Dynamic Pricing Schemes in Combinatorial Markets

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

In combinatorial markets where buyers are self-interested, the buyers may make purchases that lead to suboptimal item allocations. As a central coordinator, our goal is to impose prices on the items of the market so that its buyers are incentivized to exclusively make optimal purchases. In this thesis, we study the question of whether dynamic pricing schemes can achieve the optimal social welfare in multi-demand combinatorial markets. This well-motivated question has been the topic of some study, but has remained mostly open, and to date, positive results are only known for extremal cases.

In this thesis, we present the current results for unit-demand, bi-demand and tri-demand markets. In the context of these results, we discuss the significance of not having a deficiency of items, which is known as the (OPT) condition. We outline an approach for handling an item deficiency, and we expose barriers to extending the known techniques to markets of larger demand.

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Chapter 1

Introduction

Markets are ubiquitous. Individual consumers, corporations, governments and computer algorithms ceaselessly interact with them. For this reason and others, abstract markets have been studied extensively for several decades in the fields of mathematics, economics and beyond. We study the open question of whether dynamic pricing schemes can achieve the optimal social welfare in combinatorial markets. To date, positive results are only known for extremal cases. We investigate how to strengthen the known techniques to answer the question in multi-demand markets and in markets with a deficiency of items.

A *combinatorial market* consists of a set of buyers and a set of indivisible items. Every buyer has a *valuation function* over subsets of items in the market, and this function indicates the benefit that each subset of items yields them. We refer to subsets of items as *bundles* for convenience. An *allocation* of a market is an assignment of bundles to buyers such that every item is assigned to at most one buyer. Given an allocation, its *social welfare* is the total benefit that it brings to all the buyers. An allocation is said to be *optimal* if it achieves the maximum social welfare among all possible allocations of the market.

As a central coordinator, our goal is to compute prices for the items of the market that force the buyers to make purchases that will ultimately lead to optimal allocations of the market. In our model, the buyers are self-interested, and are only concerned with maximizing their own utility. The hope is that, despite this individualistic behaviour, we may nevertheless be able to cleverly compute prices for the items of the market such that the buyers will universally be steered towards purchases that lead to optimal allocations of the market. Given prices for the items of the market, the *utility* of a bundle with respect to a buyer is the buyer's valuation of that bundle minus its total cost. A bundle is said to be *in demand* for a buyer if it achieves the maximum utility among all possible bundles that the buyer may purchase. In our model, every buyer will always purchase a bundle that is in demand for them.

In our purchasing model, the buyers arrive sequentially to the market in some unknown order. Upon arrival, each buyer purchases an arbitrary bundle that is in demand for them. Then, they exit the market along with their purchased bundle. At the end of this process, we obtain an allocation of the market. Our goal is then to compute a pricing scheme for the items of the market such that the resulting allocation of the market is always optimal, regardless of the order in which the buyers arrive and regardless of the specific bundles in demand that they purchase. Such a pricing scheme is said to be optimal.

The setting described above is quite general, and it boasts a comparatively small number of reasonable assumptions pertaining to the behaviour of buyers in a broad, economic setting. Thus, it should come as little surprise that these aforementioned pricing schemes in combinatorial markets are anything but a new concept in the field. In fact, the 19th-century mathematician Léon Walras has studied such problems as far back as 1896 [10]. In this work, Walras pioneered the notion of a *Walrasian equilibrium*. A Walrasian equilibrium in a combinatorial market is an allocation together with an item pricing scheme such that, for every buyer in the market, the bundle that is assigned to them by the allocation is also a bundle that is in demand for them with respect to the pricing scheme. If granted *tie-breaking authority*; that is, the ability to decide on the specific bundle that a given buyer shall purchase when multiple bundles are in demand for them, then a Walrasian pricing scheme is also an optimal pricing scheme for the market. Moreover, previous work by Kelso and Crawford in [7] has shown that Walrasian equilibria exist in arbitrary markets that admit gross substitutes valuation functions. However, in our model, we are not granted the power to reconcile ties. In [4], Cohen-Addad et al. show that without this tie-breaking power, Walrasian prices can lead to arbitrarily bad social welfare.

In addition, the work presented in [4] demonstrates that without tie-breaking authority, any *static pricing scheme*; that is, a pricing scheme that is computed once and cannot be updated, cannot yield more than 2/3 of the optimal social welfare in general. To illustrate this phenomenon, we present a detailed example that exposes the shortcomings of these static pricing schemes. However, we first need to introduce the concept of a *multi-demand* market.

A multi-demand market is a combinatorial market in which each buyer has a *demand*, and this demand indicates the maximum number of items that the buyer may purchase from the market. In addition, each buyer values single items in this setting and, for each bundle, the buyer's valuation of that bundle is the sum of their valuations of the individual items in the bundle. Remark that in this setting, every allocation of the market cannot

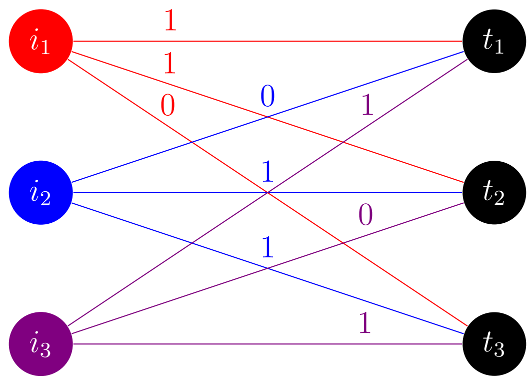

Figure 1.0.1: A market that does not admit an optimal static pricing scheme.

assign more items to a buyer than their demand allows for.

Now, returning to our aforementioned example, we consider a multi-demand combinatorial market with buyers $\{i_1, i_2, i_3\}$, items $\{t_1, t_2, t_3\}$ and singleton valuations as given in Figure 1.0.1. Also, every buyer has demand one. That is, every buyer can purchase a maximum of one item. It follows that market has exactly two optimal allocations, and each of them achieves a social welfare of 3. Indeed, one optimal allocation assigns item t_j to buyer i_j for all $j \in [3]$, and the other optimal allocation assigns item t_{j+1} to buyer i_j for all $j \in [3]$, where $t_4 := t_1$. Suppose for a contradiction that p is an optimal static pricing scheme for this market. Without tie-breaking authority, it follows that we must have $p_{t_j} < 1$ for all $j \in [3]$ to incentivize the buyers to purchase non-empty bundles. Without loss of generality, suppose $p_{t_1} \geq p_{t_2} \geq p_{t_3}$. Now, suppose the buyer i_3 arrives first to the market. Then, they may purchase the bundle $\{t_3\}$, since this bundle is in demand for them. Similarly, if the buyer i_1 arrives next, then they may purchase the bundle $\{t_2\}$. But then, regardless of whether the buyer i_2 purchases the bundle $\{t_1\}$ or purchases the bundle $\emptyset$, it follows that the resulting allocation achieves a sub-optimal social welfare of 2. Hence, this market does not admit an optimal static pricing scheme.

The above example reveals that, even in very simple markets such as the one presented above, static pricing schemes may fail to achieve more that $2/3$ of the optimal social welfare. To address this limitation, we instead consider *dynamic* pricing schemes, in which we may update the prices of the items after each buyer completes their purchase. In the previous example, after i_3 purchases $\{t_3\}$, for instance, then we can set $p_{t_1} := 0.5$ and $p_{t_2} := 0.9$. Then, these updated prices will steer the next buyer toward an optimal allocation of the market. In fact, the market in Figure 1.0.1 does admit an optimal dynamic pricing scheme.

Currently, it is an open question whether dynamic pricing schemes can achieve the optimal social welfare in general combinatorial markets that admit gross substitutes valuation functions. Informally, we say that a buyer's valuation function is *gross substitutes* if increasing the prices of some of the items does not cause the buyer to lose interest in purchasing their previously-preferred items whose prices have not been increased. For further discussion on gross-substitutes functions, their formal definition and their properties, we refer the reader to [8]. The restriction to gross substitutes functions is crucial, as Gul and Stacchetti in [6] and Berger et al. in [3] show that dynamic pricing schemes cannot be optimal, in general, whenever the market contains a non-gross substitutes valuation function.

Within the framework of multi-demand markets, positive results are known only for specific cases: every buyer has demand at most three; there are at most four buyers in the market; and there are at most two optimal allocations of the market [9]. Furthermore, these results assume the (OPT) condition, which is the requirement that in every optimal allocation, no buyer may receive less than their demand. The (OPT) condition is formally presented in Definition 2.0.2. An open question that our research failed to answer and which continues to interest us is whether optimal dynamic pricing schemes exist in multi-demand markets in which every buyer has demand at most four. Another direction to consider is to remove the (OPT) condition. In [5], Szögi gives a reduction to markets which satisfy the (OPT) condition for the case where every buyer has demand two. Their algorithm relies on a specific method of obtaining optimal prices for the reduced (OPT) case. Hence, we refer to this reduction as a "white-box" reduction. In contrast, a "black-box" reduction does not rely on any particular method of obtaining optimal prices in the reduced (OPT) case. The existence of such a black-box reduction to the (OPT) case remains an open problem.

The remainder of this thesis is organized as follows: In Chapter 2, we formally introduce the setting of our problem, and we present some results from the current literature. Next, in Chapter 3, we present a simplification of our problem that was first devised by Bérczi et al. in [1], and we use this reduction to prove the existence of optimal dynamic pricing schemes in three classes of multi-demand markets that satisfy the (OPT) condition: unit-demand markets, bi-demand markets and tri-demand markets. The unit-demand case was first solved by Cohen-Addad et al. in [4], the bi-demand case was first solved by Bérczi et al. in [1] and the tri-demand case was first solved by Pashkovich and Xie in [9]. Our aim is to provide a coherent, unified view of these three independent results and to expose the challenges associated with increasing the buyers' demands. Then, in Chapter 4, we investigate the consequences of removing the (OPT) condition. We present some tools that can help us handle this new setting and we give a black-box reduction for a specific case of multi-demand markets. Finally, in Chapter 5, we briefly discuss three open problems, and we present some barriers that naturally arise when trying to extend the known techniques.

Chapter 2

Preliminaries

In this chapter, we present some results from the current literature on dynamic pricing schemes. We also lay out our notation and explain the conventions that we will adopt.

Formally, we consider a multi-demand combinatorial market M with buyers I and items T. For each buyer $i \in I$, their demand is $b_i \in \mathbb{Z}^+$ and their valuation function over singleton sets of items is $v_i : T \to \mathbb{R}^+$. Then, for a bundle $X \subseteq T$, we have $v_i(X) = \sum_{t \in X} v_i(t)$. An allocation $\boldsymbol{\mathcal{A}}$ of the market M is an assignment of bundles to buyers $\boldsymbol{\mathcal{A}} = \{A_i \subseteq T : i \in I\}$ such that $|A_i| \leq b_i$ for every buyer $i \in I$ and such that $A_i \cap A_j = \emptyset$ for all buyers $i, j \in I$ such that $i \neq j$. In other terms, every buyer $i \in I$ receives at most b_i items, and every item is assigned to at most one buyer. The social welfare of an allocation is then $SW(\boldsymbol{\mathcal{A}}) := \sum_{i \in I} v_i(A_i)$. This is the total benefit that the allocation brings to all the buyers. Given a pricing scheme $p : T \to \mathbb{R}$ for the items of the market, the utility of a bundle $X \subseteq T$ with respect to a buyer $i \in I$ is $u_i(X, p) := v_i(X) - p(X)$, where $p(X) := \sum_{t \in X} p(t)$ is the total cost of the bundle X. Then, a bundle X is in demand for a buyer i if $X \in \operatorname{argmax}_{\substack{Y \subseteq T \\ |Y| \leq b_i}} u_i(Y, p)$.

We proceed to present an LP-based approach for computing optimal dynamic pricing schemes that is used in both [1] and in [9]. To begin, we form the *market graph* of the market M. The market graph of the market M, denoted by H, is the complete bipartite graph with partition sets I and T and whose edge weights are given by the buyers' valuations. That is, $w_{it} := v_i(t)$ for every buyer $i \in I$ and for every item $t \in T$. Note that Figure 1.0.1 is an example of a market graph. Then, we formulate the problem of maximizing the social welfare in the market M as the following LP relaxation together with its dual:

$$
\begin{array}{lll|lll}
\max & \sum_{i\in I}\sum_{t\in T} w_{it}x_{it} & & \min & \sum_{i\in I} b_i y_i + \sum_{t\in T} y_t & \\
\text{s.t.} & \sum_{t\in T} x_{it} \leq b_i & \forall i\in I & \text{s.t.} & y_i + y_t \geq w_{it} & \forall i\in I, t\in T \\
& \sum_{i\in I} x_{it} \leq 1 & \forall t\in T & & \boldsymbol{y} \geq 0 & \\
& 0 \leq \boldsymbol{x} \leq 1 & & & &
\end{array}
$$

The above problem is the maximum weight b-matching problem, where we set $b_t := 1$ for all items $t \in T$. The constraints ensure that every buyer $i \in I$ receives at most b_i items and that every item is allocated to at most one buyer. Thus, optimal integral solutions to the above LP are in a one-to-one correspondence with optimal allocations of the market M. We proceed to introduce the concept of *item legality*:

Definition 2.0.1. *For every buyer $i \in I$ and for every item $t \in T$, we say that the edge $it \in E(H)$ is legal if there exists an optimal allocation $\boldsymbol{\mathcal{A}} = \{A_j \subseteq T : j \in I\}$ of the market M such that $t \in A_i$. Moreover, we say that the item t is legal for the buyer i, and we write $\mathscr{L}_M(i) := \{r \in T :$ item r is legal for buyer $i\}$ to denote the set of items in the market M that are legal for the buyer i.*

Next, we show how to encode the legality of every buyer-item pair in the market M with a specific dual solution:

Claim 2.0.1. *There exists an optimal solution $\tilde{\boldsymbol{y}}$ to the dual LP presented above such that the following two conditions hold:*

(i) For every buyer $i \in I$ and for every item $t \in T$, the edge $it \in E(H)$ is legal if and only if $\tilde{y}_i + \tilde{y}_t = w_{it}$.

(ii) For every vertex $v \in I \cup T$, we have $\tilde{y}_v = 0$ if and only if there exists a maximum-weight b-matching F of the market graph H such that $\deg_F(v) < b_v$. Equivalently, if there exists an optimal allocation of the market M such that either the buyer v receives less than b_v items, or such that the item v is not allocated to any buyer.

Proof. The result follows by applying strict complementarity to the above primal-dual pair of LP's. The complementary slackness conditions are ($x_{it} = 0$ or $y_i + y_t = w_{it}$) and ($y_i = 0$ or $\sum_{t\in T} x_{it} = b_i$) and ($y_t = 0$ or $\sum_{i\in I} x_{it} = 1$.) Strict complementarity ensures the existence of a primal-dual pair of optimal solutions that satisfy the above complementary slackness conditions where every "or" is replaced with "exclusive or." This yields the desired properties. □

Next, we define the (OPT) condition:

Definition 2.0.2. *A multi-demand market M satisfies the (OPT) condition if every buyer $i \in I$ receives exactly b_i items in every optimal allocation of the market M.*

The (OPT) condition is useful because it implies $\tilde{y}_i > 0$ for all buyers $i \in I$, where $\tilde{\boldsymbol{y}}$ is the dual solution from Claim 2.0.1. The practicality of this will become apparent in Section 3.1, as it plays a key role in the reduction we present. In Chapter 4, we discuss some approaches to handling multi-demand markets that do not satisfy the (OPT) condition.

In [1], Bérczi et al. give a polynomial-time algorithm for computing the dual solution $\tilde{\boldsymbol{y}}$ in Claim 2.0.1. Then, assuming the (OPT) condition, these authors reduce the problem of finding an optimal dynamic pricing scheme to that of finding an *adequate ordering*. This an ordering $\sigma : T \to \{1, \ldots, |T|\}$ of the items of the market M such that, for each buyer $i \in I$, there exists an optimal allocation which assigns to the buyer i the first b_i items (according to the ordering σ) that are legal for them. We present this reduction in Section 3.1, but with a different perspective that is based on properties of the buyers' valuation functions.

In the case where every buyer is unit-demand; that is, when $b_i = 1$ for all buyers $i \in I$, then the reduction presented in [1] instantly gives an optimal dynamic pricing scheme for the market M. This is a much simpler proof than the one formerly presented in [4]. Indeed, in the unit-demand case, every ordering of the items is adequate because every legal edge is contained in an optimal allocation of the market by definition of item legality. However, when buyers' demands are increased to two, computing an adequate ordering is significantly more difficult because not every pair of legal edges is necessarily contained in an optimal allocation of the market. The authors of [1] still manage to find such an adequate ordering, and their high-level approach is to use the *legality graph* of the market M. The legality graph of the market M, denoted by G, is the bipartite graph with partition sets I and T and whose edges are precisely the legal edges of the market graph H. Assuming the (OPT) condition, and assuming that the number of items in the market coincides with the buyers' total demand, an assumption that is justified by Claim 2.0.2, it follows that optimal allocations of the market M are in a one-to-one correspondence with b-factors of the legality graph G. The authors of [1] proceed to consider structures in the legality graph G which arise from pairs of legal edges that are not contained in any b-factor of the graph G. By Hall's theorem, these edges enable a splitting of the market into two smaller markets. The authors apply induction on these smaller markets, and then combine the two smaller adequate orderings into one global adequate ordering of the original market.

The authors of [9] prove the existence of optimal dynamic pricing schemes in the case where every buyer has demand at most three. Their high-level approach is very similar to the one used in [1]. However, increasing the buyers' demands to three requires a stronger induction hypothesis: that for any item $t \in T$, there exists an adequate ordering in which the item t is placed first. Without this strengthening of the induction hypothesis, the adequate orderings that are obtained from the two smaller markets may not be able to be combined into a global adequate ordering of the original market.

To continue, we note the following convention that we will adopt: Whenever we prove the optimality of a dynamic pricing scheme p, we simply prove that for every buyer $i \in I$, every bundle that is in demand for the buyer i with respect to the prices p extends to an optimal allocation of the market. This is sufficient because, after the first buyer arrives and completes their purchase, we obtain a smaller instance of a multi-demand market in which item prices may be recomputed. Then, induction may be applied to obtain the subsequent item prices.

Next, we introduce the following notation that will be convenient when we discuss how to combine two smaller pricing schemes into a larger pricing scheme:

Definition 2.0.3. *Given a pricing scheme $p : T \to \mathbb{R}$ and given two disjoint subsets of items $S_1, S_2 \subseteq T$, we write $S_1 <_p S_2$ to indicate $p_{s_1} < p_{s_2}$ for all items $s_1 \in S_1, s_2 \in S_2$.*

Next, given an allocation $\boldsymbol{\mathcal{A}}$ of the market M, we define two properties that we essentially always want the allocation to have:

Definition 2.0.4. *Given an allocation $\boldsymbol{\mathcal{A}} = \{A_j \subseteq T : j \in I\}$ of the market M, we say that the allocation $\boldsymbol{\mathcal{A}}$ respects legality if $A_j \subseteq \mathscr{L}_M(j)$ for all buyers $j \in I$. We say that the allocation $\boldsymbol{\mathcal{A}}$ respects demand if $|A_j| \leq b_j$ for all buyers $j \in I$.*

Now, we introduce two definitions that will be referenced throughout the technical chapters to come. These definitions may seem mysterious at the moment, but they will be motivated in Chapter 5 when we discuss possible generalizations of the arguments to be presented:

Definition 2.0.5. *A bundle $B \subseteq T$ is said to be flexible in the market M if every partial assignment of items in B to buyers in I that respects both legality and demand in the market M extends to an optimal allocation of the market M.*

Definition 2.0.6. *We write $k_M := \min_{B \subseteq T}\{|B| : B$ is not flexible in the market $M\}$ to denote the smallest size of a bundle that is not flexible in the market M. Note that $k_M \geq 2$. Also, note that it is possible to have $k_M = \infty$.*

Next, we show that for a multi-demand market M, we may assume every item is allocated to some buyer in each of its optimal allocations:

Claim 2.0.2. *Let M be a multi-demand market. If there exist optimal dynamic pricing schemes for multi-demand markets where all items are allocated in every optimal allocation, then there exists an optimal dynamic pricing scheme for the market M.*

Proof. Let $\boldsymbol{\mathcal{A}}$ be an optimal allocation of the market M, and let T^* denote the set of items which are not assigned by the allocation $\boldsymbol{\mathcal{A}}$ to any buyer in the market M. Let M' be the market with buyers I, items $T \setminus T^*$ and the same demands b and valuations v as the original market M. Observe that every optimal allocation of the market M' is an optimal allocation of the market M. We iterate this process as many times as necessary until we obtain a market M'', say with items $T \setminus T^\dagger$, which satisfies the requirement that each of its items is allocated to some buyer in each of its optimal allocations. Then, we again have that every optimal allocation of the market M'' is an optimal allocation of the market M. Now, suppose p'' is an optimal dynamic pricing scheme for the market M''. Then, we construct a pricing scheme p for the original market M as follows:

$$p_t := \begin{cases} p''_t & \text{if } t \in T \setminus T^\dagger \\ \max_{\substack{i \in I \\ t \in T}} \{v_i(t)\} + 1 & \text{if } t \in T^\dagger \end{cases} \qquad \forall t \in T.$$

By construction, we have $u_i(t, p) < 0$ for every buyer $i \in I$ and for every item $t \in T^\dagger$. Hence, for every buyer $i \in I$, the bundles that are in demand for the buyer i with respect to the prices p are precisely the bundles that are in demand for the buyer i with respect to the prices p''. Thus, if we impose the prices p on the market M and the buyer i arrives first, then they will purchase a bundle which, by optimality of the dynamic pricing scheme p'', extends to an optimal allocation of the market M''. Since every optimal allocation of the market M'' is an optimal allocation of the market M, it follows that the dynamic pricing scheme p is an optimal dynamic pricing scheme for the original market M, as desired. □

Chapter 3

Dynamic Pricing in Multi-Demand Markets With (OPT)

In this chapter, we investigate the existence of optimal dynamic pricing schemes in multi-demand markets which satisfy the (OPT) condition. Recall that the (OPT) condition is the requirement that every optimal allocation of the market M assigns exactly b_i items to every buyer $i \in I$. Remark that if a market M satisfies the (OPT) condition, then it follows that $|T| \geq b(I)$, where $b(I) := \sum_{i \in I} b_i$ is the total demand of all the buyers. By Claim 2.0.2, we may further assume $|T| = b(I)$. In Section 3.1, we present a reduction to multi-demand markets which have very specific valuation functions. This reduction enables us to link buyers' valuations to optimal allocations of the market, which will prove to be quite useful in our task of proving the existence of optimal dynamic pricing schemes. A version of this reduction was first shown by Bérczi et al. in [1]. After proving the correctness of this reduction, in Section 3.2, we proceed to prove the existence of optimal dynamic pricing schemes in "unit-demand" markets, i.e. multi-demand markets in which every buyer's demand is equal to one. This result was first proved by Cohen-Addad et al. in [9] using a different method. The result follows immediately from the aforementioned reduction in Section 3.1. Next, in section 3.3, we prove the existence of optimal dynamic pricing schemes in "bi-demand" markets, i.e. multi-demand markets in which every buyer's demand is at most two. This result was first proved by Bérczi et al. in [1]. This case is more intricate, and it involves subdividing the market into two smaller submarkets which admit optimal dynamic pricing schemes by induction, and then combining these two smaller pricing schemes in a suitable way to obtain an optimal dynamic pricing scheme for the original market. Finally, in Section 3.4, we prove the existence of optimal dynamic pricing schemes in "tri-demand" markets, i.e. multi-markets in which every buyer's demand is at most three. This result was first proved by Pashkovich and Xie in [9]. This case is more complicated than the bi-demand case because it admits two distinct scenarios that lead to the aforementioned splitting of the market. In addition, the tri-demand case requires a stronger induction hypothesis. Although more complicated than the bi-demand case, almost all of the proofs of the intermediate results that we present in the tri-demand case are natural extensions of their analogues in the bi-demand case. Hopefully, these parallelisms succeed at illustrating how one could hope to generalize the method we will present to allow for arbitrary buyer demands in multi-demand markets. However, our method also exposes some barriers to such a generalization.

3.1 Reduction to Markets With the $(*)$ Condition

In this section, we present a reduction of the problem of finding optimal dynamic pricing schemes in multi-demand markets satisfying the (OPT) condition to multi-demand markets which have very specific valuation functions. This criterion, which we will refer to as the $(*)$ condition, is presented below:

Definition 3.1.1. *A market M satisfies the $(*)$ condition if its valuations are as follows:*

$$v_i(t) = \begin{cases} 1 & \text{if } t \in \mathscr{L}_M(i) \\ 0 & \text{if } t \notin \mathscr{L}_M(i) \end{cases} \quad \forall i \in I, t \in T.$$

Intuitively, these values correspond to the support of the legality graph G. As a preliminary step to the reduction, we prove the following claim:

Claim 3.1.1. *If there exists an optimal dynamic pricing scheme for a multi-demand market M satisfying the (OPT) condition, then there exists an optimal dynamic pricing scheme for the market M such that every item has a unique price.*

Proof. Let p be an optimal dynamic pricing scheme for the market M, and suppose there exist items $x, y \in T$ such that $x \neq y$ and $p_x = p_y$. The idea of our proof is to slightly increase the value of p_x so that it becomes unique, but also to increase p_x by a small enough value so that we preserve the optimality of the pricing scheme p.

Next, we define the following quantity:

$$\epsilon := \min_{\substack{i \in I \\ t \in T}} \{u_i(x,p) - u_i(t,p) : u_i(x,p) > u_i(t,p)\}.$$

Intuitively, the value of ϵ is the minimum amount by which a buyer $i \in I$ strictly prefers the item x to another item in the market M with respect to the prices p.

We also define the following quantity:

$$\delta := \begin{cases} 1 & \text{if } p_x \in \text{argmax}_{t \in T}\{p_t\} \\ \min_{t \in T}\{p_t - p_x : p_t > p_x\} & \text{otherwise} \end{cases}.$$

Intuitively, the value of δ is the minimum amount by which an item $t \in T$ is priced higher than the item x, if such a value exists, and it is equal to one otherwise. Note that $\epsilon, \delta > 0$. We proceed to consider two cases:

Case 1 $\epsilon = \infty$. That is, no buyer in the market M strictly prefers the item x to another item in the market M. In this case, we define the following modified pricing scheme:

$$p_t^{new} := \begin{cases} p_t & \text{if } t \neq x \\ p_t + \delta/2 & \text{if } t = x \end{cases} \quad \forall t \in T.$$

Let $i \in I$ be a buyer. We may assume $|T| > b_i$, as otherwise, we have $I = \{i\}$, and computing an optimal dynamic pricing scheme with unique prices is trivial in this case. Next, since $\epsilon = \infty$, it follows by the (OPT) condition that there exist b_i distinct items $\{t_1, \dots, t_{b_i}\} \subseteq T \setminus \{x\}$ such that $u_i(t_j, p) \geq u(x, p)$ for all $j \in [b_i]$. To prove that this new pricing scheme p^{new} is optimal for the market M, we show that every bundle in demand for the buyer i with respect to the new prices p^{new} is also a bundle in demand for the buyer i with respect to the original (optimal) prices p. The optimality of the pricing scheme p^{new} follows.

Let $B \subseteq T$ be a bundle in demand for the buyer i with respect to the prices p^{new}. Recall that for every index $j \in [b_i]$, we have:

$$u_i(t_j, p^{new}) = u_i(t_j, p) \geq u_i(x, p) > u_i(x, p^{new}).$$

Hence, we have $x \notin B$. It follows that $u_i(B, p^{new}) = u_i(B, p)$. Suppose for a contradiction that the bundle B is not in demand for the buyer i with respect to the original prices p. Then, there exists a bundle $C \subseteq T$ such that $u_i(C, p) > u_i(B, p)$. For every index $j \in [b_i]$ we have $u_i(t_j, p) \geq u_i(x, p)$, so we may assume $x \notin C$. Thus, we have:

$$u_i(B, p^{new}) = u_i(B, p) < u_i(C, p) = u_i(C, p^{new}).$$

This contradicts our assumption that the bundle B is in demand for the buyer i with respect to the new prices p^{new}. In conclusion, the bundle B is in demand for the buyer i with respect to the original prices p. Hence, the modified pricing scheme p^{new} is an optimal dynamic pricing scheme for the market M. Moreover, the modified pricing scheme p^{new} has exactly one more item with a unique price than the original pricing scheme p does because the value of p_x^{new} is unique by our choice of δ.

Case 2 $0 < \epsilon < \infty$. In this case, we define the following modified pricing scheme:

$$p_t^{new} := \begin{cases} p_t & \text{if } t \neq x \\ p_t + \min\{\epsilon/2, \delta/2\} & \text{if } t = x \end{cases} \quad \forall t \in T.$$

Let $i \in I$ be a buyer. We proceed as in the previous case, and we again show the correspondence between bundles in demand for the buyer i with respect to the original prices p and bundles in demand for the buyer i with respect to the new prices p^{new}.

Let $B \subseteq T$ be a bundle in demand for the buyer i with respect to the new prices p^{new}. Suppose for a contradiction that the bundle B is not in demand for the buyer i with respect to the original prices p. Then, there exists a bundle $C \subseteq T$ such that $u_i(C,p) > u_i(B,p)$. Suppose first that $x \in B$. Then, we have:

$$\begin{aligned} u_i(B,p^{new}) &= u_i(B,p) - \min\{\epsilon/2, \delta/2\} \\ &< u_i(C,p) - \min\{\epsilon/2, \delta/2\} \\ &\leq u_i(C,p^{new}). \end{aligned}$$

This contradicts our assumption that the bundle B is in demand for the buyer i with respect to the new prices p^{new}. Thus, we have $x \notin B$. It follows that there exist b_i distinct items $\{t_1, \ldots, t_{b_i}\} \subseteq T \setminus \{x\}$ such that $u_i(t_j, p^{new}) \geq u_i(x, p^{new})$ for all $j \in [b_i]$. Moreover, for each index $j \in [b_i]$, we have:

$$\begin{aligned} u_i(t_j,p) &= u_i(t_j,p^{new}) \\ &\geq u_i(x,p^{new}) \\ &= u_i(x,p) - \min\{\epsilon/2, \delta/2\} \\ &> u_i(x,p) - \epsilon. \end{aligned}$$

Thus, we have $u_i(x,p) - u_i(t_j,p) < \epsilon$. If $u_i(x,p) > u_i(t_j,p)$, then by definition of the quantity ϵ, we have $u_i(x,p) - u_i(t_j,p) \geq \epsilon$. Hence, we must have $u_i(x,p) \leq u_i(t_j,p)$. Since this inequality holds for each index $j \in [b_i]$, we may assume $x \notin C$. Then, we obtain:

$$u_i(B,p^{new}) = u_i(B,p) < u_i(C,p) = u_i(C,p^{new}).$$

This again contradicts our assumption that the bundle B is in demand for the buyer i with respect to the new prices p^{new}. In conclusion, the bundle B is in demand for the buyer i with respect to the original prices p. Hence, the modified pricing scheme p^{new} is an optimal dynamic pricing scheme for the market M. Moreover, the modified pricing scheme p^{new} again has exactly one more item with a unique price than the original pricing scheme p does because the value of p_x^{new} is again unique by our choice of δ.

In each of the above cases, we may update the initial optimal dynamic pricing scheme p to obtain a new optimal dynamic pricing scheme p^{new} that has exactly one more item with a unique price than the original pricing scheme p does. Repeating this process as many times as necessary, we ultimately obtain an optimal dynamic pricing scheme for the market M such that every item has a unique price, completing the proof of the claim. $\square$

Next, we proceed to reduce the problem of finding optimal dynamic pricing schemes in multi-demand markets satisfying the (OPT) condition to the problem of finding optimal dynamic pricing schemes in markets that satisfy the $(*)$ condition:

Lemma 3.1.1. *Let M be a multi-demand market satisfying the (OPT) condition. If there exist optimal dynamic pricing schemes for markets satisfying the $(*)$ condition, then there exists an optimal dynamic pricing scheme for the market M.*

Proof. Let M be a multi-demand market satisfying the (OPT) condition. By Claim 2.0.2, we may assume $|T| = b(I)$. Let $\tilde{\boldsymbol{y}}$ be a dual solution for the market graph H as provided by Claim 2.0.1. Then, we form an auxiliary market M' with the same buyers I, the same items T, the same demands b and the following valuations v' :

$$v_i'(t) = \begin{cases} 1 & \text{if } t \in \mathscr{L}_M(i) \\ 0 & \text{if } t \notin \mathscr{L}_M(i) \end{cases} \quad \forall i \in I, t \in T.$$

Recall that the legality of every buyer-item pair can be computed via $\tilde{\boldsymbol{y}}$. Note that optimal allocations of the auxiliary market M' are precisely optimal allocations of the original market M. Thus, the auxiliary market M' satisfies both the (OPT) condition and the $(*)$ condition.

Now, suppose p' is an optimal dynamic pricing scheme for the auxiliary market M'. By Claim 3.1.1, we may assume the pricing scheme p' has unique values, so we can order the items of our market in increasing order of their prices. Let us call this ordering $\sigma : T \to \{1, \dots, |T|\}$. That is, $\sigma(x) > \sigma(y)$ if and only if $p'_x > p'_y$. Now, we define the following quantity:

$$\alpha := \min\left\{\min_{\substack{i\in I\\ t\in T}}\{\tilde{y}_i + \tilde{y}_t - w_{it} : it \text{ is not legal}\}, \min_{v\in I\cup T}\{\tilde{y}_v : \tilde{y}_v > 0\}\right\}.$$

Recall that $\tilde{y}_v \neq 0$ for all $v \in I \cup T$ by the (OPT) condition, so the requirement $\tilde{y} > 0$ is not necessary in the above definition of the quantity α. However, this requirement will become relevant in Chapter 4 when we no longer assume the (OPT) condition. Note also that $\alpha > 0$. Next, we define prices p for the market M as follows:

$$p_t := \tilde{y}_t + \frac{\alpha}{|T|+1}\sigma(t) \quad \forall t \in T.$$

For the remainder of the proof of Lemma 3.1.1, we write u for the utility with respect to the market M and we write u' for the utility with respect to the auxiliary market M'. In the following claim, we show that legal items have strictly positive utility for every buyer $i \in I$:

Claim 3.1.2. *Let $i \in I$ be a buyer and let $t \in T$ be an item such that $t \in \mathscr{L}_M(i)$. Then, we have $u_i(t,p) > 0$.*

Proof.

$$\begin{aligned}
u_i(t,p) &= w_{it} - \left(\tilde{y}_t + \frac{\alpha}{|T|+1}\sigma(t)\right) \\
&= \tilde{y}_i - \frac{\alpha}{|T|+1}\sigma(t) && \text{by Claim 2.0.1 (i)} \\
&\geq \tilde{y}_i - \alpha\frac{|T|}{|T|+1} && \text{since } \sigma(t) \leq |T| \\
&> \tilde{y}_i - \alpha && \text{since } \alpha > 0 \\
&\geq 0 && \text{by (OPT), Claim 2.0.1 (ii) and our choice of } \alpha.
\end{aligned}$$

In conclusion, we have $u_i(t,p) > 0$, as desired. □

Next, we show that every buyer $i \in I$ strictly prefers legal items to non-legal items in the market M:

Claim 3.1.3. *Let $i \in I$ be a buyer and let $t_1, t_2 \in T$ be items such that $t_1 \in \mathscr{L}_M(i)$ and $t_2 \notin \mathscr{L}_M(i)$. Then, we have $u_i(t_1,p) > u_i(t_2,p)$.*

Proof.

$$
\begin{aligned}
& u_i(t_1,p) - u_i(t_2,p) \\
& = \left[w_{it_1} - \left(\tilde{y}_{t_1} + \frac{\alpha}{|T|+1}\sigma(t_1)\right)\right] \\
& - \left[w_{it_2} - \left(\tilde{y}_{t_2} + \frac{\alpha}{|T|+1}\sigma(t_2)\right)\right] \\
& = (\tilde{y}_i + \tilde{y}_{t_2} - w_{it_2}) + \frac{\alpha}{|T|+1}(\sigma(t_2) - \sigma(t_1)) && \text{by Claim 2.0.1 (i)} \\
& \geq \alpha + \frac{\alpha}{|T|+1}(\sigma(t_2) - \sigma(t_1)) && \text{by definition of } \alpha \\
& > \alpha - \alpha\frac{|T|}{|T|+1} && \text{since } \sigma(t_2) \geq 1 \text{ and } \sigma(t_1) < |T|+1 \\
& > 0 && \text{since } \alpha > 0.
\end{aligned}
$$

In conclusion, we have $u_i(t_1,p) > u_i(t_2,p)$, as desired. □

Lastly, we show that the preferences between legal items for every buyer $i \in I$ are the same in both the market M and the market M' :

Claim 3.1.4. *Let $i \in I$ be a buyer and let $t_1, t_2 \in T$ be distinct items such that $t_1, t_2 \in \mathscr{L}_M(i)$. Then, we have $u_i(t_1,p) > u_i(t_2,p)$ if and only if $u'_i(t_1,p') > u'_i(t_2,p')$.*

Proof. Consider:

$$
\begin{aligned}
& u_i(t_1,p) > u_i(t_2,p) \\
& \iff w_{it_1} - \left(\tilde{y}_{t_1} + \frac{\alpha}{|T|+1}\sigma(t_1)\right) \\
& > w_{it_2} - \left(\tilde{y}_{t_2} + \frac{\alpha}{|T|+1}\sigma(t_2)\right) \\
& \iff \tilde{y}_i - \frac{\alpha}{|T|+1}\sigma(t_1) > \tilde{y}_i - \frac{\alpha}{|T|+1}\sigma(t_2) && \text{by Claim 2.0.1 (i)} \\
& \iff \sigma(t_1) < \sigma(t_2) \\
& \iff p'_{t_1} < p'_{t_2} \\
& \iff v'_i(t_1) - p'_{t_1} > v'_i(t_2) - p'_{t_2} && \text{since } v'_i(t_1) = v'_i(t_2) = 1 \\
& \iff u'_i(t_1,p') > u'_i(t_2,p').
\end{aligned}
$$

Thus, we have $u_i(t_1, p) > u_i(t_2, p)$ if and only if $u_i'(t_1, p') > u_i'(t_2, p')$, as desired. $\square$

Now, let $B \subseteq T$ be a bundle in demand for the buyer i with respect to the prices p. By the previous three claims, it follows that the bundle B consists of the first b_i items (with respect to the ordering σ) that are legal for the buyer i. Since the auxiliary market M' satisfies the (OPT) condition, it follows that the bundle B is the unique bundle in demand for the buyer i with respect to the prices p'. By optimality of the pricing scheme p', it follows that the assignment of the bundle B to the buyer i extends to an optimal allocation $\boldsymbol{\mathcal{A}}$ of the auxiliary market M'. Moreover, the allocation $\boldsymbol{\mathcal{A}}$ is also an optimal allocation of the original market M. Hence, the pricing scheme p is an optimal pricing scheme for the original market M, completing the proof of the lemma. $\square$

To close Section 3.1, we present a specific example of a multi-demand market, and we apply the reduction in Lemma 3.1.1 to it. Consider the multi-demand market M_1 shown in Figure 3.1.1. In this instance, every buyer has demand two. An optimal dual solution, as provided by Claim 2.0.1, is given in Figure 3.1.2. Here, every node's label is replaced by its dual value and the tight edges of the market graph are thickened. One can verify that the market M_1 has exactly two optimal allocations, and these are both shown in Figure 3.1.3. Each optimal allocation achieves a social welfare of 31, and this is equal to the objective value of the aforementioned dual solution. Finally, the auxiliary market M_1' for the market M_1 is shown in Figure 3.1.4.

Next, we give the following optimal dynamic pricing scheme p' for the auxiliary market M_1' in Figure 3.1.4:

$$p'_{t_1} = 0.4 < p'_{t_5} = 0.5 < p'_{t_2} = 0.6 < p'_{t_6} = 0.7 < p'_{t_3} = 0.8 < p'_{t_4} = 0.9.$$

One can verify the optimality of the above pricing scheme p' for the market M_1'. Then, using the dual solution from Figure 3.1.2, we have $\alpha = 1$. Finally, employing the formula $p_t := \tilde{y}_t + \frac{\alpha}{|T|+1}\sigma(t)$, we compute the corresponding dynamic pricing scheme p for the market M_1 :

$$p_{t_1} = \frac{15}{7} < p_{t_5} = \frac{16}{7} < p_{t_2} = \frac{24}{7} < p_{t_6} = \frac{25}{7} < p_{t_4} = \frac{27}{7} < p_{t_3} = \frac{33}{7}.$$

One can verify the optimality of the above pricing scheme p for the market M_1.

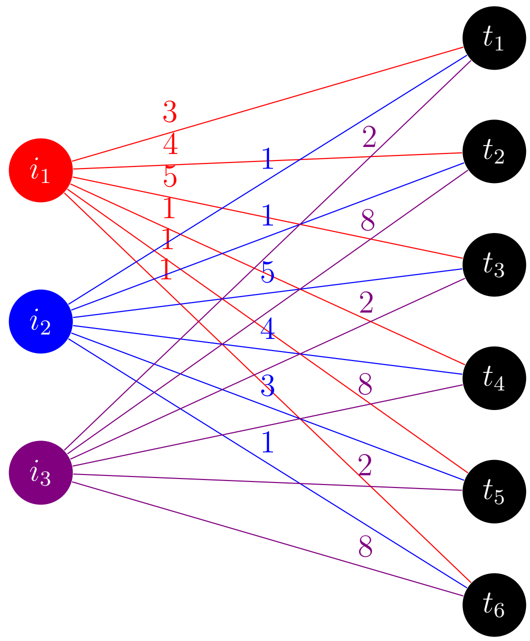

Figure 3.1.1: A multi-demand market M_1 with $b_{i_1} = b_{i_2} = b_{i_3} = 2$.

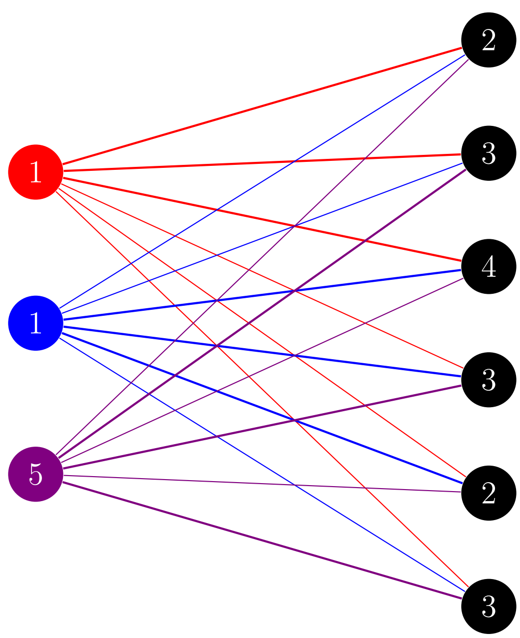

Figure 3.1.2: An optimal dual solution for the market M_1 as provided by Claim 2.0.1.

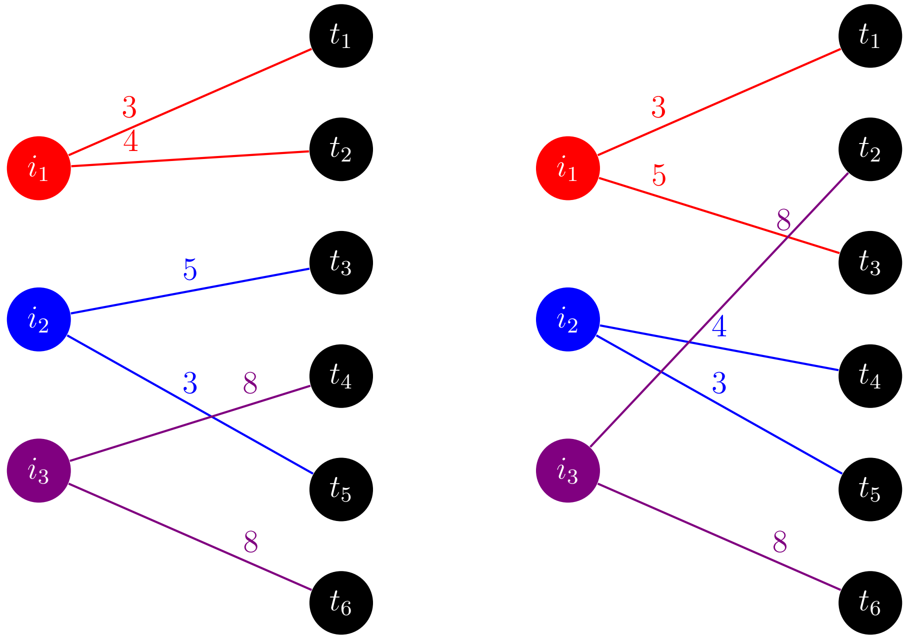

Figure 3.1.3: Optimal allocations of the market M_1.

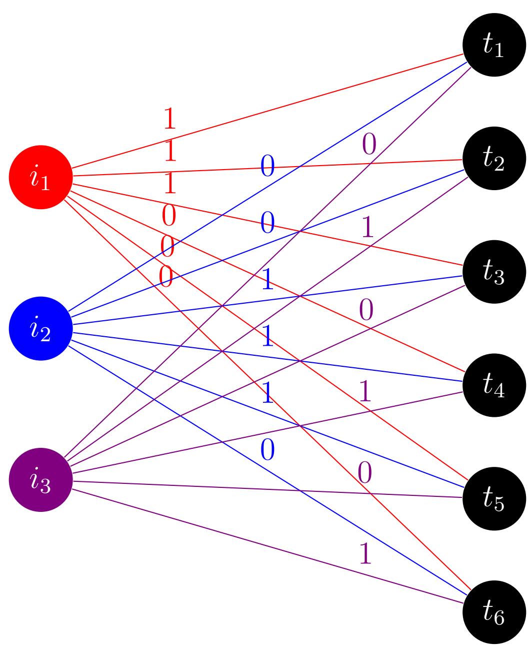

Figure 3.1.4: The auxiliary market M_1' for the market M_1.

3.2 Unit-Demand Markets

In this section, we show that optimal dynamic pricing schemes exist in unit-demand markets.

Lemma 3.2.1. *Let M be a multi-demand market satisfying the (OPT) condition and suppose $b_i = 1$ for all buyers $i \in I$. Then, there exists an optimal dynamic pricing scheme for the market M.*

Proof. First, we may assume the market M satisfies the $(*)$ condition by Lemma 3.1.1. Then, we define the following pricing scheme p for the market M :

$$p_t := 0.5 \quad \forall t \in T.$$

Suppose the buyer $i \in I$ arrives first to the market. Then, we have:

$$u_i(t, p) = \begin{cases} 0.5 & \text{if } t \in \mathscr{L}_M(i) \\ -0.5 & \text{if } t \notin \mathscr{L}_M(i) \end{cases} \quad \forall t \in T.$$

Since $b_i = 1$, it follows that the buyer i will purchase an arbitrary item $t \in T$ that is legal for them. Note that such an item exists by the (OPT) condition. Moreover, the assignment of the item t to the buyer i extends to an optimal allocation of the market M by definition of legality. Thus, the pricing scheme p is optimal, as desired. $\square$

3.3 Bi-Demand Markets

In this section, we show that optimal dynamic pricing schemes exist in bi-demand markets.

Lemma 3.3.1. *Let M be a multi-demand market satisfying the (OPT) condition and suppose $b_i \leq 2$ for all buyers $i \in I$. Then, there exists an optimal dynamic pricing scheme for the market M.*

Proof. To begin, we may assume $|T| = b(I)$ and that the market M satisfies the $(*)$ condition by the previous results.

Suppose first that $k_M \geq 3$. Then, we define the following pricing scheme:

$$p_t := 0.5 \quad \forall t \in T.$$

As in the unit-demand case, if the buyer $i \in I$ arrives first to the market, then they will purchase an arbitrary bundle $B \subseteq T$ such that $|B| = b_i$ and such that $B \subseteq \mathscr{L}_M(i)$. Again, note that such a bundle B exists by the (OPT) condition. Moreover, since $k_M > 2$, it follows that every bundle of size at most two is flexible in the market M. As $b_i \leq 2$, it follows that the assignment of the items in the bundle B to the buyer i extends to an optimal allocation of the market M. Hence, p is an optimal dynamic pricing scheme for the market M, completing the proof of Lemma 3.3.1.

Thus, we may assume $k_M = 2$. In this case, we proceed by induction on the number of buyers in the market and, subject to this quantity, we proceed by induction on the number of items in the market.

Since $k_M = 2$, it follows that there exists a bundle of size two that is not flexible in the market M. Let $\{t_1^*, t_2^*\} \subseteq T$ be such a bundle, and let $i_1^*, i_2^* \in I$ be buyers such that the assignment of item t_1^* to buyer i_1^* and item t_2^* to buyer i_2^* respects both legality and demand in the market M but does not extend to an optimal allocation of the market M. Note that it may be the case that $i_1^* = i_2^*$. Let G' be the graph obtained from the legality graph G by decreasing the b-values of each of the vertices $i_1^*, i_2^*, t_1^*, t_2^*$ by one and removing them if their updated b-value is equal to zero. Let b' denote the updated b-values of the vertices in the graph G'. Recall that optimal allocations of the market M are precisely b-factors of the graph G by the (OPT) condition. Since the above assignment does not extend to an optimal allocation of the market M, it follows that the graph G' does not have a b'-factor. Then, by Hall's theorem, there exists a set of buyers $S \subseteq I$ such that $|N_{G'}(S)| < b'(S)$. Among all possible choices of bundles B of size two that are not flexible in the market M, assignments of the items in the bundle B to buyers in the set I that do not extend to an optimal allocation of the market M, and sets S satisfying the above inequality in the resulting graph G'; we select a triple such that the set $N_G(S)$ is maximal. This last condition will be used in the proof of Claim 3.3.4.

At a high level, the above assignment of items to buyers that does not extend to an optimal allocation of the market M enables a "splitting" of the market M, as portrayed in Figure 3.3.1. This splitting yields an "upper" market M^U (see Figure 3.3.2) and one of two "lower" markets $M^{L'}$ (see Figure 3.3.3) or M^L (see Figure 3.3.4.) As we will demonstrate, these smaller markets are useful because they preserve the structure of the initial market M. Moreover, since these submarkets are strictly smaller than the original market M, we may apply induction on them to obtain their respective optimal dynamic pricing schemes. Finally, we combine the smaller pricing schemes into a global pricing scheme for the original market M.

Next, we proceed to establish some useful properties of the set S :

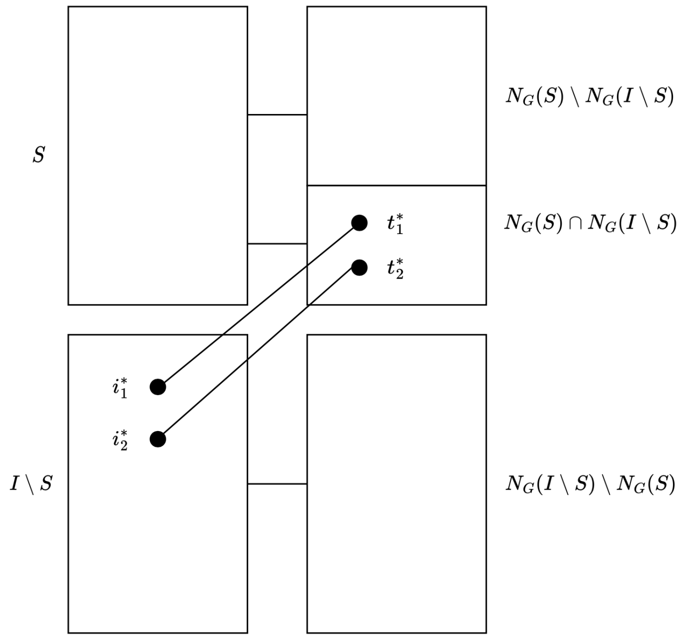

Figure 3.3.1: The splitting of the market M.

Claim 3.3.1. *We have* $|N_G(S)| = b(S) + 1$ *and* $\{t_1^*, t_2^*\} \subseteq N_G(S)$. *Furthermore, we have* $i_1^*, i_2^* \notin S$.

Proof. Since the graph G has a b-factor, it follows by Hall's theorem that $|N_G(S)| \geq b(S)$. Suppose for a contradiction that $i_1^*, i_2^* \in S$. Then, we have:

$$|N_{G'}(S)| \geq |N_G(S)| - 2 \geq b(S) - 2 = b'(S).$$

This contradicts our assumption that $|N_{G'}(S)| < b'(S)$. Hence, we have $i_1^* \notin S$ or $i_2^* \notin S$. Without loss of generality, suppose $i_1^* \notin S$.

Suppose for a contradiction that $t_1^* \notin N_G(S)$. If $i_2^* \in S$, then we have:

$$|N_{G'}(S)| \geq |N_G(S)| - 1 \geq b(S) - 1 = b'(S).$$

This contradicts our assumption that $|N_{G'}(S)| < b'(S)$. Hence, we have $i_2^* \notin S$. Then, the inequality $|N_{G'}(S)| < b'(S)$ implies $t_2^* \in N_G(S)$ and $|N_G(S)| = b(S)$. However, if $|N_G(S)| = b(S)$, then it follows by the (OPT) condition that every optimal allocation of the market M assigns every item in $N_G(S)$ to buyers in S. This contradicts our assumption

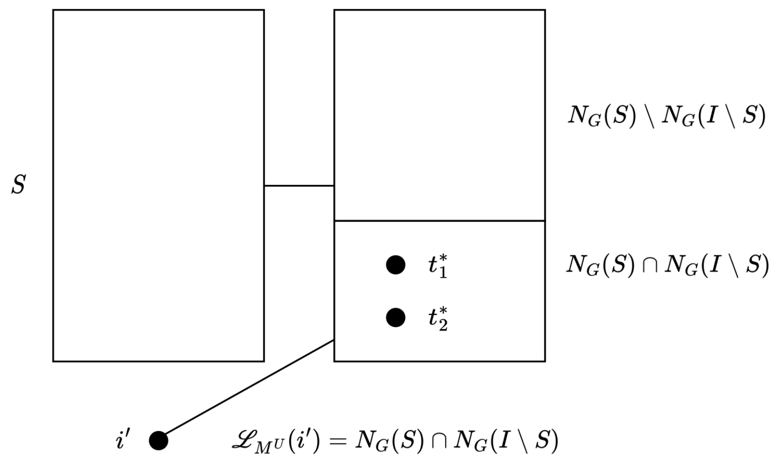

Figure 3.3.2: The upper market M^U.

that the item $t_2^* \in N_G(S)$ is legal for the buyer $i_2^* \notin S$. Hence, we have $t_1^* \in N_G(S)$.

Next, since $i_1^* \notin S$ and since the item $t_1^* \in N_G(S)$ is legal for the buyer i_1^*, we have $|N_G(S)| > b(S)$. If $|N_G(S)| \geq b(S) + 2$, then we have:

$$|N_{G'}(S)| \geq |N_G(S)| - 2 \geq b(S) \geq b'(S).$$

This contradicts our assumption that $|N_{G'}(S)| < b'(S)$. Thus, we have $|N_G(S)| = b(S)+1$. Finally, the inequality $|N_{G'}(S)| < b'(S)$ implies $i_2^* \notin S$ and $t_2^* \in N_G(S)$, completing the proof of the claim. □

Our next step is to define an auxiliary buyer i' with demand $b_{i'} := 1$ and valuations $v_{i'}(t) := 1$ if $t \in N_G(S) \cap N_G(I \setminus S)$ and $v_{i'}(t) := 0$ otherwise, for all items $t \in T$.

Then, we form the "upper" market M^U with buyers $S \cup \{i'\}$, items $N_G(S)$, and the same demands b and valuations v as in the original market M, albeit with an additional entry for the auxiliary buyer i'. Note that $b(S \cup \{i'\}) = b(S)+1 = |N_G(S)|$, i.e. the number of items in the upper market M^U coincides with its total demand. A schema for the upper market M^U is shown in Figure 3.3.2.

We proceed to show that the legality of every buyer-item pair for each buyer $i \neq i'$ in the upper market M^U is the same as its legality in the original market M :

Claim 3.3.2. *For every buyer $i \in S$ and for every item $t \in N_G(S)$, we have $t \in \mathscr{L}_M(i)$ if and only if $t \in \mathscr{L}_{M^U}(i)$.*

Proof. ($\Longrightarrow$) Suppose $t \in \mathscr{L}_M(i)$. Then, there exists an optimal allocation $\boldsymbol{\mathcal{A}} = \{A_j \subseteq T : j \in I\}$ of the market M such that $t \in A_i$. Since $|N_G(S)| = b(S) + 1$, it follows that the allocation $\boldsymbol{\mathcal{A}}$ assigns exactly one item in the set $N_G(S) \cap N_G(I \setminus S)$ to a buyer in the set $I \setminus S$. Let x be this item. Then, we define an allocation $\boldsymbol{\mathcal{A}}^U := \{A_j^U \subseteq N_G(S) : j \in S \cup \{i'\}\}$ of the upper market M^U as follows:

$$A_j^U := \begin{cases} A_j & \text{if } j \in S \\ \{x\} & \text{if } j = i' \end{cases} \quad \forall j \in S \cup \{i'\}$$

Then, we have $SW(\boldsymbol{\mathcal{A}}^U) = b(S) + 1$, which is equal to the total demand of the buyers in the upper market M^U. Since every valuation in the upper market M^U is at most one, it follows that the allocation $\boldsymbol{\mathcal{A}}^U$ is an optimal allocation of the upper market M^U. Note that this also implies that the upper market M^U satisfies the (OPT) condition. Moreover, we have $t \in A_i^U$, so it follows that $t \in \mathscr{L}_{M^U}(i)$, as required.

($\Longleftarrow$) Suppose $t \in \mathscr{L}_{M^U}(i)$. Then, there exists an optimal allocation $\boldsymbol{\mathcal{A}}^U = \{A_j^U \subseteq N_G(S) : j \in S \cup \{i'\}\}$ of the upper market M^U such that $t \in A_i^U$. By the proof of the forward direction, we have $SW(\boldsymbol{\mathcal{A}}^U) = b(S) + 1$, which is equal to the total demand of the upper market M^U. Since every valuation in the upper market M^U is at most one, it follows that $v_i(t) = 1$. Moreover, since the market M satisfies the $(*)$ condition, it follows that $t \in \mathscr{L}_M(i)$, completing the proof of the claim. $\square$

Next, we show that the upper market M^U satisfies the $(*)$ condition:

Claim 3.3.3. *The upper market M^U satisfies the $(*)$ condition.*

Proof. Let $i \in S$ and let $t \in N_G(S)$. Then, by the previous claim, we have $t \in \mathscr{L}_{M^U}(i) \iff t \in \mathscr{L}_M(i) \iff v_i(t) = 1$, as required. It remains to show that for the auxiliary buyer i', we also have $t \in \mathscr{L}_{M^U}(i') \iff v_{i'}(t) = 1$.

($\Longrightarrow$) If $t \in \mathscr{L}_{M^U}(i')$, then there exists an optimal allocation $\boldsymbol{\mathcal{A}}^U = \{A_j^U \subseteq N_G(S) : j \in S \cup \{i'\}\}$ of the upper market M^U such that $A_{i'}^U = \{t\}$. Again, we have $SW(\boldsymbol{\mathcal{A}}^U) = b(S)+1$, which is equal to the total demand of the upper market M^U. It follows that $v_{i'}(t) = 1$, as required.

($\Longleftarrow$) Suppose $v_{i'}(t) = 1$. Then, $t \in N_G(S) \cap N_G(I \setminus S)$ by definition. As $t \in N_G(I \setminus S)$, there exists a buyer $h \in I \setminus S$ such that $t \in \mathscr{L}_M(h)$. Furthermore, there exists an optimal allocation $\boldsymbol{\mathcal{A}} = \{A_j \subseteq T : j \in I\}$ of the original market M such that $t \in A_h$. Since $|N_G(S)| = b(S) + 1$, it follows that the item t is the unique item in the set $N_G(S)$ that is assigned by the allocation $\boldsymbol{\mathcal{A}}$ to a buyer in the set $I \setminus S$. Thus, we define an allocation $\boldsymbol{\mathcal{A}}^U := \{A_j^U \subseteq N_G(S) : j \in S \cup \{i'\}\}$ of the upper market M^U as follows:

$$A_j^U := \begin{cases} A_j & \text{if } j \in S \\ \{t\} & \text{if } j = i' \end{cases} \quad \forall j \in I.$$

Then, $SW(\boldsymbol{\mathcal{A}}^U) = b(S) + 1$, so the allocation $\boldsymbol{\mathcal{A}}^U$ is an optimal allocation of the upper market M^U. Moreover, we have $t \in A_{i'}^U$, so it follows that $t \in \mathscr{L}_{M^U}(i')$, completing the proof of the claim. □

Next, given that the upper market M^U satisfies both the (OPT) condition and the $(*)$ condition and preserves the buyer-item legalities from the original market M, we proceed to show that we may apply induction to obtain an optimal dynamic pricing scheme for the upper market M^U. Since $i_1^*, i_2^* \in I \setminus S$, it follows that $|I \setminus S| \geq 1$. If $|I \setminus S| > 1$, then the upper market M^U has fewer buyers than the original market M, and we may apply induction. Otherwise, if $|I \setminus S| = 1$, then we have $i_1^* = i_2^*$ and $I \setminus S = \{i_1^*\}$. Moreover, we have $b_{i_1^*} = 2$. Consider the assignment of item t_1^* to buyer i_1^*. This assignment extends to an optimal allocation $\boldsymbol{\mathcal{A}} = \{A_i \subseteq T : i \in I\}$ of the market M, and $|A_{i_1^*}| = b_{i^*} = 2$ by the (OPT) condition. Also, the item t_1^* is the unique item in the set $N_G(S) \cap N_G(I \setminus S)$ that is assigned by the allocation $\boldsymbol{\mathcal{A}}$ to a buyer in the set $I \setminus S$, so there must exist an item $x \in N_G(I \setminus S) \setminus N_G(S)$ to fulfill the demand of the buyer i_1^*. Hence, the upper market M^U has the same number of buyers as the original market M, but it has fewer items than the original market M, as it does not include the item x. Thus, we may again apply induction.

By induction, there exists an optimal dynamic pricing scheme p^U for the upper market M^U. By Claim 3.1.1, we may assume the pricing scheme p^U has unique values. Next, let $\{x\} := \operatorname{argmin}\{p_t^U : t \in N_G(S) \cap N_G(I \setminus S)\}$, i.e. the item x is the lowest-priced item according to the pricing scheme p^U that is legal for both a buyer in the set S and a buyer in the set $I \setminus S$.

Claim 3.3.4. *For every item $y \in N_G(I \setminus S) \setminus N_G(S)$, every assignment of the items $\{x, y\}$ to buyers in the set $I \setminus S$ that respects both legality and demand in the market M extends to an optimal allocation of the market M.*

Proof. Suppose for a contradiction that there is such an assignment that is not extendable. Suppose item x is assigned to buyer i and item y is assigned to buyer j, where $i, j \in I \setminus S$. Let G'' be the graph obtained from G by decreasing the b-values of each of the vertices i, j, x, y by one and removing them if their b-value is equal to one. Let b'' denote the b-values of the vertices in G''. By Hall's theorem, it follows that there exists a subset of buyers $S' \subseteq I$ such that $|N_{G''}(S')| < b''(S')$. By the same argument that is presented in the proof of Claim 3.3.1, we have $|N_G(S')| = b(S') + 1$ and $\{x, y\} \subseteq N_G(S')$. Also, we have $i, j \notin S'$. We proceed to show that $|N_G(S \cup S')| = b(S \cup S') + 1$, contradicting the maximality of the set $N_G(S)$.

First, by Hall's theorem, we have $|N_G(S \cup S')| \geq b(S \cup S')$. If equality holds, then every optimal allocation of the market M assigns every item in the set $N_G(S \cup S')$ to buyers in the set $S \cup S'$. However, this contradicts our assumption that the item $x \in N_G(S \cup S')$ is legal for the buyer $i \notin S \cup S'$. Hence, we have $|N_G(S \cup S')| \geq b(S \cup S') + 1$.

Next, we show the reverse inequality. Observe that $|N_G(S) \cap N_G(S')| \geq |N_G(S \cap S')| \geq b(S \cap S')$, where the second inequality follows by Hall's theorem. Moreover, we have $x \in N_G(S) \cap N_G(S')$. If $x \notin N_G(S \cap S')$, then we obtain $|N_G(S) \cap N_G(S')| \geq |N_G(S \cap S')| + 1 \geq b(S \cap S') + 1$ (†). Otherwise, if $x \in N_G(S \cap S')$, then suppose for a contradiction that $|N_G(S \cap S')| = b(S \cap S')$. It follows that every optimal allocation of the market M assigns every item in the set $N_G(S \cap S')$ to buyers in the set $S \cap S'$. However, this contradicts our assumption that the item $x \in N_G(S \cap S')$ is legal for the buyer $i \notin S \cap S'$. Thus, we again have $|N_G(S) \cap N_G(S')| \geq |N_G(S \cap S')| \geq b(S \cap S') + 1$ (†). Now, consider:

$$\begin{aligned}
|N_G(S \cup S')| &= |N_G(S) \cup N_G(S')| \\
&= |N_G(S)| \cup |N_G(S')| - |N_G(S) \cap N_G(S')| \\
&= (b(S) + 1) + (b(S') + 1) - |N_G(S) \cap N_G(S')| \\
&= b(S \cup S') + b(S \cap S') + 2 - |N_G(S) \cap N_G(S')| \\
&\leq b(S \cup S') + b(S \cap S') + 2 - b(S \cap S') - 1 && \text{by (†)} \\
&= b(S \cup S') + 1.
\end{aligned}$$

In conclusion, we have $|N_G(S \cup S')| = b(S \cup S') + 1$. Moreover, since $i, j \notin S \cup S'$ and since $\{x, y\} \subseteq N_G(S \cup S')$, it follows that $|N_{G''}(S \cup S')| < b''(S \cup S')$. Furthermore, we have $N_G(S) \subseteq N_G(S \cup S')$ and $y \in N_G(S \cup S') \setminus N_G(S)$. Now, consider our three choices of the bundle $\{x, y\}$ that is not flexible in the market M; of the assignment of item x to buyer i and item y to buyer j that does not extend to an optimal allocation of the market M; and of the set $S \cup S'$ satisfying the above inequality in the resulting graph G''. Together, they contradict the maximality of $N_G(S)$. In conclusion, every assignment of the items $\{x, y\}$ to buyers in the set $I \setminus S$ that respects both legality and demand in the market M extends to an optimal allocation of the market M, as desired. □

We proceed to consider two cases based on the legality of the item x with respect to buyers in the set $I \setminus S$; each leading to the construction of a slightly different "lower" market:

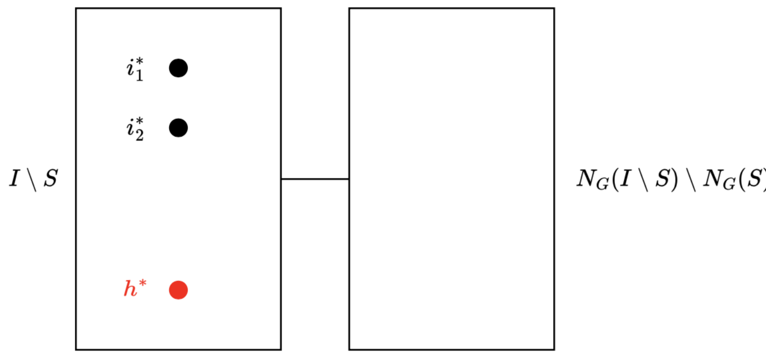

Figure 3.3.3: The lower market $M^{L'}$.

Case 1: The item x is legal for a unique buyer $h^* \in I \setminus S$ and $b_{h^*} = 1$. In this case, we form the lower market $M^{L'}$ with buyers $I^{L'} := I \setminus (S \cup \{h^*\})$, items $T^{L'} := N_G(I \setminus S) \setminus N_G(S)$ and the same demands b and valuations v as in the original market M. Note that $|T^{L'}| = b(I^{L'})$. A schema for the lower market $M^{L'}$ is given in Figure 3.3.3, where the red buyer h^* is to be deleted.

We proceed to show that the legality of every buyer-item pair in the lower market $M^{L'}$ is the same as its legality in the original market M :

Claim 3.3.5. *For every buyer $i \in I^{L'}$ and for every item $t \in T^{L'}$, we have $t \in \mathscr{L}_M(i)$ if and only if $t \in \mathscr{L}_{M^{L'}}(i)$.*

Proof. ($\Longrightarrow$) Suppose $t \in \mathscr{L}_M(i)$. Consider the assignment of item t to buyer i and item x to buyer h^*. By Claim 3.3.4, this assignment extends to an optimal allocation $\boldsymbol{\mathcal{A}} = \{A_j : j \in I\}$ of the market M. Furthermore, as $|N_G(S)| = b(S) + 1$, it follows that the item x is the unique item in the set $N_G(S)$ that is assigned to a buyer in the set $I \setminus S$. Hence, we define an allocation $\boldsymbol{\mathcal{A}}^{L'} := \{A_j^{L'} \subseteq T^{L'} : j \in I^{L'}\}$ of the lower market $M^{L'}$ as follows:

$$A_j^{L'} := A_j \quad \forall j \in I^{L'}.$$

Then, we have $SW(\boldsymbol{\mathcal{A}}^{L'}) = b(I^{L'})$, so the allocation $\boldsymbol{\mathcal{A}}^{L'}$ is an optimal allocation of the market $M^{L'}$. Note that this also implies that the market $M^{L'}$ satisfies the (OPT) condition. Moreover, we have $t \in A_i^{L'}$, so it follows that $t \in \mathscr{L}_{M^{L'}}(i)$, as required.

($\Longleftarrow$) Suppose $t \in \mathscr{L}_{M^{L'}}(i)$. Then, there exists an optimal allocation $\boldsymbol{\mathcal{A}}^{L'} = \{A_j^{L'} \subseteq T^{L'} : j \in I^{L'}\}$ of the lower market $M^{L'}$ such that $t \in A_i^{L'}$. By the proof of the forward direction, we have $SW(\boldsymbol{\mathcal{A}}^{L'}) = b(I^{L'})$. Since every valuation in the lower market $M^{L'}$ is at most one, it follows that $v_i(t) = 1$. Moreover, since the market M satisfies the $(*)$ condition, it follows that $t \in \mathscr{L}_M(i)$, completing the proof of the claim. $\square$

Next, note that it follows immediately that the market $M^{L'}$ satisfies the $(*)$ condition. Indeed, by the previous claim, for every buyer $i \in I^{L'}$ and for every item $t \in T^{L'}$, we have $t \in \mathscr{L}_{M^{L'}}(i) \iff t \in \mathscr{L}_M(i) \iff v_i(t) = 1$, as required. Moreover, since $h^* \notin I^{L'}$, it follows that the market $M^{L'}$ has fewer buyers than the original market M, so we may apply induction to obtain an optimal dynamic pricing scheme $p^{L'}$ for the lower market $M^{L'}$.

Now, we construct a pricing scheme p for the original market M with prices in the following order:

$$\{0\} <_p T^{L'} <_p N_G(S) <_p \{1\}.$$

The prices of the items in the set $N_G(S)$ are ordered according to the pricing scheme p^U and the prices of the items in the set $T^{L'}$ are ordered according to the pricing scheme $p^{L'}$. We proceed prove the optimality of this pricing scheme:

Claim 3.3.6. *The pricing scheme p defined above is an optimal dynamic pricing scheme for the market M.*

Proof. By Claim 3.1.1, we may assume the pricing schemes p^U and $p^{L'}$ each have unique values. We proceed to consider cases based on which buyer arrives first to the market M :

Case 1.1: A buyer $i \in S$ arrives first to the market M. Then, the bundle in demand for the buyer i with respect to the prices p is the bundle in demand for the buyer i with respect to the prices p^U. Let R be this bundle. By optimality of the pricing scheme p^U, it follows that there exists an optimal allocation $\boldsymbol{\mathcal{A}}^U := \{A_j^U \subseteq N_G(S) : j \in S \cup \{i'\}\}$ of the upper market M^U such that $R = A_i^U$. By the proof of Claim 3.3.2, we have that the market M^U satisfies the (OPT) condition. Thus, let $\{z\} := A_{i'}^U$ be the singleton bundle that is assigned by the allocation $\boldsymbol{\mathcal{A}}^U$ to the artificial buyer i'. By Claim 3.3.3, we have $z \in N_G(S) \cap N_G(I \setminus S)$, so there exists a buyer $f \in I \setminus S$ such that $z \in \mathscr{L}_M(f)$. Thus, there exists an optimal allocation $\boldsymbol{\mathcal{A}} = \{A_j \subseteq T : j \in I\}$ of the original market M such that

$z \in A_f$. Now, we define an allocation $\boldsymbol{\mathcal{A}}' := \{A'_j \subseteq T : j \in I\}$ of the original market M as follows:

$$A'_j := \begin{cases} A^U_j & \text{if } j \in S \\ A_j & \text{if } j \in I \setminus S \end{cases} \quad \forall j \in I.$$

Remark that the item z is the unique item in the set $N_G(S)$ that is assigned by the allocation $\boldsymbol{\mathcal{A}}$ to a buyer in the set $I \setminus S$. Thus, the allocation $\boldsymbol{\mathcal{A}}'$ indeed assigns every item to exactly one buyer. Moreover, by Claim 3.3.2, it follows that $SW(\boldsymbol{\mathcal{A}}') = SW(\boldsymbol{\mathcal{A}})$, so the allocation $\boldsymbol{\mathcal{A}}'$ is an optimal allocation of the market M. Furthermore, we have $R = A'_i$. Hence, the assignment of the bundle R to the buyer i extends to an optimal allocation $\boldsymbol{\mathcal{A}}'$ of the market M, as required.

Case 1.2: A buyer $i \in I^{L'}$ arrives first to the market M. Then, the bundle in demand for the buyer i with respect to the prices p is the bundle in demand for the buyer i with respect to the prices $p^{L'}$. Let R be this bundle. By optimality of the pricing scheme $p^{L'}$, it follows that there exists an optimal allocation $\boldsymbol{\mathcal{A}}^{L'} = \{A^{L'}_j \subseteq T^{L'} : j \in I^{L'}\}$ of the lower market $M^{L'}$ such that $R = A^{L'}_i$. Moreover, since $x \in \mathscr{L}_M(h^*)$, it follows that there exists an optimal allocation $\boldsymbol{\mathcal{A}} = \{A_j \subseteq T : j \in I\}$ of the original market M such that $\{x\} = A_{h^*}$. Now, we define an allocation $\boldsymbol{\mathcal{A}}' := \{A'_j \subseteq T : j \in I\}$ of the original market M as follows:

$$A'_j := \begin{cases} A^{L'}_j & \text{if } j \in I^{L'} \\ A_j & \text{if } j \in I \setminus I^{L'} \end{cases} \quad \forall j \in I.$$

Remark that the item x is the unique item in the set $N_G(S)$ that is assigned by the allocation $\boldsymbol{\mathcal{A}}$ to a buyer in the set $I \setminus S$. Thus, the allocation $\boldsymbol{\mathcal{A}}'$ indeed assigns every item to exactly one buyer. Moreover, by Claim 3.3.5, it follows that $SW(\boldsymbol{\mathcal{A}}') = SW(\boldsymbol{\mathcal{A}})$, so the allocation $\boldsymbol{\mathcal{A}}'$ is an optimal allocation of the market M. Furthermore, we have $R = A'_i$. Hence, the assignment of the bundle R to the buyer i extends to an optimal allocation $\boldsymbol{\mathcal{A}}'$ of the market M, as desired.

Case 1.3: The buyer h^* arrives first to the market M. Since we have $b_{h^*} = 1$ by assumption, it follows that the bundle in demand for the buyer h^* with respect to the prices p is a singleton set, say $\{z\}$, such that $z \in \mathscr{L}_M(h^*)$. Thus, the assignment of the item z to the buyer h^* extends to an optimal allocation of the market M by definition of legality. Overall, the pricing scheme p is an optimal dynamic pricing scheme for the market M, completing the proof of the claim. □

Case 2: The item x is legal for a unique buyer in the set $I \setminus S$ whose demand is at least two, or the item x is legal for at least two distinct buyers in the set $I \setminus S$. In this case,

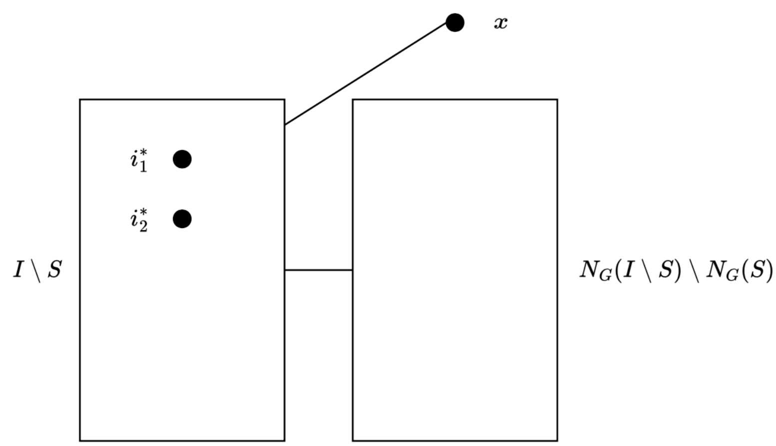

Figure 3.3.4: The lower market M^L.

we form the lower market M^L with buyers $I \setminus S$, items $T^L := (N_G(I \setminus S) \setminus N_G(S)) \cup \{x\}$, and the same demands b and valuations v as in the original market M. A schema for the lower market M^L is shown in Figure 3.3.4.

We proceed to show that the legality of every buyer-item pair in the lower market M^L is the same as its legality in the original market M :

Claim 3.3.7. *For every buyer $i \in I \setminus S$ and for every item $t \in T^L$, we have $t \in \mathscr{L}_M(i)$ if and only if $t \in \mathscr{L}_{M^L}(i)$.*

Proof. ($\Longrightarrow$) Suppose $t \in \mathscr{L}_M(i)$. First, suppose $t = x$. Then, there exists an optimal allocation $\boldsymbol{\mathcal{A}} = \{A_j \subseteq T : j \in I\}$ of the market M such that $x \in A_i$. Furthermore, the item x is the unique item in the set $N_G(S)$ that is assigned by the allocation $\boldsymbol{\mathcal{A}}$ to a buyer in the set $I \setminus S$. Hence, we define an allocation $\boldsymbol{\mathcal{A}}^L := \{A_j^L \subseteq T^L : j \in I \setminus S\}$ of the lower market M^L as follows:

$$A_j^L := A_j \quad \forall j \in I \setminus S.$$

Then, we have $SW(\boldsymbol{\mathcal{A}}^L) = b(I \setminus S)$, so the allocation $\boldsymbol{\mathcal{A}}^L$ is an optimal allocation of the market M^L. Note that this also implies that the market M^L satisfies the (OPT) condition. Moreover, we have $x \in A_i^L$, so it follows that $x \in \mathscr{L}_{M^L}(i)$, as required.

Next, suppose $t \neq x$. Then, since $x \in N_G(I \setminus S)$, it follows that there exists a buyer $h \in I \setminus S$ such that $x \in \mathscr{L}_M(h)$. By the assumption of Case 2, we can select such a buyer h so that the assignment of item t to buyer i and item x to buyer h respects demand.

Note that it may be the case that $h = i$. By Claim 3.3.4, it follows that the above assignment extends to an optimal allocation $\boldsymbol{\mathcal{A}} = \{A_j : j \in I\}$ of the market M such that $t \in A_i$ and $x \in A_h$. Moreover, the item x is the unique item in the set $N_G(S)$ that is assigned by the allocation $\boldsymbol{\mathcal{A}}$ to a buyer in the set $I \setminus S$. Hence, we define an allocation $\boldsymbol{\mathcal{A}}^L := \{A_j^L \subseteq T^L : j \in I \setminus S\}$ of the lower market M^L as follows:

$$A_j^L := A_j \quad \forall j \in I \setminus S.$$

Then, we have $SW(\boldsymbol{\mathcal{A}}^L) = b(I \setminus S)$, so the allocation $\boldsymbol{\mathcal{A}}^L$ is an optimal allocation of the market M^L. Note that this also implies that the market M^L satisfies the (OPT) condition. Moreover, we have $t \in A_i^L$, so it follows that $t \in \mathscr{L}_{M^L}(i)$, as required.

($\Longleftarrow$) The proof is analogous to the proof of the backward direction of Claim 3.3.5. $\square$

As in Case 1, it follows immediately that the market M^L satisfies the $(*)$ condition. Moreover, since $|N_G(S)| = b(S) + 1$, we have $S \neq \emptyset$. Hence, the market M^L, which has buyers $I \setminus S$, has fewer buyers than the original market M. Thus, we may apply induction to obtain an optimal dynamic pricing scheme p^L for the lower market M^L.

Next, we define the following bipartition of the set $N_G(S) \setminus \{x\}$:

$$N_G^{<x}(S) := \{t \in N_G(S) : p_t^U < p_x^U\}$$
$$N_G^{>x}(S) := \{t \in N_G(S) : p_t^U > p_x^U\}$$

Then, we construct a pricing scheme p for the original market M with prices in the following order:

$$\{0\} <_p N_G^{<x}(S) <_p T^L <_p N_G^{>x}(S) <_p \{1\}$$

The prices of the items in the set $N_G^{<x}(S) \cup N_G^{>x}(S)$ are ordered according to the pricing scheme p^U and the prices of the items in the set T^L are ordered according to pricing scheme p^L. By Claim 3.1.1, we may assume the pricing schemes p^U and p^L each have unique values. Then, for a buyer $i \in S$, the bundle in demand for the buyer i with respect to the prices p is the bundle in demand for the buyer i with respect to the prices p^U. In addition, for a buyer $i \in I \setminus S$, the bundle in demand for the buyer i with respect to the prices p is the bundle in demand for the buyer i with respect to the prices p^L. By an argument that is analogous to the proof of Claim 3.3.6, we conclude that the pricing scheme p is an optimal dynamic pricing scheme for the market M, completing the proof of Lemma 3.3.1. $\square$

To close Section 3.3, we revisit the example of the multi-demand market M_1 from Figure 3.1.1, and we apply the methods from the proof of Lemma 3.3.1 to the auxiliary

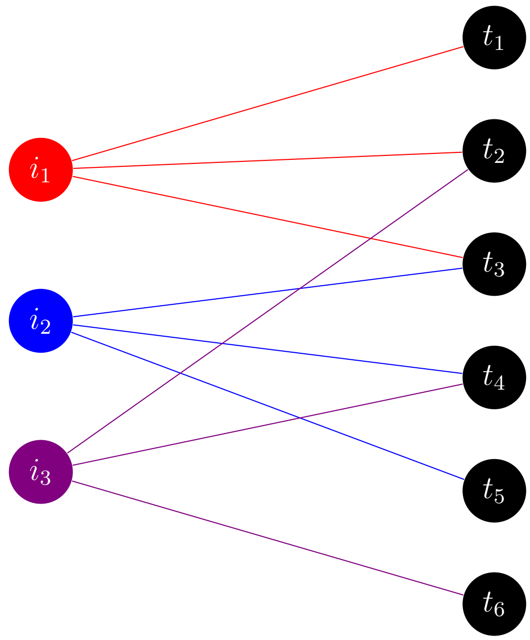

Figure 3.3.5: The legality graph of the market M_1'.

market M_1' of the market M_1, as given in Figure 3.1.4. In Figure 3.3.5, we present the legality graph of the market M_1'. Observe that $k_{M_1'} = 2$. Moreover, the assignment of the items $\{t_2, t_4\}$ to the buyer i_3 does not extend to an optimal allocation of the market M_1'. In addition, selecting the set $S := \{i_1, i_2\}$ results in the maximality of the set $N_{G_{M_1'}}(S)$. This particular splitting of the market M_1' is shown in Figure 3.3.6, where the thick edges represent the non-extendable assignment of size two. Note that the items in Figure 3.3.6 are presented in a different order to improve readability.

We present the upper market $M_1'^U$ obtained from this splitting in Figure 3.3.7.

Next, we give the following optimal dynamic pricing scheme for its upper market $M_1'^U$ in Figure 3.3.7:

$$p^U_{t_1} = 0.5 < p^U_{t_5} = 0.6 < p^U_{t_2} = 0.7 < p^U_{t_3} = 0.8 < p^U_{t_4} = 0.9.$$

One can verify the optimality of the above pricing scheme p^U for the upper market $M_1'^U$.

Using the optimal pricing scheme p^U above, we have $\{x\} = \operatorname{argmin}\{p^U_t : t \in N_G(S) \cap N_G(I \setminus S)\} = \{t_2\}$. We present the lower market $M_1'^L$ obtained from this splitting and this pricing scheme in Figure 3.3.8.

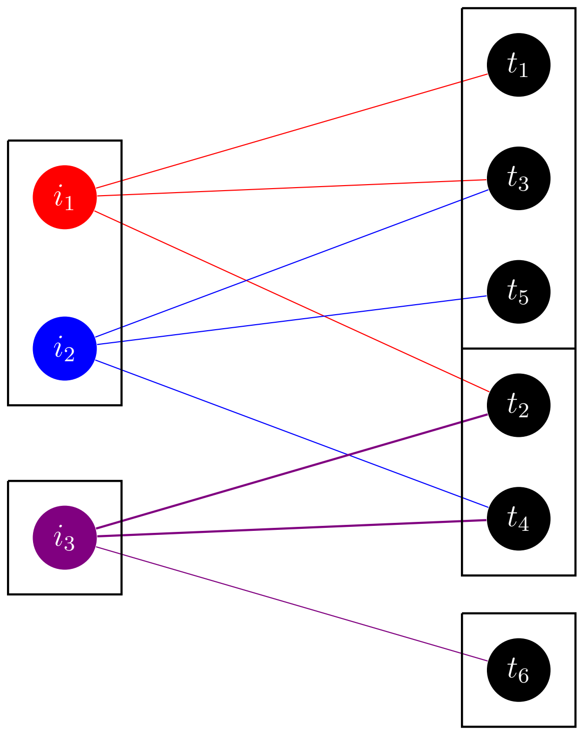

Figure 3.3.6: The splitting of the market M_1'.

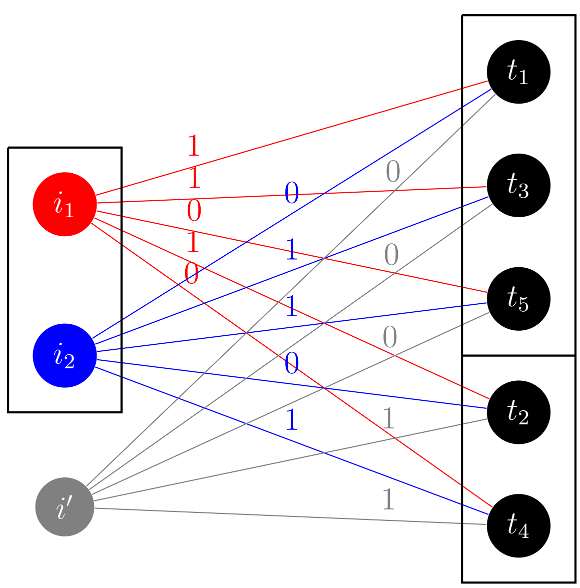

Figure 3.3.7: The upper market $M_1'^U$ obtained from the splitting in Figure 3.3.6.

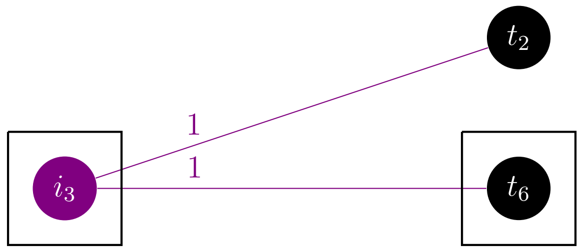

Figure 3.3.8: The lower market $M_1'^L$ obtained from the splitting in Figure 3.3.6.

Recall the optimal pricing scheme p^U for the upper market $M_1'^U$, and observe that $N_G^{<x}(S) = \{t_1, t_5\}$ and $N_G^{>x}(S) = \{t_3, t_4\}$. Moreover, an optimal dynamic pricing scheme p^L for the lower market $M_1'^L$ is given by $p^L_{t_2} = 0.5 < p^L_{t_6} = 0.6$. Finally, we combine these two pricing schemes into a pricing scheme p' for the market M_1' as follows:

$$p'_{t_1} = 0.4 < p'_{t_5} = 0.5 < p'_{t_2} = 0.6 < p'_{t_6} = 0.7 < p'_{t_3} = 0.8 < p'_{t_4} = 0.9.$$

Note that this is the same optimal pricing scheme that we gave for the market M_1' in Section 3.1.

3.4 Tri-Demand Markets

In this section, we show that optimal dynamic pricing schemes exist in tri-demand markets. The high-level approach is the same as the one used in the previous section: If $k_M \geq 4$, then any buyer's purchase of any legal bundle is guaranteed to extend to an optimal allocation of the market. If $k_M = 2$, then the splitting of the market M into the two smaller "upper" and "lower" markets M^U and $M^{L'}$ or M^L that was employed in the previous section can again be used in this new setting. Finally, if $k_M = 3$, then we first show that unit-demand buyers can be eliminated from the market, which will prove to be crucial in the proof of optimality of the combined pricing schemes. Next, observe that in the case where $k_M = 2$, the lower market M^L requires just one item $x \in N_G(S) \cap N_G(I \setminus S)$ to be added to its item set to fulfill its demand. However, in the case where $k_M = 3$, the lower market M^L requires two items $x, y \in N_G(S) \cap N_G(I \setminus S)$ to fulfill its demand. This poses a potential problem: If the pricing scheme p^U for the upper market M^U satisfies $p^U_x < p^U_y$ and if the pricing scheme p^L for the lower market M^L satisfies $p^L_x > p^L_y$, then it is not possible to combine the pricing schemes p^U and p^L into an optimal pricing scheme for the original market M, as was done in the case where $k_M = 2$. To resolve this issue, we prove a stronger result than that of the existence of optimal dynamic pricing schemes. In fact, we show that for any item t^* of the market M, there exists an optimal dynamic pricing scheme for the market M in which the item t^* is the lowest-priced item:

Lemma 3.4.1. *Let M be a multi-demand market satisfying the (OPT) condition and suppose $b_i \leq 3$ for all buyers $i \in I$. Then, for every item $t^* \in T$, there exists an optimal dynamic pricing scheme p for the market M such that $\{t^*\} = \operatorname{argmin}\{p_t : t \in T\}$.*

The proof of Lemma 3.4.1 is organized as follows:

- First, we prove that such a pricing scheme exists if $k_M \geq 4$.
- Next, in Claim 3.4.1, we prove that such a pricing scheme exists if $k_M = 2$. We use the same splitting of the market as in Section 3.3. There are three cases, each with two subcases as follows:
  - 1.1: $t^* \in N_G(S) \cap N_G(I \setminus S)$ and we form the lower market $M^{L'}$.
  - 1.2: $t^* \in N_G(S) \cap N_G(I \setminus S)$ and we form the lower market M^{L}.
  - 2.1: $t^* \in N_G(S) \setminus N_G(I \setminus S)$ and we form the lower market $M^{L'}$.
  - 2.2: $t^* \in N_G(S) \setminus N_G(I \setminus S)$ and we form the lower market M^{L}.
  - 3.1: $t^* \in N_G(I \setminus S) \setminus N_G(S)$ and we form the lower market $M^{L'}$.
  - 3.2: $t^* \in N_G(I \setminus S) \setminus N_G(S)$ and we form the lower market M^{L}.
- Next, assuming $k_M = 3$, we prove that such a pricing scheme exists if there is a unit-demand buyer $i^* \in I$ in the market M. This is Claim 3.4.2. There are two cases as follows:
  - 1: $\mathscr{L}_M(i^*) = \{t^*\}$.
  - 2: There exists an item $x \in \mathscr{L}_M(i^*) \setminus \{t^*\}$.
- Next, assuming $k_M = 3$ and $b_i \geq 2$ for all buyers $i \in I$, we prove that such a pricing scheme exists if the item t^* is not contained in any 3-subset of items that is not flexible in the market M.
- Finally, assuming $k_M = 3$, $b_i \geq 2$ for all buyers $i \in I$ and the item t^* is contained in some 3-subset of items $\{t^*, r_1^*, r_2^*\}$ that is not flexible in the market M, the remainder of the proof of Lemma 3.4.1 is analogous to the proof of Lemma 3.3.1 in the case where $k_M = 2$.

Proof. Let $t^* \in T$ be an arbitrary item of the market M that we desire to price the lowest. To begin, we may assume $|T| = b(I)$ by Claim 2.0.2 and that the market M satisfies the $(*)$ condition by Lemma 3.1.1.

First, suppose $k_M \geq 4$. Then, we define the following pricing scheme:

$$p_t := \begin{cases} 0.4 & \text{if } t = t^* \\ 0.5 & \text{otherwise} \end{cases} \quad \forall t \in T.$$

As in the bi-demand case, if the buyer $i \in I$ arrives first to the market, then they will purchase a bundle $B \subseteq T$ such that $|B| = b_i$ and such that $B \subseteq \mathscr{L}_M(i)$. Note that such a bundle B exists by the (OPT) condition. As $k_M > 3$, it follows that every bundle of size at most three is flexible in the market M. Since $b_i \leq 3$, we conclude that the assignment of the items in the bundle B to the buyer i extends to an optimal allocation of the market M. Hence, the pricing scheme p is an optimal dynamic pricing scheme for the market M. Moreover, $\{t^*\} = \text{argmin}\{p_t : t \in T\}$, as desired.

Thus, we may assume $k_M \leq 3$. We again proceed by induction on the number of buyers in the market and, subject to this quantity, we proceed by induction on the number of items in the market.

Claim 3.4.1. *Suppose $k_M = 2$. Then, there exists an optimal dynamic pricing scheme p for the market M such that $\{t^*\} = \text{argmin}\{p_t : t \in T\}$.*

Proof. We proceed as we did in the bi-demand case, and we select a bundle B of size two that is not flexible; an assignment of the items in the bundle B to buyers in the set I that does not extend to an optimal allocation of the market M; and a set S violating Hall's condition in the resulting graph G' such that the set $N_G(S)$ is maximal. Remark that none of the claims that are presented in Section 3.3 use the fact that $b_i \leq 2$ for every buyer $i \in I$. Thus, each of these claims holds in our new setting where $b_i \leq 3$ for every buyer $i \in I$. We now consider cases based on the location of the item t^* with respect to the set $N_G(S)$ and based on whether we form the lower market $M^{L'}$ or the lower market M^L :

Case 1: $t^* \in N_G(S) \cap N_G(I \setminus S)$.

We form the upper market M^U as in the previous section. By induction, there exists an optimal dynamic pricing scheme p^U for the upper market M^U such that $\{t^*\} = \text{argmin}\{p_t^U : t \in N_G(S)\} = \text{argmin}\{p_t^U : t \in N_G(S) \cap N_G(I \setminus S)\}$. We consider two subcases based on which lower market we form:

Subcase 1.1: The item t^* is legal for a unique buyer $h^* \in I \setminus S$ and $b_{h^*} = 1$. Then, we form the lower market $M^{L'}$ as in the previous section. By induction, there exists an optimal dynamic pricing scheme $p^{L'}$ for the lower market $M^{L'}$. Now, we construct a pricing scheme p for the original market M with prices in the following order:

$$\{0\} <_p \{t^*\} <_p T^{L'} <_p N_G(S) \setminus \{t^*\} <_p \{1\}.$$

The prices of the items in the set $N_G(S) \setminus \{t^*\}$ are ordered according to the pricing scheme p^U and the prices of the items in the set $T^{L'}$ are ordered according to the pricing scheme $p^{L'}$. By Claim 3.1.1, we may assume the pricing schemes p^U and $p^{L'}$ each have unique values. Then, for a buyer $i \in S$, the bundle in demand for the buyer i with respect to the prices p is the bundle in demand for the buyer i with respect to the prices p^U. Moreover, for a buyer $i \in I^{L'} = I \setminus (S \cup \{h^*\})$, we have $t^* \notin \mathscr{L}_M(i)$ by assumption. Hence, the bundle in demand for the buyer i with respect to the prices p is the bundle in demand for the buyer i with respect to the prices $p^{L'}$. Finally, the bundle in demand for the buyer h^* is $\{t^*\}$, and this extends to an optimal allocation of the market M. Then, by the proof of Claim 3.3.6, we have that the pricing scheme p is an optimal dynamic pricing scheme for the market M and $\{t^*\} = \operatorname{argmin}\{p_t : t \in T\}$, as desired.

Subcase 1.2: The item t^* is legal for a unique buyer in the set $I \setminus S$ whose demand is at least two, or the item t^* is legal for at least two distinct buyers in the set $I \setminus S$. Then, we form the lower market M^L as in the previous section. By induction, there exists an optimal dynamic pricing scheme p^L for the lower market M^L such that $\{t^*\} = \operatorname{argmin}\{p^L_t : t \in T^L\}$, where $T^L = (N_G(I \setminus S) \setminus N_G(S)) \cup \{t^*\}$. Now, we construct a pricing scheme p for the original market M with prices in the following order:

$$\{0\} <_p T^L <_p N_G(S) \setminus \{t^*\} <_p \{1\}.$$

The prices of the items in the set $N_G(S) \setminus \{t^*\}$ are ordered according to the pricing scheme p^U and the prices of the items in the set T^L are ordered according to the pricing scheme p^L. By Claim 3.1.1, we may assume the pricing schemes p^U and p^L each have unique values. Then, for a buyer $i \in S$, the bundle in demand for the buyer i with respect to the prices p is the bundle in demand for the buyer i with respect to the prices p^U. Moreover, for a buyer $i \in I \setminus S = I^L$, the bundle in demand for the buyer i with respect to the prices p is the bundle in demand for the buyer i with respect to the prices p^L. Hence, by an argument that is analogous to the proof of Claim 3.3.6, we have that the pricing scheme p is an optimal dynamic pricing scheme for the market M and $\{t^*\} = \operatorname{argmin}\{p_t : t \in T\}$, as desired.

Case 2: $t^* \in N_G(S) \setminus N_G(I \setminus S)$.

We form the upper market M^U as in the previous section. By induction, there exists an optimal dynamic pricing scheme p^U for the upper market M^U such that $\{t^*\} = \text{argmin}\{p_t^U : t \in N_G(S)\}$. By Claim 3.1.1, we may assume the pricing scheme p^U has unique values. Let $\{x\} = \text{argmin}\{p_t^U : t \in N_G(S) \cap N_G(I \setminus S)\}$. We again consider two subcases based on which lower market we form:

Subcase 2.1: The item x is legal for a unique buyer $h^* \in I \setminus S$ and $b_{h^*} = 1$. Then, we form the lower market $M^{L'}$ as in the previous section. By induction, there exists an optimal dynamic pricing scheme $p^{L'}$ for the lower market $M^{L'}$. Now, we construct a pricing scheme p for the original market M with prices in the following order:

$$\{0\} <_p \{t^*\} <_p T^{L'} <_p N_G(S) \setminus \{t^*\} <_p \{1\}.$$

The prices of the items in the set $N_G(S) \setminus \{t^*\}$ are ordered according to the pricing scheme p^U and the prices of the items in the set $T^{L'}$ are ordered according to the pricing scheme $p^{L'}$. Recall that for every buyer $i \in I^{L'}$, we have $t^* \notin \mathscr{L}_M(i)$ by the assumption of Case 2. Hence, one can verify the optimality of the pricing scheme p for the market M and $\{t^*\} = \text{argmin}\{p_t : t \in T\}$, as desired.

Subcase 2.2: The item x is legal for a unique buyer in the set $I \setminus S$ whose demand is at least two, or the item x is legal for at least two distinct buyers in the set $I \setminus S$. Then, we form the lower market M^L as in the previous section. By induction, there exists an optimal dynamic pricing scheme p^L for the lower market M^L. Define the sets $N_G^{<x}(S)$ and $N_G^{>x}(S)$ as in the previous section. Now, we construct a pricing scheme p for the original market M with prices in the following order:

$$\{0\} <_p N_G^{<x}(S) <_p T^L <_p N_G^{>x}(S) <_p \{1\}.$$

The prices of the items in the set $N_G^{<x}(S) \cup N_G^{>x}(S)$ are ordered according to the pricing scheme p^U and the prices of the items in the set T^L are ordered according to the pricing scheme p^L. Moreover, the optimality of this pricing scheme p for the market M was proved in the previous section and $\{t^*\} = \text{argmin}\{p_t : t \in T\}$, as desired.

Case 3: $t^* \in N_G(I \setminus S) \setminus N_G(S)$.

We form the upper market M^U as in the previous section. By induction, there exists an optimal dynamic pricing scheme p^U for the upper market M^U. By Claim 3.1.1, we may assume the pricing scheme p^U has unique values. Let $\{x\} = \text{argmin}\{p_t^U : t \in N_G(S) \cap N_G(I \setminus S)\}$. We again consider two cases based on which lower market we form:

Subcase 3.1: The item x is legal for a unique buyer $h^* \in I \setminus S$ and $b_{h^*} = 1$. Then, we form the lower market $M^{L'}$ as in the previous section. By induction, there exists an optimal dynamic pricing scheme $p^{L'}$ for the lower market $M^{L'}$ such that $\{t^*\} = \text{argmin}\{p_t^{L'} : t \in$

$T^{L'}\}$. Now, we construct a pricing scheme p for the original market M with prices in the following order:

$$\{0\} <_p T^{L'} <_p N_G(S) <_p \{1\}.$$

The prices of the items in the set $N_G(S)$ are ordered according to the pricing scheme p^U and the prices of the items in the set $T^{L'}$ are ordered according to the pricing scheme $p^{L'}$. Moreover, the optimality of the pricing scheme p for the market M was proved in the previous section and $\{t^*\} = \operatorname{argmin}\{p_t : t \in T\}$, as desired.

Subcase 3.2: The item x is legal for a unique buyer in the set $I \setminus S$ whose demand is at least two, or the item x is legal for at least two distinct buyers in the set $I \setminus S$. Then, we form the lower market M^L as in the previous section. By induction, there exists an optimal dynamic pricing scheme p^L for the lower market M^L such that $\{t^*\} = \operatorname{argmin}\{p_t^L : t \in T^L\}$. Define the sets $N_G^{<x}(S)$ and $N_G^{>x}(S)$ as in Subcase 2.2. Now, we construct a pricing scheme p for the original market M with prices in the following order:

$$\{0\} <_p \{t^*\} <_p N_G^{<x}(S) <_p T^L \setminus \{t^*\} <_p N_G^{>x}(S) <_p \{1\}.$$

The prices of the items in the set $N_G^{<x}(S) \cup N_G^{>x}(S)$ are ordered according to the pricing scheme p^U and the prices of the items in the set $T^L \setminus \{t^*\}$ are ordered according to the pricing scheme p^L. Recall that for every buyer $i \in S$, we have $t^* \notin \mathscr{L}_M(i)$ by the assumption of Case 3. Hence, one can verify the optimality of the pricing scheme p for the market M and $\{t^*\} = \operatorname{argmin}\{p_t : t \in T\}$, completing the proof of the claim. □

Thus, we may assume $k_M = 3$. Our next step is to eliminate unit-demand buyers from the market:

Claim 3.4.2. *Suppose there exists a buyer $i^* \in I$ such that $b_{i^*} = 1$. Then, there exists an optimal dynamic pricing scheme p for the market M such that $\{t^*\} = \operatorname{argmin}\{p_t : t \in T\}$.*

Proof. We consider two cases based on whether or not the item t^* is the unique item that is legal for the buyer i^*. Each case leads to the construction of a submarket that is obtained by removing the buyer i^* and by removing one of the items that is legal for them.

Case 1: $\mathscr{L}_M(i^*) = \{t^*\}$.

We form the submarket M' with buyers $I \setminus \{i^*\}$, items $T \setminus \{t^*\}$ and the same demands b and valuations v as in the original market M. We proceed to show that the legality of every buyer-item pair in the submarket M' is the same as its legality in the original market M :

Let $i \in I \setminus \{i^*\}$ and let $t \in T \setminus \{t^*\}$.

($\Longrightarrow$) Suppose $t \in \mathscr{L}_M(i)$. Then, there exists an optimal allocation $\boldsymbol{\mathcal{A}} = \{A_j \subseteq T : j \in I\}$ of the market M such that $t \in A_i$. Since t^* is the only item that is legal for the buyer i^*, it follows by the (OPT) condition that $A_{i^*} = \{t^*\}$. Hence, the allocation $\boldsymbol{\mathcal{A}}' := \{A_j \subseteq T \setminus \{t^*\} : j \in I \setminus \{i^*\}\}$ satisfies $SW(\boldsymbol{\mathcal{A}}') = b(I \setminus \{i^*\})$, and thus, the allocation $\boldsymbol{\mathcal{A}}'$ is an optimal allocation of the market M'. Moreover, we have $t \in A_i$, so $t \in \mathscr{L}_{M'}(i)$, as desired.

($\Longleftarrow$) Suppose $t \in \mathscr{L}_{M'}(i)$. Then, there exists an optimal allocation $\boldsymbol{\mathcal{A}}' = \{A'_j \subseteq T \setminus \{t^*\} : j \in I \setminus \{i^*\}\}$ of the submarket M' such that $t \in A'_i$. Then, let $A_{i^*} := \{t^*\}$ and let $\boldsymbol{\mathcal{A}} := \boldsymbol{\mathcal{A}}' \cup \{A_{i*}\}$. It follows that $SW(\boldsymbol{\mathcal{A}}) = b(I)$. Thus, $\boldsymbol{\mathcal{A}}$ is an optimal allocation of the market M and $t \in A'_i$, so $t \in \mathscr{L}_M(i)$, as desired.

It follows that the submarket M' satisfies both the (OPT) condition and the ($*$) condition. By induction, there exists an optimal dynamic pricing scheme p' for the submarket M'. Now, we construct a pricing scheme p for the original market M with prices in the following order:

$$\{0\} <_p \{t^*\} <_p T \setminus \{t^*\} <_p \{1\}.$$

The prices of the items in the set $T \setminus \{t^*\}$ are ordered according to the pricing scheme p'. By Claim 3.1.1, we many assume the pricing scheme p' has unique values. Moreover, by the assumption of Case 1 and by the (OPT) condition, it follows that every optimal allocation of the market M assigns the item t^* to the buyer i^*. Hence, we have $t^* \notin \mathscr{L}_M(i)$ for every buyer $i \in I \setminus \{i^*\}$. Thus, the bundle in demand for the buyer i with respect to the prices p is the bundle in demand for the buyer i with respect to the prices p'. Thus, the pricing scheme p is an optimal dynamic pricing scheme for the market M and $\{t^*\} = \operatorname{argmin}\{p_t : t \in T\}$, as required.

Case 2: There exists an item $x \in \mathscr{L}_M(i^*) \setminus \{t^*\}$.

We form the submarket M'' with buyers $I \setminus \{i^*\}$, items $T \setminus \{x\}$ and the same demands b and valuations v as in the original market M. Again, we show that the legality of every buyer-item pair in the submarket M'' is the same as its legality in the original market M :

Let $i \in I \setminus \{i^*\}$ and let $t \in T \setminus \{x\}$.

($\Longrightarrow$) Suppose $t \in \mathscr{L}_M(i)$. Consider the assignment of item t to buyer i and item x to buyer i^*. This assignment respects both legality and demand in the market M. Moreover, as $k_M > 2$, it follows that the aforementioned assignment extends to an optimal allocation $\boldsymbol{\mathcal{A}} = \{A_j \subseteq T : j \in I\}$ of the market M with $t \in A_i$ and $A_{i^*} = \{x\}$. Hence, the allocation $\boldsymbol{\mathcal{A}}'' := \{A_j \subseteq T \setminus \{x\} : j \in I \setminus \{i^*\}\}$ satisfies $SW(\boldsymbol{\mathcal{A}}'') = b(I \setminus \{i^*\})$, and thus, the allocation $\boldsymbol{\mathcal{A}}''$ is an optimal allocation of the market M''. Moreover, we have $t \in A_i$, so $t \in \mathscr{L}_{M''}(i)$, as desired.

($\Longleftarrow$) The proof is identical to the proof of the backward direction of Case 1 if we replace the item t^* with the item x.

It follows that the submarket M'' satisfies both the (OPT) condition and the $(*)$ condition. By induction, there exists an optimal dynamic pricing scheme p'' for the submarket M'' such that $\{t^*\} = \text{argmin}\{p''_t : t \in T \setminus \{x\}\}$. Now, we construct a pricing scheme p for the original market M with prices in the following order:

$$\{0\} <_p T \setminus \{x\} <_p \{x\} <_p \{1\}.$$

The prices of the items in the set $T \setminus \{x\}$ are ordered according to the pricing scheme p''. Then, for a buyer $i \in I \setminus \{i^*\}$, the bundle in demand for the buyer i with respect to the prices p is the bundle in demand for the buyer i with respect to the prices p''. Thus, the pricing scheme p is an optimal dynamic pricing scheme for the market M and $\{t^*\} = \text{argmin}\{p_t : t \in T\}$, completing the proof of the claim. □

Thus, we may assume $b_i > 1$ for all buyers $i \in I$. Now, suppose the item t^* is not contained in any 3-subset of items that is not flexible in the market M. By Claim 2.0.2, we may assume every item in the market M is assigned to some buyer in every optimal allocation of the market M, so it follows that there exists a buyer $i^* \in I$ such that $t^* \in \mathscr{L}_M(i^*)$. Then, we form the submarket M' with buyers $I \setminus \{i^*\}$, items $T \setminus \{t^*\}$ and the same demands b and valuations v as in the original market M. Using the same argument as in Case 2 of Claim 3.4.2, we have that the legality of every buyer-item pair in the submarket M' is the same as its legality in the original market M and that the submarket M' satisfies both the (OPT) condition and the $(*)$ condition. By induction, there exists an optimal dynamic pricing scheme p' for the submarket M'. Then, we construct a pricing scheme p for the original market M with prices in the following order:

$$\{0\} <_p \{t^*\} <_p T \setminus \{t^*\} <_p \{1\}.$$

The prices of the items in the set $T \setminus \{t^*\}$ are ordered according to the pricing scheme p'. Let $i \in I$ be a buyer. If $t^* \in \mathscr{L}_M(i^*)$, then the bundle in demand for the buyer i with respect to the prices p contains the item t^*. By our assumption that the item t^* is not contained in any 3-subset of items that is not flexible in the market M, it follows that the assignment of the aforementioned bundle to the buyer i extends to an optimal allocation of the market M. Otherwise, if $t^* \notin \mathscr{L}_M(i^*)$, then the bundle in demand for the buyer i with respect to the prices p is the bundle in demand for the buyer i with respect to the prices p'. Thus, the pricing scheme p is optimal for the market M and $\{t^*\} = \text{argmin}\{p_t : t \in T\}$, as desired.

Now, we may assume the item t^* is contained in some 3-subset of items $\{t^*, r_1^*, r_2^*\} \subseteq T$ that is not flexible in the market M. The remainder of the proof of this lemma is analogous to the proof of Lemma 3.3.1 in the case where $k_M = 2$. However, we present all of its details to show how one might hope to generalize the argument, and to expose the areas in the proof in which barriers to this potential generalization seem to arise naturally. These ideas are discussed further in Chapter 5.

To begin, we let $i^*, h_1^*, h_2^* \in I$ be buyers such that the assignment of item t^* to buyer i^*, item r_1^* to buyer h_1^* and item r_2^* to buyer h_2^* respects both legality and demand but does not extend to an optimal allocation of the market M. Let G' be the graph obtained from the legality graph G by decreasing the b-values of each of the vertices $i^*, h_1^*, h_2^*, t^*, r_1^*, r_2^*$ by one and removing them if their updated b-value is equal to zero. Let b' denote the b-values of the vertices in the graph G'. Since optimal allocations of the market M are precisely b-factors of the legality graph G, it follows that the graph G' does not have a b'-factor. By Hall's theorem, there exists a set of buyers $S \subseteq I$ such that $|N_{G'}(S)| < b'(S)$. We again choose to maximize the same quantity as in the proof of Lemma 3.3.1: Among all possible choices of bundles B of size three containing the item t^* that are not flexible in the market M, assignments of the items in the bundle B to buyers in the set I that do not extend to an optimal allocation of the market M, and sets S satisfying the above inequality in the resulting graph G', we select a triple such that the set $N_G(S)$ is maximal. We proceed to prove the analogue of Claim 3.3.1; establishing some useful properties of the set S :

Claim 3.4.3. *We have $|N_G(S)| = b(S) + 2$ and $\{t^*, r_1^*, r_2^*\} \subseteq N_G(S)$. Furthermore, we have $i^*, h_1^*, h_2^* \notin S$.*

Proof. Since G has a b-factor, it follows by Hall's theorem that $|N_G(S)| \geq b(S)$. Suppose for a contradiction that $i^*, h_1^*, h_2^* \in S$. Then, we have:

$$|N_{G'}(S)| \geq |N_G(S)| - 3 \geq b(S) - 3 = b'(S).$$

This contradicts our assumption that $|N_{G'}(S)| < b'(S)$. Hence, we have $i^* \notin S$, $h_1^* \notin S$ or $h_2^* \notin S$. Without loss of generality, suppose $i^* \notin S$.

Suppose for a contradiction that $t^* \notin N_G(S)$. Consider the assignment of item r_1^* to buyer h_1^* and item r_2^* to buyer h_2^*. Let G'' be the graph obtained from the graph G by decreasing the b-values of each of the vertices $h_1^*, h_2^*, r_1^*, r_2^*$ by one and removing them if their updated b-value is equal to zero. Let b'' denote the b-values of the vertices in the graph G''. Since $k_M > 2$, it follows that the above assignment, which respects both legality and demand in the market M, extends to an optimal allocation of the market M. Thus, the resulting graph G'' has a b''-factor and by Hall's condition, it follows that $|N_{G''}(S)| \geq b''(S)$. Since $i^* \notin S$ and since $t^* \notin N_G(S)$ by assumption, we have:

$$|N_{G'}(S)| = |N_{G''}(S)| \geq b''(S) = b'(S).$$

This contradicts our assumption that $|N_{G'}(S)| < b'(S)$. Thus, we have $t^* \in N_G(S)$.

Since $i^* \notin S$, $t^* \in N_G(S)$ and $t^* \in \mathscr{L}_M(i^*)$, it follows that $|N_G(S)| > b(S)$.

Suppose for a contradiction that $|N_G(S)| = b(S) + 1$. Then, the inequality $|N_{G'}(S)| < b'(S)$ implies $r_1^* \in N_G(S)$ or $r_2^* \in N_G(S)$. Without loss of generality, suppose $r_1^* \in N_G(S)$. If $h_1^* \notin S$, then the assignment of item r_1^* to buyer h_1^* and item t^* to buyer i^* respects both legality and demand but does not extend to an optimal allocation of the market M, contradicting $k_M > 2$. Thus, $h_1^* \in S$. Consequently, the inequality $|N_{G'}(S)| < b'(S)$ implies $r_2^* \in N_G(S)$ and $h_2^* \notin S$. But then, the assignment of item r_2^* to buyer h_2^* and item t^* to buyer i^* respects both legality and demand but does not extend to an optimal allocation of the market M, again contradicting $k_M > 2$. Overall, we have $|N_G(S)| \neq b(S) + 1$.

Suppose for a contradiction that $|N_G(S)| \geq b(S) + 3$. Then, we have:

$$|N_{G'}(S)| \geq |N_G(S)| - 3 \geq b(S) \geq b'(S).$$

This contradicts our assumption that $|N_{G'}(S)| < b'(S)$. Hence, we have $|N_G(S)| \leq b(S)+2$, and thus, it follows that $|N_G(S)| = b(S) + 2$.

Suppose for a contradiction that $h_1^* \in S$. Then, we obtain:

$$|N_{G'}(S)| \geq |N_G(S)| - 3 = b(S) - 1 \geq b'(S).$$

This contradicts our assumption that $|N_{G'}(S)| < b'(S)$. Thus, we have $h_1^* \notin S$. This argument can also be used to show $h_2^* \notin S$.

Finally, suppose for a contradiction that $r_1^* \notin N_G(S)$. Then, we obtain:

$$|N_{G'}(S)| \geq |N_G(S)| - 2 \geq b(S) \geq b'(S).$$

This contradicts our assumption that $|N_{G'}(S)| < b'(S)$. Thus, we have $r_1^* \in N_G(S)$. This argument can also be used to show $r_2^* \in N_G(S)$, completing the proof of the claim. □

Following the proof of Lemma 3.3.1, our next step is to define an auxiliary buyer i' with demand $b_{i'} = 2$ and valuations $v_{i'}(t) := 1$ if $t \in N_G(S) \cap N_G(I \setminus S)$ and $v_{i'}(t) := 0$ otherwise.

Then, we for the upper market M^U with buyers $S \cup \{i'\}$, items $N_G(S)$ and the same demands b and valuations v as in the original market M, albeit with an additional entry for the auxiliary buyer i'. Note that $b(S \cup \{i'\}) = b(S) + 2 = |N_G(S)|$, i.e. the number of items in the upper market M^U coincides with its total demand.

Next, we prove the analogue of Claim 3.3.2. We show that the legality of every buyer-item pair for each buyer $i \neq i'$ in the upper market M^U is the same as its legality in the original market M :

Claim 3.4.4. *For every buyer $i \in S$ and for every item $t \in N_G(S)$, we have $t \in \mathscr{L}_M(i)$ if and only if $t \in \mathscr{L}_{M^U}(i)$.*

Proof. ($\Longrightarrow$) Suppose $t \in \mathscr{L}_M(i)$. Then, there exists an optimal allocation $\boldsymbol{\mathcal{A}} = \{A_j \subseteq T : j \in I\}$ of the market M such that $t \in A_i$. Since $|N_G(S)| = b(S) + 2$, it follows that the allocation $\boldsymbol{\mathcal{A}}$ assigns exactly two items in $N_G(S) \cap N_G(I \setminus S)$ to a buyer in the set $I \setminus S$. Let x_1, x_2 be these items. Then, define an allocation $\boldsymbol{\mathcal{A}}^U = \{A_j^U \subseteq N_G(S) : j \in S \cup \{i'\}\}$ of the upper market M^U as follows:

$$A_j^U := \begin{cases} A_j & \text{if } j \in S \\ \{x_1, x_2\} & \text{if } j = i' \end{cases} \quad \forall j \in S \cup \{i'\}.$$

Then, we have $SW(\boldsymbol{\mathcal{A}}^U) = b(S) + 2$, so the allocation $\boldsymbol{\mathcal{A}}^U$ is an optimal allocation of the market M^U. Note that this also implies that the market M^U satisfies the (OPT) condition. Moreover, we have $t \in A_i^U$, so it follows that $t \in \mathscr{L}_{M^U}(i)$, as required.

($\Longleftarrow$) Suppose $t \in \mathscr{L}_{M^U}(i)$. Then, there exists an optimal allocation $\boldsymbol{\mathcal{A}}^U = \{A_j^U \subseteq N_G(S) : j \in S \cup \{i'\}\}$ of the upper market M^U such that $t \in A_i^U$. By the proof of the forward direction, we have $SW(\boldsymbol{\mathcal{A}}^U) = b(S) + 2$, which is equal to the total demand of the upper market M^U. Since every valuation in the upper market M^U is at most one, it follows that $v_i(t) = 1$. Moreover, since the market M satisfies the $(*)$ condition, it follows that $t \in \mathscr{L}_M(i)$, completing the proof of the claim. $\square$

Next, we prove the analogue of Claim 3.3.3. We show that the upper market M^U satisfies the $(*)$ condition:

Claim 3.4.5. *The upper market M^U satisfies the $(*)$ condition.*

Proof. Let $i \in S$ and let $t \in N_G(S)$. Then, by the previous claim, we have $t \in \mathscr{L}_{M^U}(i) \iff t \in \mathscr{L}_M(i) \iff v_i(t) = 1$, as required. It remains to show that for the auxiliary buyer i', we also have $t \in \mathscr{L}_{M^U}(i') \iff v_{i'}(t) = 1$:

($\Longrightarrow$) If $t \in \mathscr{L}_{M^U}(i')$, then there exists an optimal allocation $\boldsymbol{\mathcal{A}}^U = \{A_j^U \subseteq N_G(S) : j \in S \cup \{i'\}\}$ of the upper market M^U such that $t \in A_{i'}^U$. Again, we have $SW(\boldsymbol{\mathcal{A}}^U) = b(S) + 2$, which is equal to the total demand of the upper market M^U. It follows that $v_{i'}(t) = 1$, as required.

($\Longleftarrow$) Suppose $v_{i'}(t) = 1$. Then, $t \in N_G(S) \cap N_G(I \setminus S)$ by definition. As $t \in N_G(I \setminus S)$, there exists a buyer $h_1 \in I \setminus S$ such that $t \in \mathscr{L}_M(h_1)$. Let $x \in N_G(S) \cap N_G(I \setminus S)$ be such that $x \neq t$. Note that such an item exists by the (OPT) condition. Moreover, since $x \in N_G(I \setminus S)$, there exists a buyer $h_2 \in I \setminus S$ such that $x \in \mathscr{L}_M(h_2)$. Now, consider the assignment of item t to buyer h_1 and item x to buyer h_2. This assignment respects legality and, by our assumption that there is no unit-demand buyer in the market M, it follows that this assignment also respects demand. Since $k_M > 2$, it follows that there exists an optimal allocation $\boldsymbol{\mathcal{A}} = \{A_j \subseteq T : j \in I\}$ of the original market M such that $t \in A_{h_1}$ and $x \in A_{h_2}$. Since $|N_G(S)| = b(S) + 2$, it follows that the items $\{t, x\}$ are the only items in the set $N_G(S)$ that are assigned by the allocation $\boldsymbol{\mathcal{A}}$ to buyers in the set $I \setminus S$. Thus, we define an allocation $\boldsymbol{\mathcal{A}}^U := \{A_j^U \subseteq N_G(S) : j \in S \cup \{i'\}\}$ of the upper market M^U as follows:

$$A_j^U := \begin{cases} A_j & \text{if } j \in S \\ \{t, x\} & \text{if } j = i' \end{cases} \quad \forall j \in I.$$

Then, $SW(\boldsymbol{\mathcal{A}}^U) = b(S) + 2$, so the allocation $\boldsymbol{\mathcal{A}}^U$ is an optimal allocation of the upper market M^U. Moreover, we have $t \in A_{i'}^U$, so it follows that $t \in \mathscr{L}_{M^U}(i')$, completing the proof of the claim. $\square$

Now, we show that we may apply induction to obtain an optimal dynamic pricing scheme for the upper market M^U. Since $i^*, h_1^*, h_2^* \in I \setminus S$, it follows that $|I \setminus S| \geq 1$. If $|I \setminus S| > 1$, then the upper market M^U has fewer buyers than the original market M, and we may apply induction. Otherwise, if $|I \setminus S| = 1$, then we have $i^* = h_1^* = h_2^*$ and $I \setminus S = \{i^*\}$. Moreover, we have $b_{i^*} = 3$. Consider the assignment of the items $\{t^*, r_1^*\}$ to the buyer i^*. This assignment respects both legality and demand in the market M. Moreover, as $k_M > 2$, it follows that this assignment extends to an optimal allocation $\boldsymbol{\mathcal{A}} = \{A_i \subseteq T : i \in I\}$ of the market M, and $|A_{i^*}| = b_{i^*} = 3$ by the (OPT) condition. Furthermore, the items $\{t^*, r_1^*\}$ are the only items in the set $N_G(S) \cap N_G(I \setminus S)$ that are assigned by the allocation $\boldsymbol{\mathcal{A}}$ to buyers in the set $I \setminus S$, so there must exist an item $x \in N_G(I \setminus S) \setminus N_G(S)$ to fulfill the demand of the buyer i^*. Hence, the upper market M^U has the same number of buyers as the original market M, but it has fewer items than the original market M, as it does not include the item x. Thus, we may again apply induction.

By induction, there exists an optimal dynamic pricing scheme p^U for the upper market

M^U such that $\{t^*\} = \text{argmin}\{p_t : t \in T\}$. By Claim 3.1.1, we may assume the pricing scheme p^U has unique values.

Let $\{x\} := \text{argmin}\{p_t^U : t \in (N_G(S) \cap N_G(I \setminus S)) \setminus \{t^*\}\}$, i.e. the item x is the lowest-priced item according to the pricing scheme p^U that is not the item t^* and that is legal for both a buyer in the set S and a buyer in the set $I \setminus S$.

Next, we prove the analogue of Claim 3.3.4:

Claim 3.4.6. *For every $y \in N_G(I \setminus S) \setminus N_G(S)$, every assignment of the items $\{t^*, x, y\}$ to buyers in the set $I \setminus S$ that respects both legality and demand in the market M extends to an optimal allocation of the market M.*

Proof. Suppose for a contradiction that there is such an assignment that is not extendable. Suppose item t^* is assigned to buyer i_1, item x is assigned to buyer i_2 and item y is assigned to buyer i_3, where $i_1, i_2, i_3 \in I \setminus S$. Let G'' be the graph obtained from the graph G by decreasing the b-values of each of the vertices i_1, i_2, i_3, t^*, x, y by one and removing them if their updated b-value is equal to one. Let b'' denote the b-values of the vertices in the graph G''. By Hall's theorem, it follows that there exists a subset of buyers $S' \subseteq I$ such that $|N_{G''}(S')| < b''(S')$. By an argument that is analogous to the proof of Claim 3.4.3, we have $|N_G(S')| = b(S') + 2$ and $\{t^*, x, y\} \subseteq N_G(S')$. Also, we have $i_1, i_2, i_3 \notin S'$. We proceed to show that $|N_G(S \cup S')| = b(S \cup S') + 2$, contradicting the maximality of the set $N_G(S)$.

First, consider the assignment of item t^* to buyer i_1 and item x to buyer i_2. This assignment respects both legality and demand in the market M and, as $k_M > 2$, it follows that this assignment extends to an optimal allocation of the market M. Since $t^*, x \in N_G(S \cup S')$ and since $i_1, i_2 \notin S \cup S'$, it follows that $|N_G(S \cup S')| \geq b(S \cup S') + 2$.

Next, we show the reverse inequality. Observe that $|N_G(S) \cap N_G(S')| \geq |N_G(S \cap S')| \geq b(S \cap S')$, where the second inequality follows by Hall's theorem. Moreover, we have $t^*, x \in N_G(S) \cap N_G(S')$. We consider four cases based on the locations of the items t^* and x with respect to the set $N_G(S \cap S')$:

Case 1: $t^*, x \notin N_G(S \cap S')$.

Then, we have $|N_G(S) \cap N_G(S')| \geq |N_G(S \cap S')| + 2 \geq b(S \cap S') + 2$ (†).

Case 2: $t^*, x \in N_G(S \cap S')$.

Consider the assignment of item t^* buyer i_1 and item x to buyer i_2. Again, this assignment extends to an optimal allocation of the market M. Since $t^*, x \in N_G(S \cap S')$ and since $i_1, i_2 \notin S \cap S'$, it follows that $|N_G(S \cap S')| \geq b(S \cap S') + 2$. Hence, we have $|N_G(S) \cap N_G(S')| \geq |N_G(S \cap S')| + 2 \geq b(S \cap S') + 2$ (†).

Case 3: $t^* \in N_G(S \cap S')$ and $x \notin N_G(S \cap S')$.

Consider the assignment of item t^* to buyer i_1. This assignment extends to an optimal allocation of the market M. Since $t^* \in N_G(S \cap S')$ and since $i_1 \notin S \cap S'$, it follows that $N_G(S \cap S')| \geq b(S \cap S') + 1$. Moreover, since $x \in (N_G(S) \cap N_G(S')) \setminus N_G(S \cap S')$, we have $|N_G(S) \cap N_G(S')| \geq |N_G(S \cap S')| + 1$. Overall, we have $|N_G(S) \cap N_G(S')| \geq |N_G(S \cap S')| + 1 \geq b(S \cap S') + 2$ (†).

Case 4: $t^* \notin N_G(S \cap S')$ and $x \in N_G(S \cap S')$.

This case is analogous to Case 3, and we again obtain the chain of inequalities (†).

Now, consider:

$$\begin{aligned}
|N_G(S \cup S')| &= |N_G(S) \cup N_G(S')| \\
&= |N_G(S)| \cup |N_G(S')| - |N_G(S) \cap N_G(S')| \\
&= (b(S) + 2) + (b(S') + 2) - |N_G(S) \cap N_G(S')| \\
&= b(S \cup S') + b(S \cap S') + 4 - |N_G(S) \cap N_G(S')| \\
&\leq b(S \cup S') + b(S \cap S') + 4 - b(S \cap S') - 2 && \text{by (†)} \\
&= b(S \cup S') + 2.
\end{aligned}$$

In conclusion, we have $|N_G(S \cup S')| = b(S \cup S') + 2$. Moreover, since $i_1, i_2, i_3 \notin S \cup S'$ and since $\{t^*, x, y\} \subseteq N_G(S \cup S')$, it follows that $|N_{G''}(S \cup S')| < b''(S \cup S')$. Furthermore, we have $N_G(S) \subseteq N_G(S \cup S')$ and $y \in N_G(S \cup S') \setminus N_G(S)$. Consider our three choices of the bundle $\{t^*, x, y\}$ that is not flexible; of the assignment of item t^* to buyer i_1, item x to buyer i_2 and item y to buyer i_3 that does not extend; and of the set $S \cup S'$ satisfying the above inequality in the resulting graph G''. Together, they contradict the maximality of the set $N_G(S)$. In conclusion, every assignment of the items $\{t^*, x, y\}$ to buyers in the set $I \setminus S$ that respects both legality and demand in the market M extends to an optimal allocation of the market M, as desired. □

We proceed, as in the bi-demand case, to consider two cases based on the legalities of the items t^* and x; each leading to the construction of a slightly different lower market:

Case 1: The items t^* and x are both legal for a unique buyer $h^* \in I \setminus S$ and $b_{h^*} = 2$.

As in the bi-demand case, we form the lower market $M^{L'}$ with buyers $I^{L'} := I \setminus (S \cup \{h^*\})$, items $T^{L'} := N_G(I \setminus S) \setminus N_G(S)$ and the same demands b and valuations v as in the original market M. Note that $|T^{L'}| = b(I^{L'})$. We proceed to show the analogue of Claim 3.3.5: The legality of every buyer-item pair in the lower market $M^{L'}$ is the same as its legality in the original market M :

Claim 3.4.7. *For every buyer $i \in I^{L'}$ and for every item $t \in T^{L'}$, we have $t \in \mathscr{L}_M(i)$ if and only if $t \in \mathscr{L}_{M^{L'}}(i)$.*

Proof. ($\Longrightarrow$) Suppose $t \in \mathscr{L}_M(i)$. Consider the assignment of item t to buyer i and items $\{t^*, x\}$ to buyer h^*. By Claim 3.4.6, this assignment extends to an optimal allocation $\boldsymbol{\mathcal{A}}$ of the market M. Furthermore, as $|N_G(S)| = b(S) + 2$, it follows that the items $\{t^*, x\}$ are the only items in the $N_G(S)$ that are assigned to buyers in the set $I \setminus S$. Hence, we define an allocation $\boldsymbol{\mathcal{A}}^{L'} := \{A_j^{L'} \subseteq T^{L'} : j \in I^{L'}\}$ of the lower market $M^{L'}$ as follows:

$$A_j^{L'} := A_j \quad \forall j \in I^{L'}.$$

Then, we have $SW(\boldsymbol{\mathcal{A}}^{L'}) = b(I^{L'})$, so the allocation $\boldsymbol{\mathcal{A}}^{L'}$ is an optimal allocation of the market $M^{L'}$. Note that this also implies that the market $M^{L'}$ satisfies the (OPT) condition. Moreover, we have $t \in A_i^{L'}$, so it follows that $t \in \mathscr{L}_{M^{L'}}(i)$, as required.

($\Longleftarrow$) Suppose $t \in \mathscr{L}_{M^{L'}}(i)$. Then, there exists an optimal allocation $\boldsymbol{\mathcal{A}}^{L'} = \{A_j^{L'} \subseteq T^{L'} : j \in I^{L'}\}$ of the lower market $M^{L'}$ such that $t \in A_i^{L'}$. By the proof of the forward direction, we have $SW(\boldsymbol{\mathcal{A}}^{L'}) = b(I^{L'})$. Since every valuation in the lower market $M^{L'}$ is at most one, it follows that $v_i(t) = 1$. Moreover, since the market M satisfies the $(*)$ condition, it follows that $t \in \mathscr{L}_M(i)$, completing the proof of the claim. $\square$

Next, note that it follows immediately that the market $M^{L'}$ satisfies the $(*)$ condition. Indeed, by the previous claim, for every buyer $i \in I^{L'}$ and for every item $t \in T^{L'}$, we have $t \in \mathscr{L}_{M^{L'}}(i) \iff t \in \mathscr{L}_M(i) \iff v_i(t) = 1$, as required. Moreover, since $h^* \notin I^{L'}$, it follows that the market $M^{L'}$ has fewer buyers than the original market M, so we may apply induction to obtain an optimal dynamic pricing scheme $p^{L'}$ for the lower market $M^{L'}$.

Now, we construct a pricing scheme p for the original market M with prices in the following order:

$$\{0\} <_p \{t^*\} <_p T^{L'} <_p N_G(S) \setminus \{t^*\} <_p \{1\}$$

The prices of the items in the set $N_G(S) \setminus \{t^*\}$ are ordered according to the pricing scheme p^U and the prices of the items in the set $T^{L'}$ are ordered according to the pricing scheme $p^{L'}$.

We proceed to prove the analogue of Claim 3.3.6:

Claim 3.4.8. *The pricing scheme p defined above is an optimal dynamic pricing scheme for the market M and $\{t^*\} = \operatorname{argmin}\{p_t : t \in T\}$.*

Proof. By Claim 3.1.1, we may assume the pricing scheme $p^{L'}$ has unique values. We proceed to consider cases based on which buyer arrives first to the market M :

Case 1.1: A buyer $i \in S$ arrives first to the market M. Recall that the pricing scheme p^U satisfies $\{t^*\} = \text{argmin}\{p^U_t : t \in N_G(S)\}$. Thus, the bundle in demand for the buyer i with respect to the prices p is the bundle in demand for the buyer i with respect to the prices p^U. Let R be this bundle. By optimality of the pricing scheme p^U, it follows that there exists an optimal allocation $\boldsymbol{\mathcal{A}}^U := \{A^U_j \subseteq N_G(S) : j \in S \cup \{i'\}\}$ of the upper market M^U such that $R = A^U_i$. By the proof of Claim 3.4.4, we have that the upper market M^U satisfies the (OPT) condition. Thus, let $\{w, z\} := A^U_{i'}$ be the bundle that is assigned by the allocation $\boldsymbol{\mathcal{A}}^U$ to the artificial buyer i'. By Claim 3.3.3, we have $w, z \in N_G(S) \cap N_G(I \setminus S)$, so there exist buyers $f, g \in I \setminus S$ such that $w \in \mathscr{L}_M(f)$ and $z \in \mathscr{L}_M(g)$. Consider the assignment of item w to buyer f and item z to buyer g. This assignment respects legality and, by our assumption that there is no unit-demand buyer in the market M, we have that this assignment also respects demand. As $k_M > 2$, it follows that this assignment extends to an optimal allocation $\boldsymbol{\mathcal{A}} = \{A_j \subseteq T : j \in I\}$ of the original market M such that $w \in A_f$ and $z \in A_g$. Now, we define an allocation $\boldsymbol{\mathcal{A}}' := \{A'_j \subseteq T : j \in I\}$ of the original market M as follows:

$$A'_j := \begin{cases} A^U_j & \text{if } j \in S \\ A_j & \text{if } j \in I \setminus S \end{cases} \quad \forall j \in I.$$

Remark that the items $\{w, z\}$ are the unique items in the set $N_G(S)$ that are assigned by the allocation $\boldsymbol{\mathcal{A}}$ to buyers in the set $I \setminus S$. Thus, the allocation $\boldsymbol{\mathcal{A}}'$ indeed assigns every item to exactly one buyer. Moreover, by Claim 3.3.2, it follows that $SW(\boldsymbol{\mathcal{A}}') = SW(\boldsymbol{\mathcal{A}})$, so the allocation $\boldsymbol{\mathcal{A}}'$ is an optimal allocation of the market M. Furthermore, we have $R = A'_i$. Hence, the assignment of the bundle R to the buyer i extends to an optimal allocation $\boldsymbol{\mathcal{A}}'$ of the market M, as required.

Case 1.2: A buyer $i \in I^{L'}$ arrives first to the market M. Recall that $t^* \notin \mathscr{L}_M(i)$, by the assumption of Case 1. Thus, the bundle in demand for the buyer i with respect to the prices p is the bundle in demand for the buyer i with respect to the prices $p^{L'}$. Let R be this bundle. By optimality of the pricing scheme $p^{L'}$, it follows that there exists an optimal allocation $\boldsymbol{\mathcal{A}}^{L'} = \{A^{L'}_j \subseteq T^{L'} : j \in I^{L'}\}$ of the lower market $M^{L'}$ such that $R = A^{L'}_i$. Moreover, since $t^*, x \in \mathscr{L}_M(h^*)$, since $b_{h^*} = 2$ and since $k_M > 2$, it follows that there exists an optimal allocation $\boldsymbol{\mathcal{A}} = \{A_j \subseteq T : j \in I\}$ of the original market M such that $\{t^*, x\} = A_{h^*}$. Now, we define an allocation $\boldsymbol{\mathcal{A}}' := \{A'_j \subseteq T : j \in I\}$ of the original market M as follows:

$$A'_j := \begin{cases} A^{L'}_j & \text{if } j \in I^{L'} \\ A_j & \text{if } j \in I \setminus I^{L'} \end{cases} \quad \forall j \in I.$$

Remark that the items $\{t^*, x\}$ are the unique items in the set $N_G(S)$ that are assigned by the allocation $\boldsymbol{\mathcal{A}}$ to buyers in the set $I \setminus S$. Thus, the allocation $\boldsymbol{\mathcal{A}}'$ indeed assigns every item to exactly one buyer. Moreover, by Claim 3.3.5, it follows that $SW(\boldsymbol{\mathcal{A}}') = SW(\boldsymbol{\mathcal{A}})$, so the allocation $\boldsymbol{\mathcal{A}}'$ is an optimal allocation of the market M. Furthermore, we have $R = A'_i$. Hence, the assignment of the bundle R to the buyer i extends to an optimal allocation $\boldsymbol{\mathcal{A}}'$ of the market M, as desired.

Case 1.3: The buyer h^* arrives first to the market M. Since we have $b_{h^*} = 2$ by assumption, it follows that the bundle in demand for the buyer h^* with respect to the prices p is a set of size two, say $\{w, z\}$, such that $w, z \in \mathscr{L}_M(h^*)$. Thus, the assignment of the items $\{w, z\}$ to the buyer h^* extends to an optimal allocation of the market M because $k_M > 2$. Overall, the pricing scheme p is an optimal dynamic pricing scheme for the market M and $\{t^*\} = \operatorname{argmin}\{p_t : t \in T\}$, completing the proof of the claim. $\square$

Case 2: $b((I \setminus S) \cap N_G(\{t^*, x\})) \geq 3$.

As in the bi-demand case, we form the lower market M^L with buyers $I \setminus S$, items $T^L := (N_G(I \setminus S) \setminus N_G(S)) \cup \{t^*, x\}$, and the same demands b and valuations v as in the original market M. Note that $|T^L| = b(I \setminus S)$. We proceed to show the analogue of Claim 3.3.7. That is, the legality of every buyer-item pair in the lower market M^L is the same as its legality in the original market M :

Claim 3.4.9. *For every buyer $i \in I \setminus S$ and for every item $t \in T^L$, we have $t \in \mathscr{L}_M(i)$ if and only if $t \in \mathscr{L}_{M^L}(i)$.*

Proof. ($\Longrightarrow$) Suppose $t \in \mathscr{L}_M(i)$. First, suppose $t = t^*$. Let $\hat{i} \in I \setminus S$ be such that $x \in \mathscr{L}_M(\hat{i})$. Consider the allocation of item t^* to buyer i and item x to buyer $\hat{i}$. This assignment respects both legality and demand in the market M and, as $k_M > 2$, it follows that this assignment extends to an optimal allocation $\boldsymbol{\mathcal{A}} = \{A_j \subseteq T : j \in I\}$ of the market M such that $t^* \in A_i$. Furthermore, the items t^* and x are the unique items in $N_G(S)$ that are assigned by the allocation $\boldsymbol{\mathcal{A}}$ to buyers in the set $I \setminus S$. Hence, we define an allocation $\boldsymbol{\mathcal{A}}^L := \{A^L_j \subseteq T^L : j \in I \setminus S\}$ of the lower market M^L as follows:

$$A^L_j := A_j \quad \forall j \in I \setminus S.$$

Then, we have $SW(\boldsymbol{\mathcal{A}}^L) = b(I \setminus S)$, so the allocation $\boldsymbol{\mathcal{A}}^L$ is an optimal allocation of the market M^L. Note that this also implies that the market M^L satisfies the (OPT) condition. Moreover, we have $t^* \in A^L_i$, so it follows that $t^* \in \mathscr{L}_{M^L}(i)$, as required.

Next, suppose $t = x$. Then, we select a buyer $\hat{i} \in I \setminus S$ such that $t^* \in \mathscr{L}_M(\hat{i})$ and proceed analogously to the previous case where $t = t^*$ to conclude $x \in \mathscr{L}_M(i)$.

Finally, suppose $t \notin \{t^*, x\}$. Then, since $t^*, x \in N_G(I \setminus S)$, it follows that there exist buyers $h_1, h_2 \in I \setminus S$ such that $t^* \in \mathscr{L}_M(h_1)$ and $x \in \mathscr{L}_M(h_2)$ By the assumption of Case 2, we can select such buyers h_1 and h_2 so that the assignment of item t to buyer i, item t^* to buyer h_1 and item x to buyer h_2 respects demand. By Claim 3.4.6, it follows that this assignment extends to an optimal allocation $\boldsymbol{\mathcal{A}} = \{A_j : j \in I\}$ of the market M such that $t \in A_i$, $t^* \in A_{h_1}$ and $x \in A_{h_2}$. Moreover, the items $\{t^*, x\}$ are the only items in the set $N_G(S)$ that are assigned by the allocation $\boldsymbol{\mathcal{A}}$ to buyers in the set $I \setminus S$. Hence, we define an allocation $\boldsymbol{\mathcal{A}}^L := \{A_j^L \subseteq T^L : j \in I \setminus S\}$ of the lower market M^L as follows:

$$A_j^L := A_j \quad \forall j \in I \setminus S.$$

Then, we have $SW(\boldsymbol{\mathcal{A}}^L) = b(I \setminus S)$, so the allocation $\boldsymbol{\mathcal{A}}^L$ is an optimal allocation of the market M^L. Moreover, we have $t \in A_i^L$, so it follows that $t \in \mathscr{L}_{M^L}(i)$, as required.

($\Longleftarrow$) The proof is analogous to the proof of the backward direction of Claim 3.4.7. $\square$

As in Case 1, it follows immediately that the market M^L satisfies the $(*)$ condition. Moreover, since $|N_G(S)| = b(S) + 2$, we have $S \neq \emptyset$. Hence, the market M^L, which has buyers $I \setminus S$, has fewer buyers than the original market M. Thus, we may apply induction to obtain an optimal dynamic pricing scheme p^L for the lower market M^L such that $\{t^*\} = \operatorname{argmin}\{p_t : t \in T^L\}$.

Next, we define the following bipartition of the set $N_G(S) \setminus \{x\}$:

$$N_G^{<x}(S) := \{t \in N_G(S) : p_t^U < p_x^U\}.$$
$$N_G^{>x}(S) := \{t \in N_G(S) : p_t^U > p_x^U\}.$$

Then, we construct a pricing scheme p for the original market M with prices in the following order:

$$\{0\} <_p N_G^{<x}(S) <_p T^L \setminus \{t^*\} <_p N_G^{>x}(S) <_p \{1\}.$$

The prices of the items in the set $N_G^{<x}(S) \cup N_G^{>x}(S)$ are ordered according to the pricing scheme p^U and the prices of the items in the set $T^L \setminus \{t^*\}$ are ordered according to the pricing scheme p^L. By Claim 3.1.1, we may assume the pricing scheme p^L has unique values. Recall that the pricing scheme p^U satisfies $\{t^*\} = \operatorname{argmin}\{p_t : t \in N_G(S)\}$. Thus, for a buyer $i \in S$, the bundle in demand for the buyer i with respect to the prices p is the bundle in demand for the buyer i with respect to the prices p^U. In addition, recall that the pricing scheme p^L satisfies $\{t^*\} = \operatorname{argmin}\{p_t : t \in T^L\}$. Thus, for a buyer $i \in I \setminus S$, the bundle in demand for the buyer i with respect to the prices p is the bundle in demand for the buyer i with respect to the prices p^L. By an argument that is analogous to the proof of Claim 3.4.8, we conclude that the pricing scheme p is an optimal dynamic pricing scheme for the market M and $\{t^*\} = \text{argmin}\{p_t : t \in T\}$, completing the proof of Lemma 3.4.1. $\square$

Chapter 4

Dynamic Pricing in Multi-Demand Markets Without (OPT)

In this chapter, we investigate the consequences of removing the (OPT) condition. First, in Section 4.1, we provide a framework that relates the problem of finding optimal dynamic pricing schemes in general multi-demand markets to the problem of finding optimal dynamic pricing schemes in multi-demand markets that satisfy both the (OPT) condition and the $(*)$ condition. In [5], Szögi uses this framework to give a white-box reduction for the case where every buyer's demand is equal to two. Next, in Section 4.2, we use provide a black-box reduction for a specific case of multi-demand markets.

4.1 A Framework for Reducing the Problem to Markets With (OPT)

Let M be a multi-demand market which does not satisfy the (OPT) condition. By Claim 2.0.2, we may assume every item is allocated to some buyer in every optimal allocation of the market M. Thus, $|T| < b(I)$. In other words, there is a deficiency of items in the market M. Let $\tilde{\boldsymbol{y}}$ be the specific dual solution for the market graph H, as provided by Claim 2.0.1. Then, we define the set $\hat{I} := \{i \in I : \tilde{y}_i = 0\}$. By 2.0.1 (ii), we have that the set $\hat{I}$ is precisely the set of buyers in the market $|M$ that receive fewer items than their demand in some optimal allocation of the market M. We refer to these buyers as "dummy" buyers.

Next, we proceed as in Section 3.1, and we form an auxiliary market M' with the same buyers I, the same items T, the same demands b and the following valuations v' :

$$v'_i(t) = \begin{cases} 1 & \text{if } t \in \mathscr{L}_M(i) \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in I, t \in T.$$

It follows that optimal allocations of the market M' are precisely optimal allocations of the market M.

Next, to compensate for the dearth of items in the market M', we artificially introduce a set $\hat{T}$ of $b(I) - |T|$ items, which we refer to as "dummy" items. The buyers' valuations of these dummy items are as follows:

$$v'_i(\hat{t}) := \begin{cases} 1 & \text{if } i \in \hat{I} \\ 0 & \text{if } i \in I \setminus \hat{I} \end{cases} \quad \forall i \in I, \hat{t} \in \hat{T}.$$

Let $\hat{M}$ be the market with buyers I, items $T \cup \hat{T}$, the same demands b and the valuations v' as defined above. It follows that every optimal allocation of the market $\hat{M}$ is the disjoint union of an optimal allocation of the market M' and an assignment of the dummy items $\hat{T}$ to dummy buyers in $\hat{I}$ which respects demand. Conversely, every optimal allocation of the market M' can be extended to include an assignment of the dummy items $\hat{T}$ to dummy buyers in $\hat{I}$ which respects demand, and this is an optimal allocation of the market $\hat{M}$. Hence, the modified market $\hat{M}$ satisfies both the (OPT) condition and the $(*)$ condition.

Next, we recall the following quantity:

$$\alpha := \min\{\min_{\substack{i \in I \\ t \in T}} \{\tilde{y}_i + \tilde{y}_t - w_{it} : it \text{ is not legal}\}, \min_{v \in I \cup T}\{\tilde{y}_v : \tilde{y}_v > 0\}\}.$$

Note that $\alpha > 0$ by our assumption that every item is allocated to some buyer in every optimal allocation.

Now, suppose p' is an optimal dynamic pricing scheme for the modified market $\hat{M}$. By Claim 3.1.1, we may assume the pricing scheme p' has unique values, so we may order the non-dummy items T in increasing order of their prices as given by p'. Let us call this ordering $\sigma : T \to \{1, \dots, |T|\}$. That is, $\sigma(x) > \sigma(y)$ if and only if $p'_x > p'_y$.

Remark that the dual solution $\tilde{\boldsymbol{y}}$ for the market graph H is not conducive to defining prices, as we did for the (OPT) case in Section 3.1. This is because $\tilde{y}_i = 0$ for all $i \in \hat{I}$, so the analogue of Claim 3.1.2 does not hold. To work around this, we define a function

$\hat{\boldsymbol{y}} : I \cup T \to \mathbb{R}$ on the vertices of the market graph H' of the auxiliary market M' as follows:

$$\hat{y}_v := \begin{cases} \tilde{y} + \alpha/2 & \text{if } v \in I \\ \tilde{y} - \alpha/2 & \text{if } v \in T \end{cases} \quad \forall v \in I \cup T.$$

We will eventually define prices for the items T using the values of $\{\hat{y}_t : t \in T\}$, but first, we list a few elementary properties of the function $\hat{\boldsymbol{y}}$ that will be useful to this end:

Claim 4.1.1. *The function $\hat{\boldsymbol{y}}$ satisfies the following:*

(i) For all buyers $i \in I \setminus \hat{I}$, we have $\hat{y}_i \geq 3\alpha/2$.

(ii) For all buyers $i \in \hat{I}$, we have $\hat{y}_i = \alpha/2$.

(iii) For all buyers $i \in I$ and for all items $t \in T$, we have $\hat{y}_i + \hat{y}_t \geq w_{it}$. Moreover, equality occurs if and only if the edge $it \in E(H)$ is legal in the market M.

Proof. Item (i) follows from Claim 2.0.1 (ii) and from the definition of α; item (ii) follows from Claim 2.0.1 (ii); and item (iii) follows from Claim 2.0.1. □

Now, if we attempt to proceed as we did Section 3.1, and we try to define prices for the items T using the values of $\{\hat{y}_t : t \in T\}$ in the analogous way, then we run into a potential issue with the dummy buyers in the market M. To see why this is the case, let us first order all of the items $T \cup \hat{T}$ of the market $\hat{M}$ in increasing order of their prices as given by p', and let us call this ordering $\tau : T \cup \hat{T} \to \{1, \ldots, |T \cup \hat{T}|\}$. Then, for a dummy buyer $\hat{i} \in \hat{T}$, we have that the assignment of the first $b_{\hat{i}}$ items (according to τ) that are legal for the buyer $\hat{i}$ to the buyer $\hat{i}$ extends to an optimal allocation of the market $\hat{M}$. However, when the buyer $\hat{i}$ arrives to the market M, they do not consider purchasing any dummy items because those items do not exist in the market M. Consequently, they will skip over the dummy items in the ordering τ and they will instead purchase the first $b_{\hat{i}}$ non-dummy items (according to the ordering σ) that are legal for them. This assignment is no longer guaranteed to extend to an optimal allocation of the market M. To remedy this issue, we seek to "cut off" dummy buyers from purchasing any items past a certain point in the ordering σ, which we call the "cutoff point." This can be achieved by increasing the prices of the items which appear after the cutoff point by a small amount which, as we will show, is small enough to not change the behaviour of the non-dummy buyers in the market M. The remainder of this section is dedicated to proving these assertions.

We proceed to select a cutoff point $C \in \{0, 1, \ldots, |T|\}$. Note that selecting $C = 0$ ensures that no dummy buyer will purchase any item whatsoever, and that selecting $C =$

$|T|$ does not prevent the dummy buyers from purchasing any items that they otherwise would be able to purchase. Then, we define prices p for the market M as follows:

$$p_t := \begin{cases} \hat{y}_t + \frac{\alpha/2}{|T|+1}\sigma(t) & \text{if } \sigma(t) \le C \\ \hat{y}_t + \frac{\alpha/2}{|T|+1}\sigma(t) + \alpha/2 & \text{if } \sigma(t) > C \end{cases}$$

For an item $t \in T$, we say that the item t appears *before* the cutoff point if $\sigma(t) \le C$, and we say that the item t appears *after* the cutoff point if $\sigma(t) > C$. Also, we write u to denote the utility with respect to the market M and we write u' to denote the utility with respect to the modified market $\hat{M}$.

Now, we show that legal items have strictly positive utility for non-dummy buyers in the market M, regardless of whether the items appear before or after the cutoff point:

Claim 4.1.2. *Let $i \in I \setminus \hat{I}$ be a non-dummy buyer and let $t \in T$ be an item such that $t \in \mathscr{L}_M(i)$. Then, we have $u_i(t, p) > 0$.*

Proof. Consider:

$$\begin{aligned} u_i(t,p) &\ge w_{it} - \left(\hat{y}_t + \frac{\alpha/2}{|T|+1}\sigma(t) + \alpha/2\right) \\ &= \hat{y}_i - \frac{\alpha/2}{|T|+1}\sigma(t) - \alpha/2 && \text{by Claim 4.1.1 (iii)} \\ &\ge \alpha - \frac{\alpha/2}{|T|+1}\sigma(t) && \text{by Claim 4.1.1 (i)} \\ &\ge \alpha - \alpha/2\frac{|T|}{|T|+1} && \text{since } \sigma(t) \le |T| \\ &> \alpha/2 && \text{since } \alpha > 0. \end{aligned}$$

Thus, $u_i(t,p) > \alpha/2 > 0$, as desired. ☐

Next, we show that legal items that appear before the cutoff point have strictly positive utility for dummy buyers in the market M, and that legal items that appear after the cutoff point have strictly negative utility for dummy buyers in the market M :

Claim 4.1.3. *Let $i \in \hat{I}$ be a dummy buyer and let $t \in T$ be an item such that $t \in \mathscr{L}_M(i)$. If $\sigma(t) \le C$, then we have $u_i(t,p) > 0$. Otherwise, if $\sigma(t) > C$, then we have $u_i(t,p) < 0$.*

Proof. First, suppose $\sigma(t) \le C$ and consider:

$$\begin{aligned}
u_i(t,p) &= w_{it} - \left(\hat{y}_t + \frac{\alpha/2}{|T|+1}\sigma(t)\right) \\
&= \hat{y}_i - \frac{\alpha/2}{|T|+1}\sigma(t) && \text{since } it \text{ is legal and by Claim 4.1.1 (iii)} \\
&= \alpha/2 - \frac{\alpha/2}{|T|+1}\sigma(t) && \text{by Claim 4.1.1 (ii)} \\
&\ge \alpha/2 - \alpha/2\frac{|T|}{|T|+1} && \text{since } \sigma(t) \le |T| \\
&> 0 && \text{since } \alpha > 0.
\end{aligned}$$

Thus, $u_i(t,p) > 0$, as required.

Next, suppose $\sigma(t) > C$ and consider:

$$\begin{aligned}
u_i(t,p) &= w_{it} - \left(\hat{y}_t + \frac{\alpha/2}{|T|+1}\sigma(t) + \alpha/2\right) \\
&= \hat{y}_i - \frac{\alpha/2}{|T|+1}\sigma(t) - \alpha/2 && \text{by Claim 4.1.1 (iii)} \\
&= -\frac{\alpha/2}{|T|+1}\sigma(t) && \text{by Claim 4.1.1 (ii)} \\
&< 0 && \text{since } \alpha > 0.
\end{aligned}$$

Thus, $u_i(t,p) < 0$, as desired. □

Next, we show that every buyer in the market M strictly prefers legal items to non-legal items:

Claim 4.1.4. *Let $i \in I$ be a buyer and let $t_1, t_2 \in T$ be items such that $t_1 \in \mathscr{L}_M(i)$ and $t_2 \notin \mathscr{L}_M(i)$. Then, $u_i(t_1,p) > u_i(t_2,p)$.*

Proof. Consider:

$$
\begin{aligned}
& u_i(t_1, p) - u_i(t_2, p) \\
& \geq \left[w_{it_1} - \left(\hat{y}_{t_1} + \frac{\alpha/2}{|T|+1}\sigma(t_1)\right) + \alpha/2\right] \\
& - \left[w_{it_2} - \left(\hat{y}_{t_2} + \frac{\alpha/2}{|T|+1}\sigma(t_2)\right)\right] \\
& = (\hat{y}_i + \hat{y}_{t_2} - w_{it_2}) + \frac{\alpha/2}{|T|+1}(\sigma(t_2) - \sigma(t_1)) - \alpha/2 && \text{by Claim 4.1.1 (iii)} \\
& = (\tilde{y}_i + \tilde{y}_{t_2} - w_{it_2}) + \frac{\alpha/2}{|T|+1}(\sigma(t_2) - \sigma(t_1)) - \alpha/2 \\
& \geq \alpha/2 + \frac{\alpha/2}{|T|+1}(\sigma(t_2) - \sigma(t_1)) && \text{by our choice of } \alpha \\
& > \alpha/2 - \alpha/2\frac{|T|}{|T|+1} && \text{since } \sigma(t_2) \geq 1 \text{ and } \sigma(t_1) < |T|+1 \\
& > 0 && \text{since } \alpha > 0.
\end{aligned}
$$

Thus, $u_i(t_1, p) > u_i(t_2, p)$, as desired. □

Next, we show that for a non-dummy buyer in the market M, their preferences between non-dummy legal items are the same in both markets M and $\hat{M}$. Moreover, we show that for a dummy buyer in the market M, their preferences between non-dummy legal items that appear before the cutoff point are the same in both markets M and $\hat{M}$. To this end, we first consider the item prices $p^* : T \to \mathbb{R}$ given by $p^*_t := \hat{y}_t + \frac{\alpha/2}{|T|+1}\sigma(t)$. That is, the prices p^* correspond to the prices p, but without increasing the prices of the items that appear after the cutoff point. We proceed to show that the buyers' preferences between legal items are the same in both markets M and $\hat{M}$ with respect to the prices p^* and p', and our desired result follows:

Claim 4.1.5. *Let $i \in I$ be a buyer and let $t_1, t_2 \in T$ be distinct items such that $t_1, t_2 \in \mathscr{L}_M(i)$. Then, $u_i(t_1, p^*) > u_i(t_2, p^*)$ if and only if $u'_i(t_1, p') > u'_i(t_2, p')$.*

Proof. Consider:

$$
\begin{aligned}
& u_i(t_1, p^*) > u_i(t_2, p^*) \\
& \iff w_{it_1} - \left(\hat{y}_{t_1} + \frac{\alpha/2}{|T|+1}\sigma(t_1)\right) \\
& > w_{it_2} - \left(\hat{y}_{t_2} + \frac{\alpha/2}{|T|+1}\sigma(t_2)\right) \\
& \iff \hat{y}_i - \frac{\alpha/2}{|T|+1}\sigma(t_1) > \hat{y}_i - \frac{\alpha/2}{|T|+1}\sigma(t_2) && \text{by Claim 4.1.1 (iii)} \\
& \iff \sigma(t_1) < \sigma(t_2) \\
& \iff p'_{t_1} < p'_{t_2} \\
& \iff v'_i(t_1) - p'_{t_1} > v'_i(t_2) - p'_{t_2} && \text{since } v'_i(t_1) = v'_i(t_2) = 1 \\
& \iff u'_i(t_1, p') > u'_i(t_2, p').
\end{aligned}
$$

Thus, $u_i(t_1, p^*) > u_i(t_2, p^*)$ if and only if $u'_i(t_1, p') > u'_i(t_2, p')$, as desired. □

In conclusion, if our initial market M does not satisfy the (OPT) condition, then we may first form the auxiliary market M', and then we may artificially introduce a set $\hat{T}$ of $b(I) - |T|$ dummy items which every dummy buyer values at 1 and which every non-dummy buyer values at 0. It follows that the resulting market $\hat{M}$ satisfies both the (OPT) condition and the $(*)$ condition. Then, if we manage to compute an optimal dynamic pricing scheme for the market $\hat{M}$, then it remains for us to determine the correct point at which to cut off the dummy buyers. If we are successful in determining such a cutoff point, then we obtain an optimal dynamic pricing scheme for the original market M. In [5], Szögi uses this technique to prove the existence of optimal dynamic pricing schemes in multi-demand markets where $b_i = 2$ for all buyers $i \in I$, without assuming the (OPT) condition. Their algorithm is a direct extension of the algorithm constructed by Bérczi et al. in [1].

Finally, remark that the so-called dummy items that are employed in this section do not play a key role in the arguments we have presented. In fact, we just need some way of relating the original market M to a modified market $\hat{M}$ that satisfies the (OPT) condition and that preserves the structure of the original market M in some useful sense. In the next section, we present a specific case of multi-demand markets in which we obtain a black-box reduction. However, instead of introducing dummy items into the market M, we alternatively *merge* the dummy buyers into a single non-dummy buyer of lower total demand to obtain our modified market.

4.2 A Black-Box Reduction to (OPT) for a Specific Case

In this section, we provide a black-box reduction for the problem of finding optimal dynamic pricing schemes in general multi-demand markets to the problem of finding optimal dynamic pricing schemes in markets that satisfy the (OPT) condition for a specific case. Loosely speaking, the condition we require on the input market is that its deficiency of items is large with respect to each of the dummy buyers' demands. Formally, the statement is as follows:

Lemma 4.2.1. *Let M be a multi-demand market such that $b(\hat{I}) - (b(I) - |T|) \leq \min\{b_i : i \in \hat{I}\}$. If there exist optimal dynamic pricing schemes for multi-demand markets satisfying the (OPT) condition, then there exists an optimal dynamic pricing scheme for the market M.*

Proof. To begin, we may assume by Claim 2.0.2 that every item is allocated to some buyer in every optimal allocation of the market M. Hence, every optimal allocation of the market M assigns exactly $b(I \setminus \hat{I})$ items to the buyers in the set $I \setminus \hat{I}$, and it distributes the remaining $|T| - b(I \setminus \hat{I}) = b(\hat{I}) - (b(I) - |T|) := q$ items among the dummy buyers in the set $\hat{I}$. Thus, the condition that we impose in the statement of this lemma is that the number of items q that are distributed among the dummy buyers in every optimal allocation of the market M is at most the demand of each dummy buyer. This condition enables to redistribute items among the dummy buyers, which we will make use of later in the proof.

To continue, we recall that the first step in constructing the (OPT) framework in the previous section is to form the auxiliary market M', which satisfies the $(*)$ condition. For convenience, we bypass this step, and we simply assume that the market M satisfies the $(*)$ condition to begin with. Then, we create a *merged* dummy buyer $\hat{i}$ with demand $b_{\hat{i}} := q$ and with valuations as follows:

$$v(\hat{i})_t := \begin{cases} 1 & \text{if } t \in \bigcup_{i \in \hat{I}} \mathscr{L}_M(i) \\ 0 & \text{otherwise} \end{cases} \qquad \forall t \in T.$$

Intuitively, the merged dummy buyer $\hat{i}$ values every item that is legal for at least one dummy buyer in the market M at 1, and they value all other items at 0.

Next, we form a modified market $\hat{M}$ with buyers $(I \setminus \hat{I}) \cup \{\hat{i}\}$, the same items T, and the same demands and valuations b and v as in the original market M, albeit with an additional entry for the merged dummy buyer $\hat{i}$. In other terms, we form the modified market $\hat{M}$ by essentially replacing the set of dummy buyers $\hat{I}$ with the single merged dummy buyer $\hat{i}$. We proceed to prove some useful properties about the modified market $\hat{M}$. First, we show that the legality of every buyer-item pair for non-dummy buyers in the market M is preserved in the modified market $\hat{M}$:

Claim 4.2.1. *For every non-dummy buyer $i \in I \setminus \hat{I}$ and for every item $t \in T$, we have $t \in \mathscr{L}_M(i)$ if and only if $t \in \mathscr{L}_{\hat{M}}(i)$.*

Proof. ($\Longrightarrow$) Suppose $t \in \mathscr{L}_M(i)$. Then, there exists an optimal allocation $\boldsymbol{\mathcal{A}} = \{A_j \subseteq T : j \in I\}$ of the market M such that $t \in A_i$. By our previous discussion, it follows that the allocation $\boldsymbol{\mathcal{A}}$ distributes exactly q items among the dummy buyers in the set $\hat{I}$. Let these items be $\{t_1, \ldots, t_q\}$. By definition of legality, it follows that for every $j \in [q]$, we have $t_j \in \mathscr{L}_M(k)$ for some dummy buyer $k \in \hat{I}$. Hence, have $v(\hat{i})_{t_j} = 1$ for all $j \in [q]$. Now, we define an allocation $\boldsymbol{\mathcal{A}}' := \{\hat{A}_j \subseteq T : j \in (I \setminus \hat{I}) \cup \{\hat{i}\}\}$ of the modified market $\hat{M}$ as follows:

$$\hat{A}_j := \begin{cases} A_j & \text{if } j \neq \hat{i} \\ \{t_1, \ldots, t_q\} & \text{if } j = \hat{i} \end{cases} \quad \forall j \in (I \setminus \hat{I}) \cup \{\hat{i}\}.$$

It follows that $SW(\boldsymbol{\mathcal{A}}') = SW(\boldsymbol{\mathcal{A}}) = b(I) - |T| = b(I \setminus \hat{I}) + q = b((I \setminus \hat{I}) \cup \{\hat{i}\})$, which is equal to the total demand of the buyers in the modified market $\hat{M}$. Since every valuation of the modified market is at most one, it follows that the allocation $\boldsymbol{\mathcal{A}}'$ is an optimal allocation of the modified market $\hat{M}$. Moreover, we have $t \in \hat{A}_i$, so $t \in \mathscr{L}_{\hat{M}}(i)$, as desired. Note that this also implies that the modified market $\hat{M}$ satisfies the (OPT) condition.

($\Longleftarrow$) Suppose $t \in \mathscr{L}_{\hat{M}}(i)$. Then, there exists an optimal allocation $\boldsymbol{\mathcal{A}}' = \{\hat{A}_j \subseteq T : j \in (I \setminus \hat{I}) \cup \{\hat{i}\}\}$ of the modified market $\hat{M}$ such that $t \in \hat{A}_i$. By the proof of the forward direction, we have $v_i(t) = 1$. Since the market M satisfies the $(*)$ condition, it follows that $t \in \mathscr{L}_M(i)$, as required. Note that it easily follows, by an argument that is analogous to the proof of Claim 3.3.3, that the modified market $\hat{M}$ also satisfies the $(*)$ condition. $\square$

Now, suppose we have an optimal dynamic pricing scheme $\hat{p}$ for the modified market $\hat{M}$. By Claim 3.1.1, we may assume that the pricing scheme $\hat{p}$ has unique values. Let $t_1, \ldots, t_q$ be the q lowest-priced items according to the pricing scheme $\hat{p}$ that are legal for the merged buyer $\hat{i}$. Then, there exists an optimal allocation of the modified market $\hat{M}$ which assigns the set of items $\{t_1, \ldots, t_q\}$ to the merged buyer $\hat{i}$. Suppose further that $\hat{p}_{t_1} > \cdots > \hat{p}_{t_q}$. Then, we claim that cutting off the dummy buyers after the item t_q results in an optimal dynamic pricing scheme for the original market M. Indeed, suppose t_q is the k-th lowest-priced item according to the prices $\hat{p}$, and let p be the pricing scheme obtained from $\hat{p}$ by setting $C = k$ as the cutoff point. We proceed to consider two cases based on whether a dummy buyer or a non-dummy buyer arrives first to the market M :

Case 1: Suppose a non-dummy buyer $i \in I \setminus \hat{I}$ arrives first to the market M. By the results in Section 4.1, it follows that the buyer i will purchase the b_i lowest-priced items according to the pricing scheme $\hat{p}$ that are legal for them in the market $\hat{M}$. Let this set of items be R. By optimality of the pricing scheme $\hat{p}$, it follows that there exists an optimal allocation $\boldsymbol{\mathcal{A}}' := \{\hat{A}_j \subseteq T : j \in (I \setminus \hat{I}) \cup \{\hat{i}\}\}$ of the modified market $\hat{M}$ such that $R = \hat{A}_i$. Let $\hat{I} := \{\hat{i}_1, \dots, \hat{i}_n\}$ be an arbitrary ordering on the set of dummy buyers in the market M. Now, we define an allocation $\boldsymbol{\mathcal{A}} = \{A_j \subseteq T : j \in I\}$ of the original market M as follows:

$$A_j := \begin{cases} \hat{A}_j & \text{if } j \in I \setminus \hat{I} \\ \hat{A}_{\hat{i}} \cap \mathscr{L}_M(\hat{i}_1) & \text{if } j = \hat{i}_1 \\ (\hat{A}_{\hat{i}} \cap \mathscr{L}_M(\hat{i}_k)) \setminus \left(\bigcup_{l=1}^{k-1} \hat{A}_{\hat{i}} \cap \mathscr{L}_M(\hat{i}_l) \right) & \text{if } j = \hat{i}_k \text{ for some } k \in \{2, \dots, n\} \end{cases} \qquad \forall i \in I.$$

Intuitively, under the allocation $\boldsymbol{\mathcal{A}}$, every non-dummy buyer is assigned the same bundle that is assigned to them by the allocation $\boldsymbol{\mathcal{A}}'$, and for each $k \in [n]$, the dummy buyer $\hat{i}_k$ is assigned all of the items that are assigned to the merged dummy buyer $\hat{i}$ by the allocation $\boldsymbol{\mathcal{A}}'$ that are not legal to each of the previous dummy buyers $\hat{i}_{k-1}, \dots, \hat{i}_1$ in our ordering of the set of dummy buyers $\hat{I}$. Since the modified market $\hat{M}$ satisfies the (OPT) condition, it follows that $|\hat{A}_{\hat{i}}| = q$. Moreover, by our assumption that $q \leq \min\{b_i : i \in \hat{I}\}$ in the statement of the lemma, it follows that the allocation $\boldsymbol{\mathcal{A}}$ indeed respects the demand of every buyer in the market M. Finally, by Claim 4.2.1, it follows that $SW(\boldsymbol{\mathcal{A}}) = b(I \setminus \hat{I}) + q = b(I) - |T|$, so the allocation $\boldsymbol{\mathcal{A}}$ is an optimal allocation of the market M. Moreover, we have $R = A_i$. Hence, the assignment of the bundle R to the buyer i extends to an optimal allocation $\boldsymbol{\mathcal{A}}$ of the market M, as required.

Case 2: Suppose a dummy buyer arrives first to the market M. Again, we let $\hat{I} := \{\hat{i}_1, \dots, \hat{i}_n\}$ be an arbitrary ordering on the set of dummy buyers in the market M. Without loss of generality, suppose the dummy buyer $\hat{i}_1$ arrives first to the market M. Since $b_{\hat{i}_1} \geq q$ by assumption, it follows by the results in Section 4.1 that the buyer $\hat{i}_1$ will purchase the bundle $\{t_1, \dots, t_q\} \cap \mathscr{L}_M(\hat{i}_1)$. By optimality of the pricing scheme $\hat{p}$, it follows that that there exists an optimal allocation $\boldsymbol{\mathcal{A}}' := \{\hat{A}_j \subseteq T : j \in (I \setminus \hat{I}) \cup \{\hat{i}\}\}$ of the modified market $\hat{M}$ such that $\{t_1, \dots, t_q\} = \hat{A}_{\hat{i}}$. Now, we define an optimal allocation $\boldsymbol{\mathcal{A}} = \{A_j \subseteq T : j \in I\}$ of the original market M in the same way that we did for Case 1. Thus, it again follows that $\boldsymbol{\mathcal{A}}$ is an optimal allocation of the market M. Moreover, recall that we have $\hat{A}_{\hat{i}} = \{t_1, \dots, t_q\}$, so it follows that $A_{\hat{i}_1} = \{t_1, \dots, t_q\} \cap \mathscr{L}_M(\hat{i}_1)$. Hence, the assignment of the bundle $\{t_1, \dots, t_q\} \cap \mathscr{L}_M(\hat{i}_1)$ to the buyer $\hat{i}_1$ extends to an optimal allocation $\boldsymbol{\mathcal{A}}$ of the market M. In conclusion, the pricing scheme p is an optimal dynamic pricing scheme for the market M, completing the proof of the lemma. $\square$

Chapter 5

Open Problems

In this final chapter, we briefly discuss some unsolved questions pertaining to dynamic pricing schemes in combinatorial markets. We present some natural extensions of the problem that we have studied, and we outline some of the barriers that arise when attempting to extend the techniques that we have presented in the previous chapters.

5.1 Larger Demands in Multi-Demand Markets

To begin, the most natural extension of the problem we have considered is to ask whether optimal dynamic pricing schemes exist in a multi-demand market M satisfying the (OPT) condition and such that $b_i \leq 4$ for all buyers $i \in I$. In this new setting, which we refer to as the "four-demand" case, we are forced to deal with assignments of items to buyers of size three that do not extend to optimal allocations of the market. Following the techniques presented in Chapter 3, we infer that these assignments may lead to a splitting of the market M into an upper market M^U and a lower market M^L such that the lower market M^L requires three items to be introduced into its base item set to fulfill its demand. In order for the resulting pricing schemes p^U and p^L to be able to be combined into an overall pricing scheme p, we would now require the induction hypothesis that any two items of the market M may be priced lowest. This may not be possible. For instance, consider the case where $k_M = 2$ and there exists a buyer $i \in I$ and items $x, y \in T$ such that the assignment of the items $\{x, y\}$ to the buyer i respects both legality and demand in the market M but does not extend to an optimal allocation of the market M. Then, the items $\{x, y\}$ cannot be the two lowest-priced items in any optimal dynamic pricing scheme for the market M. This is what motivates Definition 4 of *flexible items*, and it is also what motivates us to consider the minimum possible size of a bundle that is not flexible in the market M. Indeed, the bundle $\{x, y\}$ presented above is not *flexible* in the market M, and it also enables the most useful splitting of the market M : a splitting in which just one item needs to be introduced into the base item set of the lower market M^L. Then, the proof of Lemma 3.3.1 can be used to show that the two smaller pricing schemes can be combined into an optimal pricing scheme for the original market M. Thus, one might hope to make use of the following induction hypothesis: *For every bundle B that is flexible in the market M, there exists an optimal dynamic pricing scheme for the market M in which the items in B are priced the lowest.* Note that we would only need to consider inclusion-wise minimal non-flexible sets in order to obtain our desired splitting of the market. The aforementioned induction hypothesis is somewhat consistent with our treatment of tri-demand markets in Section 3.4, as singleton item sets are always flexible, and we were able to price them the lowest. However, we were not able to show that any two items of the market M can be priced the lowest in tri-demand markets in the case where $k_M = 3$. One difficulty that arises is that a bundle B which is flexible in the market M may not be flexible in the lower market M^L. It is worth noting, however, that the requirement that certain items be priced the lowest is not strictly necessary to guarantee the compatibility of the two smaller pricing schemes. Instead, we may alternatively require that certain items may be priced in any order. This requirement seems more reasonable; however, we were not able to exploit it in any meaningful way in our research to date.

In addition to the difficulties already outlined when considering the four-demand case, another complication presents itself in relation to the buyers' demands. Recall that in the tri-demand case, we had to eliminate unit-demand buyers from the market in order to guarantee that we could reassign sets of two items to arbitrary buyers. This was crucial in the proof of Claim 3.4.8, where we proved the optimality of the combined pricing scheme p. When allowing for buyers' demands to attain four, we would analogously need some method of eliminating bi-demand buyers from the market. We were unable to do so while also preserving the lowest-price item requirement of the pricing scheme. Overall, the difficulty of having to keep track of the prices of two arbitrary items together with the complication of having to eliminate bi-demand buyers from the market resulted in us being unsuccessful in trying to devise an optimal dynamic pricing scheme for the four-demand case.

5.2 Removing the (OPT) Condition in Multi-Demand Markets

As discussed in Chapter 4, the problem of finding optimal dynamic pricing schemes in general multi-demand markets is closely related to the problem of finding optimal dynamic pricing schemes in multi-demand markets which satisfy the (OPT) condition. Ideally, one hopes to find a black-box reduction in the general case, but we were unable to find such an algorithm in our research to date. The difficulty is in determining the correct point at which to cut off the dummy buyers from buying any further legal items, and it is not clear that such a cutoff point always exists. On the positive side, however, our black-box reduction in Section 4.2 introduces the potentially useful technique of merging buyers to obtain a market which satisfies the (OPT) condition. Although we were unable to make good use of the merging of dummy buyers without the assumption outlined in Lemma 4.2.1, it is worth noting that any technique -whether it be introducing dummy items, merging buyers, or otherwise- that relates the original market M to a modified market $\hat{M}$ that satisfies the (OPT) condition and that preserves the structure of the original market M in some useful sense, can potentially be useful in devising such a black-box reduction.

Another closely related problem is that of producing a white-box reduction from general tri-demand markets to tri-demand markets which satisfy the (OPT) condition. We did not investigate this question, and it is possible that the proof presented in Section 3.4 can be modified to accommodate general tri-demand markets.

5.3 Other Classes of Valuation Functions

As mentioned in the introduction, the existence of optimal pricing schemes in combinatorial markets is primarily of interest when the valuation functions are gross substitutes functions. In a sense, multi-demand markets admit the simplest class of gross substitutes valuation functions. In [2], Bérczi et al. investigate combinatorial markets under matroid rank valuations, which constitute a slightly more complicated class of gross substitutes valuation functions. In this setting, each buyer of the market has a matroid over the same ground set of items. Their valuation over subsets of items is then given by the rank of that bundle with respect to their matroid. These authors show the existence of efficient algorithms for computing optimal dynamic pricing schemes in the specific case where there are two buyers in the market and where either one of the matroids is a simple partition matroid or both matroids are strongly base orderable. In general, the case of matroid rank valuations is still open. In addition, there are several other classes of gross substitutes valuation functions for which it is not known whether optimal dynamic pricing schemes exist in general combinatorial markets.

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