Lower Tail Independence of Hitting Times of Two-Dimensional Diffusions

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Abstract

The coefficient of tail dependence is a quantity that measures how extreme events in one component of a bivariate distribution depend on extreme events in the other component. It is well-known that the Gaussian copula has zero tail dependence, a shortcoming for its application in credit risk modeling and quantitative risk management in general. We show that this property is shared by the joint distributions of hitting times of bivariate (uniformly elliptic) diffusion processes.

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1 Introduction

Let Y and Z be continuous random variables, with distribution functions F_Y and F_Z respectively. The coefficients of lower tail dependence, λ_L and upper tail dependence λ_U, of Y and Z are defined to be:

$$\lambda_L = \lim_{\alpha \downarrow 0} \mathbb{P}(Y \leqslant F_Y^-(\alpha) | Z \leqslant F_Z^-(\alpha)),$$

and

$$\lambda_U = \lim_{\alpha \uparrow 1} \mathbb{P}(Y \geqslant F_Y^-(\alpha) | Z \geqslant F_Z^-(\alpha))$$

where F^- denotes the generalized inverse of F: $F^-(y) = \inf\{x \in \mathbb{R} : F(x) \geqslant y\}$.

The coefficients of tail dependence can also be expressed in terms of the copula C of Y and Z as (McNeil et al. [2005], page 209):

$$\lambda_L = \lim_{\alpha \downarrow 0} \frac{C(\alpha, \alpha)}{\alpha}, \quad \lambda_U = \lim_{\alpha \uparrow 1} \frac{1 - 2\alpha + C(\alpha, \alpha)}{\alpha}$$

It is well-known that when Y and Z have a bivariate Gaussian distribution with correlation ρ, $|\rho| < 1$, or more generally, when Y and Z have a Gaussian copula with this correlation, $\lambda_L = \lambda_U = 0$ (McNeil et al. [2005], page 211).

In this paper, we show that the property of zero lower tail dependence of the Gaussian copula is shared by the hitting times of the components of two-dimensional diffusion processes. In particular, we show that if X is a two-dimensional diffusion process with generator L, acting on smooth functions f as:

$$Lf = \sum_i b_i(x) \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j} a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}$$

where L is uniformly elliptic with smooth and bounded coefficients, and if the hitting times τ_i are defined as $\tau_i = \inf\{t > 0 | X^i(t) = c_i\}$, then the coefficient of lower tail dependence between τ_1 and τ_2 is $\lambda_L = 0$. For upper tail independence, the fact that $\lambda_U = 0$ was proved for the case when b and a are constant in Metzler [2008].

The main technical result employed in our proof of the lower tail independence ($\lambda_L = 0$) for the hitting times of a non-degenerate two-dimensional diffusion is the large deviations principle for small time for such processes due to Varadhan [1967]. More refined results are available, see, for example, Azencott [1985]. There is a large literature on the application of large deviations to the problem of the exit of a diffusion from a domain, see, for example, Freidlin and Wentzell [2012], Fleming and James [1992], and the references therein. Exact asymptotics of hitting times of Gaussian processes are presented in Dębicki et al. [2010].

We have not aimed for maximum generality with respect to the coefficients in presenting our results. Large deviation principles generalizing the result in Varadhan [1967] are known (see, e.g. Baldi and Chaleyat-Maurel [1988], or the aforementioned work of Azencott [1985]), and under further conditions, these results can be extended (with the same rate function) to inhomogeneous diffusion processes (e.g. Herrmann et al. [2006]). Smoothness of the coefficients is also used to obtain smoothness of minimizing geodesics based on classical results in the calculus of variations (used in Theorem 4), and also leads to increased regularity of the distributions of hitting times (see Remark 1). It is straightforward to extend the results of this paper to show that the hitting times of any two components of a uniformly elliptic n-dimensional diffusion process with smooth bounded coefficients have zero tail dependence.

There is a large literature on the extremes of multivariate distributions. Resnick [2007] provides a motivated introduction to multivariate extreme value analysis, with a number of applications. Balkema and Embrechts [2007] provide mathematical and statistical techniques for finding the asymptotic behaviour of multivariate extremes. Multivariate notions of tail dependence have been studied by some authors (e.g. Li [2009]). Furman et al. [2016] study generalized notions of tail dependence. The tail order of Ledford and Tawn [1996] (see also Hua and Joe [2011]) provides a more refined measure of dependence in the tail. Using the large deviation results employed in this paper (or perhaps the extensions of Azencott [1985]), it may be possible to evaluate these tail dependence measures for diffusion hitting times. Consideration of extensions of the results in this paper to such notions is left for future research.

It is possible to construct models with positive lower tail dependence, using auxiliary variables (e.g. latent variables or subordinators) that affect both components of the diffusion simultaneously. As a simple example, let W be a standard two-dimensional Brownian motion, and N_t a standard Poisson process. Then a straightforward calculation shows that the hitting times of $X_t = W(N_t)$ exhibit positive tail dependence. The situation is somewhat analogous to the positive coefficient of tail dependence for multivariate Student-t random vectors, which are variance mixtures of multivariate Gaussians.

The analysis of the asymptotic behaviour of joint hitting times is of interest in a number of applied fields. Bivariate hitting time densities arise in applications in neuroscience, where they may be used to model the joint distributions of firing times of action potentials of coupled neurons (Iyengar [1985]). Sacerdote et al. [2014] provide a numerical method for the computation of the joint density of τ_1 and τ_2 based on the solution of a system of integral equations, and prove convergence of their method. Hitting times of multidimensional diffusion processes often arise in credit risk modelling in mathematical finance. See, for example, Bielecki and Rutkowski [2002]. In this context, our results may be interpreted as showing that the tail independence property of the Gaussian copula, which has received so much criticism in the context of the application of the model of Li [2001] to the pricing of CDOs, is shared by multivariate versions of the credit risk model of Black and Cox [1976].

The remainder of this paper is structured as follows. The second section sets notation and reviews some results from large deviations theory needed in subsequent sections. The third section considers the case of a Brownian motion, i.e. when $b \equiv 0$ and a is a constant matrix, and proves lower tail independence of the hitting times using Schilder's Theorem on large deviations of the Brownian sample paths. This special case is included here as its proof is particularly simple, and contains all the main ideas behind the proof of the general case. The fourth section shows lower tail independence of τ_1 and τ_2 in the general case, using results on the small-time behaviour of diffusion processes due to Varadhan [1967].

2 Notation and Background

Denote by $C([0,1],\mathbb{R}^2)$ the set of all continuous functions ω from $[0,1]$ to $\mathbb{R}^2$, with $\|\omega\| = \sup_{t\in[0,1]} |\omega(t)|$, and $C_x([0,1],\mathbb{R}^2)$ the subset with $\omega(0) = x$. Let $b : \mathbb{R}^2 \to \mathbb{R}$ and $\sigma : \mathbb{R}^{2\times 2} :\to \mathbb{R}$ be bounded, C^2 functions with bounded derivatives, and suppose that there exist $\kappa, K > 0$ such that $\kappa\|\xi\|^2 \leqslant \xi' a(x)\xi \leqslant K\|\xi\|^2$ for all $x, \xi \in \mathbb{R}^2$, where $a = \sigma\sigma'$. Let L_ε be the operator:

$$L_\varepsilon f = \varepsilon \sum_i b_i(x)\frac{\partial f}{\partial x_i} + \frac{\varepsilon}{2}\sum_{i,j} a_{ij}(x)\frac{\partial f}{\partial x_i \partial x_j} \tag{1}$$

acting on smooth functions $f : \mathbb{R}^2 \to \mathbb{R}$, and let $L = L_1$. Let X^ε (with $X = X^1$) be the Markov process associated with L_ε, and let $\mathbb{P}^\varepsilon_x$ be the measure on $C([0,1],\mathbb{R}^2)$ giving the distribution of the process X^ε started at the point x at time 0. This may be realized as the distribution of the unique strong solution to the stochastic differential equation:

$$dX^\varepsilon(t) = \varepsilon b(X^\varepsilon(t))dt + \sqrt{\varepsilon}\cdot\sigma(X^\varepsilon(t))\,dW_t, \quad X^\varepsilon(0) = x_0 \tag{2}$$

where W_t is a two-dimensional standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,1]}, \mathbb{P})$ satisfying the usual conditions. We assume, without loss of generality, that $x_0 = 0$. For $i = 1, 2$, let $c_i > 0$, $\tau_i = \inf\{t > 0 | X_i(t) \geqslant c_i\}$, and let F_i be the distribution of τ_i, and note the time-scaling property:

$$\begin{aligned}\mathbb{P}(\tau_i \leqslant \varepsilon) &= \mathbb{P}\left(\sup_{t\in[0,\varepsilon]} X^i(t) \geqslant c_i\right)\\ &= \mathbb{P}\left(\sup_{t\in[0,1]} X^i(\varepsilon t) \geqslant c_i\right) = \mathbb{P}\left(\sup_{t\in[0,1]} X^{\varepsilon,i}(t) \geqslant c_i\right)\end{aligned} \tag{3}$$

Let H^1 be the subset of $C([0,1],\mathbb{R}^2)$ consisting of absolutely continuous functions ω with square-integrable derivative, and H^1_x the subset of H^1 with $\omega(0) = x$. Similarly, for any $A \subseteq C([0,1],\mathbb{R}^2)$, let $A_x = A \cap C_x([0,1],\mathbb{R}^2)$. Let $g = a^{-1}$, and define $I : C([0,1],\mathbb{R}^2) \to [0,\infty]$ by:

$$I(\omega) = \begin{cases} \frac{1}{2}\int_0^1 \dot{\omega}(t)' g(\omega(t))\dot{\omega}(t)\,dt & \omega \in H^1 \\ \infty & \omega \notin H^1 \end{cases} \tag{4}$$

and recall that I is lower-semicontinuous, with compact lower level sets (see, e.g. Friedman [2006], pages 326-332).

Recall the distance $d(x, y)$ on $\mathbb{R}^2$, defined through the length function $l : H^1 \to \mathbb{R}_+$:

$$l(\omega) = \int_0^1 \sqrt{\dot{\omega}(t)' g(\omega(t)) \dot{\omega}(t)}\, dt \tag{5}$$

by:

$$d(x, y) = \inf\{l(\omega) | \omega(0) = x, \omega(1) = y\} \tag{6}$$

If $\omega \in H^1$ and $\omega_{\alpha,\beta}(s) : [\alpha, \beta] \to \mathbb{R}^2$ is defined by $\omega_{\alpha,\beta}(s) = \omega((\beta-\alpha)^{-1}(s-\alpha))$ then

$$\frac{1}{2}\int_\alpha^\beta \dot{\omega}_{\alpha,\beta}(s)' g(\omega_{\alpha,\beta}(s)) \omega_{\alpha,\beta}(s)\, ds = (\beta - \alpha)^{-1} I(\omega) \tag{7}$$

This leads to the following two facts stated in Varadhan [1967]:

Lemma 1. *Let* $0 \leqslant \alpha \leqslant \beta$*,*

$$\inf\{I(\omega) | \omega(\alpha) = x, \omega(\beta) = y\} = \frac{d^2(x, y)}{2(\beta - \alpha)} \tag{8}$$

Lemma 2. *For* $0 \leqslant t_1 < t_2 < t_3 \cdots < t_n \leqslant 1$*,*

$$\inf\{I(\omega) | \omega(t_j) = x_j, j = 1, \ldots, n\} = \frac{1}{2} \sum_{j=1}^{n-1} \frac{d^2(x_{j+1}, x_j)}{t_{j+1} - t_j} \tag{9}$$

The following large deviation principle is the main result required to derive the tail independence of the hitting times of X, and is due to Varadhan [1967], which generalized the result for the special case of Brownian motion, often referred to as Schilder's Theorem:

Theorem 1. *For $G \subseteq C([0,1],\mathbb{R}^2)$ open and $F \subseteq C([0,1],\mathbb{R}^2)$ closed:*

$$\liminf_{\varepsilon\downarrow 0, y\to x} \varepsilon \log \mathbb{P}_y^{\varepsilon}(G) \geqslant -\inf_{\omega\in G_x} I(\omega) \tag{10}$$

$$\limsup_{\varepsilon\downarrow 0, y\to x} \varepsilon \log \mathbb{P}_y^{\varepsilon}(F) \leqslant -\inf_{\omega\in F_x} I(\omega) \tag{11}$$

For $k_1, k_2 \in [0,\infty)$, $t_1, t_2 \in [0,1]$, define:

$$B(k_1,t_1,k_2,t_2) = \{\omega \in C_0([0,1],\mathbb{R}^2) | \sup_{t\in[0,t_1]} \omega_1(t) \geqslant k_1, \sup_{t\in[0,t_2]} \omega_2(t) \geqslant k_2\} \tag{12}$$

That is, $B(k_1,t_1,k_2,t_2)$ is the set of paths that start at 0, cross k_1 by time t_1 and cross k_2 by t_2. For $k_1, k_2 \in [0,\infty)$ define:

$$B_{k_1,k_2} = \{\omega \in C_0([0,1],\mathbb{R}^2) | \sup_{t\in[0,1]} \omega_1(t) \geqslant k_1, \sup_{t\in[0,1]} \omega_2(t) \geqslant k_2\} \tag{13}$$

so that $B_{k_1,k_2} = B(k_1,1,k_2,1)$.

Define the constants:

$$J_1 = \inf\{I(\omega), \omega \in B_{c_1,0}\} \tag{14}$$

$$J_2 = \inf\{I(\omega), \omega \in B_{0,c_2}\} \tag{15}$$

$$J_{1,2} = \inf\{I(\omega), \omega \in B_{c_1,c_2}\} \tag{16}$$

Since $B_{c_1,c_2} \subseteq B_{c_1,0}$, and $B_{c_1,c_2} \subseteq B_{0,c_2}$, we have $J_{1,2} \geqslant J_1$ and $J_{1,2} \geqslant J_2$.

Also, notice that the time-scaling property (7) implies that with $t < 1$,

$$\inf \{I(\omega) | \omega \in B(c_1, t, c_2, t)\} = t^{-1} J_{1,2} \tag{17}$$

3 Brownian Motion Case

In this section, we will consider tail independence in the special case where $b \equiv 0$ and a is a constant matrix with $a_{ii} = \sigma_i^2$, and $a_{12} = a_{21} = \sigma_1 \sigma_2 \rho$ for $\sigma_i > 0$, $i = 1, 2$, and $\rho \in (-1, 1)$. In this case X_t may be taken as the solution of the SDE (2) with $b \equiv 0$ and

$$\sigma = \begin{pmatrix} \sigma_1 \sqrt{1 - \rho^2} & \rho \sigma_1 \\ 0 & \sigma_2 \end{pmatrix}$$

with $\sigma_i > 0, i = 1, 2$ and $\rho \in (-1, 1)$. Thus X_t^1 is σ_1 multiplied by a Brownian motion $Z_t^1 = \sqrt{1 - \rho^2} W_t^1 + \rho W_t^2$ and X_t^2 is σ_2 multiplied by a Brownian motion $Z_t^2 = W_t^2$, and Z_t^1 and Z_t^2 have correlation ρ. Denote $\tilde{c}_i = \frac{c_i}{\sigma_i}$ for $i = 1, 2$. Without loss of generality, we may assume that $\tilde{c}_2 \leqslant \tilde{c}_1$, so that $\kappa = (\tilde{c}_2 / \tilde{c}_1)^2 \leqslant 1$.

Here, we summarize a proof that the coefficient of lower tail dependence is equal to zero in the Brownian motion case, using Schilder's Theorem. The extension to the case of variable coefficients based on Theorem 1 from Varadhan [1967] is presented in the next section. We may assume $\rho \neq 0$, since zero tail dependence follows immediately from the independence of τ_1 and τ_2 in the case of zero correlation. The argument proceeds as follows:

1. **Use properties of marginal hitting time distributions to remove inverse CDFs from the ratio in the definition of the coefficient of tail dependence:** Let β_t be a standard Brownian motion and let $\kappa = (\tilde{c}_2/\tilde{c}_1)^2 \leqslant 1$. By Brownian scaling:

$$F_1(t) = \mathbb{P}\left(\sup_{s\leqslant t} \beta_s \geqslant \tilde{c}_1\right) = \mathbb{P}\left(\sup_{s\leqslant \kappa t} \beta_s \geqslant \tilde{c}_2\right) = F_2(\kappa t) \tag{18}$$

which can also be seen from the explicit formula for the hitting time distribution of Brownian motion. Recall the definition of λ_L, and make the substitution $F_1(\varepsilon) = \alpha$

$$\lambda_L = \lim_{\alpha\downarrow 0} \frac{\mathbb{P}(\tau_1 \leqslant F_1^{-1}(\alpha), \tau_2 \leqslant F_2^{-1}(\alpha))}{\mathbb{P}(\tau_1 \leqslant F_1^{-1}(\alpha))} = \lim_{\varepsilon\downarrow 0} \frac{\mathbb{P}(\tau_1 \leqslant \varepsilon, \tau_2 \leqslant \kappa\varepsilon)}{\mathbb{P}(\tau_1 \leqslant \varepsilon)} \tag{19}$$

2. **Apply time scaling and large deviation results to approximate probabilities:** Using Brownian scaling

$$\begin{aligned}\mathbb{P}(\tau_1 \leqslant \varepsilon) = \mathbb{P}\left(\sup_{s\leqslant \varepsilon} \beta_s \geqslant \tilde{c}_1\right) = \mathbb{P}\left(\sup_{s\leqslant 1} \varepsilon^{-1/2}\beta_{\varepsilon s} \geqslant \varepsilon^{-1/2}\tilde{c}_1\right)& \\ = \mathbb{P}(\sup_{s\leqslant 1} \sqrt{\varepsilon}\beta_s \geqslant \tilde{c}_1) = \mathbb{P}_0^{\varepsilon}(B_{\tilde{c}_1,0})& \end{aligned} \tag{20}$$

and similarly:

$$\mathbb{P}(\tau_1 \leqslant \varepsilon, \tau_2 \leqslant \kappa\varepsilon) = \mathbb{P}_0^{\varepsilon}(B(\tilde{c}_1, 1, \tilde{c}_2, \kappa)) \tag{21}$$

Applying Theorem 1 yields:[1]

$$-\lim_{\varepsilon\downarrow 0}\varepsilon\log(\mathbb{P}(\tau_1\leqslant\varepsilon))=\inf_{\omega\in B_{\tilde{c}_1,0}}I(\omega)=\tilde{J}_1 \tag{22}$$

$$-\lim_{\varepsilon\downarrow 0}\varepsilon\log(\mathbb{P}(\tau_1\leqslant\varepsilon,\tau_2\leqslant\kappa\varepsilon)=\inf_{\omega\in B(\tilde{c}_1,1,\tilde{c}_2,\kappa)}I(\omega)=\tilde{J}_{1,2} \tag{23}$$

Heuristically, $\mathbb{P}(\tau_1\leqslant\varepsilon)\approx\exp(-\varepsilon^{-1}\tilde{J}_1)$ and $\mathbb{P}(\tau_1\leqslant\varepsilon,\tau_2\leqslant\kappa\varepsilon)\approx\exp(-\varepsilon^{-1}\tilde{J}_{1,2})$.

3. **Show that $\tilde{J}_{1,2}>\tilde{J}_1$, and conclude that $\lambda_L=0$.**

   Let $\omega\in H_0^1$ be any path such that $\sup_{t\in[0,1]}\omega_1(t)\geqslant\tilde{c}_1$, i.e. $\omega\in B_{\tilde{c}_1,0}$, with $\omega^1(t')=\tilde{c}_1$. Then, by the Cauchy-Schwartz inequality:

$$I(\omega)\geqslant\frac{1}{2(1-\rho^2)}\int_0^{t'}(\rho\cdot\dot{\omega}_1(t)-\dot{\omega}_2(t))^2\,dt+\frac{1}{2}\int_0^{t'}\dot{\omega}_1(t)^2\,dt\geqslant\frac{\tilde{c}_1^2}{2t'}$$

   with equality if and only if $\omega=\omega^*$ with $\omega_1^*(t)=\tilde{c}_1t$, $\omega_2^*(t)\equiv\rho\tilde{c}_1t<\tilde{c}_2$ for $t\leqslant\kappa$. So $\tilde{J}_1=\frac{\tilde{c}_1^2}{2}$. Since I is lower-semi-continuous with compact level sets, and $F=B(\tilde{c}_1,1,\tilde{c}_2,\kappa)$ is closed, I attains its minimum on F at some $\tilde{\omega}$. The above argument shows that $\tilde{J}_{1,2}=I(\tilde{\omega})>\tilde{J}_1$ (since $\omega^*\notin F$, so $\tilde{\omega}\neq\omega^*$). Thus:

$$0>\tilde{J}_1-\tilde{J}_{1,2}=\lim_{\varepsilon\downarrow 0}\varepsilon\log\left(\frac{\mathbb{P}(\tau_1\leqslant\varepsilon,\tau_2\leqslant\kappa\varepsilon)}{\mathbb{P}(\tau_1\leqslant\varepsilon)}\right) \tag{24}$$

[1] Here we use the fact that $B_{\tilde{c}_1,0}$ and $B(\tilde{c}_1,1,\tilde{c}_2,\kappa)$ are both *continuity sets*, meaning that the upper and lower bounds given by Theorem 1 are equal. This is shown below in Proposition 3.

implying that $\lambda_L = 0$.

4 General Case

In this section, we present a proof of the tail independence of the hitting times of diffusions in the general two-dimensional case. Throughout, we assume without loss of generality that $J_1 \geqslant J_2$ where J_1 and J_2 are defined in (14) and (15) respectively.

4.1 Properties of the Hitting Time Distributions

In this section, we derive properties of the distributions of the hitting times τ_1 and τ_2 that are used later in the paper. While many of the results are likely well-known, we include proofs when we are unaware of a precise reference.

First of all, we have that $F_i(0) = 0$ since X has continuous sample paths and $c_i > x_0^i = 0$. Also, as a distribution function, F_i is right-continuous with left-hand limits, and is increasing (not necessarily strictly increasing). We would like to show that F_i are continuous and strictly increasing, as this justifies the use of expressions such as $F_1^{-1}(\alpha)$ and $F_2^{-1}(F_1(\varepsilon))$ (as well as the continuity of these inverses), below. The fact that the distribution F_i is strictly increasing can be derived as a consequence of the following result from Bass [1998], for $d = 2$, and referred to there as the *support theorem* for X_t.

Theorem 2. *Suppose σ and b are bounded, σ^{-1} is bounded, $x \in \mathbb{R}^d$, and X_t satisfies (2) with $X_0 = x$, $\varepsilon = 1$. Suppose $\psi : [0,t] \to \mathbb{R}^d$ is continuous, with $\psi(0) = x$ and $\delta > 0$. There exists $k > 0$, depending only on δ, t, the modulus of continuity of ψ, and the bounds on b and σ such that*

$$\mathbb{P}\left(\sup_{s \leqslant t} |X_s - \psi(s)| < \delta\right) \geqslant k \tag{25}$$

Proof. See Bass [1998], pages 25-27. □

Quoting Bass [1998], page 26 (substituting our notation) "This can be phrased as saying the graph of X_s stays inside an δ-tube about ψ. By this we mean, if $G_\psi^\delta = \{(s,y) : |y - \psi(s)| < \delta, s \leqslant t\}$, then $\{(s, X_s) : s \leqslant t\}$ is contained in G_ψ^δ with positive probability."

Proposition 1. *F_i is strictly increasing, $i = 1, 2$.*

Proof. By reordering the indices, the result only needs to be proved for $i = 1$. Let $\delta > 0$, and $r > 0$. Applying the above theorem with $\psi_1(s) = (c_1 + 2\delta)s/r$ gives that $F_1(r) > 0 = F(0)$. Now let $t > r$, and consider a smooth ψ such that $\psi_1(s) \leqslant c_i - 2\delta$ for $s \leqslant r$ and $\psi_1(t) > c_i + 2\delta$. Then

$$F_1(t) - F_1(r) = \mathbb{P}(r < \tau_1 \leqslant t) \geqslant \mathbb{P}\left(\sup_{s \leqslant t} |X_s - \psi(s)| < \delta\right) > 0 \tag{26}$$

□

Next, we consider whether $F_i(t)$ are continuous, i.e. whether $F_i(t-) =$

$F_i(t)$. To prove this, we use two results. The first is another result from Bass [1998]. Throughout, K_j will be strictly positive constants.

Theorem 3. *Suppose X_t solves (2) with $\varepsilon = 1$, σ and b bounded. There exist K_1 and K_2 depending only on $|\sigma|$ such that:*

$$\mathbb{P}\left(\sup_{s\leqslant t}|X_s - X_0| > \lambda + \|b\|_\infty t\right) \leqslant K_1 \exp(-K_2\lambda^2/t) \tag{27}$$

Proof. See Bass [1998], page 23. □

Let $\Gamma(t,x;y)$ denote the density of X_t given that $X_0 = y$, which is the fundamental solution of the operator $L - \partial_t$. Bounds on the fundamental solution, given by Friedman [1964] (page 24), imply that:

$$0 \leqslant \Gamma(t,x;y) \leqslant \frac{K_3}{t}\exp\left(-K_4\frac{\|x-y\|^2}{t}\right) \tag{28}$$

(where we have specialized the results in the reference to the case $d = 2$).

Proposition 2. *For all $t > 0$, $F_i(t-) = F_i(t)$.*

Proof. Let $t > 0$, and suppose to the contrary that $F_1(t) - F_1(t-) = \xi > 0$. Let $\delta_n = n^{-1}\|b\|_\infty$, and $t_n = t - \frac{1}{n}$, and note that:

$$F_1(t) - F_1(t_n) = \mathbb{P}(t_n < \tau \leqslant t) > \xi \tag{29}$$

But

$$\mathbb{P}(t_n < \tau \leqslant t) \leqslant \mathbb{P}(X_1(t_n) \in [c_1 - \delta_n, c_1))$$
$$+ \mathbb{P}\left(X_1(t_n) < c_1 - \delta_n, \sup_{s \in (t_n, t]} X_1(s) \geqslant c_1\right) = P_1(n) + P_2(n) \quad (30)$$

We will control each term separately.

$$\begin{aligned} P_1(n) &= \int_{c_1-\delta_n}^{c_1} \int_{-\infty}^{\infty} \Gamma(t_n, 0, y) dy_2 dy_1 & (31) \\ &\leqslant \frac{K_3}{t_n} \int_{c_1-\delta_n}^{c_1} \int_{-\infty}^{\infty} \exp\left(-K_4 \frac{(y_1^2 + y_2^2)}{t_n}\right) dy_2 dy_1 & (32) \\ &\leqslant \frac{K_5 \delta_n}{t_n} \leqslant K_6 \delta_n & (33) \end{aligned}$$

Using the Markov property, and the bound (27):

$$\begin{aligned} P_2(n) &\leqslant \int_{-\infty}^{c_1-\delta_n} \int_{-\infty}^{\infty} \Gamma(t_n, 0, y) \mathbb{P}\left(\sup_{0 < s \leqslant n^{-1}} |X_s - y| \geqslant c_1 - y_1\right) dy_2 dy_1 \quad (34) \\ &= \int_{-\infty}^{c_1-\delta_n} \int_{-\infty}^{\infty} \Gamma(t_n, 0, y) \mathbb{P}\left(\sup_{0 < s \leqslant n^{-1}} |X_s - y| \geqslant c_1 - y_1 - \delta_n + n^{-1}\|b\|_\infty\right) dy_2 dy_1 \\ &\leqslant K_1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{1}_{\{y_1 < c_1 - \delta_n\}} \Gamma(t_n, 0, y) \exp\left(-K_2 n (c_1 - y_1 - n^{-1}\|b\|_\infty)^2\right) dy_2 dy_1 \\ &\leqslant K_1 K_3 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{1}_{\{y_1 < c_1 - \delta_n\}} \exp\left(-K_4 \frac{\|y\|^2}{t_n}\right) \exp\left(-K_2 n (c_1 - y_1 - n^{-1}\|b\|_\infty)^2\right) dy_2 dy_1 \end{aligned}$$

$P_1(n)$ clearly tends to zero, and $P_2(n)$ tends to zero by the Dominated Convergence Theorem. But then $F_1(t) - F_1(t_n)$ tends to zero, contradicting (29). $\square$

Remark 1. According to Elliott et al. [2012], $u(x,t) = \mathbb{P}(\tau_1 > t | X_0 = x)$ solves the partial differential equation $\partial_t u = Lu$. where It should be noted that the boundary data is not continuous (the initial condition is $u(x,0) = 1_{\{x_1 < c_1\}}$, but the boundary condition is $u(c_1, x_2, t) = 0$). Nonetheless, a suitable notion of weak solution exists that is smooth on the interior of the domain. Let $v(x,t) = 1 - u(x,t) = \mathbb{P}(\tau_1 \leqslant t | X_0 = x)$, so that $F_1(t) = v(0,t)$. v is as smooth as u, so F_1 should inherit the maximum smoothness (in time) of the solution to the PDE. One can also argue that the density of the hitting time is strictly positive for $t > 0$ as follows. Let $f_1(t) = F_1'(t) = \partial_t v(0,t) > 0$. Let $w(x,t) = \partial_t v(x,t)$, so that $f_1(t) = w(0,t)$, and note that, by differentiating the PDE: $\partial_t u = Lu \Rightarrow \partial_t v = Lv \Rightarrow \partial_t w = Lw$. Since v is increasing in t, $w \geqslant 0$. Now, suppose that $f_i(t_0) = 0$ for some $0 < t_0 < T$. Then $w(0,t) = 0$. But $w \geqslant 0$, so this means that w attains a minimum at $(0, t_0)$, and by the Strong Maximum Principle for parabolic PDEs (Evans [2010], pages 396-397), w is constant on $U_{t_0} = U \times (0, t_0]$. In particular, this implies that $f_1(t) = w(0,t) = w(0,t_0) = 0$ for all $0 < t < t_0$, contradicting the fact that F_1 is strictly increasing. (In order to meet the hypotheses in the reference, we can consider the w over $U \times (0,T]$ where $U = (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)$ to meet the requirements that U be bounded, and w smooth on U_T and continuous on $\bar{U}_T$.)

It is clear that in general we cannot expect a result as precise as $F_1(\varepsilon) = F_2(\kappa\varepsilon)$, as held in the Brownian motion case. Nonetheless, when $J_1 > J_2$ we can show that for $\kappa > \frac{J_2}{J_1}$ and ε small enough, $F_1(\varepsilon) \leqslant F_2(\kappa\varepsilon)$, and when $J_1 = J_2$, the logarithmic asymptotics of $\mathbb{P}(\tau_1 \leqslant \varepsilon, \tau_2 \leqslant F_2^{-1}(F_1(\varepsilon)))$ and $\mathbb{P}(\tau_1 \leqslant \varepsilon, \tau_2 \leqslant \varepsilon)$ are the same. These results are given in Proposition (4), which requires the following consequences of Theorem (1).

Proposition 3. *For J_i defined as in (14), (15):*

$$\lim_{\varepsilon \downarrow 0} -\varepsilon \log(F_i(\varepsilon)) = J_i \tag{35}$$

Furthermore, for any $\gamma \leqslant 1$:

$$\lim_{\varepsilon \downarrow 0} -\varepsilon \log(\mathbb{P}(\tau_1 \leqslant \varepsilon, \tau_2 \leqslant \gamma\varepsilon)) = \inf\{I(\omega) | \omega \in B(c_1, 1, c_2, \gamma)\} \tag{36}$$

$$\lim_{\varepsilon \downarrow 0} -\varepsilon \log(\mathbb{P}(\tau_1 \leqslant \gamma\varepsilon, \tau_2 \leqslant \varepsilon)) = \inf\{I(\omega) | \omega \in B(c_1, \gamma, c_2, 1)\} \tag{37}$$

Proof. We will consider the result for F_1. The proof of the other results is similar. By (3):

$$F_1(\varepsilon) = \mathbb{P}_x^1(B(c_1, \varepsilon, 0, \varepsilon)) = \mathbb{P}_x^\varepsilon(B_{c_1,0})$$

Noting that $\mathring{B}_{c_1,0}$ consists of all $\omega \in C([0,1], \mathbb{R}^2)$ such that the supremum of the first component is strictly greater than c_1 we have by Theorem 1:

$$\begin{aligned} -\inf\{I(\omega) | \omega \in \mathring{B}_{c_1,0}\} &\leqslant \liminf_{\varepsilon \downarrow 0} \varepsilon \log(F_1(\varepsilon)) \\ &\leqslant \limsup_{\varepsilon \downarrow 0} \varepsilon \log F_1(\varepsilon) \leqslant -\inf\{I(\omega) | \omega \in B_{c_1,0}\} = -J_1 \end{aligned} \tag{38}$$

Now let $\omega^* \in B_{c_1,0}$ be such that $I(\omega^*) = J_1$ (existence of ω^* follows from the fact that $B_{c_1,0}$ is closed, and I has compact lower level sets). Take $\omega^n(s) = \omega^*(s) + \frac{s}{n}$ for $n \geqslant 1$. Then $\omega^n \in \mathring{B}_{c_1,0}$ (the interior of $B_{c_1,0}$), $\omega^n \to \omega$, and $I(\omega_n) \to I(\omega)$ by Dominated Convergence, so the two bounds in (38) coincide and the result follows. □

Notice that a special case of the last limit in the above proposition is that:

$$\lim_{\varepsilon \downarrow 0} -\varepsilon \log(\mathbb{P}(\tau_1 \leqslant \varepsilon, \tau_2 \leqslant \varepsilon)) = \inf\{I(\omega) | \omega \in B_{c_1,c_2}\} = J_{1,2} \tag{39}$$

Proposition 4.

i) Suppose that $J_1 > J_2$. Then for any $\gamma > J_2/J_1$ there exists $t_0(\gamma) > 0$ such that for all $t \leqslant t_0$, $F_1(t) \leqslant F_2(\gamma t)$. In particular, for t small enough, $F_2^{-1}(F_1(t)) \leqslant \gamma t$.

ii) Suppose that $J_1 = J_2$. Then

$$-\lim_{\varepsilon \downarrow 0} \varepsilon \log \mathbb{P}(\tau_1 \leqslant \varepsilon, \tau_2 \leqslant F_2^{-1}(F_1(\varepsilon)) = J_{1,2} \tag{40}$$

Proof. i) We may assume $\gamma < 1$. Let $c < 1$ be such that $c^2 \geqslant J_2/(J_1\gamma)$. By Proposition 3, there is a t_0 small enough so that for all $t \leqslant t_0$,

$$F_1(t) \leqslant \exp\left(-\frac{J_1 c}{t}\right) \leqslant \exp\left(-\frac{J_2}{c(\gamma t)}\right) \leqslant F_2(\gamma t) \tag{41}$$

ii) Let $J = J_1 = J_2$ and $\eta > 0$. By Proposition 3 for $r < r(\eta)$, and $i = 1, 2$, we have that

$$\exp\left(-\frac{J\sqrt{1+\eta}}{r}\right) \leqslant F_i(r) \leqslant \exp\left(-\frac{J}{r\sqrt{1+\eta}}\right) \tag{42}$$

so for ε small enough:

$$F_1(\varepsilon) \leqslant \exp\left(-\frac{J}{\varepsilon\sqrt{1+\eta}}\right) = \exp\left(-\frac{J\sqrt{1+\eta}}{(1+\eta)\varepsilon}\right) \leqslant F_2((1+\eta)\varepsilon) \tag{43}$$

and we have that $F_2^{-1}(F_1(\varepsilon)) \leqslant (1+\eta)\varepsilon$. A similar argument shows that for ε small enough, $F_2^{-1}(F_1(\varepsilon)) \geqslant (1+\eta)^{-1}\varepsilon$. Using (17) then implies:

$$\begin{aligned}
(1+\eta)J_{1,2} &= \inf\{I(\omega) | \omega \in B(c_1, (1+\eta)^{-1}, c_2, (1+\eta)^{-1})\} \\
&\geqslant \inf\{I(\omega) | \omega \in B(c_1, 1, c_2, (1+\eta)^{-1})\} \\
&= \lim_{\varepsilon \downarrow 0} \left[-\varepsilon \log\left(\mathbb{P}(\tau_1 \leqslant \varepsilon, \tau_2 \leqslant (1+\eta)^{-1}\varepsilon)\right)\right] \\
&\geqslant \limsup_{t \downarrow 0} \left[-\varepsilon \log\left(\mathbb{P}(\tau_1 \leqslant \varepsilon, \tau_2 \leqslant F_2^{-1}(F_1(\varepsilon)))\right)\right] \qquad (44) \\
&\geqslant \liminf_{\varepsilon \downarrow 0} \left[-\varepsilon \log\left(\mathbb{P}(\tau_1 \leqslant \varepsilon, \tau_2 \leqslant F_2^{-1}(F_1(\varepsilon)))\right)\right] \\
&\geqslant \lim_{\varepsilon \downarrow 0} \left[-\varepsilon \log\left(\mathbb{P}(\tau_1 \leqslant \varepsilon, \tau_2 \leqslant (1+\eta)\varepsilon)\right)\right] \\
&= (1+\eta)^{-1} \inf\{I(\omega) | \omega \in B(c_1, (1+\eta)^{-1}, c_2, 1)\} \\
&\geqslant (1+\eta)^{-1} J_{1,2}
\end{aligned}$$

The result now follows by letting $\eta \downarrow 0$. □

4.2 Properties of the Variational Problems

Lemma 3. *Suppose that* $\omega \in B_{c_1,0}$ *is such that* $I(\omega) = J_1$. *Then* $\omega_1(1) = c_1$, *and* $\omega_1(t) < c_1$ *for* $t < 1$. *Similarly, if* $\omega \in B_{0,c_2}$ *is such that* $I(\omega) = J_2$, *then* $\omega_2(1) = c_2$ *and* $\omega_2(t) < c_2$ *for* $t < 1$.

Proof. Suppose to the contrary that $\omega \in B_{c_1,0}$, $I(\omega) = J_1$, $\omega_1(\tilde{t}) = c_1$ for some $\tilde{t} < 1$. Using Lemma 1:

$$J_1 \geqslant \int_0^{\tilde{t}} \dot{\omega}(s)' g(\omega(s)) \dot{\omega}(s)\, ds \geqslant \frac{d^2(0, \omega(\tilde{t}))}{2\tilde{t}} > \frac{d^2(0, \omega(\tilde{t}))}{2} \geqslant J_1 \tag{45}$$

since a path that reaches the point $\omega(\tilde{t})$ at time 1 is in $B_{c_1,0}$. This is a contradiction, so we must have $\omega_1(t) < c_1$ for all $t < 1$. The proof when $I(\omega) = J_2$ is similar. □

Lemma 4. *If* $J_1 > J_2$ *then there exists* $\gamma \in (\frac{J_2}{J_1}, 1)$ *such that*

$$\inf\{I(\omega) | \omega \in B(c_1, 1, c_2, \gamma)\} > J_1 \tag{46}$$

Proof. Since $B(c_1, 1, c_2, \gamma) \subseteq B_{c_1,0}$, weak inequality in (46) is immediate for any $\gamma \geqslant 0$. Suppose to the contrary that equality holds for all $\gamma \in (J_2/J_1, 1)$. Let $\gamma_n \downarrow J_2/J_1$, and let $\omega^n \in B(c_1, 1, c_2, \gamma_n)$ and $t_n \in (0, \gamma_n]$ be such that $I(\omega^n) = J_1$, and $\omega_2^n(t_n) = c_2$. Passing to a subsequence if necessary, and using the fact that I is lower semi-continuous with compact level sets, we obtain $\omega^n \to \omega^*$, $t_n \to t^* \leqslant J_2/J_1$, and $\omega_2^*(t^*) = c_2$, with $I(\omega^*) = J_1$. Now:

$$\begin{aligned} J_1 = I(\omega^*) &= \frac{1}{2}\int_0^{J_2/J_1} \dot{\omega}^*(t)' g(\omega^*(t))\dot{\omega}^*(t)\,dt + \frac{1}{2}\int_{J_2/J_1}^1 \dot{\omega}^*(t)' g(\omega^*(t))\dot{\omega}^*(t)\,dt \\ &\geqslant J_1 + \frac{1}{2}\int_{J_2/J_1}^1 \dot{\omega}^*(t)' g(\omega^*(t))\dot{\omega}^*(t)\,dt \\ &\geqslant J_1 + \frac{d^2(\omega^*(J_2/J_1), \omega^*(1))}{2(1 - J_2/J_1)} \end{aligned}$$

where the first line follows by applying (7), and the second line from Lemma 1. Using Lemma 3 and the fact that $I(\omega^*) = I(\omega^n) = J_1$, $\omega_1^*(1) = \lim_{n\to\infty} \omega_1^n(1) = c_1$, and we must have that $\omega_1^*(1) < c_1$ for $t < 1$, so $\omega^* \in B(c_1, 1, c_2, J_2/J_1)$, and $d^2(\omega^*(J_2/J_1), \omega^*(1)) > 0$, yielding a contradiction. □

4.3 Tail Independence

In this section, we prove that τ_1 and τ_2 have zero lower tail dependence. In order to do so, we need to recall some facts about the (scaled) squared-distance function:

$$u(z) = \tfrac{1}{2}d^2(0, z) : \mathbb{R}^2 \to \mathbb{R}_+ \tag{47}$$

We begin with some standard definitions. Here $G \subseteq \mathbb{R}^2$ is an open set.

Definition 1. *Let $u \in C(G)$, $x \in G$. The sets $D^+u(x)$ and $D^-u(x)$ are* *defined to be:*

$$D^+u(x) = \left\{ p \in \mathbb{R}^2 : \limsup_{y \to x, y \in \Omega} \frac{u(y) - u(x) - p \cdot (y - x)}{|x - y|} \leqslant 0 \right\} \tag{48}$$

$$D^-u(x) = \left\{ p \in \mathbb{R}^2 : \liminf_{y \to x, y \in \Omega} \frac{u(y) - u(x) - p \cdot (y - x)}{|x - y|} \geqslant 0 \right\} \tag{49}$$

$D^+u(x)$ is referred to as the (viscosity) super-differential of u at x and $D^-(x)$ is the (viscosity) sub-differential of u at x.

As noted earlier, $u(z)$ is the optimal value of the optimization problem:

$$u(z) = \inf_{\omega \in C_0[0,1], \omega(1) = z} I(\omega) = \inf_{\omega \in H_0^1, \omega(1) = z} \frac{1}{2} \int_0^1 \dot{\omega}(t)' g(\omega(t)) \dot{\omega}(t)\, dt \tag{50}$$

Results in the calculus of variations can be used to show that minimizers ω of (50) are Lipschitz continuous (Clarke [2013], Theorem 16.18, pages 330-332), and indeed since $\lambda(t, x, p) = \frac{1}{2} p' g(x) p$ is smooth, any Lipschitz minimizer ω^* satisfies the integral Euler equation (the Theorem of du-Bois-Reymond, Clarke [2013], Theorem 15.2, pages 308-309). The positive definiteness of Λ_{pp} then implies higher order regularity of ω^* given higher regularity of Λ, by a Theorem of Hilbert and Weierstrass (Clarke [2013], Theorem 15.7, page 313), which in turn implies that ω^* is a smooth classical solution of the Euler equation for the problem.

Optimal solutions of (50) are geodesics connecting 0 and z in the Riemannian metric on $\mathbb{R}^2$ defined by the distance (6). As such, they have constant, nonzero speed, i.e. $\dot{\omega}^*(t)' g(\omega^*(t)) \omega^*(t) = k > 0$ (this may also be seen from the point of view of the calculus of variations as a consequence of the Erdmann condition, Clarke [2013], Proposition 14.4, pages 290-291). Using the formula for the first variation of the energy I (see do Carmo [1992], pages 192-196), it can be shown that u is super-differentiable, and $g(z)\dot{\omega}^*(1) \in D^+(u(z))$ (for a sketch of the proof, see Figalli and Villani [2011], page 178).

Theorem 4. *The hitting times τ_1 and τ_2 have zero lower tail dependence:*

$$\lambda_L = \lim_{\alpha \downarrow 0} \frac{\mathbb{P}(\tau_1 \leqslant F_1^{-1}(\alpha), \tau_2 \leqslant F_2^{-1}(\alpha))}{\alpha} = 0 \tag{51}$$

Proof. Let $\varepsilon = F_1^{-1}(\alpha)$, so that $\alpha = F_1(\varepsilon)$ and

$$\lambda_L = \lim_{\varepsilon \downarrow 0} \frac{\mathbb{P}(\tau_1 \leqslant \varepsilon, \tau_2 \leqslant F_2^{-1}(F_1(\varepsilon))}{F_1(\varepsilon)} = \lim_{t \downarrow 0} L(\varepsilon) \tag{52}$$

It is enough to show that $\limsup_{\varepsilon \downarrow 0} \Lambda(\varepsilon) < 0$ where:

$$\Lambda(\varepsilon) = \varepsilon \log(L(\varepsilon)) = \varepsilon \log\left(\mathbb{P}(\tau_1 \leqslant \varepsilon, \tau_2 \leqslant F_2^{-1}(F_1(\varepsilon))\right) - \varepsilon \log(F_1(\varepsilon)) \tag{53}$$

i) Suppose $J_1 > J_2$. Using Proposition 3 we need to show that:

$$\liminf_{\varepsilon \downarrow 0} -\varepsilon \log\left(\mathbb{P}(\tau_1 \leqslant \varepsilon, \tau_2 \leqslant F_2^{-1}(F_1(\varepsilon))\right) > J_1 \tag{54}$$

Let γ be as in Lemma 4. Then, applying Proposition 4 and Proposition 3:

$$\begin{aligned}\liminf_{\varepsilon\downarrow 0} -\varepsilon \log\left(\mathbb{P}(\tau_1 \leqslant \varepsilon, \tau_2 \leqslant F_2^{-1}(F_1(\varepsilon))\right) &\geqslant \lim_{t\downarrow 0} -\varepsilon \log\left(\mathbb{P}(\tau_1 \leqslant \varepsilon, \tau_2 \leqslant \gamma\varepsilon)\right) \\ &= \inf\{I(\omega)|\omega \in B(c_1, 1, c_2, \gamma)\} \\ &> J_1 \end{aligned} \tag{55}$$

ii) Suppose $J_1 = J_2$. Then

$$\begin{aligned}\Lambda(\varepsilon) = \varepsilon &\left(\log\left(\mathbb{P}(\tau_1 \leqslant \varepsilon, \tau_2 \leqslant F_2^{-1}(F_1(\varepsilon))\right) - \log\left(\mathbb{P}(\tau_1 \leqslant \varepsilon, \tau_2 \leqslant \varepsilon)\right)\right) \\ &+ \varepsilon \log(\mathbb{P}(\tau_1 \leqslant \varepsilon, \tau_2 \leqslant \varepsilon)) - \varepsilon \log(\mathbb{P}(\tau_1 \leqslant \varepsilon))\end{aligned} \tag{56}$$

Using Propositions 4 and 3, the result will follow if we can show that $J_{12} > J_1$. Since $B_{c_1,c_2} \subseteq B_{c_1,0}$, we immediately have $J_{12} \geqslant J_1$. Suppose that $J_{12} = J_1 = J_2$. Let $\omega^* \in B_{c_1,c_2}$ be such that $I(\omega^*) = J_{12} = u(c)$, where $c = (c_1, c_2)'$ (under the assumptions, $J_{12} = u(c)$ follows immediately from Lemma 3). By Lemma 3, $\omega^*(1) = (c_1, c_2)'$ and $\omega_1^*(t) < c_1$, $\omega_2^*(t) < c_2$ for all $t < 1$, and in particular, $\dot{\omega}^*(1) \geqslant 0$. Let $p \in D^+u(c)$. Considering the sequence $y_n = (c_1 - \frac{1}{n}, c_2)$ and using the fact that $u(y_n) \geqslant J_2 = u(c)$ yields:

$$0 \geqslant \limsup_{y_n \to c} \frac{u(y_n) - u(c) - p \cdot (y_n - c)}{|y_n - c|} \geqslant \limsup_{y_n \to c} \frac{-p \cdot (y_n - c)}{|y_n - c|} = p_1 \tag{57}$$

Similarly, we have $p_2 \leqslant 0$. Since $g(c)\dot{\omega}^*(1) \in D^+u(c)$, $g(c)\dot{\omega}^*(1) \leqslant 0$. But then $\dot{\omega}^*(1)' g(c)\dot{\omega}^*(1) \leqslant 0$. □

5 Conclusion

By utilizing the large deviations results of Varadhan [1967], we have shown that the hitting times τ_1 and τ_2 of the components of a two-dimensional uniformly elliptic diffusion process have coefficient of lower tail dependence equal to zero.

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