# A Generalization of the Turán Theorem and its Applications

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**Abstract**

We axiomatize the main properties of the classical Turán Theorem in order to apply it to a general context. We provide applications in the cases of number fields, function fields, and geometrically irreducible varieties over a finite field.
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## 1 Introduction.

Let $m \in \mathbb{N}$ and define $\omega(m)$ to be the number of distinct prime divisors of $m$. Hardy and Ramanujan [3] proved in 1917 that the normal order of $\omega(m)$ is $\log\log m$. In other words, given any $\epsilon > 0$, we have

$$\#\{m \leq x, |\omega(m) - \log\log m| > \epsilon \log\log m\} = \mathrm{o}(x).$$

The method they used was rather complicated and seemed difficult to generalize. In 1934, Turán [12] gave a greatly simplified proof of the Hardy-Ramanujan result by showing that

$$\sum_{m \leq x} (\omega(m) - \log\log x)^2 \ll x \log\log x.$$

His proof was essentially probabilistic and concealed in it an elementary sieve method [4]. Because of its simplicity and importance, this result is now known as the Turán Theorem. At the end of [12], Turán also stated that

$$\sum_{m \leq x} (\omega(m) - \log\log x)^2 = x \log\log x + \mathrm{o}(x \log\log x)$$

can be obtained and the proof of it is at [1]. Recently, Saidak [11] improved the Turán Theorem by proving the asymptotic formula

$$\sum_{m \leq x} (\omega(m) - \log\log x)^2 = x \log\log x + Cx + \mathrm{O}\Big(\frac{x \log\log x}{\log x}\Big),$$

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where $C$ is an explicit constant. Indeed, the setting of the Turán Theorem can be generalized. The purpose of this paper is to axiomatize the main properties in order to apply the results in a more general context. We will see applications in Section 4 in the cases of number fields, function fields, and geometrically irreducible varieties over a finite field.

We now formulate the general setting of the Turán's Theorem. Let $P$ be a set of elements with a map

$$N\colon P \to \mathbb{N}\backslash\{1\},\ p \mapsto N(p).$$

Let $M$ be a free abelian monoid generated by elements of $P$. For each $m \in M$, we write

$$m = \sum_{p\in P} n_p(m)p,$$

with $n_p(m) \in \mathbb{N}\cup\{0\}$ and $n_p(m) \neq 0$ for only finitely many $p$. We extend the map $N$ on $M$ as follows:

$$\begin{aligned} N : M &\longrightarrow \mathbb{N} \\ m = \sum_{p\in P} n_p(m)p &\longmapsto N(m) := \prod_{p\in P} N(p)^{n_p(m)}, \end{aligned}$$

i.e., N is a monoid homomorphism from $(M,+)$ to $(\mathbb{N},\cdot)$. Let $X$ be a subset of $\mathbb{N}$ that contains the image $\mathrm{Im}(N(M))$. We choose either $X = \mathbb{N}$ or $X = \{q^{rn}, n \in \mathbb{N}\cup\{0\}\}$ for some fixed $q, r \in \mathbb{N}\setminus\{1\}$.

Given $P$, $M$, and $X$ as above, for each (sufficiently large) $x \in X$, we assume that the following two conditions hold: let $m \in M$ and $p \in P$, we have

(A) (Cardinality of elements) $\displaystyle\sum_{N(m)\le x} 1 = \kappa x + \mathrm{O}(x^{\theta})$, for some $\kappa > 0$ and $0 \le \theta < 1$.

(B) (Cardinality of primes) $\displaystyle\sum_{N(p)\le x} 1 = \mathrm{O}\Big(\frac{x}{\log x}\Big)$.

For each $m \in M$, we define

$$\omega(m) = \sum_{\substack{p\in P\\ n_p(m)\ge 1}} 1,$$

the number of elements of $P$ that generate $m$, counted without multiplicity. Then we have a generalization of the Turán Theorem.

**Theorem 1** *Given $P$, $M$, and $X$ satisfying* (A) *and* (B)*, for $x \in X$, we have*

$$\sum_{N(m)\le x} (\omega(m) - \log\log x)^2 = \kappa x \log\log x + Cx + \mathrm{O}\Big(\frac{x\log\log x}{\log x}\Big).$$

*Here $\kappa$ is the same constant as in* (A) *and $C$ is a constant that depends only on $P$.*

As an immediate corollary of Theorem 1, we obtain a generalization of the Hardy-Ramanujan Theorem on the normal order of $\omega(m)$.

**Corollary 1** *Let $P$, $M$, and $X$ satisfy* (A) *and* (B). *For $\epsilon > 0$ and $x \in X$, we have*

$$\#\big\{m \in M, N(m) \le x, |\omega(m) - \log\log N(m)| > \epsilon \log\log N(m)\big\} = \mathrm{o}(x).$$

## 2 Technical lemmas.

To prove Theorem 1, we need the following lemmas.

**Lemma 1** *Given $P$, $M$, and $X$ satisfying* (A) *and* (B), *we have*

$$(1) \sum_{N(p)\le x} \frac{1}{N(p)^\alpha} \ll \frac{x^{1-\alpha}}{\log x} \qquad \text{if } 0 \le \alpha < 1,$$

$$(2) \sum_{N(m)\le x} \frac{1}{N(m)^\alpha} \ll 1 \qquad \text{if } \alpha > 1.$$

*In particular,* (2) *implies that*

$$\sum_{N(p)\le x} \frac{1}{N(p)^\alpha} \ll 1 \quad \text{if } \alpha > 1.$$

*Proof:* These results follow from the technique of partial summation [8](p17-18).

The next lemma is a generalization of Mertens' theorem [7].

**Lemma 2** *Given $P$, $M$, and $X$ satisfying* (A) *and* (B), *we have*

$$\sum_{N(p)\le x} \frac{1}{N(p)} = \log\log x + A + \mathrm{O}\Big(\frac{1}{\log x}\Big)$$

*for some constant $A$ that depends only on $P$.*

*Proof:* Consider $\sum_{N(m)\le x} \log N(m)$. Applying (A) and partial summation, we have

$$\sum_{N(m)\le x} \log N(m) = \kappa x \log x + \mathrm{O}(x).$$

On the other hand, for $p \in P$, we can write

$$\sum_{N(m)\le x} \log N(m) = \sum_{\substack{N(p)^s \le x \\ s\ge 1}} \Bigg( \sum_{N(m')\le \frac{x}{N(p)^s}} 1 \Bigg) \log N(p) \quad (\text{here } m' = m - sp)$$

$$= \kappa x \sum_{\substack{N(p)^s \le x \\ s\ge 1}} \frac{\log N(p)}{N(p)^s} + \mathrm{O}\Bigg( \sum_{\substack{N(p)^s \le x \\ s\ge 1}} \frac{x^\theta \log N(p)}{N(p)^{s\theta}} \Bigg).$$

By Lemma 1, we have

$$\sum_{\substack{N(p)^s \le x \\ s\ge 1}} \frac{\log N(p)}{N(p)^{s\theta}} \ll x^{1-\theta}$$

and

$$\sum_{\substack{N(p)^s \le x \\ s\ge 2}} \frac{\log N(p)}{N(p)^{s}} \ll 1.$$

It follows that

$$\sum_{N(p)\le x} \frac{\log N(p)}{N(p)} = \log x + \mathrm{O}(1).$$

Let $X = \mathbb{N}$ and $z \in \mathbb{N}$. Define

$$S(z) := \sum_{N(p)\le z} \frac{\log N(p)}{N(p)} = \log z + \tau(z), \quad \text{where } \tau(z) = \mathrm{O}(1).$$

We have

$$\sum_{N(p)\le x} \frac{1}{N(p)} = \frac{S(x)}{\log x} + \int_2^x \frac{\log t + \tau(t)}{(\log t)^2 t} dt$$

$$= 1 + \int_2^x \frac{1}{t \log t} dt + \int_2^\infty \frac{\tau(t)}{t(\log t)^2} dt - \int_x^\infty \frac{\tau(t)}{t(\log t)^2} dt + \mathrm{O}\Big(\frac{1}{\log x}\Big)$$

$$= \log\log x + \Bigg(1 - \log\log 2 + \int_2^\infty \frac{\tau(t)}{t(\log t)^2} dt\Bigg) + \mathrm{O}\Big(\frac{1}{\log x}\Big).$$

If $X = \{q^{rn}, n \in \mathbb{N}\cup\{0\}\}$, define

$$S'(z) := \sum_{N(p)\le q^{rz}} \frac{\log N(p)}{N(p)} = z \log(q^r) + \tau(z), \quad \text{where } \tau(z) = \mathrm{O}(1).$$

For $x = q^{rx'}$, we have

$$\sum_{N(p)\le x = q^{rx'}} \frac{1}{N(p)} = \frac{S'(x')}{\log q^{rx'}} + \int_1^{x'} \frac{t \log q^r + \tau(t)}{t^2 \log q^r} dt$$

$$= \log\log x + \Bigg(1 - \log\log q^r + \int_1^\infty \frac{\tau(t)}{t^2 \log q^r} dt\Bigg) + \mathrm{O}\Big(\frac{1}{\log x}\Big).$$

This completes the proof of Lemma 2

**Lemma 3** *Given $P$, $M$, and $X$ satisfying* (A) *and* (B),
(1) *If $X = \mathbb{N}$, we have*

$$\sum_{N(p)\le \frac{x}{2}} \frac{1}{N(p)} \log\log \frac{x}{N(p)} = (\log\log x)^2 + A\log\log x + B + \mathrm{O}\Big(\frac{\log\log x}{\log x}\Big).$$

(2) *If $X = \{q^{rn}, n \in \mathbb{N}\cup\{0\}\}$, we have*

$$\sum_{N(p)\le \frac{x}{q^r}} \frac{1}{N(p)} \log\log \frac{x}{N(p)} = (\log\log x)^2 + A\log\log x + B + \mathrm{O}\Big(\frac{\log\log x}{\log x}\Big).$$

*Here $A$ is the same constant as in* Lemma 2 *and $B$ is some other constant. Both depend only on $P$.*

*Proof:* (1) Let $X = \mathbb{N}$. By Lemma 2 and partial summation, we have

$\mathrm{O}\Big(\frac{1}{\log x}\Big)$

$$\sum_{N(p)\le \frac{x}{2}} \frac{1}{N(p)} \log\log \frac{x}{N(p)} = (\log\log 2)\log\log x + A\log\log 2 + \mathrm{O}\Big(\frac{1}{\log x}\Big) + \int_2^{\frac{x}{2}} \frac{\log\log t + A + \mathrm{O}\big(\frac{1}{\log t}\big)}{\log x - \log t}\frac{dt}{t}.$$

By elementary integrations, we see that

$$\frac{dt}{\log t(\log x - \log t)t} \ll \frac{\log\log x}{\log x}$$

and

$$\int_2^{\frac{x}{2}} \frac{1}{\log x - \log t}\frac{dt}{t} = \log\log x - \log\log 2 + \mathrm{O}\Big(\frac{1}{\log x}\Big).$$

By change of variables, we write

$$\begin{aligned}\int_2^{\frac{x}{2}} \frac{\log\log t}{\log x - \log t}\frac{dt}{t} &= \int_{\log 2}^{\log\frac{x}{2}} \frac{\log\big(\log x(1-\frac{u}{\log x})\big)}{u} du \\ &= (\log\log x)^2 - \log\log 2\cdot\log\log x + \mathrm{O}\Big(\frac{\log\log x}{\log x}\Big) \\ &\quad + \int_{\frac{\log 2}{\log x}}^{1-\frac{\log 2}{\log x}} \frac{\log(1-s)}{s} ds.\end{aligned}$$

Since $\log(1-s) \ll s$ and $\int_0^1 \frac{\log(1-s)}{s}ds = \frac{\pi^2}{6}$ for $0 < s < 1$, we have

$$\int_2^{\frac{x}{2}} \frac{\log\log t}{\log x - \log t}\frac{dt}{t} = (\log\log x)^2 - \log\log 2\cdot\log\log x - \frac{\pi^2}{6} + \mathrm{O}\Big(\frac{\log\log x}{\log x}\Big).$$

Combining all the above results, we obtain

$$\sum_{N(p)\leq \frac{x}{2}} \frac{1}{N(p)} \log\log \frac{x}{N(p)} = (\log\log x)^2 + A\log\log x - \frac{\pi^2}{6} + \mathrm{O}\Big(\frac{\log\log x}{\log x}\Big).$$

(2) For $X = \{q^{rn}, n \in \mathbb{N}\cup\{0\}\}$, replace $z$ in the above proof by $q^{rz}$. Using similar arguments as before, we obtain

$$\sum_{N(p)\leq \frac{x}{q^l}} \frac{1}{N(p)} \log\log \frac{x}{N(p)} = (\log\log x)^2 + A\log\log x + \Big((\log\log q^r)^2 - \frac{\pi^2}{6}\Big) + \mathrm{O}\Big(\frac{\log\log x}{\log x}\Big).$$

**Lemma 4** *Given* $P$, $M$, *and* $X$ *satisfying* (A) *and* (B),
(1) *If* $X = \mathbb{N}$, *we have*

$$\sum_{N(p)\leq \frac{x}{2}} \frac{1}{N(p)\log\frac{x}{N(p)}} \ll \frac{\log x}{\log\log x}.$$

(2) *If* $X = \{q^{rn}, n \in \mathbb{N}\cup\{0\}\}$, *we have*

$$\sum_{N(p)\leq \frac{x}{q^r}} \frac{1}{N(p)\log\frac{x}{N(p)}} \ll \frac{\log x}{\log\log x}.$$

*Proof:* (1) Divide $[1, \frac{x}{2}]$ as $I_j = [e^j, e^{j+1}]$. We have

$$\begin{aligned}\sum_{N(p)\leq \frac{x}{2}} \frac{1}{N(p)\log\frac{x}{N(p)}} &\leq \sum_{j=0}^{\log\frac{x}{2}} \frac{1}{\log\frac{x}{e^{j+1}}} \sum_{e^j<N(p)\leq e^{j+1}} \frac{1}{N(p)} \\ &= \sum_{j=0}^{\log\frac{x}{2}} \frac{1}{(\log x-(j+1))}\Big(\log\frac{j+1}{j} + \mathrm{O}\Big(\frac{1}{j}\Big)\Big).\end{aligned}$$

The last inequality follows from Lemma 2. Since $\log\big(1+\frac{1}{x}\big) \ll \frac{1}{x}$ for $|x|<1$, we have

$$\begin{aligned}\sum_{N(p)\leq \frac{x}{2}} \frac{1}{N(p)\log\frac{x}{N(p)}} &\ll \sum_{j=1}^{\log\frac{x}{2}} \frac{1}{(\log x-j)}\frac{1}{j} \\ &= \frac{1}{\log x}\Big(\sum_{j=1}^{\log\frac{x}{2}}\Big(\frac{1}{j} + \frac{1}{\log x - j}\Big)\Big) \\ &\ll \frac{\log\log x}{\log x}.\end{aligned}$$

(2) The proof is exactly the same as above except replacing all $\frac{x}{2}$ by $\frac{x}{q^r}$.

## 3 Proof of Theorem 1.

Now, we are ready to prove Theorem 1. Our goal is to get an asymptotic formula for

$$\begin{aligned}
&\sum_{N(m)\le x}(\omega(m)-\log\log x)^2\\
&=\sum_{N(m)\le x}\omega^2(m)-2\log\log x\sum_{N(m)\le x}\omega(m)+(\log\log x)^2\sum_{N(m)\le x}1.
\end{aligned}$$

By (A), the third term is

$$\kappa x(\log\log x)^2+\mathrm{O}(x^\theta(\log\log x)^2).$$

By Lemmas 1 and 2, the sum of the second term is equal to

$$\begin{aligned}
\sum_{N(m)\le x}\omega(m)&=\sum_{N(p)\le x}\sum_{\substack{N(m)\le x\\ n_p(m)\ge 1}}1\\
&=\kappa x\sum_{N(p)\le x}\frac{1}{N(p)}+\mathrm{O}\Big(x^\theta\sum_{N(p)\le x}\frac{1}{N(p)^\theta}\Big)\\
&=\kappa x\log\log x+A\kappa x+\mathrm{O}\Big(\frac{x}{\log x}\Big).
\end{aligned}$$

Now, we consider

$$\begin{aligned}
\sum_{N(m)\le x}\omega^2(m)&=\sum_{\substack{N(p)N(q)\le x\\ p\ne q}}\sum_{\substack{N(m)\le x\\ n_p(m),n_q(m)\ge 1}}1+\sum_{N(p)\le x}\sum_{\substack{N(m)\le x\\ n_p(m)\ge 1}}1\\
&=\sum_{N(p)N(q)\le x}\sum_{N(m')\le\frac{x}{N(p)N(q)}}1-\sum_{N(p)\le x^{1/2}}\sum_{N(m'')\le\frac{x}{N(p)^2}}1\\
&\quad+\kappa x\log\log x+A\kappa x+\mathrm{O}\Big(\frac{x}{\log x}\Big).
\end{aligned}$$

Here $m'=m-p-q$ and $m''=m-2p$.

The first sum of the last equation is

$$\sum_{N(p)N(q)\le x}\sum_{N(m')\le\frac{x}{N(p)N(q)}}1=\kappa x\sum_{N(p)N(q)\le x}\frac{1}{N(p)N(q)}+\mathrm{O}\Big(x^\theta\sum_{N(p)N(q)\le x}\frac{1}{N(p)^\theta N(q)^\theta}\Big).$$

If $X=\mathbb{N}$, Lemmas 2, 3, and 4 implies that

$$
\begin{aligned}
\sum_{N(p)N(q)\le x}\frac{1}{N(p)N(q)} &= \sum_{N(p)\le \frac{x}{2}}\frac{1}{N(p)}\Bigg(\sum_{N(q)\le \frac{x}{N(p)}}\frac{1}{N(q)}\Bigg)\\
&= \sum_{N(p)\le \frac{x}{2}}\frac{1}{N(p)}\Bigg(\log\log\frac{x}{N(p)}+A+\mathrm{O}\Big(\frac{1}{\log\frac{x}{N(p)}}\Big)\Bigg)\\
&= \sum_{N(p)\le \frac{x}{2}}\frac{1}{N(p)}\log\log\frac{x}{N(p)}+A\Bigg(\log\log\frac{x}{2}+A+O\Big(\frac{1}{\log x}\Big)\Bigg)\\
&\quad+\mathrm{O}\Bigg(\sum_{N(p)\le \frac{x}{2}}\frac{1}{N(p)}\frac{1}{\log\frac{x}{N(p)}}\Bigg)\\
&= (\log\log x)^2+2A\log\log x+A^2+B+\mathrm{O}\Big(\frac{\log\log x}{\log x}\Big).
\end{aligned}
$$

Moreover, by Lemmas 1 and 2, we have

$$
\begin{aligned}
\sum_{N(p)N(q)\le x}\frac{1}{N(p)^\theta N(q)^\theta} &= \sum_{N(p)\le \frac{x}{2}}\frac{1}{N(p)^\theta}\Big(\sum_{N(q)\le \frac{x}{N(p)}}\frac{1}{N(q)^\theta}\Big)\\
&\ll \sum_{N(p)\le \frac{x}{2}}\frac{1}{N(p)^\theta}\frac{\big(\frac{x}{N(p)}\big)^{1-\theta}}{\log x}\\
&\ll \frac{x^{1-\theta}}{\log x}\sum_{N(p)\le \frac{x}{2}}\frac{1}{N(p)}\\
&\ll \frac{x^{1-\theta}\log\log x}{\log x}.
\end{aligned}
$$

By replacing $\frac{x}{2}$ by $\frac{x}{q^r}$, we obtain the same results for $X=\{q^{rn}, n\in\mathbb{N}\cup\{0\}\}$. Hence, we have

$$
\sum_{N(p)N(q)\le x}\;\sum_{N(m')\le \frac{x}{N(p)N(q)}}1=(\log\log x)^2+2A\log\log x+A^2+B+\mathrm{O}\Big(\frac{\log\log x}{\log x}\Big).
$$

Now, consider

$$
\begin{aligned}
\sum_{N(p)\le x^{1/2}}\;\sum_{N(m'')\le \frac{x}{N(p)^2}}1 &= \sum_{N(p)\le x^{1/2}}\Big(\frac{\kappa x}{N(p)^2}+\mathrm{O}\Big(\frac{x^\theta}{N(p)^{2\theta}}\Big)\Big)\\
&= \kappa x\sum_{p\in P}\frac{1}{N(p)^2}-\kappa x\sum_{N(p)>x^{1/2}}\frac{1}{N(p)^2}\\
&\quad+\begin{cases}\mathrm{O}\Big(x^\theta\frac{x^{\frac{1}{2}(1-2\theta)}}{\log x}\Big) & \text{if } 0\le\theta<1/2,\\ \mathrm{O}(x^\theta) & \text{if } \theta\ge 1/2.\end{cases}
\end{aligned}
$$

By (B) and partial summation, we have

$$\sum_{N(p)>x^{1/2}} \frac{1}{N(p)^2} \ll \frac{1}{\sqrt{x}\log x}.$$

Combining all the above results, we obtain

$$\begin{aligned}\sum_{N(m)\le x} \omega^2(m) &= \kappa x(\log\log x)^2 + (2A+1)\kappa x\log\log x \\ &\quad + \Big(A - \sum_{p\in P}\frac{1}{N(p)^2} + A^2 + B\Big)\kappa x + \mathrm{O}\Big(\frac{x\log\log x}{\log x}\Big).\end{aligned}$$

It follows that

$$\begin{aligned}&\sum_{N(m)\le x} (\omega(m) - \log\log x)^2 \\ &= \kappa x\log\log x + \Big(A - \sum_{p\in P}\frac{1}{N(p)^2} + A^2 + B\Big)\kappa x + \mathrm{O}\Big(\frac{x\log\log x}{\log x}\Big),\end{aligned}$$

which completes the proof of Theorem 1.

**Remark** We restrict $X = \mathbb{N}$ or $X = \{q^{rn}, n \in \mathbb{N}\cup\{0\}\}$ in our general setting to obtain Theorem 1. If we allow $X$ to be any subset of $\mathbb{N}$, we can still get a weaker result

$$\sum_{N(m)\le x} (\omega(m) - \log\log x)^2 = \kappa x\log\log x + \mathrm{O}(x)$$

by using a similar method. If we replace condition (B) by a much weaker condition,

$$(\mathrm{B}') \sum_{N(p)\le x} \frac{1}{N(p)} = \log\log x + \mathrm{O}(1),$$

With condition (A), we obtain

$$\sum_{N(m)\le x} (\omega(m) - \log\log x)^2 = \kappa x\log\log x + \mathrm{o}(x\log\log x).$$

# 4 Applications of the general setting.

In this section, we provide some examples where the general setting applies. Thus analogues of the Turán Theorem hold in these cases.

**Example 1** In the case of rational number, let $P$ be the set of primes of $\mathbb{N}$ and $M = \mathbb{N}$. Take $N : M \to \mathbb{N}$ to be the identity map and choose $X = \mathbb{N}$. Conditions (A) and (B) are satisfied with $\kappa = 1$. Hence, Theorem 1 implies the classical Turán Theorem and we recover the asymptotic formula of Saidak [11].

**Example 2** Let $K/\mathbb{Q}$ be a number field of degree $[K:\mathbb{Q}]$ and $\mathcal{O}_K$ its ring of integers. Let $P$ be the set of prime ideals of $\mathcal{O}_K$ and $M$ the set of ideals of $\mathcal{O}_K$. Take $N: M \to \mathbb{N}$ to be the standard norm map, i.e., $\mathfrak{m} \mapsto N(\mathfrak{m}) := |\mathcal{O}_K/\mathfrak{m}|$ and choose $X = \mathbb{N}$. For $\mathfrak{m} \in M$, it was proved by Weber that [13]

$$\sum_{N(\mathfrak{m}) \leq x} 1 = \kappa x + \mathrm{O}\left(x^{1-\frac{1}{[K:\mathbb{Q}]}}\right) \text{ where } \kappa = \frac{2^{r_1}(2\pi)^{r_2} h R}{\omega\sqrt{|d_K|}},$$

$$\begin{aligned} \text{with} \quad r_1 &= \text{ number of real embeddings of } K, \\ 2r_2 &= \text{ number of complex embeddings,} \\ h &= \text{ class number,} \\ R &= \text{ regulator,} \\ \omega &= \text{ number of roots of unity,} \\ d_K &= \text{ discriminant of } K. \end{aligned}$$

Notice that there are at most $[K:\mathbb{Q}]$ many prime ideals $\mathfrak{p}$ lying above $p\,\mathcal{O}_K$ for a prime $p$. Hence, the Chebyshev Theorem [8](p36-37) implies (B). Prachar [9] proved in 1952 that

$$\sum_{N(\mathfrak{m}) \leq x} (\omega(\mathfrak{m}) - \log\log x)^2 \ll x \log\log x.$$

Theorem 1 implies his result with a stronger estimate.

In the examples of function fields and varieties, to verify conditions (A) and (B), it suffices to get the cardinalities of elements of $P$ and $M$ with fixed image in $\mathbb{N}$. Using elementary geometric sums and integration techniques, we have

**Lemma 5** *Let $P, M, X$ be defined as before with $X = \{q^{rn}, n \in \mathbb{N} \cup \{0\}\}$. Define*

$$a_d := \#\{m \in M,\ N(m) = q^{rd}\},\ d \in \mathbb{N} \cup \{0\}$$

*and*

$$b_d := \#\{p \in P,\ N(p) = q^{rd}\},\ d \in \mathbb{N}.$$

(1) *If for all $d \in \mathbb{N} \cup \{0\}$,*

$$a_d = \kappa' q^{rd} + \mathrm{O}(q^{(r-\theta')d}), \quad \textit{for some } \kappa' > 0 \textit{ and } \theta' > 0,$$

*we have*

$$\sum_{N(m) \leq x} 1 = \frac{\kappa' q^r}{q^r - 1} x + \mathrm{O}(x^\theta),$$

*where $\theta = 1 - \frac{\theta'}{r}$.*
(2) *If for all $d \in \mathbb{N}$,*

$$b_d = \frac{q^{rd}}{d} + \mathrm{O}\left(q^{(r-\frac{1}{2})d}\right),$$

*we have*
$$\sum_{N(p)\leq x} 1 = \mathrm{O}\Big(\frac{x}{\log x}\Big).$$

**Example 3** Let $\mathbb{F}_q[t]$ be the ring of 1-variable polynomials over a finite field $\mathbb{F}_q$. Take $P$ to be the set of monic irreducible polynomials in $\mathbb{F}_q[t]$ and $M$ the set of monic polynomials. We define the map $N$ as follows:

$$N\colon M \to \mathbb{N},\ m := m(t) \mapsto q^{\deg m(t)},$$

where $\deg m(t)$ is the degree of the polynomial $m(t)$. Since $\mathrm{Im}(N(M))$ only contains non-negative powers of $q$, we take $X = \{q^n, n \in \mathbb{N} \cup \{0\}\}$. In this case, we have [10] (p 6)

$$a_d = q^d$$

and
$$b_d = \frac{q^d}{d} + \mathrm{O}(q^{\frac{d}{2}}).$$

These satisfy the assumptions of Lemma 5 with $r = 1$. Hence, condition (A) and (B) are verified and we have an analogue of the Turán Theorem in $\mathbb{F}_q[t]$.

**Example 4** Let $V/\mathbb{F}_q$ be a geometrically irreducible variety of dimension $r$ in a projective space. Let $P$ be the set of closed points of $V/\mathbb{F}_q$, which is in bijection with the set of orbits of $V(\overline{\mathbb{F}_q})$ under the action of $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ [5](p259). For each $p \in P$, we define $\deg p$ to be the length of the corresponding orbit. The monoid of effective 0-cycles $M$ of $V/\mathbb{F}_q$ is defined by

$$M = \Big\{ m = \sum_{p\in P} n_p(m)p, n_p(m) \in \mathbb{N} \cup \{0\} \text{ and } n_p(m) \neq 0 \text{ for only finitely many } p \Big\}.$$

For $m \in M$, we define
$$\deg\ m = \sum_{p\in P} n_p(m) \deg\ p.$$

The map $N$ is defined by
$$N\colon M \to \mathbb{N},\ m \mapsto q^{r\deg m}.$$

We take $X = \{q^{rn}, n \in \mathbb{N} \cup \{0\}\}$.

The zeta function of $V/\mathbb{F}_q$ is defined by

$$Z(T) = \exp\left( \sum_{n=1}^{\infty} \frac{|V(\mathbb{F}_{q^n})|}{n} T^n \right).$$

Let $a_d$ and $b_d$ be defined as in Lemma 5. Using the fact that [5](p259)

$$|V(\mathbb{F}_{q^n})| = \sum_{d|n} d b_d,$$

we have

$$Z(T) = \prod_{d=1}^{\infty}(1-T^d)^{-b_d} = \sum_{d=1}^{\infty} a_d T^d.$$

It was proved by Lang and Weil [6] in 1954 that

$$|V(\mathbb{F}_{q^n})| = q^{rn} + \mathrm{O}\Big(q^{(r-\frac{1}{2})n}\Big).$$

Applying the Möbius inversion formula, we get

$$\begin{aligned} db_d &= \sum_{n|d} \mu\Big(\frac{d}{n}\Big)\Big(q^{rn} + \mathrm{O}\big(q^{(r-\frac{1}{2})n}\big)\Big) \\ &= q^{rd} + \mathrm{O}\big(dq^{(r-\frac{1}{2})d}\big) \end{aligned}$$

Hence, we have

$$b_d = \frac{q^{rd}}{d} + \mathrm{O}\big(q^{(r-\frac{1}{2})d}\big).$$

The computation of $a_d$ is much more involved. Using the result of Lang-Weil, we have

$$Z(T) = \exp(-\log(1-q^rT))\exp\Bigg(\sum_{n=1}^{\infty}\frac{\mathrm{O}\big(q^{(r-\frac{1}{2})n}\big)}{n}T^n\Bigg).$$

From the theory of the $l$-adic cohomology of Grothendieck [2], we can write

$$Z(T) = \Big(\frac{1}{1-q^rT}\Big)\frac{f_1(T)f_3(T)\cdots f_{2r-1}(T)}{f_0(T)f_2(T)\cdots f_{2r-2}(T)},$$

where $f_i(T)$ are polynomials. Write

$$f_i(T) = \prod_{j=1}^{B_i}(1-\omega_{i,j}T),$$

where $B_i$ is the ith Betti number and $\omega_{i,j}$ are eigenvalues of the ith cohomology group. By taking logarithms on both expressions of $Z(T)$, we have

$$\sum_{i,j}(-1)^i\omega_{i,j}^n = \mathrm{O}\Big(q^{(r-\frac{1}{2})n}\Big).$$

Since there are only finitely many $\omega_{i,j}$ and the big O notation above is independent from $n$, we have

$$|\omega_{i,j}| \le q^{r-\frac{1}{2}},$$

for all $i, j$.

To consider the coefficients $a_d$ of $Z(T)$, we need the following lemmas.

**Lemma 6** *Let $Z(T)$ be the zeta function of a geometrically irreducible variety $V/\mathbb{F}_q$ of dimension $r$. We define*

$$H(T) = Z(T)(1 - q^r T) = \frac{f_1(T) f_3(T) \cdots f_{2r-1}(T)}{f_0(T) f_2(T) \cdots f_{2r-2}(T)} = \sum_{i=0}^{\infty} c_i T^i.$$

*Then, we have*

$$c_i \ll q^{(r-\frac{1}{2})i} i^s,$$

*where $s = B_0 + B_2 + \cdots + B_{2r-2} - 1$.*

*Proof:* If $i$ is odd, we write

$$f_i(T) = \sum_{j=0}^{\infty} c_{i,j} T^j.$$

Since $f_i(T)$ is a polynomial, it follows that

$$|c_{i,j}| \ll 1.$$

If $i$ is even, we write

$$\frac{1}{f_i(T)} = \frac{1}{\prod_{j=1}^{B_i} (1 - \omega_{i,j} T)} = \sum_{j=0}^{\infty} c_{i,j} T^j.$$

For a fixed $i$, the largest absolute value of $c_{i,j}$ appears when all $\omega_{i,j}$ are the same. Notice that the coefficient of $T^j$ of the rational function

$$\frac{1}{(1 - \omega T)^B} = (1 + \omega T + \omega^2 T^2 + \cdots + \omega^j T^j + \cdots)^B$$

is $\leq (j+1)^{B-1} |\omega|^j$. Hence, by the above upper bound of $|\omega_{i,j}|$, we have

$$c_{i,j} \ll j^{B_i - 1} q^{(r-\frac{1}{2})j}.$$

Notice that for $\alpha, \beta$, and $a \in \mathbb{R}$, suppose $|d_j| \ll j^\alpha q^{aj}$, $|e_k| \ll k^\beta q^{ak}$ for all $j, k \in \mathbb{N} \cup \{0\}$. Write

$$\left( \sum_{j=0}^{\infty} d_j T^j \right) \left( \sum_{k=0}^{\infty} e_k T^k \right) = \sum_{s=0}^{\infty} c_s T^s.$$

Then we have

$$|c_s| \ll q^{as} s^{\alpha+\beta+1}.$$

It follows that the coefficient $c_i$ of $T^i$ of $H(T)$ is bounded by

$$c_i \ll q^{(r-\frac{1}{2})i} i^s,$$

where $s = B_0 + B_2 + \cdots + B_{2r-2} - 1$.

**Lemma 7** *Let $c_i$ be the coefficient of $T^i$ of $H(T)$ defined in Lemma 6. For $z \in \mathbb{N} \cup \{0\}$, define*

$$C(z) = \sum_{i \leq z} \frac{c_i}{q^{(r-\frac{1}{2})i}}.$$

*For any $\epsilon > 0$, we have*

$$\sum_{i=0}^{d} \frac{c_i}{q^{ri}} = \kappa' + \mathrm{O}\left(\frac{1}{q^{(\frac{1}{2}-\epsilon)d}}\right),$$

*where $\kappa' = \sum_{z=0}^{\infty} C(z)\left(\frac{1}{q^{\frac{1}{2}z}} - \frac{1}{q^{\frac{1}{2}(z+1)}}\right)$.*

*Proof:* By Lemma 7, we have

$$\frac{c_i}{q^{(r-\frac{1}{2})i}} \ll i^s.$$

It implies that

$$C(z) \ll z^{s+1}.$$

Using partial summation, we obtain

$$\begin{aligned}\sum_{i=0}^{d} \frac{c_i}{q^{ri}} &= \frac{C(d)}{q^{\frac{1}{2}d}} - \sum_{z=0}^{d-1} C(z)\left(\frac{1}{q^{\frac{1}{2}(z+1)}} - \frac{1}{q^{\frac{1}{2}z}}\right) \\ &= \kappa' + \mathrm{O}\left(\frac{d^{s+1}}{q^{\frac{1}{2}d}} + \sum_{z=d}^{\infty} z^{s+1}\left(\frac{1}{q^{\frac{1}{2}z}} - \frac{1}{q^{\frac{1}{2}(z+1)}}\right)\right).\end{aligned}$$

For any $\epsilon > 0$, choose $z_0$ large enough such that $z^{s+1} \leq q^{\epsilon z}$ for $z \geq z_0$. Then for $d \geq z_0$, we have

$$\begin{aligned}\frac{d^{s+1}}{q^{\frac{1}{2}d}} + \sum_{z=d}^{\infty} z^{s+1}\left(\frac{1}{q^{\frac{1}{2}z}} - \frac{1}{q^{\frac{1}{2}(z+1)}}\right) &\leq \frac{1}{q^{(\frac{1}{2}-\epsilon)d}} + \sum_{z=d}^{\infty} \frac{1}{q^{(\frac{1}{2}-\epsilon)z}} \\ &\ll \frac{1}{q^{(\frac{1}{2}-\epsilon)d}}.\end{aligned}$$

This completes the proof of this Lemma.

Now, we write

$$Z(T) = H(T)\frac{1}{1-q^rT} = \left(\sum_{i=0}^{\infty} c_iT_i\right)\left(\sum_{j=0}^{\infty} q^{rj}T^j\right) = \sum_{d=0}^{\infty} a_dT^d.$$

Hence, we have

$$a_d = \sum_{i=0}^{d} c_i q^{r(d-i)}$$

By Lemma 7, we obtain the following theorem.

**Theorem 2** *Let $V/\mathbb{F}_q$ be a geometrically irreducible variety of dimension $r$. Let $P$ be the set of closed points and $M$ the set of effective 0-cycles. We define the map $N : M \to \mathbb{N}, m \mapsto q^{r \deg m}$. For any $\epsilon > 0$, we have*

$$(1)\ a_d = \#\{\, m \in M,\, \deg m = d \,\} = \kappa' q^{rd} + \mathrm{O}\big(q^{(r-\frac{1}{2}+\epsilon)d}\big),$$

*where $\kappa'$ is the same constant as in* Lemma 7. *We also have*

$$(2)\ b_d = \#\{\, p \in P,\, \deg p = d \,\} = \frac{q^{rd}}{d} + \mathrm{O}\big(q^{d(r-\frac{1}{2})}\big).$$

Theorem 2 and Lemma 5 imply that condition (A) and (B) are satisfied in this setting. Thus we obtain an analogue of the Turán Theorem for a geometrically irreducible variety.

**Remark 1** By Lemma 5 and Theorem 2, we have

$$\sum_{N(m)\le x} 1 = \frac{\kappa' q^r}{q^r - 1} x + \mathrm{O}\big(x^{1-\frac{1}{2r}+\epsilon}\big).$$

We see from the above proof that the $x^\epsilon$ term can be replaced by $\log x$. If we apply the fact from the cohomology theory that

$$|\omega_{i,j}| \le q^{\frac{i}{2}},$$

where $\omega_{i,j}$ are the eigenvalues of the ith cohomology group, we can improve the above estimation to

$$\sum_{N(m)\le x} 1 = \frac{\kappa' q^r}{q^r - 1} x + \mathrm{O}\big(x^{1-\frac{1}{r}} \log x\big).$$

This is a similar result to the case of number fields where $r = [K : \mathbb{Q}]$ except the extra $\log x$ factor. It will be nice if we can eliminate it.

**Remark 2** In the case of smooth projective curve $C/\mathbb{F}_q$, $M$ is the set of effective divisors. Using Weil's result on the zeta function of $C$ [5](Ch VIII), we have

$$a_d = \kappa' q^d + \mathrm{O}(1).$$

Moreover, the constant $\kappa$ can be written explicitely. We have

$$\kappa = \frac{\kappa' q}{q-1} = \frac{h}{q^g}\Big(\frac{q}{q-1}\Big)^2,$$

where $h$ is the order of $\mathrm{Pic}^0(C/\mathbb{F}_q)$ and $g$ is the genus of $C$. It will be an interesting projective to study $\kappa$ and express it explicitely in terms of geometric objects in a general case.

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## References

[1] P.D.T.A. Elliott, *Probabilistic number theory*, Vol. I. & II., Springer Verlag (1979).

[2] R. Hartshorne, *Algebraic geometry*, Appendix C, Springer Verlag (1977).

[3] G.H. Hardy & S. Ramanujan, *The normal number of prime factors of a number $n$*, Quar. J. Pure. Appl. Math., 48 (1917), 76-97.

[4] Y.-R. Liu & M. R. Murty , *The Turán sieve method and some of its applications*, J. Ramanujan Math. Soc., 14 (1999), 21-35.

[5] D. Lorenzini, *An invitation to arithmetic geometry*, Grad. Stud. in Math. Vol. 9, AMS (1996).

[6] S. Lang & A. Weil, *Number of points of varieties in finite fields*, Am. J. of Math. 76 (1954), 819-827.

[7] F. Mertens, *Ein Beitrag zur analytischen Zahlentheorie*, J. für die Reine und Angew. Math., 78(1874), 46-62.

[8] M.R. Murty, *Problems in analytic number theory*, Springer Verlag (2001).

[9] K. Prachar, *Verallgemeinerung eines satzes von Hardy and Ramanujan auf algebraische zahlkörper*, Monatsh. Math 56 (1952), 229-232.

[10] M. Rosen, *Number theory in Function Fields*, Lecture notes given at K.A.I.S.T., Taejon, Korea (1994).

[11] F. Saidak, *On a theorem of Hardy-Ramanujan-Turán I.*(to appear)

[12] P. Turán, *On a theorem of Hardy and Ramanujan*, J. London Math. Soc. 9 (1934), 274-276.

[13] H. Weber, *Über Zahlengruppen in algebraischen Körpern*, Math. Ann., Vol 49 (1897), 83-100.

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