Near-optimal quantum strategies for nonlocal games, approximate representations, and BCS algebras

by

Connor Paul-Paddock

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Combinatorics and Optimization - Quantum Information

Waterloo, Ontario, Canada, 2023

Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

External Examiner:	Ivan Todorov Professor, Dept. of Mathematics, University of Delaware
Supervisor(s):	Jon Yard Professor, Dept. of Combinatorics & Optimization, University of Waterloo
	William Slofstra Professor, Dept. of Pure Mathematics, University of Waterloo
Internal Member:	Ashwin Nayak Professor, Dept. of Combinatorics & Optimization, University of Waterloo
Internal-External Member:	Michael Brannan Professor, Dept. of Pure Mathematics, University of Waterloo
Other Member(s):	Richard Cleve Professor, Dept. of Computer Science, University of Waterloo

Author's Declaration

This thesis consists of material all of which I authored or co-authored: see **Statement of Contributions** included in this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Chapters 1, 2, 3 and 4 comprise of work in which I am the sole author. Some of the results in Chapter 3 and 4 have been made previously available in the arXiv preprint [Pad22]. The material in Chapter 5 was co-written with William Slofstra and is to appear in a upcoming work [PS23].

Abstract

Quantum correlations can be viewed as particular abstract states on the tensor product of operator systems which model quantum measurement scenarios. In the paradigm of nonlocal games, this perspective illustrates a connection between optimal strategies and certain representations of a finitely presented $*$-algebra affiliated with the nonlocal game. This algebraic interpretation of quantum correlations arising from nonlocal games has been valuable in recent years. In particular, the connection between representations and strategies has been useful for investigating and separating the various frameworks for quantum correlation as well as in developing cryptographic primitives for untrusted quantum devices. However to make use of this correspondence in a realistic setting one needs mathematical guarantees that this correspondence is robust to noise.

We address this issue by considering the situation where the correlations are not ideal. We show that near-optimal finite-dimensional quantum strategies using arbitrary quantum states are approximate representations of the affiliated nonlocal game algebra for synchronous, boolean constraint systems (BCS), and XOR nonlocal games. This result robustly extends the correspondence between optimal strategies and finite-dimensional representations of the nonlocal game algebras for these prominent classes of nonlocal games. We also show that finite-dimensional approximate representations of these nonlocal game algebras are close to near-optimal strategies employing a maximally entangled state. As a corollary, we deduce that near-optimal quantum strategies are close to a near-optimal quantum strategy using a maximally entangled state.

A boolean constraint system B is *pp*-definable from another boolean constraint system B' if there is a *pp*-formula defining B over B'. There is such a *pp*-formula if all the constraints in B can be defined via conjunctions of relations in B' using additional boolean variables if needed. We associate a finitely presented $*$-algebra, called a BCS algebra, to each boolean constraint system B. We show that *pp*-definability can be interpreted algebraically as $*$-homomorphisms between BCS algebras. This allows us to classify boolean constraint languages and separations between various generalized notions of satisfiability. These types of satisfiability are motivated by nonlocal games and the various frameworks for quantum correlations and state-independent contextuality. As an example, we construct a BCS that is C^*-satisfiable in the sense that it has a representation on a Hilbert space H but has no tracial representations, and thus no interpretation in terms of commuting operator correlations.

Acknowledgements

I would like to thank all the people who made this thesis possible. In particular, my wonderful supervisors Jon Yard, and William Slofstra, for their support and guidance throughout this journey. I would also like to thank several friends and colleagues for their help, advice, and support throughout my time at Waterloo: John Schank, Sam Winnick, Yuming Zhao, Ben Lovitz, Joan Arrow, Junqiao Lin, Dan Puzzuoli, Vincent Russo, Kieran Mastel, and Turner Silverthorne.

Dedication

To my wife, Madeline, this would not have been possible without your endless love, patience, and support.

Table of Contents

Chapter 1

Introduction

This thesis establishes results in areas related to the study of nonlocal games. First, we explore the correspondence between perfect quantum strategies and the representations of synchronous and boolean constraint system (BCS) algebras. In addition, we examine the correspondence between optimal strategies for XOR nonlocal games and representations of the affiliated XOR algebra. Using techniques from approximate representation theory we show that this correspondence is robust, in the sense that near-perfect (near-optimal in the XOR case) quantum strategies correspond to approximate representations of these game algebras. This thesis builds on the work of Slofstra and Vidick [SV18] who used similar techniques to establish that near-perfect strategies for $\mathbb{Z}_2$-linear constraint system (LCS) games corresponded to approximate representations of the solution group. We also improve the robustness result for the XOR game case previously established in [Slo11] by removing the dependence on the dimension of the Hilbert space in the strategy. Secondly, we provide the first systematic treatment of boolean constraint systems (BCS) algebras. Boolean constraint systems offer a rich array of associated nonlocal games such as the Mermin-Peres magic square and their associated algebras. Also, we show that every synchronous algebra is $*$-isomorphic to a BCS algebra. Based on observations in [Ji13, CM14, AKS19, KPS18, HMPS19], we illustrate that BCS-algebra provides examples that separate several interesting satisfiability generalizations inspired by the various frameworks for quantum correlations. We also show that BCS-algebras can be classified in terms of boolean constraint languages and definability, a concept from computer science and logic. Lastly, we answer an open question of Sam Harris [Har21] by giving a synchronous nonlocal game whose synchronous algebra has a representation in $\mathcal{B}(H)$ but no tracial states.

1.1 Strategies for nonlocal games and representations

In a nonlocal game, two spatially separated parties, Alice and Bob, receive questions from a referee. The questions or inputs x and y are drawn from finite sets X and Y according to a probability distribution known to both players. Prohibited from communicating with the other player, they must each reply to the referee with an answer a and b from finite sets A and B. To win the game, the player's answer pair (a, b) must satisfy the winning predicate. The winning predicate is a function $V : \mathsf{X} \times \mathsf{Y} \times \mathsf{A} \times \mathsf{B} \to \{0, 1\}$ such that if (a, b) is a winning answer to (x, y), then $V(x, y, a, b) = 1$, and otherwise $V(x, y, a, b) = 0$. Although Alice and Bob cannot communicate during the game, they can decide on a strategy beforehand. Loosely speaking, a *strategy* is simply a means by which each player chooses their answer based on their given question. This process could be deterministic or probabilistic, employing some kind of shared randomness. We call a strategy with shared randomness a classical strategy. A strategy is *perfect* if it allows the players to satisfy the predicate $V(x, y, a, b)$ on every question pair (x, y). It is not hard to see that there are nonlocal games for which no classical strategy can be perfect. Consider a $\mathbb{Z}_2$-linear system of equations $Ax = b$, where A is an $n \times m$ matrix with $\{0, 1\}$-entries. Consider the nonlocal game where Alice receives the ith equation and is required to reply with a solution x to the equation $A_i \cdot x = b_i$. Meanwhile, Bob is given an index j and must give a $\{0, 1\}$-assignment to the jth entry of $x \in \mathbb{Z}_2^n$. The winning condition for the game is a consistency check. That is, Alice and Bob must have assigned the same $\{0, 1\}$-value to x_j, otherwise they lose. Because Alice and Bob are unaware of which question are asked to the other player, it is not hard to see that they will be unable to guarantee consistency. For example, if $Ax = b$ does not have a solution, then it can be shown that there will always be some question pair on which they could lose with non-zero probability (provided that every question has a non-zero probability of being asked). On the other hand, if $Ax = b$ has a solution, then there is always a perfect strategy. Alice and Bob simply decide on a solution ahead of time and use that assignment to consistently satisfy any question pair.

Although the players in a nonlocal games cannot communicate, they can share a quantum state, which gives them access to the larger set of quantum correlations. In particular, they can use a quantum state as a resource in their strategy by making local quantum measurements to obtain their answers. They can even choose which state they want to share prior to the game. A collection of local measurements for each player and a choice of quantum state make up a *quantum strategy* for the nonlocal game. What is surprising is that there are nonlocal games with no perfect classical strategy but a perfect quantum strategy. A famous example is the Mermin-Peres magic square [Mer90, Per90], which consists of a 3×3 grid, where each row and column has a specified ± 1-valued constraint. The object of the game is, given a row or column, reply with ±1-values that multiply to the row or column constraint. If each player gets a row or column, their assigned values overlap on at least one value of the grid. The object of the game is to give a consistent assignment on this overlapping square. For certain configurations, namely when the total number of -1 constraints is odd, no perfect classical strategy exists. Yet, there is a assignment of ±1-values observables and a quantum state which enables the players to win on every input.

Quantum strategies do not always provide such shocking improvements over classical strategies. For example, in the well-known CHSH nonlocal game there is neither a perfect quantum nor classical strategy. However, the optimal quantum strategy does provide a significant improvement over the best classical strategy [CHTW10, CHSH69]. It should also be said that quantum strategies often provide no advantage over classical strategies. Determining if a nonlocal game has a quantum advantage is extremely hard computationally. In particular, for an arbitrary nonlocal game, and any $0 < \epsilon < 1$, deciding (in the promise case) if there is a quantum strategy that wins with probability 1 or $1 - \epsilon$ is complete for RE [JNV$^+$22]. Despite the hardness of this problem, one can still attempt to characterize classes of games for which a quantum advantage is present. From a more practical perspective, understanding which games require highly entangled quantum states provides a means of certifying the presence of an entangled state. This is a concept critical to the area of device-independent cryptography and self-testing, which is markedly related to the theory of nonlocal games, for instance, see [RUV13].

Some of the first progress toward characterizing scenarios that required large amounts of entanglement was made by Tsirelson. Tsirelson was investigating specific quantum correlations produced in simple Bell scenarios, like the CHSH scenario of [CHSH69], which we now call XOR correlations. He showed that the set of these XOR correlations was related to a closed convex set of real symmetric matrices, and that the extreme points of this set corresponded to representations of the Clifford algebra with tracial states [Tsi87, Tsi85]. We now know that these points also correspond to unique optimal strategies for XOR nonlocal games [Weh06]. Hence, Tsirelson described a correspondence between unique optimal quantum strategies for certain XOR games and representations of the Clifford algebra. That is, the measurement operators employed by Alice and Bob in any unique optimal strategy for an XOR game must be equivalent to representations of the Clifford algebra, and in fact, their employed quantum state must come from an abstract tracial state on the Clifford algebra. In [Slo11], Slofstra associated a finitely presented (XOR) algebra to the optimal strategy of a XOR game. The approach of Tsirelson and Slofstra are complementary in the sense that the Clifford algebras correspond to the extreme points of these correlations, while the XOR algebras[1] are associated with supporting hyperplanes of Tsirelson's XOR correlations. It follows that whenever there is a unique supporting hyperplane, the XOR algebra is isomorphic to a Clifford algebra [Slo11].

Based on the structure of the Mermin Peres magic square example, several authors [CM14, Ji13, Ark12] began the study of linear constraint system (LCS) nonlocal games. Their main observation was that perfect quantum strategies for these LCS nonlocal games correspond to *matrix solutions* of the $m \times n$ linear system $Ax = b$ with a maximally entangled state. By thinking of a system of $\mathbb{Z}_2$-linear equations multiplicatively, a matrix solution to $Ax = b$ is a collection of ± 1-valued observables $\{X_1, \ldots, X_n\}$ such that

(a) $\prod_{j \in K_i} X_j = (-\mathbb{1})^{b_i}$ for all $1 \leq i \leq m$, and

(b) $X_j X_k = X_k X_j$ whenever $k, j \in K_i$, for all $1 \leq i \leq m$.

Where K_i is the set of variables $\{x_1, \ldots, x_n\}$ appearing (i.e. having non-zero coefficient) in the ith equation. Satisfying condition (a) ensures that the observables satisfy each of the linear constraints, whereas satisfying condition (b) ensures that the observables are jointly measurable within the context defined by the row equation. Although implicit in the earlier works, it was in [Slo19c, CLS17] that the existence of a perfect quantum strategy was tied to the algebraic structure of a finitely-presented group called the *solution group* $\Gamma(A, b)$ associated to a linear system $Ax = b$. Conditions (a) and (b) are some of the abstract relations in this group, and hence are automatically satisfied by any representation. Hence, perfect quantum strategies for the linear constraint system games were matrix representations of this group, and vice versa.

Around the same time, another group of researchers introduced an interesting family of nonlocal games called *synchronous* games. The prototypical synchronous nonlocal game is the *graph k-colouring game*, which was first given in [CHTW10], and has been the basis for many subsequent works [CMN$^+$07, MR16, PSS$^+$16, AMR$^+$19, PT13, MR18] to list a few. Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertices $\mathcal{V}$ and edges $\mathcal{E}$, the referee sends vertices u to Alice and v to Bob, each drawn according to some a distribution on $\mathcal{V} \times \mathcal{V}$. The players reply to the referee with one of the k colours for their given vertex. If they are given adjacent vertices and they return the same colour, then they lose the game. They also lose, if they reply with different colours when given the same vertex. It is clear, that there is a perfect classical strategy for the game if and only if there is a k-colouring of the graph $\mathcal{G}$. In a synchronous game, like graph k-colouring, both the players' input and output sets are the same, that

[1]Slofstra called these *solution algebras* but to avoid confusion with the algebra of the solution group we will use the term XOR algebra.

is $\mathsf{A} = \mathsf{B}$ and $\mathsf{X} = \mathsf{Y}$. The synchronous condition states that, whenever the players are given the same input, they lose if they give different outputs (i.e. a lack of consistency). In the graph colouring game, the synchronous condition is enforced by the condition that they lose whenever they give a different colour to the same vertex. Remarkably, it was shown that there are graphs, and a value of $k \in \mathbb{N}$, which cannot be k-coloured, yet a perfect quantum strategy exists for the corresponding graph k-colouring nonlocal game. Additionally, the existence of perfect quantum strategies for synchronous games correspond to representations of an abstract finitely-presented $*$-algebra called the *synchronous algebra*, which can be associated to every synchronous nonlocal game. There are many results about synchronous nonlocal games in the literature, in particular due to the connection with the Connes' Embedding Problem, see for instance [DP16, Rus20, KPS18].

There is a natural way to generalize $\mathbb{Z}_2$-linear constraint system (LCS) nonlocal games. Rather than having a system of linear constraint over a set of variables, one could allow a system of arbitrary constraint formulae in the boolean variables. Such games are called boolean constraint system (or BCS) nonlocal games. A boolean constraint system consists of a set of global variables and a set of constraints in those variables. These games were investigated in [CM14, Ji13], but there has not been much exposition on them in the literature since. In the past, BCS games have been called "binary" constraint system nonlocal games, but we prefer the term boolean. Use of the term "binary" rather than boolean in the naming could lead to conflation with the notion of a system of "2-ary" constraints, which is unintended, as we do not wish to restrict the arity of the constraining relations.

In a BCS nonlocal game, Alice receives a constraint C_i, over variables $X = \{x_1, \ldots, x_n\}$, from a set $\{C_1, \ldots, C_m\}$ and must reply with an assignment $\phi : X \to \{\pm 1\}$ that restricts to a satisfying assignment on the variables in the scopes of the ith constraint. Like in an LCS game, Bob receives a single variable and must return a $\mathbb{Z}_2$-assignment $\varphi(x_j)$ to that variable. The winning condition is again consistency check between Alice and Bob's assignment. That is, they lose whenever $\phi(x_j) \neq \varphi(x_j)$. For a BCS nonlocal game, one can associate a finitely presented $*$-algebra to the game, called the BCS algebra. Unlike in the case of LCS games, the more general constraints allowed in a BCS cannot be encoded in group relations but rather as $*$-polynomials. Despite this difference, similar arguments to the LCS case show that every perfect quantum strategy for a BCS game is a representation of the BCS algebra [CM14, Ji13]. We will return to discuss BCS algebras again later.

1.2 Near-optimal strategies and approximate representations

In the previous section, we highlighted several examples the correspondence between finite-dimensional representations of finitely-presented $*$-algebras and perfect (or optimal) quantum strategies for some classes of nonlocal games. This correspondence has been very fruitful for finding interesting nonlocal games and uncovering curious features of the quantum correlation set. However, the results stemming from this idealized picture is too abstract for those who want to make statements about physical quantum correlations. One fundamental issue is that quantum measurement and state preparation is inherently noisy, and it's unlikely we will be able to eliminate these effects anytime soon. For instance, when we record the outcome probabilities from some experiment, these values may differ from the idealized values we expect from the model. In the context of nonlocal games, if our strategy is no longer perfect (or optimal) but only near optimal, do we still have any guarantees that our strategy is close to the representations we have in the ideal case?

The issue of noise matters in the context of the widely used application of nonlocal games known as self-testing. Generally speaking, a correlation can have many different quantum models, which are indistinguishable to an observer. In particular, the optimal value of a nonlocal game could be achieved my more than one quantum strategy. Remarkably, there are certain correlations, called self-tests, which are unique up to the addition of auxiliary quantum systems and under a local change of basis [MY04, ŠB20]. Self-tests are highly relevant in the field of quantum device-independent verification protocols [RUV13, BŠCA18a, BŠCA18b]. Suppose you are given a black-box "quantum device" and want to certify that it is employing features of entanglement. You have the device perform a self-testing Bell scenario, and if you see these self-testing correlations, you can ensure that the device is performing the measurements and utilizing the state from that unique quantum model. If the self-test requires a maximally entangled state, you can confidently conclude that the device is quantum. However, if there is noise in the state or the measurements you might not obtain the self-testing correlation but an approximation.

The study of self-testing in the approximate regime is called *robust self-testing*, for more, see [MYS12]. We will not get into the specifics of self-testing in this thesis, but it was recently shown [PSZZ23] that self-testing of extremal quantum correlations for several natural classes of quantum models is equivalent to there being a unique (finite-dimensional) abstract state on the bipartite measurement algebra. Hence, self-testing can be viewed as a statement about the uniqueness of the representations from scenarios and games. We believe that the methods in this thesis, although not specifically self-testing results, can be used to establish a more mathematical theory of robust self-testing.

In the context of nonlocal games, we are concerned with the following question: If a nonlocal game achieves a value near the optimal value, is the strategy still close to a representation of the associated game algebra? This concept is related to notions of *weak* self-testing, or approximate rigidity, see for instance [Kan20]. To tackle this question we need to adopt tools of approximate representation theory, which provides us with a rigorous notion of a near-representation. For the class of linear-constraint systems (LCS) nonlocal games, Slofstra and Vidick [SV18] employed approximate representation theoretic results to show that the correspondence between representations and perfect quantum strategies is robust, in the sense that near-perfect strategies correspond to approximate representations of the solution group. This work builds on the results in [SV18] by examining the robustness of the correspondence between quantum strategies and representations of the game algebra for synchronous, boolean constraint systems, and XOR nonlocal games. The results in [SV18] for LCS games, in combination with the dilation stability results for groups [GH17, DCOT19], allow for strong theory of robust self-testing for LCS games [CS17].

Before we present our main results, we need to discuss two notions in more detail, near-perfect (near-optimal in the XOR case) quantum strategies and near-representations. For an $\epsilon > 0$, a strategy is ϵ-optimal if the probability of winning with the strategy is less than ϵ-away from the optimal value of the game. Similarly, a strategy is ϵ-perfect if the probability of winning is at least $1-\epsilon$. On the other hand, the notion of near-representations comes from approximate representation theory, typically studied in the setting of groups and C^*-algebras. For a finitely presented $*$-algebra an approximate representation with parameter $\epsilon > 0$, or ϵ-representation, is a map sending the abstract generators to matrices where the relations hold approximately. More precisely, the parameter $\epsilon > 0$ measures how closely the matrices in the approximate representation satisfy the relations. This is measured using some chosen matrix norm. The notion of an ϵ-representation comes from the fact that in a genuine representation the norm of any relation will be zero.

One of the main contributions of this thesis is a quantitative version of the correspondence between near-perfect (near-optimal in the XOR case) quantum strategies and approximate representations of the synchronous, BCS, and XOR games algebras. We show that near-perfect (near-optimal) quantum strategies correspond to near-representations acting on finite-dimensional Hilbert space H with respect to the little Frobenius norm $\|\cdot\|_f$. We will formally define a projective quantum strategy and the little Frobenius norm in Chapter 2.

Theorem 1.3.

(1) If $\mathcal{S}$ is an ϵ-perfect strategy for a synchronous nonlocal game G_{sych}, then Bob's

measurement operators are an $O(\epsilon^{1/8})$-representation of the synchronous algebra $\mathscr{A}(G_{sync})$ on a non-zero subspace of H_B.

(2) If $\mathcal{S}$ is an ϵ-perfect strategy for a BCS nonlocal game G_{bcs}, then Bob's measurement operators are an $O(\epsilon^{1/4})$-representation of the BCS algebra $\mathscr{B}(G_{bcs})$ on a non-zero subspace of H_B.

(3) If $\mathcal{S}$ is an ϵ-optimal strategy for a XOR nonlocal game G_{xor}, then Bob's measurement operators are an $O(\epsilon^{1/8})$-representation of the XOR solution algebra $\mathscr{C}(G_{xor})$ on a non-zero subspace of H_B.

Importantly, the estimates are independent of the dimension of the supporting Hilbert space H_B.

An independent but similar result to part (1) of Theorem 1.3 was recently established in [Vid22] for approximate synchronous correlations. An advantage of the result in [Vid22] is that it applies more generally to correlations and applies to synchronous games as a special case. Both results are based on ideas from [SV18], but we emphasize that this work takes a more algebraic perspective and additionally covers the XOR games case, which is far from the case of synchronous correlations. Additonally, in [Slo11] it was shown that near-optimal strategies for XOR-games do correspond to approximate representations of XOR algebras. However, the estimate of the approximate representation was dependent of the dimension of the underlying Hilbert space. For a subclass of XOR games, a dimension independent bound for approximate representations from ϵ-perfect strategies can be deduced from a result of Ostrev and Vidick [OV16], who used an indirect approach employing mathematical ideas from the theory of robust self-testing. Our new results employs a novel averaging procedure which allows our conclusion to be independent of the Hilbert space dimension for all XOR games.

Although there is no dependence on the Hilbert space dimension, the estimates do depend on the parameters of the game such as the size of the question and answer sets. This means that care is required when applying these results to situations where question and answer sets are an essential parameter. Our second main result is complementary to our first theorem: near-representations are close to near-optimal strategies with a maximally-entangled state.

Theorem 1.4.

(1) If ϕ is an ϵ-representation of the synchronous algebra $\mathscr{A}(G_{synch})$ on a Hilbert space H with respect to $\|\cdot\|_f$, then ϕ is close to an $O(\epsilon^2)$-perfect strategy $\mathcal{S}$ for the synchronous

nonlocal game G_{synch}, in which the players employ a maximally entangled state on $H \otimes H$.

(2) If ϕ is an ϵ-representation of the BCS algebra $\mathscr{B}(G_{bcs})$ on a Hilbert space H with respect to $\|\cdot\|_f$, then ϕ is close to an $O(\epsilon^2)$-perfect strategy $\mathcal{S}$ for the BCS nonlocal game G_{bcs}, in which they employ a maximally entangled state on $H \otimes H$.

(3) If ϕ is an ϵ-representation of the XOR solution algebra $\mathscr{C}(G_{xor})$ on a Hilbert space H with respect to $\|\cdot\|_f$, then ϕ is close to an $O(\epsilon)$-optimal strategy $\mathcal{S}$ for the XOR nonlocal game G_{xor}, in which the players employ a maximally entangled state on $H \otimes H$.

Here *close* means that each measurement operator in $\mathcal{S}$ is $O(\epsilon)$-away from the corresponding element of a representation $\phi : \mathscr{A}(G) \to M_d(\mathbb{C})$ with respect to the little Frobenius norm $\|\cdot\|_f$. Theorem 1.4 is a consequence of an important concept in approximate representation theory called *stability*, which roughly is the property that there is a genuine representation whenever there is an ϵ-representation with ϵ sufficiently small. Many of algebraic relations required for quantum measurement are stable. We will discuss stability more in Chapter 2. As a corollary to the proofs of Theorems 1.3 and 1.4 we show that each near-optimal quantum strategy is close to a near-optimal quantum strategy using a maximally entangled state.

Corollary 1.5. Near-optimal quantum strategies with arbitrary states are close to those with maximally entangled states:

(1) For any ϵ-optimal synchronous quantum strategy $\mathcal{S}$ for a synchronous nonlocal game, there is an $O(\epsilon^{1/4})$-optimal quantum strategy $\tilde{\mathcal{S}}$ using a maximally entangled state $|\tilde{\psi}\rangle$, such that each measurement operator in $\tilde{\mathcal{S}}$ is at most $O(\epsilon^{1/8})$-away from the measurement operator in $\mathcal{S}$ with respect to $\|\cdot\|_f$ on the support of $|\tilde{\psi}\rangle$.

(2) For any ϵ-perfect quantum strategy $\mathcal{S}$ for a BCS nonlocal game there is an $O(\epsilon^{1/2})$-optimal quantum strategy $\tilde{\mathcal{S}}$ using a maximally entangled state $|\tilde{\psi}\rangle$, such that each measurement operator in $\tilde{\mathcal{S}}$ is at most $O(\epsilon^{1/4})$-away from the measurement operator in $\mathcal{S}$ with respect to $\|\cdot\|_f$ on the support of $|\tilde{\psi}\rangle$.

(3) For any ϵ-optimal quantum strategy $\mathcal{S}$ for an XOR nonlocal game, there is an $O(\epsilon^{1/8})$-optimal quantum strategy $\tilde{\mathcal{S}}$ using a maximally entangled state $|\tilde{\psi}\rangle$, such that each measurement operator in $\tilde{\mathcal{S}}$ is at most $O(\epsilon^{1/8})$-away from the measurement operator in $\mathcal{S}$ with respect to $\|\cdot\|_f$ on the support of $|\tilde{\psi}\rangle$.

1.6 BCS algebras and generalized satisfiability

So far we have focussed on the case where a quantum strategy consists of measurement operators and a vector state on a tensor-product of finite-dimensional Hilbert spaces. However, this is not the only mathematical framework for "quantum correlations". Alternatively, we could consider strategies to be collections of measurement operators and vector states on a potentially infinite-dimensional Hilbert space H, where Alice and Bob measurement operators both act on this space H, but need to commute, rather than act only on the different tensor factors. This gives the class of commuting-operator strategies, and quantum-commuting correlations. Furthermore, just as quantum strategies allowed us to find perfect strategies, where there were no perfect classical ones, there are nonlocal games with perfect commuting operator strategies, but no perfect (finite-dimensional) quantum strategies [Slo19c]. Hence, the quantum-commuting correlation set is strictly larger than the set of quantum correlations.

Recall that for an LCS nonlocal game, a perfect classical strategy was equivalent to a solution to the $\mathbb{Z}_2$-linear system $Ax = b$, and a perfect quantum strategy was equivalent to a *matrix solution* to the linear system, that is, a matrix representation of the solution group $\Gamma(A, b)$. The existence of a perfect commuting operator strategy for an LCS game can also be expressed in terms of a property of the solution group [Slo19c, CLS17]. To be exact, the existence of a perfect commuting operator strategy comes down to whether there is a nontrivial central element of order 2, commonly denoted as the J element in $\Gamma(A, b)$. In particular, an LCS nonlocal game G_{lcs} has a perfect commuting operator strategy if and only if there is a $*$-homomorphism from the $*$-algebra $\mathbb{C}\Gamma(A, b)/\langle\langle J = -1\rangle\rangle$ to a C^*-algebra with a tracial state[2]. One could say that rather than $Ax = b$ having a solution, or matrix solution, it has a *tracial solution*. Because there is a trace on every finite-dimensional matrix algebra, this notion of a tracial solution for $Ax = b$ does indeed generalize the notion of a matrix solution.

Linear constraint systems are an important family of boolean constraint satisfaction problems. One interesting aspect is that the complexity of deciding whether there is a satisfying assignment, i.e. a solution to the linear system, is polynomial in the size of the linear system. Similarly, one might wonder about the complexity of deciding whether there is a quantum solution, whether a matrix or a tracial solution to $Ax = b$. However, using the group theoretic connection, Slofstra showed that this problem is undecidable [Slo19c]. Specifically, deciding if J is trivial in $\Gamma(A, b)$ is equivalent to the word problem for arbitrary

[2] A tracial state on a $*$-algebra $\mathcal{A}$ is a positive, normalized, linear functional $f : \mathcal{A} \to \mathbb{C}$ with the property that $f(ab) = f(ba)$ for all $a, b \in \mathcal{A}$.

finitely-presented groups. For some sub-classes of LCS nonlocal games, such as *incidence games*, the complexity of deciding whether there is a tracial solution is not only decidable but decidable in polynomial time [PRSS22].

Every $\mathbb{Z}_2$-linear constraint system $Ax = b$ is a boolean constraint system B, where all the constraints are *linear* (or affine), and the BCS algebra $\mathscr{A}(B)$ of an LCS is $*$-isomorphic to the quotient of the solution group algebra $\mathbb{C}\Gamma(A, b)/\langle\langle J = -1\rangle\rangle$ by the ideal generated by $J = -1$ [KPS18, Gol21, Fri20]. Every BCS with only linear constraints belongs to the boolean constraint language LIN. For a BCS over LIN, that is a BCS where all the constraints are linear, a satisfying assignment for the BCS is equivalent to a solution to the associated linear system of equations.

With this in mind, we say that a BCS has a satisfying matrix assignment if there is a set of commuting ± 1-observables which satisfy the constraint polynomials of the BCS, since any constraint is equivalent to a real multilinear polynomial [O'D14]. By definition of the BCS algebra, a matrix satisfying assignment for B is equivalent to the BCS algebra having a matrix representation. We can also ask whether a BCS is tracially satisfiable, that is, has a $*$-representation to a tracial C^*-algebra. Other notions of satisfiability are related to subtle algebraic properties of the BCS algebra. For instance, we say that a BCS is $\mathcal{R}^{\mathcal{U}}$-satisfiable if the BCS algebra has a $*$-homomorphism to the ultrapower $\mathcal{R}^{\mathcal{U}}$ of the type II_1-factor $\mathcal{R}$. In fact, by the results of [KPS18], this type of satisfiability for a BCS is related to whether there is a perfect strategy for the associated game using the closure C_{qa} of the set of quantum correlations C_q, since C_q is not closed [Slo19b]. However, we do not restrict ourselves to types of satisfiability that necessarily correspond to strategies for nonlocal games. For instance, given a BCS B, we could ask whether the BCS algebra is nontrivial. This notion of algebraic-satisfiable does not have a meaning in terms of correlations, but it is still an interesting property, particularly for those more interested in notions like contextuality and less about correlations, for instance see [AFLS15].

Unlike a BCS over LIN, it is NP-hard to determine whether a BCS B has a satisfying assignment. Although we know that the problem of whether a given BCS is tracially satisfiability is undecidable. One may wonder about the complexity of finding an approximation to the optimal tracial satisfying assignment to a BCS. However, by the results of [JNV+22], we know this problem is complete for RE. In establishing this result the authors provided a BCS that is tracially-satisfiable but not $\mathcal{R}^{\mathcal{U}}$-satisfiable, which settled the Connes Embedding Problem from [Con76].

For all the types of satisfiability we have mentioned so far, there is a BCS B which illustrates that each type of satisfiability is distinct. Not every $*$-algebra is nontrivial as a C^*-algebra. In [HMPS19], it was shown that there is a nontrivial synchronous $*$-algebra that is trivial as a C^*-algebra. By the construction of Gel'fand-Naimark-Segal (GNS), this can happen if a nontrivial $*$-algebra has no states[3]. Additionally, not every state is a tracial state. In [Har21], Sam Harris asked whether there is a synchronous algebra that is non-trivial as a C^*-algebra but has no tracial state? We provide an explicit construction of a BCS-algebra with a representation on $\mathcal{B}(H)$ and no tracial states. We prove that BCS algebras and synchronous algebras are isomorphic as $*$-algebras, and therefore our example answers the question in the affirmative.

One of the motivations to study BCS algebras is based on the work [AKS19]. Although they do not work at the level of BCS algebras, the authors were interested in which types of boolean constraint systems exhibited separations between conventional satisfiability, matrix satisfiability, and C^*-satisfiability (which they call operator-satisfiability). In particular, the authors were interested whether there were other boolean constraint languages like LIN, where there were separations. All of the current examples at the time came from BCS over LIN. One of their starting points was Schaefer's dichotomy theorem [Sch78], which characterized the boolean constraint languages which, like LIN, have polynomial time algorithms for deciding classical satisfiability. In addition to LIN, they consist of the classes 2SAT, HORN, DUAL-HORN, 0-VALID, and 1-VALID. For convenience, let $\mathcal{K}$ be the collection of these languages. Given a BCS over any other boolean constraint language (i.e. one not in $\mathcal{K}$) the decidability problem is NP-hard. However, it was shown in [Ji13, AKS19] that each class in $\mathcal{K}$ (except for LIN) is algebraically-satisfiable if and only if they are satisfiable, so no separations exist over these languages. Despite this, using boolean constraint languages to classify BCS-algebras into families is appealing. In particular, one could be optimistic that it could help understand the BCS systems that arise in [JNV+22] and probabilistically checkable proofs (PCPs). One of the main tools for characterizing BCS systems from boolean constraint languages is the notion of definability[4]. We show that definability is natural in the language of BCS algebras:

Lemma 1.7. If B is definable from a boolean constraint language $\mathcal{L}$ then there is a BCS B' over $\mathcal{L}$, a natural inclusion of $*$-algebras $\iota : \mathscr{A}(B) \hookrightarrow \mathscr{A}(B')$, and a surjection $\pi : \mathscr{A}(B') \twoheadrightarrow \mathscr{A}(B)$, such that $\mathbb{1} = \pi \circ \iota$.

This lemma provides simple algebraic proofs for many of the results in [AKS19] and suggests that using the structure of boolean constraint languages coming from Post's lattice could be a way to classify BCS-algebras. Using this classification could lead to a systematic understanding of how different constraint languages lead to specific algebraic properties of

[3]The GNS correspondence is a bijection between states on $*$-algebras and cyclic representations on a Hilbert space H.

[4]This is often called definability by positive primitives or *pp*-definability in some literature

BCS algebras. For example, if $\mathcal{L}$ is not contained in any of the classes from $\mathcal{K}\backslash$LIN, then LIN is definable from $\mathcal{L}$ [AKS19].

Lastly, it is very interesting that LIN provides such an ample class of separations, but the others in $\mathcal{K}$ do not. Moreover, there is still much we could learn by studying this boolean constraint language in this algebraic framework. For instance, there is an algebraically satisfiable but not $\mathcal{R}^{\mathcal{U}}$-satisfiable BCS B over LIN if and only if there is a non-hyperlinear group. The existence of a non-hyperlinear group is a significant open problem in group theory. Every sofic group is hyperlinear, hence the existence of a non-hyperlinear group would imply the existence of a non-sofic group.

Chapter 2

Background and preliminaries

In this chapter, we review some mathematical preliminaries and definitions. Although we cannot fully cover all the required background, we will attempt to cover the core concepts and material necessary for the following chapters. We have split this chapter into three sections. In Section 2.1, we cover Hilbert spaces and semi-pre-C^*-algebras. In Section 2.8, we cover quantum states, measurements, and correlations. We recommend the texts [Bla06, Wat18] for more background. Section 2.19 is devoted to the background of approximate representation theory. As the concepts and results related to approximate representation theory are likely new to many readers, we cover them in greater detail.

2.1 Hilbert spaces and semi-pre-$*$-algebras

2.1.1 Hilbert spaces

Let $\mathbb{C}$ be the field of complex numbers. Given a set X the **free (complex) vector space** over X is denoted $\mathbb{C}X$, where the elements $\{|x\rangle\}_{x\in X}$ are a basis for $\mathbb{C}X$. Elements of $\mathbb{C}X$ are finite linear combinations of the elements in X, that is every element $|v\rangle \in \mathbb{C}X$ can be expressed as $|v\rangle = \sum_{x\in S} c_x|x\rangle$, with coefficients $c_x \in \mathbb{C}$, and $S \subseteq X$. The ℓ_1-norm of $|v\rangle \in \mathbb{C}X$ is the sum of the absolute value of the coefficients, $\| \, |v\rangle \, \|_{\ell_1} = \sum_{x\in S} |c_x|$. If X is finite, then $\mathbb{C}X \cong \mathbb{C}^X$, and $\{\langle x|\}_{x\in X}$ is a basis for the **dual space** of linear functionals $\mathbb{C}X \to \mathbb{C}$. This convention is known as bra-ket notation and is commonly used in quantum physics.

An **inner-product space** is a vector space V with a **sesquilinear form** $\langle \cdot \mid \cdot \rangle : V \times V \to \mathbb{C}$ such that for all $|u\rangle, |v\rangle, |w\rangle \in V$ and $a, b \in \mathbb{C}$ the following hold:

(1) $\langle u|v\rangle = \overline{\langle u|v\rangle}$ (conjugate symmetric),

(2) $\langle u, av + bw\rangle = a\langle u, v\rangle + b\langle u, w\rangle$ (linear in the second argument),

(3) if $v \neq 0$ then $\langle v|v\rangle > 0$ (positive definite).

Every inner-product space is a **normed vector space** with norm $\||v\rangle\|^2 = \langle v|v\rangle$ for all $|v\rangle \in V$. In addition to being a normed space, in an inner-product space we have the **Cauchy-Schwarz** inequality,

$$|\langle u|v\rangle|^2 \leq \langle u|u\rangle \langle v|v\rangle ,$$

for all $|u\rangle, |v\rangle \in V$. Where in the case of equality, it must hold that $|u\rangle = \lambda|v\rangle$ for some scalar $\lambda \in \mathbb{C}$. Additionally, in an inner-product space we have the **Parallelogram identity**. That is, for every $|u\rangle, |v\rangle \in V$ it holds that

$$\||u\rangle + |v\rangle\|^2 + \||u\rangle - |v\rangle\|^2 = 2\||u\rangle\|^2 + \||v\rangle\|^2.$$

In fact, a normed vector space is an inner-product space if and only if the parallelogram identity holds.

A sequence $|v_1\rangle, |v_2\rangle, \ldots$ in a normed vector space is said to be **Cauchy**, if for every positive real number $r > 0$, there exists an $n \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ with $m, n > N$, such that

$$\||v_n\rangle - |v_m\rangle\| < r.$$

An inner-product space V is **complete** if every Cauchy sequence in V converges to an element in V. A **Hilbert space** H is a complete inner-product space.

Example 2.2. The canonical example of a Hilbert space is the space $\ell^2(\mathbb{N})$, whose elements are **square summable sequences**. That is $\mathbf{z} = z_1, z_2, \ldots \in \ell^2(\mathbb{N})$, if $\sum_{i=1}^{\infty} |z_i|^2 < \infty$. If $\mathbf{w}, \mathbf{z} \in \ell^2(\mathbb{N})$ the inner-product is defined by $\langle \mathbf{w}|\mathbf{z}\rangle = \sum_{i=1}^{\infty} w_i \bar{z}_i$.

Every Hilbert space has an orthonormal basis. A Hilbert H space is **separable** if it contains a countable dense subset. A Hilbert space is **finite-dimensional** if there is a finite basis. If the basis for a Hilbert space H has cardinality d, then H is isomorphic to the complex **Euclidean space** $\mathbb{C}^d$, where d is the dimension of H, i.e. the cardinality of the basis, otherwise, H is said to be **infinite dimensional**. Every separable infinite-dimensional Hilbert space is isomorphic to $\ell^2(\mathbb{N})$. If H and K are Hilbert spaces with $|h_1\rangle, |h_2\rangle \in H$ and $|k_1\rangle, |k_2\rangle \in K$, we define the inner product on the vector space $H \otimes K$ by

$$\langle h_1 \otimes k_1|h_2 \otimes k_2\rangle = \langle h_1|h_2\rangle\langle k_1|k_2\rangle.$$

Taking the completion of $H \otimes K$ with respect to this inner product defines the **tensor product of Hilbert spaces**.

If H and K are Hilbert spaces, a linear operator is a map $T : H \to K$ such $a|v_1\rangle + b|v_2\rangle \in H$ then $T(a|v_1\rangle + b|v_2\rangle) = aT|v_1\rangle + bT|v_2\rangle \in K$. We let $Lin(H,K)$ denote the set of linear operators from H to K. If H is a Hilbert space then we denote the set of linear operators $H \to H$ by $Lin(H)$. When $H \cong \mathbb{C}^d$ is finite-dimensional we note that $Lin(H) \cong Lin(\mathbb{C}^d) \cong M_d(\mathbb{C})$ where $M_d(\mathbb{C})$ is the set of $d \times d$ complex matrices. We write $\mathbb{1}$ to denote the identity operator in $Lin(H,K)$.

When H is finite-dimensional, a **trace** is a linear functional $\text{Tr} : Lin(H) \to \mathbb{C};\ A \mapsto \text{Tr}(A)$ for $A \in Lin(H)$, with the **cyclic** property $\text{Tr}(AB) = \text{Tr}(BA)$ for $A, B \in Lin(H)$. This defines a trace uniquely up to scalar multiplication. If $\{|u_i\rangle\}_{i\in I}$ is an orthonormal basis for H, then we can define a trace on $Lin(H)$ via $\text{tr}(A) = \sum_{i\in I} \langle u_i| A |u_i\rangle$. Since $\text{tr}(\mathbb{1}) = dim(H)$, it is also common to take the **normalized trace** which is defined by $\widetilde{\text{tr}}(A) = \text{tr}(A)/dim(H)$, for all $A \in Lin(H)$.

2.2.1 Bounded operators on Hilbert space

The **bounded linear operators** from a Hilbert H to a Hilbert space K are the operators

$$\mathcal{B}(H,K) = \{T : H \to K \mid \exists\ C > 0 \text{ such that } \|T|h\rangle\|_K \leq C\||h\rangle\|_H, \text{ for all } |h\rangle \in H\}$$

For convenience, we write $\mathcal{B}(H) := \mathcal{B}(H,H)$. The operators in $\mathcal{B}(H)$ inherit their norm structure from the underlying Hilbert space. That is for $T \in Lin(H)$, we define the operator norm of T by

$$\|T\|_{op} := \sup\{\|T|h\rangle\| : |h\rangle \text{ a unit vector in } H\}.$$

The algebra $\mathcal{B}(H)$ is equipped with an **antilinear involution** $* : \mathcal{B}(H) \to \mathcal{B}(H)$ sending $A \mapsto A^*$, with the property that that $(A^*)^* = A$, $(AB)^* = B^*A^*$, and $(\lambda A)^* = \bar{\lambda}A^*$ for all $\lambda \in \mathbb{C}$, and all $A, B \in \mathcal{B}(H)$. This $*$-operation defines the **adjoint** map. If $T \in \mathcal{B}(H,K)$, then $T^* : K \to H$ is the adjoint map, and satisfies the property $\langle k|Th\rangle_K = \langle T^*k|h\rangle_H$ for all $k \in K$, $h \in H$. The adjoint gives a convenient way to classify many types of operators on Hilbert space, for example:

- $T \in \mathcal{B}(H)$ such that $T^* = T$ (self-adjoint operators),
- $N \in \mathcal{B}(H)$ such that $N^*N = NN^*$ (normal operators),

- $P \in \mathcal{B}(H)$ such that $P = P^2 = P^*$ (orthogonal projections),
- $V \in \mathcal{B}(H)$ such that $V^*V = \mathbb{1}$ (isometries),
- $W \in \mathcal{B}(H)$ such that $WW^* = \mathbb{1}$ (co-isometries), and
- $U \in \mathcal{B}(H)$ such that $UU^* = U^*U = \mathbb{1}$, (unitaries).

An operator $A \in \mathcal{B}(H)$ is **positive**, written $A \geq 0$, if $A = B^*B$ for some $B \in \mathcal{B}(H)$, or equivalently, if $\langle h|A|h\rangle \geq 0$ for all $|h\rangle \in H$. The subset of positive elements define a translationally invariant partial order on the self-adjoint elements of $\mathcal{B}(H)$. For $A, B \in \mathcal{B}(H)$ with $A^* = A$ and $B^* = B$, we have $A \leq B$ if and only if $B - A \geq 0$.

2.2.2 Useful decompositions for operators on Hilbert spaces

The **spectrum** of bounded operator $T \in \mathcal{B}(H)$ is the set

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda\mathbb{1} \text{ does not have a bounded inverse}\}.$$

Any operator $A \in \mathcal{B}(H)$ has a **polar decomposition** $A := U|A|$, where U is a partial isometry and $|A| = \sqrt{A^*A} \geq 0$. In the case that H is finite dimensional, we can take U to be unitary. For any compact self-adjoint operator $T \in \mathcal{B}(H)$ there is an orthonormal family of orthogonal projections $\{P_i\}_{i\in I}$ with $P_i : H \to H$, for all $i \in I$, such that

$$T = \sum_{i\in I} \lambda_i P_i, \tag{2.1}$$

where λ_i is an eigenvalue of T and P_i is the (spectral) projection onto the corresponding eigenspace for each $i \in I$. The decomposition in Equation (2.1) is called the **spectral decomposition**. Moreover, if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $X : \mathbb{C}^d \to \mathbb{C}^d$ is a self-adjoint operator with spectral decomposition $X = \sum_{i=1}^d \lambda_i |\phi_i\rangle\langle\phi_i|$, then $f(X) = \sum_{i=1}^d f(\lambda_i)|\phi_i\rangle\langle\phi_i|$. This defines the functional calculus of continuous functions for self-adjoint operators on finite-dimensional Hilbert spaces.

2.2.3 Positive cones and semi-pre-C^*-algebras

An **algebra** $\mathcal{A}$ is a vector space equipped with a multiplication operation $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ sending $(a, b) \mapsto a \cdot b$, for all $a, b \in \mathcal{A}$. In this work, all of our algebras are **unital** (i.e. contain a multiplicative unit) and are vector spaces over the field of complex numbers. A **$*$-algebra** is an algebra $\mathcal{A}$ equipped with an antilinear involution $* : \mathcal{A} \to \mathcal{A}$; $a \mapsto a^*$. That satisfies the properties $(\alpha xy + \beta z)^* = \bar{\alpha} y^* x^* + \bar{\beta} z^*$, where $x, y, z \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$. A **$*$-subalgebra $\mathcal{S} \subset \mathcal{A}$** is a vector subspace of $\mathcal{A}$ that is closed under multiplication and the $*$-operation. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism if $\varphi(ab) = \varphi(a)\varphi(b)$, for all $a, b \in \mathcal{A}$, and $\varphi(a^*) = \varphi(a)^*$, for all $a \in \mathcal{A}$. A **two-sided $*$-ideal** is a $*$-subalgebra $\mathcal{I} \subseteq \mathcal{A}$ such that $a\mathcal{I} \subset \mathcal{I}$ and $\mathcal{I}b \subset \mathcal{I}$, for all $a, b \in \mathcal{A}$, and $a^*\mathcal{I} \subset \mathcal{I}$, whenever $a\mathcal{I} \subset \mathcal{I}$. Given a finite collection of elements $R \subset \mathcal{A}$, the **ideal generated by** R, denoted $\langle\langle R \rangle\rangle$, is the smallest two-sided ideal containing R. Elements $b \in \mathcal{A}$ contained in the ideal generated by R can be expressed as $b = \sum_{j \in J} a_j r_j b_j$, for all $a_j, b_j \in \mathcal{A}$, $r_j \in R$. If R is an ideal, then $\langle R \cup R^* \rangle$ is a $*$-ideal in $\mathcal{A}$. If $\mathcal{I} \subset \mathcal{A}$ is a two-sided ideal in an algebra $\mathcal{A}$, then the **quotient algebra $\mathcal{A}/\mathcal{I}$** is the algebra with multiplication $(a + \mathcal{I}) \cdot (b + \mathcal{I}) = ab + \mathcal{I}$, where $a + \mathcal{I}$, $b + \mathcal{I}$ are (cosets) elements of $\mathcal{A}/\mathcal{I}$.

The following is based on the exposition of semi-pre-C^*-algebras found in [Oza13a]. Given a $*$-algebra $\mathcal{A}$, we can identify the set of **hermitian** (i.e. self-adjoint) elements $\mathcal{A}_h := \{a \in \mathcal{A} : a = a^*\}$. We then say that $a \in \mathcal{A}$ is **positive**, written $a \geq 0$, if it is a **sum of hermitian squares** (SOS), that is $a = \sum_{i \in I} b_i^* b_i$, with $b_i \in \mathcal{A}$ for all $i \in I$. Like in the case of $\mathcal{B}(H)$, this notion of positivity induces a translationally invariant partial order on the self-adjoint elements $\mathcal{A}$. We write $a \geq b$ if $a - b$ is positive (i.e. a sum of hermitian squares). We denote the set of positive elements in $\mathcal{A}$ by

$$\mathcal{A}_+ := \{a \in \mathcal{A} : a \geq 0\}.$$

More generally, the positive elements of a $*$-algebra defined as sums of squares are an example of a **$*$-positive cone**. That is, the following holds for elements of $\mathcal{A}_+$:

(i) $\lambda \cdot 1 \in \mathcal{A}_+$ for all $\lambda \in \mathbb{R}_{\geq 0}$,

(ii) $\lambda(a + b) \in \mathcal{A}_+$ for all $a, b \in \mathcal{A}_+$ and $\lambda \in \mathbb{R}_{\geq 0}$, and

(iii) $x^* a x \in \mathcal{A}_+$ whenever $x \in \mathcal{A}$ and $a \in \mathcal{A}_+$.

The reason for defining a $*$-positive cone on $\mathcal{A}$ is to define an abstract notion of bounded elements in $\mathcal{A}$. To do this, we let

$$\mathcal{A}_{bdd} = \{a \in \mathcal{A} : \; \exists R > 0 \text{ such that } a^* a \leq R1\},$$

denote the set of bounded elements in $\mathcal{A}$. Not that $\mathcal{A}_{bdd}$ is a $*$-subalgebra of $\mathcal{A}$. Hence, if $\mathcal{A}$ is generated as a $*$-algebra by $\mathcal{S}$, then $\mathcal{S} \subset \mathcal{A}_{bdd}$ implies that $\mathcal{A} = \mathcal{A}_{bdd}$. If a $*$-algebra has a positive cone such that $\mathcal{A} = \mathcal{A}_{bdd}$ then $\mathcal{A}$ is said to be **archimedean**[1] [Oza13a]. A $*$-algebra with a $*$-positive cone satisfying the archimedean condition is called a **semi-pre-C^*-algebra**. Note that $\|a\| = \inf\{R > 0 : a^*a - R^2\mathbb{1} \leq 0\}$ is a semi-norm on $\mathcal{A}$. The elements for which $\|a\| = 0$ form the **ideal of infinitesimal elements**

$$\mathcal{I}(\mathcal{A}) = \{a \in \mathcal{A} : a^*a \leq \epsilon\mathbb{1} \text{ for all } \epsilon > 0\}.$$

The **archimedean closure** of $\mathcal{A}_+$ are the hermitian elements

$$arch(\mathcal{A}_+) = \{a \in \mathcal{A}_h : a + \epsilon\mathbb{1} \in \mathcal{A}_+ \text{ for all } \epsilon > 0\}$$

A **C^*-algebra** $\mathcal{A}$ is a norm closed $*$-subalgebra of $\mathcal{B}(H)$, and every C^*-algebra $\mathcal{A}$ is a semi-pre-C^*-algebra with infinitesimal ideal $\mathcal{I}(\mathcal{A}) = 0$, and an archimedean closed cone

$$arch(\mathcal{A}_+) = \{a \in \mathcal{A}_h : \langle h|a|h\rangle \geq 0 \text{ for all } |h\rangle \in H\}.$$

This is quite different than the typical definition of a C^*-algebra as a Banach $*$-algebra $\mathcal{A}$ with a C^*-norm. That is, a Banach $*$-algebra with a submultiplicative norm that satisfies the C^*-identity $\|a^*a\| = \|a\|^2$, for all $a \in \mathcal{A}$ [Bla06].

2.2.4 States and $*$-representations

A **state** on a semi-pre-C^*-algebra $\mathcal{A}$ is a linear functional $f : \mathcal{A} \to \mathbb{C}$, such that

(1) $f(a) \geq 0$ for all positive elements $a \in \mathcal{A}_+$,

(2) $f(1) = 1$.

A state $f : \mathcal{A} \to \mathbb{C}$ is **tracial** if $f(ab) = f(ba)$ for all $a, b \in \mathcal{A}$. Another important concept in the theory of operator algebras is the commutant. The **commutant** of $\mathcal{S} \subseteq \mathcal{A}$ is

$$\mathcal{S}' = \{b \in \mathcal{A} : ba = ab \text{ for all } a \in \mathcal{S}\}.$$

A **representation** of a semi-pre-C^*-algebra is a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(H)$, for some Hilbert space H. By Schur's lemma, a representation is **irreducible** if the commutant of the image contains only scalars, that is $\pi(\mathcal{A})' = \{\lambda \cdot \mathbb{1} : \lambda \in \mathbb{C}\}$. A **vector state** $|\psi\rangle$ is a unit vector in H. Given a representation $\pi : \mathcal{A} \to \mathcal{B}(H)$ and a vector state $|\psi\rangle$, we obtain a concrete state on the image $\pi(\mathcal{A})$ via $a \mapsto \langle\psi|\pi(a)|\psi\rangle$, since π preserves positivity

[1] This is sometimes also called the *Combes axiom*.

and $\langle\psi|\psi\rangle = 1$. A vector state $|\psi\rangle \in H$, is a **cyclic vector** for $\pi : \mathcal{A} \to \mathcal{B}(H)$ if $\pi(\mathcal{A})\,|\psi\rangle$ is dense (topologically) in H. That is $\overline{\pi(\mathcal{A})\,|\psi\rangle} = H$, where the closure is taken with respect to the norm topology on H. A vector state $|\psi\rangle \in H$, is a **separating vector** for $\pi : \mathcal{A} \to \mathcal{B}(H)$, if $\pi(a)\,|\psi\rangle = 0$ implies $\pi(a) = 0$, for all $a \in \mathcal{A}$. It is well known that if $\pi : \mathcal{A} \to \mathcal{B}(H)$ is a representation, then a state vector $|\psi\rangle$ is cyclic for $\pi(\mathcal{A})$ if and only if it is separating for $\pi(\mathcal{A})'$ [Bla06].

A tuple consisting of a Hilbert space H, a representation $\pi : \mathcal{A} \to \mathcal{B}(H)$, and a cyclic vector $|\psi\rangle \in H$, is called a **cyclic or (GNS) representation** of $\mathcal{A}$. It's clear that every cyclic representation gives rise to a state on $\mathcal{A}$ via $f(a) = \langle\psi|\pi(a)|\psi\rangle$, for all $a \in \mathcal{A}$. On the other hand, the fundamental **GNS construction** of Gel'fand, Naimark, and Segal shows that every state $f : \mathcal{A} \to \mathbb{C}$, gives rise to a cyclic representation $(\pi, H, |\psi\rangle)$ of $\mathcal{A}$. We will not prove this fundamental correspondence as it can be found in many standard texts, for example [Bla06, Dav96].

The importance of defining a positive (sum of squares) cone on abstract $*$-algebras can now be made apparent.

Proposition 2.3. If $\mathcal{A}$ is a semi-pre-C^*-algebra, then each $a \in \mathcal{A}$ is bounded in every representation.

Proof. Consider the norm

$$\|a\| = \sup\{\|\pi(a)\|_{op} : \ *\text{-homomorphism } \pi : \mathcal{A} \to Lin(H)\},$$

for all $a \in \mathcal{A}$. Let π be a representation, and suppose $|v\rangle \in H$ is the unit vector such that $\|\pi(a)\|_{op}^2 = \langle v|\pi(a^*a)|v\rangle$. Let $\phi : \mathcal{A} \to \mathbb{C}$ be the state $a \mapsto \langle v|\pi(a)|v\rangle$. Now, by the archimedean property there exists $R > 0$ such that $a^*a \leq R1$. However, this implies that $\phi(a^*a) \leq R$, hence a is bounded in any representation. □

Lemma 2.4. If $\tau : \mathcal{A} \to \mathbb{C}$ is tracial state with GNS construction $(H, \pi, |v\rangle)$, then $|v\rangle \in H$ is separating for $\pi(\mathcal{A})$.

Proof. Suppose $\pi(a)|v\rangle = 0$. Then for any $b \in \mathcal{A}$, we have $\pi(b)\pi(a)|v\rangle = 0$, which implies that

$$\begin{aligned}
\langle v|\pi(a)^*\pi(b)^*\pi(b)\pi(a)|v\rangle &= \tau(a^*b^*ba) \\
&= \tau(baa^*b^*) \\
&= \langle v|\pi(b)\pi(aa^*)\pi(b)^*|v\rangle \\
&= \langle w|\pi(a)\pi(a)^*|w\rangle \\
&= 0,
\end{aligned}$$

where $\pi(b)^*|v\rangle = |w\rangle$. Since $b \in \mathcal{A}$ was arbitrary and $\overline{\pi(\mathcal{A})|v\rangle} = H$ this implies that $\|\pi(a^*)\| = \|\pi(a)\| = 0$, and it follows that $\pi(a) = 0$, as desired. □

In this work, we say that a state $f : \mathcal{A} \to \mathbb{C}$ is **finite-dimensional** if the Hilbert space in the GNS representation of f is finite-dimensional. Whenever we have a representation $\pi : \mathcal{A} \to \mathcal{B}(H)$ on a finite-dimensional Hilbert space H, we have the following structure theorem for H and the representation π.

Theorem 2.5 (Double commutant decomposition)**.** If $\mathcal{A}$ is a C^*-algebra and $\pi : \mathcal{A} \to \mathcal{B}(H)$ is a $*$-representation with H finite-dimensional, then there is an isometric isomorphism

$$H \cong \bigoplus_{i=1}^{k} \mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i}, \tag{2.2}$$

for positive integers n_i, m_i for $i \in \{1, \ldots, k\}$, and the decompositions

$$\pi(\mathcal{A}) \cong \bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C}) \otimes \mathbb{1}_{m_i}, \text{ and } \pi(\mathcal{A})' \cong \bigoplus_{i=1}^{k} \mathbb{1}_{n_i} \otimes M_{m_i}(\mathbb{C}). \tag{2.3}$$

In particular $\pi(\mathcal{A})$ and $\pi(\mathcal{A})'$ are direct sums of matrix algebras $M_{n_i}(\mathbb{C})$ and $M_{m_i}(\mathbb{C})$, each with multiplicities m_i and n_i respectively.

We omit the proof. The statement can be found in classic text such as [Dav96]. The following lemma is an easy consequence of this structure theorem:

Lemma 2.6. If f is a tracial state on $\mathcal{A} \cong \oplus_{i=1}^{k} M_{n_i}(\mathbb{C})$ then $f(\cdot) = \sum_{i=1}^{k} \lambda_i \widetilde{tr}_{n_i}(\cdot)$, where $\widetilde{tr}_{n_i}$ is the normalized trace on $M_{n_i}(\mathbb{C})$, and $\sum_{i=1}^{k} \lambda_i = 1$.

We leave the proof to the reader. Note that if $\mathcal{A} \cong \oplus_{i=1}^{k} M_{n_i}(\mathbb{C})$, then the trace is not faithful whenever there is a $\lambda_i = 0$ for some $1 \leq i \leq k$.

Lemma 2.7. If e and f are projections in a C^*-algebra $\mathcal{A}$ and τ is a faithful tracial state with $s(ef) = 0$, then $ef = 0$.

Proof. Use the tracial condition and self-adjointness of e to write $0 = \tau(ef) = \tau(e^2 f) = \tau(e^* f e) = \tau((f^{1/2}e)^*(f^{1/2}e)) = 0$, then $f^{1/2}e = 0$, implies $ef = 0$. □

2.8 Quantum states, measurements, and correlations

The following is a brief review of quantum probability.

2.8.1 Quantum states and the Schmidt decomposition

A **quantum state** (or state vector) is a unit vector $|\psi\rangle$ in a (complex) Hilbert space H. A **bipartite quantum state** is a unit vector $|\psi\rangle \in H_A \otimes H_B$. A bipartite state is called a **product state** (or product vector) if $|\psi\rangle = |\phi\rangle \otimes |\xi\rangle$, for some $|\phi\rangle \in H_A$ and $|\xi\rangle \in H_B$. If a state is not a product state, then it is said to be **entangled** (or an entangled quantum state).

A **density operator** is a positive (semidefinite) operator $\rho \in Lin(H)$, with the additional property that $\mathrm{tr}(\rho) = 1$. Each quantum state $|\psi\rangle$ gives rise to density operators ρ via the outer product $|\psi\rangle \mapsto |\psi\rangle\langle\psi| := \rho$. The unit norm condition on $|\psi\rangle$ ensures that any density operator ρ has $\mathrm{tr}(\rho) = 1$. A density operator $\rho \in Lin(H)$ is said to be a **pure state**, if $\rho = |\psi\rangle \langle\psi|$, for some state vector $|\psi\rangle \in H$ (i.e. ρ has rank 1). Density operators are convenient for representing probabilistic combinations (i.e. statistical mixtures) of pure states.

If H_A and H_B are Hilbert spaces and $|\psi\rangle \in H_A \otimes H_B$ then $|\psi\rangle$ has a **Schmidt decomposition**

$$|\psi\rangle = \sum_{i \in I} \alpha_i |u_i\rangle \otimes |v_i\rangle,$$

where the (positive) coefficients, $\alpha_i > 0$, for all $i \in I$, are called the **Schmidt coefficients**, and the collections $\{|u_i\rangle\}_{i\in I}$ and $\{|v_i\rangle\}_{i\in I}$ are orthonormal subsets of H_A and H_B respectively. The **Schmidt rank** of $|\psi\rangle$ is the cardinality of I. The **support of** $|\psi\rangle$ is the image of the projector $\sum_{i\in I} |u_i\rangle\langle u_i| \otimes |v_i\rangle\langle v_i| \in Lin(H_A \otimes H_B)$. Often we restrict to the image of the local projections $\sum_{i\in I} |u_i\rangle\langle u_i| \in Lin(H_A)$ or $\sum_{i\in I} |v_i\rangle\langle v_i| \in Lin(H_B)$ to the tensor factors H_A and H_B, which we refer to as the local support projections.

Given an operator $X = X_A \otimes X_B$ acting on a bipartite Hilbert space $H_A \otimes H_B$ the **partial trace** is the unique linear operator $\mathrm{tr}_A : Lin(H_A \otimes H_B) \to Lin(H_B)$ for which $\mathrm{tr}_A(X) = \mathrm{tr}(X_A)X_B$, where tr is the standard trace on $Lin(H_B)$. The partial trace is ofen employed to "trace out" the one of the bipartite systems, resulting in the local operator on one of the system. A quantum state $|\psi\rangle \in H_A \otimes H_B$ is said to be **maximally entangled** if its reduced density matrix $\mathrm{tr}_{H_A}(\rho) = \rho_B$ on H_B (or $\rho_A = \mathrm{tr}_{H_B}(\rho)$ on H_A) is $\mathbb{1}/\dim(H_B)$ (or $\mathbb{1}/\dim(H_B)$). The denstity matrix $\mathbb{1}/\dim(H)$ is called **maximally mixed**.

2.8.2 Measurements and quantum probability

In the following, let X, Y, A, and B, be finite sets. A **positive operator-valued measure** (or **POVM**) over a set of outcomes A, is a collection of positive operators $\{M_a\}_{a\in A}$ acting on a Hilbert space H such that $\sum_a M_a = \mathbb{1}_H$. A **projective (operator) valued measure** (or **PVM**) over a set of outcomes A, is a POVM $\{P_a\}_{a\in A}$, with the additional property that each P_a is an orthogonal projection, for each $a \in A$. If the state of a quantum mechanical system is in the state $|\psi\rangle \in H$, then the probability $p(a)$, of obtaining outcome $a \in A$, is given by

$$p(a) = \langle\psi|M_a|\psi\rangle.$$

A (bipartite) **correlation** is a function $p : A\times B\times X\times Y \to \mathbb{R}_{\geq 0}$ such that $\sum_{a,b} p(a, b|x, y) = 1$, for all $(x, y) \in X \times Y$. Thus, a correlation can be thought of as an element inside the real quadrant $\mathbb{R}_{\geq 0}^{X\times Y\times A\times B}$.

Definition 2.9. A **POVM quantum model** for a bipartite correlation p is a pair of finite dimensional Hilbert spaces H_A and H_B, collections of POVMs $\{\{M_a^x\}_{a\in A} : x \in X\}$ and $\{\{N_b^y\}_{b\in B} : y \in Y\}$, and a state vector $|\psi\rangle \in H_A \otimes H_B$ such that

$$p(a, b|x, y) = \langle\psi|M_a^x \otimes N_b^y|\psi\rangle,$$

for all $(a, b, x, y) \in A \times B \times X \times Y$.

If p has a quantum model $\mathcal{M}$, with H_A and H_B finite-dimensional, then p is said to be a **quantum correlation**.

Definition 2.10. A **PVM quantum model** for a bipartite correlation is a POVM model with the additional constraint that each measurement operator is a projection.

The following results show that every quantum correlation has a projective quantum model.

Theorem 2.11 (Naimark dilation theorem)**.** Let $\{M_a\}_{a\in A}$ be a POVM on a (complex) Hilbert space H and $|\psi\rangle \in H$, then there is a Hilbert space K, an isometry $V : H \to K$, and a projective valued measure (PVM) $\{P_a\}_{a\in A}$ acting on K, such that $M_a = V^*P_aV$.

Proof. Let $K = H \otimes \mathbb{C}^A$, and define $V : H \to H \otimes \mathbb{C}^A$ via $|h\rangle \mapsto \sum_{a\in A} M_a^{1/2}|h\rangle \otimes |a\rangle$, for $|h \in H$, where $\{|a\rangle\}_{a\in A}$ is the standard orthonormal basis for $\mathbb{C}^A$. Now, define $P_a :=$

$\mathbb{1} \otimes |a\rangle\langle a|$. Clearly, $\sum_{a\in A} P_a = \mathbb{1}_H \otimes \mathbb{1}_{\mathbb{C}^A} = \mathbb{1}_K$ and $P_a^2 = P_a = P_a^*$. Lastly we observe,

$$\begin{aligned}\langle h|V^* P_a V|h\rangle &= \langle h|V^* \left(\mathbb{1} \otimes |a\rangle\langle a|\right) V|h\rangle \\ &= \sum_{b,c\in A} \langle h|M_b^{1/2^*} M_c^{1/2}|h\rangle_H \langle b|a\rangle_{\mathbb{C}^A} \langle a|c\rangle_{\mathbb{C}^A} \\ &= \langle h|M_a|h\rangle,\end{aligned}$$

for all $|h\rangle \in H$, and hence $M_a = V^* P_a V$ as desired. □

Corollary 2.12. If $\{M_a\}_{a\in A}$ a POVM on H and $|v\rangle$ is a state on H such that $p(a) = \langle v|M_a|v\rangle$, then there is a Hilbert space K, a vector $|u\rangle \in K$, and projective (PVM) measurement $\{P_a\}_{a\in A}$ on K, such that $p(a) = \langle u|P_a|u\rangle$. Moreover, if H is finite-dimensional, then we can pick K to be finite-dimensional.

Proposition 2.13 (Simultaneous Naimark dilation)**.** Let $\{\{M_a^x\}_{a\in A} : x \in X\}$ be a collection of POVMs on H, there is an Hilbert space K, an isometry $V : H \to K$ (independent of $x \in X$), and a collection of projective measurements $\{\{P_a^x\}_{a\in A} : x \in X\}$, such that $V^* M_a^x V = P_a^x$ for all $a \in A, x \in X$.

The idea in Proposition 2.13 is to iteratively construct Naimark dilations. We refer the reader to the proof in [PSS+16, HPV16].

Corollary 2.14. If $\{M_a\}_{a\in A}$ is a POVM on H_A, and $|v\rangle$ is a state on $H_A \otimes H_B$, such that $p(a) = \langle v|M_a \otimes N_b|v\rangle$, then there is a Hilbert space K, a vector $|u\rangle \in K_A \otimes K_B$, and projective (PVM) measurement $\{P_a\}_{a\in A}$ on K_A and $\{Q_b\}_{b\in B}$ such that $p(a, b) = \langle u|P_a \otimes Q_b|u\rangle$ for all $(a, b) \in A \times B$. Moreover, if $H_A \otimes H_B$ is finite-dimensional, then so is $K_A \otimes K_B$.

Proof. Suppose $\mathcal{M}$ is a model achieving (or realizing) p, then there are Hilbert spaces H_A and H_B, POVMs $\{M_a^x\}_{a\in A}$ acting on H_A, $\{N_b^y\}_{b\in B}$ acting on H_B, and a vector state $|\psi \in H_A \otimes H_B$, such that

$$p(a, b|x, y) = \langle\psi|M_a^x \otimes N_b^y|\psi\rangle,$$

or all $x, y, a, b \in X \times Y \times A \times B$. For each $x \in X$ and $y \in Y$, consider the corresponding projections $\{P_a^x\}_{a\in A}$ and $\{Q_b^y\}_{b\in B}$ acting on $\tilde{H}_A = H_A \otimes \mathbb{C}^A$ and $\tilde{H}_B = H_B \otimes \mathbb{C}^B$ and state $|\tilde{\psi}\rangle = V_A \otimes V_B|\psi\rangle \in \tilde{H}_A \otimes \tilde{H}_B$ obtained via Naimark dilation (Theorem 2.11), then

$$\begin{aligned}\langle\tilde{\psi}|P_a^x \otimes N_b^y|\tilde{\psi}\rangle &= \langle\psi|(V_A \otimes V_B)^* P_a^x \otimes Q_b^y (V_A \otimes V_B)|\psi\rangle \\ &= \langle\psi|V_A^* P_a^x V_A \otimes V_B^* Q_b^y V_B)|\psi\rangle \\ &= \langle\psi|M_a^x \otimes N_b^y|\psi\rangle \\ &= p(a, b|x, y).\end{aligned}$$

It follows that $\tilde{H_A}$ and $\tilde{H_B}$ are finite-dimensional only if H_A and H_B are finite-dimensional respectively. $\square$

2.14.1 State induced matrix semi-norms

By a matrix algebra, we mean the $*$-algebra $M_d(\mathbb{C})$ of complex $d \times d$ matrices equipped with an antilinear involution. For $A \in M_d(\mathbb{C})$ this antilinear involution, mapping $A \mapsto A^*$, is the conjugate transpose $A^* := \overline{A}^\top$, where the transpose $\top$ is taken with respect to a basis for $\mathbb{C}^d$. For $A \in M_d(\mathbb{C})$, a matrix semi-norm $\| \cdot \|_{M_d(\mathbb{C})} : M_d(\mathbb{C}) \to \mathbb{R}$, satisfies

(1) $\|cA\| = |c|\|A\|$ for all $c \in \mathbb{C}$, $A \in M_d(\mathbb{C})$,

(2) $\|A + B\| \leq \|A\| + \|B\|$ for all $A, B \in M_d(\mathbb{C})$,

(3) $\|A\| \geq 0$, for all $A \in M_d(\mathbb{C})$.

If in addition, $\|A\| = 0$ if and only if $A = 0$ (the definiteness property), then $\| \cdot \|_{M_d(C)}$ is a matrix norm. Two common matrix norms are the operator norm $\|A\|_{op}$ and the Frobenius (or Hilbert Schmidt) norm $\|A\|_F$. For a finite-dimensional Hilbert space H, the set of linear operators $Lin(H)$ is equipped with the Frobenius (Hilbert-Schmidt) inner-product

$$\langle A, B\rangle_F := \mathrm{tr}(A^* B), \quad \text{for all } A, B \in Lin(H).$$

We also often use the little Frobenius (or normalized Hilbert-Schmidt) norm, denoted by

$$\|A\|_f^2 := \mathrm{tr}(A^* A)/d = \|A\|_F^2/d,$$

for all $A \in M_d(\mathbb{C})$. The normalization in the little Frobenius norm ensures that $\|\mathbb{1}\|_f = 1$, in contrast to $\|\mathbb{1}\|_F = \sqrt{d}$.

Lemma 2.15. For A, B in $M_d(\mathbb{C})$ we recall the following standard results:

1. $\|A\|_{op} \leq \|A\|_F \leq \sqrt{d}\|A\|_{op}$ (matrix 2-norm inequality).
2. $\|AB\|_F \leq \|A\|_{op}\|B\|_F \leq \|A\|_F\|B\|_F$ (submultiplicativity of the F-norm).
3. $\|A\|_f \leq \|A\|_{op} \leq \sqrt{d}\|A\|_f$. (normalized 2-norm inequality)
4. $\|UAV\|_f = \|AV\|_f = \|A\|_f$ for all unitaries U, V (unitary invariance).

5. $\|ABC\|_f \leq \|A\|_{op}\|B\|_f\|C\|_{op}$ (bimodule property of f-norm).
6. $\|A^*\|_f = \|A\|_f$ ($*$-norm identity).
7. If $0 \leq A \leq B$, then $\|A\|_f \leq \|B\|_f$ (respects positive order).
8. If $AA^* \leq \mathbb{1}$ (or $A^*A \leq \mathbb{1}$), then $\|AB\|_F \leq \|B\|_F$ (similarly $\|BA\|_F \leq \|B\|_F$) for any $B \in M_d(\mathbb{C})$.

These well known results about $\|\cdot\|_{op}$, $\|\cdot\|_F$, and $\|\cdot\|_f$ can be deduced from facts in the wonderful book [Bha13].

Remark 2.16. Unlike its unnormalized counterpart $\|\cdot\|_F$, the *little* Frobenius norm $\|\cdot\|_f$ is not submultiplicative.

Definition 2.17. For a density operators $\rho \in Lin(H)$ the **state induced semi-norm** (or ρ-norm) is given by

$$\|T\|_\rho := \|T\rho^{1/2}\|_F$$

for all $T \in Lin(H)$. The non positive-definiteness of the ρ-norm is the result of any degeneracies (0-eigenspaces) in ρ.

Remark 2.18. In the case where $\rho = \mathbb{1}/d$ is the maximally mixed state, the ρ-norm $\|\cdot\|_\rho$ coincides with the little Frobenius norm $\|\cdot\|_f = \|\cdot\|_{\mathbb{1}/d}$. Therefore, starting from a maximally entangled state the induced ρ-norm on H_B (or H_A) is the little Frobenius norm $\|\cdot\|_f$.

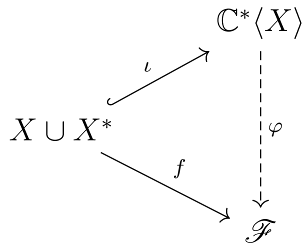
2.19 Approximate representation theory for $*$-algebras

Much of this section is based on the lectures from William Slofstra's graduate course on Approximate Representation Theory [Slo19a]. Many statements in this section are generalizations of results[Slo19b] generalized from the group case.

2.19.1 Finitely-presented $*$-algebras

Let X be a finite set of variables and define the set $X^* = \{x^* : x \in X\}$. We let $\mathbb{C}^*\langle X\rangle$ denote the **free $*$-algebra generated by** X. The free $*$-algebra $\mathbb{C}^*\langle X\rangle$ has the following universal property. For any $*$-algebra $\mathscr{F}$, suppose $f : X \cup X^* \to \mathscr{F}$ is a function, and

$\iota : X \cup X^* \hookrightarrow \mathbb{C}^*\langle X \rangle$ is the natural inclusion, then there is a unique homomorphism $\varphi : \mathbb{C}^*\langle X \rangle \to \mathscr{F}$, such that $\varphi|_{(X \cup X^*)} = f$, that is, the restriction of φ to $X \cup X^*$ is f. This universal property is summarized by the following commutative diagram:



(2.4)

The elements of $\mathbb{C}^*\langle X \rangle$ are finite linear combinations of monomials (i.e $*$-polynomials) in the variables $x \in X \cup X^*$, with complex coefficients. Let $multi(X \cup X^*)$ be the ordered multisets of $X \cup X^*$, then every $p \in \mathbb{C}^*\langle X \rangle$ can be expressed as a non-commutative polynomial

$$p = \sum_{\alpha \subset multi(X \cup X^*)} c_\alpha \prod_{x \in \alpha} x,$$

with $c_\alpha \in \mathbb{C}$, for all $\alpha \subset multi(X \cup X^*)$. The convection is that the empty product gives the unit, that is we write $\prod_\emptyset = 1$.

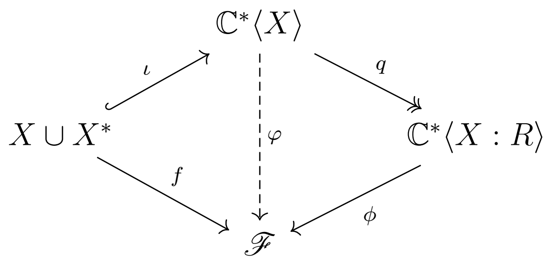
If R a finite collection of non-commutative $*$-polynomials in the variables $X \cup X^*$. The finitely presented $*$-algebra $\mathbb{C}^*\langle X : R \rangle$ is the quotient of $\mathbb{C}^*\langle X \rangle$ by the smallest $*$-ideal containing R, also known as $\langle\langle R \rangle\rangle$ (the $*$-ideal generated by R). Since X and R are finite, we call

$$\mathbb{C}^*\langle X : R \rangle := \mathbb{C}^*\langle X \rangle / \langle\langle R \rangle\rangle.$$

a finitely-presented $*$-algebra. If $\mathscr{A} \cong \mathbb{C}^*\langle X : R \rangle$, then we call the choice of generators and relations (X, R) a **presentation** of $\mathscr{A}$. It can be shown that every $*$-algebra has a presentation, however, we are not guaranteed that the existing X and R are finite in this presentation. To motivate our definition of approximate representation we note the following fact.

Proposition 2.20. Let $\mathscr{F}$ be an arbitrary $*$-algebra. $*$-homomorphisms $\phi : \mathbb{C}^*\langle X : R \rangle \to \mathscr{F}$ are in bijection with functions $f : X \cup X^* \to \mathscr{F}$ such that $f(r) = 0$, for all $r \in R$.

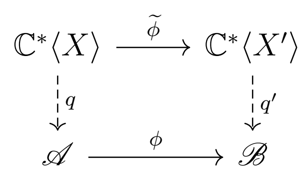
Proof. Given a $*$-homomorphism $\phi : \mathbb{C}^*\langle X : R\rangle \to \mathscr{F}$, we have the following diagram:



(2.5)

where $q : \mathbb{C}^*\langle X\rangle \twoheadrightarrow \mathbb{C}^*\langle X : R\rangle$ is the quotient map induced by the ideal $\langle\langle R\rangle\rangle$. By the universal property, the homomorphism φ factors as $\phi \circ q$, and is 0 on the kernel of q. Hence, $\varphi|_{X\cup X^*}(r) = 0$ implies that $f(r) = 0$, for all relations $r \in R$. □

Definition 2.21. Let $\mathscr{A} := \mathbb{C}^*\langle X : R\rangle$ and $\mathscr{B} = \mathbb{C}^*\langle X' : R'\rangle$ be finitely presented $*$-algebras. A homomorphism $\phi : \mathscr{A} \to \mathscr{B}$ of presented algebras is a **pair** $(\phi, \widetilde{\phi})$, such that $\widetilde{\phi} : \mathbb{C}^*\langle X\rangle \to \mathbb{C}^*\langle X'\rangle$. The terminology is motivated by the following commutative diagram:



where q and q' are the canonical projections induced by the quotient of the $*$-ideals $\langle\langle R\rangle\rangle$ and $\langle\langle R'\rangle\rangle$ respectively. With this is mind we refer to $\widetilde{\phi}$ as the **lift** of ϕ (or equivalently that $\widetilde{\phi}$ **descends**) to ϕ.

By Proposition 2.20 $*$-homomorphisms of finitely presented $*$-algebras $\phi : \mathscr{A} \to \mathscr{B}$ are in bijection with lifts $\widetilde{\phi} : \mathbb{C}^*\langle X\rangle \to \mathbb{C}^*\langle X'\rangle$ such that $\widetilde{\phi}(r) = 0$, for all $r \in R$.

Remark 2.22. Assume X is finite, a relation r is non-commutative $*$-polynomial $(x_1, \ldots, x_n) \mapsto r(x_1, \ldots, x_n)$, that is

$$r = \sum_\alpha c_\alpha \prod_{x\in\alpha} x,$$

where α is a subset of the multiset of $X \cup X^*$. Functions $f : X \cup X^* \to \mathscr{F}$ can be viewed as elements $\underline{y} \in \mathscr{F}^{X\cup X^*}$. The evaluation map $\mathrm{eval}_{\underline{y}} : \mathbb{C}^*\langle X\rangle \to \mathscr{F}$ is the homomorphism taking polynomials $p \mapsto p(\underline{y})$. Hence, we can identify $\mathrm{Hom}(\mathbb{C}^*\langle X : R\rangle, \mathscr{F})$ with the set $\{\underline{y} \in \mathscr{F}^{X\cup X^*} : r(\underline{y}) = 0,\ \forall\ r \in R\}$. In particular, such $*$-homomorphisms send

$$p(X \cup X^*) = \sum_\alpha c_\alpha \prod_{x\in\alpha} x \mapsto p(f(X \cup X^*)) = \sum_\alpha c_\alpha \prod_{x\in\alpha} f(x).$$

2.22.1 Approximate representations of finitely presented $*$-algebras and stability in matrix algebras

The final remark in the previous subsection suggests a notion of approximate representation. Suppose that $\mathscr{A}$ is a finitely presented $*$-algebra $\mathscr{A} \cong \mathbb{C}^*\langle X : R\rangle$. Letting $\mathscr{F} = M_d(\mathbb{C})$ with a matrix norm (or seminorm) $\|\cdot\|_{M_d(\mathbb{C})}$, we can identify the space $\mathrm{Hom}(\mathscr{A}, M_d(\mathbb{C}))$ with matrix representations $\mathrm{Rep}(\mathscr{A}, M_d(\mathbb{C}))$. Let $\epsilon > 0$, we denote the space of approximate representations by

$$\mathrm{Approx}(\mathcal{A}, M_d(\mathbb{C}), \epsilon) = \{\underline{m} \in M_d(\mathbb{C})^{X\cup X^*} : \|r(\underline{m})\|_{M_d(\mathbb{C})} \leq \epsilon,\ \forall\ r \in R\}$$

With that in mind, consider the following definition:

Definition 2.23. Let $\mathscr{A} = \mathbb{C}^*\langle X : R\rangle$ be a finitely presented $*$-algebra and let $\|\cdot\|$ be a semi-norm on $M_d(\mathbb{C})$. An **ϵ-representation** of $\mathscr{A}$ is a $*$-homomorphism $\phi : \mathbb{C}^*\langle X\rangle \to M_d(\mathbb{C})$ such that

$$\|\phi(r)\| \leq \epsilon,$$

for all $r \in R$.

Remark 2.24. Whenever $\|\cdot\|$ is a norm, a 0-representation corresponds to a genuine homomorphism $\mathbb{C}^*\langle X : R\rangle \to M_d(\mathbb{C})$ by Proposition 2.20. In this work, will primarily focus on the case where the matrix norm $\|\cdot\|$ is a state induced semi-norm $\|\cdot\|_\rho$. That being said, the study of ϵ-representations is certainly not limited to this particular family of matrix norms, nor even matrix algebras. In fact, it is extremely interesting to contemplate a general theory of the subject.

Sometimes we may abuse notation and write $\phi : \mathscr{A} \to M_d(\mathbb{C})$ for an ϵ-representation, or say that ϕ is an ϵ-representation of $\mathscr{A}$, with the understanding that the ϵ-representation refers to the $*$-homomorphism $\tilde{\phi} : \mathbb{C}^*\langle X\rangle \to M_d(\mathbb{C})$, the lift of ϕ. Although one can study approximate representation theory with respect to any family of normed algebras. Because our focus here is on proving results concerning quantum strategies and nonlocal games, we will be restricting ourselves to the family of matrix algebras with the little Frobenius norm $\|\cdot\|_f$. This norm has a natural motivation in the sense that if $\phi : \mathcal{A} \to M_d(\mathbb{C})$ and $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle \otimes |i\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ then the state $f(a) = \langle\psi|a \otimes \mathbb{1}|\psi\rangle$ has the property that

$$f(a^*a) = \langle\psi|\phi(a^*a) \otimes \mathbb{1}|\psi\rangle = \|\phi(a) \otimes \mathbb{1}|\psi\rangle\|^2 = \widetilde{\mathrm{tr}}(a^*a) = \|\phi(a)\|_f^2.$$

We will also be interested in proving results for other state induced semi-norms. In particular, in the context of nonlocal games, we will obtain a norm that may not be induced by a tracial state but will have a certain approximate tracial property. The following technical definitions will come in handy when working with approximate representations:

Definition 2.25. An ϵ-representation ϕ of a finitely presented algebra $\mathbb{C}^*\langle X : R\rangle$ is a **mixed approximate representation** if for a nonempty subset $T \subset R$ of the relations we have that $\phi(r') = 0$ for all $r' \in T$. We call the subset of relations in T the **exact** relations of ϕ.

Definition 2.26. A finite-dimensional **unitary ϵ-representation** of $\mathbb{C}^*\langle X : R\rangle$ is a map $\phi : \mathbb{C}^*\langle X\rangle \to \mathbf{U}(\mathbb{C}^d)$ such that in addition to $\|\phi(r)\| \leq \epsilon$, for all $r \in R$, we have $\phi(x)\phi(x)^* = \phi(x)^*\phi(x) = \mathbb{1}$ for all $x \in X$.

Here, we mention an elementary lemmas about approximate representation that may be of interest to the reader regarding tensor products and direct sums of approximate representations.

Lemma 2.27. Suppose $\phi : \mathbb{C}\langle X\rangle \to M_d(\mathbb{C})$ and $\psi : \mathbb{C}\langle X\rangle \to M_{d'}(\mathbb{C})$ are ϵ- and ϵ'-representations of $\mathscr{A} = \mathbb{C}^*\langle X : R\rangle$ respectively with $\epsilon, \epsilon' < 1$. Then $\phi \oplus \psi$ is a $\max(\epsilon, \epsilon')$-representation, and $\phi \otimes \psi$ is an $(\epsilon + \epsilon')$-representation, with respect to $\|\cdot\|_f$.

Proof. For the first claim, we consider

$$\begin{aligned}
\|(\phi \oplus \psi)(r)\|_f^2 &= \left\| \begin{pmatrix} \phi(r) & 0 \\ 0 & \psi(r) \end{pmatrix} \right\|_f^2 \\
&= \frac{\mathrm{tr}}{d+d'} \begin{pmatrix} \phi(r)^* & 0 \\ 0 & \psi(r)^* \end{pmatrix} \begin{pmatrix} \phi(r) & 0 \\ 0 & \psi(r) \end{pmatrix} \\
&= \frac{1}{d+d'} \left(\mathrm{tr}(\phi(r)^*\phi(r)) + \mathrm{tr}(\psi(r)^*\psi(r)) \right) \\
&= \frac{\|\phi(r)\|_f^2}{(1+d/d')} + \frac{\|\psi(r)\|_f^2}{(1+d'/d)}.
\end{aligned}$$

Without loss of generality, suppose $\|\phi(r)\|_f \geq \|\psi(r)\|_f$, then

$$\begin{aligned}
\|(\phi \oplus \psi)(r)\|_f^2 &\leq \|\phi(r)\|_f^2 \frac{(1+d/d'+1+d'/d)}{(1+d/d')(1+d'/d)} \\
&= \|\phi(r)\|_f^2 \leq \epsilon^2.
\end{aligned}$$

It follows that $\phi \oplus \psi$ is a $\max(\epsilon, \epsilon')$-representation of $\mathcal{A}$.

For the second claim, we have

$$\begin{aligned}\|(\phi\otimes\psi)(r)\|_f^2 &= \frac{\mathrm{tr}}{d\cdot d'}\left[(\phi(r)\otimes\psi(r))(\phi(r)\otimes\phi(r))^*\right]\\ &= \frac{\mathrm{tr}(\phi(r)^*\phi(r))}{d}\frac{\mathrm{tr}(\psi(r)^*\psi(r))}{d'}\\ &= \|\phi(r)\|_f^2\|\psi(r)\|_f^2.\end{aligned}$$

Hence $\|\phi\otimes\psi(r)\|_f \leq \epsilon+\epsilon'$ provided both $\epsilon,\epsilon' < 1$. □

The assumptions that $\epsilon < 1$ may seem strong, but for the most part we are interested in the case when $\epsilon \to 0$. When doing asymptotic analysis it is often convenient to ignore or hide constants, focussing on the functional dependence or relationship in terms of ϵ. This is done using the notion of big-O notation. For strictly positive functions $f, g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq}$, we write $f(x) = O(g(x))$ as $x \to 0$, to mean that there exists a $\delta > 0$ and an $N > 0$, such that $f(x) \leq N \cdot g(x)$, for all $x \leq \delta$.

2.27.1 Finitely-presented ℓ_1-bounded $*$-algebras

A fundamental issue when doing approximate representation theory that we must address is that the free algebra is not bounded. That is, there are $*$-representations of a finitely presented $*$-algebra $\mathscr{A}$ where the elements have unbounded norm. There are a few ways to deal with this issue. Here we recall that, if $p \in \mathbb{C}^*\langle X : R\rangle$, then we recall that the ℓ_1-norm of p is $\|p\|_{\ell_1} = \sum_\alpha |c_\alpha|$. Now, we will equip $\mathbb{C}^*\langle X : R\rangle$ with the $*$-positive cone of hermitian squares. Recall that an element $a \in \mathscr{A}$ is SOS (or a sum of hermitian squares) if $a = \sum_{i\in I} b_i^* b_i$, with $b_i \in \mathcal{A}$ for all $i \in I$. That is $\mathscr{A}_+ = \{p \in \mathscr{A} : p \text{ is a SOS}\}$, and $\mathscr{A}$ is a semi-pre-C^*-algebra. With a notion of positivity we can now make the following definition.

Definition 2.28. A finitely-presented $*$-algebra $\mathscr{A} \cong \mathbb{C}^*\langle X : R\rangle$ is an **ℓ_1-bounded $*$-algebra** if $0 \leq x^*x \leq \mathbb{1}$, for all $x \in X$

By including this relation we are effectively restricting ourselves to the representations of $\mathbb{C}^*\langle X\rangle$ where the image of each generator x has operator norm at most one. The above definition is motivated by the following observation:

Lemma 2.29. If $\mathscr{A}$ is an ℓ_1-bounded $*$-algebra and $\phi : \mathbb{C}^*\langle X\rangle \to Lin(H)$ is a representation, then for every $p \in \mathbb{C}^*\langle X\rangle$ we see that

$$\|\phi(p)\|_{op} \leq \|p\|_{\ell_1}.$$

Proof. By the GNS construction $\|\pi(x)\|^2 = \langle v|\pi(x^*x)|v\rangle$ for some $v \in H$, corresponds to a state $\phi(x^*x)$. But note by the ℓ_1-bound property we see that $x^*x \leq \mathbb{1}$, which implies $\phi(x^*x) \leq 1$, hence x is bounded in representations. The result follows by noting that p is a linear combination of monomials in the elements of X, hence

$$\|\phi(p)\|_{op} \leq \sum_{\alpha \subseteq multi(X)} |c_\alpha| \prod_{j \in \alpha} \|\phi(x_j)\|_{op} \leq \sum_{\alpha} |c_\alpha| = \|p\|_{\ell_1},$$

as desired. □

We mention some examples of finitely presented ℓ_1-bounded $*$-algebras.

Example 2.30. That is the semi-pre-C^*-algebra generated by $\mathbb{C}^*\langle X\rangle$ with relations $x^* = x$ and $0 \leq x^*x \leq 1$, for all $x \in X$. This gives rise to the universal C^*-algebra of contractions.

Example 2.31. The semi-pre-C^*-algebra generated by $\mathbb{C}^*\langle X\rangle$ with relations $xx^* = x^*x = 1$, for all $x \in X$. This gives rise to the universal C^*-algebra generated by a unitary .

Example 2.32. The complete orthogonal projection algebra

$$PVM(m) = \mathbb{C}^* \left\langle \{p_1, \ldots, p_m\} : p_j^2 = p_j^* = p_j, p_j p_k = 0 \text{ for } k \neq j, \sum_{j=1}^{m} p_j = 1 \right\rangle.$$

Example 2.33. The (unitary) group algebra of $\mathbb{Z}_2^k$,

$$\mathbb{C}\mathbb{Z}_2^k = \mathbb{C}^*\langle \{z_1, \ldots, z_k\} : z_i^* = z_i, z_i^2 = 1, z_i z_j = z_j z_i, \text{ for } i \neq j\rangle.$$

2.33.1 ϵ-representations of ℓ_1-bounded $*$-algebras under different presentations

Lemma 2.34. Let $\mathscr{A} = \mathbb{C}^*\langle X : R\rangle$ and $\mathscr{B} = \mathbb{C}^*\langle X' : R'\rangle$ be finitely presented $*$-algebras generated by contractions and suppose that $\psi : \mathscr{A} \to \mathscr{B}$ is a homomorphism. If $\widetilde{\psi} : \mathbb{C}^*\langle X\rangle \to \mathbb{C}^*\langle X'\rangle$ is a lift of ψ, then there exists a constant $C > 0$, such that if φ is an ϵ-representation of $\mathscr{B}$, then $\varphi \circ \widetilde{\psi}$ is a $C\epsilon$-representation of $\mathscr{A}$ with respect to $\|\cdot\|_f$.

Proof. Consider the lifts of $\mathscr{A}$ and $\mathscr{B}$ as in Definition 2.21 with quotient maps $q : \mathbb{C}^*\langle X\rangle \to \mathscr{A}$ and $q' : \mathbb{C}^*\langle X'\rangle \to \mathscr{B}$. Let $r \in R$, since $\psi \circ q(x) = q' \circ \psi(x)$ for all $x \in X$, we see that $q' \circ \psi(r) = 0$. It follows that $\psi(r) \in \langle R' \rangle$ for all $r \in R$, and $\widetilde{\psi}(r) = \sum_{i \in I} x_i' r_i' y_i'$ for $p_i', s_i' \in \mathbb{C}\langle X' \rangle$, $r_i' \in R'$ for all $i \in I$. Hence, we see that

$$\begin{aligned}
\|\varphi \circ \widetilde{\psi}(r)\|_f &= \|\sum_{i \in I} p_i' r_i' s_i'\|_f \\
&\leq \sum_{i \in I} \|\varphi(p_i')\|_{op} \|\varphi(r_i')\|_f \|\varphi(s_i)\|_{op} \\
&\leq |I| (\max_i \{\|\varphi(p_i')\|_{op}, \|\varphi(s_i')\|_{op}\})^2 \epsilon \\
&\leq |I| (\max_i \{\|p_i'\|_{\ell_1}, \|s_i'\|_{\ell_1}\})^2 \epsilon,
\end{aligned}$$

and the result follows by setting $C = |I|(\max_i\{\|p_i'\|_{\ell_1}, \|s_i'\|_{\ell_1}\})^2$. $\square$

Of particular interest is when $\mathscr{A}$ and $\mathscr{B}$ are $*$-isomorphic as finitely presented $*$-algebras (i.e. they are different presentations for the same $*$-algebra).

Corollary 2.35. Suppose $\mathscr{A} = \mathbb{C}^*\langle X : R \rangle$ and $\mathscr{B} = \mathbb{C}^*\langle X' : R' \rangle$ are ℓ_1-bounded $*$-algebras and there is an $*$-isomorphism $\psi : \mathscr{A} \to \mathscr{B}$. If $\phi : \mathbb{C}^*\langle X \rangle \to M_d(\mathbb{C})$ is an ϵ-representation with respect to $\|\cdot\|_f$, then there exists a constant $C > 0$ for which $\phi \circ \widetilde{\psi}$ is an $C\epsilon$-representation of $\mathscr{B}$ with respect to $\|\cdot\|_f$, where $\widetilde{\psi}$ is descends to ψ.

It follows that ϵ-representations of finitely presented $*$-algebras with different presentations are $O(\epsilon)$-equivalent, where the constant depends on the presentation of the algebra.

2.35.1 Stable relations and replacement

For some relations an approximate representation has a nearby exact representation. Informally, relations for which this holds are called *stable relations*. The stability of relations with respect to matrices and a norm $\|\cdot\|$ is a property of finitely presented $*$-algebras that we now describe formally:

Definition 2.36. Let $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a function. A finitely presented complex $*$-algebra $\mathbb{C}^*\langle X : R \rangle$ is **(g, C)-stable with respect to matrices and $\|\cdot\|$**, if for every ϵ-representation $\phi : \mathbb{C}^*\langle X \rangle \to M_d(\mathbb{C})$, with $\epsilon \leq C$, there is a $*$-representation $\psi : \mathbb{C}^*\langle X : R \rangle \to M_d(\mathbb{C})$ such that

$$\|\phi(x) - \psi(x)\| \leq g(\epsilon),$$

for all $x \in X \cup X^*$. We say that $\mathbb{C}^*\langle X : R \rangle$ is **matrix stable with respect to $\|\cdot\|$**, if for every ϵ-representation with $\epsilon \leq C$, there exists a constant $\tilde{C} > 0$ such that $\mathbb{C}^*\langle X : R \rangle$ is $(\tilde{C}\epsilon, C)$-stable (i.e. whenever $g(\epsilon)$ is $O(\epsilon)$).

The definition above is not the only definition of stability found in the literature. This definition of stability is sensitive to the choice of matrix norm. In the following results, we restrict to the little Frobenius norm, as it's the most useful for our applications. The following "replacement" lemma is a key tool in approximate representation theory and stable relations.

Lemma 2.37 (Replacement)**.** Let $\mathscr{A} = \mathbb{C}^*\langle X : R\rangle$, $\tilde{R} \subset R$, such that $\mathbb{C}^*\langle X : \tilde{R}\rangle$ is an ℓ_1-bounded $*$-algebra. There exists a constant $C > 0$, such that if $\psi : \mathbb{C}^*\langle X\rangle \to M_d(\mathbb{C})$ is an ϵ-representation and $\varphi : \mathbb{C}^*\langle X : \tilde{R}\rangle \to M_d(\mathbb{C})$ is a representation with

$$\|\varphi(x) - \psi(x)\|_f \leq \delta,$$

for all $x \in X \cup X^*$, then φ is a $(C\delta + \epsilon)$-representation of $\mathscr{A}$ with respect to $\|\cdot\|_f$.

Proof. Since each relation $r \in R$ is a polynomial in $X \cup X^*$, there is a finite set of monomials $w_\alpha = \prod_{x\in\alpha} x$, for $\alpha \in A$, such that $r = \sum_{\alpha\in A} c_\alpha w_\alpha$, with $c_\alpha \in \mathbb{C}$ for all $\alpha \in A$. First we observe that on the level of monomials

$$\begin{aligned}
\|\varphi(w_\alpha) - \psi(w_\alpha)\|_f &= \|\varphi(x_1 \dots x_k) - \psi(x_1 \cdots x_k)\|_f \\
&\leq \sum_{i=1}^{k} \|\varphi(x_i) - \psi(x_i)\|_f \prod_{j\neq i} \|\varphi(x_j)\|_{op} \\
&\leq \sum_{i=1}^{k} \|\varphi(x_i) - \psi(x_i)\|_f,
\end{aligned}$$

since $\|\varphi(x)\|_{op} \leq 1$, for all $x \in X \cup X^*$. So

$$\|\varphi(w) - \phi(w)\|_f \leq k\delta,$$

for monomials of length k. Now,

$$\begin{aligned}
\|\varphi(r)\|_f \leq& \|\varphi(r) - \psi(r)\|_f + \|\psi(r)\|_f \\
\leq& \|\varphi(r) - \phi(r)\|_f + \epsilon \\
\leq& \sum_{\alpha\in A} |c_\alpha| \|\varphi(w_\alpha) - \phi(w_\alpha)\|_f + \epsilon.
\end{aligned}$$

If l_α denotes the length of the monomial w_α, then we see that

$$\|\varphi(r) - \psi(r)\|_f \leq |A| \max_\alpha\{|c_\alpha| \cdot l_\alpha\}\delta + \epsilon,$$

and the result follows by setting $C = |A| \max_\alpha\{|c_\alpha| \cdot l_\alpha\}$. □

In particular, Lemma 2.37 shows that if we have an ϵ-representation of $\mathscr{A}$, and $\mathscr{A}$ has a subset of matrix stable relations, then we can replace the ϵ-representation by one satisfying the stable relations, and the resulting "stabilized" representation will remain an $O(\epsilon)$-representation with respect to the remaining relations $R \setminus \tilde{R}$.

2.38 Some stability results for matrices and $\|\cdot\|_f$

We now state some elementary stability results that hold in the case of the $\|\cdot\|_f$-norm. These results will be used in later Chapters. Many of these results hold for other choices of matrix norm but we leave the specifics to the interest of the reader.

Proposition 2.39. For any matrix X, there is a self-adjoint matrix Y with

$$\|X - Y\|_f \leq \frac{1}{2}\|X - X^*\|_f.$$

Proof. Let $Y := \frac{1}{2}(X^* + X)$, then $Y^* = \frac{1}{2}(X + X^*) = Y$ and

$$\|X - Y\|_f = \left\|\frac{2X - X^* - X}{2}\right\|_f = \frac{1}{2}\|X - X^*\|_f.$$

□

Proposition 2.40. For any matrix X with $\|X\|_{op} \leq 1$, there is a unitary W such that

$$\|X - W\|_f \leq \|X^*X - \mathbb{1}\|_f.$$

Proof. Take the singular value decomposition $X = U\Sigma V$ so that Σ is a diagonal matrix with non-negative singular values $s_j \in [0,1]$, for all $1 \leq j \leq d$. Define the unitary $W := UV$ and observe that

$$\begin{aligned} &\|X - UV\|_f = \|U(\Sigma - \mathbb{1})V\|_f = \||\Sigma - \mathbb{1}|\|_f \\ \leq&\||(\Sigma - \mathbb{1})(\Sigma + \mathbb{1})|\|_f = \|\Sigma^2 - \mathbb{1}\|_f = \|V^*(\Sigma U^* U\Sigma - \mathbb{1})V\|_f \\ =&\|(V^*\Sigma^*U^*)U\Sigma V - V^*V\|_f = \|X^*X - \mathbb{1}\|_f, \end{aligned}$$

since $1 - s_j^2 = (1 - s_j)(1 + s_j) \geq 1 - s_j$, for all $s_j \in [0,1]$. □

Remark 2.41. If X is self-adjoint, then in Proposition 2.40 it suffices to pick the unitary $W = U$ from the polar decomposition $X = U|X|$ of X, where $|X| := \sqrt{X^*X}$ is a positive-semidefinite matrix.

Proposition 2.42. For any diagonal matrix X, there is another diagonal matrix Y, with $Y^2 = \mathbb{1}$ and

$$\|X - Y\|_f \leq \left(1 + \frac{1}{\sqrt{2}}\right) \|X^2 - \mathbb{1}\|_f .$$

We refer the reader to the proof of this proposition from [Slo19c].

Lemma 2.43. Let X be a matrix with $\|X\|_{op} \leq C$. If $\|X - X^*\|_f \leq \epsilon$ and $\|X^2 - \mathbb{1}\|_f \leq \epsilon$, then there exists a constant $\tilde{C} > 0$, and a unitary Z such that $Z^2 = \mathbb{1}$, and

$$\|Z - X\|_f \leq \tilde{C}\epsilon.$$

Proof. Let $Y = (X^* + X)/2$ so that Y is self-adjoint and by Proposition 2.39 $\|X - Y\|_f \leq \epsilon/2$. Since Y is self-adjoint there is a unitary U that diagonalizes Y. By Proposition 2.42 there is diagonal matrix W with $W^2 = \mathbb{1}$ and

$$\begin{aligned}
\|W - UYU^*\|_f &= \|U^*WU - Y\|_f \\
&\leq \left(1 + \frac{1}{\sqrt{2}}\right) \|UY^2U^* - \mathbb{1}\|_f \\
&= \left(1 + \frac{1}{\sqrt{2}}\right) \|Y^2 - \mathbb{1}\|_f.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\|Y^2 - \mathbb{1}\|_f &\leq \|Y^2 - X^2\|_f + \|X^2 - \mathbb{1}\|_f \\
&\leq \|Y^2 - XY\|_f + \|XY - X^2\|_f + \epsilon \\
&= \|(Y - X)Y\|_f + \|X(Y - X)\|_f + \epsilon \\
&\leq \|Y\|_{op}\epsilon/2 + C\epsilon/2 + \epsilon \\
&\leq (C + 1)\epsilon,
\end{aligned}$$

hence, letting $Z = U^*WU$ we see that $Z^2 = U^*WUU^*WU = \mathbb{1}$ and $Z^* = (U^*WU)^* = U^*W^*U = Z$, since $W^* = W$. Putting it all together, we see that

$$\begin{aligned}
\|Z - X\|_f &\leq \|Z - Y\|_f + \|Y - X\|_f \\
&\leq (1 + \frac{1}{\sqrt{2}})\|Y^2 - \mathbb{1}\|_f + \epsilon/2 \\
&\leq (1 + \frac{1}{\sqrt{2}})(C + 1)\epsilon + \epsilon/2 \\
&\leq \tilde{C}\epsilon,
\end{aligned}$$

as desired. □

Corollary 2.44. The $*$-algebra generated by $\{x_1, \ldots, x_n\}$ with relations $x_i = x_i^*$ and $x_i^2 = 1$, for all $1 \le i \le n$, is matrix stable with respect to $\| \cdot \|_f$.

Proof. This follows directly from Lemma 2.43 and the observation that any diagonal matrix W is self-adjoint, hence $Z^* = U^* W^* U = U^* W U = Z$, as desired. □

Proposition 2.45. Let $P \in Lin(H)$. If there exists a constants $0 < C_0$, and $0 < \epsilon \le 1$ such that $\|P\|_{op} \le C_0$, $\|P - P^*\|_f \le \epsilon$, and $\|P^2 - P\|_f \le \epsilon$, then there is a constant $\tilde{C} > 0$ and an orthogonal projection $\tilde{P}$ such that

$$\|P - \tilde{P}\|_f \le C'\epsilon.$$

Proof. Let $X = \mathbb{1} - 2P$ so that $\|X\|_{op} \le 2C_0 + 1 = C$. Since $P = (\mathbb{1} - X)/2$, we see that $\|X^2 - \mathbb{1}\|_f \le 4\epsilon$ and $\|X - X^*\|_f \le 4\epsilon$. Hence by Lemma 2.43 there exists a self-adjoint unitary Z such that $Z^2 = \mathbb{1}$ and $\|X - Z\|_f \le \tilde{C}4\epsilon$.Now let $\tilde{P} = (\mathbb{1} - Z)/2$ so that $\tilde{P}^2 = \tilde{P} = \tilde{P}^*$. Lastly, we see that

$$\|P - \tilde{P}\|_f = \frac{1}{2}\|Z - X\|_f \le \tilde{C}2\epsilon,$$

and letting $C' = 2\tilde{C}$ completes the proof. □

The proposition above demonstrates that if we have a matrix that is almost an orthogonal projection in the little Frobenius norm, then there is always an orthogonal projection nearby.

Definition 2.46. Let

$$\mathcal{A}_{PVM} = C^*\langle e_a, a \in A, : e_a = e_a^2 = e_a^* \ \forall a \in A, \sum_{a \in A} e_a = \mathbb{1}\rangle$$

denote the PVM algebra. That is the universal C^*-algebra generated by orthogonal projections with the completeness property.

Lemma 2.47. Given an $\epsilon > 0$, and a collection of positive contractions $A_1, \ldots, A_m$ acting on a finite dimensional Hilbert space H such that $\|A_i^2 - A_i\|_f < \epsilon$, $\|A_i A_j\|_f < \epsilon$ and $\|\sum_i A_i - \mathbb{1}\|_f \le \epsilon$, for all $1 \le i \ne j \le m$. There exists a constant $N > 0$ and a collection of mutually orthogonal projections $P_1, \ldots, P_m$ such that $\|P_i - A_i\|_f \le N\epsilon$, for all $1 \le i \le m$. In addition, we can pick the projections so that $\sum_i P_i = \mathbb{1}$.

The proof is based on a result in [KPS18] which relies on the following observation.

Lemma 2.48. Let C be a positive contraction in $M_d(\mathbb{C})$. If $\|C^2 - C\|_f \leq \epsilon$, then there is an orthogonal projection Π such that $\|\Pi - C\|_f \leq 2\sqrt{2}\epsilon$.

Proof. Let $\Pi_C = \chi_{[1/2,1]}(C)$. We claim that $\|\Pi_C - C\|_f \leq 2\sqrt{2}\epsilon$. Let $Q = (\mathbb{1} - \Pi_C)C$ and $P = \Pi_C C$ so that $C = P + Q$. We begin by noting that $\|Q - Q^2\|_f \leq \|C - C^2\|_f$ and $\|P - P^2\|_f \leq \|C - C^2\|_f$. Since $0 \leq Q \leq \frac{\mathbb{1}}{2}$, we see that

$$Q - Q^2 = Q(\mathbb{1} - Q) \geq \frac{1}{2}Q,$$

from which we conclude that $\|Q\|_f \leq 2\|Q - Q^2\|_f$. Similarly, from the fact that $\frac{1}{2}\Pi_C \leq P \leq \mathbb{1}$, we see that

$$P - P^2 = P(\mathbb{1} - P) \geq \frac{1}{2}\Pi_C(\mathbb{1} - P) = \frac{1}{2}(\Pi_C - P),$$

hence $\|\Pi_C - P\|_f \leq 2\|P - P^2\|$. Since Q and $\Pi_C(\mathbb{1} - P)$ are orthogonal, by the pythagorean identity, we see that

$$\|C - \Pi_C\|_f^2 = \|(P + Q) - \Pi_C\|_f^2 = \|P - \Pi_C\|_f^2 + \|Q\|_f^2 \leq 8\|C - C^2\|_f^2,$$

and the result follows. □

Proof of Lemma 2.47. We proceed by induction on $m \in \mathbb{N}$. The $m = 1$ case is mostly covered by Lemma 2.48. It only remains to see that the projection $\|\Pi_1 - \mathbb{1}\|_f \leq \epsilon$. However, since we have $\|C_1 - \mathbb{1}\|_f \leq \epsilon$ and $\|\Pi_1 - C_1\|_f \leq 2\sqrt{2}\epsilon$, the result follows. Now, by the induction hypothesis suppose it holds for all $1 \leq k \leq m$. Let $C_{m+1} = C$ and define the projection $R = \mathbb{1} - \sum_{i=1}^m \Pi_i$. Define $\Pi = \chi_{[1/2,1]}(RCR)$, so that our objective is to bound the quantity $\|\Pi - C\|_f$ in terms of ϵ. Let $N_m > 0$ be the constant obtained in the inductive step. First, we note that

$$\|\Pi_i C\|_f \leq \|C_i C\|_f + \|\Pi_i - C_i\|_f \|C\|_{op} \leq (N_m + 1)\epsilon, \tag{2.6}$$

that for all $1 \leq i \leq m$. Now, observe that

$$\begin{aligned}
\|RCR - C\|_f &\leq \|RCR - RC\|_f + \|RC - C\|_f \\
&\leq \|R\|_{op}\|CR - C\|_f + \|RC - C\|_f \\
&\leq \|R^*C^* - C^*\|_f + \|RC - C\|_f \\
&\leq 2\|RC - C\|_f,
\end{aligned}$$

using self-adjointness of the norm, as well as of C and R. Hence,

$$\|RCR - C\|_f \leq 2\|RC - C\|_f \leq 2\sum_{i=1}^{m} \|\Pi_i C\|_f \leq 2(N_m + 1)\epsilon \tag{2.7}$$

using equation (2.6). Since RCR commutes with C, we see that

$$\|(RCR)^2 - C^2\|_f = \|(RCR + C)(RCR - C)\|_f \leq 2\|RCR - C\|_f. \tag{2.8}$$

It follows that

$$\begin{aligned}\|(RCR)^2 - RCR\|_f &\leq \|(RCR)^2 - C^2\|_f + \|C^2 - C\|_f + \|C - RCR\|_f \\ &\leq 3\|RCR - C\|_f + \|RCR - C\|_f \\ &\leq 6(N_m + 1)\epsilon + \epsilon \\ &\leq 6(N_m + 2)\epsilon\end{aligned}$$

Putting it all together, we deduce that

$$\begin{aligned}\|\Pi - C\|_f &\leq 2\sqrt{2}\|(RCR)^2 - RCR\|_f + \|RCR - C\|_f \\ &\leq 12\sqrt{2}(N_m + 2)\epsilon + 2(N_m + 1)\epsilon \\ &\leq \left[(12\sqrt{2} + 2)N_m + (24\sqrt{2} + 2)\right]\epsilon.\end{aligned}$$

Π is clearly an orthogonal projection, and since R is orthogonal to each Π_i, it follows that $\Pi\Pi_i = 0$ for all $1 \leq i \leq m$. Lastly, define $\tilde{\Pi}_1 = \Pi_1 + \mathbb{1} - \sum_{i=1}^{m+1} \Pi_i = \mathbb{1} - \sum_{i=2}^{m+1} \Pi_i$ so that $\tilde{\Pi}_1 + \sum_{i=2}^{m+1} \Pi_i = \mathbb{1}$. Then $\tilde{\Pi}_1, \Pi_2, \ldots, \Pi_{m+1}$ is a family of complete mutually orthogonal projections and we see that

$$\begin{aligned}\|\tilde{\Pi}_1 - C_1\|_f &\leq \|\mathbb{1} - \sum_{i=1} C_i\|_f + \|\sum_{i=2} C_i - \sum +i = 2^{m+1}\Pi_i\|_f \\ &\leq \epsilon + \sum_{i=2}^{m+1} \|C_i - \Pi_i\|_f \\ &\leq \epsilon + (m-1)N_m\epsilon \\ &= [(m-1)N_m + 1]\epsilon,\end{aligned}$$

and the result follows. ∎

Corollary 2.49. The PVM algebra is stable with respect to matrices and $\|\cdot\|_f$.

Chapter 3

Perfect strategies for nonlocal games and game algebras

In this chapter, we formalize the theory of nonlocal games and nonlocal game algebras for synchronous, boolean constraint systems (BCS), and XOR nonlocal games. We begin in Section 3.1, with a review of the theory of quantum correlations from Bell scenarios and show that the value of a nonlocal game is a positive linear functional on the space of correlations. In Section 3.2, we define perfect and optimal finite-dimensional quantum strategies. In Sections 3.7 and 3.13, we show that in the case of projective quantum strategies, perfect quantum strategies for synchronous and BCS nonlocal games correspond to representations of the synchronous and BCS-algebras with maximally entangled states. In Section 3.24, we show that BCS and synchronous algebras are isomorphic as $*$-algebras. We end the chapter with Section 3.27, where we elaborate on the correspondence between optimal strategies for XOR games and tracial states on the XOR-algebra.

3.1 Quantum correlations from Bell scenarios

In a Bell scenario, two spatially separated parties, Alice and Bob, are given inputs x and y from finite sets X and Y. Upon receiving their inputs, the parties, which are not allowed to communicate with each other, reply with outputs a and b from finite sets A and B. The probability of observing outcomes $a, b \in \mathsf{A} \times \mathsf{B}$ given inputs $x, y \in \mathsf{X} \times \mathsf{Y}$ is denoted by the expression $p(a, b|x, y)$. The collection of probabilities

$$p = \{p(a, b|x, y\}_{a,b,x,y \in \mathsf{A}\times\mathsf{B}\times\mathsf{X}\times\mathsf{Y}}$$

is called a correlation or behaviour. Indeed, if each party selects their outcomes from probability distributions $p_x : \mathsf{A} \to \mathbb{R}_{\geq 0}$ for each $x \in \mathsf{X}$ and $q_y : \mathsf{B} \to \mathbb{R}_{\geq 0}$ for each $y \in \mathsf{Y}$, then their behaviour $p(a,b|x,y) = p_x(a)q_y(b)$, for all $a,b,x,y \in \mathsf{A}\times\mathsf{B}\times\mathsf{X}\times\mathsf{Y}$, is entirely classical. However, this expression does capture all the possible classical behaviours. To obtain all classical correlations, one must allow for some shared randomness between the parties. More formally, we say that a correlation p is a classical correlations, or an element of the set $C_c(\mathsf{X},\mathsf{Y},\mathsf{A},\mathsf{B})$, if there are distributions $p_x^{(i)}(a)$ and $q_y^{(i)}(b)$ and probabilities $0 \leq \lambda_i \leq 1$ for $1 \leq i \leq k$, such that

$$p(a,b|x,y) = \sum_{i=1}^{k} \lambda_i p_x^{(i)}(a) q_y^{(i)}(b) \tag{3.1}$$

for all $a,b,x,y \in \mathsf{A}\times\mathsf{B}\times\mathsf{X}\times\mathsf{Y}$. Given a classical correlation p, the collection of probabilities $\{\lambda_i : 1 \leq i \leq k\}$ and the distributions $\{p_x^{(i)}(a) : 1 \leq i \leq k\}$, $\{q_y^{(i)}(b) : 1 \leq i \leq k\}$, for all $x,y \in \mathsf{X}\times\mathsf{Y}$, are a classical model for p. Hence, a correlation is classical if and only if it has a classical model. The set of classical correlations $C_c(\mathsf{X},\mathsf{Y},\mathsf{A},\mathsf{B})$ is a closed convex subset of $\mathbb{R}^{\mathsf{X}\times\mathsf{Y}\times\mathsf{A}\times\mathsf{B}}$.

Quantum mechanics provides a more general way to describe the probabilities of a Bell scenario. If H_A and H_B are finite-dimensional Hilbert spaces, and $\{\{M_a\}_{a\in\mathsf{A}} : x \in \mathsf{X}\}$, $\{\{N_b\}_{b\in\mathsf{B}} : y \in \mathsf{Y}\}$ are collections of measurement operators (POVMs) acting on H_A and H_B, and ρ is the density matrix describing the state of the joint system $H_A \otimes H_B$, then the probability of observing outcome a,b on inputs x,y is given by the quantum expectation value

$$p(a,b|x,y) = \operatorname{tr}\big((M_a^x \otimes N_b^y)\rho\big), \tag{3.2}$$

for all $a,b,x,y \in \mathsf{A}\times\mathsf{B}\times\mathsf{X}\times\mathsf{Y}$. We call the pair of Hilbert spaces, collection of measurement operators, and density matrix, a quantum model for p. Note that if the density matrix ρ has a separable decomposition, then Equation (3.2) is of the same form as Equation (3.1), with the *local* distributions given by the inner-product of the separable eigenstates with the POVMs. Therefore, the quantum mechanical description fully captures the set of classical correlations. The natural question is what happens if the state is *entangled*, that is, when ρ is not separable?

Before we discuss whether the set of correlations captured by a quantum model is larger than the set of classical correlations, we note that a simple purification argument shows that any correlation obtained with a mixed state ρ can be obtained from a pure state $|\psi\rangle$ in a larger (but still finite-dimensional) bipartite Hilbert space. Since the local measurement operators do not act on this auxiliary space, we restrict ourselves to considering quantum models with pure states, and therefore correlations of the form

$$p(a,b|x,y) = \langle\psi|M_a^x \otimes N_b^y|\psi\rangle, \tag{3.3}$$

for all $a, b, x, y \in \mathsf{A} \times \mathsf{B} \times \mathsf{X} \times \mathsf{Y}$. A correlation p that has a realization as in Equation (3.3) is called a quantum correlation. The set of quantum correlations is denoted $C_q(\mathsf{X}, \mathsf{Y}, \mathsf{A}, \mathsf{B})$, or just C_q. Hence a correlations is a quantum correlation if and only if it has a quantum model, and from the above we note that $C_c \subset C_q$.

However, the notion of "quantum model" above is not the only way to characterize "quantum correlations". There is no a priori reason from we need to assume the Hilbert spaces H_A and H_B are finite-dimensional. Removing this assumption gives us a potentially larger set of tensor-product (or spatial) quantum models, and a correlation belongs to this set of correlations C_{qs}, if it has a tensor-product model. Furthermore, rather than a tensor-product of Hilbert spaces, why not consider a single (potentially infinite-dimensional) Hilbert space where the compatibility of the measurement operators is enforced by the restriction that each N_b^y commutes each M_a^x. This framework is called the commuting operator framework, and a correlation p is in the set of commuting operator correlations C_{qc} if there is a commuting operator model for p. In the tensor-product (resp. commuting operator) model, the quantum states can be any unit vector in $H_\mathsf{A} \otimes H_\mathsf{B}$ (resp. H). All of these frameworks, although each one more general than the other, are consistent when restricted to the set of classical correlations.

We also mention that the sets C_c and C_{qc} are closed, however, it was shown by Slofstra [Slo19b] that neither C_q nor C_{qs} are closed. Therefore, we consider their closure C_{qa} (sometimes called the quantum approximable correlations). This gives us a chain inclusions:

$$C_c \subset C_q \subset C_{qs} \subset C_{qa} \subset C_{qc} \ . \tag{3.4}$$

Bell's celebrated result [Bel64] is the separation of C_c and C_q by demonstrating that there are quantum correlations that do not have a classical model. Although not obvious at all, we now know that all the inclusions in Equation (3.4) are strict, meaning there are correlations in each set, not contained in the latter. The separation between C_q and C_{qs} was given by Coladangelo and Stark [CS18], between C_{qs} and C_{qa} via the non-closure result of Slofstra [Slo19b] (several other proofs now exist, for instance [DPP19, Bei21, Col20]), and finally the separation of C_{qa} and C_{qc} by the celebrated construction in the [JNV⁺22], which also resolved the Conne's Embedding Problem via the equivalence shown in [JNP⁺11, Fri12, Oza13b].

Each of these sets is a convex subset of $\mathbb{R}^{\mathsf{X} \times \mathsf{Y} \times \mathsf{A} \times \mathsf{B}}$. In this language, a Bell inequality is a separating hyperplane on the set of quantum correlations, and a violation of the inequality is achieved by correlations that have a quantum but no classical model. A refinement of Bell's inequality, known as the CHSH inequality given by [CHSH69], provides a more concrete notion of this separation in terms of a concrete scenario Bell scenario and values of the corresponding classical and quantum correlations.

Inspired by the similarity of a Bell scenarios and multiprover interactive proof systems, a concept from complexity theory. Cleve, Hoyer, Toner, and Watrous introduced the notion of a two-player nonlocal game, a framework which unified several pre-existing examples and families of Bell-like inequalities [CHTW10]. A two-player nonlocal game G is a Bell scenario with another party called the referee (or verifier). The role of the referee is to distribute inputs to the players from the predetermined distribution $\varrho : \mathsf{X}\times\mathsf{Y} \to \mathbb{R}_{\geq 0}$ and to verify the outputs. This verification is specified by a predicate $V : \mathsf{X}\times\mathsf{Y}\times\mathsf{A}\times\mathsf{B} \to \{0,1\}$, where a 0 is a loss, and 1 is a win. The aim of the players is to devise a strategy $\mathcal{S}$ that achieves the highest probability of winning. Given a strategy $\mathcal{S}$, the probability of winning the game G under $\mathcal{S}$ is called the "value" of the game under $\mathcal{S}$. The value of the game is expressed by the quantity

$$\omega(G;\mathcal{S}) = \sum_{x,y\in X\times Y} \varrho(x,y) \sum_{a,b\in A\times b} V(x,y,a,b)p(a,b|x,y).$$

Note that a strategy $\mathcal{S}$ realizes the correlations $p(a,b|x,y)$, for all $a,b,x,y \in \mathsf{X}\times\mathsf{Y}\times\mathsf{A}\times\mathsf{B}$, and so often we talk about correlations and strategies interchangeably.

Just as we associated a class of models with a set of correlations, there is an analogous class of strategies for that set of correlations. Given a class of strategies $\mathscr{C}$, we can view the object of the game as finding a strategy $\mathcal{S}\in\mathscr{C}$ that wins with probability $\sup_{\mathcal{S}\in\mathscr{C}}\{\omega(G;\,\mathcal{S})\}$. This optimal probability is called the $\mathscr{C}$-value of the game and is denoted by $\omega_{\mathscr{C}}(G)$. For example is $\mathscr{C}$ is the class of classical strategies, then the *classical* or c-value of G is expressed as

$$\omega_c(G) = \max_{p\in C_c(\mathsf{X},\mathsf{Y},\mathsf{A},\mathsf{B})} \left\{ \sum_{x,y\in X\times Y} \varrho(x,y) \sum_{a,b\in A\times b} V(x,y,a,b)p(a,b|x,y) \right\}.$$

The reason we no longer need a supremum is because the set C_c is a closed convex set, and computing the c-value of the game is the maximization of a positive linear functional over a convex set, and therefore the optimal value is achieved at an extreme point of $C_c(\mathsf{X},\mathsf{Y},\mathsf{A},\mathsf{B})$. However, finding the optimal c-value is in general NP-hard [CHTW10].

Analogously, the quantum or q-value of G is expressed by

$$\omega_q(G) = \sup_{p\in C_q(\mathsf{X},\mathsf{Y},\mathsf{A},\mathsf{B})} \left\{ \sum_{x,y\in X\times Y} \varrho(x,y) \sum_{a,b\in A\times b} V(x,y,a,b)p(a,b|x,y) \right\}.$$

Here, we cannot replace the supremum with a maximum because the set of quantum correlations C_q is not closed [Slo19b]. In the case of computing the q-value, it is known that for any $0 < \epsilon < 1$, it is undecidable (in the promise case) to determine whether $\omega_q(G) = 1$ or $\omega_q(G) < 1 - \epsilon$, that is, it is equivalent to the halting problem [JNV+22]. Although this implies that computing $\omega_q(G)$ is in general infeasible, there are in fact concrete methods to finding optimal strategies through the semi-definite programming hierarchy of [NPA07, NPA08], or by optimizing iteratively in each local dimension $d = 1, 2, \ldots$ and so on, though these approaches require extensive computational resources.

3.2 Perfect quantum strategies and tracial states

As mentioned in Chapter 1, a two-player nonlocal game $G = (\mathsf{X}, \mathsf{Y}, \mathsf{A}, \mathsf{B}, \varrho)$ is a Bell scenario with finite sets of questions (or inputs) X, Y and answers (or outputs) A, B, a distribution on the questions, and a winning predicate V. The referee gives each player is given an input $x \in \mathsf{X}$ and $y \in \mathsf{Y}$, according to the distribution $\varrho(x, y)$. Unable to communicate the players' return outputs a and b to the referee, who determines whether they win by evaluating the predicate $V(x, y, a, b) \in \{0, 1\}$. The winning probability of the game is related to the ability of the players to generate the correlations $\{p(a, b|x, y)\}_{(a,b,x,y)}$ that achieve the games q-value, by employing a particular quantum strategy or model for correlations $p \in C_q$. Since any quantum correlation can be achieved by projective measurement (through Naimark dilation), we restrict ourselves to these strategies.

Definition 3.3. A **projective quantum strategy** $\mathcal{S}$ for a nonlocal game G is a tuple

$$\mathcal{S} = (H_A, H_B, \{\{P_a^x\}_{a \in \mathsf{A}} : x \in \mathsf{X}\}, \{\{Q_b^y\}_{b \in \mathsf{B}} : y \in \mathsf{Y}\}, |\psi\rangle),$$

where H_A and H_B are finite-dimensional Hilbert spaces, $\{\{P_a^x\}_{a \in \mathsf{A}} : x \in \mathsf{X}\}$ are PVMs in $Lin(H_A)$, $\{\{Q_b^y\}_{b \in \mathsf{B}} : y \in \mathsf{Y}\}$ are PVMs in $Lin(H_B)$, and $|\psi\rangle$ a quantum state (unit vector) in $H_A \otimes H_B$.

The probability of winning the game with a quantum strategy $\mathcal{S}$ is given by

$$\omega(G; \mathcal{S}) = \sum_{x,y} \varrho(x, y) \sum_{a,b} V(x, y, a, b) \langle \psi | P_a^x \otimes Q_b^y | \psi \rangle$$

Definition 3.4. A quantum strategy $\mathcal{S}$ is **perfect** if $\omega(G; \mathcal{S}) = 1$.

For perfect strategies we observe that if $p_{x,y} := \sum_{a,b:V(a,b|x,y)=1} p(a, b|x, y)$ denotes the probability of winning on input (x, y), then a quantum strategy $\mathcal{S}$ is perfect if and only if it generates the quantum correlation $\{p(a, b|x, y)\}$ for which $p_{x,y} = 1$, for all $x, y \in \mathsf{X} \times \mathsf{Y}$.

This rest of this chapter will focus on some known results regarding the representations of game algebras and their connection to perfect or optimal strategies for nonlocal games. With the exception of Proposition 3.26, many of the results in this section are not new, and are likely known to most experts. That being said, it is illustrative to see how the techniques in the exact case work in comparison the approximate case, which will be our focus in Chapter 4. We also note that all of the finitely-presented "nonlocal game" $*$-algebras are examples of ℓ_1-bounded algebras, under the standard positive sum of square cone from the preliminaries.

Before we begin we present some key lemmas:

Lemma 3.5. If $X \in Lin(\mathbb{C}^d)$ and $|\psi\rangle$ is a vector state in $\mathbb{C}^d \otimes \mathbb{C}^d$, then there is a linear operator $Y \in Lin(\mathbb{C}^d)$ such that

$$X \otimes \mathbb{1}|\psi\rangle = \mathbb{1} \otimes Y|\psi\rangle.$$

Moreover if $X = X^*$, then X commutes with the support of ρ (the reduced density matrix of $|\psi\rangle$) in the first tensor factor $\mathbb{C}^d$.

Proof. Let $\sum_{i=1}^r \lambda_i |u_i\rangle \otimes |v_i\rangle$ be the Schmidt decomposition of $|\psi\rangle$. Let $\lambda : \mathbb{C}^d \to \mathbb{C}^d$ be the map sending $|u_i\rangle \mapsto \lambda_i |v_i\rangle$, with kernel $span\{|u_j\rangle : d \geq j > i\}$, so that we can write $|\psi\rangle = \sum_{i=1}^d |u_i\rangle \otimes \lambda |u_i\rangle$. Letting $|\eta\rangle = \sum_{i=1}^d |u_i\rangle \otimes |u_i\rangle$, we see that

$$X \otimes \mathbb{1}|\eta\rangle = \mathbb{1} \otimes X^\top |\eta\rangle,$$

where the transpose is taken with respect to the basis $\{|u_i\rangle\}_{i=1}^d$ for $\mathbb{C}^d$. Then $|\psi\rangle = \mathbb{1} \otimes \lambda |\eta\rangle$, and if λ^- is the pseudo-inverse of λ, we see that

$$X \otimes \mathbb{1}|\psi\rangle = X \otimes \lambda|\eta\rangle = \mathbb{1} \otimes \lambda X^\top |\eta\rangle = \mathbb{1} \otimes \lambda X^\top \lambda^- |\psi\rangle. \tag{3.5}$$

Hence, letting $Y := \lambda X^\top \lambda^-$ establishes the claim. Note that if $|\psi\rangle$ is maximally entangled, then $\lambda = \frac{1}{\sqrt{d}}\mathbb{1}$ and we see that $Y = X^\top$.

For the second part, let $\Pi = \sum_{i=1}^r |u_i\rangle \langle u_i| \in Lin(\mathbb{C}^d)$ be the support of ρ (or of $|\psi\rangle$) on the first tensor factor. It follows from the first part that

$$\Pi X \otimes \mathbb{1} |\psi\rangle = \Pi \otimes Y |\psi\rangle = \mathbb{1} \otimes Y |\psi\rangle = X \otimes \mathbb{1} |\psi\rangle .$$

However, $0 = (\Pi X - X) \otimes \mathbb{1} |\psi\rangle = \sum_{i=1}^r \lambda_i (\Pi X - X) |u_i\rangle \otimes |v_i\rangle$ implies that $(\Pi X - X) |u_i\rangle = 0$ for all $1 \leq i \leq r$, and it follows that $\Pi X \Pi = X \Pi$. Now if $X = X^*$, then $\Pi X = (X\Pi)^* = (\Pi X \Pi)^* = \Pi X \Pi = X\Pi$ as desired. $\square$

Lemma 3.6. If X and Y are self-adjoint operators on Hilbert spaces H_A and H_B respectively with $\rho = \lambda^*\lambda$ the reduced density matrix of $|\psi\rangle$ on H_B, then

$$\|(X \otimes \mathbb{1} - \mathbb{1} \otimes Y)|\psi\rangle\| = \|\lambda\overline{X} - Y\lambda\|_F, \tag{3.6}$$

where $\overline{X}$ is the entry-wise conjugate taken with respect to the basis of $\lambda \in Lin(H_B)$.

Proof. Let $|\psi\rangle = \sum_t |t\rangle \otimes \lambda|t\rangle$, where t indexes an orthonormal basis for H_A (the Schmidt basis for λ), then we have $(X \otimes \mathbb{1})|\psi\rangle = \sum_t |t\rangle \otimes \lambda\overline{X}|t\rangle$ and $(\mathbb{1} \otimes Y)|\psi\rangle = \sum_t |t\rangle \otimes Y\lambda|t\rangle$. Now,

$$\begin{aligned}
\|(X \otimes \mathbb{1} - \mathbb{1} \otimes Y)|\psi\rangle\|^2 &= \left\|\sum_t |t\rangle \otimes (\lambda\overline{X} - Y\lambda)|t\rangle\right\|^2 \\
&= \left\langle \sum_t |t\rangle \otimes (\lambda\overline{X} - Y\lambda)|t\rangle, \sum_{t'} |t'\rangle \otimes (\lambda\overline{X} - Y\lambda)|t'\rangle \right\rangle \\
&= \sum_{t,t'} \langle t|t'\rangle \left\langle (\lambda\overline{X} - Y\lambda)|t\rangle, (\lambda\overline{X} - Y\lambda)|t'\rangle \right\rangle \\
&= \sum_t \langle t|(\lambda\overline{X} - Y\lambda)^*(\lambda\overline{X} - Y\lambda)|t\rangle \\
&= \mathrm{tr}\left((\lambda\overline{X} - Y\lambda)^*(\lambda\overline{X} - Y\lambda)\right) \\
&= \|\lambda\overline{X} - Y\lambda\|_F^2,
\end{aligned}$$

as desired. □

3.7 Synchronous algebras and perfect strategies

A **synchronous** nonlocal game G_{sync} is a nonlocal game where the questions and answers sets are the same for each player (i.e. $\mathsf{A} = \mathsf{B}$ and $\mathsf{X} = \mathsf{Y}$). Additionally, the winning predicate must satisfy the following *synchronous* property:

$$V(x, x, a, b) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b, \end{cases} \quad \text{for all } x \in \mathsf{X}.$$

This condition ensures that if the players are given the same question, then they lose whenever they give different answers. To every synchronous nonlocal game G_{sync} there is an associated synchronous algebra.

Definition 3.8. The **synchronous algebra** $\mathscr{A}(G_{sync})$ is the $*$-algebra generated by elements $\{\{e_a^x\}_{a\in\mathsf{A}} : x \in \mathsf{X}\}$ subject to the relations:

(1) $(e_a^x)^2 = (e_a^x)^* = e_a^x$, for all $a \in \mathsf{A}$, $x \in \mathsf{X}$,

(2) $[e_a^x, e_b^x] = 0$, for all $a, b \in \mathsf{A}$,

(3) $\sum_{a\in\mathsf{A}} e_a^x = 1$, for all $x \in \mathsf{X}$,

(4) $e_a^x e_b^y = 0$, whenever $V(x, y, a, b) = 0$.

There is a correspondence between perfect quantum strategies for synchronous games and $*$-representations of the synchronous algebra which is made precise by the following theorem.

Theorem 3.9. Let G_{sync} be a synchronous nonlocal games with associated synchronous algebra $\mathscr{A}(G_{sync})$.

(i) Every perfect quantum strategy for G_{sync} is a representation of $\mathscr{A}(G_{sync})$ with a finite-dimensional tracial state.

(ii) Every finite-dimensional tracial state $f : \mathscr{A}(G_{sync}) \to \mathbb{C}$ for which

$$\sum_{\substack{a,b\in\mathsf{A}\times\mathsf{A} \\ V(x,y,a,b)=0}} f(e_a^x \cdot e_b^y) = 0,$$

for all $x, y \in \mathsf{X} \times \mathsf{X}$, gives a weighted direct sum of perfect quantum strategies for G_{sync} with maximally entangled states.

This result is actually a consequence of a more general theorem in [PSS+16][Theorem 5.5] establishing a correspondence between arbitrary tracial states (i.e. not just finite dimensional ones) and commuting operator strategies for synchronous nonlocal games, as commuting operator strategies generalize quantum strategies.

Before we give the proof of Theorem 3.9, we establish some elementary results.

Proposition 3.10 ([PSS+16])**.** If $\mathcal{S}$ is a perfect strategy for a synchronous nonlocal game, then the state defined by $f(e_a^x) = \langle\psi|P_a^x \otimes \mathbb{1}|\psi\rangle$ for all $a \in \mathsf{A}$, $x \in \mathsf{X}$, is a tracial state on $\mathscr{A}(G_{sync})$ by extending linearly and multiplicatively to polynomials in the generators.

The proof of this is well-known but we include it here for completeness.

Proof. Let $\mathcal{S}$ be a perfect projective quantum strategy for the synchronous game G_{sync}. Then we must have that

$$\begin{aligned}
1 = \sum_{a,b} p(a,b|x,x) = \sum_a p(a,a|x,x) = \sum_a \langle\psi|P_a^x \otimes Q_a^x|\psi\rangle \\
\leq \sum_a \|P_a^x \otimes \mathbb{1}|\psi\rangle\| \|\mathbb{1} \otimes Q_a^x|\psi\rangle\| \\
\leq \left(\sum_a \|P_a^x \otimes \mathbb{1}|\psi\rangle\|^2\right)^{1/2} \left(\sum_a \|\mathbb{1} \otimes Q_a^x|\psi\rangle\|^2\right)^{1/2} \\
= \left(\sum_a \langle\psi|P_a^x \otimes \mathbb{1}|\psi\rangle\right)^{1/2} \left(\sum_a \langle\psi|\mathbb{1} \otimes Q_a^x|\psi\rangle\right)^{1/2} \\
= 1.
\end{aligned}$$

It follows from the Cauchy-Schwarz inequality that

$$P_a^x \otimes \mathbb{1}|\psi\rangle = \mathbb{1} \otimes Q_a^x|\psi\rangle \tag{3.7}$$

for all $a \in A$, $x \in X$.

Next we show that $f(e_a^x) = \langle\psi|P_a^x \otimes \mathbb{1}|\psi\rangle$ is a tracial state. Suppose $w_0 = e_{i_1} \cdots e_{i_k}$ and $w_1 = e_{j_1} \cdots e_{j_\ell}$ are monomials in generator, hence

$$\begin{aligned}
f(w_0 w_1) &= \langle\psi|P_{i_1} \cdots P_{i_k} \cdot P_{j_1} \cdots P_{j_\ell} \otimes \mathbb{1}|\psi\rangle \\
&= \langle\psi|P_{i_1} \cdots P_{i_k} \otimes Q_{j_\ell} \cdots Q_{j_1}|\psi\rangle \\
&= \langle\psi|(\mathbb{1} \otimes Q_{j_1} \cdots Q_{j_\ell})^*(P_{i_1} \cdots P_{i_k} \otimes \mathbb{1})|\psi\rangle \\
&= \langle\psi|(P_{j_\ell} \cdots P_{j_1} \otimes \mathbb{1})^*(P_{i_1} \cdots P_{i_k} \otimes \mathbb{1})|\psi\rangle \\
&= \langle\psi|P_{j_1} \cdots P_{j_\ell} \cdot P_{i_1} \cdots P_{i_k} \otimes \mathbb{1})|\psi\rangle \\
&= f(w_1 w_0),
\end{aligned}$$

this extends to all of $\mathscr{A}(G_{sync})$ by linearity. □

The following propositions can also be deduced from the work of [PSS+16] but we include if for completeness.

Proposition 3.11. If $\mathcal{S}$ is a perfect strategy for a synchronous nonlocal game G_{sync}, then the measurement operators $\{P^x_a\}_{a\in\mathsf{A}}$ (resp. $\{Q^x_a\}_{a\in\mathsf{A}}$), for all $x \in \mathsf{X}$, restricted to the support of the reduced density matrices ρ_A (resp. ρ_B) of $|\psi\rangle$, are a representation of $\mathscr{A}(G_{sync})$.

Proof. Since $\mathcal{S}$ is perfect, we have that

$$0 = \langle\psi|P^x_a \otimes Q^y_b|\psi\rangle = \langle\psi|P^x_a P^y_b \otimes \mathbb{1}|\psi\rangle, \tag{3.8}$$

for all a, b, x, y such that $V(a, b|x, y) = 0$. Let Π be the support projection of ρ_A, and define $\tilde{P^x_a} = \Pi P^x_a \Pi$. It follows that $\langle\psi|P^x_a P^y_b \otimes \mathbb{1}|\psi\rangle = 0$, implies that $\tilde{P^x_a}\tilde{P^y_b} = 0$ whenever $V(a, b, x, y) = 0$, as desired. However, by the second part of Lemma 3.5 the projection Π commutes with each P^x_a and therefore we have that $\tilde{P^x_a} = (\tilde{P^x_a})^* = (\tilde{P^x_a})^2$, and $\sum_a \tilde{P^x_a} = \widetilde{\mathbb{1}}$ for all $x \in \mathsf{X}$, where $\widetilde{\mathbb{1}}$ is the identity on the support of ρ_A. Hence, we see that $\{\{\tilde{P^x_a}\}_{a\in\mathsf{A}} : x \in \mathsf{X}\}$ is a representation of $\mathscr{A}(G_{sync})$ on the support of $|\psi\rangle$. A similar argument works for Bob's measurement operators. □

Proposition 3.12. Let $\mathscr{A}(G_{synch})$ be the synchronous algebra. If $f : \mathscr{A}(G_{synch}) \to \mathbb{C}$ is a finite-dimensional tracial state on the such that

$$\sum_{\substack{a,b\in\mathsf{A}\times\mathsf{A}\\ V(x,y,a,b)=0}} f(e^x_a \cdot e^y_b) = 0,$$

for all $x, y \in \mathsf{X} \times \mathsf{X}$, then there are is an isometric isomorphism and local change of basis for the GNS representation such that

$$f(e^x_a \cdot e^y_b) = \sum_{i=1}^{k} |\lambda_i|^2 \langle\psi_i|\pi_i(e^x_a)\pi_i(e^y_b) \otimes \mathbb{1}_{d_i}|\psi_i\rangle,$$

where each $\pi_i : \mathscr{A}(G_{synch}) \to M_{d_i}(\mathbb{C})$ is an irreducible representation of $\mathscr{A}(G_{synch})$ and $|\psi_i\rangle = \frac{1}{\sqrt{d_i}} \sum_{j=1}^{d_i} |j\rangle \otimes |j\rangle$ for all $1 \leq i \leq k$. Moreover,

$$\sum_{\substack{a,b\in\mathsf{A}\times\mathsf{A}\\ V(x,y,a,b)=0}} \langle\psi_i|\pi_i(e^x_a)\pi_i(e^y_b) \otimes \mathbb{1}_{d_i}|\psi_i\rangle = \sum_{\substack{a,b\in\mathsf{A}\times\mathsf{A}\\ V(x,y,a,b)=0}} \langle\psi_i|\pi_i(e^x_a) \otimes \overline{\pi_i(e^y_b)}|\psi_i\rangle = 0,$$

for all $1 \leq i \leq k$. Hence, for each for $1 \leq i \leq k$, $(\{\{\pi_i(e^x_a)\}_{a\in\mathsf{A}} : x \in \mathsf{X}\}, \{\{\overline{\pi_i(e^y_b)}\}_{b\in\mathsf{A}} : y \in \mathsf{X}\}, |\psi_i\rangle \in \mathbb{C}^{d_i} \otimes \mathbb{C}^{d_i})$ is a perfect quantum strategy for G_{sync}.

We do not give a proof of Proposition 3.12 as it can be found several places in the literature. In the context of synchronous games, the result can be found in [PSS+16], where the idea is often attributed to [SW08]. A more detailed description of the decomposition can also be found in [PRSS22][Corollary 2.10].

Proof of Theorem 3.9. Part (i) follows from Proposition 3.10 and part (ii) follows from Proposition 3.12. □

3.13 BCS games and perfect strategies

We now give a quick overview of boolean constraint system (BCS) nonlocal games and BCS algebras. In several previous works, these have been referred to as *binary* constraint system games or algebras [CM14, Ji13, KPS18, AKS19]. We believe that the term "binary" could be misleading, in the sense that a binary constraint could refer to a 2-ary constraint (i.e. constraints containing two variables), rather than that the domain of the variables is $\mathbb{Z}_2$. Since constraints over the $\mathbb{Z}_2$ domain are more aptly known as boolean constraints, we suggest calling these boolean constraint systems moving forward, and will do so in this work.

We now give some basic definitions before presenting boolean constraint systems. We will take a much closer look at boolean constraint systems and boolean constraint systems algebras in Chapter 5. In this work, we represent $\mathbb{Z}_2$ in multiplicative form. That is, rather than $\{0, 1\}$, we will used $\{+1, -1\}$. This means that over the boolean domain, we associate -1 with TRUE and 1 with FALSE. A k-ary **boolean relation** R is a subset of $\{\pm 1\}^k$, for $k > 0$. Given a set of variables $X = \{x_1, \ldots, x_n\}$, a **constraint** C is a pair (S, R), where the **scope** $S = (s_1, \ldots, s_k)$ is a sequence with each $s_i \in X \cup \{\pm\}$ and R is a k-ary boolean relation. We let $K = X \cap S$ denote the subset of variables that appear in the constraint C. A **(perfect) satisfying assignment** to a constraint C, is a function $\phi : X \to \{\pm 1\}$ such that $\phi(S) \in R$, where $\phi(s_i) = s_i$ whenever $s_i \in \{\pm 1\}$ and $\phi(s_i) = \phi(x_i)$ for all $s_i \in X$ for $1 \leq i \leq k$.

A **boolean constraint system** B is a pair $(X, \{C_i\}_{i=1}^m)$, where X is a set of variables and $\{C_i\}_{i=1}^m$ is a collection of constraints. A satisfying assignment to a BCS B is a function $\phi : X \to \{\pm 1\}$ such that $\phi(S_i) \in R_i$, for all $1 \leq i \leq m$. A BCS is **satisfiable** if it has a satisfying assignment. If $\phi : X \to \{\pm 1\}$ is a satisfying assignment to C_i, let $sat(C_i) = \{\underline{a} = (a_1, \ldots, a_\ell) : \phi(S_i) \in R_i \text{ and } \phi(S_i)|_X = \underline{a}\}$ be the set of satisfying assignments for C_i.

Given a BCS B we can define a **BCS nonlocal game** G_{bcs}. In the game Alice receives a constraint C_i for some $1 \leq i \leq m$, and must reply with a satisfying assignment $\phi(X)$ to C_i.

Meanwhile, Bob receives a single variable x_j for $1 \leq j \leq n$, and replies with an assignment $\varphi(x_j) \in \{\pm 1\}$. They lose the game if $x_j \in X \cap S_i$ and $\phi(x_j) \neq \varphi(x_j)$, otherwise they win. The probability distribution on the inputs for the game determines the probability of Alice obtaining the ith constraint, and Bob obtaining the jth variable. It is not hard to see that if B has a perfect satisfying assignment $\phi(X)$, then the players can always win by employing the strategy where they both employ $\phi(X)$. To each k-ary relation R, we can associate the **indicator function** $f_R : \{\pm 1\}^k \to \{\pm 1\}$ that is -1, whenever $x \in R$, and 1 otherwise. Given an indicator function f_R, for a k-ary relation R and a set of variables X, we can define the **indicator polynomial**

$$P_R(K) = \sum_{v \in \{\pm 1\}^k} f_R(v) \prod_{i=1}^{k} \frac{(1 + v_i x_i)}{2},$$

for $K = \{x_1, \ldots, x_k\}$. The indicator polynomial for a constraint C is a real multilinear polynomial. Unlike general polynomials, the monomials of a multilinear polynomial in k variables are indexed by subsets of $[k]$. That is every real multilinear polynomial in commuting variables can be written in the form

$$P(x_1, \ldots, x_n) = \sum_{\alpha \subseteq [n]} r_\alpha \prod_{j \in \alpha} x_j,$$

where $r_\alpha \in \mathbb{R}$ for all α.

For a constraint C, the indicator polynomial $P_R(K)$ equals -1 whenever the variables $K = X \cap S$ are evaluated at a satisfying assignment for C. It is well known that every propositional formula over the $\{0, 1\}$-domain has a corresponding polynomial representation, and the same is true over the ± 1-domain. We give a few simple examples:

Example 3.14. For $a \in \{0, 1\}$, the formula $\mathsf{NOT}(a) = 1 - a$ goes to the ± 1-valued polynomial $\widetilde{\mathsf{NOT}}(x) = -x$. Similarly for $a_1, a_2 \in \{0, 1\}$, the XOR polynomial $\mathsf{XOR}(a_1, a_2) = a_1 \oplus a_2$ is given by $\widetilde{\mathsf{XOR}}(x_1, x_2) = x_1 x_2$, the $\mathsf{AND}(a_1, a_2) = a_1 a_2$ becomes the ± 1 values polynomial $\widetilde{\mathsf{AND}}(x_1, x_2) = \frac{1}{2}(1 + x_1 + x_2 - x_1 x_2)$, and the $\mathsf{OR}(a_1, a_2)$ is given by the polynomial $\widetilde{\mathsf{OR}}(x_1, x_2) = \frac{1}{2}(x_1 x_2 + x_1 + x_2 - 1)$.

Let us remind ourselves how we take polynomials of matrices. In the case that we have a polynomial in commuting self-adjoint elements, we can simply take the polynomial of the joint spectral eigenvalues. We call a matrix $X \in M_d(\mathbb{C})$ a **± 1-valued observable** if $X^* = X$ and $X^2 = \mathbb{1}$. We note that this condition implies that X is unitary with ± 1-eigenvalues.

Lemma 3.15. If $P : \{\pm 1\}^k \to \{\pm 1\}$ is a real multilinear polynomial in k-variables $\{x_1, \dots, x_k\}$, where each $x_i \in \{\pm 1\}$ written $P(x_1, \dots, x_k)$ and $X = \{X_1, \dots, X_n\}$ is a collection of commuting ± 1-valued observables in $M_d(\mathbb{C})$, then

$$P(X_1, \dots, X_k) = \sum_{i=1}^{d} P(\lambda_{i_1}, \dots, \lambda_{i_k}) |\phi_i\rangle\langle\phi_i|,$$

where each $\lambda_{i_j} \in \{\pm 1\}$ is the ith eigenvalue of the matrix X_j. In particular, this shows that $\|P(X)\| = 1$.

Proof. Since, $X_1, \dots, X_k$ are commuting self-adjoint operators, they are jointly diagonalizable in the basis $\{|\phi_i\rangle\}_{i=1}^{d}$. By the functional calculus, the real multilinear polynomial P acts on each of the d-eigenspaces, hence

$$\begin{aligned} P(X_1, \dots, X_n) &= \sum_{\alpha \subset [n]} c_\alpha \prod_{j \in \alpha} X_j \\ &= \sum_{\alpha \subset [n]} c_\alpha \prod_{j \in \alpha} \left(\sum_{i=1}^{d} \lambda_{i_j} |\phi_i\rangle\langle\phi_i| \right) \\ &= \sum_{i=1}^{d} \left(\sum_{\alpha \subset [n]} c_\alpha \prod_{j \in \alpha} \lambda_{i_j} \right) |\phi_i\rangle\langle\phi_i| \\ &= \sum_{i=1}^{d} P(\lambda_{i_1}, \dots, \lambda_{i_n}) |\phi_i\rangle\langle\phi_i|, \end{aligned}$$

as desired. This also shows that $P_(X_1, \dots, X_n)$ is self-adjoint, since all the X_i's pairwise commute and are self-adjoint, and each c_α is real. □

With this in mind we have the following definition:

Definition 3.16. A **quantum satisfying assignment** to a BCS B is a collection of ± 1-valued observables $\{X_1, \dots, X_n\}$ such that:

(i) $X_{j'} X_j = X_j X_{j'}$ whenever $x_j, x_{j'} \in K_i$, for all $1 \leq i \leq m$, and

(ii) $P_{R_i}(X_{\ell_1}, \dots, X_{\ell_k}) = -\mathbb{1}$, for all $1 \leq i \leq m$.

In [CM14, Ark12], the authors showed that the existence of a quantum satisfying assignment is equivalent to the existence of a perfect quantum strategy for the BCS nonlocal game G_{bcs}. We define the following finitely presented $*$-algebra based on this correspondence.

Definition 3.17. The **BCS algebra** $\mathscr{B}(B)$ of a boolean constraint system B is generated by $\mathbb{C}^*\langle\{x_1,\ldots,x_n\}\rangle$ subject to the following algebraic relations:

(1) $x_j^* = x_j$ and $x_j^2 = 1$ for all $1 \le j \le n$,

(2) $P_i = -1$ for all $1 \le i \le m$, and

(3) $x_{j'}x_j = x_j x_{j'}$ whenever $x_j, x_{j'} \in K_i$, for all $1 \le i \le m$.

Where each $P_i = P_{R_i}(K_i) = P_{R_i}(x_{\ell_1},\ldots,x_{\ell_k})$ is the multilinear indicator polynomial for the constraint C_i.

In subsequent arguments we may abuse notation and write $P_R(x_1,\ldots,x_n)$, even when we know that P_i may actually only be a polynomial on a subset of the variables.

Theorem 3.18. Let $G_{bcs} := G(B)$ be the BCS nonlocal game of a BCS B. The following are equivalent:

(1) $\mathcal{S}$ is a perfect strategy for BCS nonlocal G_{bcs} game,

(2) there is a representation of $\mathscr{B}(G_{bcs})$ on a finite-dimensional Hilbert space with a tracial state.

The proof of Theorem 3.18 will follow from the following lemmas.

Lemma 3.19. $\mathcal{S}$ is a perfect quantum strategy for a BCS nonlocal game $G(B)$ if and only if there are collections of ± 1-valued observables $\{Y_{ij}\}_{i,j\in[m]\times[n]}$ in $Lin(H_A)$ and $\{X_j\}_{j=1}^n$ in $Lin(H_B)$ such that $[Y_{ij}, Y_{ik}] = 0$ for all j,k, and $\langle\psi|Y_{ij}\otimes X_j|\psi\rangle = 1$ for all $i,j \in [m]\times[n]$, where $|\psi\rangle \in Lin(H_A)\otimes Lin(H_B)$ is the state from the perfect strategy $\mathcal{S}$.

Proof. For a BCS nonlocal game we have $\mathsf{X} = [m]$, $\mathsf{B} = [n]$, and $\mathsf{A} = \{\underline{a} \in sat(C_i),\ \text{for each } 1 \le i \le m\}$, $\mathsf{B} = \{\pm 1\}$. Hence, a projective quantum strategy consists of PVMs $\{\{P^i_{\underline{a}}\}_{\underline{a}\in sat(C_i)} : 1 \le i \le m\}$ and $\{\{Q^j_b\}_{b\in\mathbb{Z}_2} : 1 \le j \le n\}$. From now on, we will use i and j as inputs for Alice and Bob. We can define the ±1-valued observables from a projective strategy $\mathcal{S}$ for a BCS nonlocal game $G(B)$ via

$$Y_{i,j} = \sum_{\underline{a}\in sat(C_i)} a_j P^i_{\underline{a}}, \text{ and } X_j = \sum_{b\in\mathbb{Z}_2} bQ^j_b.$$

Since $\{P^i_{\underline{a}}\}_{\underline{a}\in sat(C_i)}$ is a PVM for each $1 \leq i \leq m$. it is not hard to see that and each $Y_{ij} = Y^*_{ij}$ and $Y^2_{ij} = \mathbb{1}$, as well as the commutation relation $[Y_{ij}, Y_{ik}] = 0$, for all j, k, and $1 \leq i \leq m$. Similarly, by construction we have that $X_j = X^*_j$ and $X^2_j = \mathbb{1}$, for all $1 \leq j \leq n$. Recall that $p_{i,j} = \sum_{a,b:V(a,b|i,j)=1} p(a, b|i, j)$ denotes the probability of winning on input (i, j). For a BCS game this can be expressed as

$$p_{i,j} = \sum_{\underline{a}\in sat(C_i),b\in\mathbb{Z}_2:a_j=b} p(a, b|i, j).$$

Let $\beta_{i,j} = \langle\psi|Y_{ij} \otimes X_j|\psi\rangle$, and observe that

$$\begin{aligned}
\langle\psi|Y_{ij} \otimes X_j|\psi\rangle &= \langle\psi| \sum_{\underline{a}\in sat(C_i)} a_j P^x_{\underline{a}} \otimes \sum_{b\in\mathbb{Z}_2} Q^y_b|\psi\rangle \\
&= \sum_{\underline{a}\in sat(C_i)} a_j \cdot b\langle\psi|P^i_{\underline{a}} \otimes Q^j_b|\psi\rangle \\
&= \sum_{\underline{a}\in sat(C_i)} a_j \cdot b\langle\psi|P^i_{\underline{a}} \otimes Q^y_b|\psi\rangle \\
&= 2\left[\sum_{\underline{a}\in sat(C_i),b\in\mathbb{Z}_2:a_j=b} p(a, b|i, j)\right] - 1 \\
&= 2p_{i,j} - 1.
\end{aligned}$$

Hence $\beta_{i,j} = 1$ if and only if $p_{i,j} = 1$. It also follows that $-1 \leq \beta_{i,j} \leq 1$. Now, the result follows as $\beta_{i,j} = 1$, for all $(i, j) \in [m]\times[n]$. Otherwise, we see that if $\beta_{i,j} < 1$ for some input pair (i, j), then $p_{x,y} = (\beta_{i,j} + 1)/2 < 1$ and $\omega(G_{bcs}; \mathcal{S}) = \sum_{i,j} \varrho(i, j)p_{i,j} < 1$, contradicting the fact that $\mathcal{S}$ is perfect for G_{bcs}. □

For BCS games the value of the game is equivalent to:

$$\omega(G(B); \mathcal{S}) = \frac{1}{2}\sum_{i,j} \rho(i, j)\beta_{i,j} + \frac{1}{2}, \tag{3.9}$$

which is why $\beta(i, j)$ is called the *bias* of the strategy $\mathcal{S}$.

Lemma 3.20. If X and Y are d-dimensional self-adjoint matrices with $-\mathbb{1} \le X, Y \le \mathbb{1}$, and $|\psi\rangle$ is a unit vector in $\mathbb{C}^d \otimes \mathbb{C}^d$ such that $\langle\psi|Y \otimes X|\psi\rangle = 1$, then

$$Y \otimes \mathbb{1}|\psi\rangle = \mathbb{1} \otimes X|\psi\rangle. \tag{3.10}$$

Proof. Since $Y \otimes \mathbb{1}$ is self-adjoint we have, by Cauchy-Schwarz, that

$$1 = \langle\psi|(Y \otimes \mathbb{1})^*(\mathbb{1} \otimes X)|\psi\rangle \le \|Y \otimes \mathbb{1}|\psi\rangle\|\|\mathbb{1} \otimes X|\psi\rangle\| \le 1,$$

since both $X \otimes \mathbb{1}$ and $Y \otimes \mathbb{1}$ have norm at most 1. Again by Cauchy-Schwarz, we have that there is a constant $z \in \mathbb{C}$ such that

$$z(Y \otimes \mathbb{1})|\psi\rangle = \mathbb{1} \otimes X|\psi\rangle$$

Taking the norm of both sides, we get that $z = e^{i\theta}$ for some $\theta \in [0, 2\pi)$, since $|z| = 1$. It follows by the conjugate symmetry of the inner product and self-adjointness of $X \otimes Y$ that

$$e^{i\theta} = \langle\psi|(X \otimes Y)|\psi\rangle = \langle\psi|(X \otimes Y)^*|\psi\rangle = \overline{\langle\psi|(X \otimes Y)|\psi\rangle} = e^{-i\theta},$$

thus $\theta = 0$, and the result follows. □

Definition 3.21. If $\mathcal{S}$ is a projective strategy for a BCS game $G(B)$, then let the corresponding operators $\{Y_{ij}\}_{i,j\in[m]\times[n]}$ and $\{X_j\}_{j=1}^n$, with $[Y_{ij}, Y_{ik}] = 0$ for all j, k, $1 \le i \le m$ be a *strategy presented as* ± 1*-valued observables* $\widetilde{\mathcal{S}}$ with the state $|\psi\rangle$.

Note that by Lemma 3.19, a strategy of ± 1-valued observables is perfect if $\langle\psi|Y_{ij} \otimes X_j|\psi\rangle = 1$ for all $i, j \in [m] \times [n]$.

Proposition 3.22. Suppose $\widetilde{\mathcal{S}}$ is a perfect strategy presented in terms of ± 1-valued observables and let ρ_B be the reduced density matrix of $|\psi\rangle$ on H_B. Then the operators $\{X_j\}_{j=1}^n$ are a representation of the BCS algebra $\mathscr{B}(B)$ on the support of ρ_B, and $f : \mathscr{B}(B) \to \mathbb{C}$ with $f(x_j) = \langle\psi|\mathbb{1} \otimes X_j|\psi\rangle$ is a tracial state on $\mathscr{B}(B)$.

Proof. By Lemma 3.20, in every perfect strategy we have that

$$Y_{ij} \otimes \mathbb{1}\,|\psi\rangle = \mathbb{1} \otimes X_j\,|\psi\rangle\,,$$

for all $1 \le j \le n$ and $1 \le i \le m$. Additionally, since the strategy is perfect, Alice's observables must satisfy the following equation

$$\langle\psi|P_i(Y_{i_1}, \ldots, Y_{i_n}) \otimes \mathbb{1}|\psi\rangle = -1,$$

for each $1 \leq i \leq m$, where P_i is the indicator polynomial for the constraint C_i. However, by expanding P_i and employing Lemma 3.20, this implies that

$$-\mathbb{1} \otimes \mathbb{1} |\psi\rangle = \mathbb{1} \otimes P_i(X_1, \ldots, X_n) |\psi\rangle \iff \mathbb{1} \otimes (P_i(X_1, \ldots, X_n) + \mathbb{1}) |\psi\rangle = 0.$$

Let $P_i(X) = P_i(X_1, \ldots, X_n)$ and recall that by Lemma 3.5 that $P_i(X) + \mathbb{1}$ commutes with the support projection Π or ρ_B on H_B. Now, letting $\widetilde{X}_j = \Pi X_j \Pi$, the operators $\{\widetilde{X}_j\}_{j=1}^n$ satisfy $\widetilde{X}_j = \widetilde{X}_j^*$ and $\widetilde{X}_j^2 = \mathbb{1}$ on ΠH_B. Additionally, the commutation relations $[\widetilde{X}_j, \widetilde{X}_k] = 0$ whenever $j, k \in K_i$, for all $1 \leq i \leq m$, still hold on ΠH_B. Lastly, on ΠH_B we have that $P_i(\widetilde{X}_1, \ldots, \widetilde{X}_n) = -\mathbb{1}$ for all $1 \leq i \leq m$, the details of the last claim are very similar to the proof of Lemma 3.5. Hence, $\{\widetilde{X}_j\}_{j=1}$ gives a representation of the BCS algebra on ΠH_B.

We now proceed with the proof the of tracial property for the state. We will argue that the tracial property holds on pairs of generators and then extends to the whole algebra. The proof is very similar to the case for synchronous algebras. Let x_j, x_k be generators of $\mathscr{B}(B)$, then

$$\begin{aligned} f(x_j x_k) &= \langle\psi|\mathbb{1} \otimes X_j X_k|\psi\rangle \\ &= \langle\psi|(\mathbb{1} \otimes X_j)(\mathbb{1} \otimes X_k)|\psi\rangle \\ &= \langle\psi|(\mathbb{1} \otimes X_j)(Y_{ik} \otimes \mathbb{1})|\psi\rangle \\ &= \langle\psi|Y_{ik} \otimes X_j|\psi\rangle \\ &= \langle\psi|(Y_{ik}^* \otimes \mathbb{1})(\mathbb{1} \otimes X_j)|\psi\rangle \\ &= \langle\psi|(\mathbb{1} \otimes X_k)(\mathbb{1} \otimes X_j)|\psi\rangle \\ &= \langle\psi|(\mathbb{1} \otimes X_k X_j|\psi\rangle \\ &= f(x_k x_j). \end{aligned}$$

This extends to monomials in the generator and then to linear combinations by the linearity of the inner-product. □

Proposition 3.23. If there is a finite-dimensional tracial state on the BCS algebra $f : \mathscr{B}(B) \to \mathbb{C}$ with the property that $f(x_j x_i) = \beta_{i,j} = 1$, for all $i, j \in [m] \times [n]$, then the GNS of f is a weighted direct sum of perfect strategies for $G(B)$ presented in terms of ± 1-valued observables with maximally entangled states.

The proof of Proposition 3.23 is essentially the same as the proof of the synchronous.

Proof of Theorem 3.18. By Lemma 3.19 every perfect strategy gives rise to a strategy presented in terms of ± 1-variables. By Proposition 3.22 these operators give a representation of the corresponding BCS-algebra on the Hilbert space supported by ρ_B, moreover, the state defined by the vector state in the strategy is a tracial state. On the other hand, Proposition 3.23 shows that any finite tracial state gives a weighted direct sum of a perfect strategies presented in terms of ± 1-observables with maximally entangled states. Any such collection can be turned back into a projective strategy by Lemma 3.19. □

3.24 The SynchBCS algebra of a synchronous nonlocal game

For BCS and synchronous nonlocal games there were a lot of similarities between the mathematical techniques used to show that perfect strategies correspond to representations of the BCS and synchronous algebras. This turns out to be no coincidence, and these games are not as unrelated as they may appear.

It is not hard to see that there is a synchronous version of any BCS game by considering the game where Alice and Bob each receive a constraint C_i and C_j and must reply with satisfying assignments to each of the variables in those constraints. In this game they win, if their assignment to all non-zero variables x_j, contained in the intersection of both constraints, agree. Any perfect strategy will correspond to representation of the BCS-algebra. However, the game must have a corresponding synchronous algebra since it is synchronous. In fact, one can show these algebras are isomorphic, although we do not establish this result here, it is analogous to the LCS case in [Fri20, Gol21].

On the other hand, given a synchronous nonlocal game G_{sync}, we consider the BCS game where each input $x \in X$ and output $a \in \mathsf{A}$ has an associated $\{\pm 1\}$-valued variable z_a^x. Whenever $V(a, b|x, y) = 0$ in the synchronous game, we add the constraint $\widetilde{\mathrm{AND}}(z_a^x, z_b^y) = 1$ to the BCS, and to ensure that each z_a^x comes from a single measurement (i.e. for any a the set of z_a^x's are jointly measurable and that exactly one of them outputs a -1). To do this, we add the constraint $\widetilde{\mathrm{XOR}}_{a\in A}(z_a^x) = 1$, for each $x \in \mathsf{X}$. This constraint prevents two different 1's from each question while ensuring at least one -1 output is given for each input $x \in \mathsf{X}$. In this new BCS game, the players can receive any one of these constraints and they must reply with a satisfying assignment to the variables. Note that we have not dealt with the probability distribution here, and so this transformation is only on the level of relations and perfect strategies (if they exist for G_{sych}). We call the BCS version of the synchronous nonlocal game the SynchBCS game $G_{synch}(B)$.

Definition 3.25. The **SynchBCS algebra** $\mathscr{B}(G_{synch})$ of a synchronous nonlocal game is the finitely presented∗-algebra

$$\mathbb{C}^*\langle z_a^x,\ \text{for each pair } (a,x) \in \mathsf{A} \times \mathsf{X}\rangle,$$

with relations:

(r.1) $z_a^x = (z_a^x)^*$,

(r.2) $(z_a^x)^2 = 1$,

(r.3) $\widetilde{\mathsf{AND}}(z_a^x, z_b^y) = 1$ whenever $V(x,y,a,b) = 0$,

(r.4) $\prod_{a \in \mathsf{A}} z_a^x = -1$, for all $x \in \mathsf{X}$, and

(r.5) $z_a^x z_b^x = z_b^x z_a^x$ whenever $a, b \in \mathsf{A}$, for all $x \in \mathsf{X}$.

By construction, finite-dimensional representations of the SynchBCS algebras give a quantum satisfying assignment to a SychBCS nonlocal game. Hence, by the correspondence with quantum satisfying assignments and perfect strategies to BCS games, there is a BCS nonlocal game for each synchronous nonlocal game. In [KPS18], the authors showed that the binary synchronous LCS game algebra, which is a central quotient of a group algebra, is isomorphic to the synchronous algebra. Here we demonstrate that there is an isomorphism in the more general BCS case. A similar result for LCS games was given in [Fri20, Gol21].

Proposition 3.26. The synchronous game algebra $\mathscr{A}(G_{sych})$ is ∗-isomorphic to the Synch-BCS algebra $\mathscr{B}(G_{sync})$.

Proof. We begin by describing the ∗-homomorphism $\phi : \mathscr{A}(G_{sych}) \to \mathscr{B}(G_{sync})$. Define the map on the generators $e_a^x \mapsto (1 - z_a^x)/2$, which extends to a ∗-homomorphism on the free algebra. We now check that it descends to a homomorphism from $\mathscr{A}_{sync}$ to $\mathscr{B}_{sync}$. First note that $\phi(e_a^x)$ is an an orthogonal projection, since $z_a^{x*} = z_a^x$, $z_a^{x2} = 1$, and $z_a^x z_b^x = z_b^x z_a^x$ for all $a, b \in A$. Moreover, the $\widetilde{\mathsf{AND}}$ relation implies that $1 - z_a^x - z_b^y + z_a^x z_b^y = 0$ whenever $V(a, b|x, y) = 0$, and thus

$$\phi(e_a^x)\phi(e_b^x) = \frac{(1 - z_a^x)}{2}\frac{(1 - z_b^y)}{2} = 0,$$

is satisfied whenever $V(a, b|x, y) = 0$.

For each $x \in \mathsf{X}$ with $|\mathsf{Y}_i| = n$, observe that the unit 1 can be expanded as the sum of indicator polynomials in the variables z_a^x, giving us

$$1 = \sum_{(v_1,\dots,v_n)\in\{\pm 1\}^n} \prod_{a\in\mathsf{Y}_i} \frac{(1+v_a z_a^x)}{2}. \tag{3.11}$$

However, upon enforcing (r.3), we notice that the product

$$\prod_{a\in Y_i} \frac{(1+v_a z_a^x)}{2} = 0,$$

whenever there is a pair $a, b \in \mathsf{A}_i$ with $v_a = v_b = -1$. Thus, there are only two cases we need to consider. Firstly, when $v_a = 1$, for all $a \in \mathsf{Y}_i$. In this case, we have the term

$$\prod_{a\in\mathsf{Y}_i} \frac{(1+z_a^x)}{2} = \frac{1}{2^n}\left(\sum_{S\subseteq[n]} \prod_{a\in S} z_a^x\right), \tag{3.12}$$

where $|\mathsf{Y}_i| = n$. Now, recall that (r.4) ensures that $\prod_{a\in\mathsf{A}} z_a^x = -1$, and observe that this has the following consequence,

$$\prod_{a\in S} z_a^x + \prod_{a\in[n]\setminus S} z_a^x = 0, \tag{3.13}$$

for any $S \subseteq [n]$, by recalling that each $(z_a^x)^2 = 1$ by (r.2). It follows that equation (3.12) is 0, because each subset $S \subseteq [n]$ is in bijection with its complementary subset $S^c = [n] \setminus S$, and so by equation (3.13) each term with an S product cancels out with the term for S^c product. In the other case, the remaining terms are those with is exactly one $a \in A$ with $v_a = -1$. In this case, let $\Pi_a^x = (1 - z_a^x)/2$ and $\Pi_b^x = (1 - z_b^x)/2$, and observe that Π_a^x and Π_b^x are self-adjoint orthogonal projections with $\Pi_a^x \Pi_b^x = 0$, therefore $\Pi_a^x(1 - \Pi_b^x) = \Pi_a^x - \Pi_a^x\Pi_b^x = \Pi_a^x$. Now, noting $1 - \Pi_b^x = (1 + z_b^x)/2$, it follows that $\Pi_a^x \left[\prod_{b\neq a}(1 - \Pi_b^x)\right] = \Pi_a^x$. With these being the only remaining terms in (3.11) we see that

$$1 = \sum_{a\in\mathsf{Y}_i} \frac{(1-z_a^x)}{2} \prod_{b\neq a} \frac{(1+z_b^x)}{2} = \sum_{a\in\mathsf{Y}_i} \frac{(1-z_a^x)}{2} = \sum_{a\in\mathsf{Y}_i} \phi(p_a^x),$$

for all $x \in \mathsf{X}$, as desired. This shows that indeed ϕ a $*$-homomorphism.

On the other hand consider the map $\varphi : \mathscr{B}_{sync} \to \mathscr{A}_{synch}$, defined by sending the generators $z_a^x \mapsto (1-2e_a^x)$. Recalling that $e_a^x e_b^x = 0$, for all $a \neq b$, hence

$$\begin{aligned}\prod_{a\in \mathsf{Y}_i} \varphi(z_a^x) &= \prod_{a\in \mathsf{Y}_i}(1-2e_a^x)\\ &= \sum_{S\subset \mathsf{Y}_i}(-2)^{|S|}\prod_{a\in S} e_a^x\\ &= 1+(-2)\sum_{a\in \mathsf{Y}_i} e_a^x\\ &= 1+(-2)\cdot 1\\ &= -1,\end{aligned}$$

by recalling relation (r.2) in $\mathscr{A}_{sync}$. Now if $V(a,b|x,y) = 0$, then we have $e_a^x e_b^y = 0$, and hence

$$\begin{aligned}\widetilde{\mathsf{AND}}(\varphi(z_a^x),\varphi(z_b^x)) &= \frac{1}{2}\big(1+(1+2e_a^x)+(1_{\mathcal{A}}+2e_b^y)\\ &- (1+2e_a^x)(1+2e_b^y)\big)\\ &= \frac{1}{2}(2\cdot 1+2e_a^x+2e_b^y-2e_a^x-2e_b^y-4e_a^x e_b^y)\\ &= 1.\end{aligned}$$

Lastly since $V(a,b|x,x) = 0$, we have that $\varphi(z_a^x)\varphi(z_b^x) = (1+2e_a^x)(1+2e_b^x) = 1+2e_a^x+2e_b^x = \varphi(z_b^x)\varphi(z_a^x)$ for all $a \neq b$, as desired.

It remains to show that φ and ϕ are mutual inverses. Observe,

$$\begin{aligned}\varphi\left(\phi(e_a^x)\right) &= \varphi\left(\frac{1-z_a^x}{2}\right)\\ &= \frac{1}{2}\left(\varphi(1)-\varphi(z_a^x)\right)\\ &= \frac{1}{2}\left(1-(1-2e_a^x)\right)\\ &= e_a^x.\end{aligned}$$

Similarly,

$$\begin{aligned}\phi\left(\varphi(z_a^x)\right) &= \phi(1_{\mathcal{A}} - 2e_a^x)\\ &= \phi(1) - 2\phi(e_a^x)\\ &= 1 - 2\frac{(1-z_a^x)}{2}\\ &= 1 - 1 + z_a^x\\ &= z_a^x,\end{aligned}$$

thus $\varphi \circ \phi = \mathbb{1}_{\mathscr{A}_{sync}}$ and $\phi \circ \varphi = \mathbb{1}_{\mathscr{B}_{sync}}$, completing the proof. □

Since the algebras are isomorphic, we know that matrix representations of the Synch-BCS algebra correspond to perfect quantum strategies for the corresponding synchronous nonlocal game and vice versa.

3.27 XOR games and solution algebras

Unlike the characterizations for BCS and synchronous games, XOR nonlocal games do not admit perfect quantum strategies. One can show that if they did, they would also admit a perfect classical strategy, a fact known in [CHTW10]. However, in some cases XOR games have finite-dimensional quantum strategies that significantly outperform the best classical strategies [Slo11]. Like BCS and synchronous game, each XOR game G_{xor} has an affiliated solution algebra $\mathscr{C}(G_{xor})$, where the optimal strategies to G_{xor} correspond to representations of $\mathscr{C}(G_{xor})$.

3.27.1 Tsirelson's XOR correlations

In [Tsi85, Tsi87] Tsirelson contributed extensively to the theory of XOR games through the lens of representation theory much before the idea of an XOR games was even around. Specifically, Tsirelson was interested in Bell scenarios with input sets $[m]$ and $[n]$, and binary outputs from each party. These scenarios produce the set of binary quantum correlations $p \in C_q([m], [n], \{0,1\}, \{0,1\})$. Tsirelson related these binary correlations to what are called XOR correlations.

Definition 3.28. A matrix c in an **XOR correlation** matrix if there is a Euclidean space E, and collections of vectors $\{|u_i\rangle\}_{i=1}^m$, $\{|v_j\rangle\}_{j=1}^n \in E$ of norm at most 1, such that c has entries $c_{ij} = \langle u_i|v_j\rangle$ for all $i, j \in [m] \times [n]$.

To each binary correlation p, Tsirelson associated an *XOR correlation* matrix c. We denote the set of all XOR correlation matrices by $Cor(m,n) \subset \mathbb{R}^{m\times n}$. Note that if $p \in C_q([m],[n],\{0,1\},\{0,1\})$ with a POVM quantum model $\{\{M_a^i\}_{a\in\{0,1\}} : 1 \leq i \leq m\}$ and $\{\{N_b^j\}_{b\in\{0,1\}} : 1 \leq j \leq n\}$, and a state $|\psi\rangle \in H_A \otimes H_B$, then there is a corresponding XOR correlation matrix c with entries

$$c_{ij} = \sum_{a,b\in\{0,1\}} (-1)^{a+b} p(a,b|i,j),$$

since $\{(M_0^x - M_1^x \otimes \mathbb{1})|\psi\rangle\}_{x\in X}$ and $\{(N_0^y - N_1^y \otimes \mathbb{1})|\psi\rangle\}_{y\in Y}$ are vectors of norm at most 1 in a space E. We note that the dimension of the real vector space E can be bounded by a constant times m and n (we refer the reader to [HPV16, Tsi87, Tsi85] for the details). The mapping from the set of binary correlations to the correlations $Cor(m,n)$ is surjective, but not injective. That being said, the map $C_q(\{0,1\},\{0,1\},m,n) \to Cor(m,n)$ restricts to an isomorphism on the subset of **unbiased correlations** $C_q^{unbiased}$, which are the binary correlations that satisfy

$$p(1,1|i,j) = p(0,0|i,j) \text{ and } p(1,0|i,j) = p(0,1|i,j), \text{ for all } i,j \in [m]\times[n].$$

As we will see, every optimal set of quantum correlations for an XOR nonlocal games can be achieved with unbiased correlations. Additionally, unlike the set of all quantum correlations C_q, the set of XOR correlations are a closed convex set (due to the isomorphism described above). This means that computing the q-value for an XOR game is more tractable than in the case of general games. In fact, there are even polynomial time optimization techniques for finding the optimal q-value for XOR games [Weh06, CHTW10, CSUU08], despite the optimization over classical strategies being NP-hard.

3.28.1 XOR games and the XOR algebra

In a two-player XOR nonlocal game G_{xor}, the players Alice and Bob are given questions i and j from sets $[m] = \mathsf{X}$ and $[n] = \mathsf{Y}$ according to a distribution $\varrho : \mathsf{X}\times\mathsf{Y} \to \mathbb{R}_{\geq 0}$. They respond to the referee with outputs $a \in \{0,1\} = \mathsf{A}$ and $b \in \{0,1\} = \mathsf{B}$. The predicate $V : \{0,1\}\times\{0,1\}\times[m]\times[n] \to \{0,1\}$ is determined by the XOR $(a \oplus b)$ of their output bits. An XOR game predicate can be concisely described by an $m \times n$, $\{0,1\}$-matrix, T, with entries $(T)_{i,j} = t_{ij}$. Where the predicate is

$$V(i,j,a,b) = \begin{cases} 1, \text{ if } a \oplus b = t_{ij}, \\ 0, \text{ otherwise} \end{cases}.$$

The **cost matrix** W for an XOR game is then defined as the $m \times n$ matrix with entries $w_{ij} = (-1)^{t_{ij}} \varrho(i,j)$. With the cost matrix for the game, we can conveniently express the **bias** of an XOR game G_{xor}. Given a quantum strategy $\mathcal{S}$ for an XOR game, consisting of ± 1-valued observables $\{Y_1, \ldots, Y_m\}$ and $\{X_1, \ldots, X_n\}$, and a vector state $|\psi\rangle$, the bias with respect to $\mathcal{S}$ is

$$\beta(G_{xor}; \mathcal{S}) = \sum_{i=1,j=1}^{m,n} w_{ij} \langle \psi | Y_i \otimes X_j | \psi \rangle.$$

The supremum over all quantum strategies $\mathcal{S}$ gives the **optimal bias**, denoted by $\beta_q(G_{xor})$, for the XOR game G_{xor}.

To recover a PVM strategy from one presented in terms of ± 1-valued observables, we note that each ± 1-observable is the difference of the 2-outcome projections. That is, in an XOR game, if Alice has a PVM strategy $\{\{P_a^i\}_{a \in \{0,1\}} : 1 \leq i \leq m\}$, we define $Y_i = P_0^i - P_1^i$ for all $1 \leq i \leq m$, and if Bob has PVMs $\{\{Q_a^i\}_{b \in \{0,1\}} : 1 \leq j \leq n\}$, we let $X_j = Q_0^j - Q_1^j$ for $1 \leq j \leq n$. Note that for binary outcome measurements, since $P_0^i + P_1^i = \mathbb{1}$ for all $1 \leq i \leq m$, the projections are uniquely determined by the observables $\{Y_i\}_{i=1}^m$ and $\{X_j\}_{j=1}^[n]$.

Like in the case of BCS games, the bias of an XOR game is related to the value by observing that

$$
\begin{aligned}
\beta(G_{xor};\mathcal{S}) &= \sum_{i=1,j=1}^{m,n} w_{ij}\langle\psi|Y_i\otimes X_j|\psi\rangle \\
&= \sum_{i=1,j=1}^{m,n} \rho(i,j)(-1)^{t_{ij}}\langle\psi|Y_i\otimes X_j|\psi\rangle \\
&= \sum_{i=1,j=1}^{m,n} \rho(i,j)\sum_{a,b\in\{0,1\}}(-1)^{a\oplus b\oplus t_{ij}}p(a,b|i,j) \\
&= \sum_{i=1,j=1}^{m,n} \rho(i,j)\left[\sum_{a\oplus b=t_{ij}}p(a,b|i,j)-\sum_{a\oplus b\neq t_{ij}}p(a,b|i,j)\right] \\
&= \sum_{i=1,j=1}^{m,n} \rho(i,j)\left[2\sum_{a\oplus b=t_{ij}}p(a,b|i,j)-1\right] \\
&= 2\sum_{i=1,j=1}^{m,n} \rho(i,j)\sum_{a,b}V(i,j,a,b)p(a,b|i,j)-1 \\
&= 2\omega(G_{xor};\mathcal{S})-1.
\end{aligned}
$$

This calculation also shows that the optimal bias for an XOR game can be expressed as an optimization over Tsirelson's set of XOR correlations, that is

$$\beta_q(G_{xor}) = \max_{c\in Cor(m,n)}\sum_{i=1,j=1}^{m,n} w_{ij}c_{ij}. \tag{3.14}$$

Note that rather than a supremum, we have a maximum in Equation (3.15). This is because, unlike the set of quantum correlations, the set $Cor(m,n)$ is closed. Optimization problems of the form Equation (3.15) are amenable to semidefinite programming (SDPs) techniques. With this in mind, we will use the following result from [Slo11] without proof:

Lemma 3.29. For every $m\times n$ XOR game G_{xor} and $1\leq i\leq m$ there is a constant $r_i\in\mathbb{R}$, called the *ith marginal row bias*, such that if $(\{Y_i\}_{i=1}^m,\{X_j\}_{j=1}^n,|\psi\rangle)$ is an optimal ±1-valued observable strategy, we have that

$$\sum_{j=1}^{n} w_{ij}(\mathbb{1}\otimes X_j)|\psi\rangle = r_i(Y_i\otimes\mathbb{1})|\psi\rangle, \tag{3.15}$$

for all $1 \leq i \leq m$.

This observation motivates the definition of the *XOR-algebra*[1], which we now define.

Definition 3.30. Let G_{xor} be an XOR game with an $m \times n$ cost matrix W and marginal row biases $\{r_1, r_2, \ldots, r_m\}$. The **XOR algebra** $\mathscr{C}(G_{xor})$ is the finitely presented algebra $\mathbb{C}^*\langle x_1, \ldots, x_n\rangle$ subject to the relations:

(1) $x_j = x_j^*$ and $x_j^2 = 1$, for all $1 \leq j \leq n$,

(2) $\left(\sum_{j=1}^n w_{ij} x_j\right)^2 = r_i^2 \cdot 1$, for all $1 \leq i \leq m$.

Theorem 3.31. Representations of the XOR-algebra $\mathscr{C}(G_{xor})$ correspond optimal strategies for G_{xor}:

1. If $\widetilde{\mathcal{S}} = (\{Y_i : 1 \leq i \leq m\}, \{X_j : 1 \leq j \leq n\}, |\psi\rangle \in H_A \otimes H_B)$ is an optimal quantum strategy for an XOR game G_{xor}, then $\{X_j : 1 \leq j \leq n\}$ is a finite-dimensional representation of the XOR-algebra $\mathscr{C}(G_{xor})$ on the support of ρ_B, and the state $f : \mathscr{C}(G_{xor}) \to \mathbb{C}$ given by $f(x_j) = \langle\psi|\mathbb{1} \otimes X_j|\psi\rangle$ is tracial.
2. Finite tracial states $f : \mathscr{C}(G_{xor}) \to \mathbb{C}$ correspond to weighted direct sums of optimal strategies for G_{xor} with maximally entangled states.

Proof. Let $\widetilde{\mathcal{S}} = (\{Y_i : 1 \leq i \leq m\}, \{X_j : 1 \leq j \leq n\}, |\psi\rangle \in H_A \otimes H_B)$ be an optimal quantum strategy presented in terms of ± 1-valued observables for the XOR nonlocal game G_{xor}. Since, $\mathcal{S}$ is optimal, Lemma 3.29 we must have that Equation (3.15) holds for all $1 \leq i \leq m$. By Lemma 3.6, this implies that

$$\frac{1}{r_i} \sum_j w_{ij} X_j \lambda = \lambda \overline{Y_i},$$

where $\rho_B^{1/2} = \lambda$, for all $1 \leq i \leq m$. Since X_j commutes with the support of λ, by the second

[1] This was called the solution algebra in [Slo11], but to avoid confusion with the solution group algebra of an LCS game, we prefer the term XOR-algebra.

part of Lemma 3.5, it follows that

$$\begin{aligned}\left(\frac{1}{r_i}\sum_j g_{ij}X_j\lambda\right)\left(\frac{1}{r_i}\sum_j g_{ij}X_j\lambda\right)^* &= \lambda\overline{Y_i}(\lambda\overline{Y_i})^* \\ \left(\frac{1}{r_i}\sum_j g_{ij}X_j\right)^2\lambda\lambda^* &= \lambda\overline{Y_i}^2\lambda^* \\ \left(\frac{1}{r_i}\sum_j g_{ij}X_j\right)^2\rho_B &= \rho_B.\end{aligned}$$

Since $X_j^2 = \mathbb{1}$, $X_j^* = X_j^*$, it follows that on the support ΠH_B of ρ_B, $\{\tilde{X}_j\}_{j=1}^n$, with $\tilde{X}_J = \Pi X_j \Pi$ for all $1 \leq j \leq n$ is a representation of $\mathscr{C}(G_{xor})$ on the support of ρ_B.

The proof that this state is tracial is identical to the proof in the BCS case. Along the same lines, the proof that any tracial state on XOR-algebra gives a weighted direct sum of optimal strategies is the same as the BCS and synchronous case, using the GNS of f along with the double commutant decomposition of the Hilbert space from the GNS representation, so we leave it to the reader. □

Theorem 3.31 is due to Slofstra [Slo11]. The correspondence between optimal strategies to XOR games and the XOR-algebra is complementary to a result established by Tsirelson for extremal XOR correlations. In particular, Tsirelson showed that every rank r extreme point in $Cor(m, n)$ has a realization as a rank r Clifford algebra [Tsi87]. It follows that if there is a unique optimal strategy for an XOR game, then the XOR-algebra is isomorphic to a Clifford algebra. Moreover, when the quantum strategy is unique an XOR game provides an example of a finite-dimensional commuting operator self-test in the language of [PSZZ23].

Chapter 4

Near optimal strategies and approximate representations

In Chapter 3, we saw that certain optimal quantum strategies for BCS, synchronous, and XOR games corresponded with representations of the affiliated BCS, synchronous, and XOR game algebras. In this chapter, we present new results about the robustness of this correspondence. In particular, we show that near-perfect (resp. optimal) strategies correspond to approximate representations of the BCS, synchronous (resp. XOR) game algebras.

The remainder of Chapter 4 is as follows. We begin in Section 4.1, with the definitions of near-optimal and near-perfect quantum strategies for nonlocal games. In Section 4.3, we introduce the *approximate tracial property* and begin the proof of theorem 1.3 by establishing that near-perfect strategies, for the family of BCS, synchronous, and XOR games, give approximate representations, with respect to a matrix semi-norm induced by the state employed in the quantum strategy. In Section 4.15, we prove Theorem 1.3 by establishing the crucial "rounding" lemma. Lemma 4.16 shows that for a particular class of finitely presented $*$-algebras, certain approximate representation with respect to these state-dependent semi-norms restrict (or round) to an approximate representation with respect to the little Frobenius norm, removing any dependence on the employed state. With Theorem 1.3 established, Section 4.21 illustrates how the approximate representation-theoretic tools developed in Section 2.22.1 are used to find a near-optimal strategy using a maximally entangled state. To do this, we give a stability result for the algebra of PVMs based on a result in [KPS18]. We conclude the chapter with some remarks regarding Corollary 1.5.

4.1 Near-optimal strategies for nonlocal games

Our first goal is to show that, given a near-optimal (resp. near-perfect) strategy to an XOR (resp. BCS) nonlocal game, that the measurement operators in the strategy are an approximate representation of the affiliated solution (resp. BCS) algebra with respect to the state-induces semi-norm. To do this, we need a few new definitions. Recall that a quantum strategy $\mathcal{S}$ for a nonlocal game G is optimal if $\omega(G;\mathcal{S}) = \omega^*(G)$ and is perfect if $\omega(G;\mathcal{S}) = 1$. With this in mind, we give the following definitions:

Definition 4.2. A quantum strategy $\mathcal{S}$ is **ϵ-optimal**, for $\epsilon > 0$, if $\omega(G;\mathcal{S}) \geq \omega^*(G) - \epsilon$. A quantum strategy $\mathcal{S}$ is **ϵ-perfect**, for $\epsilon > 0$, if $\omega(G;\mathcal{S}) \geq 1 - \epsilon$.

Recall that a strategy $\mathcal{S}$ is perfect if and only if the probability $p(a,b|x,y)$ is 0 on every tuple $(x,y,a,b) \in V^{-1}(\{0\})$. Formally this means, that if $p_{xy}(\mathcal{S})$ denotes the probability of winning with strategy $\mathcal{S}$ on inputs (x,y), then $\mathcal{S}$ is perfect if and only if $p_{xy}(\mathcal{S}) = 1$, for all $(x,y) \in \mathsf{X} \times \mathsf{Y}$. However, in an ϵ-perfect strategy it only holds that $p_{xy}(\mathcal{S}) \geq 1 - \frac{\epsilon}{\varrho(x,y)}$, for all inputs $(x,y) \in \mathsf{X} \times \mathsf{Y}$. This means, that in the case $\varrho(x,y) = 1/|\mathsf{X}||\mathsf{Y}|$ is the uniform probability measure on $\mathsf{X} \times \mathsf{Y}$ our results will depend on the size of X and Y. We will discuss the consequences of this when they come up in the context of our main results.

4.3 Approximate representations of game algebras from ϵ-optimal strategies

In this section, we show that near-optimal strategies correspond to near-representations with respect to the state-induced ρ-norm. Where ρ is the reduced density matrix of the state in the employed quantum strategy. We also establish that each approximate representation derived from a near-optimal strategy has a certain *approximate tracial property*. This property is a crucial requirement for establishing theorem 1.3 via lemma 4.16. To do this, we first need an important lemma from [SV18].

Lemma 4.4 (Proposition 5.4 in [SV18])**.** Let X and Y be self-adjoint unitary operators on finite-dimensional Hilbert space H and let $|\psi\rangle \in H \otimes H$.

$$\langle\psi|X \otimes Y|\psi\rangle \geq 1 - O(\epsilon),$$

if and only if

$$\|Y\lambda - \lambda\overline{X}\|_F \leq O(\epsilon^{1/2}).$$

Moreover, if the above holds we have that:

1. $\|Y\lambda - \lambda Y\|_F \leq O(\epsilon^{1/2})$,

2. $\|\overline{X}\lambda - \lambda\overline{X}\|_F \leq O(\epsilon^{1/2})$

also hold. Here λ is the square-root of the reduced density matrix ρ of the state $|\psi\rangle$ on H.

Proof. Without loss of generality fix an orthonormal basis $\{|t\rangle\}_{t=1}^{dimH}$ for H so that $|\psi\rangle = \sum_{t=1}^{\dim(H)} |t\rangle \otimes \lambda|t\rangle$, with $\rho^{1/2} = \lambda$. Recalling that $(X \otimes \mathbb{1})|\sum_t |t\rangle \otimes |t\rangle = (\mathbb{1} \otimes \overline{X})|\sum_t |t\rangle \otimes |t\rangle$. Observe that

$$1 - O(\epsilon) \leq \langle\psi|X \otimes Y|\psi\rangle = \mathrm{tr}(\overline{X}\lambda Y\lambda),$$

where $\overline{X}$ is the complex-conjugate of X in the basis $\{|t\rangle\}_{t=1}^{\dim(H)}$ for H. By the Cauchy-Schwarz inequality for the Frobenius inner-product $\mathrm{tr}(A^*A) \leq \|A^*\|_F\|A\|_F$, we see that

$$\mathrm{tr}(\overline{X}\lambda Y\lambda) = \mathrm{tr}\left((\lambda^{1/2}\overline{X}\lambda^{1/2})(\lambda^{1/2}Y\lambda^{1/2})\right) \leq \|\lambda^{1/2}\overline{X}\lambda^{1/2}\|_F\|\lambda^{1/2}Y\lambda^{1/2}\|_F.$$

Again by Cauchy-Schwarz, we observe that

$$\|\lambda^{1/2}\overline{X}\lambda^{1/2}\|_F^2 = \mathrm{tr}(X\lambda\overline{X}\lambda) \leq \|X\lambda\overline{X}\|_F\|\lambda|_F = \|\lambda\|_F^2 = \mathrm{tr}(\rho) = 1.$$

Hence, if $1 - O(\epsilon) \leq \mathrm{tr}(V^*\lambda V\lambda)$, then $1 - \|\lambda^{1/2}Y\lambda^{1/2}\|_F^2 \leq O(\epsilon)$. Finally, we have that

$$\|Y\lambda - \lambda Y\|_F^2 = 2 - 2\mathrm{tr}(Y^*\lambda Y\lambda) = 2\left(1 - \|\lambda^{1/2}Y\lambda^{1/2}\|_F^2\right) \leq O(\epsilon),$$

as desired. Since, $Y\lambda$ and λX^* are unit vectors in the Hilbert space of operators with $\|\cdot\|_F$ means that $\|Y\lambda - \lambda\overline{X}\|_F^2 \leq O(\epsilon)$ follows from the parralellogram law.

On the other hand one can see that if

$$\|Y\lambda - \lambda\overline{X}\|_F^2 \leq O(\epsilon),$$

then

$$\begin{aligned}
&\|Y\lambda - \lambda\overline{X}\|_F^2 \\
=&\mathrm{tr}\left((\lambda Y - \overline{X}\lambda)(Y\lambda - \lambda\overline{X})\right) \\
=&\mathrm{tr}(\lambda Y^2\lambda) - 2\mathrm{tr}(\lambda Y\lambda\overline{X}) + \mathrm{tr}(\overline{X}\lambda^2\overline{X}) \\
=&2\mathrm{tr}(\rho) - 2\mathrm{tr}(\lambda Y\lambda\overline{X}) \\
=&2(1 - \langle\psi|X \otimes Y|\psi\rangle) \\
=&2(1 - \mathrm{tr}(\overline{X}\lambda Y\lambda)) \leq O(\epsilon),
\end{aligned}$$

and the result follows. ∎

The following concept is key to our main argument:

Definition 4.5. Given a positive semi-definite operator $\rho = \lambda^*\lambda \in Lin(H)$ and a constant $\delta > 0$, a representation $\phi : \mathbb{C}^*\langle S : R\rangle \to Lin(H)$ is (δ, λ)**-tracial** if

$$\|\phi(s)\lambda - \lambda\phi(s)\|_F \leq \delta,$$

for all $s \in S$. Moreover, an ϵ-representation $\psi : \mathbb{C}^*\langle S : R\rangle \to Lin(H)$ in the ρ-norm has the **approximate tracial property** (or is δ**-ATP**) if it is $(\delta, \sqrt{\rho})$-tracial.

Remark 4.6. If a representation $\phi : \mathbb{C}^*\langle S\rangle \to Lin(H)$ is $(0, \lambda)$-tracial, then there is a state $|\psi\rangle = \sum_{t\in H} \lambda|t\rangle \otimes |t\rangle \in H \otimes H$, where $\{t\}_{i=1}^{\dim(H)}$ indexes an orthonormal basis for H, such that

$$(\phi(s) \otimes \mathbb{1})|\psi\rangle = (\mathbb{1} \otimes \phi(s)^{op})|\psi\rangle, \quad \text{for all } s \in S.$$

Where *op* here denotes the *opposite* representation with multiplication $(a \cdot b)^{op} = b \cdot a$. Lastly, the representations $\phi(s) \otimes \mathbb{1}$ and $\mathbb{1} \otimes \phi(s)^{op}$ commute on the subspace of $H \otimes H$ spanned by $|\psi\rangle$. A similar property to being δ-ATP arose in the work of [MPS21] while investigating *robust self-testing* with finite-dimensional algebras.

4.6.1 Almost perfect strategies give approximate representations

We recall that Lemma 3.6 has an important corollary:

Corollary 4.7. If X and Y are self-adjoint matrices and $\|(X \otimes \mathbb{1} - \mathbb{1} \otimes Y)|\psi\rangle\|_{H_A\otimes H_B} \leq \epsilon$, then $\|X\lambda - \lambda\overline{Y}\|_F \leq \epsilon$.

With this simple observation we have the following:

Proposition 4.8. If $\{E_a^x\}_{x,a\in\mathsf{X}\times\mathsf{A}}$ and $\{F_a^x\}_{x,a\in\mathsf{X}\times\mathsf{A}}$ are Alice and Bob PVM's from an ϵ-perfect strategy for a synchronous nonlocal game G_{sync} with the state $|\psi\rangle$, and the uniform distribution $\varrho(x, y) = 1/(|\mathsf{X}||\mathsf{Y}|)$, then we have that $\|E_a^x\lambda - \lambda\overline{F_a^x}\|_F \leq O(\epsilon^{1/2})$ for all $x \in \mathsf{X}$, $a \in \mathsf{A}$.

Proof. If $\mathcal{S}$ is an ϵ-perfect strategy then for any pair of inputs (x, y), then

$$\sum_{a,b\in A\times B} V(x, y, a, b)p(a, b|x, y) \geq 1 - nm\epsilon,$$

where $|\mathsf{X}| = n$, and $|\mathsf{Y}|$=m. In particular, $V(x, y, a, b) = 0$ whenever $a \neq b$, for all $x \in \mathsf{X}$, $y \in \mathsf{X}$. So for $\mathcal{S}$ to be ϵ-perfect we must have $\sum_{a\neq b}\langle\psi|E_a^x \otimes F_b^x|\psi\rangle < nm\epsilon$, for all $x \in \mathsf{X}$.

Hence, we see that

$$\begin{aligned}
&\|(E_a^x \otimes \mathbb{1} - \mathbb{1} \otimes F_a^x)|\psi\rangle\|^2 \\
=&\langle\psi|E_a^x \otimes \mathbb{1}|\psi\rangle + \langle\psi|\mathbb{1} \otimes F_a^x|\psi\rangle - 2\langle\psi|E_a^x \otimes F_a^x|\psi\rangle \\
=&\sum_{b\in B}\langle\psi|E_a^x \otimes F_b^x|\psi\rangle + \sum_{a'\in A}\langle\psi|E_{a'}^x \otimes F_a^x|\psi\rangle - 2\langle\psi|E_a^x \otimes F_a^x|\psi\rangle \\
=&\sum_{b\neq a}\langle\psi|E_a^x \otimes F_b^x|\psi\rangle + \sum_{a'\neq a}\langle\psi|E_{a'}^x \otimes F_a^x|\psi\rangle \\
\leq&2nm\epsilon \\
\leq&O(\epsilon).
\end{aligned}$$

□

Proposition 4.9. Let $|\psi\rangle \in H_A \otimes H_B$ be a quantum state, and suppose ρ is the reduced density matrix of $|\psi\rangle$ on H_B. If $\mathcal{S} = (\{E_a^x\}_{(x,a)\in \mathsf{X}\times\mathsf{A}}, \{F_a^x\}_{(x,a)\in\mathsf{X}\times\mathsf{A}}, |\psi\rangle)$ is an ϵ-perfect for G_{sync} employing $|\psi\rangle$, then $\{F_a^x\}_{(x,a)\in\mathsf{X}\times\mathsf{A}}$ is an $O(\epsilon^{1/4})$-representation of the synchronous game algebra in $Lin(H_B)$ with respect to the state induced semi-norm $\|\cdot\|_\rho$. Moreover, the $O(\epsilon^{1/4})$-representation is $O(\epsilon^{1/2}, \lambda)$-tracial.

Proof. Since $\mathcal{S}$ is ϵ-perfect, whenever $V(x, y, a, b) = 0$ we have

$$\langle\psi|E_a^x \otimes F_b^y|\psi\rangle = \mathrm{tr}(\overline{E_a^x}\lambda F_b^y\lambda) < nm\epsilon.$$

Hence, by proposition 4.8 and Cauchy-Schwarz we have that

$$\begin{aligned}
\|F_a^x F_b^y\|_\rho^2 &= \|F_a^x F_b^y\lambda\|_F^2 \\
&= \mathrm{tr}(\lambda F_b^y F_a^x F_b^y \lambda) \\
&= \mathrm{tr}(\lambda F_b^y F_a^x (F_b^y\lambda - \lambda\overline{E_b^y})) + \mathrm{tr}(\lambda F_b^y F_a^x \lambda \overline{E_b^y}) \\
&\leq \mathrm{tr}(\lambda F_b^y F_a^x \lambda\overline{E_b^y}) + \|F_a^x F_b^y \lambda\|_F \|F_b^y\lambda - \lambda\overline{E_b^y}\|_F \\
&= \mathrm{tr}(F_a^x \lambda \overline{E_b^y}\lambda F_b^y) + O(\epsilon^{1/2}) \\
&= \mathrm{tr}(F_a^x\lambda\overline{E_b^y}(\lambda F_b^y - \overline{E_b^y}\lambda)) + \mathrm{tr}(F_a^x\lambda\overline{E_b^y}^2\lambda) + O(\epsilon^{1/2}) \\
&\leq \mathrm{tr}(\overline{E_b^y}\lambda F_a^x\lambda) + \|\overline{E_b^y}\lambda F_a^x\|_F \|\lambda F_b^y - \overline{E_b^y}\lambda\|_F + O(\epsilon^{1/2}) \\
&\leq nm\epsilon + O(\epsilon^{1/2}) \\
&\leq O(\epsilon^{1/2}).
\end{aligned}$$

The result follows since each F_a^x is an orthogonal projection and $\sum_{a\in\mathsf{A}} F_a^x = \mathbb{1}_{H_B}$, for each $x \in \mathsf{X}$ by definition of being a PVM. The fact that the approximate representation is $O(\epsilon^{1/2}, \lambda)$-tracial follow from the second part of Proposition 4.4 since $Z_a^x = \mathbb{1} - 2E_a^x$ and $W_a^x = \mathbb{1} - 2F_a^x$ for all $x \in \mathsf{X}$ and $a \in \mathsf{A}$ are self-adjoint unitaries satisfying

$$\|W_a^y \lambda - \lambda \overline{Z}_a^x\|_F \leq O(\epsilon^{1/2}).$$

□

Corollary 4.10. If the state $|\psi\rangle$ in the strategy $\mathcal{S}$ for the synchronous nonlocal game $\mathcal{G}$ is maximally entangled then we see that $\{\{F_a^x\}_{a\in\mathsf{A}} : x \in \mathsf{X}\}$ generates an $O(\epsilon^{1/4})$-representation of the synchronous game algebra on $Lin(H_B)$ with respect to $\|\cdot\|_f$.

We now show that given any ϵ-perfect strategy to the BCS game, Bob's operators give an approximate representation in the state-induced norm, by showing that the relations of the BCS algebra $\mathscr{B}(G_{bcs})$ are bounded by $O(\epsilon^{1/2})$ in this ρ-norm. In the remainder of the section we restrict the operators in each quantum strategy $\mathcal{S}$ to the support of the employed state $|\psi\rangle$ to ensure that the ρ-seminorm is proper norm on H_B (or H_A).

Proposition 4.11. If $(\{Y_{ij}\}_{i,j=1}^{m,n}, \{X_j\}_{j=1}^n, |\psi\rangle \in H_A \otimes H_B)$ is an ϵ-perfect strategy for the BCS game G_{bcs}, where the state $|\psi\rangle$ has reduced density matrix $\rho = \lambda^*\lambda$ on H_B, then the operators $\{X_j\}_{j=1}^n$ give an $O(\epsilon^{1/2})$-approximate representation of the BCS algebra $\mathscr{B}(G_{bcs})$ with respect to the ρ-norm. Moreover, the approximate representation is $O(\epsilon^{1/2})$-ATP.

Proof. First, it is clear that each X_j is a self-adjoint unitary, so it only remains to establish that

(1) $\|P_i(K_i) + \mathbb{1}\|_\rho \leq O(\epsilon^{1/2})$, for all $1 \leq i \leq m$, and

(2) $\|X_k X_j - X_j X_k\|_\rho \leq O(\epsilon^{1/2})$, for all $1 \leq i \leq m$, $j, k \in K_i$,

where K_i is the subset of variables contained in the constraint polynomial for C_i, for $1 \leq i \leq m$. Let $Z_{i_j} = \overline{Y_{ij}}$ for all $i, j \in [m] \times [n]$. Toward (1), we claim that

$$\left\|\prod_{j\in S} X_j \lambda - \lambda \prod_{j\in S} Z_{i_j}\right\|_F \leq \sum_{j=1}^{|S|} \|X_j\lambda - \lambda Z_{i_j}\|_F,$$

for all $1 \leq i \leq m$. We proceed by induction, when $|S| = 2$ we have that

$$\begin{aligned}\|X_2X_1\lambda - \lambda Z_{i_2}Z_{i_1}\|_F &\leq \|X_2X_1\lambda - X_2\lambda Z_{i_1}\|_F + \|X_2\lambda Z_{i_1} - \lambda Z_{i_2}Z_{i_1}\|_F \\ &\leq \|X_2(X_1\lambda - \lambda Z_{i_1})\|_F + \|(X_2\lambda - \lambda Z_{i_2})Z_{i_1}\|_F \\ &\leq \|X_1\lambda - \lambda Z_{i_1}\|_F + \|X_2\lambda - \lambda Z_{i_2}\|_F,\end{aligned}$$

as desired. Now suppose the result holds for all $1 \leq k \leq |S| - 1$ and observe that

$$\begin{aligned}&\|X_{k+1}\cdots X_1\lambda - \lambda Z_{i_{k+1}}\cdots Z_{i_1}\|_F \\ \leq&\|X_{k+1}\cdots X_1\lambda - X_{k+1}\lambda Z_{i_k}\cdots Z_{i_1}\|_F + \|X_{k+1}\lambda Z_{i_k}\cdots Z_{i_1} - \lambda Z_{i_{k+1}}\cdots Z_{i_1}\|_F \\ \leq&\|X_k\cdots X_1\lambda - \lambda Z_{i_k}\cdots Z_{i_1}\|_F + \|X_{k+1}\lambda - \lambda Z_{i_{k+1}}\|_F \\ \leq&\sum_{j=1}^{k}\|X_j\lambda - \lambda Z_{i_j}\|_F + \|X_{k+1}\lambda - \lambda Z_{i_{k+1}}\|_F,\end{aligned}$$

as desired. Since the strategy is ϵ-perfect, by Lemma 4.4 and the proof of Lemma 3.19, we see that $\|X_j\lambda - \lambda Z_{i_j}\|_F \leq O(\epsilon^{1/2})$, for all $1 \leq j \leq n$, and $1 \leq i \leq n$. Lastly, noting that $P_i(Z_{i_1}, \ldots, Z_{i_n}) = -\mathbb{1}$, we see that the semi-norm properties of $\|\cdot\|_\rho$ suffice to obtain

$$\begin{aligned}\|f_i(K_i) + \mathbb{1})\|_\rho &\leq \left\|\sum_{S\subset K_i}\hat{f}_S^{(i)}\prod_{j\in S}X_j\lambda - \lambda(-\mathbb{1})\right\|_F \\ &\leq \sum_{S\subset K_i}|\hat{f}_S^{(i)}|\left\|\prod_{j\in S}X_j\lambda - \lambda\prod_{j\in S}Z_{i_j}\right\|_F \\ &\leq O(\epsilon^{1/2})\end{aligned}$$

as desired. To see that (2) holds, observe that

$$\|X_kX_j - X_jX_k\|_\rho \leq \|X_kX_j\lambda - \lambda Z_{i_j}Z_{i_k}\|_F + \|X_jX_k\lambda - \lambda Z_{i_k}Z_{i_j}\|_F,$$

since the Z_{i_j}'s all commute for all $j, k \in K_i$, and the result follows along the lines of (1). For the tracial property, we see that if $\mathcal{S}$ is ϵ-perfect then $\|X_j\lambda - \lambda X_j\|_F^2 \leq O(\epsilon)$, for all $1 \leq j \leq n$, follows directly from the second statement of Lemma 4.4. □

Corollary 4.12. If the state $|\psi\rangle$ in the ϵ-perfect strategy $\mathcal{S}$ for a BCS game G_{bcs} is maximally entangled we see that the operators $\{X_j\}_{j=1}^n$ generate an $O(\epsilon^{1/2})$-representation of $\mathscr{B}(G_{bcs})$ on $Lin(H_B)$ with respect to $\|\cdot\|_f$.

4.12.1 Optimal strategies for XOR games and solution algebras

In [Slo11] it was shown that an ϵ-optimal strategy for an XOR game using quantum state supported on a d-dimensional Hilbert space H gives a $O(d^{2/3}\epsilon^{1/8})$-representations of the XOR-algebra for an XOR game. Using the techniques in this paper, we are able to eliminate the dependence on the dimension of the supporting Hilbert space. As noted in the introduction, the removal of the dependence on the dimension was achieved for a certain family of XOR games in [OV16], using different techniques.

In the case of XOR games we will use a slightly weaker definition of ϵ-optimal strategies than in the BCS case. The notion used for XOR strategies comes from the strong duality statement for the semi-definite program (SDP) achieving the optimal value of an XOR game [CSUU08]. The definition we use here is based on [Slo11, Theorem 3.1], which gives a relation between ϵ-optimal strategies and the observables in the employed strategy for G_{xor}. Formally, it states that for every XOR game G_{xor}, there exists a collection of constants $r_i \geq 0$ (called the marginal row biases) such that if $\mathcal{S} = (\{Y_i\}_{i=1}^m, \{X_j\}_{j=1}^n, |\psi\rangle)$ is ϵ-optimal strategy of ± 1-valued observables, with $0 \leq \epsilon \leq \frac{1}{4}(m+n)$, then

$$\left\| \left(\sum_{j=1}^{n} w_{ij} (\mathbb{1} \otimes X_j) - r_i (Y_i \otimes \mathbb{1}) \right) |\psi\rangle \right\| \leq O(\epsilon^{1/4}), \tag{4.1}$$

for all $1 \leq i \leq m$, and the constants hidden in the $O(\epsilon^{1/4})$ depend only on the size of the input sets m and n. Starting from equation (4.1), Lemma 4.4 shows that

$$\left\| \sum_{j=1}^{n} w_{ij} X_j \lambda - \lambda r_i \overline{Y_i} \right\|_F \leq O(\epsilon^{1/4}). \tag{4.2}$$

With this fact and Proposition 4.11 we can establish the following:

Proposition 4.13. Let $\mathcal{S} = (\{Y_i\}_{i=1}^n, \{X_j\}_{j=1}^n, |\psi\rangle \in H_A \otimes H_B)$ be an ϵ-optimal strategy for an XOR game $\mathcal{G}$ where $|\psi\rangle$ has reduced density matrix $\rho = \lambda^2 \in Lin(H_B)$, then the operators $\{X_j\}_{j=1}^n$ generate an $O(\epsilon^{1/4})$-representation of the solution algebra $\mathscr{C}(G_{xor})$ with respect to $\|\cdot\|_\rho$. Moreover, the $O(\epsilon^{1/4})$-representation is $(O(\epsilon^{1/4}), \lambda)$-tracial.

Proof. If $\mathcal{S}$ is an ϵ-optimal strategy for the XOR game G_{xor} and $\rho = \lambda^*\lambda$ is the reduced density matrix of the state on H_B, then since each X_j is a self-adjoint unitary, it only remains to show that

$$\left\| r_i^2 \mathbb{1} - \left(\sum_{j=1}^{n} w_{ij} X_j \right)^2 \right\|_\rho \leq O(\epsilon^{1/4}),$$

for all $1 \leq i \leq m$. By invoking Equation (4.2), we see that

$$
\begin{aligned}
&\left\| r_i^2 \mathbb{1} - \left(\sum_{j=1}^n w_{ij} X_j \right)^2 \right\|_\rho \\
=&\left\| r_i^2 \lambda - \left(\sum_{j=1}^n w_{ij} X_j \right)^2 \lambda \right\|_F \\
\leq &\left\| r_i^2 \lambda - \sum_{j=1}^n w_{ij} X_j r_i \lambda \overline{Y}_i \right\|_F + \left\| \sum_{j=1}^n w_{ij} X_j r_i \lambda \overline{Y}_i - \left(\sum_{j=1}^n w_{ij} X_j \right)^2 \lambda \right\|_F \\
\leq &\left\| r_i^2 \lambda \overline{Y}_i - \sum_{j=1}^n w_{ij} X_j r_i \lambda \right\|_F + \sum_{j=1}^n |w_{ij}| \left\| X_j \left(r_i \lambda \overline{Y}_i - \sum_{j=1}^n w_{ij} X_j \lambda \right) \right\|_F \\
\leq &|r_i| \left\| r_i \lambda \overline{Y}_i - \sum_{j=1}^n w_{ij} X_j \lambda \right\|_F + \sum_{j=1}^n |w_{ij}| \left\| r_i \lambda \overline{Y}_i - \sum_{j=1}^n w_{ij} X_j \lambda \right\|_F \\
\leq &O(\epsilon^{1/4}),
\end{aligned}
$$

as desired. To see that this approximate representation is $(O(\epsilon^{1/4}), \lambda)$-tracial we note that $\|X_j\lambda - \lambda X_j\|_F \leq O(\epsilon^{1/4})$, for all $1 \leq j \leq n$, follows from Lemma 4.4 and Equation (4.2). □

Corollary 4.14. If the state $|\psi\rangle$ in the ϵ-optimal strategy $\mathcal{S}$ for the XOR nonlocal game $\mathcal{G}$ is maximally entangled, then $\{X_j\}_{j=1}^n$ is an $O(\epsilon^{1/4})$-representation of $\mathcal{C}(\mathcal{G})$) on H_B with respect to $\| \cdot \|_f$.

4.15 Rounding to the little Frobenius norm

In this section, we show that any ϵ-representation in a state-dependent semi-norm arising from a near-optimal quantum strategy gives rise to an $O(\epsilon^{1/2})$-representation of the game algebra in the $\| \cdot \|_f$-norm. To do this, we rely on the following "rounding" lemma, which works for a specific class of finitely presented $*$-algebras and ϵ-representations where a suitable approximate tracial condition holds.

Lemma 4.16. Let $\rho = \lambda^* \lambda$ be a density operator on a finite-dimensional Hilbert space H and let $\mathscr{G} = \mathbb{C}^*\langle X : R \rangle$ be a finitely presented $*$-algebra generated by self-adjoint unitaries. If $\varphi : \mathbb{C}^*\langle X \rangle \to Lin(H)$ is an ϵ-representation of $\mathscr{G}$ with respect to $\| \cdot \|_\rho$ that is $O(\epsilon)$-tracial, then there is a non-zero projection $P \in Lin(H)$, and a finite-dimensional unitary $O(\epsilon^{1/2})$-representation ϕ' on $Im(P) = \widetilde{H} \subseteq H$ with respect to the $\|\cdot\|_f$-norm on $\widetilde{H}$.

Before we give the proof, we require several intermediate results.

Let χ_I be the indicator function for the real interval $I \subseteq \mathbb{R}$. For a compact self-adjoint operator T, and measurable subset $I \subseteq \mathbb{R}$, we let $\chi_I(X)$ be the **spectral projection onto** $I \cap \sigma(T)$, where $\sigma(T)$ is the spectrum of T. In particular, if T is positive semi-definite, then by the spectral theorem $\sigma(T) \subset [0, +\infty)$.

Lemma 4.17. (Connes's "joint distribution trick" [Con76]) Let λ and λ' be positive semi-definite operators on a finite-dimensional Hilbert space.

$$\int_0^{+\infty} \|\chi_{\geq\sqrt{\alpha}}(\lambda) - \chi_{\geq\sqrt{\alpha}}(\lambda')\|_F^2 d\alpha \leq \|\lambda - \lambda'\|_F \|\lambda + \lambda'\|_F. \tag{4.3}$$

We give the proof here for completeness. The argument we present here is found in [SV18] for the finite-dimensional case. Readers wishing to see the more general case can consult the seminal work of Connes' [Con76].

Proof. Let λ, λ' be positive semi-definite operators on a finite dimensional Hilbert space H. Consider the spectral decompositions $\lambda = \sum_i \lambda_i |u_i\rangle\langle u_i|$ and $\lambda' = \sum_j \mu_j |v_j\rangle\langle u_j|$. Define the discrete measure $\nu : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ via $\nu(x,y) = \sum_{i,j} \delta_{(\lambda_i,\mu_j)}(x,y) |\langle u_i|v_j\rangle|^2$. Then, for any functions $f, g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ we see that

$$\begin{aligned}
&\|f(\lambda) - g(\lambda')\|_F^2 \\
=&\mathrm{tr}\left[\sum_{i,j} f(\lambda_i)^2 |v_i\rangle\langle v_i| - f(\lambda_i) g(\mu_j) \langle u_i|v_j\rangle |u_i\rangle\langle v_j| - g(\mu_j) f(\lambda_i) \langle v_j|u_i\rangle |v_j\rangle\langle u_i| + g(\mu_j)^2 |v_j\rangle\langle v_j|\right] \\
=&\sum_{i,j} f(\lambda_i)^2 - 2 f(\lambda_i) g(\mu_j) |\langle u_i|v_j\rangle|^2 + g(\mu_j)^2 \\
=&\sum_i f(\lambda_i)^2 \langle u_i| \sum_j |v_j\rangle\langle v_j||u_i\rangle - 2\sum_{i,j} f(\lambda_i) g(\mu_j) |\langle u_i|v_j\rangle|^2 + \sum_j g(\mu_j)^2 \langle v_j| \sum_i |u_i\rangle\langle u_i||v_j\rangle \\
=&\sum_{i,j} f(\lambda_i)^2 |\langle u_i|v_j\rangle|^2 - 2\sum_{i,j} f(\lambda_i) g(\mu_j) |\langle u_i|v_j\rangle|^2 + \sum_{i,j} g(\mu_j)^2 |\langle u_i|v_j\rangle|^2 \\
=&\int_{(x,y)} f(x)^2 d\nu - \int_{(x,y)} 2 f(x) g(y) d\nu + \int_{(x,y)} g(y)^2 d\nu \\
=&\int_{(x,y)} |f(x) - g(y)|^2 d\nu \; .
\end{aligned}$$

Now, consider non-negative real numbers $x \leq y$, and note that $\chi_{\geq\sqrt{\alpha}}(x) = \begin{cases} 1, x^2 \geq \alpha \\ 0, x^2 < \alpha \end{cases}$, hence we also see that

$$\begin{aligned}
&\int_0^{+\infty} |\chi_{\geq\sqrt{\alpha}}(x) - \chi_{\geq\sqrt{\alpha}}(y)| d\alpha \\
=&\int_0^{x^2} |\chi_{\geq\sqrt{\alpha}}(x) - \chi_{\geq\sqrt{\alpha}}(y)| d\alpha + \int_{x^2}^{y^2} |\chi_{\geq\sqrt{\alpha}}(x) - \chi_{\geq\sqrt{\alpha}}(y)| d\alpha + \int_{y^2}^{+\infty} |\chi_{\geq\sqrt{\alpha}}(x) - \chi_{\geq\sqrt{\alpha}}(y)| d\alpha \\
=&\int_0^{x^2} |1-1| d\alpha + \int_{x^2}^{y^2} |0-1| d\alpha + \int_{y^2}^{+\infty} |0-0| d\alpha \\
=&\int_{x^2}^{y^2} 1 d\alpha \\
=&|x^2 - y^2| \\
=&|x-y||x+y|.
\end{aligned}$$

Putting the above calculations together and using Fubini's theorem to swap the order of integration we see that

$$\begin{aligned}
&\int_0^{+\infty} \|\chi_{\geq\sqrt{\alpha}}(\lambda) - \chi_{\geq\sqrt{\alpha}}(\lambda')\|_F^2 d\alpha \\
=&\int_{(x,y)} \int_0^{\infty} |\chi_{\geq\sqrt{\alpha}}(x)^2 - \chi_{\geq\sqrt{\alpha}}(y)^2| d\alpha d\nu \\
=&\int_{(x,y)} |x-y||x+y| d\nu \\
\leq&\left(\int_{(x,y)} |x-y|^2 d\nu\right)^{1/2} \left(\int_{(x,y)} |x+y|^2 d\nu\right)^{1/2} \\
=&\|\lambda - \lambda'\|_F \|\lambda + \lambda'\|_F.
\end{aligned}$$

□

We also require the following simple observation:

Proposition 4.18. Let λ be a positive semi-definite operator on a finite-dimensional Hilbert space. Then

$$\int_0^{+\infty} \chi_{\geq\sqrt{\alpha}}(\lambda) d\alpha = \lambda^2$$

and for any self-adjoint operator $T = T^*$ we have

$$\int_0^{+\infty} \operatorname{tr}\left(T\chi_{\geq\sqrt{\alpha}}(\lambda)\right) d\alpha = \operatorname{tr}(T\lambda^2).$$

Proof. Since every positive operator on a finite dimensional Hilbert space has a spectral decomposition it suffices to prove the result for $\lambda = t|v\rangle\langle v|$, where $|v\rangle$ a unit vector and $t \geq 0$. Since

$$\chi_{\geq\sqrt{\alpha}}(\lambda) = \begin{cases} |v\rangle\langle v| \text{ if} & \alpha \leq t^2, \\ 0 \text{ if} & \alpha > t^2 \end{cases} \tag{4.4}$$

we see that

$$\int_0^{+\infty} \chi_{\geq\sqrt{\alpha}}(\lambda) d\alpha = \int_0^{t^2} |v\rangle\langle v| d\alpha = t^2|v\rangle\langle v| = \lambda^2. \tag{4.5}$$

The second part follows from linearity of the trace. □

We are now ready to establish Lemma 4.16.

Proof of lemma 4.16. Let $\varphi : \mathbb{C}^*\langle X\rangle \to Lin(H)$ sending $x_j \mapsto X_j$ be an $O(\epsilon)$-representation in the ρ-norm that is $O(\epsilon)$-tracial, with exact relations $X_j^2 = \mathbb{1}$ and $X_j^* = X_j$. We begin by showing that there is a non-zero orthogonal projection P on H for which

$$\|X_jP - PX_j\|_F \leq O(\epsilon)\operatorname{tr}(P)^{1/2} \text{ for } 1 \leq j \leq n \text{ and} \tag{4.6}$$

$$\|\varphi(r)P\|_F \leq O(\epsilon)\operatorname{tr}(P)^{1/2} \text{ for all } r \in R, \tag{4.7}$$

where $\varphi(r)$ is the image of the polynomial relations $r \in R$ in the approximate representation φ. To establish these claims we employ Lemma 4.17. For each $1 \leq j \leq n$, we have the representative $\varphi(x_j) = X_j$, hence

$$\begin{aligned}
&\int_0^{+\infty} \|X_j\chi_{\geq\sqrt{\alpha}}(\lambda) - \chi_{\geq\sqrt{\alpha}}(\lambda)X_j\|_F^2 d\alpha \\
&= \int_0^{+\infty} \|\chi_{\geq\sqrt{\alpha}}(\lambda) - X_j^*\chi_{\geq\sqrt{\alpha}}(\lambda)X_j\|_F^2 d\alpha \\
&\leq \|\lambda - X_j^*\lambda X_j\|_F \|\lambda + X_j^*\lambda X_j\|_F \\
&= \|X_j\lambda - \lambda X_j\|_F \|X_j\lambda + \lambda X_j\|_F \\
&\leq 2\|X_j\lambda - \lambda X_j\|_F \\
&\leq O(\epsilon),
\end{aligned}$$

by using the ATP property. The fact that $\|X_j\lambda + \lambda X_j\|_F \leq 2$ follows from the triangle inequality, since $\lambda^*\lambda$ is a density operator $\|\lambda\|_F \leq 1$. Likewise by Proposition 4.18, for each of the relations $r \in R$ we have that

$$\int_0^{+\infty} \|\varphi(r)\chi_{\geq\sqrt{\alpha}}(\lambda)\|_F^2 d\alpha = \|\varphi(r)\|_\rho^2 \leq O(\epsilon^2).$$

Therefore on average (when summing over all the relations) we observe that

$$\int_0^{+\infty} \left(\sum_{j=1}^{n} \|\chi_{\geq\sqrt{\alpha}}(\lambda) - X_j^*\chi_{\geq\sqrt{\alpha}}(\lambda)X_j\|_F^2 + \sum_{r\in R} \|\varphi(r)\chi_{\geq\sqrt{\alpha}}(\lambda)\|_F^2 \right) d\alpha$$
$$\leq O(\epsilon) \int_0^{+\infty} \operatorname{tr}\left(\chi_{\geq\sqrt{\alpha}}(\lambda)\right) d\alpha,$$

holds for any $\alpha \geq 0$. From the definition of $\chi_{\geq\sqrt{\alpha}}(\lambda)$, we observe that each integrand is zero if $\alpha > \|\lambda\|_{op}^2$. Therefore, there exists an α_0 with $0 \leq \alpha_0 \leq \|\lambda\|_{op}^2$ such that $P := \chi_{\geq\sqrt{\alpha_0}}(\lambda)$ is a non-zero projection. It follows that

$$\sum_{j=1}^{n} \|X_jP - PX_j\|_F^2 + \sum_{r\in R} \|\varphi(r)P\|_F^2 \leq O(\epsilon)\operatorname{tr}(P). \tag{4.8}$$

This bounds each summand on the LHS of (4.8) by $O(\epsilon)\operatorname{tr}(P)$, as all the terms are positive, establishing the claims in equations (4.6) and (4.7).

We now show that the following holds for each $1 \leq j \leq n$:

(a) The operator $\widetilde{X}_j := PX_jP|PX_jP|^{-1}$ is unitary on $Im(P)$ and

$$\|\widetilde{X}_j - X_jP\|_F \leq O(\epsilon^{1/4})\operatorname{tr}(P). \tag{4.9}$$

(b) If $X_{j_1}\cdots X_{j_k}$, for $1 \leq j_1, \ldots, j_k \leq n$, is a word of length k, then

$$\|X_{j_1}\cdots X_{j_k}P - \widetilde{X}_{j_1}\cdots\widetilde{X}_{j_k}\|_F \leq O(\epsilon^{1/4})\operatorname{tr}(P)$$

where the constant depends only on $k \in \mathbb{N}$.

We begin by proving (a). Since $(PX_jP)^* = PX_jP$, we have that

$$\begin{aligned}\|(PX_jP)^2 - P\|_F &= \|PX_jPX_jP - P^3\|_F \\ &\leq \|X_jPX_j - P\|_F \\ &= \|PX_j - X_jP\|_F \\ &\leq O(\epsilon^{1/4})\operatorname{tr}(P)^{1/2},\end{aligned}$$

by recalling that $P = P^*P \leq \mathbb{1}$ and using the unitary invariance of $\|\cdot\|_F$. Observe that $\|PX_jP\|_{op} \leq \|X_j\|_{op} \leq 1$, hence by Proposition 2.40 we conclude that $\|\widetilde{X}_j - PX_jP\|_F \leq O(\epsilon^{1/4})\text{tr}(P)$, thus $\widetilde{X}_j$ is a unitary (by noting it's the unitary part of PX_jP). Before continuing, remark that we can rewrite the above equation to show that

$$\|PX_jP - X_jP\|_F \leq O(\epsilon^{1/4})\text{tr}(P)^{1/2}.$$

Therefore, by the triangle inequality

$$\|\widetilde{X}_j - X_jP\|_F \leq \|\widetilde{X}_j - PX_jP\|_F + \|PX_jP - X_jP\|_F \leq O(\epsilon^{1/4})\text{tr}(P)^{1/2}, \tag{4.10}$$

so the result follows since $(PX_jP)^*(PX_jP)$ is almost the identity on $Im(P)$. For (b), remark that $\widetilde{X}_j = P\widetilde{X}_j$ for all $1 \leq j \leq n$, and therefore

$$\begin{aligned}
&\|X_{j_1}\cdots X_{j_k}P - \widetilde{X}_{j_1}\cdots\widetilde{X}_{j_k}\|_F \\
&\leq \|X_{j_1}\cdots X_{j_k}P - X_{j_1}\cdots X_{j_{k-1}}P\widetilde{X}_{j_k}\|_F + \|X_{j_1}\cdots X_{j_{k-1}}P\widetilde{X}_{j_k} - \widetilde{X}_{j_1}\cdots\widetilde{X}_{j_k}\|_F \\
&\leq \|X_{j_k}P - \widetilde{X}_{j_k}\|_F + \|X_{j_1}\cdots X_{j_{k-1}}P - \widetilde{X}_{j_1}\cdots\widetilde{X}_{j_{k-1}}\|_F \\
&\leq O(\epsilon^{1/4})\text{tr}(P)^{1/2} + \|X_{j_1}\cdots X_{j_{k-1}}P\widetilde{X}_{j_k} - \widetilde{X}_{j_1}\cdots\widetilde{X}_{j_{k-1}}\|_F,
\end{aligned}$$

from which the result follows.

We now conclude the proof by showing that the function $\phi : \mathbb{C}^*\langle X\rangle \to Lin(Im(P))$, sending $x_j \mapsto \widetilde{X}_j$ is an $O(\epsilon^{1/4})$-representation of the game algebra $\mathscr{G}$ on $Im(P) \subset H$ with respect to $\|\cdot\|_f$. We have already seen that $\widetilde{X}_j$ is unitary on $Im(P)$, so all that remains to show is that $\|\widetilde{X}_j^2 - P\|_f \leq O(\epsilon^{1/4})$. This follows from the observation that

$$\begin{aligned}
\|\widetilde{X}_j^2 - (PX_jP)^2\|_F &\leq \|\widetilde{X}_j - PX_jP\|_F\|\widetilde{X}_j + PX_jP\|_F \\
&\leq O(\epsilon^{1/4})\text{tr}(P)^{1/2},
\end{aligned}$$

and that $\|(PX_jP)^2 - P\|_F \leq O(\epsilon^{1/4})\text{tr}(P)^{1/2}$ by Equation (4.9). Hence we have that

(i) $P_i(\widetilde{X}_1, \ldots, \widetilde{X}_n) = -P$ for each $1 \leq i \leq m$, and

(ii) $\widetilde{X}_k\widetilde{X}_j = \widetilde{X}_j\widetilde{X}_k$ for each pair $1 \leq j < k \leq n$ in K_i, for all $1 \leq i \leq m$.

For the remaining relations, the result follows from parts (a) and (b), since each polynomial relation $r \in R$ is a finite sum of monomials. By the triangle inequality, and recalling that $\|X\|_f = \frac{1}{\mathrm{tr}(P)^{1/2}}\|X\|_F$ on $Im(P)$, we see that

$$\begin{aligned}\|\phi(r)\|_f &= \frac{1}{\mathrm{tr}(P)^{1/2}}\|\phi(r)\|_F \\ &\leq \frac{1}{\mathrm{tr}(P)^{1/2}}\left(\|\phi(r)-\varphi(r)P\|_F + \|\varphi(r)P\|_F\right) \\ &\leq O(\epsilon^{1/4})\end{aligned}$$

completing the proof. □

Recall that in the synchronous game case, starting from an ϵ-perfect strategy with an arbitrary state, the rounding lemma ensures that there is an ϵ-representation of the SynchBCS algebra in the $\|\cdot\|_f$-norm. So to apply our rounding result in the synchronous algebra case, we need to ensure that under the $*$-isomorphism in Proposition 3.26, that an ϵ-representation of the synchronous algebra $\mathscr{A}(G_{sync})$ is an $O(\epsilon)$-representation of the SynchBCS algebra in a ρ-norm.

Proposition 4.19. Recall the isomorphism $\phi : \mathscr{A}(G_{sync}) \to \mathscr{B}(G_{sync})$ (from Proposition 3.26). If ψ is an ϵ-representation of $\mathscr{A}(G_{sync})$ in a ρ-norm, then $\phi \circ \psi(p_a^x)$ is an $O(\epsilon)$-representation of $\mathscr{B}(G_{sync})$ with respect $\|\cdot\|_\rho$.

Proof. We let $\{F_a^i\}_{a\in\mathsf{A}} : 1 \leq i \leq m\}$ be the approximate representation of the synchronous algebra. It is easy to see that if $\|{F_a^i}^2 - F_a^i\|_\rho \leq \epsilon$ and $\|{F_a^i}^* - F_a^i\|_\rho \leq \epsilon$, then $\|{F_a^i}^2 - \mathbb{1}\|_\rho \leq 4\epsilon$ and $\|{X_a^i}^* - X_a^i\|_\rho \leq 2\epsilon$. It remains to show that the remaining synchBCS algebra relations hold approximately.

If $V(a,b|i,j) = 0$ and $\|F_a^iF_b^j\|_\rho \leq \epsilon$, we have $X_i^a = \mathbb{1} - 2F_a^x$, hence $\|\widetilde{\mathsf{AND}}(z_a^i, z_b^j) - 1\|_\rho = \|1 - z_a^i - z_b^j + z_a^iz_b^j\|_\rho$, and

$$\begin{aligned}\|\mathbb{1} - X_a^i - X_b^j + X_a^iX_b^j\|_\rho =& 4\left\|\frac{(\mathbb{1}-X_a^i)}{2}\frac{(\mathbb{1}-X_b^j)}{2}\right\|_\rho \\ =& 4\|F_a^iF_b^j\|_\rho \\ \leq& 4\epsilon.\end{aligned}$$

Next, we have

$$
\begin{aligned}
\left\|\prod_{a\in A} X_a^i + \mathbb{1}\right\|_\rho &= \left\|\prod_{a\in A}(\mathbb{1} - 2F_a^i) + \mathbb{1}\right\|_\rho \\
&= \left\|\sum_{\alpha\subset A}(-2)^{|\alpha|}\prod_{a\in\alpha} F_a^i + \mathbb{1}\right\|_\rho \\
&= \left\|\sum_{|\alpha|=1}(-2)F_a^i + 2\mathbb{1} + \sum_{|\alpha|>1}(-2)^{|\alpha|}\prod_{a\in\alpha} F_a^i\right\|_\rho \\
&\leq 2\|\mathbb{1} - \sum_a F_a^i\|_\rho + \sum_{|\alpha|>1} 2^{|\alpha|}\|\prod_{a\in\alpha} F_a^i\|_\rho \\
&\leq 2\epsilon + \sum_{|\alpha|>1} 2^{|\alpha|} \prod_{a''\in\alpha\setminus\{a,a'\}} \|F_{a''}^i\|_{op}\|F_{a'}^i F_a^x\|_\rho \\
&\leq 2\epsilon + \sum_{|\alpha|>1} |2|^{|\alpha|} C^{|\alpha|-1}\|F_a^i F_{a'}^i\|_\rho \\
&\leq 2\epsilon + 2^{2|A|}C^{|A|-1}\epsilon \\
&\leq O(\epsilon),
\end{aligned}
$$

where C is the constant that bounds the operator norm of each F_a^i. Lastly, we ensure that the commutation relation holds.

$$
\begin{aligned}
\|X_a^i X_{a'}^i - X_{a'}^i X_a^i\|_\rho &= \|(\mathbb{1} - 2F_a^x)(\mathbb{1} - 2F_{a'}^i) - (\mathbb{1} - 2F_{a'}^i)(\mathbb{1} - 2F_a^i)\|_\rho \\
&\leq 4(\|F_a^i F_{a'}^i\|_\rho + \|F_{a'}^i F_a^i\|_\rho) \\
&\leq 8\epsilon,
\end{aligned}
$$

as desired. □

Although the above is for arbitrary ϵ-representations, recall that in our case the approximate representation of $\mathscr{A}(G_{sync})$ is *exact* on several of the relations because they are projective measurements. Moreover, if $X_a^i = \mathbb{1} - 2F_a^i$ is the ±1-valued observable assigned to the orthogonal projection onto outcome (i, a) then under the isomorphism in Proposition 3.26 the collection of observables $\{X_a^i\}_{(i,a)\in X\times A}$ are an $O(\epsilon^{1/4})$-representation the SynchBCS algebra $\mathscr{B}(G_{sync})$. It is clear that the relations (r.1), (r.2), (r.4), and (r.5) in Definition 3.25 hold exactly in this approximate representation, since $\mathcal{S}$ is a PVM strategy. Therefore, it only remains to check relation (r.3) in Definition 3.25 holds approximately, which we leave to the reader to verify. Lastly, since the projections satisfy the property in Proposition 4.8 their corresponding observables satisfy the hypothesis of Lemma 4.4 and we have the following corollary.

Corollary 4.20. If $\mathcal{S}$ is an ϵ-perfect strategy to an synchronous nonlocal game game G_{sync}, then the corresponding $O(\epsilon^{1/4})$-representation of the SynchBCS algebra is $O(\epsilon^{1/2}, \lambda)$-tracial.

We can now give the proof of Theorem 1.3.

Proof of Theorem 1.3. The proof follows from Propositions 4.9, 4.11 and 4.13 as we can apply Lemma 4.16 to the approximate representations of the SynchBCS, BCS, and XOR game algebras which arise from optimal strategies for their respective games. In each of these cases we showed that the approximate representations are $O(\epsilon^{1/\gamma_G})$-ATP for suitable constants γ_G that depend on the class of nonlocal game G. In particular, $\gamma_{G_{synch}} = 2$, $\gamma_{G_{bcs}} = 2$, and $\gamma_{G_{xor}} = 4$, which results in the $O(\epsilon^{1/(2\gamma_G)})$-representation for each of the corresponding classes of game. □

4.21 Rounding ϵ-representations to near-optimal strategies

In this section, we will show that the relations forcing the representatives of the game algebra to be quantum measurements are stable. Since BCS and XOR algebras are ℓ_1-bounded, we can use Lemma 2.37 to replace an approximate representation with one satisfying the measurement requirements without significantly altering the quality of the other relations. After making these corrections to the approximate representation, we compute the value for the game G using a maximally entangled state. We begin with the case of ϵ-representations of the synchronous nonlocal game algebra.

Proposition 4.22. Given an ϵ-representation of the synchronous algebra $\mathscr{A}(G_{synch})$, there are is an $O(\epsilon^2)$-perfect strategy $\tilde{\mathcal{S}}$ using a maximally entangled state, and each projective measurement operators is at most $O(\epsilon)$ from $\phi(p_a^x)$ for all $a \in \mathsf{A}$, $x \in \mathsf{X}$ with respect to $\|\cdot\|_f$.

Proof. By the stability of the PVM algebra in Corollary 2.49, for any ϵ-representation of the synchronous algebra there is an $O(\epsilon)$-representation where all "approximate projections" for inputs $x, y \in X$ can be replaced by self-adjoint projections that sum to the identity. Hence,

$$\begin{aligned}
\|\overline{\tilde{P}^x_a}\tilde{Q}^y_b\|_f &\leq \|\overline{\tilde{P}^x_a}\tilde{Q}^y_b - P^i_a\tilde{Q}^y_b\|_f + \|P^x_a\tilde{Q}^y_b - P^x_aQ^y_b\|_f + \|P^x_aQ^y_b\|_f \\
&\leq \|\overline{\tilde{P}^x_a} - P^x_a\|_f\|\tilde{Q}^y_b\|_{op} + \|P^x_a\|_{op}\|\tilde{Q}^y_b - Q^y_b\|_f + \|P^x_aQ^y_b\|_f \\
&\leq \|\overline{\tilde{P}^x_a} - P^x_a\|_f + \|\tilde{Q}^y_b - Q^y_b\|_f + \epsilon \\
&\leq 3\epsilon.
\end{aligned}$$

Therefore, $\|\overline{\tilde{P}^x_a}\tilde{Q}^y_b\|_f \leq O(\epsilon)$ whenever $V(x, y, a, b) = 0$.

It follows that the strategy with a maximally entangled state $\mathcal{S} = (\{\{\tilde{P}^x_a\}_{a\in\mathsf{A}} : x \in \mathsf{X}\}, \{\{\tilde{Q}^y_b\}_{b\in\mathsf{B}} : y \in \mathsf{Y}\}, |\psi\rangle)$ is $O(\epsilon^2)$-perfect, since the probability of losing on each input is

$$\begin{aligned}
\sum_{a,b:V(x,y,a,b)=0} p(a, b|x, y) &= \sum_{a,b:V(x,y,a,b)=0} \langle\psi|\tilde{P}^x_a \otimes \tilde{Q}^y_b|\psi\rangle \\
&= \sum_{a,b:V(x,y,a,b)=0} \frac{1}{d}\mathrm{tr}(\overline{\tilde{P}^x_a}\tilde{Q}^y_b) \\
&= \sum_{a,b:V(x,y,a,b)=0} \frac{1}{d}\mathrm{tr}\left((\overline{\tilde{P}^x_a})^*\overline{\tilde{P}^x_a}\tilde{Q}^y_b(\tilde{Q}^y_b)^*\right) \\
&= \sum_{a,b:V(x,y,a,b)=0} \|\overline{\tilde{P}^i_a}\tilde{Q}^y_b\|^2_f \\
&\leq |A|^2 9\epsilon^2 \\
&\leq O(\epsilon^2).
\end{aligned}$$

□

Remark 4.23. If we applied our rounding result to the SynchBCS algebra and we wanted to correct an approximate representation of the synchronous algebra in the $\|\cdot\|_f$-norm to a strategy. We would first need to ensure that the isomorphism described in Proposition 3.26 does not significantly alter the quality of the approximate representation $\varphi : \mathscr{B}(G_{synch}) \to \mathscr{A}(G_{synch})$. That is we need to check that any ϵ-representation of the SynchBCS algebra in the $\|\cdot\|_f$-norm remains an $O(\epsilon)$-representation under the isomorphism $\varphi : \mathscr{B}(G_{synch}) \to \mathscr{A}(G_{synch})$.

Proposition 4.24. If ψ is a unitary ϵ-representation of $\mathscr{B}(G_{synch})$ in the $\|\cdot\|_f$-norm, then under the isomorphism φ in Proposition 3.26 $\psi \circ \varphi$ is an $O(\epsilon)$-representation of $\mathscr{A}(G_{synch})$ with respect to $\|\cdot\|_f$.

Before we prove Proposition 4.24, we require a few technical results.

Lemma 4.25. Let $X_1, \ldots, X_n \in M_d(\mathbb{C})$. If $\|X_i\|_{op} \le C$ and $\|X_iX_j - X_jX_i\|_f \le \epsilon$ for all $1 \le i < j \le n$. Then there exists a constant $C_1 > 0$ (depending on C) such that the monomial

$$\|X_{i_1} \cdots X_{i_j} \cdots X_{i_k} \cdots X_{i_n} - X_{i_1} \cdots X_{i_j} X_{i_k} \cdots X_{i_n}\|_f \le C_1\epsilon$$

for all $1 \le j \le k \le n$

Proof. We proceed by induction on $|k-j|$. When X_j and X_k only differ by 2 positions in the monomial let $k = j+2$. In this case, we see that

$$\begin{aligned}
&\|X_{i_1} \cdots X_{i_j} X_{i_{j+1}} X_{i_{j+2}} \cdots X_{i_n} - X_{i_1} \cdots X_{i_{j+1}} X_{i_j} X_{i_{j+2}} \cdots X_{i_n}\|_f \\
\le&\|X_{i_1} \cdots X_{i_{j-1}}\|_{op} \|X_{i_j} X_{i_{j+1}} - X_{i_{j+1}} X_{i_j}\|_f \|X_{i_{j+2}} \cdots X_{i_n}\|_{op} \\
\le&C^{n-2}\epsilon.
\end{aligned}$$

Now suppose it holds for $1 \le j \le n$ and $k = j + \ell - 1$ with constant $C^{n-\ell-1} > 0$, and consider

$$\begin{aligned}
&\|X_{i_1} \cdots X_{i_j} \cdots X_{i_{j+\ell}} \cdots X_{i_n} - X_{i_1} \cdots X_{i_{j+\ell-1}} X_{i_j} X_{i_{j+\ell}} \cdots X_{i_n}\|_f \\
\le&\|X_{i_1} \cdots X_{i_j} \cdots X_{i_{j+\ell}} \cdots X_{i_n} - X_{i_1} \cdots X_{i_j} X_{i_{j+\ell-1}} \cdots X_{i_n}\|_f \\
+&\|X_{i_1} \cdots X_{i_j} X_{i_{j+\ell-1}} X_{i_{j+\ell}} \cdots X_{i_n} - X_{i_1} \cdots X_{i_{j+\ell-1}} X_{i_j} X_{i_{j+\ell}} \cdots X_{i_n}\|_f \\
\le&C^{n-\ell-1}\epsilon + \|X_{i_1} \cdots X_{i_{j-1}}\|_{op} \|X_{i_j} X_{i_{j+\ell-1}} - X_{i_{j+\ell-1}} X_{i_j}\|_f \|X_{i_{j+\ell}} \cdots X_{i_n}\|_{op} \\
\le&C^{n-\ell-1}\epsilon + C^{n-\ell}\epsilon.
\end{aligned}$$

Letting $C_1 = 2\max\{C^{n-\ell-1}, C^{n-\ell}\}$ completes the proof. ∎

Proposition 4.26. Let $X_1, \ldots, X_n \in M_d(\mathbb{C})$ such that $\|X_i\|_{op} \le C$, $\|X_i^2 - \mathbb{1}\|_f \le \epsilon$ for $1 \le i \le n$, and $\|X_iX_j - X_jX_i\|_f \le \epsilon$ for all $1 \le i < j \le n$. Then, there exists a $\widetilde{C} > 0$ such that

$$\|(X_1 \cdots X_n)^2 - \mathbb{1}\|_f \le \widetilde{C}\epsilon.$$

Proof. Our proof follows from two claims. The first is that there is a constant C_0 such that

$$\|(X_1 \cdots X_n)^2 - X_1^2 \cdots X_n^2\|_f \le C_0\epsilon.$$

Consider,

$$\|X_1 \cdots X_n X_1 \cdots X_n - X_1^2 \cdots X_n^2\|_f$$

by lemma 4.25 we have

$$\begin{aligned}
&\|X_1 \cdots X_n X_1 \cdots X_n - X_1^2 \cdots X_n^2\|_f \\
\leq &\|X_1 \cdots X_n X_1 \cdots X_n - X_1 \cdots X_{n-1} X_1 \cdots X_n^2\|_f \\
+ &\|X_1 \cdots X_{n-1} X_1 \cdots X_{n-1} - X_1^2 \cdots X_{n-1}^2\|_f \|X_n^2\|_{op} \\
\leq &C_2 \epsilon + C^2 \epsilon \|X_1 \cdots X_{n-1} X_1 \cdots X_{n-1} - X_1^2 \cdots X_{n-1}^2\|_f.
\end{aligned}$$

Iteratively applying Lemma 4.25 establishes the first claim. Secondly, we claim that there is a constant $C_2 > 0$ such that

$$\|X_1^2 \cdots X_n^2 - \mathbb{1}\|_f \leq C_2 \epsilon.$$

This follows by noting that

$$\begin{aligned}
&\|X_1^2 \cdots X_n^2 - \mathbb{1}\|_f \\
\leq &\|X_1^2 \cdots X_n^2 - X_1^2 \cdots X_{n-1}^2\|_f + \|X_1^2 \cdots X_{n-1}^2 - \mathbb{1}\|_f \\
\leq &\|X_1^2 \cdots X_{n-1}^2\|_{op} \|X_n^2 - \mathbb{1}\|_f + \|X_1^2 \cdots X_{n-1}^2 - \mathbb{1}\|_f \\
\leq &C^{2(n-1)} \epsilon + \|X_1^2 \cdots X_{n-1}^2 - \mathbb{1}\|_f,
\end{aligned}$$

and therefore an inductive argument works here also. Combining our two claims we see that

$$\|(X_1 \cdots X_n)^2 - \mathbb{1}\|_f \leq \|(X_1 \cdots X_n)^2 - X_1^2 \cdots X_n^2\|_f + \|X_1^2 \cdots X_n^2 - \mathbb{1}\|_f \leq \widetilde{C} \epsilon,$$

as desired. □

Proposition 4.27. For a set A, let X_a be a unitary such that $\|X_a^2 - \mathbb{1}\|_f \leq \epsilon$, and $\|X_a X_b - X_b X_a\|_f \leq \epsilon$ for all $a \neq b \in A$. If

$$\left\| \prod_{a \in A} X_a + \mathbb{1} \right\|_f \leq \epsilon,$$

then there exists a constant $\tilde{C} > 0$ such that

$$\left\| \prod_{a \in S} X_a + \prod_{a \in A \setminus S} X_a \right\|_f \leq \tilde{C} \epsilon$$

for any $S \subseteq A$.

Proof. Let S be a subset of A. By the unitary invariance of $\|\cdot\|_f$ we see that

$$\begin{aligned}\left\|\prod_{a\in S}X_a+\prod_{a\in A\setminus S}X_a\right\|_f &= \left\|\prod_{a\in S}X_a\prod_{a\in A\setminus S}X_a+\left(\prod_{a\in A\setminus S}X_a\right)^2\right\|_f \\ &= \left\|\prod_{a\in A}X_a+\mathbb{1}\right\|_f+\left\|\left(\prod_{a\in A\setminus S}X_a\right)^2-\mathbb{1}\right\|_f \\ &\leq \epsilon+C'\epsilon,\end{aligned}$$

by Proposition 4.26, and the result follows with letting $\tilde{C}=(C'+1)$. □

Proof of Proposition 4.24. It is straightforward to verify that if ψ is an ϵ-representation of $\mathscr{B}(G_{sync})$ with respect to $\|\cdot\|_f$, then the relations in the synchronous game algebra are bounded by a constant times ϵ in $\|\cdot\|_f$ under $\varphi:\mathscr{B}(G_{sync})\to\mathscr{A}(G_{sync})$ sending $\psi(z_a^i)\mapsto(1-\psi(z_a^i))/2$. It only remains to ensure that the 3rd (completeness) relation holds approximately.

Let $\phi(z_a^i)=X_a^i$ so that $\varphi\circ\phi(z_a^i)=(\mathbb{1}-X_a^i)/2$, then

$$\begin{aligned}\left\|\mathbb{1}-\sum_{a\in A}\varphi(\psi(z_a^i))\right\|_f &= \left\|\mathbb{1}-\sum_{a\in A}\frac{(\mathbb{1}-X_a^i)}{2}\right\|_f \\ &= \left\|\sum_{(e_1,\ldots,e_n)\in\{\pm1\}^n}\prod_{a\in A}\frac{(1+v_aX_a^i)}{2}-\sum_{a\in A}\frac{(\mathbb{1}-X_a^i)}{2}\right\|_f \\ &= \left\|\sum_{(v_1,\ldots,v_n)\in\{\pm1\}^n}\prod_{a\in A}\frac{(1+v_aX_a^i)}{2}-\sum_{a\in A}\frac{(\mathbb{1}-X_a^i)}{2}\right\|_f.\end{aligned}$$

There are three cases to consider in the sum of the elements $v=(v_1,\ldots,v_n)\in\{\pm1\}^n$: when v is completely trivial, when v has exactly one nontrivial v_i, and when there is more than one nontrivial v_i. In the first and last case, we will see that these contributions are bounded by $\tilde{C}\epsilon$. For the first case, we follow the same approach as in Proposition 3.26, expanding and then applying Proposition 4.27 to obtain

$$\left\|\frac{1}{2^n}\sum_{S\subseteq[n]}\prod_{a\in S}z_a^i\right\|_f \leq \frac{1}{2^n}\sum_{S\subseteq[n/2]}\left\|\prod_{a\in S}z_a^i+\prod_{a\in A\setminus S}z_a^i\right\|_f$$
$$\leq \frac{2^{n/2}}{2^n}C\epsilon$$
$$\leq \frac{1}{2^{n/2}}C\epsilon.$$

We now consider the case where v contains at least one nontrivial v_i. Let $F_a^i = (1 - X_a^i)/2$ so that $\tilde{F}_a^i = 1 - F_a^i = (1 + X_a^i)/2$, and recall that the $\widetilde{\mathsf{AND}}$ relation implies that $\|F_a^i F_{a'}^i\|_f \leq 4\epsilon$. Since for nearly orthogonal projections we have that $\|F_a^i(1 - F_a^i)\|_f = \|F_a^i - F_a^i F_{a'}^i\|_f \leq \|F_a^i\|_f + 4\epsilon$. In any term where there are at least two nontrivial elements in v, we have a monomial where at least two projections are almost orthogonal. Remarking that the projections F_a^i almost commute, the result follows from Lemma 4.25 by noting that each term is $O(\epsilon)$ away from a term in the $\|\cdot\|_f$-norm (where the two almost orthogonal projections are adjacent in the monomial). Then by the bimodule property, we see that these terms are at most $O(\epsilon)$.

For the remaining case, the terms are sufficiently close to those of the form

$$\sum_{a\in A}\frac{(1-X_a^i)}{2}\prod_{a'\neq a\in A}\frac{(1+X_{a'}^i)}{2}$$

and therefore we only need to show that

$$\left\|\sum_{a\in A}\frac{(1-X_a^i)}{2}\prod_{a'\neq a\in A}\frac{(1+X_{a'}^i)}{2}-\sum_{a\in A}\frac{(\mathbb{1}-X_a^i)}{2}\right\|_f \leq C'\epsilon.$$

Hence,

$$
\begin{aligned}
&\left\| \sum_{a\in A} \frac{(1-X_a^i)}{2} \prod_{a'\neq a\in A} \frac{(1+X_{a'}^i)}{2} - \sum_{a\in A} \frac{(1-X_a^i)}{2} \right\|_f \\
&\leq \sum_{a\in A} \left\| F_a^i \prod_{a'\neq a\in A} (1-F_{a'}^i) - F_a^i \right\|_f \\
&\leq \sum_{a\in A} \left\| F_a^i \sum_{S\subseteq A\setminus\{a\}} \prod_{a'\in S} F_{a'}^i - F_a^i \right\|_f \\
&\leq \sum_{\emptyset\neq S\subseteq A\setminus\{a\}} \left\| F_a^i \prod_{a'\in S} F_{a'}^i \right\|_f \\
&\leq \sum_{\emptyset\neq S\subseteq A\setminus\{a\}} C^{|S|-1} \|F_{a''}^i F_{a'}^i\|_f \\
&\leq \sum_{\emptyset\neq S\subseteq A\setminus\{a\}} C^{|S|-1} 4\epsilon,
\end{aligned}
$$

and the result follows. □

Our next case is that ϵ-representations of the BCS algebra in the $\|\cdot\|_f$-norm give near-perfect strategies using the maximally entangled state. For this result we rely on the stability of the group algebra $\mathbb{C}[\mathbb{Z}_2^k]$, which was shown in [Slo19b].

Proposition 4.28. If ϕ is an ϵ-representation of the BCS algebra $\mathscr{B}(G_{bcs})$ on a finite-dimensional Hilbert space H_B, then there is a $O(\epsilon^2)$-perfect strategy to the BCS game using a maximally entangled state $|\tilde{\psi}\rangle \in H_A \otimes H_B$.

Proof. Let ϕ be an ϵ-representation of $\mathscr{B}(G_{bcs})$. For a fixed $1 \leq i \leq m$ consider the ϵ-representation restricted to K_i. On the subset ϕ is an ϵ-representation of $\mathbb{Z}_2^k$ for some $k = |K_i|$. Since the group algebra $\mathbb{C}[\mathbb{Z}_2^k]$ is stable with respect to unitary matrices in the little Frobenius norm, there is a unitary representation φ of $\mathbb{Z}_2^k$ such that $\|\varphi(x_j) - \phi(x_j)\|_f \leq O(\epsilon)$ for all $1 \leq j \leq k$. Therefore by Lemma 2.37, we can replace the $K_i \subset S$ of our representation ϕ to be exact on all the $x_j \in K_i$. This new approximate representation η is an $\tilde{C}C\epsilon + \epsilon = (\tilde{C}C + 1)\epsilon$-representation of $\mathscr{B}(G_{bcs})$, where $\tilde{C}$ depends on $k \leq n$.

Now if φ is the map sending $x_j \mapsto Y_j$ and η maps $x_j \mapsto Z_{ij}$ for all K_i, then $\|Y_j - Z_{ij}\|_f \leq O(\epsilon)$ for all $x_j \in K_i$. If Alice's strategy consists of these Z_{ij}'s, while Bob employs the strategy consisting of the Y_j's, the ϵ-representation $x_j \mapsto Y_j$ can be replaced with a self-adjoint unitary X_j such that $\|X_j - Y_j\|_f \leq O(\epsilon)$. Hence by Lemma 4.4 the strategy is $O(\epsilon^2)$-perfect, since $\|X_j - Z_{ij}\|_f \leq \|X_j - Y_j\|_f + \|Y_j - Z_{ij}\|_f \leq 2\epsilon$. □

Our last objective is to determine the optimality of strategies arising from ϵ-representations in the $\|\cdot\|_f$-norm of the XOR algebra.

Proposition 4.29. Let G_{xor} be an XOR game. Given a ϵ-representations $\phi : \mathscr{C}(G_{xor}) \to Lin(H_B)$ of the XOR algebra $\mathscr{C}(G_{xor})$ with respect to $\|\cdot\|_f$, there is an $O(\epsilon)$-optimal strategy for the corresponding XOR game using the maximally entangled state on $H_A \otimes H_B$.

Proof. Let ϕ be an ϵ-represenation of $\mathscr{C}(G_{xor})$. Let $\{X_j\}_{1\leq j\leq n}$ be the measurement operators arising from the representatives $\{\phi(x_j)\}_{1\leq j\leq n} \in Lin(H_B)$ in the sense that for each $1 \leq j \leq n$. By Lemma 2.43 we can find a self-adjoint unitary X_j such that $\|X_j - \phi(x_j)\|_f \leq O(\epsilon)$.

Define

$$\overline{Y_i} = \frac{1}{r_i}\sum_j w_{ij} X_j$$

for each $1 \leq i \leq m$. Since

$$\|\overline{Y_i}^2 - \mathbb{1}\|_f = \frac{1}{|r_i^2|}\left\|\left(\sum_j w_{ij}X_j\right)^2 - r_i^2\mathbb{1}\right\|_f \leq O(\epsilon),$$

we can again use Lemma 2.43 to find a self-adjoint unitary $\overline{Z}_i$ such that $\|\overline{Z}_i - \overline{Y}_i\|_f \leq O(\epsilon)$. Then the strategy where Alice employs the operators $\{Z_i\}_{1\leq i\leq m} \in Lin(H_A)$ and Bob employs $\{X_j\}_{1\leq j\leq n} \in Lin(H_B)$ and $|\psi\rangle \in H_A \otimes H_B$ is maximally entangled we have that

$$\begin{aligned}
|\beta^*(G_{xor}) - \beta(\mathcal{S}; G_{xor})| &= \left| \sum_i r_i - \sum_{ij} w_{ij} \langle\psi| Z_i \otimes X_j |\psi\rangle \right| \\
&= \left| \sum_i r_i - \sum_i r_i \langle\psi| Z_i \otimes \overline{Y_i} |\psi\rangle \right| \\
&\leq \sum_i |r_i| \left| 1 - \langle\psi| \mathbb{1} \otimes \overline{Y}_i \overline{Z}_i |\psi\rangle \right| \\
&\leq \sum_i |r_i| \left| \langle\psi| \mathbb{1} \otimes \mathbb{1} - \mathbb{1} \otimes \overline{Y}_i \overline{Z}_i |\psi\rangle \right| \\
&\leq \sum_i |r_i| \left\| \mathbb{1} - \overline{Y}_i \overline{Z}_i \right\|_f \\
&= \sum_i |r_i| \left\| \overline{Z}_i - \overline{Y}_i \right\|_f \\
&\leq O(\epsilon),
\end{aligned}$$

as desired. □

Proof of Theorem 1.4. The result is established by combining Propositions 4.22, 4.28, and 4.29. □

4.29.1 Distance between the rounded strategy and the original strategy

We end with a discussion of Corollary 1.5. In particular we are interested in the following question. In what sense is the "rounded" strategy with a maximally entangled state *close* to the original strategy with an arbitrary quantum state? First, let us write the proof.

Proof of Corollary 1.5. The result follows from the Theorem 1.3, which shows that near-optimal strategies can be rounded to approximate representation in the little Frobenius norm. The quality of the resulting near-optimal strategy using the maximally entangled state then follows from Theorem 1.4. It only remains to show, that under the support of the employed maximally entangled state $|\tilde{\psi}\rangle$, the operators are close in the $\|\cdot\|_f$-norm.

In the proof of Lemma 4.16, we see that on the support of P, each unitary $\widetilde{X}_j$ is close to the starting unitary X_j (see Equation (4.10)). It follows that the measurement operators in the "rounded" quantum strategy is $O(\epsilon^{1/4})$ away from the initial measurement operators with respect to the little Frobenius norm on the subspace $\widetilde{H}$. Since each measurement operator remains close to the operators in the approximate representation, the result follows. □

A limitation of our technique is that it does not seem to ensure a dimension independent ϵ-dependence on the distance between the states $|\psi\rangle$ and $|\tilde{\psi}\rangle$. We conclude with a simple upper bound on the distance between the states that depends on the dimension.

Proposition 4.30. Let G be a nonlocal game and $\mathcal{S}$ an ϵ-optimal strategy employing the quantum state $|\psi\rangle \in H \otimes H$, and $H \cong \mathbb{C}^d$ for some d. If $\tilde{\mathcal{S}}$ is the resulting $O(\epsilon^{1/\gamma})$-optimal strategy with the maximally entangled state $|\tilde{\psi}\rangle \in \widetilde{H} \otimes \widetilde{H}$, then

$$\||\psi\rangle - |\tilde{\psi}\rangle\| = \|\lambda - \tilde{P}\|_F \leq \sqrt{2}\left(1 - \sqrt{\frac{r}{d}}\right)^{\frac{1}{2}}$$

where $\tilde{P} = \frac{1}{\sqrt{\dim(\widetilde{H})}} P$ is the normalized projection onto $\widetilde{H}$.

Proof. It is clear that $\||\psi\rangle - |\tilde{\psi}\rangle\|^2 = \|\lambda - \tilde{P}\|_F^2$ as λ and $\tilde{P}$ are the reduced density matrices of $|\psi\rangle$ and $|\tilde{\psi}\rangle$ respectively. The proof proceeds as it would for any rank r spectral projection of λ with $\mathrm{tr}(\lambda^2) = 1$. If $rank(\lambda) = d$ and $rank(\tilde{P}) = r \leq d$, with r eigenvalues equal to $1/\sqrt{r}$, then

$$\mathrm{tr}(\lambda\tilde{P}) = \frac{1}{\sqrt{r}} \sum_{k=1}^{r} \lambda_k \geq \frac{1}{\sqrt{r}} \frac{r}{\sqrt{d}} = \sqrt{\frac{r}{d}}.$$

Hence, we see that

$$\begin{aligned}\|\lambda - \tilde{P}\|_F^2 =& \mathrm{tr}(\lambda^2) - 2\mathrm{tr}(\lambda\tilde{P}) + \mathrm{tr}(\tilde{P}^2) \\ =& 2(1 - \mathrm{tr}(\lambda\tilde{P})) \\ \leq& 2\left(1 - \sqrt{\frac{r}{d}}\right).\end{aligned}$$

□

Chapter 5

Satisfiability problems and algebras of binary constraint system games

This Chapter contains results that are based on [PS23].

5.1 Introduction

In this chapter we revisit boolean constraint systems algebras. Section 5.2 reviews some algebraic preliminaries, particularly the representation theory of $\mathbb{C}\mathbb{Z}_2^k$. In Section we give a different definition of BCS algebras than the one that appeared in Chapter 3 and show that these two definitions are equivalent. We also introduce the types of generalized satisfiability for BCS algebras and review some known examples. In Section we explain definability for a BCS with contexts and a boolean constraint language. In this section we also prove our main lemma: that definability induces algebraic transformations between BCS-algebras from boolean constraint languages. We explain the connection and consequences of this result in the context of Schaefer's dichotomy theorem and the class of LIN BCS. Section 5.26 provides an example of BCS that is C^*-satisfiable but not tracially satisfiable. The construction is based on the Mermin-Peres magic square and the idea that in BCS algebras, it is easy to find relations that enforce specific global algebraic properties on the algebra.

5.2 Preliminaries and notation

5.2.1 Finitely-presented $*$-algebras

For a set X, let $\mathbb{C}^*\langle X\rangle$ be the free unital complex $*$-algebra generated by X. If $R \subseteq \mathbb{C}^*\langle X\rangle$, let $\mathbb{C}^*\langle X : R\rangle$ denote the quotient of $\mathbb{C}^*\langle X\rangle$ by the two-sided ideal $\langle\langle R\rangle\rangle$ generated by R. If X and R are finite, then $\mathbb{C}^*\langle X : R\rangle$ is said to be a finitely-presented $*$-algebra.

If W is a complex vector space, then $\mathrm{End}(W)$ will denote the space of linear operators from W to itself. We use $\mathbb{1}_W$ for the identity operator on W. A representation of a $*$-algebra $\mathscr{A}$ is an algebra homomorphism $\phi : \mathscr{A} \to \mathrm{End}(W)$ for some vector space W. A subrepresentation is a non-zero subspace $K \subseteq W$ such that $\phi(a)K \subseteq K$ for all $a \in \mathscr{A}$, and a representation is irreducible if it has no subrepresentations. A $*$-representation of $\mathscr{A}$ is a $*$-homomorphism $\phi : \mathscr{A} \to \mathcal{B}(H)$, where H is a Hilbert space, and $\mathcal{B}$ is the $*$-algebra of bounded operators on H.

If $\mathscr{A} = \mathbb{C}^*\langle X : R\rangle$ is a presentation of a $*$-algebra, and $\mathscr{B}$ is another $*$-algebra, then $*$-homomorphisms $\mathscr{A} \to \mathscr{B}$ correspond to $*$-homomorphisms $\phi : \mathbb{C}^*\langle X\rangle \to \mathscr{B}$ such that $\phi(r) = 0$ for all $r \in R$. Hence a $*$-representation of $\mathscr{A}$ is an assignment of operators to the elements of X, such that the operators satisfy the defining $*$-relations in R, and we often work with representations in these terms. An element x of a $*$-algebra algebra $\mathscr{A}$ is said to be **positive**, written $x \geq 0$, if $x = \sum_{i=1}^{k} s_i^* s_i$ for some $k \geq 1$ and $s_1, \ldots, s_k \in \mathscr{A}$. The algebra $\mathscr{A}$ is said to be a **semi-pre-C^*-algebra** if for all $x \in \mathscr{A}$, there is a scalar $\lambda \geq 0$ such that $x^*x \leq \lambda$ [Oza13a]. All the $*$-algebras we work with will be semi-pre-C^*-algebras. A **state** on a semi-pre-C^*-algebra $\mathscr{A}$ is a linear functional $f : \mathscr{A} \to \mathbb{C}$ such that $f(x^*) = \overline{f(x)}$ for all $x \in \mathscr{A}$, $f(x) \geq 0$ for all $x \geq 0$, and $f(1) = 1$. If $\phi : \mathscr{A} \to \mathcal{B}(H)$ is a $*$-representation of $\mathscr{A}$, and $|v\rangle \in H$ is a unit vector, then $x \mapsto \langle v|\phi(x)|v|v|\phi(x)|v\rangle$ is a state. Conversely, if f is a state then by the GNS representation theorem, there is a $*$-representation $\phi : \mathscr{A} \to \mathcal{B}(H)$ and a unit vector $|v\rangle \in H$ such that $f(x) = \langle v|\phi(x)|v|v|\phi(x)|v\rangle$ for all $x \in \mathscr{A}$. Hence a semi-pre-C^*-algebra $\mathscr{A}$ has a state if and only if it has a $*$-representation. A state f on $\mathscr{A}$ is **tracial** if $f(ab) = f(ba)$.

5.2.2 The joint spectrum and representations of $\mathbb{C}\mathbb{Z}_2^k$

Consider the finitely presented group

$$\mathbb{Z}_2^k = \langle z_1, \ldots, z_k : z_i^2 = 1, 1 \leq i \leq k, z_i z_j = z_j z_i, 1 \leq i \neq j \leq k\rangle.$$

The group algebra $\mathbb{C}\mathbb{Z}_2^k$ is the $*$-algebra generated by $z_1, \ldots, z_k$, with defining relations from the group presentation of $\mathbb{Z}_2^k$ above, along with the relations $z_i^* z_i = z_i z_i^* = 1$ for all $1 \leq i \leq k$. Hence a $*$-representation of $\mathbb{C}\mathbb{Z}_2^k$ is a collection of unitary operators $Z_1, \ldots, Z_k$ such that $Z_i^2 = \mathbb{1}$ and $Z_i Z_j = Z_j Z_i$ for all $1 \leq i, j \leq k$. The irreducible $*$-representations λ of $\mathbb{C}\mathbb{Z}_2^k$ are one-dimensional, and are determined via the vectors $v = (\lambda(z_1), \ldots, \lambda(z_k)) \in \{\pm 1\}^k$. Conversely, for any vector $v \in \{\pm 1\}$ there is a representation $\lambda_v : \mathbb{Z}_2^k \to \mathbb{C}$ with $\lambda(x_i) = v_i$, so $\mathbb{Z}_2^k$ has 2^k irreducible representations (up to isomorphism). If $v \in \{\pm 1\}^k$, we let

$$\Pi_v = \prod_{i=1}^{k} \tfrac{1}{2}(1 + v_i z_i) = \frac{1}{2^k} \sum_{x \in \mathbb{Z}_2^k} \lambda_v(x) x$$

be the central projection in $\mathbb{C}\mathbb{Z}_2^k$ corresponding to λ_v. These projections satisfy the identities $\Pi_v^* = \Pi_v = \Pi_v^2$ and $\sum_{v \in \{\pm 1\}^k} \Pi_v = 1$ in $\mathbb{C}\mathbb{Z}_2^k$. Hence if $\psi : \mathbb{C}\mathbb{Z}_2^k \to \mathrm{End}(W)$ is a $*$-representation, the operators $\{\psi(\Pi_v)\}_{v \in \{\pm 1\}^k}$ form a complete orthonormal family of projections. In particular,

$$W = \bigoplus_{v \in \{\pm 1\}^k} \psi(\Pi_v) W$$

as an orthogonal direct sum. If $x \in \mathbb{C}\mathbb{Z}_2^k$ and $w \in W_v := \psi(\Pi_v) W$, then $\psi(x) w = \lambda(x) w$. Hence $\psi(x) = \oplus_{v \in \{\pm 1\}^k} \lambda_v(x) \mathbb{1}_{W_v}$ for all $x \in \mathbb{C}\mathbb{Z}_2^k$, so ψ is diagonal with respect to this subspace decomposition. This leads to the following definition:

Definition 5.3. Let $\psi : \mathbb{C}\mathbb{Z}_2^k \to \mathrm{End}(W)$ be a $*$-representation of $\mathbb{C}\mathbb{Z}_2^k$ on a vector space W. The **joint spectrum** of ψ is the set

$$\mathcal{J}_\psi = \{v \in \{\pm 1\}^k : \psi(\Pi_v) \neq 0\},$$

In other words, $\mathcal{J}_\psi$ is the set of vectors $v \in \{\pm 1\}^k$ for which the subspace W_v is non-zero.

5.3.1 Binary constraint systems

To match with conventions from the previous section, in this paper we represent $\mathbb{Z}_2$ in multiplicative form as $\{\pm 1\}$, rather than $\{0, 1\}$. We also use this convention for boolean truth values, meaning we think of -1 as TRUE and 1 as FALSE. A **binary relation of arity** $k > 0$ is a subset of $\{\pm 1\}^k$. The **indicator function** of a relation $R \subset \{\pm 1\}^k$ is the function $f_R : \{\pm 1\}^k \to \{\pm 1\}$ sending $x \mapsto -1$ if $x \in R$, and $x \mapsto 1$ otherwise. Given a set of variables $X = \{x_1, \ldots, x_n\}$, a **constraint** C on X is a pair (S, R), where the **scope** $S = (s_1, \ldots, s_k)$ is a sequence of length $k \geq 1$ over $X \cup \{\pm 1\}$, and R is a k-ary relation.

A **binary constraint system (BCS)** is a pair $(X, \{C_i\}_{i=1}^m)$, where X is a finite set of variables, and $\{C_i\}_{i=1}^m$ is a finite set of constraints on X.

For practical reasons, we will often write relations and constraints informally in the standard short-hand, using $\vee$ for logical OR and $\wedge$ for logical AND. For instance, $x_1 \vee x_2 \vee x_3 = \text{TRUE}$ could refer to the relation $R = \{\pm 1\}^3 \setminus \{(0,0,0)\}$, or to the constraint $((x_1, x_2, x_3), R)$. Also, if S is a scope, we abuse notation slightly and use $X \cap S$ to refer to the set of variables listed in S. Note a peculiarity of using multiplicative notation for $\mathbb{Z}_2$ is that the XOR $x_1 \oplus x_2$ is written as the product $x_1 x_2$, so for instance $\{x_1 x_2 x_3 = -1, x_2 x_3 = 1, x_1 x_2 = 1\}$ is actually a linear system, despite initial appearances.

An **assignment** to a set of variables X is a function $\phi : X \to \{\pm 1\}$. If $S = (s_1, \ldots, s_k)$ is a sequence over $X \cup \{\pm 1\}$, we set $\phi(S) = (\phi(s_1), \ldots, \phi(s_k)) \in \{\pm 1\}^k$, where we extend ϕ to $\{\pm 1\}$ as the identity function. If $(X, \{(S_i, R_i)\}_{i=1}^m)$ is a BCS, then an assignment ϕ to X is a **satisfying assignment** if $\phi(S_i) \in R_i$ for all $1 \leq i \leq m$, or equivalently if $f_{R_i}(\phi(S_i)) = -1$ for all $1 \leq i \leq m$. An assignment which is not a satisfying assignment will be called a **non-satisfying assignment**. A BCS is said to be **satisfiable** if it has a satisfying assignment.

A **boolean constraint language** $\mathcal{L}$ is a collection of relations with possibly different arity's. We say that a BCS B is a **BCS over** $\mathcal{L}$ if every relation in B belongs to $\mathcal{L}$. Constraint languages allow us to talk about constraint systems where the relations are of a certain form.

5.4 Binary constraint system algebras and games

To define boolean constraint system algebras, we first extend the definition of a binary constraint system slightly:

Definition 5.5. A **boolean constraint system (BCS) with contexts** is a tuple $(X, \{(U_i, V_i)\}_{i=1}^{\ell})$, where X is a finite set of variables, and (U_i, V_i) is a constraint system on variables $U_i \subseteq X$ for all $1 \leq i \leq \ell$. The sets U_i, $1 \leq i \leq \ell$ are called the **contexts** of the system.

If $\mathcal{L}$ is a constraint language, then a **BCS with contexts over** $\mathcal{L}$ is a BCS with contexts $(X, \{(U_i, V_i)\}_{i=1}^{\ell})$ in which (U_i, V_i) is a BCS over $\mathcal{L}$ for all $1 \leq i \leq \ell$.

Intuitively, the idea behind this definition is that variables are grouped into contexts, and constraints can only be placed on variables in the same context. Given a BCS $B =$

(X, V), we can always add contexts to make it into a BCS with contexts. Typically there is more than one way that this can be done. For instance, we could group all the variables together into a single context to get $(X, \{(X, V)\})$. At the other end of the spectrum, we add a separate context for each constraint, containing only the variables in that constraint. In this case, if $V = \{(S_i, R_i)\}_{i=1}^{\ell}$, then the BCS with contexts is $(X, \{(U_i, V_i)\})$, where $U_i \subseteq X$ is the set of variables appearing in S_i, and $V_i = \{(S_i, R_i)\}$. We use this option as the default option when thinking of a BCS as a BCS with contexts.

Definition 5.6. Let $B = (X, \{U_i, V_i\}_{i=1}^{\ell})$ be a BCS with contexts. Let $\mathscr{A}_{con}(B)$ be the finitely-presented $*$-algebra generated by X and subject to the relations

1. $x^2 = 1$ and $x^* = x$ for all $x \in X$ and
2. $xy = yx$ for all $x, y \in U_i$, $1 \leq i \leq m$.

For $1 \leq i \leq \ell$ and ϕ an assignment to U_i, let $\Pi_{U_i,\phi}$ denote the projection

$$\prod_{x \in U_i} \tfrac{1}{2}(1 + \phi(x)x)$$

in $\mathscr{A}_{con}(B)$. The **boolean constraint system algebra** $\mathscr{A}(B)$ is the quotient of $\mathscr{A}_{con}(B)$ by the relations

3. $\Pi_{U_i,\phi} = 0$ for all $1 \leq i \leq \ell$ and non-satisfying assignments ϕ for (U_i, V_i).

The relations (1) imply that $\mathscr{A}(B)$ is a semi-pre-C^*-algebra. As mentioned in the preliminaries, a $*$-representation of $\mathscr{A}_{con}(B)$ is an assignment $x \mapsto \psi(x)$ of operators to every variable $x \in B$, such that $\psi(x)^* = \psi(x)$ and $\psi(x)^2 = 1$ for all $x \in X$, and $\psi(x)\psi(y) = \psi(y)\psi(x)$ for all $x, y \in U_i$ and $1 \leq i \leq \ell$. If ψ is a $*$-representation of $\mathscr{A}_{con}(B)$, then for any $1 \leq i \leq \ell$, the operators $\psi(x)$, $x \in U_i$ form a $*$-representation of $\mathbb{Z}_2^{U_i}$. The $*$-subalgebra of $\mathscr{A}_{con}(B)$ generated by U_i is isomorphic to the group algebra of $\mathbb{Z}_2^{U_i}$. An assignment ϕ to U_i is equivalent to an irreducible representation of $\mathbb{Z}_2^{U_i}$, and $\Pi_{U_i,\phi}$ is the central projection in $\mathbb{C}\mathbb{Z}_2^{U_i}$ corresponding to ϕ. Hence a representation ψ of $\mathscr{A}_{con}(B)$ induces a representation of $\mathscr{A}(B)$ if and only if for all $1 \leq i \leq \ell$, the joint spectrum $\mathcal{J}_{\psi_{U_i}}$ is contained in the set of satisfying assignments for (U_i, V_i). In particular, a one-dimensional $*$-representation of $\mathscr{A}(B)$ is the same thing as a satisfying assignment for B.

More generally, if $B = (X, (X, V))$ is a BCS with exactly one context, then $\mathscr{A}(B)$ is commutative, and a $*$-representation of $\mathscr{A}(B)$ is a direct sum of satisfying assignments for (X, V). However, with more contexts it's possible to have $*$-representations even when there are no one-dimensional representations. The Mermin-Peres magic square is a famous example:

Example 5.7. The Mermin-Peres magic square is the constraint system B over $X = \{x_1, \dots, x_9\}$ with constraints $x_1x_2x_3 = 1$, $x_4x_5x_6 = 1$, $x_7x_8x_9 = 1$, $x_1x_4x_7 = -1$, $x_2x_5x_8 = -1$, and $x_3x_6x_9 = -1$. These constraints arise from putting the variables $x_1, \dots, x_9$ in a 3×3 grid

x_1	x_2	x_3
x_4	x_5	x_6
x_7	x_8	x_9

,

and requiring that the row products are 1 and the column products are -1. If we think of this BCS as a BCS with contexts in the default way (adding a context for each constraint), then $\mathscr{A}(B)$ does not have a one-dimensional $*$-representation, but does have a $*$-representation in dimension 4.

Note that if we allow arbitrary relations, then it is somewhat redundant to explicitly specify contexts. Indeed, given a BCS with contexts $B = (X, \{(U_i, V_i)\}_{i=1}^{\ell})$, let C_i be the constraint (S_i, R_i), where S_i is an enumeration of U_i, and R_i is the set of satisfying assignments to (U_i, V_i). Consider $B' = (X, \{C_i\}_{i=1}^{\ell})$, an ordinary BCS without contexts. If we regard B' as a BCS with contexts in the default way mentioned above then $\mathscr{A}(B) = \mathscr{A}(B')$, so we can always assume that the contexts are defined implicitly from the relations. However, as the following example shows, being able to explicitly specify contexts is convenient when working over more restrictive constraint languages:

Example 5.8. Let $X = \{x_1, x_2, x_3, x_4\}$, and let B be the 3SAT instance (X, V), where $V = (\{x_1 \vee x_2 \vee x_3 = \text{TRUE}, x_2 \vee x_3 \vee x_4 = \text{TRUE}\})$. Then x_1 and x_4 do not commute in $\mathscr{A}(B)$. If we want them to commute, we can instead use the BCS with contexts $B' = (X, \{(X, V)\})$. We also have $\mathscr{A}(B') = \mathscr{A}(B'')$ where $B'' = (X, \{(x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee x_3 \vee x_4)\}$ is a BCS without contexts, but B'' is not a 3SAT system.

There is an alternative presentation of $\mathscr{A}(B)$ that is also useful. Recall that if S is the scope of a constraint over variables X, then $X \cap S$ refers to the set of variables in X.

Lemma 5.9. Suppose $B = (X, \{(U_i, V_i)\}_{i=1}^{\ell})$ is a BCS with contexts. Given a constraint $C = (S, R) \in V_i$, $1 \leq i \leq \ell$ and an assignment ϕ to $X \cap S$, let

$$\Pi_{C,\phi} = \prod_{x \in S \cap R} \tfrac{1}{2}(1 + \phi(x)x).$$

Then $\mathscr{A}(B)$ is the quotient of $\mathscr{A}_{con}(B)$ by the relations $\Pi_{C,\phi} = 0$ for all $C = (S, R) \in V_i$, $1 \leq i \leq \ell$, and ϕ an assignment to $X \cap S$ such that $\phi(S) \notin R$.

Proof. If ϕ is a non-satisfying assignment to (U_i, V_i), then $\phi(S) \notin R$ for some $C = (S, R) \in V_i$, and $\Pi_{U_i,\phi}$ is in the ideal generated by $\Pi_{C,\phi|_{X\cap S}}$.

Conversely, suppose $C = (S, R) \in V_i$ and ϕ is an assignment to $X \cap S$ such that $\phi(S) \notin R$. If $\widetilde{\phi}$ is an assignment to U_i such that $\widetilde{\phi}|_{X\cap S} = \phi$, then $\widetilde{\phi}$ is a non-satisfying assignment to (U_i, V_i). Hence

$$\Pi_{C,\phi} = \sum_{\widetilde{\phi}|_{X\cap S}=\phi} \Pi_{U_i,\widetilde{\phi}}$$

is in the ideal generated by $\Pi_{U_i,\widetilde{\phi}}$ for $\widetilde{\phi}$ a non-satisfying assigment to (U_i, V_i). $\square$

By thinking of the elements of $\mathbb{C}\mathbb{Z}_2^k$ as polynomials in the variables $z_1, \ldots, z_k$, we can identify $\mathbb{C}\mathbb{Z}_2^k$ with the space of functions $\{\pm 1\}^k \to \mathbb{C}$. Specifically, if $f : \{\pm 1\}^k \to \mathbb{C}$ is a function, then the corresponding element of $\mathbb{C}\mathbb{Z}_2^k$ is

$$P = \sum_{v\in\{\pm 1\}^k} f(v) \prod_{i=1}^{k} \frac{(1 + v_i z_i)}{2}.$$

When f is the indicator function f_R of a relation $R \subseteq \{\pm 1\}$, we denote this element by P_R. If $C = (S, R)$ is a constraint over variables X, and $S = (s_1, \ldots, s_k)$, the evaluation $P_R(S)$ of P_R at S is the element of $\mathbb{C}\mathbb{Z}_2^{X\cap S}$ that we get from replacing x_i in P_R with s_i for all $1 \leq i \leq k$. It is not hard to see that

$$P_R(S) = \sum_{\phi(S)\notin R} \Pi_{C,\phi} - \sum_{\phi(S)\in R} \Pi_{C,\phi},$$

where the sums are over assignments ϕ to $X \cap S$, and $\Pi_{C,\phi}$ is defined in Lemma 5.9. Since $\sum_\phi \Pi_{C,\phi} = 1$, we conclude that $1 + P_R(S) = 2\sum_{\phi(S)\notin R} \Pi_{C,\phi}$.

Corollary 5.10. Suppose $B = (X, \{(U_i, V_i)\}_{i=1}^{\ell})$ is a BCS with contexts. Then $\mathscr{A}(B)$ is the quotient of $\mathscr{A}_{con}(B)$ by the relations $P_R(S) = -1$ for all $C = (S, R) \in V_i$, $1 \leq i \leq \ell$.

Proof. The argument above shows that $1+P_R(S)$ is in the ideal generated by the projections $\Pi_{C,\phi}$ with ϕ an assignment to $X \cap S$ such that $\phi(S) \notin R$. Conversely, the projections $\Pi_{C,\phi}$, ϕ an assignment to $X \cap S$ are orthogonal. Hence if $\phi(S) \notin R$ then $\Pi_{C,\phi} = \frac{1}{2}\Pi_{C,\phi}(1+P_R(S))$ is in the ideal generated by $1 + P_R(S)$. The Corollary follows from Lemma 5.9. $\square$

Example 5.11. If R is the AND relation $x \wedge y =$ TRUE, then $P_R(z_1, z_2) = \frac{1}{2}(1 + z_1 + z_2 - z_1 z_2)$. If B is the BCS with two variables z_1, z_2 and the single relation R, then $\mathscr{A}(B) = \mathbb{C}^*\langle z_1, z_2 : z_i^* = z_i, z_i^2 = 1, i = 1, 2, P_R(z_1, z_2) = -1\rangle$.

Example 5.12. Suppose $Ax = b$ is an $m \times n$ linear system over $\mathbb{Z}_2$, written in additive notation (a.k.a. the normal way of writing linear systems). This gives a BCS with contexts $(X, \{(U_i, \{C_i\})\}_{i=1}^m)$, where $X = \{x_1, \ldots, x_n\}$ is the set of variables, $U_i = \{x_j : A_{ij} \neq 0\}$ is the set of variables in equation i, and $C_i = (S_i, R_i)$ is the ith equation of the system, which written multiplicatively is $x_1^{A_{i1}} \cdots x_n^{A_{in}} = (-1)^{b_i}$. Then $P_{R_i}(S_i) = (-1)^{b_i+1} x_1^{A_{i1}} \cdots x_n^{A_{in}}$, and $\mathscr{A}(B)$ is the finitely-presented $*$-algebra generated by $x_1, \ldots, x_n$ subject to the relations

(a) $x_i^2 = 1$ and $x_i^* = x_i$ for all $1 \leq i \leq n$,

(b) $x_j x_k = x_k x_j$ for all $x_j, x_k \in U_i$, $1 \leq i \leq m$, and

(c) $x_1^{A_{i1}} \cdots x_n^{A_{in}} = (-1)^{b_i}$ for all $1 \leq\leq m$.

Note that these relations are very close to the relations for a group algebra, and indeed $\mathscr{A}(B)$ is the quotient $\mathbb{C}\Gamma(A,b)/\langle J = -1\rangle$, where $\Gamma(A,b)$ is the *solution group* of $Ax = b$ [CLS17, Slo19c].

5.12.1 Contextuality scenarios and nonlocal games

Let $B = (X, \{(U_i, V_i)\}_{i=1}^\ell)$ be a BCS with contexts, and suppose ψ is a $*$-representation of $\mathscr{A}_{con}(B)$ on some Hilbert space H. For any $1 \leq i \leq \ell$, the operators $\psi(x)$, $x \in U_i$ are jointly-measurable ± 1-valued observables, with joint outcomes corresponding to assignments ϕ to $X \cap S$. This type of measurement scenario, in which a bunch of observables are grouped into contexts, where observables in the same context are jointly measurable, observables from different contexts are not necessarily jointly measurable, and observables can belong to more than one context, is called a **contextuality scenario** (see, e.g. [AFLS15]). This physical interpretation of $*$-representations of $\mathcal{A}_{con}(B)$ goes back to the original papers of Mermin and Peres [Mer90, Per90]. If ψ is a $*$-representation of $\mathscr{A}(B)$, then the outcome of measuring the operators $\psi(x)$, $x \in U_i$ with respect to any state is always a satisfying assignment to (U_i, V_i). If B does not have a satisfying assignment (or in other words, a one-dimensional $*$-representation), then the operators $\psi(x), x \in X$ are said to be **contextual**, since the behaviour of $\psi(x)$ seems to depend on what context it is measured in.

Another physical interpretation of $*$-representations of $\mathscr{A}(B)$ is provided by the **BCS nonlocal game** $\mathcal{G}(B)$ associated to B. In this game, two players (commonly called Alice and Bob) are each given an input $1 \leq i \leq \ell$, and must respond with a satisfying assignment ϕ for (U_i, V_i). If Alice and Bob receive inputs i and j respectively, and respond with outputs ϕ_A and ϕ_B, then they win if $\phi_A|_{U_i \cap U_j} = \phi_B|_{U_i \cap U_j}$. If this condition is not satisfied, then they lose. The players are cooperating to win, and they know the rules and can decide on a strategy ahead of time. However, they are not able to communicate once the game is in progress (so in particular, Alice does not know which context Bob received, and vice-versa).

There are different types of strategies Alice and Bob might use, depending on what physical resources they have access to. A strategy is **classical** if Alice and Bob have access to shared randomness; **quantum** if Alice and Bob share a finite-dimensional bipartite entangled quantum state; **quantum-approximable** if the strategy is a limit of quantum strategies; and **commuting-operator** if Alice and Bob share a quantum state in a possibly infinite-dimensional Hilbert space, and rather than using separate Hilbert spaces to model the no-communication requirement, Alice's measurement operators just have to commute with Bob's operators. We refer to [LMP+20] for more background on quantum strategies.

A **perfect strategy** for a nonlocal game is a strategy which wins on every pair of inputs. If B has a satisfying assignment, then $\mathcal{G}(B)$ has a classical perfect strategy. Indeed, Alice and Bob can agree on a satisfying assignment ϕ ahead of time, and respond with $\phi|_{U_i}$ on input i. It turns out that $\mathcal{G}(B)$ has a perfect classical strategy if and only if B has a satisfying assignment. The following theorem describes the relationship between perfect strategies for $\mathcal{G}(B)$, and $*$-representations of the BCS algebra $\mathscr{A}(B)$.

Theorem 5.13 ([CM14, KPS18])**.** Let B be a BCS with contexts. Then:

(1) $\mathcal{G}(B)$ has a perfect classical strategy if and only if there is a $*$-homomorphism $\mathscr{A}(B) \to \mathbb{C}$,

(2) $\mathcal{G}(B)$ has a perfect quantum strategy if and only if there is a $*$-homomorphism $\mathscr{A}(B) \to M_d(\mathbb{C})$ for some $d \geq 1$,

(3) $\mathcal{G}(B)$ has a perfect quantum-approximable strategy if and only if there is a $*$-homomorphism $\mathscr{A}(B) \to \mathcal{R}^{\mathcal{U}}$, where $\mathcal{R}^{\mathcal{U}}$ is an ultrapower of the hyperfinite II_1 factor $\mathcal{R}$.

(4) $\mathcal{G}(B)$ has a perfect commuting-operator strategy if and only if $\mathscr{A}(B)$ has a tracial state.

Although BCS algebras hadn't been invented at that point, parts (1) and (2) of Theorem 5.13 were essentially proved in [CM14, Ji13]. Parts (3) and (4) were proved in [KPS18]. We note that the conditions are ordered in decreasing strength. The existence of a $*$-homomorphism $\mathscr{A}(B) \to \mathbb{C}$ implies the existence of a $*$-homomorphism $\mathscr{A}(B) \to M_d(\mathbb{C})$, which implies the existence of a homomorphism $\mathscr{A}(B) \to \mathcal{R}^{\mathcal{U}}$. Finally, if $\mathcal{R}^{\mathcal{U}}$ has a tracial state, there is a $*$-homomorphism $\mathscr{A}(B) \to \mathcal{R}^{\mathcal{U}}$ and $\mathscr{A}(B)$ has a tracial state (see [CLP15] for background on $\mathcal{R}^{\mathcal{U}}$).

A BCS with contexts is said to be **satisfiable** if it has a satisfying assignment. The interpretations in terms of $*$-representations of $\mathscr{A}(B)$ in Theorem 5.13 above suggest the following definition for generalized satisfiability.

Definition 5.14. A BCS with contexts B is:

(i) **satisfiable** if there is a $*$-representation $\mathscr{A}(B) \to \mathbb{C}$,

(ii) **matrix-satisfiable** if there is a $*$-representation $\mathscr{A}(B) \to M_d(\mathbb{C})$ for some $d \geq 1$,

(iii) $\mathcal{R}^{\mathcal{U}}$**-satisfiable** if there is a $*$-representation $\mathscr{A}(B) \to \mathcal{R}^{\mathcal{U}}$,

(iv) **tracially-satisfiable** if $\mathscr{A}(B)$ has a tracial state,

(v) C^***-satisfiable** if $\mathscr{A}(B)$ there is a $*$-representation $\mathscr{A}(B) \to \mathcal{B}(H)$ for some Hilbert space H, and

(vi) **algebraically-satisfiable** if $1 \neq 0 \in \mathscr{A}(B)$.

With the above definitions there is a chain of implications from the strongest notion of satisfiability to the weakest, in the sense that

$$\text{(i)} \implies \text{(ii)} \implies \text{(iii)} \implies \text{(vi)} \implies \text{(v)} \implies \text{(vi)}.$$

However, it is much less obvious whether there are examples of BCS B which are satisfiable in each sense and not satisfiable in the stronger sense. In Definition 5.14 the notions of satisfiability (i)-(iv) can be thought of operationally within the paradigm of strategies for nonlocal games and the different various frameworks for correlations. However, we do not know of any such interpretation for (v) and (vi). That being said, these two notions do appear to make sense in the more general framework of contextuality.

For each type of satisfiability in Definition 5.14 we can consider the problem t-SAT(B), for a BCS B. That is the problem of deciding whether B is t-satisfiable. For some results on the computational complexity of the t-SAT(B) problems we refer the reader to [MNY22]. Here, we focus on the existence of seperating examples for these problems.

Example 5.15 ([Mer90, Per90, CM14])**.** The Mermin-Peres magic square is an LCS over $\mathbb{Z}_2$ that is matrix-satisfiable, but not satisfiable.

Example 5.16 ([Slo18]). There is a LCS over $\mathbb{Z}_2$ which is $\mathcal{R}^{\mathcal{U}}$-satisfiable, but not matrix-satisfiable. The idea is that there is a solution group Γ with $J \neq 1$, which is both hyperlinear and non-residually finite.

Example 5.17 ([JNV$^+$22]). There is a BCS algebra which is tracially-satisfiable, but not $\mathcal{R}^{\mathcal{U}}$-satisfiable. In the MIP*=RE work, the authors give a description of a synchronous nonlocal game with qc-value of 1 but a qa-value < 1.

Example 5.18 ([HMPS19]). There is a BCS which is algebraically-satisfiable, but not C^*-satisfiable. The synchronous algebra of 4-colouring K_5 is nontrivial, but the algebra has no abstract states.

5.18.1 Connection with synchronous games

Another very prominent class of nonlocal games are the synchronous nonlocal games $\mathcal{G}(S)$. One reason they are nice mathematically is, like BCS nonlocal games, perfect t-strategies for synchronous games correspond to certain representations of the finitely-presented $*$-algebra, called the **synchronous algebra** $\mathscr{A}(S)$ associated to the synchronous game $\mathcal{G}(S)$.

Let O and I be finite sets and $\lambda : I^2 \times O^2 \to \{0,1\}$ be a function with the synchronous property, that is

$$\lambda(a,b,x,x) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b, \end{cases} \text{ for all } x \in I.$$

The synchronous algebra is the $*$-algebra generated by $\mathbb{C}\langle e_a^x : a \in O, x \in I\rangle$ subject to the relations:

(i) $(e_a^x)^* = (e_a^x)^2 = e_a^x$, for all $a \in O$, $x \in I$,

(ii) $\sum_{a \in A} e_a^x = 1$ for all $x \in I$,

(iii) $e_a^x e_b^y = 0$, whenever $\lambda(x,y,a,b) = 0$.

Like BCS nonlocal games, synchronous games can encode interesting computational problems. For instance, an interesting class of synchronous games are the proper k-colouring games for a graph G. The existence of a perfect classical strategy to the k-colouring game is equivalent to a proper k-colouring of the vertices in G.

It was shown in [Pad22] (see Proposition 3.26) that every synchronous algebra is $*$-isomorphic to a BCS algebra for a BCS B_{sync}. Therefore any synchronous algebra in which any of the separations exist, there is a an example of a BCS-algebra with that separation and vice versa.

5.19 Constraint system languages and definability

Definition 5.20. A relation $R \subseteq \{\pm 1\}^k$ is **definable** (or *pp*-**definable**) from a binary constraint language $\mathcal{L}$ if there is a BCS B over $\mathcal{L}$ with (potentially additional) variables $\{x_1, \ldots, x_k\} \cup \{y_1, \ldots, y_\ell\}$ such that $(a_1, \ldots, a_k) \in R$ if and only if there is a satisfying assignment ϕ for B with $\phi(x_i) = a_i$.

We say that a BCS B (resp. BCS with contexts) is definable from $\mathcal{L}$ if every relation in B is definable from $\mathcal{L}$.

Our insight is that this definability is reflected in the algebraic characterization of BCS-algebras in the following sense:

Lemma 5.21. If a BCS with contexts B is definable from the constraint language $\mathcal{L}$, then there exists a BCS with contexts B' over $\mathcal{L}$ and $*$-homomorphisms

$$\mathscr{A}(B) \underset{\pi}{\overset{\iota}{\rightleftarrows}} \mathscr{A}(B')$$

such that $\pi \circ \iota = \mathbb{1}_{\mathscr{A}(B)}$.

Proof. Let $B = (X, \{U_i, V_i\}_{i=1}^{\ell})$. For each BCS (U_i, V_i), there is a BCS $(U_i \cup Y_i, W_i)$ defined over $\mathcal{L}$ such that ϕ is a satisfying assignment for (U_i, V_i) if and only if there is a satisfying assignment $\widetilde{\phi}$ for $(U_i \cup Y_i, W_i)$ with $\widetilde{\phi}|_{U_i} = \phi$. Let Y be the disjoint union of sets Y_i, and consider the BCS with contexts $B' = (X \cup Y, \{(U_i \cup Y_i, W_i)\}_{i=1}^{\ell})$. Since U_i is contained inside a context of B' for all $1 \leq i \leq \ell$, there is a $*$-homomorphism $\widetilde{\iota} : \mathscr{A}_{con}(B) \to \mathscr{A}_{con}(B')$ sending $x \mapsto x$ for all $x \in X$. If ϕ is an assignment to $U_i \cup Y_i$, then

$$\Pi_{U_i \cup Y_i, \widetilde{\phi}} = \prod_{x \in U_i} \left(\frac{1 + \tilde{\phi}(x) x}{2} \right) \prod_{y \in Y_i} \left(\frac{1 + \tilde{\phi}(y) y}{2} \right) = \widetilde{\iota}(\Pi_{U_i, \phi}) \Pi_{Y_i, \widetilde{\phi}},$$

where $\phi = \widetilde{\phi}|_{U_i}$, and $\Pi_{Y_i, \widetilde{\phi}} := \prod_{y \in Y_i} \frac{1}{2}(1 + \widetilde{\phi}(y) y)$. For every assignment ϕ to U_i, we have

$$\sum_{\widetilde{\phi}|_{U_i} = \phi} \Pi_{Y_i, \widetilde{\phi}} = 1,$$

where the sum is over assignments $\widetilde{\phi}$ to $U_i \cup Y_i$ with $\widetilde{\phi}|_{U_i} = \phi$. If ϕ is a non-satisfying assignment for (U_i, V_i), then every assignment $\widetilde{\phi}$ to $U_i \cup Y_i$ with $\widetilde{\phi}|_{U_i} = \phi$ is a non-satisfying assignment to $(U_i \cup Y_i, W_i)$. Therefore

$$\widetilde{\iota}(\Pi_{U_i,\phi}) = \sum_{\widetilde{\phi}|_{U_i}=\phi} \widetilde{\iota}(\Pi_{U_i,\phi})\Pi_{Y_i,\widetilde{\phi}} = \sum_{\widetilde{\phi}|_{U_i}=\phi} \Pi_{U_i \cup Y_i,\widetilde{\phi}}$$

vanishes in $\mathscr{A}(B')$ for all non-satisfying assignments ϕ to (U_i, V_i). We conclude that $\widetilde{\iota}$ induces a homomorphism $\iota : \mathscr{A}(B) \to \mathscr{A}(B')$ sending $x \mapsto x$ for all $x \in X$.

For the other direction, for each $1 \leq i \leq \ell$ and assignment ϕ to U_i, choose an assignment h_ϕ to $U_i \cup Y_i$, such that $h_\phi|_{U_i} = \phi$, and if ϕ is satisfying for (U_i, V_i) then h_ϕ is satisfying for $(U_i \cup Y_i, W_i)$. Define $\widetilde{\pi} : \mathscr{A}_{con}(B') \to \mathscr{A}_{con}(B)$ by $\widetilde{\pi}(x) = x$ for all $x \in X$, and

$$\widetilde{\pi}(y) = \sum_{\phi} h_\phi(y)\Pi_{U_i,\phi}$$

if $y \in Y_i$, $1 \leq i \leq \ell$. Since $\widetilde{\pi}(y)^* = \widetilde{\pi}(y)$ and $\widetilde{\pi}(y)^2 = 1$ for all $y \in Y$, and $\widetilde{\pi}(z)\widetilde{\pi}(w) = \widetilde{\pi}(w)\widetilde{\pi}(z)$ commute for all $z, w \in U_i \cup Y_i$, the homomorphism $\widetilde{\pi}$ is well-defined. For any $a \in \{\pm 1\}$, we have that

$$\tfrac{1}{2}(1 + a\pi(y)) = \sum_{\phi : h_\phi(y) = a} \Pi_{U_i,\phi}.$$

Suppose $\widetilde{\phi}$ is an assignment to $U_i \cup Y_i$, and let $\phi_0 = \widetilde{\phi}|_{U_i}$. Then

$$\widetilde{\pi}(\Pi_{U_i \cup Y_i,\widetilde{\phi}}) = \Pi_{U_i,\phi_0} \cdot \prod_{y \in Y_i} \left(\sum_{\phi : h_\phi(y) = \widetilde{\phi}(y)} \Pi_{U_i,\phi} \right) = \begin{cases} \Pi_{U_i,\phi_0} & \widetilde{\phi} = h_{\phi_0} \\ 0 & \text{otherwise} \end{cases}.$$

By construction, if h_ϕ is a non-satisfying assignment to $(U_i \cup Y_i, W_i)$, then ϕ is a non-satisfying assignment to (U_i, V_i). Hence if $\widetilde{\phi}$ is a non-satisfying assignment to $(U_i \cup Y_i, W_i)$, then $\widetilde{\pi}(\Pi_{U_i \cup Y_i,\widetilde{\phi}})$ vanishes in $\mathscr{A}(B)$. We conclude that $\widetilde{\pi}$ induces a homomorphism $\pi : \mathscr{A}(B') \to \mathscr{A}(B)$ with the property that $\pi \circ \iota = \mathbb{1}_{\mathscr{A}(B)}$. □

Corollary 5.22. If $\mathcal{L}$ is definable from $\mathcal{L}'$, then any separation of satisfiability that holds for $\mathcal{L}$ also holds for $\mathcal{L}'$.

Boolean constraints are typically represented by boolean propositional formulae. We review some basic terminology about boolean propositional formulae which will make defining the relevant boolean constraint languages much easier. A **term** is just a boolean variable or constant. A **literal** is a term or its negation. A literal is said to be negative if it is the negation of a term and positive otherwise. A **clause** is a disjunction (OR) of literals. Any constraint C can be expressed as a conjunction (AND) of clauses. This is known as the conjunctive normal form (CNF) of C. With these terms in mind, we can describe some well known boolean constraint languages. The constraint language k-SAT is the set of relations expressible by CNF-formulae where each clause contains at most k literals.

Schaefer's dichotomy theorem is a statement about the complexity of boolean satisfiability problems under several boolean constraint languages. There are a number of boolean constraint languages that arise in this classification. We briefly review them here. A constraint C is said to be:

(i) Bijunctive if each clause contains at most two literals (also known as 2-SAT). The language of all bijunctive relations is denoted 2SAT.

(ii) 0-valid (resp. 1-valid) if the formula is true on the string consisting of all FALSE (resp. TRUE) assignments to each variable. The language of all 0-valid (resp. 1-valid) relations is denoted 0-VALID (resp. 1-VALID).

(iii) Horn (resp. dual-Horn) if each clause contains at most one negative (resp. positive) literal. The language of all Horn (resp. dual-Horn) relations is denoted HORN (resp. DUAL-HORN).

(iv) Linear (or affine) if each clause consists only exclusive disjunctions (XORs) of literals. In this case, the constraint is equivalent to a linear system of equations over $\mathbb{Z}_2$. The language of all linear relations is denoted LIN.

The satisfiability problem SAT(B) for a BCS B is determining if there is a satisfying assignment to B. The following theorem provides a surprising dichotomy about the computational complexity for the satisfiability problem for a BCS over the above class of constraint languages:

Theorem 5.23 (Schaefer's dichotomy theorem [Sch78])**.** Let $\mathcal{L}$ be any of the constraint languages (i)-(iv) fom the above list. If B is a BCS over $\mathcal{L}$ then the problem SAT(B) is in P, otherwise SAT(B) is NP-complete.

Based on the knowledge that LIN contains examples of BCS's, like the magic square, which are matrix-satisfiable but not satisfiable as well as a BCS that is $\mathcal{R}^{\mathcal{U}}$-satisfiable but not matrix-satisfiable [Slo19c]. A line of enquiry initiated in [Ji13, AKS19], was to understand whether the other boolean constraint languages in Schaefer's classification provide separations between these types of satisfiability. However, it turns out that with the exception of LIN, all the other constraint languages in Schaefer's dichotomy theorem have no separating examples. In [Ji13], Ji was the first to establish results in this direction by proving that if B is a BCS over the languages 2SAT, HORN, and DUAL-HORN, then B is matrix-satisfiable only if B is satisfiable. Later, using different methods the authors of [AKS19] established that if B is a BCS over the languages 0-VALID, 1-VALID, 2SAT, HORN, DUAL-HORN, then B is C^*-satisfiable only if B is satisfiable. Upon inspection, the arguments in [AKS19] do not appear to rely on any specific properties of the underlying Hilbert space or C^*-algebra and thus extends to abstract BCS $*$-algebras:

Corollary 5.24. If B is a BCS over the constraint languages 0-VALID, 1-VALID, HORN, DUAL-HORN, or 2SAT, then B is algebraically-satisfiable only if B is satisfiable.

The class of BCS with contexts over LIN has been well studied in the setting of $\mathbb{Z}_2$-linear constraint system ($\mathbb{Z}_2$-LCS) nonlocal games. An LCS is a BCS described by linear system of equations of the form $Ax = b$. Moreover, every BCS-algebra over LIN is isomorphic to a quotient of a group algebra by a central element of order 2. Given an $m \times n$ linear system $Ax = b$ over $\mathbb{Z}_2$ let V_i be the set of variables appearing in the ith equation (i.e. a row of A). The **solution group** $\Gamma(A, b)$ is the finitely presented group generated by $\{x_1, \dots, x_n\} \cup \{J\}$ subject to the relations:

(1) $x_j^2 = 1$ for all $1 \leq j \leq n$ and $J^2 = 1$,

(2) $[x_j, J] = 1$, for all $1 \leq j \leq n$,

(3) $[x_j, x_k] = 1$ whenever $x_j, x_k \in V_i$, for some $1 \leq i \leq m$.,

(4) $\prod_{j \in V_i} x_j = J^{b_i}$ for all $1 \leq i \leq m$,

where $[x, y] = xyx^{-1}y^{-1}$ is the group commutator.

Like in the case of BCS nonlocal games, representations of the solution group correspond to perfect t-strategies. It follows that if B is a BCS over LIN, then the BCS-algebra $\mathscr{A}(B)$ is the quotient of the group algebra $\mathbb{C}\Gamma(A, b)$ by the ideal $\langle\langle J = -1 \rangle\rangle$. In [Slo19c] it was shown that any finitely presented group can be embedded into a solution group, which perhaps explains why LIN gives rise to such a rich family of BCS-algebras.

Since the Mermin Peres magic square example from earlier is an example of a BCS over LIN, its BCS can be interpreted in as a group algebra. As mentioned previously, there are separations for tracial-satisfiability and matrix satisfiability with BCS from LIN. Whether there are examples of more separations in LIN for other types of satisfiability is a very interesting problem. That being said, we do not believe that finding such an example is a straightforward task.

Proposition 5.25. If there exists a BCS B over LIN that is algebraically-satisfiable and not $\mathcal{R}^{\mathcal{U}}$-satisfiable, then there exists a non-hyperlinear group.

Proof. All group algebras modulo central elements have tracial states, thus if B is a BCS over LIN that is algebraically-satisfiable, then $\mathscr{A}(B)$ has a tracial state and is therefore tracially-satisfiable. It is well known that a group is hyperlinear if and only if it has a $*$-homomorphism to $\mathcal{R}^{\mathcal{U}}$. Since we assumed B is not $\mathcal{R}^{\mathcal{U}}$-satisfiable, the result follows. □

Note that for a BCS over LIN, the polynomial time algorithm for SAT(B) reduced to solving the linear system $Ax = b$. However, a result of the second author [Slo19c] implies that given a BCS over LIN it is **undecidable** to determine if B is tracially-satisfiable. Although it is generally undecidable to determine whether constraint systems in LIN are quantum satisfiable, it was shown in [Ark12, PRSS22] that for the subclass of LIN where the constraints can be represented by the incidence matrix of a graph G. The matrix SAT problem for these (graph) LCS can be reduced to finding certain graph minors in the graph G, which by the Roberston-Seymour algorithm is in P. We note that both the magic square and magic pentagram games belong in this class. It would be interesting to know whether there are other nice subclasses of LIN where the problem is decidable and there are separating examples for the different types of satisfiability.

5.26 A BCS that is C^*-satisfiable and not tracially-satisfiable

The only remaining separation remaining in Definition 5.14 is a BCS that is C^*-satisfiable but not tracially-satisfiable. This question was asked (in the case of synchronous algebras) by Sam Harris in [Har21]. We now construct explicit BCS which is C^*-satisfiable but not tracial-satisfiable. It follows that we have a strict chain of implications

$$\text{(i)} \not\Leftarrow \text{(ii)} \not\Leftarrow \text{(iii)} \not\Leftarrow \text{(vi)} \not\Leftarrow \text{(v)} \not\Leftarrow \text{(vi)}.$$

Before we proceed, we remind the reader of some basic algebraic properties of the BCS-algebra for the magic square:

Lemma 5.27. Let $\mathscr{M}$ be the BCS algebra associated to the constraints described by the Mermin-Peres magic square (as in Example 5.7).

(a) In any C^*-satisfying assignment the operators corresponding to pairs of variables x_1, x_5 and x_2, x_4 both anticommute. Moreover, the assignments to x_1 and x_5 each commute with the operators assigned to x_2 and x_4,

(b) For any pair of anticommuting order 2 unitaries A, B acting on a Hilbert space H there is a C^*-satisfying assignment to the Mermin-Peres magic square on $H \otimes \mathbb{C}^2$ with $\phi(x_1) = A \otimes \mathbb{1}$, $\phi(x_5) = B \otimes \mathbb{1}$, $\phi(x_2) = \mathbb{1} \otimes A$, and $\phi(x_4) = \mathbb{1} \otimes B$.

We now proceed with the example.

Example 5.28. Consider the finitely presented $*$-algebra:

$$\mathscr{X} = \mathbb{C}^*\langle X_1, X_2, Z_1, Z_2, W, T\rangle,$$

where each generator is self-adjoint $Z_i^* = Z_i$, $X_i^* = X_i$, $W^* = W$, $T^* = T$ and has order 2 (i.e. $Z_i^2 = X_i^2 = W^2 = T^2 = 1$ for $i = 1, 2$). Let $\mathscr{R}$ be the $*$-ideal generated by the following relations:

(1) $X_iZ_i = -Z_iX_i$ for $i = 1, 2$, and $X_iZ_j = Z_jX_i$ for $i \neq j$,

(2) $X_1X_2 = X_2X_1$,

(3) $Z_1Z_2 = Z_2Z_1$,

(4) $2W = \mathbb{1} + Z_1 + Z_2 - Z_1Z_2$,

(5) $TW = -WT$.

Lastly, let $\mathscr{B} = \mathscr{X}/\mathscr{R}$.

Proposition 5.29. We make three claims about $\mathscr{B}$:

(i) $\mathscr{B}$ has no trace,

(ii) there is a $*$-representation $\mathscr{B} \to \mathcal{B}(H)$, and

(iii) there is a BCS B for which $\mathscr{B} \hookrightarrow \mathscr{A}(B)$ and any $*$-representation of $\mathscr{B}$ extends to a $*$-representation of $\mathscr{A}(B)$.

Proof. Toward (i), suppose that a tracial state $\tau : \mathscr{B} \to \mathbb{C}$ exists. Now, consider the relations (1)-(3). Since each Z_i anticommutes with an X_i for $i = 1, 2$, one can show that

$$\tau(Z_1) = \tau(Z_2) = \tau(Z_1 Z_2) = 0.$$

So now, upon enforcing relation (4) we see that $\tau(W) = 1/2$, since $\tau(1) = 1$. However by enforcing relation (5) we see that

$$\tau(W) = \tau(WT^2) = \tau(TWT) = \tau(-WT^2) = -\tau(W),$$

and so $\tau(W) = 0$, a contradiction and we conclude no tracial states exists.

For point (ii), consider the 2×2 Pauli matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and let $\psi : \mathscr{X} \to M_4(\mathbb{C})$ be the function sending:

$$\begin{aligned} X_1 &\mapsto X \otimes \mathbb{1}_{\mathbb{C}^2}, \\ X_2 &\mapsto \mathbb{1}_{\mathbb{C}^2} \otimes X, \\ Z_1 &\mapsto Z \otimes \mathbb{1}_{\mathbb{C}^2}, \\ Z_2 &\mapsto \mathbb{1}_{\mathbb{C}^2} \otimes Z. \end{aligned}$$

Hence, ψ extends to a map on $\mathscr{B}$ where X_1, X_2, Z_1, Z_2 and W (by relation (4)) are self-adjoint unitaries acting on $\mathbb{C}^4$ and relations (1)-(4) are satisfied. For ψ to be a $*$-homomorphism, all that remains is to find an order 2 unitary representative for T acting on $\mathbb{C}^4$ which anticommutes with $\psi(W)$. This is not possible, since in any finite-dimensional representation $\mathrm{tr}(\cdot)$ is a trace on $M_d(\mathbb{C})$. In particular $\mathrm{tr}(\psi(W)) \neq 0$, so if there was such a $\psi(T)$, then

$$\mathrm{tr}(\psi(W)) = \mathrm{tr}\left(\psi(T)^* \psi(W) \psi(T)\right) = \mathrm{tr}\left(-\psi(T)^* \psi(T) \psi(W)\right) = -\mathrm{tr}(\psi(W)),$$

giving a contradiction.

With that in mind we consider the map $\varphi : \mathscr{B} \to \mathcal{B}(\mathbb{C}^4 \otimes \ell^2(\mathbb{N}))$; $x \mapsto \psi(x) \otimes \mathbb{1}_{\ell^2(\mathbb{N})}$ for all $x \in \{X_1, X_2, Z_1, Z_2\}$ with the same ψ as above. Consider the image Z_1 and Z_2 under φ. Now, by enforcing relation (4), we obtain a description of $\varphi(W)$. In particular, up to a change of basis we see that

$$\varphi(W) \cong \begin{pmatrix} \mathbb{1}_{\ell^2(\mathbb{N})^{\oplus 3}} & 0 \\ 0 & -\mathbb{1}_{\ell^2(\mathbb{N})} \end{pmatrix}.$$

Now let T_0 be the vector space isomorphism $T_0 : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})^{\oplus 3}$ with adjoint $T_0^* : \ell^2(\mathbb{N})^{\oplus 3} \to \ell^2(\mathbb{N})$ and consider the assignment

$$T \mapsto \begin{pmatrix} 0 & T_0 \\ T_0^* & 0 \end{pmatrix} := \varphi(T).$$

Observe that $\varphi(T)$ is a self-adjoint unitary with the property that

$$\begin{aligned} \varphi(T)\varphi(W)\varphi(T) &= \begin{pmatrix} 0 & T_0 \\ T_0^* & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1}_{\ell^2(\mathbb{N})^{\oplus 3}} & 0 \\ 0 & -\mathbb{1}_{\ell^2(\mathbb{N})} \end{pmatrix} \begin{pmatrix} 0 & T_0 \\ T_0^* & 0 \end{pmatrix} \\ &= \begin{pmatrix} -T_0 \mathbb{1}_{\ell^2(\mathbb{N})} T_0^* & 0 \\ 0 & T_0^* \mathbb{1}_{\ell^2(\mathbb{N})^{\oplus 3}} T_0 \end{pmatrix} \\ &= \begin{pmatrix} -\mathbb{1}_{\ell^2(\mathbb{N})^{\oplus 3}} & 0 \\ 0 & \mathbb{1}_{\ell^2(\mathbb{N})} \end{pmatrix} \\ &= -\varphi(W). \end{aligned}$$

We conclude that relations (4) and (5) hold under the map φ. We conclude that the map $\psi(x) \mapsto \psi(x) \otimes \mathbb{1}_{\ell^2(\mathbb{N})}$ is a $*$-homomorphism, hence relations (1)-(3), that held under ψ on $\mathbb{C}^4 \otimes \ell^2(\mathbb{N})$ still hold under φ. Hence, $\varphi = \psi \otimes \mathbb{1}_{\ell^2(\mathbb{N})}$ is a $*$-representation of $\mathscr{B}$ on $\mathcal{B}(\mathbb{C}^4 \otimes \ell^2(\mathbb{N}))$ and $\mathscr{B}$ has no tracial states.

For part (iii), we argue that there is BCS nonlocal game B where (i) and (ii) hold for $\mathscr{A}(B)$. For this we rely on the fact that we can uniquely enforce the desired relations in $\mathscr{B}$ between pairs of variables using a magic squares. Consider the BCS nonlocal games with constraints described by the magic square with variables Z_i, X_i, and Y_i for $i = 1, 2, 3$ with the $+1$-row constraints and -1-column constraints

X_1	Z_1	Y_1
X_2	Z_2	Y_2
X_3	Z_3	Y_3

call this BCS B_0. Now, we "add" another disjoint magic square to B_0 introducing additional variables T_i, W_i and L_i for $i = 1, 2, 3$, while keeping the same row and column constraints in the second square

T_1	W_1	L_1
T_2	W_2	L_2
T_3	W_3	L_3

Call the resulting BCS B_1. Now letting $T = T_1$ and $W = W_2$, by Lemma 5.27 part (a) we see that relations (1)-(3) hold in $\mathscr{A}(B)$. Note that relation (4) is equivalent to W_2 being the AND of Z_1 and Z_2, hence it's a valid BCS constraint and can be added to the BCS B_1 resulting in a new BCS B. Moreover, Z_1 and Z_2 already commute so the addition of this constraint doesn't add any more algebraic relations. Now, the argument in parts (i) and (ii) will hold for any representation of $\mathscr{A}(B)$, in particular by Lemma 5.27 part (b) the representation ψ in part (ii) extends to a C^*-satisfying assignment for B, and the result follows. □

Corollary 5.30. There is a BCS B with the following equivalent properties:

1. B is C^*-satisfiable but not tracially-satisfiable,
2. there is a BCS nonlocal game $\mathcal{G}(B)$ with no perfect qc-strategy, but $\mathscr{A}(B)$ has a $*$-representation,
3. $\mathscr{A}(B)$ has a state but does not have any tracial states.

The idea of using muliple magic squares to force interesting algebraic relations is not a new idea, see for instance [CMMN20].

References

[AFLS15] Antonio Acín, Tobias Fritz, Anthony Leverrier, and Ana Belén Sainz. A combinatorial approach to nonlocality and contextuality. *Communications in Mathematical Physics*, 334:533–628, 2015.

[AKS19] Albert Atserias, Phokion G Kolaitis, and Simone Severini. Generalized satisfiability problems via operator assignments. *Journal of Computer and System Sciences*, 105:171–198, 2019.

[AMR+19] Albert Atserias, Laura Mančinska, David E Roberson, Robert Šámal, Simone Severini, and Antonios Varvitsiotis. Quantum and non-signalling graph isomorphisms. *Journal of Combinatorial Theory, Series B*, 136:289–328, 2019.

[Ark12] Alex Arkhipov. Extending and characterizing quantum magic games. *arXiv:1209.3819 [quant-ph]*, Sep 2012.

[Bei21] Salman Beigi. Separation of quantum, spatial quantum, and approximate quantum correlations. *Quantum*, 5:389, 2021.

[Bel64] John S Bell. On the einstein podolsky rosen paradox. *Physics Physique Fizika*, 1(3):195, 1964.

[Bha13] Rajendra Bhatia. *Matrix Analysis*, volume 169. Springer Science & Business Media, 2013.

[Bla06] Bruce Blackadar. *Operator Algebras: Theory of C^*-Algebras and von Neumann Algebras*, volume 122. Springer Science & Business Media, 2006.

[BŠCA18a] Joseph Bowles, Ivan Šupić, Daniel Cavalcanti, and Antonio Acín. Device-independent entanglement certification of all entangled states. *Physical review letters*, 121(18):180503, 2018.

[BŠCA18b] Joseph Bowles, Ivan Šupić, Daniel Cavalcanti, and Antonio Acín. Self-testing of Pauli observables for device-independent entanglement certification. *Physical Review A*, 98(4):042336, 2018.

[CHSH69] John F Clauser, Michael A Horne, Abner Shimony, and Richard A Holt. Proposed experiment to test local hidden-variable theories. *Physical review letters*, 23(15):880, 1969.

[CHTW10] Richard Cleve, Peter Hoyer, Ben Toner, and John Watrous. Consequences and limits of nonlocal strategies. *arXiv:quant-ph/0404076*, Jan 2010.

[CLP15] Valerio Capraro, Martino Lupini, and Vladimir Pestov. *Introduction to sofic and hyperlinear groups and Connes' embedding conjecture*, volume 1. Springer, 2015.

[CLS17] Richard Cleve, Li Liu, and William Slofstra. Perfect commuting-operator strategies for linear system games. *Journal of Mathematical Physics*, 58(1):012202, Jan 2017.

[CM14] Richard Cleve and Rajat Mittal. Characterization of binary constraint system games. In *Automata, Languages, and Programming: 41st International Colloquium, ICALP 2014, Copenhagen, Denmark, July 8-11, 2014, Proceedings, Part I 41*, pages 320–331. Springer, 2014.

[CMMN20] David Cui, Arthur Mehta, Hamoon Mousavi, and Seyed Sajjad Nezhadi. A generalization of CHSH and the algebraic structure of optimal strategies. *Quantum*, 4:346, 2020.

[CMN$^+$07] Peter J Cameron, Ashley Montanaro, Michael W Newman, Simone Severini, and Andreas Winter. On the quantum chromatic number of a graph. *The Electronic Journal of Combinatorics*, 14(1):R81, 2007.

[Col20] Andrea Coladangelo. A two-player dimension witness based on embezzlement, and an elementary proof of the non-closure of the set of quantum correlations. *Quantum*, 4:282, 2020.

[Con76] Alain Connes. Classification of injective factors cases ii_1, ii_∞, $iii_{\lambda,\lambda\neq1}$. *Annals of Mathematics*, pages 73–115, 1976.

[CS17] Andrea Coladangelo and Jalex Stark. Robust self-testing for linear constraint system games. *arXiv:1709.09267*, 2017.

[CS18] Andrea Coladangelo and Jalex Stark. Unconditional separation of finite and infinite-dimensional quantum correlations. *arXiv:1804.05116*, 2018.

[CSUU08] Richard Cleve, William Slofstra, Falk Unger, and Sarvagya Upadhyay. Strong parallel repetition theorem for quantum XOR proof systems. *arXiv:quant-ph/0608146*, Apr 2008.

[Dav96] Kenneth R Davidson. *C*-Algebras by Example*, volume 6. American Mathematical Soc., 1996.

[DCOT19] Marcus De Chiffre, Narutaka Ozawa, and Andreas Thom. Operator algebraic approach to inverse and stability theorems for amenable groups. *Mathematika*, 65(1):98–118, 2019.

[DP16] Kenneth J Dykema and Vern Paulsen. Synchronous correlation matrices and Connes' embedding conjecture. *Journal of Mathematical Physics*, 57(1):015214, 2016.

[DPP19] Ken Dykema, Vern I Paulsen, and Jitendra Prakash. Non-closure of the set of quantum correlations via graphs. *Communications in Mathematical Physics*, 365:1125–1142, 2019.

[Fri12] Tobias Fritz. Tsirelson's problem and Kirchberg's conjecture. *Reviews in Mathematical Physics*, 24(05):1250012, Jun 2012.

[Fri20] Tobias Fritz. Quantum logic is undecidable. *Archive for Mathematical Logic*, Sep 2020. arXiv: 1607.05870.

[GH17] William Timothy Gowers and Omid Hatami. Inverse and stability theorems for approximate representations of finite groups. *Sbornik: Mathematics*, 208(12), 2017.

[Gol21] Adina Goldberg. Synchronous linear constraint system games. *Journal of Mathematical Physics*, 62(3):032201, 2021.

[Har21] Samuel J Harris. Synchronous games with $*$-isomorphic game algebras. *arXiv:2109.04859*, 2021.

[HMPS19] J William Helton, Kyle P Meyer, Vern I Paulsen, and Matthew Satriano. Algebras, synchronous games, and chromatic numbers of graphs. *New York J. Math*, 25:328–361, 2019.

[HPV16] Samuel Harris, Satish Pandey, and Paulsen Vern. Entanglement and nonlocality. PMATH 990/QIC 890 course notes, University of Waterloo, 2016.

[Ji13] Zhengfeng Ji. Binary constraint system games and locally commutative reductions. *arXiv:1310.3794*, 2013.

[JNP+11] M. Junge, M. Navascues, C. Palazuelos, D. Perez-Garcia, V. B. Scholz, and R. F. Werner. Connes' embedding problem and Tsirelson's problem. *Journal of Mathematical Physics*, 52(1):012102, Jan 2011.

[JNV+22] Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen. MIP* =RE. *arXiv:2001.04383v3*, 2022.

[Kan20] Jedrzej Kaniewski. Weak form of self-testing. *Physical Review Research*, 2, 2020.

[KPS18] Se-Jin Kim, Vern Paulsen, and Christopher Schafhauser. A synchronous game for binary constraint systems. *Journal of Mathematical Physics*, 59(3):032201, 2018.

[LMP+20] Martino Lupini, Laura Mančinska, Vern I Paulsen, David E Roberson, Giannicola Scarpa, Simone Severini, Ivan G Todorov, and Andreas Winter. Perfect strategies for non-local games. *Mathematical Physics, Analysis and Geometry*, 23(1):7, 2020.

[Mer90] N David Mermin. Simple unified form for the major no-hidden-variables theorems. *Physical review letters*, 65(27):3373, 1990.

[MNY22] Hamoon Mousavi, Seyed Sajjad Nezhadi, and Henry Yuen. Nonlocal games, compression theorems, and the arithmetical hierarchy. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing*, pages 1–11, 2022.

[MPS21] Laura Mančinska, Jitendra Prakash, and Christopher Schafhauser. Constant-sized robust self-tests for states and measurements of unbounded dimension. *arXiv:2103.01729*, 2021.

[MR16] Laura Mančinska and David E Roberson. Quantum homomorphisms. *Journal of Combinatorial Theory, Series B*, 118:228–267, 2016.

[MR18] Laura Mančinska and David E Roberson. Oddities of quantum colorings. *arXiv:1801.03542 [quant-ph]*, Jan 2018.

[MY04] Dominic Mayers and Andrew Yao. Self testing quantum apparatus. *Quantum Info. Comput.*, 4(4):273–286, jul 2004.

[MYS12] Matthew McKague, Tzyh Haur Yang, and Valerio Scarani. Robust self-testing of the singlet. *Journal of Physics A: Mathematical and Theoretical*, 45(45), 2012.

[NPA07] Miguel Navascués, Stefano Pironio, and Antonio Acín. Bounding the set of quantum correlations. *Physical Review Letters*, 98(1):010401, 2007.

[NPA08] Miguel Navascués, Stefano Pironio, and Antonio Acín. A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations. *New Journal of Physics*, 10(7):073013, 2008.

[O'D14] Ryan O'Donnell. *Analysis of Boolean Functions*. Cambridge University Press, 2014.

[OV16] Dimiter Ostrev and Thomas Vidick. Entanglement of approximate quantum strategies in XOR games. *arXiv:1609.01652*, 2016.

[Oza13a] Narutaka Ozawa. About the Connes embedding conjecture: algebraic approaches. *Japanese Journal of Mathematics*, 8(1):147–183, 2013.

[Oza13b] Narutaka Ozawa. Tsirelson's problem and asymptotically commuting unitary matrices. *Journal of Mathematical Physics*, 54(3):032202, 2013.

[Pad22] Connor Paddock. Rounding near-optimal quantum strategies for nonlocal games to strategies using maximally entangled states. *arXiv:2203.02525*, 2022.

[Per90] Asher Peres. Incompatible results of quantum measurements. *Physics Letters A*, 151(3-4):107–108, 1990.

[PRSS22] Connor Paddock, Vincent Russo, Turner Silverthorne, and William Slofstra. Arkhipov's theorem, graph minors, and linear system nonlocal games. *arXiv:2205.04645*, 2022.

[PS23] Connor Paddock and William Slofstra. Satisfiability problems and algebras of boolean constraint systems games. *in preparation*, 2023.

[PSS+16] Vern I Paulsen, Simone Severini, Daniel Stahlke, Ivan G Todorov, and Andreas Winter. Estimating quantum chromatic numbers. *Journal of Functional Analysis*, 270(6):2188–2222, Mar 2016.

[PSZZ23] Connor Paddock, William Slofstra, Yuming Zhao, and Yangchen Zhou. An operator-algebraic formulation of self-testing. *arXiv:2301.11291*, 2023.

[PT13] Vern I Paulsen and Ivan G Todorov. Quantum chromatic numbers via operator systems. *arXiv:1311.6850 [math]*, Nov 2013.

[Rus20] Travis Russell. Two-outcome synchronous correlation sets and Connes' embedding problem. *Quantum Information and Computation*, 20(5&6):361, 2020.

[RUV13] Ben W Reichardt, Falk Unger, and Umesh Vazirani. Classical command of quantum systems. *Nature*, 496(7446):456–460, 2013.

[ŠB20] Ivan Šupić and Joseph Bowles. Self-testing of quantum systems: a review. *Quantum*, 4:337, 2020.

[Sch78] Thomas J Schaefer. The complexity of satisfiability problems. In *Proceedings of the tenth annual ACM symposium on Theory of computing*, pages 216–226, 1978.

[Slo11] William Slofstra. Lower bounds on the entanglement needed to play XOR non-local games. *Journal of Mathematical Physics*, 52(10):102202, 2011.

[Slo18] William Slofstra. A group with at least subexponential hyperlinear profile. *arXiv:1806.05267*, 2018.

[Slo19a] William Slofstra. Approximate representation theory. PMATH 990 course notes, University of Waterloo, 2019.

[Slo19b] William Slofstra. The set of quantum correlations is not closed. *Forum of Mathematics, Pi*, 7:e1, 2019.

[Slo19c] William Slofstra. Tsirelson's problem and an embedding theorem for groups arising from non-local games. *Journal of the American Mathematical Society*, 33(1):1–56, Sep 2019. arXiv: 1606.03140.

[SV18] William Slofstra and Thomas Vidick. Entanglement in non-local games and the hyperlinear profile of groups. *Annales Henri Poincaré*, 19(10):2979–3005, 2018.

[SW08] Volkher B Scholz and Reinhard F Werner. Tsirelson's problem. *arXiv:0812.4305*, 2008.

[Tsi85] Boris S Tsirelson. Quantum analogues of bell's inequalities. the case of two spatially divided domains. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.(LOMI)*, 142:174–194, 1985. In Russian.

[Tsi87] Boris S Tsirelson. Quantum analogues of bell's inequalities. the case of two spatially divided domains. *Journal of Soviet Mathematics*, 36(4):557–570, 1987. Translated from Russian.

[Vid22] Thomas Vidick. Almost synchronous quantum correlations. *Journal of mathematical physics*, 63(2):022201, 2022.

[Wat18] John Watrous. *The Theory of Quantum Information*. Cambridge university press, 2018.

[Weh06] Stephanie Wehner. Tsirelson bounds for generalized Clauser-Horne-Shimony-Holt inequalities. *Physical Review A*, 73(2):022110, 2006.

APPENDICES

Appendix A

Other works by the author

In addition to the work included in this thesis, I also spent substantial time during my Ph.D. collaborating on the following research projects:

Graph minors and linear system nonlocal games: Based on the work of Alex Arkhipov [Ark12], Vincent Russo, Turner Silverthorne, William Slofstra, and myself studied a subclass of linear system nonlocal games where the linear system are the incidence system of 2-coloured connected graphs. We showed that unlike for general linear system nonlocal games, the problem of deciding if there is a perfect quantum strategy for these games can be done in polynomial time, with respect to the size of the graph.

- [PRSS22] Paddock, Connor, Vincent Russo, Turner Silverthorne, and William Slofstra. "Arkhipov's theorem, graph minors, and linear system nonlocal games." *arXiv preprint arXiv:2205.04645* (2022) (to appear in Algebraic Combinatorics).

Self-testing and operator algebras: With William Slofstra, Yuming Zhao, and Yangchen Zhou, we presented an alternative definition for the concept of a self-test that is operator algebraic in nature. We showed that for the class of finite-dimensional quantum correlations our definition is equivalent to the original definition of self-testing with respect to local dilations of Meyers and Yao [MY04].

- [PSZZ23] Paddock, Connor, William Slofstra, Yuming Zhao, and Yangchen Zhou. "An operator-algebraic formulation of self-testing." *arXiv preprint arXiv:2301.11291* (2023) (submitted to Annales Henri Poincaré).