Enumerating matroid extensions

by

Shayla Redlin Hume

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Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

External Examiner:	Rudi Pendavingh Assistant Professor, Dept. of Mathematics and Computer Science Eindhoven University of Technology
Supervisor:	Peter Nelson Associate Professor, Dept. of Combinatorics & Optimization, University of Waterloo
Internal Member:	Jim Geelen Professor, Dept. of Combinatorics & Optimization, University of Waterloo
Internal Member:	Sophie Spirkl Assistant Professor, Dept. of Combinatorics & Optimization, University of Waterloo
Internal-External Member:	Matthew Satriano Associate Professor, Dept. of Pure Mathematics University of Waterloo

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

This thesis investigates the problem of enumerating the extensions of certain matroids. A matroid M is an extension of a matroid N if M delete e is equal to N for some element e of M. Similarly, a matroid M is a coextension of a matroid N if M contract e is equal to N for some element e of M. In this thesis, we consider extensions and coextensions of matroids in the classes of graphic matroids, representable matroids, and frame matroids. We develop a general strategy for counting the extensions of matroids which translates the problem into counting stable sets in an auxiliary graph. We apply this strategy to obtain asymptotic results on the number of extensions and coextensions of certain graphic matroids, projective geometries, and Dowling geometries.

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Dedication

To Jesse.

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Chapter 1

Introduction

Matroids are combinatorial objects that were first introduced in 1935 by Whitney [51] and, independently, by Nakasawa [32, 33, 34]. These objects generalize both graphs and matrices and are increasingly prevalent in modern combinatorics research. However, many natural matroid enumeration questions are unanswered or have gone unanswered until recently. This thesis aims to solve several open problems related to enumerating certain classes of matroids. One way to approach matroid enumeration is by considering the number of ways to add an element to a matroid, or a set of matroids. We say a matroid M is an *extension* of a matroid N if M delete e is equal to N for some element e of M. Similarly, we say a matroid M is a *coextension* of a matroid N if M contract e is equal to N for some element e of M.

The broad goal of this research is to contribute to the understanding of extensions and coextensions of matroids in the following classes: graphic matroids, frame matroids, and $GF(q)$-representable matroids. Understanding extensions in general is difficult, so our focus is on dense, highly symmetric matroids, such as cycle matroids of complete graphs, Dowling geometries, and projective geometries.

In 1965, Crapo [10] showed that the extensions of a matroid M correspond to the "linear subclasses" of M, which are collections of hyperplanes that satisfy a certain property. The dual of this equivalence implies that coextensions of M correspond to "colinear subclasses" of M, which correspond to collections of circuits satisfying the dual version of the property. Using this dual characterization, one can show that the coextensions of graphic matroids correspond to "biased graphs," which consist of a graph and a collection of its cycles that satisfy a certain property. Biased graphs are independently well-studied and can be used to define three classes of matroids, including frame matroids [52, 53, 54].

The cycle matroid $M(G)$ of a graph G is the matroid whose ground set is the edge set of G and a subset X of edges is a circuit if and only if the subgraph of G induced on X is a cycle. In [39], Nelson and Van der Pol proved that the number of biased graphs on a complete graph K_{n+1} is $2^{\frac{1}{2}n!(1+o(1))}$. Using Crapo's characterization, this implies that the number of coextensions of the cycle matroid of a complete graph $M(K_{n+1})$ is $2^{\frac{1}{2}n!(1+o(1))}$. Using some similar techniques and some new techniques, we prove the following similar results for the complete bipartite graphs $K_{n,n}$ and $K_{n,n-1}$ in Chapter 6. Note that $o(1)$ denotes an unspecified function of n that goes to 0 as n goes to infinity and log denotes the base-2 logarithm.

Theorem 1.0.1 (Theorem 6.0.1)**.** $\log \operatorname{coext}(M(K_{n,n})) = \frac{n!^2}{2n}(1 + o(1))$.

Theorem 1.0.2 (Theorem 6.0.2)**.** $\log \operatorname{coext}(M(K_{n,n-1})) = \frac{n!(n-2)!}{4}(1 + o(1))$.

We also prove the following similar result for the extensions of the cycle matroid of a complete graph in Chapter 9.

Theorem 1.0.3 (Theorem 9.2.1)**.** $\log \operatorname{ext}(M(K_{n+1})) = \binom{n}{n/2}(1 + o(1))$.

Dowling geometries are a subclass of frame matroids which are defined using certain graphs whose edges are labelled by the elements of a finite group. They are objects that resemble complete graphs, but have additional symmetry arising from groups. A Dowling geometry with rank n over a finite group Γ is denoted $DG(n, \Gamma)$. Dowling geometries play a role among frame matroids similar to that of the cycle matroids of complete graphs among simple graphic matroids. A natural starting point when studying extensions and coextensions of frame matroids would be to consider those of Dowling geometries. We would expect to obtain results similar to those for complete graphs and, thus, we might expect this problem to be straightforward. However, determining the number of coextensions of a Dowling geometry is a significantly different problem than that for cycle matroids of complete graphs. In Chapter 7, we determine asymptotic upper and lower bounds on the number of coextensions of a Dowling geometry. These bounds imply the following asymptotic result.

Theorem 1.0.4 (Corollary 7.0.2)**.** $\log(\log(\operatorname{coext}(DG(n, \Gamma)))) = n \log n(1 + o(1))$.

Partial progress towards determining the asymptotic number of extensions of Dowling geometries appears in Chapter 10.

Similar to how complete graphs are related to simple graphs or Dowling geometries are related to frame matroids, projective geometries are related to representable matroids.

Specifically, projective geometries are the densest simple representable matroids. (See Chapter 2 for the definitions of these matroids.) A projective geometry of rank n over a finite field $GF(q)$ is denoted $PG(n-1,q)$. In this case, the problem of counting extensions is straightforward, since each extension of a projective geometry corresponds to a flat. On the other hand, the coextensions of projective geometries are not so easily determined. In Chapter 8 we give asymptotic upper and lower bounds for the number of coextensions of a projective geometry. These bounds imply the following asymptotic result.

Theorem 1.0.5 (Corollary 8.0.2)**.** $\log(\log(\text{coext}(PG(n-1,q)) = n^2(\log(q)+o(1))$.

1.1 Background

Beginning with matroid enumeration in general, the first natural problem is to determine the number of matroids on an n-element ground set, for each positive integer n. This is still an open problem; however, progress has been made for small n and for some classes of matroids. In 1973, Blackburn, Crapo, and Higgs [7] explicitly determined all non-isomorphic simple matroids on at most eight elements. This list of matroids was expanded in 1984 when Acketa [1] added all non-isomorphic matroids on at most eight elements, including a complete sublist of the non-isomorphic binary matroids on at most eight elements. In 2008, Mayhew and Royle [29] created an online database of all non-isomorphic n-element matroids where $n \leq 9$. They estimate that the number on 10 elements exceeds 2.5×10^{12}, so determining a precise list for $n \geq 10$ might be intractable. Since exact enumeration seems out of reach for larger n, our interest turns to finding good bounds, perhaps for specific matroid classes.

Let $m(n)$ denote the number of matroids on ground set $[n] = \{1,\ldots,n\}$. Some general lower and upper bounds on $m(n)$ are known. In 1965, Crapo [10] used analysis of single element extensions of matroids to show that $m(n) \geq 2^n$. Knuth [22] improved this lower bound in 1974 by showing that $\log\log m(n) \geq n - \frac{3}{2}\log n + O(\log\log n)$, where log is the base-2 logarithm. This lower bound is obtained by constructing a large set of "sparse paving" matroids. In 2013, Mayhew and Welsh [30] showed that the asymptotics in Knuth's bound can be improved slightly by adapting his argument with a coding theory result of Graham and Sloane [17].

Since there are 2^n subsets of an n-set and each collection of these subsets could be an independent set of a matroid, there are at most 2^{2^n} matroids on n elements. This trivial upper bound was first improved by Piff and Welsh [46] in 1971. Piff [45] improved the upper bound again in 1973 when he showed that $\log\log m(n) \leq n - \log n + O(\log\log n)$. In a breakthrough result from 2015, Bansal, Pendavingh, and Van der Pol [6] made significant progress towards finding matching upper and lower bounds for $m(n)$. They proved that $\log \log m(n) \leq n - \frac{3}{2} \log n + \frac{1}{2} \log \frac{2}{\pi} + 1 + o(1)$, which almost matches the best known lower bound. These upper and lower bounds show that $m(n)$ grows doubly exponentially with n, and is much closer to 2^{2^n} than to Crapo's earlier lower bound of 2^n; however, studying extensions has the potential to contribute to lower bounds for the number of matroids in a certain class or with a certain structure.

Similar to the proof of the lower bound for $m(n)$, Bansal, Pendavingh, and Van der Pol [6] use analysis of "sparse paving" matroids to obtain their upper bound on $m(n)$. A matroid M is *paving* if every circuit has size at least the rank of M. A matroid is *sparse paving* if it and its dual are paving or, equivalently, if every non-spanning circuit is a hyperplane. Pendavingh and Van der Pol [41, 42, 43, 44, 50], sometimes together with Bansal [5, 6], have made several important contributions to the enumeration of sparse paving matroids, matroids of a fixed rank, and matroids in general. Throughout their work, they represent sparse paving matroids as stable sets in an auxiliary graph and determine matroid enumeration bounds by first determining bounds on the number of stable sets in this graph. This strategy of representing the combinatorial objects they are counting by stable sets in an auxiliary graph is very similar to the underlying strategy used in this thesis, as we will see in Chapter 3.

The work of Bansal, Pendavingh, and Van der Pol suggests that sparse paving matroids dominate the set of all matroids, which gives support to the conjecture of Mayhew, Newman, Welsh, and Whittle [27] that almost all matroids are paving. This conjecture appears in Oxley's book "Matroid theory" [40] alongside several other open problems in matroid enumeration. In addition to the breakthrough towards proving almost all matroids are paving, other important progress has been made towards solving a number of the open enumeration problems listed in [40] since it was published in 2011. For example, Oxley conjectured that almost all matroids are not representable over any field and an exciting result from Nelson [36] in 2018 proved that this is true.

Representable matroids have been omnipresent in the study of matroid theory since its inception. We also focus on representable matroids, for the most part, in this thesis. This may seem surprising since Nelson proved in [36] that almost all matroids are not representable; however, note that the extensions of a representable matroid are not necessarily all representable. In fact, we show in this thesis that, for some n-element representable matroids, the number of extensions is significantly larger than the number of $(n+1)$-element representable matroids. Enumerating the extensions of a matroid is very difficult in general, so it is natural to focus on familiar classes of matroids, such as representable matroids or graphic matroids.

Since biased graphs parameterize coextensions of graphic matroids and define frame matroids, they are of interest in this thesis. In 1991, Zaslavsky introduced three matroid classes that arise from biased graphs: frame, lift, and complete lift [52, 53, 54]. Note that *complete lift* matroids are extensions of lift matroids, and frame matroids are called "bias matroids" in Zaslavsky's papers. Frame matroids that arise from group-labelled graphs are also discussed, under a different name, in Kahn and Kung's seminal 1982 paper on combinatorial geometries [18]. Another connection between biased graphs and group-labelled graphs is given by Nelson and Park in [37]. They prove a Ramsey theorem for biased graphs which shows that if $(G, \mathcal{B})$ is a biased graph where G is a very large complete graph, then G contains a large complete subgraph H with balanced circuits $\mathcal{B}'$ such that $(H, \mathcal{B}')$ arises from a Γ-labelled graph for some finite cyclic group Γ. Thus, frame matroids, and specifically those that arise from group-labelled graphs, are familiar classes of matroids for us to focus on in this thesis.

Even within a familiar class, enumerating the extensions of a matroid is difficult. Therefore, our goal is to enumerate the extensions of specific matroids within the classes of graphic matroids, representable matroids, and frame matroids. Since all simple graphs are subgraphs of a complete graph, it is natural to first consider the cycle matroids of complete graphs. In [18], Kahn and Kung show that Dowling geometries are the universal models for the collection of frame matroids from group-labelled graphs and projective geometries are the universal models for the collection of all geometries coordinatizable over the finite field $GF(q)$. Importantly, each simple representable matroid is a restriction of a projective geometry and each simple frame matroid that arises from a group-labelled graph is a restriction of a Dowling geometry. Therefore, within the classes of representable and frame matroids, it is natural to first consider projective geometries and Dowling geometries.

Studying the extensions of matroids within a certain minor-closed class has the potential to reveal common structures in the excluded minors of the class. Minor-closed classes of matroids are a popular topic of study, prompted by a desire to generalize Robertson and Seymour's Graph-Minors Project. In the 2020 paper by Mayhew, Newman, and Whittle [28], the concept of "fractal" classes of matroids is introduced. Let $\mathcal{M}$ be a minor-closed class of matroids and let $\mathcal{EX}$ be the class of excluded minors for $\mathcal{M}$, which are the minor-minimal matroids not in the class. Let m_n and x_n denote the number of non-isomorphic n-element matroids in $\mathcal{M}$ and $\mathcal{EX}$, respectively. The class $\mathcal{M}$ is *strongly fractal* if $\lim_{n\to\infty} \frac{x_n}{m_n+x_n} = 1$. Their example of a strongly fractal class is the collection of sparse paving matroids with at most k circuit-hyperplanes. Essentially, the "boundary," or set of excluded minors, of such a class is eventually larger than the class itself. Following this idea, it is natural to ask what the "boundaries" of other classes of matroids look like. Since the excluded minors are the minor-minimal matroids not in the class, we can think of them as the extensions and coextensions of matroids in the class that fall out of the class. This thesis could potentially be used to shed light on the relationship between minor orders and enumeration.

When dealing with minor-closed classes of matroids, we can gain some intuition about this boundary by focusing on the extensions and coextensions of the most dense matroids of the class. For each minor-closed class $\mathcal{M}$ and positive integer n, define the *extremal function*, denoted $h_{\mathcal{M}}(n)$, to be the maximum number of elements in simple rank-r matroid in $\mathcal{M}$, where $r \leq n$. Let $h_{\mathcal{M}}(n) = \infty$ if such a maximum does not exist. We say that matroids are "dense" if they have close to $h_{\mathcal{M}}(n)$ elements. Determining $h_{\mathcal{M}}(n)$ is a widely studied problem. The Growth Rate Theorem proved by Geelen, Kung, and Whittle [13] shows that either $\mathcal{M}$ contains $U_{2,n}$ for all $n \geq 2$, or the extremal function is linear, quadratic, or exponential in n.

Upper bounds for the extremal function have been determined for many interesting classes. Kung [25] proved in 1993 that for each integer $\ell \geq 2$, each simple rank-r matroid M with no $U_{2,\ell+2}$-minor satisfies $|E(M)| \leq \frac{\ell^r-1}{\ell-1}$. In 2010, Geelen and Nelson [14] showed that if r is large enough and q is the largest prime power at most ℓ, then $|E(M)| \leq \frac{q^r-1}{q-1}$. Additionally, they showed that equality holds if and only if M is a projective geometry $PG(r-1, q)$. That is, projective geometries are the most dense matroids in the minor-closed class of simple matroids omitting a fixed rank-2 uniform minor.

Projective geometries are not representable over all fields, so it is natural to wonder which matroids are most dense in the minor-closed class of $\mathbb{K}$-representable matroids omitting a fixed rank-2 uniform minor, where $\mathbb{K}$ is a field of characteristic zero. Nelson conjectures that the most dense members of this class are no denser than Dowling geometries [35]. Recently, Geelen, Nelson, and Walsh [15] showed that the conjecture is true if $\mathbb{K} = \mathbb{C}$. Since projective and Dowling geometries are the dense examples in some interesting minor-closed classes, understanding their extensions and coextensions will contribute to our understanding of the boundaries of these classes.

1.2 Summary of results

In this thesis, we will determine the asymptotic behaviour on the log scale of the number of extensions of certain matroids M in terms of the rank of M. Here, it is important to note that the results in this thesis are all joint work with Peter Nelson and Jorn van der Pol. In this section, we summarize these results in Tables 1.1 and 1.2. In order to more easily compare these results with each other and with the number of matroids on a fixed ground set, we express the results both in terms of rank and ground set size.

First, we define some asymptotic notation. For functions $f, g : \mathbb{Z}_{>0} \to \mathbb{R}$, we say $f(n) = O(g(n))$ if $g(n) > 0$ and $\limsup_{n\to\infty} \frac{f(n)}{g(n)} < \infty$. We say $f(x) = \Theta(g(x))$ if $f(x) = O(g(x))$ and $g(x) = O(f(x))$.

	matroid	rank	# elements	log log (# extensions)
1	$PG(n-1, q)$	n	$N = \dfrac{q^n - 1}{q - 1}$	$\log\log \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q = \Theta(\log\log N)$
2	$M(K_{n+1})$	n	$N = \dbinom{n+1}{2}$	$\log \dbinom{n}{n/2} = \Theta(\sqrt{N})$

Table 1.1: Summary of extension results.

A summary of the results in this thesis is given in Tables 1.1 and 1.2, where for each matroid M, the number of extensions or coextensions is expressed on the double log scale both in terms of the rank of M and number N of elements in M. To simplify the table, lower order terms (that is, factors of $(1 + o(1))$) are ignored. To compare these results to the number of matroids on a fixed ground set, recall that $m(N)$ denotes the number of matroids on the ground set $[N]$ and $\log\log m(N) = N(1+o(1))$. Another interesting point of reference is the number of representable matroids with ground set $[N]$, which is at most $3 \log N$ on the double log scale [36].

The result in Row 1 of Table 1.1 is well known and we discuss it in Proposition 9.1.1. Row 2 of Table 1.1 is implied by Theorem 9.2.1. Nelson and Van der Pol [39] proved the result in Row 1 of Table 1.2, which is discussed in Theorem 6.1.4. Theorems 6.0.1 and 6.0.2 imply Rows 2 and 3 of Table 1.2. The result in Row 4 of Table 1.2 is obtained in Corollary 7.0.2. Corollary 8.0.2 contains the result in Row 5 of Table 1.2.

Although the results in Tables 1.1 and 1.2 are expressed on the double log scale, the main results in this thesis are on the log scale. The corresponding log scale results for Row 2 in Table 1.1 and Rows 1, 2, and 3 in Table 1.2 are all exact, up to lower order terms, which is why the left hand side of the rightmost column is sometimes more precise than it needs to be for the double log scale. The corresponding log scale results for Rows 4 and 5 in Table 1.2 are in the form of upper and lower bounds that differ. Conjectures for the exact values, up to lower order terms, can be found in Chapters 7 and 8, respectively.

	matroid	rank	# elements	log log (# coextensions)
1	$M(K_{n+1})$	n	$N = \binom{n+1}{2}$	$\log(\frac{1}{2}n!) = \Theta(\sqrt{N}\log N)$
2	$M(K_{n,n})$	$2n-1$	$N = n^2$	$\log \frac{n!(n-1)!}{2} = \Theta(\sqrt{N}\log N)$
3	$M(K_{n,n-1})$	$2n-2$	$N = n^2 - n$	$\log \frac{n!(n-2)!}{4} = \Theta(\sqrt{N}\log N)$
4	$DG(n,\Gamma)$	n	$N = \binom{n}{2}q + n$	$n\log n = \Theta(\sqrt{N}\log N)$
5	$PG(n-1,q)$	n	$N = \frac{q^n - 1}{q - 1}$	$n^2 \log q = \Theta(\log^2 N)$

Table 1.2: Summary of coextension results.

1.3 Outline

This thesis begins with preliminary definitions and results in Chapter 2. In Chapter 3, we describe a method for bounding the number of extensions of any matroid M. This method begins in Section 3.1 by using a result of Crapo [10] to parameterize the extensions as "linear subclasses." A *linear subclass* of a matroid M is a set $\mathcal{H}'$ of hyperplanes where if two intersect in a rank-$(r(M)-2)$ flat F, then all hyperplanes that contain F are in $\mathcal{H}'$ as well. If, for each rank-$(r(M)-2)$ flat F of M, at most one hyperplane that contains F is in $\mathcal{H}'$, then we say $\mathcal{H}'$ is a *scarce* linear subclass. In Section 3.2, we represent the scarce linear subclasses of a matroid M as stable sets in an auxiliary graph whose vertices are the hyperplanes of M where two vertices are adjacent if and only if they intersect in a rank-$(r(M)-2)$ flat. Also in Section 3.2, we show that the number of extensions of M can be bounded by the stable sets in this auxiliary graph and the size of a set of "small" hyperplanes. The methods from Chapter 3 are then applied to specific matroids in Chapters 6, 7, 8, 9, and 10.

Chapter 6 discusses coextensions of graphic matroids. In Section 6.1, the connection between biased graphs and graphic matroids is reviewed. In Section 6.2, we determine the asymptotic number of coextensions of the cycle matroid of a complete bipartite graph $K_{n,n}$ or $K_{n,n-1}$. Chapter 7 determines asymptotic upper and lower bounds on the number of coextensions of a Dowling geometry. Continuing with coextensions, we determine upper and lower bounds for the number of coextensions of a projective geometry in Chapter 8.

In Chapter 9, we discuss the extensions of some representable matroids, mainly projective geometries and cycle matroids of complete graphs. The asymptotic number of extensions of the cycle matroid of a complete graph is given in Section 9.2. In Chapter 10, we discuss the number of extensions of Dowling geometries.

Before these chapters that apply the theory developed in Chapter 3, however, are two chapters with results that are applied later in the thesis, but that are also interesting independent of their applications here. In Chapter 4, we prove bounds on the number of stable sets in certain generalized Hamming graphs. The auxiliary graph in the case of Dowling geometry coextensions is comparable to these Hamming graphs, so the results of Chapter 4 are used in Chapter 7. In Chapter 5, we give a brief introduction to container methods for bounding the number of stable sets in graphs. This chapter also contains the details of two container methods that will be used in Chapters 6 and 8.

Chapter 2

Preliminaries

2.1 Basic definitions and notation

This thesis deals with finite graphs and matroids, so sets are assumed to be finite unless otherwise indicated. The cardinality of a set X is denoted $|X|$ and its collection of subsets is denoted 2^X. The collection of k-subsets of a set X is denoted $\binom{X}{k}$. The sets of positive integers, nonnegative integers, integers, and real numbers are denoted $\mathbb{Z}_{>0}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}$, and $\mathbb{R}$. A finite field (sometimes called a Galois field) of order q, where q is a prime power, is denoted $GF(q)$. For $m, n \in \mathbb{Z}_{\geq 0}$ where $m \leq n$, let

$$[m, n] = \{m, m+1, \ldots, n\}.$$

If $m = 1$, then we write $[n] = [1, n] = \{1, 2, \ldots, n\}$.

When naming sets, we usually use uppercase Latin letters for sets (e.g. X), uppercase script Latin letters for sets of sets (e.g. $\mathcal{X}$), and uppercase blackboard-bold Latin letters for sets of sets of sets (e.g. $\mathbb{X}$). We also use set and collection interchangeably, often to emphasize when a set has sets as elements.

If X and Y are sets, then $X \setminus Y = \{x \in X : x \notin Y\}$. If X is a subset of a set Y, then the *complement* of X is the set $Y \setminus X$. We say sets X and Y *intersect* in a set W if $X \cap Y = W$. If X and Y are sets, then their *union* is $X \cup Y = \{w : w \in X \text{ or } w \in Y\}$. The Cartesian product of sets X and Y, denoted $X \times Y$, is the set $\{(x, y) : x \in X,\ y \in Y\}$. For a positive integer n, let X^n denote the Cartesian product of n copies of a set X. Let S be a set and let $\mathcal{X}$ be a collection of subsets of S. We say a set $L \subseteq S$ is *linear with respect to* $\mathcal{X}$ if $|L \cap X| \in \{0, 1, |X|\}$ for all $X \in \mathcal{X}$.

A *tuple* $\tau = (t_1, t_2, \ldots, t_n)$ is a finite collection of elements, called *coordinates*, in which repetition is allowed and order matters. A tuple is called a n-tuple if it contains n coordinates. Each coordinate of a tuple has an index which describes its position in the tuple. We assume that the first coordinate in the sequence has index 1 unless otherwise specified. In some cases, it will be convenient to let the first coordinate have index 0. However, assuming that the first index is 1, if $\tau = (t_1, t_2, \ldots, t_n)$, then the index of t_i is i, for each $i \in [n]$. We may use $\tau(i)$ to denote the coordinate in the tuple τ with index i.

An *infinite sequence* is a function with domain $\mathbb{Z}_{>0}$ or $\mathbb{Z}_{\geq 0}$ and a *finite sequence* is a function with domain $[n]$ or $[0, n-1]$ for some positive integer n. In this thesis, we assume sequences are finite unless otherwise specified. If a sequence σ has domain $[n]$ or $[0, n-1]$, then we say σ has n elements or has length n, and we often denote this sequence $\sigma(1)\sigma(2)\ldots\sigma(n)$ or $\sigma(0)\sigma(1)\ldots\sigma(n-1)$. We refer to the elements in the range of a sequence σ as the elements in σ. Note that tuples are equivalent to finite sequences.

A *permutation* is a bijection from a set to itself. For a set X, define $\mathcal{S}(X)$ to be the set of permutations of X. In this thesis, we restrict our attention to the permutations of $[n]$. A permutation of $[n]$ is an n-element sequence whose elements are in $[n]$ and each element is in the sequence exactly once. If a permutation in cycle notation can be expressed as the product of an even number of transpositions, then it is an *even permutation*; otherwise, it is an *odd permutation*.

If $f : X \to Y$ is a function from X to Y and $X' \subseteq X$, then $f(X') = \{f(x) : x \in X'\}$. The support of a function $f : X \to Y$, denoted $\mathrm{supp}(f)$, is the set of elements x in X where $f(x) \neq 0$. If $f : X \to Y$ is a function, then the restriction of f to $X' \subseteq X$ is the function $f|_{X'} : X' \to Y$ where $f|_{X'}(x) = f(x)$ for each $x \in X'$. If $f : X \to Y$ and $g : W \to Z$ are functions and for all $x \in X \cap W$ we have $f(x) = g(x)$, then $f \cup g$ is defined as a function from $X \cup W$ to $Y \cup Z$ where

$$(f \cup g)(x) = \begin{cases} f(x) & \text{if } x \in X, \\ g(x) & \text{otherwise.} \end{cases}$$

In this thesis, we use log to denote the base-2 logarithm and ln to denote the natural logarithm. For each real number x, we define $\binom{n}{x}$ to be $\binom{n}{\lfloor x \rfloor}$.

Since the main results of this thesis are *asymptotic*, we define the following asymptotic notation. Consider functions $f, g : \mathbb{Z}_{>0} \to \mathbb{R}$. Recall that $f(n) = O(g(n))$ if $g(n) > 0$ and $\limsup_{n\to\infty} \frac{f(n)}{g(n)} < \infty$. We say $f(n) = o(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. Thus, if $f(n) = o(1)$, then $\lim_{n\to\infty} f(n) = 0$. We say f is *on the order* of g if $f(n) = g(n) + o(g(n)) = (1 + o(1))g(n)$. That is, f is on the order of g if it can be expressed as a finite sum of functions where one is equal to g and the rest are functions that, when divided by g, go to 0 as n goes to infinity. We refer to these functions that go to 0 as n goes to infinity as *lower order* terms.

In general, we use $o(1)$ ("little-o of 1") to denote a function of n that goes to 0 as n goes to infinity. If we want to use little-o notation to describe a function over a different variable, say q, that goes to 0 as q goes to infinity, we use $o_q(1)$.

2.2 Useful bounds and sums

Binomial coefficients appear often in this thesis, so we identify some useful bounds for binomial coefficients and sums of binomial coefficients here. The following standard bounds will be used throughout the thesis:

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \sum_{i=0}^{k} \binom{n}{i} \leq \left(\frac{\mathrm{e}n}{k}\right)^k.$$

The first inequality follows from the observation that $\frac{x}{y} \leq \frac{x-1}{y-1}$ for all $1 \leq y \leq x$. The middle inequality follows from the observation that $\binom{n}{k}$ is a term in the sum. The last inequality follows from the binomial theorem and the bound $1 + x < \mathrm{e}^x$ for $x \neq 0$. More precisely, notice that $\sum_{i=0}^{k} \binom{n}{i} \leq \sum_{i=0}^{k} \binom{n}{i} \frac{(k/n)^i}{(k/n)^k} = \frac{n^k}{k^k} \sum_{i=0}^{n} \binom{n}{i} (k/n)^i 1^{n-i}$, which is at most $\frac{(1+k/n)^n n^k}{k^k}$ by the binomial theorem. Using the bound $1 + x < \mathrm{e}^x$ for $x \neq 0$ with $x = k/n$, we get the last inequality.

Now we focus on better bounds for the central binomial coefficient. First, we give the following version of *Stirling's approximation* for $n!$ [47]:

$$\sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n \mathrm{e}^{\frac{1}{12n+1}} < n! < \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n \mathrm{e}^{\frac{1}{12n}}. \tag{2.1}$$

These bounds imply the following bounds for the central binomial coefficient almost directly. Using the inequality $1 + x < \mathrm{e}^x$ for $x \neq 0$ for the lower bound, we find:

$$\left(1 - \tfrac{1}{4n}\right)\sqrt{2/\pi}\frac{2^n}{\sqrt{n}} \leq \binom{n}{n/2} \leq \sqrt{2/\pi}\frac{2^n}{\sqrt{n}}. \tag{2.2}$$

When k is "close" to $n/2$, we can get a better bound for $\sum_{i=0}^{k} \binom{n}{k}$ using the Chernoff Bound for binomial distributions, which is stated in the next theorem. We use $\mathbb{P}(A)$ to denote the probability of an event A and we refer to the book by Alon and Spencer [3] for the prerequisite definitions for the following result.

Theorem 2.2.1 (Chernoff Bound for Binomial Distribution [3])**.** *Let n be a positive integer and let $0 \leq p \leq 1$. Let $X \sim \mathrm{Bin}(n,p)$. If $0 < \varepsilon < 1$, then $\mathbb{P}(X \leq (1-\varepsilon)np) \leq \mathrm{e}^{-\frac{\varepsilon^2 np}{2}}$.*

Corollary 2.2.2. *Let n be a positive integer. If $0 < \varepsilon < 1$, then*

$$\sum_{i=0}^{\frac{n}{2}(1-\varepsilon)} \binom{n}{i} \leq 2^n \mathrm{e}^{-\frac{\varepsilon^2 n}{4}}.$$

Proof. Let $p = \frac{1}{2}$ and let $X \sim \mathrm{Bin}(n,p)$. The expected value of X is $\frac{n}{2}$. The probability that X is at most $\frac{n}{2}(1-\varepsilon)$ is

$$\mathbb{P}\left(X \leq \frac{n}{2}(1-\varepsilon)\right) = \sum_{i=0}^{\frac{n}{2}(1-\varepsilon)} \binom{n}{i} p^i (1-p)^{n-i} = \sum_{i=0}^{\frac{n}{2}(1-\varepsilon)} \binom{n}{i} \left(\frac{1}{2}\right)^n.$$

Furthermore, Theorem 2.2.1 implies that $\mathbb{P}(X \leq (1-\varepsilon)np) \leq \mathrm{e}^{-\frac{\varepsilon^2 np}{2}}$. Therefore, we find $\sum_{i=0}^{\frac{n}{2}(1-\varepsilon)} \binom{n}{i} \leq 2^n \mathrm{e}^{-\frac{\varepsilon^2 n}{4}}$. □

Bounds for sums of reciprocals of factorials will also be useful, so we make note of the following identities:

$$\sum_{k \geq 0} \frac{1}{k!^2} = I_0(2) \approx 2.28 \qquad \text{and} \qquad \sum_{k \geq 1} \frac{1}{k!(k-1)!} = I_1(2) \approx 1.59,$$

where $I_n(z)$ is the modified Bessel function of the first kind. The specific values of these sums are not of concern except that they are clearly less than e.

The following lemma, which appears in [9], gives a useful bound for the size of a collection of subsets of $[n]$ that intersect with the subsets in another collection, with certain properties, in a known amount. The proof makes use of the entropy function (from information theory), which can also be used to bound partial sums of binomial coefficients. We do not give bounds that use the entropy function here, as they are not needed in this thesis, but note that they have a similar flavour to the other bounds given.

Lemma 2.2.3 (Combinatorial Shearer's Lemma [9])**.** *Let $\mathcal{F}$ be a family of subsets of $[n]$ with each $i \in [n]$ included in at least t members of $\mathcal{F}$. Let $\mathcal{A}$ be another set of subsets of $[n]$. Then*

$$|\mathcal{A}| \leq \prod_{F \in \mathcal{F}} |\{A \cap F : A \in \mathcal{A}\}|^{1/t}.$$

2.3 Graph theory

In this section, we give a brief introduction to the terminology and notation of graph theory and then state or prove some preliminary results. The graphs in this thesis are encountered in two fundamentally different contexts. There are graphs that give rise to matroids, which may have parallel edges or loops, and there are auxiliary graphs used to represent the extensions of a matroid, which are always simple graphs.

2.3.1 Basic definitions

A *graph* G consists of a set $V(G)$ of *vertices* and a set $E(G)$ of *edges* and an incidence function $f_G : E(G) \to \binom{V(G)}{2} \cup V(G)$, where $\binom{V(G)}{2}$ is the collection of 2-subsets of $V(G)$. If $e \in E(G)$ and $f_G(e) = \{u, v\}$, then we may say $e = uv$ or $e = vu$. If $e = uv$, then we refer to u and v as the *endpoints* or *ends* of the edge e and we say e is an edge *between* u and v. If $e \in E(G)$ and $f_G(e) = \{u\}$, then e is called a *loop* or, specifically, a *loop on* u. If $e = uv$ and $e' = uv$, then we say e and e' are *parallel edges*. In some cases, we are interested in graphs that do not have loops or parallel edges. A *simple* graph G is a graph that does not contain loops or parallel edges. If G is a simple graph, it is convenient to think of $E(G)$ as a set of unordered pairs of distinct vertices. Notice that every simple graph is a graph, so all following definitions made for graphs hold for simple graphs as well.

If $e = uv$ is an edge of a graph G, then we say u and v are *adjacent* in G. If it is clear from context which graph u and v are adjacent in, then we simply say that u and v are adjacent. A vertex u is a *neighbour* of a vertex v if u and v are adjacent. If v is a vertex in a graph G, we define the *neighbourhood* of v, denoted $N_G(v)$, to be the set of vertices in G that are adjacent to v. If $e = uv$ is an edge of a graph G, then we say e is *incident* with u and v. If $e' = wu$ is another edge in G, then we say e and e' are *incident* edges.

We call $V(G)$ the *vertex set* and $E(G)$ the *edge set* of the graph G. If $V(G)$ is empty, then $E(G)$ is as well and G is called the *empty graph*. In this thesis, we assume that all graphs are not the empty graph.

The *degree* of a vertex v in a graph G, denoted $\deg_G(v)$, is the number of edges incident with v, where a loop on v is considered to be incident twice. That is, $\deg_G(v) = |\{uv \in E(G) : u \in V(G) \text{ and } u \neq v\}| + 2|\{e \in E(G) : f_G(e) = \{v\}\}|$. If the graph is clear from context, then we write $\deg(v)$ instead of $\deg_G(v)$. A graph G is *regular* if every vertex in G has the same degree. A graph G is *d-regular* if $\deg(v) = d$ for every $v \in V(G)$.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and $f_H = f_G|_{E(H)}$. A subgraph H of a graph G is called an *induced subgraph* if $V(H) \subseteq V(G)$ and

$E(H) = \{uv \in E(G) : u, v \in V(H)\} \cup \{e \in E(G) : f_G(e) = \{v\} \text{ and } v \in V(H)\}$. That is, an induced subgraph H of G has a vertex set V that is a subset of $V(G)$ and contains all edges in G whose endpoints are both in V. If H is an induced subgraph of G, then we say H is the subgraph of G *induced* on $V(H)$, which is denoted $G[V(H)]$. For a graph G with vertex subset V, if e is an edge in $G[V]$, we say e is *spanned* by V. Usually, when we refer to induced subgraphs, we mean subgraphs induced on a set of vertices; however, it is sometimes convenient to talk about subgraphs induced on a set of edges. If E is a subset of the edges in a graph G, then the subgraph of G induced on E, denoted $G[E]$, has edge set E and vertex set $\{v \in V(G) : v \in f_G(e) \text{ for some } e \in E\}$.

A graph G is *isomorphic* to a graph H if there exist bijections $\phi : V(G) \to V(H)$ and $\psi : E(G) \to E(H)$ such that $v \in V(G)$ and $e \in E(G)$ are incident in G if and only if $\phi(v)$ and $\psi(e)$ are incident in H. Equivalently, if G and H are simple graphs, then they are isomorphic if there exists a bijection $\phi : V(G) \to V(H)$ such that $u, v \in V(G)$ are adjacent in G if and only if $\phi(u), \phi(v)$ are adjacent in H.

A *path* P is a simple graph with vertex set $\{v_0, v_1, \ldots, v_k\}$ and edge set $\{e_1, e_2, \ldots, e_k\}$ where $e_i = v_{i-1}v_i$ for each $i \in [k]$. This path P may be denoted by the sequence $v_0e_1v_1e_2 \ldots v_{k-1}e_kv_k$ or $v_ke_kv_{k-1} \ldots e_2v_1e_1v_0$. We say this path is a (v_0, v_k)-path or (v_k, v_0)-path and that v_0, v_k are the *ends* of the path. The *length* of a path is the number of edges it contains. We say P is a path in a graph G if P is a subgraph of G and P is a path. Since the edge set and incidence function of a path determines the path, we may denote the path $P = v_0e_1v_1e_2 \ldots v_{k-1}e_kv_k$ by the sequence of edges $e_1e_2 \ldots e_k$. If a path $P = v_0e_1v_1e_2 \ldots v_{k-1}e_kv_k$ is in a simple graph, then we may say $P = v_0v_1 \ldots v_k$. In the case where $v_0e_1v_1e_2 \ldots v_{k-1}e_kv_k$ is a path in a graph G that has parallel edges and it does not matter which edges are in the path, we may still refer to this path by the sequence of vertices $v_0v_1 \ldots v_k$.

A *cycle* C is a graph with vertex set $\{v_1, v_2, \ldots, v_k\}$ and edge set $\{e_1, e_2, \ldots, e_k\}$ where $e_1 = v_kv_1$ and $e_i = v_{i-1}v_i$ for $i \in [2, k]$. We say $C = v_ke_1v_1e_2 \ldots v_{k-1}e_kv_k$ if $V(C) = \{v_1, v_2, \ldots, v_k\}$ and $E(C) = \{e_1, e_2, \ldots, e_k\}$ where $e_1 = v_kv_1$ and $e_i = v_{i-1}v_i$ for $i \in [2, k]$. We say C is a cycle in a graph G if C is a subgraph of G and C is a cycle. The *length* of a cycle is the number of edges it contains or, equivalently, the number of vertices it contains. If a cycle contains n vertices, then we call it an n-cycle. A graph that does not contain a cycle is called a *forest*. Note that, since loops and parallel edges are cycles, forests are simple graphs. A *Hamiltonian path* P of a graph G is a path where $V(P) = V(G)$. A *Hamiltonian cycle* C of a graph G is a cycle where $V(C) = V(G)$.

A graph G is *connected* if there exists a (u, v)-path in G for each $u, v \in V(G)$. A graph is *disconnected* if it is not connected. The maximally connected subgraphs of a graph are called *components*. A graph that is connected and does not contain a cycle is called a *tree*. If T is a subgraph of a graph G where $V(T) = V(G)$ and T is a tree, then T is a *spanning tree* of G. For any tree T, the number of edges in T is one less than the number of vertices.

If V is a subset of the vertex set of a graph G, then we define $G \setminus V$ to be the graph $G[V(G) \setminus V]$. If H is a subgraph of G, then we define $G \setminus H$ to be $G \setminus V(H)$. If $E \subseteq E(G)$, then we define $G \setminus E$ to be the subgraph of G with vertex set $V(G)$ and edge set $E(G) \setminus E$. It is sometimes convenient to describe removing one vertex or one edge from a graph, rather than a set of vertices or edges. If x is a vertex or an edge of G, then we say $G - x$ is the graph $G \setminus \{x\}$.

If the incidence functions of two graphs G and H agree for each edge in $E(G) \cap E(H)$, then the *union* of G and H, denoted $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$, edge set $E(G) \cup E(H)$, and incidence function $f_G \cup f_H$. The graph obtained from a graph G by *subdividing* the edge $e \in E(G)$ is $G \setminus \{e\} \cup P$ where P is a (u, v)-path of length at least 2 whose vertices other than u and v are not in G. A *subdivision* of a graph G is a graph obtained from G by a sequence of edge-subdivisions.

A *matching* M of a graph G is a set of edges, none of which are loops, of G where no two edges in M have an endpoint in common. A *perfect matching* M of a graph G is a matching of G where every vertex in G is an endpoint of an edge in M.

An important concept in this thesis is that of stable sets. A *stable set* of a graph G is a set S of vertices of G such that $G[S]$ contains no edges. Stable sets are usually called independent sets, but since independent set has a different meaning in matroid theory, we use stable sets in this thesis. The size of a largest stable set in a graph G is often denoted $\alpha(G)$ and is called the *independence number* of G.

In this thesis, we will investigate the cycle matroids of complete graphs and complete bipartite graphs. A *complete graph* is a simple graph in which every pair of vertices is adjacent. We say K_n is the complete graph with vertex set $[n]$. A graph G is *bipartite* if its vertex set has a partition (A, B) such that every edge in G has one end in A and one in B. A *complete bipartite graph* G is a bipartite simple graph with vertex set bipartition (A, B) where ab is an edge in G for each $a \in A$ and $b \in B$. We say $K_{n,m}$ is the complete bipartite graph with vertex set $[n + m]$ and bipartition $([n], [n + m] \setminus [n])$.

A structure within these graphs that will play a significant role in determining the number of coextensions is a theta graph. A *theta graph* is a graph with precisely two vertices u, v of degree 3 and three internally disjoint (u, v)-paths. Equivalently, a theta graph is a subdivision of the graph on exactly two vertices with exactly three edges between them. Importantly, observe that the intersection of cycles C and C' is a path of length at least 1 if and only if $C \cup C'$ is a theta graph.

Hamming graphs play an important role in determining the number of coextensions of a Dowling geometry, so we give their definition here. Let $t, n_1, \ldots, n_t \in \mathbb{Z}_{>0}$ and let $S_1, \ldots, S_t$ be sets where S_i contains n_i elements for each $i \in \{1, \ldots, t\}$. The *generalized Hamming graph* $H(n_1, \ldots, n_t)$ is the graph whose vertices are t-tuples in $S_1 \times \cdots \times S_t$ where two vertices are adjacent if and only if they differ in exactly one coordinate. Hamming graphs can also be described as a Cartesian product of graphs. The *Cartesian product* of two simple graphs G and H, denoted $G \square H$, is the graph with vertex set $V(G) \times V(H)$ where vertices $(u, v), (x, y)$ are adjacent if and only if either $u = x$ and v is adjacent to y in H, or $v = y$ and u is adjacent to x in G. One can show that the Hamming graph $H(n_1, \ldots, n_t)$ is isomorphic to $K_{n_1} \square K_{n_2} \square \ldots \square K_{n_t}$.

2.3.2 Preliminary results

The first preliminary result is one that could be found in most introductions to graph theory. We mention it here, but it will be used without reference throughout this thesis.

Lemma 2.3.1 (Handshaking Lemma)**.** *If G is a graph, then $2|E(G)| = \sum_{v \in V(G)} \deg(v)$.*

An example of how this lemma could be used is given in the following lemma, which will itself be used in Chapter 6.

Lemma 2.3.2. *If G is a connected graph where $|E(G)| = |V(G)| + 1$ and C, C' are cycles of G that share an edge, then C, C' are in a theta subgraph of G.*

Proof. Let H be obtained from G by repeatedly removing vertices with degree 1 until none remain. Since G contains cycles, this process ends in a nonempty graph where every vertex has degree at least 2. Since removing a vertex of degree 1 decreases the number of edges and the number of vertices each by 1, we have $|E(H)| = |V(H)| + 1$. Furthermore, since removing vertices of degree 1 does not remove cycles, the cycles C and C' are contained in H.

By the Handshaking Lemma 2.3.1,

$$\sum_{v \in V(H)} \deg(v) = 2|E(H)| = 2|V(H)| + 2.$$

Since each vertex in H has degree at least 2, the degree sequence of H is either $4, 2, 2, \ldots, 2$ or $3, 3, 2, 2, \ldots, 2$. If the former, then H is the union of two cycles that intersect precisely in a vertex of degree 4. These cycles must be C and C', so C and C' do not share an edge, which is a contradiction. Therefore, the degree sequence of H is $3, 3, 2, 2, \ldots, 2$ and, since C and C' share an edge, it follows that H is a theta graph that contains C and C'. $\square$

Graphs that are 2-connected come up briefly in Chapter 6, so we mention the following theorem about 2-connected graphs here. A connected graph G that is not a complete graph is *k-connected* if k is the size of a smallest subset $V \subseteq V(G)$ such that $G \backslash V$ is disconnected. We say an *ear decomposition* of a graph G is a sequence of graphs $(G_0, G_1, \ldots, G_k)$ where G_0 is a cycle, $G_k = G$, and $G_i = G_{i-1} \cup P$ where P is a path whose ends are distinct vertices in G_{i-1} and no other vertices of P are in G_{i-1}.

Theorem 2.3.3 ([11])**.** *A graph G is 2-connected if and only if G has an ear decomposition.*

Stable set preliminaries

The remaining preliminaries in this section have to do with counting stable sets. As we will see in Chapter 3, the problem of counting extensions or coextensions can be reduced to the problem of counting stable sets in an auxiliary graph. Let $i(G)$ denote the number of stable sets of a graph G. We now prove a variety of propositions that bound $i(G)$ in terms of the stable sets of various substructures in G.

Proposition 2.3.4. *If H is an induced subgraph of a graph G, then $i(H) \leq i(G)$.*

Proof. Consider a stable set $S \subseteq V(H)$ of H. If S is not a stable set of G, then there exists an edge in G between two vertices in S, which contradicts the assumption that H is an induced subgraph of G; therefore, S is a stable set of G as well. □

Proposition 2.3.5. *If G is a graph and E is a subset of edges of G, then $i(G) \leq i(G \setminus E)$.*

Proof. Note that $G \setminus E$ is the subgraph of G with vertex set $V(G)$ and edge set $E(G) \setminus E$. Consider a stable set $S \subseteq V(G)$ of G. Since $G \setminus E$ is obtained from G by removing a set of edges, the set S is a stable set of $G \setminus E$ as well. □

Proposition 2.3.6. *If G is a graph whose components are $G_1, G_2, \ldots, G_k$, where $k \geq 1$, then $i(G) = \prod_{j=1}^{k} i(G_j)$.*

Proof. If S is a stable set of G, then $S \cap V(G_j)$ is a stable set of G_j for each $j \in [k]$. If S_j is a stable set of G_j for each $j \in [k]$, then $S_1 \cup \cdots \cup S_k$ is a stable set of G. Thus, each stable set of G is uniquely described by choosing a stable set of each G_j. There are $i(G_j)$ choices for a stable set of G_j, so there are $\prod_{j=1}^{k} i(G_j)$ stable sets of G. □

Proposition 2.3.7. *If G is a graph with induced subgraphs $G_1, G_2, \ldots, G_k$, where $k \geq 1$, such that $V(G) = \bigcup_{j=1}^{k} V(G_j)$, then $i(G) \leq \prod_{j=1}^{k} i(G_j)$.*

Proof. For each $i \in [k]$, let $V_i = V(G_i) \setminus \bigcup_{j=1}^{i-1} V(G_j)$. Observe that $\bigcup_{j=1}^{k} V_k = V(G)$. Thus, since $V_i \cap V_j = \emptyset$ for all $i \neq j \in [k]$, the graphs $G[V_1], \ldots, G[V_k]$ are disjoint induced subgraphs of G that span $V(G)$. Therefore, there exists a set $E \subseteq E(G)$ such that the graph $G' = (V(G), E(G) \setminus E)$ is the disjoint union of $G[V_1], \ldots, G[V_k]$. That is, the graphs $G[V_1], \ldots, G[V_k]$ are the components of G'. Since $V(G) = V(G')$, Proposition 2.3.5 implies that $i(G) \leq i(G')$. By Proposition 2.3.6, we have $i(G') = \prod_{j=1}^{k} i(G[V_j])$. For each $j \in [k]$, since the graph $G[V_j]$ is an induced subgraph of G_j, we have $i(G[V_j]) \leq i(G_j)$, and the claim follows. □

Proposition 2.3.8. *If H is an induced subgraph of a graph G, then*

$$i(G) \leq i(H) \cdot 2^{|V(G)|-|V(H)|}.$$

Proof. The number of pairs $(I, \mathcal{X})$ where I is a stable set of H and $\mathcal{X} \subseteq V(G) \setminus V(H)$ is $i(H) \cdot 2^{|V(G) \setminus V(H)|}$. Since each stable set of G can be partitioned into a stable set of H and a set of vertices in $V(G) \setminus V(H)$, there is at most one stable set of G for each such pair $(I, \mathcal{X})$. Therefore, the number of stable sets of G is at most $i(H) \cdot 2^{|V(G) \setminus V(H)|} = i(H) \cdot 2^{|V(G)|-|V(H)|}$. □

The next theorem gives a useful lower bound for the number of stable sets in a regular graph. It is obtained by greedily constructing stable sets.

Theorem 2.3.9. *If G is a $(d-1)$-regular graph with N vertices, where $d \geq 1$ and $N \geq 1$, then*

$$i(G) \geq d^{\lfloor \frac{N}{d} \rfloor}.$$

Proof. Let $k = \lfloor \frac{N}{d} \rfloor$ and let $\mathcal{S}$ be the set of stable sets of G with size k. Let $\mathcal{T}$ be the set of all tuples $(v_1, \ldots, v_k)$ for which $\{v_1, \ldots, v_k\}$ is in $\mathcal{S}$. Since each tuple in $\mathcal{T}$ is a set in $\mathcal{S}$ whose elements have been ordered, we have $|\mathcal{T}| = |\mathcal{S}| \cdot k!$.

Consider constructing a tuple $(v_1, v_2, \ldots, v_k) \in \mathcal{T}$ by adding v_i to the tuple $(v_1, \ldots, v_{i-1})$ at each step $i = 1, 2 \ldots, k$. There are N choices for v_1. At each step $i \geq 2$, there are $i-1$ vertices in the tuple and since each vertex in G has $d-1$ neighbours, there are at least $N-(i-1)-(i-1)(d-1) = N-(i-1)d$ choices for v_i. So there are at least $N(N-d)(N-2d)\ldots(N-(k-1)d)$ ways to construct an element of $\mathcal{T}$, which implies $|\mathcal{T}| \geq N(N-d)(N-2d)\ldots(N-(k-1)d)$.

Since $\mathcal{S}$ is a subset of the collection of stable sets of G, the size of $\mathcal{S}$ is a lower bound for $i(G)$. Therefore

$$\begin{aligned} i(G) &\geq |\mathcal{S}| = |\mathcal{T}|/k! \\ &\geq \frac{N(N-d)(N-2d)\dots(N-(k-1)d)}{\frac{N}{d}(\frac{N}{d}-1)(\frac{N}{d}-2)\dots(\frac{N}{d}-(k-1))} \\ &= d^{\lfloor \frac{N}{d} \rfloor}. \end{aligned}$$

□

Interestingly, Theorem 2.3.9 is best possible, as a complete graph on n vertices is an $(n-1)$-regular graph with exactly n stable sets.

2.4 Matroid theory

In this section, we give a brief introduction to matroid theory. The definitions and conventions used in this thesis mostly follow those of Oxley [40]. Note, though, that we use '$\backslash$' for both set deletion and matroid deletion, unlike in [40].

2.4.1 Basic definitions

There are several equivalent definitions of a matroid; in this section, we focus on two. A *matroid* M consists of a set $E(M)$ and a nonempty collection $\mathcal{B}(M)$ of subsets of $E(M)$ with the following property:

> For each $B, B' \in \mathcal{B}(M)$ and $e \in B \setminus B'$,
> there exists $e' \in B' \setminus B$ such that $(B \setminus \{e\}) \cup \{e'\} \in \mathcal{B}(M)$.

This property is called the *basis exchange axiom*. The set $E(M)$ is called the *ground set* of M and each subset in $\mathcal{B}(M)$ is called a *basis* of M. We say that the elements in $E(M)$ are the elements of the matroid M.

A subset of $E(M)$ is called an *independent set* of M if it is a subset of a basis, otherwise it is called a *dependent set*. The descriptions of subsets as independent or dependent comes from an equivalent definition of a matroid M as a ground set $E(M)$ and a collection $\mathcal{I}(M)$ of subsets of $E(M)$ (independent sets) with certain properties. This definition clearly shows the motivation of defining matroids to abstract the notion of linear independence, but since circuits play an important role in this thesis, we move on to the definition of matroids by their circuits.

A minimal dependent set in a matroid M is called a *circuit* of M. That is, every proper subset of a circuit is an independent set. The collection of circuits of a matroid M is denoted $\mathcal{C}(M)$. If a circuit has n elements, then we call it an n-circuit. A matroid is uniquely defined by its collection of bases, collection of independent sets, or collection of circuits. One can show [40] that a collection $\mathcal{C}$ of subsets of E is the collection of circuits of a matroid on E if and only if $\mathcal{C}$ has the following properties:

(C1) $\emptyset \notin \mathcal{C}$;

(C2) if $C, C' \in \mathcal{C}$ and $C \subseteq C'$, then $C = C'$; and

(C3) if C and C' are distinct members of $\mathcal{C}$ and $e \in C \cap C'$, then there is a member $C'' \in \mathcal{C}$ such that $C'' \subseteq (C \cup C') \setminus \{e\}$.

Property (C3) is known as the *circuit elimination axiom*. A circuit of a matroid M that contains exactly $r(M) + 1$ elements contains a basis of M, so we call it a *spanning* circuit. A circuit containing exactly three elements is called a *triangle*. A circuit containing exactly one element is called a *loop*. If a circuit contains exactly two elements, then we say those elements are *parallel*. A maximal set of pairwise parallel elements is called a *parallel class*. A matroid is *simple* if it contains no loops or parallel classes of size at least 2.

Much of the terminology of matroids is borrowed from graph theory and linear algebra, which also provide the fundamental examples of matroids. Using the definition of matroids by their circuits, it is straightforward to show that the collection of edge sets of cycles of a graph G are the circuits of a matroid with ground set $E(G)$. This matroid is called the *cycle matroid* of G, which is denoted $M(G)$. A matroid that is isomorphic to the cycle matroid of some graph is called a *graphic matroid*. The independent sets of $M(G)$ correspond to the subgraphs of G that do not contain any cycles.

The other fundamental example of a matroid comes from linear algebra. The set of bases of a matrix A corresponds to the set of bases of a matroid with ground set consisting of the column indices of A. This matroid, denoted $M[A]$, is called the *column matroid* of A. A matroid that is isomorphic to the column matroid of some matrix is called a *representable* matroid. A matroid M is $\mathbb{F}$-*representable* if there exists a matrix A over the field $\mathbb{F}$ such that M is isomorphic to $M[A]$.

Motivated by the concept of rank in linear algebra, the *rank* of a matroid M is the size of a basis of M. One can show that all bases of M have the same size, so this value is well-defined. The rank of M is denoted $r(M)$. The rank of a set X of elements of a matroid M, denoted $r_M(X)$, is the size of a largest independent set contained in X. When the matroid M is clear from context, we use $r(X)$ instead of $r_M(X)$. If a subset X of the ground set of a matroid M where $|X| = r(M)$ is not a basis, then it is called a *non-basis* of M.

The *closure* of a set $X \subseteq E(M)$ in a matroid M, denoted $\text{cl}_M(X)$, is the set $\{x \in E(M) : r(X \cup \{x\}) = r(X)\}$. If the matroid is clear from context, then we use $\text{cl}(X)$ instead of $\text{cl}_M(X)$. A *flat* of a matroid M is a subset F of $E(M)$ that is equal to its closure; that is, $F = \text{cl}(F)$. A *hyperplane* of M is a flat with rank $r(M)-1$. The collection of hyperplanes of a matroid M is denoted $\mathcal{H}(M)$. A *line* of M is a flat with rank 2.

An elementary argument shows that if M is a matroid, then $\{E(M)\setminus B : B \in \mathcal{B}(M)\}$ is the set of bases of a matroid with ground set $E(M)$. This matroid is called the *dual* matroid of M and is denoted M^*. The rank of M^* is $|E(M)| - r(M)$ and it is straightforward to show that $(M^*)^* = M$. The properties of sets in M^* are called "co-properties" in M. For example, a circuit in M^* is called a *cocircuit* in M and the rank of a set X in M^* is called the *corank* of X in M. This naming convention applies to loops, independent sets, bases, hyperplanes, and other properties as well.

Hyperplanes and circuits play a significant role in this thesis, as we will see that extensions and coextensions can be described by certain sets of these objects. Importantly, one can show that a set $X \subseteq E(M)$ is a hyperplane of a matroid M if and only if $E(M) \setminus X$ is a circuit of M^*. That is, X is a hyperplane if and only if $E(M) \setminus X$ is a cocircuit.

Corank is also an important concept here. When the matroid M is clear from context, we use $r^*(X)$ to denote the corank of $X \subseteq E(M)$ in M, which is equivalently the rank of X in M^*. The corank of M is the rank of M^*, which is denoted $r^*(M)$ or $r(M^*)$. For each subset X of the ground set of a matroid M, one can show with an elementary argument that $r^*(X) = r(E(M) \setminus X) + |X| - r(M)$.

The *deletion* of $X \subseteq E(M)$ from a matroid M, denoted $M \setminus X$, is the matroid with ground set $E(M)\setminus X$ whose bases are the maximal members of the set $\{B\setminus X : B \in \mathcal{B}(M)\}$. Note that the independent sets of $M\setminus X$ are the subsets of $E(M)\setminus X$ that are independent in M. The *simplification* of M, denoted $\text{si}(M)$, is the matroid obtained from M by deleting from M all loops and all but one element from each parallel class.

The *restriction* of a matroid M to a subset X of its ground set, denoted $M|X$, is the matroid $M\setminus(E(M)\setminus X)$. The *contraction* of $X \subseteq E(M)$ in a matroid M, denoted M/X, is the matroid $(M^*\setminus X)^*$. One can show that this matroid has ground set $E(M)\setminus X$ and basis set $\{B' \subseteq E(M) \setminus X : M|X$ has a basis B such that $B' \cup B \in \mathcal{B}(M)\}$. A matroid N is a *minor* of a matroid M if there exist disjoint sets $C, D \subseteq E(M)$ such that $N = M/C \setminus D$.

Although defined earlier, a matroid M is an *extension* of a matroid N if there exists $X \subseteq E(M)$ such that $M \setminus X = N$. If M^* is an extension of N^*, then M is called a *coextension* of N. So, M is a coextension of N if there exists $X \subseteq E(M)$ such that $M^* \setminus X = N^*$. Since $(N^*)^* = N$, this implies $M/X = N$. Extensions and coextensions are essentially the opposite of deletion and contraction, respectively. Recall, though, that we use extension and coextension to mean single-element extension or coextensions in this thesis. For a matroid M, we let $\mathrm{ext}(M)$ and $\mathrm{coext}(M)$ denote the number of extensions and coextensions of M, respectively.

We will show in the next chapter that the connected corank-2 restrictions of a matroid are important to enumerating coextensions. A matroid is *connected* if and only if, for every pair of distinct elements of $E(M)$, there is a circuit containing both. Although we have defined corank and restriction, we emphasize that a *corank-2 restriction* of a matroid M is a matroid $M|X$ where $X \subseteq E(M)$ and $r^*(M|X) = r((M|X)^*) = 2$. That is, 2 is the corank of the restriction $M|X$, not the corank of the set X. In particular, this means that $r_M(X) = |X| - 2$. It is also important to note that if C is a circuit in a restriction $M|X$, then C is a circuit in M as well.

2.4.2 Preliminary results

The main goal of this matroid preliminary section is to establish some properties of rank-$(r(M) - 2)$ flats and corank-2 restrictions of a matroid M. As we will see in Chapter 3, these structures are important in the definitions of linear and colinear subclasses, which parameterize extensions and coextensions.

Proposition 2.4.1. *Let M be a matroid and let X be a set of elements of M. The intersection of all of the flats in M that contain X is equal to the closure of X.*

Proof. Since $\mathrm{cl}(X)$ is a flat that contains X, the intersection of all flats in M that contains X is a subset of $\mathrm{cl}(X)$. Consider a flat F that contains X. Since F is a flat, $\mathrm{cl}(F) = F$. By Lemma 1.4.3 (CL2) in [40], if $X \subseteq Y$, then $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$, thus $\mathrm{cl}(X) \subseteq \mathrm{cl}(F) = F$. Therefore, every flat that contains X also contains $\mathrm{cl}(X)$. □

Proposition 2.4.2. *Two spanning circuits C_1, C_2 are contained in a corank-2 restriction of a matroid M if and only if the intersection of C_1 and C_2 is a basis of M.*

Proof. Let r be the rank of M. If C_1 and C_2 intersect in a basis B, then $C_1 \cup C_2 = B \cup \{e_1, e_2\}$ where, for each $i \in [2]$, the edge e_i is in $C_i \setminus C_{3-i}$. Thus, the corank of $M|(C_1 \cup C_2)$ is $|C_1 \cup C_2| - r(M|(C_1 \cup C_2)) = r + 2 - r = 2$.

Now, suppose C_1 and C_2 are contained in a corank-2 restriction N of M. Suppose towards a contradiction that the largest independent set I in $C_1 \cap C_2$ has rank $n < r$. If there exists $e \in (C_1 \cap C_2) \setminus I$, then $I \cup \{e\}$ is a circuit that is a subset of C_1, which is a contradiction, so $C_1 \cap C_2 = I$. Thus, $|C_1 \cup C_2| = 2(r+1) - n$ and $r(N) \leq r$. By definition of corank, we know $2 = |E(N)| - r(N) \geq 2(r+1) - n - r = r - n + 2$. This implies $r \leq n$, which is a contradiction. □

Proposition 2.4.3. *Let M be a matroid with rank r where there are at most k circuits in each corank-2 restriction of M. If C is a smallest circuit in a corank-2 restriction of M, then $|C| \leq \frac{k-1}{k}(r+2)$.*

Proof. Let X be a set of elements of M such that $M|X$ has corank 2. Therefore, the size of X is at most $r+2$. Let $Y = E(M) \setminus X$ and observe that $(M|X)^* = M^*/Y$. Let $\mathcal{C}$ be the set of circuits of $M|X$ and let C be a smallest circuit in $M|X$. We are given that $|\mathcal{C}| \leq k$. For each circuit $C' \in \mathcal{C}$, the set $H' = E(M) \setminus C'$ is a hyperplane of M^* that contains Y, hence

$$\begin{aligned} r_{M^*/Y}(H' \setminus Y) &= r_{(M|X)^*}(H' \setminus Y) = r_{M|X}(X \setminus (H' \setminus Y)) + |H' \setminus Y| - r(M|X) \\ &= r_{M|X}(C') + |H' \setminus Y| - r(M|X) = |C'| - 1 + |H'| - |Y| - r(M|X) \\ &= |X| - 1 - |X| + r^*(M|X) = 2 - 1 = 1, \end{aligned}$$

which implies that $H' \setminus Y$ is a parallel class in M^*/Y. If Z is a parallel class in M^*/Y, then since $r(M^*/Y) = 2$, the set Z is a hyperplane of M^*/Y. Thus, since $M^*/Y = (M|X)^*$, the set $X \setminus Z$ is a circuit of $M|X$. Since there are at most k circuits in $M|X$, there are at most k parallel classes in M^*/Y.

Since C is a smallest circuit in $M|X$, a largest hyperplane in M^* that contains Y is $H = E(M) \setminus C$. Thus, the set $H \setminus Y$ is a largest parallel class in M^*/Y. Since there are $|X|$ elements in M^*/Y, there are at least $\frac{|X|}{k}$ elements in $H \setminus Y$. Since $|C| = |X| - |H \setminus Y|$, the size of C is at most $\frac{k-1}{k}|X| \leq \frac{k-1}{k}(r+2)$. □

Proposition 2.4.4. *A corank-2 matroid M on at most t elements has at most t circuits.*

Proof. Let $\mathcal{C}$ be the set of circuits of M and let $\mathcal{H} = \{E(M) \setminus C : C \in \mathcal{C}\}$. Notice that $\mathcal{H}$ is the set of hyperplanes of M^*. Since M^* has rank 2, each hyperplane in M^* has rank 1. The number of rank-1 flats of M^* is at most the number of elements of M^*. Since M and M^* contain at most t elements, we have $|\mathcal{C}| = |\mathcal{H}| \leq |E(M)| \leq t$. □

2.5 Frame matroids

Recall that a theta graph is a subdivision of the graph on exactly two vertices with exactly three edges between them. A *biased graph* is a pair $(G, \mathcal{B})$ where G is a graph and $\mathcal{B}$ is a collection of cycles of G where no theta subgraph of G contains precisely two cycles in $\mathcal{B}$. The cycles in $\mathcal{B}$ are called *balanced* cycles and cycles not in $\mathcal{B}$ are called *unbalanced*.

We will see that biased graphs can be used to count the coextensions of graphic matroids, but for now, we focus on another application. Biased graphs are used to define two classes of matroids, *frame* and *lift* matroids, which were introduced by Zaslavsky [52, 53, 54]. First, we define each of the following graphs to be a *cuff*:

(i) A *hinged* cuff is a theta graph.

(ii) A *tight* cuff is a subdivision of the graph on one vertex with two loops.

(iii) A *loose* cuff is a subdivision of the graph on two vertices with one edge between them and one loop on each vertex.

See Figure 2.1 for the three graphs whose subdivisions result in a hinged, tight, or loose cuff. Additionally, we define a *broken-cuff* to be a subdivision of the graph on two disconnected vertices with one loop on each vertex. The name "cuff" is motivated by the common name for graphs of type (ii) or (iii): handcuff. However, in this thesis, we refer to these graphs as cuffs so that graphs of type (i) can be included.

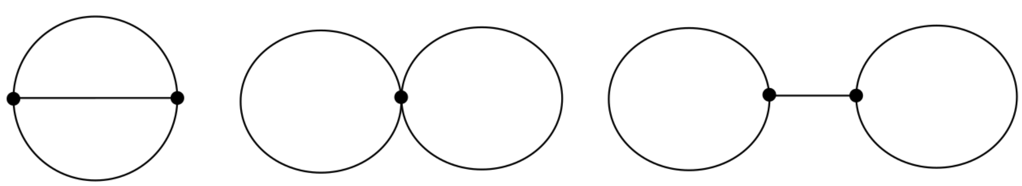

Figure 2.1: Hinged, tight, and loose cuffs are subdivisions of these graphs.

Now were are ready to define frame and lift matroids. Given a biased graph $\mathcal{G} = (G, \mathcal{B})$, the *frame matroid* of $\mathcal{G}$ has as elements the edges of G and as circuits the edge sets of subgraphs of the following forms: balanced cycles; and hinged, tight, and loose cuffs that contain no balanced cycles. The *lift matroid* of $\mathcal{G}$ has as elements the edges of G and as circuits the edge sets of subgraphs of the following forms: balanced cycles; hinged and tight cuffs that contain no balanced cycles; and broken-cuffs that contain no balanced cycles. Zaslavsky [52, 53] proved that these are indeed matroids, although when they were introduced, frame matroids were called bias matroids. He also proved that the class of frame matroids is minor-closed [53]. Notice that graphic matroids are the frame or lift matroids of biased graphs where all cycles are balanced. Thus, frame and lift matroids generalize graphic matroids.

Since corank-2 restrictions play such an important role in this thesis, we prove the following preliminary lemma for frame matroids.

Lemma 2.5.1. *Let $(G, \mathcal{B})$ be a biased graph and let M be the frame matroid of $(G, \mathcal{B})$. There are at most 6 circuits in a corank-2 restriction N of M.*

Proof. Let H be the subgraph of G induced on $E(N)$. Since N is a corank-2 restriction, H has at least one vertex of degree at least 3. Obtain the graph H' from H by iteratively contracting the the edges incident with a vertex of degree 1 until no vertices of degree 1 remain and contract the edges in each component that is a cycle to a vertex. Obtain the graph H'' from H' by contracting the edges in each path or cycle between (not necessarily distinct) vertices u, v of degree at least 3 until only the edge uv remains. Notice that every vertex in H'' has degree at least 3. Let X be the set of edges in H that are contracted to obtain H''. Let N' be the matroid obtained from N by contracting X.

Since X does not contain any circuits of N, it is an independent set of N, which implies that $E(N) \setminus X$ is spanning in N^*. Thus, we have $r_{N^*}(E(N) \setminus X) = r(N^*) = 2$. Since $N' = N/X$ and $(N/X)^* = N^* \backslash X$, it follows that $r^*(N') = r(N^* \backslash X) = r_{N^*}(E(N) \backslash X) = 2$.

Claim 2.5.1.1. *H'' has at most 6 edges.*

Proof. Let v be the number of vertices in H''. Since frame matroids are minor-closed, the matroid N' is also a frame matroid. Since N' is a frame matroid, the rank of N' is at most v. Therefore, we have $|E(H'')| = r(N') + r^*(N') \leq v + 2$.

By the Handshaking Lemma, the number of edges in H'' is equal to $\frac{1}{2} \sum_{u \in V(H'')} \deg(u)$. Since each vertex in H'' has degree at least 3, it follows that $|E(H'')| \geq \frac{3}{2}v$. Therefore, since $|E(H'')| \leq v + 2$, we have $v \leq 4$, which implies $|E(H'')| \leq 6$. □

Observe that a set C is a circuit in N if and only if $C \setminus X$ is a circuit in N'. Therefore, the number of circuits in N is at most the number of circuits in N'. By Claim 2.5.1.1, the graph H'' has at most 6 edges; thus, by Proposition 2.4.4, the edge set of H'' contains at most 6 circuits, which implies that N' contains at most 6 circuits. □

2.5.1 Dowling geometries

Although defined before frame matroids, Dowling geometries are a class of frame matroids. Dowling geometries were defined by Dowling in 1973 [12] and later identified as frame matroids by Zaslavsky [52, 53].

Let n be a positive integer. Let Γ be a finite (multiplicative) group with identity element 1 and let $q = |\Gamma|$. We will define a Dowling geometry $DG(n, \Gamma)$ similarly to Oxley [40], although note that Dowling geometries are denoted $Q_n(\Gamma)$ in [40]. First, we construct a graph K_n^Γ on vertex set $[n]$. The edge set of K_n^Γ is $\Gamma \times \binom{[n]}{2} \cup \{\beta_u : u \in [n]\}$ and the incidence function f of K_n^Γ is defined as follows. For each $(\gamma, \{u, v\}) \in \Gamma \times \binom{[n]}{2}$, let $f((\gamma, \{u, v\})) = \{u, v\}$ and for each $u \in [n]$, let $f(\beta_u) = \{u\}$. Informally, the graph K_n^Γ has vertex set $[n]$, an edge labelled γ between each pair $\{u, v\} \in \binom{[n]}{2}$ for each $\gamma \in \Gamma$, and a loop labelled β_u on each vertex $u \in [n]$. The graph $K_4^{GF(3)^*}$ is shown as an example in Figure 2.2. The ground set of $DG(n, \Gamma)$ is $E(K_n^\Gamma)$.

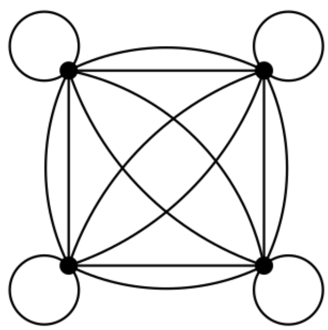

Figure 2.2: The graph $K_4^{GF(3)^*}$.

Define a function $\psi : \Gamma \times \mathbb{Z}^2_{>0} \to \Gamma$ where, for each $(\gamma, x, y) \in \Gamma \times \mathbb{Z}^2_{>0}$,

$$\psi((\gamma, x, y)) = \begin{cases} \gamma & \text{if } x \leq y \\ \gamma^{-1} & \text{if } y < x \end{cases}.$$

Let C be a cycle of K_n^Γ with at least two edges and arbitrarily assign an orientation to it. Let the vertices and edges of C, beginning with a vertex, be $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$, where $v_{k+1} = v_1$ and $e_i = (\gamma_i, \{v_i, v_{i+1}\})$ for each $i \in [k]$. We say C is *balanced* if $\prod_{i=1}^k \psi((\gamma_i, v_i, v_{i+1})) = 1$. Note that the definition of a balanced cycle does not depend on the chosen cyclic ordering of the cycle [40]. A cycle is *unbalanced* if it either has a single edge or is not balanced. Let $\mathcal{B}$ be the collection of balanced cycles of K_n^Γ. The circuits of

$DG(n,\Gamma)$ consist of the edge sets of all of the balanced cycles together with the edge sets of all of the hinged, tight, and loose cuffs in which none of the cycles are balanced.

By investigating the products of edge labels in the cycles of a theta graph, one can conclude the following proposition.

Proposition 2.5.2 ([40]). *If two of the cycles in a theta subgraph of K_n^Γ are balanced, then so is the third.*

Thus, the Dowling geometry $DG(n,\Gamma)$ is the frame matroid represented by $(K_n^\Gamma, \mathcal{B})$. The rank of $DG(n,\Gamma)$ is n and the independent sets are the edge sets of K_n^Γ that contain no balanced cycles and at most one unbalanced cycle.

2.6 Projective geometries

For a positive integer n and a prime power q, let $V(n,q)$ denote an n-dimensional vector space over the finite field $GF(q)$. Let M be the matroid whose elements are the equivalence classes of $V(n,q)\setminus\{0\}$ where vectors are equivalent if one is a nonzero scalar multiple of the other, and a set of elements is independent if and only if the corresponding set of non-equivalent vectors of $V(n,q)$ is linearly independent. The rank-n *projective geometry* over $GF(q)$, denoted $PG(n-1,q)$, is a matroid isomorphic to M.

Equivalently, the projective geometry $PG(n-1,q)$ is isomorphic to $\mathrm{si}(M[A])$ where A is the matrix whose columns are the vectors in $V(n,q)$. This means $PG(n-1,q)$ is a $GF(q)$-representable matroid. In fact, all simple rank-n $GF(q)$-representable matroids are isomorphic to a restriction of $PG(n-1,q)$.

The number of flats is important to the number of extensions of a projective geometry and, in order to bound the number of coextensions, we will use the number of bases and k-element circuits. Thus, we prove the following proposition. First, we define a product that will be useful in representing the number of flats, independent sets, and circuits. The *p-shifted factorial*, denoted $(a;p)_n$, is the product $\prod_{k=0}^{n-1}(1-ap^k)$, where $(a;p)_0=1$ and $(a;p)_\infty$ is the infinite product $\prod_{k=0}^{\infty}(1-ap^k)$. If $|p|<1$, which is always the case in this thesis, then the infinite product converges. We also define the *q-binomial coefficient*, which is defined for all integers n and k where $0\le k\le n$ by

$$\begin{aligned}\begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})} = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)} \\ &= q^{k(n-k)}\cdot\frac{(q^{-1};q^{-1})_n}{(q^{-1};q^{-1})_k(q^{-1};q^{-1})_{n-k}}.\end{aligned}$$

Proposition 2.6.1 (Proposition 6.1.4 in [40]). *Let k be a nonnegative integer.*

(i) The number of k-element independent sets of $PG(n-1,q)$ is

$$\frac{q^{nk}}{k!(q-1)^k} \cdot \frac{(q^{-1};q^{-1})_n}{(q^{-1};q^{-1})_{n-k}}.$$

(ii) The number of rank-k flats in $PG(n-1,q)$ is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q.$$

(iii) The number of k-element circuits of $PG(n-1,q)$ is 0 for $k<2$ and, for $k \geq 3$, is

$$\frac{q^{n(k-1)}}{k!(q-1)} \cdot \frac{(q^{-1};q^{-1})_n}{(q^{-1};q^{-1})_{n+1-k}}.$$

In order to prove this proposition, we use the following two lemmas.

Lemma 2.6.2 (Lemma 6.1.5 in [40]). *The number of ordered k-tuples $(v_1, v_2, \ldots, v_k)$ of distinct vectors in $V(n,q)$ such that $\{v_1, v_2, \ldots, v_k\}$ is linearly independent is*

$$q^{nk} \cdot \frac{(q^{-1};q^{-1})_n}{(q^{-1};q^{-1})_{n-k}}.$$

Proof. There are q^n vectors in $V(n,q)$. Since the zero vector is not linearly independent, there are $q^n - 1$ choices for the vector v_1. Consider $j \geq 1$ and suppose that $v_1, \ldots, v_j$ have been chosen and $\{v_1, \ldots, v_j\}$ is linearly independent. Thus, there are q^j vectors in the span of $\{v_1, \ldots, v_j\}$. This implies that there are $q^n - q^j$ choices for v_{j+1}. Therefore, the number of choices for $v_1, \ldots, v_k$ is

$$(q^n-1)(q^n-q)\ldots(q^n-q^{k-1}) = q^{nk} \cdot \prod_{i=n-k}^{n} \left(1 - \frac{1}{q^{i+1}}\right) = q^{nk} \cdot \frac{(q^{-1};q^{-1})_n}{(q^{-1};q^{-1})_{n-k}}. \qquad \square$$

Lemma 2.6.3 (Lemma 6.1.6 in [40]). *Let $n \geq 2$ be an integer. For each basis B of $PG(n-1,q)$, there are precisely $(q-1)^{n-1}$ elements e of $PG(n-1,q)$ such that $B \cup \{e\}$ is a circuit.*

Proof. Let $B = \{b_1, \ldots, b_n\}$ and consider the elements of B as vectors in the vector space $V(n, q)$. Let E be the set of vectors e of $V(n, q)$ such that, for the corresponding element e of $PG(n-1, q)$, the set $B \cup \{e\}$ is a circuit of $PG(n-1, q)$. Consider $e \in E$. Since $B \cup \{e\}$ is a circuit, the set $\{b_1, \ldots, b_n, e\}$ of vectors is linearly dependent. Thus, for each $i \in [n]$, there exists $\alpha_i \in GF(q)$ such that $\alpha_1 b_1 + \cdots + \alpha_n b_n = e$. If $\alpha_i = 0$ for some $i \in [n]$, then $B \cup e$ is not minimally dependent, which contradicts $B \cup \{e\}$ being a circuit. Therefore, for each $i \in [n]$, we have $\alpha_i \neq 0$. Since there are $(q-1)$ choices for each α_i, there are $(q-1)^n$ elements in E. Each multiple of e by a scalar $\gamma \neq 0$ is in E. Therefore, the number of elements e of $PG(n-1, q)$ such that $B \cup \{e\}$ is a circuit is $(q-1)^{n-1}$. $\square$

Now we are ready to prove Proposition 2.6.1.

Proof of Proposition 2.6.1. Since there are $k!$ ways to order k elements, it follow from Lemma 2.6.2 that the number of k-element linearly independent sets in $V(n, q)$ is $\frac{q^{nk}}{k!} \cdot \frac{(q^{-1};q^{-1})_n}{(q^{-1};q^{-1})_{n-k}}$. Since each element of $PG(n-1, q)$ corresponds to an equivalence class of $q-1$ vectors of $V(n, q)$, each k-element independent set of $PG(n-1, q)$ corresponds to $(q-1)^k$ k-element linearly independent sets of $V(n, q)$. Therefore, the number of k-element independent sets of $PG(n-1, q)$ is $\frac{q^{nk}}{k!(q-1)^k} \cdot \frac{(q^{-1};q^{-1})_n}{(q^{-1};q^{-1})_{n-k}}$, which proves (i).

Since the closure of a vector v in $V(n, q)$ contains all vectors in the equivalence class of v (that is, an element of $PG(n-1, q)$) the flats of $V(n, q)$ correspond to the flats of $PG(n-1, q)$. Consider a rank-k linearly independent set I of $V(n, q)$. It is a basis of exactly one rank-k flat of $V(n, q)$. Since every rank-k flat of $V(n, q)$ is isomorphic to $V(k, q)$, the number of rank-k linearly independent sets is equal to the number of rank-k flats multiplied by the number of bases of $V(k, q)$. Now it follows from Lemma 2.6.2 that the number of rank-k flats of $V(n, q)$ is

$$\frac{\frac{q^{nk}}{k!} \frac{(q^{-1};q^{-1})_n}{(q^{-1};q^{-1})_{n-k}}}{\frac{q^{k^2}}{k!}(q^{-1};q^{-1})_k} = q^{k(n-k)} \frac{(q^{-1};q^{-1})_n}{(q^{-1};q^{-1})_k (q^{-1};q^{-1})_{n-k}},$$

which proves (ii).

Let $\mathcal{C}_{n,k}$ denote the set of k-element circuits of $PG(n-1, q)$. Since projective geometries have no loops or parallel elements, there are no k-element circuits of $PG(n-1, q)$ for $k < 3$. Suppose $k \geq 3$. Since every k-element circuit is in a unique rank-$(k-1)$ flat, and each rank-$(k-1)$ flat is isomorphic to $PG(k-2, q)$,

$$|\mathcal{C}_{n,k}| = \begin{bmatrix} n \\ k-1 \end{bmatrix}_q |\mathcal{C}_{k-1,k}|,$$

where $\begin{bmatrix} n \\ k-1 \end{bmatrix}_q$ is the number of flats isomorphic to $PG(k-2,q)$ and $|\mathcal{C}_{k-1,k}|$ is the number of k-element circuits in such a flat. Let $\mathcal{X}$ denote the collection of ordered pairs (B,C) where B is a basis of $PG(k-2,q)$ and C is a k-element circuit in $PG(k-2,q)$ that contains B. Let $\mathcal{B}$ denote the collection of bases of $PG(k-2,q)$. We now determine the size of $\mathcal{X}$ in the following two ways. Since there are k bases in a k-element circuit,

$$|\mathcal{X}| = \sum_{C \in \mathcal{C}_{k-1,k}} \sum_{\substack{B \subset C, \\ B \in \mathcal{B}}} 1 = \sum_{C \in \mathcal{C}_{k-1,k}} k = k|\mathcal{C}_{k-1,k}|.$$

Since there are $(q-1)^{k-2}$ circuits of $PG(k-2,q)$ that contain a specific basis of $PG(k-2,q)$,

$$|\mathcal{X}| = \sum_{B \in \mathcal{B}} \sum_{\substack{C \supset B, \\ C \in \mathcal{C}_{k-1,k}}} 1 = \sum_{B \in \mathcal{B}} (q-1)^{k-2} = (q-1)^{k-2}|\mathcal{B}|.$$

By part (i) of this proposition, there are $\frac{1}{(k-1)!(q-1)^{k-1}}(q^{k-1}-1)(q^{k-1}-q)\dots(q^{k-1}-q^{k-2})$ bases in $PG(k-2,q)$. Thus,

$$|\mathcal{C}_{k-1,k}| = \frac{1}{k!(q-1)}(q^{k-1}-1)(q^{k-1}-q)\dots(q^{k-1}-q^{k-2}).$$

Multiplying this by $\begin{bmatrix} n \\ k-1 \end{bmatrix}_q$ gives

$$|\mathcal{C}_{n,k}| = \frac{1}{k!(q-1)}(q^n-1)(q^n-q)\dots(q^n-q^{k-2}) = \frac{q^{n(k-1)}}{k!(q-1)} \cdot \frac{(q^{-1};q^{-1})_n}{(q^{-1};q^{-1})_{n+1-k}},$$

which proves (iii). $\square$

Chapter 3

Extensions of matroids

The goal of this chapter is to translate the problem of counting the extensions of a matroid to a problem of counting the stable sets in a graph. To do this, we show that the number of extensions can be bounded above and below by the number of "scarce" extensions and the number of "small" hyperplanes. We then show that "scarce" extensions can be described by stable sets in an auxiliary graph. This translation of extensions to stable sets will then be applied in Chapters 6, 7, 8, 9, and 10.

One of the major conjectures in matroid enumeration is the conjecture by Mayhew, Newman, Welsh, and Whittle [27] which suggests that asymptotically almost all matroids are "paving." Pendavingh and Van der Pol [42] showed that "sparse" paving matroids dominate the problem of counting matroids. In order to do this, they represent sparse paving matroids as stable sets in a Johnson graph: a graph $J(n, r)$ whose vertices are the r-sets of $[n]$ and vertices are adjacent if and only if they intersect in $r - 1$ elements. Vertices in the Johnson graph represent the non-bases of a matroid and, since each stable set clearly satisfies the basis exchange axiom, each stable set corresponds to the non-bases of a matroid, specifically a sparse paving matroid. Pendavingh and Van der Pol showed that, while there are matroids that are not represented by stable sets in the Johnson graph, their number is "small" compared to the number of sparse paving matroids. In a similar vein, we show in this thesis that, for some matroids M, the extensions of M that can be represented as stable sets in an auxiliary graph dominate the problem of counting extensions.

3.1 Linear subclasses

An extension of a matroid M by a new element e can be described by the effect of e on the flats of M. When e is added to a flat F, the result $F \cup \{e\}$ is either a flat with the same rank as F, a flat with rank one more than F, or not a flat.

The set of flats that remain flats of the same rank when e is added to them is called a *modular cut* [40]. These sets of flats have two properties which identify them. First, if a flat F is in a modular cut $\mathcal{M}$, then each flat F' containing F is in $\mathcal{M}$ as well. Second, if two flats F_1, F_2 are in a modular cut $\mathcal{M}$ where $r(F_1) + r(F_2) = r(F_1 \cap F_2) + r(F_1 \cup F_2)$ (that is, F_1 and F_2 are *modular*), then $F_1 \cap F_2$ is in $\mathcal{M}$ as well.

Each modular cut gives rise to a unique extension, so we may count extensions by counting modular cuts. However, instead of parameterizing extensions by modular cuts, Crapo [10] showed that it is sufficient to consider the effect of the new element on the hyperplanes. Specifically, extensions can be parameterized by the sets of hyperplanes that remain hyperplanes when the new element e is added to them. These sets $\mathcal{H}'$ of hyperplanes have the property that if two hyperplanes H_1, H_2 in the set are modular, that is, they intersect in a rank-$(r(M) - 2)$ flat F, then all hyperplanes that contain F are in $\mathcal{H}'$ as well. We can see that each set of hyperplanes that has this property corresponds to an extension by considering a rank-$(r(M) - 2)$ flat F contained in hyperplanes H_1, H_2, H_3 in a matroid M. If N is an extension of M by the element e and $H_1 \cup \{e\}, H_2 \cup \{e\}$ are hyperplanes of N while $H_3 \cup \{e\}$ is not, then contracting F in N results in a line where e is in two different parallel classes, which is impossible. Crapo [10] proved the more difficult direction of showing that each extension corresponds to a set of hyperplanes that remain hyperplanes when the new element is added to them.

Recall that $\mathcal{H}(M)$ is the set of hyperplanes of a matroid M. A *linear subclass* of a matroid M is a subset $\mathcal{H}' \subseteq \mathcal{H}(M)$ such that if two hyperplanes $H, H' \in \mathcal{H}'$ intersect in a flat F of rank $r(M) - 2$, then all hyperplanes H'' that contain F are in $\mathcal{H}'$ as well. Figure 3.1 shows three hyperplanes H_1, H_2, H_3 of a rank-n matroid M that intersect in a rank-$(n-2)$ flat. If $\mathcal{H}'$ is a linear subclass of M, then $\mathcal{H}'$ contains at most one or all three of H_1, H_2, H_3.

The following theorem states the correspondence between extensions and linear subclasses. Recall that $\mathrm{ext}(M)$ denotes the number of extensions of the matroid M.

Theorem 3.1.1 ([10]). *If $\mathbb{L}$ is the collection of linear subclasses of a matroid M, then*

$$\mathrm{ext}(M) = |\mathbb{L}|.$$

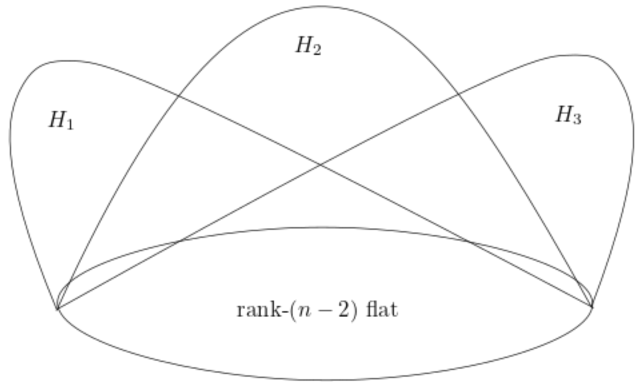

Figure 3.1: Three hyperplanes of a rank-n matroid that intersect in a rank-$(n-2)$ flat.

Now that we have this correspondence between extensions and linear subclasses, we focus on linear subclasses. We say a set $\mathcal{H}' \subseteq \mathcal{H}(M)$ of hyperplanes of a matroid M has the *hyperplane property* if, for all pairs of hyperplanes $H, H' \in \mathcal{H}'$ that intersect in a flat F of rank $r(M) - 2$, all hyperplanes H'' that contain F are in $\mathcal{H}'$ as well. Thus, a linear subclass of M is a set of hyperplanes of M that has the hyperplane property.

The following definitions identify certain linear subclasses that will be represented as stable sets in an auxiliary graph. We say a set $\mathcal{H}' \subseteq \mathcal{H}(M)$ of hyperplanes of a matroid M has the *scarce* hyperplane property if, for each rank-$(r(M) - 2)$ flat F of M, at most one hyperplane that contains F is in $\mathcal{H}'$. We define a *scarce linear subclass* of M to be a set of hyperplanes of M that has the scarce hyperplane property. As we will see, scarce linear subclasses play a role in counting matroid extensions similar to the role of sparse paving matroids in counting matroids. Note that a *(scarce) colinear subclass* of a matroid M is a set of hyperplanes $\mathcal{H}'$ of M^* that has the (scarce) hyperplane property.

The complement of a hyperplane is a cocircuit, so it is convenient to consider sets of circuits instead of hyperplanes of the dual when dealing with colinear subclasses. With this in mind, we define a "dual" property to the hyperplane property that applies to sets of circuits. We say a set $\mathcal{B}$ of circuits of a matroid M has the *circuit property* if for each pair of distinct circuits $C, C' \in \mathcal{B}$ that are contained in a corank-2 restriction N of M, all circuits in N are in $\mathcal{B}$ as well. We say a set $\mathcal{B}$ of circuits of a matroid M has the *scarce* circuit property if $\mathcal{B}$ contains at most one circuit from each corank-2 restriction of M.

In the following propositions, we establish the correspondence between colinear subclasses and sets of circuits with the circuit property.

Proposition 3.1.2. *Let M be a matroid and let (X, Y) be a partition of $E(M)$. Then*

$r((M|X)^*) = 2$ *if and only if* $r_{M^*}(Y) = r(M^*) - 2$.

Proof. Since $r((M|X)^*) = r(M^*/Y) = r(M^*) - r_{M^*}(Y)$, the corank of $M|X$ is 2 if and only if the rank of Y in M^* is $r(M^*) - 2$. □

Proposition 3.1.3. *A set of hyperplanes* $\mathcal{H}'$ *of* M^* *is a colinear subclass of* M *if and only if* $\mathcal{B} = \{E(M) \setminus H : H \in \mathcal{H}'\}$ *has the circuit property.*

Proof. Let $X \subseteq E(M)$ and let $Y = E(M) \setminus X$. By Proposition 3.1.2, the restriction $M|X$ has corank 2 if and only if $\mathrm{cl}(Y)$ is a rank-$(r(M^*) - 2)$ flat of M^*. A set $H \subseteq E(M)$ is a hyperplane of M^* that contains $\mathrm{cl}(Y)$ if and only if $E(M) \setminus H$ is a circuit of $M|X$. Therefore, there exists a rank-$(r(M^*) - 2)$ flat of M^* contained by hyperplanes $H_1, H_2, \ldots, H_k$ such that at least 2 but not all are in $\mathcal{H}'$ if and only if there exists a corank-2 restriction of M containing circuits $E(M) \setminus H_1, \ldots, E(M) \setminus H_k$ such that at least 2 but not all are in $\mathcal{B}$. Thus, the set $\mathcal{H}'$ has the hyperplane property if and only if $\mathcal{B}$ has the circuit property. □

In the case of graphic matroids, it will be convenient to focus on circuits contained in connected corank-2 restrictions. Therefore, we say a set $\mathcal{C}$ of circuits of a matroid M has the *theta property* if for all pairs of distinct circuits $C, C' \in \mathcal{C}$ that are contained in a connected corank-2 restriction N of M, all circuits in N are in $\mathcal{C}$ as well. The reason we call this the theta property is because two circuits of a graphic matroid that are contained in a connected corank-2 restriction correspond to two cycles in a theta subgraph. A set $\mathcal{C}$ of circuits of a matroid M has the *scarce theta property* if, for each connected corank-2 restriction N of M, at most one circuit of N is in $\mathcal{C}$.

Proposition 3.1.4. *A set of hyperplanes* $\mathcal{H}'$ *of* M^* *is a colinear subclass of* M *if and only if* $\mathcal{B} = \{E(M) \setminus H : H \in \mathcal{H}'\}$ *has the theta property.*

Proof. If $\mathcal{H}'$ is a colinear subclass, then it has the hyperplane property, so $\mathcal{B}$ has the circuit property by Proposition 3.1.3. Since a set with the circuit property also has the theta property, it follows that $\mathcal{B}$ has the theta property.

Now suppose towards a contradiction that $\mathcal{B}$ has the theta property, but $\mathcal{H}'$ does not have the hyperplane property. Therefore, there exists a rank-$(r(M^*) - 2)$ flat Y contained by $k \geq 3$ hyperplanes $H_1, H_2, \ldots, H_k$ and $2 \leq p < k$ are in $\mathcal{H}'$. For each $i \in [k]$, let $C_i = E(M) \setminus H_i$. Let $X = E(M) \setminus Y$. By Proposition 3.1.2, the restriction $M|X$ has corank 2. A set $H \subseteq E(M)$ is a hyperplane of M^* that contains Y if and only if $E(M) \setminus H$ is a circuit of $M|X$. Since all hyperplanes of M^* that contain Y pairwise intersect in Y, there are no elements of X in two or more of $H_1, \ldots, H_k$. Consider elements $e \in H_i$ and

$e' \in H_j$ where $i < j \in [k]$, which we now know are not in any other hyperplanes that contain Y. Both e and e' are in the circuit C_ℓ where $\ell \in [k] \setminus \{i, j\}$. Since the choice of e and e' was arbitrary, every pair of elements in X is in a circuit, hence $M|X$ is a connected corank-2 restriction of M. Therefore, there are $2 \leq p < k$ circuits from a connected corank-2 restriction in $\mathcal{B}$, which contradicts $\mathcal{B}$ having the theta property. □

Proposition 3.1.5. *A set of hyperplanes $\mathcal{H}'$ of M^* is a scarce colinear subclass of M if and only if $\mathcal{B} = \{E(M) \setminus H : H \in \mathcal{H}'\}$ has the scarce circuit property.*

Proof. Let $X \subseteq E(M)$ and let $Y = E(M) \setminus X$. By Proposition 3.1.2, the restriction $M|X$ has corank 2 if and only if $\mathrm{cl}(Y)$ is a rank-$(r(M^*) - 2)$ flat of M^*. A set $H \subseteq E(M)$ is a hyperplane of M^* that contains $\mathrm{cl}(Y)$ if and only if $E(M) \setminus H$ is a circuit of $M|X$. Therefore, there exists a rank-$(r(M^*) - 2)$ flat of M^* contained by hyperplanes $H_1, H_2, \ldots, H_k$ such that at most one is in $\mathcal{H}'$ if and only if there exists a corank-2 restriction of M containing circuits $E(M) \setminus H_1, \ldots, E(M) \setminus H_k$ such that at most one is in $\mathcal{B}$. Thus, the set $\mathcal{H}'$ has the scarce hyperplane property if and only if $\mathcal{B}$ has the scarce circuit property. □

Proposition 3.1.6. *If a set $\mathcal{B}$ of circuits of M has the scarce theta property, then $\mathcal{H}' = \{E(M) \setminus C : C \in \mathcal{B}\}$ is a colinear subclass of M.*

Proof. If $\mathcal{B}$ has the scarce theta property, but $\mathcal{H}'$ does not have the hyperplane property, then there exists a rank-$(r(M^*) - 2)$ flat Y of M^* contained by $k \geq 3$ hyperplanes $H_1, H_2, \ldots, H_k$ such that at least two but not all are in $\mathcal{H}'$. Let $X = E(M) \setminus Y$ and let $C_i = E(M) \setminus H_i$ for all $i \in [k]$. Since all hyperplanes of M^* that contain Y pairwise intersect in Y, there are no elements of X in two or more of $H_1, \ldots, H_k$. Consider elements $e \in H_i$ and $e' \in H_j$ where $i < j \in [k]$, which we now know are not in any other hyperplanes that contain Y. Both e and e' are in the circuit C_ℓ where $\ell \in [k] \setminus \{i, j\}$. Since the choice of e and e' was arbitrary, every pair of elements in X is in a circuit, hence $M|X$ is a connected corank-2 restriction of M. Therefore, there are at least two circuits of $M|X$, a connected corank-2 restriction, in $\mathcal{B}$, which contradicts $\mathcal{B}$ having the scarce theta property. □

The last result in this section plays a very important role in translating the problem of counting extensions to counting stable sets. Theorem 3.1.7 uses a totally ordered set and a collection of its subsets to define a graph G. The size of the collection of "linear" sets is then upper bounded using the number of stable sets in G. Let S be a set and let $\mathcal{X}$ be a collection of subsets of S. Recall that a set $L \subseteq S$ is *linear with respect to $\mathcal{X}$* if $|L \cap X| \in \{0, 1, |X|\}$ for all $X \in \mathcal{X}$. The extensions of a matroid M are parameterized by its linear subclasses, and the choice to name linear sets as such was intentional. In Section 3.2, we will apply this theorem to bound the number of linear subclasses of a matroid.

It is important to note that the proof strategy of Theorem 3.1.7 is not entirely new. It is a generalization of part of the proof of Theorem 4.1 in [39]. Interestingly, Theorem 4.1 in [39] was used to upper bound the number of biased graphs on a complete graph. As we will see in Chapter 6, these biased graphs correspond to the coextensions of the cycle matroid of a complete graph, so it is actually not surprising that a generalization applies to counting extensions in general.

Theorem 3.1.7. *Let S be a set with total ordering $<$. Let $\mathcal{K}$ be a collection of subsets of S. Let G be a graph with vertex set S such that $G[K]$ is a complete graph for each $K \in \mathcal{K}$ and each edge is induced by at least one $K \in \mathcal{K}$. Let $X_0 \subseteq V(G)$ be a set of vertices of G such that, for each $K \in \mathcal{K}$, the smallest vertex in K is in X_0. If $\mathcal{L}$ denotes the collection of sets $U \subseteq S$ that are linear with respect to $\mathcal{K}$, then*

$$|\mathcal{L}| \leq i(G) \cdot 2^{|X_0|}.$$

Proof. Let $\mathcal{T}$ denote the collection of all tuples $(v_1, v_2, \ldots, v_j)$ for which $v_1 < v_2 < \cdots < v_j$ and $\{v_1, v_2, \ldots, v_j\}$ is an element of $\mathcal{K}$. Since v_1 is the smallest element in a set in $\mathcal{K}$, it is in X_0.

For each $L \in \mathcal{L}$, let $\psi(L)$ be obtained from L by simultaneously removing all elements except v_2 for each tuple $(v_1, v_2, \ldots, v_j) \in \mathcal{T}$ for which $\{v_1, v_2, \ldots, v_j\} \subseteq L$. For each tuple T in $\mathcal{T}$, the set L contains either zero, one, or all of the elements in T, hence the number of elements in T and $\psi(L)$ is at most 1, which implies $\psi(L)$ is a stable set of G. Now we claim that L can be determined by $\psi(L)$ and the elements in $X_0 \cap L$, as follows.

Claim 3.1.7.1. *Let L' be a stable set of G and let $X \subseteq X_0$. There exists at most one $L \in \mathcal{L}$ where $\psi(L) = L'$ and $L \cap X_0 = X$.*

Proof. Suppose towards a contradiction that there exist two sets $L_1, L_2 \in \mathcal{L}$ such that $\psi(L_1) = \psi(L_2) = L'$ and $L_1 \cap X_0 = L_2 \cap X_0 = X$. Let $v \in \mathcal{S}$ be the minimum element with respect to $<$ that is in exactly one of L_1, L_2. Without loss of generality, say $v \in L_1$ and $v \notin L_2$. Since $L' = \psi(L_2) \subseteq L_2$, we know v is not in L'. The set L' is also equal to $\psi(L_1)$, so v is in L_1, but not in $\psi(L_1)$. Therefore, there is a tuple $(v_1, v_2, \ldots, v_j) \in \mathcal{T}$ for which $\{v_1, v_2, \ldots, v_j\} \subseteq L_1$ and $v \in \{v_i : i \in [j] \setminus \{2\}\}$. The element v_1 is in X_0, and L_1 and L_2 intersect X_0 in the same set, so we know $v \neq v_1$; hence $v \in \{v_3, \ldots, v_j\}$. Since $v_1 < v_2 < v$ and v is the minimum element with respect to $<$ that is in exactly one of L_1 and L_2, it follows that v_2 is in L_2. Therefore, since $v_1, v_2 \in L_2$ and $v \notin L_2$, at least two but not all of the elements in the tuple $(v_1, v_2, \ldots, v_j) \in \mathcal{T}$ are in L_2, which contradicts the assumption that L_2 is linear with respect to $\mathcal{K}$. □

The number of pairs (L', X) where L' is a stable set of G and $X \subseteq X_0$ is $i(G) \cdot 2^{|X_0|}$. This is an upper bound on the number of sets in $\mathcal{L}$ by Claim 3.1.7.1. $\square$

3.2 Comparing extensions to stable sets

Now, we determine the auxiliary graph whose stable sets describe the scarce linear subclasses of a matroid. Counting the number of stable sets in certain graphs is an interesting and well-studied problem itself, so the bounds we find throughout this thesis for the number of extensions of certain matroids may be of interest outside of the study of matroid extensions as well.

Definition 3.2.1 (Hyperplane graph $\Pi(M)$)**.** For a rank-n matroid M, we define the *hyperplane graph of M*, denoted $\Pi(M)$, to be the graph with vertex set $\mathcal{H}(M)$ where two vertices (hyperplanes) H and H' are adjacent if and only if they intersect in a rank-$(n-2)$ flat of M.

Proposition 3.2.2. *A set $\mathcal{H}' \subseteq \mathcal{H}(M)$ of hyperplanes is a scarce linear subclass of M if and only if it is a stable set in $\Pi(M)$.*

Proof. The set $\mathcal{H}'$ is a stable set of $\Pi(M)$ if and only if, for each rank-$(r(M)-2)$ flat F in M, at most one hyperplane that contains F is in $\mathcal{H}'$. Thus, by definition, the set $\mathcal{H}'$ is a stable set of $\Pi(M)$ if and only if it is a scarce linear subclass of M. $\square$

Consider a matroid M. In order to bound the number of extensions of M, we will use Theorem 3.1.7 with $\Pi(M)$. First, we define the sets that will play the roles of $\mathcal{K}$ and X_0 in the statement of Theorem 3.1.7. Let $\prec$ be a total ordering of $\mathcal{H}(M)$. For each flat F of M, let $\mathcal{H}_F$ be the set of hyperplanes of M that contain F. Let $\mathbb{K}(M)$ be the collection of subsets $\mathcal{H}'$ of $\mathcal{H}(M)$ such that there exists a rank-$(r(M)-2)$ flat F of M where $\mathcal{H}' = \mathcal{H}_F$. Let $\mathcal{H}_{\min}(M, \prec)$ be the collection of hyperplanes $H \in \mathcal{H}(M)$ such that, for some $\mathcal{H}' \in \mathbb{K}(M)$, the hyperplane H is the minimum element of $\mathcal{H}'$ with respect to $\prec$. If $\prec$ is a total ordering of $\mathcal{H}(M)$ that refines the preorder by size (that is, $|H| \prec |H'|$ whenever $|H| < |H'|$), we will refer to the hyperplanes in this set as the *small* hyperplanes of M. Except when the following lemma is applied to Corollary 3.2.6, it is always the case in this thesis that $\prec$ refines the preorder by size.

Lemma 3.2.3. $\log i(\Pi(M)) \leq \log \mathrm{ext}(M) \leq \log i(\Pi(M)) + |\mathcal{H}_{\min}(M, \prec)|$.

Proof. Since each scarce linear subclass is a linear subclass, the lower bound follows from Proposition 3.2.2 and Theorem 3.1.1. We will prove the upper bound using Theorem 3.1.7. First, we need to prove that $\Pi(M)$ and $\mathcal{H}_{\min}(M, \prec)$ satisfy the conditions of the theorem.

Claim 3.2.3.1. *For each $\mathcal{H}' \in \mathbb{K}(M)$, the subgraph of $\Pi(M)$ induced by $\mathcal{H}'$ is a complete graph.*

Proof. Let F be the rank-$(r(M) - 2)$ flat of M that is a subset of each hyperplane H in $\mathcal{H}'$. Consider $H, H' \in \mathcal{H}'$. Since $F \subseteq H$ and $F \subseteq H'$, it follows that $F \subseteq H \cap H'$. Since hyperplanes of M intersect in a flat of rank at most $r(M) - 2$, the intersection of H and H' is F, which implies that H and H' are adjacent in $\Pi(M)$. □

Claim 3.2.3.2. *Each edge in $\Pi(M)$ is in $\Pi(M)[\mathcal{H}']$ for at least one set $\mathcal{H}' \in \mathbb{K}$.*

Proof. Two hyperplanes H and H' are adjacent in $\Pi(M)$ if and only if there exists $\mathcal{H}' \in \mathbb{K}(M)$ such that $H, H' \in \mathcal{H}'$. Thus, each edge in $\Pi(M)$ is in $\Pi(M)[\mathcal{H}']$ for at least one set $\mathcal{H}' \in \mathbb{K}$. □

Applying Theorem 3.1.7 with $S = \mathcal{H}(M)$, $\mathcal{K} = \mathbb{K}(M)$, and $X_0 = \mathcal{H}_{\min}(M, \prec)$, we find that the number of sets of hyperplanes that are linear with respect to $\mathbb{K}(M)$ is at most $i(\Pi(M)) \cdot 2^{|\mathcal{H}_{\min}(M,\prec)|}$. By definition, the collection of subsets of $\mathcal{H}(M)$ that are linear with respect to $\mathbb{K}(M)$ is precisely the collection of linear subclasses of M. Now the upper bound follows from Theorem 3.1.1. □

In the following chapters, we will bound the number of stable sets in the hyperplane graphs of certain matroids and the number of small hyperplanes, and then apply Lemma 3.2.3. Since we are also interested in coextensions, and it is sometimes easier to study the circuits of a matroid rather than the hyperplanes of its dual, we prove a corollary that restates Lemma 3.2.3 for coextensions. First we define the dual concept of the hyperplane graph. Recall that $\mathcal{C}(M)$ is the set of circuits of a matroid M.

Definition 3.2.4 (Circuit graph $\Omega(M)$)**.** For a matroid M, we define the *circuit graph of* M, denoted $\Omega(M)$, to be the graph with vertex set $\mathcal{C}(M)$ where two vertices (circuits) C and C' are adjacent if and only if they are contained in a corank-2 restriction of M.

Proposition 3.2.5. *A set $\mathcal{C}' \subseteq \mathcal{C}(M)$ of circuits has the scarce circuit property if and only if it is a stable set of $\Omega(M)$.*

Proof. The set $\mathcal{C}'$ is a stable set of $\Omega(M)$ if and only if, for each corank-2 restriction N of M, at most one circuit in N is in $\mathcal{C}'$. Thus, by definition, the set $\mathcal{C}'$ is a stable set of $\Omega(M)$ if and only if it has the scarce circuit property. $\square$

Let $\prec$ be a total ordering of $\mathcal{C}(M)$. For each corank-2 restriction N of M, let $\mathcal{C}_N \subseteq \mathcal{C}(M)$ be the set of circuits of N. Let $\mathbb{K}_C(M)$ be the collection of subsets $\mathcal{C}'$ of $\mathcal{C}(M)$ such that there exists a corank-2 restriction N of M where $\mathcal{C}' = \mathcal{C}_N$. Let $\mathcal{C}_{\min}(M, \prec)$ be the collection of circuits $C \in \mathcal{C}(M)$ such that, for some $\mathcal{C}' \in \mathbb{K}_C(M)$, the circuit C is the minimum element of $\mathcal{C}'$ with respect to $\prec$. If $\prec$ is a total ordering of $\mathcal{C}(M)$ that refines the preorder by size, which is always the case in this thesis, then we refer to the circuits in this set as the *small* circuits of M.

Corollary 3.2.6. $\log i(\Omega(M)) \leq \log \operatorname{coext}(M) \leq \log i(\Omega(M)) + |\mathcal{C}_{\min}(M, \prec)|$.

Proof. Let $\prec_H$ be a total ordering of $\mathcal{H}(M)$ such that, for $H, H' \in \mathcal{H}(M)$, if $|E(M) - H| < |E(M) - H'|$, then $H \prec_H H'$. Note that, in this case, minimal with respect to $\prec_H$ corresponds to largest with respect to size. Since $\operatorname{coext}(M) = \operatorname{ext}(M^*)$, it follows from Lemma 3.2.3 that

$$\log i(\Pi(M^*) \leq \log \operatorname{coext}(M) \leq \log i(\Pi(M^*)) + |\mathcal{H}_{\min}(M^*, \prec_H)|.$$

By Proposition 3.1.5, a set $\mathcal{B}$ of circuits of M has the scarce circuit property if and only if $\mathcal{H}' = \{E(M) \setminus C : C \in \mathcal{B}\}$ is a scarce colinear subclass of M. Thus, by Proposition 3.2.2, the stable sets of $\Pi(M^*)$ correspond to sets of circuits of M that have the scarce circuit property. By definition of the circuit graph, a set of circuits with the scarce circuit property is a stable set of $\Omega(M)$. Therefore, the stable sets of $\Pi(M^*)$ correspond to the stable sets of $\Omega(M)$, which implies that $i(\Pi(M^*)) = i(\Omega(M))$.

Now we claim that $|\mathcal{H}_{\min}(M^*, \prec_H)| = |\mathcal{C}_{\min}(M, \prec)|$, which will complete the proof. By definition, a hyperplane H is in $\mathcal{H}_{\min}(M^*, \prec_H)$ if and only if there exists a rank-$(r(M^*)-2)$ flat F of M^* such that H is the minimum hyperplane with respect to $\prec_H$ that contains F. By Proposition 3.1.2, the restriction $M|(E(M) \setminus F)$ has corank 2 if and only if F has rank $r(M^*) - 2$ in M^*. Therefore, there exists a rank-$(r(M^*) - 2)$ flat F of M^* such that H is the minimum hyperplane with respect to $\prec_H$ that contains F if and only if $C = E(M) \setminus H$ is the minimum circuit with respect to $\prec$ that is contained in $M|(E(M) \setminus F)$. Since a circuit C is the minimum with respect to $\prec$ that is contained in a corank-2 restriction of M if and only if C is in $\mathcal{C}_{\min}(M, \prec)$, it follows that H is in $\mathcal{H}_{\min}(M, \prec_H)$ if and only if $E(M) \setminus H$ is in $\mathcal{C}_{\min}(M, \prec)$. Thus, $|\mathcal{H}_{\min}(M^*, \prec_H)| = |\mathcal{C}_{\min}(M, \prec)|$. $\square$

In some cases, it will be useful to investigate only *connected* corank-2 restrictions instead of all corank-2 restrictions. Therefore, we define a variation of the circuit graph, as follows.

Definition 3.2.7 (Overlap graph $\Theta(M)$)**.** For a matroid M, we define the *overlap graph of* M, denoted $\Theta(M)$, to be the graph with vertex set $\mathcal{C}(M)$ where two vertices (circuits) C and C' are adjacent if and only if they are contained in a connected corank-2 restriction of M.

Observe that a set of circuits of M has the scarce theta property if and only if it is a stable set of $\Theta(M)$. The name of the overlap graph is motivated by the overlap graph of graphic matroids. In this setting, the circuits are cycles of a graph and two cycles are adjacent if and only if they are contained in a theta subgraph; that is, they overlap in a path.

Corollary 3.2.8. $\log i(\Theta(M)) \leq \log \operatorname{coext}(M) \leq \log i(\Theta(M)) + |\mathcal{C}_{\min}(M, \prec)|$.

Proof. If two circuits C, C' are in a connected corank-2 restriction, then they are in a corank-2 restriction, so there exists a set E of edges in $\Omega(M)$ such that $\Omega(M) \setminus E = \Theta(M)$. Thus, by Proposition 2.3.5, $i(\Omega(M)) \leq i(\Theta(M))$. Now the upper bound follows from Corollary 3.2.6.

By Proposition 3.1.6, if $\mathcal{B}$ is a stable set of $\Theta(M)$, then $\mathcal{B}$ has the scarce theta property. If a set $\mathcal{B}$ of circuits of M has the scarce theta property, then $\mathcal{H}' = \{E(M) \setminus C : C \in \mathcal{B}\}$ is a colinear subclass of M. Therefore, by Theorem 3.1.1, the number of coextensions of M is at least $i(\Theta(M))$. $\square$

3.3 Extensions of uniform matroids

In this section, we give an example of using the hyperplane graph to count the extensions of a specific matroid: the uniform matroid. For non-negative integers r and n where $r \leq n$, the *uniform matroid* of rank r on an n-element set, denoted $U_{r,n}$, is the matroid with ground set $[n]$ where every set of size at most r is independent. The hyperplanes of $U_{r,n}$ are precisely the subsets of $[n]$ of size $r-1$ and the rank-$(r-2)$ flats of $U_{r,n}$ are precisely the subsets of $[n]$ of size $r-2$. Therefore, the hyperplane graph of $U_{r,n}$ has as vertices the $(r-1)$-subsets of $[n]$ where two vertices are adjacent if and only if they intersect in a $(r-2)$-set.

For non-negative integers m and k where $k \leq m$, the *Johnson graph* $J(m,k)$ is the graph whose vertices are the k-subsets of a fixed m-set where two vertices are adjacent if and only if they intersect in $k-1$ elements. Thus, the hyperplane graph $\Pi(U_{r,n})$ is isomorphic to $J(n, r-1)$. In [17] by Graham and Sloane, Theorem 1 implies that $J(m, k)$ contains a stable set of size $\frac{1}{m}\binom{m}{k}$. Since each subset of a stable set is itself a stable set and $J(n, r-1)$ is isomorphic to $\Pi(U_{r,n})$, we immediately get the following proposition.

Proposition 3.3.1. $\log i(\Pi(U_{r,n})) \geq \frac{1}{n}\binom{n}{r-1}$.

The rank of an extension of a rank-r matroid is r, except for one extension which has rank $r+1$ (see page 269 in [40]). Therefore, the number of extensions of $U_{r,n}$ is at most one plus the number of rank-r matroids on $n+1$ elements.

Theorem 3.3.2 (Theorem 5.1.1 in [50] and Theorem 5 in [43])**.** *Let $m(n, r)$ denote the number of rank-r matroids with ground set $[n]$. For all $r \geq 3$ and $n \geq r+12$,*

$$\log m(n, r) \leq \frac{1}{n-r+1}\binom{n}{r}\log(\mathrm{e}(n-r+1)).$$

Now Proposition 3.3.1 and Lemma 3.2.3 imply the lower bound and Theorem 3.3.2 implies the upper bound of the following theorem.

Theorem 3.3.3. $\frac{1}{n}\binom{n}{r-1} \leq \log \mathrm{ext}(U_{r,n}) \leq \frac{1}{n-r+2}\binom{n+1}{r}\log(\mathrm{e}(n-r+2)) + 1$.

Chapter 4

Stable sets in Hamming graphs

It turns out that Hamming graphs are very useful in enumerating the number of coextensions of Dowling geometries. As we will see in Chapter 7, certain Hamming graphs are "similar" to certain subgraphs of the circuit graph of a Dowling geometry. Since bounding the number of stable sets in the circuit graph is a good strategy for bounding coextensions, we are now interested in bounding the number of stable sets in certain Hamming graphs. In fact, this is an interesting problem outside of the applications to matroid theory. Hamming graphs are highly structured graphs with connections to areas of study such as error-correcting codes and association schemes.

4.1 Introduction

Let $t, n_1, \ldots, n_t \in \mathbb{Z}_{>0}$ and let $S_1, \ldots, S_t$ be sets where S_i contains n_i elements for each $i \in \{1, \ldots, t\}$. Recall that the *generalized Hamming graph* $H(n_1, \ldots, n_t)$ is the graph whose vertices are t-tuples in $S_1 \times \cdots \times S_t$ where two vertices are adjacent if and only if they differ in exactly one coordinate. Although generalized Hamming graphs seem like natural graphs to study, much more commonly studied are Hamming graphs, which are generalized Hamming graphs where $S_1 = \cdots = S_t$. The problem of enumerating d-dimensional order-n permutations is equivalent to counting maximum stable sets in $H(\underbrace{n, n, \ldots, n}_{d+1})$.

Therefore, a result of Linial and Luria [26] implies that the number of maximum stable sets in $H(\underbrace{n, n, \ldots, n}_{d+1})$ is at most $((1 + o(1)\frac{n}{e^d}))^{n^d}$.

Hamming graphs where $S_1 = \cdots = S_t = 2$ are also known as Hamming cubes or hypercubes and are denoted Q_t. Enumerating stable sets in hypercubes is well studied. Incredibly, the number of stable sets in Q_t is known up to lower order terms, rather than lower order terms on the log scale. This was first shown by Korshunov and Sapozhenko [24] in 1983 who proved that $i(Q_t) = 2\sqrt{e}2^{2^{t-1}}(1 + o_t(1))$. Since then, several refinements of the asymptotics have been made.

On the other hand, enumerating all stable sets in Hamming graphs or generalized Hamming graphs has not been as widely studied. Thus, even though the original goal of the results in this chapter was to assist in counting coextensions of Dowling geometries, we see these results as a nice contribution to the study of Hamming graphs as well.

In the rest of this thesis, we will usually use "Hamming graph" to refer to a generalized Hamming graph. Furthermore, without loss of generality, we assume $S_i = [n_i]$ for each i and $n_1 \leq \cdots \leq n_t$, unless otherwise specified. Note that, equivalently, the Hamming graph $H(n_1, \ldots, n_t)$ is isomorphic to the Cartesian product $K_{n_1} \square \ldots \square K_{n_t}$.

4.2 Preliminaries

The independence numbers of Hamming graphs are well known, but we give a proof in the following proposition.

Proposition 4.2.1. *Let $t, n_1, \ldots, n_t \in \mathbb{Z}_{>0}$ such that $n_1 \leq \cdots \leq n_t$. The independence number of $H = H(n_1, \ldots, n_t)$ is $\alpha(H) = \prod_{i=1}^{t-1} n_i$.*

Proof. Consider a set of vertices $X = \{(v_1, \ldots, v_t) \in V(H) : \sum_{i=1}^{t} v_i \mod n_t = 0\}$. We claim that X is a stable set. Suppose towards a contradiction that $(v_1, \ldots, v_t), (u_1, \ldots, u_t) \in X$ are adjacent in H. Thus, there exists $k \in [t]$ such that $v_k \neq u_k$ and $v_i = u_i$ for all $i \in [t] \setminus \{k\}$. Since $\sum_{i=1}^{t} v_i \mod n_t = 0 = \sum_{i=1}^{t} u_i \mod n_t$, we know $v_k \equiv v_k - \sum_{i=1}^{t} v_i \equiv u_k - \sum_{i=1}^{t} u_i \equiv u_k \pmod{n_t}$. Since $v_k, u_k \in [n_k]$ and $n_k \leq n_t$, it follows that $v_k = u_k$, which is a contradiction. Now we have that X is a stable set. For each $i \in [t-1]$, there are n_i choices for coordinate v_i, and there is one choice for coordinate v_t; hence $|X| = \prod_{i=1}^{t-1} n_i$ and this is a lower bound for $\alpha(H)$.

Since H is isomorphic to a Cartesian product of cliques, we have $\prod_{i=1}^{t-1} n_i$ copies of K_{n_t} and a stable set can have at most one vertex from each of these copies; thus $\alpha(H) \leq \prod_{i=1}^{t-1} n_i$. Now it follows that $\alpha(H) = \prod_{i=1}^{t-1} n_i$. $\square$

Hamming graphs with two parameters (i.e. $H(n_1, n_2)$) are sometimes known as rook graphs. The well-known problem of finding the number of ways to place non-attacking rooks on an $n_1 \times n_2$ chessboard can be modelled by the stable sets of this graph. In particular, each stable set in $H(n_1, n_2)$ of size k corresponds to a placement of k non-attacking rooks on an $n_1 \times n_2$ chessboard. The rook graph is also isomorphic to the line graph of a complete bipartite graph $L(K_{n_1,n_2})$, which is a graph with vertex set $E(K_{n_1,n_2})$ where vertices e_1, e_2 are adjacent if and only if e_1 and e_2 are incident in K_{n_1,n_2}. The stable sets of a line graph $L(G)$ correspond to matchings in G, so the number of stable sets in $H(n_1, n_2)$ is also equal to the number of matchings in K_{n_1,n_2}.

In the next lemma, we give the asymptotic number of stable sets in $H(n, n)$, which is also the number of ways to place a set of non-attacking rooks on an $n \times n$ chessboard and the number of matchings in $K_{n,n}$. Recall that log denotes the base-2 logarithm.

Lemma 4.2.2. *If $n \geq 1$ and $H = H(n, n)$, then $\log i(H) = n \log n(1 + o(1))$.*

Proof. Let $\mathcal{S}$ be the collection of stable sets of H. For each $k \in [0, n]$, let $\mathcal{S}_k = \{S \in \mathcal{S} : |S| = k\}$. By Proposition 4.2.1, the largest stable sets in H have size n, so $\mathcal{S} = \bigcup_{k=0}^{n} \mathcal{S}_k$. Thus, we have $i(H) = \sum_{k=0}^{n} |\mathcal{S}_k|$.

Note that $|S_0| = 1$. Let $k \in [n]$ and consider a stable set S in $\mathcal{S}_k$. Recall that each vertex of H is a pair (x, y) where $x, y \in [n]$. Let $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$ be the vertices in S where $x_1 \leq x_2 \leq \cdots \leq x_k$. Since S does not contain pairs that have the same first coordinate or that have the same second coordinate, it follows that $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_k\}$ are k-subsets of $[n]$. Thus, there are $\binom{n}{k}$ choices for each of the sets X and Y. Since there are $k!$ ways to order the elements of Y, there are $k!$ ways to pair the elements in Y with those in X. Therefore, there are $\binom{n}{k}^2 k!$ stable sets in $\mathcal{S}_k$.

Claim 4.2.2.1. $\sum_{k=1}^{n} \binom{n}{k}^2 k! \leq n^{n+1}$.

Proof.

$$\begin{aligned}
\sum_{k=1}^{n} \binom{n}{k}^2 k! &= \sum_{k=1}^{n} \frac{n!^2}{k!(n-k)!^2} \\
&= \sum_{k=1}^{n} \frac{\Big((n)(n-1)\ldots(n-k+1)\Big)\Big((n)(n-1)\ldots(k+1)\Big)}{(n-k)!} \\
&\leq \sum_{k=1}^{n} n^k n^{n-k} \\
&= n^{n+1}.
\end{aligned}$$

□

Since $i(H) = \sum_{k=0}^{n} |\mathcal{S}_k|$, it follows from Claim 4.2.2.1 that $i(H) \leq 1 + n^{n+1} = 2^{n \log n(1+o(1))}$. Since $|\mathcal{S}_n| = n!$, we know $\log i(H) \geq \log n! \geq n \log n - n \log e = n \log n(1 + o(1))$. Combining the lower and upper bounds gives $\log i(H) = n \log n(1 + o(1))$. $\square$

4.3 The main results

In this section, we first prove bounds on the number of stable sets in a standard Hamming graph (where each coordinate of each vertex is taken from the same set). Then, we prove bounds on the number of stable sets of a particular generalized Hamming graph, which will be useful in Chapter 7.

For a tuple $v = (v_1, \ldots, v_k)$ whose coordinates are integers, let $\phi(v) = \sum_{i=1}^{k} v_i$. For $\ell, m \in [k]$, let $\phi_\ell^m(v) = \sum_{i=\ell}^{m} v_i$.

Theorem 4.3.1. *If $n \geq 2$ and $t \geq 2$, then*

$$\tfrac{1}{t} n^{t-1} \log(t(n-1)) \leq \log i(H(\underbrace{n, \ldots, n}_{t})) \leq n^{t-1} \log n(1 + \tfrac{2}{n}).$$

Proof. Let $H = H(\underbrace{n, \ldots, n}_{t})$. If $t = 2$, then the result follows from Lemma 4.2.2. Thus, we may assume that $t \geq 3$.

Since H is a $t(n-1)$-regular graph on n^t vertices, it follows from Theorem 2.3.9 that $\log i(H) \geq \lfloor \frac{n^t}{t(n-1)} \rfloor \log(t(n-1)) \geq \frac{1}{t} n^{t-1} \log n(1 + \frac{\log t}{\log n} + o(1))$.

Let $H' = H(n, n)$. For each $(t-2)$-tuple T whose coordinates are in $[n]$, let $V_T = \{v = (v_1, v_2, \ldots, v_t) \in V(H) : (v_1, v_2, \ldots, v_{t-2}) = T\}$. Since the first $t-2$ coordinates of each vertex in V_T are fixed, the graph $H[V_T]$ is isomorphic to H'. Since each vertex of H is in V_T for exactly one tuple T, these vertex sets partition $V(H)$. Since there are n^{t-2} tuples in $[n]^{t-2}$, there are n^{t-2} sets V_T. By Proposition 2.3.7, we have $i(H) \leq i(H')^{n^{t-2}}$. By the proof of Lemma 4.2.2, we have $i(H') \leq 1 + n^{n+1} \leq n^{n+2}$. Therefore,

$$\begin{aligned} \log i(H) &\leq n^{t-2} \log i(H') \\ &\leq n^{t-1} \log(n)(1 + \tfrac{2}{n}) \end{aligned}$$

This establishes the upper bound for $i(H)$. $\square$

Theorem 4.3.2. *If $(a_n)_{n\in\mathcal{Z}_{>0}}, (b_n)_{n\in\mathcal{Z}_{>0}}$ are increasing sequences of positive integers such that $\lim_{n\to\infty}\frac{\log a_n}{\log b_n} = 0$ and $(s_n)_{n\in\mathcal{Z}_{>0}}, (t_n)_{n\in\mathcal{Z}_{>0}}$ are non-decreasing sequences of positive integers, then*

$$\tfrac{1}{t}a_n^{s_n}b_n^{t_n-1}\log(b_n)(1+o(1)) \le \log i(H(\underbrace{a_n,\dots,a_n}_{s_n},\underbrace{b_n,\dots,b_n}_{t_n})) \le a_n^{s_n}b_n^{t_n-1}\log(b_n)(1+o(1)).$$

Proof. Assume n is sufficiently large enough to ensure that $a_n \le b_n$. Let $a = a_n$, $b = b_n$, $s = s_n$, and $t = t_n$. For each $i \in [s]$, let $X_i = [0, a-1]$. For each $i \in [t]$, let $Y_i = [0, b-1]$. Let $H = H(\underbrace{a,\dots,a}_{s},\underbrace{b,\dots,b}_{t})$ where, without loss of generality, the vertices of H are the $(s+t)$-tuples of $X_1\times\cdots\times X_s\times Y_1\times\cdots\times Y_t$. Let $H' = H(\underbrace{b,\dots,b}_{t})$.

For each s-tuple S whose coordinates are in $[a]$, let $V_S = \{v = (v_1, v_2, \dots, v_{s+t}) \in V(H) : (v_1, v_2, \dots, v_s) = S\}$. Since the first s coordinates of each vertex in V_i are fixed, the graph $H[V_i]$ is isomorphic to H'. Since each vertex of H is in V_S for exactly one tuple S, these vertex sets partition $V(H)$. Since there are a^s tuples in $[a]^s$, there are a^s sets V_S. By Proposition 2.3.7, we have $i(H) \le i(H')^{a^s}$. By Theorem 4.3.1, we have $i(H') \le b^{t-1}\log(b)(1+\frac{2}{b})$ Therefore,

$$\begin{aligned}\log i(H) &\le a^s\log i(H')\\ &= a^sb^{t-1}\log(b)(1+\tfrac{2}{b}).\end{aligned}$$

This establishes the upper bound for $i(H)$. We now focus on a lower bound.

For each $i \in [0, b-1]$, let $U_i = \{v \in V(H) : \phi(v) \equiv i \pmod b\}$. We claim that each U_i is a stable set in H. Let $i \in [0, b-1]$ and suppose towards a contradiction that $u = (u_1, \dots, u_{s+t}) \in U_i$ and $v = (v_1, \dots, v_{s+t}) \in U_i$ are adjacent in H. Thus, there exists $k \in [s+t]$ such that $u_k \ne v_k$ and $u_j = v_j$ for all $j \in [s+t]\setminus\{k\}$. Therefore, since $\phi(u) \equiv \phi(v) \pmod b$, it follows that $u_k \equiv v_k \pmod b$. Since $0 \le u_k, v_k < b$, we find $u_k = v_k$, which is a contradiction.

To construct a vertex $(u_1, \dots, u_{s+t})$ in some U_i, there are a choices for each of $u_1, \dots, u_s$ and b choices for each of $u_{s+1}, \dots, u_{s+t-1}$. There is one choice for u_{s+t} since the sum must be equivalent to $i \pmod b$. Hence $|U_i| = a^sb^{t-1}$.

Let $d = \lceil\frac{b}{a}\rceil$. Let H_0 be the graph induced on $U_0 \cup U_a \cup U_{2a} \cup \cdots \cup U_{(d-1)a}$. Notice that $|V(H_0)| = a^sb^{t-1}d$, since it is made up of d disjoint sets, each of size a^sb^{t-1}. For each s-tuple $\sigma \in [0, a-1]^s$, let H_0^σ be the subgraph of H_0 induced on $\{v = (v_1, \dots, v_{s+t}) \in V(H_0) : (v_1, \dots, v_s) = \sigma$ and $\phi_{s+1}^{s+t}(v) \equiv 0 \pmod a\}$. Let $\sigma_0 \in [0, a-1]^s$ be the s-tuple whose coordinates are all 0.

Claim 4.3.2.1. *Let $i \in \{0, a, 2a, \dots, (d-1)a\}$, and let $\sigma \in [0, a-1]^s$ where $\phi(\sigma) \equiv i \pmod{b}$. The graph H_0^σ is isomorphic to $H_0^{\sigma_0}$.*

Proof. Define a function $f : V(H_0^\sigma) \to V(H_0^{\sigma_0})$, as follows. For each $(v_1, \dots, v_{s+t}) \in V(H_0^\sigma)$, let $f((v_1, \dots, v_{s+t})) = \sigma_0 \cdot (v_{s+1}, \dots, v_{s+t}) = (0, \dots, 0, v_{s+1}, \dots, v_{s+t})$. Now define a function $g : V(H_0^{\sigma_0}) \to V(H_0^\sigma)$ where, for each $(0, \dots, 0, v_{s+1}, \dots, v_{s+t}) \in V(H_0^{\sigma_0})$, we let $g((0, \dots, 0, v_{s+1}, \dots, v_{s+t})) = \sigma \cdot (v_{s+1}, \dots, v_{s+t})$. Since $(v_1, \dots, v_s) = \sigma$ for all tuples $(v_1, \dots, v_{s+t})$ in H_0^σ, the function g is the inverse of f, which implies f is a bijection between the vertices in H_0^σ and the vertices in $H_0^{\sigma_0}$.

Consider $u, v \in V(H_0^\sigma)$. Let $u = (u_1, \dots, u_{s+t})$ and $v = (v_1, \dots, v_{s+t})$. Since u and v are different vertices, they differ in at least one coordinate. Let $D \subseteq [s+t]$ be the set of indices such that, for $k \in D$, we have $u_k \neq v_k$. Since u and v are in H_0^σ, their first s coordinates are the same. Therefore, we know $D \subseteq [s+1, s+t]$. Since $f(u) = (0, \dots, 0, u_{s+1}, \dots, u_{s+t})$ and $f(v) = (0, \dots, 0, u_{s+1}, \dots, u_{s+t})$, it follows that $f(u)$ and $f(v)$ differ by $|D|$ coordinates. If u and v are adjacent, then $|D| = 1$, so $f(u)$ and $f(v)$ are adjacent. Similarly, if u and v are not adjacent, then $|D| > 1$, so $f(u)$ and $f(v)$ are not adjacent. $\square$

Since $V(H_0^{\sigma_0}) = \{v = (v_1, \dots, v_{s+t}) \in V(H_0) : (v_1, \dots, v_s) = (0, \dots, 0)$ and $\phi_{s+1}^{s+t}(v) \equiv 0 \pmod{a}\}$, for a tuple $v = (v_1, \dots, v_{s+t})$ in $V(H_0^{\sigma_0})$, there is one choice for each of $v_1, \dots, v_s$ and there are b choices for each of $v_{s+1}, \dots, v_{s+t-1}$. Since $\phi_{s+1}^{s+t}(v) \equiv 0 \pmod{a}$, it follows that $\phi_{s+1}^{s+t}(v) = aj$ for some $j \in \mathbb{Z}$. Since $v_{s+t} \in [0, b-1]$,

$$\phi_{s+1}^{s+t-1}(v) \leq aj \leq \phi_{s+1}^{s+t-1}(v) + (b-1).$$

Therefore, there are at most $\lfloor \frac{b-1}{a} \rfloor + 1 \leq \lceil \frac{b}{a} \rceil = d$ choices for j and hence for v_{s+t}. Now it follows that

$$|V(H_0^{\sigma_0})| \leq b^{t-1} d. \tag{4.1}$$

Claim 4.3.2.2. *Let $i \in \{0, a, 2a, \dots, (d-1)a\}$, and let $\sigma \in [0, a-1]^s$ where $\phi(\sigma) \equiv i \pmod{b}$. The graph H_0^σ is a union of components of H_0.*

Proof. Let $u = (u_1, \dots, u_{s+t}) \in V(H_0^\sigma)$ and let $v = (v_1, \dots, v_{s+t}) \in V(H_0) \setminus V(H_0^\sigma)$. Suppose towards a contradiction that u and v are adjacent in H_0. Therefore, there exists $k \in [s+t]$ such that $u_k \neq v_k$ and $u_j = v_j$ for all $j \in [s+t] \setminus \{k\}$. Since v is not in H_0^σ, the tuple $(v_1, \dots, v_s)$ is not equal to $(u_1, \dots, u_s)$. That is, the tuples u and v differ in at least one coordinate, which implies that $k \in [s]$. For each $i \in [0, b-1]$, the set of vertices U_i is a stable set, so u and v are not both in U_i. Therefore, there exist integers $i \neq j \in [0, d-1]$

such that $\phi(u) \equiv ai \pmod{b}$ and $\phi(v) \equiv aj \pmod{b}$. Without loss of generality, assume that $i > j$. This implies that

$$u_k - v_k = \phi(u) - \phi(v) \equiv a(i-j) \pmod{b},$$

where $(i-j) \in [0, d-1]$. Since $u_k \neq v_k$, we have $(i-j) \in [d-1]$. Thus, since $a \leq a(i-j) < b$, it follows that $u_k - v_k = a(i-j)$. That is, we have $u_k = v_k + a(i-j) \geq v_k + a$, which is a contradiction, since $u_k, v_k \in [0, a-1]$. $\square$

Let $d' = \lfloor \frac{b}{a} \rfloor$ and define integers $d_1, d_2, \ldots, d_t$ such that $d_1 = d_2 = \cdots = d_t = d'$. Next, we will show that $H_0^{\sigma_0}$ contains a^{t-1} disjoint induced subgraphs isomorphic to $H' = H(d_1, d_2, \ldots, d_t)$. Let T be the collection of tuples $\tau = (\tau_1, \ldots, \tau_t) \in [0, a-1]^t$ where $\phi(\tau) \equiv 0 \pmod{a}$. For each tuple $\tau = (\tau_1, \ldots, \tau_t) \in T$, let H'_τ denote the subgraph of $H_0^{\sigma_0}$ induced on the vertex set

$$\{v = (v_1, \ldots, v_{s+t}) \in V(H_0^{\sigma_0}) : v_{s+i} = \tau_i + aj \text{ where } j \in [0, d'-1], \text{ for each } i \in [t]\}.$$

Consider a vertex $v = (v_1, \ldots, v_{s+t})$ in $H_0^{\sigma_0}$. Recall that $\phi_{s+1}^{s+t}(v) \equiv 0 \pmod{a}$. For each $i \in [t]$, we have $v_{s+i} \in [0, b-1]$. If $r \in [0, a-1]$ and $j \in [0, d'-1]$, then $r + aj \in [0, a\lfloor b/a \rfloor - 1]$. Thus, if each $v_{s+i} \leq a\lfloor b/a \rfloor - 1$, then v is in H'_τ for some $\tau \in T$. Let G be the subgraph of $H_0^{\sigma_0}$ induced on $\bigcup_{\tau \in T} V(H'_\tau)$. Note that by Proposition 2.3.4, we have $i(H_0^{\sigma_0}) \geq i(G)$.

Claim 4.3.2.3. *For each $\tau = (\tau_1, \ldots, \tau_t) \in T$, the graph H'_τ is a union of components of G.*

Proof. Consider $\tau \in T$. Let $u = (u_1, \ldots, u_{s+t}) \in V(H'_\tau)$ and $v = (v_1, \ldots, v_{s+t}) \in V(G) \setminus V(H'_\tau)$. Suppose towards a contradiction that u and v are adjacent in G. Therefore, there exists $k \in [s+t]$ such that $u_k \neq v_k$ and $u_j = v_j$ for all $j \in [s+t] \setminus \{k\}$. Since u and v are both in G, it follows that $(u_1, \ldots, u_s) = (v_1, \ldots, v_s)$, so we know $k \in [s+1, s+t]$. Also, since u and v are both in G, we have $\phi_{s+1}^{s+t}(u) \equiv \phi_{s+1}^{s+t}(v) \equiv 0 \pmod{a}$. Since u is in H'_τ, there exists $j \in [0, d'-1]$ such that $u_k = \tau_{k-s} + aj$. Therefore,

$$\begin{aligned} v_k &= \phi_{s+1}^{s+t}(v) - \phi_{s+1}^{s+t}(u) + u_k \\ &= \phi_{s+1}^{s+t}(v) - \phi_{s+1}^{s+t}(u) + \tau_{k-s} + aj \\ &\equiv \tau_{k_s} \pmod{a}. \end{aligned}$$

This implies that $v_k = \tau_{k-s} + ai$ for some integer i. Since v is in G, we have $v_k \in [0, ad'-1]$, which implies that $i \in [0, d'-1]$. Therefore, the vertex v is in H'_τ, which is a contradiction. $\square$

Claim 4.3.2.4. *If $\tau \in T$, then the graph H'_τ is isomorphic to the Hamming graph H'.*

Proof. Let $\tau = (\tau_1, \ldots, \tau_t)$. Define a function $f : V(H'_\tau) \to V(H')$, as follows. For each $v = (v_1, \ldots, v_{s+t}) \in V(H'_\tau)$, let $f(v) = (x_1, \ldots, x_t)$ where $x_i = (v_{s+i} - \tau_i)/a$ for each $i \in [t]$. By definition of H'_τ, each x_i is an integer in $[0, d' - 1]$, which implies that $(x_1, \ldots, x_t) \in V(H')$.

Now define a function $g : V(H') \to V(H'_\tau)$, as follows. For each $x = (x_1, \ldots, x_t) \in V(H')$, let $g(x) = \sigma_0 \cdot (v_{s+1}, \ldots, v_{s+t})$ where $v_{s+i} = \tau_i + a x_i$ for each $i \in [t]$. By definition of H'_τ and since each $x_i \in [0, d' - 1]$, we know $\sigma_0 \cdot (v_{s+1}, \ldots, v_{s+t}) \in V(H'_\tau)$. Since g is the inverse of f, the function f is a bijection.

Consider $u, v \in V(H'_\tau)$. Let $u = (u_1, \ldots, u_{s+t})$ and $v = (v_1, \ldots, v_{s+t})$. Since u and v are different vertices, they differ in at least one coordinate. Let $D \subseteq [s+t]$ be the set of indices such that, for $k \in D$, we have $u_k \neq v_k$. Since u and v are in $V(H'_\tau) \subseteq V(H_0^\sigma)$, their first s coordinates are the same. Therefore, we know $D \subseteq [s+1, s+t]$. Let $f(u) = (x_1, \ldots, x_t)$ and let $f(v) = (y_1, \ldots, y_t)$. Let $D' \subseteq [s+1, s+t]$ be the set of indices such that, for $k \in D'$, we have $x_{k-s} \neq y_{k-s}$.

Consider $k \in [s+1, s+t]$. Observe that $u_k = \tau_{k-s} + a x_{k-s}$ and $v_k = \tau_{k-s} + a y_{k-s}$. If $k \in D$, then $\tau_{k-s} + a x_{k-s} \neq \tau_{k-s} + a y_{k-s}$. Thus, $x_{k-s} \neq y_{k-s}$, which implies $k \in D'$. If $k \notin D$, then $\tau_{k-s} + a x_{k-s} = \tau_{k-s} + a y_{k-s}$. Thus, $x_{k-s} = y_{k-s}$, which implies $k \notin D'$. Therefore, since $k \in D$ if and only if $k \in D'$, it follows that $D = D'$. Thus, the number of coordinates that u and v differ by is equal to the number of coordinates that $f(u)$ and $f(v)$ differ by. Hence u and v are adjacent if and only if $f(u)$ and $f(v)$ are adjacent. $\square$

By Claim 4.3.2.3 and Proposition 2.3.6, we have $\log i(G) = \sum_{\tau \in T} \log i(H'_\tau)$. By Claim 4.3.2.4 and Theorem 4.3.1, it follows that $\log i(H'_\tau) \geq \frac{1}{t}(d')^{t-1} \log(t(d'-1))$. Since $|T| = a^{t-1}$,

$$\log i(G) \geq \sum_{\tau \in T} \tfrac{1}{t}(d')^{t-1} \log(t(d'-1)) = \tfrac{1}{t}(ad')^{t-1} \log d'(1 + o(1)).$$

Since $ad' \geq b - a$ and $\lim_{n\to\infty} \frac{\log a}{\log b} = 0$,

$$\log i(G) \geq \tfrac{1}{t}(b-a)^{t-1} \log b(1 + \tfrac{\log a}{\log b} + o(1)) \geq \tfrac{1}{t} b^{t-1} \log b(1 + o(1)).$$

By Claims 4.3.2.2 and 4.3.2.1, the graph H_0 is made up of unions of components that are isomorphic to $H_0^{\sigma_0}$. Using Equation 4.1, we find that the number of these unions of components is:

$$\frac{|V(H_0)|}{|V(H_0^{\sigma_0})|} \geq \frac{a^s b^{t-1} d}{b^{t-1} d} = a^s.$$

By Proposition 2.3.6, it follows that $\log i(H_0) \geq a^s \log i(H_0^{\sigma_0})$. Now, since $i(H_0^{\sigma_0}) \geq i(G)$,

$$\log i(H_0) \geq a^s \log i(H_0^{\sigma_0}) \geq \tfrac{1}{t} a^s b^{t-1} \log b(1 + o(1)).$$

Finally, since H_0 is an induced subgraph of H, it follows from Proposition 2.3.4 that

$$\log i(H) \geq \tfrac{1}{t} a^s b^{t-1} \log b(1 + o(1)). \qquad \square$$

Chapter 5

Container methods

In this chapter, we start by introducing *container methods*, which are novel methods that bound the number of stable sets in a graph. We then describe a container method for sufficiently dense graphs that we will use as a black box later in this thesis. Last, we prove our own variation of an existing container method for regular graphs.

The work in this chapter is partially based on the course on container methods taught by Jorn van der Pol in 2020.

5.1 Introduction

Many structures in mathematics can be expressed as stable sets in graphs, which means strategies for counting stable sets have a wide range of applications. We have already seen that scarce linear subclasses can be represented as stable sets in a graph. Other applications include counting Sidon sets, sum-free sets, codes, antichains in posets, graph colourings, triangle-free graphs, and H-free graphs. Container methods for graphs were first used in the 1980's by Kleitman and Winston [19, 21]. Since then, container methods have been developed and used by various authors to solve a variety of problems that can be reduced to counting stable sets in certain graphs (see the survey by Samotij [48]). There is also a body of research on container methods for hypergraphs, which began in the 2010's with Balogh, Morris, and Samotij [4] and Saxton and Thomason [49].

The broad strategy of a container method is to construct a collection of vertex subsets, called *containers*, such that each stable set is contained within one, bound the size of each container, and then bound the number of containers. In order to construct the containers, we usually start by proving a lemma, called a *supersaturation lemma*, which establishes a density property of the graph. An algorithm, usually called a *scythe algorithm*, uses the supersaturation lemma to greedily construct containers for stable sets of a certain size. The algorithm associates each container with another smaller vertex set, called a *fingerprint*. Using the number of fingerprints, the number of subsets of each container, and a bound on the number of stable sets smaller than those considered in the scythe algorithm, an elementary argument then gives a bound on the total number of stable sets.

5.2 Sufficiently dense graphs

In 2015, Kohayakawa, Lee, Rödl, and Samotij [23] developed a container method for graphs with a certain density condition, which they applied to the problem of enumerating Sidon sets. Their application to Sidon sets is not relevant to us, but we will use their container method. The following theorem is Lemma 3.1 in [23]. Let $e_G(X)$ denote $|E(G[X])|$, the number of edges induced by a subset X of vertices of a graph G. Let $i(G, m)$ denote the number of m-element stable sets in a graph G.

Theorem 5.2.1 (Lemma 3.1 in [23]). *Let $q, N \in \mathbb{Z}_{>0}$, $R \in \mathbb{R}_{>0}$, and $0 \leq \beta \leq 1$ be such that*

$$R \geq e^{-\beta q} N.$$

Let G be an N-vertex graph with the property that

$$e_G(U) \geq \beta \binom{|U|}{2}$$

for every $U \subseteq V(G)$ containing at least R vertices. For every $m \geq q$,

$$i(G, m) \leq \binom{N}{q} \binom{R}{m-q}.$$

In the following corollary, we use Theorem 5.2.1 to bound the total number of stable sets in a graph G that satisfies the property in the theorem.

Corollary 5.2.2. *Let $q, N \in \mathbb{Z}_{>0}$, $R \in \mathbb{R}_{>0}$, and $0 \leq \beta \leq 1$ be such that*

$$R \geq e^{-\beta q} N.$$

Let G be an N-vertex graph with the property that

$$e_G(U) \geq \beta \binom{|U|}{2}$$

for every $U \subseteq V(G)$ containing at least R vertices. The number of stable sets in G is at most

$$\left(\frac{\mathrm{e}N}{q}\right)^q \cdot 2^R.$$

Proof. Since the graph G contains at most $\sum_{j=0}^{q} \binom{N}{j}$ vertex subsets of size at most q, the number of stable sets of G of size at most q is at most $\sum_{j=0}^{q} \binom{N}{j} \leq \left(\frac{\mathrm{e}N}{q}\right)^q$. By Theorem 5.2.1, and since $\binom{N}{q} \leq \left(\frac{\mathrm{e}N}{q}\right)^q$,

$$\begin{aligned}
i(G) &= \sum_{j=0}^{q} i(G,j) + \sum_{m=q+1}^{N} i(G,m) \\
&\leq \left(\frac{\mathrm{e}N}{q}\right)^q + \sum_{m=q+1}^{N} \binom{N}{q}\binom{R}{m-q} \\
&\leq \left(\frac{\mathrm{e}N}{q}\right)^q \left(1 + \sum_{j=1}^{R} \binom{R}{j}\right) \\
&= \left(\frac{\mathrm{e}N}{q}\right)^q \cdot 2^R.
\end{aligned}$$

□

This corollary will be used in Subsection 6.2.5. In order to apply it to a graph G, we will first need to prove that subsets of $V(G)$ of a certain size induce a sufficient number of edges; that is, G is sufficiently dense. This is done with a supersaturation lemma, which we will need for each application of this corollary.

5.3 Spectral method for regular graphs

In this section, we develop a container method for regular graphs. Our method is a variation of that used in Alon, Balogh, Morris, and Samotij's 2014 paper [2]. However, we use the same supersaturation lemma. In this section, and when applying spectral methods, we assume all graphs are simple.

Spectral graph theory refers to the study of graphs through parameters, such as eigenvalues, of their associated matrices, such as adjacency matrices. The *adjacency matrix* of a graph G with n vertices is the $n \times n$ matrix A whose rows and columns are indexed by $V(G)$ such that the entries corresponding to vertices u and v is 1 if u and v are adjacent and 0 otherwise. The *eigenvalues* of a graph G are the eigenvalues of its adjacency matrix.

The following result is the supersaturation lemma from [2].

Lemma 5.3.1 (One-sided Expander Mixing Lemma [2])**.** *Let G be a d-regular N-vertex graph with smallest eigenvalue $-\lambda$. For any vertex subset $U \subseteq V(G)$, the number of edges induced by U satisfies*

$$e(U) \geq \frac{d|U|^2}{2N} - \frac{1}{2}\lambda|U|\left(1 - \frac{|U|}{N}\right).$$

We now prove our own version of a container method using the following algorithm. This algorithm is based on the one in [2].

For a graph G, let $\text{Ind}(G)$ denote the collection of stable sets of G. The *maximum-degree ordering* of a set $A = \{u_1, u_2, \ldots, u_{|A|}\}$ of vertices in a graph G is the sequence $(v_1, v_2, \ldots, v_{|A|})$ of the vertices in A such that $v_i = u_j$ where u_j has maximum degree in $G[A \setminus \{v_1, \ldots, v_{i-1}\}]$ and ties are broken by indices; that is, $j \leq k$ for all vertices u_k that have maximum degree in $G[A \setminus \{v_1, \ldots, v_{i-1}\}]$.

Algorithm 1: Adaptation of the algorithm in [2]

```
Input: (G, I, ε)
S_0 ← ∅
A_0 ← V(G) where the vertices of G are labelled using the set [|V(G)|]
for i = 1, ..., d − ε(d + λ) do
    S_i ← S_{i−1}
    A_i ← A_{i−1}
    while |A_i| > ((d+λ−i)/(d+λ)) N do
        v ← first vertex in the maximum degree ordering of A_i
        if v ∉ I then
            A_i ← A_i \ {v}
        else
            S_i ← S_i ∪ {v}
            A_i ← A_i \ ({v} ∪ N_G(v))
        end
    end
end
Output: (S_{d−ε(d+λ)}, A_{d−ε(d+λ)})
```

Theorem 5.3.2. *For each d-regular N-vertex graph G with smallest eigenvalue $-\lambda$ and $\varepsilon > 0$, there exist functions $S : \mathrm{Ind}(G) \to 2^{V(G)}$ and $A : 2^{V(G)} \to 2^{V(G)}$ such that for every $I \in \mathrm{Ind}(G)$:*

(i) $(S(I), A(I))$ *is the output of Algorithm 1 on input* (G, I, ε)*;*

(ii) $S(I) \subseteq I$ *and* $I \subseteq S(I) \cup A(S(I))$*;*

(iii) $|S(I)| \leq \frac{N \ln(\varepsilon^{-1})}{d+\lambda} + d$*; and*

(iv) $|A(S(I))| \leq \left(\frac{\lambda}{d+\lambda} + \varepsilon\right) N$*.*

Before we prove Theorem 5.3.2, consider Algorithm 1. The input of Algorithm 1, is a d-regular N-vertex graph G with smallest eigenvalue $-\lambda$, a stable set I of G, and a positive real number ε. The algorithm defines two sequences of sets $S_0, S_1, \ldots, S_{d-\varepsilon(d+\lambda)}$ and $A_0, A_1, \ldots, A_{d-\varepsilon(d+\lambda)}$. The algorithm starts by setting S_0 to be the empty set and A_0 to be the vertex set of G. Note that we arbitrarily label the vertices of G so that there is exactly one maximum degree ordering of each subset of vertices.

Algorithm 1 constructs its output sets in a sequence of steps, which is its main difference from the algorithm in [2]. At each step $i \in \{1, \ldots, d - \varepsilon(d + \lambda)\}$, the set S_i is initially equal to S_{i-1} and A_i is initially equal to A_{i-1}. The final sets S_i and A_i are obtained by repeating the following process. If the first vertex v in a maximum degree ordering of A_i is not in the given stable set I, then remove v from A_i; otherwise, add v to S_i and remove v and the neighbours of v from A_i. This process is repeated as long as the size of A_i is greater than $\frac{d+\lambda-i}{d+\lambda}N$. After the two sequences of sets are created, the algorithm outputs $(S_{d-\varepsilon(d+\lambda)}, A_{d-\varepsilon(d+\lambda)})$.

The idea is to create a subset S of I that has bounded size, while keeping track of the number of remaining vertices A. This way I is a superset of S and a subset of $S \cup A$. In the setting of container methods, the set S is referred to as the fingerprint of I and the set $S \cup A$ is referred to as the container of I. We then claim that if (S, A) is the output of the algorithm on input (G, I, ε), then (S, A) is also the output on input (G, S, ε). That is, the set S is the fingerprint of both I and S, and the set $S \cup A$ is the container of both S and I. Since every stable set of G receives a fingerprint and a container from the algorithm, we can upper bound the number of stable sets using upper bounds on the size and number of fingerprints and containers.

Proof of Theorem 5.3.2. Let I be a stable set of G and suppose $(S(I), A(I))$ is the output of Algorithm 1 on input (G, I, ε). Thus, statement (i) is satisfied. Each vertex $v \in I$ begins in A_0 and at each step $i \in \{1, \ldots, d - \varepsilon(d + \lambda)\}$, either remains in A_i or is moved to S_i. Thus, the set $S(I)$ is a subset of I and I is a subset of $S(I) \cup A(I)$.

Claim 5.3.2.1. *If (S, A) is the output of Algorithm 1 on input (G, I, ε), then (S, A) is the output of the algorithm on input (G, S, ε).*

Proof. Let $(v_1, \ldots, v_r)$ and $(v'_1, \ldots, v'_{r'})$ be the vertices selected as v by the algorithm on input (G, I, ε) and (G, S, ε), respectively. Let (S', A') be the output of the algorithm on input (G, S, ε). We will show that $S = S'$ and $A = A'$. Let $X_0 = V(G)$ and for $i \in [r]$, let $X_i = X_{i-1} - \{v_i\}$ if $v_i \notin I$ and $X_i = X_{i-1} - (\{v_i\} \cup N_G(v_i))$ otherwise. Similarly, let $X'_0 = V(G)$ and for $i \in [r']$, let $X'_i = X'_{i-1} - \{v'_i\}$ if $v'_i \notin S$ and $X'_i = X'_{i-1} - (\{v'_i\} \cup N_G(v'_i))$. otherwise. The X_i's and X'_i's in this proof are constructed in the same way that the A_i sets are constructed in the algorithm, although here a new X_i or X'_i is defined for each selected vertex v, rather than for each of $d - \varepsilon(d + \lambda)$ steps. Thus, the set X_r is equal to A and similarly, the set $X'_{r'}$ is equal to A'.

Let $v_0 = v'_0 = 0$ be defined for convenience in the following induction argument. Suppose $v_{i-1} = v'_{i-1}$ and $X_{i-1} = X'_{i-1}$ for all $1 \leq i \leq k$ for some $k \in [r]$. Since $X_{k-1} = X'_{k-1}$, the first vertex in the maximum degree ordering of X_{k-1} is the same as the first vertex in the maximum degree ordering of X'_{k-1}, hence $v_k = v'_k$. Since S is a subset of I, if v_k is not in I, then v_k is not in S. If v_k is in I, then the algorithm adds v_k to the first output set S. Therefore, the vertex v_k is in I if and only if it is in S. It now follows from the way X_k and X'_k are constructed that $X_k = X'_k$. Thus, by induction, we have $v_i = v'_i$ and $X_i = X'_i$ for all $i \in [r]$.

This implies that the same vertices are selected and the same sets are created at each step in the algorithm on inputs (G, I, ε) and (G, S, ε). In particular, $r = r'$ and $A = X_r = X'_{r'} = A'$ and $S = \{v_i \in I : i \in [r]\} = \{v'_i \in S : i \in [r]\} = S'$. ☐

By Claim 5.3.2.1, $A(I) = A(S(I))$. Therefore, the set $S(I)$ is a subset of I and I is a subset of $S(I) \cup A(S(I))$, so (ii) is satisfied.

Let phase i be the ith iteration of the for loop. Consider phase i for some $1 \leq i \leq d - \varepsilon(d+\lambda)$. In phase i, we obtain S_i from S_{i-1} and A_i from A_{i-1}. Throughout this phase, the set A_i has size greater than $\frac{d+\lambda-i}{d+\lambda}N$. Therefore, by the Expander Mixing Lemma 5.3.1, the average degree in $G[A_i]$ during phase i is

$$\begin{aligned}
\frac{2e(A_i)}{|A_i|} &\geq \frac{d|A_i|}{N} - \lambda\left(1 - \frac{|A_i|}{N}\right) \\
&> \frac{d \cdot \frac{d+\lambda-i}{d+\lambda}N}{N} - \lambda\left(1 - \frac{\frac{d+\lambda-i}{d+\lambda}N}{N}\right) \\
&= d \cdot \frac{d+\lambda-i}{d+\lambda} + \lambda \cdot \frac{d+\lambda-i}{d+\lambda} - \lambda \\
&= d + \lambda - i - \lambda \\
&= d - i.
\end{aligned} \tag{5.1}$$

That is, the first vertex in the maximum degree ordering of $G[A_i]$ has degree at least $d - i + 1$.

At the beginning of phase i, we set $A_i = A_{i-1}$ and due to the previous $(i-1)$ phase, we know $|A_{i-1}| \leq \frac{d+\lambda-i+1}{d+\lambda}N$. In the last iteration of the while loop during phase i, the set A_i has size greater than $\frac{d+\lambda-i}{d+\lambda}N$ and then either 1 or at least $d-i+1$ vertices are removed from it. Since at least $d - i + 1$ vertices are removed from A_i each time a vertex is added to S_i, the number of vertices added to S_i during phase i is at most

$$\begin{aligned}
|S_i| - |S_{i-1}| \leq \frac{|A_{i-1}| - |A_i|}{d-i+1} + 1 &< \frac{\frac{d+\lambda-i+1}{d+\lambda}N - \frac{d+\lambda-i}{d+\lambda}N}{d-i+1} + 1 \\
&= \frac{N}{(d+\lambda)(d-i+1)} + 1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|S_{d-\varepsilon(d+\lambda)}| &= \sum_{i=1}^{d-\varepsilon(d+\lambda)} (|S_i| - |S_{i-1}|) \\
&\leq \sum_{i=1}^{d-\varepsilon(d+\lambda)} \left(\frac{N}{(d+\lambda)(d-i+1)} + 1\right) \\
&= \frac{N}{d+\lambda} \sum_{i=1+\varepsilon(d+\lambda)}^{d} \frac{1}{i} + d.
\end{aligned} \tag{5.2}$$

The sum $\sum_{i=1+\varepsilon(d+\lambda)}^{d} \frac{1}{i}$ is a left method Riemann sum approximation of $\int_{\varepsilon(d+\lambda)}^{d} \frac{1}{x}dx$, which evaluates to $\ln(d) - \ln(\varepsilon(d+\lambda))$. Since $\frac{d}{\varepsilon(d+\lambda)} \leq \frac{1}{\varepsilon}$, the sum $\sum_{i=1+\varepsilon(d+\lambda)}^{d} \frac{1}{i}$ is at most

$\ln(\varepsilon^{-1})$. Applying this to Equation 5.2 gives

$$|S_{d-\varepsilon(d+\lambda)}| \leq \frac{N}{d+\lambda} \ln\left(\frac{1}{\varepsilon}\right) + d.$$

Since $S(I) = S_{d-\varepsilon(d+\lambda)}$, it follows from the equation above that $S(I)$ has size at most $\frac{N}{d+\lambda} \ln\left(\frac{1}{\varepsilon}\right) + d$, which satisfies (iii).

Finally, we see from the bound in the while loop that the output $A(I)$ of the algorithm has size at most $\frac{d+\lambda-(d-\varepsilon(d+\lambda))}{d+\lambda} N = (\frac{\lambda}{d+\lambda} + \varepsilon)N$. Since $A(S(I)) = A(I)$, statement (iv) is satisfied. $\square$

Theorem 5.3.3. *For each d-regular N-vertex graph G with smallest eigenvalue $-\lambda$ and $\varepsilon > 0$,*

$$i(G) \leq \sum_{i=0}^{\frac{N}{d+\lambda}\ln(\varepsilon^{-1})+d} \binom{N}{i} \cdot 2^{(\frac{\lambda}{d+\lambda}+\varepsilon)N}.$$

Proof. For each stable set $I \in \mathrm{Ind}(G)$, let $(S(I), A(I))$ be the output of Algorithm 1 on input (G, I, ε) and define $\mathcal{S} = \{S(I) : I \in \mathrm{Ind}(G)\}$. By Theorem 5.3.2 (ii), each stable set I of G is a superset of $S(I)$ and $I \setminus S(I)$ is a subset of $A(S(I))$. Therefore, for each set $S \in \mathcal{S}$, there are at most $2^{|A(S)|}$ stable sets that contain S as a subset. Thus,

$$i(G) \leq \sum_{S \in \mathcal{S}} 2^{|A(S)|}. \tag{5.3}$$

By Theorem 5.3.2 (iii), each set $S \in \mathcal{S}$ has size at most $\frac{N\ln(\varepsilon^{-1})}{d+\lambda} + d$, so there are at most $\sum_{i=0}^{\frac{N\ln(\varepsilon^{-1})}{d+\lambda}+d} \binom{N}{i}$ sets in $\mathcal{S}$. By Theorem 5.3.2 (iv), the set $A(S)$ has size at most $\left(\frac{\lambda}{d+\lambda} + \varepsilon\right)N$ for each $S \in \mathcal{S}$. Combining these bounds with Equation 5.3 gives $i(G) \leq \sum_{i=0}^{\frac{N\ln(\varepsilon^{-1})}{d+\lambda}+d} \binom{N}{i} \cdot 2^{(\frac{\lambda}{d+\lambda}+\varepsilon)N}$. $\square$

Chapter 6

Coextensions of graphic matroids

It is well-known that coextensions of graphic matroids correspond to biased graphs, which are independently well-studied [52, 53, 54]. In 2019, Nelson and Van der Pol [39] determined the asymptotic number of biased graphs on a complete graph. In Section 6.1, we explain how this determines the number of coextensions of the cycle matroid of a complete graph. We also prove some preliminary results about graphic matroids. In Section 6.2, we prove the following two theorems about the number of coextensions of the cycle matroid of a complete bipartite graph. Recall that $o(1)$ denotes an unspecified function of n which goes to 0 as n goes to infinity, log denotes the base-2 logarithm, and $\mathrm{ext}(M)$ denote the number of extensions of a matroid M.

Theorem 6.0.1. $\log \mathrm{coext}(M(K_{n,n})) = \frac{n!^2}{2n}(1 + o(1))$.

Theorem 6.0.2. $\log \mathrm{coext}(M(K_{n,n-1})) = \frac{n!(n-2)!}{4}(1 + o(1))$.

6.1 Background

Recall that the *cycle matroid* of a graph G, denoted $M(G)$, has ground set $E(G)$ where the circuits of $M(G)$ are the edge sets of cycles of G. Since the ground set of $M(G)$ is the edge set of G, we often refer to a subgraph of G as a subset of matroid elements when we technically mean to refer to the edge set of this subgraph. For example, if C is a cycle of G, then we say C is a circuit of $M(G)$, even though technically $E(C)$ is the circuit of $M(G)$ we are referencing.

Recall that the *overlap graph* of a matroid M, denoted $\Theta(M)$ is the graph whose vertices are the circuits of M and two vertices C, C' are adjacent if and only if they are contained in a connected corank-2 restriction of M. By Corollary 3.2.8,

$$\log i(\Theta(M)) \leq \log \operatorname{coext}(M) \leq \log i(\Theta(M)) + |\mathcal{C}_{\min}(M, \prec)|.$$

Recall that $\mathcal{C}_{\min}(M, \prec)$ is the collection of circuits $C \in \mathcal{C}(M)$ such that, for some corank-2 restriction N of M, the minimum circuit in N with respect to $\prec$ is C.

Now we consider applying this result to graphic matroids. First, we establish that, for a graph G, the connected corank-2 restrictions of $M(G)$ are precisely the theta subgraphs of G.

Lemma 6.1.1. *A subgraph T of a graph G is a theta graph if and only if $r^*(M(G)|E(T)) = 2$ and $M(G)|E(T)$ is a connected matroid.*

Proof. First, note that $M(G)|E(T) = M(T)$. If T is a theta graph, then $M(T)$ has corank 2 and, since each pair of edges is in a cycle, the matroid $M(T)$ is connected.

Now we claim that if $r^*(M(T)) = 2$ and $M(T)$ is a connected matroid, then T is a theta graph. Since $M(T)$ is connected, the graph T is 2-connected. Since T is connected, we have $r(M(T)) = |V(T)| - 1$. Additionally, since the corank of $M(T)$ is 2, we know $r(M(T)) = |E(T)| - 2$. Therefore, $|V(T)| = |E(T)| - 1$.

By Theorem 2.3.3, since T is 2-connected, it has an ear decomposition $(G_1, G_2, \ldots, G_k)$ where G_1 is a cycle and $G_k = T$. Since G_1 is a cycle, we know that $|V(G_1)| = |E(G_1)|$. For each $i \geq 2$, the graph G_i is the union of G_{i-1} and a path P, which contributes $|E(P)|$ edges and $|E(P)| - 1$ vertices to G_i. Thus, by induction, we have $|V(G_i)| = |E(G_i)| - (i - 1)$. Therefore, since $|V(T)| = |E(T)| - 1$, it follows that $k = 2$ and $G_2 = T$. This implies that T contains a cycle and a path between two distinct vertices of the cycle. That is, the graph T is a theta graph. $\square$

Although we define the overlap graph with respect to connected corank-2 restrictions, we bound the number of small circuits in $M(G)$ by considering all corank-2 restrictions. We do this by first bounding the size of a small circuit using Proposition 2.4.3 and the following lemma.

Lemma 6.1.2. *For each graph G, there are at most 3 circuits in a corank-2 restriction N of $M(G)$.*

Proof. Since N is a corank-2 restriction of $M(G)$, the rank of the matroid N^* is 2. Since $M(G)$ is a $GF(2)$-representable matroid, the matroid N^* is as well. Thus, the matroid N^* is a restriction of $PG(1,2)$. By Proposition 2.6.1(ii), the number of hyperplanes in $PG(1,2)$ is $\begin{bmatrix}2\\1\end{bmatrix}_2 = \frac{2^2-1}{2^1-1} = 3$. Therefore, the matroid N^* has at most 3 hyperplanes. Since the hyperplanes of N^* are the circuits of N, there are at most 3 circuits in N. $\square$

Now we can find bounds for the number of coextensions of a graphic matroid $M(G)$ that depend on the number of stable sets in the overlap graph of G and the number of small cycles in G.

Lemma 6.1.3. *If G is a graph and s is the number of cycles of G that have length at most $\frac{2}{3}(r(M(G))+2)$, then*

$$\log i(\Theta(M(G))) \leq \log \operatorname{coext}(M(G)) \leq \log i(\Theta(M(G))) + s.$$

Proof. Let $\prec$ be a total ordering of $\mathcal{C}(M(G))$ that refines the preorder by size. By Lemma 6.1.2, there are at most 3 circuits in a corank-2 restriction of $M(G)$. Thus, by Proposition 2.4.3, the smallest circuit in a corank-2 restriction of $M(G)$ has size at most $\frac{2}{3}(r(M(G))+2)$. Therefore, the size of $\mathcal{C}_{\min}(M(G), \prec)$ does not exceed the number of cycles of length at most $\frac{2}{3}(r(M(G))+2)$. Now the lemma follows from Corollary 3.2.8. $\square$

6.1.1 Coextensions of the cycle matroid of a complete graph

Recall that a biased graph is a pair $(G, \mathcal{B})$ where G is a graph and $\mathcal{B}$ is a set of cycles of G such that if two cycles $C, C' \in \mathcal{B}$ are in a theta subgraph H of G, then the third cycle in H is in $\mathcal{B}$ as well. By Lemma 6.1.1, the connected corank-2 restrictions of $M(G)$ are precisely the theta subgraphs of G. This means that $\mathcal{B}$ has the theta property. Thus, by Proposition 3.1.4, the biased graphs $(G, \mathcal{B})$ correspond to colinear subclasses of $M(G)$. Since colinear subclasses correspond to coextensions (Theorem 3.1.1), the number of coextensions of $M(G)$ is equal to the number of sets of cycles $\mathcal{B}$ such that $(G, \mathcal{B})$ is a biased graph.

In [39], Nelson and Van der Pol proved that the number of biased graphs on the complete graph K_{n+1} is $2^{\frac{1}{2}n!(1+o(1))}$. Now, the analysis above implies the following theorem.

Theorem 6.1.4 ([39])**.** $\log \operatorname{coext}(M(K_{n+1})) = \frac{1}{2}n!(1+o(1))$.

6.2 Coextensions of the cycle matroid of a complete bipartite graph

Expanding on the ideas and preliminaries in Section 6.1, we find asymptotic results for the number of coextensions of balanced and almost balanced complete bipartite graphs. Recall that, for positive integers n, m, we let $K_{n,m}$ denote the complete bipartite graph with bipartition (A, B) where $A = [n]$ and $B = [n+m] \setminus [n]$.

The proofs of Theorems 6.0.1 and 6.0.2 broadly follow the same structure as the proof of Theorem 6.1.4 in [39]. We prove Theorems 6.0.1 and 6.0.2 by establishing asymptotically matching lower and upper bounds in the log scale. The lower bounds come from taking all subsets of a largest stable set. The upper bounds are determined using applications of a container method. In both cases, we apply an established container method for sufficiently dense graphs. In order to apply this method, we prove supersaturation lemmas for the overlap graphs of $M(K_{n,n})$ and $M(K_{n,n-1})$. Supersaturation lemmas are standard elements of container methods; they aim to show that a set of vertices that is significantly larger than a maximum stable set induces a significant number of edges. For more information, see Chapter 5.

6.2.1 Preliminaries

Recall that $\mathcal{C}(M(K_{n,m}))$ is the collection of circuits of $M(K_{n,m})$, which are cycles of $K_{n,m}$. Furthermore, recall that $\Theta(M(K_{n,m}))$ is the overlap graph of $M(K_{n,m})$ whose vertex set is $\mathcal{C}(M(K_{n,m}))$ where two vertices C, C' are adjacent if and only if C and C' are contained in a connected corank-2 restriction of $M(K_{n,m})$. By Lemma 6.1.1, two vertices C, C' in the overlap graph are adjacent if and only if they are in a theta subgraph of $K_{n,m}$.

Lemma 6.2.1. *The number of $2k$-cycles in $K_{n,m}$ is $\frac{1}{2}\binom{n}{k}\binom{m}{k}k!(k-1)!$ if $k \in [2, m]$ and zero otherwise.*

Proof. Each $2k$-cycle contains k vertices from A and k from B. There are $\binom{n}{k}$ ways to choose the vertices from A and $\binom{m}{k}$ ways to choose the vertices from B. There are $k!^2$ ways to order the vertices from A and B in a sequence $a_1b_1a_2b_2\dots a_kb_k$ so that a vertex in A is first. If a $2k$-cycle is described by such a sequence, there are k choices for the first vertex a_1 and 2 choices for the following vertex b_1, so each cycle corresponds to $2k$ sequences. Thus, there are $\frac{1}{2}k!(k-1)!$ ways to order these vertices in a cycle. Therefore, the number of $2k$-cycles in $K_{n,m}$ is $\frac{1}{2}\binom{n}{k}\binom{m}{k}k!(k-1)!$. $\square$

Corollary 6.2.2. *The number of cycles in $K_{n,m}$ is*

$$|\mathcal{C}(M(K_{n,m}))| = \sum_{i=0}^{m-2} \frac{n!m!}{2(m-i)i!(n-m+i)!}.$$

Proof. By Lemma 6.2.1,

$$\begin{aligned}
|\mathcal{C}(M(K_{n,m}))| &= \sum_{k=2}^{m} \frac{1}{2}\binom{n}{k}\binom{m}{k} k!(k-1)! \\
&= \sum_{k=2}^{m} \frac{n!m!k!(k-1)!}{2k!(n-k)!k!(m-k)!} \\
&= \sum_{i=0}^{m-2} \frac{n!m!}{2(m-i)i!(n-m+i)!}.
\end{aligned}$$

□

Bounds on the number of cycles in $K_{n,m}$ will be used to bound the number of stable sets in the overlap graph of $K_{n,m}$, so we define

$$S_{n,m} = \sum_{k=0}^{m-2} \frac{1}{(m-k)k!(n-m+k)!}.$$

The constants $I_0(2)$ and $I_1(2)$ appear in the following lemmas. Recall that $I_0(2)$ is approximately 2.28 and $I_1(2)$ is approximately 1.59.

Lemma 6.2.3. $\frac{I_0(2)}{n} + \frac{I_1(2)}{n^2} < S_{n,n} < \frac{I_0(2)}{n} + \frac{I_1(2)}{n^2} + \frac{4}{n^3}$ *for all* $n \geq 5$.

Proof. We can check that the statement is true for $n = 5$. Suppose the statement holds for some $n - 1 \geq 5$ and consider $n^2 S_{n,n}$.

$$\begin{aligned}
n^2 S_{n,n} &= \sum_{k=0}^{n-2} \frac{(n-k)(n+k)}{(n-k)k!^2} + \sum_{k=0}^{n-2} \frac{k^2}{(n-k)k!^2} \\
&= \sum_{k=0}^{n-2} \frac{(n-k)(n+k)}{(n-k)k!^2} + \sum_{k=1}^{n-2} \frac{1}{(n-k)(k-1)!^2} \\
&= \sum_{k=0}^{n-2} \frac{(n+k)}{k!^2} + \sum_{k=0}^{n-3} \frac{1}{(n-1-k)k!^2} \\
&= n\sum_{k=0}^{n-2} \frac{1}{k!^2} + \sum_{k=0}^{n-2} \frac{1}{k!(k-1)!} + S_{n-1,n-1}
\end{aligned}$$

Recall $\sum_{k=0}^{\infty}\frac{1}{k!^2}=I_0(2)$ and $\sum_{k=0}^{\infty}\frac{1}{k!(k-1)!}=I_1(2)$. Since $\sum_{k=n-1}^{\infty}\frac{1}{k!^2}<\frac{1}{(n-1)!^2}\sum_{k=0}^{\infty}\frac{1}{2^k}=\frac{2}{(n-1)!^2}$ and $\sum_{k=n-1}^{\infty}\frac{1}{k!(k-1)!}<\frac{1}{(n-2)!^2}\sum_{k=0}^{\infty}\frac{1}{2^k}=\frac{2}{(n-2)!^2}$, we get the following bounds:

$$n^2 S_{n,n} > nI_0(2)-\frac{2n}{(n-1)!^2}+I_1(2)-\frac{2}{(n-2)!^2}+S_{n-1,n-1},$$

and

$$n^2 S_{n,n} < nI_0(2)+I_1(2)+S_{n-1,n-1}.$$

Since $\frac{2n}{(n-1)!^2}+\frac{2}{(n-2)!^2}<\frac{I_0(2)}{n-1}+\frac{I_1(2)}{(n-1)^2}$ and we have assumed the statement is true for $n-1$, we find $n^2S_{n,n}>nI_0(2)+I_1(2)$, so the lower bound holds.

Also by induction, we find $n^2S_{n,n}<nI_0(2)+I_1(2)+\frac{I_0(2)}{n-1}+\frac{I_1(2)}{(n-1)^2}+\frac{4}{(n-1)^3}$. Since $n\geq 6$, we can check that $\frac{I_0(2)}{n-1}+\frac{I_1(2)}{(n-1)^2}+\frac{4}{(n-1)^3}<\frac{4}{n}$; thus, $n^2S_{n,n}<nI_0(2)+I_1(2)+\frac{4}{n}$, which gives us the upper bound. □

Lemma 6.2.4. *$\frac{I_0(2)}{n}+\frac{I_1(2)}{n^2}-\sum_{k=0}^{n-m-1}\frac{1}{(n-k)k!^2}<S_{n,m}<\frac{I_0(2)}{m}+\frac{I_1(2)}{m^2}+\frac{4}{m^3}$ for all $n\geq 6$.*

Proof. Note that $\frac{1}{i!}<\frac{1}{j!}$ for all nonnegative integers $j<i$. Thus, by Lemma 6.2.3,

$$S_{n,m}=\sum_{k=0}^{m-2}\frac{1}{(m-k)k!(k+(n-m))!}<\sum_{k=0}^{m-2}\frac{1}{(m-k)k!^2}=S_{m,m}<\frac{I_0(2)}{m}+\frac{I_1(2)}{m^2}+\frac{4}{m^3},$$

which gives the upper bound. Similarly,

$$\begin{aligned}S_{n,m}&=\sum_{k=0}^{m-2}\frac{1}{(m-k)k!(k+(n-m))!}>\sum_{k=0}^{m-2}\frac{1}{(n-(k+(n-m)))(k+(n-m))!^2}\\&=\sum_{k=n-m}^{n-2}\frac{1}{(n-k)k!^2}=S_{n,n}-\sum_{k=0}^{n-m-1}\frac{1}{(n-k)k!^2}.\end{aligned}$$ □

Corollary 6.2.5. *If $n\geq 6$, then $\frac{I_0(2)\cdot n!^2}{2n}<|\mathcal{C}(M(K_{n,n}))|<\frac{\mathrm{e}\cdot n!^2}{2n}$.*

Proof. Since $\frac{I_0(2)}{n}+\frac{I_1(2)}{n^2}+\frac{4}{n^3}<\frac{\mathrm{e}}{n}$ for $n\geq 6$, and $|\mathcal{C}(M(K_{n,n}))|=\frac{n!^2}{2}S_{n,n}$ by Corollary 6.2.2, the bounds follow from Lemma 6.2.3. □

Corollary 6.2.6. *If $m\geq 6$ and $n\leq m+2$, then $\frac{(I_0(2)-2)\cdot n!m!}{2n}<|\mathcal{C}(M(K_{n,m}))|<\frac{\mathrm{e}\cdot n!m!}{2m}$.*

Proof. Since $m \geq n-2$, the sum $\sum_{k=0}^{n-m-1} \frac{1}{(n-k)k!^2}$ is at most $\frac{2}{n}$. Since $\frac{I_0(2)}{m} + \frac{I_1(2)}{m^2} + \frac{4}{m^3} < \frac{\mathrm{e}}{m}$ for $m \geq 6$, and $|\mathcal{C}(M(K_{n,m}))| = \frac{n!m!}{2} S_{n,m}$ by Corollary 6.2.2, the bounds follow from Lemma 6.2.4. □

In Chapter 3, we described a general method of bounding the number of coextensions of a matroid M using the number of stable sets in the overlap graph of M and the number of *small* circuits of M. Recall that a circuit of M is small with respect to some total ordering of $\mathcal{C}(M)$ if it is the minimum circuit in a corank-2 restriction of M. The following propositions will be used in Subsection 6.2.6 to establish upper bounds on the number of small circuits in $M(K_{n,n})$ and in $M(K_{n,n-1})$.

Proposition 6.2.7. *If $n \geq 16$, then the number of cycles in $K_{n,n}$ with length at most $\frac{2}{3}(2n+3)$ is at most $\frac{n!^2}{2n^2}$.*

Proof. Let r be the number of cycles in $K_{n,n}$ with length at most $\frac{2}{3}(2n+3)$. By Lemma 6.2.1,

$$\begin{aligned} r &= \sum_{k=2}^{\lfloor \frac{2n+3}{3} \rfloor} \frac{n!^2}{2k(n-k)!^2} \\ &\leq \frac{n!^2}{2n} \sum_{k=2}^{n} \frac{n}{2(n-\lfloor \frac{2n+3}{3} \rfloor)!^2} \\ &\leq \frac{n!^2}{2n} \cdot \frac{n^2}{2(\lceil \frac{n}{3} \rceil - 1)!^2}. \end{aligned}$$

Since $\frac{n^2}{2(\lceil \frac{n}{3} \rceil -1)!^2} \leq \frac{1}{n}$ for $n \geq 16$, we have that $r \leq \frac{n!^2}{2n} \cdot \frac{1}{n}$. □

Proposition 6.2.8. *If $n \geq 18$, then the number of cycles in $K_{n,n-1}$ with length at most $\frac{2}{3}(2n+2)$ is at most $\frac{n!(n-2)!}{4n}$.*

Proof. Let r be the number of cycles in $K_{n,n-1}$ with length at most $\frac{2}{3}(2n+2)$. By Lemma

6.2.1,

$$\begin{aligned}
r &= \sum_{k=2}^{\lfloor\frac{2n+2}{3}\rfloor} \frac{n!(n-1)!}{2k(n-k)!(n-1-k)!} \\
&\leq \frac{n!(n-2)!}{4} \sum_{k=2}^{\lfloor\frac{2n+2}{3}\rfloor} \frac{2(n-1)}{2\lceil\frac{n-2}{3}\rceil(\lceil\frac{n-2}{3}\rceil-1)!^2} \\
&\leq \frac{n!(n-2)!}{4} \cdot \frac{2n}{3} \cdot \frac{3(n-1)}{(n-2)(\lceil\frac{n-2}{3}\rceil-1)!^2} \\
&\leq \frac{n!(n-2)!}{4} \cdot \frac{2n(n-1)}{(n-2)(\lceil\frac{n-2}{3}\rceil-1)!^2}.
\end{aligned}$$

Since $\frac{2n(n-1)}{(n-2)(\lceil\frac{n-2}{3}\rceil-1)!^2} \leq \frac{1}{n}$ for $n \geq 18$, we have that $r \leq \frac{n!(n-2)!}{4} \cdot \frac{1}{n}$. □

6.2.2 Largest stable sets

Recall that, for a matroid M, the overlap graph $\Theta(M)$ has vertex set $\mathcal{C}(M)$ where vertices C, C' are adjacent if and only if they are contained in a connected corank-2 restriction of M. By Lemma 6.1.1, a subset of the ground set of $M(G)$ is a connected corank-2 restriction if and only if it induces a theta subgraph of G. Thus, a set $\mathcal{C}'$ of circuits of $M(G)$ has the scarce theta property if, for each theta subgraph H of G, at most one cycle in H has its edge set in $\mathcal{C}'$.

In the following lemma, we upper bound the size of a set of cycles of $K_{n,n}$ that has the scarce theta property. Note that the proof is quite similar to the proof of Lemma 2.2 in [39].

Lemma 6.2.9. *Let $n \geq 3$ be an integer. If $\mathcal{B} \subseteq \mathcal{C}(M(K_{n,n}))$ has the scarce theta property, then $|\mathcal{B}| \leq \frac{n!^2}{2n}$. Moreover, if equality holds, then $\mathcal{B}$ is the set of Hamiltonian cycles of $K_{n,n}$.*

Proof. Recall that $\mathcal{S}_n$ is the set of permutations of $[n]$. We will define two functions that map cycles C in $\mathcal{B}$ to sets of permutations that encode C. Let $\Psi_1, \Psi_2 : \mathcal{B} \to 2^{\mathcal{S}_n^2}$ be these functions, where for each cycle C with length $2k < 2n$ in $\mathcal{B}$, we define

$$\begin{aligned}
\Psi_1(C) = \{&(\sigma_A, \sigma_B) \in \mathcal{S}_n^2 : \\
&(\sigma_A(1), \sigma_B(1), \dots, \sigma_A(k), \sigma_B(k)) \text{ is a cyclic ordering of } C\}, \text{ and} \\
\Psi_2(C) = \{&(\sigma_A, \sigma_B) \in \mathcal{S}_n^2 : \\
&(\sigma_A(2), \sigma_B(2), \dots, \sigma_A(k+1), \sigma_B(k+1)) \text{ is a cyclic ordering of } C\}.
\end{aligned}$$

Note that $\Psi_1(C)$ and $\Psi_2(C)$ are disjoint for all $2k$-cycles C where $k < n$. For each $2n$-cycle C, define $\Psi_1(C)$ as above and $\Psi_2(C) = \Psi_1(C)$. Now define another function $\Psi : \mathcal{B} \to 2^{\mathcal{S}_n^2}$ where $\Psi(C) = \Psi_1(C) \cup \Psi_2(C)$.

We claim that the Ψ-images of distinct cycles in $\mathcal{B}$ are disjoint. If not, then there are cycles C_1, C_2, integers $1 \leq i \leq j \leq 2$, and a pair of permutations (σ_A, σ_B) such that $(\sigma_A, \sigma_B) \in \Psi_i(C_1) \cap \Psi_j(C_2)$. Let $|V(C_1)| = 2k_1$ and $|V(C_2)| = 2k_2$.

First consider the case where $i = j \in \{1, 2\}$. If $k_1 = k_2$, then $C_1 = C_2$, so without loss of generality $k_1 < k_2$. The cycles C_1 and C_2 intersect precisely in the path $\sigma_A(i)\sigma_B(i) \ldots \sigma_A(k_1 - 1 + i)\sigma_B(k_1 - 1 + i)$. Thus, the cycles C_1 and C_2 are contained in a theta subgraph of $K_{n,n}$, which contradicts the assumption that $\mathcal{B}$ has the scarce theta property.

The remaining case is where $(i, j) = (1, 2)$. Let $\ell = \min\{k_1, k_2 + 1\}$ and note that $\ell \geq 2$. The cycles C_1 and C_2 intersect precisely in the path $\sigma_A(2)\sigma_B(2) \ldots \sigma_A(\ell)\sigma_B(\ell)$. Thus, the cycles C_1 and C_2 are contained in a theta subgraph of $K_{n,n}$, which contradicts the assumption that $\mathcal{B}$ has the scarce theta property.

Since Ψ encodes each cycle as a collection of pairs of permutations, and these collections are pairwise disjoint, it follows that $\sum_{C \in \mathcal{B}} |\Psi(C)| \leq n!^2$. Let $\mathcal{B}_k$ denote the set of $2k$-cycles in $\mathcal{B}$ for each $2 \leq k \leq n$. For a cycle $C \in \mathcal{B}_k$ where $k < n$, the number of pairs of permutations in $\Psi(C)$ is $4k(n-k)!^2$ since there is a choice of two functions Ψ_1 and Ψ_2 with disjoint ranges, two directions, k elements for $\sigma_A(1)$, and $(n-k)!^2$ ways to order the remaining $n - k$ elements in each part of $K_{n,n}$. For a cycle $C \in \mathcal{B}_n$, the number of pairs of permutations in $\Psi(C)$ is $2n$. Therefore,

$$n!^2 \geq \sum_{C \in \mathcal{B}} |\Psi(C)| = \sum_{k=2}^{n-1} 4k(n-k)!^2 |\mathcal{B}_k| + 2n|\mathcal{B}_n|.$$

Since $4k(n-k)!^2 \geq 4k(n-k) \geq 4(n-1) > 2n$ for all $2 \leq k < n$ and $n \geq 3$, it follows from the equation above that $n!^2 \geq 2n \sum_{k=2}^{n} |\mathcal{B}_k| = 2n|\mathcal{B}|$. Therefore, $|\mathcal{B}| \leq \frac{n!^2}{2n}$. If equality holds, then we know $\mathcal{B} \subseteq \mathcal{B}_n$. Since there are exactly $\frac{n!^2}{2n}$ cycles of length $2n$ in $K_{n,n}$ by Lemma 6.2.1, it follows that $\mathcal{B} = \mathcal{B}_n$. □

The largest cycles in $K_{n,n-1}$ have length $2n - 2$, so $K_{n,n-1}$ is not Hamiltonian, unlike K_n and $K_{n,n}$. This raises a unique challenge with proving results about sets of cycles that have the scarce theta property. In K_n and $K_{n,n}$, the set of Hamiltonian cycles has the scarce theta property and is, thus, a stable set in the respective overlap graph. However, the largest cycles in $K_{n,n-1}$ are not Hamiltonian and are not a stable set in the overlap graph. Observe that two $(2n-2)$-cycles C, C' of $K_{n,n-1}$ are in a theta subgraph of $K_{n,n-1}$ if C' can be obtained from the C by swapping a vertex of A in C with the one vertex in A that is not in C.

In the following lemma, we upper-bound the size of a set of cycles of $K_{n,n-1}$ that has the scarce theta property. The proof is similar to that of Lemma 6.2.9, although there is an extra step to address the difficulty of the set of $(2n-2)$-cycles not being a stable set in the overlap graph. We will give an example of a specific set of cycles that is a maximum stable set in the next section, while determining an upper bound for the number of coextensions.

Lemma 6.2.10. *Let $n \geq 4$ be an integer. If $\mathcal{B} \subseteq \mathcal{C}(M(K_{n,n-1}))$ has the scarce theta property, then $|\mathcal{B}| \leq \frac{n!(n-2)!}{4}$. Moreover, if equality holds, then $\mathcal{B}$ is a set of $(2n-2)$-cycles.*

Proof. Recall that $\mathcal{S}_n$ is the set of permutations of $[n]$. Let $\Psi_1, \Psi_2 : \mathcal{B} \to 2^{\mathcal{S}_n \times \mathcal{S}_{n-1}}$ be functions where, for each cycle C of length $2k < 2(n-1)$ in $\mathcal{B}$, we define

$$\begin{aligned}\Psi_1(C) = \{&(\sigma_A, \sigma_B) \in \mathcal{S}_n \times \mathcal{S}_{n-1} : \\ &(\sigma_A(1), \sigma_B(1), \ldots, \sigma_A(k), \sigma_B(k)) \text{ is a cyclic ordering of } C\}, \text{ and} \\ \Psi_2(C) = \{&(\sigma_A, \sigma_B) \in \mathcal{S}_n \times \mathcal{S}_{n-1} : \\ &(\sigma_A(2), \sigma_B(2), \ldots, \sigma_A(k+1), \sigma_B(k+1)) \text{ is a cyclic ordering of } C\}.\end{aligned}$$

Note that $\Psi_1(C)$ and $\Psi_2(C)$ are disjoint for all $2k$-cycles C where $k < n-1$. For each $2(n-1)$-cycle C, define $\Psi_1(C)$ as above and $\Psi_2(C) = \Psi_1(C)$. Now define another function $\Psi : \mathcal{B} \to 2^{\mathcal{S}_n \times \mathcal{S}_{n-1}}$ where $\Psi(C) = \Psi_1(C) \cup \Psi_2(C)$.

We claim that the Ψ-images of distinct cycles in $\mathcal{B}$ are disjoint. If not, then there are cycles C_1, C_2, integers $1 \leq i \leq j \leq 2$, and a pair of permutations (σ_A, σ_B) such that $(\sigma_A, \sigma_B) \in \Psi_i(C_1) \cap \Psi_j(C_2)$. Let $|V(C_1)| = 2k_1$ and $|V(C_2)| = 2k_2$.

First, consider the case where $i = j \in \{1, 2\}$. If $k_1 = k_2$, then $C_1 = C_2$, so without loss of generality $k_1 < k_2$. The cycles C_1 and C_2 intersect precisely in the path $\sigma_A(i)\sigma_B(i) \ldots \sigma_A(k_1 - 1 + i)\sigma_B(k_1 - 1 + i)$. Thus, the cycles C_1 and C_2 are contained in a theta subgraph of $K_{n,n-1}$, which contradicts the assumption that $\mathcal{B}$ has the scarce theta property.

The remaining case is where $(i, j) = (1, 2)$. Let $\ell = \min\{k_1, k_2 + 1\}$ and note that $\ell \geq 2$. The cycles C_1 and C_2 intersect precisely in the path $\sigma_A(2)\sigma_B(2) \ldots \sigma_A(\ell)\sigma_B(\ell)$. Thus, the cycles C_1 and C_2 are contained in a theta subgraph of $K_{n,n}$, which contradicts the assumption that $\mathcal{B}$ has the scarce theta property.

Since Ψ encodes each cycle as a collection of pairs of permutations, and these collections are pairwise disjoint, it follows that $\sum_{C \in \mathcal{B}} |\Psi(C)| = |\Psi(\mathcal{B})|$. Let $\mathcal{B}_k$ denote the set of $2k$-cycles in $\mathcal{B}$ for each $2 \leq k \leq n-1$.

Claim 6.2.10.1. $|\Psi(\mathcal{B})| \leq n!(n-1)! - |\Psi(\mathcal{B}_{n-1})|$.

Proof. Define a function $f : \mathcal{S}_n \times \mathcal{S}_{n-1} \to \mathcal{S}_n \times \mathcal{S}_{n-1}$ where $f((\sigma_A, \sigma_B)) = (\sigma'_A, \sigma_B)$ and σ'_A is defined as follows. We define $\sigma'_A(1) = \sigma_A(n)$ and $\sigma'_A(n) = \sigma_A(1)$ and $\sigma'_A(i) = \sigma_A(i)$ for all $i \in \{2, 3, \ldots, n-1\}$. Note that $f((\sigma'_A, \sigma_B)) = (\sigma_A, \sigma_B)$, so f is an involution.

Suppose a permutation pair (σ_A, σ_B) is in $\Psi(\mathcal{B}_{n-1})$. Therefore, there is a cycle C in $\mathcal{B}_{n-1}$ such that $(\sigma_A, \sigma_B) \in \Psi(C)$. Let $f((\sigma_A, \sigma_B)) = (\sigma'_A, \sigma_B)$ and notice that, by definition, (σ'_A, σ_B) is not in $\Psi(C)$. First, we claim that (σ'_A, σ_B) is not in $\Psi(\mathcal{B}_{n-1})$. Suppose towards a contradiction that $(\sigma'_A, \sigma_B) \in \Psi(C')$ for some $C' \in \mathcal{B}_{n-1}$. Both C and C' contain the path $\sigma_B(1)\sigma_A(2)\sigma_B(2)\ldots\sigma_A(n-1)\sigma_B(n-1)$ and while C contains the path $\sigma_B(n-1)\sigma_A(1)\sigma_B(1)$, the cycle C' contains the path $\sigma_B(n-1)\sigma_A(n)\sigma_B(1)$. Thus, $C \cup C'$ is a theta subgraph where $\sigma_B(n-1)$ and $\sigma_B(1)$ are the degree-3 vertices. Hence $C' \notin \mathcal{B}$, which is a contradiction.

We also claim that (σ'_A, σ_B) is not in $\Psi(\mathcal{B}_k)$ for all $k \in [0, n-2]$. Suppose towards a contradiction that $(\sigma'_A, \sigma_B) \in \Psi(C')$ for some $C' \in \mathcal{B}_k$ where $k \in [0, n-2]$. If $(\sigma'_A, \sigma_B) \in \Psi_1(C')$, then both C and C' contain the path $\sigma_B(1)\sigma_A(2)\sigma_B(2)\ldots\sigma_A(k)\sigma_B(k)$ and while C contains the path $\sigma_B(k)\sigma_A(k+1)\sigma_B(k+1)\ldots\sigma_B(n-1)\sigma_A(1)\sigma_B(1)$, the cycle C' contains the path $\sigma_B(k)\sigma_A(n)\sigma_B(1)$. Thus, $C \cup C'$ is a theta subgraph where $\sigma_B(k)$ and $\sigma_B(1)$ are the degree-3 vertices. However, if $(\sigma'_A, \sigma_B) \in \Psi_2(C')$, then both C and C' contain the path $\sigma_A(2)\sigma_B(2)\ldots\sigma_A(k+1)\sigma_B(k+1)$ and while C contains the path $\sigma_B(k+1)\sigma_A(k+2)\sigma_B(k+2)\ldots\sigma_B(n-1)\sigma_A(1)\sigma_B(1)\sigma_A(2)$, the cycle C' contains the path $\sigma_B(k+1)\sigma_A(2)$. Therefore, if $(\sigma'_A, \sigma_B) \in \Psi(C')$, then $C \cup C'$ is a theta graph, which implies that $C' \notin \mathcal{B}$, which is a contradiction.

Now it follows that for each permutation pair (σ_A, σ_B) in $\Psi(\mathcal{B}_{n-1})$, the corresponding pair $f((\sigma_A, \sigma_B))$ is not in $\Psi(\mathcal{B})$. Thus, there are at least $|\Psi(\mathcal{B}_{n-1})|$ permutation pairs that are not in $\Psi(\mathcal{B})$. Since there are $n!(n-1)!$ permutation pairs in $\mathcal{S}_n \times \mathcal{S}_{n-1}$, the result follows. $\square$

Observe that for a cycle $C \in \mathcal{B}_k$ where $k < n-1$, the number of pairs of permutations in $\Psi(C)$ is $4k(n-k)!(n-1-k)!$ since there is a choice of two functions Ψ_1 and Ψ_2 with disjoint ranges, two directions, k elements for $\sigma_A(1)$, and $(n-k)!(n-1-k)!$ ways to order the remaining elements in each part of $K_{n,n-1}$. For a cycle $C \in \mathcal{B}_{n-1}$, the number of pairs of permutations in $\Psi(C)$ is $2(n-1)$ since there is a choice of two directions and $n-1$

elements for $\sigma_A(1)$. Now it follows from these observations and Claim 6.2.10.1 that,

$$\begin{aligned} n!(n-1)! &\geq \sum_{k=2}^{n-2} |\Psi(\mathcal{B}_k)| + 2|\Psi(\mathcal{B}_{n-1})| \\ &\geq \sum_{k=2}^{n-2} 4k(n-k)!(n-1-k)!|\mathcal{B}_k| + 4(n-1)|\mathcal{B}_{n-1}|. \end{aligned}$$

Notice that $4k(n-k)!(n-1-k)! \geq 4k(n-k) > 4(n-1)$ for all $2 \leq k < n-2$ and $n \geq 4$. Thus, it follows from the equation above that $n!(n-1)! \geq 4(n-1)\sum_{k=2}^{n-1} |\mathcal{B}_k| = 4(n-1)|\mathcal{B}|$. Therefore, $|\mathcal{B}| \leq \frac{n!(n-1)!}{4(n-1)} = \frac{n!(n-2)!}{4}$. If equality holds, then we know $\mathcal{B} \subseteq \mathcal{B}_{n-1}$. □

6.2.3 Supersaturation for $K_{n,n}$

In the following lemma, we show that a set of vertices of the overlap graph that is significantly larger than a maximum stable set induces a significant number of edges. The proof of this lemma is quite similar to that of Lemma 2.3 in [39].

Lemma 6.2.11 (Supersaturation Lemma for $\Theta(M(K_{n,n}))$). *For all integers $n \geq 6$ and $\alpha \geq \frac{8}{n}$, if $\mathcal{B} \subseteq V(\Theta(M(K_{n,n})))$ with $|\mathcal{B}| \geq (1+\alpha)\frac{n!^2}{2n}$, then $\mathcal{B}$ spans at least $\frac{\alpha n!^2}{4}$ edges in $\Theta(M(K_{n,n}))$.*

Proof. Let $\Theta = \Theta(M(K_{n,n}))$ and let $\mathcal{C} = V(\Theta)$. Suppose towards a contradiction that there exists a set $\mathcal{B} \subseteq \mathcal{C}$ where $|\mathcal{B}| \geq (1+\alpha)\frac{n!^2}{2n}$ and $\mathcal{B}$ spans less than $\frac{\alpha n!^2}{4}$ edges in Θ.

We define functions $\Phi, \Phi_1, \Phi_2 : \mathcal{C} \to 2^{\mathcal{S}_n^2}$ similarly to those in the proof of Lemma 6.2.9, where for each $C \in \mathcal{C}$ with length $2k$:

$$\begin{aligned} \Phi_1(C) = \{&(\sigma_A, \sigma_B) \in \mathcal{S}_n^2 : \\ &(\sigma_A(1), \sigma_B(1), \ldots, \sigma_A(k), \sigma_B(k)) \text{ is a cyclic ordering of } C\}, \text{ and} \\ \Phi_2(C) = \{&(\sigma_A, \sigma_B) \in \mathcal{S}_n^2 : \\ &(\sigma_A(n-k+1), \sigma_B(n-k+1), \ldots, \sigma_A(n), \sigma_B(n)) \text{ is a cyclic ordering of } C\}, \end{aligned}$$

and $\Phi(C) = \Phi_1(C) \cup \Phi_2(C)$. If $k < n$, then $\Phi_1(C)$ and $\Phi_2(C)$ are disjoint, so $|\Phi(C)| = 4k(n-k)!^2$. If $k = n$, then $\Phi_1(C)$ and $\Phi_2(C)$ are equal, so $|\Phi(C)| = 2n$. Note that for all permutation pairs $(\sigma_A, \sigma_B) \in \mathcal{S}_n^2$, there exist exactly two $2k$-cycles whose Φ-images contain (σ_A, σ_B) when $k < n$ and exactly one $2n$-cycle whose Φ-image contains (σ_A, σ_B). This implies that $|\Phi(\mathcal{C}')| = 2n|\mathcal{C}'|$ for all sets $\mathcal{C}'$ of $2n$-cycles.

For each $2 \leq k \leq n$, let $\mathcal{B}_k$ denote the set of $2k$-cycles in $\mathcal{B}$.

Claim 6.2.11.1. *If $C, C' \in \mathcal{B}_k$ are distinct cycles where $2k > n$, then $|\Phi(C) \cap \Phi(C')| \leq 2$.*

Proof. Since C and C' are different cycles with the same length, it follows that $\Phi_1(C) \cap \Phi_1(C') = \emptyset$ and $\Phi_2(C) \cap \Phi_2(C') = \emptyset$. Note that C and C' have at least two vertices in common, so $C \cup C'$ contains at least three cycles. If $V(C \cup C') \neq V(K_{n,n})$, then $\Phi_i(C) \cap \Phi_j(C') = \emptyset$ for $i \neq j \in \{1, 2\}$, so we now assume $V(C \cup C') = V(K_{n,n})$. If $C \cup C'$ contains at least four cycles, then C and C' intersect in at least two paths, hence $\Phi_i(C) \cap \Phi_j(C') = \emptyset$ for $i \neq j \in \{1, 2\}$. However, if $C \cup C'$ contains exactly three cycles, then it is a theta subgraph of $K_{n,n}$ and C and C' intersect in one path P. Since $C \cup C'$ contains all $2n$ vertices of $K_{n,n}$, we know P contains $2p$ vertices for some integer $p \geq 1$. If $(\sigma_A, \sigma_B) \in \Phi_1(C) \cap \Phi_2(C')$, then $\sigma_A(k-p)\sigma_B(k-p) \ldots \sigma_A(k)\sigma_B(k)$ is the path P, beginning with the end of P in A, which fixes the rest of the permutations σ_A and σ_B. Therefore, $|\Phi_1(C) \cap \Phi_2(C')| = 1$ and similarly $|\Phi_2(C) \cap \Phi_1(C')| = 1$. $\square$

For each $2 \leq k \leq n$ and each $i \in \{0, 1, 2\}$, let $P_{k,i}$ be the set of permutation pairs $(\sigma_A, \sigma_B) \in \mathcal{S}_n^2$ where $|\{C \in \mathcal{B}_k : (\sigma_A, \sigma_B) \in \Phi(C)\}| = i$. Since each $(\sigma_A, \sigma_B) \in \mathcal{S}_n^2$ is in $\Phi_1(C)$ for at most one $C \in \mathcal{B}_k$ and in $\Phi_2(C)$ for at most one $C \in \mathcal{B}_k$, the sets $P_{k,0}, P_{k,1}, P_{k,2}$ partition $\mathcal{S}_n^2$.

Claim 6.2.11.2. $|P_{k,2}| \leq \frac{\alpha n!^2}{2}$ *for all* $k \geq \frac{n+1}{2}$.

Proof. Consider $(\sigma_A, \sigma_B) \in P_{k,2}$. Thus, the permutation pair (σ_A, σ_B) is in $\Phi(C) \cap \Phi(C')$ for distinct cycles $C, C' \in \mathcal{B}_k$. Hence C and C' intersect precisely in a path of at least $4k - 2n \geq 2$ vertices, so C and C' are adjacent in Θ. Since $|\Phi(C) \cap \Phi(C')| \leq 2$ by Claim 6.2.11.1, there is at least one edge in Θ for every two permutation pairs in $P_{k,2}$. Thus, the set $\mathcal{B}_k$ of $2k$-cycles in $\mathcal{B}$ spans at least $|P_{k,2}|/2$ edges. Since the number of edges in $\Theta[\mathcal{B}]$ is at most $\frac{\alpha n!^2}{4}$, the claim follows. $\square$

Claim 6.2.11.3. *If $C, C' \in \mathcal{C}$ are distinct cycles where C has length $2k$ and C' has length $2n$, then $|\Phi(C) \cap \Phi(C')| \leq 2$.*

Proof. Since C' has length $2n$, we have $\Phi(C') = \Phi_1(C') = \Phi_2(C')$. Since C' is a Hamiltonian cycle, C' and C intersect in at least one path. If C' and C intersect in at least two paths, then $\Phi_i(C) \cap \Phi(C') = \emptyset$ for $i \in \{1, 2\}$. Now consider the case where C' and C intersect in exactly one path P. It follows that P contains exactly $2k$ vertices since every vertex in C is in C' as well. Therefore, if $(\sigma_A, \sigma_B) \in \Phi(C) \cap \Phi(C')$, then P is either $\sigma_A(1)\sigma_B(1) \ldots \sigma_A(k)\sigma_B(k)$ or $\sigma_A(n-k+1)\sigma_B(n-k+1) \ldots \sigma_A(n)\sigma_B(n)$ and the remaining elements of σ_A and σ_B are fixed since $(\sigma_A, \sigma_B) \in \Phi(C)$. Therefore, there are at most two permutation pairs (σ_A, σ_B) in $\Phi(C)$ and $\Phi(C')$. $\square$

For a set $\mathcal{A} \subseteq \mathcal{C}$, let $\Phi(\mathcal{A}) = \bigcup_{C \in \mathcal{A}} \Phi(C)$.

Claim 6.2.11.4. $|\Phi(\mathcal{B}_k) \cap \Phi(\mathcal{B}_n)| \leq \frac{\alpha n!^2}{2}$ *for all* $2 \leq k < n$.

Proof. Let $C \in \mathcal{B}_k$ and $C' \in \mathcal{B}_n$ and consider a permutation pair (σ_A, σ_B) in $\Phi(C) \cap \Phi(C')$. The sequence $\sigma_A(1)\sigma_B(1) \ldots \sigma_A(n)\sigma_B(n)$ is a cyclic ordering of the vertices of C' and either $\sigma_A(1)\sigma_B(1) \ldots \sigma_A(k)\sigma_B(k)$ or $\sigma_A(n-k+1)\sigma_B(n-k+1) \ldots \sigma_A(n)\sigma_B(n)$ is a cyclic ordering of C, hence C and C' are adjacent in Θ. By Claim 6.2.11.3, there are at most two permutation pairs in $\Phi(C) \cap \Phi(C')$, so there is at least one edge in Θ for every two permutation pairs in $\Phi(\mathcal{B}_k) \cap \Phi(\mathcal{B}_n)$. Since the number of edges in $\Theta[\mathcal{B}]$ is at most $\frac{\alpha n!^2}{4}$, the claim follows. □

The number of $2k$-cycles in $K_{n,n}$ is $\frac{n!^2}{2k(n-k)!^2}$ by Lemma 6.2.1. For $2 \leq k \leq n$, let $\beta_k = |\mathcal{B}_k| \frac{2k(n-k)!^2}{n!^2}$. That is, β_k is the fraction of $2k$-cycles of $K_{n,n}$ in $\mathcal{B}$; hence $0 \leq \beta_k \leq 1$. Note that $|\Phi(\mathcal{B}_n)| = 2n|\mathcal{B}_n| = n!^2\beta_n$.

Claim 6.2.11.5. $\beta_k \leq \frac{1}{2}(1 + \alpha - \beta_n)$ *for all* $\frac{n+1}{2} \leq k < n$.

Proof. By Claim 6.2.11.4, we know

$$\begin{aligned}
|\Phi(\mathcal{B}_k)| &= |\Phi(\mathcal{B}_k) \cup \Phi(\mathcal{B}_n)| + |\Phi(\mathcal{B}_k) \cap \Phi(\mathcal{B}_n)| - |\Phi(\mathcal{B}_n)| \\
&\leq n!^2 + \frac{\alpha n!^2}{2} - n!^2\beta_n \\
&= \left(1 + \frac{\alpha}{2} - \beta_n\right) n!^2.
\end{aligned}$$

Now, note that

$$|P_{k,1}| + 2|P_{k,2}| = \sum_{i=0}^{2} i|P_{k,i}| = \sum_{C \in \mathcal{B}_k} |\Phi(C)| = 4k(n-k)!^2|\mathcal{B}_k|.$$

Furthermore, note that since each permutation pair in $\mathcal{S}_n^2$ is in the Φ-image of at most two $2k$-cycles, a permutation pair in one $2k$-cycle Φ-image is in $P_{k,1}$ and a permutation pair in more than one $2k$-cycle Φ-image is in $P_{k,2}$. Since $\Phi(\mathcal{B}_k)$ is the collection of all permutation pairs in a $2k$-cycle Φ-image, we have that $|P_{k,1}| + |P_{k,2}| = |\Phi(\mathcal{B}_k)|$. Now by Claim 6.2.11.2, it follows that

$$\begin{aligned}
2\beta_k n!^2 &= 4k(n-k)!^2|\mathcal{B}_k| \\
&= |P_{k,1}| + 2|P_{k,2}| \\
&= |\Phi(\mathcal{B}_k)| + |P_{k,2}| \\
&\leq \left(1 + \frac{\alpha}{2} - \beta_n\right)n!^2 + \frac{\alpha n!^2}{2} \\
&= (1 + \alpha - \beta_n)n!^2,
\end{aligned}$$

and the result follows by dividing both sides of the above equation by $2n!^2$. □

Since $\beta_k \leq 1$ for all $k \in [2, n]$, it follows from the assumption on the size of $\mathcal{B}$ that

$$\begin{aligned}
(1+\alpha)\frac{n!^2}{2n} \leq |\mathcal{B}| &= \sum_{0 \leq k \leq n-2} |\mathcal{B}_{n-k}| \\
&= \frac{n!^2}{2} \sum_{0 \leq k \leq n-2} \frac{\beta_{n-k}}{k!^2(n-k)} \\
&= \frac{n!^2}{2}\left(\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-2} \frac{1}{k!^2(n-k)} + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor - 1} \frac{\beta_{n-k}}{k!^2(n-k)} + \frac{\beta_n}{n}\right).
\end{aligned}$$

If $\frac{n}{2} \leq k \leq n-2$ and $n \geq 6$, then $(k+1)^2 \geq 16$, so $\frac{n-k-1}{n-k} \geq \frac{1}{2} \geq \frac{8}{(k+1)^2}$. This implies that $\frac{1}{(k+1)!^2(n-(k+1))} \leq \frac{1}{8} \cdot \frac{1}{k!^2(n-k)}$, so $\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-2} \frac{1}{k!^2(n-k)} \leq \frac{1}{\lfloor \frac{n}{2} \rfloor !^2 \frac{n}{2}} \sum_{k \geq 0} \frac{1}{8^k} \leq \frac{8}{7} \cdot \frac{1}{\lfloor \frac{n}{2} \rfloor !^2 \frac{n}{2}}$. Now, it follows from Claim 6.2.11.5, Corollary 6.2.5, and the inequality block above that

$$\begin{aligned}
(1+\alpha)\frac{n!^2}{2n} &\leq \frac{n!^2}{2}\left(\frac{8}{7}\frac{1}{\lfloor \frac{n}{2} \rfloor !^2 \frac{n}{2}} + \frac{1+\alpha-\beta_n}{2}\left(S_{n,n} - \frac{1}{n}\right) + \frac{\beta_n}{n}\right) \\
&\leq \frac{n!^2}{2}\left(\frac{16}{7}\frac{1}{n \cdot \lfloor \frac{n}{2} \rfloor !^2} + \frac{1+\alpha-\beta_n}{2} \cdot \frac{\mathrm{e}-1}{n} + \frac{\beta_n}{n}\right).
\end{aligned}$$

Using $\frac{16}{7}\frac{1}{\lfloor \frac{n}{2} \rfloor !^2} < \frac{1}{n}$ for $n \geq 6$ and the above inequality, we find

$$\begin{aligned}
1 + \alpha &< \frac{1}{n} + \frac{(1+\alpha-\beta_n)(\mathrm{e}-1)}{2} + \beta_n \\
&\leq \frac{1}{n} + \frac{(1+\alpha)(\mathrm{e}-1)}{2} + \frac{3-\mathrm{e}}{2}.
\end{aligned}$$

Rearranging to isolate α, we get $\alpha < \frac{2}{n(3-\mathrm{e})}$, which contradicts the assumption that $\alpha \geq \frac{8}{n}$ since $\frac{1}{3-\mathrm{e}} < 4$. □

6.2.4 Supersaturation for $K_{n,n-1}$

In this subsection, we prove a supersaturation lemma for $\Theta(M(K_{n,n-1}))$. Since $K_{n,n-1}$ is not Hamiltonian and the set of largest cycles in $K_{n,n-1}$ is not a stable set, the proof does not have the same structure as the proofs of Lemma 6.2.11 and Lemma 2.3 in [39]. The proof of this lemma requires a completely different approach and is much more involved.

For the remainder of this subsection, let $n \geq 6$ unless otherwise specified. Also, let $\Theta = \Theta(M(K_{n,n-1}))$ and $\mathcal{C} = \mathcal{C}(M(K_{n,n-1}))$. For any $V \subseteq \mathcal{C}$, let $e(V)$ denote the number of edges induced by V in Θ. That is, let $e(V) = |E(\Theta[V])|$. We partition $V(\Theta)$ into layers based on cycle length. For each $d \in [n-2]$, let $L_d \subset \mathcal{C}$ denote the set of cycles of $K_{n,n-1}$ of length $2(n-d)$. We say a cycle in L_d is at *depth* d in Θ. Let $\alpha = \frac{n!(n-2)!}{4}$. By Lemma 6.2.1, the number of $(2(n-1))$-cycles is $|L_1| = \frac{n!(n-1)!}{2(n-1)} = 2\alpha$.

Now we define a sequence for each cycle C at depth at most $n-3$ which gives a canonical cyclic ordering of C. For each $d \in [n-3]$ and for each cycle $C \in L_d$, let $\sigma_C = \sigma_C(0)\sigma_C(1)\ldots\sigma_C(2(n-d)-1)$ be a sequence of vertices in $K_{n,n-1}$ such that $C = \sigma_C(0)\sigma_C(1)\ldots\sigma_C(2(n-d)-1)\sigma_C(0)$ where $\sigma_C(0) \in A$ and $\sigma_C(1) \leq \sigma_C(2i-1)$ for all $i \in [n-d]$ and $\sigma_C(3) < \sigma_C(2(n-d)-1)$. Now we show that, for each cycle, this sequence is unique.

Proposition 6.2.12. *Let $d \in [n-3]$ and let $C \in L_d$. If $\nu = v_0v_1\ldots v_{2(n-d)-1}$ is a sequence of vertices in $K_{n,n-1}$ such that $C = v_0v_1\ldots v_{2(n-d)-1}v_0$ where $v_0 \in A$ and $v_1 \leq v_{2i-1}$ for all $i \in [n-d]$, while $v_3 < v_{2(n-d)-1}$, then $\nu = \sigma_C$.*

Proof. Since $v_0 \in A$, the vertex v_1 is in B. If v_1 is not the smallest element in $B \cap V(C)$, then there exists $v_{2i-1} < v_1$ for some $i \in [2, n-d]$, which is a contradiction, so we know $v_1 = \sigma_C(1)$. The vertices v_3 and $v_{2(n-d)-1}$ are the only two vertices at distance 2 from v_1 in C and $v_3 < v_{2(n-d)-1}$, so $v_3 = \sigma_C(3)$ and $v_{2(n-d)-1} = \sigma_C(2(n-d)-1)$. The vertex $\sigma_C(0)$ is the only vertex adjacent to $\sigma_C(1)$ and $\sigma_C(2(n-d)-1)$, so $v_0 = \sigma_C(0)$. Similarly, the vertex $\sigma_C(2)$ is the only vertex adjacent to $\sigma_C(1)$ and $\sigma_C(3)$, so $v_2 = \sigma_C(2)$. For each $i \in [4, 2(n-d)-2]$, the vertex $\sigma_C(i)$ is the only vertex where the path in C from $\sigma_C(1)$ to $\sigma_C(i)$ that contains $\sigma_C(2)$ has length $i-1$, so $v_i = \sigma_C(i)$. Therefore $v_i = \sigma_C(i)$ for all $i \in [0, 2(n-d)-1]$. $\square$

For each $C \in L_1$, there is exactly one vertex v in $V(K_{n,n-1}) \setminus V(C)$ and $v \in A$. Thus, the sequence $\sigma_C(0)\sigma_C(2)\ldots\sigma_C(2n-4)v$ is a permutation of $A = [n]$. Define $\mathrm{sgn}(\sigma_C)$ to be the sign of the permutation $\sigma_C(0)\sigma_C(2)\ldots\sigma_C(2n-4)v$. That is,

$$\operatorname{sgn}(\sigma_C) = \begin{cases} 1 & \text{if } \sigma_C(0)\sigma_C(2)\ldots\sigma_C(2n-4)v \text{ is an even permutation, and} \\ -1 & \text{if } \sigma_C(0)\sigma_C(2)\ldots\sigma_C(2n-4)v \text{ is an odd permutation.} \end{cases}$$

Proposition 6.2.13. *Let $C, C' \in L_1$. If $\sigma_C(1)\sigma_C(3)\ldots\sigma_C(2n-3) \neq \sigma_{C'}(1)\sigma_{C'}(3)\ldots\sigma_{C'}(2n-3)$ or $\operatorname{sgn}(\sigma_C) = \operatorname{sgn}(\sigma_{C'})$, then C and C' are not adjacent.*

Proof. Suppose that C and C' are adjacent. Hence $C \cup C'$ is a theta graph. We will show that $\sigma_C(1)\sigma_C(3)\ldots\sigma_C(2n-3) = \sigma_{C'}(1)\sigma_{C'}(3)\ldots\sigma_{C'}(2n-3)$ and $\operatorname{sgn}(\sigma_C) \neq \operatorname{sgn}(\sigma_{C'})$. Since C and C' each contain $2n-2$ vertices and their union contains at most all $2n-1$ vertices of $K_{n,n-1}$, the intersection of C and C' is a path with at least $2n-3$ vertices. The cycles C and C' contain all vertices in B, so the path $C \cap C'$ contains $n-1$ vertices in B and $n-2$ vertices in A. Thus $C \setminus C'$ and $C' \setminus C$ each contain exactly one vertex in A. Therefore, the sequence $\sigma_C(1)\sigma_C(3)\ldots\sigma_C(2n-3)$ is equal to the sequence $\sigma_{C'}(1)\sigma_{C'}(3)\ldots\sigma_{C'}(2n-3)$ and the sequences σ_C and $\sigma_{C'}$ differ in exactly one even indexed entry. Let $2k$ be the index such that $\sigma_C(2k) \neq \sigma_{C'}(2k)$. Since $\sigma_C(i) = \sigma_{C'}(i)$ for all indices other than $2k$, the vertex $\sigma_{C'}(2k)$ is the one vertex v in $A \setminus V(C)$ and the vertex $\sigma_C(2k)$ is the one vertex v' in $A \setminus V(C')$. Hence the permutations $\sigma_C(0)\sigma_C(2)\ldots\sigma_C(2n-4)v$ and $\sigma_{C'}(0)\sigma_{C'}(2)\ldots\sigma_{C'}(2n-4)v'$ can be obtained from each other by one 2-element transposition. Therefore $\operatorname{sgn}(\sigma_C) \neq \operatorname{sgn}(\sigma_{C'})$. ☐

Lemma 6.2.14. *The graph $\Theta[L_1]$ is bipartite with bipartition $(X, L_1 \setminus X)$ where $X = \{C \in L_1 : \operatorname{sgn}(\sigma_C) = 1\}$. Furthermore $|X| = |L_1 \setminus X| = \alpha$.*

Proof. Observe that $X \subseteq L_1$ is the set of cycles C such that $\operatorname{sgn}(\sigma_C) = 1$ and that $\operatorname{sgn}(\sigma_{C'}) = -1$ for all $C' \in L_1 \setminus X$. By Proposition 6.2.13, there are no edges between vertices in X or between vertices in $L_1 \setminus X$. Thus $(X, L_1 \setminus X)$ is a bipartition of $\Theta[L_1]$.

There are $\frac{(n-2)!}{2}$ sequences σ such that $\sigma = \sigma_C(1)\sigma_C(3)\ldots\sigma_C(2n-3)$ for some $C \in L_1$. Since there are $\frac{n!}{2}$ even (or odd) permutations of length n and $\frac{(n-2)!}{2}$ choices for the sequence $\sigma_C(1)\sigma_C(3)\ldots\sigma_C(2n-3)$ for some $C \in L_1$, we have $|X| = \frac{n!(n-2)!}{4} = \alpha$ (and $|L_1 \setminus X| = \alpha$). ☐

Let $\mathcal{S}$ be the collection of sequences σ such that $\sigma = \sigma_C$ for some $C \in \mathcal{C}$. Let $\mathcal{S}_1 \subseteq \mathcal{S}$ be the collection of sequences σ_C for some $C \in L_1$.

We will now define a collection of functions that will each give rise to a perfect matching of the vertices in L_1. Informally, each function will map each cycle $C \in L_1$ to an adjacent cycle $C' \in L_1$ where the vertex of A not in C is swapped with a vertex of C to obtain C'. For each $k \in [0, n-2]$, define a function $f_k : \mathcal{S}_1 \to \mathcal{S}_1$ where $f_k(\sigma) = \sigma'$ and σ' is defined as follows. First, let v be the element of A that is not in $\{\sigma(2i) : 0 \leq i \leq n-2\}$. Now, we define $\sigma'(2k) = v$ and $\sigma'(i) = \sigma(i)$ for all $i \in [0, 2n-3] \setminus \{2k\}$. The element of A that is not in σ' is $\sigma(2k)$, hence $f_k(\sigma') = \sigma$. Furthermore, since $\sigma(2k) \neq v$, we know $f_k(\sigma) \neq \sigma$. Therefore, each f_k is an involution. For each $k \in [0, n-2]$, since $C \cup C'$ is a theta graph for any two cycles $C, C' \in L_1$ where $f_k(\sigma_C) = \sigma_{C'}$, the function f_k determines pairs of cycles that are adjacent in Θ. For each $k \in [0, n-2]$, define M_k to be the set of edges CC' in $\Theta[L_1]$ such that $f_k(\sigma_C) = \sigma_{C'}$. Observe that each M_k is a perfect matching of L_1. Also, observe that $M_i \cap M_j = \emptyset$ for $i \neq j$.

Proposition 6.2.15. *The independence number of* $\Theta[L_1]$ *is* α.

Proof. Since $\Theta[L_1]$ is bipartite and each part has size α by Lemma 6.2.14, the independence number is at least α. Since $\Theta[L_1]$ has a perfect matching, the independence number is at most $|L_1|/2 = \alpha$. □

Since $\Theta[L_1]$ is a subgraph of Θ, the following proposition clearly follows from the previous proposition.

Proposition 6.2.16. *The independence number of* Θ *is at least* α. □

For each $k \in [0, n-2]$, define N_k to be a set of edges between a vertex in L_1 and a vertex not in L_1, as follows. For each adjacent pair $C_1 \in L_1$ and $C \in L_d$ where $d \geq 2$, we say the edge C_1C is in N_k if and only if there exists a theta subgraph of $K_{n,n-1}$ whose cycles are C, C_1, and some cycle $C_2 \in \mathcal{C}$ such that C_2 and C intersect in exactly one edge, and $\sigma_{C_1}(2k) \in V(C_2) \setminus V(C)$. Since $|V(C_2)| = |V(C_1)| - |V(C)| + 2 = 2(n-1) - 2(n-d) + 2 = 2d$, we know that $C_2 \in L_{n-d}$. Observe that $C \cap C_1$ is a subpath of C_1 that contains all vertices of C, the set $E(C) \setminus E(C_1)$ contains exactly one edge e, and e is a chord of C_1 in $C_1 \cup C$.

Proposition 6.2.17. *Let* $k \in [0, n-2]$ *and let* $C_1, C_1' \in L_1$ *where* $C_1C_1' \in M_k$. *Let* $C \in L_d$ *where* $d \geq 2$. *If* $CC_1 \in N_k$, *then* $CC_1' \in N_k$.

Proof. Since $CC_1 \in N_k$, we know there is a theta subgraph of $K_{n,n-1}$ whose cycles are C, C_1 and some $C_2 \in \mathcal{C}$ such that C_2 and C intersect in exactly one edge and $\sigma_{C_1}(2k) \in V(C_2) \setminus V(C)$. Note that $V(C), V(C_2) \subseteq V(C_1)$. Since $C_1C_1' \in M_k$, we have $\sigma_{C_1}(i) = \sigma_{C_1'}(i)$ for all $i \in [0, 2n-3] \setminus \{2k\}$ and $\sigma_{C_1'}(2k) \notin V(C_1)$. Let C_2' be the cycle obtained from C_2 by deleting the vertex $\sigma_{C_1}(2k)$ and adding the vertex $\sigma_{C_1'}(2k)$ and edges from $\sigma_{C_1'}(2k)$ to the neighbours of $\sigma_{C_1}(2k)$ in C_2. Observe that C, C_1', and C_2' are the three cycles of a theta subgraph of $K_{n,n-1}$. Since $\sigma_{C_1}(2k) \notin V(C)$, the edges in $E(C_2) \cup E(C_2') \setminus (E(C_2) \cap E(C_2'))$ are not in C, hence C and C_2' intersect in exactly one edge. Since $\sigma_{C_1'}(2k) \notin V(C_1)$ and $V(C) \subseteq V(C_1)$, we know $\sigma_{C_1'}(2k) \notin V(C)$; therefore $\sigma_{C_1'}(2k) \in V(C_2') \setminus V(C)$. Now it follows that $CC_1' \in N_k$. $\square$

Proposition 6.2.18. *Let $C_1 \in L_1$ and $C \in L_d$ where $2 \leq d \leq n-3$. If $CC_1 \in N_k$ for some $k \in [d+1, n-1-d]$, then the vertices $\sigma_{C_1}(2n-3)$, $\sigma_{C_1}(1)$, and $\sigma_{C_1}(3)$ are in C.*

Proof. Since $CC_1 \in N_k$, there is a theta subgraph of $K_{n,n-1}$ whose cycles are C, C_1, and some cycle $C_2 \in \mathcal{C}$ such that C_2 and C intersect in exactly one edge and $\sigma_{C_1}(2k) \in V(C_2) \setminus V(C)$.

First we claim that the vertex $\sigma_{C_1}(0)$ is in C. If $\sigma_{C_1}(0)$ is not in C, then $\sigma_{C_1}(0)$ and $\sigma_{C_1}(2k)$ are both in $V(C_1) \setminus V(C)$. Since $C_1 \setminus C$ is a path with $2d-2$ vertices, it follows that the distance between $\sigma_{C_1}(0)$ and $\sigma_{C_1}(2k)$ in C_1 is less than $2d-2$. Therefore, we have $0 \leq 2k \leq 2d-4$ or $2(n-d)+1 \leq 2k \leq 2n-3$. This implies that $0 \leq k \leq d-2$ or $n-d+1 \leq k \leq n-2$, so $k \notin [d+1, n-1-d]$, which is a contradiction. Therefore, the vertex $\sigma_{C_1}(0)$ is in C.

Now we claim that the vertex $\sigma_{C_1}(1)$ is in C. If not, then since $\sigma_{C_1}(0)$ is in C and $C \cap C_1$ is a path, the vertex $\sigma_{C_1}(2n-3)$ is in C as well, while the vertex $\sigma_{C_1}(3)$ is not in C. Therefore C_2 is the union of the edge $\sigma_{C_1}(0)\sigma_{C_1}(2d-1)$ and the subpath of C_1 from $\sigma_{C_1}(0)$ to $\sigma_{C_1}(2d-1)$ that contains $\sigma_{C_1}(1)$. That is, $C_2 = \sigma_{C_1}(0)\sigma_{C_1}(1)\ldots\sigma_{C_1}(2d-1)\sigma_{C_1}(0)$. Since the vertex $\sigma_{C_1}(2k)$ is in $C_2 \setminus C$, we have $2 \leq 2k \leq 2(d-1)$, so $k \notin [d+1, n-1-d]$, which is a contradiction. Therefore, the vertex $\sigma_{C_1}(1)$ is in C.

Next we claim that the vertex $\sigma_{C_1}(2n-3)$ is in C. If not, then since $\sigma_{C_1}(0)$ and $\sigma_{C_1}(1)$ are in C, the vertex $\sigma_{C_1}(3)$ is in C as well. Therefore C_2 is the union of the edge $\sigma_{C_1}(0)\sigma_{C_1}(2(n-d)-1)$ and the subpath of C_1 from $\sigma_{C_1}(0)$ to $\sigma_{C_1}(2(n-d)-1)$ that contains $\sigma_{C_1}(2n-3)$. That is, $C_2 = \sigma_{C_1}(0)\sigma_{C_1}(2(n-d)-1)\sigma_{C_1}(2(n-d))\ldots\sigma_{C_1}(2n-3)\sigma_{C_1}(0)$. Since the vertex $\sigma_{C_1}(2k)$ is in $C_2 \setminus C$, we have $2(n-d) \leq 2k \leq 2(n-2)$, so $k \notin [d+1, n-1-d]$, which is a contradiction. Therefore, the vertex $\sigma_{C_1}(2n-3)$ is in C.

Finally, we claim that the vertex $\sigma_{C_1}(3)$ is in C. If not, then $C_2 \cap C_1$ is either the subpath of C_1 from $\sigma_{C_1}(1)$ to $\sigma_{C_1}(2d)$ or $\sigma_{C_1}(2)$ to $\sigma_{C_1}(2d+1)$ that contains $\sigma_{C_1}(3)$. That is, $C_2 = \sigma_{C_1}(1)\sigma_{C_1}(2)\ldots\sigma_{C_1}(2d)\sigma_{C_1}(1)$ or $C_2 = \sigma_{C_1}(2)\sigma_{C_1}(3)\ldots\sigma_{C_1}(2d+1)\sigma_{C_1}(2)$. Since the vertex $\sigma_{C_1}(2k)$ is in $C_2 \setminus C$, we have $2 \leq 2k \leq 2d$, so $k \notin [d+1, n-1-d]$, which is a contradiction. Therefore, the vertex $\sigma_{C_1}(3)$ is in C. $\square$

Lemma 6.2.19. *Let $C_1 \in L_1$ and $C \in L_d$ where $2 \leq d \leq n-3$. If $CC_1 \in N_k$ for some $k \in [0, n-2]$, then the number of integers $i \in [0, n-2]$ such that $CC_1 \in N_i$ is exactly $d-1$.*

Proof. Since $CC_1 \in N_k$, there is a theta subgraph of $K_{n,n-1}$ whose cycles are C, C_1, and some cycle $C_2 \in L_{n-d}$ such that C_2 and C intersect in exactly one edge and $\sigma_{C_1}(2k) \in V(C_2) \setminus V(C)$. Since C_2 is the graph induced on $(E(C_1) \setminus E(C)) \cup (E(C) \setminus E(C_1))$, the cycle C_2 is the only cycle in a theta graph with C and C_1. Therefore, the edge CC_1 is in N_i if and only if $\sigma_{C_1}(2i) \in V(C_2) \setminus V(C)$. Let $P = C_2 \setminus C$ and notice that P is a subpath of C_1 that contains exactly $2d-2$ vertices, where $d-1$ of them are in A and $d-1$ are in B. The vertices of A in C_1 have even indices in σ_{C_1}, therefore, there are $d-1$ integers $i \in [0, n-2]$ such that the vertex $\sigma_{C_1}(2i)$ is in P. That is, there are $d-1$ integers $i \in [0, n-2]$ such that the vertex $\sigma_{C_1}(2i)$ is in $V(C_2) \setminus V(C)$. Thus, there are exactly $d-1$ sets N_i where $i \in [0, n-2]$ such that $CC_1 \in N_i$. □

Lemma 6.2.20. *Let $C \in L_d$ where $2 \leq d \leq n-3$ and let $k \in [d+1, n-1-d]$. If $CC_1 \in N_k$ for some $C_1 \in L_1$, then N_k contains at least $d!(d-1)!$ edges incident with C.*

Proof. Consider a theta subgraph of $K_{n,n-1}$ whose cycles are C, C_1, and some $C_2 \in L_{n-d}$ such that C_2 and C intersect in exactly one edge and $\sigma_{C_1}(2k) \in V(C_2) \setminus V(C)$. By Proposition 6.2.18, we know that $\sigma_{C_1}(2n-3)$, $\sigma_{C_1}(1)$, and $\sigma_{C_1}(3)$ are in C. Since $C \cup C_1$ is a theta graph, the intersection $C \cap C_1$ is a path. Let $p \in [3, 2n-2-2d]$ such that $C \cap C_1$ is the path in C_1 from $\sigma_{C_1}(p+2d-1)$ to $\sigma_{C_1}(p)$ that contains $\sigma_{C_1}(2n-3)$, $\sigma_{C_1}(1)$, and $\sigma_{C_1}(3)$. Since the vertex $\sigma_{C_1}(2k)$ is not in C, we know $p+1 \leq 2k \leq p+2d-2$.

Let $N_1(C)$ be the set of cycles $D \in L_1$ such that D is the union of the path $C \cap C_1$ and a path P in $K_{n,n-1}$ on $2d$ vertices with endpoints $\sigma_{C_1}(p+2d-1), \sigma_{C_1}(p)$ and all other vertices not in C. There are d vertices in $A \setminus V(C)$ and $d-1$ vertices in $B \setminus V(C)$, so there are $d!$ ways to choose and order the A vertices in P and $(d-1)!$ ways to order the B vertices in P. Therefore, there are $d!(d-1)!$ choices for P, and thus for D, which implies $|N_1(C)| = d!(d-1)!$.

Consider a cycle D in $N_1(C)$ such that D is the union of $C \cap C_1$ and P. Since cycles in L_1 contain all vertices in B, we know by definition that $\sigma_{C_1}(1) = \sigma_D(1)$. Furthermore, since $C \cap C_1$ is a subpath of C_1 and D, it follows that $\sigma_{C_1}(\ell) = \sigma_D(\ell)$ for all $\ell \in [0, p] \cup [p+2d-1, 2n-3]$. Let D_2 be the union of the path P and the edge $\sigma_D(p+2d-1)\sigma_D(p)$. Since $\sigma_D(p+2d-1), \sigma_D(p)$ are the endpoints of P, the graph D_2 is a cycle. The graph $C \cup D \cup D_2$ is a theta subgraph of $K_{n,n-1}$ where D_2 and C intersect in exactly one edge $(\sigma_D(p+2d-1)\sigma_D(p))$. Furthermore, since $p+1 \leq 2k \leq p+2d-2$, the vertex $\sigma_D(2k)$ is in $V(D_2) \setminus V(C)$. Thus CD is in N_k.

Therefore, each $D \in N_1(C)$ is a cycle where $CD \in N_k$. That is, the set N_k contains at least $|N_1(C)| = d!(d-1)!$ edges incident to C. □

Proposition 6.2.21. *Let $C \in L_d$ and $C' \in L_{d'}$ where $2 \leq d \leq d' \leq \frac{n}{2}$ and let $k \in [0, n-2]$. If $CC_1, C'C_1 \in N_k$ for some $C_1 \in L_1$, then C and C' are adjacent in Θ.*

Proof. Since $CC_1, C'C_1 \in N_k$, all vertices in C and C' are also in C_1. Now, since the vertex $\sigma_{C_1}(2k)$ is not in C or C', the number of vertices in $C \cup C'$ is at most $|V(C_1)| - 1 = 2n - 3$. Since C contains $2(n-d)$ vertices and C' contains $2(n-d')$ vertices, the number of vertices in the intersection $C \cap C'$ is at least $2n - 2d - 2d' + 3 \geq 3$. Since C and C' both intersect C_1 in a path that contains all of their vertices, the intersection $C \cap C'$ is either one or two paths. Thus, at least one path in $C \cap C'$ contains at least two vertices, so the cycles C and C' share at least one edge.

Since the vertex $\sigma_{C_1}(2k)$ is not in C or C', the edges incident with $\sigma_{C_1}(2k)$ in C_1 are not in C or C'. Let e be an edge incident with $\sigma_{C_1}(2k)$ in C_1 and let G be the connected graph $C \cup C' \cup C_1 - e$. Since $E(C) \setminus E(C_1)$ and $E(C') \setminus E(C_1)$ each contain a different edge that is a chord of C_1 in $C_1 \cup C \cup C'$, we have $|E(C_1 \cup C \cup C')| = |V(C_1 \cup C \cup C')| + 2$. Since G has the same vertex set as $C_1 \cup C \cup C'$, the number of edges in G is $|V(G)| + 1$. Now, it follows from Lemma 2.3.2 that C and C' are contained in a theta subgraph of G. Therefore, the cycles C and C' are adjacent in Θ. $\square$

For each $k \in [0, n-2]$, define R_k to be a set of edges between vertices in $\mathcal{C} \setminus L_1$, as follows. For each adjacent pair $C \in L_d$ and $C' \in L_{d'}$ where $2 \leq d \leq d'$, we say the edge CC' is in R_k if and only if there exists $C_1 \in L_1$ such that $CC_1, C'C_1 \in N_k$. In other words, the set R_k contains all edges in $\Theta \setminus L_1$ that are in a triangle with two edges in N_k. By Proposition 6.2.21, every pair of edges CC_1 and $C'C_1$ in N_k with a common end C_1 in L_1 is contained in such a triangle, so CC' is in R_k.

Lemma 6.2.22. *Let $C_1 \in L_{d_1}$ and $C_2 \in L_{d_2}$ where $2 \leq d_1 \leq d_2 \leq \frac{n}{2}$. If $C_1C, C_2C \in N_k$ for some $k \in [d_2+1, n-1-d_2]$ and some $C \in L_1$, then the number of integers $i \in [0, n-2]$ such that $C_1C_2 \in R_i$ is at most $d_2 - 1$.*

Proof. By Proposition 6.2.18, the vertices $\sigma_C(2n-3)$, $\sigma_C(1)$, and $\sigma_C(3)$ are in C_1 and C_2. By Proposition 6.2.21, the cycles C_1 and C_2 are adjacent in Θ; thus, their union $C_1 \cup C_2$ is a theta graph. Let $P = C_1 \cap C_2$ be the intersection of C_1 and C_2. Since $C_1 \cup C_2$ is a theta graph, the graph P is a path. The path P contains at least five vertices, as it contains $\sigma_C(2n-3)$, $\sigma_C(0)$, $\sigma_C(1)$, $\sigma_C(2)$, and $\sigma_C(3)$. Furthermore, since $V(C_1), V(C_2) \subseteq V(C)$, the vertices in P are also in C. Let s and t be integers such that $3 \leq s < t \leq 2n-3$ and the vertices $\sigma_C(s), \sigma_C(t)$ are the endpoints of P. For convenience, we also let $u = \sigma_C(s)$ and $v = \sigma_C(t)$. (Figure 6.1 shows the theta graph $C_1 \cup C_2$ where C_2 is the outer cycle.)

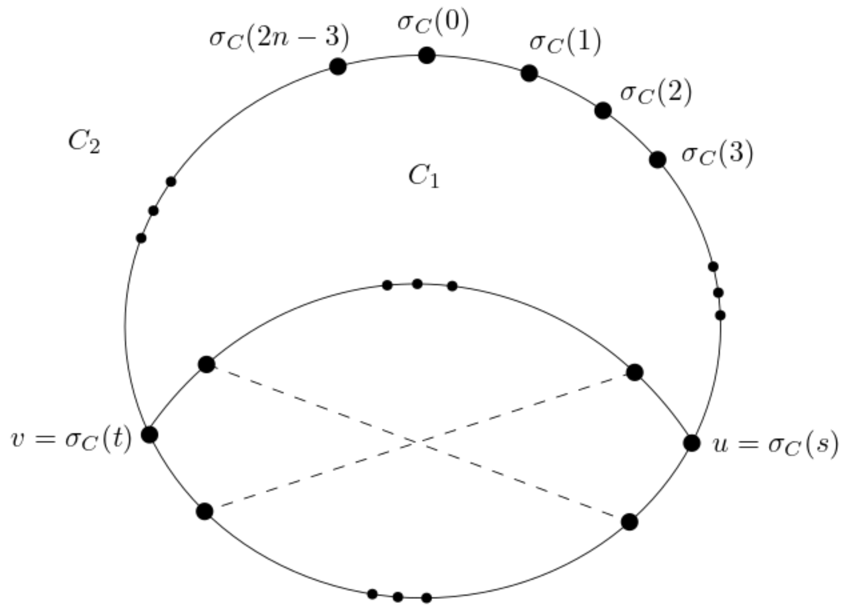

Figure 6.1: The cycles C_1 and C_2 in Lemma 6.2.22.

Let Z be the set of integers i such that $C_1C_2 \in R_i$. Since $C_1C_2 \in R_i$ if and only if there exists a cycle $D \in L_1$ such that $C_1D, C_2D \in N_i$, the set Z is also the set of integers i such that $C_1D, C_2D \in N_i$ for some $D \in L_1$. Let i be an integer in Z. We will show that $2i \in [t - 2d_2 + 2, 2d_2 - 2 + s]$. Since $i \in Z$, there exists a cycle $D \in L_1$ such that $C_1D, C_2D \in N_i$.

For each $j \in \{1, 2\}$, since $C_jD \in N_i$, the intersection $C_j \cap D$ is a path that contains all vertices of C_j and the set $E(C_j) \setminus E(D)$ contains exactly one edge, say e_j. Thus, the graph $C_j \setminus \{e_j\}$ is the path $C_j \cap D$. Therefore, the graph $C_1 \cup C_2 \setminus \{e_1, e_2\}$ is a subgraph of D.

Claim 6.2.22.1. *The path P is a subpath of D.*

Proof. If not, then since e_1, e_2 are the only edges in $E(C_1 \cup C_2) \setminus E(D)$, at least one of the edges e_1, e_2 is in P. Say e_1 is in P, which implies e_1 is in C_1 and C_2. Thus, the edge e_1 is in C_2, but is not in D. Since the set $E(C_2) \setminus E(D)$ contains exactly the edge e_2, it follows that $e_1 = e_2$. Let x and y be the endpoints of e_1. Since $C_1 \neq C_2$, each cycle intersects D in a different path; however, both paths have endpoints x and y. Since D is a cycle, there are two paths Q_1, Q_2 from x to y in D. Therefore, without loss of generality, for each $j \in \{1, 2\}$, the path $C_j \setminus \{e_j\}$ is Q_j. This implies that $V(C_1) \cup V(C_2) = V(D)$. Since the vertex $\sigma_D(2i)$ is not in C_1 or C_2, we have a contradiction. $\square$

For each $j \in \{1,2\}$ and $w \in \{u,v\}$, let e_j^w be the edge in $E(C_j) \setminus E(C_{3-j})$ incident with w.

Claim 6.2.22.2. *Let $w \in \{u,v\}$. The cycle D contains at most one of e_1^w, e_2^w.*

Proof. If D contains both e_1^w, e_2^w, then w has degree 3 in D, which is a contradiction. $\square$

Let P_1 be the graph $C_1 \cup C_2 \setminus \{e_1^u, e_2^v\}$ and let P_2 be the graph $C_1 \cup C_2 \setminus \{e_1^v, e_2^u\}$. Claim 6.2.22.2 implies that all edges in $C_1 \cup C_2$ are in D other than at most one of e_1^u, e_2^u and at most one of e_1^v, e_2^v. Additionally, by definition of N_i, the cycle D contains at least one of e_1^u, e_1^v and at least one of e_2^u, e_2^v. Therefore, the cycle D contains either P_1 or P_2 as a subgraph.

For each $j \in \{1,2\}$ and $w \in \{u,v\}$, let x_j^w be the endpoint of e_j^w other than w. Observe that P_1 is a path with endpoints x_1^u and x_2^v, and P_2 is a path with endpoints x_2^u and x_1^v. Therefore, the cycle D is either the union of P_1 and an x_1^u, x_2^v-path in $K_{n,n-1} \setminus (C_1 \cup C_2)$ or the union of P_2 and an x_2^u, x_1^v-path in $K_{n,n-1} \setminus (C_1 \cup C_2)$.

Recall that s and t are the integers in $[3, 2n-3]$ such that the endpoints of P are $u = \sigma_C(s)$ and $v = \sigma_C(t)$. Thus, the path P contains $2n-1-t+s$ vertices.

Claim 6.2.22.3. $2d_2 \leq t - s + 1$.

Proof. The cycle C_2 contains $2(n-d_2)$ vertices and the path P contains $2n-1-t+s$ vertices. Since P is a subpath of C_2, it follows that $2(n-d_2) \geq 2n-1-t+s$. Simplifying the inequality gives $2d_2 \leq t-s+1$. $\square$

Since cycles in L_1 contain all vertices in B, we know by definition that $\sigma_C(1) = \sigma_D(1)$. Furthermore, since P is a subpath of C and D, it follows that $\sigma_C(\ell) = \sigma_D(\ell)$ for all $\ell \in [0,s] \cup [t, 2n-3]$. The following two claims will determine the remaining indices of $V(C_1 \cup C_2)$ in σ_D, which we will use to determine the range of possible values for i. Recall that the vertex $\sigma_D(2i)$ is not in $C_1 \cup C_2$ since $C_1D, C_2D \in N_i$. Also, since P contains $2n-1-t+s$ vertices, note that, for each $j \in \{1,2\}$, the number of vertices in the path $C_j \setminus C_{3-j} = C_j \setminus P$ is $2(n-d_j) - (2n-1-t+s) = t-s+1-2d_j$.

Claim 6.2.22.4. *If P_1 is a subpath of D, then $2i \in [t-2d_2+2, 2d_1-2+s]$.*

Proof. The vertex x_2^u of $C_2 \setminus C_1$ is incident with u in P_1 and the distance from $\sigma_D(0)$ to u is less than the distance from $\sigma_D(0)$ to x_2^u. Therefore, since $u = \sigma_D(s)$, we have $x_2^u = \sigma_D(s+1)$. The path $C_2 \setminus C_1$ contains $t-s+1-2d_2$ vertices and has endpoints x_2^u and x_2^v. Thus, it follows that $x_2^v = \sigma_D(t+1-2d_2)$.

Similarly, the vertex x_1^v of $C_1 \setminus C_2$ is incident with v in P_1 and the distance from $\sigma_D(0)$ to v is less than the distance from $\sigma_D(0)$ to x_1^v. Therefore, since $v = \sigma_D(t)$, we have $x_1^v = \sigma_D(t-1)$. The path $C_1 \setminus C_2$ contains $t - s + 1 - 2d_1$ vertices and has endpoints x_1^v and x_1^u. Thus, it follows that $x_1^u = \sigma_D(t - (t - s + 1 - 2d_1)) = \sigma_D(s - 1 + 2d_1)$.

Now we know the path P_1 contains $\sigma_D(\ell)$ for all $\ell \in [0, t+1-2d_2] \cup [s-1+2d_1, 2n-3]$. Since the vertex $\sigma_D(2i)$ is not in P_1, the integer $2i$ is in $[t + 2 - 2d_2, s - 2 + 2d_1]$. $\square$

In the following claim, we consider the case where P_2 is in D instead of P_1. The proof is very similar to the proof of Claim 6.2.22.4.

Claim 6.2.22.5. *If P_2 is a subpath of D, then $2j \in [t - 2d_1 + 2, 2d_2 - 2 + s]$.*

Proof. The vertex x_1^u of $C_1 \setminus C_2$ is incident with u in P_2 and the distance from $\sigma_D(0)$ to u is less than the distance from $\sigma_D(0)$ to x_1^u. Therefore, since $u = \sigma_D(s)$, we have $x_1^u = \sigma_D(s+1)$. The path $C_1 \setminus C_2$ contains $t - s + 1 - 2d_1$ vertices and has endpoints x_1^u and x_1^v. Thus, it follows that $x_1^v = \sigma_D(t + 1 - 2d_1)$.

Similarly, the vertex x_2^v of $C_2 \setminus C_1$ is incident with v in P_2 and the distance from $\sigma_D(0)$ to v is less than the distance from $\sigma_D(0)$ to x_2^v. Therefore, since $v = \sigma_D(t)$, we have $x_2^v = \sigma_D(t-1)$. The path $C_2 \setminus C_1$ contains $t - s + 1 - 2d_2$ vertices and has endpoints x_2^v and x_2^u. Thus, it follows that $x_2^u = \sigma_D(t - (t - s + 1 - 2d_2)) = \sigma_D(s - 1 + 2d_2)$.

Now we know the path P_2 contains $\sigma_D(\ell)$ for all $\ell \in [0, t+1-2d_1] \cup [s-1+2d_2, 2n-3]$. Since the vertex $\sigma_D(2i)$ is not in P_2, the integer $2i$ is in $[t + 2 - 2d_1, s - 2 + 2d_2]$. $\square$

By Claims 6.2.22.4 and 6.2.22.5, and since $d_1 \le d_2$, the integer $2i$ is at least $t - 2d_2 + 2$ and at most $2d_2 - 2 + s$. Therefore, the even integer $2i$ is in $[t - 2d_2 + 2, 2d_2 - 2 + s]$. The number of even integers in $[t - 2d_2 + 2, 2d_2 - 2 + s]$ is at most

$$\left\lceil \tfrac{1}{2}(2d_2 - 2 + s - (t - 2d_2 + 2) + 1) \right\rceil = d_2 - 1 + \left\lceil \tfrac{1}{2}(s - t + 2d_2 - 1) \right\rceil.$$

By Claim 6.2.22.3, the expression $s - t + 2d_2 - 1$ is at most 0, hence the number of even integers in $[t - 2d_2 + 2, 2d_2 - 2 + s]$ is at most $d_2 - 1$. Therefore, the size of the set Z is at most $d_2 - 1$. $\square$

At this point, for each $k \in [0, n-2]$, we have three disjoint sets M_k, N_k, R_k of edges. Edges in M_k are between vertices in L_1, edges in N_k are between a vertex in L_1 and a vertex in L_d for some $d \ge 2$, and edges in R_k are between a vertex in L_d and a vertex in $L_{d'}$ where $d, d' \ge 2$. A sketch of the first few layers and some examples of edges in

M_k, N_k, and R_k can be found in Figure 6.2. The supersaturation lemma for Θ (Lemma 6.2.29), proved at the end of this subsection, relies only on the edges in these sets. Each edge between vertices in L_1 is in M_k for at most one value of k; however, edges with at least one endpoint not in L_1 may be in N_k or R_k for more than one value of k. In order to prove the supersaturation lemma, we start by considering the edges in $M_k \cup N_k \cup R_k$ individually for each $k \in [0, n-2]$. To avoid over-counting the edges in $M_k \cup N_k \cup R_k$ for multiple values of k, we define a fractional colouring so that the sum of fractional colours for each edge is at most 1.

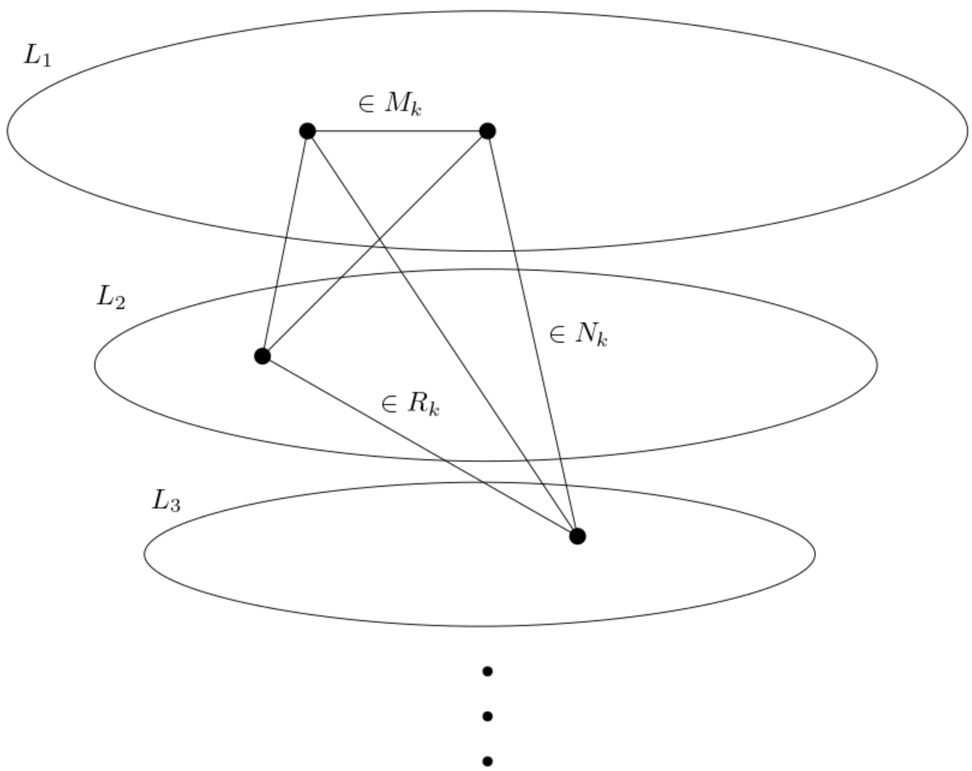

Figure 6.2: A sketch of the overlap graph of $M(K_{n,n-1})$.

First, we define a weight function $\phi_k : E(\Theta) \to \mathbb{R}$ for each colour $k \in [0, n-2]$, as follows. For each $e \in E(\Theta)$, let

$$\phi_k(e) = \begin{cases} |\{i \ : \ e \in M_i \cup N_i \cup R_i\}|^{-1} & \text{if } e \in M_k \cup N_k \cup R_k \\ 0 & \text{otherwise.} \end{cases}$$

We say each edge $e \in E(\Theta)$ has colour $k \in [0, n-2]$ with weight $\phi_k(e)$. If $e \in M_k \cup N_k \cup R_k$, then e has colour $k \in [0, n-2]$ with positive weight, so we say that C and C' are *k-neighbours*, which is denoted by $C \sim_k C'$. Furthermore, if an edge e has colour $k \in [0, n-2]$ with positive weight, then we say e is a *k-edge*.

Note that if $e \in M_k$, then since e is in M_i for at most one i, we know $\phi_k(e) = 1$. In the following two corollaries, we determine the weight $\phi_k(e)$ for each $e \in N_k$ and a lower bound on the weight $\phi_k(e)$ for each $e \in R_k$.

Corollary 6.2.23. *Let $C_1 \in L_1$ and $C \in L_d$ where $d \geq 2$. If $C \sim_k C_1$ for some $k \in [0, n-2]$, then $\phi_k(CC_1) = \frac{1}{d-1}$.*

Proof. Since $C_1 \in L_1$ and $C \in L_d$ and $C \sim_k C'$, the edge CC_1 is in N_k. Now it follows from Lemma 6.2.19 and the definition of ϕ_k that $\phi_k(CC_1) = \frac{1}{d-1}$. □

Corollary 6.2.24. *Let $C \in L_d$ and $C' \in L_{d'}$ where $2 \leq d \leq d'$. If $C \sim_k C'$ for some $k \in [d'+1, n-1-d']$, then $\phi_k(CC') \geq \frac{1}{d'-1}$.*

Proof. Since $C \in L_d$ and $C' \in L_{d'}$ and $C \sim_k C'$, the edge CC' is in R_k. Now it follows from Lemma 6.2.22 and the definition of ϕ_k that $\phi_k(CC') \geq \frac{1}{d'-1}$. □

For each $k \in [0, n-2]$ and $V \subseteq \mathcal{C}$, define $e_k(V) = \sum_{e \in E(\Theta[V])} \phi_k(e)$. The following proposition establishes that the number of edges induced by a subset V of vertices in the overlap graph is at least the sum of the weights of the k-edges induced by V.

Proposition 6.2.25. *For any $V \subseteq \mathcal{C}$, we have $e(V) \geq \sum_{k=0}^{n-2} e_k(V)$.*

Proof. For each edge $CC' \in E(\Theta)$, if CC' is in $M_k \cup N_k \cup R_k$ for some $k \in [0, n-2]$, then

$$\sum_{k=0}^{n-2} \phi_k(CC') = |\{i \ : \ CC' \in M_i \cup N_i \cup R_i\}|^{-1} |\{i \ : \ CC' \in M_i \cup N_i \cup R_i\}| = 1.$$

However, if CC' is not in $M_k \cup N_k \cup R_k$ for any $k \in [0, n-2]$, then $\sum_{k=0}^{n-2} \phi_k(CC') = 0$. Since

$$\sum_{k=0}^{n-2} e_k(V) = \sum_{k=0}^{n-2} \sum_{e \in E(\Theta[V])} \phi_k(e) = \sum_{e \in E(\Theta[V])} \sum_{k=0}^{n-2} \phi_k(e),$$

it follows that

$$\sum_{k=0}^{n-2} e_k(V) \leq \sum_{e \in E(\Theta[V])} 1 = e(V).$$ □

Next, our goal is to show that the sum of weights of k-edges induced by a set V of vertices within a certain depth in Θ, where $|V| \leq |L_1|$, is minimized when $V \subseteq L_1$. To prove this, we consider a counterexample set V of vertices and then show that we can replace a vertex in $V \setminus L_1$ with a vertex in $L_1 \setminus V$ without increasing the sum of k-edge weights.

Lemma 6.2.26. *Let $\delta \in [n-2]$ and $\gamma > 0$ and $k \in [\delta+1, n-1-\delta]$. For each set $V \subseteq \bigcup_{i=1}^{\delta} L_i$, there exists a set $V' \subseteq L_1$ such that $|V'| = \min(|V|, |L_1|)$ and $e_k(V') \leq e_k(V)$.*

Proof. Suppose not. Let $V \subseteq \bigcup_{i=1}^{\delta} L_i$ be a counterexample such that $e_k(V)$ is minimized and, subject to that, $|V \cap L_1|$ is maximized. Since V is a counterexample, we may assume that there exists at least one vertex in V that is not in L_1. Also, if $L_1 \subseteq V$, then $V' = L_1$ satisfies the lemma, so we may assume that there exists at least one vertex in L_1 that is not in V.

Claim 6.2.26.1. *For every pair $C_1, C_1' \in L_1$ such that $C_1 \sim_k C_1'$, at least one of C_1, C_1' is in V.*

Proof. Suppose towards a contradiction that there exist $C_1, C_1' \in L_1 \setminus V$ such that $C_1 \sim_k C_1'$. If there does not exist a k-neighbour $C \in V$ of C_1, then $V' = V \setminus \{D\} \cup \{C_1\}$ for any $D \in V \setminus L_1$ is a set of the same size as V that induces at most as many k-edges as V and $|V' \cap L_1| > |V \cap L_1|$, which is a contradiction.

Let C be a k-neighbour of C_1 in $V \cap L_d$ such that $2 \leq d \leq \delta$ is minimum. Consider any other k-neighbour $C' \in V \cap L_{d'}$ of C_1 where $d \leq d' \leq \delta$. By Corollary 6.2.23, the weight $\phi_k(C'C_1)$ is $\frac{1}{d'-1}$. By Proposition 6.2.21, the edge CC' is in R_k and thus $C \sim_k C'$. Now by Corollary 6.2.24, the weight $\phi_k(CC')$ is at least $\frac{1}{d'-1}$. Therefore $\phi_k(C'C_1) = \frac{1}{d'-1} \leq \phi_k(CC')$. Consider the set $V' = V \cup \{C_1\} \setminus \{C\}$. First observe that

$$\begin{aligned} e_k(V') &= e_k(V) + \sum_{C' \in V \,:\, C' \sim_k C_1} \big(\phi_k(C'C_1) - \phi_k(CC')\big) \\ &\leq e_k(V). \end{aligned}$$

Additionally, notice that $|V'| = |V|$ and $|V' \cap L_1| > |V \cap L_1|$. Therefore V' is a set with the same size as V where $e_k(V') \leq e_k(V)$ and $|V' \cap L_1| > |V \cap L_1|$, which is a contradiction. □

Claim 6.2.26.2. *For some $d \geq 2$, there exists a cycle $C \in L_d \cap V$ with a k-neighbour $C_1 \in L_1 \setminus V$.*

Proof. Suppose not. Let $d \geq 2$ and $C \in L_d \cap V$. So by assumption, all neighbours of C in L_1 are in V. Thus, by Lemma 6.2.20, the cycle C has at least $d!(d-1)!$ k-neighbours in $L_1 \cap V$. By Corollary 6.2.23, the weight $\phi_k(CD)$ is $\frac{1}{d-1}$ for all k-neighbours $D \in L_1$ of C. Therefore $\sum_{D \in L_1 \cap V : C \sim_k D} \phi_k(CD) \geq d!(d-1)! \cdot \frac{1}{d-1} \geq 2$. Since $L_1 \setminus V$ is not empty, there exists a cycle $C_1 \in L_1 \setminus V$. By assumption, there is no cycle in $V \setminus L_1$ with a k-neighbour in $L_1 \setminus V$, so C_1 has no k-neighbours in $V \setminus L_1$. Let C_1' be the k-neighbour of C_1 in L_1. By Claim 6.2.26.1, the cycle C_1' is in V. Consider the set $V' = V \cup \{C_1\} \setminus \{C\}$. Observe that

$$\begin{aligned} e_k(V') &\leq e_k(V) + \phi_k(C_1C_1') - \sum_{D \in L_1 \cap V : C \sim_k D} \phi_k(CD) \\ &\leq e_k(V) + 1 - 2 \\ &< e_k(V). \end{aligned}$$

Therefore, the set V' has the same size as V and $e_k(V') < e_k(V)$, which is a contradiction. $\square$

Now consider a pair of cycles $C_1, C_1' \in L_1$ where $C_1 \sim_k C_1'$ and $C_1 \notin V$ such that there exists $C \in V \setminus L_1$ where $C \sim_k C_1$. Such cycles C, C_1, C_1' exist by Claim 6.2.26.2. By Claim 6.2.26.1, it follows that $C_1' \in V$.

Let C be a k-neighbour of C_1 in $V \cap L_d$ such that $2 \leq d \leq \delta$ is minimum. By Lemma 6.2.20, the cycle C has at least $d!(d-1)!/2$ pairs of k-neighbours $D, D' \in L_1$ where $D \sim_k D'$. Therefore, by Claim 6.2.26.1, we have $C \sim_k D$ for at least $d!(d-1)!/2$ vertices $D \in V \cap L_1$. By Corollary 6.2.23, the weight $\phi_k(CD)$ is $\frac{1}{d-1}$ for any $D \in L_1$ where $D \sim_k C$. Thus,

$$\sum_{D \in V \cap L_1} \phi_k(CD) \geq \frac{d!(d-1)!}{2} \cdot \frac{1}{d-1} = \frac{d!(d-2)!}{2} \geq 1.$$

Since $\phi_k(C_1C_1') = 1$, it follows that $\sum_{D \in V \cap L_1} \phi_k(CD) \geq \phi_k(C_1C_1')$.

Consider any other k-neighbour $C' \in V \cap L_{d'}$ of C_1 where $d \leq d' \leq \delta$. By Corollary 6.2.23, the weight $\phi_k(C'C_1)$ is $\frac{1}{d'-1}$. By Proposition 6.2.21, the edge CC' is in R_k and thus $C \sim_k C'$. Now by Corollary 6.2.24, it follows that $\phi_k(CC') \geq \frac{1}{d'-1}$. Therefore $\phi_k(C'C_1) = \frac{1}{d'-1} \leq \phi_k(CC')$.

Consider the set $V' = V \cup \{C_1\} \setminus \{C\}$. Observe that

$$\begin{aligned} e_k(V') &= e_k(V) + \phi_k(C_1C_1') - \sum_{D \in V \cap L_1} \phi_k(CD) + \sum_{C' \in V \setminus L_1 \,:\, C' \sim_k C_1} \big(\phi_k(C'C_1) - \phi_k(CC')\big) \\ &\leq e_k(V). \end{aligned}$$

Therefore V' is a set with the same size as V where $e_k(V') \leq e_k(V)$ and $|V' \cap L_1| > |V \cap L_1|$, which is a contradiction. $\square$

Now we know that the sum of weights of k-edges induced by a set V of vertices within a certain depth in Θ, where $|V| \leq |L_1|$, is minimized when $V \subseteq L_1$. In the following lemma, we determine a lower bound on the sum of weights of k-edges induced by a subset V of vertices from L_1, where the size of V is sufficiently large.

Lemma 6.2.27. *Let $\gamma > 0$ and $k \in [0, n-2]$. If $V \subseteq L_1$ where $|V| \geq (1+\gamma)\alpha$, then $e_k(V) \geq \gamma\alpha$.*

Proof. Let $Z = \{C \in L_1 : \operatorname{sgn}(\sigma_C) = 1\}$. By Lemma 6.2.14, the pair $(Z, L_1 \setminus Z)$ is a bipartition of $\Theta[L_1]$ where $|Z| = |L_1 \setminus Z| = \alpha$. Let $X = Z \cap V$ and $Y = V \cap (L_1 \setminus Z)$. Since $X \subseteq Z$ and $Y \subseteq L_1 \setminus Z$, the pair (X, Y) is a bipartition of $\Theta[V]$. Now let $Y_k = \{C' \in L_1 : C \sim_k C' \text{ for some } C \in X\}$ and notice that $Y_k \subseteq L_1 \setminus Z$. Since M_k induces a perfect matching of $\Theta[L_1]$, the number of k-edges induced by V is at least the size of the intersection of Y_k and Y. Since $|Y_k| = |X|$ and the union of Y_k and Y is at most the size of $L_1 \setminus Z$, it follows that:

$$\begin{aligned} e_k(V) &\geq |Y_k \cap Y| \\ &= |Y_k| + |Y| - |Y_k \cup Y| \\ &\geq |X| + |Y| - |L_1 \setminus Z| \\ &= |V| - \alpha. \end{aligned}$$

By assumption, we have $|V| \geq (1+\gamma)\alpha$, hence $e_k(V) \geq (1+\gamma)\alpha - \alpha = \gamma\alpha$. $\square$

Since many of the results in this subsection apply only to sets of vertices above a certain depth in Θ, we prove the following lemma to show that the number of vertices below a certain depth is relatively small compared to the number of vertices in L_1.

Lemma 6.2.28. *For $n \geq 17$,*

$$\sum_{d=\lfloor \frac{n-1}{4} \rfloor + 1}^{n-2} |L_d| \leq \frac{64}{(n-1)^2}\alpha.$$

Proof. For each $d \in [n-2]$, the number of cycles in L_d is $\frac{n!(n-1)!}{2(n-d)d!(d-1)!}$ by Lemma 6.2.5. Therefore,

$$\sum_{d=\lfloor \frac{n-1}{4} \rfloor + 1}^{n-2} |L_d| \leq 2(n-1)\alpha \sum_{d=\lfloor \frac{n-1}{4} \rfloor + 1}^{n-2} \frac{1}{(n-d)d!(d-1)!}.$$

Since $d(n-d) \geq n-1$ for all $d \in [n-2]$, we have

$$\begin{aligned}\sum_{d=\lfloor \frac{n-1}{4} \rfloor+1}^{n-2} |L_d| &\leq 2\alpha \sum_{d=\lfloor \frac{n-1}{4} \rfloor+1}^{n-2} \frac{1}{(d-1)!^2} \\ &= 2\alpha \cdot \frac{1}{\lfloor \frac{n-1}{4} \rfloor!} \sum_{d=\lfloor \frac{n-1}{4} \rfloor+1}^{n-2} \frac{1}{(d-1)!} \\ &\leq 2\alpha \cdot \frac{1}{\lfloor \frac{n-1}{4} \rfloor!} \cdot \frac{2}{\lfloor \frac{n-1}{4} \rfloor!} \\ &= \frac{4\alpha}{\lfloor \frac{n-1}{4} \rfloor!^2}.\end{aligned}$$

Since $\lfloor \frac{n-1}{4} \rfloor! \geq \frac{n-1}{4}$ for $n \geq 17$, it follows that

$$\sum_{d=\lfloor \frac{n-1}{4} \rfloor+1}^{n-2} |L_d| \leq \frac{64\alpha}{(n-1)^2}.$$

□

Finally, we are ready to prove a supersaturation lemma for Θ, which is the main result of this subsection and will be used in an application of a container method in Subsection 6.2.5. Recall that Θ is the overlap graph of $M(K_{n,n-1})$. The vertex set of Θ is $\mathcal{C}$, the set of circuits of $M(K_{n,n-1})$. Two vertices C, C' in Θ are adjacent if and only if they are in the same theta subgraph of Θ.

Lemma 6.2.29 (Supersaturation Lemma for Θ)**.** *For each integer $n \geq 17$ and each real number $\gamma \geq \frac{e^2}{\sqrt{n-1}}$, if $\mathcal{B} \subseteq \mathcal{C}$ with $|\mathcal{B}| \geq (1+\gamma)\frac{n!(n-2)!}{4} = (1+\gamma)\alpha$, then $\mathcal{B}$ spans at least $\frac{\gamma n!(n-1)!}{16} = \frac{\gamma\alpha(n-1)}{4}$ edges in Θ.*

Proof. Let $\delta = \lfloor \frac{n-1}{4} \rfloor$ and notice that δ is in $[n-2]$. Let $\mathcal{B}' = \mathcal{B} \cap \bigcup_{d=1}^{\delta} L_d$. By Lemma 6.2.28, the set $\mathcal{B}'$ has size at least $\left(1+\gamma-\frac{64}{(n-1)^2}\right)\alpha$. Let $\gamma' = \gamma - \frac{64}{(n-1)^2}$. Since $n \geq 17$, we have $(n-1)^{1/2} \leq \frac{(n-1)^2}{16^{3/2}} \leq \frac{(n-1)^2}{64}$. Now we know $\frac{64}{(n-1)^2} \leq \frac{1}{\sqrt{n-1}} \leq \frac{1}{2} \cdot \frac{e^2}{\sqrt{n-1}} \leq \gamma/2$. Therefore $\gamma' \geq \gamma/2$.

Since $\mathcal{B}'$ is a subset of $\mathcal{B}$, the number of edges spanned by $\mathcal{B}$ is at least the number of edges spanned by $\mathcal{B}'$. Thus, it is sufficient to prove that the number of edges induced by $\mathcal{B}'$ is at least $\frac{\gamma\alpha(n-1)}{4}$.

For each $k \in [\delta+1, n-1-\delta]$, there exists a set $\mathcal{B}'' \subseteq L_1$ where $|\mathcal{B}''| = \min(|\mathcal{B}'|, |L_1|)$ and $e_k(\mathcal{B}'') \leq e_k(\mathcal{B}')$ by Lemma 6.2.26. The minimum of $|\mathcal{B}'|$ and $|L_1|$ is at least $(1+\gamma')\alpha$, hence $|\mathcal{B}''| \geq (1+\gamma')\alpha$. Now by Lemma 6.2.27, it follows that $e_k(\mathcal{B}'') \geq \gamma'\alpha$. Therefore $e_k(\mathcal{B}') \geq \gamma'\alpha$.

The size of the set $[\delta+1, n-1-\delta]$ is $n-1-2\delta = n-1-2\lfloor\frac{n-1}{4}\rfloor \geq \frac{n-1}{2}$. By Proposition 6.2.25, we have $e(\mathcal{B}') \geq \sum_{k\in[0,n-2]} e_k(\mathcal{B}')$. The sum of $e_k(\mathcal{B}')$ over all colours k is at least as much as the sum of $e_k(\mathcal{B}')$ for $k \in [\delta+1, n-1-\delta]$. Therefore,

$$\begin{aligned} e(\mathcal{B}') &\geq \sum_{k\in[\delta+1,n-1-\delta]} e_k(\mathcal{B}') \\ &\geq \frac{\gamma'\alpha(n-1)}{2}. \end{aligned}$$

Since $\gamma' \geq \gamma/2$, it follows that $e(\mathcal{B}') \geq \frac{\gamma\alpha(n-1)}{4}$. □

6.2.5 Applying a container method

First, we determine an upper bound for the number of stable sets in $\Theta(M(K_{n,n}))$ by applying the Supersaturation Lemma 6.2.11 to Corollary 5.2.2, a result implied by a container method proved by Kohayakawa, Lee, Rödl, and Samotij in [23]. The corollary states that for $q, N \in \mathbb{Z}^+$, $R \in \mathbb{R}^+$, and $\beta \in [0,1]$ where $R \geq \mathrm{e}^{-\beta q}N$, if G is an N-vertex graph where $e_G(U) \geq \beta\binom{|U|}{2}$ for every $U \subseteq V(G)$ containing at least R vertices, then the number of stable sets of G is at most $(\frac{\mathrm{e}N}{q})^q \cdot 2^R$. For more information about container methods and Corollary 5.2.2, see Section 5.2.

Theorem 6.2.30. $\log i(\Theta(M(K_{n,n}))) \leq \frac{n!^2}{2n}\left(1 + O\left(\frac{\log n}{\sqrt{n}}\right)\right)$.

Proof. First, in this proof, let Θ denote $\Theta(M(K_{n,n}))$. Let $R = \left(1 + \frac{\mathrm{e}^2}{\sqrt{n}}\right)\frac{n!^2}{2n}$. Let $q = \frac{n!^2}{2n\sqrt{n}}$ and $\beta = \frac{2n\sqrt{n}}{n!^2} = \frac{1}{q}$. Finally, let $N = |V(\Theta)|$. Since $N < \frac{\mathrm{e}n!^2}{2n}$ by Corollary 6.2.6, we have $\mathrm{e}^{-\beta q}N < \frac{n!^2}{2n} < R$. Let $U \subseteq V(\Theta)$ be a set of vertices of size $(1+\gamma)\frac{n!^2}{2n}$ where $\gamma \geq \frac{\mathrm{e}^2}{\sqrt{n}}$. Since $\gamma \geq \frac{\mathrm{e}^2}{\sqrt{n}} \geq \frac{8}{n}$ for $n \geq 1$, it follows from Lemma 6.2.11 that the number of edges spanned by U is at least $\frac{\gamma n!^2}{4}$. Note that since $|V(\Theta)| < \frac{\mathrm{e}n!^2}{2n}$, we know $\gamma < \mathrm{e}-1$. Since $\frac{\mathrm{e}^2}{\sqrt{n}} \leq \gamma < \mathrm{e}-1$,

$$\beta\binom{|U|}{2} \leq \beta \cdot \frac{|U|^2}{2} \leq \frac{2n\sqrt{n}}{n!^2} \cdot \frac{(1+\gamma)^2 n!^4}{8n^2} = \frac{(1+\gamma)^2 n!^2}{4\sqrt{n}} \leq \frac{\gamma n!^2}{4},$$

so the number of edges spanned by U is at least $\beta\binom{|U|}{2}$. Thus, the conditions of Corollary 5.2.2 are satisfied and it follows that Θ contains at most $\left(\frac{eN}{q}\right)^q \cdot 2^R$ stable sets.

Since $N < \frac{en!^2}{2n}$,

$$
\begin{aligned}
\log\left(\left(\frac{eN}{q}\right)^q \cdot 2^R\right) &= q\log\left(\frac{eN}{q}\right) + R \\
&\leq \frac{n!^2}{2n\sqrt{n}}\log(e^2\sqrt{n}) + \left(1+\frac{e^2}{\sqrt{n}}\right)\frac{n!^2}{2n} \\
&= \frac{n!^2}{2n}\left(1+\frac{e^2}{\sqrt{n}}+\frac{\log(e^2\sqrt{n})}{\sqrt{n}}\right) \\
&= \frac{n!^2}{2n}\left(1+\frac{\log(e^2\sqrt{n})+e^2}{\sqrt{n}}\right).
\end{aligned}
$$

Therefore, the number of stable sets in Θ is at most $2^{\frac{n!^2}{2n}\left(1+O\left(\frac{\log n}{\sqrt{n}}\right)\right)}$. $\square$

By a very similar proof, we obtain a similar upper bound for the number of stable sets in $\Theta(M(K_{n,n-1}))$.

Theorem 6.2.31. $\log i(\Theta(M(K_{n,n-1}))) \leq \frac{n!^2}{2n}\left(1+O\left(\frac{\log n}{\sqrt{n}}\right)\right)$.

Proof. First, in this proof, let Θ denote $\Theta(M(K_{n,n-1}))$. Let $R = \left(1+\frac{e^2}{\sqrt{n-1}}\right)\frac{n!(n-2)!}{4}$. Let $q = \frac{n!(n-2)!}{\sqrt{n-1}}$ and $\beta = \frac{2\sqrt{n-1}}{n!(n-2)!} = \frac{2}{q}$. Finally, let $N = |V(\Theta)|$. Since $N < \frac{en!(n-1)!}{2(n-1)}$ by Corollary 6.2.6, we have $e^{-\beta q}N < \frac{n!(n-1)!}{2(n-1)} < R$. Let $U \subseteq V(\Theta)$ be a set of vertices of size at least $(1+\gamma)\frac{n!(n-2)!}{4}$ where $\gamma \geq \frac{e^2}{\sqrt{n-1}}$. Note that since $|V(\Theta)| < \frac{en!^2}{2n}$, we know $\gamma < e-1$. By Lemma 6.2.29, the number of edges spanned by U is at least $\frac{\gamma n!(n-1)!}{16}$. Since $\frac{e^2}{\sqrt{n-1}} \leq \gamma < e-1$, we have

$$
\begin{aligned}
\beta\binom{|U|}{2} &\leq \beta\cdot\frac{|U|^2}{2} \leq \frac{2\sqrt{n-1}}{n!(n-2)!}\cdot\frac{(1+\gamma)^2 n!^2(n-2)!^2}{32} = \frac{(1+\gamma)^2 n!(n-1)!}{16\sqrt{n-1}} \\
&\leq \frac{\gamma n!(n-1)!}{16},
\end{aligned}
$$

so the number of edges spanned by U is at least $\beta\binom{|U|}{2}$. Thus, the conditions of Corollary 5.2.2 are satisfied and it follows that Θ contains at most $\left(\frac{eN}{q}\right)^q \cdot 2^R$ stable sets.

Since $N < \frac{\mathrm{e}n!(n-2)!}{2}$,

$$\begin{aligned}\log\left(\left(\frac{\mathrm{e}N}{q}\right)^q \cdot 2^R\right) &= q\log\left(\frac{\mathrm{e}N}{q}\right) + R \\ &\le \frac{n!(n-2)!}{\sqrt{n-1}}\log\left(\frac{\mathrm{e}^2}{2}\sqrt{n-1}\right) + \left(1+\frac{\mathrm{e}^2}{\sqrt{n-1}}\right)\frac{n!(n-2)!}{4} \\ &= \frac{n!(n-2)!}{4}\left(1+\frac{\mathrm{e}^2}{\sqrt{n-1}} + \frac{4\log(\mathrm{e}^2\sqrt{n-1}/2)}{\sqrt{n-1}}\right) \\ &= \frac{n!(n-2)!}{4}\left(1+\frac{4\log(\mathrm{e}^2)+2\log(n-1)-4\log(2)+\mathrm{e}^2}{\sqrt{n-1}}\right)\end{aligned}$$

Therefore, the number of stable sets in Θ is at most $2^{\frac{n!(n-2)!}{4}\left(1+O\left(\frac{\log n}{\sqrt{n}}\right)\right)}$. ☐

6.2.6 The main theorems

We are now ready to prove the main results of this section.

Theorem 6.0.1. $\log \operatorname{coext}(M(K_{n,n})) = \frac{n!^2}{2n}(1+o(1))$.

Proof. Since $K_{n,n}$ has $\frac{n!^2}{2n}$ Hamiltonian cycles by Lemma 6.2.1, there are $2^{\frac{n!^2}{2n}}$ distinct sets of Hamiltonian cycle. Since no theta subgraph of $K_{n,n}$ contains more than one Hamiltonian cycles, each set of Hamiltonian cycles is a stable set of $\Theta(M(K_{n,n}))$. Therefore, we have $\log i(\Theta(M(K_{n,n}))) \ge \frac{n!^2}{2n}$.

By Proposition 6.2.7, if $n \ge 16$, then the graph $K_{n,n}$ has at most $\frac{n!^2}{2n^2}$ cycles with length at most $\frac{2}{3}(2n+3)$. By Theorem 6.2.30, we have $\log i(\Theta(M(K_{n,n}))) \le \frac{n!^2}{2n}\left(1+O\left(\frac{\log n}{\sqrt{n}}\right)\right)$. Thus, by Lemma 6.1.3,

$$\frac{n!^2}{2n} \le \log \operatorname{coext}(M(K_{n,n})) \le \frac{n!^2}{2n}\left(1+O\left(\frac{\log n}{\sqrt{n}}\right)+\frac{1}{n}\right),$$

which implies that $\log \operatorname{coext}(M(K_{n,n})) = \frac{n!^2}{2n}(1+o(1))$. ☐

Theorem 6.0.2. $\log \operatorname{coext}(M(K_{n,n-1})) = \frac{n!(n-2)!}{4}(1+o(1))$.

Proof. By Proposition 6.2.16, we have $\log i(\Theta(M(K_{n,n-1}))) \ge \frac{n!(n-2)!}{4}$.

By Proposition 6.2.8, if $n \geq 18$, then the graph $K_{n,n-1}$ has at most $\frac{n!(n-2)!}{4n}$ cycles with length at most $\frac{2}{3}(2n+2)$. By Theorem 6.2.31, we have $\log i(\Theta(M(K_{n,n-1}))) \leq \frac{n!(n-2)!}{4}\Big(1+O\Big(\frac{\log n}{\sqrt{n}}\Big)\Big)$. Thus, by Lemma 6.1.3,

$$\frac{n!(n-2)!}{4} \leq \log \operatorname{coext}(M(K_{n,n-1})) \leq \frac{n!(n-2)!}{4}\left(1+O\left(\frac{\log n}{\sqrt{n}}\right)+\frac{1}{n}\right),$$

which implies that $\log \operatorname{coext}(M(K_{n,n-1})) = \frac{n!(n-2)!}{4}(1+o(1))$. $\square$

Chapter 7

Coextensions of Dowling geometries

Dowling geometries are frame matroids, which are defined using biased graphs. As we saw in Chapter 6, biased graphs correspond to coextensions of graphic matroids. In particular, the connected corank-2 restrictions of a graphic matroid $M(G)$ are the theta subgraphs of G. For frame matroids, theta subgraphs with all cycles unbalanced are circuits; that is, they are connected corank-1 restrictions. The connected corank-2 restrictions of a Dowling geometry are difficult to describe concisely, so instead of defining them precisely, we describe adjacencies in the circuit graph as needed. In this chapter, we do not need to restrict ourselves to only considering connected corank-2 restrictions, so we use the circuit graph $\Omega(DG(n,\Gamma))$ instead of the overlap graph, which was used in the previous chapter. Recall that $\Omega(DG(n,\Gamma))$ has vertex set $\mathcal{C}(DG(n,\Gamma))$ where two vertices are adjacent if and only if they are contained in a corank-2 restriction of $DG(n,\Gamma)$.

In this chapter, we prove the following bounds on the number of coextensions of the Dowling geometry $DG(n,\Gamma)$, which is denoted $\mathrm{coext}(DG(n,\Gamma))$. Recall that the Dowling geometry $DG(n,\Gamma)$ is defined using the finite (multiplicative) group Γ, which has order q. Also, recall that $o(1)$ denotes an unspecified function of n which goes to 0 as n goes to infinity and log denotes the base-2 logarithm.

Theorem 7.0.1.

$$\frac{1}{8}n!(n-4)q^{n-1}(q-1) \le \log(\mathrm{coext}(DG(n,\Gamma))) \le \frac{\sqrt[q]{\mathrm{e}}}{8}(n+1)!q^{n-1}(q-1)\log(n)(1+o(1)).$$

If we include a $(1+o(1))$ term in the lower bound, then the lower and upper bounds in Theorem 7.0.1 almost match, but they differ by a factor of $\sqrt[q]{\mathrm{e}}\log(n)$, which is logarithmic in n. If we instead asymptotically bound the double log of $\text{coext}(DG(n,\Gamma))$, then this logarithmic factor becomes a term in the function represented by $o(1)$. This is shown in the following corollary, which is directly implied by Theorem 7.0.1.

Corollary 7.0.2.

$$\log(\log(\text{coext}(DG(n,\Gamma)))) = n\log n(1+o(1)).$$

In order to prove the bounds in Theorem 7.0.1, we determine bounds on the number of stable sets in the circuit graph of $DG(n,\Gamma)$ and use Corollary 3.2.6. The lower bound is established by finding a large stable set in the circuit graph. To prove the upper bound, we describe a subgraph of $\Omega(DG(n,\Gamma))$ on a particular edge set which we show is isomorphic to a subgraph of a Hamming graph. Bounds determined in Chapter 4 for the number of stable sets in certain Hamming graphs are then used.

Notice in Theorem 4.3.2 that the lower bound for the number of stable sets in the Hamming graphs relevant to this chapter contain a log factor, similar to the upper bound. Since the lower bound in Theorem 7.0.1 does not use analysis of Hamming graphs, it seems likely that it can be improved by using Hamming graphs. If lower bound version of Shearer's Lemma 2.2.3 exists, it seems hopeful that it could be used with the lower bound in Theorem 4.3.2 to find a bound matching the upper bound in Theorem 7.0.1, up to lower order terms.

Conjecture 7.0.3. *For some fixed real number c,*

$$\log(\text{coext}(DG(n,\Gamma))) = c(n+1)!q^{n-1}(q-1)\log(n)(1+o(1)).$$

7.1 Preliminaries

First, we recall the definition of a Dowling geometry $DG(n,\Gamma)$. A similar definition and a relevant discussion on Dowling geometries can be found in [40], although note that Dowling geometries are denoted $Q_n(\Gamma)$ in [40]. Let n be a positive integer. Let Γ be a finite (multiplicative) group with identity element 1 and let $q = |\Gamma|$. Recall the construction of the graph K_n^Γ, which has vertex set $[n]$. The edge set of K_n^Γ is $\Gamma \times \binom{[n]}{2} \cup \{\beta_u : u \in [n]\}$ and the incidence function f of K_n^Γ is defined as follows. For each $(\gamma, \{u,v\}) \in \Gamma \times \binom{[n]}{2}$, let $f((\gamma,\{u,v\})) = \{u,v\}$ and for each $u \in [n]$, let $f(\beta_u) = \{u\}$. Informally, the graph K_n^Γ has vertex set $[n]$, an edge labelled γ between each pair $\{u,v\} \in \binom{[n]}{2}$ for each $\gamma \in \Gamma$, and a loop labelled β_u on each vertex $u \in [n]$. For an edge $e = (\gamma, \{u,v\})$, let γ be called the *edge label* of e and recall that u and v are called the *endpoints* of e. The ground set of $DG(n,\Gamma)$ is $E(K_n^\Gamma)$.

Recall the function $\psi : \Gamma \times \mathbb{Z}^2_{>0} \to \Gamma$ where, for each $(\gamma, x, y) \in \Gamma \times \mathbb{Z}^2_{>0}$,

$$\psi((\gamma, x, y)) = \begin{cases} \gamma & \text{if } x \leq y \\ \gamma^{-1} & \text{if } y < x \end{cases}.$$

Let C be a cycle of K_n^Γ with at least two edges and arbitrarily assign an orientation to it. Let the vertices and edges of C, beginning with a vertex, be $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_1$, where $e_i = (\gamma_i, \{v_i, v_{i+1}\})$ for each $i \in [k]$. We say C is *balanced* if $\prod_{i=1}^{k} \psi((\gamma_i, v_i, v_{i+1})) = 1$. Note that the definition of a balanced cycle does not depend on the chosen cyclic ordering of the cycle [40]. A cycle is *unbalanced* if it either has a single edge or is not balanced. Let $\mathcal{B}$ be the collection of balanced cycles of K_n^Γ. The circuits of $DG(n,\Gamma)$ consist of the edge sets of all of the balanced cycles together with the edge sets of all of the hinged, tight, and loose cuffs in which none of the cycles are balanced. Recall that the Dowling geometry $DG(n,\Gamma)$ is the frame matroid represented by $(K_n^\Gamma, \mathcal{B})$.

Now we are ready to prove some preliminary results about Dowling geometries. These results establish the number of certain cuffs or cycles in K_n^Γ. Note that when we refer to a loop in a cuff, we are referring to a graphic loop, not a matroid loop.

Proposition 7.1.1. *For each $t \in [0, n-2]$, the number of $(n+1-t)$-cuffs in K_n^Γ is at most $\frac{(n+2)!}{8t!} q^{n-t-1}(q-1)^2$.*

Proof. Let $M = DG(n,\Gamma)$. Let $\mathcal{C}$ denote the collection of $(n+1-t)$-cuffs in K_n^Γ. Let $\mathcal{L} \subseteq \mathcal{C}$ denote the collection of $(n+1-t)$-cuffs in K_n^Γ that contain a loop. Define a function $\Phi : \mathcal{C} \to 2^{E(M)^{n+1-t}}$ as follows. For each $C \in \mathcal{C}$, let

$$\begin{aligned}\Phi(C) = \{&(e_1, e_2, \ldots, e_{n-1-t}, f_1, f_2) : e_1 e_2 \ldots e_{n-1-t} \text{ is a Hamiltonian path of } C \text{ and} \\ &f_1, f_2 \in V(C) \setminus \{e_1, \ldots, e_{n-1-t}\} \text{ each have an end with degree at least 3 in } C\}.\end{aligned}$$

Each cuff is determined by its edges, so each tuple in $E(M)^{n+1-t}$ appears in the Φ-image of at most one cuff. For $\mathcal{A} \subseteq \mathcal{C}$, let $\Phi(\mathcal{A}) = \bigcup_{C \in \mathcal{A}} \Phi(C)$. Observe that $|\Phi(\mathcal{C})|$ is the number of tuples $(e_1, e_2, \ldots, e_{n-1-t}, f_1, f_2)$ in $E(M)^{n+1-t}$ where $K_n^\Gamma[\{e_1, \ldots, e_{n-1-t}\}]$ is a Hamiltonian path of some cuff C.

Claim 7.1.1.1. $|\Phi(\mathcal{C} \setminus \mathcal{L})| \leq \frac{2n!}{t!} q^{n-t-1}(n-t-1)^2(q-1)^2$.

Proof. Consider a tuple $T = (e_1, \ldots, e_{n-1-t}, f_1, f_2)$ in $\Phi(\mathcal{C} \setminus \mathcal{L})$. Thus, there exists a cuff $C \in \mathcal{C}$ such that $T \in \Phi(C)$. Since there are $\binom{n}{n-t}(n-t)!q^{n-1-t}$ paths of length $n-t-1$ in K_n^Γ where one endpoint of the path is marked, there are $\binom{n}{n-t}(n-t)!q^{n-1-t}$ choices for the elements $e_1, \ldots, e_{n-t-1}$.

Since C is a cuff, it has no vertex of degree 1, so at least one of f_1, f_2 is incident with e_1 and at least one is incident with e_{n-1-t}. Since f_1, f_2 each have an endpoint with degree at least 3 in C, there exist $i \neq j \in \{1, n-t-1\}$ such that the edge f_1 is incident with e_i and the edge f_2 is incident with e_j. Let v_i be the vertex f_1 and e_i share. Let v_j be the vertex f_2 and e_j share. Since no cuff in $\mathcal{C} \setminus \mathcal{L}$ has a loop, there are $(n-t-1)$ choices for the endpoint of f_1 other than v_i and at most $(q-1)$ choices for the edge label. Similarly, there are $(n-t-1)$ choices for the endpoint of f_2 other than v_j and at most $(q-1)$ choices for the edge label. Therefore, there are at most $2(n-t-1)^2(q-1)^2$ choices for the elements f_1, f_2. □

Claim 7.1.1.2. $|\Phi(\mathcal{L})| \leq \frac{2n!}{t!}q^{n-t-1}(n-t-1)(q-1)$.

Proof. Consider a tuple $T = (e_1, \ldots, e_{n-1-t}, f_1, f_2)$ in $\Phi(\mathcal{L})$. Thus, there exists a cuff $C \in \mathcal{C}$ such that $T \in \Phi(C)$. Since there are $\binom{n}{n-t}(n-t)!q^{n-1-t}$ paths of length $n-t-1$ in K_n^Γ where one endpoint of the path is marked, there are $\binom{n}{n-t}(n-t)!q^{n-1-t}$ choices for the elements $e_1, \ldots, e_{n-t-1}$.

Since C is a cuff, it has no vertex of degree 1, so at least one of f_1, f_2 is incident with e_1 and at least one is incident with e_{n-1-t}. Since f_1, f_2 each have an endpoint with degree at least 3 in C, there exist $i \neq j \in \{1, n-t-1\}$ such that the edge f_1 is incident with e_i and the edge f_2 is incident with e_j. Since each cuff in $\mathcal{L}$ has a loop, there is one choice for the edge in $\{f_1, f_2\}$ that is a loop and there are at most $(n-t-1)(q-1)$ choices for the other edge. Therefore, there are at most $2(n-t-1)(q-1)$ choices for the elements f_1, f_2. □

Claim 7.1.1.3. *If $C \in \mathcal{C} \setminus \mathcal{L}$, then $|\Phi(C)| \geq 16$.*

Proof. Since C has no loops, there exist distinct edges e_1, e_2 in a cycle D of C and distinct edges e_3, e_4 in another cycle D' of C where each of e_1, e_2, e_3, e_4 is incident with a vertex of degree at least 3. Observe that, for $i \in \{1, 2\}$ and $j \in \{3, 4\}$, the graph $C \setminus \{e_i, e_j\}$ is a Hamiltonian path of C. There are four choices for i and j, two ways to order e_i and e_j, and two ways to order the Hamiltonian path $C \setminus \{e_i, e_j\}$; therefore, there are at least 16 tuples in $\Phi(C)$. □

Claim 7.1.1.4. *If $C \in \mathcal{L}$, then $|\Phi(C)| \geq 4$.*

Proof. Let e_1 be a loop in C. Since C contains at least two unbalanced cycles, there exists an edge $e_2 \neq e_1$ such that $C \setminus \{e_1, e_2\}$ is a Hamiltonian path of C. There are two ways to order e_1, e_2 and two ways to order the Hamiltonian path, so there are at least 4 tuples in $\Phi(C)$. $\square$

Since each tuple is in the Φ-image of at most one cuff,

$$|\mathcal{C}| \leq \frac{|\Phi(\mathcal{C} \setminus \mathcal{L})|}{\min_{C \in \mathcal{C} \setminus \mathcal{L}} |\Phi(C)|} + \frac{|\Phi(\mathcal{L})|}{\min_{C \in \mathcal{L}} |\Phi(C)|}.$$

By Claims 7.1.1.1, 7.1.1.2, 7.1.1.3, and 7.1.1.4,

$$\begin{aligned} |\mathcal{C}| &\leq \frac{n!}{t!} q^{n-t-1}(n-t-1)(q-1)\left(\frac{1}{8}(n-t-1)(q-1) + \frac{1}{2}\right) \\ &\leq \frac{n!}{t!} q^{n-t-1}(q-1)^2\left(\frac{1}{8}(n-t-1)^2 + \frac{1}{2}(n-t-1)\right) \\ &= \frac{n!}{t!} q^{n-t-1}(q-1)^2\left(\frac{1}{8}\Big((n-t)^2 + 2(n-t) - 3\Big)\right). \end{aligned}$$

Since $t \geq 0$ and $n^2 + 2n - 3 \leq n^2 + 3n + 2 = (n+1)(n+2)$, we have

$$\begin{aligned} |\mathcal{C}| &\leq \frac{n!}{8t!} q^{n-t-1}(q-1)^2(n+1)(n+2) \\ &\leq \frac{(n+2)!}{8t!} q^{n-t-1}(q-1)^2 \end{aligned}$$ $\square$

Proposition 7.1.2. *For each $t \in [1, n-1]$, the number of balanced $(n+1-t)$-cycles in K_n^Γ is $\frac{n!}{(t-1)!(n+1-t)} q^{n-t}$.*

Proof. Consider a balanced $(n+1-t)$-cycle C in K_n^Γ, which has $n+1-t$ vertices. Let the vertices and edges of C, beginning with a vertex, be $v_1, e_1, v_2, e_2, \ldots, v_{n+1-t}, e_{n+1-t}, v_1$. There are $\binom{n}{n+1-t}$ ways to choose the $n+1-t$ vertices in C and $(n-t)!$ ways to order them in a cycle. Since C is balanced, we know $\prod_{i=1}^{n+1-t} \psi(e_i) = 1$, which implies $\psi(e_{n+1-t}) = \left(\prod_{i=1}^{n-t} \psi(e_i)\right)^{-1}$. Thus, since the edge endpoints are known, there are q^{n-t} choices for the edges $e_1, \ldots, e_{n-t}$ and one choice for the edge e_{n+1-t}. Therefore, there are $\frac{n!}{(t-1)!(n+1-t)} q^{n-t}$ balanced $(n+1-t)$-cycles in K_n^Γ. $\square$

Proposition 7.1.3. *For each $t \in [0, n-2]$, the number of $(n+1-t)$-cuffs in K_n^Γ that contain a loop or parallel edges is at most $\frac{n!}{t!}(n-t-1)q^{n-t-1}(q-1)^2$.*

Proof. Let $\mathcal{C}$ denote the collection of $(n+1-t)$-cuffs in K_n^Γ that contain a loop or parallel edges. Define a function $\Phi : \mathcal{C} \to 2^{E(DG(n,\Gamma))^{n+1-t}}$ as follows. For each $C \in \mathcal{C}$, let

$$\begin{aligned}\Phi(C) = \{(e_1, e_2, \ldots, e_{n-1-t}, f_1, f_2) : \ & e_1 e_2 \ldots e_{n-1-t} \text{ is a Hamiltonian path of } C \text{ and} \\ & f_1, f_2 \in V(C) \setminus \{e_1, \ldots, e_{n-1-t}\} \text{ each have an end with degree at least 3 in } C\}.\end{aligned}$$

Each cuff is determined by its edges, so each tuple in $E(DG(n,\Gamma))^{n+1-t}$ appears in the Φ-image of at most one cuff. For $\mathcal{A} \subseteq \mathcal{C}$, let $\Phi(\mathcal{A}) = \bigcup_{C \in \mathcal{A}} \Phi(C)$. Observe that $|\Phi(\mathcal{C})|$ is the number of tuples $(e_1, e_2, \ldots, e_{n-1-t}, f_1, f_2)$ in $E(DG(n,\Gamma))^{n+1-t}$ where $K_n^\Gamma[\{e_1, \ldots, e_{n-1-t}\}]$ is a Hamiltonian path of some cuff C.

Claim 7.1.3.1. $|\Phi(\mathcal{C})| \leq \frac{4n!}{t!}(n-t-1)q^{n-t-1}(q-1)^2$.

Proof. Consider a tuple $T = (e_1, \ldots, e_{n-1-t}, f_1, f_2)$ in $\Phi(\mathcal{C})$. Thus, there exists a cuff $C \in \mathcal{C}$ such that $T \in \Phi(C)$. Since there are $\binom{n}{n-t}(n-t)!q^{n-1-t}$ paths of length $n-t-1$ in K_n^Γ where one endpoint of the path is marked, there are $\binom{n}{n-t}(n-t)!q^{n-1-t}$ choices for the elements $e_1, \ldots, e_{n-t-1}$.

Since C is a cuff, it has no vertex of degree 1, so at least one of f_1, f_2 is incident with e_1 and at least one is incident with e_{n-1-t}. Since f_1, f_2 each have an endpoint with degree at least 3 in C there exist $i \neq j \in \{1, n-t-1\}$ such that the edge f_1 is incident with e_i and the edge f_2 is incident with e_j. Since each cuff in $\mathcal{C}$ has a loop or parallel edges, one of $\{f_1, f_2\}$ is a loop or has the same endpoints as e_1 or e_{n-t-1}. Since there are $q-2$ edges that have the same endpoints as e_1 or e_{n-t-1}, there are $q-1$ choices for the edge in $\{f_1, f_2\}$ that is a loop or parallel edge and there are at most $(n-t-1)(q-1)$ choices for the other edge. Since there are two choices for i and j, there are at most $2(n-t-1)(q-1)^2$ choices for the elements f_1, f_2. $\square$

Claim 7.1.3.2. *If $C \in \mathcal{C}$, then $|\Phi(C)| \geq 4$.*

Proof. Let e_1 be a loop or parallel edge in C. Since C contains at least two unbalanced cycles, there exists an edge $e_2 \neq e_1$ such that $C \setminus \{e_1, e_2\}$ is a Hamiltonian path of C. There are two ways to order e_1, e_2 and two ways to order the Hamiltonian path, so there are at least 4 tuples in $\Phi(C)$. $\square$

Since each tuple is in the Φ-image of at most one cuff,

$$|\mathcal{C}| \leq \frac{|\Phi(\mathcal{C})|}{\min_{C \in \mathcal{C}} |\Phi(C)|}.$$

By Claims 7.1.3.1 and 7.1.3.2,

$$|\mathcal{C}| \leq \frac{n!}{t!}(n-t-1)q^{n-t-1}(q-1)^2. \qquad \square$$

The following lemma upper bounds the number of circuits in $DG(n,\Gamma)$ with size at most $\frac{5n}{6}+1$. We will see in Section 7.4 that this upper bounds the number of small circuits in $DG(n,\Gamma)$.

Lemma 7.1.4. *If $\mathcal{C}_t$ denotes the set of $(n+1-t)$-circuits of $DG(n,\Gamma)$, then*

$$\sum_{t=\lfloor n/6\rfloor-1}^{n-1} |\mathcal{C}_t| = o((n+1)!q^{n-1}).$$

Proof. The set $\mathcal{C}_t$ contains precisely the $(n+1-t)$-cuffs and the balanced $(n+1-t)$-cycles of K_n^Γ. Therefore, by Propositions 7.1.1 and 7.1.2,

$$\begin{aligned}
\sum_{t=\lfloor n/6\rfloor-1}^{n-1} |\mathcal{C}_t| &\leq \sum_{t=\lfloor n/6\rfloor-1}^{n-1} \frac{1}{8}\frac{n!}{t!}q^{n-t-1}\left((n+2)^2(q-1)^2+\frac{8tq}{n+1-t}\right)\\
&\leq \frac{n!q^{n+1}}{8}\sum_{t=\lfloor n/6\rfloor-1}^{n-1}\frac{1}{t!q^t}(n+2)^2(1+o(1))\\
&\leq \frac{n!q^{n+1}}{8}\left(\frac{5n}{6}+2\right)\frac{n^2}{(\lfloor n/6\rfloor-1)!q^{\lfloor n/6\rfloor-1}}(1+o(1))\\
&= n!nq^{n-1}\left(\frac{5q^4}{6\cdot 8}\cdot\frac{n^2}{(n/6-2)!q^{n/6}}\right)(1+o(1))\\
&= o((n+1)!q^{n-1}). \qquad \square
\end{aligned}$$

7.2 Lower bound

In this section, we determine a lower bound for the number of stable sets in the circuit graph of $DG(n,\Gamma)$, based on a large stable set.

Consider a tight k-cuff C which contains cycles C_a and C_b, with lengths a and b, respectively. For each $i \in \{a,b\}$, let u_i be the vertex of degree 4 in C_i and let w_i be the minimum neighbour of u_i in C_i. Let $v_1e_1v_2e_2\dots v_ae_av_{a+1}$ and $v'_1e'_1v'_2e'_2\dots v'_be'_bv'_{b+1}$, where

$v_{a+1} = v_1 = u_a$, $v_2 = w_a$, $v'_{b+1} = v'_1 = u_b$, and $v'_2 = w_b$, be the vertices and edges of C_a and C_b. For each edge e in K_n^Γ, let $\varphi(e)$ denote the edge label of e. We say C is *symmetric* if $\varphi(e_1)\varphi(e_2)\ldots\varphi(e_a) = \varphi(e'_1)\varphi(e'_2)\ldots\varphi(e'_b)$. Note that tight cuffs with loops are not symmetric.

Proposition 7.2.1. *If $n \geq 5$ and $\mathcal{T}_0$ denotes the set of tight symmetric $(n+1)$-cuffs in K_n^Γ, then $|\mathcal{T}_0| \geq \frac{1}{8}n!(n-4)q^{n-1}(q-1)$.*

Proof. Let $M = DG(n,\Gamma)$. Define a function $\Phi : \mathcal{T}_0 \to 2^{E(M)^{n+1}}$ as follows. For each $C \in \mathcal{T}_0$, let

$$\Phi(C) = \{(e_1, e_2, \ldots, e_{n-1}, f_1, f_2) : e_1e_2\ldots e_{n-1} \text{ is a Hamiltonian path of } C \text{ and } f_1, f_2 \in V(C) \setminus \{e_1, \ldots, e_{n-1}\} \text{ each have an end with degree at least 3 in } C\}.$$

Each cuff is determined by its edges, so each tuple in $E(M)^{n+1}$ appears in the Φ-image of at most one cuff. For $\mathcal{A} \subseteq \mathcal{T}_0$, let $\Phi(\mathcal{A}) = \bigcup_{C\in\mathcal{A}} \Phi(C)$. Observe that $|\Phi(\mathcal{T}_0)|$ is the number of tuples $(e_1, e_2, \ldots, e_{n-1}, f_1, f_2)$ in $E(M)^{n+1}$ where $K_n^\Gamma[\{e_1, \ldots, e_{n-1}\}]$ is a Hamiltonian path of some tight symmetric $(n+1)$-cuff C.

Claim 7.2.1.1. $|\Phi(\mathcal{T}_0)| \geq 2n!q^{n-1}(n-4)(q-1)$.

Proof. Consider a tuple $T = (e_1, \ldots, e_{n-1}, f_1, f_2)$ in $\Phi(\mathcal{T}_0)$. Thus, there exists a cuff $C \in \mathcal{T}_0$ such that $T \in \Phi(C)$. Since there are $n!q^{n-1}$ paths of length $n-1$ in K_n^Γ where one endpoint of the path is marked, there are $n!q^{n-1}$ choices for the elements $e_1, \ldots, e_{n-1}$.

Since C is a tight cuff, one of f_1, f_2 is incident with e_1, the other is incident with e_{n-1}, and both are incident with a vertex u of degree 4. If u is chosen to be a vertex in C that is not an endpoint of e_1 or e_{n-1}, then there are $n-4$ choices for u and $q-1$ choices for the edge label of f_1. Since C is symmetric, there is then one choice for the edge label of f_2. Since there are two ways to order f_1, f_2, there are at least $2(n-4)(q-1)$ choices for the elements e_n, e_{n+1}. □

Claim 7.2.1.2. *If $C \in \mathcal{T}_0$, then $|\Phi(C)| = 16$.*

Proof. Since C has no loops, there exist edges e_1, e_2 in a cycle D of C and edges e_3, e_4 in another cycle D' of C where each of e_1, e_2, e_3, e_4 is incident with the vertex of degree 4 in C. Observe that, for $i \in \{1,2\}$ and $j \in \{3,4\}$, the graph $C \setminus \{e_i, e_j\}$ is a Hamiltonian path of C, and there are no other Hamiltonian paths of C. There are four choices for i and j, two ways to order e_i and e_j, and two ways to order the Hamiltonian path $C \setminus \{e_i, e_j\}$; hence $|\Phi(C)| = 16$. □

Since each tuple is in the Φ-image of at most one cuff, it follows from claims 7.2.1.1 and 7.2.1.2 that

$$|\mathcal{T}_0| \geq \frac{|\Phi(\mathcal{T}_0)|}{\max_{C\in\mathcal{T}_0}|\Phi(C)|} \geq \frac{1}{8}n!q^{n-1}(n-4)(q-1). \qquad \square$$

Lemma 7.2.2. *The set $\mathcal{T}_0$ is a stable set of $\Omega(DG(n,\Gamma))$.*

Proof. Suppose towards a contradiction that cuffs $C_1, C_2 \in \mathcal{T}_0$ are adjacent. For each $i \in [2]$, let u_i be the vertex of degree 4 and let D_i, D_i' be the cycles in C_i. By Proposition 2.4.2, the edge sets of C_1 and C_2 differ in exactly one element. For each $i \in [2]$, let e_i be the edge in C_i that is not in C_{3-i}. Without loss of generality, say e_i is in D_i. Therefore, the cycles D_1, D_2 differ in exactly one edge and $D_1' = D_2'$. This means that $u_1 = u_2$. For each $i \in [2]$, let w_i be the minimum neighbour of u_i in D_i and let γ_i be the product of edge labels $\varphi(e)$ of edges e in D_i, following the cycle, beginning with the edge u_iw_i and ending with the other edge incident with u_i.

Note that the degree sequence of C_1 and C_2 is $4, 2, 2, \ldots, 2$. If the endpoints of e_1 both have degree 2, then e_2 has the same endpoints as e_1. If one endpoint of e_1 has degree 4, then $C_1 \setminus \{e_1\}$ has degree sequence $3, 2, \ldots, 2, 1$, where the vertices of degree 3 and degree 1 are incident with e_1 in C_1. Since $C_1 \setminus \{e_1\}$ is a subgraph of C_2 and C_2 has degree sequence $4, 2, 2, \ldots, 2$, the endpoints of e_2 are also the vertices with degree 3 and degree 1 in $C_1 \setminus \{e_1\}$. That is, the edges e_1 and e_2 have the same endpoints. Since e_1 and e_2 have the same endpoints, the cycles D_1 and D_2 contain the same vertices, in the same cyclic order. In particular, this means $w_1 = w_2$. Thus, the product of edges labels of D_1 is different than the product of the edge labels of D_2. That is, $\gamma_1 \neq \gamma_2$, which is a contradiction. $\square$

Corollary 7.2.3. $\log i(\Omega(DG(n,\Gamma))) \geq \frac{1}{8}n!(n-4)q^{n-1}(q-1)$.

Proof. By Lemma 7.2.2, the set $\mathcal{T}_0$ of cuffs is a stable set in $\Omega(DG(n,\Gamma))$. Since every subset of a stable set is itself a stable set, the result follow from Proposition 7.2.1. $\square$

7.3 Upper bound

In this section, we determine an upper bound for the number of stable sets in the circuit graph of $DG(n,\Gamma)$. This is done by comparing certain subsets of the circuit graph with Hamming graphs.

Lemma 7.3.1. *Let $n \geq 3$ be an integer and let $t \in [0, \lfloor n/2 \rfloor]$. Let $\mathcal{C}_t$ denote the collection of $(n+1-t)$-cuffs of K_n^Γ. If $\Omega_t(n,\Gamma)$ denotes the subgraph of $\Omega(DG(n,\Gamma))$ induced on $\mathcal{C}_t$, then $\log i(\Omega_t(n,\Gamma)) \leq \frac{1}{8t!} n!(n-t)q^{n-t-1}(q-1)\log(n-t)(1+o(1))$.*

Proof. Let $\mathcal{C}$ denote the set of $(n+1-t)$-cuffs C of K_n^Γ where each cycle in C has length at least 3. That is, cuffs in $\mathcal{C}$ have no loops or parallel edges. Let Ω_t denote the subgraph of $\Omega_t(n,\Gamma)$ induced on $\mathcal{C}$.

Recall that $\mathcal{S}([n])$ denotes the set of permutations of $[n]$. Define a function $\Phi : V(\Omega_t) \to 2^{\mathcal{S}([n])}$ as follows. For each $C \in V(\Omega_t)$, let

$$\begin{aligned}\Phi(C) = \{&\sigma \in \mathcal{S}_n : \sigma(1)\sigma(2)\ldots\sigma(n-t) \text{ is a Hamiltonian path of } C \\ &\text{and } \sigma(1), \sigma(n-t) \text{ are adjacent to a vertex with degree at least 3 in } C\}.\end{aligned}$$

Claim 7.3.1.1. *For each $C \in V(\Omega_t)$, the size of $\Phi(C)$ is at least $8t!$.*

Proof. First, consider a loose or tight cuff C. Since all cycles in C have length at least 3, there are exactly four vertices x_1, x_2, y_1, y_2 in the cycles of C that are adjacent to the vertex or vertices with degree at least 3. Let x_1, x_2 be in one cycle of C and let y_1, y_2 be in the other. For each $w \in \{x, y\}$ and $i \in [2]$, let $P_{w_i}^{w_{3-i}}$ be the path from w_i to w_{3-i} that does not contain a vertex with degree at least 3 in C. Let P be the path between vertices of degree 3 if C is a loose cuff or the vertex of degree 4 if C is a tight cuff. For each $w \in \{x, y\}$ and $i, j \in [2]$, if $v \in \{x, y\} \setminus \{w\}$, then $P_{w_i}^{w_{3-i}}\ P\ P_{v_j}^{v_{3-j}}$ is a Hamiltonian path of C where w_i and v_{3-j} are adjacent to a vertex with degree at least 3. There are eight choices for w, i, and j, and $t!$ ways to order the elements in $[n] \setminus V(C)$, so $|\Phi(C)| \geq 8t!$.

The remaining case to consider is a hinged cuff. Let C be a hinged cuff and let u_1, u_2 be the vertices of degree 3 in C. Since all cycles in C have length at least 3, at most one (u_1, u_2)-path in C has length 1. Let P be the shortest (u_1, u_2)-path in C. For each $i \in [2]$, let x_i be the neighbour of u_i in one (u_1, u_2)-path that is not P, and let y_i be the neighbour of u_i in the other (u_1, u_2)-path that is not P. Note that for $w \in \{x, y\}$, we may have $w_1 = w_2$. For each $w \in \{x, y\}$ and $i \in [2]$, let $P_{w_i}^{w_{3-i}}$ be the path from w_i to w_{3-i} that does not contain u_1 or u_2. For each $w \in \{x, y\}$ and $i \in [2]$, if $v \in \{x, y\} \setminus \{w\}$, then $P_{w_i}^{w_{3-i}}\ u_{3-i} P u_i\ P_{v_i}^{v_{3-i}}$ is a Hamilton path of C where w_i and v_{3-i} are adjacent to a vertex with degree at least 3 in C. Also, for $w \in \{x, y\}$ and $i \in [2]$, if $v \in \{x, y\} \setminus \{w\}$, then $P_{w_i}^{w_{3-i}}\ u_{3-i}\ P_{v_{3-i}}^{v_i}\ u_i(P - u_{3-i})$ is a Hamiltonian path of C where the first and last vertices of the path are adjacent to a vertex with degree at least 3 in C. There are four choices for w and i, two distinct ways to construct a desired Hamiltonian path, and $t!$ ways to order the elements in $[n] \setminus V(C)$, so $|\Phi(C)| \geq 8t!$. $\square$

We use sets of vertices in Ω_t that contain a certain permutation in their Φ-image to find subgraphs of Ω_t that are similar to Hamming graphs. In fact, we show that subgraphs of Ω_t are isomorphic to induced subgraphs of a Hamming graph, which helps us determine an upper bound on the number of stable sets in Ω_t. For each permutation $\sigma \in \mathcal{S}([n])$, define a set $H_\sigma = \{C \in V(\Omega_t) : \sigma \in \Phi(C)\}$. By Claim 7.3.1.1, each vertex in Ω_t is in at least $8t!$ of these sets.

Let σ be a permutation in $\mathcal{S}([n])$.

Claim 7.3.1.2. *There exists a set of edges $E \subseteq E(\Omega_t[H_\sigma])$ such that $\Omega_t[H_\sigma] \setminus E$ is isomorphic to an induced subgraph of $H(\underbrace{q, \ldots, q}_{n-1-t}, (n-t)(q-1), (n-t)(q-1))$.*

Proof. Recall that 1 is the identity of the group Γ. For each $i \in [n-1-t]$, let $S_i = \Gamma$ and for $j \in \{n-t, n+1-t\}$, let $S_j = \{(\gamma, k) : \gamma \in \Gamma \setminus \{1\}, k \in [n-t]\}$. Let $H = H(\underbrace{q, \ldots, q}_{n-1-t}, (n-t)(q-1), (n-t)(q-1))$ be the Hamming graph whose vertices are $(n+1-t)$-tuples in $S_1 \times S_2 \times \cdots \times S_{n+1-t}$, where two vertices are adjacent if and only if they differ in exactly one coordinate.

We define a function $\phi : H_\sigma \to V(H)$, as follows. Let C be a cuff in $\Omega_t[H_\sigma]$. Thus, the sequence $\sigma(1)\sigma(2)\ldots\sigma(n-t)$ is a Hamiltonian path of C and $\sigma(1), \sigma(n-t)$ are adjacent to a vertex with degree at least 3. Since C contains no loops or parallel edges, each edge can be identified by two distinct vertices in C. For each $i \in [n-1-t]$, let $(\gamma_i, \{\sigma(i), \sigma(i+1)\})$ be the edge with endpoints $\sigma(i), \sigma(i+1)$ in C. Let $k_1, k_2 \in [n-t]$ and $\delta_1, \delta_2 \in \Gamma$ such that $(\delta_1, \{\sigma(1), \sigma(k_1)\}), (\delta_2, \{\sigma(n-1-t), \sigma(k_2)\})$ are the two edges in $E(C) \setminus \{(\gamma_i, \{\sigma(i), \sigma(i+1)\}) : i \in [n-t-1]\}$. Let γ_{n-t} and γ_{n+1-t} be defined, as follows:

$$\gamma_{n-t} = \prod_{j=1}^{k_1-1} \psi((\gamma_j, \sigma(j), \sigma(j+1))) \cdot \psi((\delta_1, \sigma(k_1), \sigma(1))) \text{ and}$$
$$\gamma_{n+1-t} = \prod_{j=k_2}^{n-1-t} \psi((\gamma_j, \sigma(j), \sigma(j+1))) \cdot \psi((\delta_2, \sigma(n-1-t), \sigma(k_2))).$$

Now, we define
$$\phi(C) = (\gamma_1, \gamma_2, \ldots, \gamma_{n-1-t}, (\gamma_{n-t}, k_1), (\gamma_{n+1-t}, k_2)).$$

Since the cycles $\sigma(1)\sigma(2)\ldots\sigma(k_1)\sigma(1)$ and $\sigma(k_2)\sigma(k_2+1)\ldots\sigma(n-1-t)\sigma(k_2)$ are unbalanced, we have $\gamma_{n-t}, \gamma_{n+1-t} \neq 1$. Therefore, the tuple $\phi(C)$ is a vertex of H. Furthermore, two cuffs differ in at least one edge, so ϕ is injective.

Consider $C_1, C_2 \in H_\sigma$. We claim that if $\phi(C_1)$ and $\phi(C_2)$ are adjacent in H, then C_1 and C_2 are adjacent in $\Omega_t[H_\sigma]$. Since C_1 and C_2 are both in H_σ, they both have vertex set $V = \{\sigma(1), \sigma(2), \ldots, \sigma(n-t)\}$. The graph C_1 is a Hamiltonian path of $K_n^\Gamma[V]$ and two additional edges. Since $\phi(C_1)$ and $\phi(C_2)$ are adjacent, they differ in exactly one coordinate, so C_2 contains one edge that is not in C_1. Therefore, the graph $C_1 \cup C_2$ is a Hamiltonian path of $K_n^\Gamma[V]$ and three additional edges. Note that each pair of edges in $C_1 \cup C_2$ is in a cycle or cuff in $C_1 \cup C_2$. Since $C_1 \cup C_2$ contains at least one unbalanced cycle, the rank of $E(C_1 \cup C_2)$ in $DG(n, \Gamma)$ is n. Thus, the corank of $DG(n, \Gamma)$ restricted to $E(C_1 \cup C_2)$ is $|E(C_1 \cup C_2)| - r(DG(n, \Gamma)|E(C_1 \cup C_2)| = n + 2 - n = 2$. That is, the circuits C_1, C_2 are contained in a connected corank-2 restriction of $DG(n, \Gamma)$. Thus, it follows that C_1 and C_2 are adjacent in $\Omega_t[H_\sigma]$.

Since $\phi(C_1)\phi(C_2) \in E(H)$ implies $C_1C_2 \in E(\Omega_t[H_\sigma])$, there exists a set $E \subseteq E(\Omega_t[H_\sigma])$ such that C_1, C_2 are adjacent in $\Omega_t[H_\sigma] \setminus E$ if and only if $\phi(C_1)$ and $\phi(C_2)$ are adjacent in H. That is, the graph $\Omega_t[H_\sigma] \setminus E$ is isomorphic to an induced subgraph of H. □

Claim 7.3.1.3. $\log i(\Omega_t[H_\sigma]) \leq q^{n-t-1}(n-t)(q-1)\log((n-t)(q-1))$.

Proof. By Claim 7.3.1.2 and Propositions 2.3.5 and 2.3.4, we have

$$i(\Omega_t[H_\sigma]) \leq i(H(\underbrace{q, \ldots, q}_{n-1-t}, (n-t)(q-1), (n-t)(q-1))).$$

By Theorem 4.3.2, it follows that $\log i(\Omega_t[H_\sigma]) \leq q^{n-t-1}(n-t)(q-1)\log((n-t)(q-1))$. □

Now, we use Shearer's Lemma (Lemma 2.2.3) to find an upper bound for $\log i(\Omega_t)$. Let $\mathcal{F} = \{H_\sigma : \sigma \in \mathcal{S}([n])\}$ and let $\mathcal{A}$ denote the collection of stable sets of Ω_t. Observe that $|\mathcal{F}| = n!$ and $|\mathcal{A}| = i(\Omega_t)$. By Claim 7.3.1.1, each vertex in Ω_t is in at least $8t!$ sets in $\mathcal{F}$. For each $\sigma \in \mathcal{S}([n])$, the intersection of H_σ and a stable set $A \in \mathcal{A}$ is a stable set in the graph $\Omega_t[H_\sigma]$, so $|\{H_\sigma \cap A : A \in \mathcal{A}\}| = i(\Omega_t[H_\sigma])$. Thus, by Shearer's Lemma 2.2.3,

$$i(\Omega_t) \leq \prod_{F \in \mathcal{F}} i(\Omega_t[H_\sigma])^{\frac{1}{8t!}} = i(\Omega_t[H_\sigma])^{\frac{n!}{8t!}}.$$

Taking the base-2 logarithm of both sides gives $\log i(\Omega_t) \leq \frac{1}{8t!} n! \log i(\Omega_t[H_\sigma])$. By Claim 7.3.1.3, it follows that $\log i(\Omega_t) \leq \frac{1}{8t!} n!(n-t)q^{n-t-1}(q-1)\log((n-t)(q-1))$.

By Proposition 2.3.8,

$$\begin{aligned} \log i(\Omega_t(n, \Gamma)) &\leq \log i(\Omega_t) + |V(\Omega_t(n, \Gamma)) \setminus \mathcal{C}| \\ &\leq \frac{1}{8t!} n!(n-t)q^{n-t-1}(q-1)\log((n-t)(q-1)) + |V(\Omega_t) \setminus \mathcal{C}|. \end{aligned}$$

The elements of $V(\Omega_t(n,\Gamma)) \setminus \mathcal{C}$ are precisely the $(n+1-t)$-cuffs with a cycle of length at most 2 and the balanced $(n+1-t)$-cycles. By Propositions 7.1.3 and 7.1.2,

$$\begin{aligned}|V(\Omega_t(n,\Gamma)) \setminus \mathcal{C}| &\le \frac{n!}{t!}q^{n-t-1}(n-t-1)(q-1)^2 + \frac{n!}{(t-1)!(n+1-t)}q^{n-t}\\ &\le \frac{n!}{t!}(n-t)q^{n-t-1}(q-1)\log(n-t)\left(\frac{q-1}{\log(n-t)} + \frac{qt}{(n-t)^2(q-1)^2\log(n-t)}\right).\end{aligned}$$

Therefore, since $t \le n/2$,

$$\log i(\Omega_t(n,\Gamma)) \le \frac{1}{8t!}n!(n-t)q^{n-t-1}(q-1)\log(n-t)(1+o(1)). \qquad \square$$

Lemma 7.3.2. $\log i(\Omega(DG(n,\Gamma))) \le \frac{\sqrt[q]{e}}{8}(n+1)!q^{n-1}(q-1)\log(n)(1+o(1))$.

Proof. For each $t \in [0, n-1]$, let $\mathcal{C}_t$ denote the set of $(n+1-t)$-circuits in $DG(n,\Gamma)$. By Proposition 2.3.7 and Lemmas 7.3.1 and 7.1.4,

$$\begin{aligned}\log i(\Omega(DG(n,\Gamma))) &\le \sum_{t=0}^{n/2} \log i(\Omega_t(n,\Gamma)) + \sum_{t=n/2}^{n-1} |\mathcal{C}_t|\\ &\le \sum_{t=0}^{n-1} \frac{n!}{8t!}(n-t)q^{n-t-1}(q-1)\log((n-t)(q-1))(1+o(1))\\ &\qquad + o((n+1)!q^{n-1})\\ &\le \frac{1}{8}(n+1)!q^{n-1}(q-1)\log(n)\sum_{t=0}^{n-1}\frac{1}{t!q^t}(1+o(1))\\ &\le \frac{\sqrt[q]{e}}{8}(n+1)!q^{n-1}(q-1)\log(n)(1+o(1)). \qquad \square\end{aligned}$$

7.4 The main theorem

Finally, we are ready to prove the main theorem of this chapter.

Theorem 7.0.1.

$$\frac{1}{8}n!(n-4)q^{n-1}(q-1) \le \log(\mathrm{coext}(DG(n,\Gamma))) \le \frac{\sqrt[q]{\mathrm{e}}}{8}(n+1)!q^{n-1}(q-1)\log(n)(1+o(1)).$$

Proof. Since $DG(n,\Gamma)$ is the frame matroid of $(K_n^\Gamma, \mathcal{B})$, Lemma 2.5.1 implies that there are at most 6 circuits in a corank-2 restriction of $DG(n,\Gamma)$. Now it follows from Proposition 2.4.3 that the smallest circuit in a corank-2 restriction of $DG(n,\Gamma)$ has size less than $\frac{5}{6}(n+2)$. Therefore, the number of small circuits in $DG(n,\Gamma)$ is at most the number of circuits in $DG(n,\Gamma)$ with size at most $\frac{5}{6}(n+2)$. Thus, by Lemma 7.1.4, the number of small circuits in $DG(n,\Gamma)$ is $o((n+1)!q^{n-1})$. Now, by Corollary 3.2.6 and Lemma 7.3.2,

$$\log \operatorname{coext}(DG(n,\Gamma)) \leq \frac{\sqrt[q]{\mathrm{e}}}{8}(n+1)!q^{n-1}(q-1)\log(n)(1+o(1)) + o((n+1)!q^{n-1}).$$

and the upper bound in the theorem follows. The lower bound follows from Corollary 3.2.6 and Corollary 7.2.3. □

Chapter 8

Coextensions of projective geometries

Recall that, for a positive integer n and a prime power q, a rank-n *projective geometry* over the finite field $GF(q)$ is denoted $PG(n-1,q)$. In this chapter, we prove the following lower and upper bounds on the number of coextensions of $PG(n-1,q)$, which is denoted $\mathrm{coext}(PG(n-1,q))$. Recall that $o(1)$ denotes an unspecified function of n which goes to 0 as n goes to infinity and log denotes the base-2 logarithm. For each real number x, let ν_x denote the shifted factorial $(x^{-1};x^{-1})_\infty$, which is equal to the infinite product $\prod_{k=1}^{\infty}(1-x^{-k})$. Note that ν_x is a constant less than 1 if $x>1$.

Theorem 8.0.1.

$$\begin{aligned}\frac{q^{n^2}}{(n+1)!(q-1)^n}&(\nu_q \log(q-1)+o(1))\\ &\le \log(\mathrm{coext}(PG(n-1,q)))\\ &\qquad\le \frac{q^{n^2}}{n!(q-1)^n}(\nu_q \log^2(q-1)+o(1)).\end{aligned}$$

The lower and upper bounds in Theorem 8.0.1 are very similar, but they differ by a factor of $n\log(q-1)$, which is linear in n. If we instead asymptotically bound the double log of $\mathrm{coext}(PG(n-1,q))$, then this linear factor becomes a term in the function represented by $o(1)$. This is shown in the following corollary, which is directly implied by Theorem 8.0.1.

Corollary 8.0.2.

$$\log(\log(\mathrm{coext}(PG(n-1,q)) = n^2(\log(q) + o(1)).$$

In Corollary 8.0.2, we get an asymptotically exact value for $\log\log(\mathrm{coext}(PG(n-1,q))$, up to lower order terms. The $o(1)$ term here seems to hide a lot of information about the number of coextensions of a projective geometry; however, since this number is so large, Corollary 8.0.2 is interesting to observe.

The lower bound in Theorem 8.0.1 is determined by a straightforward greedy argument, so it seems likely that it can be improved.

Conjecture 8.0.3. *For some fixed real number c,*

$$\log(\mathrm{coext}(PG(n-1,q))) = \frac{q^{n^2}}{n!(q-1)^n}(c + o(1)).$$

In this chapter, we also consider the special case of binary projective geometries $PG(n-1,2)$. In this case, we find the asymptotic number of coextensions of $PG(n-1,2)$ on the log scale, which is given in the following theorem.

Theorem 8.0.4.

$$\log(\mathrm{coext}(PG(n-1,2))) = \frac{2^{n^2}}{(n+1)!}(\nu_2 + o(1)).$$

This chapter begins with preliminary results in Section 8.1. We prove preliminary results about shifted factorials, collections of circuits, and the circuit graph. In this chapter, we define $\mathcal{C}_t(n,q)$ to be the set of $(n+1-t)$-circuits of $PG(n-1,q)$. Note that $\mathcal{C}_0(n,q)$ is the set of spanning circuits of $PG(n-1,q)$. In order to prove the main results of this chapter, we make use Corollary 3.2.6 and bounds on the number of stable sets in the circuit graph of $PG(n-1,q)$. Recall the circuit graph of $PG(n-1,q)$, denoted $\Omega(PG(n-1,q))$, is the graph with vertex set $\mathcal{C}(PG(n-1,q))$ where two circuits C, C' are adjacent if and only if they are contained in a corank-2 restriction of $PG(n-1,q)$. Of particular use is the subgraph of the circuit graph induced on the set of spanning circuits of $PG(n-1,q)$. Therefore, we define $\Omega_0(n,q)$ to be the induced subgraph $\Omega(PG(n-1,q))[\mathcal{C}_0(n,q)]$. In Sections 8.2 and 8.3, we prove lower and upper bounds for $i(\Omega_0(n,q))$, the number of stable sets in the subgraph of the circuit graph of $PG(n-1,q)$ induced on the set of spanning circuits.

8.1 Preliminaries

Shifted factorials

Recall that the *p-shifted factorial*, denoted $(a;p)_n$, is the product $\prod_{k=0}^{n-1}(1-ap^k)$, where $(a;p)_0 = 1$ and $(a;p)_\infty$ is the infinite product $\prod_{k=0}^{\infty}(1-ap^k)$. For each real number q, the shifted factorial $(q^{-1};q^{-1})_\infty$, which is equal to the infinite product $\prod_{k=1}^{\infty}(1-q^{-k})$, is denoted ν_q. Note that ν_q is a constant less than 1 if $q > 1$.

Proposition 8.1.1. $(q^{-1};q^{-1})_n = \nu_q(1+o(1))$.

Proof.

$$(q^{-1};q^{-1})_n = (q^{-1};q^{-1})_\infty \cdot \frac{1}{\prod_{k=n+1}^{\infty}\left(1-\frac{1}{q^k}\right)} = \nu_q \prod_{k=n+1}^{\infty} \frac{q^k}{q^k-1} = \nu_q(1+o(1)). \qquad \square$$

Proposition 8.1.2. *The value of $(q^{-1};q^{-1})_k$ is at least $\frac{1}{2}$, for all integers $k \geq 1$ and $q \geq 3$.*

Proof.

$$\begin{aligned}(q^{-1};q^{-1})_k &= \prod_{i\geq 1}\left(1-\frac{1}{q^i}\right) \cdot \frac{1}{\prod_{i=k+1}^{\infty}\left(1-\frac{1}{q^k}\right)} \\ &\geq \prod_{i\geq 1}\left(1-\frac{1}{q^i}\right) \geq 1 - \sum_{i\geq 1}\frac{1}{q^i} \geq 1 - \frac{1}{q-1} = \frac{q-2}{q-1} \geq \frac{1}{2}. \qquad \square\end{aligned}$$

Circuit bounds

Proposition 8.1.3. *There are at most $q+1$ circuits in a corank-2 restriction of $PG(n-1,q)$.*

Proof. Let $M = PG(n-1,q)$ and let X be a set of elements of M such that $r^*(M|X) = 2$. Let $Y = E(M) \setminus X$ and observe that $(M|X)^* = M^*/Y$. Thus, since M is $GF(q)$-representable, the matroid M^*/Y is a rank-2 $GF(q)$-representable matroid. By Proposition 2.6.1, the matroid M^*/Y has at most $\begin{bmatrix}2\\1\end{bmatrix}_q = \frac{q^2-1}{q-1} = q+1$ hyperplanes. Since the hyperplanes of M^*/Y correspond to the circuits in $M|X$, the dual of M^*/Y, there are at most $q+1$ circuits in $M|X$. $\square$

In order to bound the number of coextensions of a projective geometry, we use the number of certain circuits. Proposition 2.6.1 immediately implies the following corollary.

Corollary 8.1.4. *For $t \in [0, n-2]$, the number of $(n+1-t)$-circuits in $PG(n-1, q)$ is*

$$\frac{q^{n^2-nt}}{(n+1-t)!(q-1)} \cdot \frac{(q^{-1};q^{-1})_n}{(q^{-1};q^{-1})_t}.$$

The following proposition gives an upper bound on the number of non-spanning circuits in $PG(n-1, q)$.

Proposition 8.1.5. $\sum_{t=1}^{n-2} |\mathcal{C}_t(n,q)| \leq \frac{q^{n^2}}{q-1}\left(\frac{1}{q^n(n+1-\sqrt{n})!} + \frac{n}{q^{n\sqrt{n}}}\right)$.

Proof. By Corollary 8.1.4,

$$\begin{aligned}
\sum_{t=1}^{n-2} |\mathcal{C}_t(n,q)| &= \sum_{t=1}^{n-2} \frac{q^{n^2-nt}}{(n+1-t)!(q-1)} \frac{(q^{-1};q^{-1})_n}{(q^{-1};q^{-1})_t} \\
&\leq \frac{q^{n^2}}{q-1} \sum_{t=1}^{n-2} \frac{1}{q^{nt}(n+1-t)!} \qquad (8.1)\\
&\leq \frac{q^{n^2}}{q-1}\left(\sum_{t=1}^{\sqrt{n}-1} \frac{1}{q^{nt}(n+1-t)!} + \sum_{t=\sqrt{n}}^{n-2} \frac{1}{q^{nt}(n+1-t)!}\right).
\end{aligned}$$

For all positive integers i, j where $i \leq j$, we have $q^{ni} \leq q^{nj}$ and $(n+1-i)! \geq (n+1-j)!$. Therefore,

$$\begin{aligned}
\sum_{t=1}^{\sqrt{n}-1} \frac{1}{q^{nt}(n+1-t)!} &\leq \sum_{t=1}^{\sqrt{n}-1} \frac{1}{q^n(n+2-\sqrt{n})!} \\
&\leq \frac{\sqrt{n}}{q^n(n+2-\sqrt{n})!} \leq \frac{1}{q^n(n+1-\sqrt{n})!},
\end{aligned} \qquad (8.2)$$

and

$$\begin{aligned}
\sum_{t=\sqrt{n}}^{n-2} \frac{1}{q^{nt}(n+1-t)!} &\leq \sum_{t=\sqrt{n}}^{n-2} \frac{1}{q^{n\sqrt{n}}(n+1-(n-2))!} \\
&\leq \frac{n-\sqrt{n}}{6 \cdot q^{n\sqrt{n}}} \leq \frac{n}{q^{n\sqrt{n}}}.
\end{aligned} \qquad (8.3)$$

Applying the bounds found in Equations 8.2 and 8.3 to Equation 8.1 gives $\sum_{t=1}^{n-2} |\mathcal{C}_t(n,q)| \leq \frac{q^{n^2}}{q-1}\Big(\frac{1}{q^n(n+1-\sqrt{n})!} + \frac{n}{q^{n\sqrt{n}}}\Big)$. □

Corollary 8.1.6. $\sum_{t=1}^{n-2} |\mathcal{C}_t(n,q)| = o\Big(\frac{q^{n^2}}{(n+1)!(q-1)^n}\Big)$.

Proof. By Proposition 8.1.5,

$$\begin{aligned}\sum_{t=1}^{n-2} |\mathcal{C}_t(n,q)| &\leq \frac{q^{n^2}}{q-1}\left(\frac{1}{q^n(n+1-\sqrt{n})!} + \frac{n}{q^{n\sqrt{n}}}\right)\\ &\leq \frac{q^{n^2}}{(n+1)!(q-1)^n}\left(\frac{(n+1)!(q-1)^{n-1}}{q^n(n+1-\sqrt{n})!} + \frac{(n+2)!(q-1)^{n-1}}{q^{n\sqrt{n}}}\right)\\ &\leq \frac{q^{n^2}}{(n+1)!(q-1)^n}\left(\frac{(n+1)^{\sqrt{n}}(q-1)^{n-1}}{q^n} + \frac{(n+2)!(q-1)^{n-1}}{q^{n\sqrt{n}}}\right)\\ &= \frac{q^{n^2}}{(n+1)!(q-1)^n} \cdot o(1).\end{aligned}$$

□

Circuit graph preliminaries

Proposition 8.1.7. *The degree of each vertex C in $\Omega_0(n,q)$ is $(n+1)((q-1)^{n-1}-1)$.*

Proof. In $\Omega_0(n,q)$, all vertices are spanning circuits of $PG(n-1,q)$ and two circuits are adjacent if and only if they are contained in a corank-2 restriction of $PG(n-1,q)$. By Proposition 2.4.2, two vertices C, C' in $\Omega_0(n,q)$ are adjacent if and only if $C \cap C'$ is a basis of $PG(n-1,q)$.

There are $n+1$ elements in C and $C \setminus \{e\}$ is a basis of $PG(n-1,q)$ for all elements e in C. By Lemma 2.6.3, for each basis B of $PG(n-1,q)$, there are precisely $(q-1)^{n-1}$ elements e' of $PG(n-1,q)$ such that $B \cup \{e'\}$ is a circuit. Therefore, there are precisely $(q-1)^{n-1}-1$ elements $e' \neq e$ of $PG(n-1,q)$ such that $(C \setminus e) \cup e'$ is a circuit, for each e in C. Thus, the degree of each vertex C in $\Omega_0(n,q)$ is at most $(n+1)((q-1)^{n-1}-1)$.

Suppose towards a contradiction that $(C \setminus \{e_1\}) \cup \{e_1'\}$ and $(C \setminus \{e_2\}) \cup \{e_2'\}$ are the same circuit, for some $e_1, e_2 \in C$ where $e_1 \neq e_2$ and $e_i' \neq e_i$ for each $i \in [2]$. Since $(C \setminus \{e_1\}) \cup \{e_1'\}$ contain the element e_2, it follows that $(C \setminus \{e_2\}) \cup \{e_2'\}$ contains e_2. Thus, the element $e_2' = e_2$, which is a contradiction. Therefore, the degree of each vertex C in $\Omega_0(n,q)$ is exactly $(n+1)((q-1)^{n-1}-1)$. □

Theorem 8.1.8. $\log i(\Omega_0(n,2)) = \frac{2^{n^2}}{(n+1)!}(\nu_2 + o(1))$.

Proof. By Proposition 8.1.7, the degree of each vertex in $\Omega_0(n,2)$ is $(n+1)((2-1)^{n-1}-1) = 0$, therefore every subset of vertices in $\Omega_0(n,2)$ is a stable set. That is, the number of stable sets in $\Omega_0(n,2)$ is $2^{|V(\Omega_0(n,2))|}$. There are $\frac{2^{n^2}}{(n+1)!}(\frac{1}{2};\frac{1}{2})_n$ vertices in $\Omega_0(n,2)$ by Corollary 8.1.4. Now the result follows from Proposition 8.1.1. $\square$

8.2 Lower bound

The following lemma gives a lower bound on the number of stable sets in $\Omega_0(n,q)$.

Lemma 8.2.1. $\log i(\Omega_0(n,q)) \geq \frac{q^{n^2}}{(n+1)!(q-1)^n}(\nu_q \log(q-1) + o(1))$.

Proof. The graph $\Omega_0(n,q)$ has $\frac{q^{n^2}\nu_q}{(n+1)!(q-1)}(1+o(1))$ vertices by Corollary 8.1.4 and Proposition 8.1.1. From Proposition 8.1.7, each vertex in $\Omega_0(n,q)$ has degree $(n+1)((q-1)^{n-1}-1)$. Therefore, by Theorem 2.3.9,

$$\begin{aligned}
\log i(\Omega_0(n,q)) &\geq \left\lfloor \frac{q^{n^2}\nu_q(1+o(1))}{(n+1)!(q-1)((n+1)(q-1)^{n-1}-n)} \right\rfloor \log((n+1)(q-1)^{n-1}-n) \\
&\geq \frac{q^{n^2}\nu_q}{(n+2)!(q-1)^n}(\log(n+1) + (n-1)\log(q-1))(1+o(1)) \\
&\geq \frac{q^{n^2}\nu_q}{(n+2)!(q-1)^n}(n-1)\log(q-1)(1+o(1)) \\
&\geq \frac{q^{n^2}\nu_q \log(q-1)}{(n+1)!(q-1)^n}(1+o(1)).
\end{aligned}$$

$\square$

8.3 Upper bound

In order to determine an upper bound for $i(\Omega_0(n,q))$, we use a spectral version of the container method for regular graphs, which depends on the smallest eigenvalue of $\Omega_0(n,q)$. Before we determine this eigenvalue, we need a few definitions and lemmas. For positive integers m and k, the *Johnson graph* $J(m,k)$ is the graph whose vertices are the k-subsets of a fixed m-set where two vertices are adjacent if and only if they intersect in $k-1$ elements. The smallest eigenvalue of $J(m,k)$ and its multiplicity is given on page 179 of the textbook [8] by Brouwer and Haemers.

Lemma 8.3.1 ([8]). *The smallest eigenvalue of $J(m,k)$ is*

$$\min_{0\leq i\leq n+1}(k-i)(m-k-i)-i$$

with multiplicity $\binom{m}{k}-\binom{m}{k-1}$.

Together with the previous lemma, we will use an interlacing result in Godsil and Royle's textbook [16] to determine the smallest eigenvalue of $\Omega_0(n,q)$.

Lemma 8.3.2 (Theorem 9.1.1 in [16]). *Let A be a real symmetric $n\times n$ matrix and let B be a principal submatrix of A with order $m\times m$. Let $\theta_1(A)\geq\theta_2(A)\geq\cdots\geq\theta_n(A)$ be the eigenvalues of A and let $\theta_1(B)\geq\theta_2(B)\geq\cdots\geq\theta_m(B)$ be the eigenvalues of B. Then, for $i=1,\ldots,m$,*

$$\theta_{n-m+i}(A)\leq\theta_i(B)\leq\theta_i(A).$$

Since adjacency matrices are real symmetric square matrices and the adjacency matrix of an induced subgraph H of a graph G is a principal submatrix of the adjacency matrix of G, Lemma 8.3.2 immediately implies the following corollary.

Corollary 8.3.3. *Let G be a graph with n vertices and let H be an induced subgraph of G with m vertices. Let $\theta_1(G)\geq\theta_2(G)\geq\cdots\geq\theta_n(G)$ be the eigenvalues of G and let $\theta_1(H)\geq\theta_2(H)\geq\cdots\geq\theta_m(H)$ be the eigenvalues of H. For $i\in\{1,\ldots,m\}$,*

$$\theta_{n-m+i}(G)\leq\theta_i(H)\leq\theta_i(G).$$

Now we are ready to determine the smallest eigenvalue of $\Omega_0(n,q)$, which is done in the following lemma.

Lemma 8.3.4. *If $n\geq 5$ and $q\geq 3$, then the smallest eigenvalue of $\Omega_0(n,q)$ is $-(n+1)$.*

Proof. The Johnson graph $J(\frac{q^n-1}{q-1},n+1)$ is the graph whose vertices are the $(n+1)$-subsets of a fixed set of $\frac{q^n-1}{q-1}$ elements where two vertices are adjacent if and only if they intersect in n elements. The $(n+1)$-circuits of $PG(n-1,q)$ are $(n+1)$-subsets of $E(PG(n-1,q))$ that are adjacent in $\Omega_0(n,q)$ if and only if they intersect in n elements, by Proposition 2.4.2. If we take the fixed set with $\frac{q^n-1}{q-1}$ elements to be $E(PG(n-1,q))$, then $\Omega_0(n,q)$ is an induced subgraph of $J(\frac{q^n-1}{q-1},n+1)$. Let $\mathcal{C}_0$ denote the set of $(n+1)$-circuits of $PG(n-1,q)$; that is, $\mathcal{C}_0$ is the vertex set of $\Omega_0(n,q)$.

Let $k = \frac{q^n-1}{q-1}$ and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ be the eigenvalues of $J(k, n+1)$. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{|\mathcal{C}_0|}$ be the eigenvalues of $\Omega_0(n, q)$. Corollary 8.3.3 gives the interlacing of the eigenvalues of an induced subgraph of some graph G with the eigenvalues of G, which implies that $\lambda_{|\mathcal{C}_0|} \geq \mu_{|\mathcal{C}_0|} \geq \lambda_k$.

By Lemma 8.3.1, the smallest eigenvalue of $J(k, n+1)$ is

$$\min_{0 \leq i \leq n+1} (n+1-i)(k-(n+1)-i) - i = -(n+1)$$

with multiplicity $\binom{k}{n+1} - \binom{k}{n}$. Therefore, the eigenvalue λ_i is equal to $-(n+1)$ for all $i \in \{\binom{k}{n}+1, \binom{k}{n}+2, \ldots, \binom{k}{n+1}\}$.

Claim 8.3.4.1. $|\mathcal{C}_0| > \binom{k}{n}$.

Proof. The binomial coefficient $\binom{k}{n}$ is equal to

$$\binom{\frac{q^n-1}{q-1}}{n} = \frac{\prod_{i=0}^{n-1}\left(\frac{q^n-1}{q-1} - i\right)}{n!} = \frac{\prod_{i=0}^{n-1}(q^n - 1 - i(q-1))}{n!(q-1)^n}.$$

Since $q^n - 1 - i(q-1) < q^n$, the value of $\binom{k}{n}$ is less than $\frac{q^{n^2}}{n!(q-1)^n}$. Therefore, the quotient $|\mathcal{C}_0|/\binom{k}{n}$ is greater than

$$\frac{\frac{q^{n^2}\nu_q}{(n+1)!(q-1)}}{\frac{q^{n^2}}{n!(q-1)^n}} = \frac{\nu_q(q-1)^{n-1}}{n+1}.$$

Since $\nu_q \geq \frac{1}{2}$ by Proposition 8.1.2 and $q \geq 3$, we have $|\mathcal{C}_0|/\binom{k}{n} > \frac{2^{n-2}}{n+1}$. Since $n \geq 5$, it follows that $\frac{2^{n-2}}{n+1} \geq 1$ and thus $|\mathcal{C}_0| > \binom{k}{n}$. $\square$

Since $\lambda_i = -(n+1)$ for all $i \in \{\binom{k}{n}+1, \binom{k}{n}+2, \ldots, \binom{k}{n+1}\}$, it follows from Claim 8.3.4.1 that $\lambda_{|\mathcal{C}_0|} = -(n+1)$. Therefore, the smallest eigenvalue $\mu_{|\mathcal{C}_0|}$ of $\Omega_0(n, q)$ is $-(n+1)$. $\square$

Now we are ready to establish an upper bound for $i(\Omega_0(n, q))$ using a spectral version of the container method, as follows.

Theorem 8.3.5. $\log i(\Omega_0(n, q)) \leq \frac{q^{n^2}}{n!(q-1)^n}(\nu_q \log^2(q-1) + o(1))$.

Proof. Let d be the degree of $\Omega_0(n,q)$ and let N be the number of vertices in $\Omega_0(n,q)$. Thus, by Proposition 8.1.7 and Corollary 8.1.4, we have $d = (n+1)((q-1)^{n-1}-1)$ and $N = \frac{q^{n^2}\nu_q}{(n+1)!(q-1)}(1+o(1))$. Let $-\lambda$ be the smallest eigenvalue of $\Omega_0(n,q)$. By Lemma 8.3.4, we have $\lambda = n+1$. Finally, let $\varepsilon = \frac{\log(d)-\lambda}{d+\lambda}$.

By Theorem 5.3.3,

$$i(\Omega_0(n,q)) \leq \sum_{i=0}^{\frac{N}{d+\lambda}\ln(\frac{d+\lambda}{\log(d)+\lambda})+d} \binom{N}{i} \cdot 2^{(\frac{\lambda}{d+\lambda}+\frac{\log(d)-\lambda}{d+\lambda})N}.$$

Using the standard bound $\sum_{i=0}^{k}\binom{m}{i} \leq \left(\frac{\mathrm{e}m}{k}\right)^k$ for positive integers $k \leq m$, we find that

$$\begin{aligned} i(\Omega_0(n,q)) &\leq \left(\frac{\mathrm{e}N(d+\lambda)}{N\ln(\frac{d+\lambda}{\log(d)+\lambda})+d(d+\lambda)}\right)^{\frac{N}{d+\lambda}\ln(\frac{d+\lambda}{\log(d)+\lambda})+d} \cdot 2^{\frac{\log(d)}{d+\lambda}N} \\ &\leq 2^{\frac{N}{d+\lambda}((\ln(\frac{d+\lambda}{\log(d)+\lambda})+d(d+\lambda)/N)\cdot\log(\mathrm{e}(d+\lambda)/\ln(\frac{d+\lambda}{\log(d)+\lambda}))+\log(d))}. \end{aligned}$$

Taking the base-2 logarithm of both sides, we have

$$\begin{aligned} \log i(\Omega_0(n,q)) &\leq \frac{N}{d+\lambda}\left(\left(\ln\left(\frac{d+\lambda}{\log(d)+\lambda}\right)+\frac{d(d+\lambda)}{N}\right)\log\left(\frac{\mathrm{e}(d+\lambda)}{\ln(\frac{d+\lambda}{\log(d)+\lambda})}\right)+\log(d)\right) \\ &\leq \frac{N}{d+\lambda}\left(\ln\left(\frac{d+\lambda}{\log(d)+\lambda}\right)\log\left(\frac{\mathrm{e}(d+\lambda)}{\ln(\frac{d+\lambda}{\log(d)+\lambda})}\right)+\log(d)\right)+d\log\left(\frac{\mathrm{e}(d+\lambda)}{\ln(\frac{d+\lambda}{\log(d)+\lambda})}\right). \end{aligned}$$

Since $d+\lambda = (n+1)(q-1)^{n-1}$, the base-2 logarithm of $d+\lambda$ is equal to $(n-1)\log(q-1)(1+o(1))$. Using this and the values of N, d, and λ in terms of q and n yields

$$\begin{aligned} \log i(\Omega_0(n,q)) &\leq \frac{q^{n^2}\nu_q}{(n+1)!(n+1)(q-1)^n}(n-1)^2\log^2(q-1)(1+o(1)) \\ &\leq \frac{q^{n^2}\nu_q\log^2(q-1)}{n!(q-1)^n}(1+o(1)). \end{aligned}$$

□

8.4 The main theorems

Before proving the two main theorems of this chapter, we prove the following lemma that is used in both. Recall that $\mathcal{C}_{\min}(M,\prec)$ is the collection of circuits C of M such that C is the minimum circuit with respect to a total ordering $\prec$ of $\mathcal{C}(M)$ in some corank-2 restriction of M.

Lemma 8.4.1. *For each positive integer n and prime power q where $n \geq q$,*

$$\log i(\Omega_0(n,q)) \leq \log(\text{coext}(PG(n-1,q))) \leq \log i(\Omega_0(n,q)) + 2\sum_{t=1}^{n-2} |\mathcal{C}_t(n,q)|.$$

Proof. In this proof, let $M = PG(n-1,q)$ and let $\prec$ be a total ordering of $\mathcal{C}(M)$ that refines the preorder by size. By Proposition 2.3.4, we have $i(\Omega_0(n,q)) \leq i(\Omega(M)$. By Proposition 2.3.8, it follows that $\log i(\Omega(M) \leq \log i(\Omega_0(n,q)) + \sum_{t=1}^{n-2} |\mathcal{C}_t(n,q)|$. Applying these bounds to Corollary 3.2.6, we find

$$\log i(\Omega_0(n,q)) \leq \log \text{coext}(M) \leq \log i(\Omega_0(n,q)) + \sum_{t=1}^{n-2} |\mathcal{C}_t(n,q)| + |\mathcal{C}_{\min}(M,\prec)|. \tag{8.4}$$

By Proposition 8.1.3, there are at most $q+1$ circuits in a corank-2 restriction of M. Thus, by Proposition 2.4.3, the smallest circuit in a corank-2 restriction of M has size at most $\frac{q}{q+1}(n+2) = n+1-\frac{n-q+1}{q+1}$. Since $n \geq q$, the smallest circuit in a corank-2 restriction of M has size less than $n+1$. Let $\mathcal{C}'$ be the collection of circuits with size less than $n+1$. Therefore, $|\mathcal{C}_{\min}(M,\prec)| \leq |\mathcal{C}'|$. Since $\mathcal{C}'$ does not contain any $(n+1)$-circuits,

$$|\mathcal{C}_{\min}(M,\prec)| \leq |\mathcal{C}'| \leq \sum_{t=1}^{n-2} |\mathcal{C}_t(n,q)|.$$

Combining this bound on the number of small circuits with Equation 8.4 completes the proof. □

We are now ready to prove the main theorems of this chapter. First, we prove the following result about binary projective geometries.

Theorem 8.0.4.

$$\log(\text{coext}(PG(n-1,2))) = \frac{2^{n^2}}{(n+1)!}(\nu_2 + o(1)).$$

Proof. Let $q = 2$ and assume $n \geq 2$. The result follows from Lemma 8.4.1, Theorem 8.1.8, and Corollary 8.1.6. □

Next, we prove the main theorem about projective geometries over fields of order at least 3.

Theorem 8.0.1.

$$\begin{aligned}\frac{q^{n^2}}{(n+1)!(q-1)^n}(\nu_q \log(q-1) + o(1)) \\ \leq \log(\operatorname{coext}(PG(n-1,q))) \\ \leq \frac{q^{n^2}}{n!(q-1)^n}(\nu_q \log^2(q-1) + o(1)).\end{aligned}$$

Proof. Since q is fixed, we may assume $n \geq q$. The result follows from Lemma 8.4.1, Theorem 8.3.5, and Corollary 8.1.6. □

Chapter 9

Extensions of representable matroids

In this chapter, we consider extensions of certain $GF(q)$-representable matroids. Since all simple $GF(q)$-representable matroids are restrictions of $PG(n-1,q)$, determining the number of extensions of a projective geometry sounds like a good place to start. However, it is well known that the extensions of a projective geometry correspond to its flats, so the extensions are easily enumerated. Although the problem of enumerating extensions is straightforward for projective geometries, it is more involved for other $GF(q)$-representable matroids, such as graphic matroids, as we will see in this chapter.

In Section 9.1, we discuss the correspondence between extensions and flats of projective geometries. In Section 9.2, we enumerate the extensions of the cycle matroid of a complete graph. Section 9.3 discusses some future directions for enumerating extensions of other $GF(q)$-representable matroids.

9.1 Extensions of projective geometries

Recall that a *projective geometry* $PG(n-1,q)$ is isomorphic to $M[A]$ where the columns of A are representatives of the equivalence classes of $V(n,q)\setminus\{0\}$, where two vectors are equivalent if they are a nonzero scaling of each other.

Given a flat F, let $\mathcal{L}$ be the collection of hyperplanes of $PG(n-1,q)$ that contain F. Consider a rank-$(n-2)$ flat F' that contains F. Each hyperplane that contains F' is in $\mathcal{L}$. If F' is a rank-$(n-2)$ flat that does not contain F, then none of the hyperplanes that contain F' are in $\mathcal{L}$. Therefore, this collection $\mathcal{L}$ is a linear subclass. One can show that these linear subclasses are distinct and the only linear subclasses of $PG(n-1,q)$. This implies that, for a projective geometry, linear subclasses are parameterized by flats.

However, a more straightforward proof follows from considering modular cuts instead of linear subclasses. Recall that a *modular cut* [40] of a matroid M is a collection $\mathcal{M}$ of flats of M with the following properties:

(i) If $F \in \mathcal{M}$ and F' is a flat of M containing F, then $F' \in \mathcal{M}$.

(ii) If $F_1, F_2 \in \mathcal{M}$ and $r(F_1) + r(F_2) = r(F_1 \cap F_2) + r(F_1 \cap F_2)$, then $F_1 \cap F_2 \in \mathcal{M}$.

We prove that the extensions of a projective geometry are parameterized by its flats using modular cuts in the following proposition.

Proposition 9.1.1.

$$\operatorname{ext}(PG(n-1,q)) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

Proof. The extensions of a matroid M are parameterized by the modular cuts of M [40]. In a projective geometry, any two flats F_1, F_2 have the property that $r(F_1) + r(F_2) = r(F_1 \cap F_2) + r(F_1 \cap F_2)$ [40]. Therefore, each modular cut of $PG(n-1,q)$ has a unique minimal element F. Thus, for each modular cut $\mathcal{M}$ of $PG(n-1,q)$, there exists a unique flat F such that $\mathcal{M}$ is precisely the set of flats F' that contain F as a subset. That is, the number of modular cuts of $PG(n-1,q)$ is equal to the number of flats of $PG(n-1,q)$. By Proposition 2.6.1, the number of flats in $PG(n-1,q)$ is $\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q$. $\square$

9.2 Extensions of the cycle matroid of a complete graph

This section is based on the paper [38], which is joint work with Nelson and Van der Pol. We prove the following result about the number of extensions of $M(K_{n+1})$, which is denoted $\operatorname{ext}(M(K_{n+1}))$. Recall that $o(1)$ denotes an unspecified function of n which goes to 0 as n goes to infinity and log denotes the base-2 logarithm.

Theorem 9.2.1. $\log \operatorname{ext}(M(K_{n+1})) = \binom{n}{n/2}(1 + o(1))$.

In order to prove this theorem, we bound the number of stable sets in the hyperplane graph of $M(K_{n+1})$ and the number of small hyperplanes in $M(K_{n+1})$, and then apply Lemma 3.2.3, which says that, for a matroid M,

$$\log i(\Pi(M)) \leq \log \operatorname{coext}(M) \leq \log i(\Pi(M)) + |\mathcal{H}_{\min}(M, \prec)|.$$

Recall that $\mathcal{H}_{\min}(M, \prec)$ is the collection of small hyperplanes; that is, hyperplanes $H \in \mathcal{H}(M)$ such that, for some rank-$(r(M) - 2)$ flat F of M, the minimum hyperplane with respect to $\prec$ that contains F is H.

It is helpful to think of the flats of $M(K_{n+1})$ as partitions of vertices. We say a k-*partition* of a set S is an unordered partition of S into k nonempty parts. A set F is a rank-$(n + 1 - k)$ flat of $M(K_{n+1})$ if and only if the subgraph of K_{n+1} induced on F has precisely k components, each of which is complete. Thus, rank-$(n+1-k)$ flats of $M(K_{n+1})$ correspond to k-partitions of $[n + 1]$. Flats F and F' satisfy $F \subseteq F'$ if and only if the partition corresponding to F refines the partition for F'.

Recall that $2^{[n]}$ denotes the collection of subsets of $[n]$. Consider the poset on $2^{[n]}$ partially ordered by the subset relation, also known as the Boolean lattice and denoted $2^{[n]}$. Note that two sets are *comparable* if one is a subset of the other. Let $\mathcal{P}(n)$ denote the graph with vertex set $2^{[n]} \setminus \{\emptyset\}$ where two vertices are adjacent if and only if they are disjoint or comparable in the Boolean lattice. Before proving that the hyperplane graph of $M(K_{n+1})$ is isomorphic to $\mathcal{P}(n)$, we prove some preliminary results about the stable sets of $\mathcal{P}(n)$.

An *antichain* of the Boolean lattice $2^{[n]}$ is a set $\mathcal{A}$ of elements such that no two elements in $\mathcal{A}$ are comparable. An *intersecting antichain* is an antichain $\mathcal{A}$ where $X \cap Y \neq \emptyset$ for all $X, Y \in \mathcal{A}$. That is, a collection $\mathcal{A}$ of sets is an intersecting antichain if and only if no two sets X, Y in $\mathcal{A}$ are comparable or disjoint. Let $\mathbb{A}_I(n)$ denote the collection of intersecting antichains $\mathcal{A} \subseteq 2^{[n]} \setminus \{\emptyset\}$.

Proposition 9.2.2. *$|\mathbb{A}_I(n)| = i(\mathcal{P}(n))$.*

Proof. Consider a subset $\mathcal{X} \subseteq 2^{[n]} \setminus \{\emptyset\}$. The set $\mathcal{X}$ is an intersecting antichain in $\mathbb{A}_I(n)$ if and only if no two sets X, Y in $\mathcal{X}$ are comparable or disjoint. The set $\mathcal{X}$ is a stable set in $\mathcal{P}(n)$ if and only if no two sets X, Y in $\mathcal{X}$ are comparable or disjoint. □

Lemma 9.2.3. *$|\mathbb{A}_I(n)| \geq 2^{\binom{n}{\lceil (n+1)/2 \rceil}}$.*

Proof. Let $\mathcal{A}$ be the collection of subsets of $[n]$ that have size $\lfloor \frac{n}{2} \rfloor + 1$. There are $\binom{n}{\lfloor n/2 \rfloor + 1}$ such subsets, hence $\mathcal{A}$ has size $\binom{n}{\lfloor n/2 \rfloor + 1}$.

Since all sets in $\mathcal{A}$ have the same size, it follows that $\mathcal{A}$ is an antichain of $2^{[n]}$. Furthermore, since $\lfloor \frac{n}{2} \rfloor + 1 > \frac{n}{2}$, each pair of sets in $\mathcal{A}$ intersect in a nonempty set. This implies that $\mathcal{A}$ is an intersecting antichain of $2^{[n]}$.

Notice that each subset of $\mathcal{A}$ is also an intersecting antichain. Therefore, there are at least $2^{\binom{n}{\lfloor n/2 \rfloor + 1}}$ intersecting antichains. Since $\lfloor \frac{n}{2} \rfloor + 1 = \lceil \frac{n+1}{2} \rceil$, the result follows. $\square$

Antichains in the Boolean lattice are well studied and each intersecting antichain is also an antichain. Thus, we use the following theorem of Kleitman to determine an upper bound on the number of intersecting antichains in $2^{[n]} \setminus \{\emptyset\}$.

Theorem 9.2.4 ([20])**.** *The number of antichains in $2^{[n]}$ is $2^{\binom{n}{n/2}(1+o(1))}$.*

It follows from Theorem 9.2.4 that the number of intersecting antichains in $2^{[n]} \setminus \{\emptyset\}$ is at most $2^{\binom{n}{n/2}(1+o(1))}$.

Now that we have upper and lower bounds for the number of stable sets in $\mathcal{P}(n)$, we prove that this graph is isomorphic to the hyperplane graph of $M(K_{n+1})$.

Lemma 9.2.5. *The graphs $\Pi(M(K_{n+1}))$ and $\mathcal{P}(n)$ are isomorphic.*

Proof. In this proof, let $M = M(K_{n+1})$. For each hyperplane H of M, let $\psi(H) \subseteq [n]$ be the vertex set of the unique component of $K_{n+1}[H]$ not containing the vertex $n+1$. Since, for each nonempty $X \subseteq [n]$, the partition $\{X, [n+1] \setminus X\}$ gives rise to a hyperplane H of M with $\psi(H) = X$, the function ψ is a bijection from the set of hyperplanes of M to $2^{[n]} \setminus \{\emptyset\}$.

Now we claim that two hyperplanes H_1, H_2 are adjacent in the hyperplane graph if and only if $\psi(H_1), \psi(H_2)$ are adjacent in $\mathcal{P}(n)$.

If H_1, H_2 are adjacent, then they intersect in a rank-$(n-2)$ flat F, which corresponds to a 3-partition of $[n+1]$, say $\{X_0, X_1, X_2\}$. Since hyperplanes correspond to 2-partitions of $[n+1]$, a hyperplane that contains F corresponds to a partition with parts X_i and $X_j \cup X_k$ where $\{i, j, k\} = \{0, 1, 2\}$. Without loss of generality, we may assume that $n + 1 \in X_2$, hence $\psi(H_1), \psi(H_2) \in \{X_0, X_1, X_0 \cup X_1\}$. Therefore, $\psi(H_1), \psi(H_2)$ are either disjoint or comparable, which means they are adjacent.

If $\psi(H_1), \psi(H_2)$ are adjacent, then they are either disjoint or comparable. Thus, there exist nonempty disjoint set $S, T \subseteq [n]$ such that $\psi(H_1), \psi(H_2) \in \{S, T, S \cup T\}$. Hence, the hyperplanes H_1 and H_2 each correspond to one of the following partitions: $\{S, [n+1] \setminus S\}$, $\{T, [n+1] \setminus T\}$, $\{S \cup T, [n+1] \setminus (S \cup T)\}$. Each of these hyperplanes contain the rank-$(n-2)$

flat that corresponds to the partition $\{S, T, [n+1] \setminus (S \cup T)\}$; therefore, the hyperplanes H_1 and H_2 are adjacent. $\square$

Lemma 9.2.6. *If F is a rank-$(n-2)$ flat of $M(K_{n+1})$, then there exists a hyperplane H of $M(K_{n+1})$ that contains F such that one of the components of $K_{n+1}[H]$ contains at most $\frac{n+1}{3}$ vertices.*

Proof. Let $\{X_0, X_1, X_2\}$ be the 3-partition of $[n+1]$ that F corresponds to, where $|X_0| \leq |X_1| \leq |X_2|$. A hyperplane that contains F corresponds to a partition with parts X_i and $X_j \cup X_k$ where $\{i, j, k\} = \{0, 1, 2\}$. There are three such partitions and each corresponds to a hyperplane of $M(K_{n+1})$.[1] Since $\{X_0, X_1, X_2\}$ is a tripartition of $[n+1]$, the smallest set X_0 has size at most $\frac{n+1}{3}$. Therefore, the hyperplane H containing F that corresponds to the bipartition $\{X_0, [n+1] \setminus X_0\}$ induces a component with vertex set X_0. That is, one of the components of $K_{n+1}[H]$ contains at most $\frac{n+1}{3}$ vertices. $\square$

We are now ready to prove Theorem 9.2.1.

Proof of Theorem 9.2.1. In this proof, let $M = M(K_{n+1})$. Let $\prec$ be a total ordering of $\mathcal{H}(M)$ such that if $H \prec H'$, then one of the components of $K_{n+1}[H]$ contains at most the same number of vertices as the smaller component of $K_{n+1}[H']$. That is, $\prec$ is a refinement of the preorder of hyperplanes by the size of the smallest component they induce. By Lemma 9.2.6, for each rank-$(n-2)$ flat F, the minimum hyperplane with respect to $\prec$ that contains F induces a component that contains at most $\frac{n+1}{3}$ vertices. Therefore, at most all hyperplanes that induce a component with at most $\frac{n+1}{3}$ vertices are in $\mathcal{H}_{\min}(M, \prec)$. Since there are $\binom{n+1}{k}$ ways to choose a k-set of vertices from $[n+1]$,

$$|\mathcal{H}_{\min}(M, \prec)| \leq \sum_{k=1}^{\lfloor \frac{n+1}{3} \rfloor} \binom{n+1}{k} \leq \left(\frac{\mathrm{e}(n+1)}{\frac{n+1}{3}} \right)^{\frac{n+1}{3}} = 2^{\frac{n+1}{3} \log(3\mathrm{e})} = o\left(\binom{n}{n/2} \right).$$

By Lemma 9.2.5 and Lemma 9.2.3, we have $\log i(\Omega(M)) \geq \binom{n}{\lceil (n+1)/2 \rceil}$. By Lemma 9.2.5 and Theorem 9.2.4, it follows that $\log i(\Omega(M)) \leq \binom{n}{n/2}(1 + o(1))$. Therefore, by Lemma 3.2.3,

$$\binom{n}{\lceil (n+1)/2 \rceil} \leq \log \operatorname{ext}(M) \leq \binom{n}{n/2}(1 + o(1)) + o\left(\binom{n}{n/2} \right),$$

which implies that $\log \operatorname{ext}(M) = \binom{n}{n/2}(1 + o(1))$. $\square$

[1] Note that not all partitions necessarily correspond to hyperplanes of $M(G)$ for a graph G that is not complete. In this case, there could be fewer hyperplanes that contain F.

9.3 Extensions of other representable matroids

We have considered extensions of projective geometries and cycle matroids of complete graphs, which are the densest simple matroids representable over a finite field and simple graphic matroids, respectively. Of course, graphic matroids are also representable, but less dense than projective geometries. Since cycle matroids of complete graphs have a much wilder set of extensions than projective geometries, it would be interesting to know if there is a way to distinguish between restrictions of $PG(n-1,q)$ with a tame set of extensions and restrictions with a wild set of extensions. Is there a particular size that a restriction of $PG(n-1,q)$ must have? Is there a representable matroid with a set of extensions wilder than that of the cycle matroid of a complete graph? To start, it is natural to consider matroids which are "almost" projective geometries or cycle matroids of complete graphs, such as $PG(n-1,q)$ or $M(K_{n+1})$ with one element deleted.

Let $e=uv$ be an edge in K_{n+1} and let $G=K_{n+1}\setminus\{e\}$. Here, we consider $M=M(G)$. In this case, all hyperplanes are the same as those in $M(K_{n+1})$, except for the one that corresponds to the bipartition with one part being $\{u,v\}$. The flats of rank $n-2$ are the same, except the bipartition with one part being $\{u,v\}$ corresponds to a flat and the tripartitions with one part being $\{u,v\}$ do not correspond to flats. If F is a rank-$(n-2)$ flat of M, then F is a flat in $M(K_{n+1})$ that does not have $\{u,v\}$ as one part or F is the bipartition with $\{u,v\}$ as one part. In the first case, the hyperplanes in $M(K_{n+1})$ that contain F are still hyperplanes in M. In the second case, the only hyperplanes that contain F correspond to $\{\{u\},[n+1]\setminus\{u\}\}$ and $\{\{v\},[n+1]\setminus\{v\}\}$. In either case, if H is a hyperplane of M that contains F, then one of the components of $G[H]$ has at most $\frac{n+1}{3}$ vertices. This seems to imply that a version of Lemma 9.2.6 is true for G.

However, the hyperplane graph of $M(G)$ is not an induced subgraph of the hyperplane graph of $M(K_{n+1})$. Consider the tripartition $\{\{u,v\},A,B\}$ of $[n+1]$. The bipartitions $\{\{u,v\}\cup A,B\}$ and $\{\{u,v\}\cup B,A\}$ are vertices in both hyperplane graphs, but they are adjacent in the hyperplane graph of $M(K_{n+1})$ and not in that of $M(G)$. This is true for every bipartition $\{A,B\}$ of $[n+1]\setminus\{u,v\}$.

We might consider simply adding edges between these types of hyperplanes, but then the analysis of small hyperplanes changes. In this case, the smallest part of a bipartition that is a coarsening of a non-flat tripartition could have size $\frac{n-2}{2}$. There are 2^{n-3} bipartitions of the set $[n+1]\setminus\{u,v\}$ and each bipartition gives rise to one small hyperplane if the parts have distinct sizes or two small hyperplanes if the parts have the same size. Thus, there are at most 2^{n-2} small hyperplanes of this form. Now our upper bound on the number of small hyperplanes is at most

$$\sum_{k=1}^{\frac{n+1}{3}} \binom{n}{k} + 2^{n-2}.$$

However, in this case, the hyperplane graph of $M(G)$ is an induced subgraph of that of $M(K_{n+1})$; therefore, using Lemma 3.2.3, we find

$$\binom{n}{n/2}(1+o(1)) \leq \log \operatorname{ext}(M(K_{n+1} \setminus \{e\})) \leq \binom{n}{n/2}(1+o(1)) + 2^{n-2}. \tag{9.1}$$

Recall that $\binom{n}{n/2} = \frac{2^n}{\sqrt{n}}(\sqrt{2/\pi} + o(1))$, so 2^{n-2} is not asymptotically bounded above by $\binom{n}{n/2}$. In fact, we could rewrite the right-hand side of Equation 9.1 as $2^n(\frac{1}{4} + o(1))$. Thus, the bounds in Equation 9.1 are not particularly close together.

It would be interesting to find out which of the upper and lower bound is closer to the truth. Since we have an idea of what the hyperplane graph of $M(G)$ looks like without adding extra edges, perhaps there is a better way to bound the number of stable sets in it.

Chapter 10

Extensions of Dowling geometries

Let an extension of a matroid be called *scarce* if it corresponds to a scarce linear subclass. In previous chapters, we determined bounds on the number of scarce (co)extensions by counting stable sets in either the hyperplane, circuit, or overlap graph, and then used Lemma 3.2.3, Corollary 3.2.6, or Corollary 3.2.8 to show that the number of (co)extensions is not "much" more than the number of those that are scarce. In this chapter, we determine a representation of the hyperplane graph of $DG(n, \Gamma)$ and make partial progress towards enumerating the extensions of $DG(n, GF(3)^*)$.

The partial progress towards counting the extensions of a Dowling geometry in this chapter focuses on counting scarce extensions, which correspond to stable sets in the hyperplane graph. The following theorem gives bounds on the number of scarce extensions of a Dowling geometry over $GF(3)^*$. Let $\mathrm{ext}_{\mathrm{sc}}(M)$ denote the number of scarce extensions of a matroid M. Recall that $o(1)$ denotes an unspecified function of n which goes to 0 as n goes to infinity and log denotes the base-2 logarithm.

Theorem 10.0.1.

$$n2^{n-1}(\tfrac{1}{2} + o(1)) \leq \log \mathrm{ext}_{\mathrm{sc}}(DG(n, GF(3)^*)) \leq n2^{n-1}(1 + o(1)).$$

The lower and upper bounds in Theorem 10.0.1 differ, but only by a constant factor of 2. On the double log scale, this factor becomes a lower order term, so

$$\log(\log(\mathrm{ext}_{\mathrm{sc}}(DG(n, GF(3)^*)))) = n(1 + o(1)).$$

In order to bound the number of extensions of $DG(n, GF(3)^*)$ with Theorem 10.0.1 and Lemma 3.2.3, we need an upper bound for the number of small hyperplanes. It would be interesting to know if the number of small hyperplanes is $o(n2^{n-1})$, as this would imply that the bounds in Theorem 10.0.1 apply to the number of extensions. Based on the analysis of the small hyperplanes of $M(K_{n+1})$, it seems likely that the number of small hyperplanes of $DG(n,\Gamma)$ is "small" compared to the number of stable sets in the hyperplane graph.

Conjecture 10.0.2. *For some fixed real number $\frac{1}{2} \le c \le 1$,*

$$\log \operatorname{ext}(DG(n, GF(3)^*)) = n2^{n-1}(c + o(1)).$$

10.1 Hyperplane graph

In this section, we start by identifying the structure of flats in $DG(n,\Gamma)$. Then, we describe a graph whose vertices are functions and prove that this graph is isomorphic to the hyperplane graph of $DG(n,\Gamma)$. Recall that the hyperplane graph of a matroid M has vertex set $\mathcal{H}(M)$ where two vertices (hyperplanes) H, H' are adjacent if and only if they intersect in a rank-$(n-2)$ flat of M.

We also recall the definition of a Dowling geometry $DG(n,\Gamma)$ here. Let n be a positive integer. Let Γ be a finite (multiplicative) group with identity element 1 and let $q = |\Gamma|$. Recall the construction of the graph K_n^Γ, which has vertex set $[n]$. The edge set of K_n^Γ is $\Gamma \times \binom{[n]}{2} \cup \{\beta_u : u \in [n]\}$ and the incidence function f of K_n^Γ is defined as follows. For each $(\gamma, \{u,v\}) \in \Gamma \times \binom{[n]}{2}$, let $f((\gamma,\{u,v\})) = \{u,v\}$ and for each $u \in [n]$, let $f(\beta_u) = \{u\}$. Informally, the graph K_n^Γ has vertex set $[n]$, an edge labelled γ between each pair $\{u,v\} \in \binom{[n]}{2}$ for each $\gamma \in \Gamma$, and a loop labelled β_u on each vertex $u \in [n]$. For an edge $e = (\gamma, \{u,v\})$, let γ be called the *edge label* of e and recall that u and v are called the *endpoints* of e. The ground set of $DG(n,\Gamma)$ is $E(K_n^\Gamma)$.

Recall the function $\psi : \Gamma \times \mathbb{Z}^2_{>0} \to \Gamma$ where, for each $(\gamma, x, y) \in \Gamma \times \mathbb{Z}^2_{>0}$,

$$\psi((\gamma,x,y)) = \begin{cases} \gamma & \text{if } x \le y \\ \gamma^{-1} & \text{if } y < x \end{cases}.$$

Let C be a cycle of K_n^Γ with at least two edges and arbitrarily assign an orientation to it. Let the vertices and edges of C, beginning with a vertex, be $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_1$, where $e_i = (\gamma_i, \{v_i, v_{i+1}\})$ for each $i \in [k]$. We say C is *balanced* if $\prod_{i=1}^k \psi((\gamma_i, v_i, v_{i+1})) = 1$. Note that the definition of a balanced cycle does not depend on the chosen cyclic ordering of the cycle [40]. A cycle is *unbalanced* if it either has a single edge or is not balanced. Let $\mathcal{B}$ be the collection of balanced cycles of K_n^Γ. The circuits of $DG(n,\Gamma)$ consist of the edge sets of all of the balanced cycles together with the edge sets of all of the hinged, tight, and loose cuffs in which none of the cycles are balanced. Recall that the Dowling geometry $DG(n,\Gamma)$ is the frame matroid represented by $(K_n^\Gamma,\mathcal{B})$.

Proposition 10.1.1. *A set X is a rank-r flat of $DG(n,\Gamma)$ if and only if the components of K_n^Γ induced on X consist of $n-r$ complete graphs in which every cycle is balanced and at most one component that is isomorphic to K_k^Γ for some $k\in[0,n]$.*

Proof. If $K_n^\Gamma[X]$ has $n-r$ components that are complete graphs in which every cycle is balanced and at most one component that is isomorphic to K_k^Γ for some $k\in[0,n]$, then X has rank $n-(n-r)=r$. Since adding an element to X would reduce the number of components in $K_n^\Gamma[X]$ or add an unbalanced cycle to one of the complete components, both of which would increase the rank of X, it follows that X is a flat.

To prove the forward direction, first, we claim that at most one component of $K_n^\Gamma[X]$ contains an unbalanced cycle. Consider two unbalanced cycles C,C' and an edge e between them. Since the edge sets of loose cuffs are circuits of $DG(n,\Gamma)$, the edge e is in the closure of $E(C)\cup E(C')$. Since X is a flat, it follows that C and C' are in the same component. If a component G contains an unbalanced cycle C, then, since every edge (including loops) is in a tight or loose cuff with C, the closure of $E(G)$ is $E(K_n^\Gamma[V(G)])$. That is, the component G is isomorphic to $K_{|V(G)|}^\Gamma$.

Consider a component G that does not contain an unbalanced cycle. Thus, G is a simple graph. Let T be a spanning tree of G.

Claim 10.1.1.1. *If T is a tree in K_n^Γ, then the closure of $E(T)$ in $DG(n,\Gamma)$ induces a complete graph on $V(T)$ in which all cycles are balanced.*

Proof. Consider $u,v\in V(T)$ that are not adjacent in T. There is one path between u and v in T, so there is exactly one uv-edge e such that the cycle in $T\cup\{e\}$ is balanced. This edge e is in $\mathrm{cl}(E(T))$. Let E be the set of edges e such that the cycle in $T\cup\{e\}$ is balanced.

Suppose towards a contradiction that not all cycles in $T\cup E$ are balanced. Let C be an unbalanced cycle in $T\cup E$ where $|E(C)\cap E|$ is minimum. Since the cycle in $T\cup\{e\}$ is balanced for all $e\in E$, we know $|E(C)\cap E|\geq 2$. Let $e_1,e_2\in E(C)\cap E$ and let P_1,P_2 be the two disjoint paths in C between the endpoints of e_1 and e_2. Since T is connected, the graph $C\setminus\{e_1,e_2\}$ is connected as well. Thus, there exists a path $P\subseteq T$ between a vertex in P_1 and a vertex in P_2 that is otherwise disjoint from $P_1\cup P_2$. It follows that $C\cup P$ is a theta graph. For each $i\in[2]$, let C_i be the cycle in $C\cup P$ that contains e_i and does not contain e_{3-i}. Since C is unbalanced, the theta graph $C\cup P$ has at least one unbalanced cycle. By Proposition 2.5.2, if one cycle in a theta graph is unbalanced, then at least two of its cycles are unbalanced. Therefore, at least one of C_1, C_2 is unbalanced; without loss of generality, say C_1 is unbalanced. Since $E(C_1) \setminus E(P)$ is a subset of $E(C)$, the edge e_2 is not in C_1, and $P \cap E = \emptyset$, it follows that $|E(C_1) \cap E| < |E(C) \cap E|$. If $|E(C_1) \cap E| \geq 2$, then this contradicts the minimality of $|E(C) \cap E|$. Otherwise, if $|E(C_1) \cap E| = 1$, then C_1 is balanced by the definition of E, which is a contradiction. Therefore, every cycle in $T \cup E$ is balanced. $\square$

By the claim, we have $\mathrm{cl}(E(T)) = E(G)$ and G is a complete graph on $V(T)$ in which all cycles are balanced. The rank of $E(G)$ in $DG(n, \Gamma)$ is $|V(G)| - 1$. Let m be the number of components of $K_n^\Gamma[X]$ in which every cycle is balanced. Thus, the rank of X in $DG(n, \Gamma)$ is $n - m$. Since we are given that the rank of X is r, there are $n - r$ components that are complete graphs in which every cycle is balanced. $\square$

An example of the components of $K_7^{GF(3)^*}$ induced on a hyperplane of $DG(7, GF(3)^*)$ can be seen in Figure 10.1.

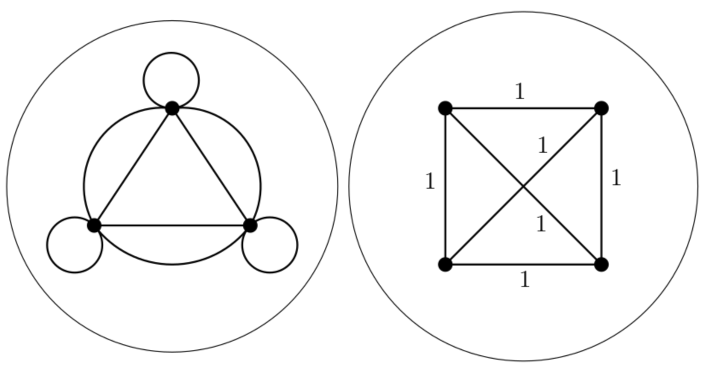

Figure 10.1: The components induced on a hyperplane of $DG(7, GF(3)^*)$.

We say a function $f : X \to Y$ is *nonzero* if $f(X) \neq \{0\}$. We say a nonzero function $f : [n] \to \Gamma \cup \{0\}$ is *canonical* if $f(v) = 1$ where $v \leq w$ for all $w \in \mathrm{supp}(f)$. That is, a nonzero function f is canonical if its first nonzero entry is 1. Note that $[n] = V(K_n^\Gamma)$. Let $\mathcal{F}$ be the collection of canonical functions $f : V(K_n^\Gamma) \to \Gamma \cup \{0\}$. We say two functions $f, f' : [n] \to \Gamma \cup \{0\}$ are *equal up to rescaling*, denoted $f \approx f'$, if there exists $\gamma \in \Gamma$ such that $f(x) = f'(x)\gamma$ for all $x \in [n]$. Let $G(n, \Gamma)$ be the simple graph with vertex set $\mathcal{F}$ and edges are defined as follows. If $f, f' \in \mathcal{F}$ where $|\mathrm{supp}(f)| \geq |\mathrm{supp}(f')|$, then f and f' are adjacent if and only if:

(i) there exists $\gamma \in \Gamma \cup \{0\}$ and $X \subseteq \text{supp}(f)$ such that $f'(x) = f(x)\gamma$ for all $x \in X$ and $f|_Y \approx f'|_Y$ where $Y = V(K_n^\Gamma) \setminus X$, or

(ii) $\text{supp}(f) \cap \text{supp}(f') = \emptyset$.

We say an edge between f and f' is of type (x) where $x \in \{\text{i}, \text{ii}\}$ if the corresponding condition above holds.

Lemma 10.1.2. *The graph $G(n, \Gamma)$ is isomorphic to the hyperplane graph $\Pi(n, \Gamma)$.*

Proof. In this proof, let $G = G(n, \Gamma)$ and let $\Pi = \Pi(n, \Gamma)$. Define a function $\phi : V(G) \to V(\Pi)$, as follows. For each $f \in V(G)$,

$$\begin{aligned}\phi(f) = \{(\gamma, \{u, v\}) : u, v \in V(K_n^\Gamma),\ u < v,\ \text{and } f(u) = \gamma f(v)\}\\ \cup \{\beta_u : u \in V(K_n^\Gamma) \text{ and } f(u) = 0\}.\end{aligned}$$

Consider $f \in V(G)$. We will show that $\phi(f) \in V(\Pi)$. Let H be the subgraph of K_n^Γ induced on the edge set $\phi(f)$. That is, $H = K_n^\Gamma[\phi(f)]$. Let $V_0 = \{v \in V(K_n^\Gamma) : f(v) = 0\}$. Since $f(u) = 0$ for every vertex $u \in V_0$, we have $f(u) = \gamma f(v)$ for every pair of vertices $u, v \in V_0$ and every group element $\gamma \in \Gamma$. Furthermore, for each $u \in V_0$, the loop on u in K_n^Γ is in $\phi(f)$. Therefore, all edges in K_n^Γ between (not necessarily distinct) vertices in V_0 are in $\phi(f)$. That is, the graph $K_n^\Gamma[V_0]$ is a subgraph of H. Let $V = V(K_n^\Gamma) \setminus V_0$ be the vertices not in V_0 and let $E = \phi(f) \setminus E(K_n^\Gamma[V_0])$ be the edges not induced by V_0. For $u \in V_0$ and $v \in V$, we know $f(u) \neq \gamma f(v)$ and $f(v) \neq \gamma f(u)$ for any $\gamma \in \Gamma$, hence $K_n^\Gamma[V_0]$ is a component of H. For $u, v \in V$ where $u < v$, since $f(u), f(v) \neq 0$, the edge $(\gamma, \{u, v\})$ is in E where $\gamma = f(u)f(v)^{-1}$, and no other uv-edge is in E. Therefore, the graph $K_n^\Gamma[V]$ is a simple, complete graph and a component of H.

Claim 10.1.2.1. *Every cycle in $K_n^\Gamma[V]$ is balanced.*

Proof. Let $C = v_1 e_1 v_2 e_2 \dots v_{|V|} e_{|V|} v_{|V|+1}$ be a cycle in $K_n^\Gamma[V]$ where $v_1 = v_{|V|+1}$ and $e_i = (\gamma_i, v_i, v_{i+1})$ for each $i \in [|V|]$. Let $\pi = \prod_{i=1}^{|V|} \psi(\gamma_i, v_i, v_{i+1})$. Consider $i \in [|V|]$. If $v_i < v_{i+1}$, then $\psi(\gamma_i, v_i, v_{i+1}) = \gamma_i = f(v_i)f(v_{i+1})^{-1}$. If $v_{i+1} < v_i$, then $\psi(\gamma_i, v_i, v_{i+1}) = \gamma_i^{-1} = (f(v_{i+1})f(v_i)^{-1})^{-1} = f(v_i)f(v_{i+1})^{-1}$. Therefore,

$$\pi = \prod_{i=1}^{|V|} f(v_i)f(v_{i+1})^{-1} = 1,$$

so the cycle C is balanced. $\square$

Now it follows that the subgraph of K_n^Γ induced on $\phi(f)$ has at most two components: $K_n^\Gamma[V_0]$, which is isomorphic to $K_{|V_0|}^\Gamma$; and $K_n^\Gamma[V]$, which is a complete graph where every cycle is balanced. Since $K_n^\Gamma[V_0]$ is connected and contains an unbalanced cycle, its edge set has rank $|V_0|$ in $DG(n,\Gamma)$. Since $K_n^\Gamma[V]$ is connected and does not contain an unbalanced cycle, its edge set has rank $|V|-1$ in $DG(n,\Gamma)$. Therefore, the edge set $\phi(f)$ has rank $|V_0|+|V|-1=n-1$. In order to complete the proof that $\phi(f)$ is a hyperplane, we argue that $\mathrm{cl}(\phi(f))=\phi(f)$.

Claim 10.1.2.2. *$\mathrm{cl}(\phi(f))=\phi(f)$.*

Proof. If an edge between $u\in V_0$ and $v\in V$ is added to $\phi(f)$, then the two components are connected, and the rank increases. Consider an edge $(\alpha,\{u,v\})$ between $u,v\in V$ that is not in $\phi(f)$. Without loss of generality, assume $u<v$. Let C be a cycle induced by $\phi(f)$ that contains the edge $(\gamma,\{u,v\})$. Let $e_1,e_2,\ldots,e_{|C|-1}$ be the edges in the path from u to v in $C\setminus(\gamma,\{u,v\})$ where $e_i=(\gamma_i,v_i,v_{i+1})$ for $i\in[|C|-1]$ and $v_1=v_{|C|}$. Since C is a balanced cycle,

$$\prod_{i=1}^{|C|-1}\psi(\gamma_i,v_i,v_{i+1})=\gamma.$$

Since the edge $(\alpha,\{u,v\})$ is not in $\phi(f)$, we know $\alpha\neq\gamma$, which implies $\prod_{i=1}^{|C|-1}\gamma_i\neq\alpha$. Therefore, the cycle $C\setminus(\gamma,\{u,v\})\cup(\alpha,\{u,v\})$ is unbalanced; hence, adding an edge between vertices in V to $\phi(f)$ increases its rank. Thus, it follows that $\mathrm{cl}(\phi(f))=\phi(f)$. $\square$

In order to prove that ϕ is a bijection, we will define a function $\zeta:V(\Pi)\to V(G)$, as follows, and then show that ζ is the inverse function of ϕ. For each hyperplane $H\in V(\Pi)$, let $\zeta(H)=f$ where $f\in\mathcal{F}$, as follows. By Proposition 10.1.1, there is a partition (V_1,V_2) of the vertex set of K_n^Γ, where $|V_1|\geq 0$ and $|V_2|\geq 1$, such the components of K_n^Γ induced by H are $K_n^\Gamma[V_1]$ and a complete graph G' with vertex set V_2 in which every cycle is balanced. If $v\in V_1$, then let $f(v)=0$. Let u be the vertex in V_2 such that $u\leq v$ for all $v\in V_2$. Let $f(u)=1$. Consider $v\in V_2$ where $v\neq u$. Since G' is simple and complete, there is one uv-edge $(\gamma,\{u,v\})$ in H. Let $f(v)=\gamma^{-1}f(u)$.

Since $|V_2|\geq 1$, not all vertices of K_n^Γ are mapped to 0 by f, hence the function f is nonzero. Since $f(u)=1$ where $u\leq v$ for all $v\in\mathrm{supp}(f)$, the function f is canonical. Therefore, the function f is in $\mathcal{F}$.

Each hyperplane H induces at most two components G_1,G_2 of K_n^Γ where G_1 is isomorphic K_k^Γ for some $k\in[0,n-1]$ and G_2 is a complete graph in which every cycle is balanced. The function ζ maps H to an assignment f of vertex labels which identify these components and, in the case of G_2, the edge labels. The function ϕ determines the components G_1, G_2 from the assignment f of vertex labels. Observe that ζ and ϕ are inverse functions. Thus, they are bijections between the hyperplanes of Π and the functions of $\mathcal{F}$.

Now we will prove that two vertices $f_1, f_2 \in V(G)$ are adjacent in G if and only if $\phi(f_1)$ and $\phi(f_2)$ are adjacent in Π. Consider two vertices $f_1, f_2 \in V(G)$. Define the following sets:

- $X_{(0,0)} = \{v \in V(K_n^\Gamma) : f_1(v) = f_2(v) = 0\}$,
- $X_{(0,1)} = \{v \in V(K_n^\Gamma) : f_1(v) = 0,\ f_2(v) \neq 0\}$,
- $X_{(1,0)} = \{v \in V(K_n^\Gamma) : f_1(v) \neq 0,\ f_2(v) = 0\}$, and
- $X_{(1,1)} = \{v \in V(K_n^\Gamma) : f_1(v) \neq 0,\ f_2(v) \neq 0\}$.

For each $i \in [2]$, let H_i^0 be the component of $K_n^\Gamma[\phi(f_i)]$ isomorphic to K_k^Γ for some $k \in [0, n-1]$ and let H_i^1 be the component of $K_n^\Gamma[\phi(f_i)]$ that is a complete graph where every cycle is balanced. Observe that $X_{(i,j)} = V(H_1^i) \cap V(H_2^j)$ for each $i, j \in \{0, 1\}$. Let H be the subgraph of K_n^Γ induced on the edges that are in $\phi(f_1) \cap \phi(f_2)$. Since $X_{(0,0)} = V(H_1^0) \cap V(H_2^0)$, the graph $H[X_{(0,0)}]$ is a connected component of H and its edge set has rank $|X_{(0,0)}|$ in $DG(n, \Gamma)$. The graphs $H[X_{(0,1)}]$ and $H[X_{(1,0)}]$ are also connected components of H, but they are induced subgraphs of complete graphs whose cycles are balanced, hence $r(E(H[X_{(0,1)}])) = |X_{(0,1)}| - 1$ and $r(E(H[X_{(1,0)}])) = |X_{(1,0)}| - 1$. Let $H' = H[X_{(0,1)} \cup X_{(1,0)} \cup X_{(1,1)}]$. By definition of ϕ, the graph H' has no unbalanced cycles.

Claim 10.1.2.3. *The functions f_1 and f_2 are adjacent if and only if the graph H' contains exactly 2 components.*

Proof. Suppose f_1 and f_2 are adjacent. If the edge between f_1 and f_2 is of type (ii), then $\text{supp}(f_1) \cap \text{supp}(f_2) = \emptyset$. Therefore, the set $X_{(1,1)}$ is empty and the sets $X_{(0,1)}$ and $X_{(1,0)}$ are not empty. Thus, the graphs $H[X_{(0,1)}]$ and $H[X_{(1,0)}]$ are connected components of H.

Suppose, without loss of generality, that $|\text{supp}(f_1)| \geq |\text{supp}(f_2)|$. If the edge between f_1 and f_2 is of type (i), then there exists $\gamma_1 \in \Gamma \cup \{0\}$, $\gamma_2 \in \Gamma$, and $X \subseteq \text{supp}(f_1)$ such that $f_2(x) = f_1(x)\gamma_1$ for all $x \in X$ and $f_1(y) = f_2(y)\gamma_2$ for all $y \in Y = V(K_n^\Gamma) \setminus X$. Thus, the set $X_{(0,1)}$ is empty. We claim that $H[X_{(1,1)}]$ is one component of H if $X_{(1,0)} \neq \emptyset$ and $H[X_{(1,1)}]$ is two components of H otherwise.

If $X_{(1,0)} \neq \emptyset$, then $\gamma_1 = 0$. Consider $u, v \in X_{(1,1)}$ where $u < v$. Since $\gamma_1 = 0$, we have that $f_2(u) = f_1(u)\gamma_2$ and $f_2(v) = f_1(v)\gamma_2$. By definition of $X_{(1,1)}$, for each $i \in [2]$, there is one edge $(\delta_i, \{u,v\})$ between u and v in H_i^1 where $\delta_i = f_i(u)f_i(v)^{-1} \in \Gamma$. Therefore,

$$\delta_1 = f_1(u)f_1(v)^{-1} = (f_2(u)\gamma_2^{-1})(f_2(v)\gamma_2^{-1})^{-1} = f_2(u)\gamma_2^{-1}\gamma_2 f_2(v)^{-1} = f_2(u)f_2(v)^{-1} = \delta_2.$$

Thus, the edge between u and v in $\phi(f_1)$ has the same label as the edge between u and v in $\phi(f_2)$. It follows that $H[X_{(1,1)}]$ is connected.

If $X_{(1,0)} = \emptyset$, then $\gamma_1 \neq 0$. For each $j \in [2]$, let $V_j \subseteq X_{(1,1)}$ such that $f_2(v) = f_1(v)\gamma_j$ for each $v \in V_j$. For each $i \in [2]$, there is one edge $(\delta_i, \{u,v\})$ between u and v in H_i^1 where $\delta_i = f_i(u)f_i(v)^{-1} \in \Gamma$. If $u, v \in V_j$ where $j \in [2]$, then

$$\delta_1 = f_1(u)f_1(v)^{-1} = (f_2(u)\gamma_j^{-1})(f_2(v)\gamma_j^{-1})^{-1} = f_2(u)\gamma_j^{-1}\gamma_j f_2(v)^{-1} = f_2(u)f_2(v)^{-1} = \delta_2.$$

So the edge between u and v in $\phi(f_1)$ has the same label as the edge between u and v in $\phi(f_2)$. It follows that $H[V_1]$ and $H[V_2]$ are connected. If $u \in V_1$ and $v \in V_2$, then

$$\delta_1 = f_1(u)f_1(v)^{-1} = (f_2(u)\gamma_1^{-1})(f_2(v)\gamma_2^{-1})^{-1} = f_2(u)\gamma_1^{-1}\gamma_2 f_2(v)^{-1} \neq f_2(u)f_2(v)^{-1} = \delta_2.$$

So the edge between u and v in $\phi(f_1)$ has a different label from the edge between u and v in $\phi(f_2)$, which implies that there are no edges in H between vertices in V_1 and vertices in V_2. That is, the graph $H[X_{(1,1)}]$ has exactly two components.

Now suppose that the graph H' contains exactly 2 components. Each component exists within $H[X_{(0,1)}]$, $H[X_{(1,0)}]$, or $H[X_{(1,1)}]$, hence one of $X_{(0,1)}, X_{(1,0)}, X_{(1,1)}$ is empty. If $X_{(1,1)}$ is empty, then $\mathrm{supp}(f_1) \cap \mathrm{supp}(f_2) = \emptyset$, which implies that f_1 and f_2 are adjacent by an edge of type (ii). If, without loss of generality, $X_{(0,1)}$ is empty, then there exists $\gamma_1 \in \Gamma \cup \{0\}$, $\gamma_2 \in \Gamma$, and $X \subseteq \mathrm{supp}(f_1)$ such that $f_2(x) = f_1(x)\gamma_1$ for all $x \in X$ and $f_1(y) = \gamma_2 f_2(y)$ for all $y \in Y = V(K_n^\Gamma) \setminus X$. Therefore, there is an edge of type (i) between f_1 and f_2. $\square$

Observe that the graph H' contains exactly 2 components if and only if $r(E(H')) = |V(H')| - 2$. Furthermore, since $r(E(H[X_{(0,0)}])) = |X_{(0,0)}|$, notice that $r(E(H')) = |V(H')| - 2$ if and only if $r(E(H)) = |V(H)| - 2$. Thus, by Claim 10.1.2.3, the functions f_1 and f_2 are adjacent in G if and only if $r(E(H)) = |V(H)| - 2$. Since $\phi(f_1)$ and $\phi(f_2)$ are hyperplanes, their intersection is a flat, hence $\phi(f_1)$ and $\phi(f_2)$ intersect in a flat of rank $|V(H)| - 2 = n - 2$ if and only if f_1 and f_2 are adjacent. Therefore, the functions f_1 and f_2 are adjacent in G if and only if $\phi(f_1)$ and $\phi(f_2)$ are adjacent in Π. $\square$

10.2 Stable set bounds

In this section, we determine upper and lower bounds on the number of stable sets in the hyperplane graph of $DG(n, GF(3)^*)$. We choose a small abelian group in order to better understand the graph defined in the previous section. For this section, let $\Gamma = GF(3)^*$. Note that $|\Gamma| = 2$ and Γ contains an identity element 1 and an element -1 that is its own inverse.

Since $\Gamma = GF(3)^*$, the graph $G(n, \Gamma)$ has vertex set $\mathcal{F}$ and edges are defined as follows. If $f, f' \in \mathcal{F}$ where $|\mathrm{supp}(f)| \leq |\mathrm{supp}(f')|$, then f and f' are adjacent if and only if:

(i) $\mathrm{supp}(f) = \mathrm{supp}(f')$,

(ii) $\mathrm{supp}(f) \subset \mathrm{supp}(f')$ and $f|_{\mathrm{supp}(f)} = \pm f'|_{\mathrm{supp}(f)}$, or

(iii) $\mathrm{supp}(f) \cap \mathrm{supp}(f') = \emptyset$.

For each $d \in [n]$, let $L_d \subseteq V(G)$ be the set of vertices $[f]$ where $|\mathrm{supp}(f)| = d$. We consider each L_d to be layer d of G. For each $d \in [n]$, let $S_d = \{V \subseteq [n] : |V| = d\}$. That is, the set S_d is the set of supports of vertices in layer d. There are $\binom{n}{d}$ possible supports for a vertex f in layer d, hence $|S_d| = \binom{n}{d}$. For each $d \in [n]$ and $V \in S_d$, let C_V denote the set of vertices $f \in L_d$ where $\mathrm{supp}(f) = V$.

Proposition 10.2.1. *For each $d \in [n]$ and $V \in S_d$, there are 2^{d-1} vertices in C_V.*

Proof. If $\mathrm{supp}(f) = V$, then $f(v) \in \Gamma$ for each $v \in V$ and $f(u) = 0$ for all $u \in [n] \setminus V$. Additionally, we have $f(w) = 1$ where $w \in \mathrm{supp}(f)$ and $w \leq v$ for all $v \in \mathrm{supp}(f)$. Thus, since $|\Gamma| = 2$, there are 2 choices for the f-image of each $v \in V \setminus \{w\}$. Since $V \in S_d$, we know that $|V| = d$. Therefore, there are 2^{d-1} functions with support V. $\square$

Proposition 10.2.2. *For each $d \in [n]$, there are $\binom{n}{d}2^{d-1}$ vertices in L_d.*

Proof. By Proposition 10.2.1, there are 2^{d-1} vertices in C_V for each $V \in S_d$. Since L_d is the disjoint union of C_V over all $V \in S_d$, the layer L_d has size $\sum_{V \in S_d} 2^{d-1} = |S_d|2^{d-1} = \binom{n}{d}2^{d-1}$. $\square$

Proposition 10.2.3. *If $V \subseteq [n]$, then the subgraph of $G(n, \Gamma)$ induced on C_V is a complete graph.*

Proof. Consider $f, f' \in C_V$. By definition of C_V, we know $\mathrm{supp}(f) = V = \mathrm{supp}(f')$. Thus, by definition of $G(n, \Gamma)$, the vertices f and f' are adjacent. Therefore, since every pair of vertices in C_V are adjacent, the subgraph of $G(n, \Gamma)$ induced on C_V is a complete graph. $\square$

The following lemma gives a lower bound on the number of stable sets in $G(n, \Gamma)$ by finding a method of constructing large stable sets. The proof is instructive, so it is included, but we find a better lower bound in Lemma 10.2.7.

Lemma 10.2.4. $\log i(G(n,\Gamma)) \geq \binom{n}{\lfloor n/2 \rfloor + 1} \cdot \lfloor n/2 \rfloor$.

Proof. In this proof, let $G = G(n, \Gamma)$. Let $d \in [\lfloor n/2 \rfloor + 1, n]$ and consider $f, f' \in L_d$. Since f and f' both have support of size d, we know $\text{supp}(f) \not\subset \text{supp}(f')$ and $\text{supp}(f') \not\subset \text{supp}(f)$. That is, edges between vertices in the same layer are not edges of type (ii). Since $d > n/2$, the supports of f and f' have at least one element in common, so $\text{supp}(f) \cap \text{supp}(f') \neq \emptyset$. That is, edges between vertices with support size at least $n/2$ are not edges of type (iii). Thus, the vertices f and f' are adjacent if and only if $\text{supp}(f) = \text{supp}(f')$. This implies that f and f' are adjacent if and only if $f, f' \in C_V$ for some $V \in S_d$. Therefore, for each $d \in [\lfloor n/2 \rfloor + 1, n]$, the subgraph $G[C_V]$ is a clique and a component of the graph $G[L_d]$ induced on the vertices in layer d.

Consider layer $d = \lfloor n/2 \rfloor + 1$. Let $k = \binom{n}{d}$, the number of elements in S_d. Let $V_1, V_2, \ldots, V_k$ be the k sets in S_d. Let $\mathcal{I} = \{X \subseteq L_d : |X \cap C_{V_i}| \leq 1 \text{ for all } i \in [k]\}$. Since, for each $i \in [k]$, the subgraph $G[C_{V_i}]$ is a component of the graph $G[L_d]$, each set $I \in \mathcal{I}$ is a stable set of $G[L_d]$, and thus is a stable set in G. Therefore, the number of stable sets in G is at least $|\mathcal{I}|$.

Consider a set $X \in \mathcal{I}$. Since, for each $i \in [k]$, the set C_{V_i} contains 2^{d-1} elements, there are $2^{d-1} + 1$ choices for the set $X \cap C_{V_i}$. Since $k = \binom{n}{d}$, there are $(2^{d-1} + 1)^{\binom{n}{d}}$ choices for the set X. Thus,

$$\log i(G) \geq \log |\mathcal{I}| \geq \binom{n}{d}(d-1). \qquad \square$$

Lemma 10.2.5. $\log i(G(n,\Gamma)) \leq n2^{n-1}$.

Proof. In this proof, let $G = G(n, \Gamma)$. Let $k = \sum_{d=0}^{n} \binom{n}{d} = 2^n$. Let $V_1, V_2, \ldots, V_k$ be the k subsets of $[n]$. Let $\mathcal{U} = \{U \subseteq V(G) : |U \cap C_{V_i}| \leq 1 \text{ for all } i \in [k]\}$. That is, let $\mathcal{U}$ be the collection of sets $U \subseteq V(G)$ obtained by choosing at most one vertex from each set C_V where $V \subseteq [n]$. Since each such set C_V induces a complete graph in G, at most one vertex from each can be in a stable set of G. Therefore, if S is a stable set of G, then $S \in \mathcal{U}$. This implies that $i(G) \leq |\mathcal{U}|$.

By Proposition 10.2.1, for each $V \subseteq [n]$ where $|V| = d$, the set C_V contains 2^{d-1} elements. Thus, there are $2^{d-1} + 1$ choices for each vertex from layer d in a set in $\mathcal{U}$. Since there are $\binom{n}{d}$ subsets of $[n]$ with size d, there are $(2^{d-1}+1)^{\binom{n}{d}}$ choices for the vertices from layer d in a set in $\mathcal{U}$. Therefore,

$$i(G) \leq |\mathcal{U}| \leq \prod_{d=0}^{n}(2^{d-1}+1)^{\binom{n}{d}}.$$

Since $2^{d-1}+1 \leq 2^d$ for $d \in [n]$,

$$\log i(G) \leq \log|\mathcal{U}| \leq \sum_{d=1}^{n}\binom{n}{d}d = n2^{n-1}. \qquad \square$$

In the next lemma, we use some probability theory to get a slightly better lower bound than that in Lemma 10.2.4. We use $\mathbb{P}(A)$ to denote the probability of an event A and $\mathbb{E}(A)$ to denote the expected value of A. We refer to Chapter 2 of the textbook by Molloy and Reed [31] for an introduction to probability theory at the level needed here. We also make note of one preliminary result, as follows.

Proposition 10.2.6 (Markov's Inequality [31])**.** *For any positive random variable X,*

$$\mathbb{P}(X \geq t) \leq \mathbb{E}(X)/t.$$

Lemma 10.2.7. *$\log i(G(n,\Gamma)) \geq n2^{n-1}(\frac{1}{2}+o(1))$.*

Proof. In this proof, let $G = G(n,\Gamma)$. Let $\ell = \frac{n}{4}(1-\varepsilon)$ where $0 < \varepsilon < \frac{1}{7}$. Let $k = \sum_{d=\lfloor n/2+1\rfloor}^{\lfloor n/2+\ell\rfloor}\binom{n}{d}$. Since there are k subsets V of $[n]$ where $|V| \in [\lfloor n/2+1\rfloor, \lfloor n/2+\ell\rfloor]$, there are k possible supports for the functions in $\bigcup_{d=\lfloor n/2+1\rfloor}^{\lfloor n/2+\ell\rfloor} L_d$. Let $V_1, V_2, \ldots, V_k$ be the k subsets V of $[n]$ where $|V| \in [\lfloor n/2+1\rfloor, \lfloor n/2+\ell\rfloor]$. Let $\mathcal{U} = \{\{v_1, v_2, \ldots, v_k\} : v_i \in C_{V_i}$ for all $i \in [k]\}$. That is, let $\mathcal{U}$ be the collection of sets $U \subseteq V(G)$ obtained by choosing a vertex from each set C_V where $V \subseteq [n]$ and $|V| \in [\lfloor n/2+1\rfloor, \lfloor n/2+\ell\rfloor]$. Note that each such C_V induces a complete graph in G. We are interested in the number of edges induced by a set U in $\mathcal{U}$, which can range from 0 to $\binom{k}{2}$. We will use bounds on the number of sets that induce a "small" number of edges to lower bound the number of stable sets in G. First, we will determine a lower bound for the size of $\mathcal{U}$.

Claim 10.2.7.1. $\log|\mathcal{U}| \geq n2^{n-1}(\frac{1}{2}+o(1))$.

Proof. Consider layer $d \in [\lfloor n/2+1\rfloor, \lfloor n/2+\ell\rfloor]$. For each $V \subseteq [n]$ where $|V| = d$, the set C_V contains 2^{d-1} elements. Thus, there are 2^{d-1} choices for each vertex from layer d in a set in $\mathcal{U}$. Since there are $\binom{n}{d}$ sets in S_d, there are $(2^{d-1})^{\binom{n}{d}}$ choices for the vertices from layer d in a set in $\mathcal{U}$. Therefore,

$$\log|\mathcal{U}| = \sum_{d=\lfloor n/2+1\rfloor}^{\lfloor n/2+\ell\rfloor} \binom{n}{d}(d-1). \tag{10.1}$$

To find a lower bound, we start by claiming that $\sum_{d=\lfloor n/2+1\rfloor}^{\lfloor n/2+\ell\rfloor}\binom{n}{d} = 2^{n-1}(1+o(1))$. We prove this by upper bounding the difference $\sum_{d=0}^{n}\binom{n}{d} - \sum_{d=\lfloor n/2+1\rfloor}^{\lfloor n/2+\ell\rfloor}\binom{n}{d}$. The partial sum $\sum_{d=0}^{\lfloor n/2\rfloor}\binom{n}{d}$ is at most

$$\sum_{d=0}^{\lfloor n/2\rfloor-1}\binom{n}{d} + \binom{n}{n/2} \le 2^{n-1} + \sqrt{\frac{2}{\pi n}}2^n.$$

The partial sum $\sum_{d=\lfloor n/2+\ell\rfloor+1}^{n}\binom{n}{d}$ is at most

$$\sum_{d=0}^{\frac{n}{4}(1+\varepsilon)}\binom{n}{d} \le \left(\frac{\mathrm{e}n}{\frac{n}{4}(1+\varepsilon)}\right)^{\frac{n}{4}(1+\varepsilon)} = 2^{\frac{n}{4}(1+\varepsilon)\log(4\mathrm{e})} \le 2^{\frac{7n(1+\varepsilon)}{8}}.$$

Using these partial sum upper bounds and the identity $\sum_{d=0}^{n}\binom{n}{d} = 2^n$, we find

$$\sum_{d=\lfloor n/2+1\rfloor}^{\lfloor n/2+\ell\rfloor}\binom{n}{d} \ge 2^n - 2^{n-1} - \sqrt{\frac{2}{\pi n}}2^n - 2^{\frac{7n(1+\varepsilon)}{8}} = 2^{n-1}(1+o(1)),$$

as desired. Since $d \ge \lfloor n/2+1\rfloor$ in Equation 10.1 we can replace $(d-1)$ with $\lfloor n/2\rfloor$. Now it follows that

$$\log|\mathcal{U}| \ge \lfloor n/2\rfloor \sum_{d=\lfloor n/2+1\rfloor}^{\lfloor n/2+\ell\rfloor}\binom{n}{d} \ge \tfrac{1}{2}n2^{n-1}(1+o(1)). \qquad \square$$

Define a function $X : \mathcal{U} \to [0, \binom{k}{2}]$ where, for each vertex subset $V \in \mathcal{U}$, let $X(V)$ be the number of edges induced by V in the graph G. Let $L = \bigcup_{d=\lfloor n/2+1\rfloor}^{\lfloor n/2+\ell\rfloor} L_d$ be the union of the vertices in layers $\lfloor n/2+1\rfloor$ to $\lfloor n/2+\ell\rfloor$. Let U be chosen from $\mathcal{U}$ uniformly at random. The probability that an edge e is induced by U is denoted $\mathbb{P}(e \in E(G[U]))$. By linearity of expectation, the expected value of $X(U)$ is $\mathbb{E}(X(U)) = \sum_{u,v\in L}\mathbb{P}(uv \in E(G[U]))$.

For each pair of distinct vertices u, v in L, let $x_{uv} = 1$ if uv is an edge in G and let $x_{uv} = 0$ otherwise. Since the probability that an edge uv is induced by U is 0 if uv is not in G and is $\mathbb{P}(u, v \in U)$ if uv is in G,

$$\mathbb{E}(X(U)) = \sum_{u,v \in L} x_{uv} \mathbb{P}(u, v \in U).$$

For each $V \subseteq [n]$ where $|V| \in [\lfloor n/2 + 1 \rfloor, \lfloor n/2 + \ell \rfloor]$, the subgraph of G induced on C_V is a complete graph and a component of the graph $G[L_{|V|}]$. Thus, since U contains exactly one vertex from each of these complete graphs, we know $x_{uv} = 0$ if u and v are in the same layer and in U. Therefore,

$$\mathbb{E}(X(U)) = \sum_{i=\lfloor n/2+1 \rfloor}^{\lfloor n/2+\ell \rfloor - 1} \sum_{u \in L_i} \sum_{j=i+1}^{\lfloor n/2+\ell \rfloor} \sum_{v \in L_j} x_{uv} \mathbb{P}(u, v \in U).$$

Claim 10.2.7.2. *For $i \in [\lfloor n/2 + 1 \rfloor, \lfloor n/2 + \ell \rfloor]$, the probability that a vertex $u \in L_i$ is in U is 2^{1-i}.*

Proof. Let $V \subseteq [n]$ such that $u \in C_V$. Since u is in L_i, we know $|V| = i$. The probability that u is in U is $\frac{1}{|C_V|}$. By Proposition 10.2.1, there are 2^{i-1} vertices in C_V. Thus, the probability that u is in U is $\frac{1}{2^{i-1}} = 2^{1-i}$. □

Since vertices in different layers are chosen to be in U independently of each other, it follows from Claim 10.2.7.2 that

$$\mathbb{E}(X(U)) = \sum_{i=\lfloor n/2+1 \rfloor}^{\lfloor n/2+\ell \rfloor - 1} \sum_{u \in L_i} \sum_{j=i+1}^{\lfloor n/2+\ell \rfloor} \sum_{v \in L_j} x_{uv} 2^{1-i} 2^{1-j}.$$

Consider vertices $u \in L_i$ and $v \in L_j$ where $i < j \in [\lfloor n/2 + 1 \rfloor, \lfloor n/2 + \ell \rfloor]$. If there is an edge between u and v, then it is an edge of type (ii). Therefore, uv is an edge in G if and only if $\mathrm{supp}(u) \subset \mathrm{supp}(v)$ and $u|_{\mathrm{supp}(u)} = \pm v|_{\mathrm{supp}(u)}$.

Claim 10.2.7.3. *Let $i < j \in [\lfloor n/2+1 \rfloor, \lfloor n/2+\ell \rfloor]$. If $u \in L_i$, then the number of functions $v \in L_j$ such that $\mathrm{supp}(u) \subset \mathrm{supp}(v)$ and $u|_{\mathrm{supp}(u)} = \pm v|_{\mathrm{supp}(u)}$ is*

$$\binom{n-i}{j-i} 2^{j-i}.$$

Proof. Consider a function $v \in L_j$ where $\mathrm{supp}(u) \subset \mathrm{supp}(v)$ and $u|_{\mathrm{supp}(u)} = \pm v|_{\mathrm{supp}(u)}$. There are $j-i$ elements of $[n] \setminus \mathrm{supp}(u)$ that are in $\mathrm{supp}(v) \setminus \mathrm{supp}(u)$, so there are $\binom{n-i}{j-i}$ choices for the elements of $\mathrm{supp}(v) \setminus \mathrm{supp}(u)$. For each $x \in \mathrm{supp}(v) \setminus \mathrm{supp}(u)$, there are two choices for the value of $v(x)$. Finally, there are two choices for $v|_{\mathrm{supp}(u)}$. Since $v(y) = 1$ for $y = \min(\mathrm{supp}(v))$, there are $\binom{n-i}{j-i}2^{j-i+1}/2 = \binom{n-i}{j-i}2^{j-i}$ choices for the function v. $\square$

By Proposition 10.2.2, the number of vertices in L_i is $\binom{n}{i}2^{i-1}$. Thus, by Claim 10.2.7.3,

$$\begin{aligned}
\mathbb{E}(X(U)) &= \sum_{i=\lfloor n/2+1 \rfloor}^{\lfloor n/2+\ell \rfloor -1} \sum_{u \in L_i} \sum_{j=i+1}^{\lfloor n/2+\ell \rfloor} \binom{n-i}{j-i} 2^{j-i}2^{1-i}2^{1-j} \\
&= \sum_{i=\lfloor n/2+1 \rfloor}^{\lfloor n/2+\ell \rfloor -1} \binom{n}{i} 2^{i-1} \sum_{j=i+1}^{\lfloor n/2+\ell \rfloor} \binom{n-i}{j-i} 2^{1-i}2^{1-i} \\
&= \sum_{i=\lfloor n/2+1 \rfloor}^{\lfloor n/2+\ell \rfloor -1} \binom{n}{i} 2^{1-i} \sum_{j=1}^{\lfloor n/2+\ell \rfloor - i} \binom{n-i}{j}.
\end{aligned}$$

Claim 10.2.7.4. *If $i \in [\lfloor n/2+1 \rfloor, \lfloor n/2+\ell \rfloor - 1]$, the expression $\binom{n}{i}2^{1-i}\sum_{j=1}^{\lfloor n/2+\ell \rfloor - i}\binom{n-i}{j}$ is maximized when $i = \lfloor n/2+1 \rfloor$.*

Proof. The maximum value of $\binom{n}{i}$ occurs when $i = n/2$. The maximum value of 2^{1-i} occurs when i is as small as possible, which, in this case, is when $i = \lfloor n/2+1 \rfloor$. Since $\binom{n-i}{j} \leq \binom{n}{j}$ for $i \geq 0$ and $j \leq \frac{n-i}{2}$, the maximum value of $\sum_{j=1}^{\lfloor n/2+\ell \rfloor - i}\binom{n-i}{j}$ also occurs when i is minimized. $\square$

By Claim 10.2.7.4,

$$\begin{aligned}
\mathbb{E}(X(U)) &\leq \ell \binom{n}{\lfloor n/2+1 \rfloor} 2^{1-\lfloor n/2+1 \rfloor} \sum_{j=1}^{\ell} \binom{\lceil n/2 \rceil - 1}{j} \\
&\leq \ell \binom{n}{n/2} 2^{1-n/2} \sum_{j=1}^{\ell} \binom{\lceil n/2 \rceil - 1}{j}.
\end{aligned}$$

By Corollary 2.2.2, and since $\ell = \frac{n}{4}(1-\varepsilon)$,

$$\sum_{j=1}^{\ell} \binom{\lceil n/2 \rceil - 1}{j} \leq \sum_{j=1}^{\frac{n}{4}(1-\varepsilon)} \binom{n/2}{j} \leq 2^{\frac{n}{2}} \mathrm{e}^{-\frac{\varepsilon^2 n}{8}}.$$

By Stirling's approximation, we know $\binom{n}{n/2} \le \sqrt{2/\pi}\frac{2^n}{\sqrt{n}}$. Therefore,

$$\begin{aligned}\mathbb{E}(X(U)) &\le \ell\binom{n}{n/2}2^{1-\frac{n}{2}}2^{\frac{n}{2}-\frac{\varepsilon^2 n}{8}\log(\mathrm{e})} \\ &\le \frac{n(1-\varepsilon)}{4}\sqrt{\frac{2}{\pi}}\frac{2^n}{\sqrt{n}}\cdot 2^{1-\frac{\varepsilon^2 n}{8}\log(\mathrm{e})} = 2^{n(1-\frac{\varepsilon^2}{8}\log(\mathrm{e}))}\sqrt{\frac{n}{2\pi}}(1-\varepsilon).\end{aligned}$$

Proposition 10.2.6 (Markov's Inequality) implies that $\mathbb{P}(X(U) < 2\mathbb{E}(X(U))) \ge \frac{1}{2}$, hence, by the equation above, it follows that

$$\mathbb{P}(X(U) < \sqrt{n}2^{n(1-\frac{\varepsilon^2}{8}\log(\mathrm{e}))}) \ge \frac{1}{2}.$$

Let a set $V \in \mathcal{U}$ be called *good* if the number of edges in $G[V]$ is less than $\sqrt{n}2^{n(1-\frac{\varepsilon^2}{8}\log(\mathrm{e}))}$. Let $\mathcal{U}_g$ denote the collection of sets V in $\mathcal{U}$ that are good. Recall that U is a set chosen from $\mathcal{U}$ uniformly at random. By the equation above it follows that $\mathbb{P}(U \text{ is good}) \ge \frac{1}{2}$. This implies that $|\mathcal{U}_g| \ge \frac{1}{2}|\mathcal{U}|$. In other words, at least half of the sets in $\mathcal{U}$ are good.

Removing one endpoint from each edge induced by a set $U \in \mathcal{U}$ gives a stable set of G. Thus, there exists a set V of at most $2\mathbb{E}(X(U)) \le \sqrt{n}2^{n(1-\frac{\varepsilon^2}{8}\log(\mathrm{e}))}$ vertices such that removing V from a good set $U \in \mathcal{U}_g$ results in a stable set. Let $\mathcal{S}_g$ denote the set of stable sets S that are obtained from a good set U by removing at most $\sqrt{n}2^{n(1-\frac{\varepsilon^2}{8}\log(\mathrm{e}))}$ vertices. Note that some stable sets in $\mathcal{S}_g$ are subsets of multiple good sets in $\mathcal{U}_g$.

For each stable set S in $\mathcal{S}_g$, let $r(S)$ denote the number of good sets U that contain S as a subset. Let $\mathcal{I}$ denote the collection of stable sets of G. Let $r = \max_{S\in\mathcal{I}} r(S)$. Since each stable set S is a subset of at most r good sets,

$$i(G) \ge \frac{|\mathcal{U}_g|}{r} \ge \frac{|\mathcal{U}|}{2r}. \tag{10.2}$$

Claim 10.2.7.5. $\log r \le 2^{n-1}(1+o(1))$.

Proof. Let S be a stable set in $\mathcal{S}_g$ such that $r = r(S)$. Let $\mathcal{V}$ be the collection of good sets $U \in \mathcal{U}_g$ such that $S \subseteq U$. Thus, we know $|\mathcal{V}| = r$. Each set $U \in \mathcal{V}$ contains the elements in S and $|U| - |S|$ elements from L. Let $m = |U| - |S|$ and let $v_1, v_2, \ldots, v_m$ be the elements of $U \setminus S$. For each $i \in [m]$, there exists a set $V_i \subseteq [n]$ where $\lfloor n/2+1\rfloor \le |V_i| \le \lfloor n/2+\ell\rfloor$ such that $v_i \in C_{V_i}$. By the definition of $\mathcal{U}$, the sets $V_1, V_2, \ldots, V_m$ are distinct.

Recall that there are k sets $V \subseteq [n]$ where $\lfloor n/2+1\rfloor \le |V| \le \lfloor n/2+\ell\rfloor$. Thus, there are at most $\binom{k}{m}$ choices for the sets $V_1, V_2, \ldots, V_m$. For each $i \in [m]$, there are at most

$|C_{V_i}| \leq 2^{\lfloor n/2+\ell \rfloor} \leq 2^{3n/4}$ choices for v_i. Therefore, there are at most $\binom{k}{m}(2^{3n/4})^m$ choices for the elements $v_1, v_2, \ldots, v_m$. This implies that $r = |\mathcal{V}| \leq \binom{k}{m}(2^{3n/4})^m$.

Since S is a stable set in $\mathcal{S}_g$ and U is a good set, the difference $m = |U| - |S|$ is at most $2\mathbb{E}(X(U)) \leq \sqrt{n}2^{n(1-\frac{\varepsilon^2}{8}\log(\mathrm{e}))}$. Since $k \leq 2^{n-1}$, we have $\binom{k}{m} \leq 2^{2^{n-1}}$. Therefore,

$$\log r \leq 2^{n-1} + \frac{3n}{4}\sqrt{n}2^{n(1-\frac{\varepsilon^2}{8}\log(\mathrm{e}))} = 2^{n-1}(1+o(1)). \qquad \square$$

By Equation 10.2 and Claims 10.2.7.5 and 10.2.7.1,

$$\begin{aligned} \log i(G) &\geq \log|\mathcal{U}| - \log r - 1 \\ &\geq n2^{n-1}(\tfrac{1}{2} + o(1)) - 2^{n-1}(1+o(1)) \\ &= n2^{n-1}(\tfrac{1}{2} + o(1)). \end{aligned} \qquad \square$$

Proof of Theorem 10.0.1. By Proposition 3.2.2, the stable sets of $\Pi(DG(n, GF(3)^*))$ correspond to the scarce extensions of $DG(n, GF(3)^*)$. Thus, the theorem follows from Lemmas 10.2.5 and 10.2.7. $\square$

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Glossary

$DG(n, \Gamma)$ Dowling geometry of rank n over Γ. 27, 95, 127

$GF(q)$ Galois field of order q. 10

$\mathcal{H}_{\min}(M, \prec)$ The "small" hyperplanes of M. 38

$K_{n,m}$ Complete bipartite graph with vertex set $[n+m]$ and bipartition $([n], [n+m] \setminus [n])$. 16

K_n Complete graph with vertex set $[n]$. 16

$M(G)$ Cycle matroid of G. 21

M/X M contract X. 22

$M \setminus X$ M delete X. 22

M^* The dual matroid of M. 22

$M|X$ M restricted to X. 22

$PG(n-1, q)$ Projective geometry of rank n over $GF(q)$ 28

$\Pi(M)$ The hyperplane graph of M. 38

$V(n, q)$ An n-dimensional vector space over $GF(q)$. 28

$(a; p)_n$ The p-shifted factorial. 28, 110

$\Omega(M)$ The circuit graph of M. 39

$\mathcal{C}(M)$ The set of circuits of M. 21

$\mathbf{cl}_M(X)$ The closure of X in M. 22

$\mathbf{coext}(M)$ The number of coextensions of M. 23

$\deg_G(v)$ Degree of v in G. 14

$\mathbb{E}(A)$ Expected value of A. 136

$\mathbf{ext}(M)$ The number of extensions of M. 23

$\mathcal{H}(M)$ The set of hyperplanes of M. 22

$\binom{X}{k}$ The set of k-subsets of X. 10

$o_q(1)$ A function of q that goes to 0 as n goes to infinity. 12

$o(1)$ A function of n that goes to 0 as n goes to infinity. 12

ln Natural (base-e) logarithm. 11

log Base-2 logarithm. 11

ν_q The shifted factorial $(q^{-1}; q^{-1})_\infty$. 110

$i(G)$ Number of stable sets in G. 18

$\Theta(M)$ The overlap graph of M. 41

$\mathbb{P}(A)$ Probability of A. 12, 136

$r_M(X)$ Rank of X in M. 22

$r(M)$ Rank of M. 22

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