# Finitary approximations of free probability, involving combinatorial representation theory

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Pure Mathematics

Waterloo, Ontario, Canada, 2023

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## Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

## Statement of Contributions

Chapter 2 presents results from two papers [13, 11]: one as Section 2.2 that I co-authored with Zhi Yin, and one as Section 2.3 and Section 2.4 that I wrote by myself. Chapter 3 presents results from a paper [12] that I co-authored with Claus Köstler and Alexandru Nica.

## Abstract

This thesis contributes to two theories which approximate free probability by finitary combinatorial structures. The first is finite free probability, which is concerned with expected characteristic polynomials of various random matrices and was initiated by Marcus, Spielman, and Srivastava in 2015. An alternate approach to some of their results for sums and products of randomly rotated matrices is presented, using techniques from combinatorial representation theory. Those techniques are then applied to the commutators of such matrices, uncovering the non-trivial but tractable combinatorics of immanants and Schur polynomials.

The second is the connection between symmetric groups and random matrices, specifically the asymptotics of star-transpositions in the infinite symmetric group and the gaussian unitary ensemble (GUE). For a continuous family of factor representations of $S_\infty$, a central limit theorem for the star-transpositions $(1, n)$ is derived from the insight of Gohm-Köstler that they form an exchangeable sequence of noncommutative random variables. Then, the central limit law is described by a random matrix model which continuously deforms the well-known traceless GUE by taking its gaussian entries from noncommutative operator algebras with canonical commutation relations (CCR). This random matrix model generalizes results of Köstler and Nica from 2021, which in turn generalized a result of Biane from 1995.

**Acknowledgements**

Alexandru Nica, for introducing me to free probability and random matrices, and constantly supporting and encouraging me even when things didn't go my way.

Kyle Harvey, Matt Kennedy, Mehrdad Kalantar, and Yuly Billig, for showing me various parts of mathematics as a high school or undergraduate student and helping me get started as a mathematician.

Mom and Dad, for their love and moral support throughout my education.

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# Introduction

The theory of *free probability* was initiated in the 1980s by D.-V. Voiculescu, motivated by the problem of distinguishing certain von Neumann algebras, the so-called free group factors. He isolated a notion of *free independence* or *freeness*, which essentially captures the lack of algebraic relations among the generators of a free group in terms of the group algebra and its trace.

In the following years, it was confirmed that freeness behaves in many ways like a form of probabilistic independence for variables which are "maximally noncommutative": there are free analogues of the classical central and Poisson limit theorems, the characteristic function, the moment-cumulant formula, the Lévy-Khinchin formula for infinitely divisible distributions, and so on.

One of the most basic facts about freeness is that it is inherently infinite-dimensional: it does not exist in any non-trivial finite-dimensional setting. For an analogy, one might think of the non-finiteness of any free product of groups. Furthermore, the operator algebras studied using free-probabilistic techniques typically fall outside the purview of any standard finite-dimensional approximation, i.e. the von Neumann algebras are not hyperfinite and the $C^*$-algebras are not nuclear.

It was a major breakthrough, then, around 1990, when it was discovered that random matrices are a rich source of freeness: for some natural and basic models, the asymptotics of their average eigenvalue distributions are described neatly by free probability. Conversely, one could say that large random matrices asymptotically approximate free products of operator algebras.

There are various approximations of free probability which are somehow related to the basic random matrix models, and this thesis contributes to two of them. The first is the new theory of *finite free probability*, initiated by A. Marcus, D. Spielman, and N. Srivastava, motivated by their 2013–2015 breakthroughs related to roots of polynomials and consequent solutions of major open problems in graph theory (existence of Ramanujan graphs) and functional analysis (Kadison-Singer problem).

To be specific: let $A$ and $B$ be $d \times d$ matrices, and let $U$ be a random $d \times d$

unitary matrix. Voiculescu showed in 1990 that $A$ and $UBU^*$ are asymptotically free as $d \to \infty$, with the consequence that $A + UBU^*$ and $AUBU^*$ are described in that limit by some operations on probability measures called free additive and multiplicative convolution, denoted by $\boxplus$ and $\boxtimes$ respectively.

In [29], it was shown that for *fixed* $d$, the expected characteristic polynomials of these random matrices behave like "finite" versions of $\boxplus$ and $\boxtimes$. In particular, there are simple formulas for

$$\mathbb{E}_U c_x(A + UBU^*) \text{ and } \mathbb{E}_U c_x(AUBU^*)$$

in terms of the individual characteristic polynomials $c_x(A)$ and $c_x(B)$.

In Chapter 2, based on [13, 11], an alternate approach to this result is presented, using tools from combinatorial representation theory, namely *Weingarten calculus* [15, 17, 16]. Moreover, this approach is applied to the problem of the commutator, i.e. the description of

$$\mathbb{E}_U c_x(AUBU^* - UBU^*A)$$

in terms of $c_x(A)$ and $c_x(B)$. This is the next natural question after addition and multiplication, especially knowing the development of free probability, in which its answer [35] was an important demonstration of the power of combinatorial methods. The key insight of the solution in the finite setting is that after using Weingarten calculus, the problem is reduced to the computation of a non-trivial *immanant*, which can be handled using a 1992 result of I. Goulden and D. Jackson.

The second part of this thesis, in Chapter 3, develops another approximation of free probability, which goes back to P. Biane's work on the asymptotics of symmetric groups in the 1990s. In [3], he showed that the star-transpositions $(1, n)$ in the infinite symmetric group algebra $\mathbb{C}[S_\infty]$, in the regular representation, satisfy a central limit theorem in which the limit law is semicircular. There is also a multivariate version with free semicircular elements in the limit. More recently, in [26], this result was extended to a wider class of characters of $S_\infty$, which are labeled by $d \in \mathbb{N}$. For each $d$, the central limit law found in [26] is the average eigenvalue distribution of a well-known random matrix model called the $d \times d$ *traceless GUE*. This recovers Biane's result in the $d \to \infty$ limit.

In Chapter 3, based on [12], this line of investigation is extended further, with a novel random matrix model whose entries come from noncommutative operator algebras. On the representation-theoretic side, we continuously deform the family of characters considered in [26]: the ones under consideration in this thesis are labeled by certain finite sequences $(\alpha_1, \dots, \alpha_d)$ in the unit interval, and the case $(1/d, \dots, 1/d)$ recovers the results of [26]. On the random matrix side, the deformation corresponds to replacing the classical gaussian variables in the GUE matrix with gaussian variables in noncommutative algebras with "canonical commutation relations" (a.k.a. CCR).

The final chapter of this thesis, Chapter 4, outlines various directions forward in relation to both of the preceding parts. The main themes are the further combinatorial development of finite free probability, and the connection of the results of Chapter 3 with random Young diagrams and the asymptotics of characters of the finite symmetric groups $S_n$.

# Chapter 1

# Background

## 1.1 Partitions and permutations

**Notation 1.1** (Permutations)**.** For $n \in \mathbb{N}$, write $S_n$ for the group of permutations of $[n] := \{1, \dots, n\}$. For $\sigma \in S_n$, we use the following notation:

- $\mathrm{Cyc}(\sigma)$ is the set of disjoint cycles in $\sigma$;
- $\#(\sigma) := |\mathrm{Cyc}(\sigma)|$;
- $t(\sigma)$ is the non-increasing sequence of sizes of the disjoint cycles in $\sigma$, called the *cycle type*;
- $c(n, k)$ is the number of $\sigma \in S_n$ with $\#(\sigma) = k$, called the $(n, k)$-th *unsigned Stirling number of the first kind.*

At one point, we will use the following generating function:

**Proposition 1.2** ([39, Proposition 1.3.7])**.** *We have*

$$\sum_{k=0}^{n} c(n,k)x^k = x^{(n)}$$

*where* $x^{(n)} := x(x+1)\cdots(x+n-1)$ *is the rising factorial.*

**Notation 1.3** (Young diagrams)**.** For $n \in \mathbb{N}$, write $\mathbb{Y}_n$ for the set of Young diagrams $\lambda$ with $n$ boxes, or equivalently the set of (integer) partitions $\lambda \vdash n$. For example, for $\sigma \in S_n$, we have $t(\sigma) \in \mathbb{Y}_n$.

Define a graph $\mathbb{Y}$ by taking $\bigsqcup_{n\in\mathbb{N}} \mathbb{Y}_n$ as the vertex set and placing an edge $\lambda \to \mu$ if and only if $\mu$ can be obtained from $\lambda$ by adding a box. In this case we may also write $\lambda \nearrow \mu$.

For $\lambda \in \mathbb{Y}$, say $\lambda = (\lambda_1, \dots, \lambda_l)$ as an integer partition, we use the following notation:

- $\ell(\lambda) = l$, i.e. the number of rows;
- $m_i(\lambda)$ for the number of times $i$ appears in $(\lambda_1, \dots, \lambda_l)$;
- $z_\lambda := \prod_{i=1}^{l} i^{m_i(\lambda)} m_i(\lambda)!$, so the number of permutations $\sigma \in S_n$ with $t(\sigma) = \lambda$ is $\frac{n!}{z_\lambda}$;
- $\lambda^T$ is the "conjugate" or "transpose" of $\lambda$, i.e. the $(i, j)$-th box of $\lambda^T$ is the $(j, i)$-th box of $\lambda$;
- $2\lambda := (2\lambda_1, \dots, 2\lambda_l)$.

**Example 1.4.** Some specific partitions which will play an outsize role in this thesis are

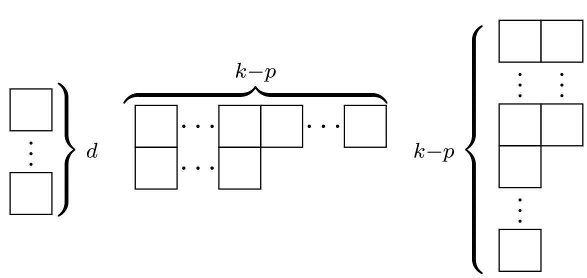
$$1^d = (\underbrace{1, \dots, 1}_{d}) \vdash d$$

and

$$2^p_k = (k-p, p)^T = (\underbrace{2, \dots, 2}_{p}, \underbrace{1, \dots, 1}_{k-2p}) \vdash k.$$

The Young diagrams of these partitions:

$1^d$ $(k-p, p)$ $2^p_k$



**Notation 1.5** (Compositions)**.** A *weak composition* of $n$ is a sequence $I = (I_1, \ldots, I_l)$ of non-negative integers with $I_1 + \cdots + I_l = n$. A *composition* is a weak composition whose parts are all positive. We will use the notation $\mathrm{WComp}(n)$ and $\mathrm{Comp}(n)$ for the weak compositions and compositions, respectively, of $n$.

For $I = (I_1, \ldots, I_l) \in \mathrm{WComp}(n)$, write $\mathrm{Orb}(I)$ for the set of distinct permutations of $I$, i.e.

$$\mathrm{Orb}(I) = \{(I_{\sigma(1)}, \ldots, I_{\sigma(l)}) : \sigma \in S_l\}.$$

On the other hand, write

$$\mathrm{Stab}(I) := \{\sigma \in S_l : (I_{\sigma(1)}, \ldots, I_{\sigma(l)}) = (I_1, \ldots, I_l)\}$$

for the subgroup of permutations which fix $I$.

**Example 1.6.** Let $I = (2, 1, 2, 0) \in \mathrm{WComp}(5)$. Then

$$\begin{aligned}\mathrm{Orb}(I) = \{&(2,1,2,0),(2,1,0,2),(2,0,1,2),(0,2,1,2),\\ &(1,2,2,0),(1,2,0,2),(1,0,2,2),(0,1,2,2),\\ &(2,2,1,0),(2,2,0,1),(2,0,2,1),(0,2,2,1)\}\end{aligned}$$

and $\mathrm{Stab}(I) = \{e, (1,3)\} \le S_4$.

**Notation 1.7** (Set partitions)**.** For $n \in \mathbb{N}$, write $P(n)$ for the lattice of set partitions of $[n]$. We will make more notation when it is needed in Chapter 3, but here are some pieces to start with:

- For $\pi \in P(n)$, write $t(\pi)$ for the non-increasing sequence of sizes of the blocks in $\pi$, which we might call the *block type*. Again, this can be seen as an element of $\mathbb{Y}_n$.
- For a map $\mathbf{r} : [n] \to \mathbb{N}$, define $\ker(\mathbf{r}) \in P(n)$ as follows: for $i, j \in [n]$, $i$ and $j$ are in the same block of $\ker(\mathbf{r})$ if and only if $\mathbf{r}(i) = \mathbf{r}(j)$.
- Write $P_2(k) := \{\pi \in P(k) : |V| = 2\, \forall V \in \pi\}$ for the set of *pair-partitions* or *pairings* of $[k]$.

## 1.2 Finite symmetric groups and symmetric functions

We will assume some basic familiarity with the representation theory of finite groups, especially the finite symmetric groups and the description of their representation theory by the Young graph. Our main reference on this topic is [14]. In this section, we will review some particular parts of the theory which play a significant role in this thesis.

**Notation 1.8.** For $\lambda \in \mathbb{Y}_n$, we use the notation $\rho^\lambda : S_n \to \mathrm{GL}(V^\lambda)$ for the irreducible representation of $S_n$ labeled by $\lambda$, and $\chi^\lambda$ for the character of $\rho^\lambda$. We will often write $\dim(\lambda)$ as shorthand for the dimension of $V^\lambda$.

An important combinatorial property of $\rho^\lambda$ is the description of its dimension in terms of so-called *hook lengths*.

**Notation 1.9.** For $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathbb{Y}_n$ and a box $(i, j) \in \lambda$, the *hook length* at $(i, j)$ is

$$h_\lambda(i,j) := (\lambda_i - j) + (\lambda_j^T - i) + 1.$$

In the expression above, $\lambda_i - j$ is the number of boxes below $(i, j)$ and $(\lambda_j^T - i)$ is the number of boxes to the right of $(i, j)$.

**Proposition 1.10** (Hook-length formula [14, Theorem 4.2.14])**.** *For $\lambda \in \mathbb{Y}_n$, we have*

$$\dim(\lambda) = \frac{n!}{\prod_{(i,j)\in\lambda} h_\lambda(i,j)}.$$

**Example 1.11.** Let $\lambda = (5, 3, 2) \vdash 10$. The hook lengths, illustrated in their respective boxes, are

| | | | | |
|---|---|---|---|---|
| 7 | 6 | 4 | 2 | 1 |
| 4 | 3 | 1 | | |
| 2 | 1 | | | |

so

$$\dim(\lambda) = \frac{10!}{7 \cdot 6 \cdot 4 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 1 \cdot 2 \cdot 1} = 450.$$

### 1.2.1 Bases of symmetric functions

**Notation 1.12.** Write Sym for the algebra of symmetric functions in an alphabet $\mathbf{x} = (x_1, x_2, \ldots)$; see e.g. [27, 40, 14] for details on its construction by various approaches. For $f \in \mathrm{Sym}$ and variables $y_1, \ldots, y_n$, we will use the shorthand $f(y_1, \ldots, y_n)$ for the evaluation $f(y_1, \ldots, y_n, 0, 0, \ldots)$.

**Notation 1.13** (Power sums)**.** For $k \geq 1$, write

$$\mathsf{p}_k(\mathbf{x}) := \sum_{i \geq 1} x_i^k.$$

This is called the $k$-th *power-sum* symmetric function, and $\{\mathsf{p}_k : k \geq 1\}$ generates Sym.

**Notation 1.14** (Monomial and elementary)**.** For $\lambda \vdash n$, consider it as an infinite sequence $(\lambda_1, \ldots, \lambda_l, 0, 0, \ldots)$ and write

$$\mathsf{m}_\lambda(\mathbf{x}) := \sum_{\substack{I \text{ distinct permutation} \\ \text{of } \lambda}} \mathbf{x}^I$$

where the product $\mathbf{x}^I := \prod_{i \geq 1} x_i^{I_i}$ makes sense because each $I$ has finitely many non-zero entries. The special cases $\mathsf{m}_{(1^n)}$, where $(1^n) := (1, \ldots, 1) \in \mathbb{Y}_n$, are denoted by $\mathsf{e}_n$, and these are called the *elementary* symmetric functions. The symmetric functions $\mathsf{e}_\lambda$, for $\lambda \vdash n$, are defined multiplicatively.

**Example 1.15.** Let $\lambda = (2, 2, 1) \vdash 5$. Then

$$\mathsf{m}_\lambda(x_1, x_2, x_3) = x_1^2 x_2^2 x_3 + x_1^2 x_2 x_3^2 + x_1 x_2^2 x_3^2$$

and

$$\begin{aligned}
\mathsf{e}_\lambda(x_1, x_2, x_3) &= \mathsf{e}_2(x_1, x_2, x_3)^2 \mathsf{e}_1(x_1, x_2, x_3) \\
&= (x_1 x_2 + x_1 x_3 + x_2 x_3)^2 (x_1 + x_2 + x_3) \\
&= (x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + 2x_1^2 x_2 x_3 + 2x_1 x_2^2 x_3 + 2x_1 x_2 x_3^2) \\
&\qquad (x_1 + x_2 + x_3) \\
&= x_1^3 x_2^2 + x_1^2 x_2^3 + x_1^3 x_3^2 + x_1^2 x_3^3 + x_2^3 x_3^2 + x_2^2 x_3^3
\end{aligned}$$

$$+ 2x_1^3x_2x_3 + 2x_1x_2^3x_3 + 2x_1x_2x_3^3$$
$$+ 5x_1^2x_2^2x_3 + 5x_1^2x_2x_3^2 + 5x_1x_2^2x_3^2.$$

The latter is

$$\mathsf{m}_{(3,2)}(x_1, x_2, x_3) + 2\mathsf{m}_{(3,1,1)}(x_1, x_2, x_3) + 5\mathsf{m}_{(2,2,1)}(x_1, x_2, x_3)$$

which is a special case of the more general relationship between the elementary and monomial bases of the algebra of symmetric functions. This will be explained in a forthcoming subsection and used in Section 2.4.2.

**Notation 1.16.** For $I = (I_1, \dots, I_l) \in \mathrm{Comp}(n)$, write

$$\mathsf{M}_I(\mathbf{x}) = \sum_{s_1 < \cdots < s_l} x_{s_1}^{I_1} \cdots x_{s_l}^{I_l}.$$

This is a sort of refinement of $\mathsf{m}_\mu$, in the sense that

$$\mathsf{m}_\mu = \sum_{I \in \mathrm{Orb}(\mu)} \mathsf{M}_I.$$

Furthermore, in the case $I = (I_1, \dots, I_l) \in \mathrm{WComp}(n)$, we will use the notation

$$\mathsf{M}_I(x_1, \dots, x_d) = \sum_{1 \le s_1 < \cdots < s_l \le d} x_{s_1}^{I_1} \cdots x_{s_l}^{I_l}$$

with a finite alphabet, while taking extra care to account for the difference between compositions and weak compositions.

**Example 1.17.** Let $I = (1, 2, 1) \in \mathrm{Comp}(4)$. Then

$$\mathsf{M}_I(x_1, x_2, x_3, x_4) = x_1^1x_2^2x_3^1 + x_1^1x_2^2x_4^1 + x_1^1x_3^2x_4^1 + x_2^1x_3^2x_4^1$$

and

$$\begin{aligned} \mathsf{m}_{(2,1,1)}(x_1, x_2, x_3, x_4) &= x_1^2x_2^1x_3^1x_4^0 + x_1^2x_2^1x_3^0x_4^1 + x_1^2x_2^0x_3^1x_4^1 + x_1^0x_2^2x_3^1x_4^1 \\ &\quad + x_1^1x_2^2x_3^1x_4^0 + x_1^1x_2^2x_3^0x_4^1 + x_1^1x_2^0x_3^2x_4^1 + x_1^0x_2^1x_3^2x_4^1 \\ &\quad + x_1^1x_2^1x_3^2x_4^0 + x_1^1x_2^1x_3^0x_4^2 + x_1^1x_2^0x_3^1x_4^2 + x_1^0x_2^1x_3^1x_4^2 \\ &= x_1^2x_2x_3 + x_1^2x_2x_4 + x_1^2x_3x_4 + x_2^2x_3x_4 \end{aligned}$$

$$
\begin{aligned}
&+ x_1x_2^2x_3 + x_1x_2^2x_4 + x_1x_3^2x_4 + x_2x_3^2x_4 \\
&+ x_1x_2x_3^2 + x_1x_2x_4^2 + x_1x_3x_4^2 + x_2x_3x_4^2 \\
= {}& \mathsf{M}_{(2,1,1)}(x_1, x_2, x_3, x_4) + \mathsf{M}_{(1,2,1)}(x_1, x_2, x_3, x_4) \\
&+ \mathsf{M}_{(1,1,2)}(x_1, x_2, x_3, x_4).
\end{aligned}
$$

**Terminology 1.18.** A *semistandard tableau* of shape $\lambda \in \mathbb{Y}$ is a filling of the Young diagram with positive integers, such that the rows are weakly increasing and the columns are strictly increasing.

**Notation 1.19** (Schur)**.** Write $\mathrm{SST}(\lambda)$ for the set of semistandard tableaux of shape $\lambda$. For $T \in \mathrm{SST}(\lambda)$, the *weight* of $T$ is defined by

$$\omega(T) := (\omega_1(T), \omega_2(T), \ldots)$$

where $\omega_i(T)$ is the number of entries of $T$ which are equal to $i$. The *Schur function* labeled by $\lambda$ is given by

$$\mathsf{s}_\lambda(\mathbf{x}) = \sum_{T \in \mathrm{SST}(\lambda)} \mathbf{x}^{\omega(T)}.$$

**Example 1.20.** Let $\lambda = (2, 1) \vdash 3$. Then $\mathsf{s}_\lambda(x_1, x_2, x_3)$ has contributions from the following semistandard tableaux:

$$
\begin{array}{cccccccc}
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} . \\
x_1^2x_2 & x_1^2x_3 & x_1x_2^2 & x_1x_2x_3 & x_1x_2x_3 & x_1x_3^2 & x_2^2x_3 & x_2x_3^2
\end{array}
$$

An important point illustrated here is that in the definition

$$
\begin{aligned}
\mathsf{s}_\lambda(x_1, x_2, x_3) &= \mathsf{s}_\lambda(x_1, x_2, x_3, 0, 0, \ldots) \\
&= \sum_{T \in \mathrm{SST}(\lambda)} x_1^{\omega_1(T)} x_2^{\omega_2(T)} x_3^{\omega_3(T)} 0^{\omega_4(T)} 0^{\omega_5(T)} \cdots,
\end{aligned}
$$

the only tableaux $T$ which can contribute non-zero monomials are the ones with $\omega_i(T) = 0$ for all $i \geq 4$, i.e. the entries are at most 3. Adding up the contributions, we have

$$\mathsf{s}_{(2,1)}(x_1, x_2, x_3) = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2.$$

**Notation 1.21.** For $\lambda \in \mathbb{Y}_n$ and a box $(i,j) \in \lambda$, the *content* at $(i,j)$ is

$$c_\lambda(i,j) := j - i,$$

i.e. the distance of $(i,j)$ from the diagonal.

**Proposition 1.22** (Hook-content formula [14, Theorem 4.3.3])**.** *For $\lambda \in \mathbb{Y}_n$ and $d \geq \ell(\lambda)$, we have*

$$\mathsf{s}_\lambda(\underbrace{1,\ldots,1}_{d},0,0,\ldots) = \frac{\dim(\lambda)}{n!} \prod_{(i,j)\in\lambda} (d + c_\lambda(i,j)).$$

**Example 1.23.** Let us return to Example 1.11, with $\lambda = (5,3,2) \vdash 10$. The contents, illustrated in their respective boxes, are

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 |
| −1 | 0 | 1 | | |
| −2 | −1 | | | |

so using the result of Example 1.11, we have

$$\begin{aligned}\mathsf{s}_\lambda(1^d) &= \frac{450}{10!} d(d+1)(d+2)(d+3)(d+4)(d-1)d(d+1)(d-2)(d-1) \\ &= \frac{1}{8064} d^2(d^2-1)^2(d^2-4)(d+3)(d+4).\end{aligned}$$

### 1.2.2 Gelfand pairs and zonal functions

At one point we will require a variation of the theory of irreducible characters of $S_n$ and Schur functions: the theory of Gelfand pairs, zonal spherical functions, and zonal polynomials. Our main reference on this topic is [27].

**Terminology 1.24.** A pair $(G,K)$ of a finite group $G$ and a subgroup $K \leq G$ is called a *Gelfand pair* if the induced representation $\mathrm{Ind}_K^G(\mathrm{triv})$ is multiplicity-free, in the sense that each irreducible representation of $K$ appears at most once in $\mathrm{Ind}_K^G(\mathrm{triv})$.

**Notation 1.25.** Denote by $H_n$ the centralizer of $(1,2)\cdots(2n-1,2n)$ in $S_{2n}$. This is called the *hyperoctahedral group*, and it turns out to be isomorphic to the group of signed permutations of $n$ letters, or the wreath product $\mathbf{Z}_2 \wr S_n$.

**Theorem 1.26** ([27, VII.2.4])**.** *$(S_{2n}, H_n)$ is a Gelfand pair. The irreducible representations of $S_{2n}$ which are contained in $\mathrm{Ind}_{H_n}^{S_{2n}}(\mathrm{triv})$ are precisely the ones labeled by $2\lambda$ for $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$.*

Associated to any Gelfand pair is its family of *zonal spherical functions*. These are defined by taking the characters of the irreducible representations contained in $\mathrm{Ind}_K^G(\mathrm{triv})$ and averaging them over $K$:

**Definition 1.27** (Zonal spherical functions)**.** For $\lambda \vdash n$, define $\omega^\lambda : S_{2n} \to \mathbb{C}$ by

$$\omega^\lambda(\sigma) = \frac{1}{|H_n|} \sum_{\zeta \in H_n} \chi^{2\lambda}(\sigma\zeta)$$

for $\sigma \in S_{2n}$. This is called the *zonal spherical function* of the Gelfand pair $(S_{2n}, H_n)$ labeled by $\lambda$.

We will use the specific form of $\omega^{1^n}$:

**Proposition 1.28** ([27, Example VII.2.2.b])**.** *We have*

$$\omega^{1^n}(\sigma) = \left(-\frac{1}{2}\right)^{n-\#(\sigma)}$$

*for $\sigma \in S_{2n}$.*

In the analogy between $\omega^\lambda$ and the irreducible characters $\chi^\lambda$ of $S_n$, the relation of $\chi^\lambda$ with the conjugacy classes of $S_n$ corresponds to the relation of $\omega^\lambda$ with the double cosets $H_n \sigma H_n$ of $H_n$ in $S_{2n}$:

**Notation 1.29.** For $\sigma \in S_{2n}$, define a graph $\Gamma(\sigma)$ as follows:

- the vertices are $1, \ldots, 2n$;
- the edges connect $2i-1$ with $2i$ and $\sigma(2i-1)$ with $\sigma(2i)$ for $1 \leq i \leq n$.

The connected components of $\Gamma(\sigma)$ are cycles of even lengths, and dividing those lengths by 2, we get an integer partition $\Xi(\sigma) \vdash n$ which is called the *coset type* of $\sigma$.

As explained in [27], the coset type labels the double cosets of $H_n$ in $S_{2n}$. Let us illustrate this by example:

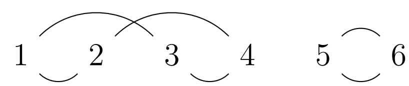



Figure 1.1: $\Gamma((1,4,3))$

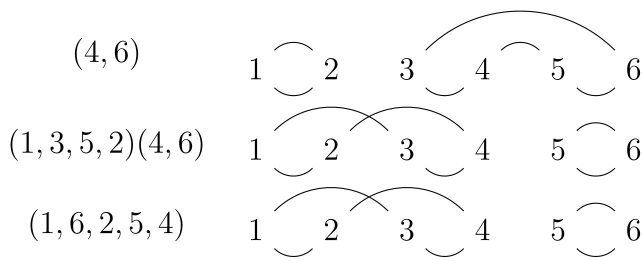



Figure 1.2: Some permutations with the same coset type as $(1,4,3)$

**Example 1.30.** Let $n = 3$ and $\sigma = (1,4,3) \in S_6$. The graph $\Gamma(\sigma)$ is illustrated in Fig. 1.1. The connected components are $\{1,2,3,4\}$ and $\{5,6\}$. The double coset $H_3(1,4,3)H_3$ has 288 elements (can be done with a computer) so it is too large to reproduce, but some randomly chosen elements are

$$(4,6),\ (1,3,5,2)(4,6),\ \text{and}\ (1,6,2,5,4).$$

Their graphs are illustrated in Fig. 1.2, and one can clearly see their shared coset type of $(2,1)$.

**Proposition 1.31** ([27, VII.2.3])**.** *For $\rho \vdash n$, write $H_\rho$ for the corresponding double coset. Then $S_{2n} = \bigsqcup_{\rho \vdash n} H_\rho$ and*

$$|H_\rho| = \frac{|H_n|^2}{z_{2\rho}} = \frac{|H_n|^2}{2^{\ell(\rho)} z_\rho}.$$

Clearly the zonal spherical functions are constant on double cosets, so we write $\omega^\lambda_\rho$ for the value of $\omega^\lambda$ on $H_\rho$. For us, the critical property of the zonal spherical functions is that they still enjoy the kind of orthogonality relations familiar from the character theory of $S_n$:

**Theorem 1.32** (Orthogonality relations [27, VII.2.15']). *For $\lambda, \mu \vdash n$, we have*

$$\sum_{\rho \vdash n} \frac{1}{z_{2\rho}} \omega_\rho^\lambda \omega_\rho^\mu = \begin{cases} \frac{h(2\lambda)}{|H_n|^2} & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases},$$

*where $h(2\lambda)$ is the product of the hook lengths in $2\lambda$.*

### 1.2.3 Permutation modules and Kostka numbers

Another somewhat particular bit of representation theory that we will need is the analysis of certain induced representations. First, a definition:

**Notation 1.33.** For $I = (I_1, \ldots, I_l) \in \mathrm{Comp}(n)$, write

$$S_I := S_{I_1} \times \cdots \times S_{I_l}$$

which is viewed as a subgroup of $S_n$ in the obvious way: $S_{I_1}$ acts on $\{1, \ldots, I_1\}$, $S_{I_2}$ acts on $\{I_1+1, \ldots, I_1+I_2\}$, and so on. This is called the *Young subgroup* labeled by $I$.

**Terminology 1.34.** For $I \in \mathrm{Comp}(n)$, the induced representation $\mathrm{Ind}_{S_I}^{S_n}(\mathrm{triv})$ is sometimes called the *permutation module* labeled by $I$. It turns out that $\mathrm{Ind}_{S_I}^{S_n}(\mathrm{triv})$ is independent (up to isomorphism) of the ordering of $I$, so typically we will restrict our attention to the case where $I$ is non-increasing, i.e. it's an integer partition.

**Theorem 1.35** ([14, Theorem 3.6.11]). *For $\lambda, \mu \vdash k$, the multiplicity of $V^\lambda$ in the permutation module $\mathrm{Ind}_{S_\mu}^{S_k}(\mathrm{triv})$ is the number of semistandard tableaux with shape $\lambda$ and weight $\mu$.*

The combinatorics involved here revolve around the following ordering of $\mathbb{Y}_n$:

**Notation 1.36.** For $\lambda, \mu \in \mathbb{Y}_n$, say $\mu \trianglelefteq \lambda$, or that $\lambda$ *dominates* $\mu$, when

$$\mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i$$

for all $i \geq 1$.

**Example 1.37.** Recall the notation $2_k^p := (2^p, 1^{k-2p}) \vdash k$ from Example 1.4. The partitions of $k$ which are dominated by $2_k^p$ are exactly $2_k^q$ for $0 \le q \le p$.

**Notation 1.38.** Denote by $K(\lambda, \mu)$ the number of semistandard tableaux with shape $\lambda$ and weight $\mu$; these are called *Kostka numbers*. Of course $K(\lambda, \mu)$ is non-negative, and it is non-zero if and only if $\mu \trianglelefteq \lambda$. Since the matrix $K := (K(\lambda, \mu))_{\lambda, \mu \vdash k}$ is upper-triangular with 1s along the diagonal, it is invertible, and $K^{-1}(\lambda, \mu)$ is the $(\lambda, \mu)$-th entry of its inverse. These so-called *inverse Kostka numbers* have a nice combinatorial interpretation [19] in terms of Young diagrams.

Another important interpretation of the Kostka numbers is that they describe transitions between different bases of the symmetric functions. The general principle can be read from e.g. [27, Section I.6]:

**Proposition 1.39.** *We have*

1. $\mathsf{e}_\lambda = \sum_{\mu \vdash k} \left( \sum_{\nu \vdash k} K(\nu, \lambda) K(\nu^T, \mu) \right) \mathsf{m}_\mu$ *and*
2. $\mathsf{m}_\lambda = \sum_{\mu \vdash k} \left( \sum_{\nu \vdash k} K^{-1}(\lambda, \nu^T) K^{-1}(\mu, \nu) \right) \mathsf{e}_\mu$

*for* $\lambda \vdash k$.

Here is the special case which will be used in this thesis:

**Corollary 1.40.** *For* $0 \le p \le k/2$*, we have*

$$\mathsf{e}_{(k-p,p)} = \sum_{0 \le q \le p} \binom{k-2q}{p-q} \mathsf{m}_{2_k^q}. \tag{1.1}$$

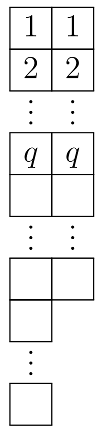
*In the other direction, we have*

$$\mathsf{m}_{2_k^q} = (-1)^q \sum_{0 \le r \le q} (-1)^r \mathsf{e}_{(k-r,r)} \left( \binom{k-q-r}{k-2q} + \binom{k-q-r-1}{k-2q} \right) \tag{1.2}$$

*for* $0 \le q \le k/2 - 1$ *and*

$$\begin{aligned} \mathsf{m}_{2_k^{k/2}} &= \mathsf{e}_{(k/2,k/2)} + 2 \cdot (-1)^{k/2} \sum_{0 \le r \le k/2-1} (-1)^r \mathsf{e}_{(k-r,r)} \\ &= (-1)^{k/2} \sum_{i+j=k} (-1)^i \mathsf{e}_i \mathsf{e}_j. \end{aligned}$$

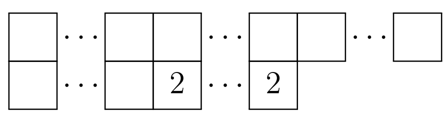
*Proof.* For Eq. (1.1), recall the interpretation of $K(\lambda,\mu)$ as the number of semistandard tableaux with shape $\lambda$ and weight $\mu$. For $\lambda = 2^r_k$ and $\mu = 2^q_k$, any such tableau must begin




so

$$K(2^r_k, 2^q_k) = \dim(2^{r-q}_{k-2q}) = \frac{(k-2q)!(k-2p+1)}{(p-q)!(k-p-q+1)!}$$

by the hook-length formula. On the other hand, a semistandard tableau of shape $(k-r,r)$ with weight $(k-p,p)$ must be of the form




since the 2s cannot go anywhere else if the other boxes are supposed to be filled with 1s. So $K((k-r,r),(k-p,p)) = 1$ if $r \leq p$, otherwise it is 0. Now, what remains is to show that

$$\sum_{q\leq r\leq p} \frac{(k-2q)!(k-2r+1)}{(r-q)!(k-r-q+1)!} = \binom{k-2q}{p-q}. \tag{1.3}$$

To this end, observe that

$$\begin{aligned}
&\binom{k-2q}{r-q} - \binom{k-2q}{r-q-1} \\
&= \frac{(k-2q)!}{(r-q)!(k-r-q)!} - \frac{(k-2q)!}{(r-q-1)!(k-r-q+1)!} \\
&= \frac{(k-2q)!(k-r-q+1) - (k-2q)!(r-q)}{(r-q)!(k-r-q+1)!}
\end{aligned}$$

$$= \frac{(k-2q)!(k-2r+1)}{(r-q)!(k-r-q+1)!}$$

so the only summand which is not cancelled out on the left-hand side of Eq. (1.3) is $\binom{k-2q}{p-q}$.

For Eq. (1.2), one can refer to [19] to find that

$$K^{-1}(2^q_k, 2^s_k) = (-1)^{q-s}\binom{k-q-s}{k-2q}$$

for $0 \leq s \leq q \leq k/2$, and

$$K^{-1}((k-r,r),(k-s,s)) = \begin{cases} 1 & \text{if } s = r \\ -1 & \text{if } s = r+1 \\ 0 & \text{otherwise} \end{cases}$$

for $0 \leq r, s \leq k/2$. □

## 1.3 Determinants and immanants

**Notation 1.41.** For a $d \times d$ matrix $X$, we will use the notation

$$c_x(X) := \det(xI - X)$$

for the characteristic polynomial of $X$. The coefficients can be arranged as follows:

$$c_x(X) = \sum_{k=0}^{d} x^{d-k}(-1)^k \mathsf{e}_k(X)$$

where $\mathsf{e}_k(X)$ is the $k$-th elementary symmetric function in the eigenvalues of $X$.

**Notation 1.42.** For an $m \times n$ matrix $X = (x_{ij})_{i,j}$ and $S \subseteq [m]$ and $T \subseteq [n]$, write

$$X(S,T) := (x_{ij})_{\substack{i \in S \\ j \in T}}$$

for the submatrix of $X$ with the rows and columns specified by $S$ and $T$ respectively.

**Proposition 1.43** (Vieta's formula [25, Theorem 1.2.16])**.** *For a $d \times d$ matrix $X$, we have*

$$\mathsf{e}_k(X) = \sum_{\substack{S \subseteq [d] \\ |S|=k}} \det(X(S,S))$$

*for $0 \leq k \leq d$.*

**Proposition 1.44** (Cauchy-Binet theorem [25, Section 0.8.7])**.** *Let $A$ be a $m \times n$ matrix and let $B$ be a $n \times p$ matrix. Then for $S \subseteq [m]$ and $T \subseteq [p]$ with $|S| = |T| = k$, we have*

$$\det((AB)(S,T)) = \sum_{\substack{U \subseteq [n] \\ |U|=k}} \det(A(S,U)) \det(B(U,T)).$$

**Notation 1.45.** For a $k \times k$ matrix $Y = (y_{ij})_{i,j}$, the *immanant* of $Y$ labeled by $\lambda \in \mathbb{Y}_n$ is

$$\mathrm{Imm}^\lambda(Y) = \sum_{\sigma \in S_k} \chi^\lambda(\sigma) \prod_{i=1}^{k} y_{i\sigma(i)}.$$

The case $\lambda = 1^k$ is the determinant, and the case $\lambda = (k)$ is the permanent. For the latter, we will use the notation $\mathrm{Per}(Y)$.

**Theorem 1.46** ([23, Equation (9)])**.** *Let $Y$ be a $k \times k$ matrix, let $z_1, \ldots, z_k$ be commuting formal variables, and write $Z := \mathrm{diag}(z_1, \ldots, z_k)$. Then for $\lambda \vdash k$, we have*

$$\mathrm{Imm}^\lambda(Y) = [z_1 \cdots z_k] \mathsf{s}_\lambda(ZY)$$

*where the Schur function $\mathsf{s}_\lambda$ is evaluated in the eigenvalues of $ZY$.*

## 1.4 Infinite symmetric group and Thoma's theorem

In this section we will establish some commonly used notation related to the infinite symmetric group $S_\infty$ and its characters.

**Notation 1.47.** Write

$$S_\infty := \{\sigma \in \mathrm{Sym}(\mathbb{N}) : \exists \text{ finite } A \subseteq \mathbb{N} \text{ s.t. } \sigma(i) = i \,\forall\, i \in \mathbb{N} \setminus A\}$$

for the group of finitely-supported permutations of $\mathbb{N}$, called the *infinite symmetric group*. Alternatively, $S_\infty$ can be described as direct limit of the finite symmetric groups $S_n$ with respect to the embeddings $S_n \hookrightarrow S_{n+1}$ in which $S_n$ acts on $\{1, \ldots, n\}$ and fixes $n + 1$.

**Notation 1.48.** Among the various generating sets of $S_\infty$, we will mostly be concerned with the *star-transpositions*, defined by

$$\gamma_n := (1, n + 1)$$

for $n \geq 1$.

The analogue for $S_\infty$ to the parameterization of irreducible representations of $S_n$ is the parameterization of extreme points in the convex set of characters of $G$. Here, characters are positive-definite class functions $\chi : S_\infty \to \mathbb{C}$ with $\chi(e) = 1$; all of these notions will be reviewed in Section 3.1. The space of parameters is continuous and infinite-dimensional, but still rather simple:

**Notation 1.49** (Thoma simplex)**.** Write

$$\begin{aligned} \Omega := \{(\alpha, \beta) \in [0,1]^{\mathbb{N}} \times [0,1]^{\mathbb{N}} : & \,\alpha_1 \geq \alpha_2 \geq \cdots \geq 0 \\ & \,\beta_1 \geq \beta_2 \geq \cdots \geq 0 \\ & \sum_{i \geq 1} \alpha_i + \sum_{j \geq 1} \beta_j \leq 1\}. \end{aligned}$$

This is called the *Thoma simplex*, and it is compact in the topology inherited from the product $[0,1]^{\mathbb{N}} \times [0,1]^{\mathbb{N}}$.

To simplify the presentation and application of the main result on characters of $S_\infty$, let us borrow some notation from [7]:

**Proposition 1.50.** *There is a unique morphism* $\mathrm{Sym} \to C(\Omega) : f \mapsto f^\circ$ *of algebras such that* $\mathsf{p}_1^\circ = 1$ *and*

$$\mathsf{p}_k^\circ(\omega) = \sum_{i \geq 1} \alpha_i^k + (-1)^{k-1} \sum_{j \geq 1} \beta_j^k$$

*for* $\omega = (\alpha, \beta) \in \Omega$*, for* $k \geq 2$*.*

Finally, here is the main theorem:

**Theorem 1.51** ([43])**.** *The extremal characters of* $S_\infty$ *are labeled as* $\chi^\omega$ *for* $\omega = (\alpha, \beta) \in \Omega$*, with*

$$\chi^\omega(\sigma) = \prod_{\substack{c \in \mathrm{Cyc}(\sigma) \\ |c|>1}} \mathsf{p}^\circ_{|c|}(\omega)$$

*for* $\sigma \in S_\infty$*.*

## 1.5 Random matrices, genus expansion, and Weingarten calculus

### 1.5.1 Gaussian self-adjoint matrices

The first random matrix model relevant to this thesis is the gaussian unitary ensemble, or GUE for short. This is a self-adjoint matrix of gaussian random variables with no dependence beyond that imposed by the self-adjointness. We will loosely follow the introduction to this model in [36, Lecture 22], which focuses on its relevance to free probability.

**Terminology 1.52.** The $d \times d$ *gaussian unitary ensemble*, or *GUE*, is the following random matrix $A = (a_{ij})_{i,j}$: the family

$$\{a_{11}, \ldots, a_{dd}\} \cup \{\mathrm{Re}(a_{ij}) : 1 \le i < j \le d\} \cup \{\mathrm{Im}(a_{ij}) : 1 \le i < j \le d\}$$

is independent and has a centered gaussian joint distribution. The variances of the (real) diagonal entries are normalized to $\frac{1}{d}$ and the variances of the (complex) off-diagonal entries are normalized to $\frac{1}{d}$ by normalizing the real and imaginary parts by $\frac{1}{2d}$.

The average eigenvalue distribution of a GUE is described by the combinatorics of pair partitions. First, the way pairings come into play is through a well-known formula for mixed moments of gaussian families:

**Proposition 1.53** (Wick formula)**.** *Let* $(X_1, \ldots, X_n)$ *be real random variables with a centered gaussian joint distribution. Then for* $k \ge 1$ *and* $\mathbf{r}$ :

$[k] \to [n]$, *we have*

$$\mathbb{E}(X_{\mathbf{r}(1)} \cdots X_{\mathbf{r}(k)}) = \sum_{\pi \in P_2(k)} \prod_{(p,q) \in \pi} \mathbb{E}(X_{\mathbf{r}(p)} X_{\mathbf{r}(q)}).$$

**Corollary 1.54.** *Let* $(a_1, \ldots, a_n)$ *be complex random variables such that*

$$\{\mathrm{Re}(a_1), \mathrm{Im}(a_1), \ldots, \mathrm{Re}(a_n), \mathrm{Im}(a_n)\}$$

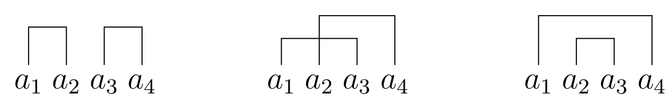
*has a centered gaussian joint distribution. Then for* $k \geq 1$, $\mathbf{r} : [k] \to [n]$, *and* $\epsilon_1, \ldots, \epsilon_k \in \{1, *\}$, *we have*

$$\mathbb{E}(a_{\mathbf{r}(1)}^{\epsilon_1} \cdots a_{\mathbf{r}(k)}^{\epsilon_k}) = \sum_{\pi \in P_2(k)} \prod_{(p,q) \in \pi} \mathbb{E}(a_{\mathbf{r}(p)}^{\epsilon_p} a_{\mathbf{r}(q)}^{\epsilon_q}).$$

Let us show a simple example of how these Wick formulas work:

**Example 1.55.** In the case $k = 4$, the Wick formula reads as

$$\mathbb{E}(a_1 a_2 a_3 a_4) = \mathbb{E}(a_1 a_2)\mathbb{E}(a_3 a_4) + \mathbb{E}(a_1 a_3)\mathbb{E}(a_2 a_4) + \mathbb{E}(a_1 a_4)\mathbb{E}(a_2 a_3).$$




The combinatorial description of the average eigenvalue distribution of a $d \times d$ GUE is through its moments, which can be written in terms of some very interesting combinatorial polynomials in $1/d$:

**Notation 1.56.** For even $k$ and $\pi \in P_2(k)$, write

$$\sigma_\pi := \prod_{(a,b) \in \pi} (a, b) \in S_k.$$

**Theorem 1.57** (Genus expansion)**.** *Let* $(A_1, \ldots, A_m)$ *be independent* $d \times d$ *GUEs. Then for* $k \geq 1$ *and* $\mathbf{p} : [k] \to [m]$, *we have*

$$\mathbb{E}\,\mathrm{tr}_d(A_{\mathbf{p}(1)} \cdots A_{\mathbf{p}(k)}) = \sum_{\substack{\pi \in P_2(k) \\ \pi \leq \ker(\mathbf{p})}} \left(\frac{1}{d}\right)^{k/2+1-\#(c_k \sigma_\pi)}$$

*when* $k$ *is even, and the above is* $0$ *when* $k$ *is odd.*

The reason for the phrase "genus expansion" is that the powers $k/2+1-\#(c_k\sigma_\pi)$ are counting the genera of certain surfaces:

**Remark 1.58.** Given $\pi \in P_2(k)$, one can arrange $\{1,\ldots,k\}$ in a clockwise circle and try to draw the arcs of $\pi$ on a surface with the circle as its boundary. As explained in e.g. [31, Section 1.9], if $g_\pi$ is the smallest possible genus of a surface on which this can be done, then

$$k/2+1-\#(c_k\sigma_\pi) = 2g_\pi.$$

The case $g_\pi = 0$, i.e. $\#(c_k\sigma_\pi) = k+1$, corresponds exactly to the *non-crossing* pair partitions.

A few examples in the single-matrix case should be helpful to understand what the polynomial in $1/d$ looks like:

**Example 1.59.** First, let $k = 4$ and $\pi = \{\{1,4\},\{2,3\}$. Then

$$c_4\sigma_\pi = (1,2,3,4)(1,4)(2,3) = (2,4)(1)(3)$$

and

$$k/2+1-\#(c_k\sigma_\pi) = 4/2+1-3 = 2\cdot 0.$$

The 0 reflects the fact that $\pi$ is non-crossing, i.e. the relevant surface has genus 0. Similarly, if $k = 6$ and $\pi = \{\{1,3\},\{2,5\},\{4,6\}\}$, then

$$c_6\sigma_\pi = (1,2,3,4,5,6)(1,3)(2,5)(4,6) = (1,4)(2,6,5,3)$$

and

$$k/2+1-\#(c_k\sigma_\pi) = 6/2+1-2 = 2\cdot 1.$$

In this case, one can see that the genus – 1 in this case – is not actually counting crossings as one might naively guess, but depends on $\pi$ in a more subtle way.

### 1.5.2 Unitary and orthogonal matrix integrals

The other random matrices which play a central role in this thesis are the ones coming from compact matrix groups with respect to their Haar measures. For these, our main tool is Weingarten calculus, which reduces Haar integration in certain cases to combinatorial representation theory.

**Unitary case**

**Theorem 1.60** ([15])**.** *There is a sequence of functions* $(\mathrm{Wg}^U_{k,d})_{k\geq 1}$ *on* $S_k$ *with the following property: for* $k, k' \geq 1$ *and* $\mathbf{i}, \mathbf{j} : [k] \to [d]$ *and* $\mathbf{i}', \mathbf{j}' : [k'] \to [d]$*, the integral*

$$\int_{U_d} u_{\mathbf{i}(1)\mathbf{j}(1)} \cdots u_{\mathbf{i}(k)\mathbf{j}(k)} \overline{u_{\mathbf{i}'(1)\mathbf{j}'(1)}} \cdots \overline{u_{\mathbf{i}'(k')\mathbf{j}'(k')}} \, dU$$

*is*

$$\sum_{\substack{\pi,\sigma \in S_k \\ \mathbf{i}=\mathbf{i}'\circ\pi \\ \mathbf{j}=\mathbf{j}'\circ\sigma}} \mathrm{Wg}^U_{k,d}(\pi^{-1}\sigma)$$

*when* $k = k'$*, and it is* $0$ *otherwise.*

**Theorem 1.61** ([15])**.** *We have*

$$\mathrm{Wg}^U_{k,d} = \frac{1}{(k!)^2} \sum_{\substack{\lambda \vdash k \\ \ell(\lambda)\leq d}} \frac{\dim(\lambda)^2}{\mathsf{s}_\lambda(1^d)} \chi^\lambda$$

*for* $k \geq 1$*. In particular,* $\mathrm{Wg}^U_{k,d}$ *is a class function, in the sense that it only depends on the conjugacy class (i.e. cycle type) of its argument.*

**Example 1.62.** For the sake of concreteness, let us evaluate the integral

$$\int_{U_d} u_{11}u_{22}\overline{u_{12}u_{21}} \, dU$$

using Theorem 1.60 and Theorem 1.61. In this case, we have

$$\mathbf{i} = (1,2),\ \mathbf{j} = (1,2),\ \mathbf{i}' = (1,2), \text{ and } \mathbf{j}' = (2,1).$$

Then the constraints $\mathbf{i} = \mathbf{i}' \circ \pi$ and $\mathbf{j} = \mathbf{j}' \circ \sigma$ on pairs of permutations $\pi, \sigma \in S_k$ amount to

$$1 = \mathbf{i}(1) = \mathbf{i}'(\pi(1)),\ 2 = \mathbf{i}(2) = \mathbf{i}'(\pi(2)),$$
$$1 = \mathbf{j}(1) = \mathbf{j}'(\sigma(1)), \text{ and } 2 = \mathbf{j}(2) = \mathbf{j}'(\sigma(2)).$$

The first pair forces $\pi(1) \in \{1\}$ and $\pi(2) \in \{2\}$, i.e. $\pi = e$, and the second pair forces $\sigma(1) \in \{2\}$ and $\sigma(2) \in \{1\}$, i.e. $\sigma = (1, 2)$. So Theorem 1.60 says

$$\int_{U_d} u_{11} u_{22} \overline{u_{12} u_{21}} \, dU = \mathrm{Wg}^U_{2,d}((1,2))$$

and Theorem 1.61 says

$$\mathrm{Wg}^U_{2,d}((1,2)) = \frac{1}{4} \sum_{\substack{\lambda \vdash 2 \\ \ell(\lambda) \leq d}} \frac{\dim(\lambda)^2}{\mathsf{s}_\lambda(1^d)} \chi^\lambda((1,2)).$$

There are only two $\lambda \vdash 2$: $\lambda = (2)$ and $\lambda = (1,1)$. They are both one-dimensional, and they label the representations triv and sgn respectively. By Proposition 1.22,

$$\mathsf{s}_{(2)}(1^d) = \frac{1}{2} d(d+1) \text{ and } \mathsf{s}_{(1,1)}(1^d) = \frac{1}{2} d(d-1),$$

and $\mathrm{sgn}((1,2)) = -1$, so

$$\mathrm{Wg}^U_{2,d}((1,2)) = \frac{1}{4} \left( \frac{2}{d(d+1)} \cdot 1 + \frac{2}{d(d-1)} \cdot (-1) \right) = -\frac{1}{d(d+1)(d-1)}$$

and

$$\int_{U_d} u_{11} u_{22} \overline{u_{12} u_{21}} \, dU = -\frac{1}{d(d^2-1)}.$$

**Orthogonal case**

There is also an orthogonal version of Weingarten calculus, where the role of $S_k$ is played by the pair partitions:

**Theorem 1.63** ([17]). *There is a sequence* $(\mathrm{Wg}^O_{k,d})_{k \geq 1}$ *of functions* $\mathrm{Wg}^O_{k,d}$ *on* $P_2(2k) \times P_2(2k)$ *with the following property: for* $k \geq 1$ *and* $\mathbf{i}, \mathbf{j} : [2k] \to [d]$, *we have*

$$\int_{O_d} u_{\mathbf{i}(1)\mathbf{j}(1)} \cdots u_{\mathbf{i}(2k)\mathbf{j}(2k)} \, dU = \sum_{\substack{\pi, \sigma \in P_2(2k) \\ \pi \leq \ker(\mathbf{i}) \\ \sigma \leq \ker(\mathbf{j})}} \mathrm{Wg}^O_{k,d}(\pi, \sigma).$$

For the orthogonal analogue of Theorem 1.61, it turns out that the right representation-theoretic object is the Gelfand pair $(S_{2k}, H_k)$. We need to set some notation relating $P_2(2k)$ and $(S_{2k}, H_k)$:

**Notation 1.64.** There is an embedding of $P_2(2k)$ in $S_{2k}$ defined as follows: for $\pi \in P_2(2k)$, write

$$\pi = \{\{a_1, b_1\}, \ldots, \{a_k, b_k\}\}$$

with $a_i < b_i$ for all $1 \leq i \leq k$ and $a_1 < \cdots < a_k$. Then the embedding is

$$\pi \mapsto \begin{pmatrix} 1 & 2 & \cdots & 2k-1 & 2k \\ a_1 & b_1 & \cdots & a_k & b_k \end{pmatrix} \in S_{2k}.$$

We also need a particular normalization constant, which conceptually comes from the theory of zonal functions:

**Notation 1.65.** For $\lambda \vdash k$, write

$$Z_{\lambda,d} := \prod_{(i,j)\in\lambda} (d + 2j - i - 1)$$

for $d \geq k$.

This notation is a special case of a certain type of symmetric function: the so-called *zonal polynomial* labeled by $\lambda$ evaluated at the partition $1^d$. This is the Gelfand-pair analogue of the Schur function and its evaluation $\mathsf{s}_\lambda(1^d)$ which appears in the unitary case.

**Theorem 1.66** ([16, Theorem 3.1])**.** *For $\pi, \sigma \in P_2(2k)$, we have*

$$\mathrm{Wg}^O_{k,d}(\pi, \sigma) = \frac{2^k k!}{(2k)!} \sum_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq d}} \frac{\dim(2\lambda)}{Z_{\lambda,d}} \omega^\lambda(\pi^{-1}\sigma)$$

*where $P_2(2k)$ is embedded into $S_{2k}$ as in* (1.64)*. In particular,* $\mathrm{Wg}^O_{k,d}(\pi, \sigma)$ *only depends on the coset type $\Xi(\pi^{-1}\sigma)$ of $\pi^{-1}\sigma$.*

## 1.6 Operator algebras and free probability

### 1.6.1 Weak and strong topologies

**Terminology 1.67.** A net $(T_i)_{i \in I}$ in $\mathcal{B}(H)$ converges to $T \in \mathcal{B}(H)$ in the *weak operator topology* if $\lim_i \langle T_i \xi, \eta \rangle = \langle T\xi, \eta \rangle$ for all $\xi, \eta \in H$, and in the *strong operator topology* if $\lim_i \|T_i \xi - T\xi\| = 0$ for all $\xi \in H$.

**Terminology 1.68.** A state $\phi$ on a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$, a.k.a. a positive unital linear functional, is said to be *normal* if it satisfies the following equivalent [6, Theorem III.2.1.4] conditions:

- $\phi$ is WOT-continuous on the closed unit ball in $\mathcal{M}$;
- $\phi$ is SOT-continuous on the closed unit ball in $\mathcal{M}$;

For our purposes, the main example of a normal state is one of the form $\phi(T) = \langle T\xi, \xi \rangle$ for a unit vector $\xi \in H$.

**Proposition 1.69.** *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra and suppose that $\tau$ is a tracial state on $\mathcal{M}$ with a unit vector $\xi_0 \in H$ such that*

- $\tau(T) = \langle T\xi_0, \xi_0 \rangle$ *for $T \in \mathcal{M}$, and*
- $\overline{\text{span}}\{T\xi_0 : T \in \mathcal{M}\} = H$.

*Let $(T_i)_{i \in I}$ be a net of self-adjoint contractions in $\mathcal{M}$ and suppose that there is a self-adjoint contraction $T \in \mathcal{M}$ such that $\lim_i T_i \xi_0 = T\xi_0$. Then $\lim_i^{\text{SOT}} T_i = T$.*

Claims to this effect, usually stated in terms of metrizability of the closed unit ball, can be found in various references such as [42], hence the following notation:

**Notation 1.70.** Let $\mathcal{M}$, $H$, $\tau$, and $\xi_0$ be as in Proposition 1.69. For $T \in \mathcal{M}$, write $\|T\|_2 := \|T\xi_0\|$.

Let us give a concrete proof of Proposition 1.69 which makes convenient use of self-adjointness and traciality:

*Proof of Proposition 1.69.* Fix $\xi \in H$ and $\varepsilon > 0$, so

- there is some $\eta \in \operatorname{span}\{S\xi_0 : S \in \mathcal{M}\}$ such that $\|\xi - \eta\| < \varepsilon$, say $\eta = \sum_j \alpha_j S_j \xi_0$, and
- there is some $i_0 \in I$ such that

  $$\|T_i\xi_0 - T\xi_0\| < \frac{\varepsilon}{\sum_j |\alpha_j| \|S_j\|}$$

  for $i \geq i_0$.

Then

$$\begin{aligned}
\|T_i(\xi) - T(\xi)\| &\leq \|T_i(\xi) - T_i(\eta)\| + \|T_i(\eta) - T(\eta)\| + \|T(\eta) - T(\xi)\| \\
&\leq (\|T_i\| + \|T\|)\|\xi - \eta\| + \|T_i(\eta) - T(\eta)\| \\
&< 2\varepsilon + \sum_j |\alpha_j| \|(T_i - T)S_j\xi_0\|.
\end{aligned}$$

We have

$$\begin{aligned}
\|(T_i - T)S_j\xi_0\|^2 &= \langle (T_i - T)S_j\xi_0, (T_i - T)S_j\xi_0 \rangle \\
&= \langle (S_j)^*(T_i - T)^*(T_i - T)S_j\xi_0, \xi_0 \rangle \\
&= \tau((S_j)^*(T_i - T)^*(T_i - T)S_j) \\
&= \tau((T_i - T)S_j(S_j)^*(T_i - T)^*) \\
&= \langle (T_i - T)S_j(S_j)^*(T_i - T)^*\xi_0, \xi_0 \rangle \\
&= \langle (S_j)^*(T_i - T)^*\xi_0, (S_j)^*(T_i - T)^*\xi_0 \rangle \\
&= \|(S_j)^*(T_i - T)^*\xi_0\|^2
\end{aligned}$$

so

$$\|(T_i - T)S_j\xi_0\| = \|(S_j)^*(T_i - T)^*\xi_0\| \leq \|S_j\| \|(T_i - T)^*\xi_0\|$$

and

$$\begin{aligned}
\|T_i(\eta) - T(\eta)\| &\leq \sum_j |\alpha_j| \|(T_i - T)S_j\xi_0\| \\
&\leq \sum_j |\alpha_j| \|S_j\| \|(T_i - T)^*\xi_0\|
\end{aligned}$$

$$= \sum_j |\alpha_j| \|S_j\| \|T_i\xi_0 - T\xi_0\|.$$

This makes

$$\begin{aligned} \|T_i(\xi) - T(\xi)\| &< 2\varepsilon + \sum_j |\alpha_j| \|S_j\| \|T_i\xi_0 - T\xi_0\| \\ &< 2\varepsilon + \sum_j |\alpha_j| \|S_j\| \frac{\varepsilon}{\sum_j |\alpha_j| \|S_j\|} = 3\varepsilon \end{aligned}$$

for $i \geq i_0$, hence the claim. □

### 1.6.2 Noncommutative probability

**Terminology 1.71.** A $*$-*probability space* is a pair $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is a $*$-algebra and $\varphi$ is a state on $\mathcal{A}$. If $\mathcal{A}$ is a $C^*$-algebra, then $(\mathcal{A}, \varphi)$ is a $C^*$-*probability space*. If $\mathcal{A}$ is a von Neumann algebra and $\varphi$ is faithful and normal, then $(\mathcal{A}, \varphi)$ is called a $W^*$-*probability space*. Of course, this is a decreasing chain of generality, and most basic notions in this area can be formulated in the purely algebraic setting.

**Terminology 1.72** (Commuting independence)**.** Let $(\mathcal{A}, \varphi)$ be a $*$-probability space. A family $\{\mathcal{A}_i : i \in I\}$ of $*$-subalgebras of $\mathcal{A}$ is said to be *commuting independent* if

- $ab = ba$ for all $a \in \mathcal{A}_i$ and $b \in \mathcal{A}_j$ with $i \neq j$, and
- for distinct $i_1, \ldots, i_k \in I$ and $a_r \in \mathcal{A}_{i_r}$ for $1 \leq r \leq k$, we have

$$\varphi(a_1 \cdots a_k) = \varphi(a_1) \cdots \varphi(a_k).$$

The prototypical example of commuting independence is components in tensor products:

**Example 1.73.** Let $(\mathcal{A}, \varphi)$ and $(\mathcal{B}, \psi)$ be $*$-probability spaces. Then in the $*$-probability space $(\mathcal{A} \otimes_{\text{alg}} \mathcal{B}, \varphi \otimes \psi)$, the $*$-subalgebras $\mathcal{A} \otimes 1$ and $1 \otimes \mathcal{B}$ are commuting independent.

The notion of *free independence* parallels commuting independence, the main example being free products instead of tensor products:

**Terminology 1.74.** Let $(\mathcal{A}, \varphi)$ be a $*$-probability space and let $\{\mathcal{A}_i : i \in I\}$ be a family of $*$-subalgebras of $\mathcal{A}$. The family $\{\mathcal{A}_i : i \in I\}$ is said to be *freely independent* or *free* if for all $k \geq 1$ and $i_1, \ldots, i_k \in I$ with $i_r \neq i_{r+1}$ for all $1 \leq r \leq k-1$, if $a_r \in \mathcal{A}_{i_r}$ and $\varphi(a_r) = 0$ for all $1 \leq r \leq k-1$, one has $\varphi(a_1 \cdots a_k) = 0$.

Furthermore, this definition applies to elements of $\mathcal{A}$: $\{a_i : i \in I\}$ is free if $\{\mathcal{A}_i : i \in I\}$ is free, where $\mathcal{A}_i$ is the $*$-algebra generated by $a_i$.

Historically, the motivation for the study of free independence came from free groups:

**Proposition 1.75** ([36, Proposition 5.11])**.** *Let $G$ be a group, let $\mathcal{A} := \mathbb{C}[G]$ be the group algebra, and let $\varphi(x) = \langle x\delta_e, \delta_e \rangle$ be the canonical trace. For a family $\{G_i : i \in I\}$ of subgroups of $G$, with $\mathcal{A}_i := \mathbb{C}[G_i]$ for $i \in I$, the following are equivalent:*

- *the subgroups $\{G_i : i \in I\}$ are* free, *in the sense that for all $k \geq 1$ and $i_1, \ldots, i_k \in I$ with $i_r \neq i_{r+1}$ for all $1 \leq r \leq k-1$, if $g_r \in G_{i_r}$ and $g_r \neq e$ for all $1 \leq r \leq k-1$, one has $g_1 \cdots g_k \neq e$;*
- *the subalgebras $\{\mathcal{A}_i : i \in I\}$ are* freely independent, *in the sense that for all $k \geq 1$ and $i_1, \ldots, i_k \in I$ with $i_r \neq i_{r+1}$ for all $1 \leq r \leq k-1$, if $a_r \in \mathcal{A}_{i_r}$ and $\varphi(a_r) = 0$ for all $1 \leq r \leq k-1$, one has $\varphi(a_1 \cdots a_k) = 0$.*

**Remark 1.76.** If $a$ and $b$ are free, then the moments of $a + b$ or $ab$ only depend on the individual moments of $a$ and $b$ [36, Lemma 5.13], and there is a theory of *free cumulants* for systematically computing either of the former in terms of the latter.

Remark 1.76 allows addition and multiplication of free variables to pass to well-defined operations on probability measures:

**Remark 1.77.** For probability measures $\mu, \nu \in \mathrm{Prob}(\mathbb{R})$ with compact support, it is easy to come up with $C^*$-probability spaces $(\mathcal{A}, \varphi)$ and $(\mathcal{B}, \psi)$ and self-adjoint elements $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that

$$\int_{\mathbb{R}} t^k \, d\mu(t) = \varphi(a^k) \text{ and } \int_{\mathbb{R}} t^k \, d\nu(t) = \psi(b^k)$$

for $k \geq 1$. For example, one could let $X = \overline{\operatorname{supp}(\mu)}$ and $\mathcal{A} = C(X)$, let $\varphi$ be the positive linear functional on $\mathcal{A}$ defined by integrating against $\mu$, and let $a$ be the identity function on $X$. (Similarly for $\nu$.)

Then, by Remark 1.76, the distribution of $a + b$ in the free product $(\mathcal{A} * \mathcal{B}, \varphi * \psi)$ only depends on $\mu$ and $\nu$. This probability measure is denoted by $\mu \boxplus \nu$ and called the *free additive convolution* of $\mu$ and $\nu$.

The *free multiplicative convolution*, defined for $\mu$ and $\nu$ with compact *positive* support and denoted by $\mu \boxtimes \nu$, is similar, modulo a small technicality related to $ab$ not necessarily being self-adjoint. (This issue is fixed by looking at $\sqrt{a} b \sqrt{a}$ instead, which has the same moments as $ab$ and is also self-adjoint.)

# Chapter 2

# Finite free probability

As explained in the introduction, the new theory of *finite free probability* starts from the observation [29] that there are simple formulas, in terms of $c_x(A)$ and $c_x(B)$, for

$$\mathbb{E}_U c_x(A + UBU^*) \text{ and } \mathbb{E}_U c_x(AUBU^*),$$

where $A$ and $B$ are $d \times d$ matrices and $U$ is a random $d \times d$ unitary matrix. This defines operations on polynomials called *finite free convolutions* which are denoted by $\boxplus_d$ and $\boxtimes_d$ respectively. After a small section setting up some more specifics concerning such random rotations and polynomials, we will see how one can use techniques from combinatorial representation theory to recover these results in manner different from [29]. The main tool is *Weingarten calculus*, which reduces the unitary matrix integrals to the combinatorics of permutations and irreducible characters of $S_n$.

Then, we apply these techniques to slightly more general polynomials in $A$ and $UBU^*$, extracting an interesting combinatorial lead: we show that the computation of

$$\mathbb{E}_U c_x(y_1 AUBU^* + y_2 UBU^* A),$$

where $y_1$ and $y_2$ are commuting formal variables, amounts in principle to the computation of the immanants

$$\mathrm{Imm}^\lambda(y_1 x_i + y_2 x_j)_{1 \le i,j \le k}$$

for commuting formal variables $(x_1, \ldots, x_k)$ and $k \le d$.

Finally, motivated by the prominent role of the commutator in the historical combinatorial development of free probability, we take $y_1 = 1$ and $y_2 = -1$ in the last paragraph and compute the relevant immanants, which turn out to be tractable in this case due to a formula of Goulden-Jackson which describes them in terms of certain Schur polynomials. The main result, Theorem 2.27, is a formula for

$$\mathbb{E}_U c_x(AUBU^* - UBU^*A)$$

in terms of $c_x(A)$ and $c_x(B)$.

## 2.1 Expected characteristic polynomials

In finite free probability, the main objects of study are expected characteristic polynomials of random matrices. In the case of randomly rotated matrices, it is very convenient to work with diagonal matrices, and we should justify why this is enough:

**Lemma 2.1.** *Let $Q(z_1, z_2)$ be a polynomial in two non-commuting variables and let $A$ and $B$ be normal $d\times d$ matrices, with diagonalizations $A = V_A D_A V_A^*$ and $B = V_B D_B V_B^*$. Then*

$$\mathbb{E}_U c_x(Q(A, UBU^*)) = \mathbb{E}_U c_x(Q(D_A, UD_BU^*))$$

*where $U$ is a $d \times d$ random unitary matrix.*

*Proof.* Write

$$Q(z_1, z_2) = \sum_{\substack{w \text{ word} \\ \text{in } \{z_1, z_2\}}} q_w w.$$

For each $w$, write $w = z_1^{p_1} z_2^{q_1} \cdots z_1^{p_k} z_2^{q_k}$ where $p_1, \ldots, p_k \geq 0$ and $q_1, \ldots, q_k \geq 0$, and observe that

$$\begin{aligned} &(A)^{p_1}(UBU^*)^{q_1} \cdots (A)^{p_k}(UBU^*)^{q_k} \\ &= (V_A D_A^{p_1} V_A^*)(UV_B D_B^{q_1} V_B^* U^*) \cdots (V_A D_A^{p_k} V_A^*)(UV_B D_B^{q_k} V_B^* U^*) \\ &= V_A D_A^{p_1} (V_A^* U V_B) D_B^{q_1} (V_A^* U V_B)^* D_A^{p_2} \cdots D_A^{p_k} (V_A^* U V_B) D_B^{q_k} (V_B^* U^*). \end{aligned}$$

Then, conjugate the above by $V_A^*(\cdot)V_A$ to get

$$D_A^{p_1}(V_A^*UV_B)D_B^{q_1}(V_A^*UV_B)^* \cdots D_A^{p_k}(V_A^*UV_B)D_B^{q_k}(V_A^*UV_B)^*.$$

This conjugation does not depend on $w$, so it can be applied uniformly to the summands in

$$\mathbb{E}_U c_x(Q(A, UBU^*)) = \mathbb{E}_U c_x \left( \sum_{\substack{w \text{ word} \\ \text{in } \{z_1, z_2\}}} q_w w(A, UBU^*) \right)$$

to make the right-hand side equal to

$$\mathbb{E}_U c_x(Q(D_A, (V_A^*UV_B)D_B(V_A^*UV_B)^*))$$

since unitary conjugation does not change the characteristic polynomial. Finally, the invariance of Haar measure shows the above is equal to

$$\mathbb{E}_U c_x(Q(D_A, UD_BU^*)),$$

hence the claim. □

## 2.2 Finite free addition and multiplication

The following operations appear, perhaps not quite obviously, in the work [45, 41] of Walsh and Szegö on the locations of roots of polynomials.

**Definition 2.2.** For polynomials $p(x)$ and $q(x)$ with degree at most $d$, say

$$p(x) = \sum_{k=0}^{d} x^{d-k}(-1)^k p_k \text{ and } q(x) = \sum_{k=0}^{d} x^{d-k}(-1)^k q_k,$$

define

$$p(x) \boxplus_d q(x) := \sum_{k=0}^{d} x^{d-k}(-1)^k \left( \sum_{i+j=k} \frac{(d-i)!(d-j)!}{d!(d-k)!} p_i q_j \right)$$

and

$$p(x) \boxtimes_d q(x) := \sum_{k=0}^{d} x^{d-k}(-1)^k \left( \frac{k!(d-k)!}{d!} p_k q_k \right).$$

These were rediscovered [29] by Marcus, Spielman, and Srivastava as the expected characteristic polynomials of sums and products of randomly rotated matrices.

**Theorem 2.3** ([29])**.** *Let $p(x)$ and $q(x)$ be monic polynomials with degree $d$, and let $A$ and $B$ be $d \times d$ diagonal matrices with $p(x) = c_x(A)$ and $q(x) = c_x(B)$. Then*

$$p(x) \boxplus_d q(x) = \mathbb{E}_U c_x(A + UBU^*)$$

*and*

$$p(x) \boxtimes_d q(x) = \mathbb{E}_U c_x(AUBU^*)$$

*where $U$ is a random $d \times d$ unitary matrix.*

The proofs in [29] use a notion of *minor-orthogonality* and a so-called *quadrature* phenomenon, replacing the continuous integrals over $U_d$ with sums over certain finite subgroups of $U_d$. In this section, this theorem is proved using techniques from combinatorial representation theory: we use *Weingarten calculus* to reduce the continuous integrals to the combinatorics of permutations, partitions, and representations of $S_n$. The key point in this approach will be to show that various subgroups of $U_d$ have the following property:

**Definition 2.4.** A compact subgroup $G$ of $U_d$ has the *quadrature property* if

$$\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \int_G u_{1\mathbf{p}(1)} \cdots u_{k\mathbf{p}(k)} \overline{u_{\sigma(1)\mathbf{p}(i)}} \cdots \overline{u_{\sigma(k)\mathbf{p}(k)}} = \begin{cases} \frac{d!}{(d-k)!} & \text{if } \mathbf{p} \text{ injective} \\ 0 & \text{otherwise} \end{cases}$$

for all $\mathbf{p} : [k] \to [d]$, for all $0 \leq k \leq d$.

**Theorem 2.5.** *The following groups have the quadrature property:*

- *the group $U_d$ of $d \times d$ unitary matrices;*
- *the group $O_d$ of $d \times d$ orthogonal matrices;*
- *the group $H_d$ of $d \times d$ signed permutation matrices.*

This quadrature property is just enough to recover the convolution formulas of Theorem 2.3:

**Theorem 2.6.** *Let $A$ and $B$ be diagonal matrices and let $G$ be a compact subgroup of $U_d$ with the quadrature property. Then*

$$\int_G \mathsf{e}_k(A + UBU^*)\, dU = \sum_{i+j=k} \frac{(d-i)!(d-j)!}{d!(d-k)!} \mathsf{e}_i(A)\mathsf{e}_j(B)$$

*and*

$$\int_G \mathsf{e}_k(AUBU^*)\, dU = \frac{1}{\binom{d}{k}} \mathsf{e}_k(A)\mathsf{e}_k(B)$$

*for $0 \le k \le d$.*

### 2.2.1 Quadrature property

To get a sense of where to begin proving Theorem 2.3, the way to go is to simply dive into the case of $A + UBU^*$ and see what Haar integrals must be handled.

**Remark 2.7.** We will assume that $A$ and $B$ are diagonal, with

$$A = \mathrm{diag}(a_1, \dots, a_d) \text{ and } B = \mathrm{diag}(b_1, \dots, b_d).$$

In the case $G = U_d$, Lemma 2.1 shows there is no loss of generality.

To ease notation, let us temporarily write $W := A + UBU^*$; by Proposition 1.43 we have

$$\mathbb{E}_U \mathsf{e}_k(A + UBU^*) = \sum_{\substack{S \subseteq [d] \\ |S| = k}} \det(W(S, S))$$

so it suffices to work with $\mathbb{E}_U \det(W(S, S))$ for a fixed choice of $S$.

**Lemma 2.8.** *We have*

$$\mathbb{E}_U \det(W(S, S)) = \sum_{R \subseteq S} \left( \prod_{i \in R} a_i \right) \sum_{\mathbf{p}: S \setminus R \to [d]} \left( \prod_{i \in S \setminus R} b_{\mathbf{p}(i)} \right)$$

$$\sum_{\sigma\in\mathrm{Sym}(S\setminus R)} \mathrm{sgn}(\sigma)\mathbb{E}_U\left(\prod_{i\in S\setminus R} u_{i\mathbf{p}(i)}\overline{u_{\sigma(i)\mathbf{p}(i)}}\right)$$

*for $S \subseteq [d]$ with $|S| = k$, for $0 \le k \le d$.*

*Proof.* We have

$$\begin{aligned}
&\det(W(S,S)) \\
&= \sum_{\sigma\in\mathrm{Sym}(S)} \mathrm{sgn}(\sigma)\prod_{i\in S}\left(a_i\delta_{i,\sigma(i)} + \sum_{p=1}^{d} u_{ip}b_p\overline{u_{\sigma(i)p}}\right) \\
&= \sum_{\sigma\in\mathrm{Sym}(S)} \mathrm{sgn}(\sigma)\sum_{R\subseteq S}\left(\left(\prod_{i\in R} a_i\delta_{i,\sigma(i)}\right)\left(\prod_{i\in S\setminus R}\left(\sum_{p=1}^{d} u_{ip}b_p\overline{u_{\sigma(i)p}}\right)\right)\right) \\
&= \sum_{R\subseteq S}\sum_{\sigma\in\mathrm{Sym}(S\setminus R)} \mathrm{sgn}(\sigma)\left(\prod_{i\in R} a_i\right)\left(\prod_{i\in S\setminus R}\left(\sum_{p=1}^{d} u_{ip}b_p\overline{u_{\sigma(i)p}}\right)\right).
\end{aligned}$$

Switching the product and sum, we have

$$\begin{aligned}
&\mathbb{E}_U\left(\prod_{i\in S\setminus R}\left(\sum_{p=1}^{d} u_{ip}b_p\overline{u_{\sigma(i)p}}\right)\right) \\
&= \sum_{\mathbf{p}:S\setminus R\to[d]} \mathbb{E}_U\left(\prod_{i\in S\setminus R} u_{i\mathbf{p}(i)}\overline{u_{\sigma(i)\mathbf{p}(i)}}\right)\left(\prod_{i\in S\setminus R} b_{\mathbf{p}(i)}\right)
\end{aligned}$$

and putting this back into the sum above, we get the desired formula. □

So we want to work with

$$\sum_{\sigma\in S_k} \mathrm{sgn}(\sigma)\mathbb{E}_U\left(\prod_{i=1}^{k} u_{i\mathbf{p}(i)}\overline{u_{\sigma(i)\mathbf{p}(i)}}\right)$$

for $\mathbf{p} : [k] \to [d]$, for $0 \le k \le d$, and the quadrature property does exactly this.

*Proof of (1) in Theorem 2.6.* We have already done a large portion of the proof; let us pick back up from the computations above. Notice that since we assume $A = \operatorname{diag}(a_1, \ldots, a_d)$ and $B = \operatorname{diag}(b_1, \ldots, b_d)$, we have

$$\mathsf{e}_i(A) = \frac{1}{i!} \sum_{\substack{\mathbf{p}:[i]\to[d] \\ \text{injective}}} a_{\mathbf{p}(1)} \cdots a_{\mathbf{p}(i)} = \sum_{\substack{\mathbf{p}:[i]\to[d] \\ \mathbf{p}(1)<\cdots<\mathbf{p}(i)}} a_{\mathbf{p}(1)} \cdots a_{\mathbf{p}(i)}$$

and similarly for $\mathsf{e}_j(B)$. So we have

$$\begin{aligned}
&\mathbb{E}_U \det(W(S,S)) \\
&= \sum_{R\subseteq S} \left(\prod_{i\in R} a_i\right) \sum_{\mathbf{p}:S\setminus R\to[d]} \left(\prod_{i\in S\setminus R} b_{\mathbf{p}(i)}\right) \\
&\qquad \sum_{\sigma\in\mathrm{Sym}(S\setminus R)} \operatorname{sgn}(\sigma)\mathbb{E}_U \left(\prod_{i\in S\setminus R} u_{i\mathbf{p}(i)}\overline{u_{\sigma(i)\mathbf{p}(i)}}\right) \\
&= \sum_{R\subseteq S} \frac{(d-|S\setminus R|)!}{d!} \left(\prod_{i\in R} a_i\right) \sum_{\substack{\mathbf{p}:S\setminus R\to[d] \\ \text{injective}}} \left(\prod_{i\in S\setminus R} b_{\mathbf{p}(i)}\right) \\
&= \sum_{R\subseteq S} \frac{|S\setminus R|!(d-|S\setminus R|)!}{d!} \det(A(R,R))\mathsf{e}_{|S\setminus R|}(B)
\end{aligned}$$

and then

$$\begin{aligned}
\mathbb{E}_U(\mathsf{e}_k(W)) &= \sum_{|S|=k} \mathbb{E}_U(\det(W(S,S))) \\
&= \sum_{|S|=k} \sum_{R\subseteq S} \frac{|S\setminus R|!(d-|S\setminus R|)!}{d!} \det(A(R,R))\mathsf{e}_{|S\setminus R|}(B) \\
&= \sum_{i+j=k} \frac{(d-i)!(d-j)!}{d!(d-k)!} \mathsf{e}_j(B) \sum_{|R|=i} \det(A(R,R)) \\
&= \sum_{i+j=k} \frac{(d-i)!(d-j)!}{d!(d-k)!} \mathsf{e}_i(A)\mathsf{e}_j(B)
\end{aligned}$$

so we are done. $\square$

*Proof of (2) in Theorem 2.6.* Write $W = AUBU^*$ and $U = (u_{ij})_{i,j}$, so the $(i,j)$-th entry of $W$ is

$$a_i \sum_{p=1}^{d} u_{ip} b_p \overline{u_{jp}}$$

and for a subset $S \subseteq [d]$ with $|S| = k$, we have

$$\begin{aligned}
\det(W(S,S)) &= \sum_{\sigma \in \mathrm{Sym}(S)} \mathrm{sgn}(\sigma) \prod_{i \in S} \left( a_i \sum_{p=1}^{d} u_{ip} b_p \overline{u_{\sigma(i)p}} \right) \\
&= \det(A(S,S)) \sum_{\sigma \in \mathrm{Sym}(S)} \mathrm{sgn}(\sigma) \prod_{i \in S} \left( \sum_{p=1}^{d} u_{ip} b_p \overline{u_{\sigma(i)p}} \right).
\end{aligned}$$

Switching the product and sum, we have

$$\begin{aligned}
\mathbb{E}_U(\det(W(S,S))) = \det(A(S,S)) &\sum_{\mathbf{p}:S \to [d]} \left( \prod_{i \in S} b_{\mathbf{p}(i)} \right) \\
&\sum_{\sigma \in \mathrm{Sym}(S)} \mathrm{sgn}(\sigma) \mathbb{E}_U \left( \prod_{i \in S} u_{i\mathbf{p}(i)} \overline{u_{\sigma(i)\mathbf{p}(i)}} \right) \\
= \det(A(S,S)) &\sum_{\substack{\mathbf{p}:S \to [d] \\ \text{injective}}} \left( \prod_{i \in S} b_{\mathbf{p}(i)} \right) \frac{(d - |S|)!}{d!} \\
= \det(A(S,S)) &\frac{k!(d-k)!}{d!} \mathsf{e}_k(B),
\end{aligned}$$

thus

$$\mathbb{E}_U \mathsf{e}_k(AUBU^*) = \frac{k!(d-k)!}{d!} \sum_{|S|=k} \det(A(S,S)) \mathsf{e}_k(B) = \frac{k!(d-k)!}{d!} \mathsf{e}_k(A) \mathsf{e}_k(B)$$

and we are done. □

### 2.2.2 Unitary rotations

**Theorem 2.9.** *$U_d$ has the quadrature property.*

To prove Theorem 2.9, we use Theorem 1.61 to reduce the computation to the following simple lemma:

**Lemma 2.10.** *We have*

$$\sum_{\sigma\in S_k} \operatorname{sgn}(\sigma)\chi^\lambda(\sigma) = \begin{cases} k! & \text{if } \lambda = 1^k \\ 0 & \text{otherwise} \end{cases}$$

*for* $\lambda \vdash k$.

*Proof.* Recall that $\chi^{1^k}$ is just the sign character of $S_k$. So if $\lambda = 1^k$, we have

$$\sum_{\sigma\in S_k} \operatorname{sgn}(\sigma)\chi^{1^k}(\sigma) = \sum_{\sigma\in S_k} 1 = k!$$

and otherwise, if $\lambda \neq 1^k$, the orthogonality relations for irreducible characters of finite groups give

$$\sum_{\sigma\in S_k} \operatorname{sgn}(\sigma)\chi^\lambda(\sigma) = \sum_{\sigma\in S_k} \chi^{1^k}(\sigma)\chi^\lambda(\sigma) = 0$$

as claimed. □

*Proof Theorem 2.9.* By Theorem 1.60 we have

$$\begin{aligned}\sum_{\sigma\in S_k} \operatorname{sgn}(\sigma) \int_{U_d} \prod_{i=1}^{k} u_{i\mathbf{p}(i)}\overline{u_{\sigma(i)\mathbf{p}(i)}}\, dU &= \sum_{\sigma\in S_k} \operatorname{sgn}(\sigma) \sum_{\substack{\pi,\tau\in S_k \\ 1=\sigma\circ\pi \\ \mathbf{p}=\mathbf{p}\circ\tau}} \mathrm{Wg}^U_{k,d}(\pi,\tau) \\ &= \sum_{\sigma\in S_k} \operatorname{sgn}(\sigma) \sum_{\substack{\tau\in S_k \\ \mathbf{p}=\mathbf{p}\circ\tau}} \mathrm{Wg}^U_{k,d}(\sigma^{-1},\tau). \quad (2.1)\end{aligned}$$

If $\mathbf{p}$ is not injective, say there are some $i, j \in [k]$ with $i \neq j$ and $\mathbf{p}(i) = \mathbf{p}(j)$, we want to identify pairs of summands which cancel each other out, i.e. for each $\sigma \in S_k$ we want a corresponding $\sigma' \in S_k$ with $\operatorname{sgn}(\sigma') = -\operatorname{sgn}(\sigma)$ and

$$\sum_{\substack{\tau\in S_k \\ \mathbf{p}=\mathbf{p}\circ\tau}} \mathrm{Wg}^U_{k,d}(\sigma^{-1},\tau) = \sum_{\substack{\tau\in S_k \\ \mathbf{p}=\mathbf{p}\circ\tau}} \mathrm{Wg}^U_{k,d}((\sigma')^{-1},\tau).$$

To this end let $\sigma' = \sigma \cdot (i,j)$, so that $\mathrm{sgn}(\sigma') = -\mathrm{sgn}(\sigma)$. Moreover, since $\mathrm{Wg}^U_{k,d}(\sigma^{-1},\tau)$ only depends on the cycle type of $\sigma\tau$, we have

$$\begin{aligned}\sum_{\substack{\tau\in S_k\\ \mathbf{p}=\mathbf{p}\circ\tau}} \mathrm{Wg}^U_{k,d}((\sigma')^{-1},\tau) &= \sum_{\substack{\tau\in S_k\\ \mathbf{p}=\mathbf{p}\circ\tau}} \mathrm{Wg}^U_{k,d}((i,j)\sigma^{-1},\tau)\\ &= \sum_{\substack{\tau\in S_k\\ \mathbf{p}=\mathbf{p}\circ\tau}} \mathrm{Wg}^U_{k,d}(\sigma^{-1},(i,j)\tau)\\ &= \sum_{\substack{\tau\in S_k\\ \mathbf{p}=\mathbf{p}\circ\tau}} \mathrm{Wg}^U_{k,d}(\sigma^{-1},\tau)\end{aligned}$$

as the condition $\mathbf{p} = \mathbf{p}\circ\tau$ is invariant under translation of $\tau$ by $(i,j)$. This shows that the summands in Eq. (2.1) cancel each other out and the sum is 0.

If $\mathbf{p}$ is injective, then the only $\tau \in S_k$ with $\mathbf{p} = \mathbf{p}\circ\tau$ is $\tau = 1$ so by Theorem 1.61 we have

$$\begin{aligned}&\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\sum_{\substack{\tau\in S_k\\ \mathbf{p}=\mathbf{p}\circ\tau}} \mathrm{Wg}^U_{k,d}(\sigma^{-1},\tau)\\ &= \sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\mathrm{Wg}^U_{k,d}(\sigma^{-1},1)\\ &= \sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\frac{1}{(k!)^2}\sum_{\substack{\lambda\vdash k\\ \ell(\lambda)\le d}}\frac{\chi^\lambda(1)^2}{s_\lambda(1^d)}\chi^\lambda(\sigma)\\ &= \frac{1}{(k!)^2}\sum_{\substack{\lambda\vdash k\\ \ell(\lambda)\le d}}\left(\frac{k!\chi^\lambda(1)^2}{\chi^\lambda(1)\prod_{(i,j)\in\lambda}(d+j-i)}\right)\left(\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\chi^\lambda(\sigma)\right)\\ &= \left(\frac{\chi^{1^k}(1)}{k!\prod_{1\le i\le k}(d+1-i)}\right)k!\\ &= \frac{1}{\prod_{1\le i\le k}(d-i+1)} = \frac{(d-k)!}{d!}.\end{aligned}$$

□

### 2.2.3 Orthogonal rotations

We can use a similar argument to handle the case of orthogonal rotations:

**Theorem 2.11.** *$O_d$ has the quadrature property.*

Similar to the unitary case, we use Theorem 1.66 to reduce the computation to the following lemma:

**Lemma 2.12.** *We have*

$$\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\omega^{\lambda}(\sigma) = \begin{cases} \frac{(k+1)!}{2^k} & \text{if } \lambda = 1^k \\ 0 & \text{otherwise} \end{cases}.$$

*for $\lambda \vdash k$.*

*Proof.* If $\lambda = 1^k$, then

$$\begin{aligned} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\omega^{1^k}(\sigma) &= \sum_{\sigma \in S_k} (-1)^{k-\#(\sigma)} \frac{(-1)^{k-\#(\sigma)}}{2^{k-\#(\sigma)}} && \text{(Proposition 1.28)} \\ &= \frac{1}{2^k} \sum_{\sigma \in S_k} 2^{\#(\sigma)} \\ &= \frac{1}{2^k} \sum_{i=1}^{k} 2^i c(k,i) \\ &= \frac{2^{(k)}}{2^k} && \text{(Proposition 1.2)} \\ &= \frac{(k+1)!}{2^k}. \end{aligned}$$

On the other hand, for any $\lambda \neq 1^k$ we have

$$\begin{aligned} 0 &= \sum_{\rho \vdash k} \frac{1}{z_{2\rho}} \omega^{\lambda}_{\rho} \omega^{1^k}_{\rho} && \text{(Theorem 1.32)} \\ &= \sum_{\rho \vdash k} \frac{1}{z_{2\rho}} \omega^{\lambda}_{\rho} \frac{(-1)^{k-\ell(\rho)}}{2^{k-\ell(\rho)}} && \text{(Proposition 1.28)} \\ &= \frac{1}{2^k} \sum_{\rho \vdash k} (-1)^{k-\ell(\rho)} 2^{\ell(\rho)} \frac{1}{z_{2\rho}} \omega^{\lambda}_{\rho} \end{aligned}$$

$$= \frac{1}{2^k k!} \sum_{\rho \vdash k} (-1)^{k-\ell(\rho)} \frac{k!}{z_\rho} \omega_\rho^\lambda$$
$$= \frac{1}{2^k k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \omega^\lambda(\sigma)$$

thus $\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\omega^\lambda_{\mu_\sigma} = 0$. □

*Proof of Theorem 2.11.* By Theorem 1.63, with $\mathbf{i}_\sigma := (1, \sigma(1), \ldots, k, \sigma(k))$ and

$$\mathbf{pp} := (\mathbf{p}(1), \mathbf{p}(1), \ldots, \mathbf{p}(k), \mathbf{p}(k)),$$

we have

$$\begin{aligned} \sum_{\sigma \in S_k} \int_{O_d} \prod_{i=1}^k u_{i\mathbf{p}(i)} u_{\sigma(i)\mathbf{p}(i)} \, dU &= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \sum_{\substack{\pi, \tau \in P_2(2k) \\ \pi \le \ker(\mathbf{i}_\sigma) \\ \tau \le \ker(\mathbf{pp})}} \operatorname{Wg}^O_{k,d}(\pi, \tau) \\ &= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \sum_{\substack{\tau \in P_2(2k) \\ \tau \le \ker(\mathbf{pp})}} \operatorname{Wg}^O_{k,d}(\ker(\mathbf{i}_\sigma), \tau) \qquad (2.2) \end{aligned}$$

since the condition $\pi \le \ker(\mathbf{i}_\sigma)$ forces equality. If $\mathbf{p}$ is not injective, say there are some $i \ne j$ with $\mathbf{p}(i) = \mathbf{p}(j)$, we want to identify pairs of summands which cancel each other out, i.e. for each $\sigma \in S_k$ we want a corresponding $\sigma' \in S_k$ with $\operatorname{sgn}(\sigma') = -\operatorname{sgn}(\sigma)$ and

$$\sum_{\substack{\tau \in P_2(2k) \\ \tau \le \ker(\mathbf{pp})}} \operatorname{Wg}^O_{k,d}(\ker(\mathbf{i}_\sigma), \tau) = \sum_{\substack{\tau \in P_2(2k) \\ \tau \le \ker(\mathbf{pp})}} \operatorname{Wg}^O_{k,d}(\ker(\mathbf{i}_{\sigma'}), \tau).$$

To this end let $\sigma' = (i,j)\sigma$, which obviously satisfies $\operatorname{sgn}(\sigma') = -\operatorname{sgn}(\sigma)$. Moreover, we have $\ker(\mathbf{j}_{\sigma'}) = (i,j)\ker(\mathbf{j}_\sigma)$ in the embedding from Notation 1.64, so with $\tau' = (i,j)\tau$, $\tau^{-1}\ker(\mathbf{j}_{\sigma'})$ and $(\tau')^{-1}\ker(\mathbf{j}_\sigma)$ have the same coset type. Since the condition $\tau \le \ker(\mathbf{pp})$ is invariant under translation of $\tau$ by $(i,j)$, by Theorem 1.66 we have

$$\sum_{\substack{\tau \in P_2 2k \\ \tau \le \ker(\mathbf{pp})}} \operatorname{Wg}^O_{k,d}(\ker(\mathbf{i}_{\sigma'}), \tau) = \sum_{\substack{\tau \in P_2(2k) \\ \tau \le \ker(\mathbf{pp})}} \operatorname{Wg}^O_{k,d}(\ker(\mathbf{i}_\sigma), \tau')$$

$$= \sum_{\substack{\tau \in P_2(2k) \\ \tau \leq \ker(\mathbf{pp})}} \mathrm{Wg}^O_{k,d}(\ker(\mathbf{i}_\sigma), \tau).$$

Thus we have shown that when $\mathbf{p}$ is not injective, the summands in (2.2) cancel each other out and the sum is 0.

If $\mathbf{p}$ is injective, then the condition $\tau \leq \ker(\mathbf{pp})$ forces equality, so since

$$\chi^{2(1^k)}(1) = \frac{(2k)!}{k!(k+1)!}$$

and $Z_{1^k}(1^d) = (d)_k$, we have

$$\begin{aligned}
&\sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \sum_{\substack{\tau \in P_2(2k) \\ \tau \leq \ker(\mathbf{pp})}} \mathrm{Wg}^O_{k,d}(\ker(\mathbf{i}_\sigma), \tau) \\
&= \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \mathrm{Wg}^O_{k,d}(\ker(\mathbf{i}_\sigma), \ker(\mathbf{pp})) \\
&= \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \frac{2^k k!}{(2k)!} \sum_{\lambda \vdash k} \frac{\chi^{2\lambda}(1)}{Z_\lambda(1^d)} \omega^\lambda_{\mu_\sigma} && \text{(Theorem 1.66)} \\
&= \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \frac{2^k k!}{(2k)!} \frac{\chi^{2(1^k)}(1)}{Z_{1^k}(1^d)} \omega^{1^k}_{\mu_\sigma} + \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \frac{2^k k!}{(2k)!} \sum_{\substack{\lambda \vdash k \\ \lambda \neq 1^k}} \frac{\chi^{2\lambda}(1)}{Z_\lambda(1^d)} \omega^\lambda_{\mu_\sigma} \\
&= \frac{2^k}{(k+1)!(d)_k} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \omega^{1^k}_{\mu_\sigma} + \sum_{\substack{\lambda \vdash k \\ \lambda \neq 1^k}} \frac{2^k k! \chi^{2\lambda}(1)}{(2k)! Z_\lambda(1^d)} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \omega^\lambda_{\mu_\sigma} \\
&= \frac{1}{(d)_k} = \frac{(d-k)!}{d!} && \text{(Lemma 2.12)}
\end{aligned}$$

and we are done. □

### 2.2.4 Some finite groups

This brief section is dedicated to the elementary proof that the *hyperoctahedral series* of groups $H^s_d = \widehat{\mathbf{Z}_s} \wr S_d$ for $2 \leq s \leq \infty$, which consist of $d \times d$ "signed" permutation matrices where the "signs" are $s$-th roots of unity (or in the case $s = \infty$, the entire circle), have the quadrature property. It is known from [29, 24] that these groups are well-behaved in this context.

What we want to show is that

$$\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \int_{H_d^s} \prod_{i=1}^{k} u_{i\mathbf{p}(i)} \overline{u_{\sigma(i)\mathbf{p}(i)}}\, dU = \begin{cases} \frac{(d-k)!}{d!} & \text{if } \mathbf{p} \text{ is injective} \\ 0 & \text{otherwise} \end{cases}$$

for $\mathbf{p} : [k] \to [d]$, for $0 \le k \le d$. For $s < \infty$, we have

$$\begin{aligned} &\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \int_{H_d^s} \prod_{i=1}^{k} u_{i\mathbf{p}(i)} \overline{u_{\sigma(i)\mathbf{p}(i)}}\, dU \\ &= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \frac{1}{s^d d!} \sum_{\epsilon_1, \ldots, \epsilon_d \in \widehat{\mathbf{Z}_s}} \sum_{\tau \in S_d} \prod_{i=1}^{k} (\epsilon_i \delta_{\mathbf{p}(i) = \tau(i)})(\overline{\epsilon_{\sigma(i)}} \delta_{\mathbf{p}(i) = \tau(\sigma(i))}) \end{aligned}$$

and the non-zero summands are the ones with $\mathbf{p}(i) = \tau(i)$ and $\mathbf{p}(i) = \tau(\sigma(i))$ for $1 \le i \le k$. If $\mathbf{p}$ is not injective, then there is no $\tau \in S_d$ with $\mathbf{p}(i) = \tau(i)$ for $1 \le i \le k$, so the sum is 0. On the other hand, if $\mathbf{p}$ is injective, there are $(d-k)!$ permutations $\tau \in S_d$ with $\mathbf{p}(i) = \tau(i)$ for $1 \le i \le k$, i.e. $\tau \in S_{d-k}$; similarly the condition $\mathbf{p}(i) = \tau(\sigma(i))$ forces $\sigma(i) = i$ for $1 \le i \le k$, i.e. $\sigma = 1$. So the sum above becomes

$$\begin{aligned} \frac{1}{s^d d!} \sum_{\epsilon_1, \ldots, \epsilon_d \in \widehat{\mathbf{Z}_s}} \sum_{\tau \in S_{d-k}} \prod_{i=1}^{k} |\epsilon_i|^2 &= \frac{(d-k)!}{s^d d!} \sum_{\epsilon_1, \ldots, \epsilon_d \in \widehat{\mathbf{Z}_s}} \prod_{i=1}^{k} 1 \\ &= \frac{(d-k)!}{d!} \end{aligned}$$

since the last sum gives $s^d$ summands, which are copies of 1. The case $s = \infty$ is similar, except with an integral over the $d$-torus $\mathbb{T}^d$ instead of a sum over $d$ copies of $\widehat{\mathbf{Z}_s}$.

## 2.3 Non-self-adjoint multiplication and immanants

In free probability, self-adjoint variables are generally the simplest to work with: if $a$ is non-self-adjoint, one must keep track of *mixed* moments of $a$ and $a^*$, not just $a$. In other words, this is the difference between

$$\{\varphi(a^{\epsilon_1} \cdots a^{\epsilon_k}) : k \ge 1, \epsilon_1, \ldots, \epsilon_k \in \{1, *\}\} \text{ and } \{\varphi(a^k) : k \ge 1\}.$$

While the basic convolution formulas of finite free probability (presented in this thesis as Definition 2.2 and Theorem 2.3) apply just as well to any polynomials and normal matrices, the theory (in its current form) really shines when one realizes that $\boxplus_d$ preserves real-rootedness and $\boxtimes_d$ preserves positive-rootedness.

That said, there is still a great interest in developing finite free probability to include non-self-adjoint variables, or more generally multiple variables. It is not clear *a priori* what the right objects even are for this, but the recent PhD thesis of B. Mirabelli [32] has started down this road by defining a certain "multivariate characteristic polynomial" and showing some of the single-variable theory carries over smoothly. This section has two purposes: to point out a potential approach to one of the main problems left open by [32], and to contextualize a related result of the author which will be presented in Section 2.4.

Here, we will just be concerned with the case of a single not-necessarily-self-adjoint variable and its adjoint:

**Notation 2.13** ([32])**.** For a $d \times d$ matrix $X$, write

$$c_{x,y_1,y_2}(X) = \det(xI - y_1X - y_2X^*)$$

which is a polynomial in three commuting formal variables $x, y_1, y_2$. Of course, this is just the characteristic polynomial of $y_1X + y_2X^*$.

**Remark 2.14.** Let $A$ and $B$ be $d \times d$ matrices and let $U$ be a random $d \times d$ unitary matrix. Then one might ask to describe

$$\mathbb{E}_U c_{x,y_1,y_2}(A + UBU^*, (A + UBU^*)^*) \text{ or } \mathbb{E}_U c_{x,y_1,y_2}(AUBU^*, (AUBU^*)^*)$$

in terms of $c_{x,y_1,y_2}(A)$ and $c_{x,y_1,y_2}(B)$. As observed in [32], addition turns out to be easy, since

$$\begin{aligned} y_1(A + UBU^*) + y_2(A + UBU^*)^* &= y_1(A + UBU^*) + y_2(A^* + UB^*U^*) \\ &= (y_1A + y_2A^*) + U(y_1B + y_2B^*)U^* \end{aligned}$$

and the formula for $\boxplus_d$ from Theorem 2.3 still applies. Multiplication, however, is not clear, and was left open in [32]. In case $A$ and $B$ are self-adjoint, this amounts to describing

$$\mathbb{E}_U c_x(y_1 AUBU^* + y_2 UBU^* A)$$

in terms of $c_x(A)$ and $c_x(B)$.

**Notation 2.15.** In this section, let $A$ and $B$ be self-adjoint $d \times d$ matrices, and let $p(x) := c_x(A)$ and $q(x) := c_x(B)$ be their characteristic polynomials. By Lemma 2.1 we can assume without loss of generality that $A = \operatorname{diag}(a_1, \ldots, a_d)$ and $B = \operatorname{diag}(b_1, \ldots, b_d)$.

In a manner similar to Section 2.2.1, we can very directly untangle the elementary symmetric function in terms of the entries of the matrix:

**Lemma 2.16.** *We have*

$$\begin{aligned}
&\mathbb{E}_U \mathsf{e}_k(y_1 AUBU^* + y_2 UBU^* A) \\
&= \frac{1}{(k!)^2} \sum_{\lambda \vdash k} \frac{\dim(\lambda)^2}{\mathsf{s}_\lambda(1^d)} \sum_{\substack{S \subseteq [d] \\ |S| = k}} \sum_{\mathbf{p}: S \to [d]} \left( \prod_{i \in S} b_{\mathbf{p}(i)} \right) \\
&\qquad \sum_{\sigma \in \mathrm{Sym}(S)} \operatorname{sgn}(\sigma) \left( \sum_{\substack{\tau \in \mathrm{Sym}(S) \\ \mathbf{p} = \mathbf{p} \circ \tau}} \chi^\lambda(\sigma\tau) \right) \prod_{i \in S} (y_1 a_i + y_2 a_{\sigma(i)})
\end{aligned}$$

*for* $0 \le k \le d$.

*Proof.* The $(i,j)$-th entry of $y_1 AUBU^* + y_2 UBU^* A$ is

$$y_1 \sum_{p=1}^{d} a_i u_{ip} b_p \overline{u_{jp}} + y_2 \sum_{p=1}^{d} u_{ip} b_p \overline{u_{jp}} a_j$$

so we have

$$\begin{aligned}
&\mathsf{e}_k(AUBU^* \pm UBU^* A) \\
&= \sum_{\substack{S \subseteq [d] \\ |S| = k}} \sum_{\sigma \in \mathrm{Sym}(S)} \operatorname{sgn}(\sigma) \prod_{i \in S} \sum_{p=1}^{d} (y_1 a_i u_{ip} b_p \overline{u_{\sigma(i)p}} + y_2 u_{ip} b_p \overline{u_{\sigma(i)p}} a_{\sigma(i)})
\end{aligned}$$

$$= \sum_{\substack{S\subseteq[d]\\|S|=k}} \sum_{\sigma\in\mathrm{Sym}(S)} \mathrm{sgn}(\sigma) \sum_{\mathbf{p}:S\to[d]}$$

$$\prod_{i\in S}(y_1 a_i u_{i\mathbf{p}(i)} b_{\mathbf{p}(i)} \overline{u_{\sigma(i)\mathbf{p}(i)}} + y_2 u_{i\mathbf{p}(i)} b_{\mathbf{p}(i)} \overline{u_{\sigma(i)\mathbf{p}(i)}} a_{\sigma(i)})$$

$$= \sum_{\substack{S\subseteq[d]\\|S|=k}} \sum_{\sigma\in\mathrm{Sym}(S)} \mathrm{sgn}(\sigma) \sum_{\mathbf{p}:S\to[d]} \prod_{i\in S}(y_1 a_i + y_2 a_{\sigma(i)}) u_{i\mathbf{p}(i)} b_{\mathbf{p}(i)} \overline{u_{\sigma(i)\mathbf{p}(i)}}$$

$$= \sum_{\substack{S\subseteq[d]\\|S|=k}} \sum_{\mathbf{p}:S\to[d]} \left(\prod_{i\in S} b_{\mathbf{p}(i)}\right)$$

$$\sum_{\sigma\in\mathrm{Sym}(S)} \mathrm{sgn}(\sigma) \prod_{i\in S}(y_1 a_i + y_2 a_{\sigma(i)}) \prod_{i\in S} u_{i\mathbf{p}(i)} \overline{u_{\sigma(i)\mathbf{p}(i)}} \tag{2.3}$$

and the claim amounts to a straightforward application of Theorem 1.60 and Theorem 1.61, with the observation that $\ell(\lambda) \leq k \leq d$ for all $\lambda \vdash k$. Namely, we have

$$\mathbb{E}_U \left(\prod_{i\in S} u_{i\mathbf{p}(i)} \overline{u_{\sigma(i)\mathbf{p}(i)}}\right) = \sum_{\substack{\pi,\tau\in\mathrm{Sym}(S)\\ 1=\sigma\circ\pi\\ \mathbf{p}=\mathbf{p}\circ\tau}} \mathrm{Wg}^U_{k,d}(\pi^{-1}\tau) \qquad \text{(Theorem 1.60)}$$

$$= \sum_{\substack{\tau\in\mathrm{Sym}(S)\\ \mathbf{p}=\mathbf{p}\circ\tau}} \mathrm{Wg}^U_{k,d}(\sigma\tau)$$

$$= \sum_{\substack{\tau\in\mathrm{Sym}(S)\\ \mathbf{p}=\mathbf{p}\circ\tau}} \frac{1}{(k!)^2} \sum_{\lambda\vdash k} \frac{\dim(\lambda)^2}{\mathsf{s}_\lambda(1^d)} \chi^\lambda(\sigma\tau)$$

(Theorem 1.61, $\ell(\lambda) \leq k \leq d$ for all $\lambda \vdash k$)

so the expectation of (2.3) is

$$\sum_{\substack{S\subseteq[d]\\|S|=k}} \sum_{\mathbf{p}:S\to[d]} \left(\prod_{i\in S} b_{\mathbf{p}(i)}\right) \sum_{\sigma\in\mathrm{Sym}(S)} \mathrm{sgn}(\sigma) \prod_{i\in S}(y_1 a_i + y_2 a_{\sigma(i)})$$

$$\sum_{\substack{\tau\in\mathrm{Sym}(S)\\ \mathbf{p}=\mathbf{p}\circ\tau}} \frac{1}{(k!)^2} \sum_{\lambda\vdash k} \frac{\dim(\lambda)^2}{\mathsf{s}_\lambda(1^d)} \chi^\lambda(\sigma\tau)$$

$$= \frac{1}{(k!)^2} \sum_{\lambda \vdash k} \frac{\dim(\lambda)^2}{\mathsf{s}_\lambda(1^d)} \sum_{\substack{S \subseteq [d] \\ |S|=k}} \sum_{\mathbf{p}:S \to [d]} \left( \prod_{i \in S} b_{\mathbf{p}(i)} \right)$$

$$\sum_{\sigma \in \mathrm{Sym}(S)} \mathrm{sgn}(\sigma) \left( \sum_{\substack{\tau \in \mathrm{Sym}(S) \\ \mathbf{p}=\mathbf{p}\circ\tau}} \chi^\lambda(\sigma\tau) \right) \prod_{i \in S} (y_1 a_i + y_2 a_{\sigma(i)})$$

hence the claim. □

**Remark 2.17.** When $\mathbf{p}$ is injective, and for the sake of clarity we take $S = \{1, \ldots, k\}$, we have

$$\sum_{\sigma \in \mathrm{Sym}(S)} \mathrm{sgn}(\sigma) \left( \sum_{\substack{\tau \in \mathrm{Sym}(S) \\ \mathbf{p}=\mathbf{p}\circ\tau}} \chi^\lambda(\sigma\tau) \right) \prod_{i \in S} (y_1 a_i + y_2 a_{\sigma(i)})$$

$$= \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \chi^\lambda(\sigma) \prod_{i=1}^{k} (y_1 a_i + y_2 a_{\sigma(i)})$$

$$= \sum_{\sigma \in S_k} \chi^{\lambda^T}(\sigma) \prod_{i=1}^{k} (y_1 a_i + y_2 a_{\sigma(i)})$$

which can be immediately recognized as the immanant $\mathrm{Imm}^{\lambda^T}(y_1 a_i + y_2 a_j)_{i,j}$.

To separate the dependence on $A$ from the dependence on $B$, in Lemma 2.16, the sum over $\mathbf{p}$ can be processed as follows:

$$\frac{1}{(k!)^2} \sum_{\lambda \vdash k} \frac{\dim(\lambda)^2}{\mathsf{s}_\lambda(1^d)} \sum_{\substack{S \subseteq [d] \\ |S|=k}}$$

$$\sum_{\mathbf{p}:S \to [d]} \sum_{\sigma \in \mathrm{Sym}(S)} \mathrm{sgn}(\sigma) \left( \sum_{\substack{\tau \in \mathrm{Sym}(S) \\ \mathbf{p}=\mathbf{p}\circ\tau}} \chi^\lambda(\sigma\tau) \right) \prod_{i \in S} (y_1 a_i + y_2 a_{\sigma(i)})$$

$$\prod_{i \in S} b_{\mathbf{p}(i)}$$

$$= \frac{1}{(k!)^2} \sum_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq d}} \frac{\dim(\lambda)^2}{\mathsf{s}_\lambda(1^d)} \sum_{\substack{S \subseteq [d] \\ |S|=k}}$$

$$\sum_{\pi\in P(S)}\left(\sum_{\sigma\in\mathrm{Sym}(S)}\prod_{i\in S}(y_1a_i+y_2a_{\sigma(i)})\left(\mathrm{sgn}(\sigma)\sum_{\substack{\tau\in\mathrm{Sym}(S)\\ \tau\leq\pi}}\chi^\lambda(\sigma\tau)\right)\right)$$
$$\left(\sum_{\substack{\mathbf{p}:S\to[d]\\ \ker(\mathbf{p})=\pi}}\prod_{i\in S}b_{\mathbf{p}(i)}\right)$$

The sum over $\mathbf{p}$ in the last line only depends on $\pi$ through the sizes of its blocks:

**Lemma 2.18.** *Let $\mu\vdash k$ and pick $\pi\in P(k)$ with $t(\pi)=\mu$. Then*

$$\sum_{\substack{\mathbf{p}:[k]\to[d]\\ \ker(\mathbf{p})=\pi}}\prod_{i=1}^{k}b_{\mathbf{p}(i)}=\frac{\ell(\mu)!}{|\mathrm{Orb}(\mu)|}\mathsf{m}_\mu(B)$$

*where $|\mathrm{Orb}(\mu)|$ is the number of distinct permutations of $\mu$.*

*Proof.* If $\pi=\{V_1,\dots,V_m\}$, then

$$\begin{aligned}\sum_{\substack{\mathbf{p}:[k]\to[d]\\ \ker(\mathbf{p})=\pi}}\prod_{i=1}^{k}b_{\mathbf{p}(i)}&=\sum_{\substack{\mathbf{p}:\pi\to[d]\\ \text{injective}}}\prod_{V\in\pi}b_{\mathbf{p}(V)}^{|V|}\\&=\sum_{\rho\in S_m}\sum_{\substack{\mathbf{p}:[m]\to[d]\\ \mathbf{p}(\rho(1))<\cdots<\mathbf{p}(\rho(m))}}b_{\mathbf{p}(1)}^{|V_1|}\cdots b_{\mathbf{p}(m)}^{|V_m|}\\&=\sum_{\rho\in S_m}\sum_{\substack{\mathbf{p}:[m]\to[d]\\ \mathbf{p}(1)<\cdots<\mathbf{p}(m)}}b_{\mathbf{p}(1)}^{|V_{\rho(1)}|}\cdots b_{\mathbf{p}(m)}^{|V_{\rho(m)}|}\end{aligned}$$

and the number of duplicate summands $b_{\mathbf{p}(1)}^{|V_{\rho(1)}|}\cdots b_{\mathbf{p}(m)}^{|V_{\rho(m)}|}$ which accumulate, for each $\mathbf{p}$, as $\rho$ varies over $S_m$, is the number of permutations in $S_m$ which fix $\mu$. So

$$|\mathrm{Stab}(\mu)|\sum_{I\in\mathrm{Orb}(\mu)}\sum_{\substack{\mathbf{p}:[m]\to[d]\\ \mathbf{p}(1)<\cdots<\mathbf{p}(m)}}b_{\mathbf{p}(1)}^{I_1}\cdots b_{\mathbf{p}(m)}^{I_m}=\frac{m!}{|\mathrm{Orb}(\mu)|}\sum_{I\in\mathrm{Orb}(\mu)}\mathsf{M}_I(B)$$

$$= \frac{\ell(\mu)!}{|\mathrm{Orb}(\mu)|}\mathsf{m}_\mu(B)$$

by the orbit-stabilizer theorem. □

**Remark 2.19.** The case $\mu = 2^q_k$ will be important later: there are $\binom{k-q}{q}$ distinct permutations of

$$(\underbrace{\overbrace{2,\ldots,2}^{q},1,\ldots,1}_{k-q})$$

so the multiple in Lemma 2.18 is $q!(k-2q)!$.

The above makes

$$\mathbb{E}_U \mathsf{e}_k(y_1AUBU^* + y_2UBU^*A) = \frac{1}{(k!)^2}\sum_{\substack{\lambda\vdash k\\ \ell(\lambda)\le d}} \frac{\dim(\lambda)^2}{\mathsf{s}_\lambda(1^d)} \sum_{\substack{S\subseteq[d]\\ |S|=k}}$$

$$\sum_{\mu\vdash k}\left(\sum_{\sigma\in\mathrm{Sym}(S)}\prod_{i\in S}(y_1a_i + y_2a_{\sigma(i)})\left(\mathrm{sgn}(\sigma)\sum_{\substack{\pi\in P(k)\\ t(\pi)=\mu}}\sum_{\substack{\tau\in\mathrm{Sym}(S)\\ \tau\le\pi}}\chi^\lambda(\sigma\tau)\right)\right) \tag{2.4}$$

$$\frac{\ell(\mu)!}{|\mathrm{Orb}(\mu)|}\mathsf{m}_\mu(B)$$

and to reach the central point of the argument, one must process the sum

$$\mathrm{sgn}(\sigma)\sum_{\substack{\pi\in P(k)\\ t(\pi)=\mu}}\sum_{\substack{\tau\in S_k\\ \tau\le\pi}}\chi^\lambda(\sigma\tau)$$

in a way which makes the bracketed portion of (2.4) into a sum of immanants. This will be done in Section 2.3.1:

**Proposition 2.20.** *For $\lambda,\mu\vdash k$, there is a constant $C_{\lambda,\mu}$ such that*

$$\sum_{\substack{\pi\in P(k)\\ t(\pi)=\mu}}\sum_{\substack{\tau\in S_k\\ \tau\le\pi}}\chi^\lambda(\sigma\tau) = C_{\lambda,\mu}\chi^\lambda(\sigma).$$

*with the following properties:*

1. *if* $\mu \ntrianglelefteq \lambda$, *then* $C_{\lambda,\mu} = 0$;

2. *if* $\lambda = 2_k^p$ *and* $\mu = 2_k^q$ *with* $0 \le q \le p \le \lfloor k/2 \rfloor$, *then*

$$C_{\lambda,\mu} = \frac{p!}{(p-q)!}\binom{k-p+1}{q}.$$

With Proposition 2.20 in hand, the bracketed portion of (2.4) can be realized as an immanant: it is equal to

$$\sum_{\sigma \in \mathrm{Sym}(S)} C_{\lambda,\mu}\mathrm{sgn}(\sigma)\chi^\lambda(\sigma)\prod_{i\in S}(y_1 a_i + y_2 a_{\sigma(i)}) = C_{\lambda,\mu}\mathrm{Imm}^{\lambda^T}(y_1 a_i + y_2 a_j)_{i,j\in S}.$$

Putting all of this together, we have

$$\begin{aligned}
&\mathbb{E}_U \mathsf{e}_k(y_1 AUBU^* + y_2 UBU^* A) \\
&= \frac{1}{(k!)^2}\sum_{\lambda\vdash k}\frac{\dim(\lambda)^2}{\mathsf{s}_\lambda(1^d)} \\
&\quad \sum_{\substack{S\subseteq[d]\\|S|=k}}\sum_{\substack{\mu\vdash k\\ \mu\trianglelefteq\lambda}} C_{\lambda,\mu}\mathrm{Imm}^{\lambda^T}(y_1 a_i + y_2 a_j)_{i,j\in S}\left(\frac{\ell(\mu)!}{|\mathrm{Orb}(\mu)|}\mathsf{m}_\mu(B)\right) \\
&= \frac{1}{(k!)^2}\sum_{\substack{\lambda\vdash k\\ \ell(\lambda^T)\le 2}}\frac{\dim(\lambda)^2}{\mathsf{s}_\lambda(1^d)} \\
&\quad \left(\sum_{\substack{S\subseteq[d]\\|S|=k}}\mathrm{Imm}^{\lambda^T}(y_1 a_i + y_2 a_j)_{i,j\in S}\right)\left(\sum_{\substack{\mu\vdash k\\ \mu\trianglelefteq\lambda}} C_{\lambda,\mu}\frac{\ell(\mu)!}{|\mathrm{Orb}(\mu)|}\mathsf{m}_\mu(B)\right) \\
&= \frac{1}{(k!)^2}\sum_{0\le p\le\lfloor k/2\rfloor}\frac{\dim(2_k^p)^2}{\mathsf{s}_{2_k^p}(1^d)} \\
&\quad \left(\sum_{\substack{S\subseteq[d]\\|S|=k}}\mathrm{Imm}^{(k-p,p)}(y_1 a_i + y_2 a_j)_{i,j\in S}\right)\left(\sum_{0\le q\le p} C_{2_k^p,2_k^q}\, q!(k-2q)!\mathsf{m}_{2_k^q}(B)\right)
\end{aligned}$$

and the problem is reduced to

1. the computation of
$$\mathrm{Imm}^\lambda(y_1x_i + y_2x_j)_{i,j}$$
for $\ell(\lambda) \le 2$, where $x_1, \dots, x_k$ are some commuting formal variables, and

2. handling the sum of the above immanants over choices of $k$ diagonal entries from $A$.

### 2.3.1 Conjugates of Young subgroups

In this subsection we will prove Proposition 2.20, which amounts to the following:

**Proposition 2.20′.** *Let $\lambda, \mu \vdash k$. Then*

$$\sum_{\substack{\pi \in P(k)\\ t(\pi)=\mu}} \sum_{\substack{\tau \in S_k\\ \tau \le \pi}} \rho^\lambda(\tau) = \frac{p_\mu |S_\mu| K(\lambda,\mu)}{\dim(\lambda)} \cdot 1$$

*where $p_\mu$ is the number of partitions $\pi \in P(k)$ with $t(\pi) = \mu$.*

The idea is that the sum is averaging over each conjugate of $S_\mu$, and then adding up all the conjugate-subgroup-sums, yielding a central element of $\mathbb{C}[S_k]$. Schur's lemma gives the scalar multiples, and then one can compute them as needed using the specifics of $S_k$. An important part of the argument is clarified by working with finite groups in general:

**Notation 2.21.** Fix a finite group $G$ and a subgroup $H$, and let

$$\{g_j H g_j^{-1} : 1 \le j \le n\}$$

be the distinct conjugates of $H$, writing $H_j := g_j H g_j^{-1}$. Fix a representation $\rho : G \to \mathrm{GL}(V)$ and write $\mathrm{Res}^G_H(\rho) = \bigoplus_{i=1}^m \rho_i$ where $\rho_1, \dots, \rho_m$ are irreducible representations of $H$.

The first general fact is that the restriction functor is invariant, up to natural isomorphism, under conjugation of subgroups:

**Lemma 2.22.** *For $g \in G$, there is an isomorphism*

$$\eta_g : \mathrm{Res}^G_H(\rho) \simeq \mathrm{Res}^G_{gHg^{-1}}(\rho).$$

*Proof.* Fix $g \in G$ and let $\rho : G \to \mathrm{GL}(V)$ be a representation of $G$. Define $\eta_g(\rho) : V \to V$ by $\eta_g(\rho)v = \rho(g)v$ for $v \in V$, which is an isomorphism of vector spaces. Moreover, for $h \in H$,

$$\begin{aligned}
\rho(ghg^{-1})\eta_g(\rho)v &= \rho(ghg^{-1})\rho(g)v \\
&= \rho(gh)v \\
&= \rho(g)\rho(h)v \\
&= \eta_g(\rho)\rho(h)v
\end{aligned}$$

for $v \in V$, so $\eta_g(\rho)$ intertwines $\mathrm{Res}^G_H(\rho)$ and $\mathrm{Res}_{gHg^{-1}}(\rho)$. □

The second general fact is that averaging a representation over a subgroup yields a projection which encodes the occurrence of the trivial representation in the restriction:

**Lemma 2.23.** *In the block-matrix decomposition with respect to $\bigoplus_{i=1}^m V_i$,*

$$\frac{1}{|H|}\sum_{h\in H}\rho(h) = \begin{pmatrix} \delta_{\rho_1=\mathrm{triv}}\cdot 1 & & 0 \\ & \ddots & \\ 0 & & \delta_{\rho_m=\mathrm{triv}}\cdot 1 \end{pmatrix}.$$

*Proof.* Clearly $\sum_{h\in H}\rho(h) \in \mathrm{End}_H(\rho)$, so in the block-matrix decomposition

$$\sum_{h\in H}\rho(h) = \begin{pmatrix} T_{11} & \cdots & T_{1m} \\ \vdots & \ddots & \vdots \\ T_{m1} & \cdots & T_{mm} \end{pmatrix}$$

we have $T_{ij} \in \mathrm{Hom}_H(\rho_j, \rho_i)$, and then by Schur's lemma we have

$$\sum_{h\in H}\rho(h) = \begin{pmatrix} t_1\cdot 1 & & 0 \\ & \ddots & \\ 0 & & t_m\cdot 1 \end{pmatrix}$$

for some scalars $t_1, \ldots, t_m$. Write $\chi_i$ for the character of $\rho_i$, so

$$\langle 1, \chi_i \rangle = \frac{1}{|H|} \sum_{h \in H} \mathrm{Tr}(\rho_i(h)) = \frac{1}{|H|} \mathrm{Tr}(t_i \cdot 1) = \frac{1}{|H|} t_i \dim(\rho_i)$$

and by the orthogonality relations we have $t_i = \delta_{\rho_i = \mathrm{triv}} \cdot |H|$. □

The final general fact combines the previous two:

**Lemma 2.24.** *The element*

$$\sum_{j=1}^{n} \sum_{h \in H_j} h \in \mathbb{C}[G]$$

*of the group algebra is central. If $\rho$ is irreducible, then*

$$\sum_{j=1}^{n} \sum_{h \in H_j} \rho(h) = \frac{n|H| \mathrm{mult}(\mathrm{triv}, \mathrm{Res}_H^G(\rho))}{\dim(\rho)} \cdot 1.$$

*Proof.* The set $\{H_1, \ldots, H_n\}$ is permuted by elements of $G$ acting by conjugation, so

$$\begin{aligned} g \left( \sum_{j=1}^{n} \sum_{h \in H_j} h \right) g^{-1} &= \sum_{j=1}^{n} \sum_{h \in H_j} g h g^{-1} \\ &= \sum_{j=1}^{n} \sum_{h \in g H_j g^{-1}} h \\ &= \sum_{j=1}^{n} \sum_{h \in H_j} h \end{aligned}$$

for $g \in G$, which is the first claim. If $\rho$ is irreducible, then by Schur's lemma,

$$\sum_{j=1}^{n} \sum_{h \in H_j} h = t \cdot 1$$

for some $t \in \mathbb{C}$. To find $t$, observe that

$$\dim(\rho) t = \mathrm{Tr} \left( \sum_{j=1}^{n} \sum_{h \in H_j} \rho(h) \right)$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \mathrm{Tr}\left(\sum_{h\in H_j} \rho(h)\right) \\
&= \sum_{j=1}^{n} |H_j| \mathrm{mult}(\mathrm{triv}, \mathrm{Res}^G_{H_j}(\rho)) \\
&= n|H| \mathrm{mult}(\mathrm{triv}, \mathrm{Res}^G_H(\rho))
\end{aligned}
$$

so the claim follows. □

Finally, let us specialize to the symmetric group $S_k$. Our key point in this argument is that the constraint $\tau \leq \pi$ in the sum

$$\sum_{\substack{\tau\in S_k \\ \tau\leq\pi}} \chi^\lambda(\sigma\tau)$$

is actually carving out conjugates of the Young subgroup:

**Notation 2.25** (Young subgroup conjugates)**.** For $\pi \in P(k)$, write $S_\pi$ for the subgroup of $S_k$ consisting of the permutations for which the blocks of $\pi$ are invariant. Clearly, if $\pi = \{V_1, \ldots, V_m\}$, then

$$S_\pi \simeq S_{|V_1|} \times \cdots \times S_{|V_m|}$$

and the right-hand side is the Young subgroup corresponding to the composition $(|V_1|, \ldots, |V_m|)$ of $k$, but the notation $S_\pi$ retains some more information about the blocks of $\pi$.

*Proof of Proposition 2.20.* In light of Lemma 2.24, the remaining tasks are to count

1. $\{\pi \in P(k) : t(\pi) = 2^q_k\}$,
2. the order of $S_{2^q_k}$,
3. the multiplicity of the trivial representation in $\mathrm{Res}^{S_k}_{S_\pi}(V^\lambda)$, and
4. the dimension of $V^{2^p_k}$;

in particular, we want the multiplicity in (3) to be 0 whenever $\mu \not\trianglelefteq \lambda$. There is a well-known formula for (1), reproduced in e.g. [2, Lemma 2.4], and the case of $2^q_k$ comes out as $\frac{k!}{2^q q!(k-2q)!}$. For (2), we already know $S_{2^q_k} \simeq \mathbf{Z}_2^q$ which has order $2^q$. For (3) and (4), we appeal to Theorem 1.35 and Proposition 1.10 respectively. □

## 2.4 Finite free commutator

In this section, we will look at

$$\mathbb{E}_U c_x(AUBU^* - UBU^*A)$$

which is essentially a special case of the last section: when $A$ and $B$ are self-adjoint, this is the case $y_1 = 1$ and $y_2 = -1$. We do not actually require self-adjointness, however, to work with the commutator. To describe our results we need a bit of notation related to $\boxplus_d$:

**Notation 2.26.** For polynomials $p(x)$ and $q(x)$ with degree $d$, write

$$p(x) \boxminus_d q(x) = \mathbb{E}_U c_x(A - UBU^*)$$

for the operation of "subtraction" with respect to $\boxplus_d$.

Here is the main result:

**Theorem 2.27.** *Let $A$ and $B$ be $d \times d$ normal matrices with $p(x) = c_x(A)$ and $q(x) = c_x(B)$. Then*

$$\mathbb{E}_U c_x(AUBU^* - UBU^*A) = (p(x) \boxminus_d p(x)) \boxtimes_d (q(x) \boxminus_d q(x)) \boxtimes_d z_d(x)$$

*where*

$$z_d(x) = \sum_{k=0}^{\lfloor d/2 \rfloor} x^{d-2k} \binom{d}{2k} (d)_k \frac{k!}{(2k)!} \frac{d+1-k}{d+1}.$$

As shown in Section 2.3, the task of computing

$$\mathbb{E}_U c_x(AUBU^* - UBU^*A)$$

can be reduced – at least in principle – to the computation of $\mathrm{Imm}^\lambda(x_i - x_j)_{i,j}$ for $\ell(\lambda) \le 2$. In this particular case, it turns out to be especially simple:

**Proposition 2.28.** *Let $X = \operatorname{diag}(x_1, \ldots, x_k)$ be a $k \times k$ diagonal matrix. Then*

$$\operatorname{Imm}^\lambda(x_i - x_j)_{i,j} = \begin{cases} (-1)^{\lambda_2} \sum_{l=0}^{k} (-1)^l (k-l)! l! \mathsf{e}_{k-l}(X) \mathsf{e}_l(X) & \text{if } \ell(\lambda) \le 2 \\ 0 & \text{otherwise} \end{cases}$$

*for $\lambda \vdash k$.*

We will prove this in Section 2.4.1 by observing that the non-zero immanants actually all have the same value, modulo a sign, so it suffices to just compute the permanant.

Then, the computation of

$$\begin{aligned} &\mathbb{E}_U \mathsf{e}_k(AUBU^* - UBU^*A) \\ &= \frac{1}{(k!)^2} \sum_{0 \le p \le k/2} \frac{\dim(2^p_k)^2}{\mathsf{s}_{2^p_k}(1^d)} \\ &\quad \sum_{\substack{S \subseteq [d] \\ |S| = k}} \sum_{0 \le q \le p} C_{2^p_k, 2^q_k} \left( (-1)^p \sum_{l=0}^{k} (-1)^l (k-l)! l! \mathsf{e}_{k-l}(A_S) \mathsf{e}_l(A_S) \right) \\ &\quad q!(k-2q)! \mathsf{m}_{2^q_k}(B) \\ &= \left( \sum_{l=0}^{k} \frac{(-1)^l}{\binom{k}{l}} \sum_{\substack{S \subseteq [d] \\ |S| = k}} \mathsf{e}_{k-l}(A_S) \mathsf{e}_l(A_S) \right) \qquad (2.5) \\ &\quad \left( \frac{1}{k!} \sum_{0 \le p \le k/2} (-1)^p \frac{\dim(2^p_k)^2}{\mathsf{s}_{2^p_k}(1^d)} \sum_{0 \le q \le p} C_{2^p_k, 2^q_k} q!(k-2q)! \mathsf{m}_{2^q_k}(B) \right) \qquad (2.6) \end{aligned}$$

amounts to some symmetric function computations, to be carried out in Section 2.4.2:

**Proposition 2.29.** *If $k$ is even, then*

1. *the expression (2.5) is equal to*

$$\frac{(k/2)!}{k!} \sum_{i+j=k} (-1)^i \frac{(d-i)!(d-j)!}{(d-k)!(d-k/2)!} \mathsf{e}_i(A) \mathsf{e}_j(A),$$

*and*

*2. the expression* (2.6) *is equal to*

$$k!\frac{d+1-k/2}{(d+1)!d!}\sum_{i+j=k}(-1)^i(d-i)!(d-j)!\mathsf{e}_i(B)\mathsf{e}_j(B).$$

*If $k$ is odd, then the expression* (2.5) *is* 0*.*

*Proof of Theorem 2.27.* All the pieces are in place by now:

$$\begin{aligned}
&\mathbb{E}_U\mathsf{e}_k(AUBU^*-UBU^*A)\\
&=\left(\frac{(k/2)!}{k!}\sum_{i+j=k}(-1)^i\frac{(d-i)!(d-j)!}{(d-k)!(d-k/2)!}\mathsf{e}_i(A)\mathsf{e}_j(A)\right)\\
&\quad\left(k!\frac{d+1-k/2}{(d+1)!d!}\sum_{i+j=k}(-1)^i(d-i)!(d-j)!\mathsf{e}_i(B)\mathsf{e}_j(B)\right)\\
&=(k/2)!\frac{(d-k)!}{(d-k/2)!}\frac{d+1-k/2}{d+1}\left(\sum_{i+j=k}(-1)^i\frac{(d-i)!(d-j)!}{(d-k)!d!}\mathsf{e}_i(A)\mathsf{e}_j(A)\right)\\
&\quad\left(\sum_{i+j=k}(-1)^i\frac{(d-i)!(d-j)!}{(d-k)!d!}\mathsf{e}_i(B)\mathsf{e}_j(B)\right)
\end{aligned}$$

for even $0\leq k\leq d$. For odd $k$, we have

$$\mathbb{E}_U\mathsf{e}_k(AUBU^*-UBU^*A)=0$$

since the expression (2.5) is equal to 0. □

### 2.4.1 Reduction of immanants to permanents

We will prove Proposition 2.28 as a special case of the following general result:

**Theorem 2.30.** *Let $Y$ be a $k\times k$ matrix with rank 2, and write $\alpha$ and $\beta$ for the non-zero eigenvalues of $ZY$. If $\beta=-\alpha$, then*

$$\mathrm{Imm}^\lambda(Y)=\begin{cases}(-1)^{\lambda_2}\mathrm{Per}(Y) & \text{if } \ell(\lambda)\leq 2\\ 0 & \text{otherwise}\end{cases}$$

*for $\lambda\vdash k$.*

**Lemma 2.31.** *We have*

$$\mathsf{s}_\lambda(\alpha, \beta, 0, \dots, 0) = \begin{cases} \alpha^k \frac{\left(\frac{\beta}{\alpha}\right)^{\lambda_2} - \left(\frac{\beta}{\alpha}\right)^{\lambda_1+1}}{1-\frac{\beta}{\alpha}} & \text{if } \ell(\lambda) \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

*for* $\lambda \vdash k$.

*Proof.* If $\ell(\lambda) > 2$, then every semistandard tableau of shape $\lambda$ has $\omega_i(T) > 0$ for some $i > 2$, so

$$\mathsf{s}_\lambda(\alpha, \beta, 0, \dots, 0) = \sum_{T \in \mathrm{SST}(\lambda)} \alpha^{\omega_1(T)} \beta^{\omega_2(T)} 0^{\omega_3(T)} \cdots = 0.$$

On the other hand, if $\ell(\lambda) \leq 2$, the only semistandard tableaux of shape $\lambda$ with $\omega_i(T) = 0$ for all $i > 2$ are of the form

$$\begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1 & \cdots & 1 & 1 & \cdots & 1 & 2 & \cdots & 2 \\ \hline 2 & \cdots & 2 \\ \cline{1-3} \end{array}$$

where the first row has $0 \leq t \leq \lambda_1 - \lambda_2$ boxes with 2s. So

$$\begin{aligned} \mathsf{s}_\lambda(\alpha, \beta, 0, \dots, 0) &= \sum_{t=0}^{\lambda_1-\lambda_2} \alpha^{k-(t+\lambda_2)} \beta^{t+\lambda_2} \\ &= \alpha^{k-\lambda_2} \beta^{\lambda_2} \sum_{t=0}^{\lambda_1-\lambda_2} \left(\frac{\beta}{\alpha}\right)^t \\ &= \alpha^k \left(\frac{\beta}{\alpha}\right)^{\lambda_2} \frac{1 - \left(\frac{\beta}{\alpha}\right)^{\lambda_1-\lambda_2+1}}{1 - \frac{\beta}{\alpha}} \end{aligned}$$

which is the non-zero expression in the claim. □

*Proof of Theorem 2.30.* By Theorem 1.46, $\mathrm{Imm}^\lambda(x_i - x_j)_{i,j}$ is the coefficient of $z_1 \cdots z_k$ in

$$\begin{aligned} \mathsf{s}_\lambda(ZY) &= \mathsf{s}_\lambda(\alpha, \beta, 0, \dots, 0) \\ &= \begin{cases} \alpha^k \frac{\left(\frac{\beta}{\alpha}\right)^{\lambda_2} - \left(\frac{\beta}{\alpha}\right)^{\lambda_1+1}}{1-\frac{\beta}{\alpha}} & \text{if } \ell(\lambda) \leq 2 \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Lemma 2.31)} \end{aligned}$$

$$= \begin{cases} (-1)^{\lambda_2}\alpha^k & \text{if } \ell(\lambda) \le 2 \\ 0 & \text{otherwise} \end{cases} \qquad (\beta = -\alpha)$$

and we observe that the expression $(-1)^{\lambda_2}\alpha^k$ does not depend on $\lambda$ except through the sign. In particular, $\mathrm{Per}(Y)$ is the coefficient of $z_1 \cdots z_k$ in $\alpha^k$, so

$$\mathrm{Imm}^\lambda(Y) = [z_1 \cdots z_k](-1)^{\lambda_2}\alpha^k = (-1)^{\lambda_2}[z_1 \cdots z_k]\alpha^k = (-1)^{\lambda_2}\mathrm{Per}(Y)$$

when $\ell(\lambda) \le 2$, and $\mathrm{Imm}^\lambda(Y) = 0$ otherwise. □

*Proof of Proposition 2.28.* First, we need to show that $Y = (x_i - x_j)_{i,j}$ satisfies the hypothesis that the two non-zero eigenvalues $\alpha, \beta$ of $ZY$ have $\beta = -\alpha$. This is clear: since

$$c_x(ZY) = x^k - \mathrm{Tr}(ZY)x^{k-1} + Ax^{k-2} \text{ and } \mathrm{Tr}(ZY) = \sum_{i=1}^{k} z_i(x_i - x_i) = 0$$

for some constant $A$, we have

$$c_x(ZY) = x^{k-2}(x + i\sqrt{A})(x - i\sqrt{A})$$

hence the eigenvalues are $\pm i\sqrt{A}$. Then, by Theorem 2.30, it suffices to compute $\mathrm{Per}(Y)$:

$$\begin{aligned}
&\mathrm{Per}(x_i - x_j)_{i,j} \\
&= \sum_{\sigma \in S_k} \prod_{i=1}^{k} (x_i - x_{\sigma(i)}) \\
&= \sum_{\sigma \in S_k} \sum_{R \subseteq [k]} (-1)^{|R|} \prod_{i \in R} x_i \prod_{i \notin R} x_{\sigma(i)} \\
&= \sum_{R \subseteq [k]} (-1)^{|R|} \det(X(R,R)) \left( \sum_{\substack{\sigma : [k] \to [k] \\ \text{injective}}} \prod_{i \in [k] \setminus R} x_{\sigma(i)} \right) \\
&= \sum_{R \subseteq [k]} (-1)^{|R|} \det(X(R,R)) |R|! \left( \sum_{\substack{\sigma : [k] \setminus R \to [k] \\ \text{injective}}} \prod_{i \in [k] \setminus R} x_{\sigma(i)} \right)
\end{aligned}$$

$$= \sum_{R \subseteq [k]} (-1)^{|R|} \det(X(R,R)) |R|!(k-|R|)! \left( \sum_{\substack{\sigma:[k]\setminus R \to [k] \\ \text{increasing}}} \prod_{i \in [k]\setminus R} x_{\sigma(i)} \right)$$

$$= \sum_{l=0}^{k} (-1)^l \sum_{\substack{R \subseteq [k] \\ |R|=l}} \det(X(R,R))(k-l)!l! \left( \sum_{1 \le i_1 < \cdots < i_{k-l} \le k} x_{i_1} \cdots x_{i_{k-l}} \right)$$

$$= \sum_{l=0}^{k} (-1)^l (k-l)!l! \mathsf{e}_{k-l}(X) \mathsf{e}_l(X).$$

□

### 2.4.2 Basis transitions

This section is dedicated to the proof of Proposition 2.29. First of all, the claim for odd $k$ is almost trivial: the summands in the expression (2.5) cancel each other out because

$$\frac{(-1)^l}{\binom{k}{l}} \mathsf{e}_{k-l}\mathsf{e}_l = -\frac{(-1)^{k-l}}{\binom{k}{k-l}} \mathsf{e}_l \mathsf{e}_{k-l}$$

for $0 \le l \le k$. So for the rest of this section, $k$ is assumed to be even.

#### Left-hand dependence

The goal of this subsection is to show that

$$\sum_{l=0}^{k} \frac{(-1)^l}{\binom{k}{l}} \sum_{\substack{S \subseteq [d] \\ |S|=k}} \mathsf{e}_{k-l}(A_S)\mathsf{e}_l(A_S)$$
$$= \frac{(k/2)!}{k!} \sum_{i+j=k} (-1)^i \frac{(d-i)!(d-j)!}{(d-k)!(d-k/2)!} \mathsf{e}_i(A)\mathsf{e}_j(A)$$

when $k$ is even. Observe that

$$\sum_{\substack{S \subseteq [d] \\ |S|=k}} \mathsf{e}_{k-l}(A_S)\mathsf{e}_l(A_S) = \sum_{1 \le s_1 < \cdots < s_k \le d} \sum_{\substack{\mathbf{i}:[k]\to[k] \\ \mathbf{i}(1)<\cdots<\mathbf{i}(k-l) \\ \mathbf{i}(k-l+1)<\cdots<\mathbf{i}(k)}} a_{s_{\mathbf{i}(1)}} \cdots a_{s_{\mathbf{i}(k)}}$$

$$= \sum_{\substack{\mathbf{i}:[k]\to[k] \\ \mathbf{i}(1)<\cdots<\mathbf{i}(k-l) \\ \mathbf{i}(k-l+1)<\cdots<\mathbf{i}(k)}} \sum_{1\le s_1<\cdots<s_k\le d} a_{s_1}^{|\mathbf{i}^{-1}(1)|}\cdots a_{s_k}^{|\mathbf{i}^{-1}(k)|}$$

$$= \sum_{\substack{\mathbf{i}:[k]\to[k] \\ \mathbf{i}(1)<\cdots<\mathbf{i}(k-l) \\ \mathbf{i}(k-l+1)<\cdots<\mathbf{i}(k)}} \mathsf{M}_{|\mathbf{i}^{-1}|}(a_1,\ldots,a_d)$$

which leads to the first piece of the proof:

**Lemma 2.32.** *The expression* (2.5) *is equal to*

$$\sum_{q=0}^{k/2} \mathsf{m}_{2_k^q}(A)\left(\frac{\binom{d-(k-q)}{q}}{\binom{k-q}{q}}\sum_{l=q}^{k-q}\frac{(-1)^l}{\binom{k}{l}}\binom{k-q}{l}\binom{l}{q}\right).$$

To prove this, let us set up some more notation:

**Notation 2.33.** Write

$$\mathrm{C}_m(k,l) := \{\mathbf{i}:[k]\to[m] : \mathbf{i}(1)<\cdots<\mathbf{i}(k-l) \text{ and } \mathbf{i}(k-l+1)<\cdots<\mathbf{i}(k)\}$$

for $0\le l\le k$. For $\mathbf{i}\in \mathrm{C}_m(k,l)$, define a weak composition $I(\mathbf{i})\in \mathrm{WComp}(m)$ by $I(\mathbf{i}) := (|\mathbf{i}^{-1}(1)|,\ldots,|\mathbf{i}^{-1}(k)|)$.

**Lemma 2.34.** *Let* $0\le q\le \frac{k}{2}$. *Then*

1. $|\mathrm{Orb}(2^q,1^{k-2q},0^q)| = \binom{k}{q}\binom{k-q}{q}$*;*
2. *for* $0\le l\le k$*, we have*

$$|\{\mathbf{i}\in \mathrm{C}_k(k,l) : I(\mathbf{i})\in \mathrm{Orb}(2^q,1^{k-2q},0^q)\}|$$
$$= \begin{cases} \binom{k}{l}\binom{k-l}{q}\binom{l}{q} & \text{if } q\le l\le k-q \\ 0 & \text{otherwise}\end{cases}.$$

*Proof.* For (1), the distinct permutations of

$$(\underbrace{2,\ldots,2}_{q},\underbrace{1,\ldots,1}_{k-2q},\underbrace{0,\ldots,0}_{q})$$

are determined by placing $q$ 2s in $k$ available entries, then placing $q$ 0s in the remaining $k-q$ available entries; the $k-2q$ 1s are then forced into the remaining $k-2q$ entries. There are of course $\binom{k}{q}\binom{k-q}{q}$ ways of doing this.

For (2), to build a $\mathbf{i}$ with $I(\mathbf{i})$ a permutation of $(2^q, 1^{k-2q}, 0^q)$, one may proceed as follows:

- start with a chain $\mathbf{i}(1) < \cdots < \mathbf{i}(k-l)$;
- choose $q$ values of the above, which will be duplicated;
- choose which of $\mathbf{i}(k-l+1), \ldots, \mathbf{i}(k)$ will be used for the duplication.

There are $\binom{k}{l}$ choices for the first, $\binom{k-l}{q}$ choices for the second, and $\binom{l}{q}$ choices for the third, hence the claim. $\square$

*Proof of Lemma 2.32.* With Notation 2.33, the expression (2.5) is equal to

$$\begin{aligned}
&\sum_{l=0}^{k} \frac{(-1)^l}{\binom{k}{l}} \sum_{\mathbf{i}\in \mathrm{C}_k(k,l)} \mathsf{M}_{I(\mathbf{i})}(A) \\
&= \sum_{l=0}^{k} \frac{(-1)^l}{\binom{k}{l}} \sum_{q=0}^{k/2} \sum_{I\in \mathrm{Orb}(2^q,1^{k-2q},0^q)} |\{\mathbf{i}\in \mathrm{C}_k(k,l) : I(\mathbf{i}) = I\}| \mathsf{M}_I(A) \\
&= \sum_{q=0}^{k/2} \sum_{I\in \mathrm{Orb}(2^q,1^{k-2q},0^q)} \left( \sum_{l=0}^{k} \frac{(-1)^l}{\binom{k}{l}} |\{\mathbf{i}\in \mathrm{C}_k(k,l) : I(\mathbf{i}) = I\}| \right) \mathsf{M}_I(A) \qquad (2.7) \\
&= \sum_{q=0}^{k/2} \sum_{I\in \mathrm{Orb}(2^q,1^{k-2q},0^q)} \mathsf{M}_I(A) \\
&\qquad \left( \frac{1}{\binom{k}{q}\binom{k-q}{q}} \sum_{l=q}^{k-q} \frac{(-1)^l}{\binom{k}{l}} \binom{k}{l}\binom{k-l}{q}\binom{l}{q} \right). \qquad \text{(Lemma 2.34)}
\end{aligned}$$

It is easy to see that

$$\frac{1}{\binom{k}{q}\binom{k-q}{q}} \sum_{l=q}^{k-q} \frac{(-1)^l}{\binom{k}{l}} \binom{k}{l}\binom{k-l}{q}\binom{l}{q} = \frac{1}{\binom{k-q}{q}} \sum_{l=q}^{k-q} \frac{(-1)^l}{\binom{k}{l}} \binom{k-q}{l}\binom{l}{q}$$

by pushing around some factorials, so the remaining task is to show that

$$\sum_{I \in \mathrm{Orb}(2^q, 1^{k-2q}, 0^q)} \mathsf{M}_I(A) = \binom{d-(k-q)}{q} \mathsf{m}_{2_k^q}(A).$$

To this end, recall the definition

$$\mathsf{m}_{2_k^q}(A) = \sum_{J \in \mathrm{Orb}(2^q, 1^{k-2q}, 0^{d-(k-q)})} a_1^{J_1} \cdots a_d^{J_d},$$

i.e. we add zeros to $2_k^q$ as "padding" in case its length is less than $d$. On the other hand, we can write

$$\begin{aligned} &\sum_{I \in \mathrm{Orb}(2^q, 1^{k-2q}, 0^q)} \mathsf{M}_I(A) \\ = &\sum_{I \in \mathrm{Orb}(2^q, 1^{k-2q}, 0^q)} \sum_{1 \le s_1 < \cdots < s_k \le d} a_{s_1}^{I_1} \cdots a_{s_k}^{I_k} \\ = &\sum_{I \in \mathrm{Orb}(2^q, 1^{k-2q}, 0^q)} \sum_{1 \le s_1 < \cdots < s_k \le d} a_1^0 \cdots a_{s_1-1}^0 a_{s_1}^{I_1} a_{s_1+1}^0 \cdots a_{s_k-1}^0 a_{s_k}^{I_k} a_{s_k+1}^0 \cdots a_d^0 \end{aligned}$$

so each summand is of the form $a_1^{J_1} \cdots a_d^{J_d}$ with

$$J = (0, \dots, 0, \underbrace{I_1}_{s_1}, 0, \dots, 0, \underbrace{I_k}_{s_k}, 0, \dots, 0) \in \mathrm{Orb}(2^q, 1^{k-2q}, 0^{d-(k-q)}).$$

Every $J \in \mathrm{Orb}(2^q, 1^{k-2q}, 0^{d-(k-q)})$ arises as such, in $\binom{d-(k-q)}{q}$ ways, since an element of the preimage is the same as a choice of $q$ 0s to keep from the $d-(k-q)$ 0s in $J$. □

For the remainder of the proof, we will require two identities of binomial coefficients, which can be found in e.g. [22]. In these identities, $y$ is a formal variable.

**Lemma 2.35** ([22, 4.8])**.** *We have*

$$\sum_{s=0}^{2n} (-1)^s \frac{\binom{2n}{s}}{\binom{2n+2y}{s+y}} = \frac{\binom{2n}{n}}{\binom{y+n}{n}\binom{2y+2n}{y+n}}$$

*for $n \geq 1$.*

**Lemma 2.36** (Rothe-Hagen identity [22, 3.146]). *We have*

$$\sum_{s=0}^{n} \frac{n}{n+s}\binom{n+s}{s}\binom{y-s}{n-s} = \binom{n+y}{n}$$

*for $n \geq 1$.*

The right-hand side of Lemma 2.32 can be re-arranged as

$$\begin{aligned}
&\sum_{q=0}^{k/2} \mathsf{m}_{2^q_k}\left(\frac{\binom{d-(k-q)}{q}}{\binom{k-q}{q}}\sum_{l=q}^{k-q}\frac{(-1)^l}{\binom{k}{l}}\binom{k-q}{l}\binom{l}{q}\right)\\
&= \sum_{0\leq q\leq k/2} \mathsf{m}_{2^q_k}\binom{d-k+q}{q}\left(\sum_{q\leq l\leq k-q}(-1)^l\frac{\binom{k-2q}{l-q}}{\binom{k}{l}}\right)\\
&= \sum_{0\leq q\leq k/2} \mathsf{m}_{2^q_k}\binom{d-k+q}{q}\left((-1)^q\frac{\binom{k-2q}{k/2-q}}{\binom{k/2}{q}\binom{k}{k/2}}\right) \qquad \text{(Lemma 2.35)}\\
&= \frac{(k/2)!}{k!}\sum_{0\leq q\leq k/2}(-1)^q \mathsf{m}_{2^q_k}(d-k+q)_q\frac{(k-2q)!}{(k/2-q)!}
\end{aligned}$$

and one can apply Eq. (1.2): the above is equal to

$$\begin{aligned}
&\frac{(k/2)!}{k!}\sum_{0\leq r\leq k/2-1}(-1)^r \mathsf{e}_{(k-r,r)}\frac{(d-k+r)!}{(d-k)!}\\
&\quad\sum_{r\leq q\leq k/2-1}(d-k+q)\cdots(d-k+r+1)\frac{(k-2r)(k-q-r-1)!}{(q-r)!(k/2-q)!}\\
&\quad+\frac{(k/2)!}{k!}(d-k/2)_{k/2}\sum_{i+j=k}(-1)^i\mathsf{e}_i\mathsf{e}_j\\
&=\frac{(k/2)!}{k!}\sum_{0\leq r\leq k/2-1}(-1)^r \mathsf{e}_{(k-r,r)}\frac{(d-k+r)!}{(d-k)!}\\
&\quad\sum_{r\leq q\leq k/2-1}\binom{d-k+q}{q-r}\frac{(k-2r)(k-q-r-1)!}{(k/2-q)!} \qquad (2.8)\\
&\quad+\frac{(k/2)!}{k!}\frac{(d-k/2)!}{(d-k)!}\sum_{i+j=k}(-1)^i\mathsf{e}_i\mathsf{e}_j.
\end{aligned}$$

By Lemma 2.36 with $n = k/2 - r$ and $y = d - k/2$, the expression (2.8) is equal to

$$2\frac{(d-r)!}{(d-k/2)!} - 2\frac{(d-k/2)!}{(d-k+r)!}$$

so the expression (2.5) is equal to

$$\begin{aligned}
&\frac{(k/2)!}{k!}\left(2\sum_{0\le r\le k/2-1}(-1)^r \mathsf{e}_{(k-r,r)}\frac{(d-r)!(d-k+r)!}{(d-k)!(d-k/2)!}\right.\\
&\quad\left.-2\frac{(d-k/2)!}{(d-k)!}\sum_{0\le r\le k/2-1}(-1)^r\mathsf{e}_{(k-r,r)} + \frac{(d-k/2)!}{(d-k)!}\sum_{i+j=k}(-1)^i\mathsf{e}_i\mathsf{e}_j\right)\\
&=\frac{(k/2)!}{k!}\left(2\sum_{0\le r\le k/2-1}(-1)^r\mathsf{e}_{(k-r,r)}\frac{(d-r)!(d-k+r)!}{(d-k)!(d-k/2)!}\right.\\
&\quad\left.+(-1)^{k/2}\frac{(d-k/2)!}{(d-k)!}\mathsf{e}_{(k/2,k/2)}\right)\\
&=\frac{(k/2)!}{k!}\sum_{i+j=k}(-1)^i\frac{(d-i)!(d-j)!}{(d-k)!(d-k/2)!}\mathsf{e}_i\mathsf{e}_j
\end{aligned}$$

hence the claim of (1) in Proposition 2.29.

### Right-hand dependence

The remaining part of Proposition 2.29 is the basis transition

$$\begin{aligned}
&\frac{1}{k!}\sum_{0\le p\le k/2}(-1)^p\frac{\dim(2^p_k)^2}{\mathsf{s}_{2^p_k}(1^d)}\sum_{0\le q\le p}C_{2^p_k,2^q_k}q!(k-2q)!\mathsf{m}_{2^q_k}(B)\\
&=k!\frac{d+1-k/2}{(d+1)!d!}\sum_{i+j=k}(-1)^i(d-i)!(d-j)!\mathsf{e}_i(B)\mathsf{e}_j(B)
\end{aligned}$$

which is much more straightforward to prove than the previous one. The left-hand side is

$$\frac{1}{k!}\sum_{0\le q\le p\le k/2}(-1)^p\frac{\dim(2^p_k)^2}{\mathsf{s}_{2^p_k}(1^d)}\frac{p!}{(p-q)!}\binom{k-p+1}{q}q!(k-2q)!\mathsf{m}_{2^q_k}$$

(Proposition 2.20)

$$= k! \sum_{0\le q\le p\le k/2} (-1)^p \frac{(d+1-p)!(d-(k-p))!}{(d+1)!d!} \frac{(k-2q)!(k-2p+1)}{(p-q)!(k-p-q+1)!} \mathsf{m}_{2^q_k}$$

(Proposition 1.10)

$$= k! \sum_{0\le q\le k/2} \mathsf{m}_{2^q_k}$$
$$\sum_{q\le p\le k/2} (-1)^p \frac{(d+1-p)!(d-(k-p))!}{(d+1)!d!} \left(\binom{k-2q}{p-q} - \binom{k-2q}{p-q-1}\right) \tag{2.9}$$

and with $Q_p(d) := \frac{(d+1-p)!(d-(k-p))!}{(d+1)!d!}$, the expression (2.9) is equal to

$$\sum_{q\le p\le k/2} (-1)^p \binom{k-2q}{p-q} (Q_p(d) + Q_{p+1}(d))$$

where for the sake of notation we say $Q_{k/2+1}(d) = 0$. Then, for $q \le p \le k/2 - 1$, we have

$$Q_p(d) + Q_{p+1}(d) = 2\frac{d+1-k/2}{(d+1)!d!}(d-p)!(d-(k-p))!$$

and

$$Q_{k/2}(d) = \frac{(d+1-k/2)!(d-k/2)!}{(d+1)!d!} = \frac{d+1-k/2}{(d+1)!d!}(d-k/2)!(d-k/2)!.$$

Putting this back into (2.9), we get

$$k! \sum_{0\le q\le k/2} \mathsf{m}_{2^q_k} \sum_{q\le p\le k/2} (-1)^p \binom{k-2q}{p-q} (Q_p(d) + Q_{p+1}(d))$$
$$= k! \frac{d+1-k/2}{(d+1)!d!} \sum_{0\le p\le k/2} (-1)^p (d-p)!(d-(k-p))! \sum_{0\le q\le p} \binom{k-2q}{p-q} \mathsf{m}_{2^q_k}$$
$$= k! \frac{d+1-k/2}{(d+1)!d!} \sum_{i+j=k} (-1)^i (d-i)!(d-j)! \mathsf{e}_i \mathsf{e}_j \qquad \text{(Eq. (1.1))}$$

hence the claim of (2) in Proposition 2.29.

# Chapter 3

# Thoma characters, star-transpositions, and random matrices

In the last few decades, an important trend in probability and combinatorics has been the connection between combinatorial structures that grow or decay, like the Young graph and the irreducible characters of $S_n$, and probabilistic models like eigenvalues of random matrices. In a certain sense, the limit of the representation theory of $S_n$ is the representation theory of $S_\infty$; on the other hand, it is well known that the asymptotics of GUE eigenvalues are described by free probability.

This context suggests that the representation theory of $S_\infty$ should be connected with free probability, and indeed it was discovered by P. Biane [3] that one can use certain elements of $\mathbb{C}[S_\infty]$ to approximate free semicircular families. Specifically, the sequence of star-transpositions $\gamma_n := (1, n+1)$ satisfies a central limit theorem where the limit law is semicircular, and one can set up a multivariate version which makes the semicirculars free. In retrospect, an important point about this result is that everything was done in the regular representation of $S_\infty$, i.e. the noncommutative probability space was the one defined using the character labeled by 0 in the Thoma classification.

More recently, this result of Biane was extended by C. Köstler and A.

Nica [26], using the Thoma characters labeled by

$$(\underbrace{1/d, \ldots, 1/d}_{d}, 0, 0, \ldots; 0, 0, \ldots)$$

for $d \geq 1$. Their central limit law is the average eigenvalue distribution of the *traceless GUE*, i.e. the random matrix $A - \mathrm{tr}(A)$ where $A$ is a $d \times d$ GUE. This recovers Biane's result in the limit $d \to \infty$.

This chapter is an exposition of the author's joint work [12] with Köstler and Nica, which extends these results further to include a continuous family of Thoma characters. Interestingly, on the random matrix side, we land on an unexpected type of random matrix model: the entries need to come from noncommutative operator algebras with canonical commutation relations (a.k.a. CCR).

## 3.1 Von Neumann algebras generated by characters

Let $G$ be a countable discrete group and let $\mathbb{C}[G]$ be the group algebra. First let us recall some standard terminology:

**Terminology 3.1.** A function $\phi : G \to \mathbb{C}$ is said to be *positive-definite* if for all $n \geq 1$, for all $g_1, \ldots, g_n \in G$ and $c_1, \ldots, c_n \in \mathbb{C}$, we have

$$\sum_{i,j=1}^{n} c_i \overline{c_j} \phi(g_j^{-1} g_i) \geq 0.$$

In other words, the matrix $(\phi(g_j^{-1} g_i))_{i,j=1}^{n}$ is positive-semidefinite. Write $\mathcal{P}$ for the set of positive-definite functions in $\ell^\infty(G)$, and $\mathcal{P}_1$ for the functions $\phi \in \mathcal{P}$ with $\phi(e) = 1$; these are both convex sets. A *character* of $G$ is a function $\chi \in \mathcal{P}_1$ which is constant on conjugacy classes (a.k.a. a class function). A positive-definite function $\phi$ on $G$ extends to a positive linear functional $\varphi$ on $\mathbb{C}[G]$; if $\phi(e) = 1$, then $\varphi(1) = 1$, and if $\phi$ is a class function, then $\varphi$ is tracial.

The GNS construction, which is fundamental to the theory of $C^*$-algebras, can also be made for positive definite functions on groups, as in e.g. [18, 20].

**Notation 3.2.** For $\phi \in \mathcal{P}$, define $\langle \cdot, \cdot \rangle_\phi$ on $\mathbb{C}[G]$ by

$$\langle x, y \rangle_\phi = \varphi(y^* x)$$

for $x, y \in \mathbb{C}[G]$. This is a positive-semidefinite symmetric sesquilinear form, and if $N_\phi := \{x \in \mathbb{C}[G] : \langle x, x \rangle_\phi = 0\}$, then the quotient space $\mathbb{C}[G]/N_\phi$ inherits an inner product. Write $H_\phi$ for the Hilbert space completion of $\mathbb{C}[G]/N_\phi$, and for $x \in \mathbb{C}[G]$, write $\widehat{x}$ for the image of $x$ in $H_\phi$.

**Theorem 3.3.** *For $\phi \in \mathcal{P}$, there is a unitary representation $\pi_\phi : G \to \mathcal{B}(H_\phi)$ such that*

1. *$\pi_\phi(x)\widehat{y} = \widehat{xy}$ for $x \in G$ and $y \in \mathbb{C}[G]$,*
2. *there is a vector $\xi_\phi \in H_\phi$ such that $\pi_\phi(x)\xi_\phi = \widehat{x}$ for $x \in \mathbb{C}[G]$, namely $\xi_\phi = \widehat{e}$ where $e$ is the identity of $G$, and*
3. *$\phi(g) = \langle \pi_\phi(g)\xi_\phi, \xi_\phi \rangle$ for $g \in G$.*

*Such a $\pi_\phi$ is unique up to unitary equivalence.*

**Terminology 3.4.** The representation $\pi_\phi$ in Theorem 3.3 is called the *GNS representation* of $\phi$, and $\xi_\phi$ is called the *cyclic vector*.

**Remark 3.5.** We will primarily be interested in the case where $\chi$ is *extremal*, i.e. an extreme point in the convex set of characters mentioned in Terminology 3.1. We should mention, then, that this case corresponds exactly to the GNS representation being irreducible, which might provide some motivation for the premise that extremality in this context is analogous to irreducibility in the context of finite groups.

**Notation 3.6.** Let $\chi$ be a character of $G$ and let $\pi : G \to \mathcal{B}(H)$ be the GNS representation with cyclic vector $\xi$. Write

$$\mathcal{M} := W^*(\pi(G)) \subseteq \mathcal{B}(H)$$

for the von Neumann algebra generated by the range of $\pi$, and define a linear functional $\tau$ on $\mathcal{M}$ by $\tau(T) = \langle T\xi, \xi \rangle$ for $T \in \mathcal{M}$. This is a faithful, normal, tracial state. In Terminology 1.71, this makes $(\mathcal{M}, \tau)$ a tracial $W^*$-probability space.

## 3.2 Star-transpositions as random variables

In this section, let $G = S_\infty$, fix a Thoma parameter $\omega \in \Omega$, write $\chi$ for its extremal character, and let $(\mathcal{M}, \tau)$ be the tracial $W^*$-probability space constructed in Notation 3.6 using $\chi$. For $\sigma \in S_\infty$, we will generally use the symbol $\sigma$ to refer to either the permutation $\sigma$ itself, or to its representation in $\mathcal{M}$; this will be clear from context and should not cause any confusion.

### 3.2.1 Law of large numbers

Recall from Notation 1.47 that we use the notation $\gamma_n := (1, n+1)$ for the so-called *star-transpositions*. This is a very interesting sequence, and they have a very interesting limit which shows the value of $(\mathcal{M}, \tau)$ as a "completion" of $\mathbb{C}[S_\infty]$:

**Theorem 3.7** ([37, 21]). *There is a self-adjoint contraction $A_0 \in \mathcal{M}$ such that*

$$A_0 = \overset{\text{WOT}}{\lim_{n\to\infty}} \gamma_n = \overset{\text{SOT}}{\lim_{n\to\infty}} \frac{1}{n} \sum_{i=1}^{n} \gamma_i.$$

*The limit is determined by*

$$\langle A_0 \widehat{\sigma}, \widehat{\tau} \rangle = \frac{\mathsf{p}^\circ_{|V|+1}(\omega)}{\mathsf{p}^\circ_{|V|}(\omega)} \langle \widehat{\sigma}, \widehat{\tau} \rangle$$

*for $\sigma, \tau \in S_\infty$, where $V$ is the cycle of $\sigma\tau^{-1}$ containing* 1.

**Remark 3.8.** An important takeaway from Theorem 3.7 is that $A_0$ has the Thoma parameter encoded in its spectrum. With some considerable work, this observation can lead to a proof of Thoma's theorem [21].

To prove Theorem 3.7, for the weak limit, we naturally want to look at the bilinear forms $\langle \gamma_n \cdot, \cdot \rangle$:

**Lemma 3.9.** *Let $\sigma, \tau \in S_m$ and let $V$ be the cycle of $\sigma\tau^{-1}$ which contains* 1. *Then*

$$\langle \gamma_n \widehat{\sigma}, \widehat{\tau} \rangle = \frac{\mathsf{p}^\circ_{|V|+1}(\omega)}{\mathsf{p}^\circ_{|V|}(\omega)} \langle \widehat{\sigma}, \widehat{\tau} \rangle$$

*for $n \geq m$.*

*Proof.* It suffices to show that $\chi(\tau^{-1}\gamma_n\sigma) = \frac{\mathsf{p}^\circ_{|V|+1}(\omega)}{\mathsf{p}^\circ_{|V|}(\omega)}\chi(\tau^{-1}\sigma)$ or equivalently that $\chi(\gamma_n\sigma\tau^{-1}) = \frac{\mathsf{p}^\circ_{|V|+1}(\omega)}{\mathsf{p}^\circ_{|V|}(\omega)}\chi(\sigma\tau^{-1})$. To this end, observe that if the disjoint cycles of $\sigma\tau^{-1}$ are

$$\sigma\tau^{-1} = (c_1, \ldots, c_{i_1})(c_{i_1+1}, \ldots, c_{i_2}) \cdots (c_{i_{l-1}+1}, \ldots, c_{i_l})$$

with $c_1 = 1$, then for $n \geq m$,

$$\begin{aligned}\gamma_n\sigma\tau^{-1} &= (1, n+1)(1, c_2, \ldots, c_{i_1})(c_{i_1+1}, \ldots, c_{i_2}) \cdots (c_{i_{l-1}+1}, \ldots, c_{i_l}) \\ &= (1, c_2, \ldots, c_{i_1}, n+1)(c_{i_1+1}, \ldots, c_{i_2}) \cdots (c_{i_{l-1}+1}, \ldots, c_{i_l}).\end{aligned}$$

The Thoma character formula then gives

$$\begin{aligned}\chi(\gamma_n\sigma\tau^{-1}) &= \mathsf{p}^\circ_{i_1+1}(\omega) \prod_{\substack{c\in\mathrm{Cyc}(\gamma_n\sigma\tau^{-1}) \\ 1\notin c}} \mathsf{p}^\circ_{|c|}(\omega) \\ &= \mathsf{p}^\circ_{i_1+1}(\omega) \prod_{\substack{c\in\mathrm{Cyc}(\sigma\tau^{-1}) \\ 1\notin c}} \mathsf{p}^\circ_{|c|}(\omega) \\ &= \frac{\mathsf{p}^\circ_{i_1+1}(\omega)}{\mathsf{p}^\circ_{i_1}(\omega)} \prod_{c\in\mathrm{Cyc}(\sigma\tau^{-1})} \mathsf{p}^\circ_{|c|}(\omega) \\ &= \frac{\mathsf{p}^\circ_{i_1+1}(\omega)}{\mathsf{p}^\circ_{i_1}(\omega)} \chi(\sigma\tau^{-1})\end{aligned}$$

hence the claim. □

*Proof of Theorem 3.7.* For the weak limit, we want to show there is a self-adjoint contraction $A_0 \in \mathcal{M}$ with

$$\lim_{n\to\infty} \langle \gamma_n\xi, \eta\rangle = \langle A_0\xi, \eta\rangle$$

for all $\xi, \eta \in H$. Since $\mathrm{span}\{\widehat{\sigma} : \sigma \in S_\infty\}$ is dense in $H$ and for $\sigma, \tau \in S_m$, we have

$$\langle \gamma_n\widehat{\sigma}, \widehat{\tau}\rangle = \frac{\mathsf{p}^\circ_{|V|+1}(\omega)}{\mathsf{p}^\circ_{|V|}(\omega)} \langle \widehat{\sigma}, \widehat{\tau}\rangle$$

for $n \geq m$, the limit $\lim_{n\to\infty}\langle \gamma_n\xi, \eta\rangle$ exists, and we can define a sesquilinear form $B$ on $H$ by

$$B(\xi, \eta) = \lim_{n\to\infty} \langle \gamma_n\xi, \eta\rangle$$

for $\xi, \eta \in H$. This form is bounded in the sense that

$$|B(\xi,\eta)| = \lim_{n\to\infty} |\langle \gamma_n \xi, \eta\rangle| \leq \lim_{n\to\infty} \|\gamma_n \xi\|\|\eta\| = \|\xi\|\|\eta\|$$

so by the well-known correspondence between bounded sesquilinear forms and bounded linear operators, there is a contraction $A_0 \in \mathcal{B}(H)$ with $B(\xi,\eta) = \langle A_0\xi, \eta\rangle$. Since $\gamma_n \in \mathcal{M}^{\mathrm{sa}}$ and $A_0 = \lim^{\mathrm{WOT}}_{n\to\infty} \gamma_n$, we have $A_0 \in \mathcal{M}^{\mathrm{sa}}$.

For the strong limit, using Notation 1.70 with the cyclic vector $\widehat{e}$, we will show that

$$\left\| \frac{1}{n} \sum_{i=1}^{n} \gamma_i - A_0 \right\|_2^2 = \frac{1 - \mathsf{p}_3^\circ(\omega)}{n}$$

for $n \geq 1$, and then use Proposition 1.69 to reach the conclusion. We have

$$\begin{aligned} \left\| \frac{1}{n} \sum_{i=1}^{n} \gamma_i - A_0 \right\|_2^2 &= \left\| \frac{1}{n} \sum_{i=1}^{n} \widehat{\gamma_i} - A_0(\widehat{e}) \right\|^2 \\ &= \frac{1}{n^2} \left\| \sum_{i=1}^{n} \widehat{\gamma_i} \right\|^2 - 2\mathrm{Re} \left\langle \frac{1}{n} \sum_{i=1}^{n} \widehat{\gamma_i}, A_0(\widehat{e}) \right\rangle + \|A_0(\widehat{e})\|^2. \end{aligned} \tag{3.1}$$

For the first part of (3.1), we have

$$\left\| \sum_{i=1}^{n} \widehat{\gamma_n} \right\|^2 = \sum_{i,j=1}^{n} \langle \widehat{\gamma_i}, \widehat{\gamma_j}\rangle = \sum_{i=1}^{n} \chi(\gamma_i^2) + \sum_{i\neq j} \chi(\gamma_i\gamma_j) = n + n(n-1)\mathsf{p}_3^\circ(\omega)$$

since

$$\gamma_i^2 = e \text{ and } (1, i+1)(1, j+1) = (1, j+1, i+1)$$

for $i \neq j$. For the second part of (3.1), we have

$$\langle \widehat{\gamma_i}, A_0(\widehat{e})\rangle = \langle A_0(\widehat{\gamma_i}), \widehat{e}\rangle = \lim_{n\to\infty} \langle \gamma_n \widehat{\gamma_i}, \widehat{e}\rangle = \lim_{n\to\infty} \chi(\gamma_n\gamma_i) = \mathsf{p}_3^\circ(\omega)$$

so

$$2\mathrm{Re} \left\langle \frac{1}{n} \sum_{i=1}^{n} \widehat{\gamma_i}, A_0(\widehat{e}) \right\rangle = \frac{2}{n} \sum_{i=1}^{n} \mathsf{p}_3^\circ(\omega) = 2\mathsf{p}_3^\circ(\omega).$$

For the third part of (3.1), we have

$$\|A_0(\widehat{e})\|^2 = \lim_{n\to\infty} \langle \gamma_n \widehat{e}, A_0(\widehat{e})\rangle = \lim_{n\to\infty} \langle \widehat{\gamma_n}, A_0(\widehat{e})\rangle = \mathsf{p}_3^\circ(\omega).$$

So in total, (3.1) is

$$\frac{1}{n^2}(n + n(n-1)\mathsf{p}_3^\circ(\omega)) - 2\mathsf{p}_3^\circ(\omega) + \mathsf{p}_3^\circ(\omega) = \frac{1 - \mathsf{p}_3^\circ(\omega)}{n}$$

hence the claim. □

**Remark 3.10.** The perspective that Theorem 3.7 is a "law of large numbers" is pursued in [21]: among other things, they studied a certain "canonical" subalgebra related to the star-transpositions, namely the tail algebra

$$\mathcal{M}_0 := \bigcap_{n\geq 1} W^*(\gamma_k : k \geq n) \subseteq \mathcal{M}.$$

This subalgebra admits a unique conditional expectation $E_0 : \mathcal{M} \to \mathcal{M}_0$, and it turns out that $\mathcal{M}_0 = W^*(A_0)$ and $A_0 = E_0(\gamma_n)$ for any $n \geq 1$. So in this operator-valued setting, $A_0$ really is the "mean" of $\gamma_n$ and Theorem 3.7 can be properly described as a noncommutative "law of large numbers".

### 3.2.2 Exchangeability and singleton blocks

**Terminology 3.11.** Let $(\mathcal{A}, \varphi)$ be a $*$-probability space. A sequence $(x_n)_{n\geq 1}$ in $\mathcal{A}$ is said to

- be *exchangeable* if

  $$\varphi(x_{\mathbf{i}(1)} \cdots x_{\mathbf{i}(k)}) = \varphi(x_{\mathbf{j}(1)} \cdots x_{\mathbf{j}(k)})$$

  for all $\mathbf{i}, \mathbf{j} : [k] \to \mathbb{N}$ with $\ker(\mathbf{i}) = \ker(\mathbf{j})$;

- have the *singleton-vanishing property* if

  $$\varphi(x_{\mathbf{i}(1)} \cdots x_{\mathbf{i}(k)}) = 0$$

  for all $\mathbf{i} : [k] \to \mathbb{N}$ such that $\ker(\mathbf{i})$ has a singleton block.

**Proposition 3.12** ([21])**.** *The sequence* $(\gamma_n)_{n\geq 1}$ *is exchangeable.*

*Proof.* Let $\mathbf{i}, \mathbf{j} : [k] \to \mathbb{N}$ and suppose that $\ker(\mathbf{i}) = \ker(\mathbf{j})$, so there is a permutation $\sigma \in S_\infty$ with $\sigma(\mathbf{i}(r) + 1) = \mathbf{j}(r) + 1$ for $r \in [k]$ and $\sigma(1) = 1$. Then

$$\begin{aligned}
\tau(\gamma_{\mathbf{j}(1)} \cdots \gamma_{\mathbf{j}(k)}) &= \chi((1, \mathbf{j}(1) + 1) \cdots (1, \mathbf{j}(k) + 1)) \\
&= \chi((1, \sigma(\mathbf{i}(1) + 1)) \cdots (1, \sigma(\mathbf{i}(k) + 1))) \\
&= \chi(\sigma(1, \mathbf{i}(1) + 1) \cdots (1, \mathbf{i}(k) + 1)\sigma^{-1}) \\
&= \chi((1, \mathbf{i}(1) + 1) \cdots (1, \mathbf{i}(k) + 1)) \\
&= \tau(\gamma_{\mathbf{i}(1)} \cdots \gamma_{\mathbf{i}(k)})
\end{aligned}$$

hence the claim. □

**Proposition 3.13.** *Fix a map* $\mathbf{i} : [k] \to \mathbb{N} \cup \{\infty\}$ *and write*

$$T_p = \begin{cases} \gamma_{\mathbf{i}(p)} & \text{if } \mathbf{i}(p) \in \mathbb{N} \\ A_0 & \text{if } \mathbf{i}(p) = \infty \end{cases}$$

*for* $1 \leq p \leq k$. *Define* $\mathbf{j} : [k] \to \mathbb{N}$ *by replacing each value of* $\infty$ *in* $\mathbf{i}$ *with a new, distinct positive integer. Then*

$$\tau(T_1 \cdots T_k) = \tau(\gamma_{\mathbf{j}(1)} \cdots \gamma_{\mathbf{j}(k)}).$$

Here, we will present the proof of Proposition 3.13 in a way which emphasizes the main conceptual point, which is the use of Theorem 3.7, at the expense of a certain degree of formality. A more formal presentation of the argument is given in [12].

*Proof.* For each $p \in [k]$ with $\mathbf{p}(i) = \infty$, since multiplication is separately continuous in the weak operator topology, we have

$$\begin{aligned}
T_1 \cdots T_{p-1} A_0 T_{p+1} \cdots T_k &= (T_1 \cdots T_{p-1}) \left( \overset{\text{WOT}}{\lim_{n\to\infty}} \gamma_n \right) (T_{p+1} \cdots T_k) \\
&= (T_1 \cdots T_{p-1}) \left( \overset{\text{WOT}}{\lim_{n\to\infty}} \gamma_n T_{p+1} \cdots T_k \right) \\
&= \overset{\text{WOT}}{\lim_{n\to\infty}} T_1 \cdots T_{p-1} \gamma_n T_{p+1} \cdots T_k.
\end{aligned}$$

Since $\tau$ is normal and we are in the unit ball, it is continuous with respect to the weak operator topology, so

$$\tau(T_1 \cdots T_k) = \lim_{n\to\infty} \tau(T_1 \cdots T_{p-1}\gamma_n T_{p+1} \cdots T_k).$$

In this manner, we can remove the $A_0$s from $T_1 \cdots T_k$ and replace them by limits of $\gamma_n$. Moreover, once this is done, the value of $\chi$ on $\gamma_{\mathbf{j}(1)} \cdots \gamma_{\mathbf{j}(k)}$ does not depend on the $n$s, as long as they are sufficiently large. □

### 3.2.3 Central limit theorem

**Theorem 3.14** ([9]). *Let $(\mathcal{A}, \varphi)$ be a $*$-probability space and let $(a_n)_{n\geq 1}$ be a sequence in $\mathcal{A}$ which is exchangeable and has the singleton-vanishing property. For $n \geq 1$, write*

$$s_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} a_i.$$

*Then*

$$\lim_{n\to\infty} \varphi(s_n^k) = \begin{cases} \sum_{\pi\in P_2(k)} \varphi(a_{\mathbf{i}(1)} \cdots a_{\mathbf{i}(k)}) & \text{if } k \text{ even} \\ 0 & \text{if } k \text{ odd} \end{cases}$$

*for $k \geq 1$, where $\mathbf{i} : [k] \to \mathbb{N}$ is chosen for each $\pi \in P_2(k)$ so that $\pi = \ker(\mathbf{i})$.*

*Idea of proof.* The first key point is to realize that when one expands

$$\varphi(s_n^k) = \frac{1}{n^{k/2}} \sum_{\mathbf{i}:[k]\to[n]} \varphi(a_{\mathbf{i}(1)} \cdots a_{\mathbf{i}(k)}),$$

the summands only depend on $\ker(\mathbf{i})$, so the above can be written as

$$\frac{1}{n^{k/2}} \sum_{\pi\in P(k)} |\{\mathbf{i} : [k] \to [n] : \ker(\mathbf{i}) = \pi\}| \cdot \varphi(\pi)$$

where $\varphi(\pi)$ is the common value of $\varphi(a_{\mathbf{i}(1)} \cdots a_{\mathbf{i}(k)})$ for any $\mathbf{i} : [k] \to [n]$ with $\pi = \ker(\mathbf{i})$. By the singleton-vanishing property, $\varphi(\pi) = 0$ for all $\pi$ with a singleton block, so the above can be written as

$$\frac{1}{n^{k/2}} \sum_{\substack{\pi\in P(k) \\ |V|\geq 2\,\forall\, V\in\pi}} |\{\mathbf{i} : [k] \to [n] : \ker(\mathbf{i}) = \pi\}| \cdot \varphi(\pi).$$

The second important point is that one can easily count

$$|\{\mathbf{i} : [k] \to [n] : \ker(\mathbf{i}) = \pi\}| = \frac{n!}{(n - |\pi|)!}$$

which is $\approx n^{|\pi|}$ as $n \to \infty$. Then

$$\varphi(s_n^k) \approx \sum_{\substack{\pi \in P(k) \\ |V| \geq 2 \, \forall \, V \in \pi}} n^{|\pi| - k/2} \varphi(\pi)$$

as $n \to \infty$, so the $\pi$ whose summands don't vanish in the limit are exactly the ones with $|\pi| \geq k/2$. But since each block of $\pi$ has at least two elements, this only leaves the $\pi$ with $|\pi| = k/2$, i.e. $\pi \in P_2(k)$. This is exactly the claim of the theorem:

$$\lim_{n \to \infty} \varphi(s_n^k) = \sum_{\pi \in P_2(k)} \varphi(\pi).$$

□

**Proposition 3.15.** *The centered sequence* $(\gamma_n - A_0)_{n \geq 1}$ *is exchangeable and has the singleton-vanishing property.*

This is the first instance in which we will need to convert mixed moments of $(\gamma_n - A_0)_{n \geq 1}$ into sums of mixed moments of $(\gamma_n)_{n \geq 1}$. The following lemma will be used again in a later section to analyze the central limit law.

**Lemma 3.16.** *Let* $\pi \in P_2(k)$ *and pick* $\mathbf{i} : [k] \to \mathbb{N}$ *such that* $\pi = \ker(\mathbf{i})$. *Then*

$$\tau((\gamma_{\mathbf{i}(1)} - A_0) \cdots (\gamma_{\mathbf{i}(k)} - A_0)) = \sum_{S \subseteq [k]} (-1)^{|S|} \tau(\gamma_{\mathbf{j}(1)} \cdots \gamma_{\mathbf{j}(k)})$$

*where* $\mathbf{j} : [k] \to \mathbb{N}$ *is chosen for each* $S \subseteq [k]$ *such that* $\ker(\mathbf{j}) = \pi \wedge \pi_S$.

*Proof.* We have

$$\tau((\gamma_{\mathbf{i}(1)} - A_0) \cdots (\gamma_{\mathbf{i}(k)} - A_0)) = \sum_{S \subseteq [k]} (-1)^{|S|} \tau(T_1^{(S)} \cdots T_k^{(S)})$$

where

$$T_j^{(S)} := \begin{cases} A_0 & \text{if } j \in S \\ \gamma_{\mathbf{i}(j)} & \text{if } j \notin S \end{cases}$$

for $1 \leq j \leq k$. By Proposition 3.13, $\tau(T_1^{(S)} \cdots T_k^{(S)}) = \tau(\gamma_{\mathbf{j}(1)} \cdots \gamma_{\mathbf{j}(k)})$ where $\mathbf{j} : [k] \to \mathbb{N}$ is chosen so that $\ker(\mathbf{j}) = \pi \wedge \pi_S$. □

*Proof of Proposition 3.15.* Lemma 3.16 shows clearly that

$$\tau((\gamma_{\mathbf{i}(1)} - A_0) \cdots (\gamma_{\mathbf{i}(k)} - A_0))$$

only depends on $\ker(\mathbf{i})$, i.e. $(\gamma_n - A_0)_{n \geq 1}$ is exchangeable. For the singleton-vanishing property, let $\mathbf{i} : [k] \to \mathbb{N}$ be a multi-index such that $\ker(\mathbf{i})$ has a singleton block $\{b\}$. Then in the sum

$$\tau((\gamma_{\mathbf{i}(1)} - A_0) \cdots (\gamma_{\mathbf{i}(k)} - A_0)) = \sum_{S \subseteq [k]} (-1)^{|S|} \tau(\gamma_{\mathbf{j}(1)} \cdots \gamma_{\mathbf{j}(k)})$$

from Lemma 3.16, to show that the right-hand side is 0, it suffices to show there is a bijection $f : 2^{[k]} \to 2^{[k]}$ with $\pi \wedge \pi_S = \pi \wedge \pi_{f(S)}$ and $(-1)^{|S|} + (-1)^{|f(S)|} = 0$ for all $S \subseteq [k]$. For this, simply define

$$f(S) = \begin{cases} S \setminus \{p\} & \text{if } p \in S \\ S \cup \{p\} & \text{if } p \notin S \end{cases}$$

for $S \subseteq [k]$. The property of changing the sign of $|S|$ is obvious, and since $p$ is not connected to anything in $\pi$, $\pi \wedge \pi_S$ does not depend on whether $p$ is in $S$. So

$$\tau((\gamma_{\mathbf{i}(1)} - A_0) \cdots (\gamma_{\mathbf{i}(k)} - A_0)) = 0,$$

i.e. $(\gamma_n - A_0)_{n \geq 1}$ has the singleton-vanishing property. □

**Corollary 3.17.** *Let*

$$s_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\gamma_i - A_0)$$

*for* $n \geq 1$. *Then*

$$\lim_{n \to \infty} \tau(s_n^k) = \begin{cases} \sum_{\pi \in P_2(k)} \tau((\gamma_{\mathbf{i}(1)} - A_0) \cdots (\gamma_{\mathbf{i}(k)} - A_0)) & \textit{if } k \textit{ even} \\ 0 & \textit{if } k \textit{ odd} \end{cases}$$

*for* $k \geq 1$.

**Corollary 3.18.** *There is a unique probability measure $\mu_\omega$ on $\mathbb{R}$ with the moments*

$$\int_{\mathbb{R}} t^k \, d\mu_\omega(t) = \begin{cases} \sum_{\pi \in P_2(k)} \tau((\gamma_{\mathbf{i}(1)} - A_0) \cdots (\gamma_{\mathbf{i}(k)} - A_0)) & \text{if } k \text{ even} \\ 0 & \text{otherwise} \end{cases},$$

*where $\mathbf{i} : [k] \to \mathbb{N}$ is chosen for each $\pi \in P_2(k)$ so that $\pi = \ker(\mathbf{i})$. Moreover, the distribution of $s_n$ converges weakly to $\mu_\omega$ as $n \to \infty$.*

*Proof.* As is well-known in probability theory [5, Theorem 30.1], to show a probability measure on $\mathbb{R}$ with finite moments $m_k$ is uniquely determined by those moments, it suffices to show the exponential MGF $\sum_{k=0}^{\infty} m_k \frac{z^k}{k!}$ has a positive radius of convergence, i.e.

$$\limsup_{k \to \infty} \left( \frac{m_k}{k!} \right)^{\frac{1}{k}} < \infty.$$

The even moments are easy to estimate: using Lemma 3.16,

$$|\tau((\gamma_{\mathbf{i}(1)} - A_0) \cdots (\gamma_{\mathbf{i}(k)} - A_0))| \leq \sum_{S \subseteq [k]} |\chi(\gamma_{\mathbf{j}(1)} \cdots \gamma_{\mathbf{j}(k)})|,$$

and to see that the above is $\leq 2^k$, observe that

$$\left| \sum_{i \geq 1} \alpha_i^k + (-1)^{k-1} \sum_{j \geq 1} \beta_j^k \right| \leq 1$$

for $k \geq 2$ and use the Thoma formula for $\chi$. Then,

$$|m_k| = \left| \sum_{\pi \in P_2(k)} \tau((\gamma_{\mathbf{i}(1)} - A_0) \cdots (\gamma_{\mathbf{i}(k)} - A_0)) \right| \leq 2^k |P_2(k)| = 2^k (k-1)!!$$

so

$$\frac{m_k}{k!} \leq \frac{2^k (k-1)!!}{k!} = \frac{2^{k/2}}{(k/2)!} \text{ and } \left( \frac{2^{k/2}}{(k/2)!} \right)^{\frac{1}{k}} = \frac{\sqrt{2}}{(k/2)!^{1/k}} \to 0$$

which shows the radius of convergence is $\infty$. With this uniqueness, it is well known [5, Theorem 30.2] that convergence in moments implies weak convergence. □

## 3.3 Combinatorics of the central limit law

The initial description of the central limit law in Theorem 3.14 is rather opaque, and we of course want to describe it more concretely. For this purpose, we will make the following assumption for the remainder of the present chapter: the Thoma parameter $\omega = (\alpha, \beta)$ has $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\sum_{i=1}^{d} \alpha_i = 1$, and $\beta = 0$.

The key observation is that

1. each
$$(\gamma_{\mathbf{i}(1)} - A_0) \cdots (\gamma_{\mathbf{i}(k)} - A_0)$$
appearing in the even moments is a sum of words in $\{\gamma_i : i \geq 1\} \cup \{A_0\}$;

2. for any word in (1), we can apply Proposition 3.13 to the trace, which reduces to words in $\{\gamma_i : i \geq 1\}$ where each $\gamma_i$ appears at most twice.

It thus suffices to understand mixed moments of the form

$$\tau(\gamma_{\mathbf{i}(1)} \cdots \gamma_{\mathbf{i}(k)})$$

where $\pi = \ker(\mathbf{i})$ has blocks with sizes at most 2.

### 3.3.1 Permutations and split-pair partitions

**Notation 3.19.** Write

$$P_{\leq 2}(k) := \{\pi \in P(k) : |V| \leq 2 \,\forall\, V \in \pi\}$$

for the set of partitions whose blocks are pairs or singletons.

**Notation 3.20.** Take $\pi \in P_{\leq 2}(k)$, say with blocks $V_i = \{a_i, b_i\}$, with $a_i \leq b_i$, for $1 \leq i \leq l$, ordered such that $b_1 > \cdots > b_l$. Define a map $\mathbf{r} : [k] \to [l]$ by $\mathbf{r}(a_i) = i$ and $\mathbf{r}(b_i) = i$. Then write

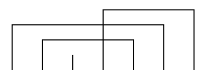
$$\sigma_\pi := \prod_{\substack{1 \leq i \leq l \\ a_i \neq b_i}} (a_i, b_i) \in S_k,$$

which extends and is consistent with the previous Notation 1.56, and

$$\tau_\pi := \gamma_{\mathbf{r}(1)} \cdots \gamma_{\mathbf{r}(k)} \in S_{l+1}.$$

Write $a_0 := k + 1$ and $b_0 := k + 1$, and $B_\pi := \{b_0, b_1, \ldots, b_l\} \subseteq [k + 1]$.

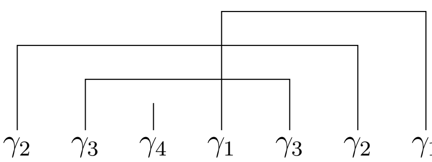
**Example 3.21.** Consider the partition

$$\pi =$$  $$\in P_{\leq 2}(7).$$

Then $\sigma_\pi = (4,7)(1,6)(2,5)$, and

$$c_{7+1}\sigma_\pi = (1,2,3,4,5,6,7,8)(4,7)(1,6)(2,5) = (1,7,5,3,4,8)(2,6),$$

while




$$\tau_\pi = (1,3)(1,4)(1,5)(1,2)(1,4)(1,3)(1,2) = (1,5,4,2)(3)\ .$$

It is not a coincidence that $c_{k+1}\sigma_\pi$ and $\tau_\pi$ have the same number of cycles, nor is the following refinement: the orbits of $c_{7+1}\sigma_\pi$ partition $B_\pi = \{3,5,6,7,8\}$ into $B_\pi = \{6\} \cup \{3,5,7,8\}$, which is conjugate to $\tau_\pi$. The present subsection is dedicated to this relation.

**Theorem 3.22.** *Let $\pi \in P_{\leq 2}(k)$, and let $R_1, \ldots, R_p$ be the orbits of $c_{k+1}\sigma_\pi$ which intersect $B_\pi$. Then*

1. *$\tau_\pi$ has exactly $p$ orbits in $[l+1]$, and their sizes are $|R_1 \cap B_\pi|, \ldots, |R_p \cap B_\pi|$;*

2. *the orbits of $c_{k+1}\sigma_\pi$ in $[k+1]$ are exactly $R_1, \ldots, R_p$.*

**Notation 3.23** (Restriction of permutations)**.** For $\sigma \in S_\infty$ and $A \subseteq \mathbb{N}$, define a permutation $\sigma|_A$ of $A$ by removing the elements of $\mathbb{N} \setminus A$ from the disjoint cycle notation (not including cycles of size 1).

**Lemma 3.24.** *If $0 \leq i,j \leq l$ and $((\sigma_\pi c_{k+1}^{-1})\big|_{B_\pi})(b_i) = b_j$, then $\tau_\pi(i+1) = j+1$.*

This consists of many cases; we will prove the lemma for $i \neq 0$ and then show that the $i = 0$ case follows.

*Proof.* First of all, fix $0 \leq i, j \leq l$ with $i \neq 0$, and suppose that

$$((\sigma_\pi c_{k+1}^{-1})\big|_{B_\pi})(b_i) = b_j.$$

This means there is some $q \geq 1$ such that

$$(\sigma_\pi c_{k+1}^{-1})^q(b_i) = b_j \text{ and } (\sigma_\pi c_{k+1}^{-1})^p(b_i) \notin B_\pi$$

for all $0 \leq p < q$. To show $\tau_\pi(i+1) = j+1$, there are two cases: $q = 1$ and $q > 1$.

For the case $q = 1$, i.e. $\sigma_\pi c_{k+1}^{-1}(b_i) = b_j$, we have $c_{k+1}^{-1}(b_i) = a_j$. Here, there are two sub-cases:

- if $b_i = 1$, then we must have $i = l$, $V_i = \{1\}$, and $a_i = b_i = 1$. Then $a_j = c_{k+1}^{-1}(1) = k+1$ so $j = 0$ and $\tau_\pi(i+1) = \tau_\pi(l+1) = 1 = j+1$ since $\tau_\pi$ only has one factor of $\gamma_l$.

- if $b_i > 1$, then $a_j = c_{k+1}^{-1}(b_i) = b_i - 1$ and

$$\tau_\pi = \underbrace{(\gamma_{\mathbf{r}(1)} \cdots \gamma_{\mathbf{r}(a_j-1)})}_{(1)} \underbrace{(\gamma_{\mathbf{r}(a_j)}\gamma_{\mathbf{r}(b_i)})}_{(2)} \underbrace{(\gamma_{\mathbf{r}(b_i+1)} \cdots \gamma_{\mathbf{r}(k)})}_{(3)}.$$

  The brackets (3) fix $i+1$ because $\mathbf{r}(m) \neq i$ for all $b_i + 1 \leq m \leq k$. The brackets (2) are the important ones:

$$(\gamma_{\mathbf{r}(a_j)}\gamma_{\mathbf{r}(b_i)}) = (\gamma_j\gamma_i)(i+1) = \gamma_j(1) = j+1.$$

  The brackets (1) fix $j+1$ because $\mathbf{r}(m) \neq j$ for all $1 \leq m \leq a_j - 1$.

In both of these sub-cases, $\tau_\pi(i+1) = j+1$.

For the case $q > 1$, the relation $(\sigma_\pi c_{k+1}^{-1})^q(b_i) = b_j$ can be broken up as follows: by minimality of $q$, for all $2 \leq p < q$, we have $(\sigma_\pi c_{k+1}^{-1})^p(b_i) \in [k+1] \setminus B_\pi$, i.e. there is a map $\mathbf{i} : [q-1] \to [k+1] \setminus B_\pi$ such that

- $(\sigma_\pi c_{k+1}^{-1})(b_i) = a_{\mathbf{i}(1)}$,
- $(\sigma_\pi c_{k+1}^{-1})(a_{\mathbf{i}(p-1)}) = a_{\mathbf{i}(p)}$ for $2 \leq p \leq q-1$, and
- $(\sigma_\pi c_{k+1}^{-1})(a_{\mathbf{i}(q-1)}) = b_j$.

These translate to

1. $c_{k+1}^{-1}(b_i) = b_{\mathbf{i}(1)}$,
2. $c_{k+1}^{-1}(a_{\mathbf{i}(p-1)}) = b_{\mathbf{i}(p)}$ for $2 \leq p \leq q-1$, and
3. $c_{k+1}^{-1}(a_{\mathbf{i}(q-1)}) = a_j$.

Since $a_{\mathbf{i}(1)}, \ldots, a_{\mathbf{i}(q-1)} \in [k+1] \setminus B_\pi$, we must have $b_{\mathbf{i}(1)}, \ldots, b_{\mathbf{i}(q-1)} \neq k+1$, so the first two bullet points are $b_i - 1 = b_{\mathbf{i}(1)}$ and $a_{\mathbf{i}(p-1)} - 1 = b_{\mathbf{i}(p)}$ for $2 \leq p \leq q-1$. For Item 3, there is

- the generic case of $a_{\mathbf{i}(q-1)} > 1$ and $j > 0$, as well as
- the case of $a_{\mathbf{i}(q-1)} = 1$ and $j = 0$, meaning $\mathbf{i}(q-1) = l$.

In the generic case, Item 3 says $a_j = a_{\mathbf{i}(q-1)} - 1$. Then

$$\begin{aligned}\tau_\pi = \big(\gamma_{\mathbf{r}(1)} \cdots \gamma_{\mathbf{r}(a_j - 1)}\big) &\underbrace{\big(\gamma_{\mathbf{r}(a_j)} \gamma_{\mathbf{r}(a_{\mathbf{i}(q-1)})}\big)}_{(q)} \big(\gamma_{\mathbf{r}(a_{\mathbf{i}(q-1)}+1)} \cdots \gamma_{\mathbf{r}(b_{\mathbf{i}(q-1)}-1)}\big) \\ &\underbrace{\big(\gamma_{\mathbf{r}(b_{\mathbf{i}(q-1)})} \gamma_{\mathbf{r}(a_{\mathbf{i}(q-2)})}\big)}_{(q-1)} \quad \cdots \quad \underbrace{\big(\gamma_{\mathbf{r}(b_{\mathbf{i}(2)})} \gamma_{\mathbf{r}(a_{\mathbf{i}(1)})}\big)}_{(2)} \\ &\big(\gamma_{\mathbf{r}(a_{\mathbf{i}(1)}+1)} \cdots \gamma_{\mathbf{r}(b_{\mathbf{i}(1)}-1)}\big) \underbrace{\big(\gamma_{\mathbf{r}(b_{\mathbf{i}(1)})} \gamma_{\mathbf{r}(b_i)}\big)}_{(1)} \big(\gamma_{\mathbf{r}(b_i+1)} \cdots \gamma_{\mathbf{r}(k)}\big)\end{aligned}$$

and the only factors which affect $i+1$ are the ones in the numbered brackets. Specifically, (1) sends $i+1$ to $\mathbf{i}(1)+1$, (2) sends $\mathbf{i}(1)+1$ to $\mathbf{i}(2)+1$, and so on, until $(q)$ sends $\mathbf{i}(q-1)+1$ to $j+1$. In the case of $a_{\mathbf{i}(q-1)} = 1$ and $j = 0$, Item 3 is vacuous, saying nothing but $c_{k+1}^{-1}(1) = k+1$. This can be handled similarly to the generic case, by following $i+1$ through the product of star-transpositions which defines $\tau_\pi$.

Finally, let us explain why our initial assumption of $i \neq 0$ is sufficient to prove the full claim of the lemma. Since $(\sigma_\pi c_{k+1}^{-1})\big|_{B_\pi}$ is a permutation of $B_\pi$, we can write

$$b_0 \mapsto b_{j_0},\ b_1 \mapsto b_{j_1},\ \ldots,\ b_l \mapsto b_{j_l}.$$

In case $i = 0$, the lemma is claiming that $\tau_\pi(1) = j_0 + 1$. Since we already know

$$\tau_\pi(1+1) = j_1 + 1,\ \tau_\pi(2+1) = j_2 + 1,\ \ldots,\ \tau_\pi(l+1) = j_l + 1,$$

the only remaining value for $\tau_\pi(0+1)$ is $j_0+1$. So we have proved the lemma in all cases. □

*Proof of Theorem 3.22.* Since $c_{k+1}\sigma_\pi = (\sigma_\pi c_{k+1}^{-1})^{-1}$, the sets $R_i \cap B_\pi$, for $1 \le i \le p$, are the orbits of the permutation $(\sigma_\pi c_{k+1}^{-1})\big|_{B_\pi}$ of $B_\pi$. Define $f : B_\pi \to [l+1]$ by $f(b_i) = i+1$ for $0 \le i \le l$. Then by Lemma 3.24,

$$f \circ (\sigma_\pi c_{k+1}^{-1})\big|_{B_\pi} = \tau_\pi \circ f$$

so the orbits of $\tau_\pi$ are $f(R_i \cap B_\pi)$ for $1 \le i \le p$, which proves the first claim.

For the second claim, it suffices to show that every orbit of $c_{k+1}\sigma_\pi$ intersects $B_\pi$. In fact, we claim that for every orbit $R$ of $c_{k+1}\sigma_\pi$, we have $(c_{k+1}\sigma_\pi)^{-1}(\min(R)) \in B_\pi$. The case $\min(R) = 1$ is easy:

$$(c_{k+1}\sigma_\pi)^{-1}(1) = \sigma_\pi(k+1) = k+1 = b_0 \in B_\pi.$$

Now suppose that $\min(R) > 1$. Observe that $\min(R) - 1 \ne \max(V_i)$ for all $1 \le i \le l$: otherwise, if $\min(R) - 1 = \max(V_i)$ for some $1 \le i \le l$, then

$$\begin{aligned}(c_{k+1}\sigma_\pi)^{-1}(\min(R)) &= \sigma_\pi c_{k+1}^{-1}(\min(R)) \\ &= \sigma_\pi(\min(R) - 1) \\ &= \sigma_\pi(\max(V_i)) \\ &= \min(V_i) \le \max(V_i) \\ &= \min(R) - 1 < \min(R)\end{aligned}$$

but $(c_{k+1}\sigma_\pi)^{-1} \in R$, which is a contradiction. This forces $\min(R) - 1 = \min(V_i)$ for some $1 \le i \le l$, so

$$(c_{k+1}\sigma_\pi)^{-1}(\min(R)) = \sigma_\pi c_{k+1}^{-1}(\min(V_i) + 1) = \sigma_\pi(\min(V_i)) = \max(V_i) \in B_\pi$$

hence the claim. □

### 3.3.2 Coloured Wick formula

#### Centering

As alluded to in the beginning of the present section, the moments of our central limit law come down to the values $\chi(\tau_\pi)$:

**Proposition 3.25.** *For even $k$, the moments of $\mu_\alpha$ are*

$$\int_{\mathbb{R}} t^k \, d\mu_\alpha(t) = \sum_{\pi \in P_{\leq 2}(k)} (-1)^{m_1(\pi)/2} (m_1(\pi) - 1)!! \chi(\tau_\pi)$$

*Proof.* By Corollary 3.17, the even moments of $\mu_\alpha$ are

$$\sum_{\pi \in P_2(k)} \tau((\gamma_{\mathbf{i}(1)} - A_0) \cdots (\gamma_{\mathbf{i}(k)} - A_0))$$

where $\mathbf{i} : [k] \to \mathbb{N}$ is chosen for each $\pi$ so that $\pi = \ker(\mathbf{i})$. For each fixed $\pi$, by Lemma 3.16, we can write

$$\tau((\gamma_{\mathbf{i}(1)} - A_0) \cdots (\gamma_{\mathbf{i}(k)} - A_0)) = \sum_{S \subseteq [k]} (-1)^{|S|} \tau(\gamma_{\mathbf{j}(1)} \cdots \gamma_{\mathbf{j}(k)})$$

where $\mathbf{j}$ is chosen so that $\ker(\mathbf{j}) = \pi \wedge \pi_S$. The above can be written as

$$\sum_{\substack{\rho \in P_{\leq 2}(k) \\ \rho \leq \pi}} \left( \sum_{\substack{S \subseteq [k] \\ \pi \wedge \pi_S = \rho}} (-1)^{|S|} \right) \tau(\gamma_{\mathbf{j}(1)} \cdots \gamma_{\mathbf{j}(k)}) = \sum_{\substack{\rho \in P_{\leq 2}(k) \\ \rho \leq \pi}} \left( \sum_{\substack{S \subseteq [k] \\ \pi \wedge \pi_S = \rho}} (-1)^{|S|} \right) \chi(\tau_\rho)$$

where, again, $\mathbf{j}$ is chosen so that $\ker(\mathbf{j}) = \rho = \pi \wedge \pi_S$. Then the even moments in question are

$$\sum_{\pi \in P_2(k)} \sum_{\substack{\rho \in P_{\leq 2}(k) \\ \rho \leq \pi}} \left( \sum_{\substack{S \subseteq [k] \\ \pi \wedge \pi_S = \rho}} (-1)^{|S|} \right) \chi(\tau_\rho) = \sum_{\rho \in P_{\leq 2}(k)} \sum_{\substack{\pi \in P_2(k) \\ \rho \leq \pi}} \left( \sum_{\substack{S \subseteq [k] \\ \pi \wedge \pi_S = \rho}} (-1)^{|S|} \right) \chi(\tau_\rho).$$

Now, all that remains is to count the portion in brackets: it suffices to show that

$$\sum_{\substack{S \subseteq [k] \\ \pi \wedge \pi_S = \rho}} (-1)^{|S|} = (-1)^{m_1(\pi)/2}$$

for $\pi \in P_2(k)$ and $\rho \in P_{\leq 2}(k)$ with $\rho \leq \pi$. To this end, write $V_1, \ldots, V_p$ for the blocks of $\pi$ which also appear in $\rho$, and write $W_1, \ldots, W_q$ for the ones which are split into singletons. Then, for $S \subseteq [k]$, the condition that $\pi \wedge \pi_S = \rho$ is equivalent to $S \cap V_i = \emptyset$ for $1 \leq i \leq p$ and $S \cap W_j \neq \emptyset$ for $1 \leq j \leq q$. So a choice of $S$ amounts to the following:

- a choice of $0 \leq r \leq q$ and $r$ elements of $\{W_1, \ldots, W_q\}$, say the ones from which only one element will be chosen to be a member of $S$, and
- a choice of one element from each of the above.

Thus

$$\begin{aligned}\sum_{\substack{S \subseteq [k] \\ \pi \vee \pi_S = \rho}} (-1)^{|S|} &= \sum_{r=0}^{q} \binom{q}{r} 2^r (-1)^{r+2(q-r)} = \sum_{r=0}^{q} \binom{q}{r} 2^r (-1)^r \\ &= (1-2)^q = (-1)^q = (-1)^{m_1(\pi)/2}\end{aligned}$$

and we are done. □

### Character values

Now, the goal is to understand $\chi(\tau_\pi)$ for $\pi \in P_{\leq 2}(k)$. For this, we use Theorem 3.22.

**Proposition 3.26.** *We have*

$$\chi(\tau_\pi) = \sum_{\substack{\mathbf{i}:[k]\to[d] \\ \text{constant on orbits} \\ \text{of } c_k\sigma_\pi}} \alpha_{\mathbf{i}(1)} (\alpha_{\mathbf{i}(b_1)} \cdots \alpha_{\mathbf{i}(b_l)})$$

*for* $\pi \in P_{\leq 2}(k)$.

The main part of the argument – the invocation of Theorem 3.22 – is simpler in a slightly different setup, which is sufficient to deduce Proposition 3.26:

**Proposition 3.26′.** *We have*

$$\chi(\tau_\pi) = \sum_{\substack{\mathbf{j}:[k+1]\to[d] \\ \textit{constant on orbits} \\ \textit{of } c_{k+1}\sigma_\pi}} \alpha_{\mathbf{j}(b_0)}(\alpha_{\mathbf{j}(b_1)}\cdots\alpha_{\mathbf{j}(b_l)})$$

*for* $\pi \in P_{\leq 2}(k)$.

*Proof.* Let $R_1, \ldots, R_p$ be the orbits of $c_{k+1}\sigma_\pi$ in $[k+1]$, so by Theorem 3.22 the orbits of $\tau_\pi$ in $[l+1]$ can be written as $Q_1, \ldots, Q_p$ with $|Q_i| = |R_i \cap B_\pi|$ for $1 \leq i \leq p$. Then

$$\begin{aligned}
\chi(\tau_\pi) &= \left(\sum_{j\geq 1}\alpha_j^{|Q_1|}\right)\cdots\left(\sum_{j\geq 1}\alpha_j^{|Q_p|}\right) \\
&= \left(\sum_{j\geq 1}\alpha_j^{|R_1\cap B_\pi|}\right)\cdots\left(\sum_{j\geq 1}\alpha_j^{|R_p\cap B_\pi|}\right) \\
&= \sum_{\mathbf{j}:[p]\to[d]} \alpha_{\mathbf{j}(1)}^{|R_1\cap B_\pi|}\cdots\alpha_{\mathbf{j}(p)}^{|R_p\cap B_\pi|} \\
&= \sum_{\substack{\mathbf{j}:[k+1]\to[d] \\ \text{constant on orbits} \\ \text{of } c_{k+1}\sigma_\pi}} \alpha_{\mathbf{j}(b_0)}(\alpha_{\mathbf{i}(b_1)}\cdots\alpha_{\mathbf{i}(b_l)})
\end{aligned}$$

and we are done. □

*Proof of Proposition 3.26.* It suffices to show that the sets

$$\mathcal{J} := \{\mathbf{j} : [k+1] \to [d] : \text{constant on orbits of } c_{k+1}\sigma_\pi\}$$

and

$$\mathcal{I} := \{\mathbf{i} : [k] \to [d] : \text{constant on orbits of } c_k\sigma_\pi\}$$

are in bijection, in such a way that

$$\alpha_{\mathbf{i}(1)}(\alpha_{\mathbf{i}(b_1)}\cdots\alpha_{\mathbf{i}(b_l)}) = \alpha_{\mathbf{j}(b_0)}(\alpha_{\mathbf{j}(b_1)}\cdots\alpha_{\mathbf{j}(b_l)})$$

when $\mathbf{i}$ and $\mathbf{j}$ are paired. For $\mathbf{j} \in \mathcal{J}$, define $\mathbf{r_j} : [k] \to [d]$ by $\mathbf{r_j} = \mathbf{j}|_{[k]}$ for $\mathbf{j} \in \mathcal{J}$.

To see that $\mathbf{r_j} \in \mathcal{I}$, take $r \in [k]$, so there are some cases.

1. If $r = a_i$ for some $1 \le i \le p$, then

$$\mathbf{r_j}((c_k\sigma_\pi)(r)) = \mathbf{j}(c_k(b_i))$$

and there are subcases:

   (a) if $b_i = k$, then $\mathbf{j}(c_k(b_i)) = \mathbf{j}(1)$

   (b) if $b_i < k$, then

$$\mathbf{j}(c_k(b_i)) = \mathbf{j}(b_i + 1) = \mathbf{j}(c_{k+1}(b_i)) = \mathbf{j}(c_{k+1}\sigma_\pi(a_i)) = \mathbf{j}(r).$$

   In either subcase, we have $\mathbf{r_j}((c_k\sigma_\pi)(r)) = \mathbf{r_j}(r)$.

2. If $r = b_i$ for some $1 \le i \le p$, then

$$\mathbf{r_j}((c_k\sigma_\pi)(r)) = \mathbf{j}(c_k(a_i)) = \mathbf{j}(a_i{+}1) = \mathbf{j}(c_{k+1}(a_i)) = \mathbf{j}((c_{k+1}\sigma_\pi)(b_i)) = \mathbf{j}(r)$$

   since $a_i < b_i \le k < k+1$. So in this case, we also have $\mathbf{r_j}((c_k\sigma_\pi)(r)) = \mathbf{r_j}(r)$.

In all cases, we have $\mathbf{r_j} \in \mathcal{I}$.

To see that $\mathbf{j} \mapsto \mathbf{r_j}$ is injective, suppose that $\mathbf{r_j} = \mathbf{r_{j'}}$ for some $\mathbf{j}, \mathbf{j}' \in \mathcal{J}$. Then of course $\mathbf{j}(r) = \mathbf{j}'(r)$ for all $r \in [k]$. For $r = k+1$, let $V$ be the block of $\pi$ which contains $k$, so $V = \{a_i, k\}$ for some $1 \le i \le p$, and we have

$$\mathbf{j}(k+1) = \mathbf{j}((c_{k+1}\sigma_\pi)(a_i)) = \mathbf{j}(a_i)$$

and similarly $\mathbf{j}'(k+1) = \mathbf{j}'(a_i)$. Since $a_i \le k$, $\mathbf{j}(a_i) = \mathbf{j}'(a_i)$, and then $\mathbf{j}(k+1) = \mathbf{j}'(k+1)$.

To show that $\mathbf{j} \mapsto \mathbf{r_j}$ is surjective, take $\mathbf{i} \in \mathcal{I}$ and define $\mathbf{j} : [k+1] \to [d]$ by

$$\mathbf{j}(r) = \begin{cases} \mathbf{i}(r) & \text{if } r \le k \\ \mathbf{i}(1) & \text{if } r = k+1 \end{cases}$$

for $r \in [k+1]$. To see that $\mathbf{j} \in \mathcal{J}$, i.e. $\mathbf{j}(c_{k+1}\sigma_\pi(r)) = \mathbf{j}(r)$ for all $r \in [k+1]$, there are again some cases.

- If $r = k+1$, then

$$\mathbf{j}(c_{k+1}\sigma_\pi(r)) = \mathbf{j}(1) = \mathbf{i}(1) = \mathbf{j}(k+1).$$

- If $r = a_i$ for some $1 \leq i \leq p$, then

  $$\mathbf{j}(c_{k+1}\sigma_\pi(r)) = \mathbf{j}(c_{k+1}(b_i)) = \mathbf{j}(b_i + 1)$$

  and there are sub-cases:

  – if $b_i + 1 \leq k$, then $a_i \leq b_i \leq k - 1$ and

  $$\mathbf{j}(b_i + 1) = \mathbf{i}(b_i + 1) = \mathbf{i}(c_k\sigma_\pi(r)) = \mathbf{i}(r) = \mathbf{j}(r);$$

  – if $b_i + 1 = k + 1$, then $b_i = k$ and

  $$\mathbf{j}(b_i + 1) = \mathbf{i}(1) = \mathbf{i}(c_k(b_i)) = \mathbf{i}(c_k\sigma_\pi(r)) = \mathbf{i}(r) = \mathbf{j}(r).$$

  In either case, $\mathbf{j}(c_{k+1}\sigma_\pi(r)) = \mathbf{j}(r)$.

- If $r = b_i$ for some $1 \leq i \leq p$, then

  $$\mathbf{j}(c_{k+1}\sigma_\pi(r)) = \mathbf{j}(c_{k+1}(a_i)) = \mathbf{j}(a_i + 1)$$

  and again there are sub-cases:

  – if $a_i + 1 \leq k$, then $a_i \leq k - 1$ and

  $$\mathbf{j}(a_i + 1) = \mathbf{i}(a_i + 1) = \mathbf{i}(c_k\sigma_\pi(r)) = \mathbf{i}(r) = \mathbf{j}(r);$$

  – if $a_i + 1 = k + 1$, then $a_i = k$ and

  $$\mathbf{j}(a_i + 1) = \mathbf{i}(1) = \mathbf{i}(c_k(a_i)) = \mathbf{i}(c_k\sigma_\pi(r)) = \mathbf{i}(r) = \mathbf{j}(r).$$

  In either case, again, $\mathbf{j}(c_{k+1}\sigma_\pi(r)) = \mathbf{j}(r)$.

Finally, after all these cases, we can see that $\mathbf{j}$ is constant on the orbits of $c_{k+1}\sigma_\pi$, i.e. $\mathbf{j} \in \mathcal{J}$. Of course $\mathbf{r_j} = \mathbf{j}|_{[k]} = \mathbf{i}$, so $\mathbf{j} \mapsto \mathbf{r_j}$ is surjective. □

**Corollary 3.27.** *If $k$ is even, then the $k$-th moment of $\mu_\alpha$ is*

$$\sum_{\mathbf{i}:[k]\to[d]} \left( \sum_{\substack{\pi \in P_{\leq 2}(k) \\ \mathbf{i}\circ c_k = \mathbf{i}\circ\sigma_\pi}} (-1)^{m_1(\pi)/2}(m_1(\pi) - 1)!!\alpha_{\mathbf{i}(1)} \prod_{V\in\pi} \alpha_{\mathbf{i}(\max(V))} \right)$$

*Proof.* This is simply the product of using Proposition 3.26 in the centering formula from Proposition 3.25. □

**Coloured partitions**

**Notation 3.28.** Write $P^{\circ\bullet}(k)$ for the set of partitions of $[k]$ with each block coloured white or black, and make similar notation for subsets of $P(k)$, such as $P_2^{\circ\bullet}(k)$. For $\pi \in P^{\circ\bullet}(k)$, write $\pi^\circ$ for the set of white blocks and $\pi^\bullet$ for the set of black blocks. For $\pi \in P_2^{\circ\bullet}(k)$, write

$$\sigma_\pi^\circ := \prod_{(a,b)\in\pi^\circ} (a,b) \in S_k.$$

**Proposition 3.29.** *For $k \geq 1$, the $k$-th moment of $\mu_\alpha$ is*

$$\sum_{\mathbf{i}:[k]\to[d]} \alpha_{\mathbf{i}(1)} \left( \sum_{\substack{\pi\in P_2^{\circ\bullet}(k) \\ \mathbf{i}\circ c_k=\mathbf{i}\circ\sigma_\pi^\circ}} \prod_{(p,q)\in\pi^\circ} \alpha_{\mathbf{i}(q)} \prod_{(p,q)\in\pi^\bullet} (-\alpha_{\mathbf{i}(p)}\alpha_{\mathbf{i}(q)}) \right).$$

This is a simple restatement of Corollary 3.27, due to the relation between $P_{\leq 2}(k)$ and $P_2^{\circ\bullet}(k)$:

**Lemma 3.30.** *If $k$ is even, then there is a surjection*

$$P_2^{\circ\bullet}(k) \twoheadrightarrow P_{\leq 2}(k)$$

*such that the preimage of each $\pi \in P_{\leq 2}(k)$ has cardinality $(m_1(\pi) - 1)!!$.*

*Proof.* For $\pi \in P_2^{\circ\bullet}$, define $\pi'$ by replacing each black pair $(p,q) \in \pi_\bullet$ with singleton blocks $\{p\}$ and $\{q\}$. This is surjective because for $\rho \in P_{\leq 2}(k)$, the number of singleton blocks is $k - 2m_2(\rho)$, which is even. Then one can colour the pairings white and pick a pairing of the remaining singletons, colouring the blocks of this pairing black. In this procedure, there are $|P_2(m_1(\rho))|$ choices which lead to different elements of $P_2^{\circ\bullet}(k)$, so the preimage of $\rho$ has cardinality $|P_2(m_1(\rho))| = (m_1(\rho) - 1)!!$. □

*Proof of Proposition 3.29.* If $k$ is odd, then $P_2^{\circ\bullet}(k)$ is empty, so the sum is empty and the claim is true since the odd moments in Corollary 3.17 are 0. If $k$ is even, then the claim follows immediately from Lemma 3.30 and Corollary 3.27. □

## 3.4 CCR deformation of the traceless GUE

**Definition 3.31.** Let $(\mathcal{A}, \varphi)$ be a $*$-probability space and let $(\omega_{(1,*)}, \omega_{(*,1)})$ be a pair of parameters in $(0, \infty)$. An element $a \in \mathcal{A}$ is *centered complex CCR-gaussian* with parameters $(\omega_{(1,*)}, \omega_{(*,1)})$ if

1. $a^*a - aa^* = (\omega_{(*,1)} - \omega_{(1,*)}) \cdot 1$, and

2. we have
$$\varphi(a^p(a^*)^q) = \begin{cases} p!\omega_{(1,*)}^p & \text{if } p = q \\ 0 & \text{otherwise} \end{cases}$$
for $p, q \geq 0$.

**Definition 3.32.** Let $(\mathcal{A}, \varphi)$ be a $*$-probability space and suppose that there are some commuting-independent unital $*$-subalgebras $\mathcal{A}_0 \leq \mathcal{A}$ and $\mathcal{A}_{i,j} \leq \mathcal{A}$ for $1 \leq i < j \leq d$, with $\mathcal{A}_0$ commutative. For $\alpha_1, \ldots, \alpha_d > 0$ with $\sum_{i=1}^d \alpha_i = 1$, a *traceless CCR-GUE matrix* with parameters $(\alpha_1, \ldots, \alpha_d)$ is a matrix $A = (a_{ij})_{1 \leq i,j \leq d} \in M_d(\mathcal{A})$ such that

1. $a_{11}, \ldots, a_{dd} \in \mathcal{A}_0$ are self-adjoint and form a centered gaussian family with covariance matrix
$$\begin{pmatrix} \alpha_1 - \alpha_1^2 & & -\alpha_i\alpha_j \\ & \ddots & \\ -\alpha_i\alpha_j & & \alpha_d - \alpha_d^2 \end{pmatrix},$$
and

2. for $1 \leq i < j \leq d$, $a_{ij} \in \mathcal{A}_{i,j}$ is a centered complex CCR-gaussian element with parameters $(\alpha_j, \alpha_i)$, and $a_{ji} = a_{ij}^*$.

The expectation functional $\varphi_\alpha$ on $M_d(\mathcal{A})$ is defined by
$$\varphi_\alpha(X) = \sum_{i=1}^d \alpha_i \varphi(x_{ii})$$
for $X = (x_{ij})_{i,j} \in M_d(\mathcal{A})$.

The main result of this chapter is the following:

**Theorem 3.33.** *For $k \geq 1$, we have*

$$\int_{\mathbb{R}} t^k \, d\mu_\alpha(t) = \varphi_\alpha(A^k) \tag{3.2}$$

*where $A$ is a traceless CCR-GUE matrix with parameters $(\alpha_1, \ldots, \alpha_d)$.*

Let us briefly outline the proof: a lot of the work has already been done in Section 3.3, and we have a combinatorial formula for the left-hand side of Eq. (3.2) in Proposition 3.29. On the other hand, we must work out the combinatorics on the random matrix side; in Theorem 3.43 we will establish a Wick formula for CCR-gaussian elements, and use it in Proposition 3.45 to compute the mixed moments of entries of a CCR-GUE matrix. Then, the proof of Theorem 3.33 will amount to an easy computation of expected traces, and comparison with Proposition 3.29.

### 3.4.1 Construction of CCR-gaussian elements

Before studying CCR-GUE matrices, we should take a moment to explain why they exist. In order to concretely build a traceless CCR-GUE matrix, it suffices to find a concrete construction of a single CCR-gaussian element:

**Remark 3.34.** Suppose that we have the following $*$-probability spaces and elements therein:

1. $(\mathcal{A}_{i,j}, \varphi_{i,j})$, with centered complex CCR-gaussian elements $a_{ij} \in \mathcal{A}_{i,j}$ with parameters $(\alpha_j, \alpha_i)$, for $1 \leq i < j \leq d$;

2. $(\mathcal{A}_0, \varphi_0)$, with $\mathcal{A}_0$ commutative, and with a centered self-adjoint gaussian family $\{a_{11}, \ldots, a_{dd}\}$ with covariance matrix

$$\begin{pmatrix} \alpha_1 - \alpha_1^2 & & -\alpha_i\alpha_j \\ & \ddots & \\ -\alpha_i\alpha_j & & \alpha_d - \alpha_d^2 \end{pmatrix}.$$

Let

$$\mathcal{A} := \mathcal{A}_0 \otimes \bigotimes_{1\le i<j\le d} \mathcal{A}_{i,j} \text{ and } \varphi := \varphi_0 \otimes \bigotimes_{1\le i<j\le d} \varphi_{i,j},$$

write $a_{ji} := a_{ij}^*$ for $1 \le i < j \le d$, and $A := (a_{ij})_{i,j} \in M_d(\mathcal{A})$. Then $A$ is a traceless CCR-GUE matrix.

Of course, of the items mentioned above, (1) is the more interesting one; for (2), we just need to verify that the covariance matrix is positive semidefinite.

**Proposition 3.35.** *For $1 \ge \alpha_1 \ge \ldots \ge \alpha_d > 0$ with $\sum_{i=1}^d \alpha_i = 1$, the matrix*

$$C := \begin{pmatrix} \alpha_1 - \alpha_1^2 & & -\alpha_i\alpha_j \\ & \ddots & \\ -\alpha_i\alpha_j & & \alpha_d - \alpha_d^2 \end{pmatrix}$$

*is positive semidefinite. Consequently, there is a centered real gaussian family with this covariance matrix.*

*Proof.* By Sylvester's criterion (see e.g. [25, Theorem 7.2.5]) it suffices to show $\det(C) \ge 0$ and $\det(C([k],[k])) > 0$ for all $1 \le k \le d-1$. From e.g. [25, Equation 0.8.5.11], for a vector $v$ and an invertible matrix $D$, we have

$$\det(D - vv^T) = \det(D) - v^T \mathrm{adj}(D) v = \det(D)(1 - v^T D^{-1} v).$$

In the case $v = (\alpha_1, \ldots, \alpha_k)$ and $D = \mathrm{diag}(\alpha_1, \ldots, \alpha_k)$, the right-hand side is

$$\alpha_1 \cdots \alpha_k (1 - (\alpha_1 + \cdots + \alpha_k))$$

which is non-negative for $k = d$ and positive for $k < d$. □

As for (1), let us first review a well-known construction: the symmetric Fock space and its creation and annihilation operators. We will mostly follow the paper [8], setting $q = 1$.

**Notation 3.36.** For a Hilbert space $H$, write

$$\mathcal{F} := \mathcal{F}(H) = \mathbb{C}\Omega \oplus \bigoplus_{n\ge1} H^{\otimes n}$$

for the full Fock space and write

$$\mathcal{F}_0 := \operatorname{span}(\{\Omega\} \cup \{\xi_1 \otimes \cdots \otimes \xi_n : \xi_1, \ldots, \xi_n \in H, n \geq 1\})$$

for the dense subspace of algebraic tensors.

**Notation 3.37.** For $\xi \in H$, define linear operators $c^*(\xi)$ and $c(\xi)$ on $\mathcal{F}_0$, called *creation* and *annihilation* operators respectively, by

$$c^*(\xi)\Omega = \xi \text{ and } c^*(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n,$$

and

$$c(\xi)\Omega = 0 \text{ and } c(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \sum_{k=1}^{n} \langle \xi_k, \xi \rangle \xi_1 \otimes \cdots \otimes \xi_{k-1} \otimes \xi_{k+1} \otimes \cdots \otimes \xi_n,$$

for $\xi_1, \ldots, \xi_n \in H$.

**Proposition 3.38.** *We have*

$$c(\xi)c^*(\eta) - c^*(\eta)c(\xi) = \langle \eta, \xi \rangle \cdot 1$$

*for* $\xi, \eta \in H$.

*Proof.* Fix $\xi, \eta \in H$. For $\xi_1, \ldots, \xi_n \in H$, we have

$$\begin{aligned}
c(\xi)c^*(\eta)(\xi_1 \otimes \cdots \otimes \xi_n) &= c(\xi)(\eta \otimes \xi_1 \otimes \cdots \otimes \xi_n) \\
&= \langle \eta, \xi \rangle \xi_1 \otimes \cdots \otimes \xi_n \\
&\qquad + \sum_{k=1}^{n} \langle \xi_k, \xi \rangle \eta \otimes \xi_1 \otimes \cdots \otimes \xi_{k-1} \otimes \xi_{k+1} \otimes \cdots \otimes \xi_n \\
&= \langle \eta, \xi \rangle \xi_1 \otimes \cdots \otimes \xi_n \\
&\qquad \eta \otimes \left( \sum_{k=1}^{n} \langle \xi_k, \xi \rangle \xi_1 \otimes \cdots \otimes \xi_{k-1} \otimes \xi_{k+1} \otimes \cdots \otimes \xi_n \right) \\
&= \langle \eta, \xi \rangle \xi_1 \otimes \cdots \otimes \xi_n + c^*(\eta)c(\xi)\xi_1 \otimes \cdots \otimes \xi_n
\end{aligned}$$

so

$$(c(\xi)c^*(\eta) - c^*(\eta)c(\xi))(\xi_1 \otimes \cdots \otimes \xi_n) = \langle \eta, \xi \rangle \xi_1 \otimes \cdots \otimes \xi_n$$

hence the claim. □

**Proposition 3.39.** *There is a unique sesquilinear form* $\langle\cdot,\cdot\rangle_{\mathcal{F}}$ *on* $\mathcal{F}_0$ *with*

$$\langle \xi_1\otimes\cdots\otimes\xi_n, \eta_1\otimes\cdots\otimes\eta_m\rangle_{\mathcal{F}} = \begin{cases} \sum_{k=1}^n \langle \xi_1,\eta_k\rangle\langle \xi_2\otimes\cdots\otimes\xi_n, \eta_1\otimes\cdots\otimes\eta_{k-1}\otimes\eta_{k+1}\otimes\cdots\otimes\eta_m\rangle_{\mathcal{F}} & \text{if } n=m \\ 0 & \text{otherwise}\end{cases}$$

*for* $\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_m\in H$, *for* $n,m\geq 1$, *and it is symmetric. Moreover,*

1. *for* $\xi\in H$, $c^*(\xi)$ *is the adjoint of* $c(\xi)$ *with respect to* $\langle\cdot,\cdot\rangle_{\mathcal{F}}$*;*
2. $\langle\cdot,\cdot\rangle_{\mathcal{F}}$ *is positive semidefinite.*

*Proof.* The prescribed values of $\langle\cdot,\cdot\rangle_{\mathcal{F}}$ define a sesquilinear form by recursion and (conjugate-)linearity. First of all, for $\zeta\in H$, we can see that

$$\begin{aligned}\langle c^*(\zeta)\xi_1\otimes\cdots\otimes\xi_n, \eta_1\otimes\cdots\otimes\eta_{n+1}\rangle_{\mathcal{F}} &= \langle \zeta\otimes\xi_1\otimes\cdots\otimes\xi_n, \eta_1\otimes\cdots\otimes\eta_{n+1}\rangle_{\mathcal{F}} \\ &= \sum_{k=1}^{n+1}\langle\zeta,\eta_k\rangle\langle \xi_1\otimes\cdots\otimes\xi_n, \eta_1\otimes\cdots\otimes\eta_{k-1}\otimes\eta_{k+1}\otimes\cdots\otimes\eta_{n+1}\rangle_{\mathcal{F}} \\ &= \left\langle \xi_1\otimes\cdots\otimes\xi_n, \sum_{k=1}^{n+1}\overline{\langle\zeta,\eta_k\rangle}\eta_1\otimes\cdots\otimes\eta_{k-1}\otimes\eta_{k+1}\otimes\cdots\otimes\eta_{n+1}\right\rangle_{\mathcal{F}} \\ &= \left\langle \xi_1\otimes\cdots\otimes\xi_n, \sum_{k=1}^{n+1}\langle\eta_k,\zeta\rangle\eta_1\otimes\cdots\otimes\eta_{k-1}\otimes\eta_{k+1}\otimes\cdots\otimes\eta_{n+1}\right\rangle_{\mathcal{F}} \\ &= \langle \xi_1\otimes\cdots\otimes\xi_n, c(\zeta)\eta_1\otimes\cdots\otimes\eta_{n+1}\rangle_{\mathcal{F}}\end{aligned}$$

which amounts to (1). To see that $\langle\cdot,\cdot\rangle_{\mathcal{F}}$ is symmetric and positive-semidefinite, observe that the given definition can be re-written as

$$\langle \xi_1\otimes\cdots\otimes\xi_n, \eta_1\otimes\cdots\otimes\eta_n\rangle_{\mathcal{F}} = \sum_{\sigma\in S_n}\prod_{k=1}^n\langle\xi_k,\eta_{\sigma(k)}\rangle$$

so in particular, one can re-index the sum and product and use the symmetry of $\langle\cdot,\cdot\rangle$ to recover

$$\overline{\langle \eta_1\otimes\cdots\otimes\eta_n, \xi_1\otimes\cdots\otimes\xi_n\rangle_{\mathcal{F}}}.$$

For positive-semidefiniteness, let $\{e_{\mathbf{i}(1)}\otimes\cdots\otimes e_{\mathbf{i}(n)} : \mathbf{i}:[n]\to\mathbb{N}\}$ be the canonical orthonormal basis of $H^{\otimes n}$ and observe that

$$\langle \eta_1\otimes\cdots\otimes\eta_n, \eta_1\otimes\cdots\otimes\eta_n\rangle_{\mathcal{F}} = \sum_{\sigma\in S_n}\langle \eta_1\otimes\cdots\otimes\eta_n, \eta_{\sigma(1)}\otimes\cdots\otimes\eta_{\sigma(n)}\rangle$$

$$
\begin{aligned}
&= \frac{1}{n!} \sum_{\sigma,\tau \in S_n} \langle \eta_{\sigma(1)} \otimes \cdots \otimes \eta_{\sigma(n)}, \eta_{\tau(1)} \otimes \cdots \otimes \eta_{\tau(n)} \rangle \\
&= \frac{1}{n!} \sum_{\sigma,\tau \in S_n} \Bigg\langle \eta_{\sigma(1)} \otimes \cdots \otimes \eta_{\sigma(n)}, \\
&\qquad \sum_{\mathbf{i}:[n]\to\mathbb{N}} \langle e_{\mathbf{i}(1)} \otimes \cdots \otimes e_{\mathbf{i}(n)}, \eta_{\tau(1)} \otimes \cdots \otimes \eta_{\tau(n)} \rangle e_{\mathbf{i}(1)} \otimes \cdots \otimes e_{\mathbf{i}(n)} \Bigg\rangle \\
&= \frac{1}{n!} \sum_{\mathbf{i}:[n]\to\mathbb{N}} \Bigg( \sum_{\sigma,\tau \in S_n} \langle \eta_{\sigma(1)} \otimes \cdots \otimes \eta_{\sigma(n)}, e_{\mathbf{i}(1)} \otimes \cdots \otimes e_{\mathbf{i}(n)} \rangle \\
&\qquad \overline{\langle \eta_{\tau(1)} \otimes \cdots \otimes \eta_{\tau(n)}, e_{\mathbf{i}(1)} \otimes \cdots \otimes e_{\mathbf{i}(n)} \rangle} \Bigg) \\
&= \frac{1}{n!} \sum_{\mathbf{i}:[n]\to\mathbb{N}} \left| \left\langle \sum_{\sigma \in S_n} \eta_{\sigma(1)} \otimes \cdots \otimes \eta_{\sigma(n)}, e_{\mathbf{i}(1)} \otimes \cdots \otimes e_{\mathbf{i}(n)} \right\rangle \right|^2 \\
&\geq 0
\end{aligned}
$$

which completes the proof that $\langle \cdot, \cdot \rangle_{\mathcal{F}}$ is a positive-semidefinite symmetric sesquilinear form on $\mathcal{F}_0$. □

**Notation 3.40.** Write $\mathcal{F}_1$ for the Hilbert space completion of $\mathcal{F}_0$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{F}}$, modulo the kernel of the latter, and let $\varphi$ be the vector state on $\mathcal{B}(\mathcal{F}_1)$ corresponding to the vacuum vector $\Omega$, i.e.

$$\varphi(T) = \langle T\Omega, \Omega \rangle_{\mathcal{F}}$$

for $T \in \mathcal{B}(\mathcal{F}_1)$.

**Proposition 3.41** (Wick formula)**.** *For $\xi \in H$, we have*

$$\varphi(c^{\epsilon_1}(\xi) \cdots c^{\epsilon_k}(\xi)) = \sum_{\pi \in P_2(k)} \prod_{(r,s) \in \pi} \varphi(c^{\epsilon_r}(\xi) c^{\epsilon_s}(\xi))$$

*for $\epsilon_1, \dots, \epsilon_k \in \{1, *\}$, with covariances*

$$\begin{pmatrix} \varphi(c(\xi)c(\xi)) & \varphi(c(\xi)c^*(\xi)) \\ \varphi(c^*(\xi)c(\xi)) & \varphi(c^*(\xi)c^*(\xi)) \end{pmatrix} = \begin{pmatrix} 0 & \|\xi\|^2 \\ 0 & 0 \end{pmatrix}.$$

With this construction, we can finally show how to construct a CCR-gaussian element:

**Theorem 3.42.** *For orthogonal* $\xi, \eta \in H$, $c(\xi) + c^*(\eta)$ *is CCR-gaussian with parameters* $(\|\xi\|^2, \|\eta\|^2)$.

*Proof.* Let $\xi$ and $\eta$ be orthogonal vectors in $H$, write $x := c(\xi)$, $y := c(\eta)$, and $a := x + y^*$. Then

$$\begin{aligned} a^*a - aa^* &= (x^* + y)(x + y^*) - (x + y^*)(x^* + y) \\ &= x^*x + x^*y^* + yx + yy^* - xx^* - xy - y^*x^* - y^*y \\ &= (x^*x - xx^*) + (yy^* - y^*y) + (x^*y^* - y^*x^*) + (yx - xy) \\ &= -\|\xi\|^2 + \|\eta\|^2 + (x^*y - y^*x^*) + (yx - xy) \end{aligned}$$

and to finish the claimed commutation relation, it suffices to show that $x$ and $y$ commute. For this, simply observe that for $\xi_1, \ldots, \xi_n \in H$, we have

$$\begin{aligned} c(\xi)c(\eta)\xi_1 \otimes \cdots \otimes \xi_n &= c(\xi) \sum_{k=1}^{n} \langle \xi_k, \eta \rangle \xi_1 \otimes \cdots \otimes \xi_{k-1} \otimes \xi_{k+1} \otimes \cdots \otimes \xi_n \\ &= \sum_{k \neq l} \langle \xi_k, \eta \rangle \langle \xi_l, \xi \rangle \bigotimes_{r \in [n] \setminus \{k,l\}} \xi_r \end{aligned}$$

and

$$\begin{aligned} c(\eta)c(\xi)\xi_1 \otimes \cdots \otimes \xi_n &= c(\eta) \sum_{k=1}^{n} \langle \xi_k, \xi \rangle \xi_1 \otimes \cdots \otimes \xi_{k-1} \otimes \xi_{k+1} \otimes \cdots \otimes \xi_n \\ &= \sum_{k \neq l} \langle \xi_k, \xi \rangle \langle \xi_l, \eta \rangle \bigotimes_{r \in [n] \setminus \{k,l\}} \xi_r \end{aligned}$$

which are equal. For the prescribed moments, we have

$$\begin{aligned} \varphi(aa^*) &= \varphi(xx^*) + \varphi(xy) + \varphi(y^*x^*) + \varphi(y^*y) \\ &= \varphi(c(\xi)c^*(\xi)) + 0 + 0 + 0 = \|\xi\|^2 \end{aligned}$$

and

$$\varphi(a^*a) = \varphi(x^*x) + \varphi(x^*y^*) + \varphi(yx) + \varphi(yy^*)$$

$$= 0 + 0 + 0 + \varphi(yy^*) = \|\eta\|^2,$$

since

$$\varphi(xy) = \langle c(\xi)c(\eta)\Omega, \Omega\rangle_{\mathcal{F}} = \langle c(\eta)\Omega, c^*(\xi)\Omega\rangle_{\mathcal{F}} = \langle 0, \xi\rangle_{\mathcal{F}} = 0$$

and then also $\varphi(y^*x^*) = \varphi((xy)^*) = \overline{\varphi(xy)} = 0$, and similarly $\varphi(yx) = \varphi(x^*y^*) = 0$. Then for $p, q \geq 0$, we have

$$\begin{aligned}
\varphi(a^p(a^*)^q) &= \varphi((c(\xi) + c^*(\eta))^p(c^*(\xi) + c(\eta))^q) \\
&= \sum_{i=1}^{p}\sum_{j=1}^{q}\binom{p}{i}\binom{q}{j}\varphi(c(\xi)^i c^*(\eta)^{p-i}c^*(\xi)^j c(\eta)^{q-j}) \\
&= \sum_{i=1}^{p}\sum_{j=1}^{q}\binom{p}{i}\binom{q}{j}\varphi(c(\xi)^i c^*(\xi)^j c^*(\eta)^{p-i} c(\eta)^{q-j}) \\
&= \sum_{i=1}^{p}\binom{p}{i}\langle c(\xi)^i c^*(\xi)^q c^*(\eta)^{p-i}\Omega, \Omega\rangle_{\mathcal{F}} \qquad (c(\eta)\Omega = 0) \\
&= \sum_{i=1}^{p}\binom{p}{i}\langle c(\xi)^i \underbrace{\xi\otimes\cdots\otimes\xi}_{q}\otimes\underbrace{\eta\otimes\cdots\otimes\eta}_{p-i}, \Omega\rangle_{\mathcal{F}}
\end{aligned}$$

since, again, $c(\xi)$ and $c(\eta)$ commute.

Notice that

$$\begin{aligned}
c(\xi)\underbrace{\xi\otimes\cdots\otimes\xi}_{q}\otimes\underbrace{\eta\otimes\cdots\otimes\eta}_{p-i} &= \sum_{j=1}^{q}\langle\xi,\xi\rangle\underbrace{\xi\otimes\cdots\otimes\xi}_{q-1}\otimes\underbrace{\eta\otimes\cdots\otimes\eta}_{p-i} \\
&\quad + \sum_{j=q+1}^{q+p-i}\langle\eta,\xi\rangle\underbrace{\xi\otimes\cdots\otimes\xi}_{q}\otimes\underbrace{\eta\otimes\cdots\otimes\eta}_{p-i-1} \\
&= \sum_{j=1}^{q}\langle\xi,\xi\rangle\underbrace{\xi\otimes\cdots\otimes\xi}_{q-1}\otimes\underbrace{\eta\otimes\cdots\otimes\eta}_{p-i}
\end{aligned}$$

since $\langle\eta,\xi\rangle = 0$. Now, if $p = q$, then

$$c(\xi)^i\underbrace{\xi\otimes\cdots\otimes\xi}_{p}\otimes\underbrace{\eta\otimes\cdots\otimes\eta}_{p-i} = \sum_{j_1=1}^{p}\cdots\sum_{j_i=1}^{p-i+1}\langle\xi,\xi\rangle^i\underbrace{\xi\otimes\cdots\otimes\xi}_{p-i}\otimes\underbrace{\eta\otimes\cdots\otimes\eta}_{p-i}$$

$$= \frac{p!}{(p-i)!}\|\xi\|^{2i}\underbrace{\xi\otimes\cdots\otimes\xi}_{p-i}\otimes\underbrace{\eta\otimes\cdots\otimes\eta}_{p-i}$$

and

$$\langle\underbrace{\xi\otimes\cdots\otimes\xi}_{p-i}\otimes\underbrace{\eta\otimes\cdots\otimes\eta}_{p-i},\Omega\rangle_{\mathcal{F}} = \begin{cases}1 & \text{if } i=p\\ 0 & \text{otherwise}\end{cases}$$

so $\varphi(a^p(a^*)^p) = p!\|\xi\|^{2p}$. On the other hand, if $p \neq q$, then for

$$c(\xi)^i\underbrace{\xi\otimes\cdots\otimes\xi}_{q}\otimes\underbrace{\eta\otimes\cdots\otimes\eta}_{p-i} \tag{3.3}$$

there are some cases:

- if $p < q$, then $i < q$, so Eq. (3.3) has degree $(q-i)+(p-i) > 0$ and is orthogonal to $\Omega$;
- if $p > q$, then for $1 \leq i \leq p$, there are two subcases:
  - if $1 \leq i < q$, then $i < p$ and

    $$c(\xi)^i\underbrace{\xi\otimes\cdots\otimes\xi}_{q}\otimes\underbrace{\eta\otimes\cdots\otimes\eta}_{p-i}$$

    has degree $(p-i)-(p-q) = q-i > 0$ and is orthogonal to $\Omega$;
  - if $q \leq i \leq p$, then when we compute

    $$c(\xi)^i\underbrace{\xi\otimes\cdots\otimes\xi}_{q}\otimes\underbrace{\eta\otimes\cdots\otimes\eta}_{p-i},$$

    there will be a factor of $\langle\eta,\xi\rangle = 0$ in each summand.

In any case,

$$\langle c(\xi)^i\underbrace{\xi\otimes\cdots\otimes\xi}_{q}\otimes\underbrace{\eta\otimes\cdots\otimes\eta}_{p-i}\Omega,\Omega\rangle_{\mathcal{F}} = 0$$

so $\varphi(a^p(a^*)^q) = 0$. □

### 3.4.2 Moments of CCR-gaussian elements

**Theorem 3.43** (Wick formula)**.** *Let $(\mathcal{A}, \varphi)$ be a $*$-probability space and let $a \in \mathcal{A}$ be a centered complex CCR-gaussian element with parameters $(\omega_{(1,*)}, \omega_{(*,1)})$. Then*

$$\varphi(a^{\epsilon_1} \cdots a^{\epsilon_k}) = \sum_{\pi \in P_2(k)} \prod_{(p,q) \in \pi} \omega_{(\epsilon_p, \epsilon_q)} \tag{3.4}$$

*for $\epsilon_1, \ldots, \epsilon_k \in \{1, *\}$, where for the sake of notation we write $\omega_{(1,1)} = \omega_{(*,*)} = 0$.*

*Proof.* We will proceed by induction on

$$\mathrm{inv}(\epsilon) = |\{(p,q) : 1 \leq p < q \leq k, \epsilon(p) = *, \epsilon(q) = 1\}|.$$

For the base case $\mathrm{inv}(\epsilon) = 0$, we have $\epsilon = (1, \ldots, 1, *, \ldots, *)$, so

$$\varphi(a^{\epsilon_1} \cdots a^{\epsilon_k}) = \varphi(a^r (a^*)^s)$$

for some $r + s = k$. On the right-hand side of Eq. (3.4), the $\pi \in P_2(k)$ whose summands are non-zero are exactly the ones with $p \in \{1, \ldots, r\}$ and $q \in \{r+1, \ldots, r+s\}$ for all $(p,q) \in \pi$. These summands are $\omega_{(1,*)}^r$, and there are $r!$ of them, so putting all this together, we have

$$\varphi(a^{\epsilon_1} \cdots a^{\epsilon_k}) = \varphi(a^r (a^*)^s) = r! \omega_{(1,*)}^r = \sum_{\pi \in P(k)} \prod_{(p,q) \in \pi} \omega_{(\epsilon_p, \epsilon_q)}.$$

Now suppose the claim Eq. (3.4) is true for all $\epsilon$ with $\mathrm{inv}(\epsilon) < l$, and take $\epsilon_1, \ldots, \epsilon_k \in \{1, *\}$ with $\mathrm{inv}(\epsilon) = l$. Pick $1 \leq j \leq k-1$ with $\epsilon_j = *$ and $\epsilon_{j+1} = 1$ and define $\epsilon'$ by $\epsilon'_j = 1$, $\epsilon'_{j+1} = *$, and $\epsilon'|_{[k] \setminus \{j,j+1\}} = \epsilon|_{[k] \setminus \{j,j+1\}}$. Then

$$\begin{aligned} a^{\epsilon_1} \cdots a^{\epsilon_k} &= (a^{\epsilon_1} \cdots a^{\epsilon_{j-1}})(a^* a)(a^{\epsilon_{j+2}} \cdots a^{\epsilon_k}) \\ &= (a^{\epsilon_1} \cdots a^{\epsilon_{j-1}})(a a^* + (\omega_{(*,1)} - \omega_{(1,*)}))(a^{\epsilon_{j+2}} \cdots a^{\epsilon_k}) \\ &= a^{\epsilon'_1} \cdots a^{\epsilon'_k} + (\omega_{(*,1)} - \omega_{(1,*)}) a^{\epsilon_1} \cdots a^{\epsilon_{j-1}} a^{\epsilon_{j+2}} \cdots a^{\epsilon_k} \end{aligned}$$

and

$$\varphi(a^{\epsilon_1} \cdots a^{\epsilon_k}) = \varphi(a^{\epsilon'_1} \cdots a^{\epsilon'_k}) + (\omega_{(*,1)} - \omega_{(1,*)}) \varphi(a^{\epsilon_1} \cdots a^{\epsilon_{j-1}} a^{\epsilon_{j+2}} \cdots a^{\epsilon_k}).$$

Both monomials on the right-hand side have inv $< l$, so with

$$\epsilon'' := (\epsilon_1, \ldots, \epsilon_{j-1}, \epsilon_{j+2}, \ldots, \epsilon_k),$$

we have

$$\varphi(a^{\epsilon_1} \cdots a^{\epsilon_k}) = \sum_{\rho \in P_2(k)} \prod_{(p,q)\in\rho} \omega_{(\epsilon'_p, \epsilon'_q)} + \left(\omega_{(*,1)} - \omega_{(1,*)}\right) \sum_{\rho' \in P_2(k-2)} \prod_{(p,q)\in\rho'} \omega_{(\epsilon''_p, \epsilon''_q)}$$

and it suffices to observe that

$$\sum_{\pi \in P_2(k)} \prod_{(p,q)\in\pi} \omega_{(\epsilon_p, \epsilon_q)} = \sum_{\rho \in P_2(k)} \prod_{(p,q)\in\rho} \omega_{(\epsilon'_p, \epsilon'_q)} + \left(\omega_{(*,1)} - \omega_{(1,*)}\right) \sum_{\rho' \in P_2(k-2)} \prod_{(p,q)\in\rho'} \omega_{(\epsilon''_p, \epsilon''_q)}.$$

by the construction of $\epsilon'$ and $\epsilon''$. □

### 3.4.3 Matrix model

The following notation will be convenient in the proof of the forthcoming Proposition 3.45:

**Notation 3.44.** For $\mathbf{i}, \mathbf{j} : [k] \to [d]$, write

$$P_2^{\circ\bullet}(\mathbf{i}, \mathbf{j}) := \{\pi \in P_2^{\circ\bullet}(k) : \mathbf{i}(p) = \mathbf{j}(q) \,\&\, \mathbf{i}(q) = \mathbf{j}(p) \,\forall\, (p,q) \in \pi_\circ \text{ and } \mathbf{i}(p) = \mathbf{j}(p) \,\&\, \mathbf{i}(q) = \mathbf{j}(q) \,\forall\, (p,q) \in \pi_\bullet\}.$$

**Proposition 3.45.** *Let $A = (a_{ij})_{i,j}$ be a $d \times d$ traceless CCR-GUE matrix with parameters $(\alpha_1, \ldots, \alpha_d)$. Then*

$$\varphi(a_{\mathbf{i}(1)\mathbf{j}(1)} \cdots a_{\mathbf{i}(k)\mathbf{j}(k)}) = \sum_{\pi \in P_2^{\circ\bullet}(\mathbf{i},\mathbf{j})} \left( \prod_{(p,q)\in\pi_\circ} \alpha_{\mathbf{i}(q)} \prod_{(p,q)\in\pi_\bullet} (-\alpha_{\mathbf{i}(p)}\alpha_{\mathbf{j}(q)}) \right)$$

*for* $\mathbf{i}, \mathbf{j} : [k] \to [d]$.

*Proof.* Write

$$P_0 := \{p \in [k] : \mathbf{i}(p) = \mathbf{j}(p)\} \text{ and } C_0 := \varphi\left(\prod_{p \in P_0} a_{\mathbf{i}(p)\mathbf{j}(p)}\right),$$

and

$$P_{r,s} := \{p \in [k] : \{\mathbf{i}(p), \mathbf{j}(p)\} = \{r, s\}\} \text{ and } C_{r,s} := \varphi\left(\prod_{p \in P_{r,s}} a_{\mathbf{i}(p)\mathbf{j}(p)}\right)$$

for $1 \leq r < s \leq d$. With this notation, by commuting-independence, we can write

$$\varphi(a_{\mathbf{i}(1)\mathbf{j}(1)} \cdots a_{\mathbf{i}(k)\mathbf{j}(k)}) = C_0 \prod_{\substack{1 \leq r < s \leq d \\ P_{r,s} \neq \emptyset}} C_{r,s}$$

and apply the various Wick formulas. On the diagonal, we have

$$\begin{aligned}
C_0 = \varphi\left(\prod_{p \in P_0} a_{\mathbf{i}(p)\mathbf{j}(p)}\right) &= \sum_{\pi \in P_2(P_0)} \prod_{(p,q) \in \pi} \varphi(a_{\mathbf{i}(p)\mathbf{j}(p)} a_{\mathbf{i}(q)\mathbf{j}(q)}) \\
&= \sum_{\pi \in P_2(P_0)} \prod_{\substack{(p,q) \in \pi \\ \mathbf{i}(p) = \mathbf{i}(q)}} (\alpha_{\mathbf{i}(p)} - \alpha_{\mathbf{i}(p)}^2) \prod_{\substack{(p,q) \in \pi \\ \mathbf{i}(p) \neq \mathbf{i}(q)}} (-\alpha_{\mathbf{i}(p)} \alpha_{\mathbf{i}(q)}) \\
&= \sum_{\pi \in P_2(P_0)} \prod_{\substack{(p,q) \in \pi \\ \mathbf{i}(p) = \mathbf{i}(q)}} (\alpha_{\mathbf{i}(q)} - \alpha_{\mathbf{i}(p)} \alpha_{\mathbf{i}(q)}) \prod_{\substack{(p,q) \in \pi \\ \mathbf{i}(p) \neq \mathbf{i}(q)}} (-\alpha_{\mathbf{i}(p)} \alpha_{\mathbf{i}(q)}) \\
&= \sum_{\substack{\pi \in P_2^{\circ\bullet}(P_0) \\ \text{every } (p,q) \in \pi \text{ with } \mathbf{i}(p) \neq \mathbf{i}(q) \\ \text{is coloured } \bullet}} \prod_{(p,q) \in \pi_\circ} \alpha_{\mathbf{i}(q)} \prod_{(p,q) \in \pi_\bullet} (-\alpha_{\mathbf{i}(p)} \alpha_{\mathbf{i}(q)})
\end{aligned}$$

and off the diagonal, we have

$$\begin{aligned}
C_{r,s} = \varphi\left(\prod_{p \in P_{r,s}} a_{\mathbf{i}(p)\mathbf{j}(p)}\right) &= \sum_{\pi \in P_2(P_{r,s})} \prod_{(p,q) \in \pi} \varphi(a_{\mathbf{i}(p)\mathbf{j}(p)} a_{\mathbf{i}(q)\mathbf{j}(q)}) \\
&= \sum_{\pi \in P_2(P_{r,s})} \prod_{(p,q) \in \pi} \delta_{\mathbf{j}(p) = \mathbf{i}(q)} \delta_{\mathbf{i}(p) = \mathbf{j}(q)} \alpha_{\mathbf{i}(q)}
\end{aligned}$$

so

$$\varphi(a_{\mathbf{i}(1)\mathbf{j}(1)} \cdots a_{\mathbf{i}(k)\mathbf{j}(k)})$$

$$= \left( \sum_{\substack{\pi \in P_2^{\circ\bullet}(P_0) \\ \text{every } (p,q)\in\pi \text{ with } \mathbf{i}(p)\neq\mathbf{i}(q) \\ \text{is coloured } \bullet}} \prod_{(p,q)\in\pi_\circ} \alpha_{\mathbf{i}(q)} \prod_{(p,q)\in\pi_\bullet} (-\alpha_{\mathbf{i}(p)}\alpha_{\mathbf{i}(q)}) \right) \tag{3.5}$$

$$\prod_{\substack{1\leq r<s\leq d \\ P_{r,s}\neq\emptyset}} \left( \sum_{\pi\in P_2(P_{r,s})} \prod_{(p,q)\in\pi} \delta_{\mathbf{j}(p)=\mathbf{i}(q)} \delta_{\mathbf{i}(p)=\mathbf{j}(q)} \alpha_{\mathbf{i}(q)} \right). \tag{3.6}$$

On the right-hand side of the claim, each $\pi \in P_2^{\circ\bullet}(\mathbf{i},\mathbf{j})$ is forced to have either $p, q \in P_0$, or else $p, q \in P_{r,s}$ for some $1 \leq r < s \leq d$, and in the latter case, $(p,q) \in \pi_\circ$. (This can be verified by tedious casework.) So we can split up the sum as

$$\sum_{\pi\in P_2^{\circ\bullet}(\mathbf{i},\mathbf{j})} \prod_{(p,q)\in\pi_\circ} \alpha_{\mathbf{i}(q)} \prod_{(p,q)\in\pi_\bullet} (-\alpha_{\mathbf{i}(p)}\alpha_{\mathbf{i}(q)})$$

$$= \left( \sum_{\pi\in P^{(0)}(\mathbf{i},\mathbf{j})} \prod_{(p,q)\in\pi_\circ} \alpha_{\mathbf{i}(q)} \prod_{(p,q)\in\pi_\bullet} (-\alpha_{\mathbf{i}(p)}\alpha_{\mathbf{i}(q)}) \right) \tag{3.7}$$

$$\prod_{\substack{1\leq r<s\leq d \\ P_{r,s}\neq\emptyset}} \left( \sum_{\pi\in P^{(r,s)}(\mathbf{i},\mathbf{j})} \prod_{(p,q)\in\pi} \alpha_{\mathbf{i}(q)} \right) \tag{3.8}$$

where

$$P^{(0)}(\mathbf{i},\mathbf{j}) := \{\pi \in P_2^{\circ\bullet}(P_0) : \mathbf{i}(p) = \mathbf{j}(q) \,\&\, \mathbf{i}(q) = \mathbf{j}(p) \,\forall\, (p,q) \in \pi_\circ$$
$$\text{and } \mathbf{i}(p) = \mathbf{j}(p) \,\&\, \mathbf{i}(q) = \mathbf{j}(q) \,\forall\, (p,q) \in \pi_\bullet\}$$

and

$$P^{(r,s)}(\mathbf{i},\mathbf{j}) := \{\pi \in P_2(P_{r,s}) : \mathbf{i}(p) = \mathbf{j}(q) \,\&\, \mathbf{i}(q) = \mathbf{j}(p) \,\forall\, (p,q) \in \pi\}.$$

Clearly (3.8) is identical to (3.6). To show (3.7) is equal to (3.5), it suffices to show that

$$\{\pi \in P_2^{\circ\bullet}(P_0) : \text{if } (p,q) \in \pi \text{ and } \mathbf{i}(p) \neq \mathbf{i}(q) \text{ then } (p,q) \in \pi_\bullet\} = P^{(0)}(\mathbf{i},\mathbf{j}).$$

To this end, take $\pi \in P^{(0)}(\mathbf{i},\mathbf{j})$, and take $(p,q) \in \pi$ with $\mathbf{i}(p) \neq \mathbf{i}(q)$. If $(p,q) \in \pi_\circ$, then $\mathbf{i}(p) = \mathbf{j}(q)$ and $\mathbf{i}(q) = \mathbf{j}(p)$, but since $p,q \in P_0$, we must also have

$$\mathbf{i}(p) = \mathbf{j}(q) = \mathbf{i}(q) = \mathbf{j}(p)$$

which is a contradiction. So we must have $(p,q) \in \pi_\bullet$. Conversely, take $\pi \in P_2^{\circ\bullet}(P_0)$ and suppose that $(p,q) \in \pi_\bullet$ whenever $(p,q) \in \pi$ and $\mathbf{i}(p) \neq \mathbf{i}(q)$. Then

- for $(p,q) \in \pi_\circ$, we must have $\mathbf{i}(p) = \mathbf{i}(q)$, so

  $$\mathbf{i}(p) = \mathbf{i}(q) = \mathbf{j}(q) \text{ and } \mathbf{i}(q) = \mathbf{i}(p) = \mathbf{j}(p),$$

  and

- for $(p,q) \in \pi_\bullet$, we have

  $$\mathbf{i}(p) = \mathbf{j}(p) \text{ and } \mathbf{i}(q) = \mathbf{j}(q)$$

  since $p,q \in P_0$,

so $\pi \in P^{(0)}(\mathbf{i},\mathbf{j})$. This finally proves the claim of the proposition. □

It may be clarifying to see a concrete example of how (3.5) and (3.7) are related in the proof of Proposition 3.45:

**Example 3.46.** Let $k = 10$ and consider

$$\varphi(a_{11}a_{12}a_{11}a_{33}a_{21}a_{31}a_{12}a_{13}a_{22}a_{21})$$

so

$$\mathbf{i} = (1,1,1,3,2,3,1,1,2,2) \text{ and } \mathbf{j} = (1,2,1,3,1,1,2,3,2,1).$$

Then $P_0 = \{1,3,4,9\}$, $P_{1,2} = \{2,5,7,10\}$, $P_{1,3} = \{6,8\}$, and $P_{r,s} = \emptyset$ for the remaining $r < s$. By commuting independence, we have

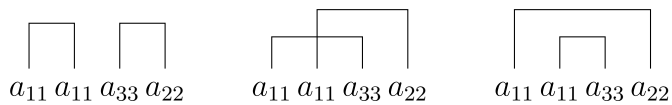
$$\varphi(a_{11}a_{12}a_{11}a_{33}a_{21}a_{31}a_{12}a_{13}a_{22}a_{21}) = \varphi(C_0)\varphi(C_{1,2})\varphi(C_{1,3})$$

where

$$C_0 = a_{11}a_{11}a_{33}a_{22},\ C_{1,2} = a_{12}a_{21}a_{12}a_{21}, \text{ and } C_{1,3} = a_{31}a_{13}.$$

On the diagonal, we have

$$\varphi(C_0) = \quad c_{1,1}c_{3,2} \quad + \quad c_{1,3}c_{1,2} \quad + \quad c_{1,2}c_{1,3}.$$

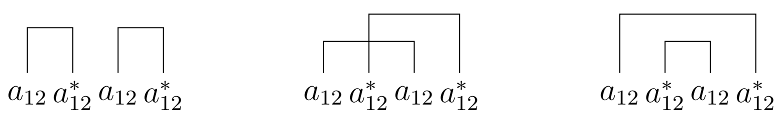



where $(c_{i,j})_{i,j}$ is the covariance matrix indicated in Definition 3.32, i.e. the above is equal to

$$(\alpha_1 - \alpha_1^2)(-\alpha_3\alpha_2) + (-\alpha_1\alpha_3)(-\alpha_1\alpha_2) + (-\alpha_1\alpha_2)(-\alpha_1\alpha_3) = (3\alpha_1^2 - \alpha_1)\alpha_2\alpha_3.$$

Off the diagonal, we have

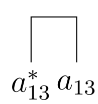
$$\varphi(C_{1,2}) = \varphi(a_{12}a_{12}^*)\varphi(a_{12}a_{12}^*) + \varphi(a_{12}a_{12})\varphi(a_{12}^*a_{12}^*) + \varphi(a_{12}a_{12}^*)\varphi(a_{12}^*a_{12})$$




$$\begin{aligned} &= \quad \alpha_2\alpha_2 \quad + \quad 0\cdot 0 \quad + \quad \alpha_2\alpha_1 \\ &= \quad \alpha_2^2 + \alpha_1\alpha_2 \end{aligned}$$

and

$$\varphi(C_{1,3}) = \varphi(a_{13}^*a_{13}) = \alpha_1.$$




Then (3.5) and (3.6) are equal to

$$(3\alpha_1^2 - \alpha_1)\alpha_2\alpha_3 \text{ and } (\alpha_2^2 + \alpha_1\alpha_2)\alpha_1$$

respectively.

For (3.7), we need to work out $P^{(0)}(\mathbf{i}, \mathbf{j})$. For the three choices of $\pi \in P_2^{\circ\bullet}(P_0)$, the only one which offers a choice of colour is $\{\{1,3\},\{4,9\}\}$. Namely, the block $\{1,3\}$ can have either colour. Any other block must be black. So

$$P^{(0)}(\mathbf{i}, \mathbf{j}) = \{\{\{1,3\}_\circ, \{4,9\}_\bullet\}, \{\{1,3\}_\bullet, \{4,9\}_\bullet\},$$

$$\{\{1,4\}_\bullet,\{3,9\}_\bullet\},\{\{1,9\}_\bullet,\{3,4\}_\bullet\}\}$$

and (3.7) is equal to

$$\begin{aligned}
&\alpha_{\mathbf{i}(3)}(-\alpha_{\mathbf{i}(4)\mathbf{i}(9)}) + (-\alpha_{\mathbf{i}(1)}\alpha_{\mathbf{i}(3)})(-\alpha_{\mathbf{i}(4)}\alpha_{\mathbf{i}(9)}) \\
&\quad + (-\alpha_{\mathbf{i}(1)}\alpha_{\mathbf{i}(4)})(-\alpha_{\mathbf{i}(3)}\alpha_{\mathbf{i}(9)}) + (-\alpha_{\mathbf{i}(1)}\alpha_{\mathbf{i}(9)})(-\alpha_{\mathbf{i}(3)}\alpha_{\mathbf{i}(4)}) \\
&= -\alpha_1\alpha_3\alpha_2 + \alpha_1\alpha_1\alpha_3\alpha_2 + \alpha_1\alpha_3\alpha_1\alpha_2 + \alpha_1\alpha_2\alpha_1\alpha_3 \\
&= (3\alpha_1^2 - \alpha_1)\alpha_2\alpha_3
\end{aligned}$$

which is equal to (3.5). This was the more tricky part of the proof, so we will omit the off-diagonal in this example.

With Proposition 3.45 in hand, along with Proposition 3.29, it is easy to prove the main theorem of this section:

*Proof of Theorem 3.33.* By Proposition 3.45, we have

$$\begin{aligned}
\varphi_\alpha(A^k) &= \sum_{\mathbf{i}:[k]\to[d]} \alpha_{\mathbf{i}(1)}\varphi(a_{\mathbf{i}(1)\mathbf{i}(2)}\cdots a_{\mathbf{i}(k)\mathbf{i}(1)}) \\
&= \sum_{\mathbf{i}:[k]\to[d]} \alpha_{\mathbf{i}(1)} \sum_{\pi\in P_2^{\circ\bullet}(k)} \prod_{(p,q)\in\pi_\circ} \delta_{\mathbf{i}(p)=\mathbf{i}(q+1)}\delta_{\mathbf{i}(q)=\mathbf{i}(p+1)}\alpha_{\mathbf{i}(q)} \\
&\qquad \prod_{(p,q)\in\pi_\bullet} \delta_{\mathbf{i}(p)=\mathbf{i}(p+1)}\delta_{\mathbf{i}(q)=\mathbf{i}(q+1)}(-\alpha_{\mathbf{i}(p)}\alpha_{\mathbf{i}(q)}) \\
&= \sum_{\mathbf{i}:[k]\to[d]} \alpha_{\mathbf{i}(1)} \sum_{\pi\in P_2^{\circ\bullet}(k)} \prod_{(p,q)\in\pi_\circ} \delta_{\mathbf{i}(\sigma_\pi^\circ(p))=\mathbf{i}(c_k(q+1))}\delta_{\mathbf{i}(\sigma_\pi^\circ(p))=\mathbf{i}(c_k(p))}\alpha_{\mathbf{i}(q)} \\
&\qquad \prod_{(p,q)\in\pi_\bullet} \delta_{\mathbf{i}(\sigma_\pi^\circ(p))=\mathbf{i}(c_k(p))}\delta_{\mathbf{i}(\sigma_\pi^\circ(q))=\mathbf{i}(c_k(q))}(-\alpha_{\mathbf{i}(p)}\alpha_{\mathbf{i}(q)}) \\
&= \sum_{\mathbf{i}:[k]\to[d]} \alpha_{\mathbf{i}(1)} \sum_{\substack{\pi\in P_2^{\circ\bullet}(k)\\ \mathbf{i}\circ\sigma_\pi^\circ=\mathbf{i}\circ c_k}} \prod_{(p,q)\in\pi_\circ} \alpha_{\mathbf{i}(q)} \prod_{(p,q)\in\pi_\bullet} (-\alpha_{\mathbf{i}(p)}\alpha_{\mathbf{i}(q)})
\end{aligned}$$

which is exactly the expression from Proposition 3.29. □

# Chapter 4

# Future work

## 4.1 Finite free commutator

In free probability, to study the commutator, it is most convenient to work with $i(ab-ba)$ to ensure self-adjointness. In the finite setting, Theorem 2.27 handles this variation as follows:

$$\mathbb{E}_U c_x(i(AUBU^* - UBU^*A)) = (p(x) \boxminus_d p(x)) \boxtimes_d (q(x) \boxminus_d q(x)) \boxtimes \widetilde{z}_d(x),$$

where

$$\widetilde{z}_d(x) := \sum_{k=0}^{\lfloor d/2 \rfloor} x^{d-2k}(-1)^k \binom{d}{2k} (d)_k \frac{k!}{(2k)!} \frac{d+1-k}{d+1}.$$

This is because

$$\begin{aligned} \mathsf{e}_{2k}(i(AUBU^* - UBU^*A)) &= i^{2k}\mathsf{e}_{2k}(AUBU^* - UBU^*A) \\ &= (-1)^k \mathsf{e}_{2k}(AUBU^* - UBU^*A). \end{aligned}$$

In terms of the wider context of finite free probability, the main question facilitated by this re-phrasing of Theorem 2.27 is about real-rootedness:

**Question 4.1.** Suppose that $p(x)$ and $q(x)$ are real-rooted (monic, degree-$d$) polynomials. Is

$$(p(x) \boxminus_d p(x)) \boxtimes_d (q(x) \boxminus_d q(x)) \boxtimes_d \widetilde{z}_d(x)$$

real-rooted?

With the assumptions above, $p(x) \boxminus_d p(x)$ and $q(x) \boxminus_d q(x)$ are real-rooted and even, but the known results related to the preservation of real-rootedness by $\boxtimes_d$ do not apply, since they require positivity. Indeed, with a computer one will quickly find that for even real-rooted $r(x)$ and $s(x)$, $r(x) \boxtimes_d s(x)$ is not necessarily real-rooted. Nonetheless, with the same computer, one will find that

$$r(x) \boxtimes_d s(x) \boxtimes_d \widetilde{z}_d(x)$$

*does* seem to be real-rooted. Proving this would answer Question 4.1, and one way to approach such a proof would be to find an applicable variation of the results on roots of polynomials which are cited in [29].

Another approach might be to write

$$r(x) = R(x^2),\ s(x) = S(x^2),\ \text{and}\ \widetilde{z}_d(x) = Z(x^2)$$

for polynomials $R(y)$, $S(y)$, and $Z(y)$ with degree $d/2$, to get rid of the negative roots, and apply the standard results to $\boxtimes_{d/2}$. The obstruction in this latter approach is in recovering a conclusion about the roots of $r(x) \boxtimes_d s(x) \boxtimes_d \widetilde{z}_d(x)$ from information about the roots of $R(y) \boxtimes_{d/2} S(y) \boxtimes_{d/2} Z(y)$. Simply substituting $x^2$ back in does not recover the right polynomial, due to the dependence of the convolution operations on the degrees.

Another important part of an answer to Question 4.1 would be a better understanding of the nature of $z_d(x)$ and $\widetilde{z}_d(x)$.

**Question 4.2.** What does $\widetilde{z}_d(x)$ represent? Is there some polynomial $r(x)$ such that $\widetilde{z}_d(x) = r(x) \boxminus_d r(x)$?

Based on the situation in free probability, one might think to look at the finite version of the free Poisson distribution, which is essentially an associated Laguerre polynomial with some normalizations [28, 2]:

$$P_{\lambda,d}(x) = \sum_{k=0}^{d} x^{d-k}(-1)^k \binom{d}{k} \frac{1}{d^k}(d\lambda)_k.$$

With $\lambda = 1 + \frac{1}{d}$, this is not too far off: the coefficient of $x^{d-2k}$ in $P_{\lambda,d}(x) \boxminus_d P_{\lambda,d}(x)$ is

$$(-1)^k \frac{1}{d^{2k}} \binom{d}{2k} (d)_k \frac{(2k)!}{k!} \frac{d+1}{d+1-k}$$

which primarily differs from $\widetilde{z}_d(x)$ by some reciprocals. It is not clear, however, what modifications might rectify this discrepancy.

## 4.2 Combinatorics of finite free probability

### 4.2.1 Review of finite free cumulants

Since [29] first appeared in 2015, the theory of finite free probability has grown somewhat in parallel with the development of free probability in the 1990s. In particular, [2] defined a sequence $(\kappa_n^{(d)})_{n\geq 1}$ of *finite free cumulants* which

- linearize $\boxplus_d$, in the sense that $\kappa_n^{(d)}(A \boxplus_d B) = \kappa_n^{(d)}(A) + \kappa_n^{(d)}(B)$;
- are related to the moments $m_n := \frac{1}{d}\mathsf{p}_n$ by

$$m_n = \frac{(-1)^{n-1}}{d^{n+1}(n-1)!} \sum_{\substack{\alpha,\beta\in S_n \\ \langle\alpha,\beta\rangle\leq S_n \text{ transitive}}} (-d)^{\#(\alpha)+\#(\beta)}\kappa_\alpha^{(d)}$$

- converge to free cumulants as $d \to \infty$.

In free probability the moment-cumulant formula

$$m_n = \sum_{\pi\in NC(n)} \kappa_\pi$$

is generalized in a certain sense by the Nica-Speicher formula [33]

$$\kappa_n(a \boxtimes b) = \sum_{\pi\in NC(n)} \kappa_\pi(a)\kappa_{K(\pi)}(b) \tag{4.1}$$

where $K(\cdot)$ is the so-called *Kreweras complement*, a special bijection on the lattice $NC(n)$ of noncrossing partitions. [1] found the finite version of this formula:

$$\kappa_n^{(d)}(A \boxtimes_d B) = \frac{(-1)^{n-1}}{d^{n+1}(n-1)!} \sum_{\substack{\alpha,\beta\in S_n \\ \langle\alpha,\beta\rangle\leq S_n \text{ transitive}}} (-d)^{\#(\alpha)+\#(\beta)}\kappa_\alpha^{(d)}(A)\kappa_\beta^{(d)}(B).$$

Moreover, they found a very nice way of organizing the complicated sum above, according to the order of $\frac{1}{d}$: for $k \geq 0$, the coefficient of $\frac{1}{d^k}$ is

$$(-1)^k \sum_{g=0}^{\lfloor k/2 \rfloor} \sum_{\substack{\zeta \vdash n \\ \ell(\zeta)=k+1-2g}} N_\zeta \sum_{\alpha \in S_{NC}^{(g)}(\gamma_\zeta)} u_\alpha v_{\alpha^{-1}\gamma_\zeta} \tag{4.2}$$

where

- $\gamma_\zeta$ is a choice of permutation with cycle type $\zeta$,
- $N_\zeta$ is the number of permutations with cycle type $\zeta$, and
- for $\gamma \in S_n$ and $g \geq 0$,

$$S_{NC}^{(g)}(\gamma) := \{\alpha \in S_n : \langle \alpha, \gamma \rangle \leq S_n \text{ transitive, genus wrt } \gamma \text{ is } g\}.$$

The first-order asymptotics straightforwardly recover the Nica-Speicher formula via the identification (see e.g. [36, Lecture 23]) of $NC(n)$ as a geodesic in $S_n$.

### 4.2.2 Cumulants of commutators

A great demonstration of the power of free cumulants was the description [35] of the commutator of free variables. The main point is that for interesting combinatorial reasons, to understand $i(ab - ba)$ for free $a, b$, one can assume without loss of generality that $a$ and $b$ are even.

**Theorem 4.3** ([35])**.** *Let $a$ and $b$ be self-adjoint, even, and free. Write $\alpha_n = \kappa_{2n}(a)$ and $\beta_n := \kappa_{2n}(b)$. Then $i(ab - ba)$ is even and*

$$\kappa_{2n}(i(ab - ba)) = 2 \sum_{\substack{\pi, \sigma \in NC(n) \\ \sigma \leq K(\pi)}} \alpha_\pi \beta_\sigma. \tag{4.3}$$

With the observation that it suffices to work with the joint distribution of $ab$ and $ba$, it turns out this falls under the general umbrella of *R-diagonality.*

Specifically, if $a$ and $b$ are even and free, then $ab$ is $R$-diagonal, so one can compute

$$\begin{aligned}\kappa_{2n}(i(ab-ba)) &= \kappa_{2n}(ab,ba,\ldots,ab,ba)+\kappa_{2n}(ba,ab,\ldots,ba,ab)\\ &= 2\kappa_{2n}(ab,ba,\ldots,ab,ba)\\ &= 2\sum_{\substack{\pi\in NC(4n)\\ \pi\vee\sqcap\cdots\sqcap=1_{4n}}}\kappa_\pi(a,b,b,a,\ldots,a,b,b,a)\\ &= 2\sum_{\substack{\pi,\sigma\in NC(2n)\\ \sigma\leq K(\pi)}}\alpha_\pi\beta_\sigma\end{aligned}$$

using some basic properties of multivariate free cumulants (namely, a certain formula for how they behave with products as arguments).

**Question 4.4.** The role of

$$\{(\pi,K(\pi)) : \pi\in NC(n)\}$$

in Eq. (4.1) is played in Eq. (4.2) by

$$\{(\alpha,\alpha^{-1}\gamma) : \alpha\in S_{NC}^{(g)}(\gamma)\}.$$

Is there a finite version of Eq. (4.3)? If so, what combinatorial indices play the role of

$$\{(\pi,\sigma) : \pi\in NC(n), \sigma\leq K(\pi)\}?$$

## 4.3 Multivariate finite free probability

The relation of free commutators to $R$-diagonality and multivariate free cumulants is hard to interpret in a finite context, and points to a general program of development in finite free probability:

**Question 4.5.** The description of the free commutator uses the notion of $R$-diagonality and multivariate free cumulants, neither of which exist yet in finite free probability. How can one remedy this?

The recent PhD thesis [32] of B. Mirabelli has started to develop a multivariate theory of finite free probability, as alluded to in Section 2.3. With these ideas in hand, one can think about where $R$-diagonality came from: it was conceived in [34] as a way of unifying Haar unitaries and circular elements. Naturally, one would guess the "finite" versions of those elements should respectively be

$$\mathbb{E}_U c_{x,y_1,y_2}(U, U^*) \text{ and } \mathbb{E}_C c_{x,y_1,y_2}(C, C^*),$$

where $U$ is a random $d \times d$ unitary matrix and $C$ is a $d \times d$ Ginibre matrix, and try to read off what they have in common. The latter was studied in [32], and the former can be naturally approached using the techniques of this thesis. These observations are a work in progress, with the forthcoming Question 4.6 pending. On the other hand, one could try to come up with multivariate finite free cumulants, perhaps using the above examples to get a sense of how the new variables $y_1, y_2$ influence the hypothetical moment-cumulant relation.

The example of $R$-diagonality which is really relevant to the commutator is the product of even elements. Section 2.3 shows that for self-adjoint $A$ and $B$, the computation of

$$\mathbb{E}_U c_{x,y_1,y_2}(AUBU^*, (AUBU^*)^*)$$

comes down to the computation of

$$\operatorname{Imm}^{(k-p,p)}(y_1 x_i + y_2 x_j)_{i,j}$$

for commuting formal variables $x_1, \ldots, x_k$ and $0 \leq p \leq k/2$.

**Question 4.6.** What is

$$\operatorname{Imm}^{(k-p,p)}(y_1 x_i + y_2 x_j)_{i,j}$$

as a symmetric function in $(x_1, \ldots, x_d)$? The answer should be given in terms of the elementary basis.

In the case of the commutator, there is a key simplification which allows one to evade the substance of this question: the immanants $\operatorname{Imm}^{(k-p,p)}(x_i -$

$x_j)_{i,j}$ do not depend on $p$ except through a sign, so it suffices to simply compute the permanent. It seems very plausible that the Goulden-Jackson formula is the way to go here, and computer experiments show that in small but substantial cases, these immanants are of the form

$$\sum_{i+j=k} (?)\, \mathsf{e}_i(\mathbf{x})\mathsf{e}_j(\mathbf{x})$$

where the unknown coefficients are some integers. To pin down what exactly they are, some new combinatorial insight is needed.

**Question 4.7.** Is the expression for

$$\mathbb{E}_U c_{x,y_1,y_2}(AUBU^*, UBU^*A)$$

in Section 2.3 made any simpler by assuming $c_x(A)$ and $c_x(B)$ are even polynomials?

## 4.4 Thoma characters

In Chapter 3, we began by working with any Thoma parameter $\omega = (\alpha, \beta) \in \Omega$, before reducing our scope at a convenient point by assuming $\alpha$ is finite and $\beta = 0$. Before the cutoff, we saw that the law of large numbers and central limit theorem apply just as well to any Thoma parameter.

Beyond the case of $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $\beta = 0$ considered in this thesis, the other simplest case is $\alpha = 0$ and $\beta = (\beta_1, \ldots, \beta_d)$. This does not seem to add much complication, except for some signs that would have to be kept track of.

**Question 4.8.** How should the central limit law in Corollary 3.18 be interpreted for more general Thoma parameters $\omega = (\alpha, \beta) \in \Omega$? Specifically, what if $\alpha, \beta$ are only assumed to be finite?

Allowing $\alpha \neq 0$ and $\beta \neq 0$ adds an immediate new layer of complication in Section 3.3.2 since expanding the product

$$\prod_{\substack{c \in \mathrm{Cyc}(\sigma) \\ |c|>1}} \left( \sum_{i \geq 1} \alpha_i^{|c|} + (-1)^{|c|-1} \sum_{j \geq 1} \beta_j^{|c|} \right)$$

will involve a sum over subsets of $\mathrm{Cyc}(\sigma)$.

**Remark 4.9.** The motivation for the specific scope of Question 4.8 is that the Young diagrams have an important natural embedding into $\Omega$ by their so-called *Frobenius coordinates*. The image of any Young diagram in $\Omega$ has finite $\alpha$ and $\beta$, but only in the most trivial cases can one of them be 0.

## 4.5 Connection with limit shapes

In the traceless CCR-GUE model from Chapter 3, the gaussian family of diagonal entries is centered and real with the covariance matrix

$$\begin{pmatrix} \alpha_1 - \alpha_1^2 & & -\alpha_i\alpha_j \\ & \ddots & \\ -\alpha_i\alpha_j & & \alpha_d - \alpha_d^2 \end{pmatrix}.$$

This covariance structure has appeared in the literature [10, 30] in an adjacent but distinct context. Besides the extremal characters of $S_\infty$, the Thoma parameters label certain probability distributions on Young diagrams, and one is interested in the asymptotics of various statistics of random Young diagrams sampled accordingly. If $r_i(\lambda)$ and $c_j(\lambda)$ are the lengths of the $i$-th row and $j$-th column of $\lambda$, respectively, then the law of large numbers is a famous one of Vershik and Kerov [44]:

$$\frac{1}{n} r_i(\lambda) \to \alpha_i \text{ and } \frac{1}{n} c_j(\lambda) \to \beta_j$$

in probability as $n \to \infty$.

The central limit theorem in this context, from [10, 30], is that in the case where $\alpha$ and $\beta$ have finite length, say $M$ and $N$ respectively, and both are strictly decreasing, the tuple

$$\left( \frac{r_1(\lambda) - \alpha_1 n}{\sqrt{n}}, \ldots, \frac{r_M(\lambda) - \alpha_M n}{\sqrt{n}}, \frac{c_1(\lambda) - \beta_1 n}{\sqrt{n}}, \ldots, \frac{c_N(\lambda) - \beta_N n}{\sqrt{n}} \right)$$

converges in joint distribution to a centered real gaussian family with covariance matrix

$$\begin{pmatrix} \alpha_1 - \alpha_1^2 & & -\alpha_i\alpha_j & & & & \\ & \ddots & & & -\alpha_i\beta_j & & \\ -\alpha_i\alpha_j & & \alpha_M - \alpha_M^2 & & & & \\ & & & \beta_1 - \beta_1^2 & & -\beta_i\beta_j & \\ & -\alpha_i\beta_j & & & \ddots & & \\ & & & -\beta_i\beta_j & & \beta_N - \beta_N^2 \end{pmatrix}.$$

**Question 4.10.** If the combinatorial work done in this thesis can be extended to finite $\alpha$ and $\beta$, does the diagonal part of the hypothetical random matrix model follow the extended covariance matrix displayed above? If so, what is the common reason for these occurrences?

In a sense, this coincidence should not really be a surprise. Our results are phrased in terms of "central limits" of star-transpositions, i.e.

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\gamma_i - A_0),$$

but as elements of $\mathbb{C}[S_\infty]$, the sums

$$\gamma_1 + \cdots + \gamma_{m-1} = (1,2) + \cdots + (1,m)$$

are conjugate to the famous *Jucys-Murphy elements*, defined by

$$X_m := (1,m) + (2,m) + \cdots + (m-1,m)$$

for $m \geq 1$. So our work could also be interpreted as studying their asymptotics in different representations. On the other hand, the Jucys-Murphy elements are often used in the study of random Young diagrams and limit shapes, for example in [4, 38].

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