbb{E}\left[(\beta_x - \hat{\beta}_x)\left(\frac{(A_{t+1} - L \times I_{t+1})_-}{\Sigma_l I_{t+1}}\right)\right] \\ &\qquad + {}_{t+1}p_x \mathbb{E}\left[\hat{\beta}_x \left(\frac{(A_{t+1} - L \times I_{t+1})_-}{\Sigma_l I_{t+1}}\right) - \hat{\beta}_x \left(\frac{(A_{t+1} + \xi_{t+1} Z_{t+1} - L \times I_{t+1})_-}{\Sigma_l I_{t+1}}\right)\right].\end{aligned}$$

Clearly, the first three terms are always positive. For the fourth term (third line), we consider four cases:

- $\xi_t > 0 \cap A_t > L \times I_t \implies \xi_{t+1} > 0$, which implies that:

$$_{t+1}p_x \mathbb{E}\left\{\left[\hat{\beta}_x\left(\frac{(A_{t+1} - L \times I_{t+1})_-}{\Sigma_l I_{t+1}}\right) - \hat{\beta}_x\left(\frac{(A_{t+1} + \xi_{t+1} Z_{t+1} - L \times I_{t+1})_-}{\Sigma_l I_{t+1}}\right)\right] \mathbf{1}_{\xi_t > 0 \cap A_t > L \times I_t}\right\} \geq 0.$$

- $\xi_t > 0 \cap A_t < L \times I_t \implies \xi_{t+1} > l_{x+t}(\beta_x - \hat{\beta}_x)\frac{A_t - L \times I_t}{\Sigma_l}$ (we have the constraint such that the participants will not share more than the portfolio risk $(1 - \Sigma_B) > 0$). Next, we combine with the second term to have:

$$\mathbb{E}\left\{\left[{}_tp_x(\beta_x - \hat{\beta}_x)\left(\frac{(A_t - L \times I_t)_-}{\Sigma_l I_t}\right) + {}_{t+1}p_x \hat{\beta}_x\left(\frac{l_{x+t}(\beta_x - \hat{\beta}_x)\frac{A_t - L \times I_t}{\Sigma_l}}{\Sigma_l I_{t+1}} Z_{t+1}\right)\right] \mathbf{1}_{\xi_t > 0 \cap A_t < L \times I_t}\right\}$$

$$= \mathbb{E}\left\{(\beta_x - \hat{\beta}_x)\frac{(A_t - L \times I_t)_-}{\Sigma_l}\left(\frac{{}_tp_x}{I_t} - \frac{{}_{t+1}p_x l_{x+t}\hat{\beta}_x}{\Sigma_l I_{t+1}} Z_{t+1}\right) \mathbf{1}_{\xi_t > 0 \cap A_t < L \times I_t}\right\}$$

$$= \mathbb{E}\left[\left(\beta_x - \hat{\beta}_x\right)\frac{(A_t - L \times I_t)_-}{\Sigma_l I_t} \mathbf{1}_{\xi_t > 0 \cap A_t < L \times I_t}\right] \times \mathcal{R}_{t+1} \times \left(\frac{{}_tp_x}{\mathcal{R}_{t+1}} - \frac{{}_{t+1}p_x l_{x+t}\hat{\beta}_x}{\Sigma_l}\right)$$

$$\geq \mathbb{E}\left[\max(-\xi_{t+1}, 0)\mathbf{1}_{\xi_t > 0 \cap A_t < L \times I_t}\right] \times \mathcal{R}_{t+1} \times \left(\prod_{j=1}^{t+1}\frac{1}{\mathcal{R}_j} - \frac{\hat{\beta}_x}{\Sigma_l}\sum_{j=1}^{t+1}{}_jp_x \prod_{i=j+1}^{t+1}\frac{1}{\mathcal{R}_i}\right).$$

  The last line is due to the fact that $\mathcal{R}_t > 0$ for all t such that

$$\left(\frac{{}_tp_x}{\mathcal{R}_{t+1}} - \frac{{}_{t+1}p_x l_{x+t}\hat{\beta}_x}{\Sigma_l}\right) > \left(\prod_{j=1}^{t+1}\frac{1}{\mathcal{R}_j} - \frac{\hat{\beta}_x}{\Sigma_l}\sum_{j=1}^{t+1}{}_jp_x \prod_{i=j+1}^{t+1}\frac{1}{\mathcal{R}_i}\right).$$

- $\xi_t < 0 \cap A_t > L \times I_t$, implies that:

$$_{t+1}p_x \mathbb{E}\left\{\left[\hat{\beta}_x\left(\frac{(A_{t+1} - L \times I_t)_-}{\Sigma_l I_{t+1}}\right) - \hat{\beta}_x\left(\frac{(A_{t+1} + \xi_{t+1} Z_{t+1} - L \times I_t)_-}{\Sigma_l I_{t+1}}\right)\right] \mathbf{1}_{\xi_t < 0 \cap A_t > L \times I_t}\right\}$$

$$\geq {}_{t+1}p_x \mathbb{E}\left\{\left[\hat{\beta}_x \frac{\xi_t Z_t Z_{t+1}\left(1 - \Sigma_B + l_{x+t}\frac{\beta_x - \hat{\beta}_x}{\Sigma_l}\right)}{\Sigma_l I_{t+1}}\right] \mathbf{1}_{\xi_t < 0 \cap A_t > L}\right\}.$$

Combining with the first term we have:

$$\mathbb{E}\left[\frac{-\xi_t Z_t}{I_t}\left(\prod_{j=1}^{t}\frac{1}{\mathcal{R}_j}-\frac{\hat{\beta}_x}{\Sigma_l}\sum_{j=1}^{t}\prod_{i=j+1}^{t}\frac{{}_jp_x}{\mathcal{R}_i}-{}_{t+1}p_x\hat{\beta}_x\frac{Z_{t+1}\left(1-\Sigma_B+l_{x+t}\frac{\beta_x-\hat{\beta}_x}{\Sigma_l}\right)}{\Sigma_l I_{t+1}/I_t}\right)\mathbf{1}_{\xi_t<0\cap A_t>L\times I_t}\right]$$

$$\geq\mathbb{E}\left\{\left[\frac{-\xi_t Z_t}{I_t}\left(\prod_{j=1}^{t}\frac{1}{\mathcal{R}_j}-\frac{\hat{\beta}_x}{\Sigma_l}\sum_{j=1}^{t}\prod_{i=j+1}^{t}\frac{{}_jp_x}{\mathcal{R}_i}-{}_{t+1}p_x\hat{\beta}_x\frac{Z_{t+1}}{\Sigma_l I_{t+1}/I_t}\right)\right]\mathbf{1}_{\xi_t<0\cap A_t>L\times I_t}\right\}$$

$$\geq\mathbb{E}\left\{\max(-\xi_{t+1},0)\mathbf{1}_{\xi_t<0\cap A_t>L\times I_t}\right\}\times\mathcal{R}_{t+1}\times\left(\prod_{j=1}^{t+1}\frac{1}{\mathcal{R}_j}-\frac{\hat{\beta}_x}{\Sigma_l}\sum_{j=1}^{t+1}{}_jp_x\prod_{i=j+1}^{t+1}\frac{1}{\mathcal{R}_i}\right).$$

- $\xi_t<0\cap A_t<L\times I_t \implies \xi_{t+1}<0$, implies that:

$${}_{t+1}p_x\mathbb{E}\left\{\left[\hat{\beta}_x\left(\frac{(A_{t+1}-L\times I_t)_-}{\Sigma_l I_{t+1}}\right)-\hat{\beta}_x\left(\frac{(A_{t+1}+\xi_{t+1}Z_{t+1}-L\times I_t)_-}{\Sigma_l I_{t+1}}\right)\right]\mathbf{1}_{\xi_t<0\cap A_t<L\times I_t}\right\}$$

$$\geq\mathbb{E}\left\{{}_{t+1}p_x\hat{\beta}_x\frac{\xi_{t+1}Z_{t+1}}{\Sigma_l I_{t+1}}\mathbf{1}_{\xi_t<0\cap A_t<L\times I_t}\right\}.$$

Combined with first and second term we have

$$\mathbb{E}\left\{\left[\frac{-\xi_t Z_t}{I_t}\left(\prod_{j=1}^{t}\frac{1}{\mathcal{R}_j}-\frac{\hat{\beta}_x}{\Sigma_l}\sum_{j=1}^{t}\prod_{i=j+1}^{t}\frac{{}_jp_x}{\mathcal{R}_i}-{}_{t+1}p_x\hat{\beta}_x\frac{Z_{t+1}\left(1-\Sigma_B+l_{x+t}\frac{\beta_x-\hat{\beta}_x}{\Sigma_l}\right)}{\Sigma_l I_{t+1}/I_t}\right)\right]\mathbf{1}_{\xi_t<0\cap A_t<L\times I_t}\right\}$$

$$+\,\mathbb{E}\left\{(\beta_x-\hat{\beta}_x)\frac{(A_t-L)_-}{\Sigma_l I_t}\left({}_tp_x-\frac{{}_{t+1}p_x l_{x+t}\hat{\beta}_x}{\Sigma_l I_{t+1}/I_t}Z_{t+1}\right)\mathbf{1}_{\xi_t<0\cap A_t<L\times I_t}\right\}$$

where the second line is positive as we have proved for the case $\xi_t>0\cap A_t<L\times I_t$, and the first time can be shown to be

$$\geq\mathbb{E}\left\{-\xi_{t+1}\mathbf{1}_{\xi_t<0\cap A_t<L\times I_t}\right\}\times\mathcal{R}_{t+1}\times\left(\prod_{j=1}^{t+1}\frac{1}{\mathcal{R}_j}-\frac{\hat{\beta}_x}{\Sigma_l}\sum_{j=1}^{t+1}{}_jp_x\prod_{i=j+1}^{t+1}\frac{1}{\mathcal{R}_i}\right).$$

Notice that since

$$\mathbb{E}\left[-\xi_{t+1}\mathbf{1}_{\xi_t<0\cap A_t<L\times I_t}\right]+\mathbb{E}\left[\max(-\xi_{t+1},0)\mathbf{1}_{\xi_t<0\cap A_t>L\times I_t}\right]+\mathbb{E}\left[\max(-\xi_{t+1},0)\mathbf{1}_{\xi_t>0\cap A_t<L\times I_t}\right]$$

$$=\mathbb{E}\left[-\xi_{t+1}\mathbf{1}_{\xi_{t+1}<0}\right],$$

together with the third term, we have:

$$\mathcal{E}^{\boldsymbol{\beta}_x,t+1}(x)-\mathcal{E}^{\hat{\boldsymbol{\beta}}_x,t+1}(x)\geq\mathbb{E}\left\{-\xi_{t+1}\mathbf{1}_{\xi_{t+1}<0}\right\}\times\mathcal{R}_{t+1}\times\left(\prod_{j=1}^{t+1}\frac{1}{\mathcal{R}_j}-\frac{\hat{\beta}_x}{\Sigma_l}\sum_{j=1}^{t+1}{}_jp_x\prod_{i=j+1}^{t+1}\frac{1}{\mathcal{R}_i}\right)$$

$$+{}_{t+1}p_x\times\mathbb{E}\left[(\beta_x-\hat{\beta}_x)\left(\frac{(A_{t+1}-L)_-}{\Sigma_l I_{t+1}}\right)\right]$$

□

The proposition shows that, if $\hat{\beta}_x$ satisfies equation (4), then any $\beta_x\geq\hat{\beta}_x$ would also satisfy the objective. In other words, a retiree aged x will always benefit when his/her corresponding deficit sharing parameter β_x is reduced. Of course this comes at a cost to other cohorts, as the risk is shared between a smaller number of cohorts, but the monotonicity enables us to recursively solve Equation (5) for each age.