# BSDE Approach to Utility Maximization with Square-root Factor Processes

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**Abstract**

We consider the utility-based portfolio selection problem in a continuous-time setting. We assume the market price of risk depends on a stochastic factor that satisfies an affine-form, square-root, Markovian model. This financial market framework includes the classical geometric Brownian motion, CEV model, and Heston's model as special cases. Adopting the BSDE approach, we obtain closed-form solutions for the optimal portfolio strategies and value functions for the logarithmic, power, and exponential utility functions.

*Keywords:* Backward Stochastic Differential Equations, Utility Maximization, Square-root Factor Process, Riccati Equation

## 1. Introduction

We consider a utility-based continuous-time portfolio selection problem in a model in which the market price of risk depends on a stochastic factor that satisfies a CIR diffusion process. This framework includes the geometric Brownian motion, constant elasticity of variance (CEV), and Heston model as special cases. We derive closed-form expressions for the optimal investment strategy and the optimal value using the backward stochastic differential equation (BSDE) approach to stochastic control for logarithmic, power, and exponential utility functions. The solutions are obtained by solving a system of ODEs involving a Riccati ODE with constant coefficients. The boundedness of the solution to this Riccati ODE is critical for this solution technique, and we show that boundedness holds in the problem.

Our paper was motivated in part by [7], in which the mean-variance investment-reinsurance problem is considered for the same market model. Another related paper is [6], in which the investment problem is formulated taking into account the effect of stochastic volatility. [1] is another related paper, which studies portfolio optimization under stochastic volatility, considered as a perturbation of the complete market constant volatility model, under both fast and slow mean-reverting volatility. Our results cover (particular cases of) both stochastic and local volatility models. We note that most of the literature concerning utility maximization investment problems adopting the BSDE approach only considers the existence and uniqueness of solutions to the corresponding BSDE without presenting closed-form solutions. Furthermore, in typical applications of the BSDE approach to portfolio optimization, the solution $(Y, Z)$ to the BSDE is considered under the requirement that $Y$ is a uniformly bounded process. Our work relaxes the boundedness assumption on $Y$.

The remainder of this paper is structured as follows. Section 2 presents the financial market model and required assumptions. Section 3 formulates and solves the utility maximization problems. Some technical lemmas and proofs are presented in the Appendix.

## 2. Model Formulation and Preliminary Analysis

### 2.1. Financial Market Model

An agent, with initial wealth $x_0 > 0$, invests capital in a risk-free bond $B$ and a risky asset $S$ with price processes as follows:

$$\begin{cases} dB_t = r_t B_t dt, \\ dS_t = \mu_t S_t dt + \sigma_t S_t dW_t^{(1)}, \end{cases}$$

where $r_t$ is the risk-free short rate at time $t$, $\mu_t$ is the growth rate of the risky asset at time $t$ and $\sigma_t$ is the

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instantaneous volatility of the risky asset at time $t$. We assume that the market price of risk $\theta_t := \frac{\mu_t - r_t}{\sigma_t}$, $0 \leqslant t \leqslant T$, is related to a stochastic factor process $\alpha := \{\alpha_t\}_{0\leqslant t\leqslant T}$ as follows:

$$\theta_t = \bar{\theta}\sqrt{\alpha_t}, \ \ \forall t \in [0,T], \ \ \bar{\theta} \in \mathbb{R}\setminus\{0\},$$

where the stochastic factor process $\{\alpha_t\}_{0\leqslant t\leqslant T}$ satisfies the following SDE

$$\begin{cases} d\alpha_t = \kappa(\phi - \alpha_t)\,dt \\ \qquad\qquad + \sqrt{\alpha_t}\left(\rho_1 dW_t^{(1)} + \rho_2 dW_t^{(2)}\right), \\ \alpha_t|_{t=0} = \alpha_0 \geqslant 0. \end{cases} \tag{1}$$

$W := \{(W_t^{(1)}, W_t^{(2)}), t \geqslant 0\}$ is a standard Brownian motion on $\mathbb{R}^2$ under the physical measure $\mathbb{P}$ defined over a probability space $(\Omega, \mathcal{F})$. We use $\mathbb{F} := \{\mathcal{F}_t, t \geqslant 0\}$ to denote the $\mathbb{P}$-augmentation of the natural filtration generated by the Brownian motion $W$. We impose the following two assumptions:

**H1.** $\kappa\phi \geqslant 0$;

**H2.** $r_t = 0$ for $0 \leqslant t \leqslant T$.

**Remark 2.1.** ***H1*** *is imposed to ensure $\alpha_t \geqslant 0$ for all $t \in [0,T]$. Notice that we do not impose the Feller condition for strict positivity of $\alpha$, i.e. $2\kappa\phi \geqslant \rho_1^2 + \rho_2^2$ in our case; for further details see Chapter 6 of [3].* ***H2*** *follows most references concerning utility maximization using the BSDE approach; see [2] and Chapter 6 in [4]. If* ***H2*** *is not imposed, the utility maximization problem can be considered in terms of the discounted wealth instead of the terminal wealth.*

The above financial model was studied in [7] in the context of solving a mean-variance investment-reinsurance problem. It covers several well-known models including the geometric Brownian motion model, the CEV model, the Heston's model, as well as other non-Markovian models.

**Example 2.1.** *(CEV Model). If $\mu_t = \mu$, $\sigma_t = \sigma S_t^{\beta}$, $r_t = r$, with $\mu > r \geqslant 0$, $\sigma > 0$ and $\beta \in \mathbb{R}$, then the risky asset price is given by the CEV model:*

$$dS_t = S_t\left[\mu dt + \sigma S_t^{\beta} dW_t^{(1)}\right],$$

*where $\beta$ is called the elasticity parameter of the risky asset. If we set $\alpha_t = S_t^{-2\beta}$, $\kappa = 2\beta\mu$, $\phi = (\beta + \frac{1}{2})\frac{\sigma^2}{\mu}$, $\rho_1 = -2\beta\sigma$, $\rho_2 = 0$ and $\bar{\theta} = \frac{\mu - r}{\sigma}$, then*

$$\begin{aligned} d\alpha_t &= dS_t^{-2\beta} \\ &= 2\beta\mu\left[\left(\beta + \frac{1}{2}\right)\frac{\sigma^2}{\mu} - S_t^{-2\beta}\right]dt - 2\beta\sigma S_t^{-\beta} dW_t^{(1)} \\ &= \kappa(\phi - \alpha_t)\,dt + \sqrt{\alpha_t}\left(\rho_1 dW_t^{(1)} + \rho_2 dW_t^{(2)}\right). \end{aligned}$$

*If we set $\beta = 0$, then the CEV model reduces to the classical geometric Brownian motion framework.*

**Example 2.2.** *(Heston's Model). If $r_t = r$, $\mu_t = r + \bar{\theta}\nu_t$, $\sigma_t = \sqrt{\nu_t}$, $\rho_1 = \sigma_0\rho$ and $\rho_2 = \sigma_0\sqrt{1-\rho^2}$ where $r \geqslant 0$, $\bar{\theta} \in \mathbb{R}\setminus\{0\}$, $\sigma_0 > 0$ and $\rho \in (-1,1)$, then the risky asset price is given by Heston's model:*

$$dS_t = S_t\left[\left(r + \bar{\theta}\nu_t\right)dt + \sqrt{\nu_t}\, dW_t^{(1)}\right],$$

*where $\nu_t = \alpha_t$ for $0 \leqslant t \leqslant T$ satisfies*

$$\begin{aligned} d\nu_t &= \kappa(\phi - \nu_t)\,dt \\ &\quad + \sigma_0\sqrt{\nu_t}\left(\rho dW_t^{(1)} + \sqrt{1-\rho^2}\, dW_t^{(2)}\right). \end{aligned}$$

**Example 2.3.** *Set $\mu_t = r_t + \bar{\theta}\sqrt{\alpha_t}\cdot\widehat{\sigma}(\alpha_{[0,t]})$ and $\sigma_t = \widehat{\sigma}(\alpha_{[0,t]})$ for some functional $\widehat{\sigma} : \mathcal{C}(0,t;\mathbb{R}) \to \mathbb{R}_+$, where $\alpha_{[0,t]} := \{\alpha_s\}_{s\in[0,t]}$ is the restriction of $\alpha(\cdot) \in \mathcal{C}(0,T;\mathbb{R})$ to $\mathcal{C}(0,t;\mathbb{R})$, i.e. the space of real-valued, continuous functions defined on $[0,t]$. Then the risky asset price is given by a path-dependent model:*

$$dS_t = S_t\left[\left(r_t + \bar{\theta}\sqrt{\alpha_t}\cdot\widehat{\sigma}(\alpha_{[0,t]})\right)dt + \widehat{\sigma}(\alpha_{[0,t]})dW_t^{(1)}\right],$$

*and $\alpha_t$ satisfies (1). This is a special case of the more general non-Markovian risky asset price model in [8].*

**Lemma 2.2.** *If two deterministic functions $m_1(t)$ and $m_2(t)$ are uniformly bounded on $[0,T]$, then the stochastic exponential process defined by*

$$\begin{aligned} \exp\Bigg\{ &-\frac{1}{2}\int_0^t \left(m_1^2(s) + m_2^2(s)\right)\alpha_s ds \\ &+ \int_0^t m_1(s)\sqrt{\alpha_s}\, dW_s^{(1)} + \int_0^t m_2(s)\sqrt{\alpha_s}\, dW_s^{(2)} \Bigg\} \end{aligned}$$

*is a $\mathcal{F}_t$-adapted martingale.*

*Proof.* See Appendix A ☐

## 3. Utility Maximization

In the sequel, the logarithmic, power, and exponential utility cases will be indicated by the subscript $i = 1, 2, 3$ respectively. We consider a finite investment time horizon $[0,T]$ with $T > 0$. Let $w_t$ denote the proportion of total wealth invested in the risky asset at time $t$, assuming the total wealth remains at a strictly positive level within the investment horizon. With the trading strategy $w := \{w_t, 0 \leqslant t \leqslant T\}$, the portfolio value process $X_t^w$ follows:

$$dX_t^w = X_t^w\left[w_t\mu_t dt + \sigma_t w_t dW_t^{(1)}\right], \ \ t \geqslant 0. \tag{2}$$

For exponential utility maximization, we work with the amount of wealth $\pi_t$ invested in the risky asset at time $t$. With the trading strategy $\pi := \{\pi_t, 0 \leqslant t \leqslant T\}$, the portfolio value process $X_t^\pi$ follows:

$$dX_t^\pi = \pi_t \mu_t dt + \sigma_t \pi_t dW_t^{(1)}, \quad t \geqslant 0. \tag{3}$$

The trading strategies $\pi$ and $w$ are $\mathbb{F}$-progressively measurable and they satisfy $\mathbb{E}\left[\int_0^T \sigma_t^2 w_t^2 dt\right] < \infty$ and $\mathbb{E}\left[\int_0^T \sigma_t^2 \pi_t^2 dt\right] < \infty$ respectively, so that unique strong solutions exist for the SDEs (2) and (3). We let $\mathcal{S}$ denote the set of trading strategies $w$ satisfying the above conditions. Further, for constant $M > 0$, we write $\mathcal{S}_M$ as the set of trading strategies $\pi$ satisfying the above conditions and such that the collection $\{e^{-\eta X_\tau^\pi + M\alpha_\tau} : \tau$ is a stopping time valued in $[0, T]\}$ is a uniformly integrable family.

**Definition 3.1.** *For power and logarithmic utility, a trading strategy $w := \{w_t, 0 \leqslant t \leqslant T\}$ is called admissible with initial wealth $x_0 > 0$ if it belongs to the following set:*

$$\mathcal{A}_1(x_0) = \mathcal{A}_2(x_0) := \{w \in \mathcal{S} : \; X_0^w = x_0, \\ \text{and } X_t^w > 0, a.s., \forall\, 0 \leqslant t \leqslant T\}.$$

*For exponential utility, a trading strategy $\pi := \{\pi_t, 0 \leqslant t \leqslant T\}$ is called admissible with initial wealth $x_0 > 0$ if it belongs to the following set:*

$$\mathcal{A}_3(x_0) := \Big\{\pi \in \bigcup_{M > M_0} \mathcal{S}_M : \; X_0^\pi = x_0, \\ \text{and } X_t^\pi \geqslant 0, a.s., \forall\, 0 \leqslant t \leqslant T\Big\},$$

*where $M_0$ is a positive constant.*

**Remark 3.1.** *A similar problem is studied in [2] in an incomplete market setting. For exponential utility, they imposed the additional regularity condition:*

$$\{e^{-\eta X_\tau^\pi} : \; \tau \text{ is a stopping time valued in } [0, T]\}$$

*is a uniformly integrable family. This assumption enables them to prove the optimality of the their obtained strategy. In our case, due to the difference between their formulation and our general framework, we consider an admissible set such that $\pi$ satisfies a stronger condition.*

The utility maximization problems for logarithmic and power utility then become

$$\begin{cases} \sup\limits_{w \in \mathcal{A}_i(x_0)} & \mathbb{E}\left[U_i\left(X_T^w\right)\right] \\ \text{subject to} & (X_t^w, w_t) \text{ satisfying (2) for } t \geqslant 0, \end{cases} \tag{4}$$

where $U_1(x) = \log(x)$ for logarithmic utility, and $U_2(x) = \frac{x^\gamma}{\gamma}$, $\gamma < 1$ and $\gamma \neq 0$ for power utility. For exponential utility, the problem is:

$$\begin{cases} \sup\limits_{\pi \in \mathcal{A}_3(x_0)} & \mathbb{E}\left[U\left(X_T^\pi\right)\right] = \mathbb{E}\left[-e^{-\eta X_T^\pi}\right], \\ \text{subject to} & (X_t^\pi, \pi_t) \text{ satisfying (3) for } t \geqslant 0. \end{cases} \tag{5}$$

Now, for $i = 1, 2, 3$, we introduce the following BSDE:

$$\begin{cases} dY_t = h_i\left(\theta_t, Z_t^{(1)}, Z_t^{(2)}\right) dt \\ \qquad + Z_t^{(1)} dW_t^{(1)} + Z_t^{(2)} dW_t^{(2)}, \\ Y_T = 0, \end{cases} \tag{6}$$

where $h_1(p, z_1, z_2) = -\frac{p^2}{2}$ for logarithmic utility,

$$h_2(p, z_1, z_2) = \frac{\gamma p^2 + 2\gamma p z_1 + z_1^2 - (\gamma - 1) z_2^2}{2(\gamma - 1)}$$

for power utility, and

$$h_3(p, z_1, z_2) = \frac{p^2}{2\eta} + p z_1 - \frac{\eta z_2^2}{2}$$

for exponential utility. In equation (6), $\theta_t$ is the market price of risk at time $t$ and is modelled by $\theta_t = \bar{\theta}\sqrt{\alpha_t}$ with $\alpha_t$ satisfying (1).

**Proposition 3.2.** *A solution pair $(Y, Z)$ to BSDE (6) is given by*

$$\begin{cases} Y_t = g_i(t)\alpha_t + c_i(t), \\ Z_t^{(1)} = \rho_1 \sqrt{\alpha_t} g_i(t), \\ Z_t^{(2)} = \rho_2 \sqrt{\alpha_t} g_i(t), \end{cases} \tag{7}$$

*where $g(t)$ and $c(t)$ satisfy*

$$\frac{dc_i(t)}{dt} + \kappa\phi g_i(t) = 0, \quad c_i(T) = 0 \tag{8}$$

*and*

$$\frac{dg_i(t)}{dt} - \kappa g_i(t) = \begin{cases} -\frac{1}{2}\bar{\theta}^2, & i = 1 \\ -\left[\frac{1}{2(1-\gamma)}\rho_1^2 + \frac{1}{2}\rho_2^2\right] g_i^2(t) \\ \qquad + \frac{\bar{\theta}\rho_1\gamma}{\gamma - 1} g_i(t) + \frac{\bar{\theta}^2\gamma}{2(\gamma-1)}, & i = 2 \\ -\frac{\eta\rho_2^2}{2} g_i^2(t) + \bar{\theta}\rho_1 g_i(t) + \frac{\bar{\theta}^2}{2\eta}, & i = 3 \end{cases} \tag{9}$$

*with $g_i(T) = 0$, and $i = 1, 2, 3$ for logarithmic, power, and exponential utility, respectively.*

*Proof.* The proof is a straightforward application of Itô's formula with (7). ☐

The following assumption is sufficient to ensure boundedness of the solutions of the Riccati equations that appear above.

**H3.** If $\gamma \in (0,1)$, we impose $\kappa + \frac{\bar{\theta}\rho_1\gamma}{\gamma-1} > 0$ and $\frac{\kappa\rho_1}{\bar{\theta}} \leq -1$.

**Proposition 3.3.** *A solution to the system of ODEs* (8) *and* (9) *is given by*

$$g(t) = g\left(t; A_i, B_i, C_i\right),$$
$$c(t) = c\left(t; A_i, B_i, C_i, \kappa\phi\right)$$

*where*

$$A_i = \begin{cases} -\frac{\bar{\theta}^2}{2} & i=1, \\ \frac{\bar{\theta}^2\gamma}{2(\gamma-1)} & i=2, \\ \frac{\bar{\theta}^2}{2\eta} & i=3, \end{cases}$$

$$B_i = \begin{cases} \kappa & i=1, \\ \kappa + \frac{\bar{\theta}\rho_1\gamma}{\gamma-1} & i=2, \\ \kappa + \bar{\theta}\rho_1 & i=3, \end{cases}$$

$$C_i = \begin{cases} 0 & i=1, \\ \frac{\rho_1^2}{2(1-\gamma)} + \frac{\rho_2^2}{2} & i=2, \\ \frac{\eta\rho_2^2}{2} & i=3, \end{cases}$$

*and* $g(t;\cdot,\cdot,\cdot)$ *and* $c(t;\cdot,\cdot,\cdot,\cdot)$ *are given in Lemmas B.1 and B.2 respectively. Furthermore,* $g(t)$ *is bounded for* $t \in [0,T]$.

*Proof.* Applying Lemmas B.1 and B.2 yields the solution. The boundedness of the solution $g(t)$ can be proved by applying Lemma C.1. □

**Remark 3.4.** *For the power utility case, the boundedness of the solution can be proved by using assumption* **H3** *and Lemma C.1. The analogous boundedness result is obtained in Lemma 3.4 in [7] by imposing other assumptions.*

### *3.1. Optimal Solutions and Optimal Values*

**Proposition 3.5.** *Let* $(Y,Z)$ *be a solution to* (6).

1. *For logarithmic utility, the optimal solution* $w_t^*$ *to* (4) *and the optimal value* $v(x_0)$ *are given by:*

$$w_t^* = \frac{\theta_t}{\sigma_t}, \quad v(x_0) = \ln(x_0) + Y_0. \tag{10}$$

2. *For power utility, the optimal solution* $w_t^*$ *to* (4) *and the optimal value* $v(x_0)$ *are given by:*

$$w_t^* = \frac{1}{1-\gamma}\left[\frac{\theta_t}{\sigma_t} + \frac{Z_t^{(1)}}{\sigma_t}\right], \quad v(x_0) = \frac{x_0^\gamma}{\gamma}e^{Y_0}. \tag{11}$$

3. *For exponential utility, the optimal solution* $\pi_t^*$ *to* (5) *and the optimal value* $v(x_0)$ *are given by:*

$$\pi_t^* = \frac{1}{\sigma_t}\left[\frac{\theta_t}{\eta} + Z_t^{(1)}\right], \quad v(x_0) = -e^{-\eta(x_0 - Y_0)}. \tag{12}$$

*Proof.* 1. Define $J_t^w := \ln(X_t^w) + Y_t$, so that $J_0^w = J_0 = v(x_0)$ is independent of $w$ and given by (10). For all $w \in \mathcal{A}_1(x_0)$,

$$J_t^w = J_0 + \int_0^t \left(w_s\mu_s - \frac{1}{2}w_s^2\sigma_s^2 - \frac{1}{2}\theta_s^2\right)ds + \int_0^t \left(w_s\sigma_s + Z_s^{(1)}\right)dW_s^{(1)} + \int_0^t Z_s^{(2)}dW_s^{(2)}.$$

Observe that:

$$\begin{aligned} &\mathbb{E}\Big[\int_0^T (\sigma_t w_t + Z_t^{(1)})^2 dt + \int_0^T (Z_t^{(2)})^2 dt\Big] \\ &\leqslant 2\mathbb{E}\Big[\int_0^T \sigma_t^2 w_t^2 dt + \int_0^T \Big((Z_t^{(1)})^2 + \frac{(Z_t^{(2)})^2}{2}\Big)dt\Big] \\ &\leqslant 2\mathbb{E}\Big[\int_0^T \sigma_t^2 w_t^2 dt + \int_0^T c\alpha_t dt\Big] \\ &= 2\mathbb{E}\Big[\int_0^T \sigma_t^2 w_t^2 dt\Big] \\ &\quad + 2c\int_0^T \left[\alpha_0 e^{-\kappa t} + \phi(1-e^{\kappa t})\right]dt \\ &< \infty \end{aligned}$$

where $c = (\rho_1^2 + \frac{\rho_2^2}{2})\sup_{t\in[0,T]} g^2(t)$, the first equality follows from Fubini's Theorem and the last inequality follows from the definition of $\mathcal{A}_1(x_0)$. Therefore, the stochastic integral $\left\{\int_0^t \left(w_s\sigma_s + Z_s^{(1)}\right)dW_s^{(1)} + \int_0^t Z_s^{(2)}dW_s^{(2)}\right\}_{t\in[0,T]}$ is a martingale.
Moreover, for all $w \in \mathcal{A}_1(x_0)$ and $w^*$ as in (10), we have that for each $t \in [0,T]$,

$$0 = w_t^*\mu_t - \frac{1}{2}(w_t^*)^2\sigma_t^2 - \frac{1}{2}\theta_t^2 \geqslant w_t\mu_t - \frac{1}{2}w_t^2\sigma_t^2 - \frac{1}{2}\theta_t^2.$$

Therefore, $\{J_t^w\}_{t\in[0,T]}$ is a supermartingale and $\{J_t^{w^*}\}_{t\in[0,T]}$ is a martingale, which implies that $\mathbb{E}[J_T^\pi] \leqslant J_0 = v(x_0) = \mathbb{E}[J_T^{\pi^*}]$.

2. Define $J_t^w := \frac{(X_t^w)^\gamma}{\gamma}e^{Y_t}$. Then $J_0^w = J_0 = v(x_0)$, where $v(x_0)$ is defined in (11). For $w \in$

$\mathcal{A}_2(x_0)$, we write $J^w_t = A^w_t M^w_t$ with

$$A^w_t = \frac{x_0^\gamma}{\gamma} \exp\Big\{ \int_0^t \Big( \gamma w_s \mu_s - \frac{1}{2}\gamma w_s^2 \sigma_s^2 - f(s, Z_s^{(1)}, Z_s^{(2)}) + \frac{1}{2}(\gamma w_s \sigma_s + Z_s^{(1)})^2 + \frac{1}{2}(Z_s^{(2)})^2 \Big) ds \Big\},$$

$$M^w_t = \exp\Big\{ \int_0^t (\gamma w_s \sigma_s + Z_s^{(1)}) dW_s^{(1)} - \frac{1}{2}\int_0^t (\gamma w_s \sigma_s + Z_s^{(1)})^2 ds \Big\} \times \exp\Big\{ \int_0^t Z_t^{(2)} dW_s^{(2)} - \frac{1}{2}\int_0^t (Z_s^{(2)})^2 ds \Big\},$$

and $f = -h_2$ is the negation of the drift coefficient term of the BSDE of $Y$ defined in (6). It can be easily verified that $\{M^w_t\}_{t\in[0,T]}$ is a local martingale. Thus, there exists a sequence of stopping times satisfying $\lim_{n\to\infty} \tau_n = T$ a.s. such that $\{M^w_{t\wedge\tau_n}\}_{t\in[0,T]}$ is a positive martingale for each $n$. For all $w \in \mathcal{A}_2(x_0)$ and $w^*$ defined in (11), if $\gamma \in (0,1)$, we have for each $t \in [0,T]$:

$$\begin{aligned} P(w_t) &:= \gamma w_t \mu_t - \frac{1}{2}\gamma w_t^2 \sigma_t^2 - f(t, Z_t^{(1)}, Z_t^{(2)}) + \frac{1}{2}(\gamma w_t \sigma_t + Z_t^{(1)})^2 + \frac{1}{2}(Z_t^{(2)})^2 \\ &\leqslant P(w_t^*) = \gamma w_t^* \mu_t - \frac{1}{2}\gamma (w_t^*)^2 \sigma_t^2 - f(t, Z_t^{(1)}, Z_t^{(2)}) + \frac{1}{2}(\gamma w_t^* \sigma_t + Z_t^{(1)})^2 + \frac{1}{2}(Z_t^{(2)})^2 = 0, \end{aligned}$$

while $\gamma < 0$, $P(w_t) \geqslant P(w_t^*) = 0$. Thus, $\{A^w_t\}_{t\in[0,T]}$ is a non-increasing process. So, for $t \geqslant s$,

$$\begin{aligned} &\mathbb{E}[J^w_{t\wedge\tau_n} | \mathcal{F}_s] = \mathbb{E}[A^w_{t\wedge\tau_n} M^w_{t\wedge\tau_n} | \mathcal{F}_s] \\ \leqslant\ &A^w_{s\wedge\tau_n} \mathbb{E}[M^w_{t\wedge\tau_n} | \mathcal{F}_s] = A^w_{s\wedge\tau_n} M^w_{s\wedge\tau_n} = J^w_{s\wedge\tau_n}. \end{aligned}$$

Note that $\{J^w_t\}_{t\in[0,T]}$ is bounded below by 0. Passing to the limit and applying Fatou's Lemma yields that $\{J^w_t\}_{t\in[0,T]}$ is a supermartingale. It remains to show that $\{J^{w^*}_t\}_{t\in[0,T]}$ is a martingale with $w^*$ as defined in (11). Note that $A^{w^*}_t = x_0^\gamma$ and

$$M^{w^*}_t = \exp\Big\{ -\frac{1}{2}\int_0^t \big(m_1^2(s) + m_2^2(s)\big) \alpha_s ds + \int_0^t m_1(s)\sqrt{\alpha_s} dW_s^{(1)} + \int_0^t m_2(s)\sqrt{\alpha_s} dW_s^{(2)} \Big\},$$

where $m_1(t) = \frac{\gamma\bar{\theta}}{(1-\gamma)} + \frac{\rho_1}{(1-\gamma)} g(t)$ and $m_2(t) = \rho_2 g(t)$. By Lemma 2.2, $\{M^{w^*}_t\}_{t\in[0,T]}$ is a positive martingale, and so is $\{J^{w^*}_t\}_{t\in[0,T]}$. Then, $\mathbb{E}[J^w_T] \leqslant J_0 = v(x_0) = \mathbb{E}[J^{w^*}_T]$.

3. Define $J^\pi_t := -e^{-\eta(X^\pi_t - Y_t)}$. Then $J^\pi_0 = J_0 = v(x_0)$, where $v(x_0)$ is defined in (12). For all $\pi \in \mathcal{A}_3(x_0)$, we write $J^\pi_t = A^\pi_t M^\pi_t$, where

$$A^\pi_t = -\exp\Big\{ -\eta \int_0^t \Big( \pi_s \mu_s + f(s, Z_s^{(1)}, Z_s^{(2)}) - \frac{\eta}{2}(\sigma_s \pi_s - Z_s^{(1)})^2 - \frac{\eta}{2}(Z_s^{(2)})^2 \Big) ds \Big\},$$

$$M^\pi_t = \exp\Big\{ -\eta \int_0^t (\sigma_s \pi_s - Z_s^{(1)}) dW_s^{(1)} - \frac{\eta^2}{2}\int_0^t (\sigma_s \pi_s - Z_s^{(1)})^2 ds \Big\} \times \exp\Big\{ \int_0^t \eta Z_t^{(2)} dW_s^{(2)} - \frac{\eta^2}{2}\int_0^t (Z_s^{(2)})^2 ds \Big\},$$

and $f = -h_3$ is the negation of the drift coefficient term of the BSDE of $Y$ defined in (6). It is easy to see that $\{M^\pi_t\}_{t\in[0,T]}$ is a local martingale. Thus, there exists a sequence of stopping times satisfying $\lim_{n\to\infty} \tau_n = T$ a.s. such that $\{M^\pi_{t\wedge\tau_n}\}_{t\in[0,T]}$ is a positive martingale for each $n$. Moreover, for all $\pi \in \mathcal{A}_3(x_0)$ and $\pi^*$ as in (12), we have for each $t \in [0,T]$,

$$\begin{aligned} 0 &= \pi_t^* \mu_t + f(t, Z_t^{(1)}, Z_t^{(2)}) - \frac{\eta}{2}(\sigma_t \pi_t^* - Z_t^{(1)})^2 - \frac{\eta}{2}(Z_t^{(2)})^2 \\ &\geqslant \pi_t \mu_t + f(t, Z_t^{(1)}, Z_t^{(2)}) - \frac{\eta}{2}(\sigma_t \pi_t - Z_t^{(1)})^2 - \frac{\eta}{2}(Z_t^{(2)})^2. \end{aligned}$$

Therefore, $\{A^\pi_t\}_{t\in[0,T]}$ is a non-increasing process. Hence, for $t \geqslant s$,

$$\begin{aligned} &\mathbb{E}[J^\pi_{t\wedge\tau_n} | \mathcal{F}_s] = \mathbb{E}[A^\pi_{t\wedge\tau_n} M^\pi_{t\wedge\tau_n} | \mathcal{F}_s] \\ \leqslant\ &A^\pi_{s\wedge\tau_n} \mathbb{E}[M^\pi_{t\wedge\tau_n} | \mathcal{F}_s] = A^\pi_{s\wedge\tau_n} M^\pi_{s\wedge\tau_n} = J^\pi_{s\wedge\tau_n}. \end{aligned}$$

That is, for any $A \in \mathcal{F}_s$, we have $\mathbb{E}[J^\pi_{t\wedge\tau_n} \mathbf{1}_A] \leqslant \mathbb{E}[J^\pi_{s\wedge\tau_n} \mathbf{1}_A]$. Further, for two constants $c$ and $M$, $|J^\pi_{t\wedge\tau_n}| \leqslant |J^\pi_{\tau_n}| \leqslant c e^{-\eta X^\pi_{\tau_n} + M\alpha_{\tau_n}}$ holds for any $\pi \in \mathcal{S}_M$. Thus, the uniform integrability of $J^\pi_{t\wedge\tau_n}$ follows from that of $e^{-\eta X^\pi_{\tau_n} + M\alpha_{\tau_n}}$. Thus, passing to the limit yields that $\{J^\pi_t\}_{t\in[0,T]}$ is a supermartingale.

It remains to show that $\{J^{\pi^*}_t\}_{t\in[0,T]}$ is a martingale with $\pi^*$ as in (12). Note that $A^{\pi^*}_t = -1$

and

$$M_t^{\pi^*} = \exp\Big\{ -\frac{1}{2}\int_0^t \left(m_1^2(s)+m_2^2(s)\right)\alpha_s ds + \int_0^t m_1(s)\sqrt{\alpha_s}dW_s^{(1)} + \int_0^t m_2(s)\sqrt{\alpha_s}dW_s^{(2)} \Big\},$$

where $m_1(t) = -\bar{\theta}$ and $m_2(t) = \eta\rho_2 g(t)$. By Lemma 2.2, $\{M_t^{\pi^*}\}_{t\in[0,T]}$ is a positive martingale, and so is $\{J_t^{\pi^*}\}_{t\in[0,T]}$. Then, $\mathbb{E}[J_T^{\pi}] \leqslant J_0 = v(x_0) = \mathbb{E}[J_T^{\pi^*}]$.

□


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## References


[1] J.-P. Fouque, R. Sircar, and T. Zariphopoulou. Portfolio optimization and stochastic volatility asymptotics. *Mathematical Finance*, 27(3):704–745, 2017.

[2] Y. Hu, P. Imkeller, and M. Müller. Utility maximization in incomplete markets. *The Annals of Applied Probability*, 15(3):1691–1712, 2005.

[3] M. Jeanblanc, M. Chesney, and M. Yor. *Mathematical models for financial markets*. Springer, 2008.

[4] H. Pham. *Continuous-time stochastic control and optimization with financial applications*. Springer, New York, 2009.

[5] W. Reid. *Riccati Differential Equations*. Academic Press, London, 1972.

[6] A. Richter. Explicit solutions to quadratic BSDEs and applications to utility maximization in multivariate affine stochastic volatility models. *Stochastic Processes and their Applications*, 124(11):3578–3611, 2014.

[7] Y. Shen and Y. Zeng. Optimal investment–reinsurance strategy for mean–variance insurers with square-root factor process. *Insurance: Mathematics and Economics*, 62: 118–137, 2015.

[8] T. Siu. Functional Itô's calculus and dynamic convex risk measures for derivative securities. *Communications on Stochastic Analysis*, 6(2), 2012.

[9] X. Zeng and M. Taksar. A stochastic volatility model and optimal portfolio selection. *Quantitative Finance*, 13(10): 1547–1558, 2013.


## A. Proof of Lemma 2.2

The proof is adapted from Lemma A1 in [7] and Lemma 4.3 in [9].

Firstly, from the boundedness of both $m_1$ and $m_2$, we can find an $M$ such that $0 < M < \infty$ and $\frac{1}{2}\left(m_1^2(t)+m_2^2(t)\right) \leqslant M$ for all $t \in [0,T]$. Then for any $T_0 \in [0,T]$, we define $\kappa_- = \max\{0, -\kappa\}$,

$$f(t) := \exp\left\{ -\left[2M + 2\kappa_- + \frac{1}{2}(\rho_1^2+\rho_2^2)\right](t-T_0)\right\},$$

and

$$\begin{aligned}
F(t) &:= \frac{1}{2}\left(m_1^2(t)+m_2^2(t)\right) + f'(t) - \kappa f(t) + \frac{1}{2}(\rho_1^2+\rho_2^2)f^2(t) \\
&\leqslant M - f(t)\left[2M + 2\kappa_- + \frac{1}{2}(\rho_1^2+\rho_2^2)\right] - \kappa f(t) + \frac{1}{2}(\rho_1^2+\rho_2^2)f^2(t) \\
&= M[1-2f(t)] - |\kappa|\cdot f(t) - \frac{1}{2}(\rho_1^2+\rho_2^2)f(t)[1-f(t)] =: H(t).
\end{aligned}$$

It is obvious that $H(t) < 0$ for $t \in [T_0, T_0+h]$ where $h = \frac{\ln 2}{2M+2\kappa_-+\frac{1}{2}(\rho_1^2+\rho_2^2)} > 0$. Therefore, $F(t) < 0$ for $t \in [T_0, T_0+h]$ as well.

Now, for $t \in [T_0, T_0+h]$, we denote

$$G(t) := \exp\left[\int_{T_0}^t \frac{1}{2}\left(m_1^2(s)+m_2^2(s)\right)\alpha_s ds + f(t)\alpha_t\right] \geqslant 0.$$

Applying Itô's formula to $G(t)$ gives

$$dG(t) = G(t)\Big[\left(\kappa\phi f(t) + F(t)\alpha_t\right)dt + \rho_1\sqrt{\alpha_t}f(t)dW_t^{(1)} + \rho_2\sqrt{\alpha_t}f(t)dW_t^{(2)}\Big].$$

Taking expectations on both sides yields

$$\begin{aligned}
&\mathbb{E}\left[G(t)|\mathcal{F}_{T_0}\right] \\
&= \mathbb{E}\Big[e^{\alpha_{T_0}}\exp\Big\{\kappa\phi\int_{T_0}^t f(s)ds + \int_{T_0}^t F(s)\alpha_s ds\Big\} \\
&\quad \times \exp\Big\{ -\frac{1}{2}(\rho_1^2+\rho_2^2)\int_{T_0}^t f^2(s)\alpha_s ds + \rho_1\int_{T_0}^t \sqrt{\alpha_s}f(s)dW_s^{(1)} + \rho_2\int_{T_0}^t \sqrt{\alpha_s}f(s)dW_s^{(2)}\Big\}\Big|\mathcal{F}_{T_0}\Big] \\
&\leqslant e^{\alpha_{T_0}}\exp\Big\{\kappa\phi\int_{T_0}^t f(s)ds\Big\} \leqslant e^{\kappa\phi(t-T_0)+\alpha_{T_0}} \\
&< \infty, \quad \text{a.s.},
\end{aligned}$$

where the last equality follows from the fact that $F<0$ on $[T_0, T_0+h]$ and the supermartingale property of stochastic exponentials. Therefore,

$$\mathbb{E}\Big[e^{\int_{T_0}^t \frac{1}{2}\left(m_1^2(s)+m_2^2(s)\right)\alpha_s ds}|\mathcal{F}_{T_0}\Big] \leqslant \mathbb{E}[G(t)|\mathcal{F}_{T_0}] \leqslant e^{\kappa\phi(t-T_0)+\alpha_{T_0}} < \infty, \text{ a.s.}$$

This means that, for $t \in [T_0, T_0+h]$, the stochastic exponential process defined by

$$\exp\Big\{-\frac{1}{2}\int_{T_0}^t \Big(m_1^2(s)+m_2^2(s)\Big)\alpha_s ds + \int_0^t m_1(s)\sqrt{\alpha_s}dW_s^{(1)} + \int_{T_0}^t m_2(s)\sqrt{\alpha_s}dW_s^{(2)}\Big\}$$

is a martingale.

Lastly, for any $t \in [0,T]$, we find a partition of the interval $[0,t]$, i.e. $0=t_0<t_1<\cdots<t_n=t$ such that $n=\lceil \frac{t}{h}\rceil$ and $t_{k+1}-t_k=\frac{t}{n}\leqslant h$ for $k=0,1,\cdots,n-1$, where $\lceil x\rceil$ is the smallest integer greater than or equal to $x$. Then

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big\{-\frac{1}{2}\int_0^t \Big(m_1^2(s)+m_2^2(s)\Big)\alpha_s ds + \int_0^t m_1(s)\sqrt{\alpha_s}dW_s^{(1)} + \int_0^t m_2(s)\sqrt{\alpha_s}dW_s^{(2)}\Big\}\Big]\\
=&\mathbb{E}\Big[\prod_{k=0}^{n-1}\exp\Big\{-\frac{1}{2}\int_{t_k}^{t_{k+1}}\Big(m_1^2(s)+m_2^2(s)\Big)\alpha_s ds + \int_{t_k}^{t_{k+1}} m_1(s)\sqrt{\alpha_s}dW_s^{(1)} + \int_{t_k}^{t_{k+1}} m_2(s)\sqrt{\alpha_s}dW_s^{(2)}\Big\}\Big]\\
=&\mathbb{E}\Big\{\mathbb{E}\Big[\prod_{k=0}^{n-1}\exp\Big\{-\frac{1}{2}\int_{t_k}^{t_{k+1}}\Big(m_1^2(s)+m_2^2(s)\Big)\alpha_s ds + \int_{t_k}^{t_{k+1}} m_1(s)\sqrt{\alpha_s}dW_s^{(1)} + \int_{t_k}^{t_{k+1}} m_2(s)\sqrt{\alpha_s}dW_s^{(2)}\Big\}\Big|\mathcal{F}_{t_{n-1}}\Big]\Big\}\\
=&\mathbb{E}\Big[\prod_{k=0}^{n-2}\exp\Big\{-\frac{1}{2}\int_{t_k}^{t_{k+1}}\Big(m_1^2(s)+m_2^2(s)\Big)\alpha_s ds + \int_{t_k}^{t_{k+1}} m_1(s)\sqrt{\alpha_s}dW_s^{(1)} + \int_{t_k}^{t_{k+1}} m_2(s)\sqrt{\alpha_s}dW_s^{(2)}\Big\}\times 1\Big]\\
=&\cdots = 1.
\end{aligned}$$

## B. Solution to ODEs involving Riccati Equation

The proof of the next lemma can be seen in [5].

**Lemma B.1.** *Consider the following Riccati equation:*

$$\frac{dg(t)}{dt} + a_2 g^2(t) - a_1 g(t) = a_0, \quad g(T)=0 \tag{B.1}$$

*where $a_0$, $a_1$ and $a_2$ are three constants. The solution has the form:*

$$g(t) = \frac{R_2(\tau)}{R_1(\tau)}$$

*where $\tau := T-t$ and the vector $(R_1(\tau), R_2(\tau))^\top$ follows the ODE:*

$$d\begin{pmatrix}R_1(\tau)\\ R_2(\tau)\end{pmatrix} = \begin{pmatrix}0 & -a_2\\ -a_0 & -a_1\end{pmatrix}\begin{pmatrix}R_1(\tau)\\ R_2(\tau)\end{pmatrix} d\tau$$

*where $R_1(\tau)|_{t=T}=1$ and $R_2(\tau)|_{t=T}=0$. More precisely, let $\Delta = a_1^2+4a_0a_2$ and $\delta=\frac{1}{2}\sqrt{|\Delta|}$. An explicit solution $g(t) =: g(t;a_0,a_1,a_2)$ is given as follows:*

$$g(t;a_0,a_1,a_2) := g(t) = \begin{cases} \dfrac{\frac{-a_0}{\delta}\sin(\delta\tau)}{\cos(\delta\tau)+\frac{a_1}{2\delta}\sin(\delta\tau)} & \text{if } \Delta<0,\\ \dfrac{-a_0\tau}{1+\frac{a_1}{2}\tau} & \text{if } \Delta=0,\\ \dfrac{\frac{-a_0}{\delta}\sinh(\delta\tau)}{\cosh(\delta\tau)+\frac{a_1}{2\delta}\sinh(\delta\tau)} & \text{if } \Delta>0.\end{cases} \tag{B.2}$$

The proof of the following result is elementary and thus omitted.

**Lemma B.2.** *Suppose $g(t)$ follows the Riccati equation in* (B.1) *and $c(t)$ satisfies:*

$$\frac{dc(t)}{dt} + a_3 g(t) = 0, \quad c(T)=0 \tag{B.3}$$

*where $a_3$ is a constant. Let $\Delta = a_1^2+4a_0a_2$ and $\delta=\frac{1}{2}\sqrt{|\Delta|}$. A solution $c(t) =: c(t;a_0,a_1,a_2,a_3)$ to* (B.3) *is given as follows:*

1. *If $a_2 \neq 0$,*

$$c(t;a_0,a_1,a_2,a_3) := c(t) = \begin{cases} \frac{-a_3}{a_2}\left[-\frac{a_1\tau}{2}+\ln\left|\cos(\delta\tau)+\frac{a_1}{2\delta}\sin(\delta\tau)\right|\right], & \text{if } \Delta<0,\\ \frac{-a_3}{a_2}\left[-\frac{a_1\tau}{2}+\ln\left|1+\frac{a_1}{2}\tau\right|\right], & \text{if } \Delta=0,\\ \frac{-a_3}{a_2}\left[-\frac{a_1\tau}{2}+\ln\left|\cosh(\delta\tau)+\frac{a_1}{2\delta}\sinh(\delta\tau)\right|\right], & \text{if } \Delta>0.\end{cases}$$

2. *If $a_2=0$ and $a_1\neq 0$,*

$$c(t;a_0,a_1,a_2,a_3) := c(t) = \frac{a_0a_3}{a_1}\Big[e^{-\frac{a_1\tau}{2}}\frac{\sinh(\delta\tau)}{\delta} - \tau\Big].$$

3. *If $a_2=0$ and $a_1=0$,*

$$c(t;a_0,a_1,a_2,a_3) := c(t) = -\frac{a_0a_3}{2}\tau^2.$$

## C. Boundedness of the Solution to the Riccati Ordinary Differential Equation

**Lemma C.1.** *For the Riccati equation* (B.1)*, the following results hold:*

1. *If* $a_2 = 0$*, then the solution* $g(t; a_0, a_1, a_2)$ *given in* (B.2) *is bounded on* $t \in [0, T]$*.*
2. *If* $a_0 \geqslant 0$ *and* $a_2 > 0$*, then the solution* $g(t; a_0, a_1, a_2)$ *given in* (B.2) *is bounded on* $t \in [0, T]$*.*

*Proof.*
1. In this case, we simply substitute $a_2 = 0$ and then the solution reduces to

$$g(t; a_0, a_1, 0) = \begin{cases} \frac{a_0}{a_1}(e^{-a_1 \tau} - 1), & \text{if } a_1 \neq 0, \\ -a_0 \tau, & \text{if } a_1 = 0, \end{cases}$$

where $\tau = T - t$. Obviously, $g(t; a_0, a_1, a_2)$ is bounded on $[0, T]$.

2. In this case, we can also verify that the solution adopts the following form:

$$\begin{aligned} & g(t; a_0, a_1, a_2) \\ = & \begin{cases} \frac{a_0}{a_1}(e^{-a_1 \tau} - 1), & \text{if } a_1 \neq 0, \;\; a_0 = 0 \\ -a_0 \tau, & \text{if } a_1 = 0, \;\; a_0 = 0, \\ \dfrac{\frac{-a_0}{\delta} \sinh(\delta \tau)}{\cosh(\delta \tau) + \frac{a_1}{2\delta} \sinh(\delta \tau)} & \text{if } a_0 \neq 0, \end{cases} \end{aligned}$$

where $\tau = T - t$ and $\delta = \frac{1}{2}\sqrt{a_1^2 + 4 a_0 a_2} > 0$. It is obvious that for the first two cases $g(t; a_0, a_1, a_2)$ is bounded on $[0, T]$. For the third case, it can be verified that

$$0 \leq |g(t; a_0, a_1, a_2)| \leq \frac{a_0}{2\delta}(e^{2\delta \tau} - 1).$$

Therefore, $g(t; a_0, a_1, a_2)$ is bounded on $[0, T]$.

□