Notions of Complexity Within Computable Structure Theory

by

Luke MacLean

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Pure Mathematics

Waterloo, Ontario, Canada, 2023

© Luke MacLean 2023

Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

External Examiner:	Valentina Harizanov Professor, Dept. of Mathematics, The George Washington University
Supervisor(s):	Barbara Csima Professor, Dept. of Pure Mathematics, University of Waterloo
Internal Members:	Jason Bell Professor, Dept. of Pure Mathematics, University of Waterloo Ross Willard Professor, Dept. of Pure Mathematics, University of Waterloo
Internal-External Member:	Nancy Day Associate Professor, School of Computer Science, University of Waterloo

Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

I am the sole author of Chapters 1 and 2 of this thesis. Chapter 3 is joint work with Rachael Alvir and Barbara Csima, and Chapter 4 is joint work with Barbara Csima and Dino Rossegger.

Abstract

This thesis covers multiple areas within computable structure theory, analyzing the complexities of certain aspects of computable structures with respect to different notions of definability.

In chapter 2 we use a new metatheorem of Antonio Montalbán's to simplify an otherwise difficult priority construction. We restrict our attention to linear orders, and ask if, given a computable linear order $\mathcal{A}$ with degree of categoricity $\boldsymbol{d}$, it is possible to construct a computable isomorphic copy of $\mathcal{A}$ such that the isomorphism achieves the degree of categoricity and furthermore, that we did not do this coding using a computable set of points chosen in advance. To ensure that there was no computable set of points that could be used to compute the isomorphism we are forced to diagonalize against all possible computable unary relations while we construct our isomorphic copy. This tension between trying to code information into the isomorphism and trying to avoid using computable coding locations, necessitates the use of a metatheorem. This work builds off of results obtained by Csima, Deveau, and Stevenson [13] for the ordinals ω and ω^2, and extends it to ω^α for any computable successor ordinal α.

In chapter 3, which is joint work with Alvir and Csima, we study the Scott complexity of countable reduced Abelian p-groups. We provide Scott sentences for all such groups, and show some cases where this is an optimal upper bound on the Scott complexity. To show this optimality we obtain partial results towards characterizing the back-and-forth relations on these groups.

In chapter 4, which is joint work with Csima and Rossegger, we study structures under enumeration reducibility when restricting oneself to only the positive information about a structure. We find that relations that can be relatively intrinsically enumerated from such information have a definability characterization using a new class of formulas. We then use these formulas to produce a structural jump within the enumeration degrees that admits jump inversion, and compare it to other notions of the structural jump. We finally show that interpretability of one structure in another using these formulas is equivalent to the existence of a positive enumerable functor between the classes of isomorphic copies of the structures.

Acknowledgements

I would like to thank Dr. Barbara Csima for advising me through two degrees that would not have happened without her help. Having you to help work through problems that I was facing was incredibly helpful, and after meeting with you weekly for almost five years, I will be hearing your advice in the back of my head through all future problem solving endeavours.

To my colleagues Dr. Rachael Alvir and Dr. Dino Rossegger it was lovely working with you both. The computability theory group at the University of Waterloo is rather small, and so having people with different approaches and different interests within the topic has helped me broaden my comfort zone. It was also nice having enough people to justify a learning seminar.

I could not give enough thanks to account for all that my parents have done for me. Your support in all ways possible has lead me to where I am now. I truly believe that your constant encouragement to try anything and everything I was interested in has resulted in me doing both what I enjoy, and what I am good at.

Finally, a big thank you to my partner, who has been keeping me grounded throughout most of my PhD. School can get very stressful at times, and having someone who is always by my side and always willing to joke and laugh with me has been a great blessing.

Table of Contents

List of Figures

Chapter 1

Introduction

Notational conventions for basic definitions on computability will be following Turing Computability by Robert Soare [30]. Results on enumeration degrees mentioned in this thesis can be found in Computability Theory by S. Barry Cooper [11]. For computable structure theoretic results the reader can consult Computable Structures and the Hyperarithmetical Hierarchy by C.J. Ash and Julia F. Knight [5] or Computable Structure Theory: Within the Arithmetic and Computable Structure Theory: Beyond the Arithmetic by Antonio Montalbán[26] [27].

1.1 Computability Theory

Mathematicians in the early 1900s were concerned with formally defining what it meant for a procedure to be algorithmic. Intuitively, it should mean that there is some finite number of simple steps which one can follow without much thought, and arrive at the desired result. Many different notions were proposed, but the one that has prevailed is Alan Turing's definition of a Turing Machine, which accepts binary strings as input and can read and alter these strings, either halting and outputting a binary string, or running forever. Using clever coding techniques of Gödel and others, we can talk about much more than binary strings using Turing Machines, and in fact we often refer to these machines as computing functions from the natural numbers to the natural numbers with the coding specifics left in the background. One such coding technique that will come up frequently throughout this paper is the Cantor pairing function $\langle\,\rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. This function assigns a unique natural number to each pair of numbers in such a way that the original elements of the pair can be computably recovered. This recoverability allows us to unambiguously write $\langle a_1, \ldots, a_n \rangle = \langle \cdots \langle a_1, a_2 \rangle, a_3 \rangle \cdots \rangle, a_n \rangle$.

1.1.1 Turing Degrees

Thankfully, we do not need to understand the complexities of Turing Machines because we choose to adopt the Church-Turing thesis, which states that a function is computable by a Turing Machine exactly when it is intuitively computable. What we do need to know about Turing programs is that they are all finite sets of symbols from countable alphabets, and so we can effectively list them under some chosen ordering $P_0, P_1, \ldots$ and associate to each P_e the partial computable function Φ_e determined by P_e. We will write $\Phi_e(x) \downarrow = y$ to say that Φ_e on input x converged to y. We can then say that a function f is computable if there is some Turing functional Φ_e which converges and mirrors f on all inputs. We likewise say that a set of natural numbers A is computable when its characteristic function χ_A is. We will often make no distinction between a set and its characteristic function.

To compare the computability theoretic properties of sets of natural numbers Church and Turing developed the notion of an oracle machine. These are Turing machines that have all the information of a certain set B and at any stage in the computation can access a finite portion of the set. We think of these extra steps as "queries to the oracle" B, and we say that B computes another set A and write $A \leqslant_T B$ if there is some Turing functional Φ_e such that $\Phi_e^B = \chi_A$. An example of this is the join of two sets A and B. We define their join to be the set $A \oplus B = \{2n : n \in A\} \cup \{2n + 1 : n \in B\}$. It is easy to see that with access to the information of the set $A \oplus B$ we can compute the members of A and B and so $A \leqslant_T A \oplus B$ and $B \leqslant_T A \oplus B$.

The work done in the previous paragraph would be meaningless if every set was computable. To this end Kleene provided the fundamental example of a non-computable set with the halting set $K = \{x \in \mathbb{N} : \Phi_x(x) \downarrow\}$. This is the set of all programs in our listing that halt on their own input. It is easily shown to be non-computable by a diagonalization argument. The issue being that we could end up waiting forever for a program to halt. Those programs that do halt will do so in finite time, and so we can enumerate the set K. Letting $W_e = dom(\Phi_e)$ we obtain an effective listing of all the computably enumerable sets. As before we can let $W_e^B = dom(\Phi_e^B)$ and say that a set A is computably enumerable in a set B if there is an e such that $A = W_e^B$. We can talk about computable approximations to c.e. sets, by saying that $W_{e,s}$ is all of the elements x such that $\Phi_e(x)\downarrow$ in fewer than s steps of the computation. We say that an element n enters W_e at stage s if $n \in W_{e,s} \backslash W_{e,s-1}$.

By looking at the halting set relative to some oracle A we produce a set $K^A = A' := \{x \in \mathbb{N} : \Phi_x^A(x)\downarrow\}$ called the jump of A which is not computable from A. By iterating this jump operator we produce a hierarchy of non-computable sets. We write $A \equiv_T B$ if $A \leqslant_T B$ and $B \leqslant_T A$ and define the Turing degree of a set A to be $\boldsymbol{a} = deg(A) := \{B : B \equiv_T A\}$. We let $\mathbf{0} = deg(\varnothing)$ and $\boldsymbol{a}' = deg(A')$. Thus $deg(K) = deg(\varnothing') = \mathbf{0}'$. For two degrees $\boldsymbol{a} = deg(A), \boldsymbol{b} = deg(B)$ we say that $\boldsymbol{a} \leqslant \boldsymbol{b}$ if $A \leqslant_T B$.

A stronger notion of reducibility between sets is that of 1-reducibility. We say that A is one-one reducible to a set B and write $A \leqslant_1 B$ if there is a one-to-one computable function f such that $x \in A \iff f(x) \in B$. A nice fact about the Turing jump operator is that $A \leqslant_T B$ if and only if $A' \leqslant_1 B'$. An additional important property shown by Schoenfeld is that we can invert the jump operator. That is, given a set B such that $\varnothing' \leqslant_T B$, we can find a set A such that $A' \equiv_T B$.

1.1.2 Enumeration Degrees

While oracle Turing reducibility allows the Turing machine to ask questions about what is and is not in the oracle set, in practice we often do not know the negative information about a set, and receive our information in a setting more akin to enumerations. This led Friedberg and Rogers to introduce the notion of enumeration reducibility in 1959.

We say that a set A is enumeration reducible to another set B and write $A \leqslant_e B$ if there is an enumeration operator Ψ_e which is just a c.e. set such that $A = \Psi_e^B = \{x : \langle x, D\rangle \in \Psi_e \text{ for } D \subseteq_{\text{fin}} B\}$. Note that using this convention we obtain from an enumeration of the members of B an enumeration of the members of A. Furthermore, the order in which B is enumerated does not matter. We then define $deg_e(A) = \{B \subseteq \mathbb{N} : A \equiv_e B\}$ as we did for the Turing degrees. Unlike the Turing degrees however, we cannot use the halting set as our token non-computable set because it is computably enumerable. In our proposed hierarchy the bottom degree $\mathbf{0}_e$ will consist of all c.e. sets.

It is easy to show that a set A is computable if and only if A and its complement $\overline{A}$ are computably enumerable. Using this result we can see that it must be the case that $\overline{K}$ is not c.e., and so it acts as a good candidate for providing the extra information that we would want an enumeration jump operator to have. Thus, we define $\mathbf{0}'_e = deg_e(\overline{K})$ and the enumeration jump of a set A to be the set $J_e(A) := A \oplus \overline{K_A}$ where $K_A = \{x \mid x \in \Psi_x^A\}$. We thus obtain a hierarchy of enumeration degrees just as before.

The enumeration degrees $\mathcal{D}_e$ differ from the Turing degrees $\mathcal{D}$, but they are related as we shall now see.

Definition 1.1.1. The totalization of a set A is the set $A^+ = A \oplus \overline{A} \equiv_e \chi_A$.

Any set such that $A \equiv_e A^+$ is said to be total and an enumeration degree is total if it contains a total set. If a set is total and c.e. then it is computable. Similarly, we have that $A \leqslant_T B$ if and only if $A^+ \leqslant_e B^+$ and A is c.e. in B if and only if $A \leqslant_e B^+$. Thus we get that the Turing degrees embed in the enumeration degrees as exactly the total degrees, and we define the embedding $\iota(deg_T(A)) = deg_e(A^+)$. On total degrees, the enumeration jump agrees with the Turing jump, i.e. $deg_e((A')^+) = deg_e(J_e(A^+))$. We also know that the range of the enumeration jump operator is exactly the total degrees $\boldsymbol{a} \geqslant \mathbf{0}'_e$, since $K_A \leqslant_e A$ and $\overline{A} \leqslant_e \overline{K}_A$ implies that $J_e(A) \equiv_e J_e(A)^+$.

Enumeration degrees also admit jump inversion. Soskov [31] showed that given a total set B we can always find a total set F such that $J_e(F) \equiv_e B$.

1.1.3 Hyperarithmetic Hierarchy

One method to study the complexity of sets is to place them on the hierarchy of degrees as we've seen above. Another method is to study their definability in the set theoretic sense.

We say that a set is Σ^0_n if it is the projection of a Σ^0_n relation, that is, a set of the form $\{x | \exists x_1, \forall x_2, \exists x_3 \ldots Q x_n R(x, x_1, \ldots, x_n)\}$ for n alternating quantifiers, $Q \in \{\forall, \exists\}$ depending on the parity of n, and R a computable relation. Similarly we say that a set A is Π^0_n if $\overline{A}$ is Σ^0_n and say A is Δ^0_n if A is Σ^0_n and Π^0_n. A formula is Σ^0_n if it is Σ^0_n when viewed as a relation of its free variables.

An important theorem of Emil Post relates the two notions by saying that a set A is computable in $\varnothing^{(n)}$ if and only if A is Δ^0_{n+1}. Thus A is Σ^0_1 if and only if it is computably enumerable and our prime example of a Σ^0_1 set remains the halting set K. It is stronger than just being Σ^0_1 because given any other Σ^0_1 set W_e there is a 1-reduction from W_e to K. We call any set with this property Σ^0_1-complete and generalize it naturally to say that $\varnothing^{(n)}$ is Σ^0_n-complete.

An issue arises when we move to transfinite ordinals because there is no single-valued system of notations for all computable ordinals. To surmount this issue, Kleene developed a system of notations for ordinals called Kleene's $\mathcal{O}$ where each computable ordinal α gets a notation $a \in \mathcal{O}$ such that $|a| = \alpha$. We build up $\mathcal{O}$ and a partial order $<_{\mathcal{O}}$ as follows:

- $1 \in \mathcal{O}$ and $|1| = 0$,
- If $i \in \mathcal{O}$ and $|i| = \alpha$ then $2^i \in \mathcal{O}$ and $|2^i| = \alpha + 1$ and $i <_{\mathcal{O}} 2^i$,
- If Φ_e halts on all inputs and $ran(\Phi_e) \subset \mathcal{O}$ and for all n, $\Phi_e(n) <_{\mathcal{O}} \Phi_e(n+1)$, then $3 \cdot 5^e \in \mathcal{O}$ and for all n, $\Phi_e(n) <_{\mathcal{O}} 3 \cdot 5^e$ and $|3 \cdot 5^e| = \lim_n |\Phi_e(n)|$.

This is why, once a notation is fixed for an infinite ordinal α there is a unique path through $\mathcal{O}$ with limit α. Now to each $a \in \mathcal{O}$ we define the set $H(a)$ by transfinite recursion.

- $H(1) = \varnothing$,
- $H(2^a) = H(a)'$,
- $H(3 \cdot 5^e) = \{\langle u, v \rangle : u \leqslant_3 \cdot 5^e \,\&\, v \in H(u)\}$.

This labeling system produces sets that still agree with our arithmetic hierarchy for finite n, for if $|a| = n$ then $H(a) = \varnothing^{(n)}$. For infinite α, there are infinitely many total computable functions with limit α, but all of their notations produce Turing equivalent sets $H(a)$ [2]. Now we can continue defining our hyperarithmetic sets by saying that a relation R is Σ^0_α if it is c.e. with respect to $H(a)$, where $|a| = \alpha$. Similarly to before, a relation is Π^0_α if its complement is Σ^0_α, and Δ^0_α if and only if it is Σ^0_α and Π^0_α.

Each limit ordinal α has a unique strictly increasing sequence of ordinals $\{\alpha(p)\}_{p\in\omega}$ with limit α which is determined by our notation for α, and so to take the jump at limit ordinals, we let $A^{(\alpha)}$ be the set $\{\langle x, p\rangle : x \in A^{(\alpha(p))}\}$. When we iterate the Turing jump transfinitely many times a discrepancy appears between the two hierarchies, and the set $\varnothing^{(\alpha+1)}$ is Σ^0_α-complete.

If we are instead working with the logic $\mathcal{L}_{\omega_1\omega}$, as there is more occasion to do in computable structure theory, then we still allow finitely many nested quantifiers, but allow conjunctions over any countable set. This adds expressive power to the language $\mathcal{L}$. For this logic, Σ^0_0 and Π^0_0 formulas are the same as in first-order logic, but a Σ^0_α formula is one of the form $\bigvee\!\!\!\bigvee_{i\in\omega} \exists\overline{x}\varphi_i(\overline{x},\overline{y})$ where each φ_i is a Π^0_β formula for $\beta < \alpha$. Similarly, Π^0_α formulas are of the form $\bigwedge\!\!\!\bigwedge_{i\in\omega} \forall\overline{x}\varphi_i(\overline{x},\overline{y})$ where each φ_i is a Σ^0_β formula for $\beta < \alpha$. We say that a formula is $d-\Sigma^0_\alpha$ if it is the conjunction of a Σ^0_α formula and a Π^0_α formula. A sentence is a formula with no free variables.

The superscript 0 in all classes of formulas denotes the fact that we are restricting our quantification to natural numbers. Since no other types of quantification will be used in this thesis, the superscript may be omitted.

1.2 Computable Structure Theory

To study the complexity of mathematical structures, we need to agree upon a notion of when such a structure is computable. The universe of the structure certainly must be computable, but we also ask that the atomic diagram of the structure be computable. We will see later how to effectively associate a set of natural numbers to the atomic diagram $D(\mathcal{A})$ of a structure $\mathcal{A}$. A structure $\mathcal{A}$ is said to be decidable if its elementary diagram $D^e(\mathcal{A})$

is computable. The elementary diagram contains all first-order sentences with parameters from the structure that are true in the structure. Since we will only be discussing structures with countable domain, we may assume that the domain is a subset of the natural numbers. We call such structures ω-models or ω-presentations.

A natural question to then ask is "given a structure $\mathcal{A}$, how difficult is can it be to compute isomorphic copies of $\mathcal{A}$". The answer to that question turns out to be "as difficult as we want", since we can encode almost any set into the domain and relations of a structure. Letting $DgSp(\mathcal{A})$ be the set of Turing degrees of all copies of $\mathcal{A}$, Knight showed that it is closed upwards in the Turing degrees. That is, for any set X such that $D(\mathcal{A}) \leqslant_T X$ there is a copy $\mathcal{B}$ of $\mathcal{A}$ such that $D(\mathcal{B}) \equiv_T X$. The question of finding a maximal complexity for isomorphic copies of a structure remains interesting if we restrict our attention to computable copies of $\mathcal{A}$ and ask how difficult it is to compute the isomorphism between copies. A structure $\mathcal{A}$ that is isomorphic to a computable structure is said to be computably presentable and the computable dimension of a computably presentable structure is the number of computable presentations up to computable isomorphism. Special interest is given to those structures with computable dimension 1 or ω.

We say that a computable structure $\mathcal{A}$ is $\boldsymbol{d}$-computably categorical if, for all computable $\mathcal{B} \cong \mathcal{A}$, there exists a $\boldsymbol{d}$-computable isomorphism between $\mathcal{B}$ and $\mathcal{A}$. Under this notation, structures of computable dimension 1 are computably categorical. We call $\boldsymbol{d}$ the degree of categoricity of $\mathcal{A}$ if $\mathcal{A}$ is $\boldsymbol{d}$-computably categorical and, for all $\boldsymbol{c}$ such that $\mathcal{A}$ is $\boldsymbol{c}$-computably categorical, we have $\boldsymbol{d} \leqslant \boldsymbol{c}$. It is not guaranteed that a structure will have a degree of categoricity. We say that $\mathcal{A}$ is Δ^0_α categorical on a cone if there is a $\boldsymbol{c} \in \mathcal{D}$ such that for all $\boldsymbol{d} \geqslant \boldsymbol{c}$, whenever $\mathcal{B}$ and $\mathcal{C}$ are $\boldsymbol{d}$-computable copies of $\mathcal{A}$, there exists a $\Delta^0_\alpha(\boldsymbol{c})$-computable isomorphism between $\mathcal{B}$ and $\mathcal{C}$. In chapter 2, we study isomorphic copies of linear orders where the isomorphism achieves the degree of categoricity of the structure, and see if there must always be a computable unary relation whose image can compute the isomorphism.

1.2.1 Scott rank and back-and-forth relations

A related notion for studying the complexity of structures is the Scott complexity of a structure. Scott proved that for every countable structure $\mathcal{A}$ there is a sentence $\varphi_{\mathcal{A}}$ in the infinitary logic $\mathcal{L}_{\omega_1\omega}$ called a Scott sentence of $\mathcal{A}$ such that $\mathcal{A} \models \varphi_{\mathcal{A}}$ and for all other countable structures $\mathcal{B}$, $\mathcal{B} \models \varphi_{\mathcal{A}}$ if and only if $\mathcal{A} \cong \mathcal{B}$ [5]. The complexity of this sentence in the hyperarithmetical hierarchy can tell us about how complicated it is to describe the structure, however if a structure has a Π^0_α Scott sentence, then it trivially has a Π^0_η Scott sentence for all $\eta \geqslant \alpha$. The Scott rank of a structure $\mathcal{A}$ has gone through many different proposed definitions, but a recent and robust notion by Montalbán [24] is to define it to be the least α such that $\mathcal{A}$ has a $\Pi^0_{\alpha+1}$ Scott sentence. As previously mentioned, this notion of complexity is related to categoricity and it was shown by Montalbán that a structure has Scott rank α if and only if $\mathcal{A}$ is Δ^0_α-categorical on a cone. This notion of Scott rank has many other applications as well such as providing an upper bound on the complexity of defining the automorphism orbits of tuples from the structure.

A slightly finer notion concerning the Scott rank introduced by Alvir, Greenberg, Harrison-Trainor, and Turetsky in [2] is to look more closely at the possible complexities for the formula. A formula that is the conjunction of a Σ^0_α and a Π^0_α formula is called a $d-\Sigma^0_\alpha$ formula and is more complex than both a Σ^0_α and a Π^0_α formula. However it is less complicated than both a $\Sigma^0_{\alpha+1}$ formula and a $\Pi^0_{\alpha+1}$ formula. Thus the complexities form a lattice hierarchy with Σ^0_α and Π^0_α being incomparable for all α. The Scott complexity of a structure is the least complexity on this hierarchy such that $\mathcal{A}$ has a Scott sentence of that complexity. In chapter 3 we produce Scott sentences for arbitrary countable reduced Abelian p-groups and seek to show that they are best possible to determine the Scott complexity of such structures.

Broadening our scope a bit, we can ask how difficult it is to tell apart different structures over a fixed language. Even broader still, is asking how difficult it is to distinguish two tuples within their respective structures. This question stems from a model-theoretic notion, and so we try to distinguish the tuples using definability, but we have seen above the relation between definability and computability.

Definition 1.2.1. Given two structures $\mathcal{A}, \mathcal{B}$, tuples $\overline{a}, \overline{b}$ of the same length from their respective structures, and a computable ordinal α, we say that $(\mathcal{A}, \overline{a})$ is α back-and-forth below $(\mathcal{B}, \overline{b})$, written $(\mathcal{A}, \overline{a}) \leqslant_\alpha (\mathcal{B}, \overline{b})$ if all of the Π^0_α formulas true of $\overline{a}$ in $\mathcal{A}$ are true of $\overline{b}$ in $\mathcal{B}$.

At the zero level, we say that $(\mathcal{A}, \overline{a}) \leqslant_0 (\mathcal{B}, \overline{b})$ if all of the quantifier-free formulas φ of Gödel number $\varphi \leqslant |\overline{a}|$ true of $\overline{a}$ in $\mathcal{A}$ are true of $\overline{b}$ in $\mathcal{B}$. If we are not interested in a specific tuple, we may write $\mathcal{A} \leqslant_\alpha \mathcal{B}$ to mean $(\mathcal{A}, \varnothing) \leqslant_\alpha (\mathcal{B}, \varnothing)$. Similarly, we shorten $(\mathcal{A}, \overline{a}) \leqslant_\alpha (\mathcal{A}, \overline{b})$ to $\overline{a} \leqslant_\alpha \overline{b}$.

The reason that they are called back-and-forth relations, is because there is an equivalent definition for ordinals $\alpha > 0$ where we say that $(\mathcal{A}, \overline{a}) \leqslant_\alpha (\mathcal{B}, \overline{b})$ if and only if for all $\beta < \alpha$ and $\overline{d} \in \mathcal{B}^{<\omega}$ there is $\overline{c} \in \mathcal{A}^{<\omega}$ such that $(\mathcal{B}, \overline{bd}) \leqslant_\beta (\mathcal{A}, \overline{ac})$ (See Ash-Knight [5]). Determining whether tuples satisfy the relations then can be viewed as going back and forth with an opponent who is trying to extend the $\overline{b}$ tuple in such a way that there is a Π^0_β formula true of their new tuple that you cannot satisfy with an extension of your $\overline{a}$ tuple. Gamification of proof methods is a common theme in computability theory that we will explore more later in chapter 2.

1.2.2 R.i.c.e. relations and the structural Turing jump

Using either of the notions of a jump operator discussed in previous sections, we can take the jump of the atomic diagram of a structure $\mathcal{A}$ and get a set that is not computable from $D(\mathcal{A})$. But, there is no guarantee that this set could be viewed as the atomic diagram of a structure. When dealing with structures, we would like to give a jump that produces a new structure that has the same complexity as if we had applied the regular jump operator. In the Turing degree setting this problem has been solved by Montalbán [26].

The structural jump of a structure $\mathcal{A}$ is formed by adding a relation that encodes all of the relations that a relatively intrinsically computably enumerable in $\mathcal{A}$.

Definition 1.2.2 (Montalbán [26])**.** A relation R on $\mathcal{A}$ is said to be relatively intrinsically computably enumerable (r.i.c.e.) if, in any isomorphic copy $\mathcal{B} = f(\mathcal{A})$, $f(R) \leqslant_T D(\mathcal{B})$.

These r.i.c.e. relations turn out to be exactly the ones that are definable by Σ_1 formulas using countable disjunctions over c.e. index sets with parameters from the structure, and so, by creating a r.i.c.e. complete relation and adding it to the language, we mimic the addition of $\overline{K}$ in the enumeration jump. Other versions of the structural jump have been proposed, but the strongest evidence that this version is the best is that it admits jump inversion. Letting $J(\mathcal{A})$ be the structural jump of $\mathcal{A}$, we have that $DgSp(J(\mathcal{A})) = \{\boldsymbol{a}' : \boldsymbol{a} \in DgSp(\mathcal{A})\}$. There has not yet been an analogue of the structural jump in the enumeration degree setting. In chapter 4 we develop a fitting definition for the enumeration jump of a structure and consider its properties.

Chapter 2

Degrees of Categoricity

2.1 Background

Let $\mathcal{N}$ be the standard copy of the structure $(\omega, <)$. We shall construct a copy $\mathcal{A}$ such that the isomorphism $f : \mathcal{A} \to \mathcal{N}$ can compute the halting set K. We construct this by stages, adding a new element at each stage, ensuring that the limit structure is computable, isomorphic to $\mathcal{N}$, and that the resulting isomorphism can compute K. For every even stage $2s$ we add a new even number to the end of our current finite linear order by declaring that $2s$ is greater than everything else on which we have defined the order. For odd stages $2s+1$ we will wait until a new number is enumerated into the halting set. If n enters the halting set at stage s, then we declare that $2n <^{\mathcal{A}} 2s+1 <^{\mathcal{A}} 2n+2$. We may assume that our enumeration is such that only one element is enumerated at each stage. It is not difficult to see that this construction will result in a structure $\mathcal{A}$ which is isomorphic to $\mathcal{N}$. Furthermore, if we know the isomorphism $f : \mathcal{A} \to \mathcal{N}$, then to decide whether n is in the halting set, we look at the images of $2n$ and $2n+2$ in $\mathcal{N}$. They are mapped to immediate successors if and only if n never entered the halting set. This is an f computable question, and the successor relation on $\mathcal{N}$ is computable. Thus, $K \leqslant_T f$.

This proof is an example of a typical computable structure theoretic proof. If we want to show that an isomorphism or other object can attain a certain degree of difficulty, we simply construct a copy to ensure that it does. Throughout the construction we keep track of certain special points, whose eventual locations under the isomorphism will compute the set of known complexity. Herein lies the issue. In the above example we did not need the full isomorphism to compute K. As long as we know where the even numbers are sent, we will still be able to compute the halting set. As is the case with many constructions that use a predetermined computable set of points. We then ask whether it is possible to construct a computable copy of ω on which there is no computable unary relation whose image will compute the isomorphism.

This question was answered by Csima, Deveau, and Stephenson [13] who were able to show the following:

Proposition 2.1.1. *Let $\mathcal{A}$ be any computable copy of $(\omega, <)$ and let $f : \mathcal{A} \to \mathcal{N}$ be the isomorphism from $\mathcal{A}$ into the standard copy. Let $U := \{m \mid (\exists n)[n <^{\mathcal{N}} m \wedge m <^{\mathcal{A}} n]\}$. Then $f(U) \equiv_T f$.*

However they also showed

Proposition 2.1.2. *There exists a computable copy $\mathcal{B}$ of $(\omega, <)$ such that the isomorphism $f : \mathcal{N} \to \mathcal{B}$ can compute the Halting set K and furthermore, there is no computable unary relation U on $\mathcal{N}$ with $f(U) \equiv_T f$.*

This provides an interesting asymmetry between maps *into* and *out of* $\mathcal{N}$ that shall be explored in greater detail below. Before we continue though, important clarifications should be made. We are only considering maps for which there is no computable *unary* relation whose image computes the isomorphism. We make this choice over n-ary relations because for any two computable copies $\mathcal{A}, \mathcal{B}$ of a structure with isomorphism $f : \mathcal{A} \to \mathcal{B}$ it is the case that $f(R) \equiv_T f$, where $R = \{(n, n+1) \mid n \in \omega\}$. Additionally, we must explain why, in the above example, we ask that the map computes K instead of a different non-computable set. This is because K is the degree of categoricity of $(\omega, <)$.

There remain two major questions at this point.

- Why does the asymmetry exist?

- Does it exist for more complicated linear orders?

The first point turns out to be rather easy to answer. If the image of the relation lies in the decidable copy, then we get extra information for free. We can perform bounded searches as in Proposition 2.1.1 because we know the entire elementary diagram of the structure.

Towards the second question, more work has been done by Csima, Deveau, and Stephenson [13]. They were able to show that

Proposition 2.1.3. *Let $\mathcal{N}^2$ be the decidable copy of $(\omega^2, <)$. There is a computable copy $\mathcal{A}$ of $(\omega^2, <)$ such that if $f : \mathcal{A} \to \mathcal{N}^2$, then for no computable unary relation U do we have $f(U) \equiv_T f$.*

as well as strengthening it to show that

Proposition 2.1.4. *Let $\mathcal{N}^2$ be the decidable copy of $(\omega^2, <)$. There is a computable copy $\mathcal{A}$ of $(\omega^2, <)$ such that if $f : \mathcal{A} \to \mathcal{N}^2$, then for no computable unary relation U do we have $f(U) \equiv_T f$, and furthermore $f \geqslant_T \varnothing''$.*

However, the proofs become increasingly difficult as more information needs to be coded into the isomorphism, and so the question with regards to the degree of categoricity for $(\omega^2, <)$, $\mathbf{0}'''$, was left open. The degrees of categoricity for all computable well-orderings are given below.

Theorem 2.1.5 (Bazhenov [7])**.** *Assume that α is a computable ordinal.*

(a) If $0 < k < \omega$ and $\omega^k \leqslant \alpha < \omega^{k+1}$ then α has strong degree of categoricity $\mathbf{0}^{(2k-1)}$.

(b) If γ is a computable infinite ordinal and $\omega^\gamma \leqslant \alpha < \omega^{\gamma+1}$ then α has strong degree of categoricity $\mathbf{0}^{(2\gamma)}$.

To push the result up to higher linear orders, we will need a general proof framework to manage the increasing amounts of non-computable information with which we have to work. To do so purely through priority constructions would be a Herculean task. Thankfully, we have metatheorems, which do exactly what we need for this situation.

2.2 Metatheorems

When tasked with doing a priority construction, the complexity of the information that we need to code in will affect the construction process. For computable information, we can simply compute the answers that we want at each stage, and code that information in without ever needing to change what we previously did. For a $\mathbf{0}'$-priority construction, we will be using computable approximations to $\varnothing'$ to answer our questions, and so as our approximation gets better, we may have to change things that we have already done. The challenge increases as we move up to a $\mathbf{0}''$ construction, as our approximation to the Σ^0_2 set might be incorrect infinitely often and we have to ensure that we devise a strategy for meeting the requirements that accounts for this, and still produces a computable structure.

Beyond $\mathbf{0}''$, the conditions become too difficult to keep track of in a purely combinatorial setting, and so many general proof frameworks for $\mathbf{0}^{(\alpha)}$-priority constructions have been proposed, the most intuitive of which is the game metatheorem of Antonio Montalbán [25]. Most metatheorems seek to construct a tree, with paths through the tree representing possible forms the constructions can take. The goal then, is to ensure that the tree is built in such a way that there is always a suitable branch above what we have already constructed, thereby ensuring that there is an infinite path that meets our requirements. The game metatheorem envisions the construction as a game between two players, with a third omniscient Oracle that can answer queries. Together, the two players take turns playing longer and longer initial segments of a structure $\mathcal{L}$. The Engineer is in charge of making sure that the structure satisfies a certain property $\mathfrak{B}$ that we would like the structure to have. She can ask questions to the Oracle, which is a $\Delta^0_{\alpha+1}$-complete set. The other player is the Extender, who is in charge of making sure that the limit structure is computable. He will not coordinate with the Engineer, and does not care if $\mathcal{L}$ satisfies property $\mathfrak{B}$. Were the Extender left to his own devices, there would be no way that the resulting structure would satisfy any properties. Thankfully, the Engineer can restrict how the Extender can play by choosing the structure in which they play. Likewise, if the Engineer did not have to respect the Extender's moves in some fashion, $\mathcal{L}$ would never be computable as she uses information from the Oracle in her moves. We will now give a formal definition of the metatheorem.

Formally, an $\eta - \mathbb{A}$-game is played on a list of structures $\mathbb{A} = \{\mathcal{A}_n\}_{n\in\omega}$ such that the back-and-forth relations between tuples are uniformly computable up to η. At each stage j of the game, the Engineer will play a triple $\langle i_j, \overline{a}_j, e_j\rangle$ with $i_j, e_j \in \mathbb{N}$ and $\overline{a}_j \in \mathcal{A}_{i_j}^{<\omega}$. The Engineer will respond by playing a tuple $\overline{b}_j \in \mathcal{A}_{i_j}^{<\omega}$ and the Oracle will play the pair $\langle n_j, \beta_j\rangle$ where the number n_j represents the answer to the e_j^{th} $\Delta^0_{\eta+1}(D(\mathcal{L}))$ question and β_j is an ordinal below η.

Engineer	$i_0, \overline{a}_0, e_0$		$i_1, \overline{a}_1, e_1$		$i_2, \overline{a}_2, e_2$		$\cdots$
Extender		$\overline{b}_0$		$\overline{b}_1$		$\overline{b}_2$	$\cdots$
Oracle		η_0, β_0		η_1, β_1		η_2, β_2	$\cdots$

For stages $j > 0$ the tuple $\overline{a}_j$ must satisfy $(\mathcal{A}_{j-1}, \overline{b}_{j-1}) \leqslant_{\beta_{j-1}} (\mathcal{A}_j, \overline{a}_j)$. The Extender's tuple must be such that $\overline{b}_{i_j} \supset \overline{a}_{i_j}$. Thus, after ω many moves a structure $\mathcal{L}$ is constructed such that $D(\mathcal{L}) = \bigcup_n D_{\mathcal{A}_n}(\overline{a}_n)$. Given $X \in 2^\omega$, we fix a $\Delta^0_{\eta+1}(X)$ Turing-complete set S^η_X as our Oracle in advance, and say that n is the answer to the e^{th} $\Delta^0_{\eta+1}(X)$ question if $n = \Phi_e^{S^\eta_X}(0)$. If η is a successor ordinal, then for each stage we may assume that $\beta_j = \eta - 1$. If η is a limit ordinal, then we may assume that the sequence $\{\beta_j\}_j$ is strongly increasing with limit η.

A strategy for the Engineer is a function that tells her what to play next, based on what the Extender has already played. It is called a valid strategy if on all possible plays by the Extender, all queries to the Oracle converge. The metatheorem can now be stated.

Theorem 2.2.1 (Montalbán [25])**.** *Let η be a computable ω presentation of an ordinal and suppose $\mathbb{A} = \{\mathcal{A}_0, \mathcal{A}_1, \dots\}$ is a list of structures where the back-and-forth relations are computable up to η. For every computable valid strategy in the η -$\mathbb{A}$-game there is a sequence of moves by the Extender, such that, if the Engineer follows her strategy and the Oracle answers truthfully, the limit ω-presentation $\mathcal{L}$ is computable.*

2.3 Isomorphisms of ω^2

Our goal is to create a computable isomorphic copy $\mathcal{A}$ of ω^2 with isomorphism $f : \mathcal{A} \to \mathcal{N}^2$ where $\mathcal{N}^2$ is the decidable copy such that $f \equiv_T \varnothing'''$ and there is no computable unary relation U on $\mathcal{A}$ such that $f(U) \equiv_T f$. To use the game metatheorem with linear orders, we must first understand the back-and-forth relations on these structures.

Lemma 2.3.1 (Montalbán [26]). *If $\overline{a}, \overline{b}$ are tuples of different lengths from the same structure, then $\overline{a} \leqslant_\alpha \overline{b}$ iff $|\overline{a}| < |\overline{b}|$ and $\overline{a} \leqslant_\alpha \overline{b}\restriction|\overline{a}|$.*

Lemma 2.3.2 (Knight [5] lem 15.8). *Suppose $\mathcal{A}$ and $\mathcal{B}$ are linear orders. Let $\overline{a} = (a_0, \ldots, a_{n-1})$ and $\overline{b} = (b_0, \ldots, b_{n-1})$ be increasing tuples from $\mathcal{A}$ and $\mathcal{B}$ respectively, and let $\mathcal{A}_i$, $\mathcal{B}_i$ be intervals such that*

$$\begin{aligned}\mathcal{A} &= \mathcal{A}_0 + \{a_0\} + \mathcal{A}_1 + \cdots + \{a_{n-1}\} + \mathcal{A}_n \\ \mathcal{B} &= \mathcal{B}_0 + \{b_0\} + \mathcal{B}_1 + \cdots + \{b_{n-1}\} + \mathcal{B}_n\end{aligned}$$

Then $(\mathcal{B}, \overline{b}) \leqslant_\alpha (\mathcal{A}, \overline{a})$ if and only if for all $0 \leqslant i \leqslant n$, $\mathcal{B}_i \leqslant_\alpha \mathcal{A}_i$.

Lemma 2.3.3 (Knight [5] lem 15.10). *Let δ be either a limit ordinal or 0, and let α and β be computable ordinals. For each ordinal ζ let*

$$\begin{aligned}\alpha &= \omega^\zeta \cdot \alpha_\zeta + \rho_\zeta, \\ \beta &= \omega^\zeta \cdot \beta_\zeta + \sigma_\zeta,\end{aligned}$$

where $\rho_\zeta, \sigma_\zeta < \omega^\zeta$, and let s_ζ, t_ζ be the coefficients of ω^ζ in the Cantor normal form expressions for α, β, respectively. Then

(a) $\alpha \leqslant_{\delta+2n+1} \beta$ if and only if one of the following holds:

- *(i) $\alpha = \beta$*
- *(ii) $\rho_{\delta+n} = \sigma_{\delta+n}$, $\alpha_{\delta+n+1} \geqslant 1$, and $\beta_{\delta+n+1} \geqslant 1$,*
- *(iii) $\rho_{\delta+n} = \sigma_{\delta+n}$, $\alpha_{\delta+n+1} = \beta_{\delta+n+1} = 0$, and $s_{\delta+n} \geqslant t_{\delta+n}$.*

(b) $\alpha \leqslant_{\delta+2n+2} \beta$ if and only if one of the following holds:

- *(i) $\alpha = \beta$*
- *(ii) $\rho_{\delta+n} = \sigma_{\delta+n}$, $\alpha_{\delta+n+1} \geqslant 1$, $\beta_{\delta+n+1} \geqslant 1$, and $s_{\delta+n} \geqslant t_{\delta+n}$.*

(c) for δ a limit ordinal, $\alpha \leqslant_\delta \beta$ if and only if one of the following holds:

(i) $\alpha = \beta$

(ii) $\rho_\delta = \sigma_\delta$, $\alpha_\delta \geqslant 1$, and $\beta_\delta \geqslant 1$.

We will use this lemma to ensure that our moves at each stage of the game respect the conditions imposed by the Extender. The specific way in which we will use the lemma is given below.

Corollary 2.3.4. *Let $k, l, m \in \omega$ such that $0 < l, m$. Then*

$$\omega^k \cdot l \leqslant_{2k} \omega^k \cdot m.$$

Proof. Take Lemma 2.3.3 (b)(ii) with $\delta = 0$, $n = k-1$. Then $\rho_{\delta+n} = \sigma_{\delta+n} = 0$, $\alpha_{\delta+n+1} = l$, $\beta_{\delta+n+1} = m$, and $s_{\delta+n} = t_{\delta+n} = 0$ $\square$

We are now ready to set up our game and describe the strategy for the Engineer.

Theorem 2.3.5. *There exists a computable isomorphic copy $\mathcal{A}$ of ω^2 with isomorphism $f : \mathcal{A} \to \mathcal{N}^2$, where $\mathcal{N}^2$ is the decidable copy, such that $f \equiv_T \varnothing'''$ and there is no computable unary relation U on $\mathcal{A}$ such that $f(U) \equiv_T f$.*

Proof. To answer our question we will use Theorem 2.2.1 with $\eta = 2$. For our pool of structures $\mathbb{A}$, we will use the single structure ω^2. The back-and-forth relations are thus computable up to 2 using Lemma 2.3.3. Each move made by the Engineer and Extender will be a pair $(\omega^2, \overline{a}_s)$ where $\overline{a}_s$ is a tuple from ω^2, and so we omit the structure as it never changes. The role of the Extender ensures that the limit structure is an ω-presentation, and so the Engineer must ensure that each point is only moved finitely often, and that every grid point is filled, so that the limit tuple $\overline{a} = \lim_s \overline{a}_s$ forms an isomorphism by sending $i \mapsto a_i$. We will denote the associated partial map of a tuple $\overline{a}$ with $p_{\overline{a}}$. The copy of ω^2 we seek will be the pullback of $\mathcal{N}^2$ through the isomorphism we construct.

Instead of playing the tuple $\overline{a}_s$, we can think of putting labeled balls on a grid presentation of ω^2. Each column of the grid represents a copy of ω within ω^2, and the point (e, i)

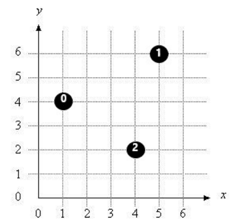

Figure 2.1: A representation of the tuple $\langle\langle 1,4\rangle,\langle 5,6\rangle,\langle 4,2\rangle\rangle$

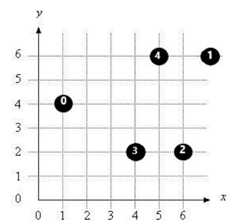

Figure 2.2: A representation of the tuple $\langle\langle 1,4\rangle,\langle 7,6\rangle,\langle 6,2\rangle,\langle 5,6\rangle,\langle 4,2\rangle\rangle$

on the grid represents the element $\omega \cdot e + i$ of ω^2. If there is the ball labeled s on position (e, i) in the grid, then we know that $\langle e, i\rangle$ will be in position s of the tuple. By the end of the construction we will have covered the grid in balls inducing a natural isomorphism with the standard presentation. The only types of changes that we will make to our grid are shifting balls to columns further to the right, and replacing balls that were shifted with new balls, so that the oracle of the computation against which we are trying to diagonalize is preserved. Both of these are demonstrated by the shift from Fig. 2.1 to Fig. 2.2, and both will be elaborated upon later. Using Lemma 2.3.1, we see that the new tuple produced by making these changes is in the same order as our old tuple when we restrict to the same length. The only difference is that the distance between the two balls closest to the insertion point is some number of copies of ω greater than in the old tuple. Corollary 2.3.4 tells us that this is a valid move.

We are allowed to ask questions to our $\varnothing''$ Oracle in this 2-$\{\omega^2\}$-game, and $\varnothing'''$ is c.e. in $\varnothing''$. Thus, at each stage we will ask the Oracle to enumerate a bit more of $\varnothing'''$, and when we see numbers enter the set we will code that into our isomorphism. We will denote the stage s approximation to $\varnothing'''$ by $K_s^{\varnothing''}$. The strategy is to sequentially pick special grid locations called coding locations. Their eventual locations in the limit structure will tell us whether or not the corresponding numbers are in $\varnothing'''$.

The way that we will code when a number n enters the halting set is by shifting a special limit point γ_n to the right by inserting copies of ω into our copy of ω^2. This is the same as moving every ball in a certain column of our grid or higher to the right, and Corollary 2.3.4 tells us that it preserves the back-and-forth relations. The eventual grid placement of a coding location will also indicate to us where to find the next coding location. For each i we have the following requirement;

Γ_i: The coding location γ_i is mapped to an even multiple of ω if and only if $i \in \varnothing'''$. Additionally, the distance between γ_{i-1} and γ_i is of the form $2\langle|\bar{b}_s|, m\rangle$ or $2\langle|\bar{b}_s|, m\rangle + 1$ for some s, m where $|\bar{b}_s| + 1$ is the label of the ball marked γ_{i+1}.

We will say that the requirement Γ_i requires attention at stage s if $i \in K_s^{\varnothing''}$.

Marking and unmarking balls is imprecise terminology. In reality we can imagine that we are building an infinite well-ordered set G alongside our actual construction. To mark a ball as γ_i, we simply ensure that the label on that ball is the ith element of G. If an injury forces us to unmark balls, we simply remove the corresponding label from G. While this is not explicitly permitted in the game construction, at each stage of the construction we can just code the finite set as a single number and ask the Oracle to remind us of the number at the next stage. In this way, at the end of the construction we will have the code for an infinite well-order whose elements tell us precisely the columns of the grid containing coding locations. It also avoids the issue of using a predetermined computable set of coding locations to code $\varnothing'''$. Each new location depends on the previous move of the Extender. If they could be computed, then we would have a way of computing $\varnothing'''$, which is impossible.

To further aid in the visualization of our construction, we can imagine that the computable unary relations on $\mathcal{A}$ colour the balls, either green if the relation holds of that ball, or red otherwise. Since our map sends the label to its grid location, the colour of the ball that rests upon a grid location will determine whether that grid location is in the image of the relation under our eventual isomorphism. This is important, because we will shift balls to other grid locations, but as long as we replace it with one of the same colour, the image of the relation will remain unchanged. Note, however, that the colour of a ball may change if we are diagonalizing against a different unary relation.

We also know that, if there is some computable unary relation U on $\mathcal{A}$ whose image will compute f, its characteristic function will be φ_e for some e and there will be some Φ_i such that $\Phi_i^{f(U)} = f$. To avoid this, we are required to diagonalize against all φ_e that could pose a threat. If φ_e is not total, or does not code a set U_e which is infinite and coinfinite, then there is no hope of $f(U_e) \equiv_T \varnothing'''$. If U_e is finite or cofinite then $f(U_e)$ will also be, and if either $f(U_e)$ or $\overline{f(U_e)}$ is finite, then $f(U_e)$ is computable and cannot compute the set $\varnothing'''$.

The plan for diagonalizing against a single pair $\langle e, i\rangle$ is to pick a witness x and wait for the partial isomorphism that we're building to have enough information to say that $\Phi_i^{f(U_e)}(x) \downarrow= f(x)$. We then seek to change $f(x)$ without changing $\Phi_i^{f(U_e)}(x)$. This means that we cannot change the oracle used for the computation (i.e. $f(U_e)$). And so, when we insert copies of ω to shift the ball that was witnessing this agreeance, we must replace any shifted green balls with green balls, and likewise red with red. This can always be done, since the only case in which we diagonalize is if U_e is infinite and coinfinite. Truthfully, we have no choice in the label of the next ball that we play. Each move must be the least label not yet used in the computation. If the colour of this ball matches the color that we are trying to replace, which is to say, they are both in the computable unary relation, then we have no problem. Otherwise, we play the ball out beyond anything that we have currently played, so that its placement does not injure any requirement, and check the colour of the next smallest ball. This gives us the following requirement, which we seek to meet for all pairs $\langle e, i\rangle$;

$R_{\langle e,i\rangle}$**:** If φ_e is total and codes an infinite and coinfinite set U_e then there exists some x such that $\Phi_i^{f(U_e)}(x) \neq f(x)$.

We will say that the requirement $R_{\langle e,i\rangle}$ requires attention as stage s if U_e is a threat, and there is an x in the column directly to the right of $\gamma_{\langle e,i\rangle}$ such that $\Phi_i^{p_{\overline{a}_s}(U_e)}(x) = p_{\overline{a}_s}(x)$. Recall that each grid location is a pair $\langle n, m\rangle$. To figure out the column of a ball, all we must do is figure out where it is sent using our finite partial isomorphism, and compute the first element of the pair.

The main issue with this construction is that our two different objectives will interfere with each other. If we have diagonalized against a certain computation somewhere in the construction, and then we are forced to move things because we saw a lower number enter the halting set, it will injure the oracle that we were trying to preserve. To fix this, we will give the requirements a priority order $\Gamma_0 > R_0 > \Gamma_1 > R_1 > \cdots$, so that a requirement of a lower priority cannot injure one of a higher priority. Thus, each requirement can only be injured finitely many times, and at a certain point will be permanently met. When an R_i requirement is injured, it will choose a new witness, and can only choose a witness once the corresponding γ_i has been marked. This will not be an issue, as we will mark a new γ_i at each step.

Construction: At stage 0 the Engineer will play the ball 0 on the grid location $(1, 0)$ and mark it as γ_0. She will also ask the Oracle whether $0 \in K_0^{\varnothing''}$, whether φ_0 is a total function which determines an infinite and coinfinite set U_0, and finally, if $\Phi_0^{p_{\bar{a}_0}(U_0)}(x) \downarrow = p_{\bar{a}_0}(x)$ for some x in the column immediately to the right of γ_0.

At stage $s + 1$ the Extender will have played a tuple $\bar{b}_s$, and the Oracle will have answered for all $j \leqslant s$ and all $\langle e, i\rangle \leqslant s$ whether $j \in K_s^{\varnothing''}$, and whether φ_e is total and determines an infinite coinfinite set U_e. Furthermore, we will know whether or not there exists an x in the column immediately to the right of $\gamma_{\langle e,i\rangle}$ such that $\Phi_i^{p_{\bar{a}_s}(U_e)}(x) \downarrow = p_{\bar{a}_s}(x)$.

At each stage the Engineer will mark the smallest coding location γ_j that hasn't been marked, and attempt to meet the highest priority requirement that requires attention.

For marking the coding location, she will take the smallest ball that hasn't been used yet and place it at the bottom of the smallest odd column to the right of everything that has been played so far. She will mark the ball as γ_j, and insert enough copies of ω directly to the left of γ_{j-1} that the number of columns between it and γ_{j-2} is $2\langle \gamma_j, \ell_j\rangle$ if requirement Γ_{j-1} has not been met, or $2\langle \gamma_j, \ell_j\rangle + 1$ if Γ_{j-1} has been met, where ℓ_{j-1} is the number of columns between γ_{j-2} and γ_{j-1} before any are inserted (chosen so that we will never end up in a situation where there are already too many columns between them). If $j - 1$ is zero, then there is no γ_{j-2}, and so the Engineer inserts columns until there are $2\langle\gamma_j, \ell_j\rangle$ (or $2\langle\gamma_j, \ell_j\rangle + 1$ as before) to the left of γ_{j-1}. If R_{j-1} has been met, then we must replace the balls shifted with ones of the same colour as described above.

The Engineer will then act to meet the highest priority requirement that requires attention. Recall that inserting columns is the same as shifting all balls in a certain column or higher to the right which is the same as increasing the first coordinate of the element $\langle n, m\rangle$ of the tuple $\overline{a}$ which represents the current location of all of the balls.

If the highest priority requirement is of the form $R_{\langle e,i\rangle}$, then we let ℓ be the largest column with a ball in it thus far, and we tell the Engineer to shift all balls in the column of our witness or greater to the right by 2ℓ and replace all balls that were shifted with new ones of the same colour as described above. Meeting $R_{\langle e,i\rangle}$ will injure all requirements of lower priority, and un-mark all γ_k for $k > \langle e, i\rangle + 1$.

If the highest priority requirement is of the form Γ_j, then the Engineer must shift every ball in γ_j's column or higher to the right by one to change the parity of γ_i from odd to even, thus meeting Γ_j. She must also shift everything in a strictly higher column by one more, to preserve the parity of γ_{j+1}. Meeting Γ_j injures all of the lower priority requirements and unmarks γ_k for $k > j + 1$.

Finally, the Engineer must place a ball at the top of every column that currently has a ball to the right of it. This completes our construction.

Claim 1: At each stage s of the game, we have that $\overline{b}_{s-1} \leqslant_2 \overline{a}_s$.

Proof. Since our only move in meeting a requirement is to insert extra copies of ω between columns of the grid, this will not change the order of elements, neither in the tuple, nor in the linear order. Thus, while $\overline{b}_s \restriction |\overline{a}_{s+1}|$ and $\overline{a}_{s+1}$ are not necessarily in increasing order, if we were to write each in increasing order, the corresponding elements of the tuples would match, and every subinterval would be identical, save for the one that we made larger. Then applying Lemma 2.3.2 and Corollary 2.3.4 we see that each subinterval for $\overline{a}_{s+1}$ is 2-above the corresponding $\overline{b}_s$ subinterval. $\square$

Claim 2: The strategy for the Engineer is computably valid.

Proof. We must ensure all of the questions to the Oracle are Δ^0_3 and will converge. The set Tot is Π^0_2, and once we know that a φ_e is total, the question of whether it is infinite or coinfinite can be phrased as $\forall x \exists y[\varphi_e(x) \neq \varphi_e(y) \wedge x < y]$. We are also only using $\varnothing''$ to enumerate $\varnothing'''$.

The strategy for the Engineer based on the responses to the question uses only computable information about the tuple $\overline{a}_s$ and $\overline{b}_s$ to construct $\overline{a}_{s+1}$. During the construction we can view any finite portion of the grid, and so looking at the number of columns between the coding locations is computable at any stage. While we are asking different questions at each stage of the construction, in the same way that we can code multiple questions into one single question, we can ask any finite number of questions to the Oracle. So at each step we have access to the answers of all previous questions to the Oracle, because if we wanted to, we could have asked all previous questions and more at each new stage. $\square$

Thus, the metatheorem (Theorem 2.2.1) tells us that there is a Δ^0_3 sequence of moves by the Extender such that, if the Engineer follows her computably-valid strategy, the limit presentation $\mathcal{A}$ is computable. The limit presentation is such that $D(\mathcal{A}) = \bigcup_{j \in \mathbb{N}} D_{A_j}(\overline{a}_j)$. We must check that each requirement is met during the construction, and that in doing so, our isomorphism has the desired properties.

Claim 3: For each requirement, there is a stage of the construction after which it is never injured.

Proof. We will prove this by induction on the priority ordering. Assume that the first s-many requirements are only injured finitely and that we are at a stage in the construction past which any of them will be injured.

If the next highest priority requirement is Γ_i for some i, then R_{i-1} is the only requirement that could injure it, because anything else would also injure higher priority requirements contradicting our hypothesis. If $i = 0$ then there is no R_{i-1} and nothing can injure Γ_i so the result holds. For $i > 0$, if R_{i-1} never acts then Γ_i is never injured again and we are done. If we must diagonalize against a computation to meet R_{i-1} then we will shift the ball γ_i outside of the use of the computation that gives us the diagonalization. Notice that γ_{i+1} is not unmarked in this case. This is important because part of the requirement Γ_i is that γ_i can tell us where to find γ_{i+1}. The number of columns between γ_{i-1} and γ_i will be of the form $2\langle\gamma_{i+1}, \ell_s\rangle(+1)$. If we were to unmark γ_{i+1} in meeting R_i, we would have to insert more copies of ω into our grid before γ_i to indicate the new location of γ_{i+1}, which could injure R_i ad infinitum. Thus γ_{i+1} is moved to a place on the grid where it will not disturb, R_i, but the label remains the same. Every lower priority requirement from then on will be met to the right of γ_i in the grid, and so nothing else will injure Γ_i. Thus Γ_i is only ever injured a finite number of times.

If the next requirement is of the form R_i for some i, then inductively it can only be injured by Γ_i. If we never see i enter our enumeration of $\varnothing'''$ then R_i will never be injured. If we do need to act to meet Γ_i then we will shift γ_i an by one and only search for witnesses in the column immediately to the right of it. Since γ_{i+1} is at least two columns to the right nothing else will injure R_i. □

Claim 4: The γ_i's are only unmarked finitely often.

Proof. A coding location γ_1 is only unmarked when a requirement of higher priority is injured. Since we have established that they are only injured finitely often, each γ_i will only be unmarked finitely often as well. □

Claim 5: $f = \lim_s p_{\overline{a}_s}$ is a well-defined isomorphism.

Proof. The Engineer is the only one who can change the location of the balls on the grid. In Claims 3 & 4 we have showed that each requirement is only injured finitely often and that each γ_i is only unmarked finitely often as well. Since we mark a new γ_i at each step and since each subsequent requirement is met further to the right of the grid, this means that the grid will eventually settle.

Since at every turn the Engineer places a ball in a new column and new balls at the tops of each current column, the grid will eventually fill in completely. □

Claim 6: $f \equiv_T \varnothing'''$.

Proof. To figure out whether or not $n \in 0'''$ one must first answer the same question for $0, \dots, n-1$. Based on the strategy above we know that the ball 0 will be on the grid point $(2\langle\gamma_1, \ell_s\rangle, 0)$ or $(2\langle\gamma_1, \ell_s\rangle + 1, 0)$ for some s. The parity of odd/even will tell us whether $0 \in \varnothing'''$ and the value γ_1 will tell us which ball to check to see if $1 \in \varnothing'''$. We then check where the ball γ_1 is sent under our isomorphism. Knowing already where γ_0 was sent, we can count the number of columns in between the two and it will be a number of the form $2\langle\gamma_2, \ell_t\rangle$ or $2\langle\gamma_2, \ell_t\rangle + 1$ for $t > s$. The parity will again tell us whether or not $1 \in \varnothing'''$ and we get the value of γ_2 exactly as above and repeat until n. The value ℓ_t is large enough that if R_1 moves the ball γ_2 the Extender can still add more columns before γ_1 to indicate where to find γ_2. □

Claim 7: There is no computable unary relation U on $\mathcal{A}$ such that $f(U) \equiv_T f$.

Proof. Assume, towards a contradiction, that there was such a relation. Furthermore, WLOG assume that $\varphi_e = \chi_U$ (a.k.a. U_e) and that $\Phi_i^{f(U_e)} = f$. By the use principle, there is a finite part of the oracle $f(U_e)$ that witnesses this computation. After a certain stage s in the construction, every requirement of a higher priority that was going to act would have acted, and so $R_{\langle e,i\rangle}$ will not be injured any further. At every stage $t > s$ we then would have asked the Oracle if there was any ball x in a certain column on which $\Phi_i^{p_t(U_e)}(x) = p_t(x)$. Since $p_t \subset f$, there must be a stage where we witness an agreement, or in the limit we would not have $\Phi_i^{f(U_e)}(x) = f(x)$ for all x in the aforementioned column. We then would have diagonalized against it and preserved the use of the computation. This is a contradiction, and so our claim is satisfied. □

Thus, we have a computable valid strategy for our Engineer and the resulting structure provided by the metatheorem has the properties we set out to construct.

□

2.4 Isomorphisms of ω^α

We now seek to extend this proof to show that for any finite $1 < k < \omega$ we can construct two copies of ω^k such that the isomorphism achieves the degree of categoricity for the structure without using a predetermined computable set of coding locations.

Theorem 2.4.1. *For $k \in \omega, k \geqslant 2$ there exists a computable isomorphic copy $\mathcal{A}$ of ω^k with isomorphism $f : \mathcal{A} \to \mathcal{N}^k$ where $\mathcal{N}^k$ is the decidable copy such that $f \equiv_T \varnothing^{(2k-1)}$ and there is no computable unary relation U on $\mathcal{A}$ such that $f(U) \equiv_T f$.*

Proof. To prove this, we play a $(2k-2)-\{\omega^k\}-$game, and use a $\varnothing^{(2k-2)}$ Oracle to enumerate $K^{\varnothing^{(2k-2)}}$. To view ω^k as a grid, we must take an ordinal in its Cantor normal form $\omega^k \cdot n_k + \omega^{k-1} \cdot n_{k-1} + \cdots + \omega \cdot n_1 + n_0$ and write it as $\langle n_k, \langle n_{k-1}, \ldots, n_0 \rangle \rangle$. This is the same as making each column in the grid of the form ω^{k-1}. Since Corollary 2.3.4 tells us that adding copies of ω^{k-1} results in a tuple that is $2(k-1)$ above the previous, we can proceed with the construction exactly as above. The questions we ask to the Oracle do not change, except, that we ask it to enumerate $K^{\varnothing^{(2k-2)}}$ instead, which is a valid question in our $(2k-2)$-game. $\square$

When we move to transfinite ordinals α, the degree of categoricity increases from $\mathbf{0}^{(2\alpha-1)}$ to $\mathbf{0}^{(2\alpha)}$, but so does the computational strength of our Oracle. Instead of using $\mathbf{0}^{(k)}$ as our Δ^0_{k+1} set, we may use $\mathbf{0}^{(\alpha+1)}$ as a $\Delta^0_{\alpha+1}$ set. This, together with Corollary 2.4.2 allows us to continue to extend the theorem to any computable infinite successor ordinal

Corollary 2.4.2. *Let $\alpha = \delta + n + 1$ where δ is a computable limit ordinal. Let $l, m \in \omega$ be such that $0 < l, m$. Then*

$$\omega^{\alpha-1} \cdot l \leqslant_{\delta+2n} \omega^{\alpha-1} \cdot m.$$

Proof. Following Lemma 2.3.3 (b)(ii) for $\delta + 2(n-1) + 2$, we see that $\rho_{\delta+n-1} = \sigma_{\delta+n-1} = 0$, $\alpha_{\delta+n} = l, \beta_{\delta+n} = m$, and $s_{\delta+n-1} = t_{\delta+n-1} = 0$. $\square$

Theorem 2.4.3. *For all computable ordinals $\alpha = \delta + n + 1$ where δ is an infinite limit ordinal and $n \in \omega$, there exists a computable isomorphic copy $\mathcal{A}$ of ω^α with isomorphism $f : \mathcal{A} \to \mathcal{N}^\alpha$, where $\mathcal{N}^\alpha$ is the decidable copy, such that $f \equiv_T \varnothing^{(2\alpha)}$ and there is no computable unary relation U on $\mathcal{A}$ such that $f(U) \equiv_T f$.*

Proof. We will be playing a $(\delta + 2n) - \{\omega^\alpha\}$-game. Since α is a successor ordinal, we may use the same grid technique as in Theorem 2.4.1. Our Oracle in this game is $\varnothing^{(\alpha+2n+1)}$, which is strong enough to enumerate $K^{\varnothing^{(\alpha)}}$ which is the degree of categoricity for the structure ω^α. Corollary 2.4.2 tells us that our insertion strategy will still provide tuples that respect the back-and-forth restraints imposed by the metatheorem. $\square$

If we are not requiring the isomorphism to compute the degree of categoricity of the structure, then our game becomes simpler. We can use an Oracle strong enough to directly compute the set we would like to code into our isomorphism, and yet the back-and-forth relations that our Oracle necessitates us to obey still allow us wiggle-room to change our play between stages. For limit ordinals, we use one final application of Lemma 2.3.3 in the following Corollary.

Corollary 2.4.4. *Let ζ, η, λ be computable ordinals such that λ is a limit ordinal and $\lambda < \eta \leqslant \zeta$. Then*

$$\omega^\eta \leqslant_\lambda \omega^\zeta.$$

Proof. We apply Lemma 2.3.3 (c)(ii) with $\delta = \lambda$, $\rho_\delta = \sigma_\delta = 0$, and $\alpha_\delta = \omega^{\eta-\beta}, \beta_\delta = \omega^{\zeta-\beta}$. $\square$

Proposition 2.4.5. *Let α, β be computable ordinals such that $\mathbf{0}^{(\beta)}$ is less than the degree of categoricity of ω^α. There exists a computable isomorphic copy $\mathcal{A}$ of ω^α with isomorphism $f : \mathcal{A} \to \mathcal{N}^\alpha$, where $\mathcal{N}^\alpha$ is the decidable copy, such that $\varnothing^{(\beta)} \leqslant_T f$ and there is no computable unary relation U on $\mathcal{A}$ such that $f(U) \equiv_T f$.*

Proof. If α is finite or an infinite successor ordinal, then this is an immediate corollary of Theorems 2.4.1 and 2.4.3.

If α is a limit ordinal, then we must make bigger changes to our strategy. We can no longer immediately meet the strongest possible level of respect to account for any β. However, since $\beta < \alpha$ we may ensure that at each stage we are playing a tuple β above the previous, so that we do not have to deal with the increasing levels of respect that would come from an α game. The big change comes from the way that we must view our grid. We can write $\omega^\alpha = \sum_{p \in \omega} \omega^{\alpha(p)}$ and view it as a grid where the $i-th$ column is of the form $\omega^{\alpha(i)}$. Having to obey the β restraint relations effectively fixes any column where $\alpha(p) < \beta$, and so our first play, instead of being on the grid spot $(1, 0)$, will be on the location $(n, 0)$

for the least odd n such that $\beta < \alpha(n)$. The rest of the strategy is as before, except that instead of inserting copies of ω^k we are inserting the appropriate columns. This changes the way that balls are moved in the copy of ω^α that we are building, but does not change the strategy, nor the grid visualization. The validity of each move follows still from Corollary 2.4.4.

□

2.5 Further Work

While the game strategy above works for successor ordinals, the case of building a computable copy of ω^α for α a limit ordinal remains open. Recall that when using the metatheorem for a limit ordinal, at each step of the construction the Oracle gives us a larger ordinal which signifies the level of back-and forth relations which we have to respect. The game that we played in Theorem 2.3.5 becomes nullified when we do not know the back-and-forth level of respect that we will have to obey before we ask the question. If $\alpha = \omega$ then at each stage we would have to obey some finite level n of back-and-forth restraint. This restraint effectively fixes the first n columns of our ω^ω grid. Not knowing what the next level of restraint will be means that any predetermined coding location could become fixed before we are able to insert columns to the left of it. It also fixes what the Extender has done in those columns, and so we lose some of our power to diagonalize. If we try limiting what can be fixed and where the Extender can play by increasing our pool of structures, then we run into the issue of the increasing levels of restraint. If we move to a bigger structure, then trying to mirror the tuple played by the Extender will always result in one subinterval that is larger (and therefore not back-and-forth below). This means that any proof of the result using the game metatheorem would have to use a drastically different approach.

We encounter issues with the metatheorem as well, if we try building a computable copy $\mathcal{B}$ of ω^α where $\alpha > 1$ such that there is no computable unary relation on $\mathcal{N}^\alpha$ whose image $f(U)$ can compute the isomorphism $f : \mathcal{N}^\alpha \to \mathcal{B}^\alpha$. Let us consider the case $\alpha = 2$, so that it may be compared to Theorem 2.3.5. Since the computable relations are now on the standard copy, as we build the computable copy $\mathcal{B}$ we are building the image of the relations. Before, if we needed to change the image of the relation, we could do so by moving a ball and replacing it with one of the same colour. In this way the image of the ball remains in the relation and yet the isomorphism has changed. This was always possible because the relations we were diagonalizing against were infinite and coinfinite, so there were always balls with which to replace.

In the case of mapping out of the standard copy, instead of colouring the balls we are colouring the grid spots. Therefore, to change the isomorphism while preserving the image of the relation, we must move a ball from a grid spot to one of the same colour. This becomes an issue as the grid fills in, because we can only insert copies of ω while respecting the 2 back-and-forth relations. So if we shift a column that has finitely many balls placed in it, we must find another column with the same coloured grid points in each location. This is not guaranteed to exist.

The main tools enabling us to employ the game metatheorem are an understanding of the back-and-forth relations on a structure, and an understanding of the degrees of categoricity. Understanding these aspects is easy to achieve when the structures are simple, of computable dimension 1 for example. A different class of structures where the back-and-forth relations are well understood and, which additionally provide interesting examples to work with, are superatomic Boolean algebras. It would be interesting to see which of the results proved in this section for linear orders carry over. Every superatomic Boolean algebra can be thought of as an interval algebra over a linear order, and so many results on the degree spectra of relations included in Hirschfeldt [18] are mentioned for linear orders and superatomic Boolean algebras together.

Chapter 3

Scott Complexity of Groups

The work done in the previous chapter cannot presently be extended to mathematical groups. This is because neither the degrees of categoricity nor the back-and-forth relations are well understood for this class of structures. Barker [6] managed to give a characterization for the back-and-forth relations between tuples from the same reduced countable Abelian primary group, but even to try and look at two different reduced Abelian p-groups becomes much more difficult. A result that will help us greatly in this class of groups is Ulm's theorem, which uniquely characterizes countable torsion groups up to isomorphism.

Ulm's theorem proves to also be a helpful tool for trying to determine the Scott complexity of such groups. Alvir and Rossegger [3] were able to construct Scott Sentences of optimal complexity for scattered linear orders. Our goal in this chapter is to characterize the back-and-forth relations for countable reduced Abelian p-groups, and use that to try and find the Scott complexity of such groups.

3.1 Background

In this section, given a group G, when we write nx for $n \in \omega, x \in G$, we mean $x + \overset{n}{\cdots} + x$. We will only be working in Abelian groups for which the group operation is commutative. For a fixed prime p, a group G is said to be primary (or p-primary) if for each group element $x \in G$ $p^n x = 0$ for some $n \geqslant 1$. We say that G is divisible if for every $x \in G$ there exists a $y \in G$ and $n \in \omega$ such that $x = ny$. A reduced group is one with no non-trivial divisible subgroups. Going forward, when we refer to a group we will mean a countable reduced Abelian primary group unless stated otherwise.

Let $G_0 = G$, and for any ordinal α we define $G_{\alpha+1} = pG_\alpha$. That is to say, $G_{\alpha+1} = \{x \in G : \exists y \in G_\alpha \, py = x\}$. We call this the set of elements of height at least α. If α is a limit ordinal, then we define $G_\alpha = \bigcap_{\beta<\alpha} G_\beta$. Notice that if $\alpha < \beta$ then $G_\alpha \supseteq G_\beta$. We define the largest α such that $x \in G_\alpha$ to be the height of x and denote it $h(x)$. The height function does not always play nicely with respect to sums. For $x, y \in G$ if $h(x) < h(y)$ then $h(x + y) = h(x)$, but if $h(x) = h(y)$ then $h(x + y) \geqslant h(x)$. Given a subgroup S of G, we say that x is proper with respect to S if $h(x) \geqslant h(x + s)$ for all $s \in S$. This is the same as the height of x being maximal in its coset $mod\, S$. For such proper elements, $h(x + s) = \min\{h(x), h(s)\}$. We also know that for all $x \neq 0$ we have $h(px) > h(x)$. We define $h(0) = \infty$ to be greater than any ordinal to ensure this property.

We similarly define, for all α, $P_\alpha = P \cap G_\alpha$, where $P = \{g \in G : pg = 0\}$. Viewing $P_\alpha/P_{\alpha+1}$ as a vector space over $\mathbb{Z}_p$, we define the α-th Ulm invariant to be its dimension, and denote it $f_G(\alpha)$. In addition, since G is countable, there must be an ordinal λ called the length of G such that $G_\lambda = G_\xi$ for all $\xi \geqslant \lambda$. Since G is reduced, we can conclude that $G_\lambda = \{0\}$.

If G is a direct sum of cyclic groups then its length is bounded by ω, and we know that $f(n)$ is exactly the number of cyclic summands of order p^{n+1} for each n.

Theorem 3.1.1 (Ulm [20])**.** *Two reduced countable primary abelian groups are isomorphic if and only if they have the same Ulm invariants.*

For groups of length greater than ω, it is harder to find explicit examples of groups with certain Ulm invariants. Additionally, determining the Ulm invariants from a group's presentation is not easy. We know, from a paper of Droste and Göbel [14], that any sequence can be realized as a group with those Ulm invariants, and so ideally we can work exclusively with Ulm sequences.

Thus, to describe a given a group G of length λ up to isomorphism, we wish to say for each α up to λ that there is a basis of $P_\alpha/P_{\alpha+1}$ of size k. Barker [6] has shown that

Proposition 3.1.2. *Let α be an ordinal. The group $G_{\omega\cdot\alpha}$ is definable by a $\Pi^0_{2\alpha}$ formula. The group $G_{\omega\cdot\alpha+n}$ for $n > 0$ is definable by a $\Sigma^0_{2\alpha+1}$ formula.*

Given an $x \in G_\alpha$ we have seen that $h(px) > h(x)$. Most of the time, it is the case that $h(px) = h(x) + 1$, but there are some x for which it increases more. Let $p^{-1}G_{\alpha+2} = \{z \in G : pz \in G_{\alpha+2}\}$. Then given a subgroup S, we define $S_\alpha = S \cap G_\alpha$ and look at the subgroup $S^*_\alpha = S_\alpha \cap p^{-1}G_{\alpha+2}$. For any $x \in S^*_\alpha$ we can find a $y \in G_{\alpha+1}$ such that $px = py$. Given such a y we may add any element of $P_{\alpha+1}$ and we will get a new element with the same desired property. Observe that the element $x - y$ is in P_α since $p(x - y) = 0$ and $h(x - y) = h(x)$. The mapping of x to $x - y$ for such a y as above, followed by the natural homomorphism from P_α to $P_\alpha/P_{\alpha+1}$ thus forms a homomorphism of S^*_α into $P_\alpha/P_{\alpha+1}$. The kernel of this map is exactly $S_{\alpha+1}$. Hence, we get an isomorphism which we shall call $u : S^*_\alpha/S_{\alpha+1} \to P_\alpha/P_{\alpha+1}$. This map does not seem to be very natural, but the usefulness comes from the following Lemma.

Lemma 3.1.3 (Kaplansky [20]). *Let S be a subgroup of the group G and u as defined above. Then the following are equivalent:*

1. *The range of u is not all of $P_\alpha/P_{\alpha+1}$.*
2. *There exists an $x \in P_\alpha$ proper with respect to S.*

This Lemma will prove useful to us in Section 3.3.

3.2 Scott Complexity

We begin this section by building off of the work done in Proposition 3.1.2. Notice that, for an element $x \in G$, $x \in P \iff p \cdot x = 0$. Thus, to say that $x \in P_\alpha$ requires the same number of quantifiers as to say that $x \in G_\alpha$. So the task becomes expressing the dimension of $P_\alpha/P_{\alpha+1}$ as a vector space over $\mathbb{Z}_p$. Letting φ_γ, ψ_γ define G_γ,P_γ respectively, the following says $P_\alpha/P_{\alpha+1}$ has dimension at least m for $m \in \omega$:

$$\theta_\alpha^m \equiv \exists g_1, \dots, g_m \left[\bigwedge_{i \leqslant m} \psi_\alpha(g_i) \wedge \bigvee_{n_1,\dots,n_m \in \mathbb{Z}_p} \left(\psi_{\alpha+1}(n_1 g_1 + \cdots + n_m g_m) \to \bigwedge_{i \leqslant m} n_i = 0 \right) \right]$$

If $\alpha = \omega \cdot \beta + n$ for $n \geqslant 0$, then $\psi_{\alpha+1}$ has complexity $\Sigma^0_{2\beta+1}$ and θ_α^m has complexity $\Sigma^0_{2\beta+2}$. To then say that $f(\alpha) = m$ we must use the formula $\theta_\alpha^m \wedge \neg\theta_\alpha^{m+1}$ which has complexity $d - \Sigma^0_{2\alpha+2}$

If $P_\alpha / P_{\alpha+1}$ has dimension ∞ over $\mathbb{Z}_p$ then we must say

$$\theta_\alpha^\infty \equiv \bigwedge\!\!\!\bigwedge_{m \in \omega} \theta_\alpha^m$$

and the above formula has complexity $\Pi^0_{2\beta+3}$.

We also need to know the length λ of the group G, so that we can distinguish it from a group G' of longer length, but with all the same Ulm invariants up to λ as G. The length of the group is the first ordinal λ such that $G_\lambda = 0$, so a formula saying this fact is

$$L_\lambda \equiv \forall x(\varphi_\lambda(x) \to x = 0) \wedge \bigwedge\!\!\!\bigwedge_{\alpha < \lambda} \exists x(\varphi_\alpha(x) \wedge x \neq 0).$$

If $\lambda = \omega \cdot \alpha$ is a limit ordinal, then this formula is $\Pi^0_{2\alpha+1}$. If $\lambda = \omega \cdot \alpha + n$ for $n > 0$ then this formula has complexity $d - \Sigma^0_{2\alpha+1}$. Thus, the group G with length λ and Ulm invariants given by f has Scott sentence

$$L_\lambda \wedge \bigwedge_{\alpha < \lambda} \theta_\alpha^{f(\alpha)} \wedge \Omega$$

Where Ω is a Π^0_2 sentence saying that G is an Abelian p-group. Any structure that is isomorphic to G will trivially satisfy this formula, and if a group satisfies this formula, then Ulm's Theorem tells us that it is isomorphic to G.

Notice that for a fixed α, the complexity of L_α will be less than that of ψ_α. This means that we will end up with cases for the complexity of the Scott sentence for G based on the length and the Ulm invariants.

Theorem 3.2.1. *Let G be a group with length $\lambda > 0$ and Ulm invariants given by f. Then for $0 < n < \omega$,*

(i) If $\lambda = \omega \cdot \alpha + n$ and $f(\omega \cdot \alpha + k) < \infty$ for $0 \leqslant k < n$ then G has a Scott Sentence of complexity $d - \Sigma^0_{2\alpha+2}$.

(ii) If $\lambda = \omega \cdot \alpha + n$ and $f(\omega \cdot \alpha + k) = \infty$ for $0 \leqslant k < n$ then G has a Scott Sentence of complexity $\Pi^0_{2\alpha+3}$.

(iii) If $\lambda = \omega \cdot \alpha + n$ and $f(\omega \cdot \alpha + i) < \infty$ and $f(\omega \cdot \alpha + j) = \infty$ for some $i \neq j$ with $0 \leqslant i, j < n$, then G has a Scott Sentence of complexity $\Pi^0_{2\alpha+3}$.

(iv) If $\lambda = \omega \cdot \alpha$ then G has a Scott Sentence of complexity $\Pi^0_{2\alpha+1}$.

Proof. (i) From our work above we note that the complexity of L_λ will be $d - \Sigma^0_{2\alpha+1}$ and the complexity of $\theta^{f(\omega\cdot\alpha+k)}_{\omega\cdot\alpha+k}$ will be $d - \Sigma^0_{2\alpha+2}$ for all $0 \leqslant k < n$. All lower Ulm invariants will be easier to describe and so the complexity of the Scott sentence becomes $d - \Sigma^0_{2\alpha+2}$.

(ii) Similarly to the previous case, the complexity of $\theta^{\infty}_{\omega\cdot\alpha+k}$ will be $\Pi^0_{2\alpha+3}$ for all $0 \leqslant k < n$.

(iii) If G has a finite and an infinite Ulm invariant withing finite distance from the length of the group then there will be a $d - \Sigma^0_{2\alpha+2}$ and a $\Pi^0_{2\alpha+3}$ appearing in the infinite conjunction of Ulm invariant formulae. The length of the group will still be $d - \Sigma^0_{2\alpha+1}$ to describe, and so the Scott sentence has complexity $\Pi^0_{2\alpha+3}$.

(iv) If the length of G is a limit ordinal $\lambda = \omega \cdot \alpha$ then the formula L_λ has complexity $\Pi^0_{2\alpha+1}$. The infinite conjunction of Ulm invariant formulae is only for those less than the limit ordinal λ. Thus their complexities are dominated by that of L_λ and the Scott sentence has complexity $\Pi^0_{2\alpha+1}$. $\square$

Note that this is not claiming to demonstrate the Scott complexities of any groups. In fact there are many cases where there exist Scott sentences of lower complexity for groups. Any finite group is expressible by a Π^0_2 formula for example. What we have done provides an upper bound on the Scott complexities of countable reduced Abelian primary groups.

Our next goal is to show when it is a tight upper bound, and to this end we need to understand better the back-and-forth relations between such groups.

Recall from Definition 1.2.1 that for two groups G, H, $G \leqslant_\alpha H$ if and only if all Σ^0_α formulas true of H are true of G if and only if all Π^0_α formulas true of G are true of H. This would include Scott sentences, and so if we can find two non-isomorphic groups that are α back-and-forth equivalent, then neither can have Scott complexity Σ^0_α nor Π^0_α. To do this we need conditions that guarantee that two groups are α equivalent and so we seek to characterize the back-and-forth relations for groups. If we are looking at tuples from the same group then the characterization is known.

Proposition 3.2.2 (Barker [6])**.** *Let $\overline{a}, \overline{b} \in G^{<\omega}$ and let $g : \overline{b} \to \overline{a}$ map corresponding members of the tuples to one another.*

1. *$\overline{a} \leqslant_{2\delta} \overline{b}$ if and only if*

   (a) g extends to an isomorphism $g : \langle \overline{b} \rangle \to \langle \overline{a} \rangle$ and

   (b) for every $b \in \langle \overline{b} \rangle$ and $a = g(b)$ we have

$$h(b) = h(a) < \omega \cdot \delta \quad \text{or} \quad h(b), h(a) \geqslant \omega \cdot \delta.$$

2. *$\overline{a} \leqslant_{2\delta+1} \overline{b}$ if and only if*

   (a) g extends to an isomorphism $g : \langle \overline{b} \rangle \to \langle \overline{a} \rangle$ and

   (b)(i) If $P_{\omega\cdot\delta+k}$ is infinite for every $k < \omega$ then for every $b \in \langle \overline{b} \rangle$ and $a = g(b)$ we have

$$h(b) = h(a) < \omega \cdot \delta \quad \text{or} \quad h(b) \geqslant \omega \cdot \delta \text{ and } h(a) \geqslant \min\{h(b), \omega \cdot \delta + \omega\}.$$

   (b)(ii) If $P_{\omega\cdot\delta+k}$ is infinite and $P_{\omega\cdot\delta+k+1}$ is finite, then for every $b \in \langle \overline{b} \rangle$ and $a = g(b)$ we have

$$h(b) = h(a) < \omega \cdot \delta \text{ or } \omega \cdot \delta \leqslant h(b) \leqslant h(a) \leqslant \omega \cdot \delta + k \text{ or } h(a) = h(b) > \omega \cdot \delta + k$$

   (b)(iii) If $P_{\omega\cdot\delta}$ is finite, then for every $b \in \langle \overline{b} \rangle$ and $a = g(b)$ we have

$$h(b) = h(a).$$

With the allowance of different groups comes many more moving pieces leading to many more cases to be considered, and before that we can turn to a different model-theoretic condition concerning the satisfaction of formulas to try and show optimality of our Scott Sentences.

Definition 3.2.3. We say that $\mathcal{A}$ is a Σ^0_α**-elementary substructure** of $\mathcal{B}$ and write $\mathcal{A} \preceq_{\Sigma^0_\alpha} \mathcal{B}$ if $\mathcal{A}$ is a substructure of $\mathcal{B}$, and $\mathcal{A}, \mathcal{B}$ satisfy the same Σ^0_α formulas with parameters from $\mathcal{A}$.

This has the following connection to back-and-forth relations.

Proposition 3.2.4. *If $\mathcal{A} \preceq_{\Sigma^0_\alpha} \mathcal{B}$ then $\mathcal{B} \leqslant_{\alpha+1} \mathcal{A}$.*

However, it is no easier to characterize when a group G is a Σ^0_α-elementary substructure of another group H than it is to characterize the back-and-forth relations. For the smallest back-and-forth relations, work of Eklof's [15] can be applied. He showed that $G \preceq_{\Sigma^0_1} H$ if and only if G is a pure subgroup of H and every Π^0_1 sentence true of G is true of H. If we combine these definitions with the following theorem we get our first usable results on pinning down Scott complexity.

Theorem 3.2.5 (Alvir [1])**.** *Suppose that $(A_i)_{i\in\omega}$ is a chain such that $A_i \preceq_{\Sigma^0_\alpha} A_{i+1}$, and $A = \bigcup_{i\in\omega} A_i$ where A is not isomorphic to any A_i. Then A has no d-$\Sigma^0_{\alpha+1}$ Scott sentence.*

The definition of a pure subgroup is unnecessary, and for this context it is sufficient to know that direct summands of primary groups are pure subgroups.

Theorem 3.2.6. *If G is a direct sum of cyclic groups and has two infinite Ulm invariants then G does not have a $d - \Sigma^0_2$ Scott Sentence.*

Proof. Assuming that G has two infinite Ulm invariants $f(n) = \infty$ and $f(m) = \infty$ for $n < m$ we can write G as $H \oplus (\mathbb{Z}_{p^n})^\omega$ or more importantly. as a chain $\bigcup_i H \oplus (\mathbb{Z}_{p^n})^i$. We seek to show that at each stage $H \oplus (\mathbb{Z}_{p^n})^i \preceq_{\Sigma^0_1} H \oplus (\mathbb{Z}_{p^n})^{i+1}$. By the remark above we trivially have that $H \oplus (\mathbb{Z}_{p^n})^i$ is a pure subgroup of $H \oplus (\mathbb{Z}_{p^n})^{i+1}$, so to apply Eklof's result we just need to show that every Π^0_1 sentence true of $H \oplus (\mathbb{Z}_{p^n})^i$ is true of $H \oplus (\mathbb{Z}_{p^n})^{i+1}$, i.e.

$H \oplus (\mathbb{Z}_{p^n})^i \leqslant_1 H \oplus (\mathbb{Z}_{p^n})^{i+1}$. This is true because we can embed $\mathbb{Z}_{p^n} \oplus (\mathbb{Z}_{p^m})^\omega$ into $(\mathbb{Z}_{p^m})^\omega$ since $\mathbb{Z}_{p^m}$ contains a subgroup isomorphic to $\mathbb{Z}_{p^n}$. Therefore we can embed $H \oplus (\mathbb{Z}_{p^n})^{i+1}$ into $H \oplus (\mathbb{Z}_{p^n})^i$ using the same trick. □

Corollary 3.2.7. *If G is a direct sum of cyclic groups with length $\lambda \geqslant 2$ and all Ulm invariants infinite then Π^0_3 is the best possible Scott Sentence.*

3.3 Back-and-Forth Relations

Barker's back-and-forth relations between tuples of the same group allowed for many simplifying assumptions, including the fact that there is only one length, one set of Ulm invariants, and one set of elements of each height. When those things can be different many possibilities open up for ways to map elements between tuples. In this section we obtain a partial result in extending the back-and-forth relations to arbitrary countable reduced Abelian p-groups. To give a full characterization of such back-and-forth relations has been a long-standing open problem [5].

Theorem 3.3.1. *Let $\mathcal{A}, \mathcal{B}$ be countable reduced Abelian p-groups of lengths $\lambda_\mathcal{A}$, $\lambda_\mathcal{B}$ respectively. Further assume that $|P^\mathcal{A}_\beta| = \infty$ for all $\beta < \min\{\lambda_\mathcal{A}, \omega \cdot \alpha + \omega\}$ for a countable ordinal α. Let $\overline{a} \in \mathcal{A}^{<\omega}, \overline{b} \in \mathcal{B}^{<\omega}$ and let $g : \overline{b} \to \overline{a}$ map corresponding members of the tuples to one another.*

1. *$(\mathcal{A}, \overline{a}) \leqslant_{2\alpha+1} (\mathcal{B}, \overline{b})$ if and only if*
   - *(a) for all $\beta < \omega \cdot \alpha$ we have $f_\mathcal{A}(\beta) = f_\mathcal{B}(\beta)$,*
   - *(b) $\lambda_\mathcal{A} = \lambda_\mathcal{B} < \omega \cdot \alpha$ or $\lambda_\mathcal{B} \geqslant \omega \cdot \alpha$ and $\lambda_\mathcal{A} \geqslant \min\{\lambda_\mathcal{B}, \omega \cdot \alpha + \omega\}$,*
   - *(c) for all $k \in \omega$, $|P^\mathcal{A}_{\omega\cdot\alpha+k}| \geqslant |P^\mathcal{B}_{\omega\cdot\alpha+k}|$ and $|G^\mathcal{A}_{\omega\cdot\alpha+k}| \geqslant |G^\mathcal{B}_{\omega\cdot\alpha+k}|$,*
   - *(d) g extends to an isomorphism $g : \langle\overline{b}\rangle \to \langle\overline{a}\rangle$, and*
   - *(e) for every $b \in \langle\overline{b}\rangle$ and $a = g(b)$ we have*

$$h(a) = h(b) < \omega \cdot \alpha \quad \text{or} \quad (h(b) \geqslant \omega \cdot \alpha \quad \text{and} \quad h(a) \geqslant \min\{h(b), \omega \cdot \alpha + \omega\}).$$

2. $(\mathcal{A}, \overline{a}) \leqslant_{2\alpha+2} (\mathcal{B}, \overline{b})$ *if and only if*

   (a) *for all* $\beta < \omega \cdot \alpha$ *we have* $f_{\mathcal{A}}(\beta) = f_{\mathcal{B}}(\beta)$,

   (b) $\lambda_{\mathcal{A}} = \lambda_{\mathcal{B}} < \omega \cdot \alpha + \omega$ *or* $\omega \cdot \alpha + \omega \leqslant \lambda_{\mathcal{A}}, \lambda_{\mathcal{B}}$,

   (c) *for all* $k \in \omega$, $|P^{\mathcal{A}}_{\omega\cdot\alpha+k}| = |P^{\mathcal{B}}_{\omega\cdot\alpha+k}|$ *and* $|G^{\mathcal{A}}_{\omega\cdot\alpha+k}| = |G^{\mathcal{B}}_{\omega\cdot\alpha+k}|$,

   (d) g *extends to an isomorphism* $g : \langle\overline{b}\rangle \to \langle\overline{a}\rangle$, *and*

   (e) *for every* $b \in \langle\overline{b}\rangle$ *and* $a = g(b)$ *we have*

$$h(b) = h(a) < \omega \cdot \alpha + \omega \quad \textit{or} \quad h(b), h(a) \geqslant \omega \cdot \alpha + \omega.$$

3. $(\mathcal{A}, \overline{a}) \leqslant_{\alpha} (\mathcal{B}, \overline{b})$ *for* α *a limit ordinal or zero if and only if*

   (a) *for all* $\beta < \omega \cdot \alpha$ *we have* $f_{\mathcal{A}}(\beta) = f_{\mathcal{B}}(\beta)$,

   (b) $\lambda_{\mathcal{A}} = \lambda_{\mathcal{B}} < \omega \cdot \alpha$ *or* $\omega \cdot \alpha \leqslant \lambda_{\mathcal{A}}, \lambda_{\mathcal{B}}$,

   (c) g *extends to an isomorphism* $g : \langle\overline{b}\rangle \to \langle\overline{a}\rangle$, *and*

   (d) *for every* $b \in \langle\overline{b}\rangle$ *and* $a = g(b)$ *we have*

$$h(b) = h(a) < \omega \cdot \alpha \quad \textit{or} \quad h(b), h(a) \geqslant \omega \cdot \alpha.$$

Proof. ($\Rightarrow$) Recall from our work at the beginning of the previous section, that the formula describing the Ulm invariant at the level $\omega\cdot\alpha$ is either $\Sigma_{2\alpha+3}$ or $\Pi_{2\alpha+3}$ depending on whether the Ulm invariant is finite or infinite. This means that for any $\beta = \omega \cdot \gamma < \omega \cdot \alpha$ we have $\gamma \leqslant (\alpha - 1)$, and so the complexity of the Ulm invariant formulas are at most $\Sigma_{2(\alpha-1)+3}$ or $\Pi_{2(\alpha-1)+3}$. Recall from Definition 1.2.1 that if $(\mathcal{A}, \overline{a}) \leqslant_{2\alpha+1} (\mathcal{B}, \overline{b})$ then any $\Pi_{2\alpha+1}$ formula true of $\mathcal{A}$ must hold of $\mathcal{B}$, and any $\Sigma_{2\alpha+1}$ true of $\mathcal{B}$ must hold of $\mathcal{A}$. Hence, any Ulm invariant of $\mathcal{A}$ that is infinite below $\omega\cdot\alpha$ forces the corresponding Ulm invariant of $\mathcal{B}$ to be infinite. Likewise any Ulm invariant of $\mathcal{B}$ that is finite and below $\omega \cdot \alpha$ can be completely described, and so the corresponding Ulm invariant of $\mathcal{A}$ must be equal. Together, this forces all Ulm invariants below $\omega \cdot \alpha$ to be equal and shows why part (a) is necessary in all conditions.

We showed earlier that, to say that the length of a group is exactly $\omega\cdot\alpha$, has complexity $\Pi_{2\alpha+1}$. To say that it has length at least $\omega\cdot\alpha+n$ we just have to say that $G_{\omega\cdot\alpha+n} \neq \{0\}$, and so this is $\Sigma_{2\alpha+1}$ as above. To say that the length of a group is at most $\omega \cdot \alpha + n$ it suffices to say that $G_{\omega\cdot\alpha+n} = \{0\}$. This would correspondingly have complexity $\Pi_{2\alpha+1}$. Thus as before, $\lambda_{\mathcal{A}} \leqslant \omega \cdot \alpha + n$ implies $\lambda_{\mathcal{B}} \leqslant \omega \cdot \alpha + n$ and $\lambda_{\mathcal{B}} \geqslant \omega \cdot \alpha + n$ implies $\lambda_{\mathcal{A}} \geqslant \omega \cdot \alpha + n$. Below the $\omega \cdot \alpha$ level they must be equivalent and above the $\omega \cdot \alpha + \omega$ level we do not need to preserve formulas of that complexity. This shows that condition 1(b) is necessary. If $(\mathcal{A}, \overline{a}) \leqslant_{2\alpha+2} (\mathcal{B}, \overline{b})$, then $(\mathcal{A}, \overline{a}) \equiv_{2\alpha+1} (\mathcal{B}, \overline{b})$. We shall use this fact again to show that an asymmetric condition in 1, must become symmetric with one more quantifier. For now, we have shown that 2(b) is necessary. If α is a limit ordinal, then $\alpha = 2\alpha$, and so the conditions look a lot like $2(\alpha - 1) + 2$ for non-limit ordinals, but there is no $\alpha - 1$ level, and so condition 2(b) would not make sense. For 3(b), (c), and (d) though, it is sufficient to show that 2(b), (d), and (e) are necessary.

Similarly as above, we recall from Proposition 3.2.2 that the subgroups $G_{\omega\cdot\alpha}$ and $G_{\omega\cdot\alpha+k}$ are describable using $\Pi_{2\alpha}$ and $\Sigma_{2\alpha+1}$ formulas respectively. Adding an existential quantifier in front of a $\Sigma_{2\alpha+1}$ formula will not change its complexity, and so to say that there are at least κ things in $G^{\mathcal{B}}_{\omega\cdot\alpha+k}$ for κ a finite cardinal is $\Sigma_{2\alpha+1}$ and this formula must then be true of $G^{\mathcal{A}}_{\omega\cdot\alpha+k}$. The subgroups $P_{\omega\cdot\alpha+k}$ have the same complexity and so this demonstrates why condition 1(c) is necessary. As before our asymmetric condition becomes symmetric when we add one more quantifier.

If g does not extend to an isomorphism between $\langle\overline{b}\rangle$ and $\langle\overline{a}\rangle$ then we can find a quantifier free formula true of $\overline{b}$ that does not hold of $\overline{a}$, so conditions 1(d), 2(d), and 3(c) are necessary.

For any $b \in \overline{b}$, $h(b) \geqslant \omega \cdot \alpha + n \iff b \in G_{\omega\cdot\alpha+n}$. From Proposition 3.2.2, we know that this is a $\Sigma_{2\alpha+1}$ statement, and so if $(\mathcal{A}, \overline{a}) \leqslant_{2\alpha+1} (\mathcal{B}, \overline{b})$ then this forces $h(a) \geqslant \omega \cdot \alpha + n$ as well. Likewise, $h(a) < \omega \cdot \alpha + n \iff a \notin G_{\omega\cdot\alpha+n}$ which is a $\Pi_{2\alpha+1}$ statement. Combining these two facts we have that whenever $\omega \cdot \alpha \leqslant h(b) \leqslant \omega \cdot \alpha + \omega$ we have $h(a) \geqslant h(b)$.

Note that when (1) holds and $h(a), h(b) < \omega\cdot\alpha$, to express that the height of an element x is $\beta < \omega \cdot \alpha$ we need only say that $x \in G_\beta \wedge x \notin G_{\beta+1}$, so we must have $h(a) = h(b)$ given that these groups are definable by low-complexity formulas. As before, anything asymmetric for $2\alpha + 1$ becomes symmetric for $2\alpha + 2$, and so this proves why conditions 1(e), 2(e) and 3(d) must hold.

($\Leftarrow$) We will proceed by induction on α. When $\alpha = 0$ and $(\mathcal{A}, \overline{a}), (\mathcal{B}, \overline{b})$ meet the conditions of 3 (really just (c), (d)), then we see that any quantifier free formula true of $\overline{a}$ in $\mathcal{A}$ will hold of $\overline{b}$ in $\mathcal{B}$. We proceed differently depending on whether α is odd, even, or a limit ordinal, and so separate into those cases below.

Case 1: Assume that $(\mathcal{A}, \overline{a}), (\mathcal{B}, \overline{b})$ satisfy the above conditions for part 1. We wish to show that for any $\overline{d} \in \mathcal{B}^{<\omega}$ we can find a $\overline{c} \in \mathcal{A}^{<\omega}$ such that by induction $(\mathcal{B}, \overline{bd}) \leqslant_{2\alpha} (\mathcal{A}, \overline{ac})$. Using a result of Kaplansky's ([20] Problem 36) if $\lambda_{\mathcal{B}} \geqslant \omega \cdot \alpha$ then $|P^{\mathcal{B}}_{\beta}| = \infty$ for all $\beta < \omega \cdot \alpha$. This allows us to apply the inductive hypothesis despite its asymmetric assumptions on $\mathcal{A}$ and $\mathcal{B}$. Notice that the conditions of 2 are more restrictive than the conditions of 3, so if 2α is a limit ordinal then if we show it meets the conditions of 2 it will suffice. Conditions 2(a) and (b) for $2(\alpha - 1) + 2$ follow from our assumptions of 1(a) and (b) respectively. If $\lambda_{\mathcal{A}} = \lambda_{\mathcal{B}} < \omega \cdot \alpha$ then by Ulm's theorem they are isomorphic and so 2(c) is satisfied. If α is a successor ordinal and $\lambda_{\mathcal{A}}, \lambda_{\mathcal{B}} \geqslant \omega \cdot \alpha$, then by (Kaplansky [20] Problem 36) $|P^{\mathcal{A}}_{\omega \cdot (\alpha-1)+k}| = |P^{\mathcal{B}}_{\omega \cdot (\alpha-1)+k}| = \infty$ and so 2(c) is satisfied. If α is a limit ordinal then we are instead trying to meet the conditions of 3 which does not mention $|P^{\mathcal{A}}_{\omega \cdot (\alpha-1)+k}|, |P^{\mathcal{B}}_{\omega \cdot (\alpha-1)+k}|$.

Let $A = \langle \overline{a} \rangle, B = \langle \overline{b} \rangle$ be subgroups of $\mathcal{A}, \mathcal{B}$ respectively. Given a $\overline{d} \in \mathcal{B}^{<\omega}$ we seek to extend the isomorphism $g : B \to A$ to an isomorphism $g : B' = \langle \overline{b}, \overline{d} \rangle \to C' = \langle \overline{a}, \overline{c} \rangle$ for a suitable $\overline{c} \in \mathcal{A}^{<\omega}$ such that it meets the conditions of part 2. We will do this by making a sequence of extensions $B = B_0 \subseteq B_1 \subseteq \cdots \subseteq B'$, and choosing at each stage i an $x \in B' \backslash B_i$ with $px \in B_i$ if possible, and extending g to $B_{i+1} = \langle x, B_i \rangle$.

Case 1(a): If $\lambda_{\mathcal{A}} = \lambda_{\mathcal{B}} < \omega \cdot \alpha$ then our assumption of condition 1(a) forces the groups to be isomorphic by Ulm's theorem. So we can certainly extend g as needed.

Case 1(b): If $\omega \cdot \alpha \leqslant \lambda_{\mathcal{B}} \leqslant \lambda_{A} < \omega \cdot \alpha + \omega$. We shall ensure that each extension of g to B_i satisfies the following condition.

$$h(a) = h(b) < \omega \cdot \alpha \quad \text{or} \quad \omega \cdot \alpha \leqslant h(b) \leqslant h(a) < \lambda_{\mathcal{A}},$$

which we will call condition $\star$. If this is met for all $i \leqslant |B' \backslash B|$, then $g : B' \to A'$ will satisfy

$$h(a) = h(b) < \omega \cdot \alpha \quad \text{or} \quad \omega \cdot \alpha \leqslant h(a), h(b).$$

Note that by our assumption of condition 1(d) g meets condition $\star$ for B_0.

As stated above we now choose an $x \in B' \backslash B_i$ such that $px \in B_i$ and x is proper with respect to B_i, and $h(px)$ is maximal among all x with this property. $\mathcal{B}$ is a p-group, and so B' being finitely generated means that it is finite. This allows us to use Lemma 3.1.3 to choose such an x.

Since g meets condition $\star_i$, we know that either $h(px) = h(y) < \omega \cdot \alpha$ or $h(px) \geqslant \omega \cdot \alpha$ and $h(y) \geqslant h(px)$.

Case 1(b)(i): If $h(y) = h(px) = h(x) + 1 < \omega \cdot \alpha$ then we choose a w such that $pw = y$ and $h(w) = h(x)$. This is possible from our assumption that $f_{\mathcal{A}}(h(x)) = f_{\mathcal{B}}(h(x))$. We know that $w \notin A_i$ because if $w = g(z)$ for some $z \in A_i$ then $pz = px$ since g is a group isomorphism. This would mean that $h(px - pz) = h(0) = \infty > h(px)$ which contradicts our choice of x.

We also get that w is proper with respect to A_i, for if there were some $a \in A_i$ such that $h(w + a) \geqslant h(w) + 1 = h(x) + 1$ and $a = g(b)$ for $b \in B_i$, then since $w + a \neq 0$ we must have $h(p(w + a)) \geqslant h(x) + 2$ which forces $h(p(x + b)) \geqslant h(x) + 2$. This also contradicts the maximal height of $px \in B_i$.

We can now extend g to $B_{i+1} = \langle x, B_i \rangle$ by mapping $g(rx + b) = rw + a$ for $0 < r < p, b \in B_i$ and $a = g(b)$. Since w was proper with respect to A_i we see that g still preserves heights for everything $\leqslant \omega \cdot \alpha$, and so g meets $\star_{i+1}$.

Case 1(b)(ii): If $h(x) = \gamma$, $h(y) > \gamma + 1$, then since $h(px) > \gamma + 1$, there is a $v \in \mathcal{B}_{\gamma+1}$ such that $pv = px$. The element $x - v$ is then in $P_{\gamma}^{\mathcal{B}}$ and it also has height γ and is proper with respect to B_i. We can apply Lemma 3.1.3 to see that the range of u is not all of $P_{\gamma}^{\mathcal{B}} / P_{\gamma+1}^{\mathcal{B}}$. By induction on i, g preserves heights for elements of height $< \omega \cdot \alpha$. Furthermore, g maps $B_i \cap \mathcal{B}_{\gamma+1}$ onto $A_i \cap \mathcal{A}_{\gamma+1}$, $B_i \cap \mathcal{B}_{\gamma}$ onto $A_i \cap \mathcal{A}_{\gamma}$ and $B_i \cap \mathcal{B}_{\gamma} \cap p^{-1}\mathcal{B}_{\gamma+2}$ onto $A_i \cap \mathcal{A}_{\gamma} \cap p^{-1}\mathcal{B}_{\gamma+2}$.

Since $(B_i \cap \mathcal{B}_{\gamma} \cap p^{-1}\mathcal{B}_{\gamma+2})/(B_i \cap \mathcal{B}_{\gamma+1})$ is finite, its dimension as a vector space over $\mathbb{Z}_p$ is strictly less than $f_{\mathcal{B}}(\gamma) = f_{\mathcal{A}}(\gamma)$. Combining these facts, we see that the dimension of $(A_i \cap \mathcal{A}_{\gamma} \cap p^{-1}\mathcal{A}_{\gamma+2})/(A_i \cap \mathcal{A}_{\gamma+1})$ is also less than $f_{\mathcal{A}}(\gamma)$, so we can apply Lemma 3.1.3 in reverse, giving us an element $w_1 \in \mathcal{A}$ such that $pw_1 = 0$, $h(w_1) = \gamma$, and which is proper with respect to A_i.

Since $h(y) > \gamma + 1$ we know that $y = pw_2$ for $w_2 \in \mathcal{A}_{\gamma+1}$. Let $w = w_1 + w_2$. Then $pw = 0 + y$, $h(w) = h(w_1) = \gamma$, and w is proper with respect to A_i. If we extend g by $g(x) = w$ as before, we see that g preserves heights $< \omega \cdot \alpha$ and so satisfies $\star_{i+1}$.

Case 1(b)(iii): If $h(x) \geqslant \omega \cdot \alpha$ then $h(x) < h(px) \leqslant h(y)$. So we can choose a $w_1 \in \mathcal{A}$ such that $pw_1 = y$ and $h(w_1) = h(y) - 1 \geqslant h(x)$. The fact that there is an element of height $\geqslant \omega \cdot \alpha$ implies that $\lambda_{\mathcal{A}} > \omega \cdot \alpha$, which justifies our use of the ordinal $\lambda_{\mathcal{A}} - 1$. By our assumptions on $\mathcal{A}$, $P^{\mathcal{A}}_{\lambda_{\mathcal{A}}-1}$ is infinite, hence we can get an element $w_2 \in P^{\mathcal{A}}_{\lambda_{\mathcal{A}}-1} \backslash A_i$. Let $w = w_1 + w_2$ and extend g to B_{i+1} by defining $g(x) = w$.

To see that g still meets condition $\star$ take an arbitrary element $rx + b$ of B_{i+1} where $0 \leqslant r < p$ since $px \in B_i$ and $b \in B_i$. We have that $g(rx + b) = rw + a$. If $h(b) < \omega \cdot \alpha$ then $h(rx + b) = h(b)$ and $h(b) = h(a)$ by our induction on i, thus $h(rw + a) = h(a)$ and g preserves heights below $\omega \cdot \alpha$. If $h(b) \geqslant \omega \cdot \alpha$ then by induction we still know that $h(a) \geqslant h(b)$. We also have that

$$h(rw + a) \geqslant \min\{h(w), h(a)\} \geqslant \min\{h(w_1), h(w_2), h(a)\} \geqslant \min\{h(x), \lambda_{\mathcal{A}} - 1, h(b)\}.$$

Since $\lambda_{\mathcal{B}} \leqslant \lambda_{\mathcal{A}}$ we must have $h(x) \leqslant \lambda_{\mathcal{A}} - 1$ and since x is proper with respect to B_i we have that $h(rx + b) = \min\{h(x), h(b)\}$. Thus g meets condition $\star$ for B_{i+1} and by induction on i and our inductive hypothesis on α we have shown that $(\mathcal{B}, \overline{bd}) \leqslant_{2\alpha} (\mathcal{A}, \overline{ac})$ for $c = g(d)$.

Case 1(c): Assume that $\lambda_{\mathcal{A}} \geqslant \omega \cdot \alpha + \omega$. We know in this case that $P^{\mathcal{A}}_{\omega \cdot \alpha + k}$ is infinite for all $k \in \omega$ (Kaplansky [20] Problem 36). Let $N_0 = |B' \backslash B|$ and $N_{i+1} = N_i - 1$.

We shall ensure that we extend g to B' such that for all $b \in B_i$ we have

$$h(a) = h(b) < \omega \cdot \alpha \quad \text{or} \quad h(b) \geqslant \omega \cdot \alpha \text{ and } h(a) \geqslant \min\{h(b), \omega \cdot \alpha + N_i\}.$$

If this is accomplished for all $i \leqslant N_0$ then we would meet condition 2(d) (or 3(d)), which would mean $(\mathcal{B}, \overline{bd}) \leqslant_{2\alpha} (\mathcal{A}, \overline{ac})$ from our inductive hypothesis. We shall call the above condition $\star_i$, and, by our assumption of condition 1(d), g meets $\star_0$.

Now by induction on i we again take an $x \in B' \backslash B_i$ such that $px \in B_i$, x is proper with respect to B_i, and $h(px)$ is maximal amongst all such elements of $x + B_i$, and let $g(px) = y$.

Case 1(c)(i): We can prove this case exactly as in 1(b)(i).

Case 1(c)(ii): We similarly follow the proof of 1(b)(ii).

Case 1(c)(iii): $h(x) \geqslant \omega \cdot \alpha$ and $h(y) \geqslant \min\{h(px), \omega \cdot \alpha + N_i\}$. Since $h(px) \geqslant h(x)+1$ we know $h(y) > \omega \cdot \alpha$ and so there exists a $w_1 \in \mathcal{A}$ such that $pw_1 = y$ and $h(w_1) \geqslant \min\{h(x), \omega \cdot \alpha + N_i - 1\}$. Since $P^{\mathcal{A}}_{\omega\cdot\alpha+N_i}$ is infinite we can find a $w_2 \in P^{\mathcal{A}}_{\omega\cdot\alpha+N_i} \backslash A_i$. Set $w = w_1 + w_2$ we extend g by sending x to w. Note that $w \notin A_i$ and $h(w) \geqslant \min\{h(w_1), h(w_2)\}$.

For all $b \in B_i$ such that $h(b) < \omega \cdot \alpha$, g preserves height as before. Let $b \in B_i$ have height $\geqslant \omega \cdot \alpha$. Then for $a = g(b)$ and $0 \leqslant r < p$ using condition $\star_i$ we have

$$\begin{aligned} h(rw + a) \geqslant \min\{h(w), h(a)\} &\geqslant \min\{h(x), \omega \cdot \alpha + N_i - 1, h(b), \omega \cdot \alpha + N_i\} \\ &\geqslant \min\{h(x), h(b), \omega \cdot \alpha + N_{i+1}\} \\ &= \min\{h(rx + b), \omega \cdot \alpha + N_{i+1}\} \end{aligned}$$

where the last equality follows from the fact that x is proper with respect to B_i. Thus g satisfies condition $\star_{i+1}$.

In all cases above g satisfies $\star_i$ for all $i < N_0$ and so we can extend g to a map which meets the conditions to show that $(\mathcal{B}, \overline{bd}) \leqslant_{2\alpha} (\mathcal{A}, \overline{ac})$ for $c = g(d)$.

Case 2: Now if $(\mathcal{A}, \overline{a}), (\mathcal{B}, \overline{b})$ meet the conditions for Part 2 of the Theorem, we must do something very similar to before to show that given any $\overline{d}$ we can find a $\overline{c}$ such that $(\mathcal{B}, \overline{bd}), (\mathcal{A}, \overline{ac})$ meet the conditions for Part 1. This time, conditions 1(a), (b), and (c) follow immediately from 2(a), (b) and (c), and so our primary concern is again extending the isomorphism g to a map $\langle \overline{b}, \overline{d} \rangle \to \langle \overline{a}, \overline{c} \rangle$. Let $\overline{d} \in \mathcal{B}^{<\omega}$ and we once again let $B_0 = B = \langle \overline{b} \rangle$, $B' = \langle \overline{bd} \rangle$, and $A_0 = A = \langle \overline{a} \rangle$.

Case 2(a): If $\lambda_{\mathcal{A}} = \lambda_{\mathcal{B}} < \omega \cdot \alpha$ then, as in case 1(a), we conclude that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.

Case 2(b): If $\lambda_{\mathcal{B}} \geqslant \omega \cdot \alpha$ and $\lambda_{\mathcal{A}} \geqslant \min\{\lambda_{\mathcal{B}}, \omega \cdot \alpha + \omega\}$, then as before we know that $|P^{\mathcal{A}}_{\omega\cdot\alpha+k}| = \infty$ for all $k < \lambda_{\mathcal{A}}$. Let

$$M = \max\{m : b' \in B', h(b') = \omega \cdot \alpha + m \text{ for } m \in \omega\} + 1,$$

$$N_0 = |B'/B| + M, \text{ and } N_{i+1} = N_i - 1.$$

For each $0 \leqslant i \leqslant N_0$ we would like to extend g to B_i in such a way that it satisfies

$$h(b) = h(a) < \omega \cdot \alpha + M \quad \text{or} \quad (h(b) \geqslant \omega \cdot \alpha + \omega \quad \text{and} \quad h(a) \geqslant \omega \cdot \alpha + N_i)$$

for all $b \in B_i$, $a = g(b)$. We will call this condition $\star_i$. If this is met for all i then g will satisfy

$$h(b') = h(a') \leqslant \omega \cdot \alpha \quad \text{or} \quad h(a') \geqslant \omega \cdot \alpha \text{ and } h(b') \geqslant \min\{h(a'), \omega \cdot \alpha + \omega\}$$

for all $b' \in B'$, which would complete our proof that $(\mathcal{B}, \overline{bd}) \leqslant_{2\alpha+1} (\mathcal{A}, \overline{ac})$. By our assumption of condition 2(d), g meets condition $\star_0$.

Now by induction on i, we take an $x \in B' \backslash B_i$ such that $px \in B_i$, x is proper with respect to B_i, and $h(px)$ is maximal amongst all such elements of $x + B_i$. Let $g(px) = y$. Since g meets condition $\star_i$ we know that either $h(px) = h(y) < \omega \cdot \alpha + M$ or $h(x) \geqslant \omega \cdot \alpha + \omega$ and $h(y) \geqslant \omega \cdot \alpha + N_i$.

Case 2(b)(i): If $h(y) = h(px) = h(x) + 1 < \omega \cdot \alpha + M$ then we proceed exactly as in case 1(b)(i).

Case 2(b)(ii): If $h(x) = \gamma < \omega \cdot \alpha + \omega$, $\gamma + 1 < h(y) < \omega \cdot \alpha$ then we similarly follow case 1(b)(ii).

Case 2(b)(iii): $h(x) \geqslant \omega \cdot \alpha + \omega$ and $h(y) \geqslant \omega \cdot \alpha + N_i$. As in case 1(b)(iii), we can find a $w_1 \in \mathcal{A}$ such that $pw_1 = y$ and $h(w_1) = h(y) - 1 \geqslant \omega \cdot \alpha + N_i - 1$. Furthermore, if we are in this case, then $\lambda_{\mathcal{B}} \geqslant \omega \cdot \alpha + \omega$ which forces $\lambda_{\mathcal{A}} \geqslant \omega \cdot \alpha + \omega$. Thus, since $P^{\mathcal{A}}_{\omega \cdot \alpha + N_i}$ is infinite, we can find a $w_2 \in P^{\mathcal{A}}_{\omega \cdot \alpha + N_i} \backslash A_i$. Set $w = w_1 + w_2$ and extend g by sending x to w. Note that $w \notin A_i$ and $h(w) \geqslant \omega \cdot \alpha + N_{i+1} > \omega \cdot \alpha + M$.

Let $b \in B_i$. If $h(b) \leqslant \omega \cdot \alpha + M$ then by assumption $h(a) = h(b)$ and so for $0 \leqslant r < p$, $h(rx + b) = h(b)$ and $h(rw + a) = h(a)$. If $h(b) \geqslant \omega \cdot \alpha + \omega$ then by induction on i, $h(a) \geqslant \omega \cdot \alpha + N_i > \omega \cdot \alpha + N_{i+1}$. Thus, $h(rx + b) \geqslant \omega \cdot \alpha + \omega$ and $h(rw + a) \geqslant \min\{h(w), h(a)\} \geqslant \omega \cdot \alpha + N_{i+1}$. This shows that g meets condition $\star_{i+1}$.

In all cases g meets $\star_i$ for all i and so we can extend it to B'. This shows that $(\mathcal{B}, \overline{bd}) \leqslant_{2\alpha+1} (\mathcal{A}, \overline{ac})$ for $c = g(d)$.

Case 3: The limit case is where we first invoke our inductive hypothesis. Assume that $(\mathcal{A}, \overline{a}), (\mathcal{B}, \overline{b})$ meet the conditions of 3 and let $\overline{d} \in \mathcal{B}^{<\omega}$ and $\beta < \alpha$.

Case 3 (a): If $\lambda_{\mathcal{A}} = \lambda_{\mathcal{B}} < \omega \cdot \alpha$ then as before the groups are isomorphic.

Case 3 (b): If $\lambda_{\mathcal{A}}, \lambda_{\mathcal{B}} \geqslant \omega \cdot \alpha$ then we know that $|P^{\mathcal{A}}_{\omega \cdot \beta + k}| = |P^{\mathcal{B}}_{\omega \cdot \beta + k}| = \infty$ (Kaplansky [20] Problem 36) and so we meet the conditions for $(\mathcal{A}, \overline{a}) \leqslant_{\beta+1} (\mathcal{B}, \overline{b})$ regardless of whether β is odd, even, or a limit ordinal. Hence, by induction there exists a $\overline{c} \in \mathcal{A}^{<\omega}$ such that $(\mathcal{B}, \overline{bd}) \leqslant_{\beta} (\mathcal{A}, \overline{ac})$.

This finishes our induction on α and so the conditions of the theorem are sufficient to imply the back-and-forth relations on countable reduced Abelian p-groups. □

3.4 Further Work

To be able to extend the back-and-forth characterization to arbitrary groups we would need to include many more cases in the proof of Theorem 3.3.1. without the assumption on the cardinality of $P^{\mathcal{A}}_{\beta}$ we cannot guarantee that, given an $x \in B' \backslash B_i$, we can always choose a corresponding $w \in \mathcal{A} \backslash A_i$ of the appropriate height. We have seen that in all cases is $\omega \cdot \alpha \leqslant h(b) \leqslant \omega \cdot \alpha + \omega$ then we must have $h(g(b)) \geqslant h(b)$, so if we have the same finite number of elements of each height in both groups, then having $h(b) < h(g(b))$ would be a problem. Purely by a counting argument, this would force g to send something of high height to something of lower height, which violates our preservation of formulas. So Theorem 3.3.1 can be partially extended under a great number of simplifying assumptions that essentially force the groups to be isomorphic, putting us in the realm of Barker's proof of 3.2.2.

The eventual goal is to extend our partial characterization of the back-and-forth relations to cover all countable reduced Abelian p-groups and then to use that characterization to show the optimality of our Scott sentences given in Theorem 3.2.1. This would characterize the Scott complexities of such groups. Our current characterization already shows promise in being able to help this goal by showing that $d - \Sigma_{2\alpha+2}$ is the best possible Scott sentence for case (i) of Theorem 3.2.1. To show that $d - \Sigma_{2\alpha+2}$ is the best possible Scott sentence for a group of length $\omega \cdot \alpha + n$, we need to find groups $\mathcal{B}$ and $\mathcal{C}$, both not isomorphic to $\mathcal{A}$, such that $\mathcal{B} \leqslant_{2\alpha+2} \mathcal{A}$ and $\mathcal{A} \leqslant_{2\alpha+2} \mathcal{C}$. Without any tuples to mention, we can only use conditions (a), (b), (c) from Theorem 3.3.1. This forces the size of the subgroups $|P_\alpha^{\mathcal{A}} + k|, |P_\alpha^{\mathcal{B}} + k|$ to be the same, but only requires the Ulm invariants to be equal $< \omega \cdot \alpha$. This affords us a little room in which to look for non-isomorphic groups, but we return to the problem of not having concrete examples of groups of fixed Ulm sequences.

Chapter 4

Positive Enumerability

While in computable structure theory countable structures are classically studied up to Turing reducibility, researchers have successfully used enumeration reducibility to both contribute to the classical study and develop a beautiful theory on its own. One example of a contribution to classical theory is Soskov's work on degree spectra and co-spectra [32] which allowed him to show that there is a degree spectrum of a structure such that the set of degrees whose ω-jump is in this spectrum is not the spectrum of a structure [33]. Another example is Kalimullin's study of reducibilities between classes of structures [19] where he studied enumeration reducibility versions of Muchnik and Medvedev reducibility between classes of structures. This topic has also been studied by Knight, Miller, and Vanden Boom [21] and Calvert, Cummins, Knight, and Miller [9]. There is a rich theory on these notions with interesting questions on the relationship between enumeration reducibility and the classical versions. In this chapter we develop a novel approach to study relations that are enumeration reducible to every copy of a given structure.

4.1 Background

Since enumeration reducibility is based only on positive information, it is natural to consider structures given by their positive atomic diagram in this setting. This is quite reasonable. When doing computability theory and looking at an infinite object, we often view it as being revealed stage by stage. It is natural to find out only about positive information, and our approach is general enough to handle negative information by considering the associated structure in the language including relation symbols for the cosets of the relations in the original language.

More formally, say that given a countable relational structure $\mathcal{A}$, a relation R is *relatively intrinsically positively enumerable* (*r.i.p.e.*) in $\mathcal{A}$ if for every copy $\mathcal{B}$ of $\mathcal{A}$ and every enumeration of the basic relations on $\mathcal{B}$, we can compute an enumeration of $R^{\mathcal{B}}$. We obtain a syntactic classification of the r.i.p.e. relations using the infinitary logic $L_{\omega_1\omega}$. Theorem 4.2.10 shows that the r.i.p.e. relations on a structure are precisely those that are definable by computable infinitary Σ^0_1 formulas in which neither negations nor implications occur.

This definition is similar to Definition 1.2.2, and we hope to show an equivalent definability concept in the same way that r.i.c.e. relations were shown to be exactly those definable by an infinitary Σ^0_1 formula. Since this result, there has been much work on r.i.c.e. relations and related concepts. For a summary, see Fokina, Harizanov, and Melnikov [16]. One particularly interesting generalization of r.i.c.e. relations is due to Montalbán [23]. He extended the definition of r.i.c.e. relations from relations on ω^n to $\omega^{<\omega}$ and to sequences of relations, obtaining a classification similar to the one given by Ash or Chisholm [4, 10]. This extension allows the development of a rich theory such as an intuitive definition of the jump of a structure, and an effective version of interpretability with a category theoretic analogue: A structure $\mathcal{A}$ is effectively interpretable in a structure $\mathcal{B}$ if and only if there is a computable functor from $\mathcal{B}$ to $\mathcal{A}$ as given in Harrison-Trainor, Melnikov, Miller, and Montalbán [17]. For a complete development of this theory see Montalbán [26]. The main goal of this chapter to develop a similar theory for r.i.p.e. relations.

Definition 4.1.1. Let $\mathcal{A}$ be a countable structure in relational language L with universe ω. We let $P(\mathcal{A})$ be the set $=^{\mathcal{A}} \oplus \neq^{\mathcal{A}} \oplus \bigoplus_{R_i \in L} R_i^{\mathcal{A}}$, which we call the *positive diagram of* $\mathcal{A}$.

The mindful reader might notice that we include inequality in the positive atomic diagram. The reason for this is that for enumerations of $\mathcal{A}$ that are not injective $=^{\mathcal{A}}$ will become an equivalence relation instead of equality and we want to take the quotient under this equivalence relation to be able to enumerate an isomorphic copy of $\mathcal{A}$ from this enumeration. Having $\neq$ in the diagram allows us to do this effectively. The positive diagram is Turing equivalent to the standard definition of the atomic diagram of a structure, which can be viewed as the set $=^{\mathcal{A}} \oplus \neq^{\mathcal{A}} \oplus \bigoplus_{R_i \in L} R_i^{\mathcal{A}} \oplus \overline{R_i}^{\mathcal{A}}$. Now, a relation $R \subseteq \omega^{<\omega}$ is r.i.c.e. if for every copy $\mathcal{B} \cong \mathcal{A}$, $R^{\mathcal{B}}$ is c.e. in $D(\mathcal{B})$. The relatively intrinsically *positively* enumerable relations are the natural analogue to the r.i.c.e. relations for enumeration reducibility. Recall that a set of natural numbers A is enumeration reducible to B, $A \leqslant_e B$ if there exists a c.e. set Ψ consisting of pairs $\langle x, D\rangle$ where D is a finite set under some fixed coding such that

$$A = \Psi^B = \{x : D \subseteq B \wedge \langle x, D\rangle \in \Psi\}.$$

Enumeration reducibility allows us to formally define the notion of a r.i.p.e. relation.

Definition 4.1.2. Let $\mathcal{A}$ be a structure. A relation $R \subseteq A^{<\omega}$ is *relatively intrinsically positively enumerable* in $\mathcal{A}$, short r.i.p.e., if, for every copy $(\mathcal{B}, R^{\mathcal{B}})$, of $(\mathcal{A}, R)$, $R^{\mathcal{B}} \leqslant_e P(\mathcal{B})$. The relation is *uniformly relatively intrinsically positively enumerable* in $\mathcal{A}$, if the above enumeration reducibility is uniform in the copies of $\mathcal{A}$, that is, if there is a fixed enumeration operator Ψ such that $R^{\mathcal{B}} = \Psi^{P(\mathcal{B})}$ for every copy $\mathcal{B}$ of $\mathcal{A}$.

The study of the computability theoretic properties of structures with respect to enumeration reducibility is an active research topic; see A. Soskova and M. Soskova [37] for a summary of results in this area.

4.2 First results on r.i.p.e. relations

In our proofs we will often build structures in stages by copying increasing pieces of finite substructures of a given structure $\mathcal{A}$. The following definitions will be useful for this.

Definition 4.2.1. Given an $\mathcal{L}$-structure $\mathcal{A}$ and $\overline{a} \in A^{<\omega}$, let $P(\mathcal{A})\restriction\overline{a}$ denote the positive diagram of the substructure of $\mathcal{A}$ with universe $\overline{a}$ in the restriction of $\mathcal{L}$ to the first $|\overline{a}|$ relation symbols.

Definition 4.2.2. Let $\mathcal{A}$ be an $\mathcal{L}$-structure and $\overline{a} = \langle a_0, \dots, a_{s-1}\rangle \in A^s$. The set $P_{\mathcal{A}}(\overline{a})$ is the pullback of $P(\mathcal{A})\restriction\overline{a}$ along the index function of $\overline{a}$, i.e.,

$$\langle i, n_0, \dots, n_{r_i}\rangle \in P_{\mathcal{A}}(\overline{a}) \Leftrightarrow \langle i, a_{n_0}, \dots, a_{n_{r_i}}\rangle \in P(\mathcal{A})\restriction\overline{a}.$$

The main feature of 4.2.2 is that if we approximate the positive diagram of a structure $\mathcal{A}$ in stages by considering larger and larger tuples, i.e., $\lim_{s \in \omega} P(\mathcal{A}) \upharpoonright \overline{a}_s = P(\mathcal{A})$, then the limit of $P_{\mathcal{A}}(\overline{a}_s)$ gives a structure isomorphic to $\mathcal{A}$. This fact is useful in constructions.

We denote by Ψ_e the e^{th} enumeration operator in a fixed computable enumeration of all enumeration operators and by $\Psi_{e,s}$ its stage s approximation. Without loss of generality we make the common assumption that $\Psi_{e,s}$ is finite and does not contain pairs $\langle n, D \rangle > s$. Notice that $\Psi_{e,s}$ itself is an enumeration operator. In our proofs we also frequently argue that a set A is enumeration reducible to a set B by using a characterization of enumeration reducibility due to Selman [29].

Theorem 4.2.3 (Selman [29])**.** *For any $A, B \subseteq \omega$*

$$A \leqslant_e B \quad \textit{iff} \quad \forall X [B \textit{ is c.e. in } X \Rightarrow A \textit{ is c.e. in } X]$$

We refer the reader to Cooper [11] for a proof of this result and further background on enumeration reducibility and enumeration degrees.

Notice that by 4.1.2 $(\mathcal{A}, R)$ is technically not a first order structure as $R \subseteq \omega^{<\omega}$. We may however still think of it as a first order structure in the language expanded by relation symbols $(Q_i)_{i \in \omega}$, each Q_i of arity i, where $Q_i^{\mathcal{A}} = \{\overline{a} \in A^n : \overline{a} \in R^{\mathcal{A}}\}$. The positive diagrams $P(\mathcal{A}, R^{\mathcal{A}})$ and $P(\mathcal{A}, (Q_i^{\mathcal{A}})_{i \in \omega})$ are enumeration equivalent.

4.2.1 Examples of r.i.p.e. relations

Example 4.2.4. *Let $\mathcal{R}$ be a countable ring. Then the relation $P = \{\overline{a} : \exists x\, [a_0 + a_1 x + \cdots + a_k x^k = 0]\}$ holding of the polynomials with a root in $\mathcal{R}$ is uniformly r.i.p.e.*

Proof. Enumerate the graphs of $+, \cdot, =$ and 0 of a copy $\hat{\mathcal{R}}$ of $\mathcal{R}$ and whenever you see $\overline{a}b$ such that $a_0 + a_1 b + \cdots + a_k b^k = 0$ enumerate $\overline{a}$. Clearly this gives an enumeration of $P^{\hat{\mathcal{R}}}$ from an enumeration of any copy of $\mathcal{R}$. ∎

Example 4.2.5. *The reachability relation $reach(x, y)$ on graphs is uniformly r.i.p.e.*

Proposition 4.2.6. *There is a graph G in the language $(=, \neq, E)$ such that $\neg E$ is not r.i.p.e. in G.*

Proof. Let K be the halting set and let G be the following graph:

1. For every $n \in \omega$, G contains a vertex v_n and an $(n+1)$-cycle with least vertex c_n and an edge $v_n \, E \, c_n$,
2. if $\langle x, y \rangle \in K$, then $v_{2x+1} \, E \, v_{2y+2}$.

Then it is not hard to see that from any enumeration of K we can enumerate $P(G)$. Notice that the set $C = \{(v_{2x+1}, v_{2y+2}) : (x, y) \in \omega \times \omega\}$ is r.i.p.e. If $\neg E$ was r.i.p.e., then so would be $C \cap \neg E$, i.e., $C \cap \neg E \leqslant_e P(G)$. But clearly $C \cap \neg E \geqslant_e \overline{K}$ and since we have $K \geqslant_e P(G)$ this would imply $K \geqslant_e \overline{K}$, a contradiction. □

4.2.2 A syntactic characterization

The main purpose of this section is to show that being r.i.p.e. in a structure $\mathcal{A}$ is equivalent to being definable by infinitary formulas in $\mathcal{A}$ of the following type.

Definition 4.2.7. A *positive computable infinitary Σ_1^p formula* is a formula of the infinitary logic $L_{\omega_1 \omega}$ of the form

$$\varphi(\overline{x}) \equiv \bigvee\!\!\!\!\bigvee_{i \in I} \exists \overline{y}_i \varphi_i(\overline{x}, \overline{y}_i)$$

where each φ_i is a finite conjunction of atomic formulas that can contain $\neq$, and the index set I is a computably enumerable subset of ω.

Notice that the above definition includes all c.e. disjunctions of finitary Σ_1^0 formulas without negation and implication symbols, as every such formula is equivalent to a finite disjunction of conjunctions with each existential quantifier occurring in front of the conjunctions.

Definition 4.2.8. A relation $R \subseteq \omega^{<\omega}$ is Σ_1^p*-definable with parameters* $\overline{c}$ in a structure $\mathcal{A}$ if there exists a uniformly computable sequence of Σ_1^p formulas $(\varphi_i(x_1, \ldots, x_i, y_1, \ldots, y_{|c|}))_{i \in \omega}$ such that for all $\overline{a} \in \omega^{<\omega}$

$$\overline{a} \in R \Leftrightarrow \mathcal{A} \models \varphi_{|\overline{a}|}(\overline{a}, \overline{c}).$$

Our goal is to show that a relation R is r.i.p.e. in a structure $\mathcal{A}$ if and only if it is Σ^p_1-definable over some parameter. We will do this by using forcing to build a generic copy of $\mathcal{A}$ and then read the syntactic definition of this copy. The right notion of genericity for this purpose is the following.

Definition 4.2.9. Let $A^* = \{\sigma \in A^{<\omega} : (\forall i \neq j < |\sigma|)[\sigma(i) \neq \sigma(j)]\}$. We say that $\gamma \in A^*$ *decides* an upwards closed subset $R \subseteq A^*$ if $\gamma \in R$ or $\sigma \notin R$ for all $\sigma \supseteq \gamma$. A $1{-}1$ function $g : \omega \to \mathcal{A}$ is a *r.i.p.e.-generic enumeration*, if for every r.i.p.e. subset $R \subseteq A^*$ there is an initial segment of g that decides R. We say that $\mathcal{B}$ is a *r.i.p.e.-generic presentation* of $\mathcal{A}$ if it is the pull-back along a r.i.p.e.-generic enumeration of $P(\mathcal{A})$.

The existence of r.i.p.e. generics follows from the Baire category theorem. Let us give a quick hands-on construction. Given a structure $\mathcal{A}$ and an enumeration $(R_e)_{e\in\omega}$ of all r.i.p.e. subsets of A^* we build an sequence $\overline{p}_0 = \varnothing \subset \overline{p}_1 \dots$ with $\lim p_s = g$. At even stages $2s$ we guarantee that g is onto by defining $\overline{p}_{2s} = \overline{p}_{2s-1}{}^\frown x$ where $x \in A$ is least such that x is not in the range of p_{2s-1}. At odd stages $2s+1$ we check whether there is $\overline{q} \supseteq \overline{p}_{2s}$ with $\overline{q} \in R_s$. If so, let $\overline{p}_{2s+1} = \overline{q}$, otherwise $\overline{p}_{2s+1} = \overline{p}_{2s}$. It is easy to see that $g = \lim p_s$ decides every upwards closed r.i.p.e. subset of A^* and is thus a r.i.p.e. generic enumeration. Furthermore, note that if R_e is dense in A^*, then there is some $p \subset g$ forcing that g meets R_e and thus $\langle 0, \dots, |p|\rangle \in R_e^{g^{-1}(\mathcal{A})}$. We will study properties of r.i.p.e. generics in 4.3.1. For now, let us obtain a syntactic characterization of r.i.p.e. relations.

Theorem 4.2.10. *Let $\mathcal{A}$ be a structure and $R \subseteq A^{<\omega}$ a relation on it. Then the following are equivalent:*

(i) R is relatively intrinsically positively enumerable in $\mathcal{A}$,
(ii) R is Σ^p_1-definable in $\mathcal{A}$ with parameters.

Proof. Assuming (ii), there is a uniformly computable sequence $(\varphi_i(\overline{x}, \overline{z}))_{i\in\omega}$ of Σ^p_1 formulas where each φ_i is of the form $\bigvee\!\!\!\bigvee_{j\in\omega} \exists \overline{y}_{i,j} \psi_{i,j}(\overline{x}, \overline{y}_{i,j}, \overline{z})$ with the property that for every $\mathcal{B} \cong \mathcal{A}$ there is a tuple $\overline{c} \in \omega^{|\overline{z}|}$ such that for all $i \in \omega$ and $\overline{a} \in \omega^i$, $\mathcal{B} \models \varphi_i(\overline{a}, \overline{c})$ if and only if $\overline{a} \in R^{\mathcal{B}}$. Recall that each $\psi_{i,j}$ is a finite conjunction of atomic formulas, i.e., $\psi_{i,j} = \theta_1(\overline{x}, \overline{y}_{i,j}, \overline{z}) \wedge \dots \wedge \theta_n(\overline{x}, \overline{y}_{i,j}, \overline{z})$ for some $n \in \omega$. For $\theta(\overline{x})$ an atomic formula, let

$\ulcorner\theta(\overline{a})\urcorner$ be the function mapping $\theta(\overline{a})$ to its code in the positive diagram of a structure. For example, if $\theta(\overline{x}) = R_i(x_3, x_5)$, then $\ulcorner\theta(\overline{a})\urcorner = \langle i+2, \langle a_3, a_5\rangle\rangle$ for $\overline{a} \in \omega^{<\omega}$. Consider the set $X_{i,j}^{\overline{a},\overline{b},\overline{c}} = \{\ulcorner\theta_k(\overline{a},\overline{b},\overline{c})\urcorner : k < n\}$. Clearly, $X_{i,j}^{\overline{a},\overline{b},\overline{c}} \subseteq P(\mathcal{A})$ if and only if $\mathcal{A} \models \psi_{i,j}(\overline{a},\overline{b},\overline{c})$ for any L-structure $\mathcal{A}$. We define an enumeration operator Ψ by enumerating all pairs $\langle \overline{a}, X_{i,j}^{\overline{a},\overline{b},\overline{c}}\rangle$ into Ψ. It follows that

$$\begin{aligned}
\overline{a} \in \Psi^{P(\mathcal{B})} &\Leftrightarrow \exists \langle \overline{a}, X_{|\overline{a}|,j}^{\overline{a},\overline{b},\overline{c}}\rangle \in \Psi \wedge X_{|\overline{a}|,j}^{\overline{a},\overline{b}\overline{c}} \subseteq P(\mathcal{B}) \\
&\Leftrightarrow \mathcal{B} \models \exists \overline{y}_{|\overline{a}|,j} \psi_{|\overline{a}|,j}(\overline{a}, \overline{y}_{|\overline{a}|,j}, \overline{c}) \\
&\Leftrightarrow \mathcal{B} \models \varphi_{|\overline{a}|}(\overline{a},\overline{c})
\end{aligned}$$

and thus R is r.i.p.e.

Let g be a r.i.p.e. generic enumeration of $\mathcal{A}$ produced as in the paragraph above the theorem and let $\mathcal{B} = g^{-1}(\mathcal{A})$. Our goal is to produce a Σ_1^p definition of $R^{\mathcal{B}}$. As $\mathcal{B} \cong \mathcal{A}$, we get that this is then also a definition for R. Towards this, consider the set

$$Q_e = \{\overline{q} \in A^* : \exists l, j_1, \dots j_l < |\overline{q}|\, [\langle j_1, \dots, j_l\rangle \in \Psi_e^{P_{\mathcal{A}}(\overline{q})} \text{ and } \langle q_{j_1}, \dots, q_{j_l}\rangle \notin R]\}.$$

We have that g meets Q_e if and only if $R^{\mathcal{B}} \neq \Psi_e^{P_{\mathcal{B}}}$. By our assumption that R is r.i.p.e., there is e_0 such that $\Psi_{e_0}^{P(\mathcal{B})} = R^{\mathcal{B}}$, and thus g does not meet Q_{e_0}.

We will use this to give a Σ_1^p definition of R with parameters $\overline{p}_s$, where $\overline{p}_s$ is the stage in the construction of g at which we decide Q_{e_0}.

Notice that if there is some $\overline{q} \supseteq \overline{p}_s$ and sub-tuple $\langle q_{j_1}, \dots, q_{j_l}\rangle$ such that $\langle j_1, \dots, j_l\rangle \in \Psi_e^{P_{\mathcal{A}}(\overline{q})}$ then we must have $\langle q_{j_1}, \dots, q_{j_l}\rangle \in R$ or else we will have that $\overline{q} \in Q_{e_0}$. We now show that R is equal to the set

$$S = \{\langle q_{j_1}, \dots, q_{j_l}\rangle \in A^* : \text{ for some } \overline{q} \in A^{<\omega} \text{ and } l, j_1, \dots, j_l < |\overline{q}| \text{ satisfying } \overline{q} \supseteq \overline{p}_s \\ \text{and } \langle j_1, \dots, j_l\rangle \in \Psi_e^{P_{\mathcal{A}}(\overline{q})}\}$$

By the previous paragraph, $S \subseteq R$. If $\overline{a} \in R$ then there are indices $j_1, \dots, j_{|\overline{a}|}$ such that $\overline{a} = \langle g(j_1), \dots g(j_{|\overline{a}|})\rangle$ and so if we take a long enough initial segment of g it will witness the fact that $\overline{a} \in S$. Fix an enumeration $(\varphi_i^{at})_{i\in\omega}$ of all atomic formulas where without loss of generality the free variables of φ_i^{at} are a subset of $\{x_0, \ldots, x_i\}$. The following is a Σ_1^p definition of S

$$\bigvee\!\!\!\bigvee_{C \subset_{\mathrm{fin}} \omega} \bigvee_{\langle j_1, \ldots, j_{|\overline{a}|}\rangle \in W_e^C} \exists\, \overline{q} \supseteq \overline{p}_s \, [\langle q_{j_1}, \ldots, q_{j_{|\overline{a}|}}\rangle = \overline{a} \,\wedge\, \bigwedge_{i \in C} [\varphi_i^{at}] \frac{q_0}{x_0} \cdots \frac{q_{|\overline{q}|}}{x_{|\overline{q}|}}]$$

where the latter half of the formula is simply saying that $C \subseteq P_{\mathcal{A}}(\overline{q})$. ☐

Corollary 4.2.11. *Let $\mathcal{A}$ be a structure and $R \subseteq A^{<\omega}$ be a relation on it. Then the following are equivalent:*

(i) R is uniformly relatively intrinsically positively enumerable in $\mathcal{A}$,
(ii) R is Σ_1^p-definable in $\mathcal{A}$ without parameters.

Proof. For $(i) \Rightarrow (ii)$ we mimic the proof of Theorem 4.2.10. Let Ψ_e be the fixed enumeration operator such that $R^{\mathcal{B}} = \Psi_e^{P(\mathcal{B})}$ and Q_e as above. Note that no $\overline{q}$ can be in Q_e and so we mimic the construction of our set S with $\overline{p}_s$ being the empty tuple.

For $(ii) \Rightarrow (i)$ we again mimic the same direction in 4.2.10 excluding the parametrizing tuple $\overline{c}$ to make the process uniform. ☐

4.2.3 R.i.p.e. completeness

Similar to the study of computably enumerable sets we want to investigate notions of completeness for r.i.p.e. relations on a given structure. Before we obtain a natural example of a r.i.p.e. complete relation we have to define a suitable notion of reduction.

Definition 4.2.12. Given a structure $\mathcal{A}$ and two relations $P, R \subseteq A^{<\omega}$, we say that P is *positively intrinsically one reducible* to R, and write $P \leqslant_{\mathrm{p1}} R$, if for all $\mathcal{B} \cong \mathcal{A}$ $P(\mathcal{B}) \oplus P^{\mathcal{B}} \leqslant_1 P(\mathcal{B}) \oplus R^{\mathcal{B}}$.

Proposition 4.2.13. *Positive intrinsic one reducibility ($\leqslant_{\mathrm{p1}}$) is a reducibility.*

Proof. Let $\mathcal{A}$ be a structure with relations $P, Q, R \subseteq A^{<\omega}$. It is easy to see that $\leqslant_{\mathrm{p1}}$ is reflexive, since for any structure $\mathcal{B} \cong \mathcal{A}$ we have that $P(\mathcal{B}) \oplus P^{\mathcal{B}} \leqslant_1 P(\mathcal{B}) \oplus P^{\mathcal{B}}$, which means $P \leqslant_{\mathrm{p1}} P$. To see that it is transitive assume that $P \leqslant_{\mathrm{p1}} R$ and $R \leqslant_{\mathrm{p1}} Q$ and let $\mathcal{B} \cong \mathcal{A}$. By assumption $P(\mathcal{B}) \oplus P^{\mathcal{B}} \leqslant_1 P(\mathcal{B}) \oplus R^{\mathcal{B}} \leqslant_1 P(\mathcal{B}) \oplus Q^{\mathcal{B}}$ and so $P \leqslant_{\mathrm{p1}} Q$. ☐

Definition 4.2.14. Fix a structure $\mathcal{A}$. A relation $R \subseteq A^{<\omega}$ is *r.i.p.e. complete* if R is r.i.p.e. and for every r.i.p.e. relation P on $\mathcal{A}$, $P \leqslant_{\mathrm{p1}} R$.

The most natural way to obtain a complete set is to follow the construction of the Kleene set in taking the computable join of all r.i.p.e. sets. The result is a relation $R \subseteq \omega \times A^{<\omega}$ which can be seen as a uniform sequence of r.i.p.e. relations in the sense that there is a enumeration operator Ψ such that $(\Psi^{\mathcal{A}})^{[i]} = R_i$. For any isomorphic copy of $\mathcal{A}$, the nth slice of its Kleene set should correspond to the nth r.i.p.e. relation. In order to accomplish this we use the following coding. Given a relation $R \subseteq \omega \times A^{<\omega}$, we can identify it with a subset $R' \subseteq A^{<\omega}$ as follows. For any two elements $b, c \in A$ let

$$\overbrace{b \dots b}^{i\times} c\overline{a} \in R' \iff \langle i, \overline{a} \rangle \in R.$$

One can now easily see that R is a uniform sequence of r.i.p.e. relations if and only if the so obtained relation R' is uniformly r.i.p.e.

We can now give a natural candidate for a r.i.p.e. complete relation.

Definition 4.2.15. Let $\varphi_{i,j}^{\Sigma_1^p}$ be the ith formula with free variables $x_1, \dots, x_j$ in a computable enumeration of all Σ_1^p formulas. The *positive Kleene predicate* relative to $\mathcal{A}$ is $\vec{K}_p^{\mathcal{A}} = \left(K_i^{\mathcal{A}}\right)_{i\in\omega}$, where

$$K_i^{\mathcal{A}} = \bigcup_{j\in\omega} \{\overline{a} \in A^j : \mathcal{A} \models \varphi_{i,j}^{\Sigma_1^p}(\overline{a})\}.$$

Notice that we defined the positive Kleene predicate as a sequence of relations instead of a single relation. It is slightly more convenient as we do not have to deal with coding, but we could write it as a single relation on $\omega \times A^{<\omega}$ and code it as above. Another alternative definition would be to break down the sequence even further and let the Kleene predicate be the sequence

$$(\{\overline{a} : \mathcal{A} \models \varphi_{i,j}^{\Sigma_1^p}(\overline{a})\})_{\langle i,j\rangle\in\omega}$$

so that $(\mathcal{A}, \vec{K}_p^{\mathcal{A}})$ is a first order structure. However, as all of these definitions are computationally equivalent these distinctions are irrelevant for our purpose.

Proposition 4.2.16. *The positive Kleene predicate $\vec{K}_p^{\mathcal{A}}$ is uniformly r.i.p.e., and r.i.p.e. complete.*

Proof. Since $\vec{K}_p^{\mathcal{A}}$ is defined in a Σ_1^p way without parameters we can use Corollary 4.2.11 to see that it is uniformly relatively intrinsically positively enumerable. Let R be any relation on $\mathcal{A}$ of arity a_R. Notice that $R^{\mathcal{B}}$ is trivially Σ_1^p definable, and so there is a formula $\varphi_{i,a_R}^{\Sigma_1^P}$ such that $\overline{a} \in R^{\mathcal{B}} \Leftrightarrow \mathcal{B} \models \varphi_{i,a_R}^{\Sigma_1^p}(\overline{a})$. This shows that $P(\mathcal{B}) \oplus R^{\mathcal{B}} \leqslant_1 P(\mathcal{B}) \oplus \vec{K}_p^{\mathcal{B}}$. $\square$

4.2.4 R.i.p.e. sets of natural numbers

Our above discussion of sequences of r.i.p.e. relations allows us to code sets of natural numbers as r.i.p.e. relations.

Definition 4.2.17. A set $X \subseteq \omega$ is *r.i.p.e. in a structure* $\mathcal{A}$ if the following relation is r.i.p.e.:

$$R_X := \{\overbrace{b \dots b}^{i\times} c : i \in X, b, c \in A\}$$

A natural question is which sets of natural numbers are r.i.p.e. in a given structure. One characterization can be derived directly from the definitions: The sets X such that $X \leqslant_e P(\mathcal{B})$ for all $\mathcal{B} \cong \mathcal{A}$. Another one can be given using co-spectra, a notion defined by Soskov [32]. Intuitively, the co-spectrum of a structure $\mathcal{A}$ is the maximal ideal in the enumeration degrees such that every member of it is below every copy of $\mathcal{A}$. More formally.

Definition 4.2.18. The *co-spectrum* of a structure $\mathcal{A}$ is

$$Co(\mathcal{A}) = \bigcap_{\mathcal{B} \cong \mathcal{A}} \{\mathbf{d} : \mathbf{d} \leqslant deg_e(P(\mathcal{B}))\}.$$

Let us point out that Soskov's definition of co-spectra appears to be different from ours. We will prove in 4.3 that the two definitions are equivalent. This definition also agrees with the definition of co-spectra on Turing degrees when we restrict our attention to total degrees.

Definition 4.2.19. A set A is said to be *total* if $A \equiv_e A \oplus \overline{A}$. An enumeration degree is said to be total if it contains a total set, and a structure $\mathcal{A}$ is total if $P(f^{-1}(\mathcal{A}))$ is a total set for every enumeration f.

Given a tuple $\overline{a}$ in some structure $\mathcal{A}$ let $\Sigma_1^p\text{-}tp_{\mathcal{A}}(\overline{a})$ be the set of positive finitary Σ_1^0 formulas true of $\mathcal{A}$. The equivalence of 1 and 3 in 4.2.20 is the analogue to a well-known theorem of Knight [22] for total structures.

Theorem 4.2.20. *The following are equivalent for every structure $\mathcal{A}$ and $X \subseteq \omega$.*

1. *X is r.i.p.e. in $\mathcal{A}$*
2. *$deg_e(X) \in Co(\mathcal{A})$*
3. *X is enumeration reducible to $\Sigma_1^p\text{-}tp_{\mathcal{A}}(\overline{a})$ for some tuple $\overline{a} \in A^{<\omega}$.*

Proof. If $deg_e(X) \in Co(\mathcal{A})$, then for all $\mathcal{B} \cong \mathcal{A}$, $X \leqslant_e P(\mathcal{B})$. Given an enumeration of X, enumerate $b^n c$ into R_X for all elements $b, c \in B$ whenever you see n enter X. The relation R_X clearly witnesses that X is r.i.p.e. in $\mathcal{A}$. This shows that 2 implies 1. On the other hand if X is r.i.p.e. in $\mathcal{A}$, then given any $B \cong \mathcal{A}$ and an enumeration of $R_X^{\mathcal{B}}$ build a set S by enumerating n into S whenever you see $b^n c$ enumerated into $R_X^{\mathcal{B}}$ for any two elements $b, c \in B$. Clearly $n \in S$ if and only if $n \in X$ and thus 1 implies 2.

To see that 3 implies 2 assume that X is enumeration reducible to the positive Σ_1 type of a tuple $\bar{b}$ in any copy $\mathcal{B}$ of $\mathcal{A}$. As the $\Sigma_1^p\text{-}tp_{\mathcal{B}}(\bar{b})$ is enumeration reducible to $P(\mathcal{B})$, by transitivity $X \leqslant_e P(\mathcal{B})$ for every $\mathcal{B} \cong \mathcal{A}$ and thus $deg_e(X) \in Co(\mathcal{A})$. At last, we show that 1 implies 3. Assume that X is r.i.p.e. in $\mathcal{A}$. Then there is a computable enumeration of Σ_1^p formulas ψ_n with parameters $\overline{p}$ such that for some $\overline{a} \in A^{<\omega}$, $n \in X \Leftrightarrow \mathcal{A} \models \psi_n(\overline{a})$. Simultaneously enumerate $\Sigma_1^p\text{-}tp_{\mathcal{A}}(\overline{a})$ and the disjuncts in the formulas ψ_n. Whenever you see a disjunct of ψ_n that is also in $\Sigma_1^p\text{-}tp_{\mathcal{A}}(\overline{a})$ enumerate n. This gives an enumeration of X given an enumeration of $\Sigma_1^p\text{-}tp_{\mathcal{A}}(\overline{a})$ and thus $X \leqslant_e \Sigma_1^p\text{-}tp_{\mathcal{A}}(\overline{a})$ as required. □

4.3 The positive jump and degree spectra

In this section we compare various definitions of the jump of a structure with respect to their enumeration degree spectra, a notion first studied by Soskov [32].

Definition 4.3.1. The *positive jump* of a structure $\mathcal{A}$ is the structure

$$J_p(\mathcal{A}) = (\mathcal{A}, \overline{\vec{K}_p^{\mathcal{A}}}) = (\mathcal{A}, (\overline{K_i^{\mathcal{A}}})_{i \in \omega}).$$

We are interested in the degrees of enumerations of $\mathcal{A}$ and $J_p(\mathcal{A})$. To be precise, let f be an enumeration of ω, that is, a surjective mapping $\omega \to \omega$, and for $X \subseteq \omega^{<\omega}$ let

$$f^{-1}(X) = \{\langle x_1, \ldots \rangle : (f(x_1), \ldots,) \in X\}.$$

Then, given a structure $\mathcal{A}$ let $f^{-1}(\mathcal{A}) = (\omega, f^{-1}(=), f^{-1}(\neq), f^{-1}(R_1^{\mathcal{A}}), f^{-1}(R_2^{\mathcal{A}}), \ldots)$. This definition differs from the definition given in Soskov [32] where $f^{-1}(\mathcal{A})$ means what we will denote as $P(f^{-1}(\mathcal{A}))$, i.e.,

$$= \oplus \neq \oplus f^{-1}(=) \oplus f^{-1}(\neq) \oplus \bigoplus_{i \in \omega} f^{-1}(R_i^{\mathcal{A}}).$$

Definition 4.3.2 (Soskov [32])**.** Given a structure $\mathcal{A}$, define the set of enumerations of a structure $Enum(\mathcal{A}) = \{P(\mathcal{B}) : \mathcal{B} = f^{-1}(\mathcal{A})$ for f an enumeration of $\omega\}$. Further, let the *enumeration degree spectrum* of $\mathcal{A}$ be the set

$$eSp(\mathcal{A}) = \{d_e(P(\mathcal{B})) : P(\mathcal{B}) \in Enum(\mathcal{A})\}$$

If $\mathbf{a}$ is the least element of $eSp(\mathcal{A})$, then $\mathbf{a}$ is called the *enumeration degree* of $\mathcal{A}$.

In classical computable structure theory many results often do not hold for structures that are structurally too simple, called the automorphically non-trivial structures. One example of this phenomenon is a basic result about Turing degree spectra due to Knight [22]: The degree spectra of automorphically non-trivial spectra are closed upwards in the Turing degrees. However, if a structure is automorphically trivial, then its degree spectrum contains a single Turing degree. Consider for example the structure $(\mathbb{N}, =, \neq)$. All its isomorphic copies are computable and in fact, there exists only one copy with universe $\mathbb{N}$ of this structure. Considering enumerations of a structure that are not bijective allows us to avoid these dichotomies when studying enumeration degree spectra. It is not hard to see that the enumeration degree spectrum of any structure is closed upwards in the Turing degrees [32, Proposition 2.6].

As mentioned after 4.2.18, Soskov's definition of the co-spectrum of a structure was slightly different [32]. He defined the co-spectrum of a structure $\mathcal{A}$ as the set

$$\{\mathbf{d} : \forall(\mathbf{a} \in eSp(\mathcal{A}))\, \mathbf{d} \leqslant \mathbf{a}\}.$$

We now show that the two definitions are equivalent.

Proposition 4.3.3. *For every structure $\mathcal{A}$, $Co(\mathcal{A}) = \{\mathbf{d} : \forall(\mathbf{a} \in eSp(\mathcal{A}))\mathbf{d} \leqslant \mathbf{a}\}$.*

Proof. First note that $\{\mathbf{d} : \forall(\mathcal{C} \cong \mathcal{A})\mathbf{d} \leqslant deg_e(P(\mathcal{C}))\} \supseteq \{\mathbf{d} : \forall(\mathbf{a} \in eSp(\mathcal{A}))\mathbf{d} \leqslant \mathbf{a}\}$ as every $P(\mathcal{C})$ is the pullback of $\mathcal{A}$ along an injective enumeration.

On the other hand for any enumeration $f : \omega \to \omega$, $f^{-1}(\mathcal{A})/f^{-1}(=) \cong \mathcal{A}$. Consider the substructure $\mathcal{B}$ of $f^{-1}(\mathcal{A})$ consisting of the least element in every $f^{-1}(=)$ equivalence class. Since $f^{-1}(\mathcal{A})/f^{-1}(=) \cong \mathcal{A}$, we get that $\mathcal{B} \cong \mathcal{A}$. As $P(f^{-1}(\mathcal{A}))$ contains both $f^{-1}(=)$ and $f^{-1}(\neq)$ we can compute an enumeration of $B \oplus \overline{B}$ from any enumeration of $P(f^{-1}(\mathcal{A}))$ and thus also the graph of its principal function p_B. Let $\mathcal{C} = p_B^{-1}(\mathcal{B})$. Then $\mathcal{C} \cong \mathcal{A}$, and $P(\mathcal{C}) \leqslant_e P(f^{-1}(\mathcal{A}))$. Thus,

$$\{\mathbf{d} : \forall(\mathbf{a} \in eSp(\mathcal{A}))\mathbf{d} \leqslant \mathbf{a}\} = \{\mathbf{d} : \forall(\mathcal{C} \cong \mathcal{A})\mathbf{d} \leqslant deg_e(P(\mathcal{C}))\} = Co(\mathcal{A}).$$

□

Using this notation we can see that for any structure $\mathcal{A}$, and any enumeration f of ω we have that $J_p(f^{-1}(\mathcal{A}))$ is the structure $(f^{-1}(\mathcal{A}), \overline{\vec{K}_p^{f^{-1}(\mathcal{A})}})$. Thus

$$P(J_p(f^{-1}(\mathcal{A}))) = f^{-1}(=) \oplus f^{-1}(\neq) \oplus \bigoplus_{j\in\omega} \overline{K_j^{f^{-1}(\mathcal{A})}} \oplus \bigoplus_{i\in\omega} R_i^{f^{-1}(\mathcal{A})}.$$

If we instead apply the enumeration to $J_p(\mathcal{A})$ we will get the structure $f^{-1}(J_p(\mathcal{A}))$. Now, for every relation R on $\mathcal{A}$, $R^{f^{-1}(\mathcal{A})} = f^{-1}(R^{\mathcal{A}})$ and thus $\overline{K_j^{f^{-1}(\mathcal{A})}} = f^{-1}\left(\overline{K_j^{\mathcal{A}}}\right)$. So $P(f^{-1}(J_p(\mathcal{A}))) = P(J_p(f^{-1}(\mathcal{A})))$.

4.3.1 Properties of R.i.p.e. generics

Recall 4.2.9 of r.i.p.e.-generic enumerations and presentations. We are now ready to study their properties. The following lemma is an analogue of the well-known result that there is a Δ^0_2 1-generic.

Lemma 4.3.4. *Every structure $\mathcal{A}$ has a r.i.p.e.-generic enumeration g such that $Graph(g) \leqslant_e P(J_p(\mathcal{A}))$. In particular, $P(g^{-1}(J_p(\mathcal{A}))) \leqslant_e P(J_p(\mathcal{A}))$.*

Proof. Recall our hands-on definition of a r.i.p.e. generic enumeration g of $\mathcal{A}$ after 4.2.9 using an enumeration $(R_e)_{e\in\omega}$ of all r.i.p.e. subsets of A^*. The set $\{\overline{p} : \exists\overline{q} \supset \overline{p}, \overline{q} \in R_e\}$ is Σ^p_1-definable in $\mathcal{A}$ and so enumerable from $P(J_p(\mathcal{A}))$, which contains $P(\mathcal{A})$. The coset $\{\overline{p} : \forall\overline{q} \supset \overline{p}, \overline{q} \notin R_e\}$ is co-r.i.p.e. and so enumerable from $\overline{\vec{K}^{\mathcal{A}}_p}$. Hence, $P(J_p(\mathcal{A}))$ will be able to decide when a tuple $\overline{p}_s$ belongs to the upward closure of R_e. Thus $Graph(g) \leqslant_e P(J_p(\mathcal{A}))$.

If we can enumerate the graph of g and also $J_p(\mathcal{A})$, then to enumerate $g^{-1}(J_p(\mathcal{A}))$, we wait for something of the form $(x, g(x)) \in Graph(g)$ to appear with $g(x) \in J_p(\mathcal{A})$ and enumerate x into the corresponding slice of $g^{-1}(J_p(\mathcal{A}))$. □

R.i.p.e. generic presentations have many useful properties. One of them is that they are minimal in the sense that the only sets enumeration below a r.i.p.e. generic presentation are the r.i.p.e. sets.

Lemma 4.3.5. *If $\mathcal{B}$ is a r.i.p.e.-generic presentation of $\mathcal{A}$, then $X \subseteq \omega$ is r.i.p.e. in $\mathcal{B}$ if and only if $X \leqslant_e P(\mathcal{B})$.*

Proof. If X r.i.p.e. then it is enumerable from $P(\mathcal{B})$ by definition. Assume X is enumerable from $P(\mathcal{B})$. Then $X = \Psi_e^{P(\mathcal{B})}$ for some e. Recall the set Q_e from 4.2.10 which we know to be r.i.p.e. because we gave a Σ^p_1 description of it.

$$Q_e = \{\overline{q} \in A^{<\omega} : \exists l, j_1, \dots j_l < |\overline{q}|\,[\langle j_1, \dots, j_l\rangle \in \Psi_e^{P_{\mathcal{B}}(\overline{q})} \text{ and } \langle q_{j_1}, \dots, q_{j_l}\rangle \notin X]\}.$$

Now because $\mathcal{B}$ is r.i.p.e.-generic, there is some tuple $\langle 0, \dots, k-1\rangle$ that decides Q_e. It must be that $\langle 0, \dots, k-1\rangle \notin Q_e$, or we would contradict the fact that $X = \Psi_e^{P(\mathcal{B})}$, and so $\langle 0, \dots, k-1\rangle$ forms the parameterizing tuple $\overline{p}_s$ from the set S in 4.2.10. □

Another useful property is that the degree of the positive jump of a r.i.p.e. generic agrees with the enumeration jump of the degree of their positive diagram.

Proposition 4.3.6. *Let $\mathcal{A}$ be a structure. For an arbitrary enumeration f of $\mathcal{A}$, let $\mathcal{B} = f^{-1}(\mathcal{A})$. Then $P(J_p(\mathcal{B})) \leqslant_e J_e(P(\mathcal{B}))$. Furthermore, if $\mathcal{B}$ is a r.i.p.e.-generic presentation then $P(J_p(\mathcal{B})) \equiv_e J_e(P(\mathcal{B}))$.*

Proof. Recall that $J_p(\mathcal{B}) = (\mathcal{B}, \overline{\vec{K}_p^{\mathcal{B}}})$ and $J_e(P(\mathcal{B})) = P(\mathcal{B}) \oplus \overline{K}_{P(\mathcal{B})}$, so to show that $P(J_p(\mathcal{B})) \leqslant_e J_e(P(\mathcal{B}))$ it is sufficient to show that $\overline{\vec{K}_p^{\mathcal{B}}} \leqslant_e \overline{K}_{P(\mathcal{B})}$. As $\vec{K}_p^{\mathcal{B}}$ is r.i.p.e., $\vec{K}_p^{\mathcal{B}} = \Psi_e^{P(\mathcal{B})}$ for some e. Thus $\overline{\vec{K}_p^{\mathcal{B}}}$ appears as a slice of $\{\langle x, i\rangle : x \notin \Psi_i^{P(\mathcal{B})}\}$ and hence $\overline{\vec{K}_p^{\mathcal{B}}} \leqslant_e \{\langle x, i\rangle : x \notin \Psi_i^{P(\mathcal{B})}\}$. The latter set is m-equivalent to $\overline{K}_{P(\mathcal{B})}$ and thus $\overline{\vec{K}_p^{\mathcal{B}}} \leqslant_e \overline{K}_{P(\mathcal{B})}$.

It remains to show that $J_e(P(\mathcal{B})) \leqslant_e P(J_p(\mathcal{B}))$ if $\mathcal{B}$ is r.i.p.e.-generic. For every e consider the set

$$R_e = \{\overline{q} \in B^* : e \in \Psi_e^{P_{\mathcal{B}}(\overline{q})}\}.$$

Note that the R_e are all trivially r.i.p.e. and are closed upwards as subsets of B^*. So since $\mathcal{B}$ is r.i.p.e.-generic, for every e there is an initial segment of B^* that either is in R_e or such that no extension of it is in R_e. The set $Q_e = \{\overline{p} \in B^* : \forall(\overline{q} \supseteq \overline{p})(\overline{q} \notin R_e)\}$ of elements in B^* that are non-extendible in R_e is co-r.i.p.e. Also note that these sets are uniform in e, that is, given e we can compute the indices of R_e and Q_e as r.i.p.e., respectively, co-r.i.p.e. subsets of $\mathcal{B}$. Thus, given an enumeration of $P(J_p(\mathcal{B}))$ we can enumerate $P(\mathcal{B})$ and the sets Q_e and R_e. By genericity for all $e \in \omega$ there is an initial segment of $\overline{b}$ of B such that either $\overline{b} \in Q_e$ or $\overline{b} \in R_e$. So whenever we see such a $\overline{b}$ in Q_e we enumerate e and thus obtain an enumeration $\overline{K}_{P(\mathcal{B})}$. □

The above properties of r.i.p.e. generics are very useful to study how the enumeration degree spectra of structures and their positive jumps relate.

Proposition 4.3.7. *For every structure $\mathcal{A}$, $J_p(\mathcal{A})$ is a total structure.*

Proof. First, note that we have the following equality for arbitrary relations R and enumerations f.

$$x \in f^{-1}(\overline{R}) \Leftrightarrow f(x) \in \overline{R} \Leftrightarrow f(x) \notin R \Leftrightarrow x \notin f^{-1}(R) \Leftrightarrow x \in \overline{f^{-1}(R)}$$

Recall that for every R_i in the language of $\mathcal{A}$, R_i is r.i.p.e. in $\mathcal{A}$ and that

$$P(f^{-1}(J_p(\mathcal{A}))) = f^{-1}(=) \oplus f^{-1}(\neq) \oplus f^{-1}(\overline{\vec{K}_p^{\mathcal{A}}}) \oplus f^{-1}(R_1^{\mathcal{A}}) \oplus \cdots$$

and we observe that all $R_i^{\mathcal{A}}$ are trivially r.i.p.e., uniformly in i, so in particular $\overline{f^{-1}(R_i^{\mathcal{A}})} \leqslant_e \overline{f^{-1}(\vec{K}_p^{\mathcal{A}})} = f^{-1}(\overline{\vec{K}_p^{\mathcal{A}}})$. Also, $f^{-1}(\vec{K}_p^{\mathcal{A}}) \leqslant_e P(f^{-1}(\mathcal{A})) \leqslant_e P(f^{-1}(J_p(\mathcal{A})))$. □

We will now see that the positive jump of a structure jumps, in the sense that the enumeration degree spectrum of the positive jump is indeed the set of jumps of the degrees in the enumeration degree spectrum of the structure. The following version of a Theorem by Soskova and Soskov [36] is essential to our proof.

Theorem 4.3.8 (Soskova, Soskova[36, Theorem 1.2])**.** *Let B be an arbitrary set of natural numbers. There exists a total set F such that*

$$B \leqslant_e F \quad \textit{and} \quad J_e(B) \equiv_e J_e(F).$$

Theorem 4.3.9. *For any structure $\mathcal{A}$,*

$$eSp(J_p(\mathcal{A})) = \{\mathbf{a}' : \mathbf{a} \in eSp(\mathcal{A})\}$$

Proof. To show that $\{\mathbf{a}' : \mathbf{a} \in eSp(\mathcal{A})\} \subseteq eSp(J_p(\mathcal{A}))$, consider an arbitrary enumeration f of ω and let $\mathcal{B} = f^{-1}(\mathcal{A})$. Then from Proposition 4.3.6 we know that $P(J_p(\mathcal{B})) \leqslant_e J_e(P(\mathcal{B}))$. Note that $P(J_p(\mathcal{B})) = P(J_p(f^{-1}(\mathcal{A}))) = P(f^{-1}(J_p(\mathcal{A})))$, and
$d_e(P(f^{-1}(J_p(\mathcal{A}))) \in eSp(J_p(\mathcal{A}))$. Since the enumeration jump is always total and enumeration degree spectra are closed upwards with respect to total degrees, $d_e(J_e(P(\mathcal{B}))) \in eSp(J_p(\mathcal{B}))$.

To show $eSp(J_p(\mathcal{A})) \subseteq \{\mathbf{a}' : \mathbf{a} \in eSp(\mathcal{A})\}$, let us look at $P(f^{-1}(J_p(\mathcal{A})))$ for some enumeration f of ω, and again write $\mathcal{B} = f^{-1}(\mathcal{A})$ so that $P(f^{-1}(J_p(\mathcal{A}))) = P(J_p(\mathcal{B}))$. Since $J_p(\mathcal{A})$ is a total structure, we know that $P(J_p(\mathcal{B}))$ is total. By 4.3.4 we can use $P(J_p(\mathcal{B}))$ to enumerate a r.i.p.e.-generic enumeration g of $\mathcal{B}$ such that $P(g^{-1}(J_p(\mathcal{B}))) \leqslant_e P(J_p(\mathcal{B}))$. Then, letting $\mathcal{C} = g^{-1}(\mathcal{B})$ and using the latter part of 4.3.6 we know that $J_e(P(\mathcal{C})) \equiv_e P(J_p(\mathcal{C}))$ which makes $J_e(P(\mathcal{C})) \leqslant_e P(J_p(\mathcal{B}))$. Using 4.3.8 there is a total set F such that $P(\mathcal{C}) \leqslant_e F$ and $J_e(F) \equiv_e J_e(P(\mathcal{C})) \leqslant_e P(J_p(\mathcal{B}))$. As F and $P(J_p(\mathcal{B}))$ are total, we have that $F' \leqslant_T P(J_p(\mathcal{B}))$.

We can now apply the relativized jump inversion theorem for the Turing degrees to get a set $Z \geqslant_T F$ such that $Z' \equiv_T P(J_p(\mathcal{B}))$. For this Z, we have that $\iota(deg(Z')) = d_e(J_e(\chi_Z)) = d_e(P(J_p((\mathcal{B}))) \in eSp(J_p(\mathcal{A}))$ and $d_e(P(\mathcal{B})) \leqslant d_e(F) \leqslant d_e(\chi_Z)$. Since χ_Z is total and enumeration degree spectra are upwards closed in the total degrees, this means $d_e(\chi_Z) \in eSp(\mathcal{B}) \subseteq eSp(\mathcal{A})$. So in particular, $d_e(J_e(\chi_Z)) = d_e(P(J_p(\mathcal{B}))) \in \{\mathbf{a}' : \mathbf{a} \in eSp(\mathcal{A})\}$. □

We now compare the enumeration degree spectrum of the positive jump of a structure $\mathcal{A}$ with the spectrum of the original structure and the spectrum of its traditional jump as was defined in Section 1.2.2.

We will also consider the structure $\mathcal{A}^+$ as defined in 1.1.1.

We will not compare the enumeration degree spectra directly but instead the sets of enumerations. This gives more insight than comparing the degree spectra as we can make use of the following analogues to Muchnik and Medvedev reducibility for enumeration degrees. Given $\mathfrak{A}, \mathfrak{B} \subseteq P(\omega)$ we say that $\mathfrak{A} \leqslant_{we} \mathfrak{B}$, $\mathfrak{A}$ is *weakly enumeration reducible* to $\mathfrak{B}$, if for every $B \in \mathfrak{B}$ there is $A \in \mathfrak{A}$ such that $A \leqslant_e B$. We say that $\mathfrak{A} \leqslant_{se} \mathfrak{B}$, $\mathfrak{A}$ is *strongly enumeration reducible* to $\mathfrak{B}$, if there is an enumeration operator Ψ such that for every $B \in \mathfrak{B}$, $\Psi^B \in \mathfrak{A}$.

It is not hard to see that given an enumeration of $\mathcal{A}^+$ one can enumerate an enumeration of $J(\mathcal{A})$ and the converse holds trivially. We thus have the following.

Proposition 4.3.10. *For every structure $\mathcal{A}$, $deg_e(J(\mathcal{A})) = deg_e(\mathcal{A}^+)$. In particular $Enum(J(\mathcal{A})) \equiv_{se} Enum(\mathcal{A}^+)$.*

Proof. By replacing every subformula of the form $\neg R_i(x_1, \ldots, x_m)$ by the formula $\overline{R}_i(x_1, \ldots, x_m)$ we get a Σ^p_1 formula in the language of $\mathcal{A}^+$. Similarly, given a Σ^p_1 formula in the language of $\mathcal{A}^+$ we can obtain a Σ^c_1 formula in the language of $\mathcal{A}$ by substituting subformulas of the form $\overline{R}_i(x_1, \ldots, x_m)$ with $\neg R_i(x_1, \ldots, x_m)$. Indeed we get a computable bijection between the Σ^p_1 formulas in the language of $\mathcal{A}^+$ and the Σ^c_1 formulas of $\mathcal{A}$. Thus $J(\mathcal{A}) \equiv_e (\mathcal{A}^+, \vec{K}^{\mathcal{A}^+}_p)$, but $\mathcal{A}^+ \equiv_e (\mathcal{A}^+, \vec{K}^{\mathcal{A}^+}_p)$ given the first equivalence. All of these equivalence are witnessed by fixed enumeration operators and thus $Enum(J(\mathcal{A})) \equiv_{se} Enum(\mathcal{A}^+)$. □

4.3.10 is not very surprising, as adding a r.i.c.e. set such as $\vec{K}^{\mathcal{A}}$ to the totalization of $\mathcal{A}$ won't change its enumeration degree. We will thus consider a slightly different notion of jump by adding the coset of $\vec{K}^{\mathcal{A}}$:

$$T(\mathcal{A}) = (\mathcal{A}, \overline{\vec{K}^{\mathcal{A}}})$$

Clearly $T(\mathcal{A}) \equiv_T J(\mathcal{A})$; however, the two sets are not necessarily enumeration equivalent as the following two propositions show.

Proposition 4.3.11. *Let $\mathcal{A}$ be a structure. For every presentation $\hat{\mathcal{A}}$ of $\mathcal{A}$, $P(\hat{\mathcal{A}}) \leqslant_e D(\hat{\mathcal{A}}) = P(\hat{\mathcal{A}}^+) \leqslant_e J_p(\hat{\mathcal{A}}) \leqslant_e T(\hat{\mathcal{A}})$. In particular*

$$Enum(\mathcal{A}) \leqslant_{se} Enum(\mathcal{A}^+) \leqslant_{se} Enum(J_p(\mathcal{A})) \leqslant_{se} Enum(T(\mathcal{A})).$$

Proof. Straightforward from the definitions. □

Proposition 4.3.12. *There is a structure $\mathcal{A}$ such that*

$$Enum(\mathcal{A}) \ngeqslant_{we} Enum(\mathcal{A}^+) \ngeqslant_{we} Enum(J_p(\mathcal{A})) \ngeqslant_{we} Enum(T(\mathcal{A})).$$

Proof. Sacks [30] showed that there is an incomplete c.e. set X of high Turing degree. Thus, X has enumeration degree $deg_e(X) = \mathbf{0}_e$ and $deg_T(X') = \mathbf{0}''$. Let $\mathcal{A}$ be the graph coding X as follows. $\mathcal{A}$ contains a single element a with a loop, i.e. $aE^{\mathcal{A}}a$ and one circle of length $n+1$ for every $n \in \omega$. Let y be the least element in $\mathcal{A}$ that is part of the cycle of length $n+1$. If $n \in X$, then connect a to y, i.e., $aE^{\mathcal{A}}y$. This finishes the construction.

As X is c.e., $\mathcal{A}$ has enumeration degree $\mathbf{0}_e$. However, $\mathbf{0}_e \notin eSp(\mathcal{A}^+)$, as $\mathcal{A}^+$ has enumeration degree $deg_e(X \oplus \overline{X}) > \mathbf{0}_e$. So, in particular $Enum(\mathcal{A}) \ngeqslant_{we} Enum(\mathcal{A}^+)$. By 4.3.9 the enumeration degree of $J_p(\mathcal{A})$ is $\mathbf{0}_e'$ and $\mathbf{0}_e' \ni \varnothing' \oplus \overline{\varnothing'} >_e X \oplus \overline{X}$, so $Enum(\mathcal{A}^+) \ngeqslant_{we} Enum(J_p(\mathcal{A}))$. For the last inequality notice that both $T(\mathcal{A})$ and $J_p(\mathcal{A})$ are total structures and that by the analogue of 4.3.9 for the traditional jumps of structures we have that the enumeration degree of $T(\mathcal{A})$ is $deg_e(\varnothing'' \oplus \overline{\varnothing''})$. As mentioned above the enumeration degree of $J_p(\mathcal{A})$ is $\mathbf{0}_e'$, $\varnothing' \oplus \overline{\varnothing'} \in \mathbf{0}_e'$, and $\varnothing' \oplus \overline{\varnothing'} <_e \varnothing'' \oplus \overline{\varnothing''}$. So, in particular, $Enum(J_p(\mathcal{A})) \ngeqslant_{we} Enum(T(\mathcal{A}))$. □

For total structures the traditional notion of the jump and the positive jump coincide.

Proposition 4.3.13. *Let $\mathcal{A}$ be a structure, then $Enum(J_p(\mathcal{A}^+)) \equiv_{se} Enum(T(\mathcal{A}))$ and $eSp(J_p(\mathcal{A}^+)) = eSp(T(\mathcal{A}))$.*

Proof. This proof is similar to the proof of 4.3.10. Mutatis mutandis. □

4.4 Functors

When comparing structures with respect to their enumerations it is natural to want to use only positive information. Csima, Rossegger, and Yu [12] introduced the notion of a positive enumerable functor which uses only the positive diagrams of structures. Reductions based on this notion preserve desired properties such as enumeration degree spectra of structures.

Recall that a functor F from a class of structures $\mathfrak{C}$ to a class $\mathfrak{D}$ maps structures in $\mathfrak{C}$ to structures in $\mathfrak{D}$ and morphisms $f : \mathcal{A} \to \mathcal{B}$ to morphisms $F(f) : F(\mathcal{A}) \to F(\mathcal{B})$ with the property that $F(id_{\mathcal{A}}) = id_{F(\mathcal{A})}$ and $F(f \circ g) = F(f) \circ F(g)$ for all morphisms f and g and structures $\mathcal{A} \in \mathfrak{C}$.

To account for non-total enumerations we consider the category $Iso^*(\mathcal{B})$, where the objects are $\{\tilde{\mathcal{B}} : \mathcal{B} \cong \tilde{\mathcal{B}}$ and $dom(\tilde{\mathcal{B}}) \subseteq \omega$ is infinite and c.e.$\}$ and the arrows are the isomorphisms among the quotient structures of the objects.

Definition 4.4.1. A functor $F : Iso^*(\mathcal{B}) \to Iso^*(\mathcal{A})$ is *positive enumerable* if there is a pair (Ψ, Ψ_*) where Ψ and Ψ_* are enumeration operators such that for all $\mathcal{B}, \tilde{\mathcal{B}} \in Iso^*(\mathcal{B})$,

1. $\Psi^{P(\mathcal{B})} = P(F(\mathcal{B}))$.
2. For all $f \in Iso^*(\mathcal{B})$ with $f : \mathcal{B} \to \tilde{\mathcal{B}}$, $\Psi_*^{P(\mathcal{B}) \oplus Graph(f) \oplus P(\tilde{\mathcal{B}})} = Graph(F(f))$.

For ease of notation, when a graph of a function occurs in an oracle, we will simply write the name of the function to represent it.

An alternative and purely syntactical method of comparing classes of structures is through the model-theoretic notion of interpretability. Since we are restricting ourselves to positive information, we introduce a new notion of interpretability that uses Σ_1^p formulas.

Definition 4.4.2. A structure $\mathcal{A} = (A, P_0^{\mathcal{A}}, \dots)$ is *positively interpretable* in a structure $\mathcal{B}$ if there exists a Σ_1^p definable sequence of relations $(Dom_{\mathcal{A}}^{\mathcal{B}}, \sim, \nsim, R_0, \dots)$ called the *interpretation of* $\mathcal{A}$ *in* $\mathcal{B}$ in the language of $\mathcal{B}$ such that

1. $Dom_{\mathcal{A}}^{\mathcal{B}} \subseteq B^{<\omega}$,
2. $\sim$ is an equivalence relation on $Dom_{\mathcal{A}}^{\mathcal{B}}$ and $\nsim$ its corelation,
3. $R_i \subseteq (B^{<\omega})^{a_{P_i}}$ [1] is closed under $\sim$ on $Dom_{\mathcal{A}}^{\mathcal{B}}$,

[1] a_{P_i} is the arity of P_i

and there exists a function $f_{\mathcal{B}}^{\mathcal{A}} : \mathcal{A} \to Dom_{\mathcal{A}}^{\mathcal{B}}$, which induces an isomorphism:

$$(Dom_{\mathcal{A}}^{\mathcal{B}}/\sim, R_0/\sim, \dots) \cong (A/=, P_0/=, \dots)$$

We seek to provide analogues to the results proven in a paper by Harrison-Trainor, Melnikov, Miller, and Montalbán [17], starting with the following.

Theorem 4.4.3. *There is a positive enumerable functor $F : Iso^*(\mathcal{B}) \to Iso^*(\mathcal{A})$ if and only if $\mathcal{A}$ is positively interpretable in $\mathcal{B}$.*

Proof. Suppose that $\mathcal{A}$ is positively interpretable in $\mathcal{B}$ using $(Dom_{\mathcal{A}}^{\mathcal{B}}, \sim, \nsim, R_0^{\mathcal{B}}, \dots)$. We want to construct a functor $F : Iso^*(\mathcal{B}) \to Iso^*(\mathcal{A})$, so let $\mathcal{B} \in Iso^*(\mathcal{B})$. We define $F(\mathcal{B})$ to be the structure $(Dom_{\mathcal{A}}^{\mathcal{B}}, \sim, \nsim, R_0^{\tilde{\mathcal{B}}}, \dots)$. Since the domain and all of the relations are Σ_1^p definable with no parameters we can apply Theorem 4.2.11 to see that they are u.r.i.p.e. and thus the positive diagram of $F(\mathcal{B})$ is as well. We can use the enumeration operator Ψ given by the corollary as our witness that F is positively enumerable.

Now to define our functor on isomorphisms we assume we have a map $f : \tilde{\mathcal{B}} \to \hat{\mathcal{B}}$. Notice that $Dom_{\mathcal{A}}^{\mathcal{B}} \subseteq \mathcal{B}^{<\omega}$ and so f induces a map which we will call $F(f) : F(\tilde{\mathcal{B}}) \to F(\hat{\mathcal{B}})$. Since $F(\tilde{\mathcal{B}})$ and $F(\hat{\mathcal{B}})$ are uniformly enumerable from $P(\tilde{\mathcal{B}})$ and $P(\hat{\mathcal{B}})$ respectively, we can uniformly enumerate the graph of an isomorphism $F(f)$ from $P(\tilde{\mathcal{B}}) \oplus Graph(f) \oplus P(\hat{\mathcal{B}})$. We let Ψ_* be the enumeration operator that witnesses the uniform enumeration. The fact that F is a functor follows easily.

Now suppose that there is a positive enumerable functor $F = (\Psi, \Psi_*)$ from $Iso^*(\mathcal{B})$ to $Iso^*(\mathcal{A})$. We want to produce the Σ_1^p sequence of relations providing the positive interpretation of $\mathcal{B}$ in $\mathcal{A}$.

In what follows, we will often write $P(\bar{b})$ instead of $P_{\mathcal{B}}(\bar{b})$ when it is clear from context which structure $\mathcal{B}$ is meant.

We can view any finite disjoint tuple $\bar{b}$ as a map $i \mapsto b_i$ for $i < |\bar{b}|$. Note that viewing $\bar{b}$ as such a map, if f is any permutation of ω extending $\bar{b}$, then $P_{\mathcal{B}}(\bar{b}) \subseteq P(\mathcal{B}_f)$ where $\mathcal{B}_f = f^{-1}(\mathcal{B})$.

Let $Dom_{\mathcal{A}}^{\mathcal{B}}$ be the set of pairs $(\bar{b}, i) \in B^{<\omega} \times \omega$ such that

$$(i, i) \in \Psi_*^{P(\bar{b}) \oplus \lambda \upharpoonright |\bar{b}| \oplus P(\bar{b})}$$

where λ is the identity function. Since the domain of $\mathcal{B}$ is u.r.i.p.e., and due to the finite oracle $Dom_{\mathcal{A}}^{\mathcal{B}}$ is u.r.i.p.e., and thus Σ_1^p-definable.

For $(\overline{b}, i), (\overline{c}, j) \in Dom_{\mathcal{A}}^{\mathcal{B}}$ we let $(\overline{b}, i) \sim (\overline{c}, j)$ exactly if there exists a finite tuple $\overline{d}$ which does not mention elements from $\overline{b}$ or $\overline{c}$, such that if $\overline{b}'$ lists the elements that occur in $\overline{b}$ but not $\overline{c}$ and $\overline{c}'$ lists the elements in $\overline{c}$ but not in $\overline{b}$, and if $\sigma = (\overline{c}\overline{b}'\overline{d})^{-1} \circ \overline{b}\overline{c}'\overline{d}$ then

$$(i, j) \in \Psi_*^{P(\overline{b}\overline{c}'\overline{d}) \oplus \sigma \oplus P(\overline{c}\overline{b}'\overline{d})} \quad \text{and} \quad (j, i) \in \Psi_*^{P(\overline{c}\overline{b}'\overline{d}) \oplus \sigma^{-1} \oplus P(\overline{b}\overline{c}'\overline{d})}.$$

It is easy to see that $\sim$ is uniformly relatively intrinsically positively enumerable. Rather than showing immediately that the complement of $\sim$ is uniformly r.i.p.e., we define a clearly uniformly r.i.p.e. relation $\nsim$, and then show that it is indeed the complement of $\sim$.

For $(\overline{b}, i), (\overline{c}, j) \in Dom_{\mathcal{A}}^{\mathcal{B}}$ we say $(\overline{b}, i) \nsim (\overline{c}, j)$ if there exist $k \neq j, l \neq i$, and a finite tuple $\overline{d}$ which does not mention elements from $\overline{b}$ or $\overline{c}$, such that if $\overline{b}'$ lists the elements that occur in $\overline{b}$ but not $\overline{c}$ and $\overline{c}'$ lists the elements in $\overline{c}$ but not in $\overline{b}$, and if $\sigma = (\overline{c}\overline{b}'\overline{d})^{-1} \circ \overline{b}\overline{c}'\overline{d}$ then

$$(i, k) \in \Psi_*^{P(\overline{b}\overline{c}'\overline{d}) \oplus \sigma \oplus P(\overline{c}\overline{b}'\overline{d})} \quad \text{or} \quad (j, l) \in \Psi_*^{P(\overline{c}\overline{b}'\overline{d}) \oplus \sigma^{-1} \oplus P(\overline{b}\overline{c}'\overline{d})}.$$

Claim 4.4.3.1. *$\nsim$ is the complement of $\sim$.*

Proof. We want to show that, for any tuples $(\overline{b}, i), (\overline{c}, j) \in Dom_{\mathcal{A}}^{\mathcal{B}}$, exactly one of $(\overline{b}, i) \sim (\overline{c}, j)$, $(\overline{b}, i) \nsim (\overline{c}, j)$ hold. Let $\overline{b}'$ list the elements in $\overline{b}$ but not $\overline{c}$, and let $\overline{c}'$ list the elements in $\overline{c}$ but not $\overline{b}$. Let $\sigma = (\overline{c}\overline{b}')^{-1} \circ \overline{b}\overline{c}'$. Let $f, g : \omega \to \mathcal{A}$ be bijections extending $\overline{b}\overline{c}', \overline{c}\overline{b}'$ respectively which agree on all inputs $k > |\overline{b}\overline{c}'| = |\overline{c}\overline{b}'|$. We can then pull back f and g to get structures $\mathcal{B}_f, \mathcal{B}_g$. Then $h = g^{-1} \circ f$ is an isomorphism extending σ which is constant on all $k > |\sigma|$. Hence

$$(i, h(i)) \in \Psi_*^{P(\mathcal{B}_f) \oplus h \oplus P(\mathcal{B}_g)} \quad \text{and} \quad (j, h(j)) \in \Psi_*^{P(\mathcal{B}_f) \oplus h \oplus P(\mathcal{B}_g)}$$

If $h(i) = j$ and $h(j) = i$, then taking a long enough initial segment of h witnesses $(\overline{b}, i) \sim (\overline{c}, j)$. If however $h(i) \neq j$ or $h(j) \neq i$, then a long enough initial segment of h witnesses $(\overline{b}, i) \nsim (\overline{c}, j)$.

We now assume towards a contradiction that both $(\overline{b}, i) \sim (\overline{c}, j)$, and $(\overline{b}, i) \nsim (\overline{c}, j)$ and that their equivalence is witnessed by $\overline{d}_1, \sigma$ and their inequivalence is witnessed by $k, l, \overline{d}_2$

and τ. Without loss of generality assume that $(i,k) \in \Psi_*^{P(\overline{b}\overline{c}'\overline{d}_2)\oplus\tau\oplus P(\overline{c}\overline{b}'\overline{d}_2)}$. We also have $(i,j) \in \Psi_*^{P(\overline{b}\overline{c}'\overline{d}_1)\oplus\sigma\oplus P(\overline{c}\overline{b}'\overline{d}_1)}$. Let

$$f_1 \supset \overline{b}\overline{c}'\overline{d}_1 \quad g_1 \supset \overline{c}\overline{b}'\overline{d}_1 \quad f_2 \supset \overline{b}\overline{c}'\overline{d}_2 \quad g_2 \supset \overline{c}\overline{b}'\overline{d}_2$$

Then we have isomorphisms as shown below.

$$\begin{tikzcd}
 & & \mathcal{B} & & \\
\mathcal{B}_{f_2} \arrow[rr, phantom] \arrow[urr, bend left, "f_2"] \arrow[r, "f_1^{-1}\circ f_2"] \arrow[rrrr, bend right, "g_2^{-1}\circ f_2"'] & \mathcal{B}_{f_1} \arrow[ur, bend left, "f_1"] \arrow[rr, "g_1^{-1}\circ f_1"] & & \mathcal{B}_{g_1} \arrow[ul, bend right, "g_1"'] \arrow[r, "g_2^{-1}\circ g_1"] & \mathcal{B}_{g_2} \arrow[ull, bend right, "g_2"']
\end{tikzcd}$$

Since F is a functor,

$$F(g_2^{-1} \circ f_2) = F(g_2^{-1} \circ g_1) \circ F(g_1^{-1} \circ f_1) \circ F(f_1^{-1} \circ f_2).$$

Since $(\overline{b}, i) \in Dom_{\mathcal{A}}^{\mathcal{B}}$ we know that $(i,i) \in \Psi_*^{P(\overline{b})\oplus\lambda\restriction|\overline{b}|\oplus P(\overline{b})}$. Notice that $f_1^{-1} \circ f_2 \supset \lambda\restriction|\overline{b}\overline{c}'|$ and $P(\mathcal{B}_{f_1}) \supset P(\overline{b}), P(\mathcal{B}_{f_2}) \supset P(\overline{b})$. Thus we get

$$(i,i) \in \Psi_*^{P(\mathcal{B}_{f_2})\oplus(f_1^{-1}\circ f_2)\oplus P(\mathcal{B}_{f_1})} = Graph(F(f_1^{-1} \circ f_2)) \Rightarrow F(f_1^{-1} \circ f_2)(i) = i.$$

Similarly, since $(\overline{c}, j) \in Dom_{\mathcal{A}}^{\mathcal{B}}$,

$$(j,j) \in \Psi_*^{P(\overline{c})\oplus\lambda\restriction|\overline{c}|\oplus P(\overline{c})} \Rightarrow (j,j) \in \Psi_*^{P(\mathcal{B}_{g_1})\oplus(g_2^{-1}\circ g_1)\oplus P(\mathcal{B}_{g_2})} = Graph(F(g_2^{-1} \circ g_1)).$$

Following from our choices for f_1, g_1, f_2, g_2 we have that $g_1^{-1} \circ f_1 \supset \sigma$ and $g_2^{-1} \circ f_2 \supset \tau$, so

$$(i,j) \in \Psi_*^{P(\overline{b}\overline{c}'\overline{d}_1)\oplus\sigma\oplus P(\overline{c}\overline{b}'\overline{d}_1)} \Rightarrow (i,j) \in \Psi_*^{P(\mathcal{B}_{f_1})\oplus(g_1^{-1}\circ f_1)\oplus P(\mathcal{B}_{g_1})} = Graph(F(g_1^{-1} \circ f_1))$$

$$(i,k) \in \Psi_*^{P(\overline{b}\overline{c}'\overline{d}_2)\oplus\tau\oplus P(\overline{c}\overline{b}'\overline{d}_2)} \Rightarrow (i,k) \in \Psi_*^{P(\mathcal{B}_{f_2})\oplus(g_2^{-1}\circ f_2)\oplus P(\mathcal{B}_{g_2})} = Graph(F(g_2^{-1} \circ f_2)).$$

The first three equations tell us that

$$F(g_2^{-1} \circ g_1) \circ F(g_1^{-1} \circ f_1) \circ F(f_1^{-1} \circ f_2)(i) = F(g_2^{-1} \circ g_1) \circ F(g_1^{-1} \circ f_1)(i) = F(g_2^{-1} \circ g_1)(j) = j$$

whereas the fourth equation tells us that $F(g_2^{-1} \circ f_2)(i) = k$. This contradicts our earlier statement of F being a functor, and so only one of $\sim, \nsim$ can hold at once. $\square$

Claim 4.4.3.2. *The relation $\sim$ is an equivalence relation on $Dom^{\mathcal{B}}_{\mathcal{A}}$.*

Proof. Let $(\overline{a}, i), (\overline{b}, j), (\overline{c}, k) \in Dom^{\mathcal{B}}_{\mathcal{A}}$. It is reflexive, since $(\overline{a}, i) \in Dom^{\mathcal{B}}_{\mathcal{A}}$ means that $(i, i) \in \Psi_*^{P(\overline{a}) \oplus \lambda \restriction |\overline{a}| \oplus P(\overline{a})}$, and so the equivalence is witnessed by the empty tuple and $\lambda \restriction |\overline{a}|$. If $(\overline{a}, i) \sim (\overline{b}, j)$ via $\overline{d}, \sigma$, then $(\overline{b}, j) \sim (\overline{a}, i)$ via $\overline{d}, \sigma^{-1}$. Now assume that $(\overline{a}, i) \sim (\overline{b}, j)$ and it is witnessed by $\overline{a}', \overline{b}'\overline{d}', \sigma$ and $(\overline{b}, j) \sim (\overline{c}, k)$ via $\overline{b}'', \overline{c}'', \overline{d}'', \tau$. Let $\overline{a}'''$ and $\overline{c}'''$ list the elements of $\overline{a} \backslash \overline{c}$ and $\overline{c} \backslash \overline{a}$ respectively. Choose bijections as follows

$$\begin{array}{lll} f_1 \supset \overline{a}\overline{b}'\overline{d}' & g_1 \supset \overline{b}\overline{c}''\overline{d}'' & h_1 \supset \overline{a}\overline{c}''' \\ f_2 \supset \overline{b}\overline{a}'\overline{d}' & g_2 \supset \overline{c}\overline{b}''\overline{d}'' & h_2 \supset \overline{c}\overline{a}''' \end{array}$$

such that h_1 and h_2 agree outside an initial segment of length $|\overline{a}| + |\overline{c}'''|$.

$$\begin{tikzcd}
 & & & \mathcal{B} & & & \\
\mathcal{B}_{h_1} \arrow[r, "f_1^{-1} \circ h_1"] \arrow[urrr, bend left, "h_1"] \arrow[rrrrrr, bend right, "h_2^{-1} \circ h_1"'] & \mathcal{B}_{f_1} \arrow[r, "f_2^{-1} \circ f_1"] \arrow[urr, bend left, "f_1"] & \mathcal{B}_{f_2} \arrow[rr, "g_1^{-1} \circ f_2"] \arrow[ur, bend left, "f_2"] & & \mathcal{B}_{g_1} \arrow[r, "g_2^{-1} \circ g_1"] \arrow[ul, bend right, "g_1"'] & \mathcal{B}_{g_2} \arrow[r, "h_2^{-1} \circ g_2"] \arrow[ull, bend right, "g_2"'] & \mathcal{B}_{h_2} \arrow[ulll, bend right, "h_2"'] \\
\overline{a}\overline{c}''' \arrow[rrrrrr, bend right, "(\cdot)_\rho"'] & \overline{a}\overline{b}'\overline{d}' & \overline{b}\overline{a}'\overline{d}' \arrow[l, "(\cdot)_\sigma"] & & \overline{b}\overline{c}''\overline{d}'' \arrow[r, "(\cdot)_\tau"] & \overline{c}\overline{b}''\overline{d}'' & \overline{c}\overline{a}'''
\end{tikzcd}$$

Since F is a functor, we have

$$F(h_2^{-1} \circ h_1) = F(h_2^{-1} \circ g_2) \circ F(g_2^{-1} \circ g_1) \circ F(g_1^{-1} \circ f_2) \circ F(f_2^{-1} \circ f_1) \circ F(f_1^{-1} \circ h_1).$$

Since $(\overline{a}, i) \in Dom^{\mathcal{B}}_{\mathcal{A}}$, and $f_1^{-1} \circ h_1 \supset \lambda \restriction |\overline{a}|$ we can show as we did in Claim 4.4.3.1 that $F(f_1^{-1} \circ h_1)(i) = i$. Similarly, $F(g_1^{-1} \circ f_2)(j) = j, F(h_2^{-1} \circ g_2)(k) = k$. By assumption $F(f_2^{-1} \circ f_1)(i) = j$ and $F(g_2^{-1} \circ g_1)(j) = k$. Thus

$$(i, k) \in Graph(F(h_2^{-1} \circ h_1)) = \Psi_*^{P(\mathcal{B}_{h_1}) \oplus (h_2^{-1} \circ h_1) \oplus P(\mathcal{B}_{h_2})}$$

Similarly, one can show that $(k, i) \in \Psi_*^{P(\mathcal{B}_{h_2}) \oplus (h_1^{-1} \circ h_2) \oplus P(\mathcal{B}_{h_1})}$. Since h_1 and h_2 agree outside of the initial segment of length $|\overline{a}| + |\overline{c}'''|$, if we take a long enough $\overline{d}'''$ and let $\rho \subset h_2^{-1} \circ h_1$ be the permutation sending $\overline{ac}'''\overline{d}'''$ to $\overline{ca}'''\overline{d}'''$, we witness that $(\overline{a}, i) \sim (\overline{c}, k)$. $\square$

Claim 4.4.3.3. *For all $i \in F(\mathcal{B})$, there is some $n \in \omega$ such that for $\overline{c} = \mathcal{B}\restriction n$, we have $(\overline{c}, i) \in Dom_{\mathcal{A}}^{\mathcal{B}}$.*

Proof. Since $i \in F(\mathcal{B}) = \Psi^{P(\mathcal{B})}$ we have that $(i, i) \in \Psi_*^{P(\mathcal{B}) \oplus \lambda \oplus P(\mathcal{B})}$ where λ is the identity function. Then, by the use principle, there is a sufficiently long initial segment $\overline{c}$ of $\mathcal{B}$ such that $(\overline{c}, i) \in Dom_{\mathcal{A}}^{\mathcal{B}}$. $\square$

Claim 4.4.3.4. *For $(\overline{b}, i) \in Dom_{\mathcal{A}}^{\mathcal{B}}$, there is an initial segment $\overline{c} = \mathcal{B}\restriction n$ of $\mathcal{B}$ and $j \in \omega$ such that $(\overline{b}, i) \sim (\overline{c}, j)$.*

Proof. Let m be greater than the maximum number in the tuple $\overline{b}$, and let $\overline{c}'$ list the numbers less than or equal to m not occurring in $\overline{b}$. Let $\overline{c} = \mathcal{B}\restriction m$, and let $f \supset \overline{c}^{-1} \circ \overline{b}\overline{c}'$ be defined by $f(n) = n$ for all $n \geqslant m$. Then since $(\overline{b}, i) \in Dom_{\mathcal{A}}^{\mathcal{B}}$ we have that $(i, j) \in Graph(F(f)) = \Psi_*^{P(\mathcal{B}_f) \oplus f \oplus P(\mathcal{B})}$ for some j. So by the use principle, there exists $\overline{d}$ such that $(i, j) \in \Psi_{*, |\overline{c}\overline{d}|}^{P(\overline{b}\overline{c}'\overline{d}) \oplus \sigma \oplus P(\overline{c}\overline{d})}$ where $\sigma = (\overline{c}\overline{d})^{-1} \circ \overline{b}\overline{c}'\overline{d}$, witnessing that $(\overline{b}, i) \sim (\overline{c}, j)$. $\square$

Claim 4.4.3.5. *If $(\overline{b}, i), (\overline{c}, j) \in Dom_{\mathcal{A}}^{\mathcal{B}}$ and $\overline{b} \subseteq \overline{c}$ then $(\overline{b}, i) \sim (\overline{c}, j)$ iff $i = j$.*

Proof. To see this we note that since $(\overline{b}, i) \in Dom_{\mathcal{A}}^{\mathcal{B}}$, we have $(i, i) \in \Psi_*^{P(\overline{b}) \oplus \lambda\restriction|\overline{b}| \oplus P(\overline{b})}$. So since $\overline{b} \subseteq \overline{c}$, by properties of enumeration operators $(i, i) \in \Psi_*^{P(\overline{c}) \oplus \lambda\restriction|\overline{c}| \oplus P(\overline{c})}$, so $(\overline{b}, i) \sim (\overline{c}, i)$. Now let $\overline{d}, \sigma$ witness that $(\overline{b}, i) \sim (\overline{c}, j)$. Then $\sigma \supseteq \lambda\restriction|\overline{b}|$ and the oracle $P(\overline{b}\overline{c}'\overline{d}) \oplus \sigma \oplus P(\overline{c}\overline{d})$ extends $P(\overline{b}) \oplus \lambda\restriction|\overline{b}| \oplus P(\overline{b})$. So again $(i, i) \in \Psi_*^{P(\overline{b}\overline{c}'\overline{d}) \oplus \sigma \oplus P(\overline{c}\overline{d})}$. As $\Psi_*^{P(\overline{b}\overline{c}'\overline{d}) \oplus \sigma \oplus P(\overline{c}\overline{d})}$ must extend to the graph of a function g, we cannot have $g(i) = i$ and $g(i) = j$. This shows that $j = i$. $\square$

We now define relations R_i on $Dom_{\mathcal{A}}^{\mathcal{B}}$. For each relation P_i of arity $p(i)$ we let $(\overline{b}_1, k_1), \ldots, (\overline{b}_{p(i)}, k_{p(i)})$ be in R_i if there is an initial segment $\overline{c} = \mathcal{B}\restriction n$ of $\mathcal{B}$ and $j_1, \ldots, j_{p(i)} \in \omega$ such that $(\overline{b}_l, k_l) \sim (\overline{c}, j_l)$ for all l and $P_i(j_1, \ldots, j_{p(i)}) \in \Psi^{P(\overline{c})}$. Note that by 4.4.3.5, it does not matter which initial segment is chosen.

Claim 4.4.3.6. *If* $(\bar{b}_1, k_1), \ldots, (\bar{b}_{p(i)}, k_{p(i)}) \in Dom_{\mathcal{A}}^{\mathcal{B}}$ *and* $(\bar{c}_1, j_1), \ldots, (\bar{c}_{p(i)}, j_{p(i)}) \in Dom_{\mathcal{A}}^{\mathcal{B}}$ *with* $(\bar{b}_l, k_l) \sim (\bar{c}_l, j_l)$ *for each* l*, then* $(\bar{b}_1, k_1), \ldots, (\bar{b}_{p(i)}, k_{p(i)})$ *is in* R_i *if and only if* $(\bar{c}_1, j_1), \ldots, (\bar{c}_{p(i)}, j_{p(i)})$ *is in* R_i.

Proof. This follows immediately from the transitivity of the equivalence relation $\sim$. □

Using Claim 4.4.3.3 we can define an isomorphism $\mathfrak{F} : F(\mathcal{B}) \to Dom_{\mathcal{A}}^{\mathcal{B}}$ by sending i to $(\bar{c}, i)$ for the initial segment $\mathcal{B}\restriction n$ that is guaranteed to exist. Claim 4.4.3.4 proves that it is surjective and Claim 4.4.3.5 tells us that $\mathfrak{F}$ is injective.

□

In the above theorem we not only show the existence of an interpretation given a positive enumerable functor, but provide a method for building the relations of the interpretation given the functor F. Using the other direction of the proof we can use these relations to enumerate a new functor. We call this new induced functor I^F. We would like I^F to agree with our original functor F in some fashion, and so we introduce the following definitions.

Definition 4.4.4. Two positive enumerable functors $F : Iso^*(\mathcal{B}) \to Iso^*(\mathcal{A})$ and $G : Iso^*(\mathcal{B}) \to Iso^*(\mathcal{A})$ are said to be *enumeration isomorphic* if there is an enumeration operator Λ such that for any $\mathcal{A} \in Iso^*(\mathcal{B})$, $\Lambda^{P(\mathcal{A})}$ is the graph of an isomorphism $F(\mathcal{A}) \to G(\mathcal{A})$. Moreover, for any morphism $h \in Hom(\mathcal{A}, \mathcal{B})$ in $Iso^*(\mathcal{B})$ when viewing $\Lambda^{P(\mathcal{A})}$, $\Lambda^{P(\mathcal{B})}$ as isomorphisms, $\Lambda^{P(\mathcal{B})} \circ F(h) = G(h) \circ \Lambda^{P(\mathcal{A})}$. That is, the diagram below commutes.

$$\begin{tikzcd} F(\mathcal{A}) \arrow[r, "\Lambda^{P(\mathcal{A})}"] \arrow[d, "F(h)"'] & G(\mathcal{A}) \arrow[d, "G(h)"] \\ F(\mathcal{B}) \arrow[r, "\Lambda^{P(\mathcal{B})}"] & G(\mathcal{B}) \end{tikzcd}$$

Proposition 4.4.5. *Let* $F : Iso^*(\mathcal{B}) \to Iso^*(\mathcal{A})$ *be positive enumerable and* $I^F : Iso^*(\mathcal{B}) \to Iso^*(\mathcal{A})$ *be the functor obtained from Theorem 4.4.3. Then* F *and* I^F *are enumeration isomorphic.*

Proof. Given a presentation $\mathcal{B} \in Iso^*(\mathcal{B})$, we can uniformly enumerate the map $\mathfrak{F}$ from $P(\mathcal{B})$ as in the proof of Theorem 4.4.3. To then turn the interpretation given by this functor into the induced functor we simply use the identity function on tuples $\langle x_0, \ldots, x_n \rangle \in Dom_{\mathcal{A}}^{\mathcal{B}}$. Both of these processes are enumberable and so we let Λ be the enumeration operator which witnesses this fact. To show Λ is an enumeration isomorphism we want to show the following diagram commutes for all $\tilde{\mathcal{B}}, \hat{\mathcal{B}} \in Iso^*(\mathcal{B})$ and all morphisms $h : \tilde{\mathcal{B}} \to \hat{\mathcal{B}}$. We extend h to a map $\tilde{\mathcal{B}}^{<\omega} \to \hat{\mathcal{B}}^{<\omega}$ and then restrict to $Dom_{\mathcal{A}}^{\tilde{\mathcal{B}}} \to Dom_{\mathcal{A}}^{\hat{\mathcal{B}}}$.

$$
\begin{tikzcd}
F(\tilde{\mathcal{B}}) \arrow[r, "\mathfrak{F}_{\tilde{\mathcal{B}}}"'] \arrow[d, "F(h)"'] \arrow[rr, bend left, "\Lambda^{P(\tilde{\mathcal{B}})}"] & Dom_{\mathcal{A}}^{\tilde{\mathcal{B}}} \arrow[r] \arrow[d, "h"'] & \hat{\mathcal{I}}^F(\tilde{\mathcal{B}}) \arrow[d, "\hat{\mathcal{I}}^F(h)"] \\
F(\hat{\mathcal{B}}) \arrow[r, "\mathfrak{F}_{\hat{\mathcal{B}}}"] \arrow[rr, bend right, "\Lambda^{P(\hat{\mathcal{B}})}"'] & Dom_{\mathcal{A}}^{\hat{\mathcal{B}}} \arrow[r] & \hat{\mathcal{I}}^F(\hat{\mathcal{B}})
\end{tikzcd}
$$

The right-hand square commutes from the first part of the proof of Theorem 4.4.3 since $I^F(h)$ is defined to be h. To see that the left-hand square commutes, take $i \in F(\tilde{\mathcal{B}})$. Then $F(h)(i) = j$ for some $j \in F(\hat{\mathcal{B}})$ and $\mathfrak{F}^{\tilde{\mathcal{B}}}(i) = (\overline{a}, i)$, $\mathfrak{F}^{\hat{\mathcal{B}}}(j) = (\overline{b}, j)$ where $\overline{a}$ and $\overline{b}$ are initial segments of ω. We want to show that $h(\overline{a}, i) = (h(\overline{a}), i) \sim^{\hat{\mathcal{B}}} (\overline{b}, j)$.

Since $(i, j) \in \Psi_*^{P(\tilde{\mathcal{B}}) \oplus h \oplus P(\hat{\mathcal{B}})}$ we can get $(i, j) \in \Psi_*^{P_{\tilde{\mathcal{B}}}(\overline{a}) \oplus h \restriction |\overline{a}| \oplus P_{\hat{\mathcal{B}}}(\overline{b})}$ by extending $\overline{a}$ and $\overline{b}$. Note that $P_{\tilde{\mathcal{B}}}(\overline{a}) = P_{\hat{\mathcal{B}}}(h(\overline{a}))$ and assume without loss of generality that $\overline{b}$ contains both $\overline{a}$ and $h(\overline{a})$. Since $\overline{b}$ is an initial segment, the map associated to it is the identity. So the map $\sigma = \overline{b}^{-1} \circ h(\overline{a})\overline{b}'$ is an initial segment of $h \restriction |\overline{a}|$. Hence

$$(i, j) \in \Psi_*^{P_{\hat{\mathcal{B}}}(h(\overline{a})) \oplus \sigma \oplus P_{\hat{\mathcal{B}}}(\overline{b})} \quad \text{and} \quad (j, i) \in \Psi_*^{P_{\hat{\mathcal{B}}}(\overline{b}) \oplus (\sigma)^{-1} \oplus P_{\hat{\mathcal{B}}}(h(\overline{a}))}$$

□

Clearly if we have a functor $F : Iso^*(\mathcal{B}) \to Iso^*(\mathcal{A})$, then every enumeration of $\mathcal{B}$ can enumerate an enumeration of $\mathcal{A}$. In order to preserve enumeration degree spectra of structures we need the relationship between the two isomorphism classes to be even stronger.

In the paper by Csima, Rossegger, and Yu [12] positive enumerable bi-transformability was introduced and it was shown that two positive enumerable bi-transformable structures have the same enumeration degree spectra. The next definition is the same as positive enumerable bi-transformability. We chose to rename it, as we learned that the notion is not new, but rather an effectivization of the highly influential notion of an adjoint equivalence of categories in category theory.

Definition 4.4.6. An *enumeration adjoint equivalence* of categories $Iso^*(\mathcal{B})$ and $Iso^*(\mathcal{A})$ consists of a tuple $(F, G, \Lambda_{Iso^*(\mathcal{B})}, \Lambda_{Iso^*(\mathcal{A})})$ where $F : Iso^*(\mathcal{B}) \to Iso^*(\mathcal{A})$ and $G : Iso^*(\mathcal{A}) \to Iso^*(\mathcal{B})$ are positive enumerable functors, $\Lambda_{Iso^*(\mathcal{B})}$ and $\Lambda_{Iso^*(\mathcal{A})}$ witness enumeration isomorphisms between the compositions of $G \circ F$ and $Id_{Iso^*(\mathcal{B})}$, respectively $F \circ G$ and $Id_{Iso^*(\mathcal{A})}$, and the two isomorphisms are mapped to each other. I.e.,

$$F(\Lambda^{P(\mathcal{B})}_{Iso^*(\mathcal{B})}) = \Lambda^{P(F(\mathcal{B}))}_{Iso^*(\mathcal{B})} \text{ and } G(\Lambda^{P(\mathcal{A})}_{Iso^*(\mathcal{A})}) = \Lambda^{P(G(\mathcal{A}))}_{Iso^*(\mathcal{A})}$$

for all $\mathcal{B} \in Iso^*(\mathcal{B})$ and $\mathcal{A} \in Iso^*(\mathcal{A})$. If there is an enumeration adjoint equivalence between $Iso^*(\mathcal{A})$ and $Iso^*(\mathcal{B})$ then we say that $\mathcal{A}$ and $\mathcal{B}$ are *enumeration adjoint*.

We will show that the following notion based on positive interpretability is equivalent to enumeration adjointness.

Definition 4.4.7. Two structures $\mathcal{A}$ and $\mathcal{B}$ are *positively bi-interpretable* if there are effective interpretations of one in the other such that the graphs of the compositions

$$f^{\mathcal{A}}_{\mathcal{B}} \circ \hat{f}^{\mathcal{B}}_{\mathcal{A}} : Dom_{\mathcal{B}}^{Dom^{\mathcal{B}}_{\mathcal{A}}} \to \mathcal{B} \quad \text{and} \quad f^{\mathcal{B}}_{\mathcal{A}} \circ \hat{f}^{\mathcal{A}}_{\mathcal{B}} : Dom_{\mathcal{A}}^{Dom^{\mathcal{A}}_{\mathcal{B}}} \to \mathcal{A}$$

are Σ^p_1 definable without parameters in $\mathcal{B}$ and $\mathcal{A}$ respectively. (Here the function $\hat{f}^{\mathcal{B}}_{\mathcal{A}}$: $(Dom^{\mathcal{B}}_{\mathcal{A}})^{<\omega} \to \mathcal{A}^{<\omega}$ is the canonical extension of $f^{\mathcal{B}}_{\mathcal{A}} : Dom^{\mathcal{B}}_{\mathcal{A}} \to \mathcal{A}$ mapping $Dom_{\mathcal{B}}^{Dom^{\mathcal{B}}_{\mathcal{A}}}$ to $Dom^{\mathcal{A}}_{\mathcal{B}}$)

Theorem 4.4.8. *$\mathcal{A}$ and $\mathcal{B}$ are positively bi-interpretable if and only if they are enumeration adjoint.*

Before proving this theorem, we give an alternate characterization of positive bi-interpretability that will prove useful.

Proposition 4.4.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be structures. If $\mathcal{A}$ is positively interpretable in $\mathcal{B}$ and $\mathcal{B}$ is positively interpretable in $\mathcal{A}$, then the following are equivalent.*

(1) $\mathcal{A}$ and $\mathcal{B}$ are positively bi-interpretable.

(2) There are uniformly r.i.p.e. isomorphisms $g : Dom_{\mathcal{A}}^{Dom_{\mathcal{A}}^{\mathcal{B}}} \to \mathcal{A}$ and $h : Dom_{\mathcal{B}}^{Dom_{\mathcal{B}}^{\mathcal{A}}} \to \mathcal{B}$ in $\mathcal{A}$ and $\mathcal{B}$ respectively, along with isomorphisms $\alpha : Dom_{\mathcal{A}}^{\mathcal{B}} \to \mathcal{B}$ and $\beta : Dom_{\mathcal{B}}^{\mathcal{A}} \to \mathcal{A}$, such that $\alpha \circ \hat{h} \circ \hat{\hat{\alpha}}^{-1} = g$ and $\beta \circ \hat{g} \circ \hat{\hat{\beta}}^{-1} = h$

(3) There are uniformly r.i.p.e. isomorphisms $g : Dom_{\mathcal{A}}^{Dom_{\mathcal{A}}^{\mathcal{B}}} \to \mathcal{A}$ and $h : Dom_{\mathcal{B}}^{Dom_{\mathcal{B}}^{\mathcal{A}}} \to \mathcal{B}$ in $\mathcal{A}$ and $\mathcal{B}$ respectively such that, for all isomorphisms $\alpha : Dom_{\mathcal{A}}^{\mathcal{B}} \to \mathcal{B}$ and $\beta : Dom_{\mathcal{B}}^{\mathcal{A}} \to \mathcal{A}$, we have $\alpha \circ \hat{h} \circ \hat{\hat{\alpha}}^{-1} = g$ and $\beta \circ \hat{g} \circ \hat{\hat{\beta}}^{-1} = h$

Proof. (3) $\Rightarrow$ (1) Let $f_{\mathcal{B}}^{\mathcal{A}}$ be any isomorphism $Dom_{\mathcal{B}}^{\mathcal{A}} \to \mathcal{B}$ and define $f_{\mathcal{A}}^{\mathcal{B}} = g \circ (\widehat{f_{\mathcal{B}}^{\mathcal{A}}})^{-1}$. Then $g = f_{\mathcal{A}}^{\mathcal{B}} \circ \widehat{f_{\mathcal{B}}^{\mathcal{A}}}$ and

$$h = f_{\mathcal{B}}^{\mathcal{A}} \circ \hat{g} \circ (\widehat{\widehat{f_{\mathcal{B}}^{\mathcal{A}}}})^{-1} = f_{\mathcal{B}}^{\mathcal{A}} \circ \widehat{f_{\mathcal{A}}^{\mathcal{B}}} \circ \widehat{\widehat{f_{\mathcal{B}}^{\mathcal{A}}}} \circ (\widehat{\widehat{f_{\mathcal{B}}^{\mathcal{A}}}})^{-1} = f_{\mathcal{B}}^{\mathcal{A}} \circ \widehat{f_{\mathcal{A}}^{\mathcal{B}}}.$$

So $f_{\mathcal{B}}^{\mathcal{A}} \circ \widehat{f_{\mathcal{A}}^{\mathcal{B}}}$ and $f_{\mathcal{B}}^{\mathcal{A}} \circ \widehat{f_{\mathcal{A}}^{\mathcal{B}}}$ are uniformly r.i.p.e. by assumption.

(1) $\Rightarrow$ (2) By assumption,

$$f_{\mathcal{B}}^{\mathcal{A}} \circ \widehat{f_{\mathcal{A}}^{\mathcal{B}}} : Dom_{\mathcal{B}}^{Dom_{\mathcal{B}}^{\mathcal{A}}} \to \mathcal{B} \quad \text{and} \quad f_{\mathcal{A}}^{\mathcal{B}} \circ \widehat{f_{\mathcal{B}}^{\mathcal{A}}} : Dom_{\mathcal{A}}^{Dom_{\mathcal{A}}^{\mathcal{B}}} \to \mathcal{A}$$

are uniformly r.i.p.e., so we let $h = f_{\mathcal{B}}^{\mathcal{A}} \circ \widehat{f_{\mathcal{A}}^{\mathcal{B}}}$ and $g = f_{\mathcal{B}}^{\mathcal{A}} \circ \widehat{f_{\mathcal{A}}^{\mathcal{B}}}$. Additionally, we let $\alpha = f_{\mathcal{B}}^{\mathcal{A}}$ and $\beta = f_{\mathcal{A}}^{\mathcal{B}}$, satisfying (2).

(2) $\Rightarrow$ (3) We assume (2) and let $\alpha' : Dom_{\mathcal{A}}^{\mathcal{B}} \to \mathcal{A}$, $\beta' : Dom_{\mathcal{B}}^{\mathcal{A}} \to \mathcal{B}$ be arbitrary isomorphisms. Then let $\delta : Dom_{\mathcal{A}}^{\mathcal{B}} \to Dom_{\mathcal{A}}^{\mathcal{B}}$ be such that $\alpha' \circ \delta = \alpha$. Hence

$$g = \alpha \circ \hat{h} \circ (\hat{\hat{\alpha}})^{-1} = \alpha' \circ \delta \circ \hat{h} \circ (\hat{\hat{\delta}})^{-1} \circ (\hat{\hat{\alpha'}})^{-1}.$$

We claim that $\delta \circ \hat{h} \circ (\hat{\hat{\delta}})^{-1} = \hat{h}$. We define the automorphism of $\mathcal{B}$, $\gamma = h \circ \hat{\delta} \circ h^{-1}$. Since h is uniformly r.i.p.e., we must have $\gamma(Graph(h)) = Graph(h)$. Thus, $\gamma \circ h \circ (\hat{\hat{\gamma}})^{-1}$ and so if we can show that $\hat{\gamma} = \delta$ we will have proved our claim. Notice that by definition of γ

$$id = \gamma^{-1} \circ h \circ \hat{\delta} \circ h^{-1} = h \circ (\hat{\hat{\gamma}})^{-1} \circ \hat{\delta} \circ h^{-1},$$

and so $\hat{\hat{\gamma}} = \hat{\delta}$. We let

$$\hat{g} = (\alpha)^{-1} \circ g \circ (\hat{\hat{\alpha}})^{-1} : Dom_{\mathcal{A}}^{Dom_{\mathcal{B}}^{Dom_{\mathcal{A}}^{\mathcal{B}}}} \to Dom_{\mathcal{A}}^{\mathcal{B}}.$$

Since $Dom_{\mathcal{A}}^{\mathcal{B}}$ is Σ_1^p-definable, $Dom_{\mathcal{A}}^{\mathcal{B}}$ is u.r.i.p.e. This, together with the fact that g is uniformly r.i.p.e. in $\mathcal{A}$, means that $\hat{g}$ is uniformly r.i.p.e. in $\mathcal{B}$. Then

$$\hat{\gamma} = \hat{g} \circ \hat{\hat{\hat{\gamma}}} \circ (\hat{g})^{-1} = \hat{g} \circ \hat{\hat{\delta}} \circ (\tilde{g})^{-1} = \delta$$

and so the claim is proven. $\square$

We are now in a position to prove Theorem 4.4.8.

Proof. ($\Rightarrow$) Using our positive interpretation of $\mathcal{B}$ in $\mathcal{A}$, we build a positively enumerable functor $F = (\Psi, \Psi_*)$ from $Iso^*(\mathcal{A})$ to $Iso^*(\mathcal{B})$ as in Theorem 4.4.3. Additionally, since the processes described in the theorem are r.i.p.e., we get an enumeration operator Ω such that $\Omega^{P(\tilde{\mathcal{A}})} : Dom_{\mathcal{B}}^{\tilde{\mathcal{A}}} \to F(\tilde{\mathcal{A}})$ is an isomorphism. Similarly, from the functor G we can get an enumeration operator Γ such that, for any $\tilde{\mathcal{B}} \in Iso^*(\mathcal{B})$ we have $\Gamma^{P(\tilde{\mathcal{B}})} : Dom_{\mathcal{A}}^{\tilde{\mathcal{B}}} \to G(\tilde{\mathcal{B}})$

By our assumption of positive bi-interpretability, we have functions $f_{\mathcal{B}}^{\mathcal{A}} : Dom_{\mathcal{A}}^{\mathcal{B}} \to \mathcal{A}$, and $f_{\mathcal{A}}^{\mathcal{B}} : Dom_{\mathcal{B}}^{\mathcal{A}} \to \mathcal{B}$. Also following from our assumptions, the function $f_{\mathcal{A}}^{\mathcal{B}} \circ \widehat{f_{\mathcal{B}}^{\mathcal{A}}} : Dom_{\mathcal{A}}^{Dom_{\mathcal{B}}^{\mathcal{A}}} \to \mathcal{A}$ is uniformly r.i.p.e., and so there is an enumeration operator Θ such that $\Theta^{P(\tilde{\mathcal{A}})}$ is an isomorphism $\tilde{\mathcal{A}} \to Dom_{\mathcal{A}}^{Dom_{\mathcal{B}}^{\tilde{\mathcal{A}}}}$. Hence we have the following diagram, which we wish to show commutes.

$$\begin{tikzcd}
\tilde{\mathcal{A}} \arrow[d, dashed, "F"'] \arrow[rr, bend left, "\Theta^{P(\tilde{\mathcal{A}})}"] & Dom_{\mathcal{B}}^{\tilde{\mathcal{A}}} \arrow[l, hook'] \arrow[dl, "\Omega^{P(\tilde{\mathcal{A}})}"] & Dom_{\mathcal{A}}^{Dom_{\mathcal{B}}^{\tilde{\mathcal{A}}}} \arrow[l, hook'] \arrow[dl, "\hat{\Omega}^{P(\tilde{\mathcal{A}})}"] \\
F(\tilde{\mathcal{A}}) \arrow[d, dashed, "G"'] & Dom_{\mathcal{A}}^{F(\tilde{\mathcal{A}})} \arrow[l, hook'] \arrow[dl, "\Gamma^{P(F(\tilde{\mathcal{A}}))}"] & \\
G(F(\tilde{\mathcal{A}})) & &
\end{tikzcd}$$

Given $\tilde{\mathcal{A}} \in Iso^*(\mathcal{A})$, let $\Lambda^{P(\tilde{\mathcal{A}})} = \Gamma^{P(F(\tilde{\mathcal{A}}))} \circ \hat{\Omega}^{P(\tilde{\mathcal{A}})} \circ \Theta^{P(\tilde{\mathcal{A}})}$. We aim to show that Λ witnesses the fact that $G \circ F$ is enumeration isomorphic to the identity functor. Assuming the above diagram commutes, for any $j : \tilde{\mathcal{A}} \to \hat{\mathcal{A}}$ we have the following diagram.

$$
\begin{tikzcd}
Dom_{\mathcal{A}}^{Dom_{\mathcal{B}}^{\tilde{\mathcal{A}}}} \arrow[rr, hook] \arrow[dr, "\hat{\Omega}^{P(\tilde{\mathcal{A}})}"] & & Dom_{\mathcal{B}}^{\tilde{\mathcal{A}}} \arrow[r, hook] \arrow[dr, "\Omega^{P(\tilde{\mathcal{A}})}"] & \tilde{\mathcal{A}} \arrow[r, "j"] \arrow[d, dashed, "F"] \arrow[dd, bend left, "\Lambda^{P(\tilde{\mathcal{A}})}"] \arrow[lll, bend right, "\Theta^{P(\mathcal{A})}"'] & \hat{\mathcal{A}} \arrow[d, dashed, "F"'] \arrow[dd, bend right, "\Lambda^{P(\hat{\mathcal{A}})}"'] \arrow[rr, bend left, "\Theta^{P(\hat{\mathcal{A}})}"] & Dom_{\mathcal{B}}^{\hat{\mathcal{A}}} \arrow[l, hook'] \arrow[dl, "\Omega^{P(\hat{\mathcal{A}})}"] & Dom_{\mathcal{A}}^{Dom_{\mathcal{B}}^{\hat{\mathcal{A}}}} \arrow[l, hook'] \arrow[dl, "\hat{\Omega}^{P(\hat{\mathcal{A}})}"] \\
& Dom_{\mathcal{A}}^{F(\tilde{\mathcal{A}})} \arrow[rr, hook] \arrow[drr, "\Gamma^{P(F(\tilde{\mathcal{A}}))}"] & & F(\tilde{\mathcal{A}}) \arrow[r, "F(j)"] \arrow[d, dashed, "G"] & F(\hat{\mathcal{A}}) \arrow[d, dashed, "G"'] & Dom_{\mathcal{A}}^{F(\hat{\mathcal{A}})} \arrow[l, hook'] \arrow[dl, "\Gamma^{P(F(\hat{\mathcal{A}}))}"] & \\
& & & G(F(\tilde{\mathcal{A}})) \arrow[r, "G(F(j))"] & G(F(\hat{\mathcal{A}})) & &
\end{tikzcd}
$$

By definition, we have that $G(F(j)) = \Gamma^{P(F(\hat{\mathcal{A}}))} \circ F\widehat{(j)} \circ (\Gamma^{P(F(\tilde{\mathcal{A}}))})^{-1}$, and by Theorem 4.2.10, we have that $F(j) = \Omega^{P(\hat{\mathcal{A}})} \circ \hat{j} \circ (\Omega^{P(\tilde{\mathcal{A}})})^{-1}$. Thus,

$$G(F(j)) \circ \Gamma^{P(F(\tilde{\mathcal{A}}))} \circ \hat{\Omega}^{P(\tilde{\mathcal{A}})} = \Gamma^{P(F(\hat{\mathcal{A}}))} \circ \hat{\Omega}^{P(\hat{\mathcal{A}})} \circ \hat{\hat{j}}.$$

Since Θ is uniformly r.i.p.e. on $\mathcal{A}$, for any $j : \tilde{\mathcal{A}} \to \hat{\mathcal{A}}$, we have that $\hat{\hat{j}} \circ \Theta^{P(\tilde{\mathcal{A}})} = \Theta^{P(\hat{\mathcal{A}})} \circ j$. Hence

$$G(F(j)) \circ \Gamma^{P(F(\tilde{\mathcal{A}}))} \circ \hat{\Omega}^{P(\tilde{\mathcal{A}})} \circ \Theta^{P(\tilde{\mathcal{A}})} = \Gamma^{P(F(\hat{\mathcal{A}}))} \circ \hat{\Omega}^{P(\hat{\mathcal{A}})} \circ \Theta^{P(\hat{\mathcal{A}})} \circ j.$$

We then see that $G(F(j)) \circ \Lambda^{P(\tilde{\mathcal{A}})} = \Lambda^{P(\hat{\mathcal{A}})} \circ j$. Therefore, $G \circ F$ is enumeration isomorphic to the identity functor via Λ. We can similarly show that $F \circ G$ is enumeration isomorphic to the identity. Let $\Lambda_{Iso^*(\mathcal{A})}$ denote the Λ obtained for $G \circ F$, and $\Lambda_{Iso^*(\mathcal{B})}$ likewise for $F \circ G$. As we did for Θ, let Υ denote the enumeration operator obtained from the uniformly r.i.p.e. isomorphism $f_{\mathcal{B}}^{\mathcal{A}} \circ \hat{f}_{\mathcal{A}}^{\mathcal{B}} : Dom_{\mathcal{B}}^{Dom_{\mathcal{A}}^{\mathcal{B}}}$. So

$$\Lambda_{Iso^*(\mathcal{B})}^{P(\tilde{\mathcal{B}})} = \Omega^{P(G(\tilde{\mathcal{B}}))} \circ \hat{\Gamma}^{P(\tilde{\mathcal{B}})} \circ \Upsilon^{P(\tilde{\mathcal{B}})}$$

whence

$$F(\Lambda_{Iso^*(\mathcal{A})}^{P(\tilde{\mathcal{A}})}) = \Omega^{P(G(F(\tilde{\mathcal{A}})))} \circ \hat{\Gamma}^{P(F(\tilde{\mathcal{A}}))} \circ \hat{\hat{\Omega}}^{P(\tilde{\mathcal{A}})} \circ \hat{\Theta}^{P(\tilde{\mathcal{A}})} \circ (\Omega^{P(\tilde{\mathcal{A}})})^{-1}.$$

We are now in a position to use Proposition 4.4.9 with $h^{-1} = \Upsilon^{P(F(\tilde{\mathcal{A}}))}$, $g^{-1} = \Theta^{P(\tilde{\mathcal{A}})}$, and $\beta = \Omega^{P(\tilde{\mathcal{A}})}$. The proposition gives us that

$$\Upsilon^{P(F(\tilde{\mathcal{A}}))} = \hat{\hat{\Omega}}^{P(\tilde{\mathcal{A}})} \circ \hat{\Theta}^{P(\tilde{\mathcal{A}})} \circ (\Omega^{P(\tilde{\mathcal{A}})})^{-1},$$

which means

$$F(\Lambda^{P(\tilde{\mathcal{A}})}_{Iso^*(\mathcal{A})}) = \Lambda^{P(F(\tilde{\mathcal{A}}))}_{Iso^*(\mathcal{B})},$$

and similarly

$$G(\Lambda^{P(\tilde{\mathcal{B}})}_{Iso^*(\mathcal{B})}) = \Lambda^{P(G(\tilde{\mathcal{B}}))}_{Iso^*(\mathcal{A})}.$$

($\Leftarrow$) Now, we assume that $\mathcal{A}$ and $\mathcal{B}$ are enumeration adjoint via $(F, G, \Lambda_{Iso^*(\mathcal{A})}, \Lambda_{Iso^*(\mathcal{B})})$. Using the functors F and G to get interpretations of $\mathcal{A}$ and $\mathcal{B}$ in each other as before, we can show the existence of enumeration operators Ω, Γ. Thus, for each $\tilde{\mathcal{A}} \in Iso^*(\mathcal{A})$ we get

$$\Theta^{P(\tilde{\mathcal{A}})} = (\hat{\Omega}^{P(\tilde{\mathcal{A}})})^{-1} \circ (\Gamma^{P(F(\tilde{\mathcal{A}}))})^{-1} \circ \Lambda^{P(\tilde{\mathcal{A}})}_{Iso^*(\mathcal{A})}$$

which is an isomorphism $\tilde{\mathcal{A}} \to Dom_{\mathcal{A}}^{Dom_{\mathcal{B}}^{\tilde{\mathcal{A}}}}$.

Let $j : \tilde{\mathcal{A}} \to \hat{\mathcal{A}}$ be any isomorphism. We want to show that $\hat{\hat{j}}(Graph(\Theta^{P(\tilde{\mathcal{A}})})) = j(Graph(\Theta^{P(\hat{\mathcal{A}})}))$, which is to show that $Graph(\Theta^{P(\tilde{\mathcal{A}})})$ is fixed under isomorphisms.

By definition of Λ, as in the other direction of the proof, we have that

$$\Lambda^{P(\hat{\mathcal{A}})}_{Iso^*(\mathcal{A})} \circ j = G(F(j)) \circ \Lambda^{P(\tilde{\mathcal{A}})}_{Iso^*(\mathcal{A})} = \Gamma^{P(F(\tilde{\mathcal{A}}))} \circ \hat{\Omega}^{P(\tilde{\mathcal{A}})} \circ \hat{\hat{j}} \circ (\hat{\Omega}^{P(\hat{\mathcal{A}})})^{-1} \circ (\Gamma^{P(F(\hat{\mathcal{A}}))})^{-1} \circ \Lambda^{P(\hat{\mathcal{A}})}_{Iso^*(\mathcal{A})},$$

and so viewing Θ as an isomorphism

$$\Theta^{P(\tilde{\mathcal{A}})} \circ j = (\hat{\Omega}^{P(\tilde{\mathcal{A}})})^{-1} \circ (\Gamma^{P(F(\tilde{\mathcal{A}}))})^{-1} \circ \Lambda^{P(\tilde{\mathcal{A}})}_{Iso^*(\mathcal{A})} \circ j = \hat{\hat{j}} \circ (\hat{\Omega}^{P(\hat{\mathcal{A}})})^{-1} \circ (\Gamma^{P(F(\hat{\mathcal{A}}))})^{-1} \circ \Lambda^{P(\hat{\mathcal{A}})}_{Iso^*(\mathcal{A})} = \hat{\hat{j}} \circ \Theta^{P(\hat{\mathcal{A}})}.$$

Since $\Theta^{P(\hat{\mathcal{A}})}$ is fixed under automorphism, it is $\mathcal{L}_{\omega_1,\omega}$-definable. We also can see that the same formula defines $\Theta^{P(\tilde{\mathcal{A}})}$. $\Theta^{P(\tilde{\mathcal{A}})}$ is enumerable from $P(\tilde{\mathcal{A}})$, and so we get that Θ is uniformly r.i.p.e. Similarly, we can define $\Upsilon^{P(\tilde{\mathcal{B}})} : \tilde{\mathcal{B}} \to Dom_{\mathcal{B}}^{Dom_{\mathcal{A}}^{\tilde{\mathcal{B}}}}$ and show that it is uniformly r.i.p.e.

By definition of $\Lambda_{Iso^*(\mathcal{B})}$, we have that for any $\tilde{\mathcal{B}} \in Iso^*(\mathcal{B})$, $\Lambda^{\tilde{\mathcal{B}}}_{Iso^*(\mathcal{B})} : \tilde{\mathcal{B}} \to F(G(\tilde{\mathcal{B}}))$ is an isomorphism. We will show that Proposition 4.4.9 (2) is satisfied with $h^{-1} = \Upsilon^{P(F(\tilde{\mathcal{A}}))}$, $g^{-1} = \Theta^{P(\tilde{\mathcal{A}})}$, $\alpha = \Gamma^{P(F(\tilde{\mathcal{A}}))}$, and $\beta = \Omega^{P(\tilde{\mathcal{A}})}$.

As before, we have that

$$\Upsilon^{P(F(\tilde{\mathcal{A}}))} = (\hat{\Gamma}^{P(F(\tilde{\mathcal{A}}))})^{-1} \circ (\Omega^{P(G(F(\tilde{\mathcal{A}})))})^{-1} \circ \Lambda^{P(F(\tilde{\mathcal{A}}))}_{Iso^*(\mathcal{B})}$$

Hence
$$\hat{\Omega}^{P(\tilde{\mathcal{A}})} \circ \hat{\Theta}^{P(\tilde{\mathcal{A}})} \circ (\hat{\Omega}^{P(\tilde{\mathcal{A}})})^{-1} = (\hat{\Gamma}^{P(F(\tilde{\mathcal{A}}))})^{-1} \circ \hat{\Lambda}^{P(\tilde{\mathcal{A}})}_{Iso^*(\mathcal{A})} \circ (\Omega^{P(\mathcal{A})})^{-1}.$$
In addition
$$F(\Lambda^{P(\tilde{\mathcal{A}})}_{Iso^*(\mathcal{A})}) = \Omega^{P(G(F(\tilde{\mathcal{A}})))} \circ \hat{\Lambda}^{P(\tilde{\mathcal{A}})}_{Iso^*(\mathcal{A})} \circ (\Omega^{P(\mathcal{A})})^{-1},$$
and so
$$\hat{\Omega}^{P(\tilde{\mathcal{A}})} \circ \hat{\Theta}^{P(\tilde{\mathcal{A}})} \circ (\hat{\Omega}^{P(\tilde{\mathcal{A}})})^{-1} = (\hat{\Gamma}^{P(F(\tilde{\mathcal{A}}))})^{-1} \circ (\Omega^{P(G(F(\tilde{\mathcal{A}})))})^{-1} \circ F(\Lambda^{P(\tilde{\mathcal{A}})}_{Iso^*(\mathcal{A})}).$$
Now since $F(\Lambda^{P(\tilde{\mathcal{A}})}_{Iso^*(\mathcal{A})}) = \Lambda^{P(F(\tilde{\mathcal{A}}))}_{Iso^*(\mathcal{B})}$ we get that $\hat{\Omega}^{P(\tilde{\mathcal{A}})} \circ \hat{\Theta}^{P(\tilde{\mathcal{A}})} \circ (\hat{\Omega}^{P(\tilde{\mathcal{A}})})^{-1} = \Upsilon^{P(F(\tilde{\mathcal{A}}))}$. Similarly we can show that $\Gamma^{P(\tilde{\mathcal{B}})} \circ \hat{\Upsilon}^{P(\tilde{\mathcal{B}})} \circ (\Gamma^{P(\tilde{\mathcal{B}})})^{-1} = \Theta^{P(G(\tilde{\mathcal{B}}))}$. Thus, Proposition 4.4.9 gives us a positive bi-interpretation. □

4.4.1 Positive reductions between classes of structures

4.4.8 shows that we can interpret positive bi-interpretability as a notion of reduction between the isomorphism classes of two structures. This can be naturally extended to arbitrary classes as follows.

Definition 4.4.10. A class of structures $\mathfrak{C}$ is *reducible via positive bi-interpretability* to a class $\mathfrak{D}$ if there are Σ^p_1 formulas defining domains, co-domains, relations and isomorphisms of a positive bi-interpretation such that every structure in $\mathfrak{C}$ is positively bi-interpretable with a structure in $\mathfrak{D}$ using this bi-interpretation.

In Csima, Rossegger, and Yu [12] the following notion of reduction between classes of structures based on enumeration adjoints was defined.

Definition 4.4.11. A class of structures $\mathfrak{C}$ is u.p.e.t. reducible to a class $\mathfrak{D}$ if there is a subclass $\mathfrak{D}' \subseteq \mathfrak{D}$ such that $\mathfrak{C}$ and $\mathfrak{D}'$ are enumeration adjoint.

By precisely the same methods used to prove 4.4.8 we get the following.

Theorem 4.4.12. *A class of structures $\mathfrak{C}$ is reducible via positive bi-interpretability to a class $\mathfrak{D}$ if and only if $\mathfrak{C}$ is u.p.e.t. reducible to $\mathfrak{D}$.*

The classical analogue of u.p.e.t. reduction are u.c.t. reductions introduced in Harrison-Trainor, Melnikov, Miller, Montalbán [17]. This reduction is defined similarly to u.p.e.t. reductions with the difference that computable functors are used instead of positive enumerable functors. It is well-known that graphs are universal for u.c.t. reducibility, i.e., every class of structures is u.c.t. reducible to the class of graphs, via standard codings used in computable structure theory.

We will present a coding that shows that graphs are complete for u.p.e.t. reducibility. The coding is based on a coding used by Rossegger in [28], but, since we only require the preservation of positive information our coding is simpler.

Proposition 4.4.13. *Any class of structures $\mathfrak{C}$ is reducible via positive bi-interpretability to the class of undirected graphs.*

Proof sketch. We may assume without loss of generality that $\mathfrak{C}$ is a class of structures in relational language $L = (R_1, \dots)$ where each R_i has arity i. Given an enumeration of $\mathcal{A} \in \mathfrak{C}$, we enumerate a graph $G_{\mathcal{A}}$ as follows. The graph $G_{\mathcal{A}}$ has a unique vertex a which is a member of the unique 3-cycle in the graph. For each element $x \in A$ we add a vertex v_x and an edge $a\,E\,v_x$. When we see $R_i^{\mathcal{A}}(x_0, \dots, x_{i-1})$ for some $x_0, \dots x_{i-1} \in A$ enumerated for the first time, we enumerate a chain $c_1^k\,E \cdots E\,c_{i+k}^k$ for every $k < i$, a vertex y, and edges $x_k E c_1^k$ and $c_{i+k}^k\,E\,y$ for all $k < i$. This finishes the construction. See 4.1 for an example. It can be shown that $Iso^*(\mathcal{A})$ and $Iso^*(G_{\mathcal{A}})$ are enumeration adjoint. $\square$

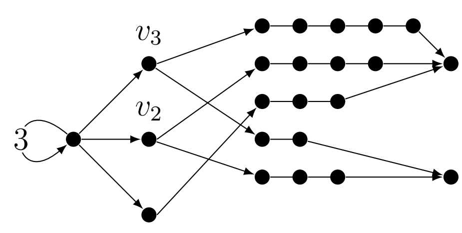

Figure 4.1: Example of $G_{\mathcal{A}}$, where $R_3^{\mathcal{A}}(1, 2, 3)$ and $R_2^{\mathcal{A}}(3, 2)$

4.5 Further Work

R.i.p.e. relations are a new and exciting addition to the study of enumeration spectra. The hope of these initial results is that they reignite interest in older results on the interplay between enumeration reducibility and countable structures achieved by Soskov and co-authors. We already start to see how the ideas developed in this section can be taken further. In upcoming work by Bazhenov, Fokina, Rossegger, Soskova, and Vatev [8] definability by positive formulas is used to prove a Lopez-Escobar theorem for continuous domains: The sets of structures definable by Σ^p_α formulas are precisely the $\mathbf{\Sigma}^0_\alpha$ sets in the Scott topology on the space of structures.

The work done by Bazhenov, Fokina, Rossegger, Soskova, and Vatev [8] on forcing conditions for Σ^p_α formulas is an important step towards extending the positive jump of a structure to transfinite ordinals. For any finite ordinal, we can simply iterate the positive jump operator, but more work is needed to go beyond the finite case.

References

[1] Rachael Alvir. *Scott Analysis of Countable Structures*. PhD thesis, University of Notre Dame, 2022.

[2] Rachael Alvir, Noam Greenberg, Matthew Harrison-Trainor, and Dan Turetsky. Scott complexity of countable structures. *Journal of Symbolic Logic*, 86(4):1706–1720, 2021.

[3] Rachael Alvir and Dino Rossegger. The complexity of Scott sentences of scattered linear orders. *The Journal of Symbolic Logic*, 85(3):1079—-1101, 2020.

[4] Chris Ash, Julia Knight, Mark Manasse, and Theodore Slaman. Generic copies of countable structures. *Annals of Pure and Applied Logic*, 42(3):195–205, 1989.

[5] C.J. Ash and J. Knight. *Computable Structures and the Hyperarithmetical Hierarchy*. ISSN. Elsevier Science, 2000.

[6] Ewan J. Barker. Back and forth relations for reduced abelian p-groups. *Annals of Pure and Applied Logic*, 75(3):223–249, 1995.

[7] Nikolay Bazhenov. Degrees of autostability for linear orders and linearly ordered abelian groups. *Algebra and Logic*, 55:257–273, 2016.

[8] Nikolay Bazhenov, Ekaterina Fokina, Dino Rossegger, Alexandra A. Soskova, and Stefan V. Vatev. A Lopez-Escobar theorem for continuous domains, 2023.

[9] Wesley Calvert, Desmond Cummins, Julia Knight, and Sara Miller. Comparing Classes of Finite Structures. *Algebra and Logic*, 43(6):374–392, 2004.

[10] John Chisholm. Effective model theory vs. recursive model theory. *The Journal of Symbolic Logic*, 55(03):1168–1191, 1990.

[11] S. Barry Cooper. *Computability Theory*. Chapman & Hall, 2003.

[12] Barbara F. Csima, Dino Rossegger, and Daniel Yu. Positive enumerable functors. In Liesbeth De Mol, Andreas Weiermann, Florin Manea, and David Fernández-Duque, editors, *Connecting with Computability - 17th Conference on Computability in Europe, CiE 2021, Virtual Event, Ghent, July 5-9, 2021, Proceedings*, volume 12813 of *Lecture Notes in Computer Science*, pages 385–394. Springer, Springer, 2021.

[13] Deveau, Michael. *Computability Theory and Some Applications*. PhD thesis, University of Waterloo, 2019.

[14] Manfred Droste and Rudiger Gobel. Countable random p-groups with prescribed Ulm-invariants. In *Proceedings of the American Mathematical Society*, volume 139, 2011.

[15] Paul C. Eklof. Some model theory of abelian groups. *Journal of Symbolic Logic*, 37:335 – 342, 1972.

[16] Ekaterina Fokina, Valentina Harizanov, and Alexander Melnikov. Computable Model Theory. *Turing's Legacy: Developments from Turing's Ideas in Logic*, 42:124–191, 2014.

[17] Matthew Harrison-Trainor, Alexander Melnikov, Russell Miller, and Antonio Montalbán. Computable functors and effective interpretability. *Journal of Symbolic Logic*, 82, 2015.

[18] Denis R. Hirschfeldt. Degree spectra of relations on computable structures in the presence of Δ_2^0 isomorphisms. *The Journal of Symbolic Logic*, 67(2):697—-720, 2002.

[19] Iskander Kalimullin. Algorithmic reducibilities of algebraic structures. *Journal of Logic & Computation*, 22(4), 2012.

[20] Irving Kaplansky. *Infinite Abelian groups*, pages 26–29. Courier Dover Publications, 1954.

[21] Julia Knight, Sara Miller, and Michael Vanden Boom. Turing Computable Embeddings. *The Journal of Symbolic Logic*, 72(3):901–918, 2007.

[22] Julia F. Knight. Degrees coded in jumps of orderings. *The Journal of Symbolic Logic*, 51(04):1034–1042, 1986.

[23] Antonio Montalbán. Rice sequences of relations. *Philosophical Transactions of the Royal Society of London A*, 370(1971):3464–3487, 2012.

[24] Antonio Montalbán. A robuster Scott rank. In *Proceedings of the American mathematical society*, volume 143, 2015.

[25] Antonio Montalbán. A new game metatheorem for Ash-Knight style priority constructions. In *Proceedings of the program Higher recursion theory and set theory*, 2019.

[26] Antonio Montalbán. *Computable Structure Theory: Within the Arithmetic*. Perspectives in Logic. Cambridge University Press, 2021.

[27] Antonio Montalbán. Computable structure theory: Beyond the arithmetic. https://math.berkeley.edu/%7Eantonio/CSTpart2_DRAFT.pdf. Accessed 2021-10-19.

[28] Dino Rossegger. Degree spectra of analytic complete equivalence relations. *The Journal of Symbolic Logic*, 87:1663–1676, 2022.

[29] Alan L. Selman. Arithmetical Reducibilities I. *Zeitschrift fur mathematische Logik und Grundlagen der Mathematik*, 17(1), 1971.

[30] Robert Irving Soare. Turing computability. In *Theory and Applications of Computability*. Springer, 2016.

[31] Ivan N. Soskov. A jump inversion theorem for the enumeration jump. *Archive for Mathematical Logic*, 39(6):417–437, 2000.

[32] Ivan N. Soskov. Degree spectra and co-spectra of structures. *Ann. Univ. Sofia*, 96:45–68, 2004.

[33] Ivan N. Soskov. A note on ω-jump inversion of degree spectra of structures. In *Conference on Computability in Europe*, pages 365—-370. Springer, 2013.

[34] Ivan N. Soskov and V. Baleva. Regular enumerations. *Journal of Symbolic Logic*, 67:1323 – 1343, 2002.

[35] Ivan N. Soskov and V. Baleva. *Ash's theorem for abstract structures*, pages 327—-341. Lecture Notes in Logic. Cambridge University Press, 2006.

[36] Alexandra Soskova and Ivan N. Soskov. A Jump Inversion Theorem for the Degree Spectra. *Journal of Logic and Computation*, 19(1):199–215, 2009.

[37] Alexandra Soskova and Mariya Soskova. Enumeration Reducibility and Computable Structure Theory. In *Computability and Complexity*, pages 271–301. Springer, 2017.

[38] Aleksei Stukachev. Degrees of presentability of structures. I. *Algebra and Logic*, 46(6):419–432, 2007.