Dynamic Hedging Strategies for Cash Balance Pension Plans

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Abstract

Cash balance pension plans with crediting rates linked to long bond yields are relatively common in the US, but their liabilities are proving very challenging to hedge. In this paper we consider dynamic hedge strategies using the one-factor and two-factor Hull White models, based on results for the liability valuation from Hardy *et al.* (2014). The strategies utilise simple hedge portfolios combining one or two zero-coupon bonds, and a money market account. We assess the effectiveness of the strategies by considering how accurately each one would have hedged a 5-year CB liability through the past 20 years, using real world returns and crediting rates, and assuming parameters calibrated using the information available at the time. We show that there is considerable impact of model and parameter uncertainty, with additional, less severe impact from discrete hedging error and transactions costs. Despite this, the dynamic hedge strategies do manage to stabilize surplus substantially, even through the turbulence of the past two decades.

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1 Introduction

Cash Balance (CB) pensions represent the fastest growing pension plan design in the US, according to Kravitz (2015). The number of CB plans has increased from around 3% to 28% of all Defined Benefit (DB) plans over the past decade. In 2014 there were 9,648 CB plans in the US, with a total of over 12 million participants.

CB plans are classified and regulated as DB plans but are presented to look very much like Defined Contribution (DC) plans. For example, participants have individual accounts showing their up-to-date accrued benefits. However, an important difference between CB and DC plans is that employee accounts in the CB plan are notional. The assets are not allocated to individuals and the amounts paid into the aggregated plan funds need not be equal to the notional contributions. The total funds invested are generally not equal to the sum of the employee accounts, and are often substantially smaller because of the actuarial funding methods used, and because DB plans are not required to be fully funded (see Hardy *et al.* (2014). The notional individual member accounts are notionally accumulated at the specified plan crediting rate, which may be a fixed rate, or a variable rate such as the current yield on government bonds of specified term. In this paper we focus on the 30-year Treasury crediting rate; it is one of the most common options, (covering around 44% of plans, according to Kravitz (2015)) and one of the hardest to hedge as the yield to maturity on long treasuries cannot be replicated year to year with standard, over the counter instruments.

The original motivation for the CB plan design was to replicate the lower risk properties of DC plans within a DB plan. CB plans started gaining popularity in the mid-1990s; Niehaus and Yu (2005) explain that in 1990s, companies switching from traditional DB plans favored CB plans over DC, to avoid the high reversion tax applied to excess pension assets. Coronado and Copeland (2003) on the other hand conclude that the conversions from traditional DB to CB were primarily driven by the labor market conditions. In 2015, new regulations came into effect clarifying and extending the permitted range of crediting rate formulae[1].

Until recently, CB plans have been considered by sponsors and most actuaries to be low risk, low volatility plans. The views expressed by the consultants at Kravitz are typical,

[1] Internal Revenue Bulletin, T.D.9743, 2015-48 I.R.B.679, "Transitional Amendments to Satisfy the Market Rate of Return Rules for Hybrid Retirement Plans".

for example:

"*Cash balance plans remove the interest rate risk that led to constantly changing value of liabilities in traditional defined benefit plans.*"(Kravitz (2012))

More recently, there has been a greater recognition of the inherent risk involved in guaranteeing returns. In a 2015 report, Segal Rogerscasey wrote:

"*Cash balance plans do not represent a low-risk arbitrage opportunity, but rather are a leveraged investment in risk assets, similar to a traditional DB plan.*" (Rogerscasey (2015)).

Hedging the cash balance liability has become a more urgent question as the low interest rate environment has persisted so long that rising costs and risks cannot be ignored. Even the traditional actuarial funding approach, which is highly unresponsive to market conditions, is beginning to generate unexpected strains and volatility.

The major objective of this paper is to derive and quantitatively explore the efficacy of a hedging strategy based on financial engineering models and principles. We base the hedge on the market valuation of the liability, rather than the actuarial valuation. There are two reasons for this. The first is that the application of traditional actuarial valuation methods to CB plans generates an 'actuarial liability' that is quite unrelated to the short or long term costs. Both Murphy (2001) and Hardy *et al.* (2014) conclude that traditional methods may significantly understate the actual liability for participants who terminate early, and tend to generate losses even where the participant exits at the assumed age. Market valuation methods are more objective than actuarial methods and the valuation and hedging formulae are inextricably linked; the valuation formula can be used to derive the hedging portfolio for a risk. In Hardy *et al.* (2014) explicit valuation formulae were derived in the case where the crediting rate is the k-year spot rate, and where interest crediting applies continuously. Thus, the valuation results give us a starting point for determining a hedge strategy.

In practice CB crediting rates based on treasury bonds use the yield to maturity (YTM), not spot rates. The same valuation framework that we used for spot rates can be applied to the YTM, but the valuation requires Monte Carlo simulation. Hardy *et al.* (2014) showed that the numerical results using the spot rate valuation formulae were a good approximation for the YTM valuation. In this paper we show that the same is true for hedging strategies based on spot rates and YTMs. We also use this paper to extend the valuation formulae in Hardy *et al.* (2014) to the case where the interest crediting is discrete.

Market consistent valuation techniques have been studied for other pension-related liabilities; examples can be found in Boyle and Hardy (2003), Marshall (2011) and Chen and Hardy (2009). Despite the fact that US public and private sector pension plans have not (yet) adopted market consistent valuation, much of the recent pensions literature has discussed the benefit of using market values to assess objectively the funding status of pension plans. Novy-Marx and Rauh (2009) and Biggs (2010) recalculate the funding ratios of US public pensions to market-consistent benchmarks, and conclude that the subjectivity of traditional actuarial techniques significantly understates the liabilities, and overstates the funding rates. Moreover, Biggs and Smetters (2013) outline common misconceptions about using a long-term market rate as the discount rate for pension liabilities. In this paper, we reinforce the advantage of market consistent valuation from a risk management perspective, as it provides a natural approach not only to the valuation, but also to the appropriate investment strategies to hedge the interest rate risk.

The CB payoff based on treasury bond crediting rates can be viewed as an interest rate derivative. Hedging interest rate derivatives tends to be more sensitive to model risk than pricing the same derivatives. Hardy *et al.* (2014) show that the price of the CB payoff evaluated using the one factor Hull-White (HW) model is very close to the price evaluated with the two-factor HW model. However, that may not be the case for the hedge strategy. Several researchers have examined how many factors are required for effective hedging of interest rate risks. Fan *et al.* (2001) studied up to four factor models in the swaption market and conclude that although low-dimensional models are capable of accurately pricing swaptions, they are not sufficient for hedging purposes. Gupta and Subrahmanyam (2005) and Driessen *et al.* (2000) draw a similar conclusion for the cap/floor market. Their work is closer to ours, as they used delta hedging strategies, whereas Fan *et al.* (2001) focus on bucket hedging.

Other research on hedging CB liabilities includes Brown *et al.* (2001), who studied a duration based hedging strategy. and Harvey (2012), who discussed the difficulties in calculating the duration of a CB plan, and outlined several practical investment strategies including credit default swaps, Treasury futures and swaptions, but without supporting theoretical or empirical analysis.

The remainder of the paper is structured as follows. Section 2 introduces the assumptions, notation and models we adopt in this paper, and also the valuation formulae. Section 3 presents the construction of hedging portfolios. Section 4 describes the accumulated hedge error, which we use to measure hedge effectiveness. Section 5 examines the hedge error arising from discrete rebalancing, without model or parameter risk. In Section 6 we consider the impact of model and parameter error by analyze the hedging effectiveness using real historical interest rates. In Section 7 we consider basis risk, by comparing the results using a spot rate crediting rate, which is the more tractable approach, with results using the YTM crediting rate which is what is used in practice. In Section 8 we briefly compare the results of hedging with results where the assets are invested in a traditional DB equity/bond portfolio, and in Section 9 we introduce early exits through a brief example. Section 10 concludes.

2 Model and Assumptions

In this section we outline the formulae and assumptions used for valuing the cash balance pension liability, following Hardy *et al.* (2014) and Zhu (2015).

2.1 The CB Accrued Benefit

This paper focuses on the interest rate risk; in this section we ignore demographic risk, but we will consider early exits in Section 9. Treating the individual CB account balance at retirement as a contingent payment, the market value of the liability is the risk neutral expectation of the discounted payoff at termination.

Our valuation approach is accrual based, and so are all the hedging strategies we construct. Under the accruals principle (equivalent to unit credit in a traditional pension plan), the accrued liability is based only on past contributions. Future notional contributions into the plan can be each regarded as a separate derivative which will be evaluated and hedged in the same way as the existing account value at the time of payment. This means that our problem is to hedge the interest rate risk arising from the application of future, unknown crediting rates to the current notional account value.

Viewing the pension plan as a whole, the hedge portfolio will comprise hedges based on a range of accrued funds and times to exit, but to illustrate we consider a single employee fund, with a T-year term to exit from the valuation date.

We denote by F_t the notional account value at time t, based on a fixed notional account value of F_0 at $t = 0$, and ignoring notional contributions after $t = 0$. The frequency of interest crediting varies between plans, but the most common choice is annual crediting. Let $i^c(s)$ denote the crediting rate applied in the year s to $s + 1$. The exit date (which is known in advance) is denoted T. The lump-sum payment at the retirement date is the random variable F_T. For $t = 0, 1, 2, ..., T$ we have

$$F_t = F_0 \prod_{s=0}^{t-1} (1 + i^c(s))$$

For the continuous crediting case we let $r^c(s)$ denote the continuously compounded crediting rate at time s, so that for $0 < t \leq T$

$$F_t = F_0 \, e^{\int_0^t r^c(s)ds}$$

The formulae for valuation and hedging are more straightforward for the continuous crediting case than for the discrete, so in the main part of the paper we present the formulae and results using continuous crediting. The formulae for the discrete case are derived alongside the continuous versions in the appendices. The results are very close.

As in Hardy *et al.* (2014), we develop our results assuming that the crediting rate at t is equal to the k-year spot rate from the US government bond yield curve at that time. This is a convenient and reasonably accurate approximation to the most common form of crediting rate, that is the yield to maturity on k-year government bonds ((Kravitz, 2015)), with $k = 30$ being the most popular choice, and this is the rate used throughout the numerical illustrations in this paper.

2.2 The Valuation Formula

Let $r(t)$ denote the continuously compounded short rate of interest at time t, let $P(t, T)$ denote the price at t of a zero coupon bond with face value of \$1, which matures at time T, and let $r_k(t)$ denote the k-year spot rate at time t. Then

$$P(t, T) = e^{-r_{T-t}(t)\,(T-t)} = \mathrm{E}_t^Q \left[e^{-\int_t^T r(s)ds} \right]$$

where E_t^Q denotes the risk neutral expectation given the information at t.

The market consistent value at time t of the payoff F_T due at time $T \geq t$ is the expected discounted payoff, using the risk neutral measure. Thus, at time $t > 0$, the value is

$$\begin{aligned} V_t &= \mathrm{E}_t^Q \left[F_T e^{-\int_t^T r(s)ds} \right] \\ &= F_t \, \mathrm{E}_t^Q \left[e^{\int_t^T r^c(s) - r(s)ds} \right] \end{aligned}$$

If $r^c(t)$ and $r(t)$ are independent, for example if the crediting rate is a constant, this becomes a simple formula. In the case where $r^c(t)$ is based on treasury rates at t, there is a strong dependence between the crediting rate and the short rate, and so we use a stochastic model of the yield curve for the joint distribution of $r^c(s)$ and $r(s)$.

We define the valuation factor $V(t, T)$ as the market value at time t per \$1 in the participant's account balance at time t, where the benefit matures at time $T \geq t$; that is

$$V(t, T) = \mathrm{E}_t^Q \left[e^{\int_t^T r^c(s) - r(s)ds} \right]$$

2.3 Short Rate Models

We use both the One-Factor and Two-Factor versions of the Hull-White (HW) model. Both of these are well recognized models; both allow a perfect match between the model and market starting yield curve, and both offer convenient analytical tractability[2]. For the one-factor HW model, the instantaneous short rate under the risk-neutral measure has the following SDE:

$$dr(t) = (\theta(t) - ar(t)) + \sigma dW(t), \qquad r(0) = r_0$$

where $a > 0$, $\sigma > 0$ are constant, $\theta(t)$ is a deterministic function chosen to match the market term structure at the starting date, $W(t)$ is a standard Brownian motion under the Q measure, and r_0 is the observed short rate at time 0.

[2]One disadvantage of these models is that the short rate has a Gaussian distribution, with the possibility of being negative, but Hardy *et al.* (2014) show that this does not have a significant impact on the valuation factor.

There are two common parameterizations for the two-factor HW model. The one we adopt here is commonly referred to as G2++ (Brigo and Mercurio (2001)). The dynamics of the instantaneous short rate under Q are

$$\begin{aligned}
r(t) &= x(t) + y(t) + \varphi(t), & r(0) &= r_0\\
dx(t) &= -a_1 x(t)dt + \sigma_1 dW_1(t), & x(0) &= 0\\
dy(t) &= -a_2 y(t)dt + \sigma_2 dW_2(t), & y(0) &= 0\\
dW_1(t)dW_2(t) &= \rho dt
\end{aligned}$$

where r_0 is the initial observed short rate, a_1, a_2, σ_1,σ_2 are all positive constants, $\varphi(t)$ is the deterministic function used to match the initial term structure, and $(W_1(t), W_2(t))$ is a two-dimensional Brownian motion under the Q measure, with correlation ρ, $-1 \leq \rho \leq 1$.

The valuation formulae for the continuous and discrete crediting cases are given in Appendices A and B respectively. Interested readers can refer to Hardy *et al.* (2014) and Zhu (2015) for more detailed derivations.

3 Constructing the Dynamic Hedging Portfolio

In this section we develop hedging strategies for a CB plan with crediting rate equal to the 30-year spot rate on Treasuries. We develop and compare three different strategies.

1. A delta hedge under the one-factor Hull-White model (requires two assets).
2. A delta-gamma hedge under the one-factor Hull-White model (requires three assets).
3. A delta hedge under the two-factor Hull-White model (requires three assets).

The hedging instruments we use are a money market account, with return equal to the short rate, and Treasury STRIPs of varying length. In practice, hedging with Treasury STRIPs may be impractical due to the size of the STRIPs market, but the techniques may be adapted to other fixed income securities, such as Treasury bonds, as we show in Section 7.2.

3.1 Greeks

The delta and gamma for a particular security are defined as the first and second order derivatives respect to the underlying risky process. Here we provide the delta and gamma for the CB liability (subscript V) and for the zero coupon bond price (subscript B) under the one-factor HW model. In the following, recall that $t > 0$ is the valuation date, $T \geq t$ is the exit date, and k is the assumed term of the spot rate used as a crediting rate under the plan.

$$\Delta_B(t) = \frac{\partial P(t,T)}{\partial r(t)} = -B(a, T-t)P(t,T) \tag{1}$$

$$\Delta_V(t) = \frac{\partial V(t,T)}{\partial r(t)} = -\gamma(a,k)B(a, T-t)V(t,T) \tag{2}$$

$$\Gamma_B(t) = \frac{\partial^2 P(t,T)}{\partial r(t)^2} = B(a, T-t)^2 P(t,T) \tag{3}$$

$$\Gamma_V(t) = \frac{\partial^2 P(t,T)}{\partial r(t)^2} = \gamma(a,k)^2 B(a, T-t)^2 V(t,T) \tag{4}$$

$$\text{where} \quad B(a,s) = \frac{1-e^{-as}}{a}$$

$$\text{and} \quad \gamma(a,k) = 1 - \frac{B(a,k)}{k}$$

Under the two-factor HW model, the deltas for the CB liability and for the zero coupon bond are the first partial derivatives with respect to each stochastic driver (x and y).

$$\Delta_B^x(t) = -B(a_1, T-t)P(t,T)$$
$$\Delta_B^y(t) = -B(a_2, T-t)P(t,T)$$
$$\Delta_V^x(t) = -\gamma(a_1,k)B(a_1, T-t)V(t,T)$$
$$\Delta_V^y(t) = -\gamma(a_2,k)B(a_2, T-t)V(t,T)$$

Analogous formulae for discrete crediting are given in Appendix D.

3.2 Position of hedging instruments

The hedging instruments we use for delta hedging under the one-factor HW model are the money market account, with return equal to the short rate, and a zero-coupon bond with maturity T_1.

For delta-gamma hedging under the one-factor HW model and for delta hedging under the two-factor HW model, we need a third asset, so we add another zero-coupon bond with maturity $T_2 > T_1$. The position in each instrument can be obtained by solving linear equation(s). For example, for delta hedging under the one-factor HW model, we solve for the position in the T_1-year zero coupon bond ($\Lambda_1^{HW}(t)$) as:

$$\Delta_V(t) = \Delta_B(t, T_1)\Lambda_1^{HW}(t) = -B(a, T-t)P(t,T)\Lambda_1^{HW}(t) \qquad \text{from (1)}$$

$$\Rightarrow -B(a, T-t)P(t,T)\Lambda_1^{HW}(t) = -\gamma(a,k)B(a,T-t)V(t,T) \qquad \text{from (2)}$$

$$\Rightarrow \Lambda_1^{HW} = \frac{\gamma(a,k)B(a,T-t)V(t,T)}{B(a,T_1-t)P(t,T_1)}$$

Similarly, for delta-gamma hedging with the one-factor HW model, the position in the T_1-year zero coupon bond (Λ_1^{HW}) and the position in the T_2-year zero coupon bond (Λ_2^{HW}), are

$$\Lambda_1^{HW} = \frac{\gamma(a,k)B(a,T-t)(\gamma(a,k)B(a,T-t) - B(a,T_2-t))V(t,T)}{B(a,T_1-t)(B(a,T_1-t) - B(a,T_2-t))P(t,T_1)}$$

$$\Lambda_2^{HW} = \frac{\gamma(a,k)B(a,T-t)(\gamma(a,k)B(a,T-t) - B(a,T_1-t))V(t,T)}{B(a,T_2-t)(B(a,T_2-t) - B(a,T_1-t))P(t,T_2)}$$

For delta- hedging with the two-factor HW model, the position in the T_1-year zero coupon bond (Λ_1^{G2}) and the position in the T_2-year zero coupon bond (Λ_2^{G2}) are

$$\Lambda_1^{G2} = \frac{\left(\gamma(a_1,k)B(a_1,T-t)B(a_2,T_2-t) - \gamma(a_2,k)B(a_2,T-t)B(a_1,T_2-t)\right)V(t,T)}{\left(B(a_1,T_1-t)B(a_2,T_2-t) - B(a_2,T_1-t)B(a_1,T_2-t)\right)P(t,T_1)}$$

$$\Lambda_2^{G2} = \frac{\left(\gamma(a_1,k)B(a_1,T-t)B(a_2,T_1-t) - \gamma(a_2,k)B(a_2,T-t)B(a_1,T_1-t)\right)V(t,T)}{\left(B(a_1,T_2-t)B(a_2,T_1-t) - B(a_1,T_1-t)B(a_2,T_2-t)\right)P(t,T_2)}$$

In each case, the amount invested in the bank account $S(t)$ equals the difference between the liability value and the total value of the position in zero coupon bonds.

For simplicity, we select the first zero coupon bond duration, T_1 to be the same maturity as the horizon of the liability, and select the longest maturity zero coupon bond available for T_2 (30-year for the U.S. Treasury Strip market).

3.3 Estimating the Parameters

In both Hardy *et al.* (2014) and Zhu (2015), the same parameters are used. For the one-factor HW model, we used $\alpha = 0.02$, $\sigma = 0.006$, and for the two-factor model, we used

$$a_1 = 0.055, \qquad a_2 = 0.108, \qquad \sigma_1 = 0.032, \qquad \sigma_2 = 0.044, \qquad \rho = -0.9999$$

Notice that the correlation ρ is close to negative one, which implies that the two-factor model is in some sense very close to an one-factor model. However, this one-factor model is not equivalent to the one-factor HW whenever $a_1 \neq a_2$ (see Brigo and Mercurio (2001)). Gurrieri *et al.* (2009), Gupta and Subrahmanyam (2005) and Brigo and Mercurio (2001) all note that ρ may become highly negative depending on the instruments used for calibration. The parameter values shown above imply a long-term unconditional standard deviation for the short rate that is close to experience over the past 30-40 years.

In Section 4, we will assess the effectiveness of delta and delta-gamma hedging strategies by applying them to CB liabilities maturing in the period 2005-2015. It would not be appropriate to use the parameters above to examine the hypothetical effectiveness of the hedging strategies over this testing period, as the parameters were derived using that same data. To test whether the hedge strategy would have been effective, we should only use information available at the time. The parameters used should be consistent with the data available at the assumed initial valuation; we also consider the possibility that the parameters could be updated between the initial valuation and the termination date.

The most common choices of market derivatives used for calibration of interest rate models are caps and swaptions. We use swaptions as they contain more information on the correlations between forward rates, which is critical in our application (see Brigo and Mercurio (2001)). The choices for the number of swaptions used in the calibration strongly affects the parameter values. For example, Gurrieri *et al.* (2009) compare the calibration results using three sets of swaptions for the one-factor HW with time-varying mean reversion and volatility. We use at-the-money swaptions with expiration dates between one year and the maturity of the CB liability. The tenor of the swaptions are chosen to be 30 years, with quarterly payments[3]. We set the model parameters by minimizing the sum of the relative squared difference between the theoretical swaption price and the market

[3]The swaption data are quoted as a Black-volatility matrix and are obtained from Bloomberg.

price, that is, minimizing

$$obj = \sum_{i=1}^{n} \left(\frac{\text{model implied swaption price}_i - \text{market swaption price}_i}{\text{market swaption price}_i} \right)^2$$

For the two-factor model, to distinguish between the parameters of the two stochastic drivers, we set constraints as

$$0 < a_2 < a_1 < 1, \qquad 0 < \sigma_1, \sigma_2 < 0.5, \qquad -1 < \rho < 1$$

There exist closed-form solutions for swaption prices under the one and two factor HW models. However, the calibration is computationally expensive. Here we adopt the approximation proposed by Schrager and Pelsser (2006), which greatly simplifies the formulae for swaption prices with sufficient accuracy (for parameter values in our study, the relative difference in the swaption prices is at most 2%).

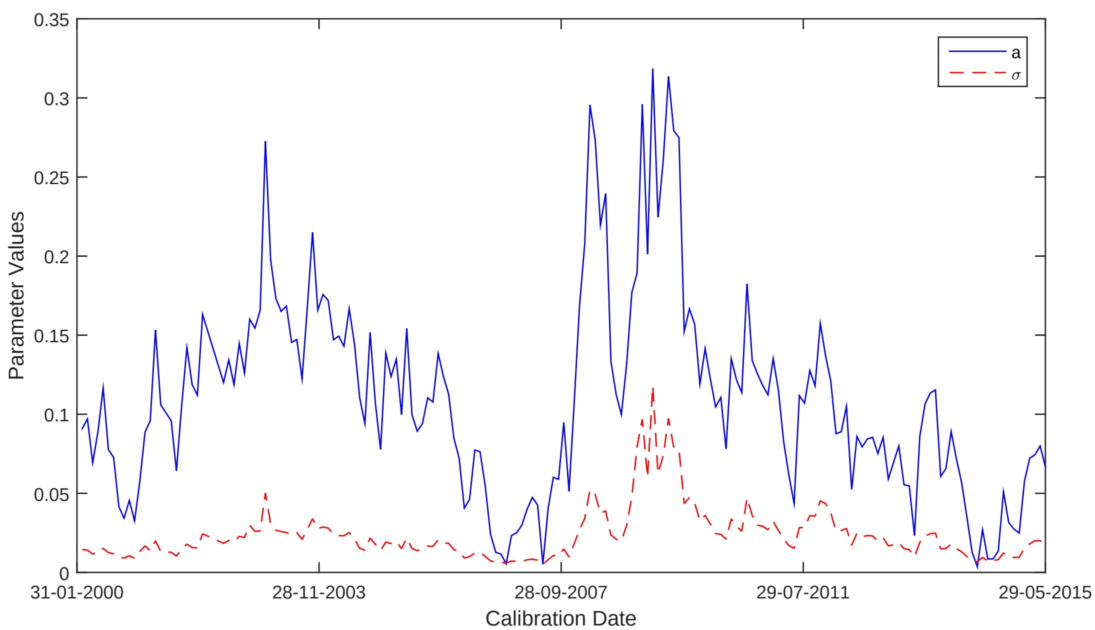

Figure 1: Calibrated Parameters for the one-factor HW based on swaption prices, 2000-2015.

Figures 1 and 2 present the calibrated parameters as a function of the calibration date. At first glance, the estimated parameters for the two-factor HW model do not match with the estimated parameters for the one-factor HW model. Indeed, the variance of future short rates under these two sets of parameters differs significantly. The reason is that the swaptions we used for calibration are focused on the long rates over the next five years. In fact, the two sets of parameters imply a close match for the variance of spot rates with long maturity in the next one to five years. Also, notice that all calibrated parameters are somewhat volatile, especially for the two-factor HW model (where parameters are often close to the constraints). This phenomenon has been observed elsewhere in the literature (for example, Enev (2011) and Gurrieri *et al.* (2009)). In Section 6 we will apply the hedging strategies derived from these models and parameters to real world interest rate paths; we will demonstrate that in spite of the volatility in the implied parameters, the hedging strategies can nevertheless be quite effective.

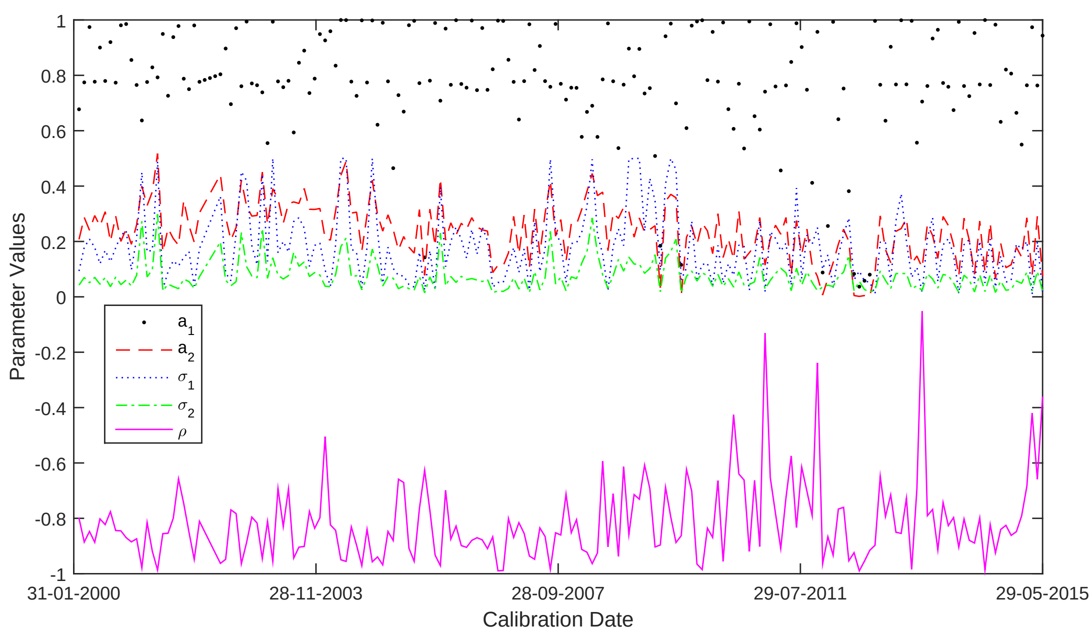

Figure 2: Calibrated Parameters for Two-Factor HW (volatility and correlation only) based on swaption prices, 2000-2015.

4 Hedging Performance Metrics

Under certain theoretical assumptions, a dynamic hedging strategy with no transaction costs can perfectly replicate the liability. However, this is impossible in practice, and the difference between the liability and the replicating instrument value is the hedge error. The hedge error comes from two main sources: the fact that hedges are rebalanced discretely, not continuously as the theory requires, and from the difference between the model and parameters assumed, and the actual real world interest rate process. There are also additional potential errors from transaction costs, but these are not very significant here because fees for trading government bonds are assumed to be very low. See Zhu (2015) for more details.

In Section 5, we quantify the discrete hedge error by considering weekly, monthly and annual re-balancing, where the monthly case serves as the benchmark, and where we eliminate the impact of model and parameter risk, by using the hedging model, adapted for the market price of risk, to generate the simulated "real world" interest paths.

In Section 6 we examine the model and parameter risk by testing the hedging strategies derived from our models, using historical paths of crediting rates and yield curves, assuming crediting rates are treasury spot rates.

The approach in Section 6 introduces basis risk, from the fact that we are assuming spot crediting rates, where the real liability is the yield to maturity (YTM). In Section 7 we investigate the basis risk by considering the difference between the spot rate hedge and the YTM liability.

4.1 Accumulated Maturity Hedging Error (MHE)

We let $\mathcal{H}(t)$ denote the value of the hedge portfolio at time t, given the participant remains in the plan. By definition, on rebalancing dates we have $\mathcal{H}(t) = V(t) = F_t V(t,T)$. Let $\mathcal{H}(t^-)$ denote the value of the hedge portfolio at time t immediately before re-balancing, and we let $\mathcal{E}(t)$ denote the hedge error at time t.

$$\mathcal{E}(t) = V(t) - \mathcal{H}(t^-)$$

Assuming that at each re-balancing date, we invest/borrow the amount equal to the hedge error at the risk-free rate, we get the accumulated hedge error at maturity (MHE)

$$\text{MHE} = \sum_{t=1}^{T} \mathcal{E}(t) e^{\int_t^T r(s)ds}$$

5 Evaluating discrete rebalancing hedging errors

The distribution of hedge errors arising from discrete rebalancing can be obtained by generating a large number of random sample paths for the future crediting rate under the real-world probability measure, then calculating the maturity hedge error for each sample path, based on a notional fund of \$1000 at the start of the simulation.

We use 10,000 paths under a P-measure model for future interest rates that is a simple shift of the assumed risk neutral model, to allow for the market price of risk. For details see Björk (2009). All other model assumptions are the same for the random paths and for the hedging and valuation model, so we are isolating the effect of discrete rebalancing without considering model, parameter or basis risk, which are all considered in subsequent sections.

Specific assumptions for both this section and Section 6 are:

- The crediting rate is the $k = 30$-year treasury spot rate
- We are hedging the payoff from a CB plan, based on the notional account $F_0 = \$1000$ at the initial valuation date, and assuming the participant exits $T = 5$ years later. We ignore additional notional contributions between the valuation date and the exit date.
- The hedge instruments at t for the one-factor HW model with delta hedging, are the money market and a pure discount bond with $T_1 = T - t$ years remaining.
- The hedge instruments at t for the one-factor HW model with delta-gamma hedging, and for the two factor HW model with delta hedging, are the money market, a pure discount bond with $T_1 = T - t$ years remaining, and another pure discount bond with $T_2 = 30$.

For the market price of risk, we follow the work of PricewaterhouseCoopers (2014), which assumed a constant market price of risk, denoted λ, estimated through historical rates from 2001 to 2012. The dynamics of the short rate under the real world measure are

$$dr(t) = (\theta(t) - ar(t) + \lambda\sigma)dt + \sigma dW^P(t)$$

We evaluate λ using historical monthly returns (from 1990 to 2015) on zero coupon bonds with maturities varying from 5 years to 30 years. The estimates we obtained range from 0.46 to 0.6. We assume that 0.5 would be a reasonable approximation for λ, and since we use 0.006 for the volatility term, the drift adjustment would be 0.003.

Figures 3 and 4 display the distribution of maturity hedge errors under simulation using the one-factor HW model, where $F_0 = \$1000$. The starting date of the plan (or the initial term structure) is chosen as at 2009-02-27, but we find that the maturity hedge error is not very sensitive to the initial term structure. Immediately, we observe that increasing the frequency of the re-balancing will reduce the overall hedge loss, as we expect. Also, for delta hedging, the effect of switching from annual re-balancing to monthly is much greater than from monthly to weekly (in both the magnitude and the shape of the distribution). Most importantly, even for annual re-balancing, the maturity hedge error for a 5-year plan is less than 0.2%, which is insignificant compared with other sources of error (as we shall illustrate in Section 6). In terms of terminal funding level, a monthly hedge frequency replicates the maturity benefit with error less than 0.01%.

We conclude that the impact of discrete hedging error is relatively minor, and in the following sections where we consider model, parameter and basis risk we may assume monthly hedging does not significantly impact the ultimate hedging errors.

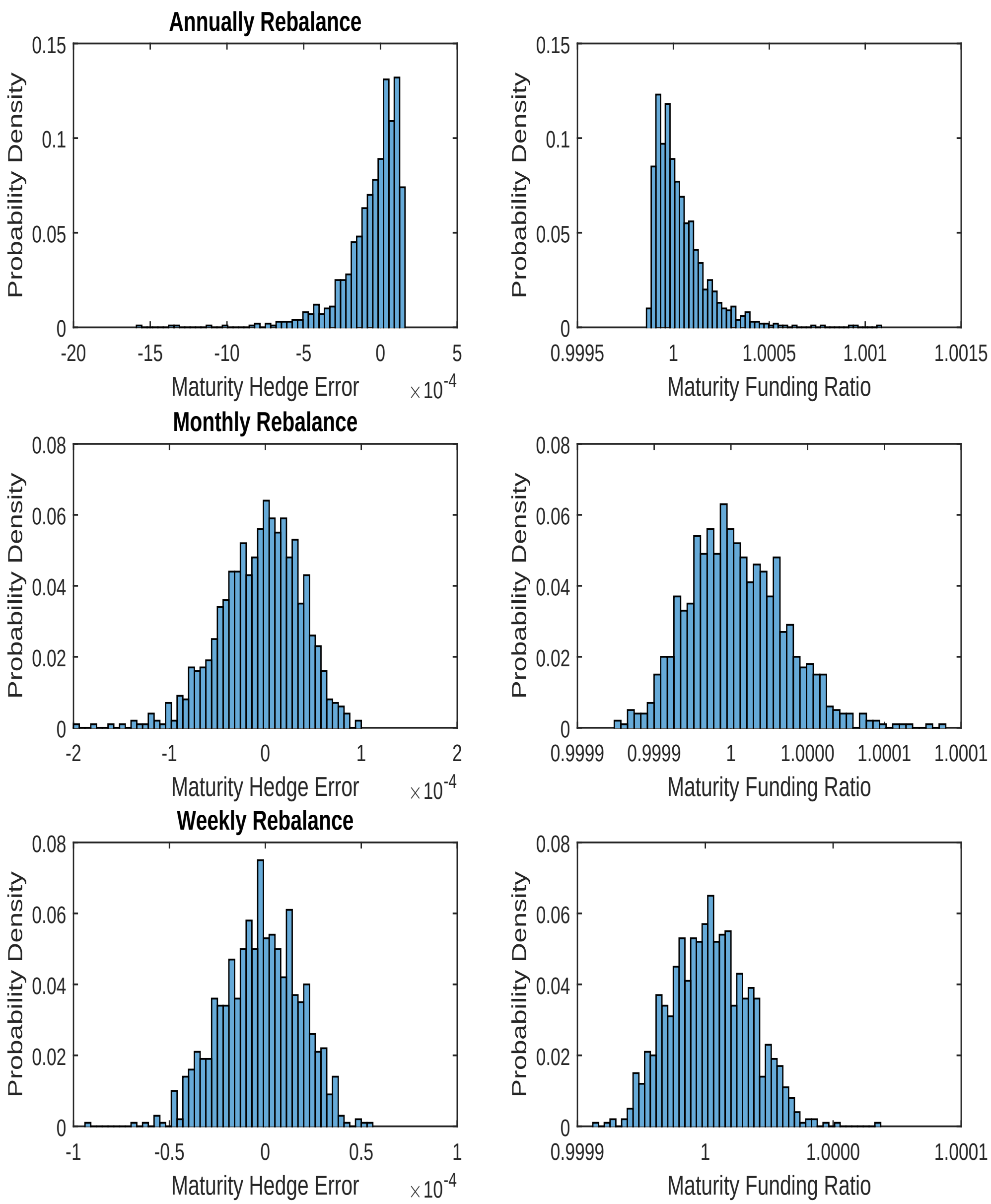

Figure 3: Simulation Results for Delta Hedging (One Factor HW), plan started at 2009-02-27, initial account value $F_0 = 1000$. Annual (top), monthly (middle) and weekly (bottom) rebalancing.

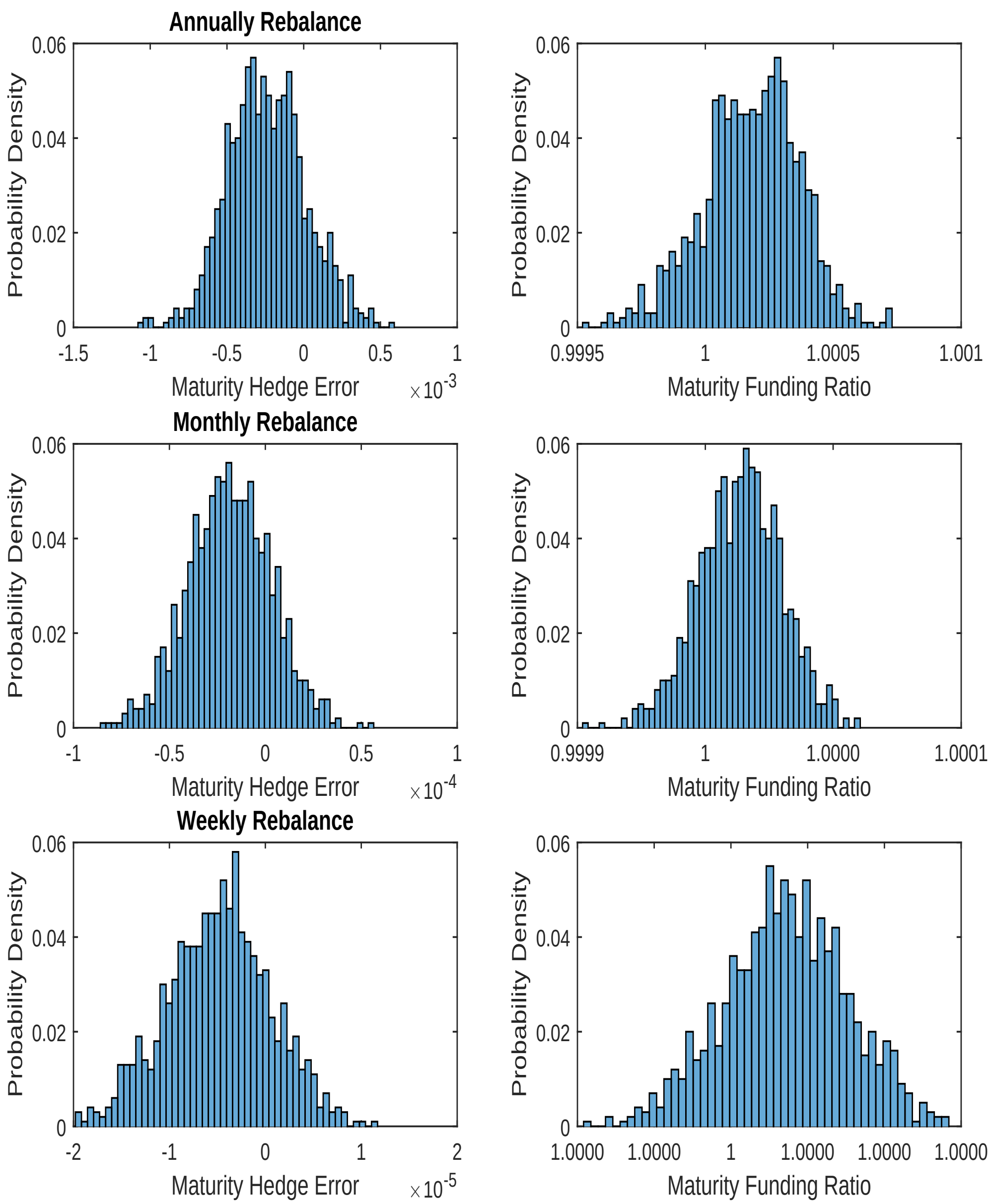

Figure 4: Simulation Results for Gamma Hedging (One Factor HW), plan started at 2009-02-27, initial account value $F_0 = 1000$. Annual (top), monthly (middle) and weekly (bottom) rebalancing.

6 Evaluating model and parameter error using the hypothetical historical hedging performance

In this section we consider the same CB plan as in Section 5, but instead of using randomly generated 5-year paths for short rates and crediting rates, we will use a succession of 5-year paths from the past 15 years, with end dates ranging from 2005 to 2015. That is, we assume the plan trustees hedge the accrued liability (i.e. past notional accumulated contributions) assuming all members leave at the end of 5-years from the valuation date[4]. We assume monthly rebalancing, based on the results of Section 5 which indicate that this will be sufficiently frequent.

To model the potential impact of different ways of updating the parameters, we show the results using three different approaches

(i) Constant parameters

(ii) Parameters calibrated at the plan starting date, and then maintained through the five year term

(iii) Parameters re-calibrated at each hedging date.

Recalibrating will, in principle, affect both the hedge and the valuation factors, but the impact is very small. In Figure 5 we present the valuation factors under different parameter calibrations and under the different models (one factor or two factors). The graph demonstrates that the valuation factors are extremely close, meaning that they are not very dependent on the parameters or on the number of factors.

In Figures 6 to 8 we show results for each of the different hedging portfolios, with different approaches to parameter recalibration. For easier comparison, the graphs are all shown on the same scale. Figure 6 shows the results for the delta strategy, using the one-factor HW model. Figure 7 shows the delta-gamma strategy hedging result, still with the one-factor HW model, and Figure 8 shows the delta strategy hedging result using the two-factor HW model.

From Figure 6 we see that the accumulated hedge errors are quite variable, and the impact of re-calibrating the model is quite significant. Using constant parameters, the

[4]This is somewhat analogous to a partly projected approach to traditional pension valuation.

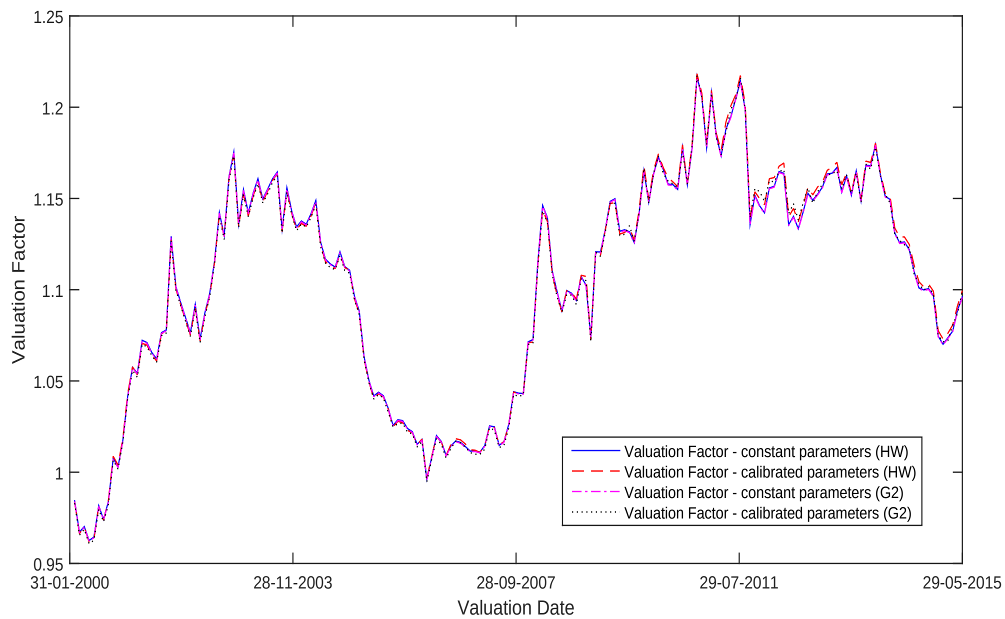

Figure 5: Valuation Factors evaluated using different parameter calibration approaches, for Hull-White models using one-factor (labelled HW) and two factors (labelled G2).

MHE ranges from around +11% (in 2005, 2012) to −4% (in 2008-9) of the terminal liability (which ranges from around \$1180 to \$1320). Using parameters that are set at the start of each 5-year term, the errors range from +12% (in 2012) to −4% (in 2008-9). Using parameters that are recalibrated monthly, the higher end of the range is reduced to around +6% of the terminal liability.

In Figure 7 we see that adding the gamma hedge to the one-factor model has not significantly improved the hedge accuracy in the constant parameter case, but has improved accuracy where the parameters are regularly recalibrated.

In contrast, Figure 8 shows that the hedge using the two-factor model is quite robust to the parameter variability, and overall losses are contained in a range from close to 0% up to around 5% of the terminal liability.

In Figure 9 we show the MHEs for each model, where the hedges were determined using constant parameters. The most immediate result is that the two factor model appears to create a much more reliable hedge than the one factor, even though the valuation results were shown in Hardy *et al.* (2014) to be very insensitive to the number of factors.

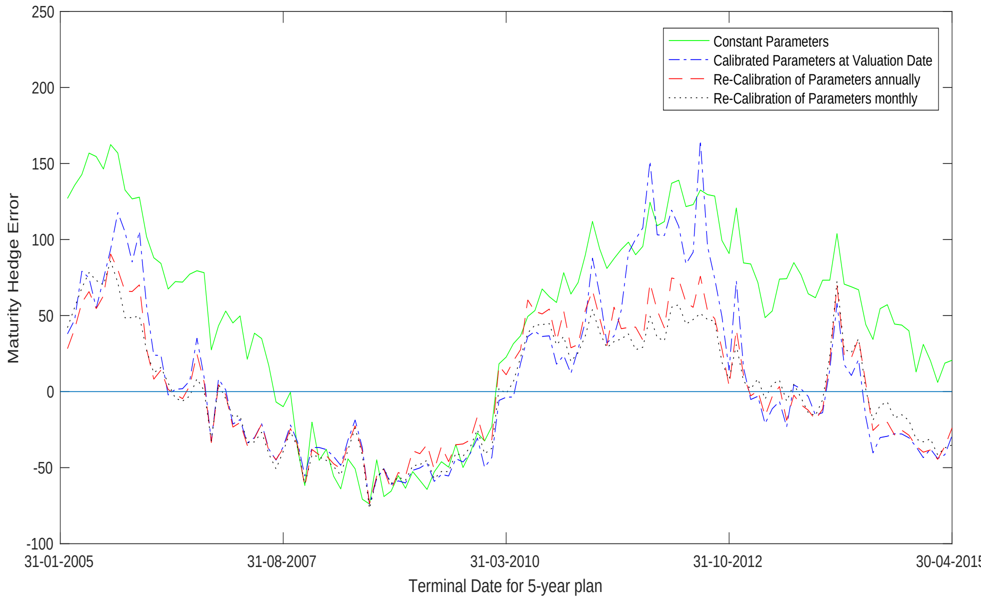

Figure 6: Maturity Hedge Loss for CB plan, 5-year horizon, maturing at 2005-2015, with initial account value $F_0 = 1000$, Delta-Hedging with the one-factor HW Model, using different parameter calibration approaches.

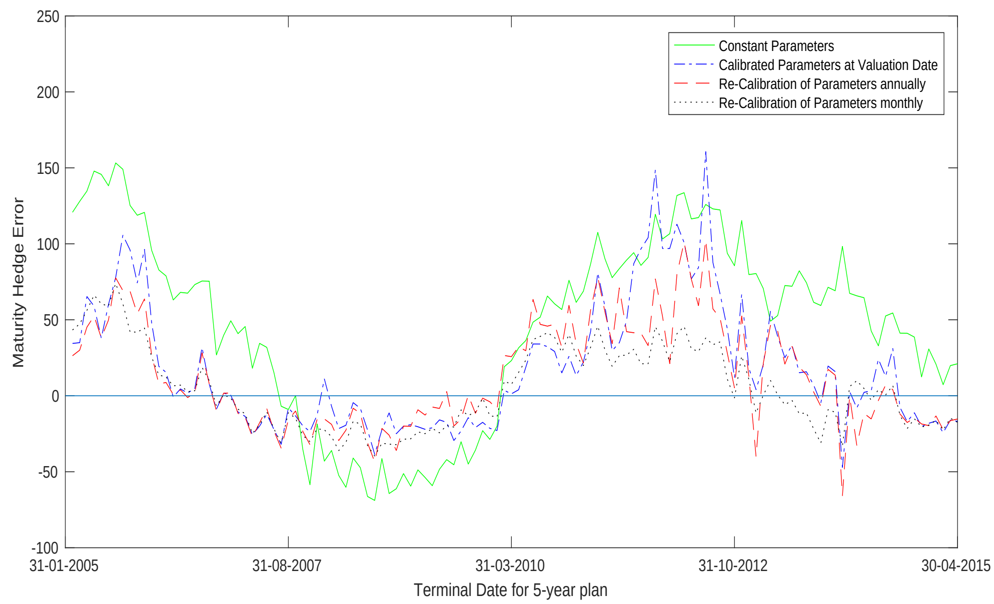

Figure 7: Maturity Hedge Loss for CB plan, 5-year horizon, maturing at 2005-2015, with initial account value $F_0 = 1000$, Gamma-Hedging with the one-factor HW Model, using different parameter calibration approaches.

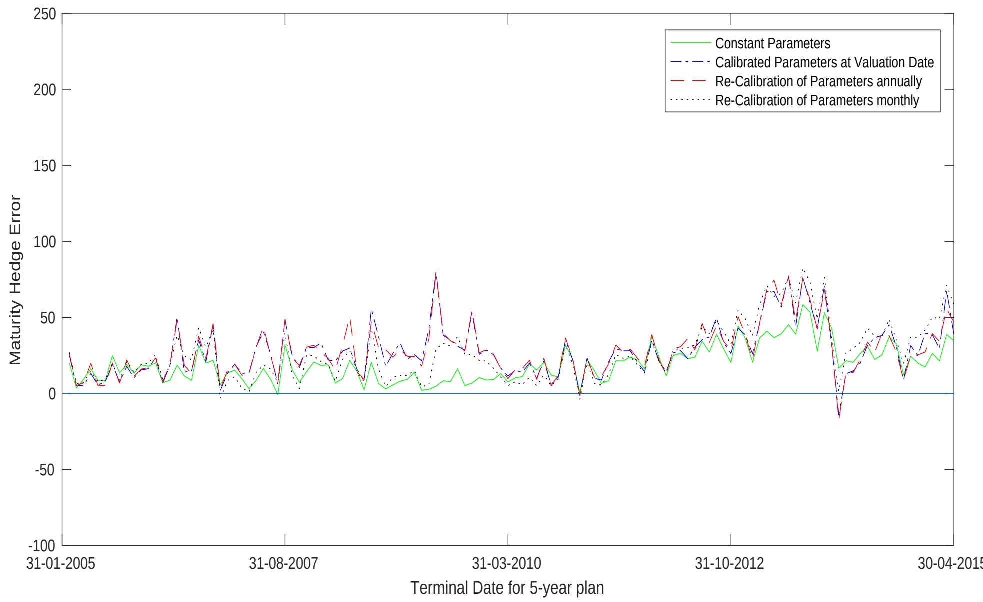

Figure 8: Maturity Hedge Loss for CB plan, 5-year horizon, maturing at 2005-2015, with initial account value $F_0 = 1000$, Delta-Hedging with the two-factor HW Model, using different parameter calibration approaches.

However, the conclusion is less clear when we compare results allowing for recalibration of the model at each hedging date, as shown in Figure 10. Here we see that, although the variability of hedging errors in the two-factor case is less than either of the one factor cases, the upper end of the errors are similar – though generated by different market conditions. The one-factor model does worst through the 2001-2006 period, with declining spreads between long and short rates. The two factor model is less sensitive to the changing market conditions up to the crisis; the worst performance covers the period from around 2008 to 2013. We note also from this graph that the addition of the gamma hedge in the one factor case does seem to generate improved hedging accuracy.

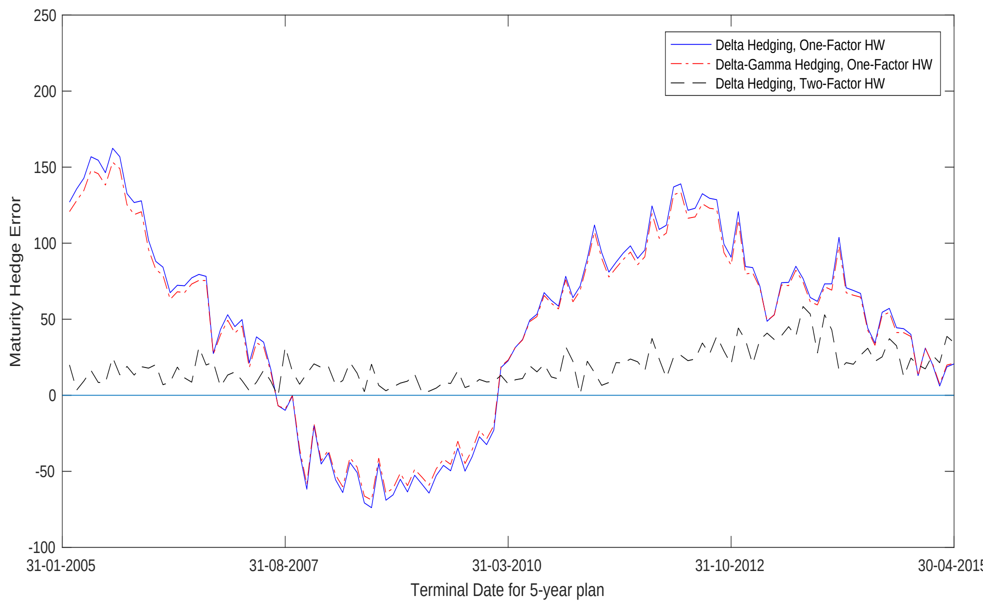

Figure 9: MHE for 5-year CB plan maturing at 2005-2015, using constant parameters.

7 Crediting with the yield to maturity

As mentioned in previous sections, in practice a treasury based crediting rate in a CB plan would use the YTM on treasuries, not the spot rate. This means that there is basis risk in the analysis in Section 6, arising from the difference between the assumed hedge liability and the actual hedge liability. The reason for using spot rates is tractability;

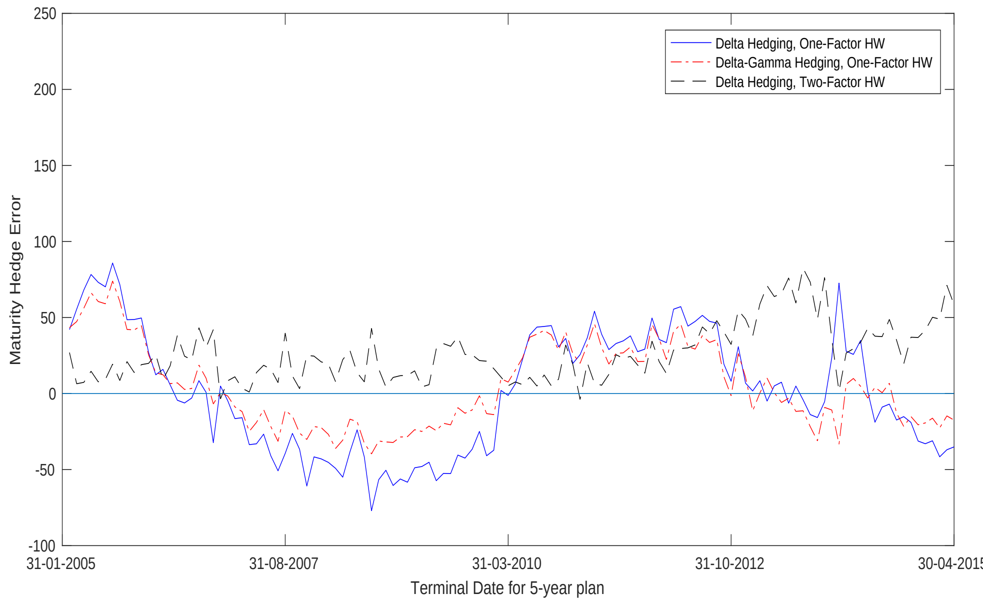

Figure 10: MHEs for 5-year CB plan maturing at 2005-2015, parameters re-calibrated monthly.

using YTMs all the hedging portfolios must be determined using repeated Monte Carlo simulation. In this section we examine the magnitude of the basis risk by considering the difference between hedging spot crediting rates and hedging YTM crediting rates, with all other assumptions as in Section 6; specifically, 30-year Treasury Bond crediting rates, 5-year hedging horizon, with Hull-White one and two factor interest rate models.

7.1 Valuation by Control Variate Method

Unfortunately, there is no closed-form solution available, thus, numerical methods such as the Monte Carlo method must be implemented. Here we denote $y_k(t)$ as the k-year YTM at time t, which is the annual coupon rate payable semi-annually on a new k-year bond issued at par at time t:

$$y_k(t) = \frac{2(1 - P(t, t+k))}{\sum_{u=1}^{2k} P(t, t+\frac{u}{2})}$$

Due to the close relationship between the YTM and the spot rate, the valuation factor for a CB plan using the spot rate can be set as a control variate in the Monte Carlo simulation for the YTM crediting rate valuation. To measure the effectiveness of the control variate, we use the variance reduction ratio (VRR), which is defined as

$$\text{VRR} = \frac{\text{Var(naive estimator)}}{\text{Var(estimator with control variate(s))}}$$

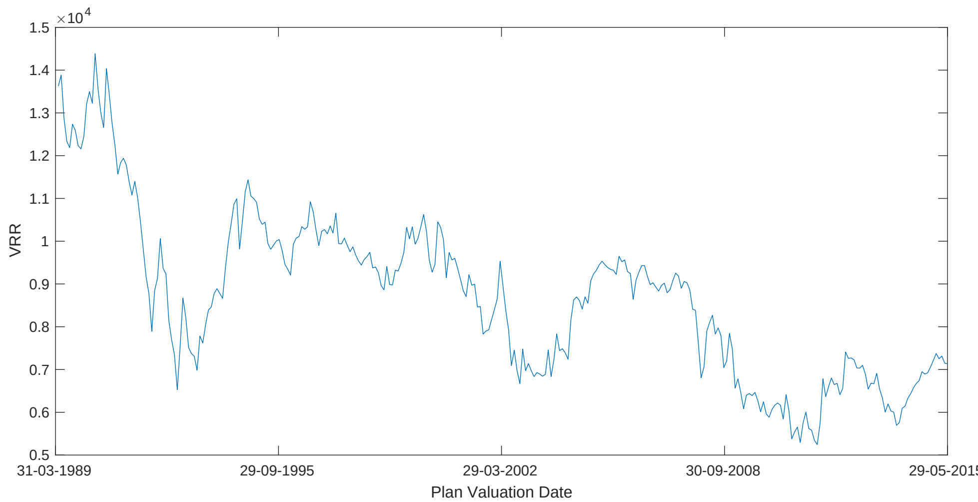

Figure 11: VRR for YTM crediting rate valuation under One-Factor HW, using spot rate as the control variable, 10,000 simulations

Figure 11 displays the VRRs for different valuation dates. The effectiveness of the control variate depends on the correlation between the spot rate and the YTM. When the yield curve is relatively flat, the YTM is close to the long-term spot rate, and the VRR is almost 15,000. Even when the yield curve is fairly steep the VRR is above 5000. This implies that the control variate reduces at least 99.98% of the variance of the naive Monte Carlo method.

Figure 12 compares the valuation factors using spot crediting rates rate and YTM crediting rates, assuming constant parameters. We note that spot crediting rates have generally produced larger valuation factors over this period; the difference is usually less than 4%, but in periods where the yield curve is particularly steep it has been as high as 9%.

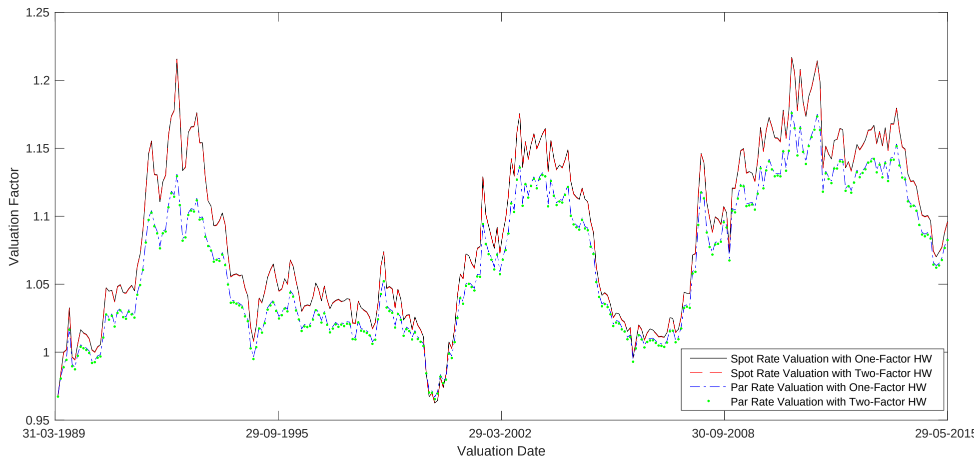

Figure 12: Compare Valuation Factors using 30-year spot and YTM crediting rates, 5-year horizon, initial fund \$1000, 10,000 simulations.

7.2 Hedging the YTM crediting rate

To calculate the delta for the YTM valuation factor we adopt the pathwise method (see Glasserman (2003)), and use the delta from equation (2) as the control variate. Although the variance reduction is not as effective as the valuation factors, VRR remains above 15 (up to 200) for all scenarios. It is important to point out that although the valuation factors for the spot rate and YTM are close, their Deltas and Gammas may differ significantly. This is because the sensitivity of the spot rate to the YTM is quite different from the sensitivity of the short rate to the long spot rate. Therefore, it is important to verify if delta hedging remains effective.

In this section, instead of using treasury STRIPS, we considered more liquid treasury bonds in our portfolio. All other settings follow exactly as in Section 6.

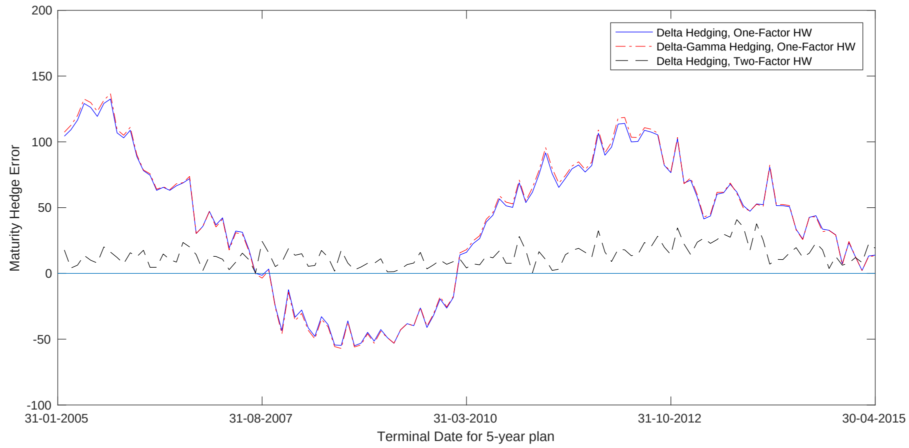

Figure 13: MHE for a CB plan with 5-year horizon, maturing at 2005-2015, initial amount \$1000, 30-year Treasury Bond YTM crediting rate, constant parameters.

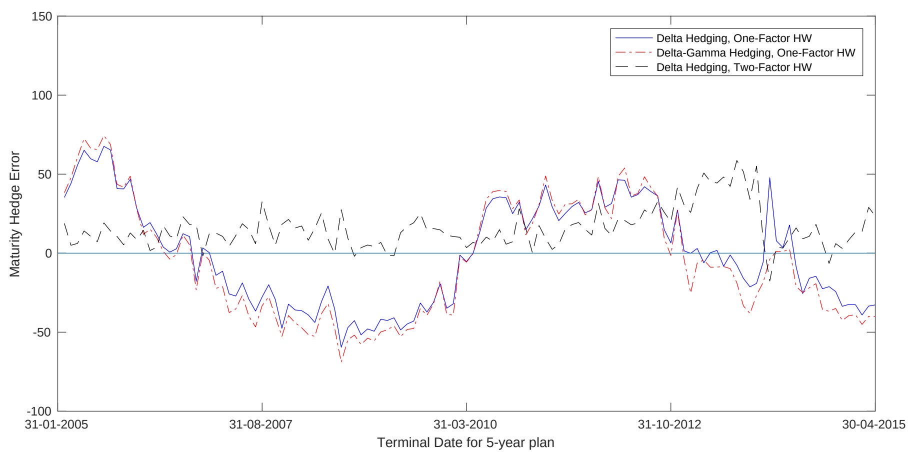

Figure 14: MHE for a CB plan with 5-year horizon, maturing at 2005-2015, initial amount \$1000, 30-year Treasury Bond YTM crediting rate, parameters recalibrated monthly.

Comparing Figure 13 and 14 with Figure 9 and 10, we can observe many similarities. The shape of the graphs are almost identical, and, as we expected from the comparison between valuation factors, the absolute value of the MHE is slightly lower for YTM crediting rates. Importantly, we see that, as for the spot rates, the delta hedge applied under the one or two factor HW models appears to offer a stable risk management strategy, with accumulated hedging errors generally less than around 5% of the fund value at maturity.

8 Comparing the Delta hedge strategy with a traditional investment strategy

One of the initial attractions of CB plans to employers is that by investing in equities, the employer could pay less than the notional contribution rate (as % of pay), since the additional return on equities would fund the crediting rate and make up the shortfall in contributions (see Gold (2001)).

Using a traditional approach to pension investing for DB plans, trustees might select a simple 60% equity/40% long bond portfolio, and it is interesting to compare the funding deficit or surplus at maturity under this strategy with the case where one of the hedging strategies is adopted. In Figure 15, we show the maturity hedging loss (or surplus) following a traditional 60/40 equity/long bond strategy, alongside the accumulated maturity hedging error using the two-factor HW hedge. Note that we have had to extend the scale dramatically compared with the hedging graphs above. We have assumed the same starting assets for each 5-year period (specifically, $1000V(t,5)$), although in practice this is significantly more than a traditionally managed CB plan might hold.

This figure clearly demonstrates how the hedged portfolio targets the terminal liability, while the unhedged portfolio may end up vastly over or under funded. On the other hand, the unhedged case does generate some very enticing profits – but this is not a good case for rejecting hedging, as the profits are unlikely to be repeated in the near future. The gains from the equity/bond portfolio are largely generated from the steady period of declining interest rates, which has generated consistent gains on long bonds that cannot continue indefinitely, and indeed will reverse in periods of rising interest rates. There is a significant possibility that the massive gains experienced in the last 10 years could be replaced by equally large losses in the next 10 years using the equity/bond investment approach. If the purpose of modern pension risk management is not to generate windfall gains, but to minimize losses and hence minimize contribution variability as much as possible, then the hedging strategy appears far more suitable.

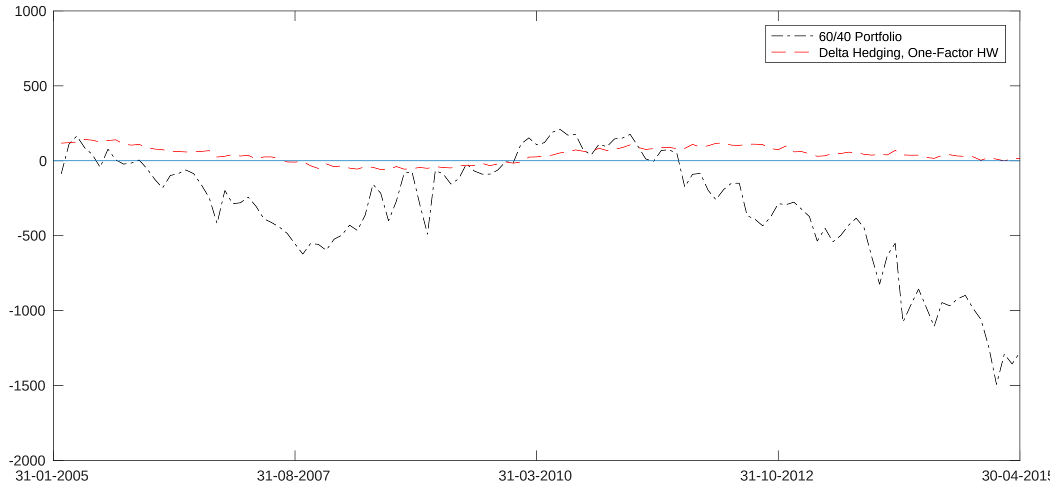

Figure 15: Maturity Hedge Loss for 5-year CB plan maturing at 2005-2015, initial account value $F_0 = 1000$, 30 year Treasury spot crediting rate, traditional equity/bond investment strategy and delta hedge strategy. Constant parameters.

9 Incorporating pre-retirement exits

In practice, employees may terminate their pension plan prior to the normal retirement date for various reasons: change of employment, disability, death or early retirement. These termination events are often assumed to be independent from the market performance. Here we construct a simple example to illustrate that delta hedging remains effective if the termination events are assumed to be independent from interest rates, and diversifiable. Let $U(t,T)$ denote the valuation factor incorporating pre-retirement exits, and assuming (for simplicity) that the individual will receive their account value at the end of the year of exit.

$$U(t,T) = \sum_{u=\lfloor t+2 \rfloor}^{T-1} V(t,u)\,{}_{\lfloor t+u-1 \rfloor - t}p_x^{00}\, q_{x+\lfloor t+u-1 \rfloor - t}^{(\tau)} + {}_{\lfloor t+1 \rfloor - t}q_x^{(\tau)} V(t, \lfloor t+1 \rfloor) + {}_{\lfloor t+T-1 \rfloor - t}p_x^{00} V(t,T)$$

where ${}_tp_x^{(\tau)}$ is the probability that a plan member age x is still in the plan at age $x+t$, and $q_x^{(\tau)}$ is the probability that a plan member age x exits the plan before age $x+1$ (Dickson *et al.* (2013)). We see that $U(t,T)$ is simply a weighted average of $V(t,S)$ where

$t \leq S \leq T$. In Figure 16 we show the surface of $V(0, T)$ for a range of horizon periods (T) and start dates. Although $V(0, T)$ is not a strictly increasing function in T it is generally increasing for $T \in [0, 20]$. Using $V(t, T)$ as an approximation for $U(t, T)$ will tend to over-price and over-hedge the risk.

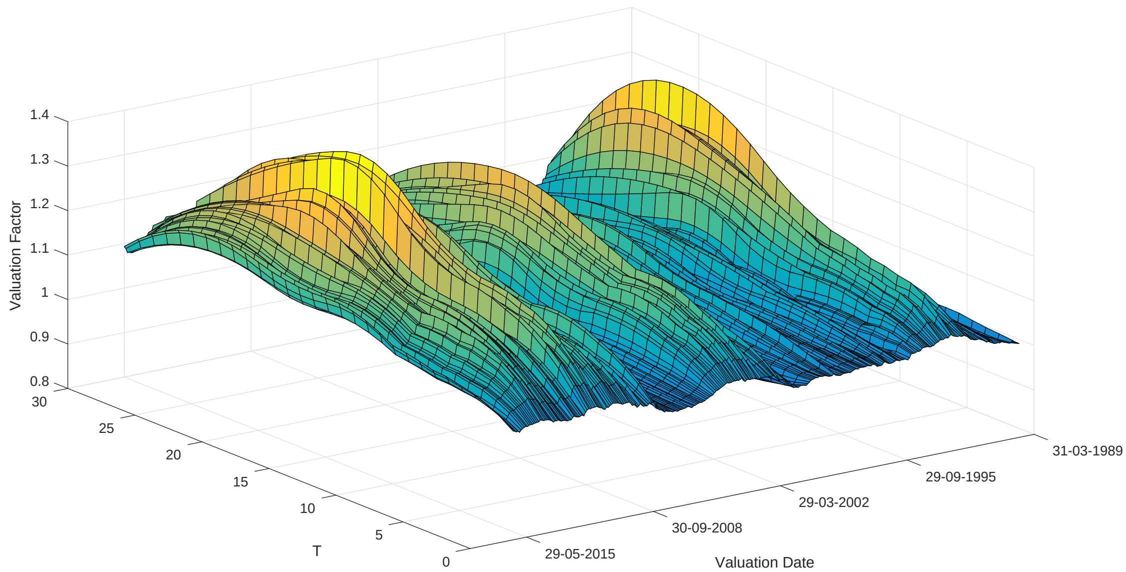

Figure 16: $V(0, T)$ using 30 year Treasury Bond YTM crediting rate, with different horizons and start dates, One-Factor HW model.

When the exit events are diversifiable the plan can be hedged by combining the exit probabilities with the appropriate hedge portfolios from Section 7.2. For a single cohort, this could be constructed as follows.

- **Step 1**: At time 0, evaluate the valuation factor $V(0, T)$ and construct the hedging portfolio based on the Greeks.
- **Step 2**: At the end of month 1, ${}_{1/12}p_x$ percent of people remain in the plan (requires $V(1/12, T)$ in liability), and ${}_{1/12}q_x$ percent of people leave the plan and will receive the full account balance at the year end. We measure the hedge loss for the period and re-construct the portfolio.
- **Repeat Step 2** for $t = [1/12, \cdots, T]$.

We illustrate this approach using demographic assumptions from the 2016 Actuarial Report for University of Toronto Pension Plan (see Hewitt (2016) for the report and Pension Experience Subcommittee (2014) for the mortality table), with a maximum horizon of 15 years. We set ${}_{\frac{1}{12}}q_x^{(\tau)}$ as the aggregate 1-month exit probability for a member age x. The probability that an employee will remain in the plan at the end of 15th year is approximately 47%. For a cohort starting on January 1st, 2000 and ending on December 31st 2014, the maturity hedge error is 10.7% of the initial account value. To show how this hedge error arises, in Figure 17 we show the monthly hedge errors that accumulate to the maturity hedge error of 10.7% of F_0 (which is set at \$1000), with and without early exits. Notice that the plan allowing for early exits has a slightly more stable hedging performance, but the hedge errors at each re-balancing date are close. Therefore, introducing early exits that are diversifiable will not affect the overall hedging results materially, and diversifiable exits can easily be incorporated into the hedging strategy.

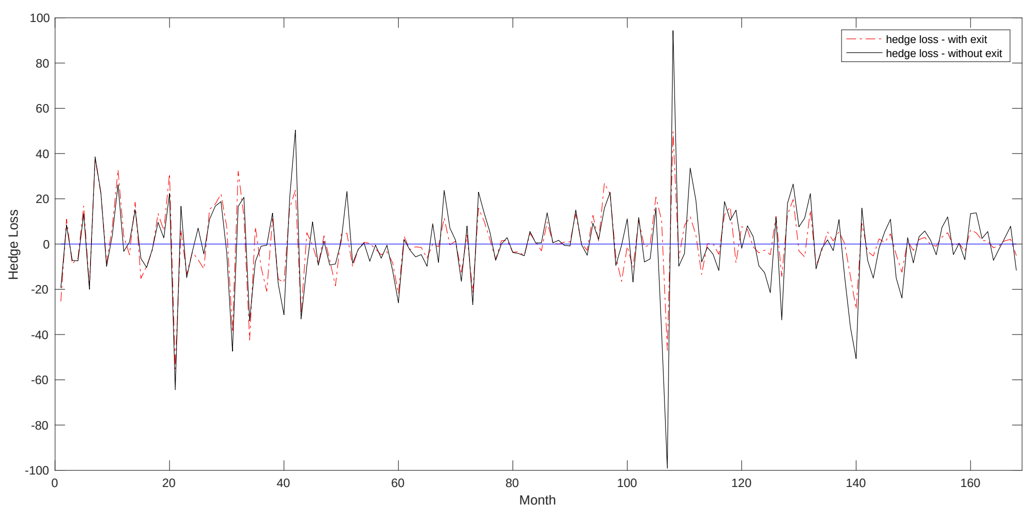

Figure 17: Monthly Hedging Errors for a 15-year CB cohort starting on 1/1/2000, with and without allowance for exits. One-Factor HW model, delta hedge, 30 year Treasury Bond YTM crediting rates.

10 Conclusion

In this article, we have studied dynamic hedging strategies for Cash Balance Pension plans. Dynamic hedging strategies are relatively simple to construct, and offer effective mitigation of the significant interest rate risk inherent in CB plans with long Treasury bond YTM crediting rates.

We have restricted the examples, for example by excluding consideration of floors, by limiting the horizon to only five years, and by limiting the hedging portfolio to simple Treasury Bonds/STRIPs. Nevertheless, the results when applied to real world scenarios do indicate that the simple delta strategy is able to provide a practical, affordable hedge against interest rate risk, even through the turmoil of the past decade.

In future work, we will explore other types of hedging strategies, including semi-static hedging, as we incorporate more complex crediting rate designs. See Marshall (2011) and Liu (2010) for discussions of semi-static hedging for different guaranteed options embedded in variable annuities. We also plan to explore different approaches to the valuation liability; in particular, we will examine the impact of a Traditional Unit Credit approach, where the liability at each date is equal to the notional account, ensuring that assets are designed to meet the liabilities whenever the participant exits the plan, once the benefits are vested.

11 Acknowledgments

We are very grateful to two anonymous referees for their insightful comments and suggestions that have contributed to strengthening and better motivating this work.

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A Continuous Crediting Based on the k-year spot rate: one-factor HW Model

This section and the following three sections provide formulae for valuation factors. Detailed proofs can be found in Hardy *et al.* (2014) and Zhu (2015). For CB plans with the k-year spot rate as the continuous crediting rate, the valuation factor using the one-factor HW model is

$$\begin{aligned} V(t,T) &= E_t^Q\left[e^{\int_0^T r_k(s)ds - \int_t^T r(s)ds + m(k)T}\right] \\ &= e^{\int_0^t r_k(s)ds} e^{-\int_t^T \frac{A(s,s+k)}{k}ds} P_\gamma(t,T) e^{m(k)T} \end{aligned}$$

where

$$\begin{aligned} P_\gamma(t,T) &= \exp\left\{A_\gamma(t,T) - B_\gamma(a,T-t)r(t)\right\} \\ A_\gamma(t,T) &= \gamma \log\left\{\frac{P^M(0,T)}{P^M(0,t)}\right\} + \left\{\frac{\sigma^2\gamma}{4a}\{\frac{2}{a}(\gamma-1)(T-t) + (e^{-2at}-\gamma)B^2(a,T-t)\right. \\ &\qquad \left. + \frac{1}{a}(2-2\gamma)B(a,T-t)\} + \gamma f^M(0,t)B(a,T-t)\right\} \\ B_\gamma(a,T-t) &= \gamma B(a,T-t) \end{aligned}$$

$$
\begin{aligned}
B(a,k) &= \frac{1-e^{-ak}}{a} \\
A(t,t+k) &= \log\frac{p(0,t+k)}{p(0,t)} + f(0,t)B(a,k) - \frac{\sigma^2}{4a}B(a,k)^2(1-e^{-2at}) \\
\gamma &= 1 - \frac{B(a,k)}{k}
\end{aligned}
$$

B Discrete Crediting Based on the k-year spot rate: one-factor HW Model

A CB plan with the k-year spot rate as the crediting rate and crediting frequency n times per year, has the valuation factor using the one-factor HW model

$$
V(t,T) = e^{\sum_{s=1}^{tn}\frac{r_k(\frac{s}{n})}{n}} e^{A(t,T)-B(a,T-t)r(t)} e^{EH(t,T,n)+\frac{1}{2}VH(t,T,n)} \tag{5}
$$

where

$$
\begin{aligned}
EH(t,T,n) &= \sum_{i=tn+1}^{Tn} \frac{-A\left(\frac{i}{n},\frac{i}{n}+k\right)}{nk} \\
&\quad + \frac{B(a,k)}{nk}\sum_{i=tn+1}^{Tn}\Bigg[r(t)e^{-a\left(\frac{i}{n}-t\right)} + \varphi\left(\frac{i}{n}\right) - \varphi(t)e^{-a\left(\frac{i}{n}-t\right)} \\
&\quad - \frac{\sigma^2}{a^2}\left(1 - \frac{1}{2}e^{-a\left(T-\frac{i}{n}\right)} - e^{-a\left(\frac{i}{n}-t\right)} + \frac{1}{2}e^{-a\left(T+\frac{i}{n}-2t\right)}\right)\Bigg] \\
VH(t,T,n) &= \left(\frac{\sigma B(a,k)}{nk}\right)^2 \sum_{j=tn+1}^{Tn}\left(\frac{e^{-\frac{a}{n}(j-1)} - e^{-aT}}{e^{\frac{a}{n}}-1}\right)^2 \frac{e^{2a\frac{j}{n}} - e^{2a\frac{j-1}{n}}}{2a} \\
\varphi(t) &= f^M(0,t) + \frac{\sigma^2}{2a^2}(1-e^{-at})^2
\end{aligned}
$$

$f^M(0,t)$ is the observed instantaneous forward rate at time t, and $A(t,T)$, $B(a,k)$ are defined in the previous section.

C Continuous Crediting Based on the k-year spot rate: two-factor HW Model

For a CB plan with the k-year spot rate as the continuous crediting rate, the valuation factor using the two-factor HW model is

$$
\begin{aligned}
V(t,T) = \exp\left\{\int_0^t r_k(s)ds\right\} \exp\left\{m(k)T\right\} \exp\left\{-\int_t^T \frac{A^G(s,s+k)}{k} ds\right\} \\
\exp\left(A^*(t,T) - \gamma(a_1,k)B(a_1,T-t)x(t) - \gamma(a_2,k)B(a_2,T-t)y(t)\right)
\end{aligned}
$$

where

$$
\begin{aligned}
A^G(t,T) &= \log\frac{p(0,T)}{p(0,t)} + \frac{1}{2}(\nu(T-t) + \nu(t) - \nu(T)) \\
\nu(k) &= \frac{\sigma_1^2}{a_1^2}(k - 2B(a_1,k) + B(2a_1,k)) + \frac{\sigma_2^2}{a_2^2}(k - 2B(a_2,k) + B(2a_2,k)) \\
&\quad + \frac{2\rho\sigma_1\sigma_2}{a_2 a_2}(k - B(a_1,k) - B(2a_2,k) + B(a_1+a_2,k)) \\
B(a,k) &= \frac{1}{a}(1 - e^{-ak}) \\
A^*(t,T) &= \log\frac{p(0,T)}{p(0,t)} + \frac{1}{2}(\nu^*(T-t) + \nu(t) - \nu(T)) \\
\nu^*(k) &= \frac{\gamma(a_1,k)^2\sigma_1^2}{a_1^2}(k - 2B(a_1,k) + B(2a_1,k)) + \frac{\gamma^2(a_2,k)\sigma_2^2}{a_2^2}(k - 2B(a_2,k) + B(2a_2,k)) \\
&\quad + \frac{2\rho\gamma(a_1,k)\gamma(a_2,k)\sigma_1\sigma_2}{a_1 a_2}(k - B(a_1,k) - B(a_2,k) + B(a_1+a_2,k)) \\
\gamma(a_j,k) &= 1 - \frac{B(a_j,k)}{k}, \qquad j = 1,2
\end{aligned}
$$

D Discrete Crediting Based on the k-year spot rate: two-factor HW Model

The CB plan with the k-year spot rate as the crediting rate and crediting frequency n times per year, has the valuation factor using the two-factor HW model

$$
V(t,T) = e^{\sum_{s=1}^{tn} \frac{r_k(\frac{s}{n})}{n}} e^{A^G(t,T) - B(a_1,T-t)x(t) - B(a_2,T-t)y(t)} e^{EG(t,T,n) + \frac{1}{2}VG(t,T,n)}
$$

where

$$EG(t,T,n) = \sum_{i=tn+1}^{Tn} \frac{-A^G\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} + \frac{B(a_1,k)}{nk}\left[x(t)e^{-a_1(\frac{i}{n}-t)} - M_x^T(t,\frac{i}{n})\right]$$
$$+ \frac{B(a_2,k)}{nk}\left[y(t)e^{-a_2(\frac{i}{n}-t)} - M_y^T(t,\frac{i}{n})\right]$$
$$VG(t,T,n) = \left(\frac{\sigma_1 B(a_1,k)}{nk}\right)^2 \sum_{j=tn+1}^{Tn}\left(\frac{e^{-\frac{a_1}{n}(j-1)} - e^{-a_1 T}}{e^{\frac{a_1}{n}}-1}\right)^2 \frac{e^{2a_1\frac{j}{n}} - e^{2a_1\frac{j-1}{n}}}{2a_1}$$
$$+ \left(\frac{\sigma_2 B(a_2,k)}{nk}\right)^2 \sum_{j=tn+1}^{Tn}\left(\frac{e^{-\frac{a_2}{n}(j-1)} - e^{-a_2 T}}{e^{\frac{a_2}{n}}-1}\right)^2 \frac{e^{2a_2\frac{j}{n}} - e^{2a_2\frac{j-1}{n}}}{2a_2}$$
$$2\rho\left(\frac{\sigma_1 B(a_1,k)}{nk}\right)\left(\frac{\sigma_2 B(a_2,k)}{nk}\right)$$
$$\sum_{j=tn+1}^{Tn}\left(\frac{e^{-\frac{a_1}{n}(j-1)} - e^{-a_1 T}}{e^{\frac{a_1}{n}}-1}\right)\left(\frac{e^{-\frac{a_2}{n}(j-1)} - e^{-a_2 T}}{e^{\frac{a_2}{n}}-1}\right)\frac{e^{(a_1+a_2)\frac{j}{n}} - e^{(a_1+a_2)\frac{j-1}{n}}}{a_1+a_2}$$
$$M_x^T(s,t) = \left(\frac{\sigma_1^2}{a_1} + \rho\frac{\sigma_1\sigma_2}{a_2}\right)B(a_1,t-s) - \frac{\sigma_1^2}{a_1}e^{-a_1(T-t)}B(2a_1,t-s) - \frac{\rho\sigma_1\sigma_2}{a_2}e^{-a_2(T-t)}B(a_1+a_2,t-s)$$
$$M_y^T(s,t) = \left(\frac{\sigma_2^2}{a_2} + \rho\frac{\sigma_1\sigma_2}{a_1}\right)B(a_2,t-s) - \frac{\sigma_2^2}{a_2}e^{-a_2(T-t)}B(2a_2,t-s) - \frac{\rho\sigma_1\sigma_2}{a_1}e^{-a_1(T-t)}B(a_1+a_2,t-s)$$