THE SHIFTED TURÁN SIEVE METHOD ON TOURNAMENTS II

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Abstract. In a previous work [Canad. Math. Bull., 62(4) (2019), 841-855], we developed the shifted Turán sieve method on a bipartite graph and applied it to problems on cycles in tournaments. More precisely, we obtained upper bounds for the number of tournaments which contain a small number of r-cycles. In this paper, we improve our sieve inequality and apply it to obtain an upper bound for the number of bipartite tournaments which contain a number of $2r$-cycles far from the average. We also provide the exact bound for the number of tournaments which contain few 3-cycles, using other combinatorial arguments.

1. Introduction

In 1934, Turán [9] gave a greatly simplified proof of a result of Hardy & Ramanujan [3] by proving that

$$\sum_{n \leq x} \big(\omega(n) - \log\log x\big)^2 \ll x \log\log x,$$

where $\omega(n)$ denotes the number of distinct prime factors of a natural number n. In particular, it implies that for most of large enough values of n, $\omega(n)$ approaches $\log\log n$, in the sense that the density of the numbers that does not approach tends to zero. In the same spirit as Turán's proof, Liu & Murty [6] introduced the Turán sieve method. This method was further generalized to a bipartite graph in [7] to investigate several combinatorial questions. In a previous paper [5], we constructed a shifted version and used it to bound the number of tournaments with few r-cycles.

Let $G = (A, B, E)$ be a bipartite graph with finite partite sets A, B and edge set E. For $a \in A$, $b \in B$, we write $a \sim b$ if there is an edge that joins a and b. For $a \in A$, $b, b_1, b_2 \in B$, we define

$$\omega(a) := \#\big\{b \in B \mid a \sim b\big\} \text{ the } \textit{degree} \text{ of } a,$$

$$\deg b := \#\big\{a \in A \mid a \sim b\big\} \text{ the } \textit{degree} \text{ of } b,$$

$$n(b_1, b_2) := \#\big\{a \in A \mid a \sim b_1 \text{ and } a \sim b_2\big\} \text{ the } \textit{number of common neighbors} \text{ of } b_1 \text{ and } b_2,$$

where both $\#S$ and $|S|$ denote the cardinality of the set S. Although $\omega(a)$ and $\deg b$ play the same role, we use different notations since the sets A and B are in general intrinsically distinct.

Liu & Murty proved the following:

Theorem 1.1. *([7, Theorem 1])*

$$\sum_{a \in A} \left(\omega(a) - \frac{1}{|A|} \sum_{b \in B} \deg b\right)^2 = \sum_{b_1, b_2 \in B} n(b_1, b_2) - \frac{1}{|A|} \left(\sum_{b \in B} \deg b\right)^2. \tag{1}$$

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The Turán sieve [7, Corollary 1] and its shifted version [5, Corollary 1.3] can be easily deduced from the theorem above.

A simple observation improves the shifted Turán sieve inequality. Let K be a non-empty subset of non-negative integers. We have that left hand side of Eq. (1) is bigger than or equal to $\#\{a \in A \mid \omega(a) \in K\} \cdot \min_{k\in K} \left(k - \frac{1}{|A|}\sum_{b\in B} \deg b\right)^2$, from where we deduce the new version of shifted Turán sieve:

Theorem 1.2.

$$\#\{a \in A \mid \omega(a) \in K\} \leq \frac{|A|^2 \sum_{b_1,b_2\in B} n(b_1,b_2) - |A|\Big(\sum_{b\in B} \deg b\Big)^2}{\min_{k\in K}\Big(|A|\cdot k - \sum_{b\in B}\deg b\Big)^2}. \tag{2}$$

A direct consequence of the above theorem is a Chebyshev-like inequality:

Corollary 1.3.

$$\#\left\{a \in A;\ \left|\omega(a) - \frac{\sum_{b\in B}\deg b}{|A|}\right| \geq t\right\} \leq \frac{1}{t^2}\left[\sum_{b_1,b_2\in B} n(b_1,b_2) - \frac{1}{|A|}\Big(\sum_{b\in B}\deg b\Big)^2\right].$$

1.1. **Tournaments and cycles.** Let $X_1, X_2, \ldots, X_t$ be t pairwise disjoint sets. A simple directed complete t-partite graph is called an $m_1 \times \cdots \times m_t$ *t-partite tournament* and we denote by $T_{m_1,\ldots,m_t}$ the collection of all such t-partite tournaments. In particular, if $t = 2$ then we have a *bipartite tournament*, and if $|X_i| = 1$ for every $1 \leq i \leq t$ then we just say *tournament*. Throughout the text, if (x, y) is a directed edge toward y, we write $x \to y$.

Now, we define the different kinds of cycles. For a t-partite tournament T, suppose that $V = \{x_1, x_2, \ldots, x_r\} \subset T$ is a set of vertices such that $x_1 \to x_2 \to \cdots \to x_r \to x_1$. We say that V is an *r-cycle* on T. We denote an r-cycle by (V, τ), where τ is a circular permutation of V.

An r-cycle (V, τ) is called a *restricted* r-cycle on a t-partite tournament T if every partite set $X_1, \ldots, X_t$ intersects V at most once. Otherwise we say that it is an *unrestricted* r-cycle. Since all cycles in tournaments are restricted and all cycles in bipartite tournaments are unrestricted, we drop the words “restricted” and “unrestricted” whenever there is no chance of misunderstanding.

In [5], the authors applied the shifted Turán sieve method to establish upper bounds for the number of tournaments and t-partite tournaments with a given number of vertices and exactly k restricted r-cycles, where k is fixed and $3 \leq r \leq t \leq n$. Joining the result obtained there with the sieve improvement, it is possible to obtain similar bounds, but allowing more cycles than a fixed quantity. The case of unrestricted cycles on t-partite tournaments is much harder than the restricted case, because the cycles can return to the same partition but cannot self-intersect, and this can cause a break in symmetry. Nevertheless, we were able to apply this method for bipartite tournaments (Theorem 1.4). The most general case, about unrestricted r-cycles on t-partite tournaments, still remains untouchable. On the other hand, several results focused on cycles of small length on t-partite tournaments are known, and for simplicity, the most studied case is $t = 2$. For instance, [8] contains a study on the average number of 4-cycles on random bipartite tournaments and a proof that the distribution of the 4-cycles satisfies the same conclusion as the Central Limit Theorem. Corollary 2.4 provides bounds for both upper and lower tails of the Gaussian curve in this case. We conjecture that, for $r \geq 3$, the distribution of the unrestricted $2r$-cycles in bipartite tournaments also satisfies the same conclusion as the Central Limit Theorem, as well as the distribution of restricted r-cycles in t-partite tournaments. For an overview on multipartite tournaments, see [10, 11] and references therein. The bound presented in Theorem 1.4 is general but not tight, in the sense that it holds for any cycle length, however the upper bound can be greatly improved. Obtaining the exact bound for the problems presented here and in [5] is an interesting and also a very difficult problem. Yet, using other combinatorial arguments, it was possible to give the precise bound for the number of tournaments with a "small" number of 3-cycles (Theorem 1.5).

Notations. As usual, we use the following asymptotic notations: If there exists a constant $C > 0$ such that $|f(x)| \le Cg(x)$ for every $x > 0$, we write $f(x) \ll g(x)$ or $f(x) = O(g(x))$. If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, we write $f(x) = o(g(x))$. If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$, we write $f(x) \sim g(x)$. Moreover, if $\{f(n)\}_{n\in\mathbb{N}}$ is a positive increasing sequence such that $\lim_{n\to\infty} f(n) = \infty$, we write $f(n) \nearrow \infty$.

In this paper, we will prove the following result, related to $2r$-cycles in bipartite tournaments:

Theorem 1.4. *Let $\varepsilon > 0$ and k, r be positive integers with $2 \le r \le \left(\frac{1}{4\log 2} - \varepsilon\right)\log(\min\{m,n\})$.*

(i) If $k < \frac{(r-1)!r!}{4^r}\binom{m}{r}\binom{n}{r}$, then

$$\#\{T \in T_{m,n} \mid T \text{ contains} \le k \ 2r\text{-cycles}\} \le \frac{2^{mn}\binom{m}{r}\binom{n}{r}m^{r-3}n^{r-3}r!^2\left\{2(m^2n + mn^2) + O\left(2^{4r}(m^2+n^2)\right)\right\}}{\left[4^r k - (r-1)!r!\binom{m}{r}\binom{n}{r}\right]^2}.$$

(ii) If $k > \frac{(r-1)!r!}{4^r}\binom{m}{r}\binom{n}{r}$, then

$$\#\{T \in T_{m,n} \mid T \text{ contains} \ge k \ 2r\text{-cycles}\} \le \frac{2^{mn}\binom{m}{r}\binom{n}{r}m^{r-3}n^{r-3}r!^2\left\{2(m^2n + mn^2) + O\left(2^{4r}(m^2+n^2)\right)\right\}}{\left[4^r k - (r-1)!r!\binom{m}{r}\binom{n}{r}\right]^2}.$$

The implicit constants in the O-terms are absolute and refer to $\min\{m,n\} \to \infty$.

The sets A and B are respectively chosen in the shifted Turán sieve to be all bipartite tournaments on $m + n$ vertices and all $2r$-cycles. The main technical difficult in applying this method lies in the sum of $n(b_1, b_2)$, that is, to count the number of tournaments $a \in A$ that associate to both $2r$-cycles $b_1, b_2 \in B$. For this, we need to first discuss how cycles b_1 and b_2 intersect each other, and this intersection can have a very complicated structure. In this paper, we use a counting method developed in [5] for estimating the sum of $n(b_1, b_2)$, that consists of "omit some existing cases" and "include some non-existing cases" to get the expected main contribution. Then we compare the "under-counting" and "over-counting" of the main contribution to get the correct estimate. Such an approach greatly simplifies many of our calculations. For example, in Section 2, for the case (ii), since one can argue that the numbers of under-counting and over-counting are the same, it is only required to estimate the expected main term.

Theorem 1.4 is useful only when $\left|k - \frac{(r-1)!r!}{4^r}\binom{m}{r}\binom{n}{r}\right| > \frac{(mn)^{r-1}\sqrt{2(m+n)}}{4^r}$, i.e., when k is "far" from the average (see Corollary 2.2). We also notice that the shifted Turán sieve method yields good upper bounds in terms of r, in the sense that it holds for every fixed $r \ge 2$, what had never been done before in the literature (except for r small).

Apart from the sieve improvement, the bounds presented in Theorem 1.4 are somewhat similar to those presented in [5] for r-cycles in tournaments. However, for small values of k, the bounds obtained there are not too tight, in view of the following result.

Theorem 1.5. *Let $k \ge 0$ be an integer. Then the number of tournaments with n vertices and exactly k 3-cycles is*

$$\frac{n!}{k! \cdot 3^k} \cdot \left[n^k + O\left((3k-2)^{3k} k! \cdot n^{k-1}\right)\right].$$

In particular, for every $\varepsilon > 0$, if $k \le (\log n)^{1-\varepsilon}$ then this number is $\sim \frac{n! \cdot n^k}{k! \cdot 3^k}$.

The latter is the only calculation that does not use shifted Turán sieve method in this paper. The number of tournaments in T_n without 3-cycles is well-known: $n!$. In Proposition 3.1, we find the exact number of tournaments with exactly one and exactly two 3-cycles. The exact bound given by theorem above is far from the upper bound given in [5, Theorem 1.4], and indicates that all other bounds such as [5, Theorem 1.5] and Theorem 1.4 are still far to be sharp. This explain why we do not strive so much to minimize the error terms.

The paper is organized as follows. In Section 2, we prove Theorem 1.4 and present some of its consequences. In Section 3, we prove Theorem 1.5.

2. $2r$-cycles on bipartite tournaments

In this section, we investigate $2r$-cycles in bipartite tournaments, proving Theorem 1.4. We also provide some consequences of this result.

Notation. We write $(x_1, \ldots, x_n) \leftrightarrow (y_1, \ldots, y_n)$ when $(x_1, \ldots, x_n)$ is a permutation of $(y_1, \ldots, y_n)$.

Let $r \geq 2$, $A = T_{m,n}$, B be the set of all $2r$-cycles, and P and Q be two disjoint sets with $|P| = m$ and $|Q| = n$. The sets P and Q will be partitions of a bipartite graph in A. Suppose we pick $\{p_{i_1}, \ldots, p_{i_r}\} \subseteq P$ and $\{q_{j_1}, \ldots, p_{j_r}\} \subseteq Q$. Note that a $2r$-cycle on these vertices means that

$$p_{i_1} \to q_{b_1} \to p_{a_1} \to q_{b_2} \to p_{a_2} \to \cdots \to p_{a_{r-1}} \to q_{b_r} \to p_{i_1}$$

where $(a_1, \ldots, a_{r-1}) \leftrightarrow (i_2, \ldots, i_r)$, and $(b_1, \ldots, b_r) \leftrightarrow (j_1, \ldots, j_r)$. Hence given $\{p_{i_1}, \ldots, p_{i_r}\} \subseteq P$ and $\{q_{j_1}, \ldots, p_{j_r}\} \subseteq Q$ there are $(r-1)!r!$ possible $2r$-cycles on these vertices. This means that

$$|A| = 2^{mn} \quad \text{and} \quad |B| = (r-1)!r!\binom{m}{r}\binom{n}{r}.$$

For a bipartite tournament $a \in A$ and a $2r$-cycle $b \in B$, we connect $a \sim b$ when a contains b. Let $V_{P,b} \subseteq P$ and $V_{Q,b} \subseteq Q$ be the sets of vertices of a fixed $2r$-cycle b in P and Q, respectively. Notice that b is determined by a circular permutation σ_b of $V_{P,b}$ and a permutation τ_b of $V_{Q,b}$, thus we write $b = (V_{P,b}, V_{Q,b}, \sigma_b, \tau_b) \in B$. Hence,

$$\begin{aligned}\omega(a) &= \#\{2r\text{-cycles } b \text{ contained in } a\},\\ \deg b &= \#\{a \in A \mid a \text{ contains } b = (V_{P,b}, V_{Q,b}, \sigma_b, \tau_b)\}.\end{aligned}$$

Since b determines $2r$ directed edges of a, there are 2^{mn-2r} other choices for the remaining $mn - 2r$ edges, hence $\deg b = 2^{mn-2r}$ for all $b \in B$ and

$$\sum_{b\in B} \deg b = (r-1)!r! \cdot 2^{mn-2r}\binom{m}{r}\binom{n}{r}.$$

Let $b_1 = (V_{P,b_1}, V_{Q,b_1}, \sigma_{b_1}, \tau_{b_1})$ and $b_2 = (V_{P,b_2}, V_{Q,b_2}, \sigma_{b_2}, \tau_{b_2}) \in B$. Consider

$$n(b_1, b_2) = \#\{a \in A \mid a \text{ contains both } b_1 \text{ and } b_2\}.$$

Let $M(r_1, r_2)$ denote the number of pairs $(b_1, b_2) \in B^2$ such that $|V_{P_{b_1}} \cap V_{P_{b_2}}| = r_1$ and $|V_{Q_{b_1}} \cap V_{Q_{b_2}}| = r_2$. As we are interested in an asymptotic upper bound, we will consider only enough cases for the pair (r_1, r_2) to obtain the leading terms of the numerator of Eq. (2). We have some cases:

(i) Case $r_1 = 0$ or $r_2 = 0$. Then b_1 and b_2 together determine $4r$ directed edges and there are $(r-1)!r!$ ways to choose each cycle. Therefore we have

$$M(0, r_2) = [(r-1)!r!]^2 \binom{m}{r}\binom{n}{r}\binom{m-r}{r}\binom{r}{r_2}\binom{n-r}{r-r_2} \quad \text{and} \quad n(b_1, b_2) = 2^{mn-4r},$$
$$M(r_1, 0) = [(r-1)!r!]^2 \binom{m}{r}\binom{n}{r}\binom{n-r}{r}\binom{r}{r_1}\binom{m-r}{r-r_1} \quad \text{and} \quad n(b_1, b_2) = 2^{mn-4r},$$

which implies

$$\sum_{\substack{b_1, b_2 \in B \\ r_1 = 0}} n(b_1, b_2) = 2^{mn-4r}[(r-1)!r!]^2 \binom{m}{r}\binom{n}{r}\binom{m-r}{r}\binom{n}{r},$$
$$\sum_{\substack{b_1, b_2 \in B \\ r_1 > 0, r_2 = 0}} n(b_1, b_2) = 2^{mn-4r}[(r-1)!r!]^2 \binom{m}{r}\binom{n}{r}\binom{n-r}{r}\left[\binom{m}{r} - \binom{m-r}{r}\right].$$

(ii) Case $(r_1, r_2) = (1, 1)$. There are $\binom{m}{r}\binom{n}{r}\binom{r}{1}^2\binom{m-r}{r-1}\binom{n-r}{r-1}$ ways to choose the vertices for b_1 and b_2. Fix a choice of vertices and suppose that $p \in P$ and $q \in Q$ are the only shared vertices between b_1 and b_2.

If b_1 and b_2 share no edges then the two cycles together determine $4r$ directed edges and $n(b_1, b_2) = 2^{mn-4r}$, and if b_1 and b_2 share one edge then $n(b_1, b_2) = 2^{mn-4r+1}$. Since $r_1 = r_2 = 1$ there is at most one shared edge between b_1 and b_2. Let N_0, N_1 be the number of ways to choose the cycles on the vertices of b_1 and b_2 *independently* so that they have 0 and 1 edges in common, respectively. There are $(r-1)!r!$ possibilities to create each cycle. Among these possibilities:

(a) There are N_0 possibilities such that b_1 and b_2 do not both contain a directed edge in $G_{p,q} = \{p \to q, q \to p\}$, since in this case b_1 and b_2 have no edges in common and the union of the two cycles represents a valid subgraph of the bipartite tournament.

(b) There are N_1 possibilities where b_1 and b_2 both contain $p \to q$ or $q \to p$, since the union of these two cycles represents a valid subgraph of the bipartite tournament. There are an additional N_1 possibilities where b_1 contains one of the edges $p \to q$ and $q \to p$ and b_2 contains the other, since the union of b_1 with the inverse of b_2 (where the orientation of every edge in b_2 is switched) represents a valid subgraph of the bipartite tournament.

Since these cases are exhaustive, we have $N_0 + 2N_1 = [(r-1)!r!]^2$ and then

$$\begin{aligned}\sum_{\substack{b_1, b_2 \in B \\ r_1 = r_2 = 1}} n(b_1, b_2) &= (2^{mn-4r}N_0 + 2^{mn-4r+1}N_1)\binom{m}{r}\binom{n}{r}\binom{r}{1}^2\binom{m-r}{r-1}\binom{n-r}{r-1} \\ &= 2^{mn-4r}[(r-1)!r!]^2\binom{m}{r}\binom{n}{r}\binom{r}{1}^2\binom{m-r}{r-1}\binom{n-r}{r-1}.\end{aligned}$$

(iii) Case $(r_1, r_2) = (1, 2)$. The case for $(r_1, r_2) = (2, 1)$ is analogous by interchanging P and Q. There are $\binom{m}{r}\binom{n}{r}\binom{r}{1}\binom{r}{2}\binom{m-r}{r-1}\binom{n-r}{r-2}$ ways to choose the vertices for b_1 and b_2. Fix a choice of vertices and suppose that $p \in P$ and $q, q' \in Q$ are the shared vertices between b_1 and b_2.

If b_1 and b_2 share no edges then the two cycles together determine $4r$ directed edges and $n(b_1, b_2) = 2^{mn-4r}$, if b_1 and b_2 share one edge then $n(b_1, b_2) = 2^{mn-4r+1}$, and if b_1 and b_2 share two edges then $n(b_1, b_2) = 2^{mn-4r+2}$. Let M_0, M_1, M_2 be the number of ways to choose the cycles on the vertices of b_1 and b_2 *independently* so that they have $0, 1$ and 2 edges in common, respectively. There are $(r-1)!r!$ ways to create each $2r$-cycle. Among these possibilities:

(a) There are M_0 possibilities such that b_1 and b_2 do not both contain an edge in $G_{p,q} = \{p \to q, q \to p\}$ or $G_{p,q'} = \{p \to q', q' \to p\}$, since in this case b_1 and b_2 have no edges in common and the union of the two cycles represents a valid subgraph of the bipartite tournament.

(b) There are M_1 possibilities where b_1 and b_2 both contain the same directed edge in $G_{p,q}$ or $G_{p,q'}$, since the union of these two cycles represents a valid subgraph of the bipartite tournament. There are an additional M_1 possibilities where b_1 contains one of the edges in $G_{p,q}$ or $G_{p,q'}$ and b_2 contains the other, since the union of b_1 with the inverse of b_2 (where the orientation of every edge in b_2 is switched) represents a valid subgraph of the bipartite tournament.

(c) There are M_2 possibilities where b_1 and b_2 both contain $q' \to p \to q$ or $q \to p \to q'$, since the union of these two cycles represents a valid subgraph of the bipartite tournament. There are an additional M_2 possibilities where b_1 contains one of $q' \to p \to q$ and $q \to p \to q'$ and b_2 contains the other, since the union of b_1 with the inverse of b_2 (where the orientation of every edge in b_2 is switched) represents a valid subgraph of the bipartite tournament.

Since these cases are exhaustive, we have $M_0 + 2M_1 + 2M_2 = [(r-1)!r!]^2$ and then

$$\begin{aligned}\sum_{\substack{b_1,b_2\in B\\(r_1,r_2)=(1,2)}} n(b_1,b_2) &= 2^{mn-4r}(M_0+2M_1+4M_2)\binom{m}{r}\binom{n}{r}\binom{r}{1}\binom{r}{2}\binom{m-r}{r-1}\binom{n-r}{r-2}\\ &= 2^{mn-4r}\left\{[(r-1)!r!]^2+2M_2\right\}\binom{m}{r}\binom{n}{r}\binom{r}{1}\binom{r}{2}\binom{m-r}{r-1}\binom{n-r}{r-2}.\end{aligned}$$

Now we need to compute M_2. Note that if b_1 and b_2 share two directed edges then they both must contain either $q \to p \to q'$ or $q' \to p \to q$. For each of b_1 and b_2 we can contract these two common edges to a single vertex in Q. Since there are $(r-2)!(r-1)!$ possible $2r$-cycles on partite sets of size $r-1$ and $r-1$, there are $[(r-2)!(r-1)!]^2$ ways to choose the $2r$-cycles b_1 and b_2 using the respective vertices. Therefore $M_2 = 2[(r-2)!(r-1)!]^2$ and

$$\sum_{\substack{b_1,b_2\in B\\(r_1,r_2)=(1,2)}} n(b_1,b_2) = 2^{mn-4r}[(r-2)!(r-1)!]^2\left[(r-1)^2r^2+4\right]\binom{m}{r}\binom{n}{r}\binom{r}{1}\binom{r}{2}\binom{m-r}{r-1}\binom{n-r}{r-2}.$$

A similar result is valid for $(r_1,r_2)=(2,1)$.

Denote the sum of the $n(b_1,b_2)$'s in (i), (ii), (iii) (and symmetrical) by S, and let $W = 2^{mn-4r}\binom{m}{r}\binom{n}{r}[(r-1)!r!]^2$, so that

$$\begin{aligned}S = W\cdot\Bigg\{&\binom{m-r}{r}\binom{n}{r}+\binom{n-r}{r}\left[\binom{m}{r}-\binom{m-r}{r}\right]+r^2\binom{m-r}{r-1}\binom{n-r}{r-1}\\ &+\frac{r^4-2r^3+r^2+4}{2(r-1)}\left[\binom{m-r}{r-1}\binom{n-r}{r-2}+\binom{m-r}{r-2}\binom{n-r}{r-1}\right]\Bigg\}.\end{aligned}$$

If $r_1+r_2 \geq 4$ with $r_1,r_2>0$, then b_1 and b_2 determine at least $2r$ directed edges, and there are at most $[(r-1)!r!]^2$ ways to complete the cycles b_1 and b_2, hence we have

$$\begin{aligned}\sum_{\substack{b_1,b_2\in B\\r_1+r_2\geq 4\\r_1,r_2>0}} n(b_1,b_2) &\leq 2^{mn-2r}[(r-1)!r!]^2\binom{m}{r}\binom{n}{r}\binom{r}{r_1}\binom{r}{r_2}\binom{m-r}{r-r_1}\binom{n-r}{r-r_2}\\ &\leq 2^{2r}W\binom{r}{r_1}\binom{r}{r_2}\left[\prod_{i=0}^{r-r_1-1}(m-r-i)\right]\left[\prod_{j=0}^{r-r_2-1}(n-r-i)\right]\\ &\leq 2^{4r}W\left[\prod_{i=0}^{r-r_1-1}(m-r-i)\right]\left[\prod_{j=0}^{r-r_2-1}(n-r-i)\right].\end{aligned}$$

Note that since $r_1, r_2 > 0$, we have $m^{r-r_1}n^{r-r_2} \leq \max(m^{r-1}n^{r-3}, m^{r-2}n^{r-2}, m^{r-3}n^{r-1})$. Since $m^{r-1}n^{r-3} + m^{r-3}n^{r-1} \geq 2m^{r-2}n^{r-2}$, for every pair (r_1, r_2) satisfying $r_1 + r_2 \geq 4$ and $r_1, r_2 > 0$ we have:

$$\sum_{\substack{b_1, b_2 \in B \\ r_1 + r_2 \geq 4 \\ r_1, r_2 > 0}} n(b_1, b_2) = W \cdot O\left(2^{4r}(m^{r-1}n^{r-3} + m^{r-3}n^{r-1})\right)$$

so that

$$\sum_{\substack{r_1+r_2 \geq 4 \\ r_1, r_2 > 0}} \sum_{\substack{b_1, b_2 \in B \\ (r_1, r_2)}} n(b_1, b_2) \leq r^2 \max_{\substack{r_1+r_2 \geq 4 \\ r_1, r_2 > 0}} \sum_{\substack{b_1, b_2 \in B \\ (r_1, r_2)}} n(b_1, b_2) = W \cdot O(2^{4r}r^2(m^{r-1}n^{r-3} + m^{r-3}n^{r-1}).$$

It follows that

$$\sum_{b_1, b_2 \in B} n(b_1, b_2) = S + W \cdot O(2^{4r}r^2(m^{r-1}n^{r-3} + m^{r-3}n^{r-1})).$$

We now return to computing the numerator of Eq. (2) (divided by $|A|^2$):

$$\begin{aligned}
\sum_{b_1, b_2 \in B} n(b_1, b_2) - \frac{1}{|A|}\left(\sum_{b \in B} \deg b\right)^2 &= \sum_{b_1, b_2 \in B} n(b_1, b_2) - W\binom{m}{r}\binom{n}{r} \\
&= W\left[\frac{S}{W} - \binom{m}{r}\binom{n}{r} + O\left(2^{4r}r^2(m^{r-1}n^{r-3} + m^{r-3}n^{r-1})\right)\right].
\end{aligned}$$

We will need the following identities to compare the degree $2r$, $2r-1$, and $2r-2$ terms of the polynomial in S with the corresponding terms of the polynomial $\binom{m}{r}\binom{n}{r}$.

Lemma 2.1. *Let $r \in \mathbb{N}$, and let $p(m) = \binom{m}{r} - \binom{m-r}{r}$. Denote the coefficient of m^k in p by $p^{(k)}$. Then p is a degree $r-1$ polynomial with*

$$p^{(r-1)} = \frac{1}{r!} \cdot r^2, \qquad p^{(r-2)} = -\frac{1}{r!} \cdot \frac{r^2(r-1)(2r-1)}{2}, \qquad |p^{(r-3)}| \leq \frac{1}{r!} \cdot 8r^6$$

Proof: The first two equalities are easily shown by the relations between coefficients and roots of polynomials. For the inequality involving $p^{(r-3)}$, we note that the coefficient of m^{r-3} in $r!\binom{m}{r}$ and $r!\binom{m-r}{r}$ are both negative, and the former is less in absolute value than the latter. The coefficient of m^{r-3} in $r!\binom{m-r}{r}$ is the sum of the products of the numbers $\{-r, -(r+1), \ldots, -(2r-1)\}$ taken three at a time, and this is at least $-r^3(2r-1)^3 > -8r^6$. The result follows. ☐

Using the lemma above, we see that

$$\begin{aligned}
D_1 &= \binom{m}{r}\binom{n}{r} - \binom{m-r}{r}\binom{n}{r} - \binom{n-r}{r}\left[\binom{m}{r} - \binom{m-r}{r}\right] \\
&= \left[\binom{m}{r} - \binom{m-r}{r}\right]\left[\binom{n}{r} - \binom{n-r}{r}\right] \\
&= \frac{1}{r!^2}\left[r^2 m^{r-1} - \frac{r^2(r-1)(2r-1)}{2}m^{r-2} + O(r^7 m^{r-3})\right] \\
&\quad \times \left[r^2 n^{r-1} - \frac{r^2(r-1)(2r-1)}{2}n^{r-2} + O(r^7 n^{r-3})\right]
\end{aligned}$$

Expanding the other terms of S/W, we have

$$\begin{aligned}\binom{m-r}{r-1}\binom{n-r}{r-1} &= \frac{1}{r!^2}\left[rm^{r-1} - \frac{r(3r-2)(r-1)}{2}m^{r-2} + O(r^6m^{r-3})\right] \\ &\quad\times\left[rn^{r-1} - \frac{r(3r-2)(r-1)}{2}n^{r-2} + O(r^6n^{r-3})\right], \\ \binom{m-r}{r-1}\binom{n-r}{r-2} &= \frac{1}{r!^2}\left[rm^{r-1} - \frac{r(3r-2)(r-1)}{2}m^{r-2} + O(r^6m^{r-3})\right]\left[(r-1)rn^{r-2} + O(r^5n^{r-3})\right], \\ \binom{m-r}{r-2}\binom{n-r}{r-1} &= \frac{1}{r!^2}\left[rn^{r-1} - \frac{r(3r-2)(r-1)}{2}n^{r-2} + O(r^6n^{r-3})\right]\left[(r-1)rm^{r-2} + O(r^5m^{r-3})\right].\end{aligned}$$

Therefore

$$\begin{aligned}D_2 &= D_1 - r^2\binom{m-r}{r-1}\binom{n-r}{r-1} \\ &= \frac{1}{r!^2}\left[\frac{r^4(r-1)^2}{2}\left(m^{r-2}n^{r-1} + m^{r-1}n^{r-2}\right) + O\left(r^9(m^{r-1}n^{r-3} + m^{r-3}n^{r-1})\right)\right].\end{aligned}$$

Finally, we have

$$\begin{aligned}\binom{m}{r}\binom{n}{r} - \frac{S}{W} &= D_2 - \frac{r^4-2r^3+r^2+4}{2(r-1)}\left[\binom{m-r}{r-1}\binom{n-r}{r-2} + \binom{m-r}{r-2}\binom{n-r}{r-1}\right] \\ &= D_2 - \frac{1}{r!^2}\left[\frac{1}{2}(r^6-2r^5+r^4+4r^2)(m^{r-1}n^{r-2}+m^{r-2}n^{r-1})\right. \\ &\qquad\left. + O\left(r^9(m^{r-1}n^{r-3}+m^{r-3}n^{r-1})\right)\right] \\ &= \frac{1}{r!^2}\left[-2r^2(m^{r-1}n^{r-2}+m^{r-2}n^{r-1})\right] + O\left(r^9(m^{r-1}n^{r-3}+m^{r-3}n^{r-1})\right).\end{aligned}$$

We now apply Theorem 1.2 with the sets $K = \{\ell \in \mathbb{Z} \mid 0 \le \ell \le k\}$ for (i) and $K = \{\ell \in \mathbb{Z} \mid \ell \ge k\}$ for (ii), and combine the above results to obtain the desired result, but before we shall notice that since $r \le \left(\frac{1}{4\log 2} - \varepsilon\right)\log(\min\{m,n\})$, the error term $O(2^{4r}(m^2+n^2))$ is negligible compared to the main term $m^2n + mn^2$. In fact,

$$\log\left(\frac{2^{4r}(m^2+n^2)}{m^2n+mn^2}\right) \le \log\left(\frac{2^{4r}}{\min\{m,n\}}\right) \to -\infty$$

as $\min\{m,n\} \to \infty$, thus $\frac{2^{4r}(m^2+n^2)}{m^2n+mn^2} \to 0$. Therefore, we obtain Theorem 1.4. □

Theorem 1.4 and its proof bring up many consequences, such as:

Corollary 2.2. *The average number of $2r$-cycles in a bipartite tournament in $T_{m,n}$ is $\frac{(r-1)!r!}{4^r}\binom{m}{r}\binom{n}{r}$.*

Corollary 2.3. *Fixed $r \ge 2$ and $\varepsilon > 0$, let $k \ge 0$ be an integer. As either $m \to \infty$ or $n \to \infty$, we have that:*

(i) If $k < (1-\varepsilon)\frac{(r-1)!r!}{4^r}\binom{m}{r}\binom{n}{r}$, then

$$\#\{T \in T_{m,n} \mid T \text{ contains} \le k\ 2r\text{-cycles}\} \ll 2^{mn} \cdot \frac{m+n}{m^2n^2}.$$

In particular, the proportion of bipartite tournaments containing at most k $2r$-cycles approaches 0.

(ii) If $k > (1+\varepsilon)\frac{(r-1)!r!}{4^r}\binom{m}{r}\binom{n}{r}$*, then*

$$\#\{T \in T_{m,n} \mid T \textit{ contains} \geq k \ 2r\textit{-cycles}\} \ll 2^{mn} \cdot \frac{m+n}{m^2n^2}.$$

In particular, the proportion of bipartite tournaments containing at most k $2r$*-cycles approaches* 1.

In the case of $r = 2$, it is possible to compute the exact constants given by Theorem 1.4.

Corollary 2.4. *(i) If* $0 \leq k < \frac{1}{8}\binom{m}{2}\binom{n}{2}$*, we have*

$$\#\{T \in T_{m,n} \mid T \textit{ contains} \leq k \ 4\textit{-cycles}\} \leq 2^{mn}\left\{\frac{\binom{m}{2}\binom{n}{2}(2m+2n-1)}{\left[8k-\binom{m}{2}\binom{n}{2}\right]^2}\right\}.$$

(ii) If $k > \frac{1}{8}\binom{m}{2}\binom{n}{2}$*, we have*

$$\#\{T \in T_{m,n} \mid T \textit{ contains} \geq k \ 4\textit{-cycles}\} \leq 2^{mn}\left\{\frac{\binom{m}{2}\binom{n}{2}(2m+2n-1)}{\left[8k-\binom{m}{2}\binom{n}{2}\right]^2}\right\}.$$

Remark 2.5. *Moon & Moser* [8] *proved that*

$$\#\left\{T \in T_{m,n} \mid T \textit{ contains} > \left\lfloor \frac{m^2}{4} \right\rfloor \cdot \left\lfloor \frac{n^2}{4} \right\rfloor \ 4\textit{-cycles}\right\} = 0.$$

3. Tournaments with few 3-cycles

One important question to consider is the tightness of the upper bounds provided by the Turán sieve for $\#\{a \in A \mid \omega(a) = k\}$, or even when this value is non-zero.

For instance, Gutin [2] calculated the number of unlabeled multipartite tournaments with zero and with one cycle, and also provided the number of unlabeled bipartite tournaments with exactly k cycles, which are pairwise vertex-disjoint. Bollobás, Frank & Karoński [1] calculated the probability of a random bipartite tournament be acyclic. Kendall & Babington Smith [4] proved that

$$\#\left\{T \in T_n \mid T \text{ contains} > \begin{cases} \dfrac{n^3-n}{4} & \text{if } n \text{ is odd} \\ \dfrac{n^3-4n}{4} & \text{if } n \text{ is even} \end{cases} \text{4-cycles}\right\} = 0.$$

So we consider

$$S_{n,r,k} := \#\{T \in T_n \mid T \text{ contains exactly } k \ r\text{-cycles}\},$$

and

$$M_{n,r} := \max\{k \geq 0 \mid S_{n,r,k} \neq 0\}.$$

For $r = 3$, we see that [5, Theorem 1.4 and Remark after Corollary 2.1] implies a tight bound on a positive proportion of the possible values of k, in the sense that if

$$A_{n,r} := \left\{0 \leq k \leq M_{n,3} \mid S_{n,3,k} = 2^{\binom{n}{2}} \cdot o\left(\frac{1}{n^3}\right)\right\},$$

then

$$\begin{aligned} 2^{\binom{n}{2}} = \sum_{k=0}^{M_{n,3}} S_{n,3,k} &= \sum_{k \in A_{n,3}} S_{n,3,k} + \sum_{k \notin A_{n,3}} S_{n,3,k} \\ &= 2^{\binom{n}{2}}\left\{|A_{n,3}| \cdot o\left(\frac{1}{n^3}\right) + (M_{n,3}+1-|A_{n,3}|) \cdot O\left(\frac{1}{n^3}\right)\right\}, \end{aligned}$$

which implies that $M_{n,3}+1-|A_{n,3}| \geq Cn^3$ for some constant $C>0$ and sufficiently large n.

For the values of $r>3$, we conjecture that $S_{n,r,k}=0$ provided $k>Cn^r$ for some constant $C>\frac{1}{r\cdot 2^r}$. For a while, using the argument of [5, Theorem 1.4] with the bound of Corollary 1.3, it is possible to prove that if $f(n)\nearrow\infty$, then the proportion of tournaments with more than $n^r\cdot f(n)$ r-cycles approaches 0, i.e.,

$$\lim_{n\to\infty}\frac{1}{2^{\binom{n}{2}}}\sum_{k>n^r\cdot f(n)} S_{n,r,k}=0.$$

On the other hand, we can explicitly compute $S_{n,3,k}$ for small values of k, say $k\leq(\log n)^{1-\varepsilon}$. For example, the tournaments which do not contain 3-cycles are exactly the *transitive tournaments* (i.e., if $x\to y$ and $y\to z$ then $x\to z$). These tournaments are precisely the ones with no cycles of any length. Therefore, for $n\geq 3$,

$$S_{n,3,0}=n!. \tag{3}$$

From the latter, we can derive the following results:

Proposition 3.1. *We have* $S_{n,3,1}=n!\cdot\frac{n-2}{3}$ *for* $n\geq 3$ *and* $S_{n,3,2}=n!\cdot\frac{n^2-7}{18}$ *for* $n\geq 4$.

Proof: First we compute $S_{n,3,1}$ for $n\geq 3$. Given a set V of n vertices, there are $2\binom{n}{3}$ ways to choose 3 vertices in V and form a 3-cycle among those 3 vertices. Denote the cycle as $a\to b\to c\to a$. In order to obtain a tournament with exactly one 3-cycle, there cannot be any cycles in the remaining directed edges. Hence there are $(n-2)!$ ways to pick the orientation of the directed edges between $V\backslash\{b,c\}$ so that there are no 3-cycles between these edges.

Fix one of these orientations. Let $d\in V\backslash\{a,b,c\}$. If $a\to d$, then it is forced that $c\to d$ and $b\to d$. If $a\leftarrow d$, then similarly we must have $b\leftarrow d$ and $c\leftarrow d$. So for all $d\in V\backslash\{a,b,c\}$, the directed edge between a,d determines the directed edge between b,d and c,d. It follows that $S_{n,3,1}=2\binom{n}{3}\cdot(n-2)!=n!\cdot\frac{n-2}{3}$, as desired.

Next we compute $S_{n,3,2}$ for $n\geq 4$. There are three possibilities for two 3-cycles:

(1) **Case 1: The 3-cycles are vertex-disjoint.** Given a set V of n vertices, there are $\frac{1}{2}\binom{n}{3}\binom{n-3}{3}$ ways to choose two subsets of 3 vertices, and 4 ways to form two 3-cycles from these vertices, giving a total of $\frac{1}{2}\cdot 4\cdot\binom{n}{3}\binom{n-3}{3}$ distinct ways to choose two 3-cycles in V. Denote the two 3-cycles as $a\to b\to c\to a$ and $a'\to b'\to c'\to a'$. In order to obtain a tournament in T_n with exactly two 3-cycles, there cannot be any cycles in the remaining directed edges. By transitivity, there are $(n-4)!$ ways to pick the orientation of the directed edges between $V\backslash\{b,c,b',c'\}$ so that there are no 3-cycles between these edges.
   Fix one of these orientations, and consider the edge between a and a'. If $a\to a'$, then to get a tournament with exactly two 3-cycles we must have $a\to b'$ and $a\to c'$. Then $c\to a$ forces $c\to c'$, $c\to b'$, $c\to a'$, and finally $b\to c$ forces $b\to c'$, $b\to b'$, $b\to a'$. The case $a\leftarrow a'$ is similar, and hence the directed edges between a,b,c,a',b',c' are completely determined by the cycles $a\to b\to c\to a$, $a'\to b'\to c'\to a'$, and the edge between a and a'. Let $d\in V\backslash\{a,b,c,a',b',c'\}$. By the same argument than $S_{n,3,1}$, the directed edges between a,b,c,a',b',c' and d are completely determined by the edge between a and d and the edge between a' and d. It follows that the number of tournaments in T_n with exactly two 3-cycles such that the cycles are disjoint is $2\binom{n}{3}\binom{n-3}{3}(n-4)!=n!\cdot\frac{n^2-9n+20}{18}$.
(2) **Case 2: The 3-cycles share a common edge.** Given a set V of n vertices, there are $\binom{n}{4}\binom{4}{2}$ ways to choose the 4 vertices and, among these, 2 vertices to form the common edge, and there are 2 ways to form two 3-cycles from these 4 vertices. Label the common edge by $a\to b$ and denote the two cycles by $a\to b\to c\to a$ and $a\to b\to c'\to a$. In order to obtain exactly two 3-cycles, there cannot be any cycles in the remaining directed edges.

By transitivity, there are $(n-3)!$ ways to pick the orientation of the directed edges between $V\backslash\{b,c,c'\}$ so that there are no 3-cycles between these edges.

Fix one of these orientations. Let $d \in V\backslash\{a,b,c,c'\}$. By the argument in Case 1, the directed edges between a,b,c,c' and d are completely determined, therefore the number of tournaments with exactly two 3-cycles such that the cycles share a common edge is $2\binom{n}{4}\binom{4}{2}(n-3)! = n! \cdot \frac{n-3}{2}$.

(3) **Case 3: The 3-cycles share only one common vertex.** We claim that in this arrangement, the tournament must have at least three 3-cycles. Suppose that vertex a is common, and the two cycles are $a \to b \to c \to a$ and $a \to b' \to c' \to a$, and assume for a contradiction that these are the only 3-cycles in the tournament. Since $c \to a \to b'$, we must have $c \to b'$ and since $c' \to a \to b$, we must have $c' \to b$. If $c \to c'$, then $c' \to b \to c \to c'$ is a 3-cycle and if $c' \to c$, then $c \to b' \to c' \to c$ is a 3-cycle, contradiction. Hence there are no tournaments with exactly two 3-cycles such that the cycles share only one common vertex.

Summing up, we obtain $S_{n,3,2} = n! \cdot \frac{n^2-7}{18}$, as desired. □

For a small value of k (compared to n), we can find an asymptotic formula for $S_{n,3,k}$ (Theorem 1.5). Since the number of possible arrangements of the 3-cycles grows quickly for large k, it would be impractical to find a precise formula for $S_{n,3,k}$.

3.1. **Proof of Theorem 1.5.** The cases $k=0$, $k=1$ and $k=2$ are already settled (by Eq. (3) and Proposition 3.1), therefore we consider $k \geq 3$.

(1) **Case 1: The 3-cycles are vertex-disjoint.** Given a set V of n vertices, there are $\frac{1}{k!}\binom{n}{3}\binom{n-3}{3}\dots\binom{n-3k+3}{3}$ ways to choose k sets of 3 vertices, and 2^k ways to form k 3-cycles from these vertices, giving a total of $\frac{1}{k!}\cdot 2^k\binom{n}{3}\binom{n-3}{3}\dots\binom{n-3k+3}{3}$ distinct ways to choose k 3-cycles among n vertices since each combination of k 3-cycles is counted exactly $k!$ times. Denote the k 3-cycles as $C_i = \{a_i \to b_i \to c_i \to a_i\}$ for $1 \leq i \leq k$. In order to obtain a tournament with exactly k 3-cycles, there cannot be any cycles in the remaining directed edges. There are $(n-2k)!$ ways to pick the orientation of the directed edges between $V\backslash\bigcup_{i=1}^{k}\{b_i,c_i\}$ so that there are no 3-cycles between these edges.

Fix one of these orientations. In order to obtain a tournament with exactly k 3-cycles, then by the same argument than Case 1 of the computation of $S_{n,3,2}$ in Proposition 3.1, the directed edges between a_i,b_i,c_i,a_j,b_j,c_j are completely determined by the edge between a_i and a_j for any $1 \leq i < j \leq k$. Let $d \in V\backslash\bigcup_{i=1}^{k}\{a_i,b_i,c_i\}$. By the same argument of the computation of $S_{n,3,1}$ in Proposition 3.1, the directed edges between b_i,d, and c_i,d are completely determined by the edge between a_i and d. It follows that the number of tournaments in T_n with exactly k 3-cycles such that the 3-cycles are disjoint is

$$\frac{2^k}{k!}(n-2k)!\prod_{i=0}^{k-1}\binom{n-3i}{3} = \frac{2^k}{k!}(n-2k)!\prod_{i=0}^{k-1}\frac{1}{6}(n-3i)(n-3i-1)(n-3i-2)$$
$$= \frac{(n-2k)!}{k!\cdot 3^k}\prod_{j=0}^{3k-1}(n-j) = \frac{n!}{k!\cdot 3^k}\prod_{j=2k}^{3k-1}(n-j) = \frac{n!}{k!\cdot 3^k}\left(n^k + O(k^3n^{k-1})\right).$$

(2) **Case 2: The 3-cycles are not vertex-disjoint.** Let $v \leq 3k-1$ be the number distinct vertices composing the 3-cycles. We actually have $v \leq 3k-2$ by the same argument than Case 3 of Proposition 3.1, otherwise if $v = 3k-1$ we would have more than k 3-cycles. On the other hand, $\binom{v}{3} \geq k$ in order to form k distinct 3-cycles, hence $v \geq \lceil\sqrt[3]{6k}\rceil \geq 3$. There are at most $\binom{n}{v}$ ways to pick the v vertices for the k 3-cycles, and at most $\left[2\binom{v}{3}\right]^k$ ways to create the k 3-cycles from these vertices. Choose an arrangement of the 3-cycles and denote the cycles as $C_i = \{a_i \to b_i \to c_i \to a_i\}$ for $1 \leq i \leq k$, as before. Without loss of generality, suppose that C_1 and C_2 share the vertex a_1, i.e., $a_1 = a_2$. The set $A = \bigcup_{i=1}^{k}\{a_i\}$ contains

at most $k-1$ vertices, then by transitivity there are at most $(n-v+k-1)!$ ways to pick the orientation of the directed edges between the vertices in $V \setminus \bigcup_{i=1}^{k}\{b_i, c_i\}$.

Fix one of these orientations, and suppose it is possible to orient the remaining edges of the tournament so that exactly k 3-cycles will be obtained; it is possible that some of these orientations may not admit such an orientation of the remaining edges. Let $d \in V \setminus \bigcup_{i=1}^{k}\{a_i, b_i, c_i\}$. By the same arguments used in Proposition 3.1, the directed edges between a_i, b_i, c_i, a_j, b_j, c_j are completely determined by the edge between a_i and a_j and the cycles C_i, C_j, and the edges between b_i, d and c_i, d are completely determined by the edge between a_i and d. It follows that there are at most $\binom{n}{v}2^k\binom{v}{3}^k(n-v+k-1)!$ tournaments in T_n with exactly k 3-cycles such that the union of the 3-cycles contains v distinct vertices. This number is asymptotic to

$$\binom{n}{v}2^k\binom{v}{3}^k(n-v+k-1)! \sim \frac{n!}{3^k}\cdot\frac{v^{3k}}{v!}\cdot\prod_{j=1}^{k-1}(n-v+j) \sim \frac{n!}{3^k}\cdot\frac{v^{3k}}{v!}\cdot n^{k-1}.$$

The latter follows from the fact that if $v \geq k$ then the product on j is $\leq n^{k-1}$, and if $v \leq k-1$, the product is $\leq (n+\log n)^{k-1} \sim n^{k-1}$.

Summing over all $\lceil \sqrt[3]{6k}\rceil \leq v \leq 3k-2$, we find that the number of tournaments in T_n with exactly k 3-cycles not all disjoint is at most

$$O\left(\frac{n!\cdot n^{k-1}}{3^k}\sum_{\sqrt[3]{6k}<v\leq 3k-2}\frac{v^{3k}}{v!}\right) = O\left(\frac{n!\cdot n^{k-1}\cdot(3k-2)^{3k}}{3^k}\right).$$

Adding the results from the two cases gives the desired result. Again, since $k \leq (\log n)^{1-\varepsilon}$, the error term $O\left((3k-2)^{3k}k!n^{k-1}\right)$ is negligible compared to the main term n^k. In fact,

$$\log\left[\frac{(3k-2)^{3k}k!n^{k-1}}{n^k}\right] \leq 4k\log k + (2+\log 3)k + \frac{1}{2} - \log n \to -\infty$$

as $n \to \infty$, thus $\frac{(3k-2)^{3k}k!n^{k-1}}{n^k} \to 0$. $\square$

Summing over small values of k, we obtain the following:

Corollary 3.2.

$$\textit{For every } \varepsilon > 0, \quad \limsup_{n\to\infty}\frac{\#\left\{T \in T_n \mid T \textit{ contains} \leq (\log n)^{1-\varepsilon} \textit{ 3-cycles}\right\}}{n!\cdot e^{n/3}} \leq 1.$$

Comparing the results of Theorem 1.4 in the case $r = 3$ with the corollary above, we see that, for a small value of k, the upper bound derived from the Turán sieve method on the number of tournaments in T_n containing at most k 3-cycles is quite weak compared to the upper bound above when $n \to \infty$. However, Theorem 1.4 allows $k = o(n^3)$, while the corollary above allows only $k \leq (\log n)^{1-\varepsilon}$. Furthermore, it is important to note that the Turán sieve is able to provide an upper bound for many related combinatorial problems, where deriving an asymptotic expression for the actual number of tournaments with a certain property may be very difficult. We plan in the future to study more general problems about unrestricted cycles on t-partite tournaments, as well as the exact bound in the case of cycles of small length.

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