Topological Manipulations of Quantum Field Theories

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Statement of Contributions

Chapter 2 of this thesis consists of material from the published paper [1], co-authored with Davide Gaiotto.

Chapter 3 of this thesis consists of material from the published paper [2].

Chapter 4 of this thesis consists of material from the published paper [3], co-authored with Ivan M. Burbano and Jonas Neuser.

Abstract

In this thesis we study some topological aspects of Quantum Field Theories (QFTs). In particular, we study the way in which an arbitrary QFT can be separated into "local" and "global" data by means of a "symmetry Topological Field Theory" (symmetry TFT). We also study how various "topological manipulations" of the global data correspond to various well-known operations that previously existed in the literature, and how the symmetry TFT perspective provides a systematic tool for studying these topological manipulations.

We start by reviewing the bijection between G-symmetric d-dimensional QFTs and boundary conditions for G-gauge theories in $(d{+}1)$-dimensions, which effectively defines the symmetry TFT. We use this relationship to study the "orbifold groupoids" which control the composition of "topological manipulations," relating theories with the same local data but different global data. Particular attention is paid to examples in $d = 2$ dimensions. We also discuss the extension to fermionic symmetry groups and find that the familiar "Jordan-Wigner transformation" (fermionization) and "GSO projection" (bosonization) appear as examples of topological manipulations. We also study applications to fusion categorical symmetries and constraining RG flows in WZW models as well.

After this, we present a short chapter showcasing an application of this symmetry TFT framework to the study of minimal models in 2d CFT. In particular, we complete the classification of 2d fermionic unitary minimal models.

Finally, we discuss how the symmetry TFT intuition can be used to classify duality defects in QFTs. In particular, we focus on $\mathbb{Z}_m$ duality defects in holomorphic Vertex Operator Algebras (VOAs) (and especially the E_8 lattice VOA), where we use symmetry TFT intuition to conjecture, and then rigorously prove, a formula relating (duality-)defected partition functions to $\mathbb{Z}_2$ twists of invariant sub-VOAs.

Acknowledgements

I would like to thank my advisor Davide Gaiotto for his guidance in becoming a physicist. Davide is the greatest physicist I have had the pleasure of meeting. His unique approach to problems, and especially his distinctive views on QFT, greatly shaped my own understanding of the subject. Oftentimes I would find myself very confused about problems before realizing the solution was the simple one that Davide had told me at the beginning. One of the most important lessons I learned from Davide was that I had to develop my own career as an independent researcher and taste for problems, something he provided a great safety net for. More broadly, his lessons about functioning as a research *physicist* were as good as his lessons on *physics*. In hindsight, I feel like I have underdelivered to him on results, atleast compared to the amount of physics problems and insight he has gifted me, and I wonder if I spent too much time not doing physics and/or working on (seemingly!) unrelated problems. This thesis is dedicated in part to him and all my other teachers for the generosity of their time and knowledge. I hope that one day I can teach them something that they don't know.

I would also like to acknowledge my co-supervisor Jaume Gomis and the other Quantum Fields & Strings and Mathematical Physics members at Perimeter Institute. Together they made PI the best place in the world to follow my dream of studying theoretical high-energy physics. I am also very grateful to professors Anton Burkov and Ben Webster for sitting on my PhD Committee; and Niayesh Afshordi, Mathew Bullimore, and Jon Yard for joining my Examining Committee.

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I had the pleasure to work on projects with a number of collaborators during my PhD, including: Latham Boyle, Kasia Budzik, Ivan Burbano, Davide Gaiotto, Diego García-Sepúlveda, Alfredo Guevara, Jonas Neuser, Sabrina Pasterski, Brian Williams, Jingxiang Wu, and Matthew Yu. Parts of the work covered in this thesis was supported by funding from a CGS-D grant from the NSERC of Canada.

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Finally, I would like to thank my family for trusting in my choices to go to graduate school and pursue studies in high energy physics. My family never questioned my decisions, despite not being familiar with the various machinations of academia, and supported me unconditionally during my year off between undergraduate and graduate school. I especially need to thank Nicole for her love during these past years and for supporting my pursuit of physics.

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Chapter 1

Introduction

This thesis discusses some modern developments in our understanding of symmetries in Quantum Field Theory (QFT). Given that symmetries are so fundamental in both the development and analysis of QFT, writing about symmetries immediately places one dangerously close to fundamental "definitional questions" in QFT, as opposed to higher level "computational questions."[1] The danger of working with such basic topics like "symmetries in QFT" is therefore that one may be tempted to try and define what QFT is (and is not), discuss various axiomatic approaches to QFT (see e.g. [4] for a list of attempts), or fall into the trap of discussing things that one finds particularly interesting around the time of writing.

In particular, the danger in trying to define QFT with this level of generality is that the writer would have some list of things that they definitely want to be quantum field theories, e.g. a free massive scalar or *maybe* the Standard Model. They would then inevitably find themselves plunged into an ever deeper list of pathologies and counterexamples for their intuitions, worrying about whether or not their definition should work for effective field theories, non-unitary theories, strange lattice models, non-Lagrangian theories, etc. Moreover, it is likely that any attempts to wade through the morass of QFTs will find one facing down additional pathological theories, cooked up simply to spoil their classification scheme.

The definition of symmetries in Quantum Field Theory are similarly complicated by desires for generality; especially as one starts to worry about internal vs spacetime symmetries, unitarity, non-Lagrangian theories, subsystem symmetries, etc. However, significant

[1] By computational questions, I mean problems like computing high loop Feynman diagrams in $\mathcal{N} = 4$ SYM, for example.

progress has been made in our understanding of symmetries which appears to be quite general and, importantly, based on simple physical principles. In modern language:

All topological observables are identified with symmetries.

On the research front, developing this new point of view has helped to both unify and rephrase some old results in QFT: providing constraints for quantum gravity, RG flows, and bootstrap; systematizing the study of anomalies, defects, and topological phases of matter; and spurring on mathematical developments in higher algebra. While influencing mathematics is usually a good sign for any theoretical physics, one thing to emphasize is the interdisciplinary nature of the subject within physics itself, having touched high energy particle physics, condensed matter physics, and even quantum computing.

However, since we don't know if this definition of symmetry will last, it may be more productive to provide some ground-up discussion of why this topological identification is a good characteristic (and possibly defining) feature of symmetries. As such, I will present a number of examples which will hopefully be convincing/informative enough to allow the reader to extrapolate to general results about symmetry. I will also provide some discussion of what constitutes a "phase of matter."

1.1 Symmetries

Let us start by recalling some standard textbook results about Noether's theorem (see e.g. [5]). Suppose we have a classical field theory, with fields collectively denoted Φ, whose action comes from a local Lagrangian density $\mathcal{S} = \int d^d x\, \mathcal{L}(\Phi, \partial_\mu \Phi)$. If we perform a general transformation:

$$x \mapsto x' = x'(x)\,, \quad \Phi \mapsto \Phi' = \mathcal{F}(\Phi)\,, \tag{1.1}$$

it induces some transformation on the action $\mathcal{S} \mapsto \mathcal{S}'$. Transformations such that $\mathcal{S} = \mathcal{S}'$ are classical symmetries of the action and clearly form a group action on the theory.

When transformations live in a continuous family, barring any pathologies, they specifically form a Lie group (for simplicity assume it is connected). In this case, we can also study general infinitesimal transformations:

$$x \mapsto x'(x) = x + \omega_a \frac{\delta x^\mu}{\delta \omega_a}\,, \quad \Phi(x) \mapsto \Phi'(x') = \Phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x)\,, \tag{1.2}$$

where the ω_a are spacetime independent.

Suppose we have some Lagrangian theory, and further suppose that its action $\mathcal{S}$ is invariant under some family of continuous symmetry transformations, i.e. $\mathcal{S}' = \mathcal{S}$. What is the response of the theory to spacetime dependent transformations $\omega_a(x)$? It is a straightforward exercise to show that the difference $\delta\mathcal{S} := \mathcal{S}' - \mathcal{S}$ becomes:

$$\delta\mathcal{S} = \int d^d x\, \partial_\mu j_a^u\, \omega_a(x) \tag{1.3}$$

where the current j_a^μ is defined

$$j_a^\mu = \left(\frac{\partial\,\mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\nu \Phi - \delta_\nu^\mu\, \mathcal{L} \right) \frac{\delta x^\nu}{\delta\omega_a} - \frac{\partial\,\mathcal{L}}{\partial(\partial_\mu \Phi)} \frac{\delta\,\mathcal{F}}{\delta\omega_a}\,. \tag{1.4}$$

Now, since we assumed that the $\omega_a = \omega_a(x)$ were a function of x, the $\omega_a(x)$ are behaving like small position dependent field variations of the Φ. But we know that field variations of the Φ around on-shell configurations of Φ leave the action invariant $0 = \delta\mathcal{S}\,\big|_{\text{on}-\text{shell}}$. Thus, when the equations of motion are satisfied, (1.3) is 0 for arbitrary $\omega_a(x)$ and we may strip away the integral, so that:

$$\partial_\mu j_a^\mu \Big|_{\text{on}-\text{shell}} = 0\,. \tag{1.5}$$

This is Noether's theorem: *For every generator of the symmetry group G, there is an independent current j_a^μ which is conserved when using the equations of motion.* We will return to the converse statement in the next section.

This is the derivation for a classical field theory, and holds true for quantum theories that do not have anomalies (see Section 1.1.2). For quantum theories without Lagrangians, one could instead take the existence of a conserved current $\partial_\mu j_a^\mu = 0$ (as an operator equation) as the definition of what it means to have a (non-anomalous) continuous symmetry.

What do symmetry operators capture in QFT? Equation (1.3) is the key ingredient to remember, it tells us that the current essentially characterizes the response of the theory to a slight G-deformation of the background. This may sound cryptic, so suppose we have a $U(1)$ flavour symmetry, the conserved current j^μ can be minimally coupled (for simplicity) to a background (classical) gauge field A_μ by adding a term to the action like:

$$\mathcal{S}_{\text{int}} = \int d^d x\, A_\mu j^\mu\,. \tag{1.6}$$

Small variations around $A_\mu = 0$ are captured by correlation functions of j_μ in the undeformed quantum theory:

$$\langle j^\mu \Phi(x_1) \cdots \Phi(x_n) \rangle_{A=0} \simeq \frac{1}{Z[0]} \frac{\delta}{\delta A_\mu} \langle \Phi(x_1) \cdots \Phi(x_n) \rangle_{A=\delta A}\,. \tag{1.7}$$

Another example is the stress tensor $T^{\mu\nu}$, which is the Noether current for translational symmetry. $T^{\mu\nu}$ couples to the metric and characterizes the response of the theory to deformations away from flat space $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Some other examples of symmetries and their background fields include fermion number $(-1)^F$, which couples to a background spin-structure; and time-reversal symmetry, which couples to the first Stiefel-Whitney class w_1.

Note that under a gauge transformation of the $U(1)$ background field $A_\mu \mapsto A_\mu + \partial_\mu \alpha$

$$\delta \mathcal{S}_{\text{int}} = \int d^d x\, \partial_\mu \alpha\, j^\mu = -\int d^d x\, \alpha\, \partial_\mu j^\mu\,. \tag{1.8}$$

We find that invariance of the action under gauge transformations corresponds to the current conservation equation (see also Section 1.1.2).

Generally, we see that a continuous symmetry group can be coupled to a background field and its local data is encoded in current correlators; meanwhile, its global data is encoded in the bundle topology. For discrete symmetry groups, connections are necessarily flat (see [6] for a review) and all data is global data. In either case, given a flat connection $A \in H^1(M, G)$, one can use Poincaré duality to effectively present it as a network of codimension-1 symmetry defects labelling G-monodromies on M. In this case, gauge transformations of the connection correspond to shifting the background field around.[2]

It is worthwhile to discuss one more elementary matter before proceeding, and that is the matter of Ward identities (see [5] again). In particular, we now know that if a theory has a continuous symmetry, then there is a conserved current $\partial_\mu j^\mu_a = 0$ as an operator equation. What is the quantum mechanical manifestation of the current conservation equation? The Ward identities are the answer.

The most pedagogical way to derive the Ward identities is again from the path integral perspective for a continuous symmetry. In particular, suppose that we have an infinitesimal transformation:

$$\Phi'(x) = \Phi(x) - i\omega_a G_a \Phi(x)\,, \tag{1.9}$$

where ω_a is once again a function of x. Note: G^a is called the generator of the symmetry transformation and characterizes the change in fields under the symmetry transformation. Anyway, let X denote some multilocal operator that can enter a correlation function $X :=$

[2]Consider the concrete example of a $U(1)$ symmetry, with background connection localized on t_0: $A = \phi\delta(t - t_0)dt$. We see that we can move the connection to a new spot t_1 by using the gauge transformation $\alpha = \phi(\theta(t - t_1) - \theta(t - t_0))$.

$\Phi(x_1)\cdots\Phi(x_n)$, then

$$\langle X\rangle := \frac{1}{Z}\int [d\Phi]\, X e^{-\mathcal{S}[\Phi]} \tag{1.10}$$

$$= \frac{1}{Z}\int [d\Phi']\, X' e^{-\mathcal{S}'[\Phi']} \tag{1.11}$$

$$= \frac{1}{Z}\int [d\Phi']\, (X+\delta X) e^{-\mathcal{S}[\Phi]-\int d^dx\,\partial_\mu j^\mu\,\omega_a}\,. \tag{1.12}$$

In the second line, we have used the fact that the path-integral just sees the fields as an integration variable, so that the change from unprimed to primed quantities is essentially a relabelling of integration variables.

Now, if we assume $[d\Phi'] = [d\Phi]$ and expand to first order, we find

$$\langle \delta X\rangle\big|_{\mathcal{O}(\epsilon)} = \int d^dx\,\partial_\mu \left\langle j^\mu_a X\right\rangle\,\omega_a(x)\,. \tag{1.13}$$

However, we can also compute δX explicitly in terms of field variations and then express the answer as an integral against delta functions:

$$\langle \delta X\rangle\big|_{\mathcal{O}(\epsilon)} = -i\int d^dx\,\omega_a(x)\sum_{i=1}^{n}\delta(x-x_i)\Phi(x_1)\cdots G_a\Phi(x_i)\cdots\Phi(x_n)\,. \tag{1.14}$$

Since $\omega_a(x)$ was an arbitrary spacetime dependent variation, we can drop the integrals to obtain the formula for the Ward identity for the the current operator j^μ_a:

$$\partial_\mu\left\langle j^\mu_a(x)\Phi(x_1)\cdots\Phi(x_n)\right\rangle = -i\sum_{i=1}^{n}\delta(x-x_i)\left\langle\Phi(x_1)\cdots G_a\Phi(x_i)\cdots\Phi(x_n)\right\rangle \tag{1.15}$$

Two comments are in order. First, it is important that the ∂_μ be outside the correlation function, often papers are careless about this point. The reason for this is because $\partial_\mu j^\mu = 0$ is an operator equation. If ∂_μ was inside the correlation function acting on j^μ_a the correlation function would be 0 on the nose. This can be seen explicitly when one works with correlation functions as time-ordered vacuum expectation values: in this case, the δ-functions (called contact terms) in the Ward identity comes from taking derivatives of the Heaviside theta-functions that implement the time ordering. The second comment is that we of course still have Ward identities even if we don't have a nice path integral derivation for intuition. In particular, the contact terms arise from singularities as the operators in correlation functions collide.

1.1.1 Topological Operators

0-Form Symmetries

Given a conserved current $\partial_\mu j^\mu_a = 0$, we can integrate the current over a codimension-1 manifold $\Sigma \subset M$ in our d-dimensional spacetime M to produce a conserved charge. To see this, it is clearest if we first switch to a coordinate free notation, representing the conserved current as a (dual Lie-algebra valued) 1-form j satisfying $d \star j = 0$. Then we define the charge $Q(\Sigma)$ as

$$Q(\Sigma) := \int_\Sigma \star j\,. \tag{1.16}$$

This charge is conserved in that $dQ(\Sigma) = 0$. This most famous example of this occurs when $\Sigma = \Sigma_t$ is a constant time slice surface, then

$$\partial_t Q_a(\Sigma_t) = \int_{\Sigma_t} d^{d-1}x\, \partial_t j^t_a = -\int_{\Sigma_t} d^{d-1}x\, \partial_i j^i_a = 0\,. \tag{1.17}$$

Clearly the charge $Q_a(\Sigma_t)$ is conserved and does not care about where it is placed in time.

Start with the Ward identity

$$d\left\langle \star j(x)\, \Phi(x_1)\cdots\Phi(x_n)\right\rangle = -i\sum_{i=1}^{n} \delta(x-x_i)\left\langle \Phi(x_1)\cdots G_a\Phi(x_i)\cdots\Phi(x_n)\right\rangle\,, \tag{1.18}$$

and integrate the expression over a solid d-dimensional volume B, with $\Sigma = \partial B$, and containing x_i but none of the other x_j, see Figure 1.1. Then

$$\int_B d\left\langle \star j\, \Phi(x_1)\cdots\Phi(x_n)\right\rangle = -i\sum_{i=1}^{n}\int_B \delta(x-x_i)\left\langle \Phi(x_1)\cdots G_a\Phi(x_i)\cdots\Phi(x_n)\right\rangle\,, \tag{1.19}$$

which implies

$$\left\langle Q_a(\Sigma)\Phi(x_1)\cdots\Phi(x_n)\right\rangle = -i\left\langle \Phi(x_1)\cdots G_a\Phi(x_i)\cdots\Phi(x_n)\right\rangle\,. \tag{1.20}$$

Here we see that the support Σ of $Q(\Sigma)$ is only topological: it does not depend on the exact position of Σ, so long as it does not "sweep past" any other insertions in the correlation functions. If it does sweep past another insertion, then Φ transforms appropriately under the symmetry. If we pick a particular quantization, so that correlation functions become time ordered expectation values, this is the statement that charges are the generators of infinitesimal symmetry transformations

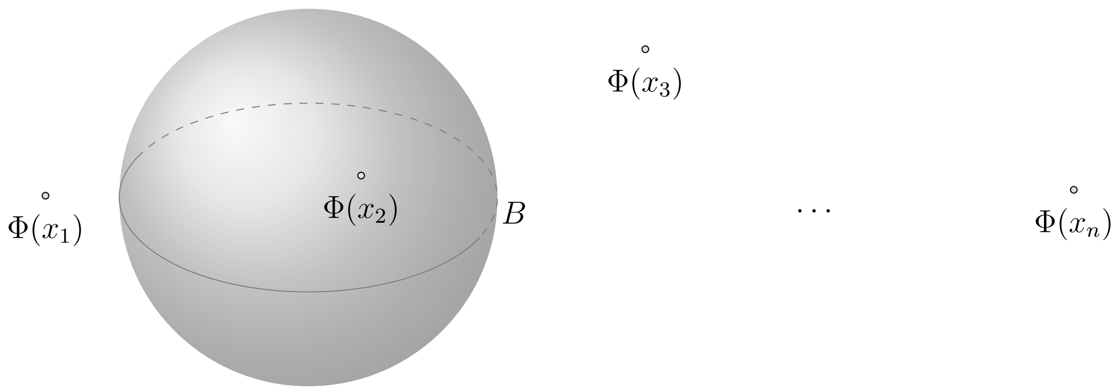


Figure 1.1: We may integrate the local operator $j(x)$ over a volume B with boundary $\Sigma = \partial B$ to produce the correlator with charge operator living on Σ. In this picture, the charge operator engulfes $\Phi(x_2)$.

$$[Q_a, \Phi] = -iG_a\Phi\,. \tag{1.21}$$

This has all been specific to continuous symmetries which have infinitesimal generators, but we can consider instead the symmetry group elements $U_g(\Sigma)$ for any $g \in G$.[3] Their support is also topological. Such an operator is defined by cutting spacetime along the codimension-1 surface Σ and inserting a g action on the complete set of states for the Hilbert space living on Σ [7]. Or, as mentioned below (1.20), we can think of the operator $U_g(\Sigma)$ living on Σ as introducing a discontinuity in the fields that pass through it, twisting them by a symmetry transformation. Thus, if the topological operator $U_g(\Sigma)$ sweeps over the operator $\mathcal{O}$, it transforms appropriately $g\cdot\mathcal{O}$. This point-of-view also works for theories which do not have Lagrangian descriptions or continuous symmetry groups.

The symmetries $U_g(\Sigma)$ defined above are called 0-form symmetries. They are supported on codimension-1 manifolds in spacetime and they can couple to 1-form background connections as in (1.6), with gauge invariance under 0-form shifts like (1.8) coming from the conservation of the 1-form current j.

[3] In the case of a continuous symmetry group, we can obtain the topological operator by exponentiating the appropriate collection of infinitesimal symmetry charges, i.e. $U_g(\Sigma) = \exp\{(i\omega^a Q_a(\Sigma))\}$.

Higher-Form Symmetries

Higher-form symmetries can be defined simply by generalizing the previous paragraph: a q-form symmetry group $G^{(q)}$ consists of symmetry operators $U_g(\Sigma)$ supported on codimension-$(q{+}1)$ surfaces. The operators couple to $(q{+}1)$-form background fields $B^{(q+1)}$ which transform under q-form gauge transformations. In the $U(1)^{(q)}$ case, this looks like

$$B^{(q+1)} \mapsto B^{(q+1)} + d\beta_B^{(q)} . \tag{1.22}$$

In the continuous case, q-form symmetries are associated to $(q+1)$-form currents $J^{(q+1)}$ satisfying

$$d \star J^{(q+1)} = 0 , \tag{1.23}$$

which is integrated to give the conserved charge just as in (1.16).

One important thing to note is that higher form symmetries must be abelian. This can be seen essentially trivially from the definition: a higher-form symmetry lives on at minimum a codimension-2 defect. Now suppose we have two such higher-form symmetry defects $U_g^{(q)}$ and $U_h^{(q)}$. Due to the dimensionality of their support, there is ambient spacetime to homotope/move the defects past each other before hitting a charged object. As a result, topologically there is no difference between hitting a charged object with $U_g^{(q)}$ or $U_h^{(q)}$ in that order, or the reverse order. So higher-form symmetry groups must be abelian, see Figure 1.2.

There is a related mathematical fact to this last point [8]. In particular, given a q-form symmetry group G, it can be shown that distinct G-bundles on spacetime M are in correspondence with (homotopy classes of) maps $\gamma : M \to B^qG$. The classifying space for the background fields $B^qG := K(G, q+1)$ is called an Eilenberg-MacLane space, and this space only makes sense for $q > 0$ when G is abelian.

Example 1 (Free Maxwell Theory)**.** Consider free $U(1)^{(0)}$ gauge theory, aka Free Maxwell Theory in 4d. This theory has *two global 1-form symmetries*, an electric symmetry $U(1)_e^{(1)}$ and a magnetic symmetry $U(1)_m^{(1)}$. To see this, start with the action

$$\mathcal{S} = -\frac{1}{2g^2} \int F_A^{(2)} \wedge \star F_A^{(2)} , \tag{1.24}$$

where $F_A^{(2)} = dA^{(1)}$ is the usual curvature 2-form associated to the $U(1)^{(0)}$ 1-form gauge field $A^{(1)}$. Now consider the shift $A^{(1)} \mapsto A^{(1)} + \alpha^{(1)}$, this leaves the action invariant if the shift is d-closed, i.e. $d\alpha^{(1)} = 0$, i.e. if we shift the gauge field $A^{(1)}$ by a flat connection.

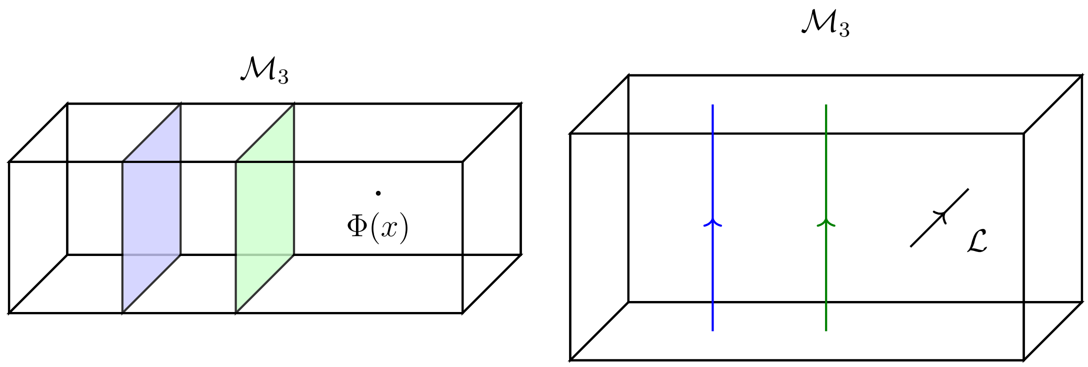


Figure 1.2: Left, 0-form symmetry operators are codimension-1 and have no freedom to move past each other before hitting the charged operator $\Phi(x)$ in the three-manifold $\mathcal{M}_3$. Right, 1-form symmetry operators are codimension-2, so there is space to move them around each other before hitting the line operator $\mathcal{L}$; for consistency, they must have abelian group composition law.

This is the origin of the $U(1)_e^{(1)}$ symmetry, but to confirm that this is a 1-form symmetry, we should confirm it acts non-trivially on line operators. The magnetic $U(1)_m^{(1)}$ symmetry is less obvious from this point-of-view unless we go to the electric-magnetic dual frame and use the dual photon field.

We can compute the associated conserved currents. They are:

$$J_e^{(2)} = \frac{1}{g^2} F^{(2)} \,, \quad J_m^{(2)} = \frac{1}{2\pi} \star F^{(2)} \,. \tag{1.25}$$

Clearly both $d \star J_e^{(2)} = 0$ and $d \star J_m^{(2)} = 0$ by the vacuum Maxwell equations, so we have two independent conserved 2-form currents, as expected.

Following the previous discussions, we can integrate and exponentiate the currents to get group elements in the electric/magnetic 1-form symmetry groups:

$$U^e_{g=e^{i\omega}}(\Sigma^{(2)}) = e^{i \frac{\omega}{g^2} \int_{\Sigma^{(2)}} \star F} \,, \tag{1.26}$$

$$U^e_{g=e^{i\lambda}}(\Sigma^{(2)}) = e^{i \frac{\lambda}{4\pi} \int_{\Sigma^{(2)}} F} \,. \tag{1.27}$$

We see that these operators correspond to the total electric/magnetic flux through the surface $\Sigma^{(2)}$ respectively, and the charged objects are Wilson and 't Hooft loops respectively. To see this last point, and confirm that these 1-form symmetries act non-trivially on some operators in the theory, consider the effect on a Wilson loop of shifting the dynamical gauge field $A^{(1)}$ by a flat connection $A^{(1)} \mapsto A^{(1)} + \alpha^{(1)}$:

$$W_{\gamma,n}(A) \mapsto W_{\gamma,n}(A) e^{in \int_\gamma \alpha^{(1)}} . \tag{1.28}$$

For the 't Hooft loops and the $U(1)^{(1)}_m$ magnetic symmetry, the corresponding equation captures the fact that the 't Hooft lines generate a magnetic flux. □

Lastly, it is worthwhile to comment on the existence of higher-group symmetries. For concreteness, we will describe the simplest case: the case of 2-groups. Roughly speaking, such symmetries arise when there is an interplay between symmetries of different form degree. In the particular case of (split) 2-groups, this occurs because the 1-form symmetry operators themselves are charged under the 0-form symmetry. As a result, the 2-group Γ is algebraically captured by a group extension

$$1 \to A^{(1)} \to \Gamma \to G^{(0)} \to 1 . \tag{1.29}$$

The splitting of the group extension is described by some cohomology class in the group cohomology $H^3(G^{(0)}, A^{(1)})$, called the Postnikov class. The Postnikov class describes the obstruction to independently coupling to both a $G^{(0)}$ and $A^{(1)}$ background.

For example, if we have a $U(1)^{(0)}$ 0-form symmetry background $A^{(1)}$, and a $U(1)^{(1)}$ 1-form symmetry background $B^{(2)}$, a non-trivial Postnikov class $\kappa \in H^3(BU(1), U(1)) \cong \mathbb{Z}$ means a general gauge transformation $(\alpha^{(0)}, \beta^{(1)})$ shifts

$$A^{(1)} \mapsto A^{(1)} + d\alpha^{(0)} , \tag{1.30}$$

$$B^{(2)} \mapsto B^{(2)} + d\beta^{(1)} + \frac{\kappa}{2\pi} \alpha^{(0)} dA^{(1)} , \tag{1.31}$$

see e.g. [9] for more details.

1.1.2 Anomalies

In general, the partition function of a theory on some manifold is both IR and UV divergent. Both types of divergences can be handled in standard ways: IR divergences can be regulated with the introduction of a compact spacetime, while UV divergences can be regulated by the introduction of a UV cutoff. Different choices of the UV regularization scheme correspond to different choices of UV counterterms in the classical/background fields. As such, we see that it's the universal IR physics that is captured by the partition function in a scheme-independent way.

Consider the partition function of a theory with a global symmetry G (0-form for simplicity), and couple the global symmetry to a (classical) background gauge field A. It is possible that the partition function $Z[A]$ is not gauge invariant in the presence of the G background

$$Z[A^\alpha] \neq Z[A]\,. \tag{1.32}$$

We will assume that the partition function transforms by a phase, which is itself determined by a functional ω that depends locally on the gauge parameter α and background connection A:

$$Z[A^\alpha] = Z[A]e^{-2\pi i \int_X \omega(\alpha,A)}\,. \tag{1.33}$$

The statement that ω is a local functional is the statement that ω obeys cutting and gluing axioms of quantum field theory. In the language of topological defect networks, $Z[A] \neq Z[A^\alpha]$ signals the breakdown of the topological nature of symmetry defects when defects are brought to coincident points/junctions of defects (see [6, 10]).

At first glance, there is nothing interesting about this local phase ambiguity of the partition function; as we will see, such phase ambiguities can always be removed by counterterms. However, a more interesting question is: can we remove the phase ambiguity using only *local* counterterms? Such local phase ambiguities are precisely what people mean by a scheme. Phase ambiguities that cannot be removed/absorbed by the addition of local counterterms, i.e. are scheme independent, are called *'t Hooft anomalies*.

Note, 't Hooft anomalies are not "bad." A theory with an 't Hooft anomaly simply has an obstruction to coupling to background gauge fields. Alternatively, you can couple to a background gauge field but there is a phase ambiguity in how the background gauge field is presented. These phase differences between otherwise gauge equivalent defect networks can be viewed as violations of group associativity. Anomalies can also be thought of as contact terms not satisfying global Ward identities. All in all, 't Hooft anomalies are actually "good" invariants for understanding QFTs, this is due in part due to their "rigidity" along RG flows and other continuous deformations.

Previously we asserted that phase ambiguities could always be removed with the addition of counterterms if we allow for non-local counterterms. This is the essence of the *anomaly inflow hypothesis* (or definition). The inflow hypothesis is the assertion that the anomalous phase $\omega(\alpha, A)$ of the d-dimensional partition function is captured by a classical invertible field theory T_{bulk} in $(d+1)$-dimensions. The name "inflow" comes from the fact that if our d-dimensional spacetime with the dynamical theory on it is called M, then we can view M as the boundary of a $(d+1)$-dimensional manifold N and extend the classical background gauge fields A into the bulk of N. Then we can construct a local Lagrangian

$\mathcal{L}_{\text{bulk}}$ on N such that

$$\exp\left(2\pi i \int_N \mathcal{L}_{\text{bulk}}(A^\alpha) - 2\pi i \int_N \mathcal{L}_{\text{bulk}}(A)\right) = \exp\left(2\pi i \int_M \omega(\alpha, A)\right). \tag{1.34}$$

On a closed manifold the classical bulk theory is gauge invariant, but on open manifolds it reproduces the anomaly of the lower-dimensional theory.

The upshot of all this is that while the theory $Z[A]$ is not gauge invariant on its own, the d-dimensional dynamical system coupled to the boundary of the $(d+1)$-dimensional classical bulk/inflow theory with classical Lagrangian $\mathcal{L}_{\text{bulk}}$ actually is gauge invariant. i.e. the combined partition functions for the bulk-boundary pairing

$$\tilde{Z}[A] := Z[A]e^{2\pi i \int_N \mathcal{L}_{\text{bulk}}(A)} \tag{1.35}$$

satisfies

$$\tilde{Z}[A^\lambda] = \tilde{Z}[A]\,. \tag{1.36}$$

It is a standard argument to show that 't Hooft anomalies are given by a cohomological classification. This cohomological classification of anomalies in d-dimensions matches precisely with the classification of SPT/invertible-phases in $(d+1)$-dimensions. And indeed, they are the same classification if we take the anomaly inflow hypothesis as defining anomalies: the invertible phases that classify anomalies are classified by bulk SPT phases in one dimension higher [11].[4]

One final philosophical comment on the topic of anomalies is that, as of time of writing this, we do not think of theories with 't Hooft anomalies as necessarily always being coupled to one-higher dimensional bulk SPT phases whose classical action is tuned to cancel the 't Hooft anomaly. However, in some systems this does happen from the natural physical setup: e.g. in a condensed matter setup studying SPT phases, the boundary has an 't Hooft anomaly (assuming no symmetry breaking).

1.1.3 Non-Invertible Symmetries

In the previous sections we discussed 0-form symmetries and their anomalies. Algebraically, 0-form symmetry operators were associated with topological codimension-1 operators in spacetime. As the supports of these operators collide, we discovered that they formed

[4]Actually, anomalies and SPT phases turn out to have a cobordism-theoretic classification, which includes the cohomological classification within it.

a group and the group could be continuous or discrete. Explicitly, the fusion of these codimension-1 operators had to have inverses and be associative, but they were not necessarily commutative. Associativity ambiguities under fusion were captured by anomalies, which had a group cohomological classification. The symmetry group also had a unit in the form of the trivial/identity defect.

After that, we discussed the existence of higher-form symmetries. These higher form symmetry operators also formed groups, exactly as in the 0-form symmetry case. The only difference was that these groups were necessarily commutative on account of the high codimension of their supports. Otherwise, everything else was the same.

Restricting our attention back to the simple starting point, a different type of generalization of the (possibly anomalous) 0-form symmetry group we are familiar with is that of a fusion category. A fusion category arises as we relax the requirement that the supports of codimension-1 topological defects "fuse like a group," and instead only require that they "fuse like a ring." The way this is usually emphasized in the physics literature, is to emphasize that not all of the codimension-1 topological defects necessarily have an inverse. Perhaps the most famous example of such a defect occurs for the Kramers-Wannier duality defect line $\mathcal{N}$ in the (1+1)d Ising model, whose self-fusion satisfies

$$\mathcal{N} \times \mathcal{N} = \mathbb{1} + \eta \tag{1.37}$$

where η is the $\mathbb{Z}_2$ symmetry line of the Ising model. We will return to this defect many times in the bulk of the text.

Note, non-invertible topological defects are not mysterious. For example, in a standard theory with only a $\mathbb{Z}_2$ symmetry, the $\mathbb{Z}_2$-projector defect $\mathcal{P} = \mathbb{1} + \eta$ is a non-invertible defect (this is clear, since it necessarily loses information when sweeping over charged operators). What is unique about the non-invertible topological objects that appear in a general fusion category is that the objects are simple, i.e. they are not just sums of other fundamental objects.

More generally, if one works physically, one can essentially "derive" the axioms of a fusion category from the intuition of the rules that should govern the fusion of topological codimension-1 defects, see e.g. [12]. We will not repeat those derivations here, but we comment more on the reasonability of these axioms in Section 4.2. Two points require commentary though:

1. **Associativity**. When one studies the axioms for a fusion category, one finds that there is "associator data" which comes in to the definition as one tries to relate various configuration of topological defects to one another. This associator data

Gadget	Topology	Inverses	Associativity	Supports
0-form Group	Disc. or Cts	Yes	Anomaly	Codim-1
p-form Group	Disc. or Cts	Yes	Anomaly	Codim-$(p+1)$
p-Group	Disc. or Cts	Yes	Anomaly + Postnikov	Codim $\leq p+1$
Fusion Category	Discrete	No	Pentagon Identity	Codim-1
Higher Fusion Category	Discrete	No	Various	Various Codim
???	Disc. or Cts.	No	Various	All Codim

Table 1.1: Essential algebraic properties of various common axiomatizations of topological symmetry operators.

straightforwardly generalizes the anomaly $\alpha \in H^{d+1}(G, U(1))$ that we encountered for a 0-form symmetry group in d-dimensions. We will comment on this more in 4.2.

2. **Topology**. The mathematical axioms of a fusion category do have one very important restriction compared to the axioms one would derive from purely physical considerations. In particular, fusion categories necessarily only have a "finite number of simple objects." Intuitively, this means that the axioms of a fusion category are really generalizing *discrete* symmetry groups.

The last point is doubly important because the discreteness of fusion categories is in part responsible for their rigidity: a fusion category cannot be deformed continuously to another fusion category. This point is often cited as the reason for fusion categories being good quantities to study e.g. under renormalization group flows. Very interesting attempts to generalize fusion categories away from having a discrete collection of simple objects have appeared recently [13].

Finally, one can then generalize this discussion to the general formalism of "higher (multi-)fusion categories," which in principle should take into account both the non-invertible and higher-codimension generalization of discrete groups. However, the lack of a continuous collection of objects leads us to suspect that the right formalism is not yet completely developed. We provide a summary of these axiomatizations in Table 1.1.

1.2 Phases of Matter

Regardless of starting point everyone would agree on two common elements in a theory of *quantum mechanics* [14, 11]:

1. A *state space* $\mathcal{H}$. A complex separable Hilbert space of states, finite or infinite dimensional.
2. A *Hamiltonian* $H \in \mathrm{End}(\mathcal{H})$. A non-negative (bounded below) self-adjoint operator inside the larger algebra of operators. The non-negativity corresponds to positivity of energy.

The unitary evolution of states is given by a one parameter group on $\mathcal{H}$ generated by H. Namely, a map $\mathbb{R} \to U(\mathcal{H})$ given by the time evolution operator

$$t \mapsto U_t = e^{-itH/\hbar}\,. \tag{1.38}$$

Since H is non-negative, t can be taken in the entire complex lower half-plane. The former Lorentzian time-evolution operator can be viewed as the boundary value of the holomorphic map $z \mapsto e^{-izH/\hbar}$. We can also consider imaginary time evolution in Euclidean time τ

$$\tau \mapsto e^{-\tau H/\hbar}\,, \quad \tau > 0\,. \tag{1.39}$$

Turning one's attention from Lorentzian time (along the boundary of the plane) to Euclidean time $t \rightsquigarrow \tau$ is called *Wick rotation*. The advantage of Wick rotation is that it turns oscillatory integrands into exponentially decaying ones, and so makes quantities well-defined. Although there are some examples where Wick rotation hides essential physics, e.g. in the study of Lorentzian CFTs.

Example 2 (Particle on a Ring)**.** The Lagrangian of a classical particle on a ring is given by

$$L = \frac{1}{2}\dot{x}^2\,, \quad x \sim x + 2\pi\,. \tag{1.40}$$

Naive quantization of this system gives

$$\hat{H} = \frac{1}{2}\partial_x^2\,, \quad \mathcal{H} = L^2(S^1;\mathbb{C})\,, \tag{1.41}$$

and so Euclidean time evolution is given by $\tau \mapsto U_\tau = e^{\frac{\tau}{2}\partial_x^2}$ (and we have set $\hbar = 1$). The eigenfunctions of this quantum system are quantized plane waves

$$\psi_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}\,, \quad E_n = \frac{n^2}{2}\,. \tag{1.42}$$

One can add a θ-angle to this system.

$$L = \frac{1}{2}\dot{x}^2 + \frac{1}{2\pi}\theta \cdot x\,. \tag{1.43}$$

Naive quantization of this system gives

$$\hat{H} = \frac{1}{2}\left(-i\partial_x - \frac{1}{2\pi}\theta\right)^2 \,, \quad \mathcal{H} = L^2(S^1; \mathcal{L}\, e^{i\theta})\,, \tag{1.44}$$

where the Hilbert space means L^2 sections of the complex line bundle on S^1 with holonomy $e^{i\theta}$. Note that θ is quantized $\theta \sim \theta + 2\pi$ and $Z[\theta] = Z[\theta + 2\pi]$. If we view this as a $(0+1)$d field theory $x : \mathbb{R} \to S^1$, then we can view θ as a coupling constant weighting different topological sectors in the path integral, i.e. different topological solitons. The eigenfunctions of this quantum system are also quantized plane waves

$$\psi_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}\,, \quad E_n = \frac{1}{2}\left(n - \frac{\theta}{2\pi}\right)^2 \,. \tag{1.45}$$

The spectrum is shown in the left of Figure 1.3. For more details on this surpisingly instructive system see [7, 11, 10]. □

The Hamiltonian in the previous example was *gapped*. The spectrum of H was bounded below and the minimum energy (aka *ground state energy*) was contained in the point spectrum of H. In other words, there was a spectral gap above the lowest eigenvalue. If a system is not gapped, it is *gapless*. Note that $\theta = \pi$ was still gapped, it just had a *ground state degeneracy* of 2 [15].

1.2.1 Gapped Phases

Now that we know what we mean by quantum mechanics, we will define a quantum system. A *quantum system* is a many-body system $(\mathcal{H}_N, H_N)$, consisting of a collection of microscopic Hilbert spaces, i.e.

$$\mathcal{H}_N = \otimes\, \mathcal{H}_i \,, \tag{1.46}$$

with each $\mathcal{H}_i$ corresponding to a finite dimensional Hilbert space, and i some finite index (typically associated with vertices/edges/faces of a lattice), and N representing some number defining the size of the index set. The Hamiltonian is given by some schematic expansion involving more and more distant lattice sites:

$$H_N = \sum_i \mathcal{O}_i + \sum_{\langle ij \rangle} \mathcal{O}_{ij} + \dots \,. \tag{1.47}$$

Locality makes the combinatorial approximation of a manifold by a lattice meaningful, emphasizing the dominance of short-range interactions between sites in the Hamiltonian.

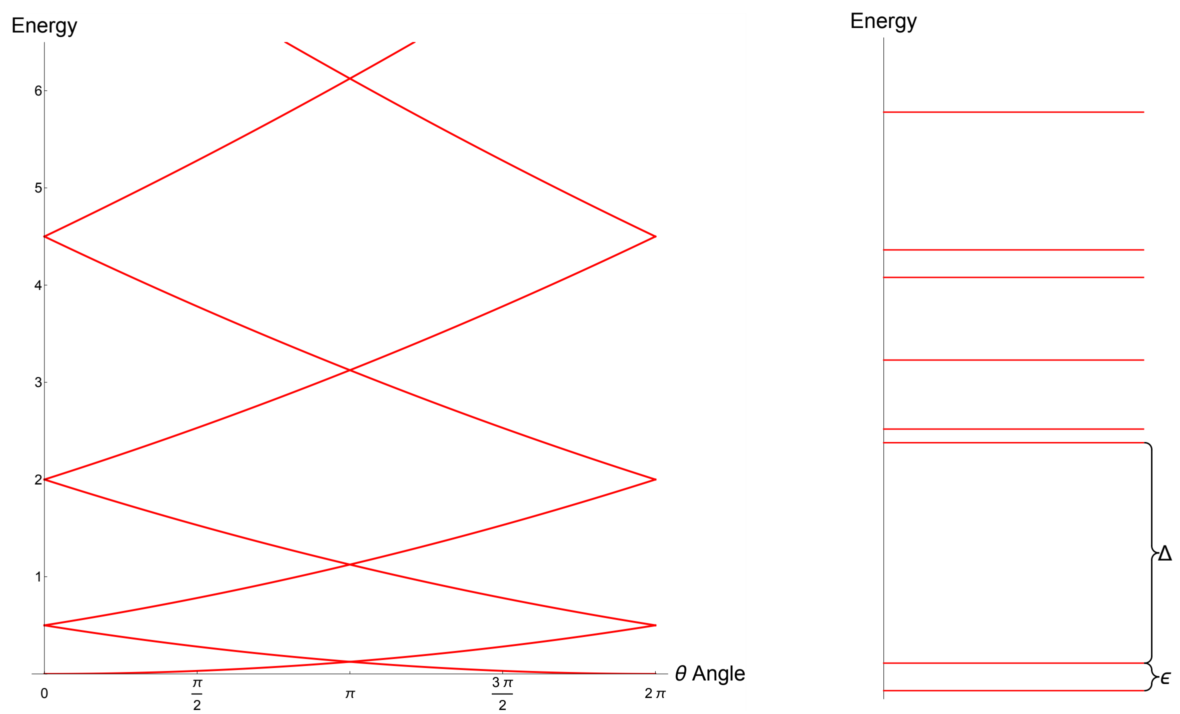


Figure 1.3: Left, the energy spectrum of a quantum mechanical particle on a ring as a function of θ angle. Right, the energy spectrum of a generic "quantum system."

The physical idea is that these "quantum systems" are supposed to describe some reasonable microscopic lattice model for materials. Then we study the macroscopic (experimentally probable) features of the quantum system. For example, superconductivity comes from microscopic electrons condensing to cooper pairs, and macroscopically this manifests in (measurable) zero electric resistance. Another example of macroscopic information are observables based on the topology of the system, e.g. a topological ground state degeneracy [16].

Naively, any "quantum system" is gapped (as a quantum mechanical model) with this definition: there are a finite number of sites, each equipped with a finite dimensional Hilbert space. Clearly we need a better notion of "gapped" for quantum systems defined in this way. The solution is to work in the "thermodynamic limit" $N \to \infty$, by defining gappedness for a quantum system as a property of a sequence.

Consider the spectrum of a generic quantum system $(\mathcal{H}_N, H_N)$, the spectrum looks like the one on the right of Figure 1.3: with a clustering of energy eigenvalues in a band of width ϵ above the actual ground state, with a gap of size Δ separating the low energy cluster from the rest of the spectrum. Intuitively, the cluster of size ϵ above the ground state characterizes finite size effects of the eventual phase we are going to study. We define a sequence of quantum systems $\{(\mathcal{H}_{N_1}, H_{N_1}), (\mathcal{H}_{N_2}, H_{N_2}), \dots\}$ to be *gapped* if each H_{N_i} is gapped with $\Delta_N \to \Delta$ and $\epsilon_N \to 0$ as $N \to \infty$. In other words, a sequence of quantum systems is gapped if all the quantum mechanical models are gapped as the spatial volume of the lattice $N \sim V \to \infty$, otherwise it is gapless [17].

This story is analogous to classical statistical mechanics. A phase transition in classical statistical mechanics is detected by a dramatic change in macroscopic response functions, which correspond to singularities (e.g. a branch cut) in free energy F. But this only occurs in the thermodynamic limit where many singularities of F condense into a branch cut. For example, the free energy

$$F = -k_B T \ln Z \,, \tag{1.48}$$

is singular if the partition function vanishes, $Z = 0$; this occurs at zeroes of $Z[T] = \sum_\mu e^{-\beta H[\mu]}$. For example, in the concrete case of the Ising model, the partition function $Z[T]$ is a degree $2N$ polynomial in the variable $z = e^{-\beta\epsilon}$ [5]. The roots necessarily lie off the real axis in the complex T plane for finite N, but condense into a branch cut which cuts the real temperature axis as $N \to \infty$. In either case, the macroscopic responses depend on the inherently "limit-based" questions that can be asked in the thermodynamic limit.

Two gapped systems are in the same *phase* if:

- They can be continuously deformed to one another.
- Their ground state wavefunctions can be related by local unitary transforms.

These are actually two different definitions of phase, but both are identifying a phase as an equivalence class of gapped systems so that the study of phases is the study of the π_0(Gapped Systems) in the "space of theories" [11]. The first definition is more classically homotopy theoretic, connected to continuous deformations of theories e.g. in the space of coupling constants. The second option is a quantum information theoretic definition and is probably more robust (although less obviously connected to standard homotopy theory, and, therefore rigorous continuum field theory theorems) [18, 15].

A gapped system is in the *trivial phase* if it can be deformed to a trivial gapped system. For example, a system $(\mathcal{H}_N, H_N)$ with $H_N = \sum_i H_i$ and each H_i having a big gap above the ground state can be continuously deformed to the trivial system, essentially because there are no interactions between lattice sites and the gaps shouldn't close due to some pathological property of the limits. Two gapped systems can be *stacked* by taking the composite system with no interaction between them: take the tensor product of Hilbert spaces and sum of Hamiltonians. Similarly, two gapped phases can be *stacked*, with the same logic, except working in the equivalence class of "gapped phases" rather than the space of gapped quantum systems. Finally, a gapped system is *invertible* if there exists another gapped system such that stacking them together results in a Hamiltonian which is in the trivial phase.

All of these definitions depend on a notion of continuous deformation. By continuous deformation we mean "smooth" things like tuning parameters of the system, such as coupling constants or masses of particles; but not discrete things, like changing dimensions or internal symmetry groups [11]. Moreover, we require that:

1. The gap remains open as we deform in the space of theories.
2. No first-order phase transitions occur. Quantum systems are still gapped at first-order phase transitions, but in this case a piece of the discrete spectrum comes down to a minimum and so changes the ground state degeneracy. In other words, on a fixed topology, ground state degeneracy is an invariant of the phase.
3. Stacking with the trivial phase leaves the phase unchanged.

This is best explained with a concrete example

Example 3 (The Ising Chain)**.** Consider the quantum mechanical Ising spin chain:

$$H = \sum_{ij} Z_i Z_j + g \sum_i X_i \,. \tag{1.49}$$

- When $|g| < 1$ the system is ordered with energy gap $\Delta = 2(1 - |g|)$.
- When $|g| = 1$ the system is gapless (at low-energies, flows to the Ising CFT with gapless excitations).
- When $|g| > 1$ the system is in the disordered phase with $\Delta = 2(|g| - 1)$.

In this example, tuning g is precisely what we mean by a continuous deformation. We see that it separates different phases because the gap closes as we tune through the value $|g| = 1$ (see [19]). We can also see that the second definition of gapped phases in terms of local unitary transformations also distinguishes the two phases (as it should): on one side of the gap the theory has a 1-dimensional ground state Hilbert space, and the theory has a 2-dimensional ground state Hilbert space on the other side, so the ground state wavefunctions are obviously not related by local unitaries. □

To work with continuum QFT, we usually work with systems where the following two constraints hold [11]:

1. We assume deformation classes characterizing macroscopic properties are determined by a low-energy effective QFT.
2. We assume the QFT for a gapped phase reduces to a TQFT in the IR limit, which describes ground states and responses to external probes (like background connections).

These assumptions/definitions make it clear what we mean when we say phrases like: the Arf theory is an invertible phase arising from the low-energy limit of the Kitaev chain. We mean that the Kitaev chain is some gapped quantum system $(\mathcal{H}_N, H_N)$ (related to the Ising chain above). The macroscopically probable IR properties are determined by an effective QFT, describing a Majorana fermion in (1+1)d, that reduces to a TQFT in the IR limit. There are two phases, roughly corresponding to whether or not the fermion mass was $m > 0$ or $m < 0$, with the gap closing at $m = 0$.[5] One choice of fermion mass flows to a TQFT with $Z[\rho] = 1$ and the other flows to a TQFT with $Z[\rho] = e^{i\,\mathrm{Arf}[\rho]}$.

1.2.2 SPT Phases

An SPT phase, or "symmetry protected topological" phase, is a gapped phase of matter (thus the word topological) that is continuously deformable to the trivial phase, but only by ignoring symmetry. In other words, to deform a gapped system corresponding to an SPT phase to the trivial system, we would have to break symmetry or undergo a phase transition.

Based on our discussion in the previous section, we know that the effective QFTs that capture such quantum systems must flow in the IR to TQFTs. Practically speaking, the partition function of these effective theories assigns a $U(1)$ number to the (classical) background field data of the theory, i.e.

$$Z_{\mathrm{SPT}}[M, A] = e^{2\pi i S_{\mathrm{SPT}}[M,A]}\,. \tag{1.50}$$

When $A = 0$ the partition function is 1, which captures the idea that the theory is indistinguishable from the trivial phase except for the effects of symmetry. The SPT phases in $(d+1)$-dimensions are exactly the classical bulk theories that classified the anomalies in d-dimensions by inflow.[6] Based on the partition function, we also see that SPT phases are

[5]There is a subtlety about saying which sign of m corresponds to which phase in the deep IR. We comment on this more in Section 2.3.1.

[6]The SPT action may actually depend on the geometry of M, which condensed matter theorists refer to this as having a "Non-trivial thermal Hall response." Such theories are simply related to gravitational Chern-Simons theories (which are classified) and act as inflow theories for gravitational anomalies. Thus we may restrict our attention to studying purely topological SPTs (no thermal Hall response) [20].

also great examples of invertible phases.

1.3 The Symmetry TFT

Given a QFT, we can split it broadly into local data and global data. Local data includes things like the S-matrix and the operator product expansion of point operators. Conversely, global data includes things like the global structure of some symmetry group (e.g. $SO(3)$ vs $SU(2)$) and 't Hooft anomalies. For example, in 2d RCFT the local data is captured by the chiral algebra (and anti-chiral algebra, if distinct) and the global data is the modular invariant.

Example 4 (2d CFT Minimal Models)**.** A 2d RCFT is specified by 3 pieces of data $(\mathcal{V}, \overline{\mathcal{V}}, \mathcal{M})$, where $\mathcal{V}$ and $\overline{\mathcal{V}}$ are the chiral and anti-chiral algebras and $\mathcal{M}$ is a modular invariant. In the case of 2d minimal models, the chiral and anti-chiral algebra are the same. The chiral algebra describes the operator product expansions of a finite number of (chiral) primary operators, which correspond to particular irreducible representations of the Virasoro algebra [21].

At generic central charge, there are 2 bosonic unitary minimal models. For example, at $c = 4/5$ one has the *tetracritical ising model* or "A-type" or "diagonal" modular invariant for the chiral algebra $\mathcal{V}(\frac{4}{5})$, while the *critical three-state Potts model* is the "D-type" modular invariant for this chiral algebra pair.

Since these two theories have the same chiral algebra, they have the same OPEs and the same local dynamics (as captured by three point functions). However, they have different modular invariants. This manifests itself in the fact that the two models have different local/twist operators, different torus partition functions, etc. The two topologically distinct theories are related by $\mathbb{Z}_2$ orbifold/gauging. □

This last example begs the question:

What are all topological manipulations of a quantum field theory?

By topological manipulations, we mean operations you can perform on a theory which change the global structure of the theory, but leave the local data unchanged. Some examples of such operations include: stacking with SPT phases, orbifolding/gauging a discrete symmetry, GSO projection (bosonization), and Jordan-Wigner transformations (fermionization). To elaborate on these more:

- **Stacking with SPTs.** Topological theories are only global data, essentially by definition. They have no local dynamics/excitations, so there is nothing to scatter. To distinguish two topological theories, we need to compute Z, $\mathcal{H}$, etc. on different topologies and in response to different backgrounds. Thus stacking with a topological theory is a good topological manipulation, as it does not talk to the original local degrees of freedom and simply changes the response of the theory to different topologies/backgrounds.
- **Orbifolding.** This operation, unlike stacking with a topological theory, does talk to the original degrees of freedom. But discrete gauge fields are necessarily flat and hence have no local dynamics themselves, hence it does not add anything to the theory.

As we will see, all topological manipulations for a theory with a discrete G global symmetry are controlled by the topological boundary conditions of a topological G gauge theory in one higher dimension. This theory now goes by the name "symmetry TFT" in the literature. This unifies the answer to the above question in a theory independent way: all topological manipulations involving a global G symmetry are probed by the same G symmetry TFT. More generally, we can replace G by any fusion category symmetry (see Section 2.4 and Appendix A.4).

Chapter 2

Orbifold Groupoids

In this section we review the properties of orbifold operations on two-dimensional quantum field theories, either bosonic or fermionic, and describe the "Orbifold groupoids" which control the composition of orbifold operations. Three-dimensional TQFT's of Dijkgraaf-Witten type will play an important role in the analysis. We briefly discuss the extension to generalized symmetries and applications to constrain RG flows.

I would like to thank Theo Johnson-Freyd, Jingxiang Wu, and Matt Yu for helpful discussions in developing the paper that this section is based on.

2.1 Introduction

Symmetries and associated anomalies are an important tool in the study of Quantum Field Theory. They increase the amount of topological data attached to a theory, are invariant under continuous deformations of the theory and, in particular, under the Renormalization Group flow.

Discrete symmetries also open avenues to important examples of "topological manipulations" in Quantum Field Theory. Indeed, gauge theories for a discrete symmetry group have no dynamics and are intrinsically topological in nature. If we couple a QFT to a dynamical discrete gauge field we will obtain a new theory with the same local dynamics, say encoded in the OPE of gauge-invariant local operators, but different global properties and correlation functions. This manipulation also commutes with RG flow.

In the context of two-dimensional quantum field theory, the operation of gauging a discrete symmetry produces an "orbifold." A surprising feature of Abelian group orbifolds is that the resulting theory is always endowed in a canonical way with some new discrete symmetry, allowing for orbifold operations to be composed in intricate ways. A basic objective of this note is to understand in detail the "composition law" of such orbifold operations, for both bosonic and fermionic systems.

An important feature of topological manipulations is that their properties are essentially independent of the actual underlying theory and only depend on the properties of the "topological hooks" employed in defining them. For example, the properties of discrete gauging operations only depend on the symmetry group and its 't Hooft anomalies. This fact can be best understood by physically separating the local degrees of freedom from their symmetry.

We will review a standard strategy to accomplish this counterintuitive feat for orbifolds with the help of a three-dimensional topological gauge theory. Such a 3d TFT setup will allow for a simple characterization of orbifold operations and their composition laws in terms of the automorphisms of the associated 3d TFT. In particular it will allow us to prove that the composition of two orbifold operations is always an orbifold.

2.1.1 Structure of the Chapter

In Section 2.2 we will discuss orbifolds of bosonic theories and describe in detail the orbifold composition law for theories with $\mathbb{Z}_p \times \mathbb{Z}_p$ symmetry, depicted schematically in Figure 2.6. We will also study the orbifolds of non-trivial Abelian extensions of cyclic groups by way of example in the case of a $\mathbb{Z}_2$ subgroup of $\mathbb{Z}_4$.

In Section 2.3 we will discuss orbifolds of fermionic theories and describe in detail the orbifold composition law for theories with $\mathbb{Z}_2 \times \mathbb{Z}_2^f$ or $\mathbb{Z}_4^f$ symmetry, depicted schematically in Figure 2.11 and Figure 2.12 respectively.

In Section 2.4 we discuss applications of the 3d setup to theories with generalized symmetries. In this section, we study the special example of current-current deformations of WZW models, with extra focus on $\mathfrak{su}(2)_k$.

We also include some Appendices reviewing: computational aspects of interfaces in 3d TFTs in Appendix A.1; the basics of spin structures in 2d in Appendix A.2; helpful identities of the Arf invariant and cup products in Appendix A.3; and a general discussion of topological aspects of QFTs in Appendix A.4.

Throughout, we use "dimensions" to mean the number of space-time dimensions (as opposed to the number of space dimensions). Hence when we say 2d we mean (1+1)d, and 3d means (2+1)d.

2.2 Bosonic Orbifolds and Symmetries of 3d Gauge Theories

Consider a (not spin) two-dimensional Quantum Field Theory T endowed with some discrete symmetry group G. We may attempt to couple the theory to a background flat G connection, but this can be obstructed by 't Hooft anomalies.

We should specify carefully what we mean by "'t Hooft anomaly" here. In principle, coupling an abstract theory with discrete G symmetry to a G flat connection can be obstructed in a variety of ways. The most serious anomalies indicate that the correct symmetry group of the theory is simply not G but some larger generalized symmetry group [22] generated by topological defects of various codimension [7].

We reserve the term 't Hooft anomaly for obstructions which can be compared between different theories and cured by adding appropriate extra degrees of freedom which are endowed with G symmetry, but are actually decoupled from the theory. In other words, invariance under G gauge transformations at most fails by invertible topological degrees of freedom [15].

It turns out that our ability to characterize the possible 't Hooft anomalies for quantum field theories in dimension d is limited by our knowledge of "invertible" quantum field theories in dimension d and lower (with no assumed symmetry) [18, 23]. If we accept the standard assumption that no non-trivial invertible bosonic theories (without extra symmetry) exist in dimension 2 or lower, except for invertible numbers in $d = 0$, then the 't Hooft anomalies relevant to our setup are encoded in a class $\mu_3(T)$ in the third group cohomology $H^3(G, U(1))$. The standard arguments for this identification are explained in e.g. [24, 7, 15, 25].

The group cohomology class economically encodes all the phase ambiguities which may occur when we attempt to couple T to a flat G connection. For example, take space to be a circle with a non-trivial G flat connection, so that the periodicity of local operators is twisted by the action of some $g \in G$. A possible manifestation of the 't Hooft anomaly is that the corresponding Hilbert space only carries a projective representation of the centralized $C(g) \in G$ of g. The possible ways a representation can be projective are labelled by a class in $H^2(C(g), U(1))$, which here can be computed as the partial integral $i_g\mu_3$ of μ_3 on a circle with holonomy g.[1]

In general, we can gauge any subgroup H of G for which the 't Hooft anomaly vanishes,

[1]See [26] for a nice discussion of the physical interpretation of this mathematical operation and generalizations to fermionic phases.

simply by making the 2d background G connection dynamical over the corresponding H subgroup. In order to gauge the H symmetry, we have to make an actual choice of how to resolve all the potential phase ambiguities, which essentially means producing an actual trivialization of the 3-cocycle μ_3 restricted to H, i.e. producing a solution ν_2 of

$$\delta\nu_2 = \mu_3|_H \,. \tag{2.1}$$

This choice is usually called a choice of "discrete torsion." Two choices are inequivalent if the difference $\nu_2' - \nu_2$ is a non-trivial class in $H^2(H, U(1))$.

Equivalently, if we identify T as a "2d theory with a non-anomalous H symmetry" for which such a choice has been made once and for all, other choices can be obtained by stacking T with a 2d SPT phase for H, labelled by a class in $H^2(H, U(1))$ [27, 28].

The orbifold operation produces a new 2d theory $[T/_{\nu_2}H]$, the orbifold of T by H.

2.2.1 Orbifolds and 3d Gauge Theory

There is a standard construction which neatly decouples topological manipulations (like orbifolds) from the local dynamics of the theory T. There is a bijection between 2d theories endowed with a G symmetry and boundary conditions for a 3d Dijkgraaf-Witten (DW) theory, which is the Topological Field Theory defined as a 3d G gauge theory $\mathrm{DW}[G]_{\mu_3}$ with "action" μ_3 [24].

The map in one direction is quite obvious: we simply couple T to the boundary value α_∂ of the dynamical 3d G flat connection α. This produces some "enriched Neumann" boundary condition $B[T]$. The 2d G 't Hooft anomaly is then cancelled by anomaly inflow between the bulk 3d G gauge theory and the 2d boundary theory [29, 30, 31, 32, 33],[2] schematically depicted in Figure 2.1.

The map in the opposite direction employs a second reference topological "Dirichlet" boundary condition D, which fixes the restriction α_∂ of α at the boundary to equal some 2d background G connection. The original theory T is obtained from a compactification on a segment with endpoints $B[T]$ and D. Notice that the Dirichlet boundary condition D is endowed with the *global* G symmetry of T while the dynamics of T is now localized at $B[T]$, as depicted in Figure 2.2. More precisely, G-invariant local operators in T map to local operators at $B[T]$, with the same OPE and local dynamics. [3]

[2]This anomaly inflow phenomena may be more recognizable in terms of the traditional example for a connected continuous group G. In this case, a d-dimensional anomaly is cancelled by adding a (d+1)-dimensional Chern-Simons action as originally described in [29]. A brief overview of the parallels and discrepancies between the continuous and discrete case are described in [33].

[3]Other local operators in T have to be attached to a Wilson line stretching all the way to D.

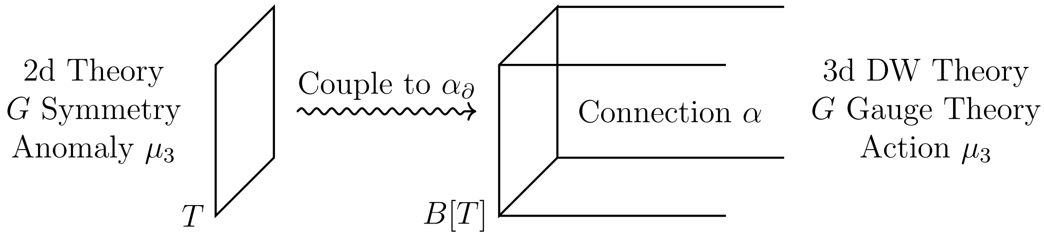


Figure 2.1: We take a 2d theory with G symmetry and anomaly μ_3 and use it to produce a boundary condition for the dynamical 3d DW theory with gauge group G and action μ_3. The anomaly of the 2d theory is cancelled by anomaly inflow from the bulk 3d theory. This picture can also be understood in terms of a boundary state as depicted in Equation 2.2.

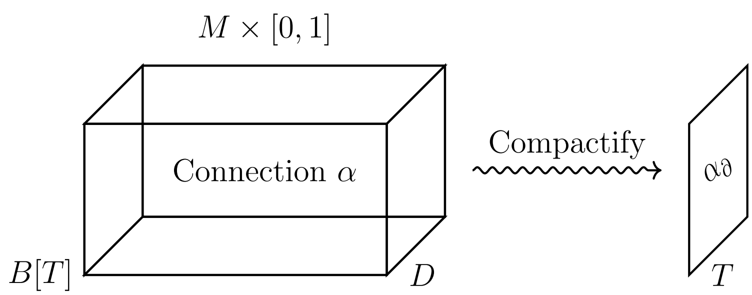


Figure 2.2: If we take the boundary condition $B[T]$ for the 3d theory we may produce T by compactifying $B[T]$ with Dirichlet boundary conditions on $M \times [0,1]$. See also Appendix A.1.

We have thus literally separated the symmetry of T from the dynamics of T. Any topological manipulation involving the G global symmetry, such as orbifolds, will only affect the D boundary condition and will not interfere with the $B[T]$ boundary condition.

For example, the orbifold theory $[T/_{\nu_2}H]$ is represented by a different segment compactification, involving a topological "partial Neumann" boundary condition N_{H,ν_2} defined by restricting the gauge group from G to H at the boundary, with a "boundary action" ν_2.

One may immediately wonder if we could define some generalization of an orbifold, where N_{H,ν_2} is replaced by some other topological boundary condition for the DW theory. Such boundary conditions have a sharp mathematical description as module categories using the theory of fusion categories.[4] Irreducible boundary conditions turn out to be classified precisely by the (H, ν_2) data, so no exotic orbifolds are available [35, 36].

At worst, some topological manipulation of T may produce a theory with multiple superselection sectors, each coinciding with some orbifold of T. We may denote such a theory as $\bigoplus_i [T/_{\nu_i} H_i]$. This corresponds to considering a boundary condition $\bigoplus_i N_{H_i,\nu_i}$ with superselection sectors, i.e. a "decomposable module category".[5]

An immediate consequence of the 3d TFT interpretation of orbifolds is that it nicely organizes the relevant manipulations of partition functions for the 2d theories: it promotes the collection $Z_T[\alpha]$ of partition functions on some reference 2d manifold M with flat G connection α to the "boundary state" for $B[T]$:

$$|T\rangle = \sum_\alpha Z_T[\alpha] \, |\alpha\rangle \ , \tag{2.2}$$

where $|\alpha\rangle$ are a natural basis of states for the 3d theory.

General 3d TFT technology provides a variety of useful alternative bases for the Hilbert space, which will be useful later on. In particular, once we choose a basis of 2-cycles, the space of states on a 2-torus has an alternative basis labelled by the anyons of the 3d TFT.

In any basis, the partition function of any orbifold theory $[T/_{\nu_2} H]$ is computed as an inner product $\langle N_{H,\nu_2}|T\rangle$ with the boundary state for N_{H,ν_2}. In the α basis, the boundary state is

$$\langle N_{H,\nu_2}| = \sum_\alpha e^{\int_M \nu_2(\alpha)} \, \langle\alpha| \ , \tag{2.3}$$

where $\nu_2(\alpha)$ is the pull-back of ν_2 to M along α.[6]

2.2.2 Emergent Symmetries and the Group of Orbifolds

It turns out that the orbifold operation never loses information: one can always find a topological manipulation of $[T/_{\nu_2} H]$ which will give back T. Generically, this manipulation

[4] See [12] for a physics introduction, or [34] for a comprehensive mathematical treatment.

[5] Physically, any topological boundary condition can be identified by the above bijection with some enriched Neumann boundary condition involving topological 2d degrees of freedom. As no non-trivial bosonic 2d order exists, the only possibility is some direct sum $\bigoplus_i N_{H_i,\nu_i}$. In higher dimensions, the classification of topological boundary conditions for the DW theory is much richer.

[6] The inner product $\langle\alpha|\beta\rangle$ has to be normalized carefully to account for gauge invariance. For Abelian H, the normalization is $\langle\alpha|\beta\rangle = \frac{1}{|H|}\delta_{\alpha\beta}$.

is not itself an orbifold. It is instead what is called a "generalized orbifold," or "2d anyon condensation."

Rather than summing over 2d flat connections for some global symmetry, a generalized orbifold involves sums over certain networks of topological line defects described by a fusion category, encoding a certain hidden "generalized symmetry" of $[T/_{\nu_2}H]$ [37, 38, 39]. Such generalized symmetries and orbifolds are quite interesting and we will return to them later in the note. For now, though, we would like to discuss situations where the orbifold theory $[T/_{\nu_2}H]$ has a standard emergent symmetry G', which can be described without the full machinery of fusion categories.

The simplest possibility is to take G to be an Abelian group A with trivial(ized) 't Hooft anomaly. Then the orbifold $[T/_{\nu_2}A]$ has an emergent, non-anomalous quantum symmetry group $\hat{A}$, the Pontryagin dual of A [40]. Notice that $\hat{A}$ is isomorphic to A, but not canonically so.

Intuitively, the new symmetry group arises from the action of Wilson lines for the A gauge fields, which are labelled by characters in $\hat{A}$. Directly gauging $\hat{A}$ gives back T. Of course, we may decide to add some extra discrete torsion $\hat{\nu}_2$ when gauging $\hat{A}$ in $[T/_{\nu_2}A]$, which will produce a new theory $[[T/_{\nu_2}A]/_{\hat{\nu}_2}\hat{A}]$ with A symmetry, and so on and so forth. Is there any relation between these new theories and orbifolds of T? How many new theories can we possibly produce that way?

In 3d terms, the $\hat{A}$ symmetry appears as an emergent symmetry of the N_{A,ν_2} Neumann boundary conditions. Gauging the two-dimensional $\hat{A}$ symmetry of N_{A,ν_2} will produce a new topological boundary condition $[N_{A,\nu_2}/_{\hat{\nu}_2}\hat{A}]$ with an emergent A symmetry, and so on. No matter what we do, the resulting boundary conditions for the 3d A gauge theory will have the form N_{B,ν_2^B} for some subgroup B of A, so the new theories we produce will all be orbifolds of T equipped with some emergent A symmetry.

We thus have some collection of topological operations acting on the space of 2d theories with non-anomalous A symmetry. We would like to characterize such operations and their composition law.

2.2.3 Emergent Symmetries and Dualities

In the example above, we encounter two different-looking ways to present the $[T/_{\nu_2}A]$ gauge theory: a slab of A gauge theory with $B[T]$ and N_{A,ν_2} boundary conditions or a slab of $\hat{A}$ gauge theory, with $B[[T/_{\nu_2}A]]$ and D boundary conditions. Inspection shows that these are two different descriptions of the *same* setup. Namely

- The A gauge theory and the $\hat{A}$ gauge theory are different dual descriptions of the same abstract 3d TFT.
- The boundary conditions N_{A,ν_2} and D are dual descriptions of the same abstract topological boundary condition.
- The boundary conditions $B[T]$ and $B[[T/_{\nu_2}A]]$ are dual descriptions of the same abstract boundary condition.

In order to understand this better, we need to recall that a 3d TFT is (conjecturally) fully captured by some categorical data, which is essentially the Modular Tensor Category $\mathcal{C}$ of topological line defects (aka "anyons"). The anyons in a discrete gauge theory $\mathrm{DW}[G]_{\mu_3}$ include a collection of Wilson lines labelled by irreps of G. Generic anyons can be presented as disorder defects carrying discrete flux as well as electric charge.

Topological boundary conditions in a 3d TFT support a fusion category $\mathcal{S}$ of boundary line defects/anyons. The specific category depends on the choice of boundary conditions, but its Drinfeld center $Z[\mathcal{S}]$ is isomorphic to the MTC $\mathcal{C}$ of bulk anyons. In particular, this isomorphism encodes which bulk lines can end at the boundary.

The only boundary lines at Dirichlet boundary conditions are the disorder defects implementing the G global symmetry, labelled by elements of G. They form the fusion category denoted as $\mathrm{Vec}_G^{\mu_3}$. Only bulk Wilson lines can end at a Dirichlet boundary condition and vice versa. We can recognize an abstract 3d TFT as a DW theory $\mathrm{DW}[G]_{\mu_3}$ by presenting a topological boundary condition with boundary anyons which fuse according to the G group law. The cocycle μ_3 is the associator for the fusion operation.

We can build a *duality groupoid* $\mathfrak{G}$ whose objects are DW theories and whose morphisms are isomorphisms of 3d TFTs. These may include non-trivial identifications of a $\mathrm{DW}[G]_{\mu_3}$ with itself, remixing the bulk anyons in a non-trivial manner, as well as different ways to identify $\mathrm{DW}[G]_{\mu_3}$ with some $\mathrm{DW}[G']_{\mu'_3}$.

Any such isomorphism in $\mathrm{Hom}(\mathrm{DW}[G]_{\mu_3}, \mathrm{DW}[G']_{\mu'_3})$ has enough information to map any anyon or boundary condition in $\mathrm{DW}[G]_{\mu_3}$ to a corresponding anyon or boundary condition in $\mathrm{DW}[G']_{\mu'_3}$. The image under this map of Dirichlet boundary conditions for $\mathrm{DW}[G]_{\mu_3}$ must always be some N_{H',ν'_2} with global G symmetry, so these maps are all orbifolds.

More precisely, we can combine these maps with the identification between boundary conditions $B[T]$ of $\mathrm{DW}[G]_{\mu_3}$ and 2d theories T with G symmetry and anomaly μ_3 to obtain an action of $\mathfrak{G}$ as a groupoid of orbifold operations acting on 2d theories.

Some of the topological operations do not really change the 2d theory: they only change the prescription of how the theory is coupled to a flat connection. From the 3d perspective,

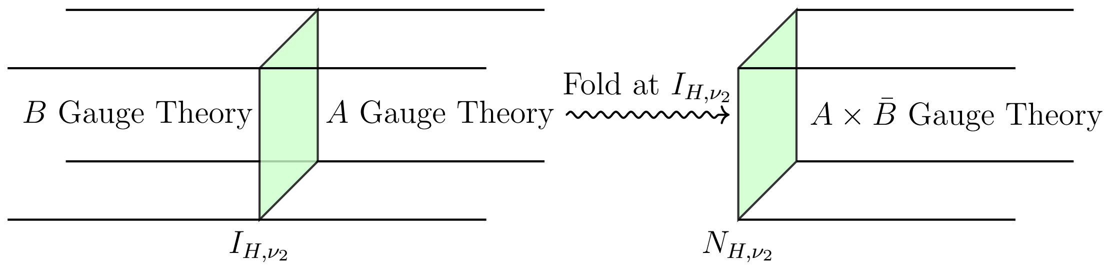


Figure 2.3: In the folding trick, we replace the setup with a gauge theory B on the left of the interface and gauge theory A on the right of the interface, by a product theory $A \times \bar{B}$ with a corresponding boundary condition.

they are automorphisms of $\mathrm{DW}[G]_{\mu_3}$ which fix the Dirichlet boundary conditions. We will thus find it useful to refine the duality groupoid to an *orbifold groupoid*, whose nodes are associated to 3d TFTs equipped with a specific topological boundary condition and whose morphisms are isomorphisms of 3d TFTs which identify the corresponding boundary conditions.

The action of these orbifold transformations on the partition functions of the 2d theories is particularly simple in an anyon basis: they simply permute the element of the basis in the same way as they permute the anyons.

Notice that most MTC's do not admit topological boundary conditions. Even if they do, they may not admit boundary conditions with a group-like fusion category of boundary anyons, or may admit only one. DW theories for Abelian gauge groups, though, have large collections of such boundary conditions and are nodes of a rich duality groupoid, which we will momentarily describe.

2.2.4 Duality Interfaces

The notion of topological interface is a natural extension of the notion of topological boundary condition. Indeed, by the folding trick, interfaces between theories A and B are precisely boundary conditions $A \times \bar{B}$, where $\bar{B}$ is the mirror image of B. See Figure 2.3.

Every theory has a trivial "identity" interface. If we have a duality between $\mathrm{DW}[G]_{\mu_3}$ and $\mathrm{DW}[G']_{\mu'_3}$, we can start from the identity interface in $\mathrm{DW}[G]_{\mu_3}$ and only apply the duality transformation to the side on the right of the interface. The result is a "duality interface" between between $\mathrm{DW}[G]_{\mu_3}$ and $\mathrm{DW}[G']_{\mu'_3}$, which can be used to implement the duality on other objects, such as boundary conditions [41, 42].

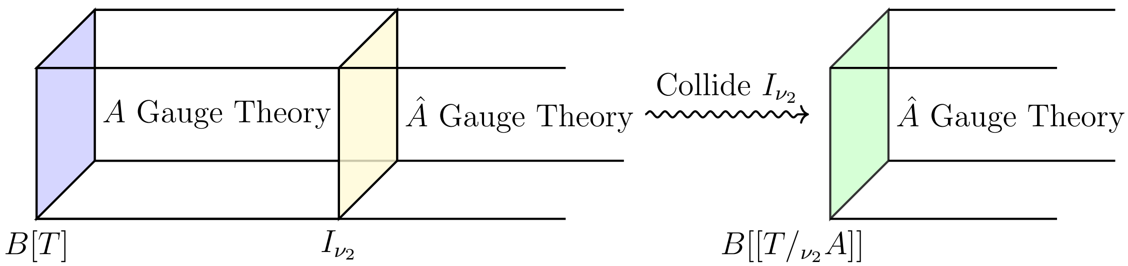


Figure 2.4: Coupling to the 3d bulk literally decouples a theory T from its topological manipulations. If T corresponds to some boundary condition $B[T]$ (in blue), and some topological manipulation corresponds to the interface I_{ν_2} (in yellow), we may produce the theory with the topological manipulation included (in green), by colliding the boundary $B[T]$ with I_{ν_2}.

A useful perspective is that the orbifold operation $T \mapsto [T/_{\nu_2}A]$ lifts to a simple operation on boundary conditions: the boundary condition $B[[T/_{\nu_2}A]]$ is obtained by the collision of $B[T]$ with the interface I_{ν_2}. The composition of orbifold operations then lifts to the composition of interfaces I_{ν_2} as depicted in Figure 2.4.[7]

On general grounds, such an interface must be labelled by some subgroup H of $G \times G'$, as well as a trivialization ν_2 of the pull-back of $\mu_3 - \mu_3'$ to H. Recovering this data from the original duality map is not an obvious operation. It must be such that the interface boundary state

$$|N_{H,\nu_2}| = \sum_{\alpha,\alpha'} e^{\int_M \nu_2(\alpha,\alpha')} |\alpha\rangle \langle\alpha'| \,, \tag{2.4}$$

agrees with the permutation of anyons in the anyon basis. We will give some explicit examples later on.

2.2.5 Specialization to Pure Abelian Gauge Theory

Consider now the case of an Abelian gauge group with no anomaly.

The group of anyons is the quantum double $A \times \hat{A}$, and the topological spin of an anyon of charges $(a, \hat{a})$ is simply the evaluation of the character $\chi_{\hat{a}}(a)$. So we expect that the group of orbifold-like topological operations relating theories equipped with an A

[7]A basic introduction on how to view and manipulate interfaces of the 3d gauge theories found in this chapter is presented in Appendix A.1.

symmetry should be the subgroup of $\mathrm{Aut}(A \times \hat{A})$ preserving the character pairing $\chi_{\cdot}(\cdot)$, which is simply $O(A \oplus \hat{A}, \chi)$ [43, 44, 45].

In [44] the authors connect the algebraic language of lines in the 3d Dijkgraaf-Witten theory to the gauge-theoretic description. In particular, for a 3d Abelian DW theory with $\mu_3 = 0$, they show that $O(A \oplus \hat{A}, \chi)$ is generated by combinations of:

1. *Universal Kinematical Symmetries*. Symmetries of the stack of A-bundles, $\mathrm{Bun}(A)$, which can be identified with $\mathrm{Aut}(A)$.
2. *Universal Dynamical Symmetries*. Symmetries of the topological action for the Dijkgraaf-Witten theory μ_3, which are elements of $H^2(A, U(1))$. This is the group of 1-gerbes on the stack of A-bundles. Recall a connection on a 1-gerbe is just a 2-form/B-field.
3. *Electric-Magnetic Dualities*. Symmetries interchanging elements of A and $\hat{A}$ at the level of anyons.

Together, the universal kinematical and dynamical symmetries have the structure $H^2(A, U(1)) \rtimes \mathrm{Aut}(A)$, which we recognize as the group of autoequivalences of the spherical fusion category Vec_A.

Moreover, we can identify these 3d symmetries with operations acting on our 2d boundary theory. The universal kinematical symmetries come from the automorphisms of A. The universal dynamical symmetries are clearly discrete torsion terms and/or stacking with a 2d SPT phase, this 2d fact was noticed in-terms of a Kalb-Ramond field in the original work by Vafa [46] and formalized by Sharpe [47]. The symmetry group of the 2d theory is just the product of the 3d kinematical and dynamical symmetry groups. Finally, the electric-magnetic dualities are not symmetries of the 2d theory, but, rather, correspond to orbifolding the 2d theory.

The authors of [44] also give explicit formulas of how these generating automorphisms of the MTC data turn into (H, ν_2) data from this 3d formalism. In the following examples we obtain the same results as the authors (in the $\mathbb{Z}_p \times \mathbb{Z}_p$ case in particular) by starting with a 2d theory.

2.2.6 Examples

In the following examples we will answer the question: how many new theories can we produce by successive orbifolds? We will warm up by starting from the traditional 2d orbifold point of view for a theory with $A = \mathbb{Z}_2$ symmetry, and then upgrade to slightly more sophisticated examples with $A = \mathbb{Z}_p \times \mathbb{Z}_p$ symmetry (p prime). Keeping in mind that the orbifold story will also be relevant for the fermionic section where a clear and organized study of orbifolds has become fruitful in the study of 2d dualities and CFT. In our final example we will investigate theories with an anomaly in the study of orbifolds of $\mathbb{Z}_4$ symmetric theories.

In each section we will interpret the results in the language of 3d interfaces. We will find that the interesting interfaces are given in the basis of connections by different cup products. In particular, SPT phases will be implemented by cup products on one side of an interface, and orbifolds by cup products across interfaces. As we will see, this similarity arises because of the folding trick. Lastly, our final example provides a formula for orbifold-interfaces for arbitrary non-anomalous Abelian groups.

Throughout, we illustrate our formulae explicitly by putting the 2d theory on $M = T^2$, although this specialization is not necessary. Appropriate generalizations can be made by replacing the two torus cycles with, say, $2g$ cycles for a genus M orientable surface.[8]

Example: Theories with $\mathbb{Z}_2$ Symmetry

Consider a 2d theory T with non-anomalous $A = \mathbb{Z}_2$ symmetry on a genus g surface M, with partition function Z_T. Coupling our $\mathbb{Z}_2$ symmetry to a background A connection allows us to identify the different twisted partition functions, labelled by the holonomies around the different cycles of M, i.e. $Z_T[\alpha]$ where α has $2g$-components with $\alpha_i \in \{0, 1\}$.

To gauge the A symmetry, we simply sum over all background flat connections, producing

$$Z_{[T/A]} = \frac{1}{|A|} \sum_{\alpha} Z_T[\alpha] \,, \tag{2.5}$$

where $\alpha \in H^1(M, A)$. Furthermore, we know that $Z_{[T/A]}$ has a quantum $\hat{A}$ symmetry arising from the action of the Wilson lines for the A gauge fields. Thus, in the same way that we identify $Z_T \sim Z_T[\alpha = 0]$, we have that $Z_{[T/A]}$ is the untwisted sector for our new $\hat{A}$ symmetry, and so we can write more generally

$$Z_{[T/A]}[\beta] = \frac{1}{|A|} \sum_{\alpha} e^{i(\beta,\alpha)} Z_T[\alpha] \,, \tag{2.6}$$

[8] Some care is needed to keep track of local curvature counterterms. As commented in Footnote 6, a good normalization for Abelian gauge theories is a factor of $|A|^{-1}$ (the dimension of the unbroken gauge group). Because of this, gauging does not always "square to the identity" because manifolds of different genus are not flat. However, if we renormalize by the curvature counterterm $|A|^{1-g}$ on a genus g surface, then we will arrive at an operation that squares to the identity by collecting a total factor of $|A|^{\chi(M)}$.

Figure 2.5: For any theory with non-anomalous finite Abelian A symmetry, we obtain a new theory with $\hat{A} \cong A$ symmetry by gauging *all* of A. These correspond to the Dirichlet and "entirely-Neumann" boundary conditions for the associated DW$[A]$. In the case of a non-anomalous $A = \mathbb{Z}_p$ (p prime) this is the complete orbifold groupoid (suppressing multi-edges and edges from a vertex to itself), as there are only two bosonic irreducible topological boundary conditions.

where $\beta \in H^1(M, \hat{A})$, and (β, α) is the intersection pairing [12].

Invertibility is a straightforward application of the formula twice:

$$Z_{[[T/A]/A]}[\gamma] = \frac{1}{|A|^2} \sum_{\beta} e^{i(\gamma,\beta)} \sum_{\alpha} e^{i(\beta,\alpha)} Z_T[\alpha] = |A|^{2g-2} Z_T[\gamma]\,. \tag{2.7}$$

Orbifolding twice gives back the original theory, up to a curvature counterterm.

The simplest examples of theories related by orbifold are the trivial theory, with $Z_T[\alpha] = 1$, and a symmetry-breaking phase, with $|A|$ trivial vacua permuted by the A action, with $Z_T[\alpha] = |A|\delta_{\alpha,0}$.

For concreteness, using the basis of flat connections around the cycles of a torus, we have

$$Z_{[T/A]}[\beta_1, \beta_2] = \frac{1}{2} \sum_{\alpha_1,\alpha_2} (-1)^{\alpha_1\beta_2 - \alpha_2\beta_1} Z_T[\alpha_1, \alpha_2]\,, \tag{2.8}$$

where α_i and β_i label the holonomies.

At this point, there are no more topological manipulations left for our $\mathbb{Z}_2$ theory. There are no nontrivial automorphisms of $\mathbb{Z}_2$, and since $H^2(\mathbb{Z}_2, U(1)) = 0$ there is no discrete torsion/SPT phase to add to the action. Indeed, the only topological manipulation is to orbifold it and produce another $\mathbb{Z}_2$ theory.

Note the fact that gauging produces an emergent $\hat{\mathbb{Z}}_2$ symmetric theory and "squares to the identity" is just capturing Kramers-Wannier duality, see [48, 49, 50, 51, 52] for recent expositions and applications. From here, we can draw a graph of the orbifold groupoid: theories correspond to vertices, and two theories are connected by an edge if they are related by orbifold as in Figure 2.5.

As previously mentioned, we can study the interface that implements the gauging operation in our 3d theory. This is clearly just our intersection pairing from above

$$I_{\text{gauge}}[\alpha; \beta] = (-1)^{\int \alpha \cup \beta}\,. \tag{2.9}$$

This interface collides with the boundary theory described by $Z_T[\alpha]$ and produces the boundary theory described by $Z_{[T/A]}[\beta]$.

If our boundary manifold is just the torus we can be more concrete and just write

$$I_{\text{gauge}}[\alpha_1, \alpha_2; \beta_1, \beta_2] = (-1)^{\alpha_1\beta_2 - \alpha_2\beta_1}\,. \tag{2.10}$$

Thus far, we've been using partition functions of the 2d theory, which can be identified with components of the boundary state for the 3d theory in a basis labelled by A-holonomy around a cycle, i.e. the basis of A connections. An alternative basis to work in when dealing with 3d TFTs is a basis of states labelled by anyons.[9]

The change of basis is a discrete Fourier transform

$$\hat{f}[\chi] = \frac{1}{|A|}\sum_{a\in A}\chi(a) f[a]\,, \tag{2.11}$$

to be applied to the holonomy label for one of the cycles of the torus.

We can thus define

$$\hat{Z}[\alpha_1, \hat{\alpha}_2] := \frac{1}{2}\sum_x (-1)^{x\hat{\alpha}_2} Z[\alpha_1, x]\,, \tag{2.12}$$

where the first index corresponds to magnetic/vortex charge describing the discrete A flux, and the second index to the electric charge (the electric charge is often labelled as "even" or "odd" in the $\mathbb{Z}_2$ case).

In the $A = \mathbb{Z}_2$ case we have the anyons of the 3d DW[$\mathbb{Z}_2$] gauge theory

$$Z_1 = \hat{Z}[0, \hat{0}]\,, \quad Z_e = \hat{Z}[0, \hat{1}]\,, \quad Z_m = \hat{Z}[1, \hat{0}]\,, \quad Z_f = \hat{Z}[1, \hat{1}]\,, \tag{2.13}$$

which are gauge theoretic realizations of the anyons $\{1, e, m, f\}$ for the toric code with trivial associator.

In this basis, our interface is simply

$$\hat{I}_{\text{gauge}}[\alpha_1, \hat{\alpha}_2; \beta_1, \hat{\beta}_2] = \delta_{\hat{\alpha}_2\beta_1}\delta_{\hat{\beta}_2\alpha_1}\,. \tag{2.14}$$

This is immediately familiar, it maps $Z_1 \mapsto Z_1$ and $Z_f \mapsto Z_f$, but swaps Z_e and Z_m. We see the famous statement that the Kramers-Wannier duality in 2d implements the 3d electric-magnetic duality and vice-versa.

[9]These are built by a solid torus geometry with an anyon running in the middle. The definition requires a choice of cycle in the torus

Moreover, when we claimed we had nothing (topological and bosonic) left to do to our 2d $\mathbb{Z}_2$-symmetric theory, we now have proof, because we have connected it to symmetries of a Dijkgraaf-Witten theory. That is, we know that $O(\mathbb{Z}_2 \oplus \hat{\mathbb{Z}}_2, \chi) = \mathbb{Z}_2$, so that we only have two distinct irreducible bosonic topological boundary conditions for $\mathrm{DW}[\mathbb{Z}_2]$. These correspond to "electric" and "magnetic" Dirichlet boundary conditions (if we identify the bulk Wilson line as being the "electric" line or "magnetic" line respectively).[10]

The duality groupoid in this case would just include a single vertex, $\mathrm{DW}[\mathbb{Z}_2]$, with a line connecting it to itself because $\mathrm{Hom}(\mathrm{DW}[\mathbb{Z}_2], \mathrm{DW}[\mathbb{Z}_2]) = \mathbb{Z}_2$.

The case for arbitrary $\mathbb{Z}_p$ (p prime) is very similar. As before, we can either orbifold all of $\mathbb{Z}_p$ or not, and $H^2(\mathbb{Z}_p, U(1)) = 0$. The automorphism group of $\mathbb{Z}_p$ is $\mathbb{Z}_{p-1}$, so the orbifold groupoid still consists entirely of two vertices joined by a line for the two topological boundary conditions (we suppress lines from a vertex to itself, or multiple lines from a vertex to another). The duality groupoid is still just a single vertex. From the orbifold groupoid it's not hard to see that the $\mathbb{Z}_2$ orbifolding operation and $\mathbb{Z}_{p-1}$ of automorphisms mix non-trivially, and that the symmetry group of $\mathrm{DW}[\mathbb{Z}_p]$ is a dihedral group, i.e.

$$\mathrm{Hom}(\mathrm{DW}[\mathbb{Z}_p], \mathrm{DW}[\mathbb{Z}_p]) = O(1,1;\mathbb{F}_p) \cong D_{2(p-1)}\,. \tag{2.15}$$

Here $O(p,q;\mathbb{F}_p)$ denotes the split orthogonal group over the finite field of order p and $D_{2(p-1)}$ denotes the dihedral group of order $2(p-1)$.

Example: Gauging $\mathbb{Z}_p$ in $\mathbb{Z}_p \times \mathbb{Z}_p$ and Discrete Torsion

We can now enhance our discussion to an example with discrete torsion. From the original 2d perspective, a choice of discrete torsion is a consistent choice of $U(1)$ weights $\epsilon_{\nu_2}(\alpha)$ for the twisted sectors

$$Z_{[T/_{\nu_2}A]}[\beta] = \frac{1}{|A|}\sum_{\alpha} e^{i(\beta,\alpha)}\epsilon_{\nu_2}(\alpha) Z_T[\alpha]\,. \tag{2.16}$$

It is known that a choice of discrete torsion is specified by an element $\nu_2 \in H^2(G, U(1))$ [46, 47, 55]. In particular, on the torus with flux given by (α_1, α_2), we have $\epsilon_{\nu_2}(\alpha_1, \alpha_2) = \nu_2(\alpha_1,\alpha_2)/\nu_2(\alpha_2,\alpha_1)$. As previously mentioned, we can interpret $\epsilon_{\nu_2}(\alpha) \sim e^{iS_{\nu_2}[\alpha]}$ as a partition function for an SPT phase, so changing discrete torsion amounts to stacking

[10] In the lattice formulation of the toric code, these topological boundary conditions manifest beautifully as "smooth" and "rough" boundaries of the lattice [53, 54], where it becomes pictorially clear that one type of anyon (say, living on plaquettes) is absorbed by the smooth boundary, and vice-versa for the dual. Superpositions of these topological boundary conditions correspond to a direct sum/reducible boundary condition.

our original theory with a 2d SPT phase. Intuitively, it is a consistent way to insert phase factors at the trivalent junctions of two (meeting and merging) topological symmetry defects.

The canonical example of discrete torsion is in a theory with non-anomalous $A = \mathbb{Z}_p \times \mathbb{Z}_p$ symmetry, then $H^2(\mathbb{Z}_p \times \mathbb{Z}_p, U(1)) = \mathbb{Z}_p$. In this case, our 2d manipulations are (generated by) the automorphisms of $\mathbb{Z}_p \times \mathbb{Z}_p$, stacking with an SPT phase, and gauging subgroups of A. From here we take p to be a prime for simplicity, extensions to non-prime order cyclic groups are investigated later.

The automorphisms of $\mathbb{Z}_p \times \mathbb{Z}_p$ form the group $GL(2;\mathbb{F}_p)$. For any matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2;\mathbb{F}_p)\,, \tag{2.17}$$

we can define the associated action π_M on (torus) partition functions

$$\pi_M : Z[\alpha_a, \alpha_b, \beta_a, \beta_b] \mapsto Z[a\alpha_a + b\beta_a, a\alpha_b + b\beta_b, c\alpha_a + d\beta_a, c\alpha_b + d\beta_b]\,. \tag{2.18}$$

Now, for any prime p, $GL(2;\mathbb{F}_p)$ is always generated by two elements. For $p = 2$ we can take the generators to be

$$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\,, \quad M_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\,. \tag{2.19}$$

For $p \neq 2$ we have to use the slightly more complicated

$$M_1 = \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}\,, \quad M_2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\,, \tag{2.20}$$

where ξ is any generator for $(\mathbb{F}_p)^\times$ [56]. We will write π_1 and π_2 for π_{M_1} and π_{M_2} respectively.

Working on the torus, our topological manipulations include the automorphisms of $\mathbb{Z}_p \times \mathbb{Z}_p$, generated by π_1 and π_2, changes of discrete torsion ($1 \leq \ell \leq p$)

$$S_\ell : Z[\alpha_a, \alpha_b, \beta_a, \beta_b] \mapsto \omega_p^{\ell(\alpha_a\beta_b - \alpha_b\beta_a)} Z[\alpha_a, \alpha_b, \beta_a, \beta_b]\,, \tag{2.21}$$

and gauging the "second" $\mathbb{Z}_p$ as

$$\mathcal{O}_2 : Z[\alpha_a, \alpha_b, \beta_a, \beta_b] \mapsto \frac{1}{p}\sum_\delta \omega_p^{\delta_a\beta_b - \delta_b\beta_a} Z[\alpha_a, \alpha_b, \delta_a, \delta_b]\,. \tag{2.22}$$

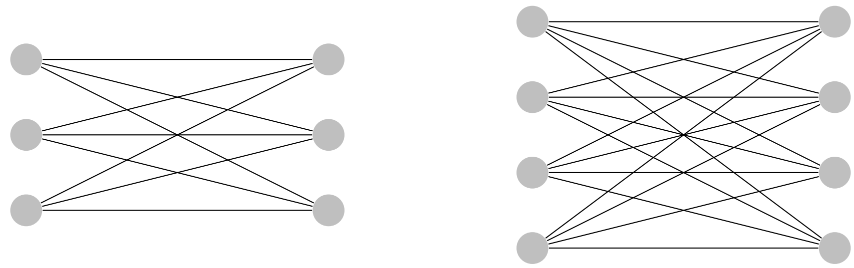


Figure 2.6: On the left is the orbifold groupoid for a theory with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. On the right is the orbifold groupoid for a theory with $\mathbb{Z}_3 \times \mathbb{Z}_3$. Edges correspond to gauging a $\mathbb{Z}_p$ subgroup. The number of vertices in a graph is $2(p+1)$ and the orbifold groupoid, with just $\mathbb{Z}_p$ gauging marked, is the complete bipartite graph $K_{p+1,p+1}$.

In this notation, an element of $\mathbb{Z}_p^2$ is given by a pair (α_i, β_i) around cycle-i, and ω_p is the principal p-th root of unity. Further note that gauging "one of the other" $\mathbb{Z}_p$ subgroups of A, can be done by applying enough of the automorphisms π_1 and π_2, and then $\mathcal{O}_2$.

Of course, we can write these operations algebraically and avoid these torus descriptions, or write them on an arbitrary genus g surface by use of the cup product. For example, we could just write the SPT phase factor as $\omega_p^{\ell \int \alpha \cup \beta}$.

We can draw our orbifold groupoid as before. Two theories live at the same vertex if they are related by any element of the group generated by the non-orbifolding operations

$$\langle S_1, \pi_1, \pi_2 \rangle \cong \mathbb{Z}_p \rtimes GL(2; \mathbb{F}_p)\,. \tag{2.23}$$

We will denote theories that are related by gauging a $\mathbb{Z}_p$ subgroup by connecting them by a line, see Figure 2.6 for $p = 2, 3$ examples. See also Example 4.3 of [57] for a discussion in terms of VOAs.

Note from the preceding discussions of 3d gauge theories that if we were also to include lines denoting gauging the entire $\mathbb{Z}_p \times \mathbb{Z}_p$, the graph would be totally connected rather than just complete bipartite. More generally, for any theory with any symmetry group, if we were to include lines for all types of orbifolds, then the graph must be totally connected by virtue of composition of the orbifold interfaces.

Additionally, from both the mathematical theorems and explicitly checking 2d partition functions, we know that the group of topological manipulations is

$$\langle S_1, \pi_1, \pi_2, \mathcal{O}_2 \rangle \cong O(\mathbb{Z}_p^4, \chi) = O(2, 2; \mathbb{F}_p)\,. \tag{2.24}$$

As before, we can interpret each of our 2d manipulations as corresponding to an interface of the 3d theory, implementing one of the symmetries of the associated 3d $\mathbb{Z}_p \times \mathbb{Z}_p$ gauge theory:

$$I_{\pi_1}[\gamma,\delta;\alpha,\beta] = p^2 \begin{cases} \delta_{\gamma,\alpha+\beta}\delta_{\delta,\beta} & \text{if } p=2 \\ \delta_{\gamma,\xi\alpha}\delta_{\delta,\beta} & \text{if } p\neq 2 \end{cases} \tag{2.25}$$

$$I_{\pi_2}[\gamma,\delta;\alpha,\beta] = p^2 \begin{cases} \delta_{\gamma,\alpha}\delta_{\delta,\beta+\alpha} & \text{if } p=2 \\ \delta_{\gamma,\beta-\alpha}\delta_{\delta,-\alpha} & \text{if } p\neq 2 \end{cases} \tag{2.26}$$

$$I_{S_\ell}[\gamma,\delta;\alpha,\beta] = p^2\, \delta_{\gamma,\alpha}\delta_{\delta,\beta}\, \omega_p^{\ell\int\alpha\cup\beta} \tag{2.27}$$

$$I_{\mathcal{O}_2}[\gamma,\delta;\alpha,\beta] = p\, \delta_{\gamma,\alpha}\, \omega_p^{\int\delta\cup\beta}\,. \tag{2.28}$$

As written, the γ and δ are short for "one of the $\mathbb{Z}_p$ connections" in a $\mathbb{Z}_p^2$ theory on one side of the interface, i.e. on a torus $\gamma \sim (\gamma_a,\gamma_b)$; and similarly for α and β on the other side of the interface.

A Fourier transform allows us to understand the results in terms of anyons. The π_1 and π_2 are trivial, and the gauging is again the electric-magnetic duality. Of particular interest is an interface (say on the torus) corresponding to adding an SPT phase,

$$\hat{I}_{S_\ell}[\gamma_1,\hat{\gamma}_2,\delta_1,\hat{\delta}_2;\alpha_1,\hat{\alpha}_2,\beta_1,\hat{\beta}_2] = p^2\delta_{\alpha_1\gamma_1}\delta_{\beta_1\gamma_1}\delta_{\hat{\alpha}_2,-\hat{\gamma}_2+\ell\delta_1}\delta_{\hat{\beta}_2,-\hat{\delta}_2-\ell\gamma_1}\,. \tag{2.29}$$

Interpreting this, the magnetic lines pass through the interface unchanged, but the electric lines get changed to some new electric lines based on the magnetic flux value. This matches the physical result in Section 3.2 of [44] after sufficient changes of notation and conventions.

In this case the duality groupoid would still contain just a single node $\mathrm{DW}[\mathbb{Z}_p^2]$, which would have $|O(2,2;\mathbb{F}_p)|$ lines to itself.

More generally, we can study $\mathbb{Z}_p^k$ theories. In this case, the group of all (irreducible bosonic) topological operations on the 2d bosonic theory would be classified by the group

$$T_B := O(k,k;\mathbb{F}_p)\,. \tag{2.30}$$

The group of operations which only include automorphisms of $\mathbb{Z}_p^k$ and stacking with SPT phases is

$$T_{B,0} := H^2(\mathbb{Z}_p^k, U(1)) \rtimes \mathrm{Aut}(\mathbb{Z}_p^k) = \mathbb{Z}_2^{\binom{k}{2}} \rtimes GL(k;\mathbb{F}_p)\,. \tag{2.31}$$

To form the orbifold groupoid for $\mathbb{Z}_p^k$, we identify vertices of the groupoid with (right) cosets of $T_B/T_{B,0}$. Given two vertices $T_{B,0}g_1$ and $T_{B,0}g_2$ they are connected by an edge iff

$$(\mathcal{O}_1 T_{B,0} g_1) \cap (T_{B,0} g_2) \neq \emptyset\,. \tag{2.32}$$

We could also include gauging of larger subgroups (i.e. $\mathbb{Z}_p^r$ $1 < r \leq k$) if we were so inclined.

Thus the number of irreducible bosonic topological boundary conditions is simply

$$\{\#\text{ Boundary Conditions}\} = \frac{|O(k,k;\mathbb{F}_p)|}{\left|H^2(\mathbb{Z}_p^k, U(1))\right||GL(2;\mathbb{F}_p)|}\,. \tag{2.33}$$

Such facts about group orders are well recorded by mathematicians (see e.g. [58]) and

$$|O(k,k;\mathbb{F}_p)| = 2p^{k(k-1)}(p^k-1)\prod_{i=1}^{k-1}(p^{2i}-1)\,, \tag{2.34}$$

$$|GL(k;\mathbb{F}_p)| = (p^k-1)\prod_{i=1}^{k-1}(p^k-p^i)\,. \tag{2.35}$$

Plugging this into our formula above tells us that[11]

$$\{\#\text{ Boundary Conditions}\} = 2\prod_{i=1}^{k-1}(p^i+1) = (-1;p)_k\,. \tag{2.36}$$

To wrap up these last two examples, we note that by the folding trick, we can go back and forth between our topological interfaces between two $\mathbb{Z}_p$ gauge theories and the irreducible boundary conditions for a $\mathbb{Z}_p^2$ gauge theory. Moreover, this explains why stacking with a 2d SPT phase and orbifolding are both given by a cup product.

For example, if we consider an interface between two non-anomalous $\mathbb{Z}_3$ theories, then by folding it must be a boundary condition for a $\mathbb{Z}_3 \times \mathbb{Z}_3$ gauge theory. We can enumerate boundary conditions for the folded theory, because they are labelled by (H, ν_2) data, and find that we get 8 agreeing with our previous discussions:

1. $H = \{0\}$. In this case $H^2(H, U(1)) = 0$, and there is only one embedding of H into $\mathbb{Z}_3^2$. Hence there is only one boundary condition of this type. In the unfolded setup this corresponds to the interface consisting of purely Dirichlet boundary conditions on both sides of the interface.
2. $H = \mathbb{Z}_3$. In this case there are four distinct embeddings of H into $\mathbb{Z}_3^2$, but $H^2(H, U(1)) = 0$ still. It can embed as $(\alpha, 0)$, $(0, \alpha)$, (α, α), or $(\alpha, 2\alpha)$. The first two boundary conditions unfold to a choice of Neumann boundary conditions on one side, and Dirichlet boundary conditions on the other. The second pair correspond to interfaces between two bulk $\mathbb{Z}_3$ gauge theories with the same connection, possibly up to some automorphism of $\mathbb{Z}_3$.

[11] Here $(a;q)_k$ denotes the q-Pochhammer Symbol.

3. $H = \mathbb{Z}_3^2$. In this case, there is only one choice of embedding: (α, β); but $H^2(H, U(1)) = \mathbb{Z}_3$. Hence we have 3 choices of topological boundary condition, two of which correspond to stacking with some non-trivial SPT phase, which is given by the cup product of the connections in the product theory. Of course, when we unfold, we have two theories with connections α and β on their respective sides of the interface, but possibly coupled by a cup product across the interface.

Again we have $2(p+1)$ boundary conditions here in the $\mathbb{Z}_p^2$ gauge theory, but only listed $2(p-1)$ interfaces in the previous $\mathbb{Z}_p$ gauge theory example. This is because the folding process produces some interfaces which are not invertible. In particular we notice that the $(0,0)$, $(\alpha,0)$, $(0,\alpha)$, and (α,β) (with no torsion), describe boundary conditions which are "completely separable." That is to say, the fields on one side don't couple to the fields on the other and the bulk slabs can be moved away from one another.

Example: Gauging $\mathbb{Z}_2$ in a Non-Anomalous $\mathbb{Z}_4$

Consider a 2d theory T with non-anomalous $G = \mathbb{Z}_4$ symmetry, and suppose we want to gauge the $H = \mathbb{Z}_2$ subgroup. $\mathbb{Z}_4$ is a non-trivial central-extension of $K = \mathbb{Z}_2^u$ by H

$$0 \to \mathbb{Z}_2 \xrightarrow{\iota} \mathbb{Z}_4 \xrightarrow{p} \mathbb{Z}_2^u \to 0\,. \tag{2.37}$$

with $\iota\alpha = 2\alpha$ and $pa = a \bmod 2$. We write u (for "ungauged") to help distinguish the $\mathbb{Z}_2$s.

In general, central extensions of K by H are characterized by cohomology classes $\kappa \in H^2(K, H)$, the trivial class corresponds to the "direct product extension" $H \times K$. In our case, we have $H^2(\mathbb{Z}_2^u, \mathbb{Z}_2) \cong \mathbb{Z}_2$, so $\mathbb{Z}_4 = \mathbb{Z}_2 \rtimes_\kappa \mathbb{Z}_2^u$ with non-trivial κ.

Gauging H leaves us with a theory with $G' = K \times \hat{H}$ symmetry, the K corresponding to the remaining ungauged symmetry, and the $\hat{H}$ corresponding to the new quantum symmetry from the gauged H.

As explained in Appendix B of [59], and very explicitly in [25],[12] the gauging of the $\mathbb{Z}_2$ subgroup of $\mathbb{Z}_4$ turns the non-triviality of κ into an anomaly in the resulting theory. A beautifully explicit and physical way to see the anomaly $\mu_3 \in H^3(G', U(1))$ from κ is illustrated in Section 2.2 of [25].

In our case, a representative for the class corresponding to our group extension is $\kappa(\alpha^u, \beta^u) = \alpha^u \beta^u$. After gauging, the anomaly $\mu_3 \in H^3(G', U(1))$ is given by

$$\mu_3((\alpha^u, \hat{\alpha}), (\beta^u, \hat{\beta}), (\gamma^u, \hat{\gamma})) = (-1)^{\hat{\gamma}\alpha^u\beta^u}\,. \tag{2.38}$$

[12] See also [60, 61].

This corresponds to the "purely mixed anomaly" in $H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$. Mixed, in that it is only non-vanishing on the "diagonal" $\mathbb{Z}_2$ subgroup of G'. Pure in that it is not a gauge-gravity anomaly.[13]

We want to know what interface implements the gauging $\mathbb{Z}_4 \mapsto \mathbb{Z}_2^u \times \hat{\mathbb{Z}}_2$. We will obtain it in two distinct ways to illustrate the power of the folding trick, and to verify it against our 2d intuition.

First, the easy way: Consider the folded theory with gauge group $\mathbb{Z}_4 \times \mathbb{Z}_2^u \times \hat{\mathbb{Z}}_2$, the topological action is given by the lift of $\mu_3 \in H^3(G', U(1))$ to $\tilde{\mu}_3 \in H^3(\mathbb{Z}_4 \times G', U(1))$ which is trivial on the $\mathbb{Z}_4$ factor. The subgroup labelling our interface must be $\mathbb{Z}_4^d \times \hat{\mathbb{Z}}_2$ embedding in $\mathbb{Z}_4 \times \mathbb{Z}_2^u \times \hat{\mathbb{Z}}_2$ through $\pi(a, \hat{\alpha}) = (a, a \bmod 2, \hat{\alpha})$. Notice that this subgroup reads in a physically meaningful way: the $\mathbb{Z}_2^u$ connection corresponds to the proper value of the $\mathbb{Z}_4$ connection that should pass through the orbifold interface, while the $\hat{\mathbb{Z}}_2$ connection is not dependent on the $\mathbb{Z}_4$ data, but will be coupled in some other way. Here, a Roman letter is used for the $\mathbb{Z}_4$ connections, while the $\mathbb{Z}_2$ connections are denoted by Greek letters with appropriate adornments to clarify which $\mathbb{Z}_2$ they represent.

On this subgroup, the topological action is given by the pullback

$$\pi^* \tilde{\mu}_3((a, \hat{\alpha}), (b, \hat{\beta}), (c, \hat{\gamma})) = (-1)^{\hat{\gamma}(a \bmod 2)(b \bmod 2)} . \tag{2.39}$$

Thus the orbifold interface will be specified by a 2-cochain ν_2 on $\mathbb{Z}_4^d \times \hat{\mathbb{Z}}_2$ satisfying

$$\delta \nu_2 = \pi^* \tilde{\mu}_3 . \tag{2.40}$$

It's not hard to find such a ν_2. If instead we were looking for a ν_2' such that $\delta \nu_2' = 1$, then the obvious choice would be the generator for $H^2(\mathbb{Z}_4^d \times \hat{\mathbb{Z}}_2, U(1)) = \mathbb{Z}_2$ given by $\nu_2'((a, \hat{\alpha}), (b, \hat{\beta})) = (-1)^{(a \bmod 2)\hat{\beta}}$. If we want to be able to produce the anomalous phase factors we can see that

$$\nu_2((a, \hat{\alpha}), (b, \hat{\beta})) = \omega_4^{(a \bmod 2)\hat{\beta}} , \tag{2.41}$$

will do. That is

$$\begin{aligned} \delta \nu_2((a, \hat{\alpha}), (b, \hat{\beta}), (c, \hat{\gamma})) &= \frac{\nu_2((b, \hat{\beta}), (c, \hat{\gamma}))\, \nu_2((a, \hat{\alpha}), (b + c, \hat{\alpha} + \hat{\gamma}))}{\nu_2((a + b, \hat{\alpha} + \hat{\beta}), (c, \hat{\gamma}))\, \nu_2((a, \hat{\alpha}), (b, \hat{\beta}))} && (2.42) \\ &= \omega_4^{\hat{\gamma}((a \bmod 2) + (b \bmod 2) - (a + b \bmod 2))} && (2.43) \\ &= \pi^* \tilde{\mu}_3 . && (2.44) \end{aligned}$$

[13] In [62], the author presents a basis of 3-cocycles classes for $H^3(\mathbb{Z}_n^k, U(1)) = \mathbb{Z}_n^{\binom{k}{1} + \binom{k}{2} + \binom{k}{3}}$, which is also used in the literature (e.g. [63]). Our anomaly corresponds to what these authors would call $\omega_{\text{II}}^{(12)}$. One can check that, $(-1)^{\hat{\gamma} \alpha^u \beta^u} = \omega_4^{\hat{\gamma}(\alpha^u + \beta^u - [\alpha^u + \beta^u])}$.

We now have our finished product, the orbifold interface must be

$$I_{\nu_2}[a; a \bmod 2, \hat{\alpha}] = 2\omega_4^{\int (a \bmod 2)\cup\hat{\alpha}} \,. \tag{2.45}$$

We can now compare this to the answer we would produce if we orbifolded by summing over the connections for the subgroup.

Naively, to get the partition function for the gauged theory, we want to sum over the $H \xrightarrow{\iota} G$ subgroup. The twisted partition function for the gauged theory must be

$$Z_{[T/\mathbb{Z}_2]}[\alpha^u, \hat{\alpha}] = \frac{1}{2} \sum_{a\in H^1(H,M)} \omega_4^{\int 2a\cup\hat{\alpha}} Z_T[2a+\alpha^u] \,, \tag{2.46}$$

$$= \frac{1}{4} \sum_{a\in H^1(G,M)} Z[a] \left(2 \sum_{x\in H^1(H,M)} \omega_4^{\int 2x\cup\hat{\alpha}} \delta_{a,2x+\alpha^u} \right) . \tag{2.47}$$

The interface interpolating from a $\mathbb{Z}_4$ to a $\mathbb{Z}_2^u \times \hat{\mathbb{Z}}_2$ theory is simply

$$I[a; \alpha^u, \hat{\alpha}] = 2 \sum_{x\in H^1(H,M)} \omega_4^{\int 2x\cup\hat{\alpha}} \delta_{a,2x+\alpha^u} \,. \tag{2.48}$$

Happily, when α^u is precisely $a \bmod 2$, then this interface is just the one we found before

$$I[a; a \bmod 2, \hat{\alpha}] = 2\omega_4^{\int (a \bmod 2)\cup\hat{\alpha}} \,. \tag{2.49}$$

We can depict the orbifold groupoid from our $\mathbb{Z}_4$ theory as in Figure 2.7.

As we can see, we have a more general result than we set out for. This interface gives us the ability to gauge any cyclic subgroup of any non-anomalous cyclic group (when 2 is replaced by $|H|$ and $\bmod 2$ is replaced by $\bmod |K|$). Moreover, since every Abelian group can be written as a product of cyclic groups, by taking appropriate products of the interface above and delta-functions we can gauge any non-anomalous Abelian subgroup of any Abelian group.

We verify that these interfaces reduce appropriately when we choose different subgroups H of G. For example, when $H = \{0\}$ then $K = G$ and we have

$$I[a; \alpha^u, \hat{\alpha}] = |G| \omega_{|G|}^{\int \alpha^u\cup\hat{\alpha}} = |G| \,, \tag{2.50}$$

which, appropriately, does nothing when we insert it. And similarly, when $H = G$, then

$$I[a; \alpha^u, \hat{\alpha}] = \omega_{|G|}^{\int a\cup\hat{\alpha}} \,. \tag{2.51}$$

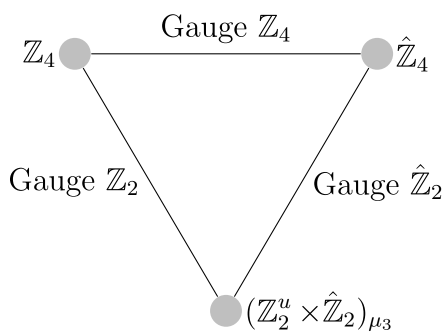


Figure 2.7: Webs of successive gaugings for a $\mathbb{Z}_4$ symmetry form the appropriate orbifold groupoid above. Gauging the $\mathbb{Z}_4$ symmetry of a $\mathbb{Z}_4$ theory produces a theory with a $\hat{\mathbb{Z}}_4$ symmetry. Gauging the $\mathbb{Z}_2$ subgroup of either produces an anomalous theory.

The orbifold interface must be invertible. It's not hard to verify the inverse interface to I is

$$J[\alpha^u, \hat{\alpha}; a] = |K| \omega_{|G|}^{\int \hat{\alpha} \cup (a - a \bmod |K|)} \delta_{\alpha^u, a \bmod |K|} \, , \tag{2.52}$$

up to a local curvature counterterm. Which reduces on the equivalent $\mathbb{Z}_4^d \times \hat{\mathbb{Z}}_2$ subspace to the cup

$$J[a \bmod |K|, \hat{\alpha}; a] = |K| \omega_{|G|}^{\int \hat{\alpha} \cup (a - a \bmod |K|)} \, . \tag{2.53}$$

In Figure 2.7 we can get to the $\mathbb{Z}_2^u \times \hat{\mathbb{Z}}_2$ node in two different ways. Either by gauging the $\mathbb{Z}_2 \leq \mathbb{Z}_4$, or by gauging the $\mathbb{Z}_4$ to $\hat{\mathbb{Z}}_4$ and then the $\hat{\mathbb{Z}}_2$ subgroup. The resulting partition functions are not the same. This makes sense because there are "two theories living at a $\mathbb{Z}_4$ node" and four at the $\mathbb{Z}_2^2$ node. The interface that interpolates between these two theories is obtained by commuting around the diagram

$$K[\alpha^u, \hat{\beta}; \hat{\gamma}^u, \delta] = \sum_{b, \hat{c}} J[\alpha^u, \hat{\beta}; b] I_{\mathbb{Z}_4}[b; \hat{c}] I[\hat{c}; \hat{\gamma}^u, \delta] \propto \delta_{\hat{\beta}, \hat{\gamma}^u} \delta_{\alpha^u, \delta} \omega_4^{\int \delta \cup \hat{\gamma}^u} \, , \tag{2.54}$$

where the proportionality constant is, again, a local curvature counterterm on $g \neq 1$. We see the corresponding 2-cochain, $\nu_{2,K} = \omega_4^{\alpha^u \hat{\beta}}$, satisfies $\delta \nu_{2,K} = \omega_4^{\hat{\gamma}(\alpha^u + \beta^u - [\alpha^u + \beta^u])}$, which is the pullback of the anomaly $\tilde{\mu}_3$ that we expect.

We see from this analysis that the $\mathbb{Z}_4$ DW theory has (unitary) symmetry group $\mathbb{Z}_2^2$ (see Table 2. of [64] for a different approach to this result), corresponding to "exchanging the 1 and 3" in $\mathbb{Z}_4$ and gauging, i.e.

$$\mathrm{Hom}(\mathrm{DW}[\mathbb{Z}_4], \mathrm{DW}[\mathbb{Z}_4]) = \mathbb{Z}_2^2 \, . \tag{2.55}$$

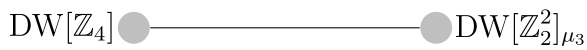


Figure 2.8: The different presentations of the $\mathbb{Z}_4$ Dijkgraaf-Witten theory (aka twisted $\mathbb{Z}_2^2$ Dijkgraaf-Witten theory) give the duality groupoid for $\mathbb{Z}_4$ (in contradistinction to the orbifold groupoid). The theories $\mathrm{DW}[\mathbb{Z}_4]$ and $\mathrm{DW}[\mathbb{Z}_2^2]_{\mu_3}$ (denoted by vertices) are dual. There is a $\mathbb{Z}_2^2$ of isomorphisms/dualities between them (collapsed into a single edge). The groups of automorphisms/symmetries of these theories are both $\mathbb{Z}_2^2$ (and edges from a vertex to itself have been suppressed).

Moreover, the existence of invertible interface(s) between $\mathrm{DW}[\mathbb{Z}_4]$ and $\mathrm{DW}[\mathbb{Z}_2^2]_{\mu_3}$ theories, tells us that

$$\mathrm{Hom}(\mathrm{DW}[\mathbb{Z}_4], \mathrm{DW}[\mathbb{Z}_2^2]_{\mu_3}) = \mathbb{Z}_2^2\,, \tag{2.56}$$

and so there are a $\mathbb{Z}_2^2$ of symmetries for the $\mathrm{DW}[\mathbb{Z}_2^2]_{\mu_3}$ theory

$$\mathrm{Hom}(\mathrm{DW}[\mathbb{Z}_2^2]_{\mu_3}, \mathrm{DW}[\mathbb{Z}_2^2]_{\mu_3}) = \mathbb{Z}_2^2\ . \tag{2.57}$$

The symmetries of the $\mathrm{DW}[\mathbb{Z}_2^2]_{\mu_3}$ theory are generated by the automorphism of $\mathbb{Z}_2^2$ that interchanges the two off-diagonal $\mathbb{Z}_2$s, and the $\mathbb{Z}_2$ of freedom in the toplogical action. We can draw the duality groupoid for the bulk theories as in Figure 2.8.

2.3 Fermionic Orbifolds and Spin-Symmetries of 3d Gauge Theories

In this section we focus on fermionic QFTs. We will assume unitarity, so that the Grassmann parity of a local operator is tied to its spin. That means we are working with QFTs which can include local operators of half-integral spin.

At first sight, that requires one to work with manifolds which are equipped with a spin structure. The correct statement is a bit more nuanced. Every fermionic theory has a "Grassmann parity" symmetry $\mathbb{Z}_2^f$ usually denoted as $(-1)^F$. This symmetry must commute with other symmetries, but the full symmetry group G_f acting on local operators may be a central extension of the form

$$0 \to \mathbb{Z}_2^f \to G_f \to G \to 0\,. \tag{2.58}$$

Unitarity requires the QFT to couple to "spin-G_f" connections, i.e. connections whose curvature equals the image of the second Stiefel-Whitney class w_2 in G_f. When $G_f =$

$\mathbb{Z}_2^f \times G$, that is the same as a choice of a spin structure η and of a G connection α. The details of the extension affect strongly the possible anomalies and SPT phases for the system. We will refer to such QFTs as spin-QFTs.

Another crucial point is that the world of spin-QFTs includes several interesting invertible theories: besides $U(1)$ phases in $d = 0$ one has Grassmann-odd one-dimensional vector spaces in $d = 1$ and the Majorana chain/Arf-invariant theory in $d = 2$, which assigns partition function $(-1)^{\mathrm{Arf}[\eta]}$ to a manifold depending on whether the spin structure η is even or odd [65, 66]. For some recent applications of the Arf-invariant see [49, 50, 67, 2, 68, 69, 70].

As a consequence, there is a rich collection of possible 't Hooft anomalies and discrete torsion for a 2d spin theory T_f with symmetry group G_f. When $G_f = \mathbb{Z}_2^f \times G$, they are classified by the "supercohomology" classes $sH^3(G)$ and $sH^2(G)$ respectively. A supercohomology 3-cocycle α consists of three pieces of data: a "Majorana layer" α_1, a "Gu-Wen layer" α_2, and a regular bosonic 't Hooft anomaly α_3.

The most dramatic 't Hooft anomaly a 2d spin theory can have occurs when some symmetry elements fail to map the theory T back to itself, but instead maps it to $T \times \mathrm{Arf}$. Such an anomaly is characterized by a homomorphism $G \to \mathbb{Z}_2$ describing which elements of G have this problem. This is the Majorana layer of the 't Hooft anomaly, and is specified by a $\mathbb{Z}_2$-valued 1-cocycle, α_1.

If the Majorana layer is trivial(ized), the next potential anomaly tells us that the group G may be extended by a $\mathbb{Z}_2$ generator which acts as ± 1 on states on a circle, depending on the circle's spin structure being even or odd. This is the Gu-Wen layer of the 't Hooft anomaly, specified by a $\mathbb{Z}_2$-valued 2-cocycle, α_2. If the Majorana layer is non-trivial, the layer is specified by a $\mathbb{Z}_2$-valued 2-cochain, α_2, such that

$$\delta\alpha_2 = \mathrm{Sq}^2\alpha_1 \,, \tag{2.59}$$

where Sq^2 denotes the Steenrod square. In fact, in such a low dimensions, we can always shift α_1 by a coboundary so that Sq^2 vanishes, so α_2 can be assumed to be a *cocycle*.

If the Gu-Wen layer is trivial(ized), then we may still be left with a standard phase anomaly $\alpha_3 \in H^3(G, U(1))$. If the Majorana layer is trivial and the Gu-Wen layer is non-trivial, we have

$$\delta\alpha_3 = (-1)^{\mathrm{Sq}^2\alpha_2} \,. \tag{2.60}$$

A simple way to think about the supercohomology class α encoding the 2d 't Hooft anomaly is that it defines an invertible 3d topological action which depends both on a G flat connection and a spin structure (or a "spin-G_f" flat connection if G_f is not split).

Similarly, the discrete torsion classes in $sH^2(G)$ can be thought of as invertible topological 2d actions.

The notion of "orbifold" should also be refined a bit. If we have a factorization $G_f = \mathbb{Z}_2^f \times G$, or at least $G_f = G'_f \times G$, we can gauge any non-anomalous subgroup H of G by coupling to a dynamical H connection. The trivialization $\hat{\nu}_2$ of the pull-back of $\hat{\mu}_3$ to H is still "super," so the available choices for such a "fermionic orbifold" are still different from those available in the bosonic setup. We can even apply a fermionic orbifold operation to a bosonic theory to produce a new fermionic theory.

If we want to gauge a more general subgroup H_f of G_f, though, we will have to employ dynamical spin-H_f connections. Effectively, we will be "gauging fermionic parity," or GSO-projecting the theory. We should call such an operation a "GSO orbifold." The resulting new theory may be bosonic or fermionic, depending on the type of topological 2d action we employ.

These subtleties carry over to the 3d setups we employ to study orbifolds. When we study fermionic orbifolds for $G_f = \mathbb{Z}_2^f \times G$, we may choose to employ a simple generalization of 3d DW theory $\mathrm{fDW}[G]_{\hat{\mu}_3}$ of 3d DW theory which employs the supercohomology class $\hat{\mu}_3$ as a topological action for a G flat connection and depends on a choice of spin structure in 3d. Such a choice will keep the whole setup fermionic, i.e. the Grassmann parity symmetry $\mathbb{Z}_2^f$ will act everywhere while G only acts at the Dirichlet boundary.

This is an intuitive setup, but it requires one to modify the standard MTC tools to allow for 3d spin-TFTs. The mathematical machinery to do so is a bit under-developed.

An alternative choice, which is necessary anyway to discuss GSO orbifolds or general G_f, is to push all symmetries, including $\mathbb{Z}_2^f$, all the way to the topological boundary. This can be done by employing a 3d theory of dynamical spin-G_f connections with action $\hat{\mu}_3$. We can denote that as $\mathrm{sDW}[G_f]_{\hat{\mu}_3}$. Crucially, this is a standard bosonic 3d TFT, described by some standard MTC.[14] It simply has the property that some of the Wilson lines will have topological spin -1 instead of 1, depending on the action of $\mathbb{Z}_2^f$ on the corresponding G_f irrep.

The Dirichlet boundary condition for $\mathrm{sDW}[G_f]_{\hat{\mu}_3}$ will be "fermionic," requiring one to specify the boundary value of the dynamical spin-G_f connection.

Fermionic topological boundary conditions for bosonic 3d TFTs are rather interesting objects. The simplest example occurs already in the toric code, aka topological $\mathbb{Z}_2$ gauge

[14]It would be interesting to identify such MTC precisely in terms of the data of $\hat{\mu}_3$. This is an instance of the more general problem of reconstructing an MTC from the data of a fermionic boundary condition, inverting the construction of [71]. Presumably this requires some variant of the Drinfeld center construction. We leave details for a future investigation.

theory. There are two non-trivial anyons e, m of topological spin $+1$ and one anyon f of topological spin -1. There are two irreducible bosonic boundary conditions B_e, B_m where either e or m can end (see Section 2.2.6), but there is also a fermionic boundary condition B_f where f can end. Secretly, the toric code is isomorphic to a topological $\mathbb{Z}_2^f$ gauge theory, such that f is the Wilson line and B_f the Dirichlet boundary.[15] We will come back to this momentarily.

Once we have translated our 2d theory T_f to a bosonic boundary condition $B[T_f]$ for a bosonic $\mathrm{sDW}[G_f]_{\hat{\mu}_3}$ gauge theory equipped with a fermionic topological Dirichlet boundary condition, we can study all types of orbifolds by varying the choice of topological boundary condition. Depending on the latter being bosonic or fermionic, the output of the orbifolds will be a bosonic or a fermionic 2d theory as well.

Our first step, then, should be to enlarge our duality groupoid. We should include both DW theories and sDW theories, as well as any isomorphisms between them as bosonic TFTs.

The simplest connected component of such a groupoid will be relevant for 2d theories which only have $\mathbb{Z}_2^f$ symmetry. The corresponding node is a 3d spin-$\mathbb{Z}_2^f$ gauge theory. This is isomorphic to the toric code. After identifying the Wilson line with the f anyon, we have two ways to identify the disorder defects with e and m, so we have a non-trivial isomorphism between the spin-$\mathbb{Z}_2^f$ gauge theory and itself. At the level of 2d theories, this is the operation of tensoring a theory with the Arf theory.

We also have two non-trivial isomorphisms to standard $\mathbb{Z}_2$ gauge theory. These map to the two possible GSO projections of a fermionic theory with $\mathbb{Z}_2^f$ symmetry to a bosonic theory with non-anomalous $\mathbb{Z}_2$ gauge symmetry. In this way, the GSO projection is a way to produce a boundary condition for a $\mathbb{Z}_2$ gauge theory from a boundary condition of the $\mathbb{Z}_2^f$ gauge theory, and the Jordan-Wigner transform is the inverse process. With all of this preceding discussion in mind, we could enhance our picture of duality groupoids as in Figure 2.9.

When $G_f = \mathbb{Z}_2^f \times G$, we should still be able to identify which of these isomorphisms correspond to fermionic orbifolds. We claim that they are these which preserve the "canonical fermion", i.e. the Wilson line labelled by the trivial representation of G and non-trivial for $\mathbb{Z}_2^f$. We will comment on this briefly in Section 2.3.2.

[15] A spin-TFT necessarily requires a choice of spin structure, so that even the trivial spin-TFT has a dependence on spin-structure. The $\mathbb{Z}_2^f$ gauge theory is the "pure spin structure gauge theory" which can be constructed by summing over spin structures in the trivial spin-TFT. In the language of [72] the $\mathbb{Z}_2^f$ gauge theory is the "shadow" of the trivial spin-TFT. We could recover the trivial spin-TFT from the $\mathbb{Z}_2^f$ gauge theory by "condensing" the f line.

Figure 2.9: The duality groupoid for the $\mathbb{Z}_2$ and $\mathbb{Z}_2^f$ gauge theories can be enhanced as above. There is a $\mathbb{Z}_2$ of symmetries for the $\mathbb{Z}_2$ gauge theory, and similarly for the $\mathbb{Z}_2^f$ theory. There are non-trivial isomorphisms between the two, generated at the level of 2d theories by the GSO/JW transformations.

2.3.1 Fermionic Examples

In the following, we upgrade our previous examples to illustrate the potentially different phenomena in orbifolds of fermionic theories. This is most interesting when there is a $\mathbb{Z}_2$ subgroup of G. Such a subgroup allows a non-trivial mixing of the $\mathbb{Z}_2$ flat connections and spin-structures, since spin-structures are "affine $\mathbb{Z}_2$ connections."

First we will review the case that T_f has just $(-1)^F$ symmetry. After that, we return to $\mathbb{Z}_p \times \mathbb{Z}_p$ but focus on the new phenomena that occurs when $p = 2$. Lastly, we complete the fermionization of our $\mathbb{Z}_4$ study from earlier, and understand it explicitly in the example of a compact boson CFT.

Fermionic Example: Theories with $\mathbb{Z}_2^f$ Symmetry

Consider a 2d spin theory T_f with only $G_f = (-1)^F$ symmetry. It is now well-known that the invertible topological phases that can be stacked with such a theory are classified by $\mathrm{Hom}(\Omega_2^{\mathrm{Spin}}(pt), U(1)) = \mathbb{Z}_2$ [66]. Furthermore, we know that the effective action for the non-trivial element in this cobordism group is given by a low energy continuum version of the Majorana-Kitaev chain

$$e^{iS[\eta]} = (-1)^{\mathrm{Arf}[\eta]}\,. \tag{2.61}$$

In Appendix A.2 we review Arf algebraically and relate it to the quadratic refinement and the mod 2 index of the Dirac operator. Another equivalent (and possibly more familiar) way to think about the theory, is as the 2d analog of the Chern-Simons term obtained when integrating out a fermion in 3d [73, 49, 74], say as

$$(-1)^{\mathrm{Arf}[\eta]} = \frac{Z_{\mathrm{Maj.}}(m \gg 0, \eta)}{Z_{\mathrm{Maj.}}(m \ll 0, \eta)}\,. \tag{2.62}$$

Some authors may say that the non-trivial Arf phase corresponds to some particular choice of $m > 0$ or $m < 0$ for the fermion. This is true in a specific renormalization scheme. It can be safer to discuss relative phases if the choice is not clear.

In the language of [75] (see also [76]), a massive Majorana fermion with $m > 0$ is isomorphic to the massive Majorana fermion with $m < 0$ as "anomalous field theories." But they are not isomorphic as "absolute field theories" (theories with well-defined partition functions and Hilbert-spaces). The obstruction to their isomorphism as absolute QFTs is given precisely by the Arf theory. That is to say, $Z_{\text{Maj.}}[-m,\eta] = (-1)^{\text{Arf}[\eta]} Z_{\text{Maj.}}[m,\eta]$.

If we were now to construct an orbifold groupoid for $(-1)^F$, we would simply have a single vertex, and inside that vertex would live two absolute theories: our T_f and $T_f \otimes \text{Arf}$. This is analogous to the bosonic case where T and $T \otimes \text{SPT}$ lived at the same vertex.

We can be more sophisticated in our discussion of orbifold groupoids and ask about gauging $(-1)^F$/GSO projection, which is obtained by summing over spin-structures [77]. In this case, each fermionic theory (the one vertex in our case) above has 2 bosonic neighbours, corresponding to summing over spin structures with or without the Arf theory stacked on top (relative to one another). These two bosonic neighbours are themselves connected by a $\mathbb{Z}_2$ orbifold. This enlarges our $\mathbb{Z}_2$ orbifold groupoid as in Figure 2.10.

Here we are assuming that the gravitational anomaly of T_f, $c_L - c_R$ in a CFT, is divisible by 8, which is necessary for the bosonic theory $[T_f/\mathbb{Z}_2^f]$ to exist as an absolute 2d theory. The gravitational anomaly of a fermionic QFT only needs to be a multiple of $\frac{1}{2}$. Looking at the example of n chiral fermions, i.e. an $SO(n)_1$ WZW model, we see that the 3d TFT which appears naturally when we "separate" $\mathbb{Z}_2^f$ from the dynamical degrees of freedom is the $Spin(n)_1$ Chern-Simons theory. This theory is bosonic, has a canonical topological fermionic boundary condition and a bosonic gapless boundary condition supporting the $Spin(n)_1$ WZW model. It is a variant of spin-$\mathbb{Z}_2^f$ gauge theory, with a different collection of topological boundary conditions. For example, $Spin(8)_1$ has three fermionic anyons and three topological fermionic boundary conditions related by a triality symmetry. In that case, GSO projections produce another fermionic theory and the orbifold groupoid has three $\mathbb{Z}_2^f$ nodes. See also [78].

To undo the process of summing over spin-structures, i.e. to re-fermionize, we can couple our 2d $\mathbb{Z}_2$ connection to a spin-structure, performing a generalized Jordan-Wigner transformation.

At the level of partition functions we can write stacking with Arf as

$$S_F : Z_{T_f}[\eta] \mapsto (-1)^{\text{Arf}[\eta]} Z_{T_f}[\eta] \,. \tag{2.63}$$

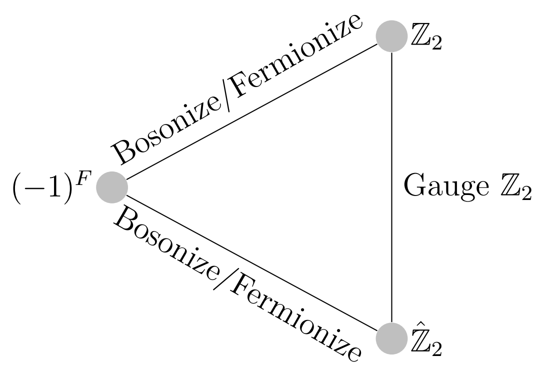


Figure 2.10: Gauging the $(-1)^F$ symmetry of a spin theory T_f produces a bosonic theory with a $\mathbb{Z}_2$ symmetry. A different bosonic theory can be produced if one first stacks with the invertible Arf theory. These two phases are related by $\mathbb{Z}_2$ orbifold. Stacking with Arf maps the $(-1)^F$ node to itself.

Similarly, we have

$$\mathcal{O}_{\text{GSO}} : Z_{T_f}[\eta] \mapsto Z_{[T_f/A]}[\alpha] \equiv \frac{1}{2}\sum_{\eta} \sigma_\eta(\alpha) Z_{T_f}[\eta]\,, \tag{2.64}$$

The inverse "Jordan-Wigner transformation" is simply

$$\mathcal{O}_{\text{JW}} : Z_{[T_f/A]}[\alpha] \mapsto Z_{T_f}[\eta] \equiv \frac{1}{2}\sum_{\alpha} \sigma_\eta(\alpha) Z_{[T_f/A]}[\alpha]\ . \tag{2.65}$$

Here $\sigma_\eta(\alpha) = (-1)^{\text{Arf}[\alpha+\eta]+\text{Arf}[\alpha]}$ is the usual quadratic form coupling $\mathbb{Z}_2$ gauge fields to spin-structures (see Appendix A.2).

The corresponding invertible interfaces are simply

$$I_{S_F}[\eta;\rho] = 2\delta_{\eta\rho}(-1)^{\text{Arf}[\eta]}\,, \tag{2.66}$$
$$I_{\text{GSO}}[\eta;\alpha] = \sigma_\eta(\alpha)\,, \tag{2.67}$$
$$I_{\text{JW}}[\alpha;\eta] = \sigma_\eta(\alpha)\,. \tag{2.68}$$

We can also revisit the folding trick once more. Suppose we are interested in interfaces between sDW$[\mathbb{Z}_2^f]$ and DW$[\mathbb{Z}_2]$. We already know there should be two of them corresponding to the two possible GSO projections at the level of 2d theories.

The folding trick tells us that studying such interfaces should be the same as studying boundary conditions for $\mathrm{sDW}[\mathbb{Z}_2^f \times \mathbb{Z}_2]$. In this case, the boundary conditions are labelled by a subgroup H_f of the finite supergroup $\mathbb{Z}_2^f \times \mathbb{Z}_2$ and an element of $sH^2(H_b)$.

From our previous experiences, we know that the H_f in question should take the data of the spin-structure on one side of an interface to the data of a connection on the other side of an interface, so we should definitely have $H_f = \mathbb{Z}_2^f \times \mathbb{Z}_2$. The relevant bosonic quotient is simply $H_b = H_f / \mathbb{Z}_2^f \cong \mathbb{Z}_2$, and because H_f is a split product of $\mathbb{Z}_2^f$ and G_b we have that

$$\begin{aligned} sH^2(H_b) &= H^2(H_b, U(1)) \times H^1(H_b, \mathbb{Z}_2) \times \mathbb{Z}_2 &(2.69)\\ &= \mathbb{Z}_2 \times \mathbb{Z}_2 \ . &(2.70)\end{aligned}$$

Where the three factors can be interpreted from left to right as giving bosonic discrete torsion factors, σ_η factors, and factors of Arf respectively [79]. Note, if the group does not split, the product is more complicated.

Now we have 4 potential boundary conditions which we can call: (η, α), $(\eta, \alpha)\sigma_\eta(\alpha)$, $(\eta, \alpha)\,\mathrm{Arf}$, and $(\eta, \alpha)\sigma_\eta(\alpha)\,\mathrm{Arf}$, labelling what the connections look like for such a boundary condition, and the associated terms in $sH^2(\mathbb{Z}_2)$. It is clear that the two boundary conditions which are not separable as interfaces are the two which actually couple the $\mathbb{Z}_2$ connection α to η in some way, in particular, the ones which include $\sigma_\eta(\alpha)$ terms.

Thus we conclude that there are two invertible interfaces from $\mathrm{sDW}[\mathbb{Z}_2^f]$ to $\mathrm{DW}[\mathbb{Z}_2]$, and they are given by

$$\begin{aligned} I_{\mathrm{GSO}_1}[\eta; \alpha] &= \sigma_\eta(\alpha)\,, &(2.71)\\ I_{\mathrm{GSO}_2}[\eta; \alpha] &= \sigma_\eta(\alpha)(-1)^{\mathrm{Arf}[\eta]}\,. &(2.72)\end{aligned}$$

Fermionic Example: Theories with $\mathbb{Z}_2 \times \mathbb{Z}_2^f$ Symmetry

In the case $G_f = \mathbb{Z}_2 \times (-1)^F$, there are a number of operations we can perform on such a theory: we can shift the spin-structure by our $\mathbb{Z}_2$ gauge field, orbifold the bosonic $\mathbb{Z}_2$, and stack with the Arf theory. Of course, we can also perform a GSO projection and continue with all the manipulations we encountered with our original $\mathbb{Z}_2 \times \mathbb{Z}_2$ theory.

In order to simplify things, we will only consider the bosonic operations, those that map our fermionic theory to a fermionic theory. Then, using the fact that each fermionic theory has two bosonic neighbours, we can construct the full orbifold groupoid. Such bosonic operations (shown for the torus) are generated by shifting the spin-structure by the $\mathbb{Z}_2$ gauge field

$$\pi_F : Z_T[\alpha_a, \alpha_b, \eta_a, \eta_b] \mapsto Z_T[\alpha_a, \alpha_b, \eta_a + \alpha_a, \eta_b + \alpha_b]\,, \qquad (2.73)$$

Fermionic	Bosonic
π_F	S_1
S_F	$\mathcal{O}_2$
$\mathcal{O}_1$	$\mathcal{O}_1$

Table 2.1: We find that shifting the spin-structure by a $\mathbb{Z}_2$ flat connection has the effect of adding a bosonic SPT phase in the bosonized 2d theory. We see again that stacking with Arf and bosonizing produces theories related by gauging.

stacking with the Arf theory

$$S_F : Z_T[\alpha_a, \alpha_b, \eta_a, \eta_b] \mapsto (-1)^{\eta_a \eta_b} Z_T[\alpha_a, \alpha_b, \eta_a, \eta_b]\,, \tag{2.74}$$

and gauging the bosonic $\mathbb{Z}_2$

$$\mathcal{O}_1 : Z_T[\alpha_a, \alpha_b, \eta_a, \eta_b] \mapsto \frac{1}{2} \sum_{\gamma} \omega_p^{\gamma_a \alpha_b - \gamma_b \alpha_a} Z[\gamma_a, \gamma_b, \eta_a, \eta_b]\,. \tag{2.75}$$

These operations correspond to the interfaces

$$I_{\pi_F}[\gamma, \eta; \alpha, \rho] = 2^2 \delta_{\gamma,\alpha} \delta_{\eta,\rho+\alpha}\,, \tag{2.76}$$

$$I_{S_F}[\gamma, \eta; \alpha, \rho] = 2^2 \delta_{\gamma,\alpha} \delta_{\eta,\rho} (-1)^{\mathrm{Arf}[\eta]}\,, \tag{2.77}$$

$$I_{\mathcal{O}_1}[\gamma, \eta; \alpha, \rho] = 2 \delta_{\eta,\rho} (-1)^{\int \gamma \cup \alpha}\,. \tag{2.78}$$

The group of interfaces here forms a group of 72 elements, in particular, $O(2, 2; \mathbb{F}_2)$. This is exactly what we would expect from the duality groupoid picture.

We can also ask what these operations bosonize to, similar to how the Arf interface became the Kramers-Wannier interface. That is, if one performs one of these operations on a theory, and then bosonizes, what effect does it have compared to just bosonizing? It's not hard to compute, and this is recorded in Table 2.1.

We can now create an orbifold groupoid of fermionic theories. Since our bosonic theory had 9 lines corresponding to gauging the "second $\mathbb{Z}_2$" (recall Figure 2.6), we expect this purely fermionic orbifold graph to have 9 nodes (one for each fermionization of a bosonic pair as in Figure 2.10). This makes sense, the manipulations acting on a node form a subgroup

$$\langle \pi_F, S_F \rangle \cong D_8\,, \tag{2.79}$$

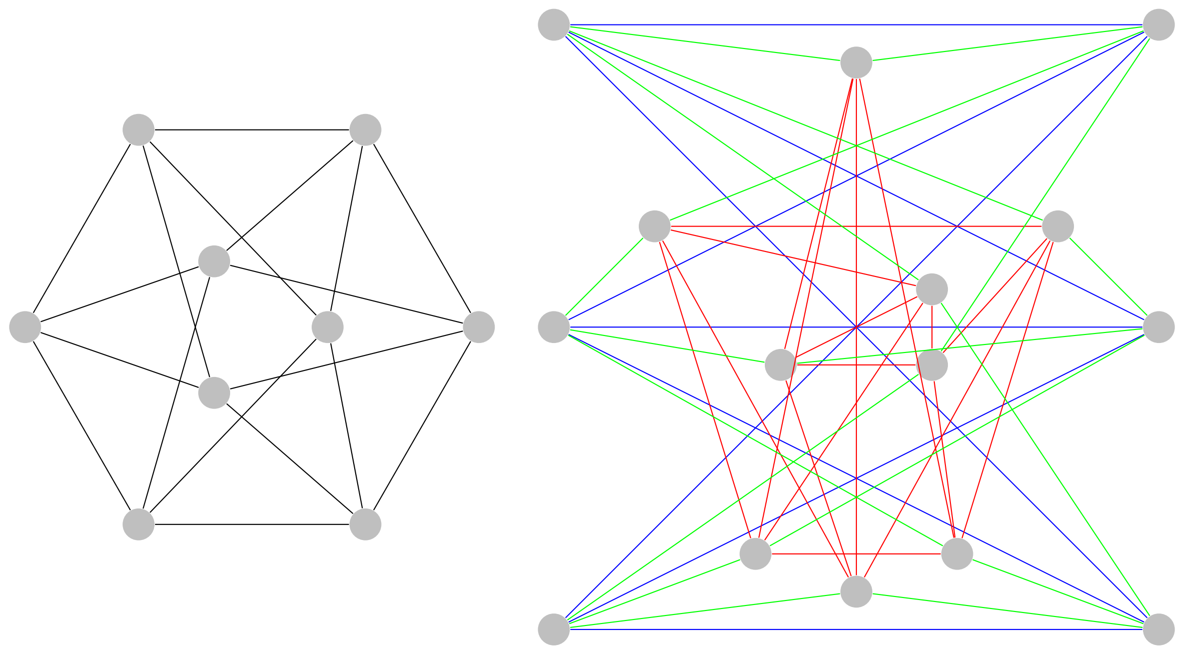


Figure 2.11: On the left, the orbifold groupoid for the bosonic topological manipulations for a theory with $\mathbb{Z}_2 \times \mathbb{Z}_2^f$ symmetry. Lines connect two theories related by gauging the bosonic $\mathbb{Z}_2$. On the right, we superimpose this graph with the results from the gauging in $\mathbb{Z}_2 \times \mathbb{Z}_2$ theories to produce the entire orbifold groupoid. The bosonic gauging is marked in blue and red depending on if it originates from a bosonic or fermionic theory respectively, GSO projections are marked in green.

of the total bosonic operations

$$\langle \pi_F, S_F, \mathcal{O}_1 \rangle \cong O(2,2;\mathbb{F}_2)\,, \tag{2.80}$$

and we see $|O(2,2;\mathbb{F}_2)|/|D_8| = 9$. We can draw this orbifold groupoid as before, producing the left diagram in Figure 2.11.

We can also combine the fermionic-fermionic orbifolds with the bosonic-bosonic orbifolds by including lines denoting GSO projections, producing the right diagram in Figure 2.11. This is investigated from a VOA perspective in [57].[16]

[16] As commented in the reference, "when there is no bosonic theory in sight," i.e. no way to distinguish vertices, the graph attains its most symmetric description where "vertices correspond to Lagrangian 2-planes inside symplectic $\mathbb{F}_2^4$." We will address this example a little more in Section 2.3.2.

Fermionic Example: Theories with $\mathbb{Z}_4$ and $\mathbb{Z}_4^f$ Symmetry

To complete our story from the bosonic Section 2.2.6, we will fermionize the $\mathbb{Z}_4$ and anomalous $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold groupoid which we encountered before.

To make points very concrete, we will phrase everything in terms of the compact boson CFT, keeping in mind that statements about orbifolds are generic to any theory with that symmetry and anomaly. Our overview will closely follow the presentation in the paper [50]. We will not review all aspects of the compact boson CFT here, just the relevant points for our discussion.

Consider the compact boson CFT with radius R, so that $X(z, \bar{z}) \sim X(z, \bar{z}) + 2\pi R$. At generic R the chiral algebra is extended from Virasoro by the $\mathfrak{u}(1)$ current generated by ∂X and the local primaries are the vertex operators

$$V_{n,w}(R) = V_{p_L p_R} = e^{ip_L X_L(z) + ip_R X_R(\bar{z})} \tag{2.81}$$

with conformal weights $h_{n,w}(R) = \frac{\alpha'}{4} p_L^2$ and $\bar{h}_{n,w}(R) = \frac{\alpha'}{4} p_R^2$, where[17]

$$p_L = \frac{n}{R} + \frac{wR}{\alpha'}\,, \quad p_R = \frac{n}{R} - \frac{wR}{\alpha'}\,. \tag{2.82}$$

Here $n, w \in \mathbb{Z}$ and are interpreted as the number quantizing momentum and winding respectively. Note that the conformal spin is $s = nw$. The partition function for this theory $S^1[R]$ is simply

$$Z(\tau) = \frac{1}{|\eta(\tau)|^2} \sum_{\substack{n \in \mathbb{Z} \\ w \in \mathbb{Z}}} q^{h_{n,w}} \bar{q}^{\bar{h}_{n,w}}\,. \tag{2.83}$$

At generic radius, the compact boson has $(U(1)_n \times U(1)_w) \rtimes \mathbb{Z}_2^C$ global symmetry, which act on the boson by

$$\begin{aligned}
\mathbb{Z}_2^C :\ & X_L(z) \mapsto -X_L(z)\,, && X_R(\bar{z}) \mapsto -X_R(\bar{z}) \\
U(1)_n :\ & X_L(z) \mapsto X_L(z) + \frac{R}{2}\theta_n\,, && X_R(\bar{z}) \mapsto X_R(\bar{z}) + \frac{R}{2}\theta_n \\
U(1)_w :\ & X_L(z) \mapsto X_L(z) + \frac{1}{2R}\theta_w\,, && X_R(\bar{z}) \mapsto X_R(\bar{z}) - \frac{1}{2R}\theta_w\,.
\end{aligned} \tag{2.84}$$

[17] The factor of α' will be left in for easy comparison to other results.

where we take $\theta_{n,w} \sim \theta_{n,w} + 2\pi$. In terms of the primaries, this says that

$$\begin{aligned} \mathbb{Z}_2^C &: V_{n,w} \mapsto V_{-n,-w} \\ U(1)_n &: V_{n,w} \mapsto e^{in\theta_n} V_{n,w} \\ U(1)_w &: V_{n,w} \mapsto e^{iw\theta_w} V_{n,w}\,. \end{aligned} \tag{2.85}$$

The $\mathbb{Z}_2^C$ symmetry is interesting and is well discussed in a number of papers, for example [49, 50, 80] as well as most classic references on CFT. Orbifolding by the $\mathbb{Z}_2^C$ symmetry produces some form of "Ashkin-Teller model," with two local Virasoro primaries σ_1 and σ_2 both with conformal weights $(\frac{1}{16}, \frac{1}{16})$. This model can be viewed as two copies of the Ising CFT deformed by a marginal operator coupling their energy densities $\varepsilon_1(z,\bar{z})\varepsilon_2(z,\bar{z})$.

We are more interested in the two $\mathbb{Z}_2$ subgroups of the $U(1)_n$ and $U(1)_w$, denoted $\mathbb{Z}_2^n$ and $\mathbb{Z}_2^w$ respectively. The $\mathbb{Z}_2^n$ symmetry shifts the compact boson half the circumference of the circle. Intuitively, orbifolding by this $\mathbb{Z}_2^n$ symmetry means shifting by half the circumference of the circle is trivial, hence we see that resultant theory is just the compact boson on a circle of radius $R/2$. The conclusion is inverted for the winding orbifold. Altogether, we have

$$[S^1[R]/\,\mathbb{Z}_2^n] = S^1[R/2]\,, \tag{2.86}$$

$$[S^1[R]/\,\mathbb{Z}_2^w] = S^1[2R]\,. \tag{2.87}$$

These two $\mathbb{Z}_2$'s may be gauged separately, but have a mixed anomaly precisely as we investigated in our earlier bosonic example of Section 2.2.6. This anomaly is manifest from our previous argument: in the $\mathbb{Z}_2^n$ twisted sector $X(z,\bar{z})$ is wound half a time so that the winding modes are shifted by a half-integer. In summary, the twisted sector operators for the $\mathbb{Z}_2^n$ subgroup have fractional winding and vice-versa

$$\mathbb{Z}_2^n \text{ twisted:} \quad n \in \mathbb{Z}\,, \qquad w \in \mathbb{Z} + \frac{1}{2}\,, \tag{2.88}$$

$$\mathbb{Z}_2^w \text{ twisted:} \quad n \in \mathbb{Z} + \frac{1}{2}\,, \quad w \in \mathbb{Z}\,. \tag{2.89}$$

In this case, the twisted partition function can be written

$$Z_{S^1[R]}[n_1, n_2; w_1, w_2] = \frac{1}{|\eta(q)|^2} \sum_{\substack{n \in \mathbb{Z} + w_1/2 \\ w \in \mathbb{Z} + n_1/2}} (-1)^{nn_2 + ww_2} q^{h_{n,w}} \bar{q}^{\bar{h}_{n,w}}\,, \tag{2.90}$$

which we can use to explicitly check all of our previous assertions.

From this presentation we can also very explicitly see how a $\mathbb{Z}_4$ symmetry appears. When we orbifold the $\mathbb{Z}_2^n$ symmetry (summing over all $n_i = 0, 1$ in the previous formula and setting $w_i = 0$) we compute

$$Z_{[S^1[R]/\mathbb{Z}_2^n]}[0,0] = \frac{1}{|\eta(q)|^2}\Big(\sum_{\substack{n\in\mathbb{Z}\\ w\in\mathbb{Z}}} \frac{1}{2}(1+(-1)^n)q^{h_{n,w}}\bar{q}^{\bar{h}_{n,w}} + \sum_{\substack{n\in\mathbb{Z}\\ w\in\mathbb{Z}+1/2}} \frac{1}{2}(1+(-1)^n)q^{h_{n,w}}\bar{q}^{\bar{h}_{n,w}}\Big) \tag{2.91}$$

$$= \frac{1}{|\eta(q)|^2}\sum_{\substack{n\in 2\,\mathbb{Z}\\ w\in\frac{1}{2}\,\mathbb{Z}}} q^{h_{n,w}}\bar{q}^{\bar{h}_{n,w}}\,. \tag{2.92}$$

It is clear how the orbifold projects out operators with $n \in 2\,\mathbb{Z}+1$, but adds operators of half-integer winding $w \in \mathbb{Z}+\frac{1}{2}$. This means that the "$\mathbb{Z}_2^w$ symmetry" is now a "$\mathbb{Z}_4^w$ symmetry" fitting into the group extension

$$1 \to \hat{\mathbb{Z}}_2^n \to \mathbb{Z}_4^w \to \mathbb{Z}_2^w \to 1\,, \tag{2.93}$$

because the term $(-1)^w$, can now act by ± 1 and $\pm i$. Repeating this analysis for $\mathbb{Z}_2^w$, we reproduce the bosonic orbifold groupoid in Figure 2.7.

We can also fermionize the $\mathbb{Z}_2^n$ symmetry by the usual generalized Jordan-Wigner transformation, which we will denote JW_n (because it fermionizes the $\mathbb{Z}_2^n$). Thus, we will define the theory

$$\mathrm{Dirac}_n[R] := \mathrm{JW}_n[S^1[R]]\,. \tag{2.94}$$

We use the name $\mathrm{Dirac}_n[R]$ because at $R = \sqrt{2\alpha'}$ the partition function is that of a free massless Dirac ($c = 1$) fermion $\Psi(z,\bar{z}) = \Psi_L(z) + \Psi_R(z)$. For other radii, it is the Dirac fermion deformed by the Thirring operator. We will defer points about conformal manifolds and deformations to the references.

As in [50], we identify the fermion operators $\Psi_{L,R}$ for the $\mathrm{Dirac}_n[R]$ theory with the primaries

$$\Psi_L(z) = V_{1,\frac{1}{2}}\,, \qquad \Psi_L^\dagger(z) = V_{-1,-\frac{1}{2}}\,, \tag{2.95}$$

$$\Psi_R(\bar{z}) = V_{1,-\frac{1}{2}}\,, \quad \Psi_R^\dagger(\bar{z}) = V_{-1,\frac{1}{2}}\,. \tag{2.96}$$

Bosonic Sector	Fermionic Sector	Range of n	Range of w	Primaries
$\mathcal{H}^+_{\text{Un.}}$	$\mathcal{H}^+_{\text{NS}}$	$2\,\mathbb{Z}$	$\mathbb{Z}$	$V_{2,0} = \Psi_L \Psi_R$
$\mathcal{H}^-_{\text{Un.}}$	$\mathcal{H}^+_{\text{R}}$	$2\,\mathbb{Z}+1$	$\mathbb{Z}$	$V_{1,0}$
$\mathcal{H}^+_{\text{Tw.}}$	$\mathcal{H}^-_{\text{R}}$	$2\,\mathbb{Z}$	$\mathbb{Z}+\frac{1}{2}$	$V_{0,\frac{1}{2}}$
$\mathcal{H}^-_{\text{Tw.}}$	$\mathcal{H}^-_{\text{NS}}$	$2\,\mathbb{Z}+1$	$\mathbb{Z}+\frac{1}{2}$	Ψ_L, Ψ_R

Table 2.2: Bosonic and fermionic Hilbert spaces and their operators for the $\mathbb{Z}_2^n$-associated theories, comparing bosonic and fermionic Hilbert spaces for the $S^1[R]$,$S^1[R]/\mathbb{Z}_2^n$ and $\text{Dirac}_n[R]$ theories, as well as some of their local primaries. Reproduced from Table 1. of [50].

From this we see that the "$\mathbb{Z}_2^w$ symmetry" is once again extended, this time to a $\mathbb{Z}_4^f$ on the fermions, that is

$$\Psi_L(z) \mapsto +i\Psi_L(z), \quad \Psi_L^\dagger(z) \mapsto -i\Psi_L^\dagger(z)\,, \tag{2.97}$$
$$\Psi_R(z) \mapsto -i\Psi_R(z), \quad \Psi_R^\dagger(z) \mapsto +i\Psi_R^\dagger(z)\,. \tag{2.98}$$

This is just the $\mathbb{Z}_4^f$ subgroup sitting in the $U(1)^f$ symmetry of the Dirac fermion

$$1 \to (-1)^F \to \mathbb{Z}_4^f \to \mathbb{Z}_2^w \to 1\,. \tag{2.99}$$

We can summarize our discussion as in Table 2.2.

We can write the $\mathbb{Z}_4^f$ twisted partition function for the $\text{Dirac}_n[R]$ theory as

$$Z_{\text{Dirac}_n[R]}[w_1, w_2] = \sum_{k\in\{0,1,2,3\}} \omega_4^{kw_2} \sum_{\substack{n\in 2\,\mathbb{Z}+w_1/2+k \bmod 2 \\ w\in 2\,\mathbb{Z}+k/2}} q^{h_{n,w}} \bar{q}^{\bar{h}_{n,w}}\,. \tag{2.100}$$

One can check explicitly that we have

$$Z_{S^1[R]}[N_1, N_2; W_1, W_2] = \frac{1}{2} \sum_w Z_{\text{Dirac}_n[R]}[w] I[w; N, W]\,. \tag{2.101}$$

where I is the interface taking us from $Z_{\text{Dirac}_n[R]}[w_1, w_2]$ to $Z_{S^1[R]}[N_1, N_2; W_1, W_2]$, and is given by

$$I[w; N, W] = \delta_{W,(w \bmod 2)} \sigma^{(4)}_{w-(w \bmod 2)}(N) \omega_4^{W_1 N_2}\,, \tag{2.102}$$

where we have written $\sigma_w^{(4)}(N) = \omega_4^{2N_1N_2+w_1N_2+w_2N_1}$.

We can produce an interface between the $\mathrm{Dirac}_n[R]$ theory and the $\mathbb{Z}_4^n$ theory by simply composing interfaces, the result is that

$$J[w;N] = \sigma_w^{(4)}(N \bmod 2)\omega_4^{\int w\cup(N-N \bmod 2)}\omega_4^{-(N_1 \bmod 2)(w_2 \bmod 2)}\,. \tag{2.103}$$

Lastly, we can compute the interface between the $Z_{\mathrm{Dirac}_n[R]}$ and $Z_{\mathrm{Dirac}_w[R]}$ partition functions

$$K[w;n] = \sigma_{w-w \bmod 2}(n \bmod 2)\sigma_{n-n \bmod 2}(w \bmod 2)\omega_4^{\int (w \bmod 2)\cup(n \bmod 2)}\,. \tag{2.104}$$

If we were to write, more suggestively, the connections w and n as combinations of $\mathbb{Z}_2$ connections $w = 2\xi + \alpha$ and $n = 2\rho + \beta$, then this interface looks like

$$\sigma_\rho(\alpha)\omega_4^{\int \alpha\cup\beta}\sigma_\xi(\beta)\,. \tag{2.105}$$

As before, we can mirror this entire discussion by swapping every statement about n and w to complete our orbifold groupoid as in Figure 2.12.

We could also phrase this in terms of Narain lattices and lattice VOAs to explicitly double check our assertions, and make contact with other presentations (e.g. lattice VOAs).

For any compact boson radius R, the spectrum of dimensionless momenta $(\ell_L, \ell_R) = \sqrt{\frac{\alpha'}{2}}(p_L, p_R)$ forms a lattice in $\mathbb{R}^2$. Single-valuedness of the OPE of two of our VOAs enforces that this ℓ lattice be integral with the diagonal inner-product of signature $(1,1)$, and modular invariance enforces that it is even and self-dual.

It is much more convenient to talk about the lattice of n and w, which is very simply $\mathbb{Z}^2$. Integrality becomes the statement that for any two (n,w) and (n',w') in the lattice

$$nw' + wn' \in \mathbb{Z}\,, \tag{2.106}$$

and the lattice being even means for any operator $V_{n,w}$ that

$$s = nw \in \mathbb{Z}\ . \tag{2.107}$$

If we want to orbifold by a non-anomalous symmetry G, we restrict this $\mathbb{Z}^2$ lattice to the appropriate invariant sub-lattice $\Lambda = (\mathbb{Z}^2)^G$ under that symmetry. Then we construct Λ^* and seek extensions of the invariant sub-lattice Λ into Λ^* that are even and self-dual. If we also want to consider fermionic theories, then we can drop the even condition (allowing $s = nw \in \frac{1}{2}\mathbb{Z}$).

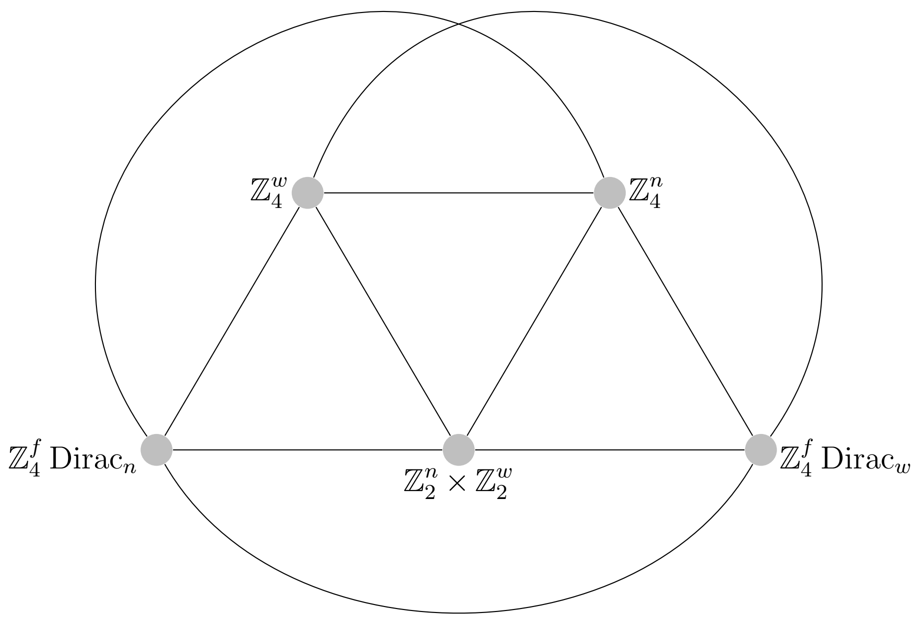


Figure 2.12: Orbifolding a non-anomalous $\mathbb{Z}_2$ subgroup of a theory with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry and mixed anomaly produces a theory with $\mathbb{Z}_4$ symmetry. We can also fermionize the non-anomalous $\mathbb{Z}_2$ symmetries to produce two theories with $\mathbb{Z}_4^f$ symmetry. By composing the intermediate interfaces, we can form the complete orbifold groupoid.

For example, to orbifold the $\mathbb{Z}_2^n$ symmetry of our compact boson, we restrict from the $\mathbb{Z}^2$ lattice to the invariant sub-lattice $\Lambda = \{n \in 2\,\mathbb{Z}, w \in \mathbb{Z}\}$, which corresponds to the shared subspace of local operators $\mathcal{H}^+_{\text{Un.}}$, and has dual lattice $\Lambda^* = \{n \in \mathbb{Z}, w \in \frac{1}{2}\,\mathbb{Z}\}$. We can extend the lattice Λ into Λ^* in three distinct ways

$$\begin{aligned} S^1[R]: &\quad \Lambda \oplus (\Lambda + (1,0))\,, && (2.108)\\ [S^1[R]/\,\mathbb{Z}_2^n]: &\quad \Lambda \oplus (\Lambda + (0,1/2))\,, && (2.109)\\ \text{Dirac}_n[R]: &\quad \Lambda \oplus (\Lambda + (1,1/2))\,. && (2.110) \end{aligned}$$

Clearly in the $S^1[R]$ case we are appending $\mathcal{H}^-_{\text{Un.}}$ to the list of local operators; in the $[S^1[R]/\,\mathbb{Z}_2^n]$ we are appending $\mathcal{H}^+_{\text{Tw.}}$; and in the fermionic case we are extending Λ to an odd self-dual lattice (by adding the spin-half operator $V_{1,\frac{1}{2}}$) which amounts to adding $\mathcal{H}^-_{\text{NS}}$. This is depicted in Figure 2.13.

2.3.2 Spin-Structure Preserving Interfaces

In the preceding bosonic and fermionic examples we computed a number of invertible interfaces in different 3d theories. Furthermore, in the bosonic examples, we saw how we could identify the 2d partition functions with anyons of the 3d bulk very explicitly. Anyons of the 3d gauge theory arise from the boundary theory partition functions (written in terms of $\mathbb{Z}_2$ connections) by Fourier transform.

Explicitly, in the case of a $\mathbb{Z}_2$-symmetric bosonic theory (on the torus), we identified the anyons in the toric code with the linear combinations

$$\hat{Z}_B[0,\hat{0}] = \frac{1}{2}(Z_B[0,0] + Z_B[0,1]) = Z_1 \tag{2.111}$$

$$\hat{Z}_B[0,\hat{1}] = \frac{1}{2}(Z_B[0,0] - Z_B[0,1]) = Z_e \tag{2.112}$$

$$\hat{Z}_B[1,\hat{0}] = \frac{1}{2}(Z_B[1,0] + Z_B[1,1]) = Z_m \tag{2.113}$$

$$\hat{Z}_B[1,\hat{1}] = \frac{1}{2}(Z_B[1,0] - Z_B[1,1]) = Z_f\,. \tag{2.114}$$

The JW/GSO process provides a way to turn states of the $\mathbb{Z}_2$ gauge theory into states

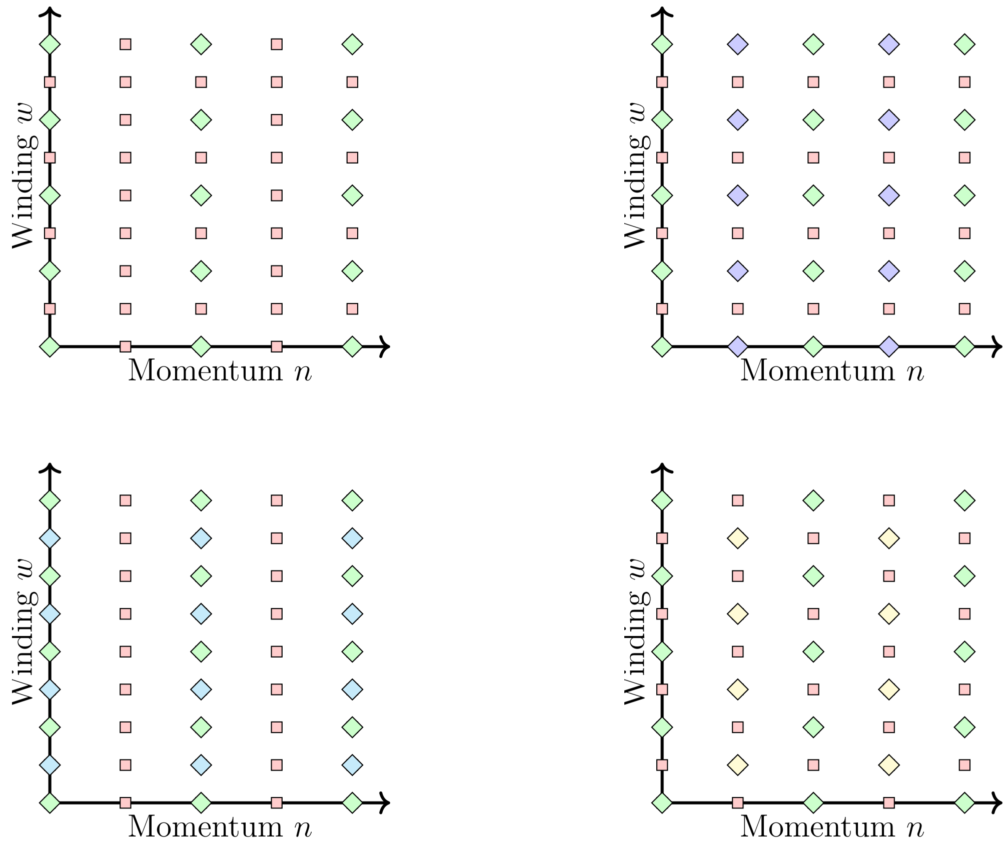


Figure 2.13: Green diamonds denote the invariant sublattice Λ under the $\mathbb{Z}_2^n$ symmetry, and the red squares denote the dual lattice Λ^* and have integral spacing along momentum and half-integral spacing along winding. We see that there are only three ways to extend Λ into Λ^*: $S^1[R]$ corresponds to the extension by the blue diamond, $S^1[R/2]$ by the cyan diamond, and $\mathrm{Dirac}_n[R]$ by the yellow diamond.

of the $\mathbb{Z}_2^f$ gauge theory. Thus we may compute

$$\hat{Z}_B[a_1, \hat{a}_2] = \frac{1}{2}\sum_{a_2}(-1)^{a_2\hat{a}_2} Z_B[a_1, a_2] \tag{2.115}$$

$$= \frac{1}{2^2}\sum_{a_2,\rho_1,\rho_2}(-1)^{a_2\hat{a}_2}\sigma_\rho(a)(-1)^{\lambda\,\mathrm{Arf}[\rho]} Z_F[\rho_1, \rho_2]$$

$$= \hat{Z}_F[a_1 + \hat{a}_2, (\lambda+1)a_1 + \lambda\hat{a}_2]\,. \tag{2.116}$$

Here $\lambda = 1$ (or 0) if we do (or do not) include Arf in our GSO projection. This tells us that

$$\hat{Z}_F[0, \hat{0}] = \frac{1}{2}(Z_F[0,0] + Z_F[0,1]) = Z_1 \tag{2.117}$$

$$\hat{Z}_F[1, \hat{1}] = \frac{1}{2}(Z_F[1,0] \mp Z_F[1,1]) = Z_{e,m} \tag{2.118}$$

$$\hat{Z}_F[1, \hat{0}] = \frac{1}{2}(Z_F[1,0] \pm Z_F[1,1]) = Z_{m,e} \tag{2.119}$$

$$\hat{Z}_F[0, \hat{1}] = \frac{1}{2}(Z_F[0,0] - Z_F[0,1]) = Z_f\,. \tag{2.120}$$

This perfectly matches what we'd expect, the Z_f line is $\hat{Z}_F[0,1]$, the Wilson line of the $\mathbb{Z}_2^f$ gauge theory. We also see there is a choice in identifying the electric and magnetic lines with the charged or uncharged fermion vortex/Ramond line, and that this factor is controlled by our choice of adding Arf into GSO projection. Moreover, we see that when such $Z_{e,m}$ lines pass through the Arf interface of the $\mathbb{Z}_2^f$ gauge theory, that their roles are interchanged.

A natural question to ask is which interfaces in the 3d theory do not change the coupling to spin-structure, i.e. do not change the spin-structure of the 2d theory upon collision. Physically, such interfaces in the 3d theory must fix the fermionic Wilson line $\hat{Z}_F[0,\hat{1}]$.

This problem is trivial in the case of a $\mathbb{Z}_2^f$ theory. We can see from Section 2.3.1 that the identity interface and Arf interface are the only two.

In the case of $\mathbb{Z}_2^f \times \mathbb{Z}_2$ symmetry, we learned in Section 2.3.1 that all of the (bosonic) topological manipulations were generated by the interfaces π_F, S_F, and $\mathcal{O}_1$. Once again, it's not hard to see explicitly (or brute-force check) that the operations which do not change the coupling to spin-structure are $\langle S_F, \mathcal{O}_1, \pi_F S_F \pi_F\rangle \cong D_{12}$.

In general, to find the group of interfaces preserving coupling to spin-structure, we are simply asking what is the stabilizer of Z_f (possibly with other restrictions we may wish to impose).

Spin-Symmetries

Presenting the group of spin-structure preserving interfaces, or at least finding the generators, is not particularly different from the $\mathbb{Z}_2^f \times \mathbb{Z}_2$ example. Especially when the group splits as $G_f = \mathbb{Z}_2^f \times G$. Morally speaking, the group will be generated by "all the operations manipulating G" (analogous to $\mathcal{O}_1$), "all the fermionic SPT-like operations" (analogous to S_F), and all the "bosonic automorphisms" and those which act on the fermionic SPT operations (analogous to $\pi_F S_F \pi_F$).

While it is easy to present the generators of the group, it's less simple to determine which group exactly is generated. Although, in individual cases, the problem is easily checked by computer.

A partial solution is offered in the slightly broader case where we consider the interfaces corresponding to "spin-symmetries." Spin-symmetries are effectively those that treat $\mathrm{DW}[\mathbb{Z}_2 \times G]$ and $\mathrm{sDW}[\mathbb{Z}_2^f \times G]$ on equal footing. That is to say, they are the symmetries of the MTC that map anyons to anyons preserving the braiding, but only preserving the square of the topological spin $\chi_\cdot(\cdot)^2$ (although this is already implied in preserving the braiding).

As an example, in the toric code this would mean that interfaces which interchanged an f line with an e or m line would be included, as opposed to just the usual (non-trivial) interface swapping e and m. We see that overall there should be 6 such interfaces, because there is an S_3 of valid ways to permute the lines $\{1, e, m, f\}$ of the toric code while preserving the braiding.

If we were to consider the duality groupoid in Figure 2.9, we would say it collapses down to a single point with an S_3 of spin-symmetries acting on the point.

This also shows us where the S_3 comes from two-dimensionally and group-theoretically. Recall that the two nodes in the figure are connected by a collection of lines (collapsed down to one line) corresponding to interfaces which implement a GSO projection (or JW transformation) in the language of 2d theories. Meanwhile, the two (suppressed) lines from a node to itself correspond to the two symmetries of $\mathrm{DW}[\mathbb{Z}_2]$ and $\mathrm{sDW}[\mathbb{Z}_2^f]$. These are generated by the identity interface, and the interface which swaps Z_e and Z_m, however it may be presented in either of the respective realizations.

So, two-dimensionally, we see that the group of spin-symmetries acting on a theory must be isomorphic to the group $\langle S_F, \mathrm{GSO}\rangle \cong S_3$ in the case of the toric code. In terms of 2d topological manipulations, one can check that the group of spin-symmetries is $Sp(4; \mathbb{F}_2)$ when $G_f = \mathbb{Z}_2^f \times \mathbb{Z}_2$, for example. Of course, we are just deriving, in 2d language, a result which is obvious in 3d. Namely that the group of spin-symmetries for, $G_f = \mathbb{Z}_2^f \times \mathbb{Z}_2^{k-1}$ say, is $Sp(2k;\mathbb{F}_2)$.

To summarize everything so far, for a 2d bosonic theory with $\mathbb{Z}_2^k$ symmetry, the (irreducible bosonic) topological operations form the group

$$T_B := O(k,k;\mathbb{F}_2)\,. \tag{2.121}$$

This includes automorphisms of $\mathbb{Z}_2^k$, stacking with SPT phases, as well as orbifolds. In terms of the duality groupoid $\mathrm{Hom}(\mathrm{DW}[\mathbb{Z}_2^k],\mathrm{DW}[\mathbb{Z}_2^k]) = O(k,k;\mathbb{F}_2)$. The group of operations leaving a phase unchanged is

$$T_{B,0} := H^2(G,U(1)) \rtimes \mathrm{Aut}(G) = \mathbb{Z}_2^{\binom{k}{2}} \rtimes GL(k;\mathbb{F}_2)\,. \tag{2.122}$$

The number of nodes in the bosonic orbifold groupoid for $\mathbb{Z}_2^k$ symmetry is 2, 6, 30, 270, 4590, ...[18]

Similarly, when we study a 2d spin theory with $G_f = \mathbb{Z}_2^f \times \mathbb{Z}_2^{k-1}$ symmetry, the bosonic topological manipulations of the fermionic theory form the group

$$T_F := O(k,k;\mathbb{F}_2) \cong T_B\,. \tag{2.123}$$

This includes automorphisms of $\mathbb{Z}_2^{k-1}$, shifting the spin-structure by $\mathbb{Z}_2$ gauge fields, bosonic orbifolds, and stacking with any fermionic SPT phases and Arf. In terms of the duality groupoid $\mathrm{Hom}(\mathrm{sDW}[\mathbb{Z}_2^f \times \mathbb{Z}_2^{k-1}],\mathrm{sDW}[\mathbb{Z}_2^f \times \mathbb{Z}_2^{k-1}]) = O(k,k;\mathbb{F}_2)$. The analogous group of operations to $T_{B,0}$ which act on a vertex is

$$T_{F,0} := (\mathbb{Z}_2^{1+(k-1)+\binom{k-1}{2}}) \rtimes (\mathbb{Z}_2^{k-1} \rtimes GL(k-1;2))\,, \tag{2.124}$$

which has a nice physical interpretation as the group of fermionic invertible phases[19] semidirect product with the group formed by shifting the spin-structure by the $k-1$ independent $\mathbb{Z}_2$ gauge fields in $\mathbb{Z}_2^{k-1}$, with an additional action of the automorphism group $GL(k-1;2)$.

[18] This is OEIS sequence `A028361` "Number of totally isotropic spaces of index n in orthogonal geometry of dimension $2n$."

[19] Note $\Omega_2^{\mathrm{Spin}}(B(\mathbb{Z}_2^k)) = \mathbb{Z}_2^{1+k+\binom{k}{2}}$ [81]. We can give each of the factors a nice physical story, 1 factor corresponds to the Arf theory, the k factor corresponds to the fSPTs that are not also bosonic SPTs which are generated by factors of σ_η in the partition function, and the $\binom{k}{2}$ comes from the fSPTs which are just bosonic SPTs. We can also "derive" this by treating the k $\mathbb{Z}_2$ gauge fields α_i and 1 spin-structure σ as being $k+1$ independent spin-structures $\sigma_0 := \sigma$ and $\sigma_i := \sigma + \alpha_i$, then we have $k+1$ independent Arf-like factors, and still the $\binom{k}{2}$ phases for the $\mathbb{Z}_2$ gauge fields viewed as differences of these spin-structures.

If we include spin-symmetries, then T_F enlarges to T_{Spin}, which is simply the collection of things preserving braidings in our gauge theory

$$T_{Spin} = Sp(2k; \mathbb{F}_2)\,. \tag{2.125}$$

Lastly, we can return to the problem of interfaces which preserve coupling to spin-structure. If we ask which interfaces from T_{Spin} do so, then we are asking what the collection of operations is that fixes a line in symplectic $\mathbb{F}_2^{2k}$ (when $G_f = \mathbb{Z}_2^f \times \mathbb{Z}_2^{k-1}$). Such a stabilizer subgroup forms a maximal parabolic subgroup of $Sp(2k; \mathbb{F}_2)$. i.e. we want to know $\mathrm{Stab}_{T_{Spin}}(Z_f)$. Finding such stabilizer subgroups is well understood for groups of Lie type (see for example the lecture notes [82]). In particular

$$\mathrm{Stab}_{T_{Spin}}(Z_f) = (\mathbb{Z}_2 \rtimes \mathbb{Z}_2^{2k-2}) \rtimes Sp(2k-2; \mathbb{F}_2)\,. \tag{2.126}$$

In the construction provided in the reference, the first $\mathbb{Z}_2$ factor corresponds exactly to stacking with the Arf interface when mapped onto our problem. However, the construction does not immediately make the interpretation of the other factors physically clear.

We conclude by mentioning that, numerically, it seems that the subgroup of T_F preserving coupling to spin-structure for a $G_f = \mathbb{Z}_2^f \times \mathbb{Z}_2^{k-1}$ is $\mathbb{Z}_2 \times Sp(2k-2; \mathbb{F}_2)$. Physically, this makes sense because for some fixed even spin-structure η the list of operations which do not change the spin-structure would be the full group of spin-symmetries $Sp(2k-2; \mathbb{F}_2)$, because here Arf acts trivially. Then for the odd spin-structures we have a non-trivial action by Arf and collect an extra $\mathbb{Z}_2$ factor. It would be nice to understand these points in more detail.

2.4 Generalized Symmetries and Applications in 2d QFTs

Some 2d QFTs, such as Rational Conformal Field Theories, are endowed with generalized symmetries, in the form of a fusion category $\mathcal{F}$ of topological line defects. Standard G symmetries with 't Hooft anomaly $\mu \in H^3(G, U(1))$ are a special case where the fusion category is group-like

$$\mathcal{F} = \mathrm{Vec}_G^{\mu} \tag{2.127}$$

with associator given by μ.

Generalized symmetries impose non-trivial constraints on RG flows. In particular, they may obstruct the existence of trivial massive RG flow endpoints and require an IR description involving multiple degenerate massive vacua (or gapless degrees of freedom), see [83] for a recent exposition, and [80] for a complementary discussion to the one here.

There is a neat trick to classify the possible massive endpoints of such RG flows: promote the 2d theory T with generalized symmetry $\mathcal{F}$ to a 2d boundary condition B for the Turaev-Viro 3d TFT described by the center $Z[\mathcal{F}]$ [84]. This is a generalization of the notion of coupling a 2d theory T to a 3d Dijkgraaf-Witten gauge theory with gauge group G and action μ.[20]

The map from boundary conditions for $Z[\mathcal{F}]$ to theories with symmetry $\mathcal{F}$ is straightforward: T is built from a segment compactification with boundary condition B at one end and the Turaev-Viro canonical boundary condition at the other end. The canonical boundary condition supports a fusion category $\mathcal{F}$ of boundary lines, which is inherited by T.

The inverse map is a bit less obvious, but still straightforward. For example, a space-like boundary condition may be described by its pairing to the states in the string-net description of the Turaev-Viro Hilbert space: a basis for the states is labelled by networks of $\mathcal{F}$ lines, and the pairing is given by the partition function of T in the presence of such a network of $\mathcal{F}$ lines.

A bit more formally, if we start from the 2d theory T and an orientation-reversed topological boundary with boundary lines $\bar{\mathcal{F}}$, we can reproduce B by a process of "2d anyon condensation", condensing products of lined from $\mathcal{F}$ and $\bar{\mathcal{F}}$.[21]

The topological coupling of T to $Z[\mathcal{F}]$ does not affect the local dynamics, and thus the RG flow of T maps to an RG flow of B. The endpoint of the B RG flow will generically be a gapped boundary condition B' for the Turaev-Viro theory. Then the corresponding endpoint of the $\mathcal{F}$-preserving RG flow of T must be the 2d theory obtained from the pairing of B and B'.

We arrive at the following claims:

- The "gapped phases with generalized symmetry $\mathcal{F}$" are classified by gapped boundary conditions B' for $Z[\mathcal{F}]$.
- Each gapped phase is a direct sum of degenerate vacua, to be obtained from a segment compactification of $Z[\mathcal{F}]$ with endpoints B and B'

[20] This generalized bulk theory is sometimes referred to as a Levin-Wen model in the condensed matter literature [85] (see also [86]), where an explicit lattice realization of the bulk 3d TFT is constructed analogous to the presentation of the toric code as a lattice gauge theory.

[21] We expect such a strategy to work in any dimension, see Appendix A.4.

- The gapped phase has an emergent $\mathcal{F} \otimes_{Z[\mathcal{F}]} \mathcal{F}'$ generalized symmetry, where $\mathcal{F}'$ is the fusion category of B' boundary lines.

When $\mathcal{F} = \mathrm{Vec}_G^\mu$, gapped boundary conditions of the Dijkgraaf-Witten gauge theory are classified by pairs (H, ν) where H is a subgroup of G and ν trivializes the pullback of μ to H. These are the usual symmetry-breaking patterns of massive theories with G symmetry.

2.4.1 Special Example: Current-Current Deformations of WZW Models

Until now, save for the compact boson CFT example, we have focused broadly on general 2d QFTs. But it is hard not to comment on RCFTs, and in particular, the oldest and most venerable: the Wess-Zumino-Witten models. We will briefly recap some important points about WZW models and then move on to an example application of our claims.[22]

The G_k WZW models are 2d RCFTs who are famously equipped with a current algebra

$$J^a(z)J^b(w) \sim \frac{k\delta_{ab}}{(z-w)^2} + \sum_c i f_c^{ab} \frac{J^c(w)}{(z-w)}\,, \tag{2.128}$$

where the f_c^{ab} are the structure constants of $\mathfrak{g}$. The Laurent modes satisfy the commutation relations of the $\mathfrak{g}_k$ affine Lie algebra. All of this is the same for the antiholomorphic sector.

To specify the full CFT, as opposed to just a chiral half, we need to specify a consistent gluing of the chiral and anti-chiral sectors, or modular invariant. This data is provided by a "mass matrix" $\mathcal{M}_{ij}$ which specifies the multiplicity of the irreps of the form $V_i \otimes \bar{V}_j$ in the Hilbert space

$$\mathcal{H} = \bigoplus_{i,j} \mathcal{M}_{ij} V_i \otimes \bar{V}_j\,. \tag{2.129}$$

Modular invariance enforces that $\mathcal{M}$ commutes with the modular S and T matrices, and we further impose uniqueness of vacuum $\mathcal{M}_{00} = 1$.[23]

[22] After this article appeared as a preprint, but before it was sent for publication, the article [87] appeared with similar ideas to those contained in this subsection.

[23] We do this without loss of substance in our understanding because any CFT with $\mathcal{M}_{00} > 1$ is just a direct sum of theories.

For $\mathfrak{su}(2)_k$ the irreps/primaries V_j are labelled by spins $j = 0, 1/2, \ldots, k/2$, and are subject to fusion rule

$$V_j \otimes V_{j'} = V_{|j-j'|} \oplus V_{|j-j'|+1} \oplus \cdots \oplus V_m \tag{2.130}$$

where $m = \min\{j + j', k - (j + j')\}$.

Moreover, a complete classification of modular invariants for $\mathfrak{su}(2)_k$ was obtained and shown to follow an ADE classification based on the level k [88, 89, 90, 91]. For convenience we record the A and D type here in their, rarely found, component form

$$k = \text{Any} \qquad \mathcal{M}_{ij}^{A_{k+1}} = \delta_{ij} \tag{2.131}$$

$$k = 4\ell \qquad \mathcal{M}_{ij}^{D_{k/2+2}} = \delta_{ij}\delta_{i \bmod 1,0}\delta_{j \bmod 1,0} + \delta_{i+j,k}\delta_{i \bmod 1,0}\delta_{j \bmod 1,0} \tag{2.132}$$

$$k = 4\ell - 2 \qquad \mathcal{M}_{ij}^{D_{k/2+2}} = \delta_{ij}\delta_{i \bmod 1,0}\delta_{j \bmod 1,0} + \delta_{i+j,k}\delta_{i \bmod 1,1/2}\delta_{j \bmod 1,1/2}\,. \tag{2.133}$$

We note that the A-type or "diagonal" modular invariants are defined for all k, while the D-type modular invariants are defined only for k even. There are also the E_6, E_7, and E_8 modular invariants at levels $k = 10, 16, 28$ respectively.

Broadly, Verlinde operators are line defects which act on the conformal blocks of a theory with some current algebra. They are in one-to-one correspondence with primaries and satisfy well-understood fusion relations in general [92, 93]. For a diagonal RCFT where the chiral and antichiral sectors are paired identically, like an A-type $\mathfrak{su}(2)_k$ theory, a Verlinde line labelled by V_i commutes with the chiral algebra(s), and acts on a primary by

$$V_i \left|\phi_j\right\rangle = \frac{S_{ij}}{S_{0j}} \left|\phi_j\right\rangle\,. \tag{2.134}$$

See [83] for an extended discussion. We point out in advance that $V_{\frac{k}{2}}$ generates a $\mathbb{Z}_2$ center symmetry, and it's only non-anomalous if k is even.

Said in 3d language, the chiral algebra of the RCFT provides a MTC describing the anyons of an associated bulk 3d TFT. If we forget the braiding relations for the bulk anyons (or push the anyons to the boundary) then this forms the fusion category associated with the Verlinde lines. The authors of [12] call this fusion category $\mathrm{Rep}(SU(2)_k)$. The $\mathbb{Z}_2$ symmetry generated by $V_{\frac{k}{2}}$ forms a subcategory $\mathrm{Vec}_{\mathbb{Z}_2}$ if k is even, and $\mathrm{Vec}_{\mathbb{Z}_2}^{[1]}$ if k is odd.

As explained in [35] (see also [12]), there is a beautiful bijection between the modular invariants of $\mathfrak{su}(2)_k$ WZW models and indecomposable module categories (irreducible boundary conditions). In particular, this means we can obtain any $\mathfrak{su}(2)_k$ WZW modular invariant by (generalized) orbifold of the diagonal model (and vice-versa by composition of orbifolds). The D-type modular invariants are obtained by the straightforward orbifold of the non-anomalous $\mathbb{Z}_2$ symmetry generated by $V_{\frac{k}{2}}$. The E_6, E_7, and E_8 orbifolds require the full power of 2d anyon condensation.

All of this may be said more three-dimensionally, to the point of our story. It is well known that given a 2d RCFT with some chiral algebra $\mathcal{A}$, the space of conformal blocks of the 2d RCFT on Σ is the space of states that a 3d TFT assigns to Σ. Mathematically, we might capture the data of chiral symmetries by some VOA, in which case this statement is essentially that the representation category of the VOA is a MTC [94, 37].

The most famous example of this relationship is the one relevant to our purposes, which says that the canonical quantization of a G Chern-Simons theory at level k on some surface $\Sigma \times \mathbb{R}$ produces the space of conformal blocks of the WZW models with matching level and group. Moreover, if the surface Σ is "punctured" by Wilson lines, then from the 2d WZW point of view, these points are corresponding operator insertions [95].

The essential mathematical work of [37], and various subsequent pieces, captures all of the 2d statements about the relationship between 2d RCFT and 3d TFT by using the mathematical language of algebra objects in the MTC associated to $\mathcal{A}$. We will not review that here, but will highlight a physical consequence first pointed out in [96] and studied further in [97].

In particular, the authors of [96] show that the invertible topological interfaces in the 3d TFT associated to some chiral algebra are in one-to-one correspondence with the modular invariants of the 2d RCFT. Said in the reverse, a full RCFT (which includes a choice of modular invariant) is specified by a choice of topological interface in the bulk 3d theory TFT. For example, the identity interface corresponds to the diagonal modular invariant. Broadly speaking, the results all originate from variations on the folding trick, by noting that $\mathcal{T} \times \bar{\mathcal{T}}$ assigns a vector space $\mathcal{H}_\Sigma \otimes \mathcal{H}_\Sigma^*$ to a 2-manifold, and so unfolding gives statements about the full CFT and original chiral algebra.

The result also gives a neat interpretation to primaries of the full CFT. A primary is labelled by a pair of representations of the chiral algebra, hence it is labelled by two line operators in the MTC. We may write it as $\phi_{i,j}$ where i and j label lines in the bulk TFT. If we insert such a primary into the CFT, then in 3d terms we must have $\mathcal{H}_\Sigma$ punctured by the line labelled by i, and $\mathcal{H}_\Sigma^*$ punctured by the line labelled by j. Thus we obtain a bijection between primaries of the full CFT and local operators which interpolate from the Wilson line i to the Wilson line j on the interface. This is depicted in Figure 2.14.

Viewing modular invariants for 2d RCFTs as interfaces in 3d Chern-Simons, we should investigate the interfaces corresponding to an $\mathfrak{su}(2)_k$ theory. If we refer to the D-type

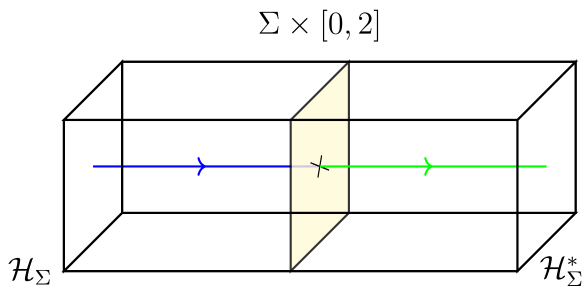


Figure 2.14: A full RCFT includes a choice of chiral algebras and modular invariant. The modular invariant of the RCFT can be understood as a choice of interface inbetween the chiral halves in the associated 3d TFT. The Hilbert space of primaries of the form $\phi_{i,j}$ are in bijection with operators on the interface which turn a Wilson line of type i into a Wilson line of type j.

modular invariants of $\mathfrak{su}(2)_k$ as D_o if $k = 4\ell - 2$, and D_e if $k = 4\ell$, we can obtain the algebra for the composition of the topological interfaces. The A-type invariant corresponds to the identity, and the D-type invariants behave as follows

$$(\mathcal{M}^{D_o})^2 = \mathcal{M}^A\,, \qquad (\mathcal{M}^{D_e})^2 = 2\mathcal{M}^{D_e}\,. \tag{2.135}$$

At $k = 10, 16, 28$ we also have the E_6, E_7, and E_8 type modular invariants respectively. These are subject to the commutative relations

$$\mathcal{M}^{E_6}\mathcal{M}^{D_o} = \mathcal{M}^{E_6}\,, \qquad (\mathcal{M}^{E_6})^2 = 2\mathcal{M}^{E_6}\,, \tag{2.136}$$

$$\mathcal{M}^{E_7}\mathcal{M}^{D_e} = 2\mathcal{M}^{E_7}\,, \qquad (\mathcal{M}^{E_7})^2 = \mathcal{M}^{E_7} + \mathcal{M}^{D_e}\,, \tag{2.137}$$

$$\mathcal{M}^{E_8}\mathcal{M}^{D_e} = 2\mathcal{M}^{E_8}\,, \qquad (\mathcal{M}^{E_8})^2 = 4\mathcal{M}^{E_8}\,. \tag{2.138}$$

Since the classification of $SU(3)$ modular invariants is now understood [98, 99], one could perform the same process for the 2-category of surface operators in the $SU(3)$ Chern-Simons theories.

Next we turn to current-current deformations of WZW, which arise by perturbing the WZW model by terms of the form $J(z)\bar{J}(\bar{z})$. We do not immediately require that the perturbation be isotropic in the Lie-algebra indices, that is to say any perturbation of the form

$$\sum_{a,b} c_{ab} J^a(z)\bar{J}^b(\bar{z}) \tag{2.139}$$

will suffice.

Such a term is obviously classically marginal, but it's subject to quantum mechanical corrections.[24] We are most interested in the case that the deformations are marginally relevant.

Before proceeding further, an obvious question is "how can we couple the two chiral halves in this 3d picture in a local way?" The answer to this is in the picture: when we couple the two halves, we quite literally couple them, gluing the segment into a circle. That is, we compactify the bulk Chern-Simons theory to $\Sigma \times S^1$.

In general, a 3d TFT on $\Sigma \times S^1$ defines an "effective" 2d TFT on Σ. In the functorial TFT language, this is a special form of "Kaluza-Klein reduction," where a 2d TFT is defined from a 3d TFT by $Z_{2\mathrm{d}}(\Sigma) = Z_{3\mathrm{d}}(\Sigma \times S^1)$ [102]. We recall that a 3d TFT assigns a Hilbert space $\mathcal{H}_\Sigma$ to Σ, and $\Sigma \times S^1$ is simply mapped to the number $\dim \mathcal{H}_\Sigma = \mathrm{Tr}_{\mathcal{H}_\Sigma}(\mathbb{1})$. Since 2d TFTs are largely characterized by their ground state degeneracy, then it would be instructional to compute this quantity, with the appropriate interfaces inserted of course.

After this compactification move and RG flow, the two joined ends must flow to some interface in the Chern-Simons theory. Since we have classified all interfaces in Chern-Simons, it must correspond to some modular invariant $\mathcal{M}^{\mathrm{IR}}$. If our original interface describing the full RCFT was called $\mathcal{M}^{\mathrm{UV}}$, then we are left with a circle-compactified Chern-Simons theory with two topological interface insertions, see Figure 2.15. Of course, away from the torus, the usual subtleties about local curvature counterterms still apply. These subtleties are nicely spelled out in the case of an Abelian Chern-Simons theory in [103] and also in [104].

Armed with our relations for the modular invariants of $\mathfrak{su}(2)_k$, we can obtain the ground state degeneracy on the torus. The number of ground states is simply the trace of the corresponding collision of our two interfaces, $\mathrm{Tr}\big(\mathcal{M}^{\mathrm{UV}}\mathcal{M}^{\mathrm{IR}}\big)$. We record these results in Table 2.3. Clearly, we have an example where the IR fixed point has multiple gapped vacua, not explained by spontaneous symmetry breaking considerations. Indeed, a spontaneous symmetry breaking may even result in an IR interface which is the direct sum of multiple irreducible interfaces, each contributing multiple vacua.

These topological considerations do not tell us, given some $J\bar{J}$ deformation of WZW, which $\mathcal{M}^{\mathrm{IR}}$ it flows to. The obvious guess is that the one which is "isotropic," i.e. of the form $\sum_a J^a(z)\bar{J}^a(\bar{z})$, flows to the diagonal modular invariant. It would be interesting to answer this question, which will depend on the specific choice of c_{ab} couplings. More precisely, there will be some phase diagram, with phases labelled by by possible $\mathcal{M}^{\mathrm{IR}}$.

[24] There exists some interesting literature (e.g. [100, 101]) studying the conditions for the theory to still be conformal after perturbations, and the properties of the resulting conformal manifolds.

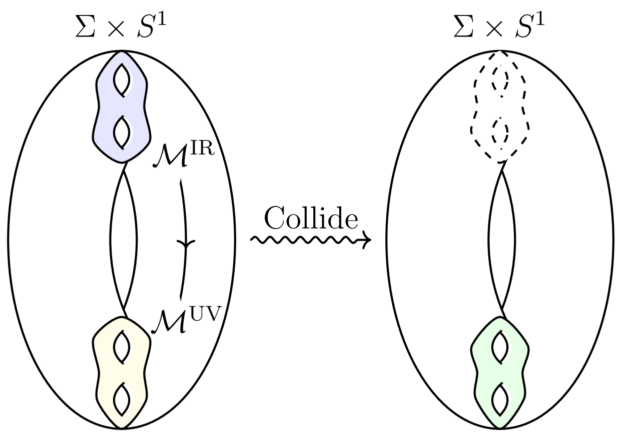


Figure 2.15: Deforming the WZW models by a relevant operator and flowing to the IR, we are left with two topological interfaces in a 3d Chern-Simons bulk on $\Sigma \times S^1$. If we let the two interfaces collide and then trace, we obtain the partition function of the relevant 2d TFT up to local curvature counterterms.

2.5 Conclusion and Open Questions

Our main conclusion is that *bosonic*, non-spin 3d TFTs naturally control the combinatorics of orbifolds and GSO projections of both bosonic or fermionic 2d QFTs. The possible results of these topological operations are labelled by topological boundary conditions for the 3d TFT, which may themselves be either bosonic or fermionic.

We have only considered in detail situations where the 3d bosonic TFT is isomorphic either to an Abelian Dijkgraaf-Witten (DW) theory, equipped with bosonic Dirichlet boundary conditions, or a spin-Dijkgraaf-Witten (sDW) theory, equipped with fermionic Dirichlet boundary conditions.

In general, topological bosonic boundary conditions in an abstract bosonic 3d TFT are described in terms of *Lagrangian algebras* in the corresponding MTC. See e.g. [105] and references within. These detail which bulk lines can end at the boundary, analogously to Wilson lines in a DW theory ending at a Dirichlet boundary.

Fermionic boundary conditions of a bosonic 3d TFT should admit a similar description

	$\mathcal{M}^A$	$\mathcal{M}^{D_o}$	$\mathcal{M}^{D_e}$	$\mathcal{M}^{E_6}$	$\mathcal{M}^{E_7}$	$\mathcal{M}^{E_8}$
$\mathcal{M}^A$	$k+1$	$\frac{k}{2}+2$	$\frac{k}{2}+2$	6	7	8
$\mathcal{M}^{D_o}$	$\frac{k}{2}+2$	$k+1$		6		
$\mathcal{M}^{D_e}$	$\frac{k}{2}+2$		$k+4$		14	16
$\mathcal{M}^{E_6}$	6	6		12		
$\mathcal{M}^{E_7}$	7		14		17	
$\mathcal{M}^{E_8}$	8		16			32

Table 2.3: Traces of products of modular invariants for the $\mathfrak{su}(2)_k$ WZW models. Interfaces in the $SU(2)_k$ Chern-Simons theory correspond to such modular invariants. Equivalently, they compute the ground state degeneracy of the effective 2d TFT on the torus when the IR and UV theories correspond to one of these interfaces.

in terms of some *Lagrangian super-algebras*. It would be nice to spell that out in detail.[25]

[25]While this work was in the final stages of preparation, it appears that such a description was indeed spelled out in detail [71].

Chapter 3

Two More Fermionic Minimal Models

In this short chapter, we comment on the existence of two more fermionic unitary minimal models not included in work by Hsieh, Nakayama, and Tachikawa [67]. These theories are obtained by fermionizing the $\mathbb{Z}_2$ symmetry of the $m = 11$ and $m = 12$ exceptional unitary minimal models. Furthermore, we explain why these are the only missing cases.

I would like to thank Davide Gaiotto for pointing out that the exceptional minimal models should be considered and for comments on the draft of the paper this section is based on.

3.1 Introduction

In a paper by Hsieh, Nakayama, and Tachikawa, it was shown that there is a fermionic unitary minimal model for each $c = 1 - 6/(m(m+1))$. In particular, it is obtained by fermionizing the $\mathbb{Z}_2$ symmetry in the A or D-type models, and can be thought of as the fermionic partner to those two bosonic theories [67].[1][2]

The goal of this note is two-fold. First, we point out that the $m = 11$ and $m = 12$ exceptional unitary minimal models, also called (A_{10}, E_6) and (E_6, A_{12}), have a non-anomalous

[1]Some of these fermionic minimal models were obtained earlier in [106], where the authors studied fermionic extensions of Virasoro minimal models.

[2]After this work was published, it was discovered that some of these theories have also appeared in [107, 108]. The author thanks V. Petkova for bringing this to his attention.

$\mathbb{Z}_2$ symmetry [109]. From general considerations about fermionization, we conclude that there are two additional unitary fermionic minimal models not included in the list of [67] and record their partition functions. Moreover, we use this note as an opportunity to illustrate some notation and simple ideas from Chapter 2.

3.2 Minimal Models

3.2.1 Review and Classification

Minimal models are (1+1)d CFTs whose Hilbert spaces are composed of a finite number of irreducible representations of the Virasoro algebra. A minimal model will generally have central charge

$$c(p,p') = 1 - \frac{6(p-p')^2}{pp'}\,, \tag{3.1}$$

where p and p' are positive co-prime integers and $p > p' \geq 2$. The potential highest weights of a minimal model at $c(p,p')$ are given by the Kac formula

$$h_{r,s} = \frac{(rp-sp')^2-(p-p')^2}{4pp'}\,, \tag{3.2}$$

where we use the symmetry $h_{r,s} = h_{p'-r,p-s}$ of the Kac formula to produce a closed operator algebra of $(p-1)(p'-1)/2$ distinct fields.

The modular invariants of minimal models are well known to have an ADE type classification [88, 89] (see also Tables 10.3 and 10.4 of [5]), which allows us to read off the highest weights/state spaces after identification with characters. More precisely, the modular invariants are in one-to-one correspondence with pairs of simply-laced Lie algebras. As a consequence of this classification, unless $p,p' = 2,4$, there is always more than one modular-invariant minimal model at $c(p,p')$. That is, there are different operator algebras constructed from the same primaries and closed under OPE.

To obtain unitary minimal models, we specify to $(p,p') = (m+1,m)$ with $m \geq 2$, and can take $1 \leq s \leq r \leq m-1$. For example, at $m=2$ we have the $c=0$ trivial CFT, at $m=3$ the $c=1/2$ critical Ising, and at $m=4$ the $c=7/10$ tricritical Ising. As promised, for $m=5$ there are A-type and D-type modular invariants, which correspond to the tetracritical Ising and critical 3-state Potts model respectively. In general, for $m \geq 5$ there is always an A-type theory and a D-type theory, and at the special values $m = 11, 12, 17, 18, 29, 30$ there is a third E-type theory, corresponding to the Dynkin diagrams E_6, E_7, and E_8 for each consecutive pair. From here out we will only discuss unitary minimal models.

As mentioned in [67], there is a temptation to call the $m = 3$ theory a free massless Majorana fermion, and the $m = 4$ theory the smallest $\mathcal{N} = 1$ supersymmetric minimal model. However, this is not strictly correct: the Ising model has only integral spin operators, and thus is bosonic, while the Majorana fermion has integral and half-integral spin operators, and thus is fermionic. It is similar for the $m = 4$ case.

3.2.2 Symmetries and $\mathbb{Z}_2$ Orbifold

Symmetries of the operator algebra of a CFT are heavily restricted by conditions of unitarity and modular invariance. For example, $h = 1$ operators are not unitary in a $c < 1$ CFT, so one can rule out continuous internal symmetries in minimal models [110]. However, there may still be discrete symmetries.

In [109], the authors determine the maximal symmetry group of all unitary minimal models by studying the theories in the presence of twisted boundary conditions as in [111, 112]. Said differently, they determine if there are twisted partition functions which can be consistently added to the theory. Summarizing, their findings are that: *All unitary minimal models have maximal symmetry group $\mathbb{Z}_2$, except 6. The critical and tricritical 3-Potts model have non-commuting $\mathbb{Z}_2$ and $\mathbb{Z}_3$ symmetry which combine to an S_3 symmetry, and the 4 E_7 and E_8 minimal models have no symmetry.* We note that these symmetries are the same as the automorphisms of the associated Dynkin diagrams.

As explained in [67], a theory T (put on $S^1 \times \mathbb{R}$ for concreteness) with non-anomalous $\mathbb{Z}_2$ symmetry can be coupled to a background $\mathbb{Z}_2$-connection. We may then consider the untwisted Hilbert space $\mathcal{H}_{\mathrm{Un.}}$, and the twisted Hilbert space $\mathcal{H}_{\mathrm{Tw.}}$, depending on whether or not the background $\mathbb{Z}_2$ is trivial, i.e. if states have holonomy around S^1. Moreover, these Hilbert spaces may be further decomposed into states that are even or odd under the $\mathbb{Z}_2$ symmetry

$$\mathcal{H}_{T,\mathrm{Un.}} = \mathcal{H}^+_{T,\mathrm{Un.}} \oplus \mathcal{H}^-_{T,\mathrm{Un.}} \tag{3.3}$$

$$\mathcal{H}_{T,\mathrm{Tw.}} = \mathcal{H}^+_{T,\mathrm{Tw.}} \oplus \mathcal{H}^-_{T,\mathrm{Tw.}}\,. \tag{3.4}$$

Intuitively, the gauged theory $[T/\mathbb{Z}_2]$ consists of both untwisted and twisted Hilbert spaces because it is a sum over all background connections, but it only has those states which are gauge-invariant (i.e. have a + superscript). Hence we have that

$$\mathcal{H}_{[T/\mathbb{Z}_2],\text{Un.}} = \mathcal{H}^+_{T,\text{Un.}} \oplus \mathcal{H}^+_{T,\text{Tw.}} \tag{3.5}$$

$$\mathcal{H}_{[T/\mathbb{Z}_2],\text{Tw.}} = \mathcal{H}^-_{T,\text{Un.}} \oplus \mathcal{H}^-_{T,\text{Tw.}}\,. \tag{3.6}$$

More generally, when $G \neq \mathbb{Z}_2$, the splitting "even" and "odd" would be promoted to the projectors $P_\chi = \frac{1}{|G|}\sum_g \chi(g)g$ for characters $\chi \in \hat{G}$, and we recover the statements that $[T/G]$ has an emergent $\hat{G}$ symmetry, and that $[[T/G]/\hat{G}] = T$ [40] (see also [48, 12] for modern discussions).

For the unitary minimal models, it is (literally) a textbook result (see 10.7 of [5]) that the $\mathbb{Z}_2$ symmetry of any of the A-type theories can be gauged to produce the D-type theory and vice-versa.[3]

3.3 Fermionic Minimal Models

3.3.1 Fermionization

Given a bosonic theory T_b with $\mathbb{Z}_2$ symmetry, it is possible to fermionize it by a generalized Jordan-Wigner transformation, turning the $\mathbb{Z}_2$ symmetry into a $(-1)^F$ Grassmann parity. Concretely, the partition function is obtained by summing over the $\mathbb{Z}_2$ connection and coupling appropriately to a spin-structure ρ. In particular, on a genus g surface Σ, we have

$$Z_{T_f}[\rho] = \frac{1}{2^g}\sum_{\alpha\in H^1(\Sigma,\mathbb{Z}_2)} (-1)^{\text{Arf}[\alpha+\rho]} Z_{T_b}[\alpha]\,. \tag{3.7}$$

The invertible topological phases that can be stacked with any fermionic theory are classified by the cobordism group $\text{Hom}(\Omega_2^{\text{Spin}}(pt), U(1)) = \mathbb{Z}_2$ [114]. The effective action for the non-trivial invertible phase is given by the low energy version of the Kitaev chain

$$e^{iS[\rho]} = (-1)^{\text{Arf}[\rho]}\,. \tag{3.8}$$

Here the Arf-invariant $\text{Arf}[\rho]$ can be thought of as the mod 2 index of the Dirac operator. For example, on the torus, $\text{Arf}[\rho] = 0$ for the 3 "even" spin-structures (NSNS, NSR, RNS), and $\text{Arf}[\rho] = 1$ for the 1 "odd" spin-structure (RR) due to the zero mode (see

[3]It is shown in [113] that the unitary minimal models do not have any 't Hooft anomalies by general MTC considerations.

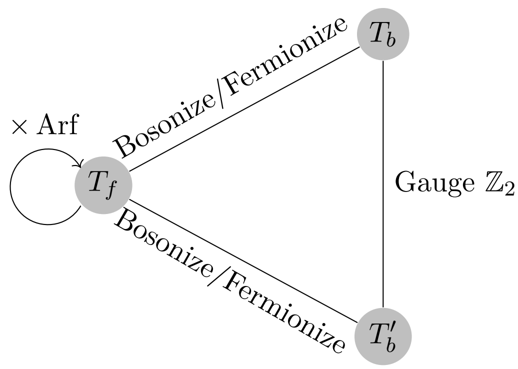


Figure 3.1: Gauging the $(-1)^F$ symmetry of a fermionic theory T_f produces a bosonic theory T_b with $\mathbb{Z}_2$ symmetry. A different bosonic theory T'_b is produced if one first stacks the fixed T_f with the Arf theory. These two theories are related by $\mathbb{Z}_2$ orbifold.

[77, 114, 115, 74, 1] and references within for further discussion). In practice, this means that stacking with this theory changes the relative sign of the even and odd partition functions in a sum over spin-structures.

Conversely, given a fermionic theory T_f, it is possible to gauge the $(-1)^F$ symmetry and produce a bosonic theory T_b with a $\mathbb{Z}_2$ symmetry in two distinct ways,[4] by summing over spin-structures without the Arf phase included

$$Z_{T_b}[\alpha] = \frac{1}{2^g} \sum_{\rho} (-1)^{\mathrm{Arf}[\alpha+\rho]} Z_{T_f}[\rho] \,, \tag{3.9}$$

or by first stacking[5] with Arf to get

$$Z_{T'_b}[\alpha] = \frac{1}{2^g} \sum_{\rho} (-1)^{\mathrm{Arf}[\alpha+\rho]+\mathrm{Arf}[\rho]} Z_{T_f}[\rho] \,. \tag{3.10}$$

It is not hard to convince oneself, with the formulas above, that the two distinct bosonizations are related by gauging the emergent $\mathbb{Z}_2$'s, forming Figure 3.1.

[4]Here we assume the gravitational anomaly, $c_L - c_R$, of T_f is divisible by 8 so that the GSO projection produces an absolute 2d bosonic theory. See Section 3 of [57] for a VOA discussion, and [1, 116] for discussions in terms of coupling to a bulk TFT.

[5]It should be stressed that it does not make sense to think of one bosonization as being more fundamental. T_f and $T_f \times$ Arf are simply related by an invertible topological phase inherently linked to the Grassmann parity. See Section 2 and 3.1.1 of [1] for further elaboration on this point.

For a very concrete example, one might take the theory with $(-1)^F$ to be a free Majorana fermion $Z_{\mathrm{Maj}}[\rho, M]$. Stacking with Arf amounts to switching the sign of the mass term, that is, $Z_{\mathrm{Maj}}[\rho, -M] = (-1)^{\mathrm{Arf}[\rho]} Z_{\mathrm{Maj}}[\rho, M]$. Bosonizing we attain the (1+1)d Ising model [115]. Tuning to criticality, we see that the $M = 0$ free Majorana fermion is the generalized Jordan-Wigner transformation of the critical Ising CFT.

General points on fermionization and gauging are discussed further from a Hilbert-space and lattice-friendly point of view in [67], and more from the point of view of partition functions in [1].

3.3.2 Fermionic Minimal Models

We now briefly outline the work done by Hsieh, Nakayama, and Tachikawa in [67].

Combining the general theory above, the authors are effectively noting that the A and D-type unitary minimal models are related by a $\mathbb{Z}_2$ orbifold. Then, as in Figure 3.1, there must be a fermionic theory "completing the triangle" between the two bosonic theories. In the case of the critical or tri-critical Ising, T_b and T_b' are not distinct theories. This will be the case for the E_6 models, and provides a consistency check for us later, because clearly if a theory is self-dual under $\mathbb{Z}_2$ orbifold its fermionization must have vanishing RR-sector.

Due to the high degree of solubility of the unitary minimal models, the authors are able to explicitly tabulate the states in each of the theories and indicate which are twisted, untwisted, and even or odd (and similarly for their fermionizations). Furthermore, because the Jordan-Wigner transformation has a very concrete implementation when the theory is presented on a lattice, the authors use it to construct explicit Majorana-chain realizations of these fermionic minimal models from the bosonic spin-chains.

3.4 The Exceptional E_6 Cases

We now come to the main point of this note, which is to point out the following: The E_6 exceptional minimal models at $m = 11$ and $m = 12$ also have a non-anomalous $\mathbb{Z}_2$ symmetry, and hence there are also fermionic minimal models associated with these.

Interestingly, the E_6 minimal models are self-dual under $\mathbb{Z}_2$ orbifold. This does make sense, there are no other unitary CFTs at the same central charge for them to transform into except possibly the $m = 11, 12$ A-type and D-type models, but clearly their $\mathbb{Z}_2$ symmetries have already "been spent" relating to one another.

We can be more explicit and list the partition functions of such theories by borrowing the expressions from [109]. In particular, we use the twisted partition functions (equations (7.7) to (7.10) in [109]) to do simple consistency checks on the statements above in the following subsections.

Moreover, the theory of invertible topological phases and fermionization provides the tools to complete this procedure for lattice and continuum descriptions of various (1+1)d theories with non-anomalous $\mathbb{Z}_2$ symmetry. However, it would still be interesting to obtain explicit statistical or Majorana-chain descriptions of these fermionic minimal models and possibly WZW models.

Lastly, now that all of the $\mathbb{Z}_2$ symmetries have been addressed, these exceptional E_6 cases, in tandem with the work in [67], cover all possible fermionic unitary minimal models. Suppose otherwise, then there would be another fermionic model whose GSO projection is a bosonic unitary minimal model with $\mathbb{Z}_2$ symmetry. As we have already determined and then fermionized all such bosonic $\mathbb{Z}_2$s, this additional fermionic model must not exist.

3.4.1 $m = 11$ Exceptional

This theory is also known as the (A_{10}, E_6) unitary minimal model. The torus partition functions are[6]

$$Z_{(A_{10},E_6)}[0,0] = \sum_{r=1,\text{odd}}^{10} \left|\chi_{(r,1)} + \chi_{(r,7)}\right|^2 + \left|\chi_{(r,4)} + \chi_{(r,8)}\right|^2 + \left|\chi_{(r,5)} + \chi_{(r,11)}\right|^2 \tag{3.12}$$

$$Z_{(A_{10},E_6)}[0,1] = \sum_{r=1,\text{odd}}^{10} \left|\chi_{(r,1)} + \chi_{(r,7)}\right|^2 - \left|\chi_{(r,4)} + \chi_{(r,8)}\right|^2 + \left|\chi_{(r,5)} + \chi_{(r,11)}\right|^2 \tag{3.13}$$

$$Z_{(A_{10},E_6)}[1,0] = \sum_{r=1,\text{odd}}^{10} \left|\chi_{(r,4)} + \chi_{(r,8)}\right|^2 + \left\{(\chi_{(r,1)} + \chi_{(r,7)})^*(\chi_{(r,5)} + \chi_{(r,11)}) + \text{c.c.}\right\} \tag{3.14}$$

$$Z_{(A_{10},E_6)}[1,1] = \sum_{r=1,\text{odd}}^{10} \left|\chi_{(r,4)} + \chi_{(r,8)}\right|^2 - \left\{(\chi_{(r,1)} + \chi_{(r,7)})^*(\chi_{(r,5)} + \chi_{(r,11)}) + \text{c.c.}\right\} \tag{3.15}$$

[6]Throughout we use the notation

$$Z_T[g,h] := \mathrm{Tr}_{\mathcal{H}_g}\, \hat{h} q^{L_0 - \frac{c_L}{24}} \bar{q}^{\bar{L}_0 - \frac{c_R}{24}} \tag{3.11}$$

for bosonic partition functions. For us, $g, h \in \mathbb{Z}_2$ and $\mathcal{H}_g$ is the g-twisted Hilbert space, so that $Z_T[0,h]$ can be thought of as the partition function with h-twist in Euclidean time. For fermionic theories, we will use the labels $NS = AP = 0$ and $R = P = 1$ for the antiperiodic and periodic sectors respectively.

It is easy to see that the spins coming from the untwisted Hilbert spaces are bosonic. With some computational work, one may verify that they take values

$$0, \pm 1, \pm 2, \dots, \pm 10, \pm 13, \pm 16, \pm 19\,. \tag{3.16}$$

One can also verify the untwisted partition function $Z_{(A_{10},E_6)}[0,0]$ is invariant under modular S and T transformations, and that the twisted partition functions transform into one another under modular S and T. It's also not hard to check that a $\mathbb{Z}_2$ orbifold returns the original partition function, or more generally that

$$Z_{(A_{10},E_6)}[\alpha_1,\alpha_2] = \frac{1}{2}\sum_{\beta_1,\beta_2} (-1)^{\alpha_1\beta_2-\beta_1\alpha_2} Z_{(A_{10},E_6)}[\beta_1,\beta_2]\,. \tag{3.17}$$

The fermionic partition functions can be obtained from equation (3.7) and are

$$\begin{aligned}
Z_{f,11}[0,0] = \sum_{r=1,\text{odd}}^{10} &\left|\chi_{(r,1)}+\chi_{(r,7)}\right|^2 + \left|\chi_{(r,5)}+\chi_{(r,11)}\right|^2 \\
&+ \left\{(\chi_{(r,1)}+\chi_{(r,7)})^*(\chi_{(r,5)}+\chi_{(r,11)}) + \text{c.c.}\right\}
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
Z_{f,11}[0,1] = \sum_{r=1,\text{odd}}^{10} &\left|\chi_{(r,1)}+\chi_{(r,7)}\right|^2 + \left|\chi_{(r,5)}+\chi_{(r,11)}\right|^2 \\
&- \left\{(\chi_{(r,1)}+\chi_{(r,7)})^*(\chi_{(r,5)}+\chi_{(r,11)}) + \text{c.c.}\right\}
\end{aligned} \tag{3.19}$$

$$Z_{f,11}[1,0] = \sum_{r=1,\text{odd}}^{10} 2\left|\chi_{(r,4)}+\chi_{(r,8)}\right|^2 \tag{3.20}$$

$$Z_{f,11}[1,1] = 0 \tag{3.21}$$

As promised, the RR-sector (the periodic-periodic sector) partition function is vanishing. We also note that the fermionic sectors have both integral and half-integral operators, the spins coming from the NSNS-sector (the antiperiodic-antiperiodic sector) partition function are

$$0, \pm\frac{1}{2}, \pm\frac{2}{2}, \dots, \pm\frac{10}{2}, \pm\frac{13}{2}, \dots, \pm\frac{17}{2}, \pm\frac{20}{2}, \pm\frac{21}{2}, \pm\frac{25}{2}, \pm\frac{26}{2}, \pm\frac{29}{2}, \pm\frac{32}{2}, \pm\frac{35}{2}, \pm\frac{38}{2}, \pm\frac{45}{2} \tag{3.22}$$

the ellipses mean all the spins appear in between (with half-integral step size).

3.4.2 $m = 12$ Exceptional

This theory is also known as the (E_6, A_{12}) unitary minimal model. The torus partition functions are

$$Z_{(E_6,A_{12})}[0,0] = \sum_{s=1,\text{odd}}^{12} \left|\chi_{(1,s)} + \chi_{(7,s)}\right|^2 + \left|\chi_{(4,s)} + \chi_{(8,s)}\right|^2 + \left|\chi_{(5,s)} + \chi_{(11,s)}\right|^2 \tag{3.23}$$

$$Z_{(E_6,A_{12})}[0,1] = \sum_{s=1,\text{odd}}^{12} \left|\chi_{(1,s)} + \chi_{(7,s)}\right|^2 - \left|\chi_{(4,s)} + \chi_{(8,s)}\right|^2 + \left|\chi_{(5,s)} + \chi_{(11,s)}\right|^2 \tag{3.24}$$

$$Z_{(E_6,A_{12})}[1,0] = \sum_{s=1,\text{odd}}^{12} \left|\chi_{(4,s)} + \chi_{(8,s)}\right|^2 + \left\{(\chi_{(1,s)} + \chi_{(7,s)})^*(\chi_{(5,s)} + \chi_{(11,s)}) + \text{c.c.}\right\} \tag{3.25}$$

$$Z_{(E_6,A_{12})}[1,1] = \sum_{s=1,\text{odd}}^{12} \left|\chi_{(4,s)} + \chi_{(8,s)}\right|^2 - \left\{(\chi_{(1,s)} + \chi_{(7,s)})^*(\chi_{(5,s)} + \chi_{(11,s)}) + \text{c.c.}\right\} \tag{3.26}$$

As before, we can see the spins coming from the untwisted Hilbert spaces are bosonic, and that the modular S and T relationships between sectors is satisfied. The fermionic partition functions can be obtained from equation (3.7) and are

$$\begin{aligned} Z_{f,12}[0,0] = \sum_{s=1,\text{odd}}^{12} & \left|\chi_{(1,s)} + \chi_{(7,s)}\right|^2 + \left|\chi_{(5,s)} + \chi_{(11,s)}\right|^2 \\ & + \left\{(\chi_{(1,s)} + \chi_{(7,s)})^*(\chi_{(5,s)} + \chi_{(11,s)}) + \text{c.c.}\right\} \end{aligned} \tag{3.27}$$

$$\begin{aligned} Z_{f,12}[0,1] = \sum_{s=1,\text{odd}}^{12} & \left|\chi_{(1,s)} + \chi_{(7,s)}\right|^2 + \left|\chi_{(5,s)} + \chi_{(11,s)}\right|^2 \\ & - \left\{(\chi_{(1,s)} + \chi_{(7,s)})^*(\chi_{(5,s)} + \chi_{(11,s)}) + \text{c.c.}\right\} \end{aligned} \tag{3.28}$$

$$Z_{f,12}[1,0] = \sum_{s=1,\text{odd}}^{12} 2\left|\chi_{(4,s)} + \chi_{(8,s)}\right|^2 \tag{3.29}$$

$$Z_{f,12}[1,1] = 0 \tag{3.30}$$

Chapter 4

Duality Defects in E_8

In this section we classify all non-invertible Kramers-Wannier duality defects in the E_8 lattice Vertex Operator Algebra (i.e. the chiral $(E_8)_1$ WZW model) coming from $\mathbb{Z}_m$ symmetries. We illustrate how these defects are systematically obtainable as $\mathbb{Z}_2$ twists of invariant sub-VOAs, compute defect partition functions for small m, and verify our results against other techniques. Throughout, we focus on taking a physical perspective and highlight the important moving pieces involved in the calculations. Kac's theorem for finite automorphisms of Lie algebras and contemporary results on holomorphic VOAs play a role. We also provide a perspective from the point of view of (2+1)d Topological Field Theory and provide a rigorous proof that all corresponding Tambara-Yamagami actions on holomorphic VOAs can be obtained in this manner. We include a list of directions for future studies.

I would like to thank Theo Johnson-Freyd for suggesting this project, for immeasurablely helpful comments throughout the research process, and feedback at various stages of the draft this section is based on. I would also like to thank Sven Möller for providing my collaborators and I with Magma code to perform twisted character calculations and for explanations of the VOA literature. I would also like to thank Davide Gaiotto for illuminating discussions on 2d CFT and (2+1)d TFT and feedback on the draft this section is based on; and Matt Yu for discussions on anyon condensation. I also would like to acknowledge the participation of Alicia Lima and Melissa Rodríguez-Zárate in the early stages of the project that led to the paper that this chapter is based on.

4.1 Introduction

A topological defect line in a 2d QFT corresponds to some oriented (possibly charged) 1d inhomogeneity in the system, which is invariant under continuous deformations, so long as the deformation does not move the inhomogeneity past any other operators. Historically, such topological defect lines have been of interest in CFT for their connection to BCFT, twisted boundary conditions, and orbifolds [112, 117, 118, 119, 37, 120, 121, 122, 123]. However, more modern applications have exploded in recent years, including: notions of generalized orbifolds and duality [38, 124, 125]; connections to anyon condensation [126, 127]; better understanding of defects in statistical physics systems [128, 129]; constraints for RG flows [130, 80, 1, 131]; constraints for 2d modular bootstrap [132, 133, 134, 135]; and even statements about confinement in (1+1)d QCD [87, 136, 137] and quantum gravity [138]. Very broadly, these ideas fit into a series of research programs studying applications of symmetries in quantum field theory and their generalizations to "higher-form symmetries" (see e.g. [7]) and "non-invertible symmetries" (see e.g. [12] in 2d).

Our goal in this chapter is to understand the construction of $\mathbb{Z}_m$ duality defects in the chiral $(E_8)_1$ WZW model, i.e. lines which separate the theory from its $\mathbb{Z}_m$ orbifold, and their associated twisted partition functions. Equivalently, we will focus on $\mathbb{Z}_m$ Tambara-Yamagami category actions on the E_8 lattice Vertex Operator Algebra (VOA). Let us briefly motivate some of these choices of specialization.

First, duality defects are an interesting object of study in the realm of non-invertible symmetries. They are non-invertible topological defect lines which separate a theory from a gauged theory, and help to explain phenomena such as Kramers-Wannier duality in statistical physics [139, 140, 141, 38] (see [142, 143, 144, 145] for related work in 4d). They also provide "spin-selection rules" (see e.g. [130]) and other constraints on CFTs, which play a role in e.g. the 2d modular bootstrap. There is also a sense (which we will review) in which duality defects are the simplest defects after symmetry defects, this makes them a good tool for understanding the physics of non-invertible defect lines.

To motivate the choice of chiral $(E_8)_1$, we compare to two other systems with duality defects: the Ising CFT and the Monster CFT. The Ising CFT has a $\mathbb{Z}_2$ duality defect, and all of the properties of this defect can be understood from the fact that it has a (2+1)d interpretation as one of the three simple anyons in the Ising TFT. By comparison, the TFT/MTC associated to chiral $(E_8)_1$ is trivial (it has no irreducible modules besides itself), so we cannot use this approach to study its duality defects. Moving on, the Monster CFT has $\mathbb{Z}_p$ duality defects (where p is a prime dividing the order of the Monster group) and in [146] the $\mathbb{Z}_2$ defects of the Monster CFT were found by fermionization (see Section

4.3.5). Generalizing such an analysis to a "parafermionization" could be possible [147, 148], but our analysis evades the technical difficulties involved with this process.

All in all, our strategy should generalize to theories with duality defects who do not lift to lines in the representation category of the theory, and which are more formidable than minimal models, such as the Monster and other Sporadic group CFTs. Additionally, we conjecture that it will work for higher n-ality defects.

The chiral $(E_8)_1$ theory is also interesting in its own right. For example, it is the unique holomorphic VOA with central charge 8, and hence describes the boundary edge modes of Kitaev's (2+1)d E_8 phase [149] (see [150] for a recent discussion). Commutant pairs inside the chiral $(E_8)_1$ theory have also appeared in general CFT contexts and the study of MLDEs and the conformal bootstrap [151, 152, 133, 135].

Note: throughout the chapter, as with the rest of this thesis, we will always write the total number of spacetime dimensions.

4.1.1 Outline of the Chapter

In this chapter we want to construct $\mathbb{Z}_m$ duality defects of the chiral E_8 theory, but a side-mission will be to carve a pathway through the literature in a ground-up way for physicists, focusing on examples and highlighting various moving pieces to enable explicit calculations. We also hope this will be of benefit to non-physicists to understand the physical point of view. We do not review any formal mathematical definitions (e.g. of fusion categories or vertex algebras) as such reviews have been done better in other places, although we do provide references to those materials throughout. When we do introduce some mathematical formalism, we try to emphasize the physical context so that the mathematical definitions seem "inevitable."

In Section 4.2 we provide background for defects in 2d CFT. We recap how topological defect lines appear in CFTs in Section 4.2.1 and how they relate to gauging symmetries (or orbifolds) in Section 4.2.2; we also set notation for the rest of this thesis. We briefly recount the story of the critical Ising model for concreteness in Section 4.2.3. We conclude in Section 4.2.4 by providing the data needed to define the fusion categories Vec_G and $\mathrm{TY}(A, \chi, \tau)$ and provide a physical story for those data.

In Section 4.3 we build all the tools necessary to compute duality defects in the chiral $(E_8)_1$ theory. Since we prefer to think of this theory from the lattice VOA viewpoint, we describe the construction of lattice VOAs in Section 4.3.1. Understanding symmetries of VOAs with all the technical details is important for our construction, so in Section 4.3.2

we recount the classic story about lifts from the underlying lattice, Kac's theorem for finite order automorphisms of Lie algebras, and a (computer-implementable) way to compute the relevant twisted characters. In Section 4.3.3 we discuss orbifolds from the lattice VOA perspective to make both calculations and the connections to physics concrete. In Section 4.3.4 we explain how to compute duality defected partition functions in the chiral E_8 theory, and in Section 4.3.5 we implement our proposal for $\mathbb{Z}_2$ symmetries and compare to the result obtained by fermionization.

In Section 4.4 we show the duality defected partition functions for $\mathbb{Z}_3$, $\mathbb{Z}_4$, and $\mathbb{Z}_5$. The purpose of this section is not just the enumeration of tables of defects, but to show-off potential complications that may arise in computing defects. For instance, we see the relevance of order-doubling in Section 4.4.1, and what happens when the invariant sub-VOA does not come from a root lattice in Section 4.4.2 and Section 4.4.3. In Section 4.4.2 we also describe a computer algorithm for systematically computing (q-expansions of) defects.

In Section 4.5 we rephrase our discussion of duality defects from the point of view of (2+1)d TFTs and gapped boundary conditions. In Section 4.5.1 we provide a brief review of the relationship between (2+1)d TFTs, Modular Tensor Categories (MTCs), and their gapped boundary conditions, then discuss how orbifolds of 2d theories (and duality defects) can be understood from this picture in Section 4.5.2. We conclude with a description of how this applies to holomorphic VOAs (like the E_8 lattice VOA) in Section 4.5.2.

We end the chapter with a list of open problems in Section 4.6. Namely those where explicit computations could give examples of the underlying physics.

In Appendix B.1 we give an alternative way to compute one of our $\mathbb{Z}_3$ duality defects from the Potts CFT, framing it from a (2+1)d point of view. In Appendix B.2 we make some technical comments on canonicality and symmetric non-degenerate bicharacters associated with the duality defects.

4.2 Topological Defects in 2d

The first example of a topological defect line in 2d occurs in the study of 0-form symmetries.[1] Given a theory with a global G symmetry, each element $g \in G$ corresponds to a topological line defect X_g in the 2d theory: if X_g sweeps past an operator insertion, it acts

[1]Since we will not discuss higher-form symmetries at all (until Section 4.5), from here out we will drop the word 0-form.

by g in the appropriate way (see e.g. [7, 48]). In the case that G is a continuous group, these topological lines are simply the (exponentials of the) Noether charges obtained by integrating the conserved currents associated to G along the line. Their topological nature can be understood as a consequence of the Ward identities for such currents.

Continuous or discrete, the topological defect lines associated with symmetries compose in a natural way: if two symmetry defects labelled by $g, h \in G$ are brought sufficiently close together, in parallel, and with the same orientation, they fuse as $X_g \otimes X_h = X_{gh}$. As a result, we find that topological defect lines associated with symmetries are *invertible*, with $X_{g^{-1}}$ the left and right inverse of the line X_g.

However, not all topological defect lines in a theory are invertible. Such non-invertible defect lines in QFTs provide a rich source of information about a theory, because they are a manifestation of some non-trivial topological data including dualities, anomalies, and other constraints on RG flows.

The classical example of a non-invertible topological defect line is the "Kramers-Wannier" duality defect in the Ising model [140, 141, 38], which relates correlators in the Ising model to disorder correlators. We briefly review this in Section 4.2.3 (see also [78, 50, 146] for recent discussions of the Kramers-Wannier duality defect).

Another source of topological defect lines are the "Verlinde lines" of Rational Conformal Field Theories (RCFTs), which are not mutually exclusive from the two previous examples. Verlinde lines are a natural consequence of the bulk-boundary relationship between a 2d RCFT and its associated (2+1)d TFT: they can be understood as 2d topological defects coming from anyons of the associated bulk (2+1)d TFT, brought to the boundary where the 2d RCFT data lives [92, 119].

In this chapter, we will focus on topological defect lines which are captured by the mathematical data of a fusion category. Fusion categories describe theories of topological defect lines which are particularly discrete in that they are defined to only have a finite number of simple (irreducible) lines. In addition to being mathematically tractable, fusion categories are also interesting because they are "rigid" and cannot be "continuously deformed" (under RG flows, say), which is mathematically captured by the concept of Ocneanu rigidity [153]. This means that they put constraints on RG flows very closely related to those implied by 't Hooft anomalies [154], see Section 7 of [130] for a modern discussion.

By now, fusion categories have appeared in a variety of ways in the physics literature, broadly in the intersection of fields that study topological aspects of quantum field theories. This includes more traditional "high-energy" contexts (e.g. [130, 87, 12, 80, 1]) and especially in the study of condensed matter systems and quantum computation (e.g. [127, 155, 156, 51, 52]).

Rather than re-define fusion categories, we recommend the enthusiastic physics reader to check out [12] for the definitions of a fusion category (called a "symmetry category" there) and to Section 3 of [130]. For the reader in a hurry, we recommend Section 5 and Appendix A of [87].

4.2.1 CFT and Defect Partition Functions

Since a defect is an inhomogeneity in our system, part of defining a line defect in 2d involves specifying boundary conditions for fields on either side of the defect line. We can do this in a 2d CFT by describing how it acts on states of the Hilbert space.

In a 2d CFT on the plane, any defect line X defines an operator $\hat{X}$ on the Hilbert space $\mathcal{H}$ of states on a circle, which acts as follows: place a state $\phi \in \mathcal{H}$ at the origin and the defect on the unit circle. Then the state that this setup provides is by definition $\hat{X}\phi \in \mathcal{H}$ [38]. Said more cylindrically, prepare the state ϕ in the infinite past and have the defect line waiting "half way" up the cylinder, then the out state is by definition $\hat{X}\phi$.

From this, we obtain a very algebraic definition of a topological defect line as one that commutes with all the modes of the stress tensor

$$[\hat{L}_n, \hat{X}] = 0 = [\hat{\bar{L}}_n, \hat{X}]\,. \tag{4.1}$$

The topological defect line X does not need to lie on a closed path. However, if we do have a defect line along an open oriented path, then we will also have to specify how the line operator ends. The Hilbert space $\mathcal{H}_X$ of operators upon which X can end is the Hilbert space of the theory on a circle, except with a single future oriented X defect piercing the circle.

To obtain the CFT partition functions, we simply periodicitize time in the cylinder setups. The partition function with the X line inserted along a spacelike slice (a twist in Euclidean time) is given by

$$Z^X(\tau,\bar{\tau}) := \mathrm{Tr}_{\mathcal{H}}\, \hat{X} q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\,, \tag{4.2}$$

where $q := e^{2\pi i\tau}$. Likewise, the partition function with the X line inserted along a timelike slice (a twist in space) is given by

$$Z_X(\tau,\bar{\tau}) := \mathrm{Tr}_{\mathcal{H}_X}\, q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\,. \tag{4.3}$$

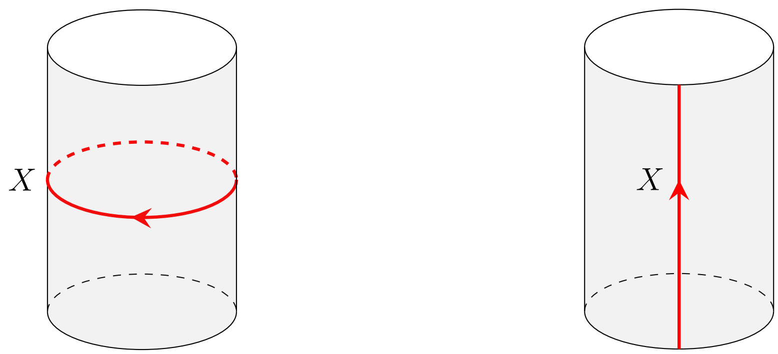


Figure 4.1: On the left, an X defect line (red) is inserted along a spatial slice, this gives a "twist" by X in the time direction. On the right, the defect line is inserted along the time direction. Periodicitizing this setup on the right, we get the trace over the X-twisted Hilbert space $\mathcal{H}_X$.

We illustrate these setups in Figure 4.1. Note that the two are related by the modular S-transformation $\tau \mapsto -\frac{1}{\tau}$ (and similarly for antiholomorphic), because the S-transformation swaps the two torus cycles.

As an almost trivial example, if X_- is a topological defect line generating a $\mathbb{Z}_2$ symmetry, then, basically by definition, if ϕ_i is charged under the $\mathbb{Z}_2$ symmetry we have

$$\hat{X}_- |\phi_i\rangle = -|\phi_i\rangle \ . \tag{4.4}$$

When tracing over the states in the Hilbert space to obtain $Z^{X_-}(\tau, \bar{\tau})$, for example, we will need to consider the signs from the charged states.

When studying a RCFT, another natural collection of topological defects comes from Verlinde line operators [92, 119, 130]. Verlinde lines are in one-to-one correspondence with primaries, and for a diagonal RCFT, a Verlinde line X_i commutes with the whole chiral algebra and acts on a primary by

$$\hat{X}_i |\phi_j\rangle = \frac{S_{ij}}{S_{0j}} |\phi_j\rangle \ , \tag{4.5}$$

where S_{ij} is the modular S-matrix of the CFT. It is a straightforward application of Verlinde's formula to show that the Verlinde lines then fuse according to the fusion rules of the associated primaries, i.e.

$$X_i \otimes X_j = \bigoplus_k N_{ij}^k X_k \, . \tag{4.6}$$

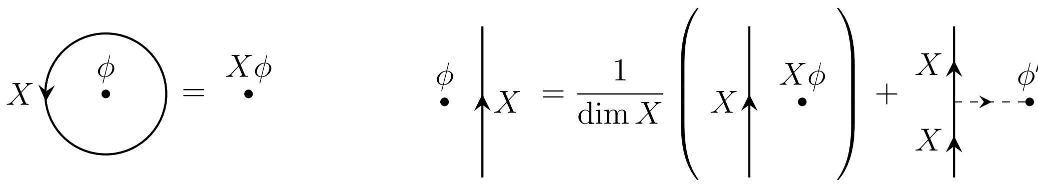


Figure 4.2: On the left, a topological defect line X encircles a ϕ insertion in the plane. Since the line is topological, this is equivalent instead to the local operator $X\phi$. On the right, a topological defect line X sweeps past the ϕ insertion leaving behind a local operator and a defect operator ϕ'. The defect operator is connected to X by a topological tail (dotted).

Given any topological defect line X encircling a local operator ϕ in the plane, we can take an equivalent view where the line X squashes down leaving us with only the local operator $X\phi$ in the plane. If, instead of a general ϕ, we had chosen the identity operator, it's clear we are just computing the expectation value of a loop of X in the plane, aka the quantum dimension of X. Throughout, we will denote this as $\dim X$ rather than $\langle X \rangle$.

More generally, when a defect line X sweeps past a local operator ϕ, the result is not necessarily a local operator, but can be written as a local operator plus a defect operator ϕ' that lies on the end of a topological tail adjoined to X, as depicted in Figure 4.2.

There are some interesting and important subtleties about all of this machinery, which are described in Section 2 of [130]. We have overlooked them because they are not particularly pressing for our discussion, but we will mention two immediate ones.

The first is that the diagram on the right of Figure 4.2 simply does not reduce to the diagram on the left, we are left with the naive answer plus a topological tadpole. The vanishing of this tadpole can be proven if one assumes that the topological defect lines act faithfully on the bulk local operators. This condition can be violated in the case the CFT has multiple ground states, but such CFTs are just a direct sum of theories and won't be relevant for us.[2]

The second subtlety is that in 2d there are local curvature counterterms which may be added to the action. Since we are only really concerned with producing torus partition functions, this will not play any role because the torus is flat. In any case, following such terms would just change overall phases.

[2]There exists some recent literature on this topic and its interplay with higher-form symmetries [87, 157, 158].

4.2.2 Orbifolding Abelian Symmetries

To warm-up, consider a 2d CFT T on M with a global finite Abelian symmetry A. If the symmetry is non-anomalous, then we can couple the theory to a background A-connection in a well-defined way. Since A is finite, this connection is necessarily flat and is specified by an $\alpha \in H^1(M, A)$ which labels the holonomies around the different cycles of M.

Framed in terms of topological defects, we can represent a background A-connection by a network of topological A-defect lines meeting at trivalent junctions (see [25] and diagrams within for an introduction). When a local operator passes the topological defect line labelled by $g \in A$, it applies the appropriate (linear) g-action.

For each background connection $\alpha \in H^1(M, A)$ we obtain a "twisted partition function" of the theory $Z_T[\alpha]$, which is like the "regular" partition function with (invertible) topological symmetry defect lines inserted around non-contractible loops. For example, on the torus, we have the $|A|^2$ twisted partition functions

$$Z_T[g, h] := \mathrm{Tr}_{\mathcal{H}_g} \hat{X}_h q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \,. \tag{4.7}$$

Here $g, h \in A$, $\mathcal{H}_g = \mathcal{H}_{X_g}$ is the g-twisted Hilbert space, and we have suppressed dependence on τ and $\bar{\tau}$.

The *orbifold* theory $[T/A]$ is the theory obtained by gauging the discrete A symmetry of T. As is familiar from the path integral, gauging a symmetry is making the background connection dynamical, so the orbifold is obtained by summing over the twisted partition functions. Moreover, when we orbifold there is an emergent dual symmetry, given by the Pontryagin dual $\hat{A}$ of A, which comes from the action of the Wilson lines for the A gauge fields [40]. This new dual symmetry can be coupled to a background flat connection $\beta \in H^1(M, \hat{A})$, and we obtain a formula relating the twisted partition functions in the different theories

$$Z_{[T/A]}[\beta] = \frac{1}{\sqrt{|H^1(M, A)|}} \sum_{\alpha \in H^1(M,A)} e^{i(\beta,\alpha)} Z_T[\alpha] \,. \tag{4.8}$$

The exponential denotes the intersection pairing between $H^1(M, A)$ and $H^1(M, \hat{A})$. Non-Abelian symmetries can be treated with only a bit more work.[3]

Equation (4.8) is not necessarily the most general possible way to orbifold by a symmetry. In particular, one can include extra $U(1)$ weights $\epsilon(\alpha)$ in front of the twisted partition

[3] Even more generally, we can gauge by "symmetric Frobenius algebras" in a theory with fusion category symmetry by inserting the corresponding triangulating trivalent mesh of topological defect lines as described in [38, 159, 12]. We discuss fusion categories more in Section 4.2.4.

functions, and still obtain a gauge invariant partition function. In the CFT language, such a choice is known as a choice of "discrete torsion" [46, 55, 47]. Such a choice of $U(1)$ phase is classified by a group cohomology class $\nu_2 \in H^2(A, U(1))$, so that most generally we have

$$Z_{[T/_{\nu_2} A]}[\beta] = \frac{1}{\sqrt{|H^1(M,A)|}} \sum_{\alpha \in H^1(M,A)} e^{i(\beta,\alpha)} \epsilon_{\nu_2}(\alpha) Z_T[\alpha] \,. \tag{4.9}$$

ϵ_{ν_2} is obtained from ν_2 by evaluating it on a triangulation of M, e.g. on the torus with fluxes in the cycles given by (α_1, α_2), so $\epsilon_{\nu_2}(\alpha_1, \alpha_2) = \nu_2(\alpha_1, \alpha_2)/\nu_2(\alpha_2, \alpha_1)$.

The two (commensurate) modern viewpoints on this discrete torsion phase are as follows: Anomalies of 2d theories manifest as phase ambiguities in coupling T to a background connection, which in turn are encoded in a cohomology class $\mu_3 \in H^3(A, U(1))$. When a theory is non-anomalous, we don't just need to know that the cohomology class is trivial, but must pick a trivialization $\nu_2 \in H^2(A, U(1))$ that consistently resolves all phase ambiguities. Alternatively, we may view the phase as the partition function for a 2d SPT phase $\epsilon_{\nu_2}(\alpha) \sim e^{iS_{\nu_2}[\alpha]}$. Then orbifolding with discrete torsion is equivalent to taking a non-anomalous theory T, with phase ambiguities already resolved, and then stacking with such a 2d SPT phase before orbifolding [28, 32] (see also [20]).

Example: $\mathbb{Z}_2$ Orbifold

To get a better appreciation of what happens at the level of the Hilbert spaces, and see where the dual symmetry comes from, consider a theory T with a non-anomalous (bosonic) $\mathbb{Z}_2$ symmetry. The untwisted orbifold partition function on the torus is

$$Z_{[T/\mathbb{Z}_2]}[0,0] = \frac{1}{2} \sum_{g,h \in \mathbb{Z}_2} Z_T[g,h] \tag{4.10}$$

$$= \sum_{g \in \mathbb{Z}_2} \mathrm{Tr}_{\mathcal{H}_g} \left[\left(\frac{\hat{X}_+ + \hat{X}_-}{2} \right) q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right] . \tag{4.11}$$

The term in round brackets just projects us onto the states which are invariant under the $\mathbb{Z}_2$ symmetry when we trace over $\mathcal{H}_g$. This shows that the operators that contribute to the untwisted orbifold partition function are those in both the untwisted and twisted Hilbert spaces, $\mathcal{H}_+$ and $\mathcal{H}_-$ respectively, but only those which are invariant under the $\mathbb{Z}_2$ action.

Said differently, we can decompose these Hilbert spaces into sums of even and odd subsectors

$$\mathcal{H}_+ = \mathcal{H}_+^{\text{Even}} \oplus \mathcal{H}_+^{\text{Odd}}\,, \tag{4.12}$$

$$\mathcal{H}_- = \mathcal{H}_-^{\text{Even}} \oplus \mathcal{H}_-^{\text{Odd}}\,. \tag{4.13}$$

Then the untwisted orbifold Hilbert space contains just even (i.e. gauge-invariant) operators, while the twisted orbifold Hilbert space contains just odd operators

$$\mathcal{H}_{\text{Orbi.}}^{\text{Even}} = \mathcal{H}_+^{\text{Even}} \oplus \mathcal{H}_-^{\text{Even}}\,, \tag{4.14}$$

$$\mathcal{H}_{\text{Orbi.}}^{\text{Odd}} = \mathcal{H}_+^{\text{Odd}} \oplus \mathcal{H}_-^{\text{Odd}}\,. \tag{4.15}$$

Hence we see the emergent $\hat{\mathbb{Z}}_2$ symmetry appear in the definition of this new twisted orbifold Hilbert space. For $A = \mathbb{Z}_m$ orbifolds we have to replace "Even" and "Odd" by the appropriate $\mathbb{Z}_m$ projectors, which are labelled by representations of A, i.e. elements of $\hat{A}$.

4.2.3 Example: The Ising CFT

Consider the (full) $c = \frac{1}{2}$ Ising CFT, which describes the Ising model at criticality. The theory has 3 Virasoro primaries: the identity operator $\mathbb{1}$, the energy operator ϵ, and the spin operator σ, with conformal weights $(h, \bar{h}) = (0,0), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{16}, \frac{1}{16})$ respectively [21]. The fusion rules are

$$[\sigma][\sigma] = [\mathbb{1}] \oplus [\epsilon]\,, \quad [\sigma][\epsilon] = [\sigma]\,, \quad [\epsilon][\epsilon] = [\mathbb{1}]\,. \tag{4.16}$$

The Ising CFT also has exactly 3 simple topological defect lines, these are the Verlinde lines naturally associated with the 3 primaries: $X_{\mathbb{1}}$, X_ϵ, and X_σ. They fuse according to the fusion rules of the primaries in Equation (4.16) above, and so it is clear that the σ line is not-invertible. The action of $X_{\mathbb{1}}$ on primaries is straightforward. X_ϵ is the $\mathbb{Z}_2$ symmetry line, so acts only non-trivially on the $\mathbb{Z}_2$ charged operator(s) (σ, ψ, $\bar{\psi}$, etc.) as in Equation (4.4). The action of X_σ on the Ising CFT primaries are depicted in Figure 4.3.

Famously, in the Ising model (not necessarily at criticality), the correlator of spins at some inverse temperature β is equal to the correlator of disorder/twist operators at the Kramers-Wannier dual inverse temperature $\tilde{\beta}$, i.e. $\langle \sigma(z_1) \cdots \sigma(z_n) \rangle_\beta = \langle \mu(z_1) \cdots \mu(z_n) \rangle_{\tilde{\beta}}$. This exhibits some duality between weak and strong couplings and can be used to understand the critical point of the Ising model [139, 140] (see also [128, 129]).

However, the disorder fields μ are not local operators in the theory alone, but have to be accompanied by a topological "line of frustration," which has to end similarly on another

Figure 4.3: Left, the X_σ defect sweeps past a σ primary, turning it into the disorder operator μ, plus a topological $\mathbb{Z}_2$ tail. Such a μ is traditionally called a "twist-field." Right, the X_σ defect sweeps past the relevant operator ϵ and turns it into $-\epsilon$.

$\mathbb{Z}_2$ "twist-field." In the CFT context above, this line of frustration is the $\mathbb{Z}_2$ symmetry line X_ϵ, and the $\mathbb{Z}_2$ twisted Hilbert space is $\mathcal{H}_\epsilon = \text{span}\{\mu, \psi, \bar{\psi}\}$.

A statistical interpretation of the X_σ line is not immediately clear, but can be realized by first studying how it acts on the primaries in the CFT picture. In particular, when we sweep the X_σ line past a σ insertion, we are left with a μ insertion and a topological X_ϵ tail. But in the orbifold CFT $[T/\mathbb{Z}_2]$, μ is a local operator (the $\mathbb{Z}_2$ line becomes invisible when we gauge), and σ must be accompanied by a topological tail.

Moreover, when X_σ sweeps past an ϵ insertion, it leaves behind a $-\epsilon$ insertion. The energy operator ϵ is a relevant operator in the Ising CFT and deformations by it flow the Ising CFT to the high-temperature (unbroken) or low-temperature (broken) phases. The action of X_σ reflects the fact that the Ising model and its orbifold see this deformation oppositely. This also explains why Kramers-Wannier duality holds away from criticality.

For these reasons, X_σ is the archetypal *duality defect.* It extends the algebra of topological defects in the Ising model from a collection of $\mathbb{Z}_2$ group-like defects to a Tambara-Yamagami category (see Section 4.2.4).

It is important to stress two basic facts about the duality defect line. The first is that it is *not* the non-simple topological defect line $X_{\text{Proj.}} := X_{\mathbb{1}} \oplus X_\epsilon$. Inserting a triangulating mesh of $X_{\text{Proj.}}$ into the original theory T just produces $[T/\mathbb{Z}_2]$. Indeed, X_σ separates the theory T from $[T/\mathbb{Z}_2]$. The second basic fact is that it is not the orbifold duality defect line $X_{\text{Proj.}}$... but it does square to it! This is important because when the X_σ line collides with itself (e.g. when deformed around a cycle of the torus) it does produce a $X_{\text{Proj.}}$ defect.

It is also instructive to compare the twisted and "defected" partition functions for this theory. The twisted partition functions can be written in terms of the usual Virasoro characters as

$$Z_T[0,0] = |\chi_0|^2 + |\chi_{\frac{1}{2}}|^2 + |\chi_{\frac{1}{16}}|^2\,, \tag{4.17}$$

$$Z_T[0,1] = |\chi_0|^2 + |\chi_{\frac{1}{2}}|^2 - |\chi_{\frac{1}{16}}|^2\,, \tag{4.18}$$

$$Z_T[1,0] = \chi_0\bar{\chi}_{\frac{1}{2}} + \chi_{\frac{1}{2}}\bar{\chi}_0 + |\chi_{\frac{1}{16}}|^2\,, \tag{4.19}$$

$$Z_T[1,1] = -\chi_0\bar{\chi}_{\frac{1}{2}} - \chi_{\frac{1}{2}}\bar{\chi}_0 + |\chi_{\frac{1}{16}}|^2\,. \tag{4.20}$$

While the defected partition function, with an X_σ inserted around the spatial S^1, is easily computable with the help of Equation (4.5) for Verlinde lines on primaries

$$Z_T[0,X_\sigma] = \sqrt{2}|\chi_0|^2 - \sqrt{2}|\chi_{\frac{1}{2}}|^2\,. \tag{4.21}$$

Such a structure for the defect partition function is typical, and can be understood from the second important fact about X_σ. Since $X_\sigma \otimes X_\sigma = X_{\mathbb{1}} \oplus X_\epsilon = X_{\text{Proj.}}$ we can compute (without identifying X_σ with the Verlinde line) that $\hat{X}_{\text{Proj.}}\,|\phi\rangle = 2\,|\phi\rangle$ if ϕ corresponds to a $\mathbb{Z}_2$ uncharged operator, and is 0 otherwise. This means that

$$\hat{X}_\sigma\,|\phi\rangle = \pm\sqrt{2}\,|\phi\rangle \tag{4.22}$$

if ϕ is uncharged under the $\mathbb{Z}_2$, and is 0 otherwise. Hence the defected partition function is just a sum over all of the uncharged primaries, and all we have to do is resolve a sign.

4.2.4 Fusion Categories for Physics

Here we will recap the pertinent properties of the 2 most important fusion categories for our purposes: the fusion category of G graded vector spaces Vec_G and Tambara-Yamagami categories $\text{TY}(A,\chi,\tau)$.

Fusion Category: Graded Vector Spaces Vec_G

Given a finite symmetry group G, the topological defects associated to the symmetry group form a unitary fusion category in a natural way. Simple objects of the category are oriented lines labelled by group elements $g,h \in G$. If two such lines are brought sufficiently close together, in parallel, with the same orientation, then they fuse according to the group multiplication law

$$X_g \otimes X_h = X_{gh} \qquad \text{for } g,h \in G. \tag{4.23}$$

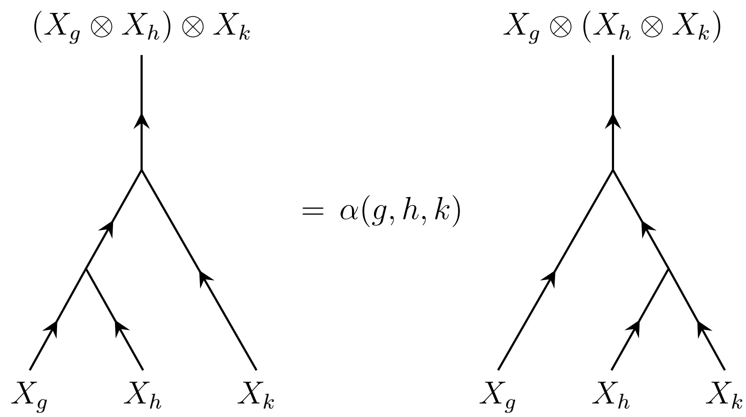


Figure 4.4: Mathematically, the associator α is an element of $\mathrm{Hom}((X_g \otimes X_h) \otimes X_k, X_g \otimes (X_h \otimes X_k))$ that defines in what way the two tensor products are equal. Physically, given a theory with non-trivializable anomaly $\alpha \in H^3(G, U(1))$, two networks of symmetry defect lines can be different up to a $U(1)$ phase.

The category is unitary because the orientation reversal of a line is (by definition) the inverse $X_g^* = X_g^{-1} = X_{g^{-1}}$.[4]

Given three simple lines labelled by $g, h, k \in G$, the data of an associator isomorphism α is needed to relate the object $(X_g \otimes X_h) \otimes X_k$ to $X_g \otimes (X_h \otimes X_k)$. In practice, this means that there can be an overall phase ambiguity between the background connection depicted with X_g and X_h fusing first or X_h and X_k fusing first as in Figure 4.4.

Such an α is required to satisfy the familiar "pentagon identity," which asserts that whatever α is, it must agree when we consider the multiple ways to fuse four G defects as in Figure 4.5.[5] In this case, such α are classified by an element of the group cohomology $H^3(G, U(1))$ (see e.g. [160, 24]). Such phase ambiguities in coupling a theory to a background G connection, which cannot be removed by adding local counterterms, are the 't Hooft anomalies from the physics literature.

All considered, such a collection of defects labelled by elements of G with a choice of group cohomology class $\alpha \in H^3(G, U(1))$ form the fusion category Vec_G^α of G-graded

[4]Since we are technically dealing with objects in a category, by $=$ we mean that there exists a natural isomorphism between objects in the category.

[5]Readers may be familiar with the data of an associator presented in the form of an F-symbol or $6j$-symbol.

$$
\begin{tikzcd}
 & (X_g \otimes X_h) \otimes (X_k \otimes X_\ell) \arrow[dr, "\alpha"] & \\
((X_g \otimes X_h) \otimes X_k) \otimes X_\ell \arrow[ur, "\alpha"] \arrow[d, "\alpha\otimes 1"'] & & X_g \otimes (X_h \otimes (X_k \otimes X_\ell)) \\
(X_g \otimes (X_h \otimes X_k)) \otimes X_\ell \arrow[rr, "\alpha"'] & & X_g \otimes ((X_h \otimes X_k) \otimes X_\ell) \arrow[u, "1\otimes\alpha"']
\end{tikzcd}
$$

Figure 4.5: Given 4 defect lines X_g, X_h, X_k, and X_ℓ, we only demand that the pentagon diagram commutes. This is familiar to those in RCFT as the pentagon identity for F-symbols.

vector spaces. In practice, we are going to be interested in the case where the symmetry in question does not have an anomaly, so that we can orbifold by that symmetry, the fusion category Vec_G.

It may not be a surprising fact, when paralleled with our discussion of coupling a CFT in the regular group symmetry framework, that the group of "autoequivalences" of the fusion category Vec_G (basically, automorphisms of Vec_G as a fusion category) is $\mathrm{Aut}(G) \ltimes H^2(G, U(1))$. The $\mathrm{Aut}(G)$ roughly corresponding to renaming of G lines, and $H^2(G, U(1))$ corresponding to stacking with 2d SPT phases [12].

Fusion Category: Tambara-Yamagami $\mathrm{TY}(A, \chi, \tau)$

In the previous subsubsection we started with a set of desired fusion rules (and unitary structure) and then were left with classifying the associators compatible with those fusion rules/unitary structure and pentagon identity. This is the same thing that Tambara and Yamagami do in a ground-up approach in their seminal paper [161] (see also [162]), except their fusion rules are minimally enriched by a duality defect.

Start with a finite Abelian group A, the simple objects of $\mathrm{TY}(A, \chi, \tau)$ are again associated with elements $a \in A$, but the fusion algebra is now extended by an additional object m of quantum dimension $\sqrt{|A|}$.[6] In addition to the familiar Vec_A fusion rules, we also

[6] In fact, such a fusion ring makes sense if the group is not Abelian, but it is only categorifiable if A is Abelian (see [34] Example 4.10.5).

demand

$$X_a \otimes X_m = X_m\,, \tag{4.24}$$

$$X_m \otimes X_m = \bigoplus_{a\in A} X_a\,. \tag{4.25}$$

What Tambara and Yamagami show, by direct computation, is that the data of potential associators is classified by a symmetric non-degenerate bicharacter $\chi : A \times A \to U(1)$, and a sign or "Frobenius-Schur indicator" $\tau = \pm 1/\sqrt{|A|}$ [161, 162] (see also [163, 12, 80, 129]). In particular, the non-trivial associators are

$$\alpha_{a,m,b} = \chi(a,b)\,\mathbb{1}_m\,, \tag{4.26}$$

$$\alpha_{m,a,m} = \bigoplus_b \chi(a,b)\,\mathbb{1}_b\,, \tag{4.27}$$

$$\alpha_{m,m,m} = (\tau\chi(a,b)^{-1}\,\mathbb{1}_m)_{a,b}\,. \tag{4.28}$$

A first point of note, which will be relevant later, is that TY categories are a $\mathbb{Z}_2$-extension of Vec_A. In general, a G-extension of a fusion category $\mathcal{D}$ is a G-graded fusion category

$$\mathcal{F} = \bigoplus_{g\in G} \mathcal{F}_g\,, \tag{4.29}$$

satisfying $\mathcal{F}_e \cong \mathcal{D}$.

Another thing to note is that $\chi : A\times A \to U(1)$, as opposed to the more familiar pairing of A with $\hat{A}$. This means χ is an assignment of (magnetic) A charges to A defect lines, in contradistinction to the natural Fourier-like pairing in orbifolds $\rho : A\times\hat{A} \to U(1)$. Pushing further, a bicharacter χ defines a homomorphism $A \to \hat{A}$, and non-degeneracy implies that this is an isomorphism. Since T and $[T/A]$ have A and $\hat{A}$ symmetry respectively, this χ reflects the freedom of choice in picking an isomorphism from A to $\hat{A}$ (or rather reflects the non-canonicality of $A \cong \hat{A}$). By the same logic, χ determines an isomorphism from $\hat{A} \to A$, and while A and its dual $\hat{A}$ are not canonically isomorphic, A *is* canonically isomorphic to its double dual $\hat{\hat{A}}$. The symmetric property ensures that whatever isomorphism χ determines from A to $\hat{A}$, and $\hat{A}$ to A, that it respects the canonicality $A = \hat{\hat{A}}$ i.e. it is honestly the identity on A and not some random automorphism of A.

The role of τ is less physically obvious, and shows up in the crossing relations for m [80]. It originates categorically as an associativity-like constraint similar in spirit to the associator in Vec_A^α [163], but could be better understood physically.

As a closing remark, the Ising Fusion category that we are familiar with forms $\mathrm{TY}(\mathbb{Z}_2, 1, +1/\sqrt{2})$ (see Section 4.1 of [80] for elaboration).

4.3 Duality Defects in $(E_8)_1$

The main goal of this chapter is to classify and construct $\mathbb{Z}_m$ duality defects in the chiral WZW-model $(E_8)_1$. Said differently, we want to understand the actions of $\mathbb{Z}_m$ Tambara-Yamagami categories on the holomorphic lattice VOA V_{E_8}.

Vertex Operator Algebras are widespread in physics as the axiomatization of "chiral algebras," which glue together to form a "full CFT" in two dimensions. We refer readers to the standard mathematical texts on VOAs [164, 165, 166, 167, 168] or any standard textbook on 2d CFT or string theory for an introduction (a nice review for both communities appeared in [169]). Lattice VOAs are also prominent in the physics literature, but understanding subtleties of their construction, automorphisms, and characters is important for our purposes, so we review them briefly in the following sections. Some particularly helpful reviews on lattice VOAs besides those already listed include [170, 171, 172, 173].

4.3.1 Lattice Vertex Operator Algebras

Given a positive definite even lattice L one can construct a lattice VOA V_L. Mathematically, $V_L = M_{\hat{\mathfrak{h}}}(1) \otimes \mathbb{C}_\epsilon[L]$ where the first factor is the "Heisenberg VOA" describing oscillator modes of chiral bosons (also called $\mathrm{Bos}(\mathfrak{h})$), and the second factor is the "twisted group algebra" describing "quantized momentum."[7]

We start the construction with the quantized momentum modes: for every vector $\alpha \in L$ we have a state $|\alpha\rangle$ which is created by the vertex operator Γ_α. By definition, the states are orthonormal $\langle\alpha|\beta\rangle = \delta_{\alpha\beta}$. There are also chiral bosonic oscillators a^i_n, $n \in \mathbb{Z}$ and $1 \le i \le \mathrm{rk}(L)$, satisfying the usual Heisenberg commutation relations

$$[a^i_m, a^j_n] = m\delta^{ij}\delta_{m+n,0}\,, \tag{4.30}$$

with $a^{i\dagger}_m = a^i_{-m}$.

Combining the two, the positive oscillator modes should annihilate the $|\alpha\rangle$

$$a^i_n\,|\alpha\rangle = 0\,, \qquad \text{if } n > 0, \tag{4.31}$$

[7] See e.g. Section 5 of [173] and references listed within for the formal mathematical construction of these objects and proof V_L is rational, C_2-cofinite, self-contregredient, and of CFT type.

and a_0^i should actually behave as the momentum operator, i.e.

$$a_0^i |\alpha\rangle = \alpha^i |\alpha\rangle . \tag{4.32}$$

The physical Hilbert space is thusly generated by the a_{-n}^i, $n > 0$, acting on the $|\alpha\rangle$, so that a general basis state is of the form

$$a_{-n_N}^{i_N} \cdots a_{-n_2}^{i_2} a_{-n_1}^{i_1} |\alpha\rangle . \tag{4.33}$$

Such a state has integral conformal weight

$$h = \frac{1}{2}(\alpha, \alpha) + \sum_{i=1}^{N} n_i , \tag{4.34}$$

where $(\, . \, , \, . \,)$ denotes the inner product on the lattice L.

As mentioned, the vertex operator Γ_α creates the state $|\alpha\rangle$, and one expects that $\Gamma_\alpha \Gamma_\beta \propto \Gamma_{\alpha+\beta}$. However, asking for Γ to be a representation of the lattice is too strong; locality requires us to consider projective representations, i.e.

$$\Gamma_\alpha \Gamma_\beta = \epsilon(\alpha, \beta) \Gamma_{\alpha+\beta} , \tag{4.35}$$

where $\epsilon : L \times L \to \{\pm 1\}$ is a (normalized) 2-cocycle satisfying $\epsilon(\alpha, \beta) = (-1)^{(\alpha,\beta)} \epsilon(\beta, \alpha)$. This skew condition on the 2-cocycle determines ϵ up to a coboundary, and so $\epsilon \in H^2(L, \{\pm 1\})$, which will play a role for us in understanding automorphisms.[8]

When L is the root lattice of a simply laced Lie algebra $\mathfrak{g}$, this construction gives a vertex-representation of the $\hat{\mathfrak{g}}_1$ WZW model (see e.g. Section 15.6.3 of [5]). In this case, the sign ambiguities which occur in trying to build V_L from the root lattice L are precisely the same as those in trying to build the Lie algebra $\mathfrak{g}$ from L. For non-simply laced algebras (or when the level $k > 1$), one must add a free fermion (or parafermion) respectively.

Up to isomorphism, the *untwisted* irreducible modules of V_L are labelled by elements of the discriminant group $D_L := L^*/L$, here L^* is the dual lattice to L [175]. The MTC of representations $\mathrm{Rep}(V_L)$ has D_L group-like fusion in the obvious way. For some $\lambda \in L^*$, the character of the irreducible V_L-module $V_{\lambda+L}$

$$\chi_{V_{\lambda+L}}(\tau) = \mathrm{tr}_{V_{\lambda+L}} \, q^{L_0 - \frac{c}{24}} = \frac{\theta_{\lambda+L}(\tau)}{\eta(\tau)^{\mathrm{rk}(L)}} , \tag{4.36}$$

[8] In other words, the Γ_α furnish, at minimum, a representation of a double cover $\hat{L}$ of the root lattice L. The freedom of choice in ϵ (due to the central extension defining the double cover not splitting) reflects the fact that there is no functor from root lattices to vertex algebras, i.e. a choice of ϵ is required [174]. Double covers of L coming from differing choices of ϵ are isomorphic, which is why one can non-canonically write $V_L = M_{\hat{\mathfrak{h}}}(1) \otimes \mathbb{C}[L]$ if they settle some signs once and for all.

where $\theta_{\lambda+L}(\tau)$ is the theta function of the shifted lattice $\lambda + L$, i.e.

$$\theta_{\lambda+L}(\tau) = \sum_{\xi\in\lambda+L} q^{\frac{1}{2}(\xi,\xi)}\,. \tag{4.37}$$

Example: V_{E_8} as a V_{E_8} Module

The E_8 root lattice is positive-definite, even, and self-dual, therefore V_{E_8} itself is the only irreducible V_{E_8} module (aka the "adjoint module"). VOAs whose only irreducible module are the adjoint module are called *holomorphic*, i.e. $\mathrm{Rep}(V_{E_8}) = \mathrm{Vec}$.

Using the description of the E_8 lattice as points in $\mathbb{Z}^8 \cup (\mathbb{Z}+\frac{1}{2})^8$, whose coordinates sum to an even integer, we obtain the E_8 partition function

$$\mathrm{tr}_{V_{E_8}}\, q^{L_0-\frac{c}{24}} = \frac{1}{\eta(\tau)^8}\sum_{\xi\in E_8} q^{\frac{1}{2}(\xi,\xi)} \tag{4.38}$$

$$= \frac{1}{2\eta(\tau)^8}\sum_{x\in\mathbb{Z}^8}\left(1+(-1)^{\sum x_i}\right)\left(q^{\frac{1}{2}\sum x_i^2} + q^{\frac{1}{2}\sum(x_i+\frac{1}{2})^2}\right) \tag{4.39}$$

$$Z_{E_8}(\tau) = \frac{1}{2\eta(\tau)^8}\left(\theta_1(\tau)^8+\theta_2(\tau)^8+\theta_3(\tau)^8+\theta_4(\tau)^8\right) \tag{4.40}$$

Note: even though this expression is not modular invariant, since $c_L - c_R \in 8\,\mathbb{Z}$ its modular non-invariance can be cured by a (2+1)d gravitational Chern-Simons term [176, 95] (see [150] for a recent discussion).

4.3.2 Automorphisms and Twisted Partition Functions

Describing Automorphisms

Recall that a VOA V has an automorphism/symmetry given by an (invertible) linear map $\hat{g} : V \to V$, if the action of $\hat{g}$ preserves the vacuum $\hat{g}\,|0\rangle = |0\rangle$, preserves the (chiral) stress-tensor $\hat{g}T(z) = T(z)$, and commutes with the state operator map

$$(\hat{g}\phi)(z) = \hat{g}\phi(z)\hat{g}^{-1}\,. \tag{4.41}$$

Suppose V_L is a lattice VOA constructed from the even lattice L. The subgroup of $\mathrm{Aut}(V_L)$ which maps the subalgebra of bosonic oscillators to itself is given by the (usually) non-split extension $T.O(L)$. Here $O(L)$ are the symmetries of the lattice itself and $T := \mathbb{R}^{\mathrm{rk}(L)}/L^*$ is the torus dual to L. T carries the natural action of $O(L)$ viewed as a subgroup of $O(\mathrm{rk}(L), \mathbb{R})$ [177].

The subgroup $T < T.O(L)$ acts trivially on the bosonic oscillators, and simply modifies the group algebra of quantized momentum modes by phases. That is, for any $x \in T$ we have the induced map

$$a_n^i \mapsto a_n^i \,, \tag{4.42}$$

$$\Gamma_\alpha \mapsto e^{2\pi i(x,\alpha)}\Gamma_\alpha \,. \tag{4.43}$$

Note again that this map is actually trivial if $x \in L^*$, and so we are only interested in $x \bmod L^*$. Conversely, the $O(L)$ factor acts on the bosons and also permutes the Γ_α [178, 177, 173, 172, 179] (see also Section 4.4 of [180]).

In this vein, suppose $g \in O(L)$ is any automorphism of the lattice L, then g preserves the skew form $(-1)^{(\cdot,\cdot)}$. Since the skew condition determines the 2-cocycle ϵ up to a 2-coboundary, we see that $\epsilon(\alpha, \beta)$ and $\epsilon(g\alpha, g\beta)$ are in the same cohomology class; and so

$$\frac{\epsilon(\alpha,\beta)}{\epsilon(g\alpha,g\beta)} = \frac{u(\alpha)u(\beta)}{u(\alpha+\beta)} \,, \tag{4.44}$$

for all $\alpha, \beta \in L$ and some 1-cocycle $u : L \to \{\pm 1\}$.

It is a classic result that such a $g \in O(L)$ and choice of 1-cocycle u define a lift to an automorphism $\hat{g}_u \in T.O(L) < \mathrm{Aut}(V_L)$ [178, 177]. This automorphism also acts in a nice way

$$\hat{g}_u(a_n) = (ga)_n \,, \tag{4.45}$$

$$\hat{g}_u\Gamma_\alpha = u(\alpha)\Gamma_{g\alpha} \,. \tag{4.46}$$

Furthermore, one can always choose "a standard lift" such that $u(\alpha) = 1$ on L^g [178]. We will not work with standard lifts right away, but with foresight we quote the fact that: any two standard lifts are conjugate in $\mathrm{Aut}(V_L)$ [172].[9]

Given a positive definite even lattice L and a lifted automorphism $\hat{g}_u \in \mathrm{Aut}(V_L)$, the irreducible $\hat{g}_u$-*twisted* modules can be constructed and classified similar to the untwisted

[9]The term standard lift is a bit of a misnomer and does not have the properties one might expect: for example, there can be more than one "standard" lift; moreover, since the extension $T.O(L)$ is *not* split (i.e. it is not a semi-direct product), the product of standard lifts is not necessarily standard. All of this richness can be traced back to the fact that there is no canonical way to construct a Lie algebra and lattice VOA from the root lattice L. For further discussion see [172, 173, 174] and references within.

modules [181, 182, 173]. In short, they are labelled by elements of the discriminant L^*/L fixed under the g action. i.e. they are of the form $V_{\lambda+L}$, where $\lambda + L \in (L^*/L)^g = \{\lambda \in L^*/L : (\mathbb{1} - g)\lambda \in L\}$.

The twisted characters are

$$\mathrm{tr}_{V_L}\, \hat{g}_u q^{L_0 - \frac{c}{24}} = \frac{\theta_{L^g,u}(\tau)}{\eta_g(\tau)}\,, \tag{4.47}$$

where we've defined the generalized theta function

$$\theta_{L^g,u}(\tau) := \sum_{\alpha \in L^g} u(\alpha) q^{(\alpha,\alpha)/2}\,. \tag{4.48}$$

The eta product $\eta_g(\tau)$ is given by

$$\eta_g(\tau) := \prod_{t|m} \eta(t\tau)^{b_t}, \tag{4.49}$$

when $g \in O(L)$ has cycle shape $\prod_{t|m} t^{b_t}$. The cycle shape can be determined by calculating the characteristic polynomial of the matrix representing the automorphism, which is of the form $\prod_{t|m}(\lambda^t - 1)^{b_t}$.

More physically, factors of $\eta(\tau)$ arise in characters/partition functions from the contributions of each independent tower of (chiral) bosonic modes. If g is, for example, a $\mathbb{Z}_2$ symmetry swapping the a^1_m and a^2_n towers, then one expects to get factors of $\eta(2\tau)$ in the vacuum character of the invariant sub-VOA due to the "extra length" of the combined $\mathbb{Z}_2$-invariant tower.

Finally, given an order m automorphism $g \in O(L)$, a standard lift $\hat{g}_u \in \mathrm{Aut}(V_L)$ has: order m if m is odd; order m if m is even and $(\alpha, g^{m/2}\alpha) \in 2\,\mathbb{Z}$ for all α; and order $2m$ otherwise (see e.g. Corollary 5.3.6 of [173]).

All of these technicalities about canonicality, lifts, and order doubling may seem overkill, but actually plays an important role in constructing as many duality defects as possible.

Finding Automorphisms

To understand how to find automorphisms of lattice VOAs, it is helpful to expand on the structure of the automorphism group of a VOA and its relationship to Lie algebras and physics.

We use the following fact for orientation: For a VOA V of "CFT-type" (like lattice VOAs or the Monster), the weight one subspace V_1 is a (possibly trivial) complex Lie algebra with bracket $[u, v] = u_0 v$. Moreover, all the weight spaces of V-modules are Lie algebra modules for V_1 (including the weight spaces of the adjoint module, i.e. the V_n), so that we can see the VOA and its modules as strata of V_1 Lie algebra modules [183]. From a physics perspective, we are not surprised that the currents form a Lie algebra under which things transform, this is just Noether's theorem and Wigner's theorem.

There are two extreme examples for the Lie algebra V_1 in relation to V: the symmetry currents V_1 generate V and $\mathrm{Aut}(V_1) \cong \mathrm{Aut}(V)$, as in affine Kac-Moody VOAs or their semi-simple quotients; or alternatively, there are no currents and $V_1 = 0$, as with the Virasoro algebra $\mathrm{Aut}(\mathrm{Vir}) = \{\mathbb{1}\}$, and the Monster CFT $\mathrm{Aut}(V^\natural) = \mathbb{M}$ [177].

The key move is to note that any automorphism/symmetry $\hat{g} \in \mathrm{Aut}(V)$ restricts to a Lie algebra automorphism of V_1.[10] Hence we ask: *to what extent can we understand* $\mathrm{Aut}(V)$ *by lifting automorphisms of the Lie algebra* $\mathrm{Aut}(V_1)$*?*

The answer to this question is known for lattice VOAs. For lattice VOAs, the weight one subspace is

$$(V_L)_1 = \mathcal{H} \oplus \bigoplus_{\alpha^2=2} \mathbb{C}\{\Gamma_\alpha\}\,, \tag{4.50}$$

where $\mathcal{H}$ is a Cartan subalgebra spanned by the weight-1 chiral bosons and the $\mathbb{C}\{\Gamma_\alpha\}$ play the role of the root spaces. The exact Lie algebra structure constants depend on the choice of ϵ, but all choices are isomorphic to a Lie algebra $\mathfrak{g}$ with root lattice L.

Using this, in [177] the authors prove that $\mathrm{Aut}(V_L)$ is a non-split product

$$\mathrm{Aut}(V_L) = K.O(\hat{L})\,, \tag{4.51}$$

where $K = \langle \{e^{u_0} : u \in (V_L)_1\}\rangle$ is the inner automorphism group generated by (exponentials of) "currents." In other words, we have that $\mathrm{der}(V_L) := \mathrm{Lie}(\mathrm{Aut}(V_L)) = \mathfrak{g}$, and so we will be interested in finite order automorphisms of $\mathfrak{g}$.[11] Note that what we called T before is the maximal toral subgroup, i.e. $T = \langle \{e^{u_0} : u \in \mathcal{H}\}\rangle$.

Beautifully, conjugacy classes of finite order automorphisms of (semi-)simple Lie algebras were classified by Kac [184, 185] (also see Section 8.3.3 of [174] and [173] for examples).

[10] A *super*symmetry would relate operators of V_1 to operators with different spins.

[11] If we were studying the derivations of a non-rational VOA, whose representations need not form an MTC and thus not be so discrete, its possible that $\mathrm{der}(V)$ isn't entirely formed by currents. In this case we could ask about infinitesimal outer automorphisms $\mathrm{out}(V) := \mathrm{der}(V)/\{\mathrm{currents}\} = \mathrm{der}(\mathrm{Rep}(V))$, which correspond to smooth deformations of the representation category. This is not exotic: consider n free chiral bosons, $V = \mathrm{Bos}(n)$, in this case $\mathrm{Aut}(\mathrm{Rep}(V)) \cong O(n)$.

This includes both inner and outer automorphisms! Even better, the classification gives an easy way to read off the fixed point Lie subalgebra under that symmetry! The theorem is as follows:

Theorem 1 (Theorem 8.6, [185]). *Let $\mathfrak{g}$ a finite dimensional simple Lie algebra with Dynkin diagram X_n, choose $k = 1, 2, 3$ and write $X_n^{(k)}$ for the corresponding affine Dynkin diagram (or "twisted Dynkin diagram" if $k > 1$) with $\ell+1$ nodes. Then choose non-negative relatively prime integers $s = (s_0, \cdots, s_\ell)$ and set*

$$m = k \sum_{i=0}^{\ell} a_i s_i \,, \tag{4.52}$$

where a_i are the marks of $X_n^{(k)}$. Then the following statements are true:

1. *The choices (k, s) define an order m automorphism $g_{k,s} \in \mathrm{Aut}(\mathfrak{g})$. For $k = 1$, if one chooses a set of simple roots and works in the usual Chevalley basis, the action is given by*
$$g_{1,s}(E^j) = e^{2\pi i s_j/m} E^j \,. \tag{4.53}$$
For $k = 2, 3$ the definition of E^j is slightly more complicated (see [185]).
2. *Up to conjugation, all order m automorphisms of $\mathfrak{g}$ are obtained this way.*
3. *Two automorphisms $g_{k,s}$ and $g_{k',s'}$ obtained in this way are conjugate by an automorphism of $\mathfrak{g}$ if and only if $k = k'$ and the sequence s can be transformed into s' by an automorphism of the diagram $X_n^{(k)}$.*
4. *The number k is the least positive integer for which $g_{k,s}^k$ is an inner automorphism. i.e. $k = 1$ automorphisms are inner, and $k = 2, 3$ automorphisms are outer.*
5. *Let $i_1, \cdots, i_p$ be all the indices such that $s_{i_1} = \cdots = s_{i_p} = 0$. Then the fixed-point Lie subalgebra $\mathfrak{g}^{g_{k,s}}$ is isomorphic to a direct sum of: the $(\ell - p)$-dimensional center $\cong \mathbb{C}^{\ell-p}$, and a semisimple Lie algebra whose Dynkin diagram is the subdiagram of the affine diagram $X_n^{(k)}$ consisting of the nodes $i_1, \cdots, i_p$.*

Kac's theorem is great for 3 purposes: enumerating finite order automorphisms of $\mathfrak{g}$, finding the fixed Lie subalgebra $\mathfrak{g}^{g_{k,s}}$, and understanding the systematic construction of $x \in T < \mathrm{Aut}(V_L)$ describing inner automorphisms of lattice VOAs like Equation (4.43). In particular, for an order m inner automorphism, x is constructed from the sequence s using the basis of fundamental weights $\omega_1, \ldots, \omega_l$, dual to the basis of simple coroots:

$$x = \frac{1}{m} \sum_{j=1}^{l} s_j \, \omega_j \,. \tag{4.54}$$

On the other hand, it is systematically easier for calculations of twisted characters of outer automorphisms to use a slightly different procedure from [179], which we describe in the remainder of this section.

Let V_L be a lattice VOA and fix a choice of Cartan subalgebra $\mathcal{H}$ of $(V_L)_1$ as in Equation (4.50), further fix a choice of simple roots Δ. Then it can be shown that: up to conjugacy, every finite-order $\hat{\sigma} \in \mathrm{Aut}(V_L)$ is of the form

$$\hat{\sigma} = e^{2\pi i (x, \cdot\,)}.\hat{g} \tag{4.55}$$

where $\hat{g}$ is a (fixed) choice of *standard lift* of $g \in H_\Delta$ and $x \in L^g \otimes \mathbb{Q}$ is some phase understood modulo $(L^g)^*$. By $H_\Delta := O(L)_{\{\Delta\}}$ we mean the setwise stabilizer of Δ in $O(L)$, note that $H_\Delta \cong O(L)/W$ which is in turn isomorphic to the automorphism group of the Dynkin diagram associated to L, its just key that we've fixed a base Δ. See [177] and Theorem 2.13 of [179] for details.

This theorem allows for a simple computation of the $\hat{\sigma}$-twisted traces

$$\mathrm{tr}_{V_L}\, \hat{\sigma}\, q^{L_0 - \frac{c}{24}} = \frac{1}{\eta_g(\tau)} \sum_{\alpha \in L^g} e^{2\pi i (x,\alpha)} q^{(\alpha,\alpha)/2}\,. \tag{4.56}$$

Importantly, if $\hat{\sigma}$ is built from the standard lift $\hat{g}$ of $g \in O(L)$ as above, then $\hat{\sigma}$ has order $|g|$ if and only if

$$x \in \begin{cases} 0 + (1/m)(L^*)^g & \text{and } |\hat{g}| = m \\ \zeta + (1/m)(L^*)^g & \text{and } |\hat{g}| = 2m \end{cases} \tag{4.57}$$

where $\zeta \in (1/2m)(L^*)^g$.

The short of all this is that: rather than studying finite order automorphisms of $(V_L)_1$ and lifting to automorphisms of the VOA, we can take a more ground-up approach. For example, suppose we want to find all $\hat{\sigma}$-twisted characters of a VOA V_L up to conjugacy, and we want $\hat{\sigma}$ to have order 2. Then we fix a set of simple roots for L and enumerate all $g \in H_\Delta$ of order 1 or 2. For each g, we make some choice of standard lift $\hat{g} \in \mathrm{Aut}(V_L)$, so $\hat{g}$ has order 1, 2, or 4. Then we enumerate all inner automorphisms of order up to 4 (by making choices of vectors x as described previously). Finally, we combine all of our choices as in (4.55), and throw away whichever do not have order 2. The theorem guarantees that we have produced all $\hat{\sigma}$ of order 2 up to conjugacy.

4.3.3 Orbifolds of V_{E_8}

Now that we understand automorphisms of lattice VOAs, we may study their associated orbifolds. Throughout, let $V := V_{E_8}$ be the holomorphic VOA[12] constructed from the E_8 root lattice L as in Example 4.3.1. V is the unique (up to isomorphism) holomorphic CFT with chiral central charge $c_L = 8$, and gauging by any non-anomalous $\mathbb{Z}_m$ symmetry will simply produce another holomorphic CFT with the same central charge, so $[V/\mathbb{Z}_m] \cong V$.

For this reason, computing orbifold partition functions isn't particularly illuminating, they are all just as in Equation (4.40). However, while the orbifold theory $[V/\mathbb{Z}_m]$ is isomorphic to V, they are not identical (just as in the Ising model). To appreciate this, we should look at what it means to orbifold V at the level of lattices.

For this, we will follow the conventions of [186]. In particular, L will be spanned by simple roots

$$\begin{aligned} \alpha_1 &= (-1, 1, 0^6), \\ \alpha_2 &= (0, -1, 1, 0^5), \\ &\vdots \\ \alpha_7 &= (0^6, -1, 1), \\ \alpha_8 &= \left(\frac{1}{2}^5, -\frac{1}{2}^3\right), \end{aligned} \tag{4.58}$$

which fit into the Dynkin diagram

$$\begin{array}{ccccccccccccc} & & & & & & & & \alpha_8 & & & & \\ & & & & & & & & | & & & & \\ \alpha_1 & — & \alpha_2 & — & \alpha_3 & — & \alpha_4 & — & \alpha_5 & — & \alpha_6 & — & \alpha_7 \end{array} \; . \tag{4.59}$$

This diagram can be extended to its affine version with the help of the highest root

$$\alpha_{\text{high}} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8 \,, \tag{4.60}$$

which yields

$$\begin{array}{ccccccccccccccc} & & & & & & & & & & \alpha_8 & & & & \\ & & & & & & & & & & | & & & & \\ (-\alpha_{\text{high}}) & — & \alpha_1 & — & \alpha_2 & — & \alpha_3 & — & \alpha_4 & — & \alpha_5 & — & \alpha_6 & — & \alpha_7 \end{array} \; . \tag{4.61}$$

[12] Also simple, rational, C_2-cofinite, self-contregredient, CFT-type. We will ignore such technical details from here out in this section.

Now, let $G = \langle g_s \rangle \cong \mathbb{Z}_m$ be a subgroup of $T < \mathrm{Aut}(V)$ generated by the sequence s as in the previous section (and with $k = 1$) i.e. G acts by a phase on the group algebra. Explicitly, write $x := \sum x_j \omega_j$ with $x_j := s_j/m$, then

$$\begin{aligned} a_n^i &\mapsto a_n^i\,, \\ \Gamma_\alpha &\mapsto e^{2\pi i (x,\alpha)}\Gamma_\alpha\,, \end{aligned} \tag{4.62}$$

with $(x, \alpha_i) \geq 0$, $(x, \alpha_{\mathrm{high}}) \leq 1$, and $x_i \in \frac{1}{m}\mathbb{Z}$.

Such a symmetry is non-anomalous and can be orbifolded if we can extend the invariant sub-VOA V^G by twisted sectors to a distinct VOA.[13] In physics terms, the orbifold Hilbert space is obtained as an extension of the subspace of unscreened/invariant operators, as in Equation (4.14) for a $\mathbb{Z}_2$ symmetry. The anomaly classifies the obstruction to this extension procedure.

The group action above leaves the chiral Cartan bosons unscreened as well as the vertex operators with $(x, \alpha) \in \mathbb{Z}$. This last condition distinguishes a sublattice $L_0 \subset L$ of index m, so that the fixed sub-VOA V^G is the lattice VOA constructed from L_0. Said in reverse, $\mathrm{rk}(L_0) = \mathrm{rk}(L)$ both define the same Heisenberg module of chiral bosons, but in V^G only the momentum states corresponding to L_0 are admissible.[14]

Our question then becomes: can we extend L_0 to an even self-dual lattice L', distinct from L, inside L_0^*?[15] This can be depicted

$$\begin{tikzcd} & L = L^* \arrow[dr, phantom, "\subseteq"] & \\ L_0 \arrow[ur, phantom, "\subseteq"] \arrow[dr, phantom, "\subseteq"] & & L_0^* \\ & L' = L'^* \arrow[ur, phantom, "\subseteq"] & \end{tikzcd} \ . \tag{4.63}$$

Such an L' exists (and is unique) if and only if

$$q(x) := \frac{(x,x)}{2} \in \frac{1}{m}\mathbb{Z}\ . \tag{4.64}$$

Generally, $q(x) \in \frac{1}{m^2}\mathbb{Z}$ because $mx \in L$, so the anomaly class is the class

$$[q(x)] \in \left(\frac{1}{m}\mathbb{Z}\right) / \left(\frac{1}{m^2}\mathbb{Z}\right) \cong \mathbb{Z}_m\ . \tag{4.65}$$

[13]Note: If V is a lattice VOA, then under an inner automorphism the fixed sub-VOA V^G is always another lattice VOA.

[14]Note: in the mathematics literature, V^G is often called the "orbifold VOA," but we will always mean the full extension.

[15]Dropping the even requirement is how one produces fermionic orbifolds.

	$[V/\mathbb{Z}_m]$	$[V/\mathbb{Z}_m]^1$	$\cdots$	$[V/\mathbb{Z}_m]^{m-1}$
V	$\mathcal{H}_0^0$	$\mathcal{H}_0^1$	$\cdots$	$\mathcal{H}_0^{m-1}$
V_1	$\mathcal{H}_1^0$	$\mathcal{H}_1^1$	$\cdots$	$\mathcal{H}_1^{m-1}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
V_{m-1}	$\mathcal{H}_{m-1}^0$	$\mathcal{H}_{m-1}^1$	$\cdots$	$\mathcal{H}_{m-1}^{m-1}$

Table 4.1: The holomorphic VOA V decomposes as a direct sum of the untwisted Hilbert spaces, where the upper grading denotes the "electric" $\mathbb{Z}_m$ charge. The orbifold VOA decomposes as a direct sum of the gauge invariant Hilbert spaces, where the lower index grading denotes the dual "magnetic" $\hat{\mathbb{Z}}_m$ charge.

Note: $H^3(\mathbb{Z}_m, U(1)) = \mathbb{Z}_m$, so the matchup with anomalies is evident.

In the language of VOAs we are just saying that the irreducible representations $\mathrm{Irr}(V^G)$ have group like fusion, with a fusion group given by a central extension of $\mathbb{Z}_m$ by $\mathbb{Z}_m$, reflecting the structure of L_0^*/L_0 when we try to factor through L. The structure of the fusion group is controlled by a class living in $H^2(\mathbb{Z}_m, \hat{\mathbb{Z}}_m) = \mathbb{Z}_m$, but should not be confused with the group of anomalies. The anomaly does influence this group, but the map between anomalies and fusion rules is not 1-to-1 (see [172]), as can be seen in following examples with $\mathbb{Z}_2$. However, in the case that G acts non-anomalously, then $\mathrm{Irr}(V^G) = \mathbb{Z}_m \times \mathbb{Z}_m$ for sure.

If we suggestively label the elements of $\mathrm{Irr}(V^G)$ as $\mathcal{H}_j^i$, mimicking the decomposition following Equation (4.12), then $\mathrm{Irr}(V^G)$ can be organized like Table 4.1. The original VOA decomposes into the "electrically charged" sectors

$$V = \bigoplus_{i=1}^{m} \mathcal{H}_0^i \,, \tag{4.66}$$

and the orbifold VOA, if it exists, decomposes into the "magnetically charged" sectors as

$$[V/\mathbb{Z}_m] = \bigoplus_{j=1}^{m} \mathcal{H}_j^0 \,. \tag{4.67}$$

Each of the $\mathcal{H}_j^i \in \mathrm{Irr}(V^G)$ is associated with a coset of L_0^*/L_0, and the discriminant $q(x)$ is simply telling us the conformal weight in that sector. In the non-anomalous case, the conformal weight of a vector in $\mathcal{H}_j^i$ is $\frac{ij}{m} + \mathbb{Z}$ which gives $\mathrm{Irr}(V^G)$ the structure of a "metric Abelian group" (see Section 4 of [173] for more details).

We record the automorphisms of V, anomaly, and corresponding Lie algebra for small m in Table 4.2. In Table 4.3 we record just the non-anomalous symmetries with the marked Dynkin diagram.

Example: $\mathbb{Z}_2$ Orbifold of V_{E_8}

Let $V := V_{E_8}$ throughout. By Kac's theorem and the discussion of Section 4.3.2, V has two automorphisms of order 2 up to conjugacy (see Table 4.2). Call the non-anomalous one 2A and the anomalous one 2B, they are generated by

$$x^A := \frac{1}{2}\omega_7 \tag{4.68}$$

$$x^B := \frac{1}{2}\omega_1\,. \tag{4.69}$$

Let us start with the 2A symmetry. In this case, our fixed sub-VOA is $V^{\mathbb{Z}_2^A} = V_{D_8}$. To see this at the level of lattices, first note $(x^A, \alpha_i) = 0$ for all i except $i = 7$, so the only nontrivial action is $\Gamma_{\alpha_7} \mapsto -\Gamma_{\alpha_7}$. Vertex operators like $\Gamma_{2\alpha_7}$ do remain invariant though, so the invariant VOA is the lattice VOA constructed from the span of $\{\alpha_1, \ldots, \alpha_6, 2\alpha_7, \alpha_8\}$. The reason why Kac's theorem works is because of the form of the highest root α_{high} in Equation (4.60), which guarantees that the fixed sublattice is also spanned by $\{\alpha_1, \ldots, \alpha_6, -\alpha_{\text{high}}, \alpha_8\}$. It is then clear from the way these vectors fit inside of the affine Dynkin diagram in Equation (4.61) that they span a D_8 lattice.

Since 2A is non-anomalous, the preceding discussion tells us that the fusion group is a metric Abelian group $\mathrm{Irr}(V^{\mathbb{Z}_2^A}) = \mathbb{Z}_2 \times \mathbb{Z}_2$ with two isotropic subgroups. To see this at the level of lattices, we note that the E_8 lattice can be presented as the span of the rows of

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \tag{4.70}$$

as in Chapter 4 of [186] (these are not roots). Similarly, the D_8 lattice and D_8^* lattice can be written as the span of the rows of

$$\begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{4.71}$$

respectively. Conveniently for us, [186] also records the following elements of D_8^* which are representatives of separate classes in D_8^*/D_8:

$$[0] := (0, 0, \ldots, 0)\,, \tag{4.72}$$

$$[1] := (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2})\,, \tag{4.73}$$

$$[2] := (0, 0, \ldots, 1)\,, \tag{4.74}$$

$$[3] := (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, -\tfrac{1}{2})\,. \tag{4.75}$$

From this we see that $D_8^*/D_8 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ with decomposition and discriminant

$$D_8^* = \begin{array}{|c|c|} \hline D_8 + [0] & D_8 + [1] \\ \hline D_8 + [3] & D_8 + [2] \\ \hline \end{array} \quad \overset{\text{Weight}}{\rightsquigarrow} \quad q^A(x) = \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 0 & \frac{1}{2} \\ \hline \end{array}\,. \tag{4.76}$$

Note that $(D_8 + [0]) \cup (D_8 + [1])$ and $(D_8 + [0]) \cup (D_8 + [3])$ both give E_8 lattices in D_8^*, and that they're not the same (their intersection is only a D_8 lattice).

The characters can be obtained as in Section 4.3.1 and are

$$\mathrm{tr}_{V_{D_8+[0]}}\, q^{L_0 - \frac{c}{24}} = \frac{1}{2\eta(\tau)^8}\big(\theta_3^8(q) + \theta_4^8(q)\big)\,, \tag{4.77}$$

$$\mathrm{tr}_{V_{D_8+[1]}}\, q^{L_0 - \frac{c}{24}} = \frac{1}{2\eta(\tau)^8}\big(\theta_2^8(q) + \theta_1^8(q)\big)\,, \tag{4.78}$$

$$\mathrm{tr}_{V_{D_8+[2]}}\, q^{L_0 - \frac{c}{24}} = \frac{1}{2\eta(\tau)^8}\big(\theta_3^8(q) - \theta_4^8(q)\big)\,, \tag{4.79}$$

$$\mathrm{tr}_{V_{D_8+[3]}}\, q^{L_0 - \frac{c}{24}} = \frac{1}{2\eta(\tau)^8}\big(\theta_2^8(q) - \theta_1^8(q)\big)\,. \tag{4.80}$$

Since $\theta_1(\tau) = 0$, we can clearly see how to decompose V into V_{D_8} modules in two distinct ways just by comparing twisted partition functions.

Looking at the 2B symmetry, $V^{\mathbb{Z}_2^B} = V_{A_1\times E_7}$ and has fusion group $\mathbb{Z}_2 \times \mathbb{Z}_2$ as well. We may write the discriminant in this case and it's

$$q^B(x) = \begin{array}{|c|c|}\hline 0 & 0 \\ \hline \frac{3}{4} & \frac{1}{4} \\ \hline \end{array}\,. \tag{4.81}$$

As expected, the only isotropic subgroup corresponds to the decomposition we started with initially.

It would be remiss not to point out that $q^A(x)$ and $q^B(x)$ are the spins of simple anyons in the $\mathbb{Z}_2$ gauge theory and twisted $\mathbb{Z}_2$ gauge theory (also called semion-antisemion) respectively.

Generally, since V is well-behaved, $\mathrm{Rep}(V^{\mathbb{Z}_m})$ is a modular tensor category (this follows rigorously from technical results of [187, 188, 189]) whose simple objects, the irreducible modules $\mathcal{H}^{n_e}_{n_m}$, correspond to simple anyons in a (2+1)d TFT which is interpretable as a (possibly twisted) $\mathbb{Z}_m$ gauge theory. If we couple $V^{\mathbb{Z}_m}$ to the bulk TFT as a 2d boundary theory, then the anyon associated to $\mathcal{H}^{n_e}_{n_m}$ may terminate on operators from the corresponding module. We will elaborate on these (2+1)d points further in Section 4.5 (see also [1]).

4.3.4 Computing Defected Partition Functions in $(E_8)_1$

In this section we use our general results about automorphisms of lattice VOAs to explain how to compute $\mathbb{Z}_m$ "duality defected" partition functions in the chiral $(E_8)_1$ WZW model, i.e. we compute twisted characters for $\mathrm{TY}(\mathbb{Z}_m)$ actions on the lattice VOA $V := V_{E_8}$.

Suppose we want to find partition functions with a duality defect line X twisting the Euclidean time direction, $Z_V[0, X]$, separating V from some $\mathbb{Z}_m$ orbifold $[V/\mathbb{Z}_m]$. In practice, we proceed as follows for order m symmetries:

1. Following Kac's theorem, we find all inner automorphisms of the E_8 Lie algebra of order m. Using Equation (4.64), we compute which symmetries of V are non-anomalous and thus can actually be gauged. Note: there may be multiple conjugacy classes of non-anomalous symmetries e.g. for $m = 5$ there are two (see Table 4.3).
2. Taking one of the non-anomalous symmetry groups $G \cong \mathbb{Z}_m$, the fixed-point sub-VOA V^G is a lattice VOA and is obtained as described in Section 4.3.3. The irreps $A := \mathrm{Irr}(V^G)$ form a metric Abelian group (A, h), with $A \cong \mathbb{Z}_m \times \hat{\mathbb{Z}}_m$ and metric

function $h(i,j) = ij/m$. V and $[V/G]$ decompose as direct sums over the isotropic subgroups with irrep labels $(i,0)$ and $(0,j)$ respectively, i.e.

$$V = \bigoplus_i \mathcal{H}_0^i \,, \tag{4.82}$$

$$[V/G] = \bigoplus_j \mathcal{H}_j^0 \,. \tag{4.83}$$

3. Finding automorphisms of V^G which "swap the axes" corresponding to V and $[V/G]$ in A, we obtain the defected partition functions as the *twisted* characters for this second automorphism.

We've already seen the first two points in detail, so we elaborate on the final point.

Write $O(A,h)$ for the group of automorphisms of A which preserve h. For simplicity, consider the case that m is prime, then we have that

$$O(A,h) = \mathbb{Z}_m^\times \rtimes \mathbb{Z}_2 \,. \tag{4.84}$$

The $\mathbb{Z}_m^\times$ factor corresponds to the freedom in redefining the action of G on V and is the normal subgroup $SO(A,h)$ which preserves the axes of A. This leaves $\mathbb{Z}_2 = O(A,h)/SO(A,h)$ to act by swapping the axes of A. The $\mathbb{Z}_2$ factor acts by electric-magnetic duality in the (2+1)d TFT. We can work this group out for more general m, but generally $O(A,h)$ is just the group of symmetries of $\mathbb{Z}_m$ Dijkgraaf-Witten theory [24] aka the group of braided auto-equivalences of $\mathcal{Z}(\mathrm{Vec}_G)$ (see [1, 64] for physics discussions, and [163, 44, 45] for mathematics discussions).

Now, any automorphism of V^G induces an automorphism of A which preserves the conformal weight h. Note: many elements of $\mathrm{Aut}(V^G)$ induce the same automorphism in $O(A,h)$.

Our principal claim for this chapter is that: $(V^G)^{\mathbb{Z}_2} = V^{\mathrm{TY}(G)}$ *if and only if the* $\mathbb{Z}_2$ *action on* V^G *switches the axes of* A. We prove this in Theorem 2. At the level of partition functions, this means if $\hat\sigma \in \mathrm{Aut}(V^G)$ induces a $\mathbb{Z}_2$ action on A which "swaps the axes" of $\mathrm{Irr}(V^G)$, then there is a $\mathbb{Z}_m$ Tambara-Yamagami line $X_{\hat\sigma}$ in V, whose partition function is the $\hat\sigma$-twisted character of V^G

$$Z_V[0, X_{\hat\sigma}] = \mathrm{tr}_{V^G}\, \hat\sigma q^{L_0 - \frac{c}{24}} \,. \tag{4.85}$$

In practice, this is where the technical details about lifts becomes relevant. We will identify $\mathbb{Z}_2$ symmetries of the underlying lattices associated to V^G which should switch the axes, and then consider their lifts to the VOA. This means even after we have picked a non-anomalous $\mathbb{Z}_m$ symmetry and a $\mathbb{Z}_2$ which switches the axes, there is *still* degeneracy in the partition function depending on how the $\mathbb{Z}_2$ lifts to a VOA automorphism, leading to different TY-lines.

Comparing to the end of Section 4.2.3, we see a 2d shadow of this equation. We already saw that the duality defected partition function involves only uncharged operators, and that we just had to resolve a sign in the partition function which tracks "if V and $[V/G]$ see the primary ϕ differently." This role of the $\hat{\sigma}$-twisted character is to track this sign.

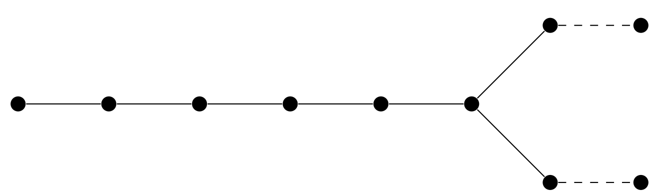
4.3.5 $\mathbb{Z}_2$ Duality Defects

Given $V := V_{E_8}$, there is one non-anomalous $\mathbb{Z}_2$ symmetry up to conjugation, and there are 4 duality defected partition functions, i.e. $\mathbb{Z}_2$ Tambara-Yamagami lines or duality defects. In this section we will compute them in detail using our VOA-theoretic techniques and then compare to the result using fermionization as in [78, 50, 146].

$\mathbb{Z}_2$ Duality Defects from Lie Theory

We start by following our general procedure outlined in the previous section. There is one non-anomalous $\mathbb{Z}_2$ action on V, generated by $x^A = \frac{1}{2}\omega^7$ in the sense of Equation (4.62), fixing the subalgebra $V^{\mathbb{Z}_2^A} \cong V_{D_8}$. We found this by taking the affine E_8 Dynkin diagram and "chopping" the 7th root to produce the D_8 Dynkin diagram.

We know from our discriminant calculations in Section 4.3.3 that there are two ways to extend the D_8 root lattice into the E_8 lattice. In fact, we can see these two extensions directly from the Dynkin diagram

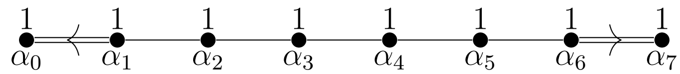


(4.86)

To obtain a TY-category, we want a $\mathbb{Z}_2$ action on $V^{\mathbb{Z}_2}$ which swaps the axes of $\mathrm{Irr}(V^{\mathbb{Z}_2})$. At the level of root lattices, this means swapping the two E_8's in D_8^*, and the $\mathbb{Z}_2$ which does this is obvious: the D_8 Dynkin diagram automorphism.

We start by looking at the conjugacy classes of $\mathbb{Z}_2$ outer automorphisms of $\mathfrak{so}(16)$, which all come from the D_8 Dynkin diagram automorphism. Using Kac's theorem with

$m = 2$ and $k = 2$, we need to study the Dynkin diagram $D_8^{(2)}$. This is given (with its marks) by



(4.87)

Kac's theorem tells us that there are four different outer $\mathbb{Z}_2$ automorphisms, which are given by sequences $(s_0, \ldots, s_7)$, where all components vanish except $s_i = 1$ with $i \in \{0, 1, 2, 3\}$.[16] The theorem also produces the fixed Lie subalgebras

$$\mathfrak{so}(1) \oplus \mathfrak{so}(15)\,, \quad \mathfrak{so}(3) \oplus \mathfrak{so}(13)\,, \quad \mathfrak{so}(5) \oplus \mathfrak{so}(11)\,, \quad \mathfrak{so}(7) \oplus \mathfrak{so}(9)\,. \tag{4.88}$$

The fixed Lie subalgebras tell us about the weight-one subspace of the four different $(V_{D_8})^{\mathbb{Z}_2}$s. Of course, $\mathfrak{so}(1)$ doesn't really exist, but the demand for its appearance in the pattern becomes clear if we recall that $\mathfrak{so}(n)_1$ is the theory of n free fermions. The patterns above reflect that the four different $(V_{D_8})^{\mathbb{Z}_2}$s are just breaking into theories of $(1, 15)$, $(3, 13)$, $(5, 11)$, and $(7, 9)$ left-moving free fermions. We address this further in Section 4.3.5.

Moving on, let's compute the appropriate $\mathbb{Z}_2$-twisted characters of V_{D_8}. By the result of [179] they all take the form $\hat{\sigma}_i = e^{2\pi i (x^i, \cdot)}.\hat{g}$ and have twisted character formula given by Equation (4.56). The first piece for the computation is the twisted eta product $\eta_g(\tau)$. Using (4.49), the D_8 Dynkin diagram automorphism gives

$$\eta_g(\tau) = \eta(\tau)^6 \eta(2\tau) = \frac{\eta(\tau)^8}{\sqrt{\theta_3(\tau)\theta_4(\tau)}}. \tag{4.89}$$

Moreover, the invariant *root lattice* L^g can be read off from the D_8 Dynkin diagram

(4.90)

Note that the fixed root lattice is actually a C_7 lattice, not the B_7 lattice that we might naively expect from looking at the fixed point Lie subalgebras.

[16] To be very explicit, if we think of $\mathfrak{so}(n)$ as being $n \times n$ antisymmetric matrices, then these four classes of outer automorphisms can be thought of as coming from the adjoint actions of the four matrices $\mathrm{diag}(+1^{2i+1}, -1^{15-2i}) \in O(16)$.

The Dynkin diagram automorphism does not experience order doubling, so we can obtain the weights $e^{2\pi i(x^i,\alpha)}$ by finding vectors in $(1/2)({D_8}^*)^g$ as described in Theorem 2.13 of [179]. We find four choices:

$$\mathfrak{so}(1) \oplus \mathfrak{so}(15) : \quad x^0 = 0\,, \tag{4.91}$$

$$\mathfrak{so}(3) \oplus \mathfrak{so}(13) : \quad x^1 = \frac{1}{2}\omega_1\,, \tag{4.92}$$

$$\mathfrak{so}(5) \oplus \mathfrak{so}(11) : \quad x^2 = \frac{1}{2}\omega_2\,, \tag{4.93}$$

$$\mathfrak{so}(7) \oplus \mathfrak{so}(9) : \quad x^3 = \frac{1}{2}\omega_3\,. \tag{4.94}$$

We can then finally sum over the C_7 root lattice, with these weights inserted, and combine with the twisted eta functions to obtain the following four duality defects:

$$\mathrm{tr}_{V_{D_8}}\, \hat{\sigma}_i\, q^{L_0-\frac{c}{24}} = \frac{\sqrt{\theta_3(\tau)\theta_4(\tau)}}{2\eta(\tau)^8}\big(\theta_3^i(\tau)\theta_4^{7-i}(\tau)+\theta_3^{7-i}(\tau)\theta_4^i(\tau)\big)\,. \tag{4.95}$$

In terms of $\mathfrak{so}(2r+1)_1$ characters

$$\chi^{(r)}_{\hat{\omega}_0}(\tau) = \frac{1}{2}\left(\frac{\theta_3(\tau)^{r+\frac{1}{2}}+\theta_4(\tau)^{r+\frac{1}{2}}}{\eta(\tau)^{r+\frac{1}{2}}}\right)\,, \tag{4.96}$$

$$\chi^{(r)}_{\hat{\omega}_1}(\tau) = \frac{1}{2}\left(\frac{\theta_3(\tau)^{r+\frac{1}{2}}-\theta_4(\tau)^{r+\frac{1}{2}}}{\eta(\tau)^{r+\frac{1}{2}}}\right)\,, \tag{4.97}$$

$$\chi^{(r)}_{\hat{\omega}_r}(\tau) = \frac{1}{\sqrt{2}}\frac{\theta_2(\tau)^{r+\frac{1}{2}}}{\eta(\tau)^{r+\frac{1}{2}}}\,, \tag{4.98}$$

we have (up to normalization)

$$Z_V[0,X_i] = \chi^{(i)}_{\hat{\omega}_0}(\tau)\chi^{(7-i)}_{\hat{\omega}_0}(\tau) - \chi^{(i)}_{\hat{\omega}_1}(\tau)\chi^{(7-i)}_{\hat{\omega}_1}(\tau)\,. \tag{4.99}$$

Fermionization

Given a bosonic theory T_b with a non-anomalous global $\mathbb{Z}_2$ symmetry, we may produce a fermionic theory T_f by turning the $\mathbb{Z}_2$ symmetry into a $\mathbb{Z}_2^f$ "Grassmann parity" i.e. $(-1)^F$ symmetry, whose background connection is the spin structure on the manifold (an affine

$\mathbb{Z}_2$ connection).[17] This is a generalization of the classic "Jordan-Wigner transformation." In practice, on a spacetime M, this Jordan-Wigner transformation is given by

$$Z_{T_f}[\rho] = \frac{1}{\sqrt{|H^1(M,\mathbb{Z}_2)|}} \sum_{\alpha\in H^1(M,\mathbb{Z}_2)} (-1)^{\mathrm{Arf}[\alpha+\rho]+\mathrm{Arf}[\rho]} Z_{T_b}[\alpha]\,, \tag{4.100}$$

where $\mathrm{Arf}[\rho]=0$ for even spin structures and $\mathrm{Arf}[\rho]=1$ for odd spin structures. In other words, it is the number of zero modes of the Dirac operator mod 2 [190, 191, 77, 115].

In modern language, the partition function $(-1)^{\mathrm{Arf}[\rho]}$ is the generator of the group $\mathrm{Hom}(\Omega_2^{\mathrm{Spin}}(pt), U(1)) = \mathbb{Z}_2$ of invertible topological phases that can be stacked with a theory with $(-1)^F$ symmetry [66]. It can also be thought of as the effective action for the continuum version of the Majorana-Kitaev chain. Practically, the effect of stacking with the Arf phase is to change the relative sign of the even and odd partition functions.

Conversely, given any fermionic theory T_f (with $c_L - c_R \in 8\,\mathbb{Z}$), it is always possible to gauge the $(-1)^F$ symmetry, i.e. sum over spin-structures, and produce a bosonic theory T_b with a non-anomalous global $\mathbb{Z}_2$ symmetry by the inverse of the Jordan-Wigner transformation, or "GSO projection" (see [77, 74]). This can be done in two ways, by

$$Z_{T_b^A}[\alpha] = \frac{1}{\sqrt{|H^1(M,\mathbb{Z}_2)|}} \sum_{\rho} (-1)^{\mathrm{Arf}[\alpha+\rho]+\mathrm{Arf}[\rho]} Z_{T_f}[\rho]\,, \tag{4.101}$$

or by first stacking with Arf

$$Z_{T_b^B}[\alpha] = \frac{1}{\sqrt{|H^1(M,\mathbb{Z}_2)|}} \sum_{\rho} (-1)^{\mathrm{Arf}[\alpha+\rho]} Z_{T_f}[\rho]\,. \tag{4.102}$$

The two bosonizations are related by gauging the emergent $\mathbb{Z}_2$ symmetry [139, 115, 78] as shown in Figure 2.10, i.e.

$$[\mathrm{GSO}[T_f]/\,\mathbb{Z}_2] = \mathrm{GSO}[T_f \times \mathrm{Arf}]\,. \tag{4.103}$$

For more examples of this procedure in action, see [146, 50, 67, 2, 68, 69, 192].

When we gauge $(-1)^F$ we can ask what happens to the chiral fermion parities $(-1)^{F_L}$ and $(-1)^{F_R}$. As explained in [78, 146, 50] they become $\mathrm{TY}(\mathbb{Z}_2)$ duality defect lines separating T_b^A from T_b^B.

[17] We assume unitarity so that spin and statistics are related.

To make all this concrete, recall the partition function(s) for a chiral (left-moving say) Majorana-Weyl fermion in it's different sectors

$$\mathrm{tr}_{\mathrm{NS}}\, q^{L_0-\frac{c}{24}} = \sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}}\,, \tag{4.104}$$

$$\mathrm{tr}_{\mathrm{R}}\, q^{L_0-\frac{c}{24}} = \sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}}\,, \tag{4.105}$$

$$\mathrm{tr}_{\mathrm{NS}}(-1)^F q^{L_0-\frac{c}{24}} = \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}\,, \tag{4.106}$$

$$\mathrm{tr}_{\mathrm{R}}(-1)^F q^{L_0-\frac{c}{24}} = \sqrt{\frac{\theta_1(\tau)}{\eta(\tau)}}\,. \tag{4.107}$$

A "full" Majorana fermion (with both left and right-moving contribution) has the spin-structure dependent partition function

$$Z_{\mathrm{Full}}[\rho] = Z_{\mathrm{Maj.}}[\rho]\bar{Z}_{\mathrm{Maj.}}[\rho]\,. \tag{4.108}$$

For example, we can GSO project the Majorana-Weyl fermion using the formulas above, and in either case we get the Ising partition function

$$Z_{T_b^{A,B}}[0,0] = \frac{1}{2}\left(\left|\frac{\theta_3}{\eta}\right| + \left|\frac{\theta_2}{\eta}\right| + \left|\frac{\theta_4}{\eta}\right| \pm \left|\frac{\theta_1}{\eta}\right|\right)\,. \tag{4.109}$$

To recover the Ising duality defected partition function, we need to gauge the diagonal spin-structure but first insert a line twisting by the chiral fermion parity i.e. we have

$$\sum_\rho Z_{\mathrm{Maj.}}[\rho_1,\rho_2+1]\bar{Z}_{\mathrm{Maj.}}[\rho_1,\rho_2] \propto \sqrt{2}|\chi_0|^2 - \sqrt{2}|\chi_{\frac{1}{2}}|^2\,. \tag{4.110}$$

More generally, all of this comes from the fact that $(-1)^{F_L}$ has one unit of the "mod 8 anomaly" coming from $\mathrm{Hom}(\Omega_3^{\mathrm{Spin}}(B\,\mathbb{Z}_2), U(1)) = \mathbb{Z}_8$. For n Majorana fermions the bosonization is the $\mathrm{Spin}(n)_1$ WZW model, and the $(-1)^{F_L}$ symmetry line bosonizes to $\mathrm{TY}(\mathbb{Z}_2, 1, +1/\sqrt{2})$ if $n = 1, 7 \bmod 8$ and $\mathrm{TY}(\mathbb{Z}_2, 1, -1/\sqrt{2})$ if $n = 3, 5 \bmod 8$. See [78, 50, 193, 194] and references within for more technical details.

Duality Defects from Fermionization

From Section 4.3.1 we see that chiral theory $V := V_{E_8}$ is just 16 chiral Majorana-Weyl fermions. From the discussion in the previous section, we see that we have 4 choices of duality defect based on if we're going to view the E_8 theory as coming from the GSO projection of

$$Z^{1}_{\text{Maj.}}[\rho]Z^{15}_{\text{Maj.}}[\rho]\,, \quad Z^{3}_{\text{Maj.}}[\rho]Z^{13}_{\text{Maj.}}[\rho]\,, \quad Z^{5}_{\text{Maj.}}[\rho]Z^{11}_{\text{Maj.}}[\rho]\,, \quad Z^{7}_{\text{Maj.}}[\rho]Z^{9}_{\text{Maj.}}[\rho]\,. \tag{4.111}$$

Let X_p ($p = 1,3,5,7$) be the duality defect obtained from bosonizing $(-1)^{F_L}$ in the $Z^{p}_{\text{Maj.}}[\rho]Z^{16-p}_{\text{Maj.}}[\rho]$ setup. Then the partition function with X_p inserted along the spatial S^1 is

$$Z_V[0,X_p] \propto \frac{1}{2}\sum_{\rho} Z^{p}_{\text{Maj.}}[\rho_1,\rho_2+1]Z^{16-p}_{\text{Maj.}}[\rho_1,\rho_2]\,, \tag{4.112}$$

$$= \frac{(\theta_3(\tau)\theta_4(\tau))^{\frac{p}{2}}}{2\eta(\tau)^8}\left(\theta_3(\tau)^{8-p}+\theta_4(\tau)^{8-p}\right)\,. \tag{4.113}$$

This precisely matches the results of Equation (4.95), where no intimate knowledge of superfusion categories, bosonization, or fermionic anomalies were required.

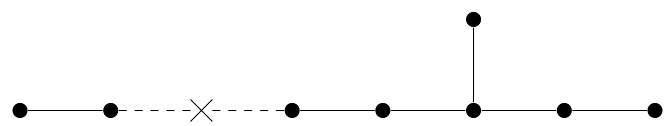
4.4 Higher Order Duality Defects

In this section, we investigate duality defects for higher order cyclic symmetries. Along the way we highlight a number of potential phenomena that one might encounter in enacting our defect hunting procedure described in Section 4.3.4, such as order doubling and $\mathfrak{u}(1)$ factors. We compute the q-expansions of the defect partition functions using Magma [195].

4.4.1 $\mathbb{Z}_3$ Duality Defects

As explained in Section 4.3.2, there is only one non-anomalous $\mathbb{Z}_3$ symmetry up to conjugation. We will see that this symmetry gives rise to seven $\mathbb{Z}_3$ Tambara-Yamagami lines, which we will investigate using both numerical and analytical methods.

In Section 4.3.3 we found that the non-anomalous $\mathbb{Z}_3$ symmetry is given by $x = \frac{1}{3}\omega_2$. Following our procedure, we obtain the fixed sub-VOA by "chopping" the second root of the affine E_8 Dynkin diagram



(4.114)

Therefore, we find that $(V_{E_8})^{\mathbb{Z}_3} \cong V_{A_2\times E_6}$.

Before starting the analysis of the Tambara-Yamagami lines, let's study $V_{A_2\times E_6}$ more carefully and verify that we can extend it into two different versions of V_{E_8}. For this, recall the following theta functions for A_2 and E_6 (for details see [186]):

$$\theta_{A_2}(\tau) = \theta_2(2\tau)\theta_2(6\tau) + \theta_3(2\tau)\theta_3(6\tau)\,, \tag{4.115}$$

$$\theta_{E_6}(\tau) = \frac{1}{4}f(\tau)^3 + \theta_{A_2}(\tau)^3\,, \tag{4.116}$$

where

$$f(\tau) = \theta_{A_2}\left(\frac{\tau}{3}\right) - \theta_{A_2}(\tau)\,. \tag{4.117}$$

The A_2 and E_6 lattices have three cosets inside their respective duals. These cosets are parametrized by shift vectors which we will call [1] and [2] in both cases. This gives two shifted theta functions for each lattice given by

$$\theta_{A_2+[1]}(\tau) = \theta_{A_2+[2]}(\tau) = \frac{1}{2}f(\tau), \tag{4.118}$$

$$\theta_{E_6+[1]}(\tau) = \theta_{E_6+[2]}(\tau) = \frac{3}{4}f(\tau)^2\,\theta_{A_2}(\tau). \tag{4.119}$$

We organize the cosets of $A_2 \times E_6$ inside its dual as a 3×3 square, where the first column and the top row give two different extensions to an E_8 lattice inside the dual of the $A_2 \times E_6$ lattice:

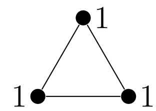
$$(A_2\times E_6)^* = \begin{array}{|c|c|c|}\hline A_2\oplus E_6 & A_2^{[1]}\oplus E_6^{[2]} & A_2^{[2]}\oplus E_6^{[1]} \\ \hline A_2^{[1]}\oplus E_6^{[1]} & A_2^{[2]}\oplus E_6 & A_2\oplus E_6^{[2]} \\ \hline A_2^{[2]}\oplus E_6^{[2]} & A_2\oplus E_6^{[1]} & A_2^{[1]}\oplus E_6 \\ \hline \end{array} \quad \overset{\text{Weight}}{\rightsquigarrow} \quad q^A(x) = \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & \frac{1}{3} & \frac{2}{3} \\ \hline 0 & \frac{2}{3} & \frac{1}{3} \\ \hline \end{array}\,. \tag{4.120}$$

This shows explicitly that this order three symmetry is non-anomalous. Summing the relevant theta functions we obtain

$$\theta_{E_8}(\tau) = \theta_{A_2}(\tau)\left(f(\tau)^3 + \theta_{A_2}(\tau)^3\right)\,. \tag{4.121}$$

Now we turn towards the duality defects of this $\mathbb{Z}_3$-symmetry. For this purpose, we need to study order two symmetries of the $A_2 \times E_6$ lattice and their possible lifts to the corresponding VOA. After that, we investigate which of the $\mathbb{Z}_2$ symmetries switch the two E_8 extensions and swap the axes of $\mathrm{Irr}(V^{\mathbb{Z}_3})$.

We actually start by looking at the automorphisms of the relevant Lie algebras. The A_2 Lie algebra has two inner automorphisms of order ≤ 2 (up to conjugation) and one outer automorphism of order 2. To see this, note the affine and twisted Dynkin diagrams are

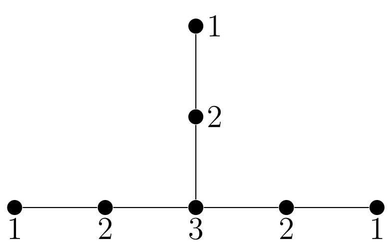
$A_2^{(1)} =$ 

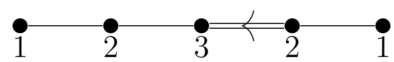
$A_2^{(2)} =$ 

(4.122)

The inner automorphisms can be read off from the affine Dynkin diagram. Kac's Theorem tells us we can either chop one node with weight $s_i = 2$, or we can chop two nodes with weights $s_i = s_j = 1$; by the symmetry of the affine Dynkin diagram, it doesn't matter which we chop. In the first case we get the trivial automorphism, and in the second case we get an inner automorphism of order 2 fixing an $A_1 \times \mathfrak{u}(1)$ Lie algebra. There is only one outer automorphism which comes from chopping the node marked 1 with weight 1, fixing an A_1 Lie algebra. Standard lifts of this automorphism experience order doubling (as we will explain in Section 4.4.1).

The E_6 case is similar, we have

$E_6^{(1)} =$ 

$E_6^{(2)} =$ 

(4.123)

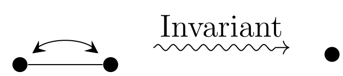
We find 3 inner automorphisms of the E_6 Lie algebra up to conjugation, given by: chopping a node marked 1 with weight 2 (the trivial automorphism); chopping a node marked 2 with weight 1, fixing $A_1 \times A_5$; or chopping two nodes marked 1 with weight 1, fixing $D_5 \times \mathfrak{u}(1)$. We find 2 outer automorphisms of E_6, fixing an F_4 or a C_4 Lie algebra.

Putting everything together, we have 15 automorphisms of the $A_2 \times E_6$ Lie algebra of order less than or equal to 2. These automorphisms can be organised as in Table 4.4. Similar to the D_8 case, we can read off automorphisms which switch the axes of $\mathrm{Irr}(V^{\mathbb{Z}_3})$ by looking at the chopped E_8 Dynkin diagram (4.114). We see that there are two ways to recombine the A_2 and E_6 into an (affine) E_8: by gluing the chopped node back in, or by acting with *one* of (the lifts of) the Dynkin diagram automorphisms of A_2 or E_6, *but not both*, and then gluing. From the off-diagonals of Table 4.4, this gives a total of seven automorphisms switching the axes and hence seven Tambara-Yamagami lines. However, before computing the defect partition functions, we need to look at order-doubling in the A_2 lattice.

Example: Order Doubling in A_2

Before moving forward, let's consider a *standard* lift of the one non-trivial outer automorphism of A_2, as it is a prototypical example of order-doubling.

So let g be the automorphism of the A_2 lattice that comes from the Dynkin diagram automorphism; switching the two simple roots α_1 and α_2. As described in Section 4.3.2, g can be lifted to a symmetry $\hat{g}$ of the lattice VOA in a "standard way": swapping the chiral bosons $a_n^1 \leftrightarrow a_n^2$, and sending $\Gamma_\alpha \mapsto u(\alpha)\Gamma_\alpha$, with $u(\alpha) = 1$ on the fixed sublattice. In this case, the fixed sublattice is a "long A_1" spanned by $\alpha_1 + \alpha_2$



(4.124)

Clearly $\hat{g}^2$ is the identity on the chiral bosons, but

$$\hat{g}^2 \Gamma_\alpha = u(\alpha) u(g\alpha) \Gamma_\alpha \,. \tag{4.125}$$

Using Equation (4.44) we have

$$u(\alpha) u(g\alpha) = \frac{\epsilon(\alpha, g\alpha)}{\epsilon(g\alpha, g^2\alpha)} u(\alpha + g\alpha) = (-1)^{(g\alpha,\alpha)} u(\alpha + g\alpha) \,. \tag{4.126}$$

Thus the order of $\hat{g}$ is doubled if and only if $(g\alpha, \alpha) \notin 2\,\mathbb{Z}$.

Better yet, we explicitly see the origin of order doubling here: we get order doubling because the Dynkin diagram automorphism folds together two simple roots which were not originally orthogonal. Hence, just by looking at Dynkin diagrams, we can tell which automorphisms will have order doubling.

At the end of Section 4.3.2 we saw all outer order 2 automorphisms of V_{A_2} can be viewed as a product of a (fixed) standard lift of this Dynkin diagram automorphism, and a twist by an "inner" automorphism (see Equation (4.55)). Now we see the importance of allowing twists by inner automorphisms of order up to 4. If we had not, we might falsely conclude that "there are no outer automorphisms of V_{A_2} of order 2," which would directly contradict the results of [177] and the fact known to physicists that the lattice automorphism $\alpha \mapsto -\alpha$ always lifts to a "charge conjugation automorphism" of order 2 in the VOA (note: $\alpha \mapsto -\alpha$ is not in the Weyl group of the A_2 lattice).

If instead we choose u so that $u(\alpha_1 + \alpha_2) = -1$ the lift has order two, and we can compute the appropriate twisted theta function. The fixed sublattice is a long A_1 lattice with theta function $\theta_3(2\tau)$, but, due to the non-trivial lift, we adjust some signs and obtain $\theta_4(2\tau)$.

$\mathbb{Z}_3$ Duality Defects from Lie Theory

The partition functions corresponding to the seven Tambara-Yamagami lines can be computed using similar techniques to the order two case. A careful study of the lattices and partition functions gives the following list of theta functions

$$\theta_{A_2,\text{inner}}(\tau) = \theta_3(2\tau)\theta_3(6\tau) - \theta_2(2\tau)\theta_2(6\tau) \tag{4.127}$$
$$\theta_{A_2,\text{outer}}(\tau) = \theta_4(2\tau) \tag{4.128}$$
$$\theta_{E_6,\text{inner }1}(\tau) = 2\theta_{A_5}(\tau)\theta_3(2\tau) - \theta_{E_6}(\tau) \tag{4.129}$$
$$\theta_{E_6,\text{inner }2}(\tau) = 1 + 8q - 50q^2 + 80q^3 - 88q^4 + \dots \tag{4.130}$$
$$\theta_{E_6,\text{outer }1}(\tau) = \tfrac{1}{2}(\theta_3^4(\tau) + \theta_4^4(\tau)) \tag{4.131}$$
$$\theta_{E_6,\text{outer }2}(\tau) = \theta_4^4(2\tau)\,. \tag{4.132}$$

The missing theta function corresponding to an inner E_6 symmetry is particularly messy, so we display the first few terms of its q-expansion. The twisted eta products are easily computed being $\eta(\tau)^6\eta(2\tau)$ for the outer A_2 symmetries and $\eta(\tau)^4\eta(2\tau)^2$ for the outer E_6 symmetries. In summary, we have the following seven defected partition functions:

$$Z_V[0, X^{(1)}_{A_2}] = \frac{\theta_{A_2,\text{outer}}(\tau)\theta_{E_6}(\tau)}{\eta(\tau)^6\eta(2\tau)}\,, \tag{4.133}$$

$$Z_V[0, X^{(2)}_{A_2}] = \frac{\theta_{A_2,\text{outer}}(\tau)\theta_{E_6,\text{inner }1}(\tau)}{\eta(\tau)^6\eta(2\tau)}\,, \tag{4.134}$$

$$Z_V[0, X^{(3)}_{A_2}] = \frac{\theta_{A_2,\text{outer}}(\tau)\theta_{E_6,\text{inner }2}(\tau)}{\eta(\tau)^6\eta(2\tau)}\,, \tag{4.135}$$

$$Z_V[0, X^{(1)}_{E_6}] = \frac{\theta_{A_2}(\tau)\theta_{E_6,\text{outer }1}(\tau)}{\eta(\tau)^4\eta(2\tau)^2}\,, \tag{4.136}$$

$$Z_V[0, X^{(2)}_{E_6}] = \frac{\theta_{A_2,\text{inner}}(\tau)\theta_{E_6,\text{outer }1}(\tau)}{\eta(\tau)^4\eta(2\tau)^2}\,, \tag{4.137}$$

$$Z_V[0, X^{(3)}_{E_6}] = \frac{\theta_{A_2}(\tau)\theta_{E_6,\text{outer }2}(\tau)}{\eta(\tau)^4\eta(2\tau)^2}\,, \tag{4.138}$$

$$Z_V[0, X^{(4)}_{E_6}] = \frac{\theta_{A_2,\text{inner}}(\tau)\theta_{E_6,\text{outer }2}(\tau)}{\eta(\tau)^4\eta(2\tau)^2}\,. \tag{4.139}$$

We have recorded the q-expansions of these defect partition functions (in the same order) with their fixed Lie subalgebras in Table 4.5. We also obtain the defect in Equation (4.136) from the Potts CFT in Appendix B.1.

As a final check, we've obtained the same q-expansions using computer algebra software Magma (in a way that we will explain in the following section) that matches with (at least) the first 11 terms of the q-expansions of the exact theta series above. This convinces us that our numerical methods work in the order three case, and justifies the computation of $\mathbb{Z}_4$ and $\mathbb{Z}_5$ defected partition functions using numerics only. In principle, exact formulas are obtainable in those cases following the techniques above.

We comment on data relevant to the symmetric non-degenerate bicharacter in Appendix B.2.

4.4.2 $\mathbb{Z}_4$ Duality Defects and Computer Implementation

Moving to the order 4 defects, our direct Lie algebra methods become less useful. As seen in Table 4.3, the only non-anomalous order 4 symmetry of E_8 has fixed Lie algebra $A_7 \times \mathfrak{u}(1)$. This is not a simple Lie algebra and thus does not have an associated root system.

An Algorithm for Computing Defects

In this case, we can still construct the fixed sublattice as in the beginning of Section 4.3.3. The symmetry is generated by the vector

$$x = \frac{1}{4}\omega_8. \tag{4.140}$$

Accordingly, the fixed sublattice is

$$L_0 = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_7 \oplus \mathbb{Z}4\alpha_8. \tag{4.141}$$

We can now compute q-expansions of our defect partition functions with the aid of Magma [195].[18] The code is effectively an implementation of Theorem 2.13 of [179]. The algorithm is as follows:

Step 0. Input an even lattice L_0. This can be done in Magma by providing a basis for L_0 as above with built-in function BasisWithLattice( ).

Step 1. Compute the automorphism group $O(L_0)$ with built-in function AutomorphismGroup( ) and the setwise stabilizer of a fixed set of roots H_Δ. To obtain H_Δ is more involved:

   (a) Enumerate a full set of simple roots (of length 2) Δ and split them into positive and negative roots. Note: this collection of simple roots may not be full rank, e.g. if there are $\mathfrak{u}(1)$'s involved and the lattice is not a root-lattice.

   (b) Find the subgroup of $O(L_0)$ stabilizing some choice of simple roots by looking at group orbits with built-in function OrbitAction(,). It is possible that $O(L_0)$ won't act faithfully on the collection Δ, in which case continually append Δ with vectors of higher norm until $O(L_0)$ acts faithfully, then obtain H_Δ.

   In view of [179], every finite-order automorphism of the VOA V_L can be constructed by considering a fixed set of representatives ν of the conjugacy classes of H_Δ. Obtained with built-in function ConjugacyClasses( ).

Step 2. Compute the centralizer $C_{O(L)}(\nu)$ of ν in $O(L)$ or, more precisely, a fixed set of orbit representatives of its action on $L^\nu \otimes_\mathbb{Z} \mathbb{Q}/\pi_\nu(L')$. Here π_ν is the projection of $L \otimes_\mathbb{Z} \mathbb{C}$ onto $L^\nu \otimes_\mathbb{Z} \mathbb{C}$, with L^ν the sublattice fixed by ν. These are our "outer $\mathbb{Z}_2$" actions. Proposition 2.15 of [179] guarantees that in order to find automorphisms of order 2 swapping the axes, we only need to look at order 1 and 2 lattice automorphisms, ν, and restrict the choice of h's according to whether or not the standard lift of ν has order doubling.

[18] We thank Sven Möller for providing us with our original Magma code and helpful Magma lessons.

Step 3. Loop through all ν and compute all possible independent h for each ν. For each ν loop over all h to create the automorphism $\hat{\nu}e^{2\pi i h(0)}$.

Step 4. Implementing the formula in Equation (4.56), compute the q-expansions of our $\mathbb{Z}_2$-twisted characters for $V^{\mathbb{Z}_m}$ by summing over weighted lattice vectors.

At this point, we have enumerated all order 2 automorphisms of our VOA V_{L_0} and the (q-expansions of the) $\mathbb{Z}_2$ twisted characters. However, not all order 2 automorphisms swap the axes of the metric Abelian group $A \cong \mathbb{Z}_m \times \hat{\mathbb{Z}}_m$. Fortunately, this is simple to check within the code described above. To do this, note that our automorphisms extend to all of $\mathbb{R}^8$ and hence to our original E_8 lattice L. Thus, by looking at the image of the basis vectors under ν, we can determine if the automorphism leaves the original E_8 invariant. In the case that it is not left invariant, then L must be mapped to a different lattice, which must be the second E_8 extension L' by orthogonality of the automorphism. A similar procedure also tells us about the symmetric non-degenerate bicharacter, see Appendix B.2.

The $\mathbb{Z}_4$ Defects

Let $b_i = \alpha_i$ for $i = 1, \dots, 7$, and let $b_8 = 4\alpha_8$. Following our algorithm above, we find three non-trivial lattice automorphisms ν of $L_0 = \mathbb{Z}\{b_i\}_{i=1}^8$. As matrices $b_i \mapsto (\nu_k)_{ij} b_j$, we have $\nu_1^{(4)}$, $\nu_2^{(4)}$ and $\nu_3^{(4)}$, which are respectively

$$\begin{pmatrix} 0&0&0&0&0&0&1&-4\\ 0&0&0&0&0&1&0&-8\\ 0&0&0&0&1&0&0&-12\\ 0&0&0&1&0&0&0&-12\\ 0&0&1&0&0&0&0&-12\\ 0&1&0&0&0&0&0&-8\\ 1&0&0&0&0&0&0&-4\\ 0&0&0&0&0&0&0&-1 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0&0&0&0&-3\\ 0&1&0&0&0&0&0&-6\\ 0&0&1&0&0&0&0&-9\\ 0&0&0&1&0&0&0&-12\\ 0&0&0&0&1&0&0&-15\\ 0&0&0&0&0&1&0&-10\\ 0&0&0&0&0&0&1&-5\\ 0&0&0&0&0&0&0&-1 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0&0&0&0&1&-1\\ 0&0&0&0&0&1&0&-2\\ 0&0&0&0&1&0&0&-3\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&3\\ 0&1&0&0&0&0&0&2\\ 1&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1 \end{pmatrix}. \tag{4.142}$$

To reiterate, these are representatives for the 3 conjugacy classes of $\mathbb{Z}_2$ automorphisms of the lattice L_0. We can check the actions of these matrices on the E_8 basis vectors spanning L, and find that $\nu_2^{(4)}$ and $\nu_3^{(4)}$ do not leave L invariant, and hence swap the two distinct E_8's in L_0^*/L_0. Neither $\nu_2^{(4)}$ nor $\nu_3^{(4)}$ experience order doubling.

Combining everything, we find six defected partition functions, with the weight one subspaces and q-expansions recorded in Table 4.6 for the $\mathbb{Z}_4$ case.

These results are plausible from Lie theory. Ignoring the $\mathfrak{u}(1)$ factor there are several symmetries of the A_7 Lie algebra. First one can leave the roots of the A_7 Dynkin diagram unchanged. The corresponding phase factors can then be chosen to obtain the fixed Lie subalgebras $A_{6-i} \times A_i \times \mathfrak{u}(1)$. The other option is to swap the roots in the A_7 Dynkin diagram. In this situation the fixed subalgebra is given by C_4. This compares well to the structure presented in Table 4.6, where the first three lines correspond to the first case and the next two lines to the second case.

More generally, in the case of $\mathbb{Z}_m$ symmetries with m not prime, its also important to keep in mind that we need to "totally swap" the axes, since there can be $\mathbb{Z}_2$ automorphisms of $\mathrm{Irr}(V^{\mathbb{Z}_m})$ which only swap a subgroup of $\mathbb{Z}_m$ with a subgroup of $\hat{\mathbb{Z}}_m$.

We record results on the symmetric non-degenerate bicharacter at the end of Appendix B.2.

4.4.3 $\mathbb{Z}_5$ Duality Defects

There are two non-anomalous $\mathbb{Z}_5$ symmetries up to conjugacy, and so two families of duality defects of order 5. We will arbitrarily call the two $\mathbb{Z}_5$ conjugacy classes $\mathbb{Z}_5^A$ and $\mathbb{Z}_5^B$ with generators

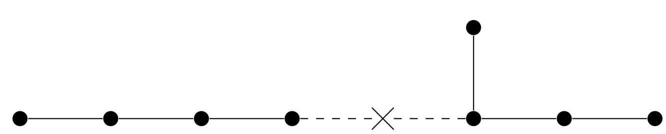
$$x^A := \frac{1}{5}\omega_4\,, \tag{4.143}$$

$$x^B := \frac{1}{5}\omega_1 + \frac{1}{5}\omega_7\,. \tag{4.144}$$

The sub-VOAs $V^{\mathbb{Z}_5^A}$ and $V^{\mathbb{Z}_5^B}$ have weight one subspaces isomorphic to $A_4 \times A_4$ and $D_6 \times \mathfrak{u}(1)^2$ respectively.

$\mathbb{Z}_5^A$ Duality Defects and $A_4 \times A_4$

The invariant sublattice L_0^A under the $\mathbb{Z}_5^A$ symmetry is a genuine root lattice spanned by the $A_4 \times A_4$ inside E_8, so we can proceed in a straightforward way. Consider the chopped Dynkin diagram



(4.145)

Focusing on just one A_4, the $A_4^{(1)}$ and $A_4^{(2)}$ Dynkin diagrams tell us that the A_4 Lie algebra has 3 inner automorphisms of order less than or equal to 2, with fixed Lie subalgebras: A_4

(trivial); $A_3\times\mathfrak{u}(1)$; and $A_2\times A_1\times\mathfrak{u}(1)$. Similarly, there is one order 2 outer automorphism with a fixed B_2 subalgebra.

Since we have two A_4 Dynkin diagrams life is slightly more complicated than the $\mathbb{Z}_3$ case. First, there are the usual commuting $\mathbb{Z}_2$ actions on each A_4 Dynkin diagram σ_1 and σ_2 say; but we can also exchange the diagrams themselves in different ways, this gives another $\mathbb{Z}_2$ action τ. The three together generate a dihedral group of order 8 $\langle\,\sigma_1,\sigma_2,\tau\,\rangle\cong \mathrm{Di}_8$.

However, we are not interested in all eight elements of this group, but in the conjugacy classes. There are three classes which do not switch the axes of $\mathrm{Irr}(V^{\mathbb{Z}_5})$ given by $\{e\}$, $\{\sigma_1\sigma_2\}$ and $\{\sigma_1\tau,\sigma_2\tau\}$ and two classes switching the axes given by $\{\sigma_1,\sigma_2\}$ and $\{\tau,\sigma_1\sigma_2\tau\}$.

As before, we can obtain explicit matrix representatives of these classes. If we represent the $A_4\times A_4$ lattice as the span of $b_i=\alpha_i$ for $i\neq 4$ and $b_4=5\alpha_4$, then representatives for the non-trivial lattice automorphisms $\nu^{(5)}_{[\sigma_1\sigma_2]}$, $\nu^{(5)}_{[\sigma_1\tau]}$, $\nu^{(5)}_{[\sigma_1]}$, and $\nu^{(5)}_{[\tau]}$ are given respectively by

$$\begin{gathered}\begin{pmatrix}1&0&0&0&0&0&1&0\\2&0&1&-5&0&0&0&0\\3&1&0&-5&0&0&0&0\\1&0&0&-1&0&0&0&0\\6&0&0&-10&0&1&0&0\\4&0&0&-10&1&0&0&0\\2&0&0&-5&0&0&0&1\\3&0&0&-5&0&0&1&0\end{pmatrix},\quad\begin{pmatrix}0&0&0&-5&0&0&1&1\\0&0&0&-10&1&0&2&0\\0&0&0&-10&0&1&3&0\\0&0&0&-2&0&0&1&0\\0&0&1&-15&0&0&6&0\\0&1&0&-10&0&0&4&0\\1&0&0&-5&0&0&2&0\\0&0&0&-5&0&0&3&0\end{pmatrix},\\ \begin{pmatrix}1&0&0&0&0&0&1&0\\2&0&1&-5&0&0&0&0\\3&1&0&-5&0&0&0&0\\1&0&0&-1&0&0&0&0\\6&0&0&-12&1&0&0&0\\4&0&0&-8&0&1&0&0\\2&0&0&-4&0&0&1&0\\3&0&0&-6&0&0&0&1\end{pmatrix},\quad\begin{pmatrix}0&0&0&-5&0&0&1&1\\0&0&0&-10&1&0&2&0\\0&0&0&-10&0&1&3&0\\0&0&0&-2&0&0&1&0\\0&1&0&-14&0&0&6&0\\0&0&1&-11&0&0&4&0\\0&0&0&-3&0&0&2&0\\1&0&0&-7&0&0&3&0\end{pmatrix}.\end{gathered}\tag{4.146}$$

The elements of this A_4 "flipping" conjugacy class $\{\sigma_1,\sigma_2\}$ flip roots of one A_4 while leaving one copy of A_4 invariant. In this case, the fixed Lie subalgebra is one of the inners multiplied by a B_2 coming from the A_4 outer automorphism, giving 3 duality defects for this conjugacy class. Note that standard lifts from this class experience order doubling because the outer automorphism folds roots together which are not orthogonal. The "exchange" conjugacy class τ fixes a "diagonal" A_4 Lie subalgebra giving 1 final duality defect. It does not experience order doubling. We enumerate these 4 cases with their q-expansions in Table 4.7.

$\mathbb{Z}_5^B$ Duality Defects and $D_6 \times \mathfrak{u}(1)^2$

For the case of $D_6 \times \mathfrak{u}(1)^2$ the solution is slightly less elegant and cannot be read off just by looking at Dynkin diagrams. Proceeding as in Section 4.4.2, we can realize the fixed sublattice as the span of $b_i = \alpha_i$ for $i \neq 1,\ 7$, and $b_1 = 5\alpha_1$ and $b_7 = 4\alpha_1 + \alpha_7$. We use our previous methods to obtain representatives of the conjugacy classes of non-trivial automorphisms $\nu_1^{(5)}$, $\nu_2^{(5)}$, $\nu_3^{(5)}$, $\nu_4^{(5)}$, given respectively by

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 1 & 0 & 0 & 0 & 0 & -9 & 0 \\ -10 & 0 & 1 & 0 & 0 & 0 & -10 & 0 \\ -10 & 0 & 0 & 1 & 0 & 0 & -11 & 0 \\ -10 & 0 & 0 & 0 & 1 & 0 & -12 & 0 \\ -5 & 0 & 0 & 0 & 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -5 & 0 & 0 & 0 & 0 & 0 & -6 & 1 \end{pmatrix}, \quad \begin{pmatrix} -5 & 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ -3 & 1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 6 & 0 & 0 & 0 & 1 & 0 & 4 & 0 \\ 6 & 0 & 0 & 0 & 0 & 1 & 4 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ -16 & 1 & 0 & 0 & 0 & 0 & -12 & 0 \\ -20 & 0 & 1 & 0 & 0 & 0 & -15 & 0 \\ -24 & 0 & 0 & 1 & 0 & 0 & -18 & 0 \\ -28 & 0 & 0 & 0 & 1 & 0 & -21 & 0 \\ -18 & 0 & 0 & 0 & 0 & 0 & -13 & 1 \\ -8 & 0 & 0 & 0 & 0 & 0 & -5 & 0 \\ -14 & 0 & 0 & 0 & 0 & 1 & -11 & 0 \end{pmatrix}, \quad \begin{pmatrix} -7 & 0 & 0 & 0 & 0 & 0 & -5 & 0 \\ 5 & 1 & 0 & 0 & 0 & 0 & 2 & 0 \\ 10 & 0 & 1 & 0 & 0 & 0 & 5 & 0 \\ 15 & 0 & 0 & 1 & 0 & 0 & 8 & 0 \\ 20 & 0 & 0 & 0 & 1 & 0 & 11 & 0 \\ 15 & 0 & 0 & 0 & 0 & 0 & 9 & 1 \\ 10 & 0 & 0 & 0 & 0 & 0 & 7 & 0 \\ 10 & 0 & 0 & 0 & 0 & 1 & 5 & 0 \end{pmatrix}. \tag{4.147}$$

The automorphisms switching the axes of $\mathrm{Irr}(V^{\mathbb{Z}_5^B})$ are $\nu_2^{(5)}$ and $\nu_3^{(5)}$. The standard lifts of both of these experience order doubling. In total, there are 7 defected partitions functions, 4 which lift from $\nu_2^{(5)}$ and 3 which lift from $\nu_3^{(5)}$. They are recorded in Table 4.8.

4.5 A (2+1)d TFT Perspective

There is a natural (2+1)d perspective which sheds light on our previous discussions and the procedure in Section 4.3.4 and Equation (4.85). This all follows from the theory of modular tensor categories, fusion categories, and their relationship to gapped boundary conditions for (2+1)d TFTs.

The relevant pieces of this formalism are well-described in mathematical physics works such as [97, 127] and are well-known in the condensed-matter literature (see e.g. [22, 196]). For this reason, we will provide only a small recap of the beautiful connection between (2+1)d TFTs, their gapped boundary conditions, and the relevant mathematics, before turning to duality defects. Helpful mathematical references include [197, 34].

4.5.1 Topological Boundaries of (2+1)d TFTs

Given a fusion category $\mathcal{A}$ it's Drinfeld center $\mathcal{Z}(\mathcal{A})$ is naturally a braided fusion category. This leads to the natural mathematical question: are all braided fusion categories the Drinfeld center of some fusion category? The answer is no, as already demonstrated by various Chern-Simons theories [198].

The next simplest question in this line of thinking is: if one has a braided (non-degenerate[19]) fusion category $\mathcal{C}$, when is $\mathcal{C} = \mathcal{Z}(\mathcal{A})$ for some fusion category $\mathcal{A}$? The answer to this question is also known: the data of a braided equivalence $\mathcal{C} \cong \mathcal{Z}(\mathcal{A})$ determines a Lagrangian algebra object $A \in \mathcal{C}$; and inversely, each Lagrangian algebra object $A \in \mathcal{C}$ determines a braided equivalence $\mathcal{C} \cong \mathcal{Z}(\mathcal{C}_A)$ where $\mathcal{C}_A$ is the category of right-A modules in $\mathcal{C}$. Finally, fixing some fusion category $\mathcal{A}$, it is known that the collection of fusion categories $\mathcal{B}$ such that there is a braided autoequivalence $\mathcal{Z}(\mathcal{A}) \cong \mathcal{Z}(\mathcal{B})$ are in bijection with indecomposable $\mathcal{A}$-module categories [202, 203]. Altogether, one has the following result:

Proposition 1 ([204], Proposition 4.8)**.** *Let $\mathcal{A}$ be a fusion category and $\mathcal{C} = \mathcal{Z}(\mathcal{A})$. There is a bijection between the sets of (isomorphism classes of) Lagrangian algebras in $\mathcal{C}$ and (equivalence classes of) indecomposable $\mathcal{A}$-module categories.*

These questions arise naturally in physics as well, as the data of a (2+1)d TFT are encoded as a Modular Tensor Category of topological line defects or anyons [205, 197, 206]. A question of interest when studying a (2+1)d TFT is: does the (2+1)d theory admit any topological boundary conditions?

[19] Non-degenerate means that the S-matrix is non-degenerate. By definition this is true for a MTC, so applies to e.g. reps of a rational VOA. More broadly, non-degeneracy is satisfied iff there are no anyons which are invisible to braiding with the entire rest of the category (Theorem 3.4 of [199]), known as "remote detectability" [200, 156, 76]. So-called "slightly degenerate" braided fusion categories are also of interest in the study of fermionic topological orders [201, 76].

In the physical case, the MTC of anyons $\mathcal{C}$ is the braided fusion category of interest, and it can be shown that if $\mathcal{C} \cong \mathcal{Z}(\mathcal{A})$ for some $\mathcal{A}$, then the bulk admits a topological boundary condition B with boundary line defects modelled by $\mathcal{A}$. So we see that the natural question for physicists is essentially the same as those studying braided fusion categories [97, 127].

More generally, the theory of "anyon condensation" allows one to describe topological interfaces between two TFTs, described by MTCs $\mathcal{C}$ and $\mathcal{D}$, in the case that the anyons of $\mathcal{C}$ "condense" to a new vacuum for $\mathcal{D}$ at the separating interface [126, 127]. The formalism allows one to easily compute condensation interfaces for $\mathcal{C}$, the new phase $\mathcal{D}$, and the excitations on the interface (see Theorem 4.7 of [127] and [207] for computational references). In particular, the search for topological boundary conditions is a condensation to vacuum, controlled by Lagrangian algebra objects $A \in \mathcal{C}$.[20]

It is common in the physics literature to talk about a particular realization of a (2+1)d TFT modeled by the MTC $\mathcal{C}$, rather than just discuss abstract MTC data. For example, in Turaev-Viro/Levin-Wen models [84, 208] one chooses a particular "input fusion category" $\mathcal{A}$ and constructs the (2+1)d TFT whose anyons are modeled by the MTC $\mathcal{C} = \mathcal{Z}(\mathcal{A})$. From this frame, the boundary condition with excitations modeled by $\mathcal{A}$ is particularly distinguished as a "reference" or "Dirichlet" boundary condition, and one identifies gapped boundaries with indecomposable $\mathcal{A}$-module categories [196].[21] By Proposition 1 these are in bijection with the Lagrangian algebras.

Similarly, if the theory admits a gauge theory description, making such distinctions are equivalent to saying "who the Wilson line is" in the TFT. The choice of a particular boundary condition B defines the Wilson line by declaring that B is the Dirichlet boundary condition for the bulk gauge fields in that frame. Mathematically, the Wilson lines are the kernel of our particular bulk-boundary map $\mathcal{Z}(\mathcal{A}) \to \mathcal{A}$, i.e. their image is a sum of copies of the vacuum operator on the boundary.

The simplest and most well known example of this relationship in physics is in the toric code [209]. Algebraically, the simple anyons of the (2+1)d TFT are described by the irreducible representations of the untwisted quantum double $D(\mathbb{Z}_2)$, or more geometrically as (2+1)d Dijkgraaf-Witten theory with gauge group $\mathbb{Z}_2$ and trivial topological action [24]. In short, the simple anyons in this model are $\{1, e, m, f\}$ with spins $\{0, 0, 0, 1/2\}$ respectively and non-trivial fusion relations

$$e \otimes m = f\,, \quad e \otimes f = m\,, \quad m \otimes f = e\,. \tag{4.148}$$

[20] We only care about bosonic anyon condensations and bosonic topological boundary conditions. For results on fermionic anyon condensation and fermionic topological orders see [71] and references within.

[21] We will use the term Dirichlet boundary condition to mean the canonical boundary condition for the Turaev-Viro theory based on $\mathcal{A}$.

Famously, such a model admits two *bosonic* indecomposable topological boundary conditions as described in [54], where they manifest in the microscopic description as "smooth" and "rough" boundaries of the square lattice. Such a description makes it pictorially clear how certain anyons of the toric code "condense" as they reach the boundary. For example, the e anyon living along a string of vertices is condensed at a rough boundary, but the m anyon gets stuck and becomes a boundary defect line (see also [125] and [86]). In other words, the "rough" topological boundary condition B_e condenses the e anyon to the vacuum on the topological boundary, and corresponds to the Lagrangian algebra object $1 \oplus e$. The topological boundary condition is populated with boundary excitations $\{1 \oplus e, m \oplus f\}$, which behave like $\mathcal{A}_e = \mathrm{Vec}_{\mathbb{Z}_2}$.

4.5.2 Duality Defects

Gauging 2d theories has a simple (2+1)d interpretation in terms of boundary conditions and interfaces in topological gauge theories (see e.g. [210, 1, 124] for an extended explanation and [211] for an application). In particular, given any 2d theory T with non-anomalous $\mathbb{Z}_m$ symmetry, the theory provides an "enriched Neumann" boundary condition $B[T]$ for a (2+1)d $\mathbb{Z}_m$ Dijkgraaf-Witten theory by coupling the boundary value of the bulk connection to T. The local operators of $B[T]$ are effectively the local operators of $T^{\mathbb{Z}_m}$.

In this case, the 2d theory T (on M say) can be understood as a compactification of a slab of this $\mathbb{Z}_m$ gauge theory on $M \times [0,1]$, with the boundary condition $B[T]$ for one boundary, and the reference (topological) Dirichlet boundary condition for the other. Similarly, $[T/\mathbb{Z}_m]$ can be understood as the same setup but with the (topological) fully-Neumann boundary condition instead.

However, by the same logic, $[T/\mathbb{Z}_m]$ can also be viewed as a sandwich of $\hat{\mathbb{Z}}_m$ gauge theory with boundary condition $B[[T/\mathbb{Z}_m]]$ and Dirichlet boundary condition for $\hat{\mathbb{Z}}_m$ on the other. These two descriptions are dual, what has changed is the identification of the bulk TFT as an "electric" $\mathbb{Z}_m$ gauge theory or a "magnetic" $\hat{\mathbb{Z}}_m$. By our previous section, the MTC data is the same, but we have presented a different topological boundary condition as "Dirichlet". The two pictures are related by an electric-magnetic duality interface I_{em}, which composes with $B[T]$ to produce $B[[T/\mathbb{Z}_m]]$ and turns the electric Dirichlet boundary condition into a magnetic Neumann boundary condition. We depict all of this in Figure 4.6.

Since a duality defect separates a theory from its orbifold, we recognize a duality defect as a potential ending line for the interface I_{em}, separating the electric and magnetic Dirichlet boundary conditions in a slab picture i.e. as a $\mathbb{Z}_2^{em}$ twist line operator, see Figure

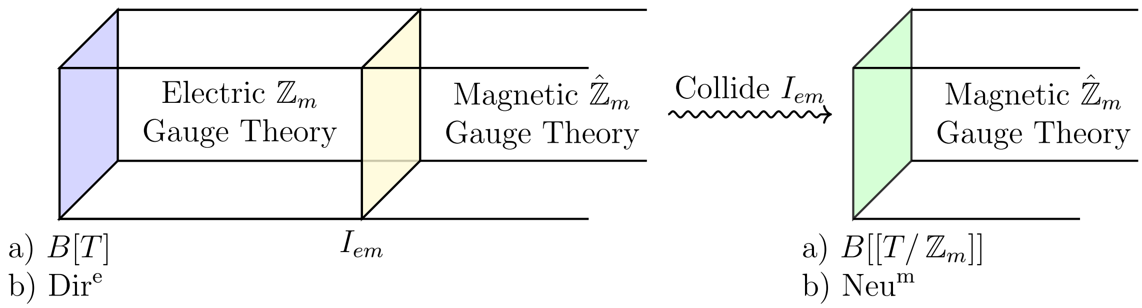


Figure 4.6: On the left, a slab of $\mathbb{Z}_m$ gauge theory and $\hat{\mathbb{Z}}_m$ gauge theory separated by an (invertible topological) electric-magnetic duality interface I_{em} (yellow). If one moves the interface I_{em} all the way to the left, then all that remains is $\hat{\mathbb{Z}}_m$ gauge theory with a new boundary condition (green). a) If the left boundary condition (blue) is an enriched Neumann boundary condition $B[T]$, then when it collides with the interface it appears as the enriched Neumann boundary condition $B[[T/\mathbb{Z}_m]]$. b) If the left boundary condition (blue) is a Dirichlet boundary condition for the $\mathbb{Z}_m$ gauge theory, then it appears as a Neumann boundary condition for the $\hat{\mathbb{Z}}_m$ gauge theory.

4.7a. Gauging the electric-magnetic $\mathbb{Z}_2^{em}$ 0-form symmetry *of the bulk TFT* (with Neumann boundary conditions on the left and Dirichlet boundary conditions on the right of Figure 4.7a), produces a new bulk TFT where the interface I_{em} "becomes invisible," leaving the twist line behind, as in Figure 4.7b. In Section XI of [22] (also Section 3.5 of [72]) the authors show explicitly how the boundary twist line left behind after gauging has the fusion rules of $\mathrm{TY}(\mathbb{Z}_m)$.

This is simply squaring two commensurate pictures. In one case, we are thinking of T as having a $\mathbb{Z}_m$ symmetry with topological line defects for the symmetry modeled by the fusion category $\mathcal{D} = \mathrm{Vec}_{\mathbb{Z}_m}$, and so we can couple it to a $\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_m})$ bulk and view the duality defect line as the endpoint of an electric-magnetic duality wall. Alternatively, we can view T as having "categorical symmetry" $\mathcal{F} = \mathrm{TY}(\mathbb{Z}_m)$, in which case it can be coupled to a bulk of $\mathcal{Z}(\mathrm{TY}(\mathbb{Z}_m))$ theory, and the duality defect line is understood simply as a condensate of bulk lines on the topological boundary.

In general, suppose $\mathcal{D}$ admits a G-extension

$$\mathcal{F} = \bigoplus_{g\in G} \mathcal{F}_g \tag{4.149}$$

with identity piece $\mathcal{F}_e \cong \mathcal{D}$, then the (2+1)d TFT Q with MTC $\mathcal{Z}(\mathcal{D})$ and reference Dirichlet boundary condition $\mathcal{D}$ necessarily has topological surfaces with a G-composition law. If we gauge this G symmetry to produce a new (2+1)d TFT $[Q/G]$, then the new MTC data is given by $\mathcal{Z}(\mathcal{F})$ with reference boundary condition $\mathcal{F}$. This follows from the work of [212, 22].

Application to Holomorphic VOAs

Following [173], we call a VOA *nice* if it is simple, rational, C_2-cofinite, self-contragredient, and of CFT-type. Examples of nice VOAs include lattice VOAs built from even lattices and the Monster $V^\natural$.

Proposition 2. *Given a conformal inclusion of nice VOAs $W \subset V$ there is a canonical fusion category $\mathcal{F}$ determined by the inclusion.*

This follows from the theory of anyon condensations described in Theorem 4.7 and Section 6.4 of [127]. In the VOA literature, the proposition is typically phrased in the reverse direction as being about extensions of the VOA W.

Proof. Since W and V are nice, their representation categories are MTCs [213], let $\mathcal{C} :=$ Rep(W) and $\mathcal{D} :=$ Rep(V). Since $W \subset V$, V can be decomposed into a finite direct sum of irreducible W modules, we will write this as $A \in \mathcal{C}$. By [214] (and the classical results Theorem 5.2 of [215] and Section 6 of [204] in the holomorphic case), A is a commutative, connected, and separable algebra object in $\mathcal{C}$.[22]

The fusion category we seek is $\mathcal{F} = \mathcal{C}_A$, the category of right A-modules in $\mathcal{C}$. In [215] this is called Rep A, the category of twisted V-modules

In the language of conformal nets, the proof runs parallel, with the algebra object A now called a "Q-system," and $\mathcal{F}$ the "category of solitons" obtained by "α-induction" [216, 217, 218]. $\square$

Generally, if $W \subset V$ as in the proposition, then we will write W as $V^{\mathcal{F}}$ and say that $\mathcal{F}$ *acts nicely* on V. Note if $\mathcal{F}$ acts nicely on V and $\mathcal{A}$ is a fusion subcategory, then $\mathcal{A}$ also acts nicely on V.

[22] In fact, when V is holomorphic A is actually a Lagrangian algebra. This can be seen since $\mathrm{FPdim}(A)^2\,\mathrm{FPdim}(\mathcal{C}_A^0) = \mathrm{FPdim}(\mathcal{C})$, but $\mathrm{FPdim}(\mathcal{C}_A^0) = \mathrm{FPdim}(\mathrm{Rep}(V)) = 1$. From the anyon-condensation point-of-view this is because the conformal inclusion is describing a gapped boundary for $\mathcal{C}$, rather than a gapped interface between two phases $\mathcal{C}$ and $\mathcal{D}$.

As an example, consider the case $W = V^G$ where G is a finite group of automorphisms of a holomorphic VOA V, then $\mathcal{F} \cong \mathrm{Vec}_G^\alpha$ as in the celebrated-results of [219, 220] (see [124, 221] for conformal nets).

Physically, $\mathcal{F}$ is a collection of topological defect lines which acts on V and W is the sub-VOA of V which commutes with the lines of $\mathcal{F}$. More explicitly, if $X \in \mathcal{F}$ is a topological defect line in $\mathcal{F}$ which acts on V, then it has a vector space of (not necessarily topological) twist operators $\mathcal{H}_X$ which X can end on. Since any element of $\mathcal{H}_X$ is not a true local operator, but must be attached to a topological X tail, a generic operator of V will collect X-monodromy if it encircles the endpoint operator; the operators of W are exactly those which collect no monodromy.

In [189], the authors prove the following

Proposition 3 ([189], Corollary 5.25)**.** *Let V be a nice VOA and G a finite solvable group of automorphisms of V then V^G is nice.*

In fact, the result they prove makes weaker "niceness assumptions" than what we've stated here. In the context of duality defects, we will be concerned with the case that G is cyclic.

So suppose V is holomorphic and $\mathcal{F} = \mathrm{TY}(\mathbb{Z}_m)$ is a Tambara-Yamagami category which acts nicely on V. Note: here we are being ambiguous about the associator data. The degree-0 subcategory of $\mathcal{F}$ defines a $\mathbb{Z}_m$ action on V which acts nicely, so that $V^{\mathcal{F}} \subset V^{\mathbb{Z}_m} \subset V$ is an inclusion of nice VOAs. By Proposition 2, $V^{\mathbb{Z}_m} \subset V$ determines a fusion category isomorphic to $\mathrm{Vec}_{\mathbb{Z}_m}$ and so $V^{\mathcal{F}} \subset V^{\mathbb{Z}_m}$ determines a fusion category of order 2. Thus we have

$$V^{\mathrm{TY}(\mathbb{Z}_m)} = (V^{\mathbb{Z}_m})^{\mathbb{Z}_2}\,, \tag{4.150}$$

where the $\mathbb{Z}_2$ action on $V^{\mathbb{Z}_m}$ may be anomalous.

We see that the data of a $\mathrm{TY}(\mathbb{Z}_m)$ action on V is defined by a choice of $\mathbb{Z}_m$ action on V and $\mathbb{Z}_2$ action on $V^{\mathbb{Z}_m}$, but not all of the former actions realize a $\mathrm{TY}(\mathbb{Z}_m)$ action. As explained in Section 4.3.4, our claim is that it does so if and only if the $\mathbb{Z}_2$ action "fully swaps the axes" of the metric Abelian group $A = \mathrm{Irr}(V^{\mathbb{Z}_m})$. By "fully," we mean the $\mathbb{Z}_2$ action truly maps all elements of A to elements of $\hat{A}$ and vice-versa, i.e., it doesn't just act non-trivially on a subgroup of order dividing m.

Theorem 2. *The inclusion $(V^{\mathbb{Z}_m})^{\mathbb{Z}_2} \subset V$ corresponds to a $\mathbb{Z}_m$ Tambara-Yamagami action on V when the $\mathbb{Z}_2$ action fully swaps the axes of the metric Abelian group $A = \mathrm{Irr}(V^{\mathbb{Z}_m})$.*

Proof. First suppose it does not fully swap the axes, i.e. that $\mathbb{Z}_2 \subset SO(A, h)$, then the $\mathbb{Z}_2$ action is only acting by automorphisms on the individual axes, permuting the contents of

$\mathbb{Z}_m$ and $\hat{\mathbb{Z}}_m$, but not amongst each other. So it is really just acting as an automorphism on $\mathbb{Z}_m$, hence there is a finite group action $G = \mathbb{Z}_m \,.\, \mathbb{Z}_2$ acting on V such that $(V^{\mathbb{Z}_m})^{\mathbb{Z}_2} = V^G$.

Now suppose we have an order 2 automorphism in $O(A, h)$ which does swap the axes. We must show that $\mathcal{F}$ determined by $(V^{\mathbb{Z}_m})^{\mathbb{Z}_2} \subset V$ is a Tambara-Yamagami category. We know that $\mathcal{F}$ is some $\mathbb{Z}_2$ extension of $\mathrm{Vec}_{\mathbb{Z}_m}$ with total dimension $2m$, so it will suffice to show that the part in degree 1 has one simple object (which will then be dimension $\sqrt{m}$). In other words, if we write $\mathcal{F} = \mathrm{Vec}_{\mathbb{Z}_m} \oplus \mathcal{A}$, then we want to show $\mathcal{A}$ has exactly one simple object.

$\mathcal{A}$ is an invertible bimodule category for $\mathrm{Vec}_{\mathbb{Z}_m}$.[23] Now, bimodule categories of $\mathrm{Vec}_{\mathbb{Z}_m}$ are in bijection with module categories for $\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_m})$. In any case, module categories for Vec_G are labelled by (H, β) where $H < G$ and $\beta \in H^2(H; U(1))$ [163].

Since $\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_m}) \cong \mathrm{Vec}_{\mathbb{Z}_m \times \hat{\mathbb{Z}}_m}$ *as fusion categories*, the module categories are labelled as above. Physically, the (H, β) data are labelling a topological boundary condition for the $\mathbb{Z}_m$ gauge theory described in our previous section. The $\mathbb{Z}_2$ action is defining an electric-magnetic duality which swaps our Dirichlet and Neumann boundary conditions, and we are convincing ourselves that at the end of the wall lives a $\mathbb{Z}_2$ twist-line with Tambara-Yamagami like properties.

There is a distinguished module category given by our $\mathbb{Z}_2$ action. In particular, the $\mathbb{Z}_2$ axis-swapping action defines an isomorphism $\chi : \mathbb{Z}_m \to \hat{\mathbb{Z}}_m$ whose graph is our subgroup $H = \mathbb{Z}_m < \mathbb{Z}_m \times \hat{\mathbb{Z}}_m$. Physically speaking, this is telling us if the electric-magnetic duality is swapping the axes and sending things of "electric charge 1" to things with "magnetic charge 1" or also rearranging the definitions of electric and magnetic charge.

Now the module category of $\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_m})$ defined by that diagonal $\mathbb{Z}_m$ (with trivial cocycle since $H^2(\mathbb{Z}_m, U(1))$ is trivial), gives us a bimodule category of $\mathrm{Vec}_{\mathbb{Z}_m}$ using the bulk-boundary functor $\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_m}) \to \mathrm{Vec}_{\mathbb{Z}_m}$. In particular, the data is simply the projection onto the first factor in $\pi : \mathbb{Z}_m \times \hat{\mathbb{Z}}_m \to \mathbb{Z}_m$. Following our particular module category $(H, 1)$, we get a $\mathrm{Vec}_{\mathbb{Z}_m}$ module category (we don't need the full bimodule category to count simple objects).

The simple objects of $\mathcal{A}$ are the simple module-objects for the $\mathrm{Vec}_{\mathbb{Z}_m}$ algebra object $\pi((H, 1))$, but these correspond to subgroups $\mathbb{Z}_m < \mathbb{Z}_m$, of which there is only 1. So $\mathcal{A}$ has one object. $\square$

[23]Note: physically, this is not saying that the TY line is invertible, but that the 2d wall which extends off of it into the (2+1)d bulk is invertible.

4.6 Conclusion

In this chapter we studied duality defects of the holomorphic E_8 lattice VOA V which separate a theory from its orbifold by a cyclic subgroup $\mathbb{Z}_m$. We computed the defect partition functions explicitly as $\mathbb{Z}_2$ twists of the invariant sub-VOAs $V^{\mathbb{Z}_m}$ and compared our results to fermionization in the $m = 2$ case and a particular conformal inclusion in the $m = 3$ case (see Appendix B.1), and obtained matching results. We also explained this computation using the (2+1)d topological field theory perspective of 2d orbifolds.

Focusing on V gave us the ability to understand the group actions very explicitly. In fact, since the non-anomalous $\mathbb{Z}_m$ symmetries of V are systematically enumerable, as well as $\mathbb{Z}_2$ actions of the invariant sub-VOA, we have actually *classified all $\mathbb{Z}_m$ Tambara-Yamagami actions on the E_8 lattice VOA* (but see the open problems list). They are given as a pair consisting of a non-anomalous $\mathbb{Z}_m$ symmetry of V and a (possibly anomalous) $\mathbb{Z}_2$ action which "swaps the axes" of the metric Abelian group of irreducible representations $\mathrm{Irr}(V^{\mathbb{Z}_m})$.

However, the (2+1)d discussion of duality defects did not actually rely very particularly on the details of V, so similar constructions should work for arbitrary holomorphic theories that are sufficiently "nice," as demonstrated in Theorem 2. Generally, by the results of [222] (see also Appendix D of [1]), we expect a modified version of this procedure to work for more complicated theories in higher dimensions (as in [142, 143, 144]), as it's all simply a manifestation of the higher-dimensional TFT perspective on these topological defect operators.

Open Problems

1. **Full associator data**. In this chapter, we do not obtain the full associator data for our Tambara-Yamagami categories. From a 2d perspective, data like the sign τ has an effect on the associativity relations for the duality defect line in the CFT. Based on our discussion, we suspect it is controlled by whether or not the $\mathbb{Z}_2$ action is anomalous or non-anomalous. Comparing to the (2+1)d TFTs, there is similarly a question of whether or not we stack with an invertible phase of matter before gauging the $\mathbb{Z}_2^{em}$ bulk 0-form symmetry, as $H^3(\mathbb{Z}_2; U(1)) = \mathbb{Z}_2$. It would be fun to work this data out explicitly and compare with the formalism of G-crossed categories [212, 22].
2. ***n*-ality defects**. In [80, 148], the authors discuss higher n-ality defects $\mathcal{N}$ with fusion rules
$$X_g \otimes \mathcal{N} = \mathcal{N} = \mathcal{N} \otimes X_g \,, \quad \mathcal{N}^n = \sum_{g \in G} g \,. \tag{4.151}$$

There is nothing particularly restrictive about our procedure for finding duality defects that could not be extended to these higher n-ality defects. In particular, for E_8, one could use Kac's theorem to construct automorphisms G so that the weight-one subspace $(V_{E_8})^G$ has an $\mathfrak{so}(8)$ subalgebra, then try twisting by (a lift of) the $\mathbb{Z}_3$ automorphism of the D_4 Dynkin diagram. This would give a triality defect of a very different origin than those discussed in [148].

3. **Matching to and from other examples**. To gain physical insight into other theories, the E_8 theory, or the structure of duality defects more broadly, it could be helpful to compare the constructions of defects in more specific cases. For example, using parafermionization or playing games with conformal inclusions as in Appendix B.1. The fact that factors of $\mathfrak{u}(1)$ appear in the weight-one subspaces of various fixed VOAs $((V^{\mathbb{Z}_m})^{\mathbb{Z}_2})_1$ indicates the possibility of explicit character decompositions mirroring those in Appendix B.1 but with free bosons attached.
4. **Other Theories.** We focused on the E_8 VOA, but our strategy should apply to arbitrary holomorphic VOAs without extra complications, although the lattice descriptions may not be available. From the discussion in Section 4.5, we actually expect it to apply to duality interfaces in more general theories.

Order	*Automorphism*	$\mathbb{Z}_m$ *Anomaly*	*Fixed Subalgebra*
2	(0,0,0,0,0,0,0,1,0)	0	D_8
	(0,1,0,0,0,0,0,0,0)	1	$A_1 \times E_7$
3	(0,0,0,0,0,0,0,0,1)	1	A_8
	(1,0,0,0,0,0,0,1,0)	2	$D_7 \times \mathfrak{u}(1)$
	(0,0,1,0,0,0,0,0,0)	0	$A_2 \times E_6$
	(1,1,0,0,0,0,0,0,0)	1	$E_7 \times \mathfrak{u}(1)$
4	(1,0,0,0,0,0,0,0,1)	0	$A_7 \times \mathfrak{u}(1)$
	(2,0,0,0,0,0,0,1,0)	2	$D_7 \times \mathfrak{u}(1)$
	(0,0,0,0,0,0,1,0,0)	3	$A_1 \times A_7$
	(0,0,0,1,0,0,0,0,0)	2	$A_3 \times D_5$
	(1,0,1,0,0,0,0,0,0)	3	$A_1 \times E_6 \times \mathfrak{u}(1)$
	(2,1,0,0,0,0,0,0,0)	1	$E_7 \times \mathfrak{u}(1)$
	(0,1,0,0,0,0,0,1,0)	1	$A_1 \times D_6 \times \mathfrak{u}(1)$
5	(2,0,0,0,0,0,0,0,1)	4	$A_7 \times \mathfrak{u}(1)$
	(3,0,0,0,0,0,0,1,0)	2	$D_7 \times \mathfrak{u}(1)$
	(0,0,0,0,0,0,0,1,1)	1	$A_7 \times \mathfrak{u}(1)$
	(1,0,0,0,0,0,0,2,0)	3	$D_7 \times \mathfrak{u}(1)$
	(1,0,0,0,0,0,1,0,0)	2	$A_1 \times A_6 \times \mathfrak{u}(1)$
	(0,0,0,0,1,0,0,0,0)	0	$A_4 \times A_4$
	(1,0,0,1,0,0,0,0,0)	1	$A_2 \times D_5 \times \mathfrak{u}(1)$
	(2,0,1,0,0,0,0,0,0)	3	$A_1 \times E_6 \times \mathfrak{u}(1)$
	(0,0,1,0,0,0,0,1,0)	4	$A_2 \times D_5 \times \mathfrak{u}(1)$
	(3,1,0,0,0,0,0,0,0)	1	$E_7 \times \mathfrak{u}(1)$
	(0,1,0,0,0,0,0,0,1)	3	$A_1 \times A_6 \times \mathfrak{u}(1)$
	(1,1,0,0,0,0,0,1,0)	0	$D_6 \times \mathfrak{u}(1)^2$
	(0,1,1,0,0,0,0,0,0)	2	$A_1 \times E_6 \times \mathfrak{u}(1)$
	(1,2,0,0,0,0,0,0,0)	4	$E_7 \times \mathfrak{u}(1)$

Table 4.2: The finite order automorphisms of the E_8 Lie algebra can be enumerated, up to conjugacy, by Theorem 1. The automorphism is given by a sequence s which describes the automorphism. In physics terms, s describes how the operators of V_{E_8} are screened and the anomaly of that symmetry.

Order	*Automorphism*	*Fixed Subalgebra*
2	$\begin{array}{ccccccccccccccc} &&&&&&&&&&0&&&& \\ &&&&&&&&&&\vert&&&& \\ 0&—&0&—&0&—&0&—&0&—&0&—&0&—&1 \end{array}$	D_8
3	$\begin{array}{ccccccccccccccc} &&&&&&&&&&0&&&& \\ &&&&&&&&&&\vert&&&& \\ 0&—&0&—&1&—&0&—&0&—&0&—&0&—&0 \end{array}$	$A_2 \times E_6$
4	$\begin{array}{ccccccccccccccc} &&&&&&&&&&1&&&& \\ &&&&&&&&&&\vert&&&& \\ 1&—&0&—&0&—&0&—&0&—&0&—&0&—&0 \end{array}$	$A_7 \times \mathfrak{u}(1)$
5	$\begin{array}{ccccccccccccccc} &&&&&&&&&&0&&&& \\ &&&&&&&&&&\vert&&&& \\ 0&—&0&—&0&—&0&—&1&—&0&—&0&—&0 \end{array}$	$A_4 \times A_4$
	$\begin{array}{ccccccccccccccc} &&&&&&&&&&0&&&& \\ &&&&&&&&&&\vert&&&& \\ 1&—&1&—&0&—&0&—&0&—&0&—&0&—&1 \end{array}$	$D_6 \times \mathfrak{u}(1)^2$

Table 4.3: List of non-anomalous symmetries of the E_8 Lie algebra and fixed point Lie subalgebras up to order $m = 5$. Instead of listing the corresponding sequence s for the automorphism, we mark s onto the (affine) E_8 Dynkin diagram, illustrating the origin of the fixed Lie subalgebras.

	E_6 inner (3)	E_6 outer (2)
A_2 inner (2)	6	4
A_2 outer (1)	3	2

Table 4.4: There are 15 automorphisms of the $A_2 \times E_6$ Lie algebra of order less than or equal to 2. The row and column labels denote the origin of the $A_2 \times E_6$ automorphisms. For example, there are 4 $\mathbb{Z}_2$ actions on $A_2 \times E_6$ which come from one of the (2) inner actions on A_2 times one of the (2) outer actions on E_6.

$((V^{\mathbb{Z}_3})^{\mathbb{Z}_2})_1$	*q-Expansion of* $\mathbb{Z}_3$ *Duality Defects*
$A_1 \times E_6$	$q^{-\frac{1}{3}}(1+76q+574q^2+3000q^3+O(q^4))$
$A_1 \times A_1 \times A_5$	$q^{-\frac{1}{3}}(1-4q+14q^2-40q^3+O(q^4))$
$A_1 \times D_5 \times \mathfrak{u}(1)$	$q^{-\frac{1}{3}}(1+12q-2q^2+56q^3+O(q^4))$
$A_2 \times F_4$	$q^{-\frac{1}{3}}(1+34q+304q^2+1446q^3+O(q^4))$
$A_1 \times F_4 \times \mathfrak{u}(1)$	$q^{-\frac{1}{3}}(1+26q+80q^2+350q^3+O(q^4))$
$A_2 \times C_4$	$q^{-\frac{1}{3}}(1+2q-16q^2+38q^3+O(q^4))$
$A_1 \times C_4 \times \mathfrak{u}(1)$	$q^{-\frac{1}{3}}(1-6q+16q^2-34q^3+O(q^4))$

Table 4.5: List of the defect partition functions of order 3, with defect twisting time. On the left are the weight one subspaces of $(V^{\mathbb{Z}_3})^{\mathbb{Z}_2}$; these are Lie algebras, not invariant lattices. On the right are q-expansions of the defect partition functions from Equations (4.133 – 4.139). The first 3 defects are lifted from outer A_2 lattice actions, the next 4 are lifted from outer E_6 lattice actions.

$((V^{\mathbb{Z}_4})^{\mathbb{Z}_2})_1$	*q-Expansion of* $\mathbb{Z}_4$ *Duality Defects*
A_7	$q^{-\frac{1}{3}}(1+62q+784q^2+5088q^3+O(q^4))$
$A_1 \times A_5 \times \mathfrak{u}(1)$	$q^{-\frac{1}{3}}(1+14q+16q^2+96q^3+O(q^4))$
$A_3 \times A_3 \times \mathfrak{u}(1)$	$q^{-\frac{1}{3}}(1-2q+16q^2-32q^3+O(q^4))$
$C_4 \times \mathfrak{u}(1)$	$q^{-\frac{1}{3}}(1+10q+80q^2+224q^3+O(q^4))$
$C_4 \times \mathfrak{u}(1)$	$q^{-\frac{1}{3}}(1+10q+16q^2+96q^3+O(q^4))$
$D_4 \times \mathfrak{u}(1)$	$q^{-\frac{1}{3}}(1-6q+16q^2-32q^3+O(q^4))$

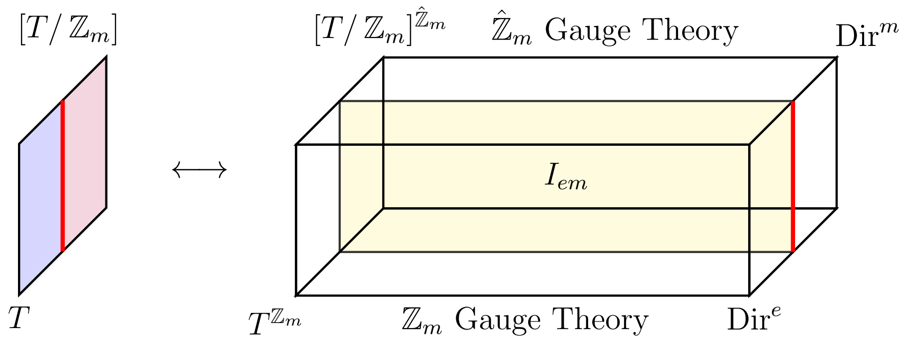
Table 4.6: List of the defect partition functions of order 4, with defect twisting time. The first three come from the lattice automorphism $\nu_2^{(4)}$ while the rest come from $\nu_3^{(4)}$.

$((V^{\mathbb{Z}_5^A})^{\mathbb{Z}_2})_1$	q-*Expansion of* $\mathbb{Z}_5^A$ *Duality Defects*
$A_4 \times B_2$	$q^{-\frac{1}{3}}(1+20q+34q^2+140q^3+O(q^4))$
$A_3 \times B_2 \times \mathfrak{u}(1)$	$q^{-\frac{1}{3}}(1+4q-14q^2+28q^3+O(q^4))$
$A_1 \times A_2 \times B_2 \times \mathfrak{u}(1)$	$q^{-\frac{1}{3}}(1-4q+10q^2-28q^3+O(q^4))$
A_4	$q^{-\frac{1}{3}}(1+24q+124q^2+500q^3+O(q^4))$

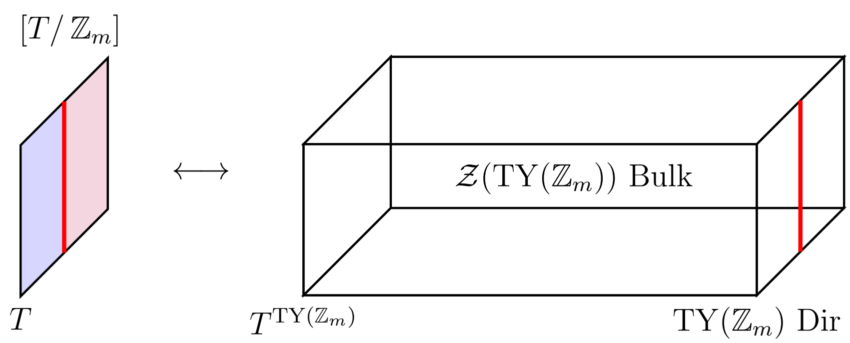
Table 4.7: List of the defect partition functions of order 5A, with defect twisting time. The first 3 defects come from the $\{\sigma_1, \sigma_2\}$ "flip" conjugacy class lattice action. The final defect lifts from the $\{\tau, \sigma_1\sigma_2\tau\}$ "exchange" conjugacy class lattice action.

$((V^{\mathbb{Z}_5^B})^{\mathbb{Z}_2})_1$	q-*Expansion of* $\mathbb{Z}_5^B$ *Duality Defects*
$D_5 \times \mathfrak{u}(1)^2$	$q^{-\frac{1}{3}}(1+26q+80q^2+352q^3+O(q^4))$
$A_5 \times \mathfrak{u}(1)^2$	$q^{-\frac{1}{3}}(1+6q+32q^2+32q^3+O(q^4))$
$A_5 \times \mathfrak{u}(1)^2$	$q^{-\frac{1}{3}}(1+6q+32q^2-44q^3+O(q^4))$
$A_3 \times A_3 \times \mathfrak{u}(1)$	$q^{-\frac{1}{3}}(1-6q+16q^2-32q^3+O(q^4))$
$B_5 \times \mathfrak{u}(1)$	$q^{-\frac{1}{3}}(1+44q+266q^2+1268q^3+O(q^4))$
$A_1 \times B_4 \times \mathfrak{u}(1)$	$q^{-\frac{1}{3}}(1+12q+10q^2+84q^3+O(q^4))$
$B_2 \times B_3 \times \mathfrak{u}(1)$	$q^{-\frac{1}{3}}(1-4q+10q^2-28q^3+O(q^4))$

Table 4.8: List of the defect partition functions of order 5B, with defect twisting time. The first four come from the automorphism $\nu_2^{(5)}$, while the rest come from $\nu_3^{(5)}$.



(a) The 2d theory T (blue) is separated from $[T/\mathbb{Z}_m]$ (purple) by a duality defect (red). This picture can be blown up to a slab of electric $\mathbb{Z}_m$ gauge theory (front) and magnetic $\hat{\mathbb{Z}}_m$ gauge theory (back) separated by an electric-magnetic duality wall (yellow). $\mathbb{Z}_m$ invariant local operators of T live at the left boundary and similarly for $[T/\mathbb{Z}_m]$. Electric and magnetic Dirichlet boundary conditions for the respective gauge theories are established at the right boundary. The duality defect is a twist line for the bulk $\mathbb{Z}_2^{em}$ symmetry.



(b) The same 2d picture can be blown up into a slab of $\mathcal{Z}(\mathrm{TY}(\mathbb{Z}_m))$ bulk with $\mathrm{TY}(\mathbb{Z}_m)$ invariant local operators on the left boundary and Dirichlet boundary conditions on the right boundary. The duality defect now lives at the end of the trivial wall. $\mathcal{Z}(\mathrm{TY}(\mathbb{Z}_m))$ has $2m$ lines with quantum dimension $\sqrt{m}$, hence many bulk lines condense to the same line at the Dirichlet boundary.

Figure 4.7: Two different (2+1)d blow-ups for a theory with a duality defect line inserted. To go from (a) to (b), one must gauge the $\mathbb{Z}_2^{em}$ 0-form symmetry generated by the duality wall I_{em}, with Neumann boundary conditions for $\mathbb{Z}_2^{em}$ on the left and Dirichlet on the right. This makes the duality wall I_{em} "become invisible," leaving behind the duality defect. To go from (b) to (a), one must gauge the bulk $\mathbb{Z}_2^{em}$ 1-form symmetry of $\mathcal{Z}(\mathrm{TY}(\mathbb{Z}_m))$.

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APPENDICES

Appendix A

Appendices for Chapter 2

A.1 Basics of 3d Interfaces

Here we will give some intuition on how to think about interfaces as used in the 3d discussions in this paper.

Suppose we are working with some 3d topological theory, then from the axioms for a TFT, a boundary condition specifies a state. For example, a Dirichlet boundary condition for a bulk 3d connection, which sets the connection equal to α at the boundary, naturally provides us with some state

$$D[\alpha] \mapsto |A| \left|\alpha\right\rangle \ . \tag{A.1}$$

Here the $|A|$ factor is required by our convention below for the normalization of states. It can be justified as following from the fact that Dirichlet boundary conditions break the A gauge symmetry, while the state is defined by fixing the connection modulo gauge transformations.

Since we will be dealing concretely with Abelian gauge theories, we normalize the inner product of these states as

$$\langle\alpha|\beta\rangle = \frac{1}{|A|}\,\delta_{\alpha\beta}\,. \tag{A.2}$$

From this, we can understand how to recover the 2d theory from the 3d picture very easily on a slab $M \times [0,1]$. T induces a boundary condition,

$$\left|T\right\rangle = \sum_{\alpha} Z_T[\alpha] \left|\alpha\right\rangle \ , \tag{A.3}$$

on one side of the slab. Now, if we put Dirichlet boundary conditions on the other side, then we are constructing some segment which computes the partition function of the 2d theory

$$ {}_T[0,1]_{D[\alpha]} = |A| \langle T|\alpha\rangle = Z_T[\alpha] \,. \tag{A.4} $$

Similarly, Neumann boundary conditions in the path integral provide us with some state $|N\rangle = \sum_\alpha |\alpha\rangle$. Hence, to recover the gauged theory, we use Neumann boundary conditions on one side

$$ {}_T[0,1]_N = \langle T|N\rangle = \frac{1}{|A|} \sum_\alpha Z_T[\alpha] \,. \tag{A.5} $$

Intuitively, an interface is like a two sided boundary condition because it interpolates between two bulks glued together. Thus in the way a boundary condition corresponds to a state, an interface corresponds to an operator.

The simplest interface we can construct is the identity interface $I_{\mathbb{1}}$. If in our given basis it is

$$ I_{\mathbb{1}} = \sum_{\alpha,\beta} I_{\mathbb{1}}[\alpha,\beta] \, |\alpha\rangle\langle\beta| \ , \tag{A.6} $$

then if we say it should be constrained to the reasonable consistency condition $I_{\mathbb{1}} = I_{\mathbb{1}} \times I_{\mathbb{1}}$, we have that

$$ I_{\mathbb{1}}[\alpha,\beta] = |A| \, \delta_{\alpha\beta} \,. \tag{A.7} $$

In general, for any interface

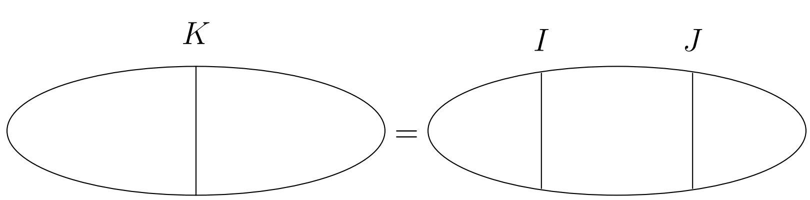
$$ I = \sum_{\alpha,\beta} I[\alpha,\beta] \, |\alpha\rangle\langle\beta| \ , \tag{A.8} $$

we have

$$ I[\alpha,\beta] = |A|^2 \langle\alpha|I|\beta\rangle \ . \tag{A.9} $$

Which corresponds to Dirichlet boundary conditions on both ends of a slab, with the topological interface I inserted somewhere in between.

Lastly, we should describe how to compose two topological interfaces. Suppose that the topological interface K is produced by fusing I and J, i.e. that



(A.10)

Or less pictorially, $K = I \times J$. In terms of the coefficients we have

$$\sum_{\alpha,\beta} K[\alpha,\beta]\,|\alpha\rangle\langle\beta| = \sum_{\alpha,\beta,\gamma,\delta} I[\alpha,\beta]\,|\alpha\rangle\langle\beta|\,J[\gamma,\delta]\,|\gamma\rangle\langle\delta| \tag{A.11}$$

$$= \sum_{\alpha,\beta}\left(\frac{1}{|A|}\sum_{\gamma} I[\alpha,\gamma]J[\gamma,\beta]\right)|\alpha\rangle\langle\beta|\ , \tag{A.12}$$

which implies that

$$K[\alpha,\beta] = \frac{1}{|A|}\sum_{\gamma} I[\alpha,\gamma]J[\gamma,\delta]\,. \tag{A.13}$$

In general, we see the product of interfaces comes with a factor of $|A|$ in components.

Let us pass through three of the simplest examples. First, we see how the identity interface functions. We know that $I_{\mathbb{1}}[\alpha,\beta] = |A|\delta_{\alpha\beta}$, so that if we hit Z_T with $I_{\mathbb{1}}$ we have the component relation

$$\frac{1}{|A|}\sum_{\alpha} Z[\alpha]I_{\mathbb{1}}[\alpha,\beta] = Z[\beta]\,. \tag{A.14}$$

The next simplest example is to see how to extract an interface (say the orbifold interface for a $\mathbb{Z}_2$ theory). Well, we know that we can write

$$Z_{[T/A]}[\beta] = \frac{1}{2}\sum_{\alpha}(-1)^{\int \alpha\cup\beta} Z_T[\alpha]\,. \tag{A.15}$$

Then we see that the orbifold interface is given by

$$I_{\text{Orbi.}}[\alpha,\beta] = (-1)^{\int \alpha\cup\beta}\,. \tag{A.16}$$

Finally, we can check that the interfaces compose properly in component form. Using the orbifold interface above, we obtain

$$\frac{1}{2}\sum_{\gamma} I_{\text{Orbi.}}[\alpha,\gamma]I_{\text{Orbi.}}[\gamma,\beta] = \frac{1}{2}\sum_{\gamma}(-1)^{\int \alpha\cup\gamma}(-1)^{\int \gamma\cup\beta} \tag{A.17}$$

$$= 2\delta_{\alpha\beta}$$

$$= I_{\mathbb{1}}[\alpha,\beta]\,. \tag{A.18}$$

A.2 Basic Facts About Spin Structures in 2d

Here we recall some basic facts about spin theories and $\mathbb{Z}_2$-structures on a 2d orientable genus g surface that will be useful in understanding examples.

The "background connection" for a spin theory is a choice of spin-structure η on the manifold, specifying the periodicity condition of the fermions around a given cycle as either Ramond (periodic) or Neveu-Schwarz (anti-periodic).

Counting, we see there are 2^{2g} spin structures on M; $2^{g-1}(2^g-1)$ of them are "odd" and $2^{g-1}(2^g+1)$ are "even." The terms odd and even refer to the number of (fixed chirality) Dirac zero modes modulo two. To count the splitting of these spin structures one just needs the fact that the number of Dirac zero modes modulo two is invariant under gluing of Riemann surfaces, i.e. it is a bordism invariant. Armed with this fact, one can build up inductively, noting that there is only one odd spin-structure (RR) on the torus, because only the purely periodic torus spin-structure could have a Majorana zero mode [77].

Now we divert our attention to $\mathbb{Z}_2$ structures. We recall that on a surface of genus g there is a symplectic basis for $H_1(M,\mathbb{Z}_2)$ given by the "a-cycle" and "b-cycle" around each hole (equivalently our $\mathbb{Z}_2$-gauge fields in $H^1(M,\mathbb{Z}_2)$ by Poincaré duality in 2d). This basis satisfies $a_i \cap b_j = \delta_{ij}$ with the cap denoting the intersection pairing.

A quadratic form on $H_1(M,\mathbb{Z}_2)$ is a function $q: H_1(M,\mathbb{Z}_2) \to \mathbb{Z}_2$ that satisfies

$$q(x+y) = q(x) + q(y) + x \cap y\,, \tag{A.19}$$

and is thusly called a "quadratic refinement" of the intersection number. For example, one particular quadratic refinement is

$$q_{\text{can}}(c_i a_i + d_j b_j) = c_i d_i\,, \tag{A.20}$$

with sums implied over repeated indices.

Given any quadratic refinement q, the Arf invariant

$$\text{Arf}[q] = \sum_{i=1}^{g} q(a_i)q(b_i) \tag{A.21}$$

is actually a basis independent quantity, uniquely classifying q up to isomorphism of quadratic forms.

Now, a result of Johnson [190] is that there is a bijection between spin structures on M and quadratic forms on $H_1(M,\mathbb{Z}_2)$. Furthermore, the bijection is simple: given a spin-structure η define

$$q_\eta(a_i) = \begin{cases} 0 & \text{if } \eta \text{ is anti-periodic around } a_i\,, \\ 1 & \text{if } \eta \text{ is periodic around } a_i\,. \end{cases} \tag{A.22}$$

And similarly for $q_\eta(b_i)$. From this, we see it makes sense to define the quantity

$$\mathrm{Arf}[\eta] := \mathrm{Arf}[q_\eta]\,. \tag{A.23}$$

As an example, on the torus equipped with spin-structure (η_1,η_2) we have

$$\mathrm{Arf}[\eta] = q_\eta(a_1)q_\eta(b_1) = \eta_1\eta_2\,. \tag{A.24}$$

Coming full circle, it is a result of Atiyah [191] that $\mathrm{Arf}[\eta]$ is precisely the mod 2 index of the Dirac operator described above.

Of course, by Poincaré duality, we have equivalently produced a quadratic form $\tilde{q}_\eta : H^1(M,\mathbb{Z}_2) \to \mathbb{Z}_2$ satisfying

$$\tilde{q}_\eta(\alpha+\beta) - \tilde{q}_\eta(\alpha) - \tilde{q}_\eta(\beta) = \int_M \alpha \cup \beta\,. \tag{A.25}$$

We will be using a multiplicative notation throughout, so it is useful to define

$$\sigma_\eta(\alpha) = (-1)^{\tilde{q}_\eta(\alpha)}\,. \tag{A.26}$$

See also [49, 74] for further discussion. Some identities for cups and Arf are included in Appendix A.3.

A.3 Identities for Cups and Arf

By construction, the term $\sigma_\eta(\alpha)$ coupling $\mathbb{Z}_2$ gauge fields to spin-structures satisfies

$$\sigma_\eta(\alpha)\sigma_\eta(\beta) = \sigma_\eta(\alpha+\beta)(-1)^{\int \alpha\cup\beta}\,. \tag{A.27}$$

Such a function can also be written in terms of the Arf invariant, which is typically the form that people present when defining GSO projection in the literature

$$\sigma_\eta(\alpha) = (-1)^{\mathrm{Arf}[\alpha+\eta]+\mathrm{Arf}[\eta]}. \tag{A.28}$$

Conversely, the Arf invariant can be written in terms of σ_η by a normalized sum over all connections

$$(-1)^{\mathrm{Arf}[\eta]} = \frac{1}{\sqrt{|H^1(M,\mathbb{Z}_2)|}} \sum_\alpha \sigma_\eta(\alpha)\,. \tag{A.29}$$

Since Arf$[\eta]$ is the number of Dirac zero modes modulo 2, then summing over $(-1)^{\mathrm{Arf}[\eta]}$ will simply count the difference in number between even and odd spin structures, hence

$$1 = \frac{1}{\sqrt{|H^1(M,\mathbb{Z}_2)|}} \sum_\eta (-1)^{\mathrm{Arf}[\eta]}\,. \tag{A.30}$$

For cyclic groups, a helpful identity for colliding interfaces with cup products is

$$\delta_{\alpha,\gamma} = \frac{1}{\sqrt{|H^1(M,\mathbb{Z}_n)|}} \sum_\beta \omega_n^{\int \alpha\cup\beta}\omega_n^{\int \beta\cup\gamma}\,, \tag{A.31}$$

where ω_n is a principal n-th root of unity.

A.4 Topological Aspects of QFTs

There are several variations on the idea of symmetry. A broad generalization of the notion of discrete symmetry involves collections of topological defects of various dimensionality, closed under fusion operations. Such collections of defects can be formalized mathematically in terms of (higher) categories. Because of the topological nature of the defects, this categorical data is also an RG flow invariant.

We will thus say that some QFT T has a *categorical symmetry* $\mathcal{S}$ if it is equipped with a collection of topological defects encoded in some higher category $\mathcal{S}$. We will leave implicit the mathematical properties required on such a *symmetry category*, which may depend sensitively on the dimensionality of spacetime, on the bosonic or fermionic nature of the QFT, etc.

An important observation is that $\mathcal{S}$ can be quite large. In particular it could be larger than the type of categories which are encountered as categorical symmetries of TQFTs. For example, a gapless 2d theory may have a categorical symmetry $\mathcal{S}$ which is too large to be described by a fusion category.

The existence of categorical symmetries may also allows one to perform certain topological manipulations on a QFT, akin to the operation of gauging a non-anomalous discrete symmetry. These manipulations produce new QFTs which have the same local dynamics as the original one, and share a large collection of local operators, but have different global properties. Such topological manipulations will commute with RG flow.

To the best of our knowledge, topological manipulations can only employ sub-collections of $\mathcal{S}$ which satisfy the axioms for categorical symmetries of TQFTs. In the discussion below, we will either restrict to the case where $\mathcal{S}$ is sufficiently finite, or only focus on a fixed sub-category of $\mathcal{S}$ which is sufficiently finite.

One may ask a variety of natural questions:

- Do the theories resulting from topological manipulations carry categorical symmetries as well?
- Are such topological manipulations invertible?
- What is the result of composing topological manipulations?
- What collection of new theories can be obtained in this manner?

The answers to these questions are independent on the dynamics of the underlying QFT. Indeed, they are expected to be independent of the specific choice of QFT as well and to only depend on the actual symmetry category $\mathcal{S}$.

Another general expectation is that the symmetry can be completely decoupled by the dynamics by a topological sandwich construction, where T is realized as a segment compactification of a topological field theory $D[\mathcal{S}]$ defined in one dimension higher. At one end of the segment we place a topological boundary condition $B[\mathcal{S}]$ supporting a symmetry category $\mathcal{S}$ of boundary defects. At the other end we place a possibly non-topological boundary theory $B[T;\mathcal{S}]$ which captures the local dynamics of T.

For a standard discrete symmetry, $D[\mathcal{S}]$ would be a Dijkgraaf-Witten discrete gauge theory and $B[\mathcal{S}]$ would be Dirichlet boundary conditions.

We expect to have a complete bijection between the collection of "absolute QFTs with symmetry category $\mathcal{S}$" and the collection of "boundary theories for $D[\mathcal{S}]$". The map from the latter to the former is the segment compactification. Invertibility of the map is not obvious, but it is expected. It is an operation analogue to the operation of coupling T to a discrete gauge theory in one dimension higher.[1] We discuss it in two dimensions in 2.4.

The higher-dimensional perspective helps answer many of the above-mentioned questions in a theory-independent manner. Topological manipulations can be applied to $B[\mathcal{S}]$ to produce new topological boundary conditions B'. The resulting QFTs will be described

[1]Mathematically, it should be a canonical condensation of $\mathcal{S}^{\mathrm{op}} \times \mathcal{S}$ in the sense of [222]

as segment compactifications involving $B[T;\mathcal{S}]$ and B'. It will have a categorical symmetry given by the category of topological defects in B'. Indeed, the collection of all possible theories which can be obtained from T by manipulating the symmetry $\mathcal{S}$ should coincide with the collection of all possible topological boundary conditions B'.

Appendix B

Appendices for Chapter 4

B.1 A $\mathbb{Z}_3$ Defect from the Potts Model

In Section 4.3.5 we obtained our $\mathbb{Z}_2$ duality defects from fermionization to contrast it to the Lie-theoretic method. We can also obtain other duality defects by playing similar games with parafermions and/or various conformal inclusions.

We will explain how to obtain one of our $\mathbb{Z}_3$ duality defects from the Potts Model. This largely just amounts to doing representation theory for 2d CFTs, but we take a (2+1)d TFT and anyon condensation viewpoint to highlight the story in Section 4.5. The final answer is given in Equation (B.20).

The (bosonic) $c = \frac{4}{5}$ minimal models are constructed by pairing data from the chiral algebra $m_{6,5}$ and the anti-chiral algebra $\overline{m}_{6,5}$ in a modular invariant way. $m_{6,5}$ has 10 chiral primaries with spins [21]

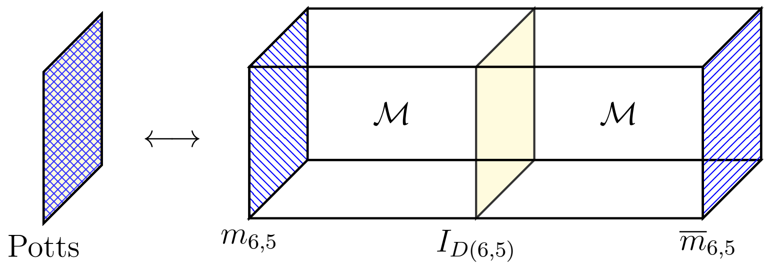
$$\{0, \tfrac{2}{5}, \tfrac{1}{40}, \tfrac{7}{5}, \tfrac{21}{40}, \tfrac{1}{15}, 3, \tfrac{13}{8}, \tfrac{2}{3}, \tfrac{1}{8}\}\,. \tag{B.1}$$

The chiral algebra data may pair up via the diagonal (or "A-type") modular invariant, which corresponds to the tetracritical Ising model, with 10 full CFT primaries and a $\mathbb{Z}_2$ symmetry group. Alternatively, there is also a non-diagonal (or "D-type") modular invariant, which corresponds to the critical three-state Potts model, with 12 primaries (we use the notation of [130])

$$\mathbb{1}_{0,0}, \epsilon_{\frac{2}{5},\frac{2}{5}}, X_{\frac{7}{5},\frac{7}{5}}, Y_{3,3}, \Phi_{\frac{7}{5},\frac{2}{5}}, \tilde{\Phi}_{\frac{2}{5},\frac{7}{5}}, \Omega_{3,0}, \tilde{\Omega}_{0,3}, \sigma_{\frac{1}{15},\frac{1}{15}}, \sigma^*_{\frac{1}{15},\frac{1}{15}}, Z_{\frac{2}{3},\frac{2}{3}}, Z^*_{\frac{2}{3},\frac{2}{3}}\,, \tag{B.2}$$

and an S_3 symmetry group. The two are related by $\mathbb{Z}_2$ orbifold.

By the construction of [96], the Potts CFT on Σ can be blown up to a slab of (2+1)d TFT on $\Sigma \times I$ with bulk $\mathcal{M} := \mathrm{Rep}(m_{6,5})$. The TFT assigns the vector space $\mathcal{H}_\Sigma$ of conformal blocks of $m_{6,5}$ to the chiral boundary (and $\overline{\mathcal{H}}_\Sigma$ to the other).[1] From this point of view, the choice of D-type modular invariant is specified by a topological interface $I_{D(6,5)}$,[2] so that we may draw a picture like



(B.3)

where "$\longleftrightarrow$" means "blows up to" in one direction and "compactifies down to" in the other.

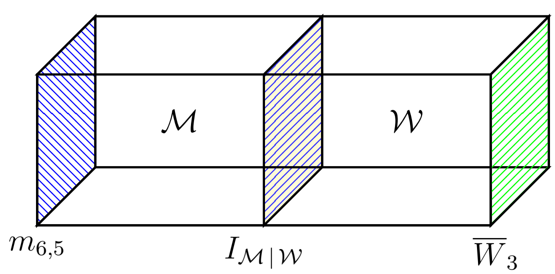
The untwisted (torus) partition function for the three-state Potts model describes this D-type pairing, it is given by

$$Z_{\mathrm{Potts}}[0,0] = |\chi_{1,1} + \chi_{4,1}|^2 + |\chi_{2,1} + \chi_{3,1}|^2 + 2|\chi_{4,3}|^2 + 2|\chi_{3,3}|^2 \,. \tag{B.4}$$

However, it is also well known that the Potts model can be realized as the diagonal modular invariant for the W_3 algebra

$$Z_{W_3}[0,0] = |\chi_{\mathbb{1}}|^2 + |\chi_\epsilon|^2 + |\chi_Z|^2 + |\chi_{Z^*}|^2 + |\chi_\sigma|^2 + |\chi_{\sigma^*}|^2 \,. \tag{B.5}$$

The fact that the Potts model can also be written as the diagonal modular invariant CFT for a W_3 algebra signals that there exists an anyon condensation wall between $\mathcal{M}$ and $\mathcal{W} := \mathrm{Rep}(W_3)$, condensing the 10 anyons of $\mathcal{M}$ down to the 6 bulk anyons of $\mathcal{W}$ (as we will confirm). Pictorially, there exists a topological interface $I_{\mathcal{M}|\mathcal{W}}$ such that



(B.6)

[1] We label the boundaries of the (2+1)d TFT by the chiral algebras to avoid overcluttering.

[2] One can quickly obtain $I_{D(6,5)}$ as one the only two (irreducible bosonic) gapped boundary conditions of $\mathcal{M} \times \overline{\mathcal{M}}$ using the Lagrangian algebra discussion outlined in Section 4.5. The interface $I_{D(6,5)}$ satisfies $I^2_{D(6,5)} = 2I_{D(6,5)}$.

The 10 lines of $\mathcal{M} = \mathrm{Rep}(m_{6,5})$ are

$$\{0_0, 1_{\frac{2}{5}}, 2_{\frac{1}{40}}, 3_{\frac{7}{5}}, 4_{\frac{21}{40}}, 5_{\frac{1}{15}}, 6_3, 7_{\frac{13}{8}}, 8_{\frac{2}{3}}, 9_{\frac{1}{8}}\}\,, \tag{B.7}$$

where the subscript denotes the chiral spin of the operator they correspond to (but will henceforth be dropped). Note that spin in the (2+1)d TFT is only meaningful $\bmod 1$ so that lines 0 and 6 have the same spin. At this point, there's only one bosonic condensation we could even try to perform; it must be that the new vacuum line is $\varphi = 0 + 6$. This makes sense since the spin-3 operator generates the W_3 algebra.

To understand the condensation at the interface $I_{\mathcal{M}|\mathcal{W}}$, we compute the fusion rules of the (non-trivial) lines in $\mathcal{M}$ with the new vacuum φ

$$\begin{aligned} &\varphi \times 1 = 1 + 3\,, && \varphi \times 2 = 2 + 4\,, && \varphi \times 3 = 3 + 1\,, \\ &\varphi \times 4 = 4 + 2\,, && \varphi \times 5 = 5_1 + 5_2\,, && \varphi \times 6 = 6 + 0\,, \\ &\varphi \times 7 = 7 + 9 && \varphi \times 8 = 8_1 + 8_2\,, && \varphi \times 9 = 9 + 7\,. \end{aligned} \tag{B.8}$$

Fusions like $\varphi \times 1 = 1 + 3$ are straightforward, as the fusion rules of lines in $\mathcal{M}$ match those of primaries in $m_{6,5}$. One subtlety is the appearance of terms like $\varphi \times 5 = 5_1 + 5_2$, where the subscript reminds us that the line formerly known as 5 splits into two simple objects on $I_{\mathcal{M}|\mathcal{W}}$. Altogether, the lines living on $I_{\mathcal{M}|\mathcal{W}}$, coming from the condensation, are

$$\{\varphi, (1+3), (2+4), 5_1, 5_2, (7+9), 8_1, 8_2\}\,. \tag{B.9}$$

We note that the lines $(2+4)$ and $(7+9)$ cannot be lifted to the bulk $\mathcal{W}$ phase because the constituent anyons have different spins and so the combined object cannot be given a braiding; in the language of [207] they are "totally confined." The anyons of the phase $\mathcal{W}$ must be

$$\mathcal{W} = \{\varphi_0, (1+3)_{\frac{2}{5}}, (5_1)_{\frac{1}{15}}, (5_2)_{\frac{1}{15}}, (8_1)_{\frac{2}{3}}, (8_2)_{\frac{2}{3}}\}\,, \tag{B.10}$$

which matches what we know about the W_3 algebra.

One can now verify that $\mathcal{W}$ has the same fusion rules, but opposite spins, as the category $\mathcal{C} := \mathrm{Rep}(\mathfrak{su}(3)_1 \times (\mathfrak{f}_4)_1)$. In particular, the spins are

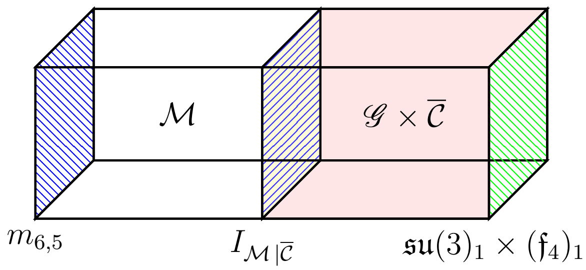
$$(\mathfrak{f}_4)_1 : \quad \{0, \tfrac{3}{5}\}\,, \tag{B.11}$$

$$\mathfrak{su}(3)_1 : \quad \{0, \tfrac{1}{3}, \tfrac{1}{3}\}\,, \tag{B.12}$$

$$\mathcal{C} : \quad \{0, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{3}{5}, \tfrac{14}{15}, \tfrac{14}{15}\}\,, \tag{B.13}$$

and we have $\mathcal{C} = \overline{\mathcal{W}}$.

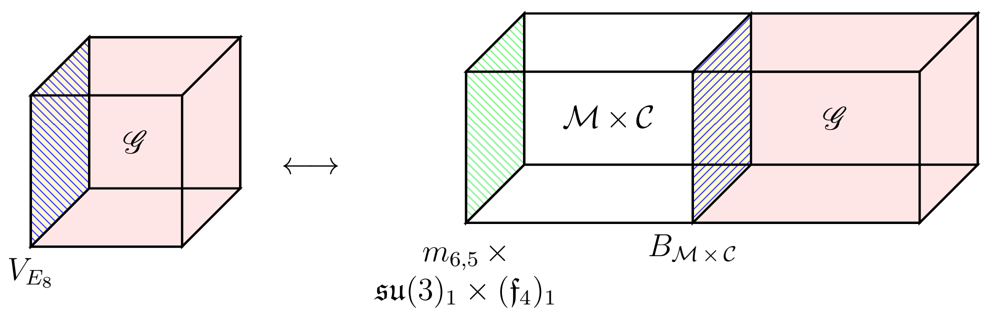
This means we can draw a picture like



(B.14)

where we have flooded the right-hand side with $\mathscr{G}$, which denotes the $k = -1$ gravitational Chern-Simons theory, to soak up the gravitational anomaly [176, 150]. Recall $c(m_{6,5}) = +4/5$, $c(\mathfrak{su}(3)_1) = +3$, and $c((\mathfrak{f}_4)_1) = +26/5$.

By folding just the $\overline{\mathcal{C}}$ phase, we obtain the way to blow up the CFT V_{E_8} as $m_{6,5} \times \mathfrak{su}(3)_1 \times (\mathfrak{f}_4)_1$ with a topological boundary condition (or an interface to a gravitational Chern-Simons phase) on the other end



(B.15)

In terms of characters, all these pictures have just been to say that

$$\begin{aligned}\chi^{E_8}(\tau) = (\chi_{1,1}(\tau) + \chi_{4,1}(\tau))\, \chi_0^{A_2}(\tau)\, \chi_0^{F_4}(\tau) + (\chi_{2,1}(\tau) + \chi_{3,1}(\tau))\, \chi_0^{A_2}(\tau)\, \chi_{\frac{3}{5}}^{F_4}(\tau) \\ + 2\, \chi_{4,3}(\tau)\, \chi_{\frac{1}{3}}^{A_2}(\tau)\, \chi_0^{F_4}(\tau) + 2\, \chi_{3,3}(\tau)\, \chi_{\frac{1}{3}}^{A_2}(\tau)\, \chi_{\frac{3}{5}}^{F_4}(\tau)\end{aligned} \tag{B.16}$$

where we recognize the shadow of the original Potts partition function from Equation (B.4). For implementation purposes, it may be helpful to know that the $(\mathfrak{f}_4)_1$ characters can similarly be written in terms of $\mathfrak{so}(9)_1$ characters (given in (4.96)) and the $c = \frac{7}{10}$ tricritical ising characters (see [223] for a cool application)

$$\chi_0^{F_4}(\tau) = \chi_{1,1}^{m_{5,4}}(\tau)\, \chi_0^{B_4}(\tau) + \chi_{2,1}^{m_{5,4}}(\tau)\, \chi_{\frac{1}{16}}^{B_4}(\tau) + \chi_{3,1}^{m_{5,4}}(\tau)\, \chi_{\frac{1}{2}}^{B_4}(\tau)\,, \tag{B.17}$$

$$\chi_{\frac{3}{5}}^{F_4}(\tau) = \chi_{3,2}^{m_{5,4}}(\tau)\, \chi_0^{B_4}(\tau) + \chi_{2,2}^{m_{5,4}}(\tau)\, \chi_{\frac{1}{16}}^{B_4}(\tau) + \chi_{3,3}^{m_{5,4}}(\tau)\, \chi_{\frac{1}{2}}^{B_4}(\tau)\,. \tag{B.18}$$

In [119] the authors enumerate the simple topological defect lines of the Potts model, with defect partition functions. We can use our character decompositions to similarly produce a $\mathbb{Z}_3$ defect partition function for E_8. There are actually two $\mathbb{Z}_3$ duality defect lines in the Potts model, related by charge conjugation (see [130]), they have the same defect partition function

$$\begin{aligned} Z_{\text{Potts}}[0, X_{\text{TY}}] = \sqrt{3}(\chi_{1,1}(\tau) + \chi_{1,5}(\tau))(\bar{\chi}_{1,1}(\bar{\tau}) - \bar{\chi}_{1,5}(\bar{\tau})) \\ + \sqrt{3}(\chi_{3,1}(\tau) + \chi_{3,5}(\tau))(\bar{\chi}_{3,1}(\bar{\tau}) - \bar{\chi}_{3,5}(\bar{\tau})) \end{aligned} \tag{B.19}$$

So that translated to V_{E_8} we have

$$\begin{aligned} Z_{V_{E_8}}[0, X_{\text{TY}}] = \sqrt{3}(\chi_{1,1}(\tau) + \chi_{1,5}(\tau))\, \chi_0^{A_2}(\tau)\, \chi_0^{F_4}(\tau) \\ + \sqrt{3}(\chi_{3,1}(\tau) + \chi_{3,5}(\tau))\, \chi_0^{A_2}(\tau)\, \chi_{\frac{3}{5}}^{F_4}(\tau) \end{aligned} \tag{B.20}$$

whose q-expansion matches our result in Equation (4.136).

With hindsight, the path to this result is somewhat obvious. Kac's theorem tells us about the weight one subspace of $(V^{\mathbb{Z}_3})^{\mathbb{Z}_2}$, one of them happens to be $A_2 \times F_4$ (where the F_4 appeared as the invariant subalgebra under the $\mathbb{Z}_2$ Dynkin diagram automorphism for E_6). If we assume these are both reflecting the presence of level-1 WZW models, then we only have to explain a missing central charge of $\frac{4}{5}$, and we know that the Potts model has a $\mathbb{Z}_3$ symmetry and is self-dual under $\mathbb{Z}_3$ orbifold (because there's no other theory for it to map to with $\mathbb{Z}_3$ symmetry).

As a related exercise, one can reobtain the results from fermionization with their associator data by noting that $\mathrm{TY}(\mathbb{Z}_2, 1, +1/\sqrt{2}) \cong$ Ising and $\mathrm{TY}(\mathbb{Z}_2, 1, -1/\sqrt{2}) \cong \mathfrak{su}(2)_2$ as fusion categories and using techniques of anyon condensation.

Such manipulations would become rather cumbersome if one wants to enumerate all duality defects. The treatment given in the main text provides a more systematic way to obtain these same results. A combined approach could be of physical interest to better understand the final expressions for the duality defects in terms of the characters of $(V^{\mathbb{Z}_m})^{\mathbb{Z}_2}$.

B.2 Symmetric Non-Degenerate Bicharacters and "Rearrangement Data"

Tambara-Yamagami categories are specified by an Abelian group G, a symmetric non-degenerate bicharacter $\chi : G \times G \to U(1)$, and a Frobenius-Schur indicator $\tau = \pm\sqrt{|G|}$.

In the main body of this paper we largely focus on the fusion rules, but the symmetric non-degenerate bicharacter is also a mostly accessible piece of data.

Recall that a symmetric non-degenerate bicharacter gives isomorphisms $\chi_G : G \to \hat{G}$ and $\chi_{\hat{G}} : \hat{G} \to G$ such that $\chi_{\hat{G}} \circ \chi_G = \mathbb{1}_G$ and $\chi_G \circ \chi_{\hat{G}} = \mathbb{1}_{\hat{G}}$. In our case, $G = \mathbb{Z}_m$, and $\mathrm{Irr}(V^{\mathbb{Z}_m}) \cong \mathbb{Z}_m \times \hat{\mathbb{Z}}_m$ is a metric Abelian group (with metric function given by the natural pairing between $\mathbb{Z}_m$ and $\hat{\mathbb{Z}}_m$).

In this paper, we look for $\mathbb{Z}_2$ symmetries of V^G which act to "swap the axes" of A. The data of a symmetric non-degenerate bicharacter describes whether or not the axes are also "rearranged" when they are swapped. To what extent can we obtain this "rearrangement data" of our $\mathbb{Z}_2$ action?

There are many "non-canonicalities" involved in this axis-swapping. For example, $\mathbb{Z}_m \cong \hat{\mathbb{Z}}_m$, but not-canonically, with $\left|\mathbb{Z}_m^\times\right|$ isomorphisms between them. Even then, when dealing with cyclic group symmetries, there isn't really a meaning of "one" unit of electric charge. We claim that we can distinguish, at minimum, if two different $\mathbb{Z}_2$ actions on V^G lead to different symmetric non-degenerate bicharacters.

We are dealing with some concrete data, we have:

1. An "original" E_8 lattice L.
2. A sublattice $L_0 \subseteq L$ which is obtained as a fixed sublattice of L under our original (non-anomalous) $\mathbb{Z}_m$ action.
3. A dual lattice L_0^* such that $L \subseteq L_0^*$ and $L_0^*/L_0 \cong A$.
4. Another E_8 lattice $L' \subseteq L_0^*$ such that $L \cap L' = L_0$.

The first piece of data in some sense explains if we can identify G or $\hat{G}$ inside of A, as we can identify if some element is in L. But this will not be of primary concern.

Let $\{\sigma_i\}$ be a collection of order 2 automorphisms of L_0 (whose actions extend to all of $\mathbb{R}^8$ by linearity), and $\{\hat{\sigma}_i\}$ the (well-defined) induced actions on $A = L_0^*/L_0$. We will only concern ourselves with those σ_i such that $\hat{\sigma}_i$ swaps the axes of A. We can determine if they correspond to different symmetric non-degenerate bicharacters as follows:

1. We start with the lattice L_0 and obtain L_0^*/L_0. In particular, we obtain a list of representatives for the m^2 equivalence classes. We can identify elements corresponding to equivalence classes "on the axes" by using the discriminant (which is well-defined on the equivalence classes), i.e. checking which representatives are even vectors.
2. Next we pick some representative ξ with discriminant 0 and order m, and declare that the coset $\xi + L_0$ generates the subgroup corresponding to L in A. We define things in the sector $\xi + L_0$ to have reference electric charge 1.

3. Now we pick one of our $\mathbb{Z}_2$ actions, σ_1 say, and declare that the operators in $\omega + L_0$ have reference magnetic charge 1, where $\omega = \sigma_1(\xi)$. In other words, $\hat{\sigma}_1$ maps the sector with electric charge 1 to the sector with magnetic charge 1. This part is non-canonical in that σ_1 picks a particular isomorphism $\hat{\sigma}_1 : G \to \hat{G}$ inside A.
4. Finally, we act with all other σ_i on ξ to find the "image magnetic charge" relative to (ξ, σ_1). i.e. $\hat{\sigma}_i(\xi + L_0) = m_i\omega + L_0$ for some $m_i \in \mathbb{Z}_m^\times$.

We see that relative to the declarations that ξ has electric charge 1 and σ_1 maps it to magnetic charge 1, m_i tells us whether or not σ_i also "rearranges" the axes when it swaps them, and so constitutes the data we want.

To what extent do our choices matter? On one hand, note that if we had chosen a different reference electric charge ξ', then σ_1 would define a different generator $\omega' + L_0 = \hat{\sigma}_1(\xi' + L_0)$ for the magnetic subgroup. But $\xi' + L_0 = k\xi + L_0$ for some invertible $k \bmod m$. Thus, multiplying both sides by k, we see $\hat{\sigma}_i(\xi + L_0) = m_i\omega + L_0$ if and only if $\hat{\sigma}_i(\xi' + L_0) = m_i\omega' + L_0$ since k is invertible. So it's not our choice of reference electric charge sector $\xi + L_0$ that matters, just σ_1.

On the other hand, m_i does depend on our original choice of σ_1 in defining the sector with one unit of magnetic charge. This is obvious since $m_i = 1$ relative to itself. We will denote this "rearrangement number" relative to σ_1 as $\mathrm{Rearr}_{\sigma_1}(\hat{\sigma}_i) = m_i$.

We can do slightly better though, as ratios of m_i are meaningful independent of σ_1. To see this is the same as before. Fix some ξ and suppose we have two reference isomorphisms σ and $\tilde{\sigma}$ with their respective ω and $\tilde{\omega}$, and suppose $\hat{\mu}$ is some $\mathbb{Z}_2$ action of interest. Then relative to σ and $\tilde{\sigma}$ we have

$$\mathrm{Rearr}_\sigma(\hat{\mu}) =: m_\mu\,, \quad \mathrm{Rearr}_{\tilde{\sigma}}(\hat{\mu}) =: \tilde{m}_\mu\,. \tag{B.21}$$

Then, by definition, this means $\hat{\mu}(\xi + L_0) = \tilde{m}_\mu\tilde{\omega} + L_0 = m_\mu\omega + L_0$. But $\tilde{\omega} + L_0 = k\omega + L_0$ for some invertible k, so $m_\mu = k\tilde{m}_\mu$ and the statement about ratios of rearrangement numbers for two $\hat{\mu}_1$ and $\hat{\mu}_2$ follows.

B.2.1 Our Results from the Text

We note that this only depends on how the $\mathbb{Z}_2$ automorphism acts on the underlying lattice, so duality defects that are lifted from the same underlying lattice action on L_0 are in the same rearrangement class.

Our data is as follows:

$\mathbb{Z}_2$ Case The fixed lattice is $L_0 = D_8$ and $L_0^*/L_0 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. There is no rearrangement data in this case.

$\mathbb{Z}_3$ Case The fixed lattice is $L_0 = A_2 \times E_6$ and $L_0^*/L_0 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. There are 7 duality defects which come from the Dynkin diagram automorphism on A_2 or on E_6, but not both. We find that these are in different rearrangement classes, and thus have rearrangement number 2 relative to each other.

$\mathbb{Z}_4$ Case The fixed lattice is given in (4.141), and there are 6 duality defects coming from lifts of 2 of the 3 non-trivial $\mathbb{Z}_2$ lattice automorphisms, namely $\nu_2^{(4)}$ and $\nu_3^{(4)}$ in Equation (4.142). Since $\left|\mathbb{Z}_4^\times\right| = 2$ there are only two possibly rearrangement classes. We find that they are in different rearrangement classes, having rearrangement number 3 relative to each other.

$\mathbb{Z}_5^A$ Case The fixed lattice is given by the $A_4 \times A_4$ root lattice inside E_8, and there are 4 duality defects coming from lifts of 2 of the 5 conjugacy classes of outer automorphisms. These correspond to the $\{\sigma_1, \sigma_2\}$ class "flipping" one A_4, and the $\{\tau, \sigma_1\sigma_2\tau\}$ class "exchanging" A_4's. We find that these are in different rearrangement classes, with the "exchange" conjugacy class having rearrangement number 2 relative to the "flip" conjugacy class, of course this means the flip-class has rearrangement number $2^{-1} = 3$ relative to the exchange-class.

$\mathbb{Z}_5^B$ Case The fixed lattice is described at the start of Section 4.4.3, and there are 7 duality defects coming from lifts of 2 lattice automorphisms $\nu_2^{(5)}$ and $\nu_3^{(5)}$ given in Equation (4.147). We find that they are in different rearrangement classes, with the $\nu_3^{(5)}$ defects having rearrangement number 2 relative to $\nu_2^{(5)}$ defects.