# Edge Modes and Carrollian Hydrodynamics on Stretched Horizons

by

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## Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

## Statement of Contributions

Chapter 3 is based on the publication [1] co-authored with Marc Geiller. An earlier version of some of the technical contents of this chapter was also presented in my Perimeter Scholars International thesis. Chapter 4 is based on the publication [2] written in collaborations with Marc Geiller, Daniele Pranzetti, and Abdulmajid Osumanu.

The problem addressed in Part II of this thesis, including Chapter 5, Chapter 6, and Chapter 7 was suggested to me by my supervisor, Laurent Freidel. I did much of the technical works and a large part of the writing. Chapter 5 is based on the publication [3].

## Abstract

The first part of this thesis is aimed at investigating the crucial role played by emergent boundary degrees of freedom, called edge modes, in gauge theories defined in spacetimes with boundaries. We propose a simple and systematic framework for including edge modes in theories with internal gauge symmetries, and argue that this is necessary in order to achieve the factorizability of the Hilbert space and the phase space. We also explain how edge modes acquire effective boundary dynamics and how they contribute to entanglement entropy using the path integral formulation. In addition, we investigate how edge modes and their corner symmetries may shed new light on the novel understanding of electromagnetic duality and explain the existence of dual magnetic charges and their centrally extended algebras with electric charges. The second part of this thesis addresses the newly discovered connection between physics at null boundaries and Carrollian hydrodynamics. We first present a new notion of symmetries, called near-Carrollian symmetries, that generalizes the Carrollian symmetries and show that they correspond to the conservation laws of Carrollian fluids. Next, we consider a local portion of a spacetime bounded by a finite distance null boundary (e.g., black hole horizons) and foliated into a series of timelike hypersurfaces, known as stretched horizons. By employing the rigging technique, we show that the Carroll geometry is naturally induced on the stretched horizon, and in turn provide a unified geometrical construction of both timelike and null surfaces. We then construct the horizon energy-momentum tensor, which correspondingly defines the dictionary between gravitational degrees of freedom and Carrollian fluid quantities, and show that its conservation laws imply the Einstein equations. Finally, we put forward a proposal that the gravitational phase space of the stretched horizon, treated as a radial expansion around the null boundary, encodes (sub-leading) informations of the null boundary phase space. Most importantly, we report the existence of spin-2 symmetries associated with the spin-2 sector of the Einstein equations on the null boundary.

## Acknowledgements

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# Chapter 1

# Introduction

This thesis revolves around the theme of symmetries and dynamics of gauge theories and gravity in the presence of boundaries. It is divided into two parts:

I. Edge Modes: Dynamics and Duality

II. Carrollian Hydrodynamics on Stretched Horizons

## 1.1 Background

The quote said by Wolfgang Pauli, *"God made the bulk; surfaces were invented by the devil"*, encapsulates well the diabolic nature of surfaces. Surfaces, as boundaries that divide a physical system from the external world, are the places where the system interacts with the environment and where the harmony of bulk physics begins to break down, enabling new boundary physics to emerge. Although it is not known what scenario this eminent physicist had in mind, many modern theoretical physicists would still agree with him.

In the past century, the status of boundaries in theoretical physics has evolved, from merely the locus at which boundary conditions are assigned to the location where abundant intriguing physical phenomena make their appearances. Countless evidence that challenged our perspective of boundaries can be found in many research fields, including the early works concerning thermodynamics of black holes [4–6] and later the membrane paradigm viewpoint of black holes [7–9], the bulk-boundary correspondence in condensed matter systems [10–12], and the area-law characteristic of entanglement entropy [13–15]. One fascinating, and perhaps the most important, aspect of boundaries is that that bulk physics seems to be entirely encoded at the boundaries, and together with all this ample evidence, it has led to the idea of the holographic principle [16, 17]. Holography was first properly realized by the breakthrough discovery of AdS/CFT duality [18–20], which describes the deep interplay between bulk dynamics of an asymptotically Anti-de Sitter spacetime and conformal field theory living at its boundary. The attempt to extend the holographic correspondence to an asymptotically flat spacetime has also led to recent developments in celestial holography [21]. It is worth mentioning that, in addition to boundaries at finite distances (e.g. black holes, condensed matter systems, or any subregions of spacetime) and asymptotic boundaries at infinity, one can add to this list boundaries in the form of defects of arbitrary codimension, and local excitations supported at these defects play an integral role in topological quantum field theories [22], condensed matter physics [23, 24], and quantum gravity [25–27].

While the study of new physics unfolded at boundaries of spacetimes has become one of the most active research trends in past decades, there exists however no unified framework to describe at once all interesting features of boundary physics. One thread that is believed will guide us to the ultimate conclusion of this discovery journey is the notion of gauge in bounded subdivisions of spacetime. Much researches developed in this direction has already started to unravel numerous striking roles of boundaries in gauge theories (including gravity) and in turn led to a new notion of holography — the local holography. Among these, the most prominent results are the unified description of edge modes and their (codimension–2) corner symmetries.

### 1.1.1 Edge Modes

The well-understood characteristic of gauge symmetries, including internal gauge symmetries and spacetime diffeomorphism, is that they are not *physical* symmetries but instead correspond to mathematical redundancies in a theory. In this regards, gauge symmetries cannot be used to label or distinguish physical states in the theory. However, this picture changes in the presence of boundaries.

Gauge theories defined on spacetimes with boundaries, be they located at asymptotic infinity or at finite distance, exhibit emergent boundary degrees of freedom, sometimes referred to as *edge modes* (also called edge states and would-be-gauge degrees of freedom in the literature). This occurrence stems from the fact that, when considering bounded regions, a subset of transformations, which are gauge in the bulk, become physical symmetries on the boundary. These symmetries, which are anchored in codimension–2 corners of spacetime, are referred to as *corner symmetries*.

Early and in-depth investigations of these emergent boundary degrees of freedom has been performed mostly for topological theories with no propagating bulk degrees of freedom. These include 3-dimensional gravity and Chern–Simons theory, where the edge modes possess an explicit boundary dynamics and encode physical properties of, for example, condensed matter systems [10–12] and black holes [28–30]. Just recently, the study of edge modes has gained ever-increasing attention in various research arenas, especially in the field of quantum gravity and holography. Many new results revealing important insights into the role of edge modes, even beyond topological theories, have been obtained, both at finite distance and at infinity. At finite distance, there have been efforts to study local subsystems of gauge theories at the level of classical phase spaces, in pursuit of the most general corner symmetry algebras spanned by the edge modes [31–41], with potentially important consequences for quantum gravity [37, 42–44]. Another important development at finite distance has been the realization that a proper treatment of the edge modes is crucial even when dealing with fictitious entangling interfaces, which has consequences in the computations of entanglement entropy [45–55]. At infinity on the other hand, a lot of work has been dedicated towards understanding the intricate infrared properties of theories with massless excitations, and there a central role is played by large gauge transformations and soft modes (see [21] and references therein). Although the relationship between the edge modes at finite distance and the soft modes at infinity is not fully understood, a unifying thread is that of having degrees of freedom supported on the boundary, and parts of this connection have been explored in [53, 56]. Other evidence, also hinting toward this connection, is that asymptotic symmetries at null infinity [57–63] can be seen as the asymptotic limit to infinity of the corner symmetry group of residual diffeomorphism associated to a generic corner of spacetime [32, 38, 64].

The understanding of edge modes is undeniably necessary to properly define gauge theories in local portions of spacetime. We highlight below some aspects of edge modes as well as their applications that we will study in the first part of the thesis.

#### Splitting, Gluing, and Entanglement

The appearance of these edge modes can be seen as an inevitable outcome when considering local subsystems of gauge theories. The reason lies on the fact that both physical phase space and Hilbert space of a gauge theory on a spatial surface fail to be factorizable due to the presence of the gauge constraints and the resulting inherent non-locality of gauge-invariant observables (e.g. Wilson loops). Aside from being a conceptual issue for the definition of local subsystems [32], this also represents an a priori technical obstruction to computing quantities such as the entanglement entropy of gauge fields across a fictitious interface between two regions [54]. This difficulty can however be bypassed by resorting to a so-called extended Hilbert space. The idea of this construction is as follows: starting from two spatial (codimension-1) slices $\Sigma$ and $\overline{\Sigma}$ sharing the same (codimension-2) boundary $S = \partial\Sigma = \partial\overline{\Sigma}$, the total Hilbert space $\mathcal{H}_{\Sigma\cup\overline{\Sigma}}$ can be factorized into factors attached to $\Sigma$ and $\overline{\Sigma}$ provided that we extend these by attaching edge modes living on $S$ and transforming under the action of a corner symmetry group $\mathcal{G}_S$. Denoting the resulting extended Hilbert space by $\mathcal{H}_{\Sigma,S}$, one can then realize the total physical Hilbert space of gauge-invariant states as a subspace $\mathcal{H}_{\Sigma\cup\overline{\Sigma}} \subset \mathcal{H}_{\Sigma,S} \otimes \mathcal{H}_{\overline{\Sigma},S}$, and is then recovered as $\mathcal{H}_{\Sigma\cup\overline{\Sigma}} = \mathcal{H}_{\Sigma,S} \otimes_{\mathcal{G}_S} \mathcal{H}_{\overline{\Sigma},S}$, where $\otimes_{\mathcal{G}_S}$ denotes an entangling product which identifies and gets rid of the extra boundary degrees of freedom. Clearly, an advantage of the extended space is that it permits the tensor product structure $\mathcal{H}_{\Sigma,S} \otimes \mathcal{H}_{\overline{\Sigma},S}$ and therefore allows for the definition of a reduced density matrix. This construction has proven very useful in computations of entanglement entropy [45–48, 54, 65–68].

The classical analog of the extended Hilbert space is the extended phase space, and it was pioneered by the authors of [32], for the case of Yang–Mills theory and metric gravity. The extension consists in adding to the bulk (covariant) phase space, for each type of gauge transformations in the theory, a corresponding edge mode field living on the boundary. This allows us to differentiate between gauge redundancies and corner symmetries, as the former has zero Hamiltonian charges while the latter, which are symmetries of edge modes, have non-trivial charges. The construction has been further explored in [34] for non-Abelian BF and Chern-Simons theories, in [35] for higher curvature gravity, in [36] for 3-dimensional gravity in first order connection-triad variables, in [69] for open string field theory, in [70] for Einstein–Maxwell theory, and in [38] for tetrad gravity in the Einstein-Cartan-Holst formulation.

Having edge modes added to the theory phase space, one natural question arises — *what is the dynamics of these edge modes?* Answering this seemingly simple question is the main objective of Part I of this thesis.

### Edge modes and Dualities

Another objective of Part I is to study the notion of dualities and the existence of dual charges from the perspective of edge modes and corner symmetries. Duality is arguably one of the most astounding concepts in modern theoretical physics, and its existence possibly signifies unexplored structures of nature. The quintessential example is electromagnetic duality, which exchanges the electric and magnetic fields in Maxwell's theory. While the classical realization of this duality has been studied extensively [71–78], it is only recently that a new picture has started to emerge. Inspired by the infrared triangle of massless theories [21], there have been efforts to connect, in particular, soft photon theorems to dual large gauge transformations and magnetic soft charges [79–85]. This development has also fostered work on dual gravitational charges [86–95], although there is no known gravitational analog of electromagnetic duality in the full theory [96–100] (see also [101, 102] for a notion of dual charges in 3-dimensional triad gravity and [103] for a gravitational duality at null infinity). Dual charges corresponding to soft theorems have also been discussed in the case of the massless scalar [104–107]. In the context of Maxwell's theory, these developments have motivated the study of (asymptotic) magnetic charges. This was done in [108] using the so-called duality-symmetric formulation, and in [109] by introducing magnetic edge modes on an extended phase space. The remarkable result of these constructions is that the electric and magnetic charges satisfy a Kač–Moody current algebra with non-vanishing central charge [82] (see however [110]).

In this thesis, we are motivated by a simple, yet important question — *In a given theory, specified by a Lagrangian and admitting asymptotic charges, how can we know if there are "hidden" dual charges?* In the gravitational case it has already been suggested that a complete description of the charges (the dual ones included) should rely on the first order formulation [38, 39, 86–90, 94]. Inspired by this idea, we will dedicate the second half of Part I to thoroughly inspect the duality of electromagnetism.

### 1.1.2 Gravity and Hydrodynamics

In Part II, we shift our attention from (internal) gauge theories to gravitational theories. Although the (extended)[1] corner symmetries [32, 35, 38, 64] of gravity are more complicated and require more subtle analysis than those in gauge theories, similar questions can still be framed. One natural question, akin to what we asked in gauge theories, is that — *what is the gravitational dynamics of boundary degrees of freedom and corner charges of the corresponding (extended) corner symmetries?* Following the perspective of local holography program, dynamics of corner charges is entirely encoded in general conservation laws, which are also referred to as flux-balance laws. This is however not a totally new question as understanding boundary dynamics and conservation laws are the subject of extended studies [41, 111, 112]. In this thesis, we look deeper into these conservation laws and scrutinize their physical interpretations as hydrodynamic conservation laws.

For the past half-century, physicists have been intrigued by the underlying connection between two apparently completely unrelated topics — gravity and hydrodynamics. This

[1] The corner symmetry group is extended in a sense that it allows transformations that move the corner along its normal directions.

connection, in hindsight, might not appear to be a totally unexpected occurrence, as both fields exhibit various similar traits:

*i*) In a similar spirit to hydrodynamics, which provides emergent effective descriptions of classical and continuum dynamics of systems of multiple fluid particles, general relativity can also be regarded as an effective theory of emergent classical and continuum dynamics of quantum "atoms of spacetime".

*ii*) The equations governing fluid dynamics (e.g. the Navier-Stokes equations) are non-linear second order differential equations much like the Einstein equations that describe the dynamics of spacetime geometry.

This connection, while starting off as a tentative analogy, is a clear reflection of a true nature of gravity, offering a completely hydrodynamic route to gravitational dynamics and opening unprecedented windows to explore some open questions in both fields.

The first-ever theoretical investigation which sparked this spectacular realization that gravity can be understood as fluid mechanics was black hole thermodynamics [4–6], which demonstrated that black hole horizons, much like fluids, can be assigned thermodynamic properties such as internal energy, pressure, temperature, and entropy. Interestingly, gravitational physics controlling dynamics of these quantities can also be represented as the standard laws of thermodynamics. Our understanding of this connection was further enhanced by the novel works of Damour [7] and, subsequently, Throne, Price, and Macdonald [8, 9]. Their developed framework would become famously known as the black hole *Membrane Paradigm*. It in particular realizes the idea that internal dynamics of a black hole, as seen from outside observers, can be modeled effectively as a membrane located in an infinitesimally close distance to the black hole horizon. The fictitious (timelike) membrane, also called a *stretched horizon*, can be viewed as arising from quantum fluctuations of geometry around the (null) horizon of the black hole, and is endowed with physical properties of fluid[2]. One of the intriguing hallmark of this membrane viewpoint is that gravitational dynamics of the stretched horizon can be fully written as the familiar equations of hydrodynamics. The fluid/gravity correspondence was put forth beyond black hole physics in the context of AdS/CFT duality [113] (see [114–116] for comprehensive reviews on this topic) and it has been since then generalized and applied in numerous works. It is also worth mentioning other works that uncovered the link between gravitational physics and fluids. Black holes, in many circumstances, actually exhibit droplet-like behaviors akin to liquid. For instance, the Gregory-Laflamme instability of higher-dimensional black

[2]The stretched horizon can also be assigned electromechanical properties such as conductance. In this circumstance, one needs to supplement the fluid equations with electromechanical equations, such as Ohm's law.

strings [117] displays similar behavior to the Rayleigh instability of liquid droplets [118]. The work [119] also showed that dynamics of a timelike surface (which they called gravitational screen) behaves like a viscous bubble with a surface tension and an internal energy. Analog models of black holes [120] illustrated the converse notion and argued that kinematic aspects of black holes can be reproduced in hydrodynamical systems and that fluids can admit sonic horizons and even the analog of Hawking temperature. Lastly, in the context of local holography, the corner symmetry group of gravity was shown to contain the symmetry group of perfect fluids as its subgroup [44].

Taking lessons from the black hole membrane paradigm, one pressing question is what type of fluids emerged at the black hole horizon. It has been a long-standing belief that the true nature of horizon fluids is either relativistic fluid or non-relativistic (Galilean) fluid, describing by the Navier-Stokes equations. This belief however was recently challenged by the authors of [121], where they demonstrated that the behavior displayed by horizon fluids are neither relativistic nor non-relativistic, but rather ultra-relativistic or Carrollian. As announced, the second part of this thesis aims to revise and formalize this proposal in full generality. To this purpose, we explain below the meaning of the term *Carrollian* and review developments in the area of Carrollian physics.

### 1.1.3 Carrollian Physics

The fascinating tale of Carroll geometries and Carrollian physics has begun purely out of the mathematical curiosity of Lévy-Leblond [122] when he first proposed a new special limit of a flat spacetime and derived its resulting isometry group. Heuristically, this limit is viewed as another end of "non-Lorentzian" limits, lying at the opposite side to the familiar Galilean (or non-relativistic) limit. This novel limit was deliberately given the name of the *Carrollian limit* (it was also referred to as the ultra-relativistic limit and the ultra-local limit by different authors) after Lewis Carroll, the author of Through the Looking-Glass. Before proceeding to recent developments in Carrollian physics, let us first briefly review the original construction of [122] and explain how the isometry of a flat spacetime, the Poincaré symmetry, can be contracted to the *Carrollian symmetry*, along with making comparisons to the well-known Galilean symmetry.

The Poincaré group, playing a pivotal role in relativistic field theories, is the isometry group of the flat Minkowski spacetime consisting of coordinate transformations that leave spacetime intervals invariant. To be more precise, starting from a flat Minkowski spacetime of general dimension, the pedestrian parameterization of the Poincaré coordinate transformations $(x^0, x^i) \to (x'^0, x'^i)$ is given by

$$x'^0 = \gamma \left(x^0 + \beta_i (Rx)^i\right) + a^0, \qquad x'^i = (Rx)^i + \frac{\gamma - 1}{|\beta|^2} \left(\beta_j (Rx)^j\right) \beta^i + \gamma \beta^i x^0 + a^i, \quad (1.1a)$$

where the Lorentz factor is defined in a standard way as $\gamma = (1 - |\vec{\beta}|^2)^{-\frac{1}{2}}$, and the norm of a spatial vector is denoted as $|\beta|^2 := \beta_i \beta^i$. The Poincaré transformations consist of spacetime translations labelled by $(a^0, a^i)$, spatial rotations given by a rotational matrix $R$, and Lorentz boosts, which mix time and space, parameterized by a spatial vector $\beta^i$. The latter two transformations together make the Lorentz transformations.

To evaluate limits of the Poincaré transformations, one then needs to introduce a parameter $c$ whose value can be varied. This parameter is nothing but the usual *speed of light*. As we have already stated, there exist two types of non-Lorentzian limit — the Galilean limit and the Carrollian limit. The former corresponds to the case where the value of the speed of light approaches infinity, $c \to \infty$[3]., and in turn reduces the the Poincaré transformations to the Galilean transformations. The Carrollian limit on the other hand, contracts the Poincaré transformations to the so-called Carrollian transformations by means of the opposite limit, $c \to 0$. Obviously, changing the speed of light affects the structure of light cones as schematically depicted in Figure 1.1. For the non-relativistic Galilean case, light cones open up as $c \to \infty$ and a particle can travel freely without any restriction on its velocity. In contrary, light cones collapse in the Carrollian limit $c \to 0$, hence freezing a particle's motion and completely inhibiting causal interactions between any events in spacetime. It is in this particular sense that the Carrollian limit is sometimes called the ultra-local limit[4]. Let us also mention that, at the level of Lie algebra, the Galilei algebra and the Carroll algebra can be obtained from the respective limits of the Poincaré group using the mathematical procedure known as the *Inönü-Wigner contraction* [123].

[3]Physically, it absolutely makes no sense to vary the value of the dimensionful parameter $c$. If one wants to be more rigorous, one rather needs to introduce a dimensionless parameter $\frac{c}{v}$ where a characteristic velocity $v$ depends on a problem under consideration. The end results however do not differ from naively using $c$ as the varying parameter.

[4]The clarification of terminology is in order here. In term of a dimensionless parameter $\frac{c}{v}$, the ultra-local limit corresponds to the case where $\frac{c}{v} \to 0$, meaning that the characteristic velocity of the problem trends to zero slower that $c$, in turn freezing the dynamics. On the other hand, the ultra-relativistic limit corresponds to the limit $\frac{c}{v} \to 1$, inferring that $v$ trends to $c$ in this limit. Unfortunately, the terminology for ultra-local limit and ultra-relativistic limit got mixed up at some point in the literature, and they were often used to refer the same thing, namely the *Carrollian limit*.

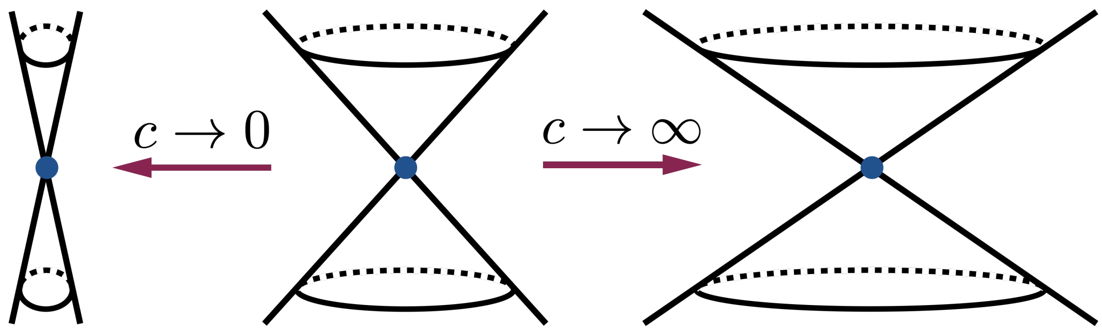


Figure 1.1: The effects of the non-Lorentzian limits on light cones. The limit $c \to \infty$ is the Galilean limit, while $c \to 0$ is the Carrollian limit.

### Galilean limit

> *"Well, in our country, you'd generally get to somewhere else, if you ran very fast for a long time, as we've been doing"*[5]
> said Alice, representing the Galilean world

To take the Galilean limit, we redefine the temporal coordinate, the boost parameter, and the time translation as follows

$$x^0 = ct, \qquad \beta^i = \frac{v^i}{c}, \qquad \text{and} \qquad a^0 = cT, \tag{1.2}$$

where $t$ now represents the Galilean notion of time, $v^i$ is the spatial velocity, also representing Galilean boosts, and lastly, $T$ parameterizes Galilean time translation. It is important to point out that the Lorentz factor becomes

$$\gamma = 1 + \frac{1}{2}\frac{|v|^2}{c^2} + \mathcal{O}(c^{-4}), \tag{1.3}$$

when considering a large value of the speed of light $c$. Taking the Galilean limit $c \to \infty$ of (1.1), while keeping $t$, $v^i$, and $T$ fixed, renders the Poincaré transformations to the Galilei transformations,

$$t' = t + T, \qquad \text{and} \qquad x'^i = (Rx)^i + v^i t + a^i. \tag{1.4}$$

These transformations in a sense reflect the core feature of Galilean theories, that is there exists the notion of absolute time.

[5]Quoted from *Through the Looking-Glass*, the infamous novel authored by Lewis Carroll and inspired the Carrollian terminology.

### Carrollian limit

> *"A slow sort of country! Now, here, it takes all the running you can do, to keep in the same place.",*
> responsed The Red Queen, representing the Carrollian world

For the Carrollian case, one needs to make the following rescalings

$$x^0 = cu, \qquad \beta^i = -cb^i, \qquad \text{and} \qquad a^0 = c\tau, \tag{1.5}$$

with $u$ representing the Carrollian notion of time and $\tau$ labels Carrollian time translations. The boost vector $b^i$ was however parameterized differently from the one chosen in the Galilean case to ensure that the transformations (1.1) has a regular limit as $c \to 0$. The Lorentz factor can be expressed in the small-$c$ expansion as[6]

$$\gamma = 1 + \frac{1}{2}c^2|b|^2 + \mathcal{O}(c^4). \tag{1.6}$$

Properly taking the $c \to 0$ limit of (1.1), we arrive at the Carrollian transformations,

$$u' = u - b_i(Rx)^i + \tau, \qquad \text{and} \qquad x'^i \ = (Rx)^i + a^i. \tag{1.7}$$

The paramount trademark of Carrollian theories, contrary to the Galilean case, is the existence of absolute space. Remarks are in order here:

*i*) It is not entirely correct to conclude that a particle has zero velocity in the Carrollian limit as one can check that the particle's velocity, $v^i = \frac{\mathrm{d}x^i}{\mathrm{d}u}$, transforms as,

$$v'^i = \frac{\mathrm{d}x'^i}{\mathrm{d}u'} = \frac{(Rv)^i}{1 - b_j(Rv)^j}. \tag{1.8}$$

This therefore dictates that if a particle starts with zero velocity, it remains at zero velocity after Carrollian transformations. In contrast, if a particle has initially non-zero velocity, it can then be boosted to any non-zero value, depending on the value of the boost parameter $b_i$. This characteristic however is not present in the Galilean case. This entails two classes of Carroll particles — those with zero velocity and those with non-zero velocity. The latter in a sense is tachyonic (see [125] for more in-depth discussions).

[6] If one were to alternatively employ the textbook definition of boost parameters in term of the relative velocity of two reference frames, $\beta^i = \frac{w^i}{c}$, one would also need to impose that the velocity goes to zero faster than the speed of light by setting $w^i = c^2 b^i$ [124].

*ii*) While it is true that a single non-tachyonic Carroll particle possesses no dynamics as it cannot move [126], there exist nonetheless situations where Carrollian systems exhibit non-trivial dynamics, e.g, when considering interacting particles and when a particle is coupled to non-trivial background fields [127].

Although Lévy-Leblond himself remarked in his original paper [122] that practical utilization of Carroll symmetry group at that time was quite problematic, interests in Carroll symmetry has recently been rejuvenated and gained ever-increasing attention in many fields of physics. This development was largely catalyzed by the connection between the (conformal) Carroll group and the symmetry group of asymptotically flat spacetime, the co-called Bondi-van der Burg-Metzner-Sachs (BMS) group [128–130], which plays a central role in understanding flat space holography, and thereby motivated the studies on Carrollian field theories [126, 131, 132]. Aspects of the Carroll symmetry in gravitational theories have also been addressed in [133–142]. In addition, hydrodynamics of Carrollian fluids that inspired the work in this thesis have previously been explored in [124], along with its applications in the field of black holes and holography [121, 143–148]. Carrollian physics has also appeared in the context of inflationary cosmology [125].

## 1.2 Outline of this Thesis

As announced, this thesis consists of two parts. The first part is dedicated to comprehensive investigations of the status of edge modes, in theories with internal gauge symmetries. We aim to explore the extended phase space of gauge theories, edge modes included, at the level of boundary Lagrangian and scrutinize corner symmetries and boundary dynamics of these edge modes, along with the applications to physical observables such as entanglement entropy. We will also argue and demonstrate that edge modes and their symmetries may help shed light on the improved understanding of dualities in physics, such as the electromagnetic duality.

We will devote the second part of this thesis to the study of gravitational dynamics of a subregion of spacetime bounded by a null horizon. A wealth of physics is however encoded on stretched horizons, timelike surfaces placed in close vicinity of the null horizon, such as the emerging connection with Carroll geometries and Carrollian hydrodynamics. This correspondence has been first investigated by the authors of [121], and we will elucidate this deep connection in full generality in this thesis.

In Chapter 2, we give a brief review of the covariant phase space formalism which we shall adopt throughout the thesis.

### 1.2.1 Part I: Edge Modes: Dynamics and Duality

Chapter 3 is devoted to the construction of extended phase space on gauge theories at the level of action and scrutinize corner symmetries and boundary dynamics of edge modes. We would like to elucidate these following points

1. Given a gauge theory, what is the action and the symplectic structure for the edge modes, and which freedom is there in their construction?
2. Can we demonstrate the role played by edge modes in splitting and seamlessly gluing of subregions, using the obtained action for edge modes?
3. What is edge modes contribution to entanglement entropy?

We will present in this chapter a framework for writing down a boundary action for edge modes, and give some preliminary examples of the subtleties and differences which arise for different gauge theories (e.g. depending on boundary conditions and Hamiltonians, and on whether the theory is topological or not). To study the dynamics of the edge modes, we will propose a new action principle which includes the edge modes in a boundary action and then naturally reproduces the extended phase space and its symplectic structure. Then, we will explain, using the path integral technique, how integrating out the bulk degrees of freedom in a subregion produces an effective boundary action which will contribute to the entanglement entropy.

In Chapter 4, we study electromagnetic duality from the perspective of corner symmetries and charges. The key message of this rather short chapter is that the electromagnetic duality and the existence of dual magnetic charges in Maxwell's theory can are better understood using the first-order formulation of electromagnetism.

### 1.2.2 Part II: Carrollian Hydrodynamics on Stretched Horizons

We dedicate Chapter 5 to first review geometrical setups, namely Carroll structures and Carroll geometries, which serve as fundamental building blocks for our studies in Part II. Armed with the geometrical setups, we will then proceed to study Carrollian fluid and corresponding conservation laws. To this end, we will present two approaches for deriving Carrollian hydrodynamic equations.

For the first approach which is rather a standard one, we follow closely the derivation already presented in [124] which was based on taking the Carrollian limit of the relativistic conservation laws of the fluid energy-momentum tensor. We however will frame our derivation in a manner that the Carroll structure becomes apparent, thereby formalize and generalize the results of [124].

For the second approach, we will give a new perspective for Carrollian hydrodynamics based on the symmetry principle. This view point has been first explored in [144]. Their derivation, which relied on Carrollian symmetries, however only reproduced parts of the full set of fluid equations. In our work, we take an inspiration from near-horizon symmetries of black holes and propose a new notion of symmetries for Carrollian fluids (we call these symmetries the near-Carrollian symmetries). We finally show that these new symmetries lead to the full set of Carrollian hydrodynamic equations.

Chapter 6 of this thesis focuses on gravitational physics in a subregion of spacetime bounded by a null boundary. The spacetime near the null boundary is foliated into a family of timelike hypersurfaces, the stretched horizons.

Our construction relies on the rigging technique of general hypersurfaces. We will show that by endowing the *null* rigged structure on the surface, the Carroll structure is naturally induced on the surface, regardless of whether the surface is null or timelike. This formalism therefore treats the timelike stretched horizons and the null boundary in the same status, and the limit from the stretched horizon to the null boundary is regular. This *null limit* can be viewed as the analog to the Carrollian limit [121, 149].

Having setup the intrinsic geometry of the surfaces, we then discuss the extrinsic geometry of the surfaces. Elements of extrinsic geometry, which are encoded in the rigged Weingarten tensor, corresponds to the Carrollian fluid momenta. We will define the energy-momentum tensor of the surface and then show that conservation laws of this tensor, which yields Carrollian hydrodynamics, infers the Einstein equations, even on the timelike stretched horizon and thus our results generalize the results of [149].

Similar to Chapter 5, we next consider in Chapter 7 conservation laws from the perspective of symmetries. We will show that the gravitational phase space of the surfaces (either null or timelike) has the same structure as the phase space of Carrollian fluids, except for some extra components arising from the embedding of the surfaces in the ambient spacetime. We then proceed to derive the Einstein equations (which are viewed as the fluid equations) from the near-horizon symmetries of the surface. We will further demonstrate that the phase space of the stretched horizons situated in extremely close proximity to the null boundary also encodes *sub-leading* information of the null boundary phase space. By treating the stretched horizons as the small-$r$ expansion around the null boundary, we will show that the sub-leading term of the pre-symplectic potential is necessary to properly derive all evolution equations (the Einstein equations) on the null boundary. Most importantly, the Einstein equations $G_{AB} = 0$ are the consequence of a certain spin-2 symmetries.

# Chapter 2

# Covariant Phase Space Formalism

This chapter provides a short review of aspects of the covariant phase space formalism (also called the covariant Hamiltonian formalism) [150–152] and introduces relevant technologies that will be adopted in this thesis. Recent development of this formalism can be found, for example in [41,153,154]. See also [111,155–157] for the situations involving null boundaries.

## 2.1 Pre-symplectic Potential and Structure

The central object in the covariant phase space formalism, the *pre-symplectic potential*, is implicitly defined through a variation of a Lagrangian. To see this, let us consider classical field theories defined in a general $d$-dimensional spacetime. The classical dynamics of a field[1] $\Phi$ is fully encoded in a classical Lagrangian $L[\Phi]$, which is a differential $d$-form. In general, any field variation of the Lagrangian has the following structure

$$\delta L[\Phi] = \mathrm{EOM}[\Phi, \delta\Phi] + \mathrm{d}\Theta[\Phi, \delta\Phi]. \tag{2.1}$$

Classical equations of motion, $\mathrm{EOM} = 0$, are usually derived by demanding that the variation of the Lagrangian vanishes up to total derivative terms (or equivalently, the variation of an action $S = \int L$ vanishes up to a boundary term). By definition, equations of motion do not contain derivatives of the variations of the field and are uniquely determined by the Lagrangian. This infers that we will always have $\mathrm{EOM} = E \wedge \delta\Phi$, where $E = 0$ is the usual Euler-Lagrange equation. The spacetime $(d-1)$-form $\Theta = \Theta[\Phi, \delta\Phi]$ is the *pre-symplectic*

[1] In this thesis, it formally represents both gauge fields and gravitational (metric) fields.

*potential current* and it encodes phase space information of the theories. Following a rather standard nomenclature, we view a field variation $\delta$ as an exterior derivative over the field space such that for a bosonic field $\Phi$, the field variations commute,

$$\delta^2_{(1,2)}\Phi = \frac{1}{2}\left(\delta_1\delta_2\Phi - \delta_2\delta_1\right)\Phi = 0. \tag{2.2}$$

The pre-symplectic potential $\Theta$ is therefore a 1-form in the space of fields[2]. Repeated action of $\delta$ is understood with anti-symmetrization. We define the pre-symplectic potential on a codimension-1 Cauchy surface $\Sigma$ to be the integral of the pre-symplectic potential current over $\Sigma$,

$$\Theta_\Sigma := \int_\Sigma \Theta[\Phi, \delta\Phi]. \tag{2.3}$$

The *pre-symplectic structure* on $\Sigma$ is the field space 2-form and it is defined as the field space differential of the pre-symplectic structure,

$$\Omega_\Sigma := \delta\Theta_\Sigma = \int_\Sigma \delta\Theta[\Phi, \delta\Phi]. \tag{2.4}$$

The pre-symplectic structure contains two variations and is thereby a closed differential form in the field space, i.e., $\delta\Omega_\Sigma = 0$. We comment that the prefix "pre" is used to indicate the fact that, at this stage, the object $\Omega_\Sigma$ is not completely qualified as being symplectic as it contains degenerate directions and the phase space is in a sense not physical. These degenerate directions correspond to gauge redundancies and they are needed to be properly quotient out in order to obtain the physical phase space.

## 2.2 Noether and Hamiltonian Charges

Having introduced the pre-symplectic potential and the pre-symplectic structure, we then proceed to describe the Noether charges and the Hamiltonian charges associated with symmetries.

To set our notations, we use $\delta_\epsilon\Phi$ to denote variations of a field $\Phi$ under some symmetries (e.g. gauge symmetries and diffeomorphism) labelled by a transformation parameter $\epsilon$. We

[2] One can knit the spacetime and the field space into a single structure called the variational bi-complex (see [158, 159] and references therein). In the literature, a spacetime $p$-form which is also a field space $n$-form is usually denoted as a $(p, n)$-form. For instance, the pre-symplectic potential current $\Theta$ is a $(d-1, 1)$-form.

regard the variation $\delta_\epsilon$ as a field space vector and denote by $I_{\delta_\epsilon}$ a field space interior product, such that $I_{\delta_\epsilon}\delta\Phi = \delta_\epsilon\Phi$. Since the thesis comprises a part that deals with internal gauge symmetries and a part that deals with spacetime diffeomorphism, we thereby consider in general gauge transformation parameter $\epsilon \in (\alpha, \xi)$ where $\alpha$ labels usual internal gauge transformations and $\xi$ parameterizes diffeomorphisms. Under gauge transformations we have $\delta_\alpha L = \mathrm{d}m_\alpha$, and $\delta_\xi L = \mathcal{L}_\xi L = \mathrm{d}(\iota_\xi L)$ for diffeomorphism[3].

Now, it follows from the standard form of variation that

$$\delta_\epsilon L = \mathrm{d}(m_\alpha + \iota_\xi L) = E \wedge \delta_\epsilon \Phi + \mathrm{d}(I_{\delta_\epsilon}\Theta), \tag{2.5}$$

leading us to

$$E \wedge \delta_\epsilon\Phi + \mathrm{d}(I_{\delta_\epsilon}\Theta - m_\alpha - \iota_\xi L) = 0. \tag{2.6}$$

The *Noether current*, which is a spacetime $(d-1)$-form, is defined as

$$J_\epsilon = I_{\delta_\epsilon}\Theta - m_\alpha - \iota_\xi L. \tag{2.7}$$

On-shell, we have the conservation of the Noether current, $\mathrm{d}J_\epsilon \approx 0$. For local symmetries, this implies that $J_\epsilon \approx \mathrm{d}Q_\epsilon$, with the spacetime $(d-2)$-form $Q_\epsilon$ being the *Noether charge*. According to the Noether's second theorem for local symmetries, we also have

$$E \wedge \delta_\epsilon\Phi = -\mathrm{d}C_\epsilon, \tag{2.8}$$

for a constraint $C_\epsilon$, that vanishes on-shell, and therefore

$$J_\epsilon = C_\epsilon + \mathrm{d}Q_\epsilon. \tag{2.9}$$

Let us now describe the Hamiltonian generators, or the *Hamiltonian charges*, of the general gauge transformations. The field variation $\delta_\epsilon\Phi$ is determined by the Poisson bracket between the field and the Hamiltonian generators $\mathcal{H}[\epsilon]$,

$$\big\{\mathcal{H}[\epsilon], \Phi\big\} = \delta_\epsilon\Phi. \tag{2.10}$$

Note however that the Hamiltonian of symmetries $\mathcal{H}[\epsilon]$ may not exist in general, due to the presence of symplectic fluxes.

The variation of the generator is given by the field space contraction of the pre-symplectic structure

$$\not{\delta}\mathcal{H}[\epsilon] = -I_{\delta_\epsilon}\Omega_\Sigma. \tag{2.11}$$

[3]This is only valid for covariant Lagrangians. In general, we will have $\delta_\xi L = \mathrm{d}(\iota_\xi L + a_\xi)$ where $a_\xi$ represented the Lagrangian anomaly (see [41]).

It is important to remark that the notation $\oint\!\!\!\!/$ is used to emphasize the aforementioned feature that the variation of Hamiltonian charges are not always integrable, and that Hamiltonians associated to the symmetries may not exist in general. The situations when the Hamiltonian charges are non-integrable are encountered especially in gravitational systems, when there are non-vanishing symplectic fluxes leaking through boundaries of the spacetime under consideration. We will come back to this point momentarily. The Hamiltonian charges (if they existed) satisfy the charge algebra,

$$\big\{\mathcal{H}[\epsilon], \mathcal{H}[\eta]\big\} = \delta_\epsilon \mathcal{H}[\eta] = -I_{\delta_\epsilon} I_{\delta_\eta} \Omega_\Sigma. \tag{2.12}$$

### 2.2.1 Relation between $J_\epsilon$ and $\mathcal{H}_\epsilon$

To finally complete our discussion on the charges, we remark the fact that the Noether charges and the Hamiltonian charges are in principle different. We will elaborate on how the difference comes about, for the simplest case where the diffeomorphism is field-independent, such that $\delta\xi = \delta\alpha = 0$, and the potential $\Theta$ is also covariant, meaning that $\delta_\xi \Theta = \mathcal{L}_\xi \Theta$. We point out for interested readers the discussion for the general case is presented in [41].

Let us first consider, for internal gauge transformations $\alpha$, the equality

$$0 = \delta\delta_\alpha L - \delta_\alpha \delta L = \mathrm{d}(\delta m_\alpha - \delta_\alpha \Theta), \tag{2.13}$$

which of course holds true because $\delta_\alpha(E \wedge \delta\Phi) = 0$. This then implies that there exists a $(d-2)$-form $M_\alpha$ such that

$$\delta m_\alpha - \delta_\alpha \Theta = \mathrm{d}M_\alpha. \tag{2.14}$$

Next, the variation of the Noether current (2.7) is given by

$$\delta J_\epsilon = \delta(I_{\delta_\epsilon}\Theta) - \delta m_\alpha - \delta(\iota_\xi L), \tag{2.15}$$

where the last term can be rewritten using

$$\begin{aligned} \delta(\iota_\xi L) = \iota_\xi \delta L &= \iota_\xi(E \wedge \delta\Phi) + \iota_\xi(\mathrm{d}\Theta) \\ &= \iota_\xi(E \wedge \delta\Phi) + \mathcal{L}_\xi \Theta - \mathrm{d}(\iota_\xi \Theta) \\ &= \iota_\xi(E \wedge \delta\Phi) + \delta_\xi \Theta - \mathrm{d}(\iota_\xi \Theta), \end{aligned} \tag{2.16}$$

where we state again that this derivation is valid for the special case when the pre-symplectic potential is covariant $\delta_\xi \Theta = \mathcal{L}_\xi \Theta$. Putting everything together leads to

$$\begin{aligned} \delta J_\epsilon &= \delta(I_{\delta_\epsilon}\Theta) - \delta m_\alpha - \iota_\xi(E \wedge \delta\Phi) - \delta_\xi \Theta + \mathrm{d}(\iota_\xi \Theta) \\ &= \delta(I_{\delta_\epsilon}\Theta) - \delta_\epsilon \Theta - \iota_\xi(E \wedge \delta\Phi) + \mathrm{d}(\iota_\xi \Theta - M_\alpha) \\ &= -I_{\delta_\epsilon}\delta\Theta - \iota_\xi(E \wedge \delta\Phi) + \mathrm{d}(\iota_\xi \Theta - M_\alpha). \end{aligned} \tag{2.17}$$

Integrating the result on the Cauchy surface $\Sigma$, we arrive at the expression for the Hamiltonian charges,

$$\begin{aligned}\not{\delta}\mathcal{H}[\epsilon] &= \int_{\Sigma} \delta C_{\epsilon} + \iota_{\xi}(E \wedge \delta\Phi) + \int_{\partial\Sigma} \delta Q - \iota_{\xi}\Theta + M \\ &\approx \int_{\partial\Sigma} \delta Q - \iota_{\xi}\Theta + M.\end{aligned} \tag{2.18}$$

It then becomes clear that the Hamiltonian charges are in general non-integrable and differ from the Noether charges due to the presence of the flux term, $-\iota_{\xi}\Theta + M$.

### 2.2.2 Noether charges of General Relativity

We will need the expression for the constraints and the Noether charges of general relativity in Part II of the thesis. Now, let us consider the gravitational case described by the Einstein-Hilbert Lagrangian[4], $L = \frac{1}{2}R\epsilon$, where $R$ represents the spacetime Ricci scalar and $\epsilon$ is the spacetime volume form. Denoting the Einstein tensor by $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$, the equations of motion are the Einstein equations,

$$\text{EOM} = \frac{1}{2}G^{ab}\delta g_{ab}\epsilon. \tag{2.19}$$

The pre-symplectic potential of the Einstein-Hilbert gravity is given by $\Theta = \Theta^a \epsilon_a$ where we defined the contraction of the volume form, $\epsilon_a := \iota_{\partial_a}\epsilon$, and the standard gravitational pre-symplectic potential current is given in terms of variation of the spacetime metric $\delta g_{ab}$ and its trace $\delta g := g^{ab}\delta g_{ab}$ by

$$\Theta^a = \frac{1}{2}\left(g^{ac}\nabla^b \delta g_{bc} - \nabla^a \delta g\right). \tag{2.20}$$

Given the bi-normal $\epsilon_{ab} := \iota_{\partial_a}\iota_{\partial_b}\epsilon$, the constraint and the Noether charges (which in this case is particularly referred to as the Komar charges) associated with the spacetime diffeomorphism parameterized by $\xi$ are

$$C_{\xi} = \xi^a G_a{}^b \epsilon_b, \qquad \text{and} \qquad Q_{\xi} = \frac{1}{2}\nabla^a \xi^b \epsilon_{ab}. \tag{2.21}$$

[4]We will only consider the case when the cosmological constant vanishes and the matter degrees of freedom are absent in this thesis

## 2.3 Anomaly Operator

Lastly, we define the anomaly[5] operator which serves as a computational tool in this thesis. Let us begin by mentioning that the variation $\delta_\xi$ under spacetime diffeomorphism can be promoted to be the field space Lie derivative which acts on any generic field space forms via the field space analog of the Cartan formula[6],

$$\delta_\xi := I_\xi \delta + \delta I_\xi. \tag{2.22}$$

Anomaly occurs when an object fails to be covariant under spacetime diffeomorphism. An object is said to be covariant under spacetime diffeomorphism if its variation under spacetime diffeomorphism agrees with its change under the action of spacetime Lie derivative. The prime example of covariant objects is the spacetime metric $g_{ab}$, meaning that

$$\delta_\xi g_{ab} = \mathcal{L}_\xi g_{ab}. \tag{2.23}$$

This property however does not necessary hold for a general field $\Phi$ and the failure specifically occurs when there exist fixed background structures in spacetime that do not change under diffeomorphism. In such circumstance, the field $\Phi$ is viewed as a function of the metric components, $\Phi = \Phi[g_{ab}]$. The change of the field under diffeomorphism is thus due to the change of the metric,

$$\delta_\xi \Phi[g_{ab}] = \Phi[g_{ab} + \delta_\xi g_{ab}] - \Phi[g_{ab}] = \frac{\delta \Phi}{\delta g_{ab}} \mathcal{L}_\xi g_{ab}, \tag{2.24}$$

and in general does not coincide with the Lie derivative $\mathcal{L}_\xi \Phi$. With this logic in mind, we define the *anomaly operator* as the difference

$$\Delta_\xi := \delta_\xi - \mathcal{L}_\xi - I_{\delta\xi}. \tag{2.25}$$

The last term $I_{\delta\xi}$ arises due to the field-dependence of the diffeomorphism vector field $\xi$. It will cancel out the $\delta\xi$ contribution that comes from the term $\delta I_\xi$ in the definition of the field space Lie derivative, in turn making the anomaly $\Delta_\xi$ completely independent of $\delta\xi$. Any object is said to be covariant if its anomaly vanishes.

[5] which was assumed to be non-existent in previous derivations

[6] Clarification is in order here. In Chapter 3, we are going to deal with two types of symmetries: gauge symmetries $\delta_\alpha$ and corner symmetries $\Delta_\alpha$. We will therefore use $I_{\delta\alpha}$ and $I_{\Delta\alpha}$ to distinguish field space contraction associated with different symmetry transformations. In Chapter 7 however, spacetime diffeomorphism $\delta_\xi$ is the only concern. As such, we will denote field space contraction simply with $I_\xi$.

In this thesis, the anomaly operator serves as a useful computation tool to compute the transformations of non-covariant objects (we will utilize this technology in Chapter 5 and Chapter 7). One of its useful properties is that it commutes with the spacetime covariant derivative, $\triangle_\xi \nabla_a = \nabla_a \triangle_\xi$. Let us also mention that this technology has been extensively utilized in the literatures [41, 111, 149, 154, 157, 160]

# Part I

# Edge Modes: Dynamics and Duality

# Chapter 3

# Extended Action and Dynamics of Edge Modes

We dedicate this chapter to a study of edge modes and their boundary dynamics. We will propose a simple and systematic framework for including edge modes in gauge theories defined on manifolds with boundaries. Starting with a boundary action containing edge modes, we then introduce a new variational principle which systematically produces a corner contribution to the symplectic structure, and thereby provides a covariant realization of the extended phase space constructions that have appeared previously in the literature. Furthermore, we demonstrate that this is necessary in order to achieve the factorizability of the path integral, the Hilbert space and the phase space, and that it explains how edge modes acquire a boundary dynamics and can contribute to observables such as the entanglement entropy.

Before diving into our proposals and detailed computations, we believe it is more beneficial to provide the readers the big picture of our construction, which is summarized schematically on figure 3.1, and to get the general idea of what we are trying to understand. Consider two spacetime manifolds $M$ and $\overline{M}$ with respective time-like boundaries $\partial M$ and $\partial\overline{M}$. A gauge theory on each manifold is defined by bulk fields, but also by boundary degrees of freedom. These edge modes are introduced via a boundary Lagrangian, which couples in a gauge-invariant manner the bulk gauge fields and the edge modes to a boundary current (which can be thought of as the edge mode's conjugate momentum). The presence of these edge modes is precisely what allows for the splitting of the path integral over $M \cup \overline{M}$ into two factors. This is the covariant analogue of the factorization in terms of extended Hilbert spaces, and it requires relaxation of the boundary conditions to allow

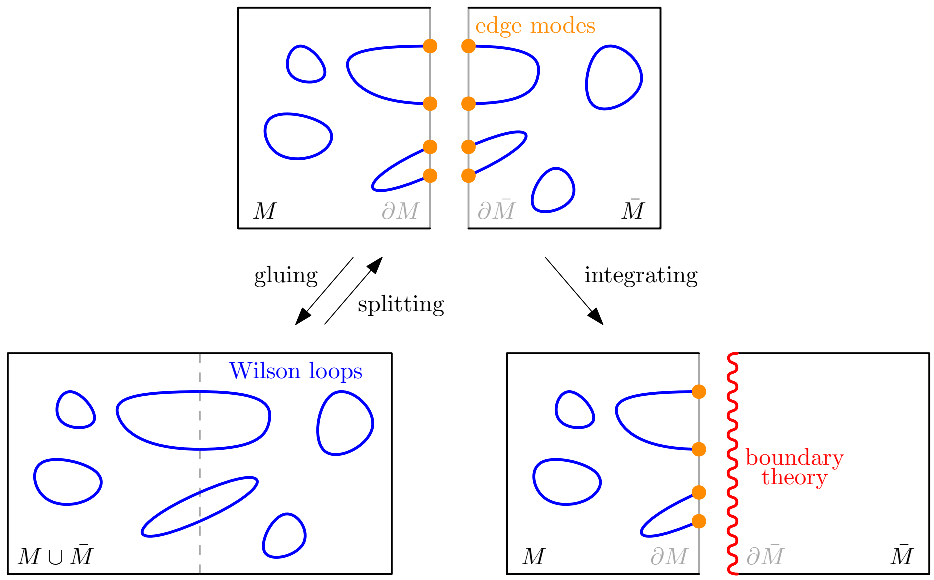


Figure 3.1: On the top, we have the same gauge theory defined on two neighboring manifolds with boundaries. Each factor contains bulk fields and edge modes. Integrating out the edge modes leads to the theory defined over $M \cup \overline{M}$ and, conversely, the splitting of $M \cup \overline{M}$ into subregions requires introducing edge modes on which Wilson lines can end. Once the theory is split into factors associated with the regions $M$ and $\overline{M}$, one bulk region can be integrated out, thereby leading to an effective boundary dynamics for the edge modes. This latter will in turn be seen by and contribute to the entanglement entropy.

for open Wilson lines to end on the boundary. In a path integral context, one can then manipulate the factorized path integrals over $M$ and $\overline{M}$ in two ways:

*i*) integrating out the edge modes living on $\partial M$ and $\partial \overline{M}$ (with suitable matching constraints) will glue the theories defined on the subregions and lead to the path integral over $M \cup \overline{M}$, while

*ii*) integrating out the bulk fields of region $\overline{M}$ will produce an effective boundary theory on $\partial \overline{M}$. This second point is very important. It means that integrating out the bulk degrees of freedom in $\overline{M}$, when taking properly into account the presence of the edge modes on the boundary $\partial M = \partial \overline{M}$, does *not* reproduce the path integral defined on $M$ only: there is a residual contribution on the boundary due to the dynamics of the edge modes, and this will contribute in particular to the entanglement entropy.

One can clearly see the fundamental role played by the edge modes in this construction:

they appear once we split a theory (i.e. its Hilbert space, phase space, or path integral), dictate how regions should be glued along an interface, and encode a leftover boundary dynamics once one of the bulk regions is integrated out. For this, we will first recall in section 3.1 how the edge modes can conveniently be parametrized in a Hamiltonian setting by using an extended phase space containing a boundary symplectic structure, then construct in section 3.2 compatible boundary actions, and finally present in section 3.3 examples and applications.

## 3.1 Extended phase spaces

For a given gauge theory, the extended phase space [32, 34–36, 38, 69, 70] is the classical analog of the extended Hilbert space. The extension consists in adding to the bulk phase space, for each type of gauge transformations in the theory, a corresponding edge mode field living on the boundary.

The construction of the extended phase space takes place in the covariant phase space formalism (see a short review in Chapter 2), and exploits a well-known corner ambiguity [161,162], which is that of supplementing the pre-symplectic potential $\Theta$ by a total exterior derivative $\mathrm{d}\vartheta$. By adding edge mode fields living on the boundary $S=\partial\Sigma$ of spatial hypersurfaces and transforming in a particular way under gauge transformations, one can construct in a minimal way an extended potential $\Theta_\mathrm{e}=\Theta+\mathrm{d}\vartheta$ such that the associated symplectic structure

$$\Omega=\int_\Sigma\delta\Theta_\mathrm{e}=\int_\Sigma\delta\Theta+\int_S\delta\vartheta \tag{3.1}$$

disentangles in a natural manner the role of gauge transformations from that of corner symmetries. This extended symplectic structure is indeed such that gauge transformations are generated by constraints that vanish on-shell and have no Hamiltonian charge, while boundary symmetries are generated by surface observables that satisfy a boundary symmetry algebra, and this without the need to impose boundary conditions on the dynamical fields or on the parameters of gauge or symmetry transformations. The role of the edge mode fields appearing with their canonical momenta in the boundary symplectic structure is two-fold:

*i*) to restore the seemingly broken gauge-invariance due to the presence of the boundary,

*ii*) to parametrize the boundary symmetries and observables.

Representing a gauge transformation by a tangent vector $\delta_\alpha$ in field space, one has in other words that the field space contraction $I_{\delta_\alpha}\Omega$ is integrable and vanishing on-shell. This is nothing but the familiar Hamiltonian generator of the transformation $\delta_\alpha$, which is however stripped from its usual boundary charge because this latter has been cancelled by the contribution of the boundary symplectic structure containing the edge modes. This is a first advantage of the extended phase space: gauge transformations are null directions of the extended symplectic structure even when they have support on the boundary. Similarly, a boundary symmetry can be represented by a tangent vector $\Delta_\alpha$, and is characterized by a generator $I_{\Delta_\alpha}\Omega$, which is integrable, gauge-invariant in the sense that $I_{\delta_\alpha}I_{\Delta_\beta}\Omega = 0$, equal to a boundary integral, and satisfies a boundary symmetry algebra $I_{\Delta_\alpha}I_{\Delta_\beta}\Omega$.

It has been shown in [34, 36] that the generators of such boundary symmetries $\Delta_\alpha$ are the usual Hamiltonian boundary observables introduced in [163–168], in which however the bulk fields are "dressed" in a gauge-invariant manner by the new edge mode fields that have been introduced on the boundary. This is a second advantage of the extended phase space: the edge modes that have been added through the boundary symplectic structure are now part of the phase space and parametrize the boundary observables and their symmetry algebra. While this description may seem formal at this point, we will provide explicit examples in section 3.3.

The natural next step is to search for a dynamical description of these edge modes, and to conceive them not as living only on the boundary $S$ of a spatial slice, but on the whole time-like boundary $S \times \mathbb{R}$. This is a familiar situation in CS theory, where the time-like boundary is known to carry a gapless chiral theory [169–171]. However, the construction of the boundary dynamics in CS theory typically relies on studying the behavior under gauge transformations of the action itself. This explains the difference of treatment which has subsisted so far between e.g. Maxwell–Yang–Mills and CS theory: the former has a gauge-invariant action while the latter does not. From this, one would (wrongly) conclude that Maxwell–Yang–Mills theory does not possess a boundary dynamics. However, as we have argued above, the study of gauge (non)-invariance should instead be carried out at the level of the symplectic structure. There, one can easily motivate the need to work with an extended phase space containing edge mode fields. Let us now describe how their boundary symplectic structure can be obtained from a boundary action.

## 3.2 Extended actions

Let us consider for simplicity that the $d$-dimensional spacetime manifold is of the form $M = \Sigma \times \mathbb{R}$, where the time-like boundary is $\partial M = S \times \mathbb{R}$. The extended symplectic structure described above can be thought of as arising from an extended field theory, where the bulk and codimension–1 boundary submanifolds each possess a Lagrangian, equations of motion, and a (pre-symplectic) potential. In order to see this, let us write the extended action and its variation in the form

$$S = \int_M L_M + \int_{\partial M} L_{\partial M}, \qquad\qquad \delta S = \int_M \mathrm{EOM}_M + \int_{\partial M} \Theta + \delta L_{\partial M}. \tag{3.2}$$

This is of course a familiar step in field theory and in the covariant phase space formalism, where it identifies the potential $\Theta$ as the total exterior derivative term arising from the integrations by parts isolating the bulk equations of motion. In usual constructions of the covariant phase space [159, 161, 172], the introduction of a boundary Lagrangian $L_{\partial M}$ is simply understood as resulting in a shift $\Theta \mapsto \Theta + \delta L_{\partial M}$ of the potential. The boundary conditions defining the variational principle are then taken to be $(\Theta + \delta L_{\partial M})\big|_{\partial M} := 0$, and one concludes that the boundary Lagrangian cannot affect the symplectic structure since upon taking a second variation to obtain the symplectic current one has $\delta\Theta \mapsto \delta\Theta + \delta^2 L_{\partial M} = \delta\Theta$ by virtue of the property $\delta^2 = 0$.

However, this viewpoint turns out to be unnecessarily restrictive, and one can be more general by realizing that this ambiguity in the boundary term fits perfectly well with the above-mentioned corner ambiguity. In other words, there is a natural way in which the boundary Lagrangian may provide a corner term. We will now explain this construction in full generality. Let us mention that similar construction to what we are about to presnt also appeared in the work of Harlow and Wu [153][1], and the works [38,39] applied the formalism to 4-dimensional first order gravity. The idea is simply to realize that acceptable boundary conditions can be more generally taken to be

$$(\Theta + \delta L_{\partial M})\big|_{\partial M} := -\mathrm{d}c. \tag{3.3}$$

Interestingly, this fits nicely with our desire to encode the dynamics of the edge mode fields in the boundary Lagrangian. Indeed, if this latter contains derivatives, upon taking a field space variation one can then integrate by parts to isolate boundary equations of motion and a boundary pre-symplectic potential. We can then suggestively rewrite the variation of the action in (3.2) as

$$\delta S = \int_M \mathrm{EOM}_M + \int_{\partial M} \mathrm{EOM}_{\partial M} - \mathrm{d}c, \tag{3.4}$$

[1]An important difference is that Harlow and Wu are not concerned with edge modes and extended phase spaces. They describe how a boundary Lagrangian can provide a corner term and discuss at length the example (among others) of Einstein–Hilbert gravity with the Gibbons–Hawking–York term, but do not consider Lagrangians which include edge mode fields. Apart from this conceptual difference, our constructions are the same.

where on the boundary we have now explicitly combined the potential $\Theta$ of $L_M$ with part of the variations of $L_{\partial M}$ to get the boundary equations of motion, and also kept the total exterior derivative containing the potential $c$ of $L_{\partial M}$. In this picture, the boundary conditions (3.3) are just rewritten as the requirement that $\text{EOM}_{\partial M} := 0$. As we will see in explicit examples below, this requirement will generally translate into several conditions, which can be fulfilled either by fixing some variables on $\partial M$ (e.g. Dirichlet boundary conditions in gravity), or by imposing boundary equations of motion.

As explained in [153], the correct potential to consider for the construction of a conserved symplectic structure is then $\Theta + \mathrm{d}c$, and therefore naturally includes a corner contribution. In our more general setup, where the boundary Lagrangian may contain edge mode fields, we will see that the correct extended symplectic potential $\Theta_\text{e} = \Theta + \mathrm{d}\vartheta$ will be obtained once we explicitly rewrite $\Theta + \mathrm{d}c$ on-shell of (some of) the boundary equations of motion which identify $c \approx \vartheta$. This is of course fine since in any case the covariant phase space formalism is on-shell, and since going on-shell of the boundary equations of motion is simply enforcing part of the boundary conditions (3.3) defining the variational principle. More precisely, we will see that in the whole set $\text{EOM}_{\partial M}$ we will have to explicitly use the boundary equations of motion involving the initial potential $\Theta$. This is a desired feature, since it means that instead of holding fixed a field configuration on the boundary (e.g. the gauge potential of Maxwell theory), we are relaxing this condition by imposing the conjugated boundary equations of motion instead. Once again, this should become crystal clear in the following section where we present concrete examples.

In summary, in order to achieve our construction relating the boundary Lagrangian $L_{\partial M}$ (which is the object we are trying to identify) to the extended symplectic structure (3.1) defining the extended phase space (which is the object we already know from the various constructions [32,34–36,69,70]), we simply have to look for a boundary Lagrangian whose potential $c$ is such that the extended potential is obtained as

$$\Theta + \mathrm{d}c \approx \Theta + \mathrm{d}\vartheta = \Theta_\text{e}. \tag{3.5}$$

Our formalism and that of [153] guarantee that this is possible, and we will give illustrative examples in the next section. A few comments are now in order before going on.

*i*) In this construction the boundary Lagrangian is more than a mere boundary term: it contains derivatives, and therefore a potential, which is then combined with the bulk potential in order to get the extended symplectic structure. As we have argued, this falls outside of the usual covariant phase space formalism of e.g. [159,161,172], and fixes unambiguously the corner contribution $c$. Furthermore, adding edge modes into the boundary Lagrangian achieves more than a simple change of polarization: it allows one to completely relax the boundary conditions by replacing them with boundary equations of motion.

*ii*) One can be puzzled by the apparent sign mismatch between the boundary potential in (3.4) and its contribution to the extended potential in (3.5). This follows of course from the compatibility of the symplectic current (more precisely the conservation of the symplectic structure) with the boundary conditions (3.3). A more heuristic way to understand this is to remember that we are trying to match the corner terms constructed in [32,34–36,69,70] by reaching the corner from the space-like hypersurface $\Sigma$, to the corner terms inherited from the boundary Lagrangian, and which therefore reach the corner from the time-like boundary $\partial M$. One can therefore understand the sign difference as coming from the sign of the bi-normal to the co-dimension 2 corner $S$, which depends on whether the corner is reached from a space-like slice or from a time-like boundary.

*iii*) We will see in the examples below that the boundary equations of motion that are used to write $c \approx \vartheta$ are, in the language of [32], gluing constraints, which determine the extended phase space by soldering together, via a classical fusion product, the boundary symplectic structure to the bulk one. The boundary Lagrangian $L_{\partial M}$ contains initially the edge mode fields and their unspecified conjugate momenta, and the boundary condition obtained through the boundary equation of motion involving $\Theta$ identifies these momenta with part of the initial bulk fields.

*iv*) The minimal requirement that we have imposed so far on the boundary Lagrangian only specifies the symplectic structure for the edge mode fields, and not their dynamics. In order to access this later, we will have to resort to an on-shell evaluation of the bulk action, thereby leading to an effective boundary action. We are also free to add to the boundary Lagrangian terms that do not change the symplectic structure and which are compatible with gauge-invariance. Such terms are in fact boundary Hamiltonians, i.e. they affect the boundary conditions (or equations of motion), but not the symplectic structure. The details of this procedure will depend on the theory under consideration, so let us now finally discuss some examples.

*v*) It is important to appreciate that there are two notions of "boundary dynamics" in the framework that we are proposing and that we have outlined above. First, there are boundary equations of motion that appear in (3.4) when varying the extended bulk + boundary action. These can be seen as continuity conditions relating the bulk and boundary fields. However, these equations alone do not determine the boundary dynamics of the edge mode fields. As we have mentioned above, this latter is obtained when evaluating the bulk fields on-shell. It will become clear in the examples discussed below that these two levels of equations of motion are different[2].

[2]One can think of this in analogy with first order theories, where one replaces a second order equation of motion by two first order equations. One can focus on one single first order equation, but this may not

## 3.3 Examples

We now present some relevant examples of extended bulk + boundary actions. This will illustrate in particular formulas (3.4) and (3.5), and reproduce the extended symplectic structures that have been studied before in the literature. It will also enable us to identify and discuss some remaining ambiguities in the characterization of the boundary dynamics, and to give more details on the factorization and the gluing of path integrals. We will also be able to establish a connection with previous results on the edge mode contributions to the entanglement entropy in various theories. We focus here on Abelian theories, and the discussion of the extended actions and phase spaces for non-Abelian theories is deferred to appendix A.4.

### 3.3.1 Chern–Simons theory

Let us focus on the Abelian case for simplicity, and describe in detail all the steps of the calculations. As usual, the theory is defined in the bulk by a connection 1-form $A$, transforming under gauge transformations as $\delta_\alpha A = \mathrm{d}\alpha$, and with curvature $F = \mathrm{d}A$. On the boundary, we now add a 0-form $a$ transforming as $\delta_\alpha a = -\alpha$, and a gauge-invariant 1-form $j$. With this field content, we can then form the action[3]

$$S = S_M + S_{\partial M} = \int_M A \wedge F + \int_{\partial M} aF + j \wedge \mathrm{D}a \pm \frac{1}{2} {*j} \wedge j, \tag{3.6}$$

where the Abelian covariant derivative is $\mathrm{D}a \coloneqq \mathrm{d}a + A$, and where $*$ is the Hodge dual on the boundary. The first term on the boundary, which is not gauge-invariant by itself, compensates for the gauge non-invariance of the bulk term, and the full action is therefore gauge-invariant. The presence of the last term, which requires use of the metric and therefore breaks the topological invariance of the theory, will be explained momentarily. This term is a boundary Hamiltonian $h[j]$, whose choice does not affect the boundary symplectic structure, but does change the boundary dynamics.

#### Extended phase space

Following the discussion of the previous section, let us now see what the introduction of the two fields $a$ and $j$ via the boundary Lagrangian implies. The variation of the action

determine completely the dynamics of a dynamical variable, which is only obtained when going on-shell of the other first order equation.

[3] We have dropped for clarity the usual coupling constant $\mathrm{k}/(4\pi)$

can be written in the form (3.2) as

$$\delta S = \left(2\int_M \delta A\wedge F + \int_{\partial M}\delta A\wedge A\right) + \delta S_{\partial M}, \tag{3.7}$$

where one can see that the potential coming from the bulk is $\Theta = \delta A\wedge A$. Writing explicitly the variation of the boundary action now leads to the form of expression (3.4), which is

$$\delta S = 2\int_M \delta A\wedge F + \int_{\partial M}\delta A\wedge(\mathrm{D}a - j) + \delta j\wedge(\mathrm{D}a \mp *j) + \delta a(\mathrm{d}j - F) - \mathrm{d}(j\delta a - a\delta A), \tag{3.8}$$

where on the boundary the first three terms identify the boundary equations of motion, and the last term identifies the boundary potential $c$. To access the bulk equations of motion, we need to impose the vanishing of the first term on the boundary. Conveniently, this can be done by imposing the boundary equation of motion $j = \mathrm{D}a$ instead of fixing the variation $\delta A$ of the gauge potential to be vanishing. This boundary equation of motion is precisely the one involving the potential $\Theta$ coming from the bulk Lagrangian. With this, the extended potential (3.5) becomes

$$\Theta_\mathrm{e} = \Theta + \mathrm{d}(j\delta a - a\delta A) = \delta A\wedge A + \mathrm{d}(j\delta a - a\delta A) \approx \delta A\wedge A + \mathrm{d}(\mathrm{D}a\delta a - a\delta A), \tag{3.9}$$

where we have been careful about the sign when including the boundary potential as our corner term, and then in the last equality used the boundary equation of motion involving $\Theta$. This result is interesting, as it reproduces precisely the extended potential that was derived in [34] for Abelian CS theory, thereby proving that the extended phase space structure can be recovered from the boundary Lagrangian introduced in (3.6) and the construction outlined in the previous section.

With this extended potential we have all the desirable properties mentioned in section 3.1. In particular, the extended symplectic structure (3.1) is given by

$$\Omega = \int_\Sigma \delta\Theta_\mathrm{e} = -\int_\Sigma \delta A\wedge\delta A + \int_S \delta(\mathrm{D}a)\delta a - \delta a\delta A, \tag{3.10}$$

and is such that for gauge transformations the generator defined by $I_{\delta_\alpha}\Omega$ is integrable and vanishing on-shell. Indeed, this is

$$I_{\delta_\alpha}\Omega = -2\int_\Sigma \mathrm{d}\alpha\wedge\delta A + 2\int_S \alpha\delta A = 2\int_\Sigma \alpha\delta F. \tag{3.11}$$

The transformation $\delta_\alpha$ is therefore a true gauge transformation, even when it has support on the boundary, and as such it has no Hamiltonian charge. In addition, the transformation acting as $\Delta_\alpha A = 0$ and $\Delta_\alpha a = \alpha$, which we will now call boundary symmetry as opposed to gauge transformation, has an integrable generator given by the manifestly gauge-invariant boundary integral

$$\mathcal{Q}[\alpha] = 2\int_S \alpha \mathrm{D}a, \tag{3.12}$$

and these generators satisfy the Abelian Kač–Moody commutation relation

$$\big\{\mathcal{Q}[\alpha], \mathcal{Q}[\beta]\big\} = I_{\Delta_\alpha} I_{\Delta_\beta} \Omega = 2\int_S \alpha \mathrm{d}\beta. \tag{3.13}$$

As is well-known, these are the boundary symmetries of CS theory on a spatial disc. One can see, as explained above, that their generator is a gauge-invariant "dressed" version of the usual Hamiltonian charge of $\delta_\alpha$, where the dressing corresponds to the finite gauge transformation of $A$ by the edge mode field $a$.

### Boundary dynamics

The Kač–Moody commutation relations which we have derived on the extended phase space result from the presence of a chiral scalar field, which is evidently the edge mode field $a$. To access the dynamics of this scalar field, we will write and manipulate the path integral for the extended action (3.6), following [170]. The key point of this derivation is to expand the components of the gauge field in the action and to carefully perform the path integral. For this, we assume that the spacetime has the topology $M = \mathbb{R} \times D$ of a cylinder, with coordinates $x^\mu = (t, r, \phi)$ such that $\varepsilon^{tr\phi} = 1$ and $\phi$ is compact, and that the space-like normal to the boundary cylinder at finite radius $r$ is $s = (0, 1, 0)$. The Hodge dual is then such that $*j \wedge j = (*j)_\phi j_t - (*j)_t j_\phi = j_t^2 - j_\phi^2$. After integrations by parts in the bulk, the total action (3.6) can be written explicitly as

$$\begin{aligned} S = &\int_M 2A_t(\partial_r A_\phi - \partial_\phi A_r) + A_\phi \partial_t A_r - A_r \partial_t A_\phi \\ &+ \int_{\partial M} a(\partial_\phi A_t - \partial_t A_\phi) + j_\phi(A_t + \partial_t a) - j_t(A_\phi + \partial_\phi a) - A_t A_\phi \pm \frac{1}{2} j_t^2 \mp \frac{1}{2} j_\phi^2. \end{aligned} \tag{3.14}$$

It is then clear that $A_t$ plays the role of a Lagrange multiplier. Path integrating[4] over

[4]As the details do not matter for our purposes so far, we will not explicitly write the path integrals and the various pre-factors coming from the integrations, but simply the resulting effective actions.

$A_t$ imposes the bulk and boundary relations

$$\partial_r A_\phi - \partial_\phi A_r = 0, \qquad\qquad j_\phi = A_\phi + \partial_\phi a, \tag{3.15}$$

which are part of the equations of motion imposed by $\delta A$ (i.e. the bulk equation of motion and the corresponding boundary condition). The first constraint can be solved by writing

$$A_r = \partial_r \alpha, \qquad\qquad A_\phi = \partial_\phi \alpha. \tag{3.16}$$

With this, after integrations by parts the bulk piece of the action becomes a boundary term, as

$$A_\phi \partial_t A_r - A_r \partial_t A_\phi = \partial_\phi \alpha \partial_t \partial_r \alpha - \partial_r \alpha \partial_t \partial_\phi \alpha = \partial_r(\partial_\phi \alpha \partial_t \alpha) - \partial_\phi(\partial_r \alpha \partial_t \alpha), \tag{3.17}$$

and (3.14) reduces to the boundary action

$$S_{\text{edge}} = \int_{\partial M} \partial_t \varphi \partial_\phi \varphi - j_t \partial_\phi \varphi \pm \frac{1}{2} j_t^2 \mp \frac{1}{2} (\partial_\phi \varphi)^2, \tag{3.18}$$

where we have introduced the gauge-invariant scalar combination $\varphi \coloneqq a + \alpha$. We recognize the first term as the canonical term of a chiral field. This is to be expected since so far the current $j$ has in a sense played no role, and we have reproduced the classic calculation of [170]. The last step is to perform the Gaussian integral over the current $j_t$ to finally obtain the effective action

$$S_{\text{edge}} = \int_{\partial M} \partial_t \varphi \partial_\phi \varphi \mp (\partial_\phi \varphi)^2, \tag{3.19}$$

which is known as the Floreanini–Jackiw action. Its equations of motion are that of a chiral field, i.e. $\partial_t \varphi = \pm \partial_\phi \varphi$. This is the boundary dynamics of Abelian CS theory, and we have recovered it from the on-shell evaluation of the path integral for the extended bulk + boundary action (3.6). The authors of [173, 174] have also presented a derivation of the edge mode dynamics of CS theory, but we believe that our presentation follows more closely the original construction presented in [173] for Maxwell theory. Moreover, we have shown explicitly the link between the extended action and the extended phase space.

The last step of the above calculation makes clear the role of the quadratic $j$ term introduced in (3.6). Without this term, the construction of the extended potential (3.9) would have of course gone through, but the derivation of the boundary dynamics would not have provided a desirable Hamiltonian for the chiral field after (3.18). This shows, as announced above, that the last term in (3.6) plays the role of a Hamiltonian: it does not affect the extended symplectic structure, but it changes the boundary dynamics. In this simple example of Abelian U(1) CS theory, this change of dynamics is equivalent to changing the velocity of the chiral bosons.

As a subtlety, one can observe in (3.8) that the two boundary equations of motion obtained by varying $A$, $j$, when combined, simply imply that $\mathrm{D}a = \pm{*\mathrm{D}a}$, meaning that $a$ is a gauged chiral field. This is essentially the same equation of motion as that derived from the effective boundary action (3.19). From this, one could conclude that the boundary dynamics is in a sense already encoded in the initial bulk + boundary action (3.6). This is however just a coincidence due to our choice of boundary Hamiltonian. Indeed, if we choose instead $h = (j_t \mp j_\phi)j_\phi$, it is easy to see that replacing the last two terms in (3.18) by $j_t\partial_\phi\varphi \mp (\partial_\phi\varphi)^2$ and then path integrating over $j_t$ leads once again to (3.19), while, however, the boundary equations of motion give $\mathrm{D}_t a = (\pm 2 - 1)\mathrm{D}_\phi a$. This last equation is once again that of a chiral field, but now the two chiralities have a different velocity. This is a known fact in CS theory and condensed matter, namely that the velocity is an external input which can be tuned by changing the Hamiltonian [12]. However, this example illustrates clearly the fact that there is a slight quantitative difference between the boundary equations of motion derived from the bulk + boundary action (3.6) and that derived from the on-shell evaluation of the action. For topological theories, these two views on the boundary dynamics are in a sense equivalent (at least qualitatively, as we have just seen), since on-shell bulk configurations are simply gauge transformations. For non-topological theories however, the on-shell evaluation of the action is crucial since it imprints on the boundary a left-over dynamics from the bulk (which is not just a gauge transformation). We will see with the example of Maxwell theory that the derivation of the boundary dynamics requires an on-shell evaluation of the action, and cannot be read off the initial extended action alone.

Finally, as a curiosity, and in order to make contact with previous literature on the subject, one can insert the boundary equation of motion $j = \mathrm{D}a$ back into the action to obtain

$$S = \int_M A \wedge F + \int_{\partial M} aF \pm \frac{1}{2}{*\mathrm{D}a} \wedge \mathrm{D}a, \tag{3.20}$$

which we recognize as the action for CS theory coupled to a gauged chiral field on the boundary [175]. As in [176], this constitutes the off-shell and gauge-invariant description of the boundary dynamics of CS theory, in the sense that it leads to the equations of motion of a chiral field without having to evaluate the bulk theory on-shell. However, a subtle yet important point is that variation with respect to $a$ on the boundary of (3.20) leads to the equation of motion $\mathrm{D}a = \mp{*\mathrm{D}a}$, which is the opposite chirality to what we have derived from (3.6). This is to be expected since the manipulations leading to (3.20) are different from that leading to the effective action (3.19). In particular, obtaining (3.20) does not require the on-shell evaluation of the bulk fields. As it will become clear below, it is indeed this on-shell evaluation which one should carry out in order to access the effective boundary dynamics, and this latter cannot simply be read off from the boundary equations of motion using (3.8) and (3.20).

### Gluing of subregions

Referring to figure 3.1, we have so far described the operations of splitting and of integrating. Splitting CS theory on $M\cup\overline{M}$ requires to consider for each subregion with boundary the extended actions (3.6). Integrating over the bulk gauge field of a subregion leads to a path integral over boundary fields only, and the dynamics of the boundary edge mode field $a$ is that of a chiral theory.

We can now describe the operation of gluing of two subregions, which will involve getting rid of the edge mode field contributions from the two boundaries. For two boundary theories on $\partial M$ and $\partial\overline{M}$ with opposite chirality, the gluing of $S[A,a,j]$ and $S[\overline{A},\overline{a},\overline{j}]$ is then given by

$$
\begin{aligned}
\mathcal{Z}=&\int\mathcal{D}[A,\overline{A},a,\overline{a},j,\overline{j}]\,\delta(a+\overline{a})\,\delta(j+\overline{j})\\
&\qquad\exp\left(\mathrm{i}\int_M A\wedge F+\mathrm{i}\int_{\overline{M}}\overline{A}\wedge\overline{F}+\mathrm{i}\int_{\partial M}aF+\overline{a}\overline{F}+j\wedge\mathrm{D}a+\overline{j}\wedge\overline{\mathrm{D}}\overline{a}\pm\frac{1}{2}{*j}\wedge j\mp\frac{1}{2}{*\overline{j}}\wedge\overline{j}\right)\\
=&\int\mathcal{D}[A,\overline{A},a,j]\exp\left(\mathrm{i}\int_M A\wedge F+\mathrm{i}\int_{\overline{M}}\overline{A}\wedge\overline{F}+\mathrm{i}\int_{\partial M}a(F-\overline{F})+j\wedge(A-\overline{A}+2\mathrm{d}a)\right)\\
=&\int\mathcal{D}[A,\overline{A},a]\,\delta(A-\overline{A}+2\mathrm{d}a)\big|_{\partial M}\exp\left(\mathrm{i}\int_M A\wedge F+\mathrm{i}\int_{\overline{M}}\overline{A}\wedge\overline{F}+\mathrm{i}\int_{\partial M}a(F-\overline{F})\right)\\
=&\int\mathcal{D}[A,a]\exp\left(\mathrm{i}\int_{M\cup\overline{M}}A\wedge F\right)\\
\propto&\int\mathcal{D}[A]\exp\left(\mathrm{i}\int_{M\cup\overline{M}}A\wedge F\right).
\end{aligned}
\tag{3.21}
$$

Here we have written the path integral over all the bulk and boundary fields coming from the two subregions and their boundaries, with two delta functions enforcing the identification of the edge modes coming from the two boundaries. After integrating over $\overline{a}$ and the two currents $j$ and $\overline{j}$, in the third equality we are left with a delta function on the boundary, imposing that the gauge fields incoming from the two subregions are equal up to a gauge transformation. Integrating over $\overline{A}\big|_{\partial M}$ then eliminates the last boundary integral, and we are left with the path integral for CS theory over $M\cup\overline{M}$. In the last step we have simply eliminated a redundant integration over $a$ by dropping a gauge volume factor. This gluing operation is the application to CS theory of the gluing described in [173].

### Entanglement entropy

Finally, let us conclude this section by discussing the role of the edge modes and the extended phase space in the computations of entanglement entropy. In general, the entanglement entropy $\mathcal{S}$ of a spatially bipartite system $\Sigma\cup\overline{\Sigma}$ receives contributions from two sources,

$$\mathcal{S}=\mathcal{S}_{\text{bulk}}+\mathcal{S}_{\text{edge}}. \tag{3.22}$$

The first piece, $\mathcal{S}_{\text{bulk}}$, comes from physical degrees of freedom in the bulk, while $\mathcal{S}_{\text{edge}}$ originates from degrees of freedom localized at the boundary, which for the bipartite system is the entangling surface $S=\partial\Sigma=\partial\overline{\Sigma}$ between the two subregions. CS theory being topological, it does not have physical bulk degrees of freedom, and therefore the sole contribution to its entanglement entropy comes from the boundary degrees of freedom, i.e. the edge modes. Although the computation of entanglement entropy in CS theory is already well understood and has been studied by many authors, it is still worth briefly reviewing the different computational techniques in order to emphasize the role of the edge modes. After all, this is the narrative which we are trying to build in the present paper: there is a unified treatment of the extended phase space for all gauge theories, and a Lagrangian description of the corresponding edge modes. In CS theory, it is well accepted (and even tested) that these edge modes have a dynamics and a contribution to entanglement entropy. This therefore strongly suggests that what is known about edge modes in CS theory is actually a generic feature of *any* gauge theory.

There are essentially three approaches for computing entanglement entropy in CS theory. The first one exploits the knowledge of the surface symmetry algebra, the second one uses a Hamiltonian quantization of the effective boundary action [177], and the third one the replica trick calculation [178]. We briefly mention the first approach below.

The computation of entanglement entropy from the surface symmetry follows from [55, 67, 68, 179, 180]. It relies on the extended Hilbert space construction, and on the factorization

$$\mathcal{H}_{\text{ext}}=\mathcal{H}_{\Sigma,S}\otimes\mathcal{H}_{\overline{\Sigma},S}, \tag{3.23}$$

where $\mathcal{H}_{\Sigma,S}$ denotes the extended Hilbert space on each subregion, containing edge states living on the entangling surface $S$. This extended Hilbert space, though it has the advantage of being factorized, contains in a sense two copies of the edge modes (one coming from each subregion) and therefore many non-physical states. The total Hilbert space of physical gauge-invariant states, which is a subspace of the factorized extended Hilbert space, is obtained as an entangling product

$$\mathcal{H}_{\Sigma\cup\overline{\Sigma}} = \mathcal{H}_{\Sigma,S} \otimes_{\mathcal{G}_S} \mathcal{H}_{\overline{\Sigma},S} \subset \mathcal{H}_{\text{ext}}, \tag{3.24}$$

and is spanned by gauge-invariant states $|\psi\rangle_{\text{phys}}$ satisfying the quantum gluing condition

$$\left(\mathcal{Q}[\alpha] \otimes \overline{\mathbb{I}} + \mathbb{I} \otimes \overline{\mathcal{Q}}[\alpha]\right) |\psi\rangle_{\text{phys}} = 0. \tag{3.25}$$

Here, the boundary symmetry generators (3.12) derived from the classical theory are promoted to quantum operators, and correspondingly the Poisson brackets are turned into operator commutators. For Abelian CS theory, the algebra is the U(1) Kač–Moody algebra (with the factor $\mathrm{k}/4\pi$ restored),

$$\left[\mathcal{Q}[\alpha], \mathcal{Q}[\beta]\right] = \frac{\mathrm{ik}}{2\pi} \int_S \mathrm{d}\phi \, (\alpha \partial_\phi \beta). \tag{3.26}$$

The fact that the boundary of CS theory carries a 2-dimensional chiral boson with corresponding Kač–Moody algebra allows us to use techniques in boundary conformal field theory. In terms of the mode expansions

$$\mathcal{Q}[\alpha] = \sum_{n\in\mathbb{Z}} \alpha_n \mathcal{J}_n, \qquad \alpha(\phi) = \sum_{n\in\mathbb{Z}} \alpha_n e^{\mathrm{i}n\phi}, \tag{3.27}$$

the algebra becomes

$$[\mathcal{J}_m, \mathcal{J}_n] = \mathrm{k} n \delta_{m+n,0}. \tag{3.28}$$

Identifying $\alpha_n = \overline{\alpha}_{-n}$, the gluing condition, which can now be rewritten as

$$(\mathcal{J}_n \otimes \overline{\mathbb{I}} + \mathbb{I} \otimes \overline{\mathcal{J}}_{-n})|\psi\rangle_{\text{phys}} = 0, \tag{3.29}$$

tell us that physical states are singlets under the action of left-moving and right-moving current operators on each side of the entangling surface (therefore the entanglement entropy in this case is known as left-right entanglement entropy). The gluing condition is solved by the conformally-invariant Ishibashi states [181]

$$|q\rangle\rangle = \sum_{N=0}^{\infty} \sum_{j=1}^{\dim(N)} |q, N, j\rangle \otimes \overline{|q, N, j\rangle}, \tag{3.30}$$

where the orthogonal states are labelled by a quasiparticle charge $q$, such that the choice $q = 0$ corresponds to the vacuum state, and $q \neq 0$ to states with Wilson lines with charge $q$ and $-q$ piercing through $\Sigma$ and $\overline{\Sigma}$. The quantum numbers $N, j$ label the descendants. The Ishibashi states are in general non-normalizable, and therefore need to be appropriately regularized. The regularized Ishibashi states are defined via the CFT modular Hamiltonian as

$$|q\rangle\rangle_{\text{reg}} = \frac{e^{-\epsilon H}}{\sqrt{n_q}}|q\rangle\rangle, \qquad H = \frac{2\pi}{\ell}\left(\mathcal{J}_0 + \overline{\mathcal{J}}_0 - \frac{c}{12}\right), \tag{3.31}$$

where $\epsilon$ is a cut-off parameter, $\ell$ is the length of the entangling surface $S$, and $c$ is the central charge of the corresponding CFT. With this, the generic edge states are linear combinations of the regularized Ishibashi states, and the entanglement entropy can be computed as the standard von-Neumann entropy. The result for the simplest case of a spherical hypersurface divided into two disks, $S^2 = D \cup D$, and without quasiparticles, is given by [182–184]

$$\mathcal{S}_{\text{CS}} = \frac{\mathbf{A}}{2\pi}\frac{\pi}{24\epsilon} - \frac{1}{2}\log \mathrm{k} + \mathcal{O}(\ell^{-1}). \tag{3.32}$$

The first term is the non-universal area law, with $\mathbf{A} = 2\pi\ell$, and the second term, which is area- and cut-off-independent, is the famous topological entanglement entropy of CS theory.

### 3.3.2 Maxwell theory

We now turn to the case of Maxwell theory. To construct the bulk + boundary action, in addition to the bulk gauge field $A$, let us consider on the boundary a scalar field $a$ transforming as $\delta_\alpha a = -\alpha$, and a gauge-invariant 2-form $j$. With this we can form the gauge-invariant action

$$S = -\frac{1}{2}\int_M \star F \wedge F + \int_{\partial M} j \wedge \mathrm{D}a + h, \tag{3.33}$$

where once again $\mathrm{D}a \coloneqq \mathrm{d}a + A$ and $h[j]$ is a boundary Hamiltonian depending on the current $j$ only. This simple action for Maxwell theory coupled to boundary currents is also the starting point of [173], where it is however introduced from the point of view of the gluing of Maxwell theory for two neighboring regions $M$ and $\overline{M}$ (this gluing is strictly analogous to what we have described in section 3.3.1 for CS theory). This action is also motivated in [52] by the need to couple Maxwell theory to currents (or matter fields) in order to achieve its factorizability. The introduction of the boundary edge mode fields allows one to factorize the theory between two neighboring subregions, and here we will furthermore show that these edge mode fields reproduce the extended phase space structure of [32]. In this thesis, we will set $h = 0$ and show that even in this case there is non-trivial boundary dynamics. We will leave the study of various other possibilities for $h$ and their physical interpretation for future work.

Following (3.4), the variation of this action reveals the bulk and boundary equations of motion, as well as the boundary potential. This variation is

$$\delta S = -\int_M \delta A \wedge \mathrm{d}{\star}F + \int_{\partial M} \delta A \wedge (j - {\star}F) + \delta j \wedge \mathrm{D}a - \delta a \mathrm{d}j + \mathrm{d}(j\delta a). \tag{3.34}$$

We can observe that the boundary equations of motion imposed by the variation of $A$ and $a$ on the boundary are together consistent with the bulk equations of motion. The boundary condition imposed by $\delta A$ identifies the edge mode momentum $j$ with the normal electric field ${\star}F$, i.e. states that[5] $j = {\star}F$. With this, the extended potential (3.5) becomes

$$\Theta_{\mathrm{e}} = \Theta - \mathrm{d}(j\delta a) = -\delta A \wedge {\star}F - \mathrm{d}(j\delta a) \approx -\delta A \wedge {\star}F - \mathrm{d}({\star}F\delta a), \tag{3.35}$$

which is the extended potential derived in [32]. The extended symplectic structure derived from this potential is such that for gauge transformations the generator defined by $I_{\delta_\alpha}\Omega$ is vanishing on-shell. Indeed, this is

$$I_{\delta_\alpha}\Omega = \int_\Sigma \mathrm{d}\alpha \wedge \delta({\star}F) - \int_S \alpha\delta({\star}F) = -\int_\Sigma \alpha\delta(\mathrm{d}{\star}F). \tag{3.36}$$

In addition, the transformation acting as $\Delta_\alpha A = 0$ and $\Delta_\alpha a = \alpha$ has an integrable generator given by the "electric charge"

$$\mathcal{Q}[\alpha] = \int_S \alpha{\star}F \tag{3.37}$$

smeared with an arbitrary function $\alpha$. We therefore see how the extended bulk + boundary action allows us to recover the extended phase space structure of Maxwell theory. We can now turn to the boundary dynamics.

The boundary dynamics for the edge mode field $a$ is obtained by integrating out the bulk degrees of freedom in the path integral. Since Maxwell theory is quadratic, the integration

[5] We should always keep in mind that equalities involving $j$ are pulled back to the boundary $\partial M$.

over the bulk gauge field $A$ produces a determinant of the bulk operator multiplying the path integral for the bulk + boundary action evaluated on-shell. Using the fact that the on-shell bulk Maxwell action is itself a boundary term, and that on the boundary the normal electric field gets identified with the boundary current $j$ according to the boundary equation of motion $j = \star F$, we have that the path integral for (3.33) is

$$\begin{aligned}\int \mathcal{D}[A,a,j]\exp(\mathrm{i}S) &= (\det\mathcal{O})^{-1/2}\int \mathcal{D}[a,j]\exp\left(\mathrm{i}\int_{\partial M} j\wedge \mathrm{D}a - \frac{1}{2}\star F\wedge A[j]\right)\\ &= (\det\mathcal{O})^{-1/2}\int \mathcal{D}[a,j]\exp\left(\mathrm{i}\int_{\partial M} j\wedge\left(\frac{1}{2}A[j]+\mathrm{d}a\right)\right). \qquad (3.38)\end{aligned}$$

In this expression, the quantity $A[j]$ refers to the boundary value of the gauge field obtained by solving the bulk Maxwell equations and the boundary conditions, i.e. the solution to

$$\mathrm{d}\star F = 0, \qquad \star F\big|_{\partial M} = j. \qquad (3.39)$$

These equations can either be interpreted in the form given here, i.e. as the free bulk equations of motion with specific boundary conditions, or alternatively as a bulk equations of motion that are not free but sourced by boundary currents. The equivalence between these viewpoints is explained in appendix A.3. The evaluation of $A[j]$ depends on the background spacetime geometry under consideration, but will always lead to a linear expression in $j$. The effective action on the right-hand side of (3.38) is therefore quadratic in $j$, and integrating this auxiliary current out will therefore produce a boundary action quadratic in the edge mode field $a$. This is the same construction as in [173], and we have now shown its generality by comparison with the CS construction of the previous section.

To be more concrete, we consider the case of 3-dimensional Minkowski spacetime[6] and solve the equations (3.39). In the radial gauge $A_r = 0$, the boundary condition $j = \star F$ translates into the two conditions

$$j_t = \sum_k \tilde{j}_t(k)e^{\mathrm{i}k\cdot x} = (\star F)_t = F_{r\phi} = \partial_r A_\phi, \qquad (3.40)$$

$$j_\phi = \sum_k \tilde{j}_\phi(k)e^{\mathrm{i}k\cdot x} = (\star F)_\phi = F_{tr} = -\partial_r A_t, \qquad (3.41)$$

which can be solved by writing

$$A_\phi = \sum_k \frac{\tilde{j}_t(k)}{\mathrm{i}k_r}e^{\mathrm{i}k\cdot x}, \qquad \text{and} \qquad A_t = -\sum_k \frac{\tilde{j}_\phi(k)}{\mathrm{i}k_r}e^{\mathrm{i}k\cdot x}. \qquad (3.42)$$

[6] The generalization to arbitrary dimension is of course straightforward, provided we keep track of more spacetime indices.

Noticing the switch of components between $A$ and $j$, one can see that the term $j \wedge A[j]$ in (3.38) is indeed quadratic in $j_t$ and $j_\phi$. In order to satisfy the bulk equations of motion, which in the Lorentz gauge are $\Box A_\mu = 0$, we simply need to restrict the sum over momenta $k$ to $k^2 = -k_t^2 + k_r^2 + k_\phi^2 = 0$. It is then clear that integrating (3.38) over $j$ produces a quadratic effective action for the edge mode $a$.

In appendix A.1 we give a more generic formula for this, and explain in details how $A[j]$ and the effective boundary action can be obtained in the case of 3-dimensional Maxwell theory in the radial gauge. The result of this calculation is that the effective path integral for the edge modes is

$$\mathcal{Z}_{\text{edge}} = \int \mathcal{D}[\varphi] \exp\left(\frac{\mathrm{i}}{2}\int \mathrm{d}^2k\, k^2 \tilde{\varphi}(k)\tilde{G}(k)^{-1}\tilde{\varphi}(-k)\right), \tag{3.43}$$

where $\varphi$ is simply a field-redefinition[7] of the initial edge modes $a$, and where $\tilde{G}(k)$ is the solution to (3.39). In appendix A.3, we explain how one can alternatively see the boundary conditions in (3.39) as boundary sources for the bulk equations of motion. By doing so we obtain an equivalent expression for the effective path integral for the edge modes, which is

$$\mathcal{Z}_{\text{edge}} = \int \mathcal{D}[\varphi] \exp\left(-\frac{\mathrm{i}}{2}\int_{\partial M} \mathrm{d}^2y\,\sqrt{|q|}\int_{\partial M} \mathrm{d}^2y'\,\sqrt{|q|}\,\partial^i\varphi(y)G(0,y-y')^{-1}\partial_i\varphi(y')\right). \tag{3.44}$$

This is the Maxwell analogue of the Poisson kernel integral obtained in [185] in the case of a scalar field. There, it was argued that properly splitting and sewing scalar field theory path integrals on manifolds with boundaries requires "scalar edge modes" in order to reproduce the Forman-Burghelea-Friedlander-Kappeler (FBFK) gluing formula for functional determinants[8] [186–190], and that the corresponding edge scalar partition function on each side of the boundary comes from the boundary term needed in order to have a well-defined variational principle for the bulk scalar field action. As such, this argument would be puzzling when transposed to Maxwell theory, since in Maxwell the bulk action already has a well-defined variational principle without the need to add a boundary term, and one does not see where the Poisson kernel contributions of [185] could come from. We have shown that these contributions come from the path integral of the edge modes, whose introduction is natural since in a gauge theory they are needed in order to even have a notion of splitting of the path integral in the first place.

[7] More precisely, we have $\varphi \coloneqq \alpha + a$, where $\alpha$ comes from the Hodge decomposition (3.55) of the 3-dimensional gauge field $A$. This variable $\varphi$ is therefore gauge-invariant. Alternatively, if one does not use the Hodge decomposition, the on-shell evaluation of the action in (3.38) requires as usual a gauge-fixing, and the resulting effective action depends on $a$ instead of $\varphi$. These two viewpoints are of course equivalent.

[8] We come back to this point in section 3.3.3.

### 3.3.3 Maxwell–Chern–Simons theory

In this section we study 3-dimensional MCS theory. This is equivalent to a theory of massive photons, where the so-called topological mass is provided by the CS term. This theory has a wide range of applications in condensed matter physics. Its boundary dynamics has been analyzed previously in [51, 191–193] in flat space, and in [194] in the context of AdS holography, and reveals the presence of a chiral edge mode, just like in pure CS theory. However, these references have conceptually different ways of introducing the edge degrees of freedom, so we believe it is useful to revisit MCS theory in the light of the general framework that we are presenting in this thesis. In particular, this will confirm the result of [51] concerning the contributions to the entanglement entropy, which will feature a contact term coming from the Maxwell part of the theory, but also a topological term coming from the CS part.

Introducing for convenience the topological mass $m = \mathrm{k}/(4\pi)$, where k is the coupling of CS theory, the bulk + boundary action is simply a combination of the extended actions (3.6) and (3.33), i.e.

$$S = S_M + S_{\partial M} = \int_M -\frac{1}{2}\star F \wedge F + mA \wedge F + \int_{\partial M} maF + j \wedge \mathrm{D}a + h, \tag{3.45}$$

where $h$ is a boundary Hamiltonian depending only on $j$, and which we leave unspecified for now. Following the same logic as in the previous sections, we are going to study the extended phase space of this theory, the boundary symmetries, the effective boundary dynamics, and the edge mode contribution to the entanglement entropy.

#### Extended phase space

The variation of the action is

$$\begin{aligned}\delta S = &\int_M \delta A \wedge (2mF - \mathrm{d}\star F) \\ &+ \int_{\partial M} \delta A \wedge (m\mathrm{D}a - \star F - j) + \delta j \wedge (\mathrm{D}a + \delta_j h) + \delta a(\mathrm{d}j + mF) - \mathrm{d}(j\delta a - ma\delta A),\end{aligned} \tag{3.46}$$

from which we can read the bulk equations of motion

$$\mathrm{d}\star F - 2mF = 0, \tag{3.47}$$

together with the boundary equations of motion

$$j = m\mathrm{D}a - \star F, \qquad \mathrm{D}a = -\delta_j h, \qquad \mathrm{d}j = -mF. \tag{3.48}$$

The first thing one can notice is that, similarly to the case of pure Maxwell theory, the first and last set of boundary equations of motion (i.e. the ones obtained by varying $A$ and $a$) are together consistent with the bulk equations of motion. Furthermore, acting on these bulk equations of motion with the operator $\star\mathrm{d}\star$ leads to

$$(\star\mathrm{d}\star\mathrm{d} - 4m^2)(\star F) = (\Box - 4m^2)(\star F) = 0, \tag{3.49}$$

which shows the equivalence of MCS theory with a massive scalar field. We will come back to the precise statement of this relationship below when deriving the effective dynamics.

We can now construct the extended potential following the prescription (3.5) and imposing the first boundary equation of motion in (3.48), which gives

$$\Theta_\mathrm{e} = \delta A \wedge (mA - \star F) + \mathrm{d}\big(j\delta a - ma\delta A\big) \approx \delta A \wedge (mA - \star F) + \mathrm{d}\big((m\mathrm{D}a - \star F)\delta a - ma\delta A\big). \tag{3.50}$$

This in turn leads to the extended symplectic structure

$$\Omega = \int_\Sigma \delta A \wedge \big(\delta(\star F) - m\delta A\big) + \int_S \big(m\delta(\mathrm{D}a) - \delta(\star F)\big)\delta a - m\delta a\delta A, \tag{3.51}$$

which as expected is that of Maxwell plus ($m$ times) that of CS theory. From this one can now easily check that the generators of gauge transformations obtained as $I_{\delta_\alpha}\Omega$ are indeed vanishing on-shell. For the boundary symmetries $\Delta_\alpha(A, a) = (0, -\alpha)$, one can compute $I_{\Delta_\alpha}\Omega$ to find that this quantity is integrable and has a manifestly gauge-invariant generator given by

$$\mathcal{Q}[\alpha] = \int_S \alpha(2m\mathrm{D}a - \star F). \tag{3.52}$$

Gauge-invariance of this generator is the statement that $\delta_\alpha \mathcal{Q}[\beta] = I_{\delta_\alpha} I_{\Delta_\beta}\Omega = 0$. Finally, the algebra of these boundary charges is again given by the Kač–Moody commutation relations

$$\big\{\mathcal{Q}[\alpha], \mathcal{Q}[\beta]\big\} = I_{\Delta_\alpha} I_{\Delta_\beta}\Omega = m\int_S \alpha\mathrm{d}\beta. \tag{3.53}$$

This shows that the surface symmetry algebra of MCS theory is identical to that of pure CS theory, even though in both cases the generators are different. This suggests the presence of a chiral boundary field, which we will now identify by evaluating the path integral.

### Boundary dynamics

We now focus on the effective boundary dynamics of the theory, which as in the previous sections will be obtained by integrating out the bulk degrees of freedom. For clarity we will proceed in three steps of increasing complexity, depending on the type of boundary. First, if the spacetime has no boundary, integrating out the bulk degrees of freedom will lead as expected to the path integral of a massive scalar field. Then, if the spacetime has an outer boundary, i.e. a boundary separating the bulk from an empty manifold, the bulk will give rise to the massive scalar field, and the boundary will carry a chiral field (for the specific Hamiltonian which we choose). Finally, for an entangling boundary within the spacetime, separating the bulk between two regions, the boundary will carry a chiral field but also an additional contact contribution due to the splitting of the path integral measure between the two subregions.

A convenient way to carry out these calculations is to use the temporal gauge $A_t = 0$ as well has a Hodge decomposition of the phase space variables. All the details are given in appendix A.2 and here we will only summarize the results. Forgetting for the moment about the boundary, we aim at computing the path integral for the bulk part $S_M$ of (3.45). Using the $2+1$ decomposition

$$S_M = \int_M \Pi_a \dot{A}_a - \frac{1}{4}(F_{ab})^2 - \frac{1}{2}(\Pi_a - m\varepsilon_{ab}A_b)^2 \tag{3.54}$$

together with the decomposition

$$A_a = \partial_a \alpha + \varepsilon_{ab}\partial_b \beta, \qquad\qquad \Pi_a = \partial_a \xi + \varepsilon_{ab}\partial_b \lambda, \tag{3.55}$$

it is explained in appendix A.2 that the path integral reduces to

$$\begin{aligned} \mathcal{Z}_M &= \int \mathcal{D}[A,\Pi]\delta(G)\exp(\mathrm{i}S_M) \\ &= (\det\Delta)^{1/2}\int \mathcal{D}[\beta]\exp\left(\frac{\mathrm{i}}{2}\int_M \beta(-\Delta)(\Box - 4m^2)\beta\right) \\ &= \big(\det(\Box - 4m^2)\big)^{-1/2}. \end{aligned} \tag{3.56}$$

Here $\delta(G)$ is imposing the Gauss constraint coming from the use of the temporal gauge, and the factor of $(\det\Delta)^{1/2}$ comes from three contributions: inserting the Hodge decomposition (3.55) in the phase space measure, in the Gauss constraint, and performing a Gaussian integral over $\lambda$. This is, as expected, the path integral for a massive scalar field, and it represents the contribution of the bulk degrees of freedom of MCS theory.

Now we have to discuss how this result will be affected by the presence of a boundary. The effective boundary dynamics will receive contributions from two sources: the boundary action $S_{\partial M}$ in the extended action (3.45), but also boundary terms coming from the the Hodge decomposition of the bulk action $S_M$. As shown in appendix A.2, carefully collecting all these terms and imposing the bulk and boundary Gauss constraints due to the temporal gauge, we find that the boundary contributions are given by

$$S_{\text{edge}} = \int_{\partial M} B[\beta] + m\partial_t\chi\partial_\phi\chi - j_t\left(\partial_\phi\chi - \frac{1}{2m}\partial_t\partial_\phi\beta - \partial_r\beta\right) + h, \tag{3.57}$$

where $B[\beta]$ is given in (A.44), and where we have defined the new field $\chi \coloneqq a+\alpha+\partial_t\beta/(2m)$. Guided by the fact that the boundary symmetries (3.53) are that of a chiral field, we can now choose the boundary Hamiltonian to be

$$h = \frac{1}{m}(j_t \mp j_\phi)j_\phi, \tag{3.58}$$

where $j_\phi$ is fixed by the temporal gauge to be (A.46). With this the boundary contributions become

$$S_{\text{edge}} = \int_{\partial M} B[\beta] + m\partial_t\chi\partial_\phi\chi + j_t\left(\frac{1}{m}\partial_t\partial_\phi\beta + 2\partial_r\beta\right) \mp \frac{1}{m}\left(m\partial_r\beta + \frac{1}{2}\partial_t\partial_\phi\beta + m\partial_\phi\chi\right)^2, \tag{3.59}$$

and path integrating over $j_t$ finally yields the chiral action

$$S_{\text{edge}} = m\int_{\partial M} \partial_t\chi\partial_\phi\chi \mp (\partial_\phi\chi)^2. \tag{3.60}$$

In the limit $m \to \infty$, which corresponds to isolating the CS piece of the action, we recover consistently (3.19). The result can be summarized by writing the total path integral for (3.45) as

$$\mathcal{Z} = \big(\det(\Box - 4m^2)\big)^{-1/2}\int \mathcal{D}[\chi]\exp(\mathrm{i}S_{\text{edge}}) = \mathcal{Z}_{\text{3d massive scalar}}\,\mathcal{Z}_{\text{2d chiral scalar}}. \tag{3.61}$$

This is consistent with what we have observed from the classical theory, namely that the bulk equations of motion describe a massive scalar field, and that the boundary symmetries are that of a chiral field.

It is interesting to notice that there is a global factor of $m$ in front of the effective boundary action (3.60). Naively, this suggests that taking the $m = 0$ limit of MCS theory, i.e. going back to pure Maxwell theory, leads to a vanishing effective boundary action. There are however several subtleties with this reasoning. First, one should remember that different boundary Hamiltonians were considered in the previous section for pure Maxwell theory (where we have chosen $h = 0$), and in this section for MCS theory (where we have chosen a chiral Hamiltonian). Second, the analysis of pure Maxwell theory in the previous section was done in the radial gauge, while here we have studied MCS theory in the temporal gauge. This means that the effective boundary dynamics (3.60) of MCS theory cannot be straightforwardly compared in the $m = 0$ limit with the effective boundary dynamics (3.44) of Maxwell theory. However, one can still go through the calculations of appendix A.2 with $h = 0$ and $m = 0$, which can then be compared to the results of appendix A.1. This provides the comparison between Maxwell theory in the radial and temporal gauges. As we will see below, it reveals that a crucial difference between the radial and temporal gauges is that in the latter there is a leftover determinant factor coming from the rewriting of the path integral measure, which is precisely the FBFK gluing factor identified in [51].

Let us now make a few important observations. The first one is that, due to our choice of boundary Hamiltonian, the constraint imposed by $j_t$ in (3.59), namely $2m\partial_r\beta + \partial_t\partial_\phi\beta = 0$, corresponds actually to the vanishing of the normal electric field to the boundary. Indeed, this latter quantity is given in the temporal gauge by $(\star F)_\phi = F_{tr} = \partial_t A_r = \partial_t\partial_r\alpha + \partial_t\partial_\phi\beta = 2m\partial_r\beta + \partial_t\partial_\phi\beta$, where for the last step we have used (A.37). In light of this, we can investigate further the boundary equations of motion given the choice of boundary Hamiltonian (3.58) we made. Explicitly, the first two sets of boundary equations of motion in (3.48) are

$$\begin{cases} j_t = m\mathrm{D}_t a - (\star F)_t = m\mathrm{D}_t a - F_{r\phi}, \\ j_\phi = m\mathrm{D}_\phi a - (\star F)_\phi = m\mathrm{D}_\phi a - F_{tr}, \end{cases} \qquad \begin{cases} m\mathrm{D}_t a = \pm 2j_\phi - j_t, \\ m\mathrm{D}_\phi a = j_\phi. \end{cases} \tag{3.62}$$

Combining the two equations on the last line leads to $F_{tr} = 0$, while combining the ones on the first line and using the boundary Gauss constraint (A.37) leads to

$$2m(\partial_t\chi \mp \partial_\phi\chi) + (\Box - 4m^2)\beta = 0, \tag{3.63}$$

which features the chiral and massive scalars. Second, let us point out that we can also use the decomposition (3.55) and the boundary Gauss constraint (A.37) into the boundary observable (3.52) (which we smear with a function $\epsilon$ since $\alpha$ is the notation used for the Hodge decomposition) to get

$$\mathcal{Q}[\epsilon] = \int_S \mathrm{d}\phi\, \epsilon\big(2m\mathrm{D}_\phi a - (\star F)_\phi\big) = \int_S \mathrm{d}\phi\, \epsilon(2m\mathrm{D}_\phi a - F_{tr}) = 2m \int_S \mathrm{d}\phi\, \epsilon\partial_\phi\chi. \tag{3.64}$$

This consistency check shows that the boundary chiral field is indeed the variable $\chi$ which we have identified in the computation of the effective boundary action.

The results of this section are in agreement with previous observations in the literature about the fact that adding a Maxwell term to CS theory does not change the boundary symmetry algebra nor affect the presence of a boundary chiral field [11, 192, 195]. However, this does not mean that the entanglement entropy of MCS will only receive a topological contribution from the CS term. As we are going to show, the Maxwell contact term also appears in MCS theory, although in the form of the FBFK gluing factor identified in [51].

### Entanglement entropy

In order to discuss contributions of the edge modes to entanglement entropy, we need to consider an inner boundary that separates the spacetime between two subregions. In this case we have the top of figure 3.1, and we want to integrate the bulk degrees of freedom of one subregion.

At first sight, one would think that the result of the previous subsection is enough, and that the entanglement entropy receives contributions from two sources: the massive scalar field in the bulk (i.e. the usual distillable part with its non-universal area law), and the chiral bosons representing the effective boundary theory and providing the same topological contribution as in the pure CS case. However, as pointed out in [51, 185, 190], one should acknowledge that there is a third contribution coming from the splitting of the path integral measure and the constraint between the two subregions. Indeed, as can be seen in (A.36), before integrating over the bulk fields the path integral written in terms of the Hodge decomposition and the temporal gauge is

$$\mathcal{Z}_M = \det\Delta \int \mathcal{D}[\alpha, \beta, \xi, \lambda]\delta(\tilde{G}) \exp(\mathrm{i}S_M). \tag{3.65}$$

The determinant factor can be traced back to the change of integration measure and the rewriting of the Gauss law in terms of the Hodge variables following (A.34) and (A.35). Importantly, one should recognize that this factor is not here in the radial gauge path integral computed in appendix A.1. Crucially, this determinant does not simply split between the two subregions. Instead, according to the FBFK gluing formula [186–189], we have that

$$\det\Delta_{M\cup\overline{M}} = \mathcal{K} \det\Delta_M \det\Delta_{\overline{M}}, \tag{3.66}$$

where the extra factor $\mathcal{K} := \det(K_M + K_{\overline{M}})$ features the so-called Poisson kernels $K_{M,\overline{M}}$, which can be expressed in terms of the normal derivatives of the Green functions for $\Delta$ restricted to $M$ and $\overline{M}$.

In fact, we could have expected the appearance of such a factor $\mathcal{K}$ on physical grounds. Indeed, in the previous subsection we have shown that, when using the Hodge decomposition, the Maxwell fields contribute in the form of (massive) scalars, and the CS term gives rise to a chiral boundary theory. Since the massive scalar is not a gauge theory and does not bring edge modes, it would naively seem that when using the Hodge decomposition we have lost track of some of the edge modes. This is not the case, and the factor of $\mathcal{K}$ precisely keeps track of the pure Maxwell edge modes. This is the contact term identified in [51]. We have already encountered it in the previous section when deriving the boundary dynamics of pure Maxwell theory (both with and without the Hodge decomposition in radial gauge), and here it resurfaces through our change of variables and the corresponding splitting of the Gauss constraint and path integral measure.

Putting all the ingredients together, we get that the pure bulk (i.e. glued) path integral $\mathcal{Z}_{M\cup\overline{M}}$ over $M\cup\overline{M}$ factorizes in terms of extended bulk + boundary actions (3.45) as[9]

$$\begin{aligned}\mathcal{Z}_{M\cup\overline{M}}[A,\Pi] &= \mathcal{Z}[A,\Pi,a,j] \times_{\text{glue}} \mathcal{Z}[\overline{A},\overline{\Pi},\overline{a},\overline{j}]\\ &= \mathcal{K}\,\mathcal{Z}[\alpha,\beta,\xi,\lambda,a,j] \times_{\text{glue}} \mathcal{Z}[\overline{\alpha},\overline{\beta},\overline{\xi},\overline{\lambda},\overline{a},\overline{j}]\\ &= \mathcal{K}\left(\mathcal{Z}_M[\alpha,\beta,\xi,\lambda]\,\mathcal{Z}_{\text{edge}}[\alpha,\beta,a,j]\right) \times_{\text{glue}} \left(\mathcal{Z}_M[\overline{\alpha},\overline{\beta},\overline{\xi},\overline{\lambda}]\,\mathcal{Z}_{\text{edge}}[\overline{\alpha},\overline{\beta},\overline{a},\overline{j}]\right).\end{aligned} \tag{3.67}$$

For the first equality, we have introduced the edge modes $(a, j)$ and $(\overline{a},\overline{j})$ on $\partial M = \partial\overline{M}$, together with the constraints enforcing that the two left and right path integrals glue together when integrating over the edge modes. This is the step that was described in (3.21) for CS theory. For the second equality, we have simply used the Hodge decomposition, which has produced the factor of $\mathcal{K}$, and for the third equality we have further split the Hodge decomposition into bulk and boundary actions. In the previous subsection we have seen that integrating out the bulk degrees of freedom in a subregion produces a chiral theory on its boundary. The contribution of this chiral theory has been computed in section 3.3.1. We can therefore conclude that the entanglement entropy in MCS theory receives contributions from three sources, i.e.

$$\mathcal{S}_{\text{MCS}} = \mathcal{S}_{\text{3d massive scalar}} + \mathcal{S}_{\text{2d left-right bosons}} + \log\mathcal{K}, \tag{3.68}$$

in agreement with [51].

[9]It should be noted that of course all the fields are integrated over in the path integrals. Here we have simply written the arguments of all the path integrals $\mathcal{Z}$ in (3.67) in order to keep track of which variables (i.e. the initial gauge fields, the fields of the Hodge decomposition, or the edge modes) are integrated over.

### 3.3.4 BF theory

We now discuss 3-dimensional Abelian BF theory. The generalization to arbitrary dimensions is straightforward, while the non-Abelian case is briefly discussed in appendix A.4. BF theory is an interesting case study because of its relevance for the description of topological phases of matter [196–198], which also involves the study of its entanglement entropy, and because in the non-Abelian case it describes 3-dimensional gravity in the first order formulation. Recently there has also been a lot of interest in a 2-dimensional BF theory model known as Jackiw–Teitelboim gravity (although there it appears with the non-Abelian gauge group $\mathrm{SL}(2,\mathbb{R})$) [174, 199–202]. We hope to apply our construction of the boundary dynamics to these more complicated cases in the future. General ideas on the boundary dynamics of 3-dimensional BF theory have already been formulated in [203, 204], where the authors have identified chiral boundary currents. Here we show that depending on the choice of boundary Hamiltonian it is possible to obtain a chiral or non-chiral boundary scalar field theory.

To construct the bulk + boundary action, in addition to the bulk 1-forms $A$ and $B$ we add on the boundary the 0-forms $a$ and $b$ and a current 1-form $j$, and consider

$$S = \int_M B \wedge F + \int_{\partial M} bF + j \wedge \mathrm{D}a + h. \tag{3.69}$$

The role of the new boundary field $b$ is to make the total action invariant under the so-called shift transformations

$$\delta_\phi B = \mathrm{d}\phi, \qquad \delta_\phi b = -\phi. \tag{3.70}$$

This is the edge mode field for the shift symmetry. Similarly to the cases studied above, this action produces the corner term that is needed for the extended phase space. To see this, consider the variation

$$\begin{aligned}\delta S = &\int_M \delta B \wedge F + \delta A \wedge \mathrm{d}B \\ &+ \int_{\partial M} \delta A \wedge (B + \mathrm{d}b - j) + \delta j \wedge (\mathrm{D}a + \delta_j h) + \delta a \mathrm{d}j + \delta b F - \mathrm{d}(j\delta a - b\delta A).\end{aligned} \tag{3.71}$$

Using the boundary equation of motion $j = B + \mathrm{d}b$, the extended potential becomes

$$\Theta_\mathrm{e} = \delta A \wedge B + \mathrm{d}(j\delta a - b\delta A) \approx \delta A \wedge B + \mathrm{d}\big((B + \mathrm{d}b)\delta a - b\delta A\big). \tag{3.72}$$

Now, notice that there exists an alternative boundary action, related to this one by a change of polarization, and which reads[10]

$$S' = \int_M B \wedge F + \int_{\partial M} B \wedge \mathrm{D}a + j \wedge (B + \mathrm{d}b) + h. \tag{3.73}$$

The bulk equations of motion are of course unchanged, and the on-shell variation is

$$\delta S' \approx \int_{\partial M} \delta B \wedge (\mathrm{D}a - j) + \delta j \wedge (B + \mathrm{d}b + \delta_j h) + \delta a \mathrm{d}B + \delta b \mathrm{d}j - \mathrm{d}(j\delta b + B\delta a), \tag{3.74}$$

from which one can clearly see the symmetry with (3.71). Using the boundary equation of motion $j = \mathrm{D}a$, the extended potential becomes

$$\Theta'_\mathrm{e} = \delta A \wedge B + \mathrm{d}(j\delta b + B\delta a) \approx \delta A \wedge B + \mathrm{d}(\mathrm{D}a\delta b + B\delta a) = \Theta_\mathrm{e} + \delta \mathrm{d}(b\mathrm{D}a). \tag{3.75}$$

Notice that with the introduction of the edge modes the boundary equations of motion in the extended action "reverse" the polarization. Indeed, in (3.71) instead of fixing $A$ on the boundary we use the boundary equation of motion to fix $(B, b)$ in terms of $j$. Conversely, in (3.74) instead of fixing $B$ on the boundary we impose a condition on $(A, a)$.

Since the potentials derived from the two extended actions differ by a total field variation, they lead to the same symplectic structure (although in a discretized setting the change of polarization can lead to inequivalent symplectic structures [205–207]), which is

$$\Omega = -\int_\Sigma \delta A \wedge \delta B - \int_S \delta(\mathrm{D}a)\delta b + \delta B \delta a, \tag{3.76}$$

in agreement with [34]. With this extended symplectic structure, we can then show as expected that the "Lorentz" and shift gauge generators $\delta_\alpha \lrcorner\, \Omega$ and $\delta_\phi \lrcorner\, \Omega$ vanish on-shell. In addition, we now also have boundary symmetries acting on the edge modes as

$$\Delta^\mathrm{g}_\alpha(a, b) = (\alpha, 0), \qquad \Delta^\mathrm{t}_\phi(a, b) = (0, \phi), \tag{3.77}$$

and generated by the boundary observables

$$\mathcal{Q}^\mathrm{g}[\alpha] = \int_S \alpha(B + \mathrm{d}b), \qquad \mathcal{Q}^\mathrm{t}[\phi] = \int_S \phi(A + \mathrm{d}a). \tag{3.78}$$

As expected from CS theory, these generators satisfy a $\mathrm{U}(1) \times \mathrm{U}(1)$ Kač–Moody algebra

$$\big\{\mathcal{Q}^\mathrm{g}[\alpha], \mathcal{Q}^\mathrm{g}[\beta]\big\} = 0, \qquad \big\{\mathcal{Q}^\mathrm{t}[\phi], \mathcal{Q}^\mathrm{t}[\chi]\big\} = 0, \qquad \big\{\mathcal{Q}^\mathrm{g}[\alpha], \mathcal{Q}^\mathrm{t}[\phi]\big\} = \int_S \phi \mathrm{d}\alpha. \tag{3.79}$$

[10] In higher-dimensional BF theory writing this action would require to change the form degree of $j$.

Based on this algebra of boundary symmetries, we can expect to find two chiral fields on the boundary.

Now, let us look at the effective boundary dynamics obtained by integrating out the bulk degrees of freedom. This calculation follows closely that in CS theory, since the actions are similar. In components, (3.69) becomes

$$\begin{aligned} S=\int_M & B_t(\partial_r A_\phi-\partial_\phi A_r)+A_t(\partial_r B_\phi-\partial_\phi B_r)+B_\phi\partial_t A_r-B_r\partial_t A_\phi \\ &+\int_{\partial M} A_t(j_\phi-B_\phi-\partial_\phi b)+A_\phi\partial_t b+j_\phi\partial_t a-j_t(A_\phi+\partial_\phi a)+h. \end{aligned} \tag{3.80}$$

As usual, the time components $A_t$ and $B_t$ are Lagrange multipliers enforcing the bulk Gauss and flatness constraints

$$\varepsilon^{ab}\partial_a B_b=0, \qquad \varepsilon^{ab}\partial_a A_b=0, \tag{3.81}$$

and on the boundary the relation

$$j_\phi=B_\phi+\partial_\phi b. \tag{3.82}$$

Path integrating over $A_t$ and $B_t$ allows us to go on-shell and to write $A_a=\partial_a\alpha$ and $B_a=\partial_a\beta$, and with this the extended action reduces to

$$S_{\text{edge}}=\int_{\partial M}\partial_t\varphi\partial_\phi\psi-j_t\partial_\phi\varphi+h, \tag{3.83}$$

where we have introduced the gauge-invariant scalars $\varphi\coloneqq a+\alpha$ and $\psi\coloneqq b+\beta$. Starting instead from the alternative action (3.73) we have

$$\begin{aligned} S'=\int_M & B_t(\partial_r A_\phi-\partial_\phi A_r)+A_t(\partial_r B_\phi-\partial_\phi B_r)+B_\phi\partial_t A_r-B_r\partial_t A_\phi \\ &+\int_{\partial M} B_t(j_\phi-A_\phi-\partial_\phi a)+B_\phi\partial_t a+j_\phi\partial_t b-j_t(B_\phi+\partial_\phi b)+h, \end{aligned} \tag{3.84}$$

and path integrating over $A_t$ and $B_t$ gives

$$S'_{\text{edge}}=\int_{\partial M}\partial_t\psi\partial_\phi\varphi-j_t\partial_\phi\psi+h. \tag{3.85}$$

The kinetic terms of the two effective boundary actions differ only by an integration by parts, and show that $\varphi$ and $\psi$ are canonically conjugated (with a derivative $\partial_\phi$). This means that we have the freedom of integrating out one of the fields (together with the current $j$) in order to obtain an effective boundary dynamics for the remaining one. This dynamics will depend on the choice of boundary Hamiltonian.

With the chiral Hamiltonian $h=(j_t\mp j_\phi)j_\phi$, which we have used previously in CS and MCS theory, we obtain

$$S_{\text{edge}}=\int_{\partial M}\partial_t\varphi\partial_\phi\psi-j_t(\partial_\phi\varphi-\partial_\phi\psi)\mp(\partial_\phi\psi)^2. \tag{3.86}$$

Integrating over $j_t$ then yields the chiral action

$$S_{\text{edge}}=\int_{\partial M}\partial_t\varphi\partial_\phi\varphi\mp(\partial_\phi\varphi)^2. \tag{3.87}$$

One can verify that using the alternative action (3.73) also leads to this chiral action. Alternatively, we can also use the boundary Hamiltonian $2h=\pm{*j}\wedge j=\pm(j_t^2-j_\phi^2)$. In CS theory, this has produced the chiral action (3.19). Here, integrating over $j_t$ leads to

$$S_{\text{edge}}=\int_{\partial M}\partial_t\varphi\partial_\phi\psi\mp\frac{1}{2}(\partial_\phi\varphi)^2\mp\frac{1}{2}(\partial_\phi\psi)^2, \tag{3.88}$$

from which the equations of motion obtained by varying $\varphi$ or $\psi$ are chiral for $\psi$ and $\varphi$ respectively. This is the form of the edge theory that was studied in a condensed matter context in [197] (although in four dimensions), where it has been shown that it can also be quantized using the Hamiltonian methods, and leads to a topological contribution to the entanglement entropy of $-\log \mathrm{k}$.

Going one step further, one may also integrate out one of the two chiral fields. For example, integrating out $\psi$ leads to

$$S_{\text{edge}}=\pm\frac{1}{2}\int_{\partial M}(\partial_t\varphi)^2-(\partial_\phi\varphi)^2, \tag{3.89}$$

which is now a single non-chiral scalar field.

# Chapter 4

# Electromagnetic Duality from First Order Formulation

In this chapter, we revisit the notion electromagnetic duality from the point of view of edge modes and corner symmetries, and sketch a speculative viewpoint on the magnetic charges and the centrally-extended electromagnetic charge algebra. This is done by exploiting the first order formulation of Maxwell's theory as a constrained topological BF theory.

The idea behind this proposal is the observation that electromagnetic duality swaps Maxwell's field equations $\mathrm{d}{*F} = 0$ and the Bianchi identity $\mathrm{d}F = 0$. These are second order equations, and as their names indicate the first one is an equation of motion while the second one is an identity. This therefore suggests to study the first order formulation, where instead of a single second order equation of motion one has two first order equations [208]. The first order formulation of Yang–Mills theories can be obtained from topological BF theory [209] supplemented by a potential [210, 211]. It is known that the BF theory admits two types of charges (which we could suggestively call electric and magnetic), arising from two independent gauge symmetries, and that these charges form a centrally-extended current algebra [165] (see also [34, 101] in the case of 3-dimensional gravity as a BF theory and [43, 212] in the case of first order 4-dimensional gravity). The idea behind this chapter is then to argue that Maxwell's theory could inherit its magnetic charge from BF theory, as well as the corresponding centrally-extended electromagnetic charge algebra.

The main message is that the magnetic charges of BF arise because of the topological nature of the theory, and the existence of so-called "translational" gauge symmetries. In Maxwell's theory, which is evidently not topological, this symmetry is broken. Depending on the dimensionality of spacetime, the translational symmetries can however be reducible

[213], and therefore admit zero-modes. Here we argue that in the 4-dimensional case these zero-modes can be identified with the magnetic gauge parameter of Maxwell's theory. The magnetic charges are then seen as arising from the reducible part of the broken translational symmetries of a topological theory. For $p$-form theory[1] in a $d$-dimensional spacetime, this is possible as long as $d-p > 1$, and in this case the translation zero-mode and the magnetic gauge parameter both have degree $d-p-2$. Consistently, this argument can also be applied to 3-dimensional Maxwell theory to show that it does not admit magnetic charges.

We organize this chapter as follows. In Section 4.1 we recall the study of the charges and charge algebra in the case of topological BF theories with Abelian gauge group. This includes the derivation of the electric and magnetic BF charges and of their centrally-extended charge algebra. For completeness and in order to describe dual scalar fields as well, we consider a BF theory of $p$-forms in $d$-dimensional spacetimes. In Section 4.2 we briefly introduce $p$-form theories. We then explain in Section 4.3 how $p$-form theories can be written in a first order formulation by adding a potential to the $p$-form BF theories. We apply this to 4-dimensional Maxwell theory and derive our observation concerning the origin of the magnetic charges as zero-modes of BF translations. We also apply this idea to 3-dimensional Maxwell theory, where it shows consistently that there are no magnetic charges. This is reinterpreted as the non-reducibility of the translations in 3-dimensional BF theory.

It is important to remark that in this chapter, unlike the previous one, we will *not* explicitly include extra edge mode fields to the theory phase space. This is however because we are only working with corner charges and corner symmetries of gauge theories, and their expressions agree with ones derived properly with edge mode and extended phase space formalism. One just has to keep in mind the philosophy that gauge symmetries are broken at boundaries, turning into physical corner symmetries, and the gauge parameters at the boundaries now label the corner charges.

### Setups

To set the stage, we work on a $d$-dimensional Lorentzian manifold $M$ with boundary $\partial M$. It is foliated by Cauchy slices $\Sigma$ with $(d-2)$-dimensional boundary $\partial\Sigma$. This boundary has poles, which can be understood as Wilson line singularities, surrounded by $(d-3)$-dimensional "circles" $\mathcal{C}$ providing a regularization of such singularities. This geometrical setup is depicted on Figure 4.1. The variables used throughout this chapter and their corresponding form degree are summarized in table 4.1 below.

[1]With our conventions $p$ denotes the degree of the field strength $F = \mathrm{d}A$ of the $(p-1)$-form $A$.

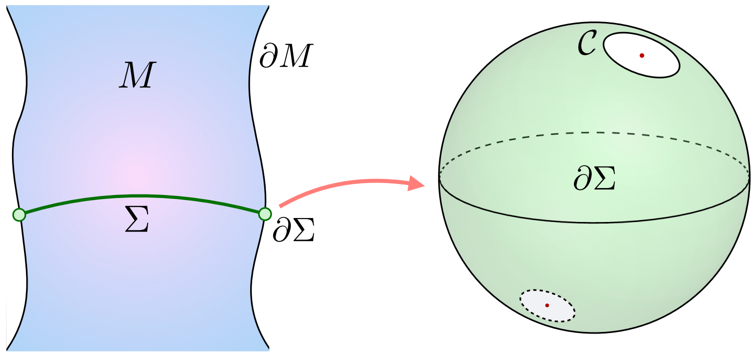


Figure 4.1: A spacetime $M$ with boundary $\partial M$ and codimension-1 Cauchy slice $\Sigma$. The boundary of this latter is $\partial\Sigma = \Sigma \cap \partial M$. In the case where $\partial\Sigma$ has poles, we can think of $\partial\Sigma \backslash \{\text{poles}\}$ as having a set $\{\mathcal{C}\}$ of codimension-3 boundaries encircling these poles. In 4 dimension, this is the usual picture of a Dirac string piercing through the north and south poles of a 2-sphere.

| variable | $A$ | $B$ | $F = \mathrm{d}A$ | $(\alpha, \beta)$ | $(\phi, \chi)$ | $\tilde{\alpha}$ |
|---|---|---|---|---|---|---|
| form degree | $p-1$ | $d-p$ | $p$ | $p-2$ | $d-p-1$ | $d-p-2$ |

Table 4.1: Variables and their associated form degree in $d$ dimension. Note that with our conventions when talking about a $p$-form theory the degree $p$ is that of the curvature $F = \mathrm{d}A$.

## 4.1 BF theories

Let us begin by reviewing the covariant phase space of Abelian BF theory in the case of $p$-forms in $d$-dimensional spacetimes, which is the general case of what we considered in Chapter 3. This is a topological field theory whose basic fields are a connection $(p-1)$-form $A$ with curvature $F = \mathrm{d}A$, and a $(d-p)$-form $B$. The Lagrangian is

$$L_{\text{BF}}[A, B] = F \wedge B, \tag{4.1}$$

where the order of wedge product has been chosen in order to minimize the amount of signs showing up below. Varying this Lagrangian gives

$$\delta L_{\text{BF}}[A, B] = F \wedge \delta B + (-1)^p \delta A \wedge \mathrm{d}B + \mathrm{d}(\delta A \wedge B), \tag{4.2}$$

which leads to the equations of motion

$$F = 0, \qquad \mathrm{d}B = 0. \tag{4.3}$$

The symplectic potential is $\Theta = \delta A \wedge B$. Given a Cauchy slice $\Sigma \subseteq M$, the corresponding symplectic structure is

$$\Omega_\Sigma = \int_\Sigma \delta\Theta = -\int_\Sigma \delta A \wedge \delta B, \tag{4.4}$$

and as usual it is independent of $\Sigma$ provided there is no symplectic flux leaking through the time-like boundary $\partial M$.

We now turn to the analysis of the Hamiltonian charges of BF theory. There are two kinds of conserved charges, associated with the two symmetries of the theory, namely the gauge symmetries and the translational symmetries (one can also use field-dependent combinations of these to describe diffeomorphisms).

**Gauge symmetry.** If $p \geq 2$, the BF Lagrangian (4.1) is invariant under the infinitesimal U(1) gauge transformations

$$\delta^{(\mathrm{g})}_\alpha A = \mathrm{d}\alpha, \qquad \delta^{(\mathrm{g})}_\alpha B = 0, \tag{4.5}$$

where $\alpha$ is a $(p-2)$-form. The Hamiltonian generator associated with this symmetry can be computed as

$$\delta\mathcal{H}^{(\mathrm{g})}[\alpha] = -I_{\delta^{(\mathrm{g})}_\alpha}\Omega_\Sigma = \int_\Sigma \mathrm{d}\alpha \wedge \delta B. \tag{4.6}$$

On-shell, the gauge charges are therefore given by

$$\mathcal{H}^{(\mathrm{g})}[\alpha] = \int_\Sigma \mathrm{d}\alpha \wedge B \approx \int_{\partial\Sigma} \alpha \wedge B. \tag{4.7}$$

These charges satisfy as expected a U(1) current algebra

$$\big\{\mathcal{H}^{(\mathrm{g})}[\alpha], \mathcal{H}^{(\mathrm{g})}[\beta]\big\} = -I_{\delta^{(\mathrm{g})}_\alpha} I_{\delta^{(\mathrm{g})}_\beta}\Omega_\Sigma = 0. \tag{4.8}$$

**Translational symmetry.** By virtue of the Bianchi identity $\mathrm{d}F = 0$, in the case $d-p \geq 1$ the BF Lagrangian (4.1) is also (quasi-)invariant under translational (or shift) symmetries, whose infinitesimal action is

$$\delta^{(\mathrm{t})}_{\phi} A = 0, \qquad \delta^{(\mathrm{t})}_{\phi} B = \mathrm{d}\phi, \tag{4.9}$$

where $\phi$ is a $(d-p-1)$-form. The Hamiltonian generator is found from

$$\delta\mathcal{H}^{(\mathrm{t})}[\phi] := -I_{\delta^{(\mathrm{t})}_{\phi}}\Omega_{\Sigma} = -\int_{\Sigma}\delta A \wedge \mathrm{d}\phi, \tag{4.10}$$

from which we get that the translational charges are

$$\mathcal{H}^{(\mathrm{t})}[\phi] = -\int_{\Sigma} A \wedge \mathrm{d}\phi \approx (-1)^{p}\int_{\partial\Sigma} A \wedge \phi. \tag{4.11}$$

These charges obey the Abelian algebra

$$\big\{\mathcal{H}^{(\mathrm{t})}[\phi], \mathcal{H}^{(\mathrm{t})}[\chi]\big\} = -I_{\delta^{(\mathrm{t})}_{\phi}} I_{\delta^{(\mathrm{t})}_{\chi}}\Omega_{\Sigma} = 0. \tag{4.12}$$

**Central extension.** The gauge and translational charges form a $\mathrm{U}(1)\times\mathrm{U}(1)$ Kač–Moody algebra, where in addition to the brackets given above we have a central term given by the mixed bracket

$$\big\{\mathcal{H}^{(\mathrm{g})}[\alpha], \mathcal{H}^{(\mathrm{t})}[\phi]\big\} = -I_{\delta^{(\mathrm{g})}_{\alpha}} I_{\delta^{(\mathrm{t})}_{\phi}}\Omega_{\Sigma} = (-1)^{p}\int_{\partial\Sigma}\mathrm{d}\alpha \wedge \phi. \tag{4.13}$$

Our goal is now to show that, when going from topological BF theory to a dynamical $p$-form theory (such as 4-dimensional Maxwell) the translational charge (4.11) can survive as a magnetic charge, which then has a centrally-extended bracket (4.13) with the electric charge.

## 4.2 $p$-form theories

We are interested in the electromagnetic duality for Abelian $p$-form theories, where $p$ is the degree of the curvature $F$. We note that asymptotic symmetries in $p$-form theories were studied in [214]. Let us first recall that the Lagrangian for such theories is

$$L_p[A] = \frac{1}{2}{*F} \wedge F, \tag{4.14}$$

where $F = \mathrm{d}A$ is again the curvature of the gauge field $A$. The equations of motion are

$$\mathrm{d}{*F} = 0, \qquad\qquad \mathrm{d}F = 0, \tag{4.15}$$

where the second equation is the Bianchi identity. The theory is invariant under the action of U(1) gauge transformations, whose finite form is $A \to A + \mathrm{d}\alpha$. The conserved charges associated with this gauge symmetry are the *electric charges*

$$\mathcal{Q}^{(\mathrm{E})}[\alpha] = \int_{\partial\Sigma} {*F} \wedge \alpha, \tag{4.16}$$

as can be worked out by computing the symplectic structure and contracting it with an infinitesimal gauge transformation.

The equations of motion (4.15) and the Lagrangian suggest that interchanging $F$ and $*F$ leaves the theory unchanged. In other words, instead of using $F = \mathrm{d}A$ we can define $*F = \mathrm{d}\tilde{A}$ and work with $\tilde{A}$. This is the duality between a $p$-form and a $(d-p)$-form theory. In the case of Maxwell theory in 4 dimensions, which is a 2-form theory, this map is the *electromagnetic duality*. In the general case of a $d$-dimensional $p$-form theory, this suggests that there must exist another type of charges, the *magnetic charges*, of the form

$$\mathcal{Q}^{(\mathrm{M})}[\tilde{\alpha}] = \int_{\partial\Sigma} \tilde{\alpha} \wedge F. \tag{4.17}$$

Since $\partial\Sigma$ is a codimension-2 manifold, a necessary condition for these magnetic charges to exist is that the form degree of the field strength $F$ be such that $p \leq (d-2)$, i.e. $d-p > 1$. This ensures that $(d-p-2)$-forms exist, so that the wedge product of $F$ with a such a $(d-p-2)$-form $\tilde{\alpha}$ produces a $(d-2)$-dimensional form which can be integrated on the codimension-2 boundary $\partial\Sigma$. This is the reason for which magnetic charges cannot exist in e.g. 3-dimensional Maxwell theory.

As it is well-known, differently than for the electric charge, the magnetic counterpart does not arise as the Noether charge of a bulk gauge transformation in the theory (4.14). It is however possible to achieve this by changing the starting theory, and working instead with the so-called dual symmetric formulation as in [108] or with an extended phase space as in [109]. Here we want to show that another understanding of these magnetic charges can be achieved from the first order formulation of the $p$-form theory, which we now present.

## 4.3 First order $p$-form theories from BF theories

The first order formulation of a $p$-form theory can be obtained as a BF theory with quadratic potential. The corresponding $d$-dimensional Lagrangian is

$$L[A,B] = F \wedge B + \frac{1}{2} {*B} \wedge B. \tag{4.18}$$

The presence of the metric in the definition of the Hodge dual breaks the topological nature of this theory. Its canonical analysis is performed in [210, 211]. To see that it indeed describes a $p$-form theory, we compute the variation

$$\delta L[A,B] = (F + {*B}) \wedge \delta B + (-1)^p \delta A \wedge \mathrm{d}B + \mathrm{d}(\delta A \wedge B), \tag{4.19}$$

which gives the first order equations of motion

$$F = -({*B}) \;\Rightarrow\; B = (-1)^{p(d-p)} {*F}, \qquad\qquad \mathrm{d}B = 0. \tag{4.20}$$

Combining these leads to the second order $p$-form Maxwell equation $\mathrm{d}{*F} = 0$. On-shell of the first equation of motion (4.20), the initial first order Lagrangian (4.18) then reduces exactly to the $p$-form Lagrangian (4.14).

Evidently, because of this on-shell equivalence, performing the analysis of the symmetries and of the charge algebra at the level of the first order $p$-form Lagrangian (4.18) cannot a priori teach us anything valuable about the magnetic charges. This Lagrangian is indeed not invariant under the translational symmetry (4.9) because of the presence of the potential term. Instead, our (admittedly non-standard) strategy will therefore be to first consider the charges and the charge algebra of BF theory alone, and only then impose in this structure the reduction to the non-topological $p$-form theory. The idea is simply to study how the reduction from BF theory to the $p$-form theory affects the charges discussed above in section 4.1. The gauge symmetry $A \to A + \mathrm{d}\alpha$ survives this reduction, since this is still a symmetry of the $p$-form theory. Using the first equation of motion in (4.20) shows that the BF gauge charge (4.7) becomes

$$\mathcal{H}^{(\mathrm{g})}[\alpha] \;\mapsto\; \mathcal{Q}^{(\mathrm{E})}[\alpha] = \int_{\partial\Sigma} {*F} \wedge \alpha, \tag{4.21}$$

which is the electric charge (4.16).

At first, the translational charges seem not to exist because, once the $B$ field is integrated out, the translation symmetry no longer survives. It is indeed evidently not a symmetry of the $p$-form theory. However, the subtlety is that the translational symmetry is reducible [213], so that one should actually think of its breaking as a constraint on the transformation parameter $\phi$. This latter must be such that

$$\mathrm{d}\phi = 0 \ \Rightarrow \begin{cases} \phi = \mathrm{d}\tilde{\alpha} & \text{when } d-p>1 \\ \phi = \text{const.} & \text{when } d-p=1 \end{cases} \tag{4.22}$$

everywhere except at poles where $\mathrm{d}^2\tilde{\alpha} \neq 0$. These restricted translations *are* symmetries of the Lagrangian (4.18). When $d-p>1$, which is precisely the condition of existence of magnetic charges as explained below (4.17), the reducible part of the translational symmetry is encoded in the $(d-p-2)$-form $\tilde{\alpha}$, which has the same form degree as the dual $*\alpha$ of the electric gauge parameter. With this identification, the translational charges become the magnetic charges as

$$\mathcal{H}^{(\mathrm{t})}[\phi] \ \mapsto \ \mathcal{Q}^{(\mathrm{M})}[\tilde{\alpha}] = (-1)^p \int_{\partial\Sigma} A \wedge \mathrm{d}\tilde{\alpha}. \tag{4.23}$$

Notice how this expression differs from the guess (4.17). To understand this difference, we should recall that when poles are present the space $\partial\Sigma$ can be seen as having boundaries by cutting out all the poles. The resulting space, $\partial\Sigma\backslash\{\text{poles}\}$, is the $(d-2)$-dimensional space with small compact boundaries $\{\mathcal{C}\}$ enclosing the poles, as in figure 4.1. This allows us to use integration by parts to obtain

$$\mathcal{Q}^{(\mathrm{M})}[\tilde{\alpha}] = \int_{\partial\Sigma\backslash\{\text{poles}\}} F \wedge \tilde{\alpha} - \sum_{\{\mathcal{C}\}} \oint_{\mathcal{C}} A \wedge \tilde{\alpha}, \tag{4.24}$$

which agrees with the magnetic charge derived in [108, 109].

The algebra of electric and magnetic charges then inherits the central extension of BF theory, and in addition to the vanishing Abelian brackets we find that (4.13) becomes

$$\big\{\mathcal{Q}^{(\mathrm{E})}[\alpha], \mathcal{Q}^{(\mathrm{M})}[\tilde{\alpha}]\big\} = (-1)^p \int_{\partial\Sigma} \mathrm{d}\alpha \wedge \mathrm{d}\tilde{\alpha} = -\sum_{\{\mathcal{C}\}} \oint_{\mathcal{C}} \mathrm{d}\alpha \wedge \tilde{\alpha}, \tag{4.25}$$

which is also in agreement with [108, 109]. It is now useful to study some explicit examples, such as Maxwell theory as well as scalar field theory and its dual.

### 4.3.1 Maxwell theory

For Maxwell theory the form degree is $p=2$. When $d=4$, we have the duality between the 2-forms $F$ and $*F$, and on the 2-sphere $\partial\Sigma = S^2$ we get the electric and magnetic charges

$$\mathcal{Q}^{(\mathrm{E})}[\alpha] = \int_{S^2} \alpha *F, \qquad \mathcal{Q}^{(\mathrm{M})}[\tilde{\alpha}] = \int_{S^2\setminus\{\mathrm{poles}\}} \tilde{\alpha} F - \sum_{\{\mathcal{C}\}} \oint_{\mathcal{C}} \tilde{\alpha} A, \tag{4.26}$$

They form the U(1) × U(1) Kač–Moody algebra

$$\left\{ \mathcal{Q}^{(\mathrm{E})}[\alpha], \mathcal{Q}^{(\mathrm{M})}[\tilde{\phi}] \right\} = -\sum_{\{\mathcal{C}\}} \oint_{\mathcal{C}} \tilde{\alpha} \mathrm{d}\alpha. \tag{4.27}$$

For other dimensions, the magnetic charges and the centrally-extended algebra exist in $p = 2$ Maxwell theory as long as $d \geq 4$, with the degree of the various fields given in table 4.1.

For $d = 3$ there is an electric charge, but since $\phi = \text{const.}$ according to (4.22) there is only a global charge

$$\mathcal{Q}^{(\mathrm{M})}_{\mathrm{global}}[\phi] = \phi \int_{S^1} A, \tag{4.28}$$

and therefore no current algebra nor central extension.

Now, recall that while 4-dimensional Maxwell theory is self-dual, 3-dimensional Maxwell theory is dual to a scalar field theory. Let us therefore study the duality and first order formulation of a $d$-dimensional scalar field.

### 4.3.2 Scalar field theory

A free massless scalar field theory in $d$ dimensions can equivalently be thought of as a 1-form theory with Lagrangian

$$L_{\mathrm{scalar}}[\Phi] = \frac{1}{2} *\mathrm{d}\Phi \wedge \mathrm{d}\Phi, \tag{4.29}$$

where the 0-form $\Phi \in \Omega^0(M)$ is the scalar field on $M$. Since this is not a gauge theory, it does not a priori admit conserved gauge charges. It is indeed obvious that the electric gauge transformations are not defined, since they require the form degree to be $p \geq 2$. As such, this theory has no electric charges (4.21).

It was however shown in [104, 215] that scalar field theories *do* admit a reformulation of the soft theorem in terms of asymptotic symmetries, and contain *scalar soft charges*.

Following the argument built from the first order formulation, these are precisely the magnetic charges (4.23), which in the scalar case become

$$\mathcal{Q}^{(\mathrm{M})}[\tilde{\alpha}] = -\int_{\partial\Sigma} \Phi \mathrm{d}\tilde{\alpha}, \tag{4.30}$$

where now $\tilde{\alpha} \in \Omega^{d-3}(M)$. This is consistent with the proposal of [105–107], which is to understand the scalar soft charges in terms of a gauge theory dual to the scalar field theory. In this reformulation, the above magnetic charges are interpreted as the electric charges of the gauge theory dual to the scalar theory (4.29). This relies on the fact that a $d$-dimensional scalar field theory is dual to a gauge theory of $(d-1)$-forms via the identification

$$F_{(d-1)} = \mathrm{d}A_{(d-2)} = *\mathrm{d}\Phi. \tag{4.31}$$

One can now apply our first order argument to this dual gauge theory, and verify by looking at the form degree that the electric charge exists while the magnetic one does not. The electric charge is given by

$$\mathcal{Q}^{(\mathrm{E})}[\alpha] = \int_{\partial\Sigma} *\big(F_{(d-1)}\big) \wedge \alpha = (-1)^d \int_{\partial\Sigma} \mathrm{d}\Phi \wedge \alpha, \tag{4.32}$$

where $\alpha \in \Omega^{d-3}(M)$. We see that the magnetic charge in the scalar field theory agrees (up to an integration by parts and a possible sign) with the electric charge in the dual gauge theory. This charge, which is either magnetic for the scalar theory or electric for its dual, is the hidden scalar soft charge.

The same conclusion is reached when using the first order BF formulation. For a 4-dimensional scalar field, the dual is a theory of 3-forms. This latter has a first order formulation where the $B$ field is a 1-form. The translational symmetry is therefore not reducible, and no zero-mode survives the reduction from BF to the 3-form theory. Consistently, we get only a single charge for the scalar field or its dual, as described above.

# Part II

# Carrollian Hydrodynamics on Stretched Horizons

# Chapter 5

# Carrollian Hydrodynamics

In the second part of the thesis, we turn our attention to the topic of gravitational physics of null boundaries and stretched horizons. Our objective is to explore and elucidate the fascinating connection between spacetime geometry near a null boundary and Carroll geometries. Inspired by the recent work of Donnay and Marteau [121], we aim to formalize their idea and thoroughly establish the correspondence between the horizon dynamics and a type of "non-relativistic" hydrodynamics, the so-called Carrollian hydrodynamics.

The entirety of this first chapter is devoted to the study of Carroll geometries and Carrollian hydrodynamic as a standalone subject. Essential elements of Carroll geometries and the characteristic of Carrollian fluids which we will carefully lay down and elaborate in this chapter will however be adopted in the next chapter when we fully discuss gravitational dynamics near a null boundary.

This chapter is structured as follows. We start in section 5.1 with the introduction of Carroll structures, which serves as the most basic building block of Carroll geometries and Carrollian physics. We will discuss Carrollian hydrodynamics in section 5.2 starting from relativistic conservation laws and then carefully consider the Carrollian limit. This closely follows the idea first explored in [124] and we formalize it using the language of Carroll structures. Finally, in section 5.3, we present a new view point on Carrollian hydrodynamics based on symmetries. We propose a new notion of symmetries, which we call near-Carrollian symmetries, that extends the usual Carroll symmetries. We will demonstrated that these symmetries are associated to the full set of Carrollian hydrodynamics.

## 5.1 Carroll Structures

Carroll geometries, which underpin the research field of Carrollian physics, are suitably studied by introducing *Carroll structures* as the starting building block. In what follows, we consider a 3–dimensional[1] Lorentzian space $H$ which can be equipped with a (weak) Carroll structure [126, 130] given by the triplet $(p, \ell, q)$. This specifically means that the space $H$ is a fiber bundle, $p : H \to S$, with a 1–dimensional fiber. The 2–dimensional base space $S$ can be chosen, for relevant physics at hand, to have a topology of a 2–sphere. We will denote local coordinates on the sphere $S$ by $\{\sigma^A\}$ and denote by $q_{AB}\mathbf{d}\sigma^A \circ \mathbf{d}\sigma^B$ a metric on $S$.

Stemming from the fiber bundle structure of the space $H$, one can naturally define the *vertical subspace*, which is denoted by $\mathbf{vert}(H)$, of the tangent space $TH$. This vertical subspace is defined to be the 1–dimensional kernel of the differential, $\mathbf{d}p : TH \to TS$, of the projection map $p$,

$$\mathbf{vert}(H) := \ker(\mathbf{d}p). \tag{5.1}$$

We will refer to a vector field that belong to $\mathbf{vert}(H)$ as being vertical. The second element of the Carroll structure is a vertical vector field $\ell \in \mathbf{vert}(H)$ which can be seen as a preferred representative of the equivalence class $[\ell]_\sim$ with the equivalence relation being rescaling that preserves the direction of $\ell$, that is $\ell \sim \mathrm{e}^\epsilon \ell$, where $\epsilon$ is any arbitrary function on the space $H$. In this regards, we will also call the Carrollian vector $\ell$ the vertical basis vector. Lastly, the third element of the Carroll structure is a null Carrollian metric $q$ whose 1–dimensional kernel coincides with the vertical subspace, inferring that $q(\ell, \cdot) = 0$. The null metric can be obtained by pulling back a metric on the sphere $S$ by the projection map $p$,

$$q = p^*(q_{AB}\mathbf{d}\sigma^A \otimes \mathbf{d}\sigma^B) = q_{AB}\boldsymbol{e}^A \otimes \boldsymbol{e}^B. \tag{5.2}$$

We have introduced the co-frame field $\boldsymbol{e}^A$[2] which is given by the pullback of the coordinate form $\mathbf{d}\sigma^A$ on the sphere $S$ by the projection map $p$,

$$\boldsymbol{e}^A := p^*(\mathbf{d}\sigma^A), \qquad \text{such that} \qquad \iota_\ell \boldsymbol{e}^A = 0. \tag{5.3}$$

Let us note that the co-frame field, by definition, is a closed form on $H$, $\mathbf{d}\boldsymbol{e}^A = 0$.

[1] In the next chapter, the space $H$ is regarded as a 3–dimensional hypersurface in a 4–dimensional ambient spacetime. We should also note that the dimension of $H$ does not affect our discussions in this part and one can easily generalize our setups to any spacetime dimension.

[2] In Part II of this thesis, we are dealing with both vectors and differential forms. To avoid confusion in notations, we will denote differential forms on spacetime by bold-face letters.

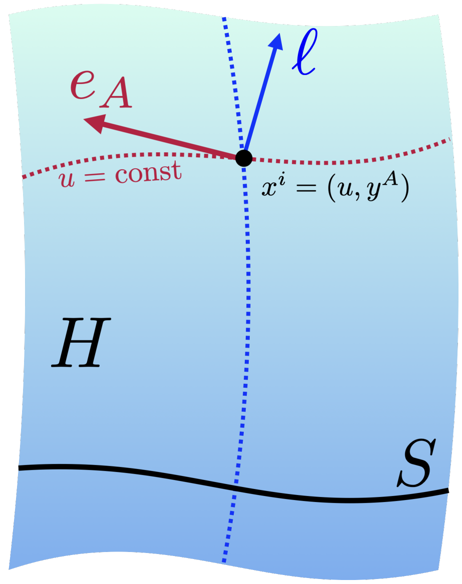


Figure 5.1: The space $H$ endowed with the Carroll structure. The general coordinates are $x^i = (u, y^A)$ where the surfaces at the cuts $u =$ constant are identified with the sphere $S$. The vertical vector $\ell$ and the horizontal vector $e_A$ span the tangent space $TH$

Provided the Carroll structure on $H$, it is then possible to have a separation of the tangent space $TH = \mathbf{vert}(H) \oplus \mathbf{hor}(H)$ into the aforementioned vertical subspace, $\mathbf{vert}(H)$, and its complement, the *horizontal subspace* denoted by $\mathbf{hor}(H)$. This splitting is achieved by introducing a connection 1–form, $\boldsymbol{k} \in T^*H$, dual to the vertical vector $\ell$,

$$\iota_\ell \boldsymbol{k} = 1. \tag{5.4}$$

The 1–form $\boldsymbol{k}$ is known as the *Ehresmann connection* in the literature. The kernel of $\boldsymbol{k}$, seen as a linear map $\boldsymbol{k} : TH \to \mathbb{R}$, thus defines the horizontal subspace. This equivalently means that

$$\mathbf{hor}(H) := \{X \in TH | \iota_X \boldsymbol{k} = 0\}. \tag{5.5}$$

We will denote a basis of the horizontal subspace by $e_A \in \mathbf{hor}(H)$ which, by definition, obeys $\iota_{e_A} \boldsymbol{k} = 0$. Furthermore, without loss of generality, we can choose these horizontal basis vectors to be ones that are dual to the co-frame field,

$$\iota_{e_A} \boldsymbol{e}^B = \delta_A^B. \tag{5.6}$$

The frame $(\ell, e_A)$ and the dual co-frame $(\boldsymbol{k}, \boldsymbol{e}^A)$ therefore serve as a complete basis for the tangent space $TH$ and the cotangent space $T^*H$, respectively (see Figure 5.1). In this basis, any vector field $X \in TH$ and any 1–forms $\boldsymbol{\omega} \in T^*H$ can therefore be uniquely decomposed as follows:

$$X = (\iota_X \boldsymbol{k})\ell + (\iota_X \boldsymbol{e}^A)e_A, \qquad \text{and} \qquad \boldsymbol{\omega} = (\iota_\ell \boldsymbol{\omega})\boldsymbol{k} + (\iota_{e_A}\boldsymbol{\omega})\boldsymbol{e}^A. \tag{5.7}$$

Similarly, a differential of a function $F$ can be expressed in this frame as

$$\mathbf{d}F = \ell[F]\boldsymbol{k} + e_A[F]\boldsymbol{e}^A. \tag{5.8}$$

Having the splitting of the tangent space $TH = \mathbf{vert}(H) \oplus \mathbf{hor}(H)$, one can naturally define the horizontal projector from the tangent space $TH$ to its horizontal components as

$$q_i{}^j := e^A{}_i e_A{}^j = \delta_i^j - k_i \ell^j, \tag{5.9}$$

and it satisfies the conditions $q_i{}^j k_j = 0$ and $\ell^i q_i{}^j = 0$.

### 5.1.1 Acceleration, Vorticity, and Expansion

Next, we introduce two important objects that are naturally inherited from the Carroll structure and will later appear when discussing Carrollian hydrodynamics [124,144]. These objects are the *Carrollian acceleration*, denoted by $\varphi_A$, and the *Carrollian vorticity*, denoted by $w_{AB}$, and they are defined to be components of the curvature of the Ehresmann connection 1–form,

$$\mathbf{d}\boldsymbol{k} := -\left(\varphi_A \boldsymbol{k} \wedge \boldsymbol{e}^A + \frac{1}{2} w_{AB} \boldsymbol{e}^A \wedge \boldsymbol{e}^B\right). \tag{5.10}$$

Let us also recall that the co-frame $\boldsymbol{e}^A$ is closed, i.e., $\mathbf{d}\boldsymbol{e}^A = 0$. We can show that the components $(\varphi_A, w_{AB})$ are also determined by the commutators of basis vector fields. This correspondence can be established by invoking the identity $[\iota_X, \mathcal{L}_Y]\boldsymbol{\omega} = \iota_{[X,Y]}\boldsymbol{\omega}$ for any vector fields $X, Y \in TH$ and any 1–form $\boldsymbol{\omega} \in T^*H$. By making use of the Cartan formula, $\mathcal{L}_X = \mathbf{d}\iota_X + \iota_X \mathbf{d}$, one can show that

$$\iota_X \iota_Y \mathbf{d}\boldsymbol{\omega} = \iota_{[X,Y]}\boldsymbol{\omega} + \mathcal{L}_Y(\iota_X \boldsymbol{\omega}) - \mathcal{L}_X(\iota_Y \boldsymbol{\omega}). \tag{5.11}$$

Using this result and the property $\mathbf{d}\boldsymbol{e}^A = 0$, we show that the commutators of the frame fields satisfy the conditions,

$$\iota_{[\ell, e_A]}\boldsymbol{e}^B = 0, \qquad \text{and} \qquad \iota_{[e_A, e_B]}\boldsymbol{e}^C = 0, \tag{5.12}$$

suggesting that both commutators $[\ell, e_A]$ and $[e_A, e_B]$ lie in the vertical subspace. Furthermore, using the definition (5.10), it follows that,

$$\varphi_A = \iota_{[\ell, e_A]}\boldsymbol{k}, \qquad \text{and} \qquad w_{AB} = \iota_{[e_A, e_B]}\boldsymbol{k}. \tag{5.13}$$

All these conditions therefore fix the commutation relations of the Carrollian frame fields,

$$\boxed{[e_A, e_B] = w_{AB}\ell, \qquad \text{and} \qquad [\ell, e_A] = \varphi_A \ell.} \tag{5.14}$$

We comment here that the Jacobi identity of the commutators determines the evolution of the Carrollian vorticity,

$$\ell[w_{AB}] = e_A[\varphi_B] - e_B[\varphi_A]. \tag{5.15}$$

It is important to appreciate that, as we have already derived, the commutator between horizontal basis vectors $[e_A, e_B]$ does not lie in the horizontal subspace $\mathbf{hor}(H)$ when the Carrollian vorticity $w_{AB}$ does not vanish. Geometrically speaking, following from the Frobenius theorem, this means that the horizontal subspace $\mathbf{hor}(H)$ is not integrable in general, meaning that it cannot be regarded as a tangent space to a 2–dimensional submanifold of $H$.

Given the metric $q_{AB}$ on the sphere $S$, we define the *expansion tensor* $\theta_{AB}$ as the change of the sphere metric along the vertical direction,

$$\theta_{AB} := \frac{1}{2}\ell[q_{AB}]. \tag{5.16}$$

The trace of the expansion tensor, called the *expansion* and denoted by $\theta$, computes the change of the are element of the sphere $S$ along the vector $\ell$,

$$\theta := q^{AB}\theta_{AB} = \ell[\ln\sqrt{q}]. \tag{5.17}$$

### 5.1.2 Horizontal Covariant Derivative

Another ingredient that is needed in order to write the Carrollian conservation laws is the notion of the horizontal covariant derivative. To this end, we introduce the Christoffel-Carroll symbols [124] defined as

$$^{(2)}\Gamma^A_{BC} := \frac{1}{2}q^{AD}\left(e_B[q_{DC}] + e_C[q_{BD}] - e_D[q_{BC}]\right). \tag{5.18}$$

It is torsion-free, ${}^{(2)}\Gamma^A_{BC} = {}^{(2)}\Gamma^A_{CB}$. We then define the *horizontal covariant derivative* (or sometimes called the Levi-Civita-Carroll covariant derivative) $\mathcal{D}_A$ which acts on a horizontal tensor $T = T^A{}_B e_A \otimes \boldsymbol{e}^B$ in the standard way,

$$\mathcal{D}_A T^B{}_C = e_A[T^B{}_C] + {}^{(2)}\Gamma^B_{DA} T^D{}_C - {}^{(2)}\Gamma^D_{CA} T^B{}_D, \tag{5.19}$$

and it can straightforwardly be generalized to a tensor of any degree. By construction, the sphere metric $q_{AB}$ is compatible with this connection, that is $\mathcal{D}_C q_{AB} = 0$.

One useful formula will be that the horizontal divergence of a horizontal vector $X = X^A e_A$ is given by

$$\mathcal{D}_A X^A = \frac{1}{\sqrt{q}} e_A \left[\sqrt{q} X^A\right]. \tag{5.20}$$

More details on this covariant derivative can be seen in Appendix B.2.

### 5.1.3 Adapted coordinates for the Carroll structure

Up until this point, we have always kept our presentation of the Carroll structure abstract and is thus completely independent of the choices of coordinates on the space $H$. We can pretty much continue this trend for the rest of this chapter. However, some physical pictures can be easily garnered when working explicitly with coordinates and, for practical purposes, some computations are conveniently carried out when expressing in coordinates. We will discuss the coordinate choices in this section.

Since the space $H$ is structured as the fiber bundle over the sphere $S$, we can, without loss of generality, choose a general coordinate system $x^i = (u, y^A)$ such that open sets of the cuts at $u =$ constant, which denoted by $S_u$, are identified with open sets of the sphere $S$ through the projection map, $S_u \to S$, which maps the coordinates $y^A$ to the coordinates on the sphere[3],

$$y^A \quad \to \quad \sigma^A = p^A(u, y^B). \tag{5.21}$$

In what follows, we will denote the Jacobian of the push-forward by $J : TS_u \to TS$, and it is explicitly given in coordinates by $J_A{}^B = \partial_A p^B$, where we have used the notation

[3]More rigorously, $p^A$ is a transition map, $p^A := (\sigma \circ p \circ x^{-1}(u, y))^A$, where $x : H \to \mathbb{R}^{D-1}$ and $\sigma : S \to \mathbb{R}^{D-2}$ provide, respectively, local coordinates on $H$ and $S$.

$\partial_A := \frac{\partial}{\partial y^A}$. In this general coordinate system, the Carroll structure is then characterized by a scale factor $\alpha$ and a velocity field $V^A$ such that

$$\ell = e^{-\alpha} D_u, \qquad \text{and} \qquad \boldsymbol{e}^A = (\mathbf{d}y^B - V^B \mathbf{d}u) J_B{}^A, \tag{5.22}$$

where we defined $D_u := (\partial_u + V^A \partial_A)$. Following from the definition of the co-frame field $\boldsymbol{e}^A := p^*(\mathbf{d}\sigma^A)$, the velocity field $V^A$ can be expressed in terms of the projection map as

$$V^A = -\partial_u p^B (J^{-1})_B{}^A, \tag{5.23}$$

where we introduced the matrix $J^{-1}$ to be the inverse of the Jacobian such that $J_A{}^C (J^{-1})_C{}^B = (J^{-1})_A{}^C J_C{}^B = \delta_A^B$. Let us also remark here that a change of the scale factor $\alpha$ preserves the Carroll structure while a variation of the velocity field $V^A$ changes the Carroll structure. It follows from the definition of the Jacobian that

$$\partial_B J_C{}^A = \partial_C J_B{}^A. \tag{5.24}$$

In addition, the property $\mathbf{d}\boldsymbol{e}^A = 0$ imposes the following constraint on the Carrollian velocity and the Jacobian,

$$D_u J_B{}^A = -(\partial_B V^C) J_C{}^A, \qquad \text{and} \qquad D_u (J^{-1})_B{}^A = (J^{-1})_B{}^C \partial_C V^A. \tag{5.25}$$

The Ehresmann connection, obeying the condition $\iota_\ell \boldsymbol{k} = 1$, is characterized by the *Carrollian connection density*, $\beta_A$, and it can be parameterized as

$$\boldsymbol{k} = \mathrm{e}^{\alpha} (\mathbf{d}u - \beta_A \boldsymbol{e}^A). \tag{5.26}$$

The choice of the Ehresmann connection also fixes the form of the horizontal basis vectors $e_A$ by the conditions, $\iota_{e_A} \boldsymbol{k} = 0$ and also $\iota_{e_A} \boldsymbol{e}^B = \delta_A^B$. In our parameterization, the horizontal basis is given by

$$e_A = (J^{-1})_A{}^B \partial_B + \beta_A D_u. \tag{5.27}$$

In this general coordinate system, we can evaluate the Carrollian commutators and in turn obtain the coordinate expression of the Carrollian acceleration $\varphi_A$ and the Carrollian vorticity $w_{AB}$ (see Appendix B.1). They are given by

$$\varphi_A = D_u \beta_A + e_A[\alpha], \tag{5.28}$$

$$w_{AB} = \mathrm{e}^{\alpha} \left( e_A[\beta_B] - e_B[\beta_A] \right). \tag{5.29}$$

In this thesis, we will always work with the general coordinates $x^i = (u, y^A)$ on the space $H$ as they are, by construction, independent of the Carroll structure. Let us, however, mention that we can also choose to work with the *adapted coordinates* $(u, \sigma^A)$ on $H$ which are such that the action of the projection is trivial, $p : (u, \sigma) \to \sigma$. With this choice, the coordinate $u$ is regarded as the fiber coordinate. By definition, the velocity field $V^A = 0$ vanishes in the adapted coordinates. These coordinates are therefore co-moving coordinates, which are such that

$$\ell = \mathrm{e}^{-\alpha}\partial_u, \qquad \text{and} \qquad \boldsymbol{e}^A = \mathbf{d}\sigma^A. \tag{5.30}$$

To connect with the previous parameterization, one can derive, given the coordinates $y^A(u, \sigma)$, the following relations

$$V^A = \frac{\partial y^A}{\partial u}, \qquad \text{and} \qquad (J^{-1})_A{}^B = \frac{\partial y^B}{\partial \sigma^A}. \tag{5.31}$$

The Ehresman connection in the adapted coordinates therefore reads

$$\boldsymbol{k} = \mathrm{e}^{\alpha}\left(\mathbf{d}u - \beta_A \mathbf{d}\sigma^A\right). \tag{5.32}$$

The expressions for the the Carrollian acceleration and the Carrollian vorticity simplifies in the co-moving coordinates becomes

$$\varphi_A = \left(\frac{\partial}{\partial \sigma^A} + \beta_A\right)\alpha + \partial_u \beta_A, \tag{5.33}$$

$$w_{AB} = \mathrm{e}^{\alpha}\left(\left(\frac{\partial}{\partial \sigma^A} + \beta_A\right)\beta_B - \left(\frac{\partial}{\partial \sigma^B} + \beta_B\right)\beta_A\right). \tag{5.34}$$

The co-moving coordinates have been widely adopted in the Carrollian literature (see for example [126, 130, 144]) as the apparent absence of the velocity field and the Jacobian factor heavily simplifies all computations. Also, this choice of coordinates works well when considering field variations that leave the Carroll structure unchanged. We will, however, be more general by considering the set of variations that can change the Carroll structure, and will therefore work with the general, field-independent, coordinates $x^i = (u, y^A)$.

### 5.1.4 Carrollian transformations

We will conclude our geometrical setup on Carroll structures by discussing Carrollian diffeomorphism. There are two types of diffeomorphism of the space $H$ — one that preserves the Carroll structure and one that changes it. Here we will focus on the former case and we will discuss the latter case when considering hydrodynamics in the next section.

Transformations that preserve the Carroll structure $(p, \ell, q)$, which has been particularly referred to as *Carrollian transformations* or Carrollian diffeomorphism are such that

$$u \to u'(u, \sigma^A), \qquad \text{and} \qquad \sigma^A \to \sigma'^A(\sigma^B). \tag{5.35}$$

As a consequence, the co-frame field $\boldsymbol{e}^A$ can only change by the diffeomorphism on the sphere $S$, inferring that the Carrollian vector $\ell$ can only change by rescaling, $\delta^{\text{Carr}}\ell \propto \ell$. In other words, the new Carrollian vector still belongs to the equivalence class $[\ell]_\sim$. This therefore means that the velocity field is unchanged under Carrollian transformations,

$$\delta^{\text{Carr}} V^A = 0. \tag{5.36}$$

We now compute how the components $(\alpha, \beta_A, q_{AB})$ of the Carroll structure change under infinitesimal Carrollian diffeomorphism generated by a vector field

$$\xi = \tau\ell + Y^A e_A, \tag{5.37}$$

where $\tau$ and $Y^A$ are, in principle, generic functions on the space $H$. It follows from

$$\delta_\xi \ell = \mathcal{L}_\xi \ell = [\xi, \ell] = -\left(\ell[\tau] + Y^A\varphi_A\right)\ell - \ell[Y^A]e_A, \tag{5.38}$$

that Carrollian diffeomorphism enforces the condition

$$\ell[Y^A] = 0. \tag{5.39}$$

Since $\delta^{\text{Carr}}\ell = -(\delta^{\text{Carr}}\alpha)\ell$, we thus obtain

$$\delta^{\text{Carr}}_{(\tau,Y)}\alpha = \ell[\tau] + Y^A\varphi_A. \tag{5.40}$$

For the Carrollian connection density $\beta_A$, we use that $\delta^{\text{Carr}}_{(\tau,Y)}\boldsymbol{k} = \mathcal{L}_\xi\boldsymbol{k}$ to read off the transformation of $\beta_A$, which is

$$-\mathrm{e}^{\alpha}\eth^{\text{Carr}}_{(\tau,Y)}\beta_A = (e_A - \varphi_A)[\tau] + w_{AB}Y^B, \tag{5.41}$$

where we defined the variation $\eth^{\text{Carr}}\beta_A := (J^{-1})_A{}^B\delta^{\text{Carr}}(J_B{}^C\beta_C)$. Lastly, we use that $\delta^{\text{Carr}}_{(\tau,Y)}q = \mathcal{L}_\xi q$ and the property $\ell[Y^A] = 0$ for Carrollian diffeomorphism to show that the sphere metric $q_{AB}$ transforms as

$$\eth^{\text{Carr}}_{(\tau,Y)}q_{AB} = 2\left(\tau\theta_{AB} + D_{(A}Y_{B)}\right), \tag{5.42}$$

where $\eth^{\text{Carr}}q_{AB} := (J^{-1})_A{}^C(J^{-1})_B{}^D\delta^{\text{Carr}}(J_C{}^E J_D{}^F q_{EF})$. Let us also note that one can consider Carrollian isometries such that $\mathcal{L}_\xi q = 0$ or conformal Carrollian isometries such that $\mathcal{L}_\xi q = \Omega q$, for a conformal factor $\Omega$. In such cases, we will have more constraints on the transformation parameters $(\tau, Y)$ (see for instance the discussions in [121, 128, 130, 144]).

## 5.2 Carrollian Hydrodynamics

Having formally established essential elements of the Carroll structure, we proceed to the discussion of hydrodynamics and its ultra-relativistic cousin, namely the *Carrollian hydrodynamics*. It has been well established fact that Galilean fluids can be derived by taking the non-relativistic limit, $c \to \infty$, of the general relativistic energy-momentum tensor $T^{ij}$ and their corresponding dynamics are therefore controlled by the non-relativistic version of the conservation laws, $\nabla_j T_i^{\ j} = 0$. The equations governing the 'Galilean' time evolution of the fluid are the continuity equation, energy conservation equation, and the Navier-Stokes equations. In a much similar spirit, taking the Carrollian limit, $c \to 0$, leads to a new, and peculiar, kind of fluid and their corresponding hydrodynamic equations that are Carrollian-covariant [124]. In this section, we will present how the Carrollian hydrodynamic equations can be obtained from the $c \to 0$ contraction of the relativistic conservation laws.

### 5.2.1 Metric on $H$

Until this stage, all introduced elements of the geometry of the space $H$ have been constructed from the Carroll structure which relied on the concept of fiber bundle. In order to discuss the conservation equations of the fluid energy-momentum tensor, $\nabla_j T_i^{\ j} = 0$, the space $H$ needs to be equipped with an additional structure: a 3–dimensional Lorentzian metric $h = h_{ij}\mathbf{d}x^i \otimes \mathbf{d}x^j$ and the Levi-Civita connection $\nabla$ compatible with it[4].

We are considering a family of Lorentzian matrices whose elements are labelled by a single real parameter, the *speed of light*[5] $c$. We also assume that a metric $h$, that belongs to this family, is constructed entirely from the data of the Carroll geometry we have already defined in the previous section. By doing so, we can ensure that the chosen metric is covariant under Carrollian diffeomorphism. We further make the following assumptions on

[4]In this thesis, we use the symbol $\nabla$ to denote the Levi-Civita connection compatible with the Lorentzian metric of the space under consideration. In the next chapter, we will also use the same symbol for the connection compatible with the 4–dimensional metric $g$. Be noted that they are different.

[5]In practice, it is the square of the speed of light, $c^2$, that will enter the metric.

the components of the metric[6],

$$h(\ell, \ell) = -c^2, \qquad h(\ell, e_A) = 0, \qquad \text{and} \qquad h(e_A, e_B) = q_{AB}. \tag{5.43}$$

These conditions infer that, when taking the limit $c \to 0$, the resulting metric on $H$ coincides with the null Carrollian metric, i.e., $h \overset{c\to 0}{=} q$. Observe that the Carrollian vector $\ell$ is timelike in general and becomes null in the Carrollian limit, $h(\ell, \ell) \overset{c\to 0}{=} 0$. The metric $h$ and its inverse $h^{-1}$ are given in the Carrollian basis by[7].

$$h = -c^2 \boldsymbol{k} \circ \boldsymbol{k} + q_{AB} \boldsymbol{e}^A \circ \boldsymbol{e}^B, \qquad \text{and} \qquad h^{-1} = -c^{-2} \ell \circ \ell + q^{AB} e_A \circ e_B. \tag{5.44}$$

The inverse metric is thus singular in the Carrollian limit $c \to 0$. This particular form of the metric is known as the *Randers-Papapetrou metric* and it has been utilized extensively in Carrollian physics literatures [124, 144–146]. Also, having the metric $h$, one can derive the relations between the basis vectors and 1–forms, which are

$$\boldsymbol{k} = -\frac{1}{c^2} h(\ell, \cdot), \qquad \text{and} \qquad \boldsymbol{e}^A = q^{AB} h(e_B, \cdot). \tag{5.45}$$

It is important to appreciate that the metric (5.44) can be viewed as the expansion in the small parameter $c^2$ around the Carrollian point, $c^2 = 0$. With this in mind, we will also make another assumption that the sphere metric $q_{AB}$ admits the expansion in the small parameter $c^2$ such that

$$q_{AB} = \mathring{q}_{AB} + 2c^2 \lambda_{AB} + \mathcal{O}(c^4), \qquad \text{and} \qquad q^{AB} = \mathring{q}^{AB} - 2c^2 \lambda^{AB} + \mathcal{O}(c^4), \tag{5.46}$$

where $\mathring{q}^{AB}$ is the inverse of $\mathring{q}_{AB}$ and we defined $\lambda^{AB} := \mathring{q}^{AC} \mathring{q}^{BD} \lambda_{CD}$ and $\lambda := \mathring{q}^{AB} \lambda_{AB}$. At first glance, doing this expansion may seem like we have introduced unnecessary complications to the problem. We will later demonstrate that this expansion is necessary to derive the hydrodynamic conservation equations from symmetries. Note also that, to properly manipulate the $c^2$–expansion, we will use the leading-order sphere metric $\mathring{q}_{AB}$ and its inverse $\mathring{q}^{AB}$ to lower and raise indices of horizontal tensors.

[6]The second condition $h(\ell, e_A) = 0$, in fact, can be relaxed by choosing $h(\ell, e_A) = c^2 \mathrm{e}^{\alpha} B_A$ for an arbitrary function $B_A$. The choice of $B_A$ is gauge as one can always absorb $B_A$ into the definition of the horizontal basis $e_A$, and correspondingly redefine the Ehresmann connection $\boldsymbol{k}$ and the sphere metric $q_{AB}$, by shifting the Carrollian connection $\beta_A \to \beta_A + B_A$. This new basis $e'_A = e_A + B_A D_u$ then satisfies the second condition $h(\ell, e'_A) = 0$.

[7]We use $\circ$ to denote the symmetric tensor product of tensors, i.e., $A \circ B = \frac{1}{2}(A \otimes B + B \otimes A)$

Since we now have the $c^2$–expansion of the metric, some objects will also inherit this similar expansion. The obvious ones are the expansion tensor and its trace, which exhibit the following expansion

$$\theta_{AB} = \mathring{\theta}_{AB} + c^2 \ell[\lambda_{AB}] + \mathcal{O}(c^4), \qquad \text{and} \qquad \theta = \mathring{\theta} + c^2 \ell[\lambda] + \mathcal{O}(c^4), \tag{5.47}$$

where the zeroth-order terms are

$$\theta_{AB} = \frac{1}{2}\ell\left[\mathring{q}_{AB}\right], \qquad \text{and} \qquad \theta = \mathring{q}^{AB}\mathring{\theta}_{AB} = \ell\left[\ln\sqrt{\mathring{q}}\right]. \tag{5.48}$$

Another object that will admits the $c^2$–expansion is the Christoffel-Carroll synbols ${}^{(2)}\Gamma^{A}_{BC}$, and we present its expansion in Appendix B.2.

In order to do integration on the space $H$, we need the volume form on $H$. We define the volume form as

$$\boldsymbol{\epsilon}_H := \frac{1}{2}\left(\varepsilon_{AB}\sqrt{q}\right)\boldsymbol{k} \wedge \boldsymbol{e}^A \wedge \boldsymbol{e}^B, \tag{5.49}$$

where $\varepsilon_{AB}$ is the standard Levi-Civita symbol (satisfying $\varepsilon_{AC}\varepsilon^{CB} = \delta_A^B$). Denoting by $\boldsymbol{\epsilon}_S$ the canonical volume form on the sphere $S$, we then have the relation

$$\iota_\ell \boldsymbol{\epsilon}_H = p^*(\boldsymbol{\epsilon}_S), \qquad \text{where we have} \qquad \boldsymbol{\epsilon}_S = \frac{1}{2}\left(\varepsilon_{AB}\sqrt{q}\right)\mathbf{d}\sigma^A \wedge \mathbf{d}\sigma^B. \tag{5.50}$$

As before, using that $\sqrt{q} = \sqrt{\mathring{q}}(1 + c^2\lambda) + \mathcal{O}(c^4)$, we thus obtain the $c^2$–expansion of the volume form,

$$\boldsymbol{\epsilon}_H = \left(1 + c^2\lambda\right)\mathring{\boldsymbol{\epsilon}}_H + \mathcal{O}(c^4), \qquad \text{where} \qquad \mathring{\boldsymbol{\epsilon}}_H = \frac{1}{2}\left(\varepsilon_{AB}\sqrt{\mathring{q}}\right)\boldsymbol{k} \wedge \boldsymbol{e}^A \wedge \boldsymbol{e}^B. \tag{5.51}$$

In the following, we will also use the notation $\mathring{\boldsymbol{\epsilon}}_S$ for the zeroth-order of the volume form on the sphere $S$. One can verify the following relations

$$\mathbf{d}(f\boldsymbol{\epsilon}_S) = \left(\ell[f] + \theta f\right)\boldsymbol{\epsilon}_H, \qquad \text{and} \qquad \mathbf{d}\left(X^A \iota_{e_A}\boldsymbol{\epsilon}_H\right) = \left(\mathcal{D}_A X^A + \varphi_A X^A\right)\boldsymbol{\epsilon}_H, \tag{5.52}$$

for a function $f$ on $H$ and for a horizontal vector $X^A e_A \in \mathbf{hor}(H)$.

One can imagine the space $H$ to have a boundary $\partial H$ situated at a certain constant value of the coordinate $u$. This boundary, in our construction, is identified under the projection map with the sphere $S$. In this setup, the Stokes theorem is written as

$$\int_H \left(\ell[f] + \mathring{\theta} f\right)\mathring{\boldsymbol{\epsilon}}_H = \int_S f\mathring{\boldsymbol{\epsilon}}_S, \tag{5.53a}$$

$$\int_H \left(\mathring{\mathcal{D}}_A X^A + \varphi_A X^A\right)\mathring{\boldsymbol{\epsilon}}_H = \int_S \mathrm{e}^{\alpha} X^A \beta_A \mathring{\boldsymbol{\epsilon}}_S. \tag{5.53b}$$

### 5.2.2 Covariant derivative

Before considering Carrollian hydrodynamics, let us compute the covariant derivative the basis vector fields, namely $\nabla_\ell \ell, \nabla_{e_A}\ell, \nabla_\ell e_A$, and $\nabla_{e_A} e_B$, as they will become handy tools when evaluating the hydrodynamic conservation equations. We will start with the covariant derivative $\nabla_\ell \ell$, which presenting the computation in full detail here. Complete derivations of the others, which are done in a similar vein, are provided for the reader in Appendix B.3. The term $\nabla_\ell \ell$, can be decomposed as

$$\nabla_\ell \ell = (k_i \nabla_\ell \ell^i)\ell + (q^{AB} e_{Bi} \nabla_\ell \ell^i) e_A. \tag{5.54}$$

Using the metric $h$ and the Leibniz rule, one can show that the vertical component vanishes as follows:

$$k_i \nabla_\ell \ell^i = -\frac{1}{c^2} h\left(\ell, \nabla_\ell \ell\right) = -\frac{1}{2c^2}\ell\left[h\left(\ell,\ell\right)\right] = 0, \tag{5.55}$$

as $h(\ell,\ell) = -c^2$ is constant. The horizontal components can be evaluated with the help of the commutation relations (5.14) as follows:

$$\begin{aligned} e_{Bi}\nabla_\ell \ell^i = h\left(e_B, \nabla_\ell \ell\right) &= -h\left(\ell, \nabla_\ell e_B\right) \\ &= -h\left(\ell, [\ell, e_B]\right) - \frac{1}{2} e_B[h\left(\ell,\ell\right)] \\ &= c^2 \varphi_B. \end{aligned} \tag{5.56}$$

Therefore, the covariant derivative of the vertical vector field along itself is given by

$$\nabla_\ell \ell = c^2 \varphi^A e_A + \mathcal{O}(c^4). \tag{5.57}$$

Observe that it vanishes in the Carrollian limit $c^2 \to 0$, dictating that the vector $\ell$ is the null generator of null geodesics on the space $H$.

The covariant derivative of the vertical vector along the horizontal vectors can be computed using the same technique. One can show that (see Appendix B.3) it is given by

$$\nabla_{e_A}\ell = \left(\mathring{\theta}_A{}^B + c^2\left(\frac{1}{2}w_A{}^B + \mathring{q}^{BC}\ell[\lambda_{AC}] - 2\lambda^{BC}\mathring{\theta}_{AC}\right)\right) e_B + \mathcal{O}(c^4). \tag{5.58}$$

The covariant derivative of the horizontal basis along the vertical basis, $\nabla_\ell e_A$, is already determined from $\nabla_{e_A}\ell$ and the commutator $[\ell, e_A]$. We are left with the remaining covariant derivative, $\nabla_{e_A} e_B$. Its vertical component, $k_i \nabla_{e_A} e_B{}^i$ can be inferred from $\nabla_{e_A}\ell$. For the horizontal components, $e^C{}_i \nabla_{e_A} e_B{}^i$, using that $q_{AB} = h(e_A, e_B)$ and the definition of the Christoffel-Carroll symbols (5.18), we can show that

$$\begin{aligned}\nabla_{e_A} e_B = & \left(\frac{1}{c^2}\mathring{\theta}_{AB} + \left(\frac{1}{2}w_{AB} + \ell[\lambda_{AB}]\right)\right)\ell + {}^{(2)}\mathring{\Gamma}^C_{AB} e_C \\ & + c^2\left(\mathcal{D}_A \lambda_B{}^C + \mathcal{D}_B \lambda_A{}^C - D^C \lambda_{AB}\right) e_C.\end{aligned} \tag{5.59}$$

With all these results, one can calculate the spacetime divergence of the basis vectors. Using the decomposition (5.9), we obtain

$$\nabla_i \ell^i = \delta_i{}^j \nabla_j \ell^i = \left(k_i \ell^j + e^B{}_i e_B{}^j\right) \nabla_j \ell^i = \mathring{\theta} + c^2 \ell[\lambda], \tag{5.60}$$

and in a similar manner,

$$\nabla_i e_A{}^i = \delta_i^j \nabla_j e_A{}^i = \left(k_i \ell^j + e^B{}_i e_B{}^j\right) \nabla_j e_A{}^i = \varphi_A + {}^{(2)}\mathring{\Gamma}^B_{AB} + c^2 e_A[\lambda]. \tag{5.61}$$

It is important to remark that the 3-dimensional metric compatible connection $\nabla_i$ diverges when taking the Carrollian limit $c \to 0$. This is to be expected since the metric (5.44) diverges in this special limit. This suggests that practical computations have to be carried out at finite $c$ and the Carrollian limit needs to be taken at the last step. In the next Chapter, when we embed the space $H$ into a higher-dimensional spacetime, there exists another notion of connection, the so-called rigged connection, which exhibits a regular limit.

### 5.2.3 Carrollian Hydrodynamics

Armed with all these tools, we are ready to discuss the hydrodynamics of Carrollian fluid. Let us start from the general form of relativistic energy-momentum tensors,

$$T^{ij} = (\mathcal{E} + \mathcal{P})\frac{\ell^i \ell^j}{c^2} + \mathcal{P} h^{ij} + \frac{q^i \ell^j}{c^2} + \frac{q^j \ell^i}{c^2} + \tau^{ij}, \tag{5.62}$$

where we chose the vertical vector $\ell$ to be the fluid velocity. The variables appeared in the fluid energy-momentum tensor consist of the fluid internal energy density $\mathcal{E}$, the fluid pressure $\mathcal{P}$, the heat current $q^i$, and the viscous stress tensor $\tau^{ij}$, which is symmetric and traceless. The latter two quantities represent dissipative effects of the fluid and, by construction, they obey the orthogonality conditions with the fluid velocity, $q_i \ell^i = 0$ and

$\tau_{ij}\ell^j = 0$. This means that, in light of Carrollian geometry we have introduced, these dissipative tensors are horizontal tensors,

$$q^i = q^A e_A{}^i, \qquad \text{and} \qquad \tau^{ij} = \tau^{AB} e_A{}^i e_B{}^j. \tag{5.63}$$

We are interested in the mixed indices version of the fluid energy-momentum tensor. Using the metric (5.44), it is given by

$$T_i{}^j = -\left(\mathcal{E}\ell^j + q^A e_A{}^j\right) k_i + \left(\frac{1}{c^2} q_{AB} q^B \ell^j + \left(q_{AC}\tau^{CB} + \mathcal{P}\delta_A^B\right) e_B{}^j\right) e^A{}_i. \tag{5.64}$$

Furthermore, we choose the following $c^2$–dependence [124, 145, 216] of the dissipative tensors,

$$q^A = \mathcal{J}^A + c^2\left(\pi^A - 2\lambda^A{}_B \mathcal{J}^B\right), \qquad \tau^{AB} = \frac{\Sigma^{AB}}{c^2} + \mathcal{S}^{AB}. \tag{5.65}$$

Note also that $q_{AB}q^B = \mathcal{J}_A + c^2\pi_A + \mathcal{O}(c^4)$. Following from this parameterization, the fluid energy-momentum tensor can be expressed as the expansion in $c^2$ as

$$T_i{}^j = \frac{1}{c^2} T_{(-1)i}{}^j + T_{(0)i}{}^j + \mathcal{O}(c^2), \tag{5.66}$$

where each term reads

$$T_{(-1)i}{}^j = \left(\mathcal{J}_A\ell^j + \Sigma_A{}^B e_B{}^j\right) e^A{}_i \tag{5.67a}$$

$$T_{(0)i}{}^j = -\left(\mathcal{E}\ell^j + \mathcal{J}^A e_A{}^j\right) k_i + \left(\pi_A\ell^j + \left(\mathcal{K}_A{}^B + \mathcal{P}\delta_A{}^B\right) e_B{}^j\right) e^A{}_i, \tag{5.67b}$$

and we defined for convenience the combination,

$$\mathcal{K}_A{}^B := \mathcal{S}_A{}^B + 2\lambda_{AC}\Sigma^{CB}. \tag{5.68}$$

The dynamics of the relativistic fluid is governed by the relativistic conservation laws, $\nabla_j T_i{}^j$. Let us first evaluate the vertical component of the conservation equations. With all the tools we derived previously, we show that

$$\begin{aligned}
\ell^i\nabla_j T_i{}^j &= \nabla_j\left(\ell^i T_i{}^j\right) - T_i{}^j\nabla_j\ell^i \\
&= -\nabla_j\left(\mathcal{E}\ell^j + q^A e_A{}^j\right) - \frac{1}{c^2} q^A\left(e_{Ai}\nabla_\ell \ell^i\right) - \left(\tau^{AB} + \mathcal{P}q^{AB}\right)\left(e_{Ai}\nabla_{e_B}\ell^i\right) \\
&= -(\ell+\theta)[\mathcal{E}] - \mathcal{P}\theta - (\mathcal{D}_A + 2\varphi_A)q^A - \tau^{AB}\theta_{AB} \\
&= \frac{1}{c^2}\mathbb{C} + \mathbb{E} + \mathcal{O}(c^2),
\end{aligned} \tag{5.69}$$

where the coefficients of the $c^2$–expansion are

$$\mathbb{E} = -(\ell + \mathring{\theta})[\mathcal{E}] - \mathcal{P}\mathring{\theta} - (\mathring{\mathcal{D}}_A + 2\varphi_A)\mathcal{J}^A - \mathcal{S}^{AB}\mathring{\theta}_{AB} - \Sigma^{AB}\ell[\lambda_{AB}], \tag{5.70}$$

$$\mathbb{C} = -\Sigma^{AB}\mathring{\theta}_{AB}. \tag{5.71}$$

Imposing $\ell^i\nabla_j T_i^{\,j} = 0$ as one taking the limit $c \to 0$ demands $\mathbb{E} = 0$ and $\mathbb{C} = 0$. The first equation is the Carrollian energy evolution equation and second equation is the constraint equation. Note that the expression $\mathbb{E}$ for the energy equation differs from the original work [124] due to the presence of the tensor $\lambda_{AB}$ and the fluid velocity $V^A$ contained implicitly in the Carrollian $\ell$. As we will discuss in the next section, these two additional variables are part of the phase space of Carrollian fluids and they are necessary when one wants to derive Carrollian conservation laws from symmetries. In this sense, our results are the generalization of [124].

In a similar manner to the vertical component, we compute the horizontal components of the conservation laws and consider the $c^2$–expansion. This is given by

$$\begin{aligned}
e_A{}^i\nabla_j T_i^{\,j} &= \nabla_j\left(e_A{}^i T_i^{\,j}\right) - T_i^{\,j}\nabla_j e_A{}^i \\
&= \nabla_j\left(\frac{1}{c^2}q_{AB}q^B\ell^j + \left(q_{AC}\tau^{CB} + \mathcal{P}\delta_A^B\right)e_B{}^j\right) + \left(\mathcal{E}k_i - \frac{1}{c^2}q^B e_{Bi}\right)\nabla_\ell e_A{}^i \\
&\quad + \left(q^B k_i - \left(q_{CD}\tau^{BD} + \mathcal{P}\delta_C^B\right)e^C{}_i\right)\nabla_{e_B}e_A{}^i \\
&= \frac{1}{c^2}(\ell+\theta)[q_{AB}q^B] + \mathcal{E}\varphi_A - w_{AB}q^B + (\mathcal{D}_B + \varphi_B)(q_{AC}\tau^{CB} + \mathcal{P}\delta_A^B) \\
&= \frac{1}{c^2}\mathbb{J}_A + \mathbb{P}_A + \mathcal{O}(c^2),
\end{aligned} \tag{5.72}$$

where the zeroth-order term is

$$\begin{aligned}
\mathbb{P}_A = (\ell + \mathring{\theta})[\pi_A] + \mathcal{E}\varphi_A - w_{AB}\mathcal{J}^B + (\mathring{\mathcal{D}}_B + \varphi_B)(\mathcal{K}_A{}^B + \mathcal{P}\delta_A^B) \\
+ \left(\ell[\lambda]\mathcal{J}_A + \Sigma_A{}^B\mathring{\mathcal{D}}_B\lambda + \Sigma^{BC}\mathring{\mathcal{D}}_A\lambda_{BC}\right),
\end{aligned} \tag{5.73}$$

while the other term is

$$\mathbb{J}_A = (\ell + \mathring{\theta})[\mathcal{J}_A] + (\mathring{\mathcal{D}}_B + \varphi_B)\Sigma^B{}_A. \tag{5.74}$$

Taking the Carrollian limit $c \to 0$ of the conservation laws, $e_A{}^i\nabla_j T_i^{\,j} = 0$, imposes the Carrollian momentum evolution, $\mathbb{P}_A = 0$ and the conservation of Carrollian current, $\mathbb{J}_A = 0$. Again, our expression for $\mathbb{P}_A$ is the more general case of [124].

Let us provide a comment here that the case when the sub-leading components of the sphere metric vanishes, $\lambda_{AB} = 0$ simplifies the Carrollian evolution equations,

$$\mathbb{E} = -(\ell + \mathring{\theta})[\mathcal{E}] - \mathcal{P}\mathring{\theta} - (\mathring{\mathcal{D}}_A + 2\varphi_A)\mathcal{J}^A - \mathcal{S}^{AB}\mathring{\theta}_{AB}, \tag{5.75a}$$

$$\mathbb{P}_A = (\ell + \mathring{\theta})[\pi_A] + \mathcal{E}\varphi_A - w_{AB}\mathcal{J}^B + (\mathring{\mathcal{D}}_B + \varphi_B)(\mathcal{S}_A{}^B + \mathcal{P}\delta^B_A), \tag{5.75b}$$

$$\mathbb{J}_A = (\ell + \mathring{\theta})[\mathcal{J}_A] + (\mathring{\mathcal{D}}_B + \varphi_B)\Sigma^B{}_A, \tag{5.75c}$$

$$\mathbb{C} = -\Sigma^{AB}\mathring{\theta}_{AB}. \tag{5.75d}$$

These are the Carrollian fluid equations given in the literature [124, 144].

## 5.3 Hydrodynamics from Symmetries

In this section, we tackle Carrollian hydrodynamics from a different, but nonetheless equivalent, perspective. Our objective is to re-derive the equations that govern Carrollian hydrodynamics (5.70), (5.71), (5.73), and (5.74) from the symmetries of the space $H$.

### 5.3.1 The Action for Carrollian Fluid

Since the metric $h$ is defined on the space $H$, we can consider the action of the fluid whose variation yields the fluid energy-momentum tensor. We will consider the fluid action that is finite when taking the Carrollian limit $c \to 0$. The variation of the fluid action we will use takes the form

$$\delta S_{\text{fluid}} = -\int_H \left(\mathcal{E}\delta\alpha - e^{\alpha}\mathcal{J}^A\delta\beta_A + e^{-\alpha}\tilde{\pi}_A\delta V^A - \frac{1}{2}\left(\tilde{\mathcal{S}}^{AB} + \mathcal{P}\mathring{q}^{AB}\right)\delta\mathring{q}_{AB} - \Sigma^{AB}\delta\lambda_{AB}\right)\mathring{\boldsymbol{\epsilon}}_H \tag{5.76}$$

where we defined the momentum conjugated to the velocity field $V^A$ and the leading-order sphere metric $\mathring{q}_{AB}$ to be

$$\tilde{\pi}_A := \pi_A + \lambda\mathcal{J}_A \tag{5.77}$$

$$\tilde{\mathcal{S}}^{AB} := \mathcal{S}^{AB} + \lambda\Sigma^{AB}. \tag{5.78}$$

We also absorbed the Jacobian factors and the velocity field variation into the definition of the variation $\delta\!\!\!\delta$ as follows,

$$\delta\!\!\!\delta\alpha := \delta\alpha + \beta_A \delta\!\!\!\delta V^A, \tag{5.79}$$

$$\delta\!\!\!\delta\beta_A := (J^{-1})_A{}^C \delta\left(J_C{}^B \beta_B\right) - (\beta \cdot \delta\!\!\!\delta V)\beta_A, \tag{5.80}$$

$$\delta\!\!\!\delta\mathring{q}_{AB} := (J^{-1})_A{}^C (J^{-1})_B{}^D \delta\left(J_C{}^E J_D{}^F \mathring{q}_{EF}\right) - 2\mathring{q}_{C(A}\beta_{B)}\delta\!\!\!\delta V^C, \tag{5.81}$$

$$\delta\!\!\!\delta\lambda_{AB} := (J^{-1})_A{}^C (J^{-1})_B{}^D \delta\left(J_C{}^E J_D{}^F \lambda_{EF}\right) - 2\lambda_{C(A}\beta_{B)}\delta\!\!\!\delta V^C, \tag{5.82}$$

and that we define

$$\delta\!\!\!\delta V^A := \left(\delta V^B\right) J_B{}^A. \tag{5.83}$$

The action (5.76) is simply derived from the fluid energy-momentum tensor $T^{ij}$ and the metric variation $\delta h_{ij}$. To see this, let us consider an action $S[h_{ij}]$ and its metric variation yields the energy-momentum tensor,

$$\delta S = \int_H \left(\frac{1}{2} T^{ij} \delta h_{ij}\right) \boldsymbol{\epsilon}_H \tag{5.84}$$

Since the fluid energy-momentum tensor (5.66) has a part that diverges when taking the limit $c \to 0$, the variation $\delta S$ also diverges in this limit. To obtain the finite action (5.76), we subtract the divergent part from $\delta S$ then take the Carrollian limit, that is

$$\delta S_{\text{fluid}} := \lim_{c\to 0}\left(\delta S - \frac{1}{c^2}\delta S_{(-1)}\right). \tag{5.85}$$

We note that the divergent part is given by

$$\delta S_{(-1)} := \lim_{c\to 0}\left(c^2 \delta S\right) = \int_H \left(\frac{1}{2} T^{ij}_{(-1)} \delta h_{(0)ij}\right) \mathring{\boldsymbol{\epsilon}}_H, \tag{5.86}$$

where we used that the metric variation is regular as $c \to 0$ and schematically expands as $\delta h_{ij} = \delta h_{(0)ij} + c^2 \delta h_{(1)ij} + \mathcal{O}(c^4)$. The fluid action (5.76) is thus

$$\delta S_{\text{fluid}} = \int_H \frac{1}{2}\left(T^{ij}_{(0)} \delta h_{(0)ij} + T^{ij}_{(-1)} \delta h_{(1)ij} + \lambda T^{ij}_{(-1)} \delta h_{(0)ij}\right) \mathring{\boldsymbol{\epsilon}}_H. \tag{5.87}$$

### 5.3.2 Near-Carrollian Diffeomorphism

To derive the Carrollian hydrodynamic equations from the variation of the action (5.76) under certain symmetries, we first need to specify those symmetries and derive the symmetry transformations for the metric components, $(\alpha, \beta_A, V^A, \mathring{q}_{AB}, \lambda_{AB})$. The seemingly obvious choice one could consider is the Carrollian diffeomorphism. However, Carrollian diffeomorphism is not sufficient to derive the complete set of hydrodynamic equations (5.70), (5.71), (5.73), and (5.74), as already shown in [144]. The reasons for this limitation are as follows:

*i*) Carrollian diffeomorphism fixes the variation of the velocity field, $\delta^{\text{Carr}} V^A = 0$, hence turning off a phase space degree of freedom conjugated to the velocity, that is the fluid momentum.

*ii*) There are only two symmetry parameters $(\tau, Y^A)$ for the Carrollian diffeomorphism, while there are four hydrodynamic equations. The symmetries labelled by the parameter $\tau$ and $Y^A$ correspond, respectively, to the the energy equation (5.70) and the momentum equation (5.73). To obtain the remaining two equations, the current conservation (5.74) and the constraint (5.71), we would need two more symmetry parameters.

We therefore need to detach our consideration from the Carrollian diffeomorphism and consider a general diffeomorphism on the space $H$. A general diffeomorphism on $H$ is labelled by vector fields of the form,

$$\xi = f\ell + X^A e_A, \tag{5.88}$$

where $f$ and $X^A$ are arbitrary functions on $H$. This general diffeomorphism will definitely change the Carroll structure. In the same fashion as our prior discussions, let us expand the transformation parameters $(f, X^A)$ in the small parameter $c^2$ as

$$f = \tau + c^2\psi + \mathcal{O}(c^4), \qquad \text{and} \qquad X^A = Y^A + c^2 Z^A + \mathcal{O}(c^4), \tag{5.89}$$

where now the parameters $(\tau, \psi, Y^A, Z^A)$ are functions on $H$. This way, we have already secured four parameters we need for four equations of Carrollian fluid. It is of extreme importance to point out that expanding the diffeomorphism $c^2 = 0$ can be regarded as the analog to the diffeomorphism of spacetime geometry in the close vicinity of a black hole horizon, the near-horizon diffeomorphism, where $c^2$ plays the same role as the distance away from the black hole horizon. We will refer to this diffeomorphism as the *near-Carrollian diffeomorphism*.

As stated previously, we need to find how the metric components vary under the near-Carrollian diffeomorphism. To carry out this task, we employ the technology of the anomaly operator $\vartriangle_\xi$. The metric $h$ is covariant under the near-horizon diffeomorphism, meaning that its anomaly $\vartriangle_\xi h := \delta_\xi h - \mathcal{L}_\xi h$ vanishes. The anomaly of the metric $h$ decomposes as

$$\begin{aligned}\vartriangle_\xi h &= -2c^2(\vartriangle_\xi \boldsymbol{k}) \circ \boldsymbol{k} + \vartriangle_\xi q \\ &= -2c^2(\iota_\ell \vartriangle_\xi \boldsymbol{k})\boldsymbol{k} \circ \boldsymbol{k} + 2\left(\vartriangle_\xi q(\ell, e_A) - c^2(\iota_{e_A}\vartriangle_\xi \boldsymbol{k})\right)\boldsymbol{k} \circ \boldsymbol{e}^A + \vartriangle_\xi q(e_A, e_B)\boldsymbol{e}^A \circ \boldsymbol{e}^B.\end{aligned} \tag{5.90}$$

Demanding covariance, $\vartriangle_\xi h = 0$, imposes the following conditions,

$$\iota_\ell \vartriangle_\xi \boldsymbol{k} = 0, \qquad \vartriangle_\xi q(\ell, e_A) = c^2(\iota_{e_A}\vartriangle_\xi \boldsymbol{k}), \qquad \text{and} \qquad \vartriangle_\xi q(e_A, e_B) = 0. \tag{5.91}$$

The problem then boils down to the computation of the anomaly of the Ehresmann connection $\boldsymbol{k}$ and the anomaly of the null Carrollian metric $q$ (we defer the derivations to the Appendix B.4). Solving the above conditions for different powers of $c^2$ gives us the transformation of the metric components under the near-Carrollian diffeomorphism,

$$\begin{aligned}
\delta_\xi \alpha &= \delta^{\text{Carr}}_{(\tau,Y)}\alpha && \text{(5.92a)}\\
\mathrm{e}^\alpha \delta_\xi \beta_A &= \mathrm{e}^\alpha \delta^{\text{Carr}}_{(\tau,Y)}\beta_A + \mathring{q}_{AB}\ell[Z^B] && \text{(5.92b)}\\
\delta_\xi \mathring{q}_{AB} &= \delta^{\text{Carr}}_{(\tau,Y)}\mathring{q}_{AB} && \text{(5.92c)}\\
\delta_\xi \lambda_{AB} &= \frac{1}{2}\delta^{\text{Carr}}_{(\psi,Z)}\mathring{q}_{AB} + \tau\ell[\lambda_{AB}] + Y^C \mathring{\mathcal{D}}_C \lambda_{AB} + 2\lambda_{C(A}\mathring{\mathcal{D}}_{B)}Y^C, && \text{(5.92d)}
\end{aligned}$$

where we recalled the functional form of the Carrollian transformations[8] (5.40), (5.41), and (5.42), and the transformation of the velocity field is given by,

$$\delta_\xi V^A = -D_u Y^A. \tag{5.93}$$

### 5.3.3 Hydrodynamics from Near-Carrollian Diffeomorphism

The Carrollian hydrodynamic equations (5.70), (5.71), (5.73), and (5.74) can be recovered by demanding invariance, up to boundary terms, of the fluid action (5.76) under the near-Carrollian transformations, $\delta_\xi S_{\text{fluid}} = 0$. Using the near-Carrollian transformations (5.92) and (5.93) and the Stokes theorem (5.53), one can show that

$$\delta_\xi S_{\text{fluid}} = -\int_H \left(\tau\mathbb{E} + \overline{\psi}\mathbb{C} + Y^A\mathbb{P}_A + \overline{Z}^A\mathbb{J}_A\right)\mathring{\boldsymbol{\epsilon}}_H + Q_\xi \tag{5.94}$$

[8] Although now there is no constraint on $Y^A$, unlike the Carrollian transformations where $\ell[Y^A] = 0$.

where we defined the combinations of the transformation parameters, $\overline{\psi} := \psi + \lambda\tau$ and $\overline{Z}^A := Z^A + \lambda Y^A$. The boundary term $Q_\xi$ is the Noether charge corresponding to the near-Carrollian diffeomorphism. We clearly see that imposing $\delta_\xi S_{\text{fluid}} = 0$ up to the boundary term yields the fluid equations.

The Noether charge of these transformations has three components associated with different sectors of the near-Carrollian symmetries,

$$Q_\xi = Q_\tau + Q_Y + Q_{\overline{Z}}, \tag{5.95}$$

where each components are given by

$$Q_\tau = -\int_S \tau \left(\mathcal{E} + \mathrm{e}^\alpha \mathcal{J}^A \beta_A\right) \mathring{\boldsymbol{\epsilon}}_S, \tag{5.96a}$$

$$Q_Y = \int_S Y^A \left(\pi_A + \mathrm{e}^\alpha \left(\mathcal{K}_A{}^B + \mathcal{P}\delta_A^B\right) \beta_B\right) \mathring{\boldsymbol{\epsilon}}_S, \tag{5.96b}$$

$$Q_{\overline{Z}} = \int_S \overline{Z}^A \left(\mathcal{J}_A + \mathrm{e}^\alpha \Sigma_A{}^B \beta_B\right) \mathring{\boldsymbol{\epsilon}}_S. \tag{5.96c}$$

As one would expect, the transformations labelled by $\overline{\psi}$ have zero Noether charges, as they are generators of the non-dynamical constraint (5.71).

It is important to appreciate that our results generalize those presented in [144] (which was only the case $V^A = 0$ and $\lambda_{AB} = 0$). In our consideration, we allow non-zero $V^A$ and $\lambda_{AB}$ and by using the proposed near-Carrollian diffeomorphism (5.89), we managed to derive the complete set of Carrollian hydrodynamic equations and identified all the Noether charges.

# Chapter 6

# Carrollian Hydrodynamics on Stretched Horizons

Having laid down the core concept of Carrollian hydrodynamics in the previous chapter, we now aim at exploring the correspondence between gravitational dynamics in a spacetime region around the null boundary and hydrodynamics of Carrollian fluids.

In section 6.1, we study a timelike foliation of a spacetime in a local region with a null surface as its boundary. We adopt the rigging technique to construct the geometry of timelike hypersurfaces, called stretched horizons, and we will show that a Carroll structure is naturally induced on the surfaces. In section 6.2, we construct the energy-momentum tensor of the surfaces and argue that it can be regarded as the energy-momentum tensor of Carrollian fluids. We also show that its conservations laws corresponds to the Einstein equations on the stretched horizons.

## 6.1 Geometries of Stretched Horizons and Null Surfaces

This first section aims to lay down relevant geometries of null and timelike hypersurfaces. As advertised, we will focus particularly on the case when the surfaces are situated at finite distances, with the prime example being event horizons of black holes (null) and fictitious membranes (timelike) located at small distances outside the black hole horizons. The reasons to consider timelike surfaces along with null boundaries are as follows:

*i*) Our study takes inspiration from the black hole membrane paradigm [8, 9], which is the statement that gravitational dynamics of a black hole as seen from outside observers is captured entirely at the membrane located at or vanishingly close to the black hole horizon. This timelike membrane placed at infinitesimal small distance, usually treated to be in the same magnitude as Planck length, outside the horizon is particularly referred to as the *stretched horizon*, and they are furnished with physical quantities such as energy, pressure, and heat flux. One fascinating feature of the membrane viewpoint is that it establishes the correspondence between gravitational dynamics of the membrane and dynamics of the fluid. The correspondence allows us to construct a dictionary between gravitational objects and fluid quantities, then study black hole physics from a hydrodynamic perspective.

*ii*) We want to elaborate the emerging connection between black hole horizons and Carrollian physics. Donnay and Marteau [121] demonstrated that the limit from the stretched horizon to the black hole horizon can be regarded as the Carrollian limit, and the corresponding hydrodynamic picture displayed at the horizon is Carrollian in nature. We therefore want to properly treat null and timelike surfaces on the same footing.

To set the stage, we consider a family of 3–dimensional timelike and null hypersurfaces $H$ embedded in an ambient 4–dimensional spacetime $M$. We indicate the embedding map by $\jmath : H \hookrightarrow M$. The spacetime is endowed with a Lorentzian metric and a Levi-Civita connection, $(g, \nabla)$, and we denote coordinates on $M$ by $\{x^a\}$. We further provide on $H$ the fiber bundle structure, $p : H \to S$. The space $S$ is a 2–sphere with coordinates $\{\sigma^A\}$ and a metric $q_{AB}\mathbf{d}\sigma^A \circ \mathbf{d}\sigma^B$. The complete Carroll structure on $H$ will be induced from another structure, the rigged structure.

Geometry of hypersurfaces can be studied through many different approaches, depending on types of hypersurfaces and problems under consideration. For example, the Arnowitt-Deser-Misner (ADM) formalism, also known as the (3+1)-decomposition of spacetime, has become a go-to tool to deal with spacelike Cauchy surfaces and timelike boundaries. This approach, however, relies on the existence of the apparent notion of time and is thus useful when one wants to tackle initial-value problems of general relativity or study Hamiltonian formulation of general relativity (see for instance [217] and references therein). The (3+1)-splitting formalism can also be applied to null hypersurfaces [218]. This "time-first" formalism instinctively imprints a Galilean nature to the considerations, rather than a Carrollian nature, which relies on "space-first" constructions. In this regard, we then refrain from adopting the ADM formalism in our study. The spacetime geometry in close vicinity to the null surface has been studied extensively using Gaussian null coordinates, which utilizes null geodesics to extend the coordinates on the null surface to the surrounding spacetime. This formalism has been used to describe the near-horizon geometry of black holes [121, 219, 220] and also the geometry of general null surfaces located at finite distances [112, 221]. Let us also mention that another type of framework suitable for studying the geometry of null hypersurfaces is the double null foliation [222], which can be perceived as a spacial, gauge fixed, case of a more general formalism, the (2+2)-formalism. The (2+2)-splitting of spacetime has been proven to be the most apt formalism for describing the geometry around codimension–2 *corner spheres*, regardless of the nature of codimension–1 boundaries, and has been tremendously utilized in the arena of local holography program [32, 44, 155]. In the context of asymptotic null infinity, the Bondi-Metzner-Sachs (BMS) formalism and its extensions are widely adopted [63, 160, 223–225]. Our construction will rely on another framework suitable for describing general hypersurfaces, the rigging technique [226].

The key point of this section is to demonstrate that by starting from a rigged structure equipped on the surface $H$, a Carroll structure is naturally induced, and together, they fully describe the intrinsic and extrinsic geometry of $H$. Our construction holds true for both timelike hypersurfaces and null hypersurfaces and therefore provides a unified description of these hypersurfaces. In this set up, the null boundary $N$ can then be properly treated as a limit of the stretched horizon $H$.

### 6.1.1 Rigged Structures

The Carroll structure we introduced in the previous chapter is purely intrinsic to the surface $H$. Describing the complete geometry of $H$ as a hypersurface embedded in the ambient spacetime $M$ requires extrinsic structures. As stated, we will utilize the rigging technique for general hypersurfaces introduced by Mars and Senovilla [226]. We dress the surface $H$ with a *rigged structure* which by definition, is given by a pair $(\boldsymbol{n}, k)$, where $\boldsymbol{n} = n_a \mathrm{d}x^a$ is the normal 1–form to $H$ and its dual vector, called the *rigging vector*, $k = k^a \partial_a$ is transverse to $H$ and thus obeys

$$\iota_k \boldsymbol{n} = 1. \tag{6.1}$$

In this construction, vectors $X$ tangent to the surface $H$ are such that $\iota_X \boldsymbol{n} = 0$. In the following, we will work with the normal form that defines a foliation of $M$. Following from the Frobenius theorem, this means $\mathbf{d}\boldsymbol{n} = \boldsymbol{a} \wedge \boldsymbol{n}$ for a 1-form $\boldsymbol{a}$ on $M$. To make this statement more precise, we foliate the spacetime $M$ into a family of null or timelike surfaces whose leaves are specified at a constant positive value of a function $r(x)$. Without loss of generality, we choose the leaves at $r(x) > 0$ to be timelike stretched horizons $H$ and situated at $r(x) = 0$ is the null boundary $N$. In this sense, the *null limit* from $H$ to $N$ corresponds to the limit $r \to 0$. In this setup, the normal form is given by

$$\boldsymbol{n} = \mathrm{e}^{\overline{\alpha}} \mathbf{d}r, \tag{6.2}$$

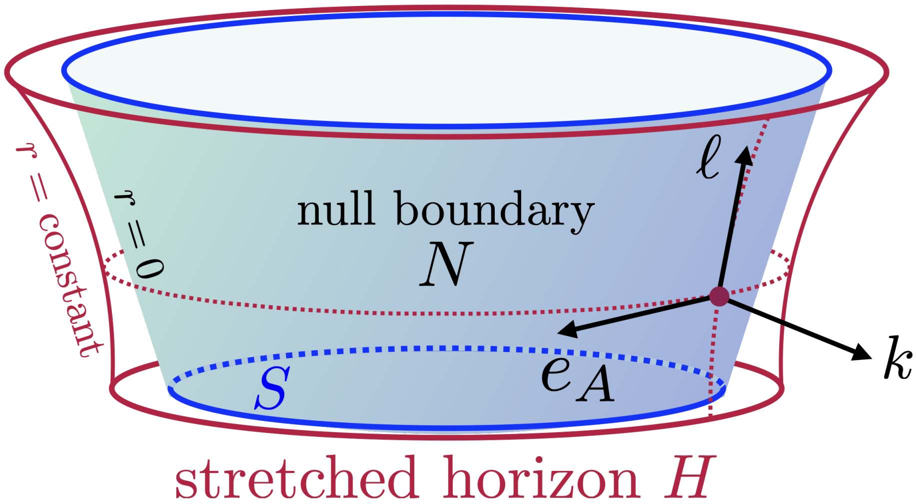


Figure 6.1: Stretched horizons $H$ are chosen to be surfaces at $r =$ constant and the null boundary $N$ can be regarded as the limit $r \to 0$ of the sequence of stretched horizons. The surface $H$ is endowed with the rigging vector $k$ and its dual form $\boldsymbol{n}$. The Carroll structure with the vertical vector $\ell$ and the horizontal vector $e_A$ is induced from the rigged structure, and together with $k$, they form a complete basis for the tangent space $TM$.

and that $\mathbf{d}\boldsymbol{n} = \boldsymbol{a} \wedge \boldsymbol{n}$ where $\boldsymbol{a} = \mathbf{d}\overline{\alpha}$ as desired.

The rigged structure allows us to define the projection tensor, $\Pi : TM \to TH$, whose components are given in terms of the rigged structure by

$$\Pi_a{}^b := \delta_a^b - n_a k^b, \qquad \text{such that} \qquad k^a \Pi_a{}^b = 0 = \Pi_a{}^b n_b. \tag{6.3}$$

This projector is designed such that, for a given vector field $X$ on $M$, the vector $\overline{X}^b := X^a \Pi_a{}^b \in TH$ is tangent to $H$ with $\overline{X}^a n_a = 0$. In the same vein, for any given 1–form $\boldsymbol{\omega} \in T^*M$, the 1–form $\overline{\omega}_a := \Pi_a{}^b \omega_b \in T^*H$ is a 1–form on $H$ such that $k^a \overline{\omega}_a = 0$.

### 6.1.2 Induced Carroll Structures

Assuming that the spacetime $M$ is equipped with a Lorentzian metric $g$, we can now use the metric and its inverse to define the 1–form $\boldsymbol{k} = g(k, \cdot)$ and the vector $n = g^{-1}(\boldsymbol{n}, \cdot)$. We restrict our consideration to the case when the rigged structure $(\boldsymbol{n}, k)$ is *null*, meaning that $\boldsymbol{k}$ is a 1–form on $H$, hence satisfying the conditions $k_a = \Pi_a{}^b k_b \in T^*H$ and thus $k_a k^a = 0$.

In addition, we denote by $\sqrt{2\rho}$ the norm of the normal 1–form. Overall this means that

$$g(k,k) = 0, \qquad g^{-1}(\boldsymbol{n},\boldsymbol{n}) = n_a n^a := 2\rho. \tag{6.4}$$

We choose the $\rho$ such that it is constant on the surface $H$, such that $\Pi_a{}^b \partial_b \rho = 0$. We can furthermore define a tangential vector field $\ell = \ell^a \partial_a \in TH$ whose components are given by $\ell^a := n^b \Pi_b{}^a$. This vector is related to the vector $n$ and $k$ by

$$n^a = 2\rho k^a + \ell^a. \tag{6.5}$$

One can easily verify that the tangential vector $\ell$ and the 1–form $\boldsymbol{k}$ obey the following properties,

$$\iota_\ell \boldsymbol{n} = 0, \qquad \text{and} \qquad \iota_\ell \boldsymbol{k} = 1. \tag{6.6}$$

The first property emphasizes the fact that $\ell$ is tangent to the space $H$ while the second one suggests that $\ell$ can be thought of as an element of a Carroll structure on $H$ where the 1–form $\boldsymbol{k}$ plays a role of the corresponding Ehresmann connection. Other elements of the Carroll geometry, including the horizontal basis $e_A$ and the co-frame field $\boldsymbol{e}^A$, follow naturally from this construction. To see this, note that the rigging projector can be further decomposed as

$$\Pi_a{}^b = q_a{}^b + k_a \ell^b, \qquad \text{with} \quad q_a{}^b k_b = 0 = \ell^a q_a{}^b. \tag{6.7}$$

The tensor $q_a{}^b = e^A{}_a e_A{}^b$ is the horizontal projector from the tangent space $TH$ to the horizontal subspace $\mathbf{hor}(H)$. The last element of the Carroll structure, the null Carrollian metric on $H$, is given by $q_{ab} = q_a{}^c q_b{}^d g_{cd}$. We will also make an additional assumption that the projection map, $p : H \to S$, is the same for all $H$, inferring that the co-frame $\boldsymbol{e}^A$ on $H$ is closed, $\mathbf{d}\boldsymbol{e}^A = 0$, throughout the spacetime $M$.

It is important to appreciate the result we have just developed. That is, the Carroll structure on the space $H$ is fully determined from the rigged structure and the spacetime metric. Let us summarize again all important bits in the box below.

**Induced Carroll structure:** Given the *null* rigged structure $(k, \boldsymbol{n})$ on any hypersurface, with the rigged vector field $k$ being null, and the spacetime metric $g$, the Carroll structure $(\pi, \ell, q)$ is naturally induced on the hypersurface. The vertical vector field $\ell$ and the Ehresmann connection $\boldsymbol{k}$ are related to the rigged structure by

$$\ell^a = n_c g^{cb} \Pi_b{}^a, \qquad \text{and} \qquad k_a = g_{ab} k^b. \tag{6.8}$$

The null Carrollian metric is $q_{ab} = q_a{}^c q_b{}^d g_{cd}$, where $q_a{}^b = \Pi_a{}^b - k_a \ell^b$ is a horizontal projector.

The basis vectors $(\ell, k, e_A)$ and the dual 1–forms $(\boldsymbol{k}, \boldsymbol{n}, \boldsymbol{e}^A)$ then span the tangent space $TM$ and the cotangent space $T^*M$, respectively (see Figure 6.1). The spacetime metric then decomposes as

$$\begin{aligned} g_{ab} &= q_{ab} + k_a \ell_b + n_a k_b \\ &= q_{ab} + 2n_{(a}k_{b)} - 2\rho k_a k_b. \end{aligned} \tag{6.9}$$

It is also important to appreciate that, in general, the Carrollian vector field $\ell$ is not null and its norm is

$$\ell_a \ell^a = -2\rho. \tag{6.10}$$

This expresses the fact that the Carroll structure is null only on the null surface $N$. Note that the metric expression is regular when $\rho = 0$. Moreover, on $N$, we have that $n_a = \ell_a$.

Armed with the induced Carroll structure on $H$, almost all analysis done in the previous chapter can be applied, including the Carrollian commutation relations (5.14) and the general coordinates $x^i = (u, y^A)$ on $H$. We will elaborate more on coordinate choices in later section. One, however, has to keep in mind that rather than considering the space $H$ only on its own, viewing $H$ as a surface embedded in the higher-dimensional spacetime benefits us with richer geometry. In our consideration, this additional geometry arises from the transverse direction, capturing by the rigged structure $(k, \boldsymbol{n})$.

To simplify our computations, we make another assumption that the null transverse vector $k$ generates null geodesics on the spacetime $M$, meaning that $\nabla_k k = 0$. This particularly infers that the curvature of the Ehresmann connection retains the form[1] (5.10),

$$\mathbf{d}\boldsymbol{k} := -\varphi_A(\boldsymbol{k} \wedge \boldsymbol{e}^A) - \frac{1}{2} w_{AB}(\boldsymbol{e}^A \wedge \boldsymbol{e}^B), \tag{6.11}$$

where $\varphi_A$ and $w_{AB}$ are the previously introduced Carrollian acceleration and the Carrollian vorticity, respectively. Let us recall that we have chosen earlier that the null normal $\boldsymbol{n} = \mathrm{e}^{\overline{\alpha}} \mathbf{d}r$ defines a foliation of the spacetime $M$, and the curvature of the normal form is

$$\mathbf{d}\boldsymbol{n} = \ell[\overline{\alpha}] \boldsymbol{k} \wedge \boldsymbol{n} - e_A[\overline{\alpha}] \boldsymbol{n} \wedge \boldsymbol{e}^A, \tag{6.12}$$

The components $\ell[\overline{\alpha}]$ and $e_A[\overline{\alpha}]$ are related to the surface gravity and the Hájíček 1-form field. They are parts of the extrinsic geometry of the surface $H$ which we will explain momentarily. Let us also mention again that the curvature $\mathbf{d}\boldsymbol{e}^A = 0$ by construction.

[1] This is equivalent to the condition $\iota_k \mathbf{d}\boldsymbol{k} = \mathcal{L}_k \boldsymbol{k} = 0$, and one can verify, using the null-ness property of $k$, that $g^{ab}(\mathcal{L}_k \boldsymbol{k})_b = \nabla_k k^a$.

Similar to the derivation carried out in the previous chapter, the curvatures of the basis 1–forms determine the commutators of their dual vector fields through the relation (5.11). In this 4–dimensional case, it follows from (6.11) and (6.12) that the non-trivial commutators of the basis vector fields are

$$[\ell, e_A] = \varphi_A \ell, \qquad [e_A, e_B] = w_{AB}\ell, \qquad [k, \ell] = \ell[\overline{\alpha}]k, \qquad [k, e_A] = e_A[\overline{\alpha}]k. \tag{6.13}$$

The first two terms again are the Carrollian commutation relations.

### 6.1.3 Rigged metric and Rigged connection

On the surface $H$, we can define the rigged metric, $H_{ab} := \Pi_a{}^c \Pi_b{}^d g_{cd}$, and its conjugate, $H^{ab} := g^{cd}\Pi_c{}^a \Pi_d{}^b$. Given any two tangential vectors $X, Y \in TH$ that, by definition, satisfy the condition $X^a n_a = Y^a n_a = 0$, we can clearly see that

$$H_{ab}X^a Y^b = g_{ab}X^a Y^b, \qquad \text{and} \qquad H_{ba}k^a = 0. \tag{6.14}$$

This shows that the rigged metric $H_{ab}$ acts on tangential vector fields the same way as the induced metric $h_{ab} = g_{ab} - \frac{1}{2\rho}n_a n_b$. The difference, however, lies in the fact that the induced metric is orthogonal to $n$, i.e., $h_{ab}n^b = 0$ while the rigged metric satisfies the transversality condition $H_{ab}k^b = 0$. Combining this definition with (6.9) we see that the rigged and induced metric on the space $H$ can be written in terms of the Carroll structure as

$$H_{ab} = q_{ab} - 2\rho k_a k_b, \qquad \text{and} \qquad h_{ab} = q_{ab} - \frac{1}{2\rho}\ell_a \ell_b, \tag{6.15}$$

Observe that the advantage of the rigged metric is that it provides an expression that is regular when taking the null limit, $\rho \to 0$, while, on the other hand, the expression for the induced metric blows up when $\rho \to 0$. In this thesis, we will only use the rigged metric in our computations. It is crucial to notice that the rigged metric is precisely in the form of the Randers-Papapetrou metric (5.44) we have discussed in the previous chapter, with the parameter $\rho$ plays a role of the *virtual* speed of light.

We next introduce a notion of a connection on the space $H$, a *rigged connection*, descended from the rigged structure. Recall that by definition, a rigged tensor field $T_a{}^b$ on $H$ is a tensor on $M$ such that $k^a T_a{}^b = 0 = T_a{}^b n_b$. We defined a rigged connection of a tensor field $T_a{}^b$ as a covariant derivative projected onto $TH$,

$$D_a T_b{}^c = \Pi_a{}^d \Pi_b{}^e (\nabla_d T_e{}^f)\Pi_f{}^c. \tag{6.16}$$

One can first check that this connection is torsionless

$$\begin{aligned} [D_a, D_b]F &= \Pi_a{}^c\Pi_b{}^d(\nabla_c\Pi_d{}^e - \nabla_d\Pi_c{}^e)\nabla_e F \\ &= -\Pi_a{}^c\Pi_b{}^d(\nabla_c n_d - \nabla_d n_c)k[F] \\ &= 0 \end{aligned} \tag{6.17}$$

where we used in the last equality the fact that $n_a$ defines a foliation $\nabla_{[a}n_{b]} = a_{[a}n_{b]}$. It is straightforward to check that the rigged connection preserves the rigged projector

$$D_a\Pi_b{}^c = \Pi_a{}^d\Pi_b{}^e(\nabla_d\Pi_e{}^f)\Pi_f{}^c = -\Pi_a{}^d\Pi_b{}^e\nabla_d(n_e k^f)\Pi_f{}^c = 0. \tag{6.18}$$

It does not, however, preserve the rigged metric. Instead, we can show that

$$\begin{aligned} D_a H_{bc} &= \Pi_a{}^d\Pi_b{}^e\Pi_c{}^f\nabla_d(\Pi_e{}^i\Pi_f{}^j g_{ij}) \\ &= \Pi_a{}^d\Pi_b{}^e\Pi_c{}^f\left((\nabla_d\Pi_e{}^j)g_{fj} + (\nabla_d\Pi_f{}^j)g_{ej}\right) \\ &= -\Pi_a{}^d\Pi_b{}^e\Pi_c{}^f\left((\nabla_d n_e)k_f + (\nabla_d n_f)k_e\right) \\ &= -K_{ab}k_c - K_{ac}k_b \end{aligned} \tag{6.19}$$

where $K_{ab} := \Pi_a{}^c\Pi_b{}^d\nabla_c n_d = \frac{1}{2}\mathcal{L}_n H_{ab}$ is the extrinsic curvature of the surface $H$ computed with the rigged metric.

### 6.1.4 Coordinates

We conclude our geometrical construction of intrinsic structure of stretched horizons with the introduction of coordinates. As we have stated that stretched horizons $H$ are defined to be hypersurfaces labelled by a parameter $r \geq 0$, we can then adapt $r$ to serve as a radial coordinate. Furthermore, since the Carroll structure is induced on $H$, this means we can use the general coordinates $(u, y^A)$ defined in Section 5.1.3 as the coordinates on $H$, and they are extended throughout the spacetime $M$ by keeping their values fixed along null geodesics generated by the transverse vector $k$. Overall, we adapt $x^a = (u, r, y^A)$ as the coordinates on $M$.

In this coordinate system the basis vector fields are expressed as follows

$$\ell = \mathrm{e}^{-\alpha}D_u, \qquad k = \mathrm{e}^{-\overline{\alpha}}\partial_r \qquad e_A = (J^{-1})_A{}^B(\partial_B + \beta_B D_u) \tag{6.20}$$

where $D_u = \partial_u + V^A\partial_A$. The dual basis 1-form are given by

$$\boldsymbol{k} = \mathrm{e}^{\alpha}(\mathbf{d}u - \beta_A\boldsymbol{e}^A), \qquad \boldsymbol{n} = \mathrm{e}^{\overline{\alpha}}\mathbf{d}r, \qquad \boldsymbol{e}^A = (\mathbf{d}y^B - V^B\mathbf{d}u)J_A{}^B. \tag{6.21}$$

The components $(\alpha, \beta_A, V^A, J_A{}^B)$ which are parts of the Carroll geometry are functions of the coordinates $(u, y^A)$ on $H$. Their independence of the radial coordinate $r$ stems from our construction that the Carroll projection $p : H \to S$ is independent of the foliation defined by the function $r(x)$, and that $k$ is tangent to null geodesics. One can indeed be more general by relax the $r$-independent conditions. Doing so would inevitably introduce more variables, i.e., radial derivatives of these components, to the consideration, thereby renders computations more complicated. We refrain from doing so and keep our analysis simple in this thesis. Let us also remark that, even though the frame $\boldsymbol{e}^A$ is set to be independent of the radial direction $r$, the null Carrollian metric $q_{ab}$ can still depend on $r$ due to the possible $r$-dependence of the sphere metric $q_{AB}$. The remaining metric components, which are the norm $\rho$ and the scale $\overline{\alpha}$, are in general functions of $(u, r, y^A)$. We will however impose in the following section that $\rho$ only depends on $r$, that is $D_a\rho = 0$ for the reason we will justify momentarily.

## 6.2 Conservation Laws on Stretched Horizons

We are now at the stage where we can discuss the Carrollian fluid energy-momentum tensor on the stretched horizon $H$ and derive its conservation laws. We will first outright define the Carrollian fluid energy-momentum tensor and how Einstein equations imply conservation laws (or vice versa). The correspondence between fluid quantities and the extrinsic geometry of stretched horizons will be discussed afterwards.

Following the definition presented in [149], the rigged energy-momentum tensor on the null surface is related to the the null Weingarten tensor $\Pi_a{}^c \nabla_c \ell^d \Pi_d{}^b$. Since the vector $n^a$ goes to $\ell^a$ on the null boundary, it suggests that the fluid energy-momentum tensor on the surface is defined as,

$$T_a{}^b = \mathcal{W}_a{}^b - \mathcal{W}\Pi_a{}^b, \tag{6.22}$$

where $\mathcal{W}_a{}^b := \Pi_a{}^c \Pi_d{}^b \nabla_c n^d$ is the rigged Weingarten tensor on $H$ and $\mathcal{W} = \mathcal{W}_a{}^a$ is its trace. This tensor agrees with the one defined in [149] on the null boundary. We will show next that the Einstein equations $G_{ab} = 0$ and the condition $D_a\rho = 0$, imply hydrodynamic conservation laws $D_b T_a{}^b = 0$ .

### 6.2.1 Conservation laws

Our goal here is to show that conservation of the energy-momentum tensor follows from the Einstein equations. In the following derivation, we will keep track of the tangential derivative of the norm of the normal form, $D_a\rho$, by allowing its value to be non-zero. We will show that the condition $D_a\rho = 0$ is necessary to have a proper definition of the energy-momentum tensor that obeys conservation laws outside the null boundary $N$, hence justifying our prior assumption.

To start with, the covariant derivative of the vector $n$ decomposes as

$$\nabla_a n^b = \mathcal{W}_a{}^b + k^b D_a\rho + n_a \nabla_k n^b, \qquad \text{and thus} \qquad \nabla_a n^a = \mathcal{W} + k[\rho], \tag{6.23}$$

where we used that $n_a\nabla_b n^a = \frac{1}{2}\nabla_b(n_a n^a) = \nabla_b\rho$. The rigged covariant derivative of the rigged Weingarten tensor can then be written as

$$D_b\mathcal{W}_a{}^b = \Pi_a{}^c\Pi_d{}^b\nabla_b\mathcal{W}_c{}^d = \Pi_a{}^c\nabla_b\mathcal{W}_c{}^b + \mathcal{W}_a{}^c\nabla_k n_c. \tag{6.24}$$

We can then show that

$$\begin{aligned}
\Pi_a{}^c\nabla_b\nabla_c n^b &= \Pi_a{}^c\nabla_b(\mathcal{W}_c{}^b + k^b D_c\rho + n_c\nabla_k n^b) \\
&= \Pi_a{}^c\nabla_b\mathcal{W}_c{}^b + (D_a\rho)(\nabla_b k^b) + \Pi_a{}^c\nabla_k(D_c\rho) + \Pi_a{}^c(\nabla_b n_c)(\nabla_k n^b) \\
&= D_b\mathcal{W}_a{}^b + (D_a\rho)(\nabla_b k^b) + \Pi_a{}^c\nabla_k(D_c\rho) + (\Pi_a{}^c\nabla_b n_c - \mathcal{W}_{ab})\nabla_k n^b \\
&= D_b\mathcal{W}_a{}^b + (D_a\rho)(D_b k^b) + \Pi_a{}^c\nabla_k(D_c\rho) - a_a k[\rho],
\end{aligned} \tag{6.25}$$

where to arrive at the last equality, we used the property that $\nabla_a k^a = D_a k^a := \Pi_a{}^b\nabla_b k^a$, provided that $k$ is the geodesic vector field, and

$$\begin{aligned}
(\Pi_a{}^c\nabla_b n_c - \mathcal{W}_{ab}) = \Pi_a{}^c(\nabla_b n_c - \nabla_c n_d\Pi^d{}_b) &= \Pi_a{}^c(\nabla_b n_c - \nabla_c n_d(\delta^d{}_b - n^d k_b)) \\
&= \Pi_a{}^c(a_b n_c - a_c n_b) + D_a\rho k_b \\
&= -a_a n_b + D_a\rho k_b.
\end{aligned} \tag{6.26}$$

Next, using the property that the Einstein tensor along the vector $n^a$ projected onto $H$ coincides with the Ricci tensor, $\Pi_a{}^c n^b G_{bc} = \Pi_a{}^c R_{nc}$, and invoking the definition of the Ricci tensor in term of the commutator, we derive

$$\begin{aligned}
\Pi_a{}^c G_{nc} = \Pi_a{}^c[\nabla_b, \nabla_c]n^b &= \Pi_a{}^c\nabla_b\nabla_c n^b - D_a(\nabla_b n^b) \\
&= D_b(\mathcal{W}_a{}^b - \mathcal{W}\Pi_a{}^b) + (D_b k^b)D_a\rho - a_a k[\rho] + \Pi_a{}^c[\nabla_k, D_c]\rho.
\end{aligned} \tag{6.27}$$

We then show that the last term can be manipulated as follows:

$$\begin{aligned}
\Pi_a{}^c[\nabla_k, D_c]\rho &= \Pi_a{}^c k^b(\nabla_b\Pi_c{}^d)\nabla_d\rho - \Pi_a{}^d(\nabla_d k^b)\nabla_b\rho \\
&= -\Pi_a{}^c k^b(\nabla_b n_c)k[\rho] - \Pi_a{}^d n_b(\nabla_d k^b)k[\rho] - \Pi_a{}^d\nabla_d k^b D_b\rho \\
&= \Pi_a{}^c k^b\left(\nabla_c n_b - \nabla_b n_c\right)k[\rho] - D_a k^b D_b\rho \\
&= a_a k[\rho] - D_a k^b D_b\rho,
\end{aligned} \tag{6.28}$$

where we used that $\nabla_{[a}n_{b]} = a_{[a}n_{b]}$ to arrive at the last equality. Finally putting everything together, the Einstein tensor can therefore be expressed as

$$\Pi_a{}^c G_{nc} = D_b \left( \mathcal{W}_a{}^b - \mathcal{W}\Pi_a{}^b + k^b D_a \rho \right). \tag{6.29}$$

It then becomes crystal clear that because the condition $D_a \rho = 0$, the energy-momentum tensor (6.61) is conserved once imposing the Einstein equations $\Pi_a{}^c G_{nc} = 0$,

$$\boxed{\Pi_a{}^b G_{nb} = D_b T_a{}^b = 0.} \tag{6.30}$$

Remarks are in order here:

*i*) Conservation laws are automatically satisfied on the null boundary $N$ without posing an extra condition on $\rho$ as its value already vanishes on $N$. This again agrees with [149].

*ii*) It is also tempting to use instead $\mathbb{T}_a{}^b = \mathbb{W}_a{}^b - \mathbb{W}\Pi_a{}^b$, where $\mathbb{W}_a{}^b := \Pi_a{}^c \nabla_c n^b = W_a{}^b + k^b D_a \rho$, as the conserved energy-momentum tensor. This however raises a problem. As the tensor $\mathbb{T}_a{}^b$ contains components in transverse direction, it does not serve as a bonafide energy-momentum tensor of the stretched horizons.

*iii*) There are in fact two possible solutions to this difficulty, that is we either require $D_a k^a = 0$ or $D_a \rho = 0$. The former is too restrictive as it deliberately kills a degree of freedom $\overline{\theta} := D_a k^a$ on the surface $H$. As such, we instead require $D_a \rho = D_a(\frac{1}{2} n_a n^a) = 0$.

*iv*) One can always reach the condition $D_a \rho = 0$ by exploiting the fact that the rigging condition $n_a k^a = 1$ only defines the normal form $\boldsymbol{n}$ and the transverse vector $k$ up to the rescaling $\boldsymbol{n} \to \Omega \boldsymbol{n}$ and $k \to \Omega^{-1} k$ for a function $\Omega$ on $M$. We will comeback to this point again shortly.

### 6.2.2 Fluid Energy-Momentum Tensor

We have already defined the energy-momentum tensor of stretched horizons and showed that it obeys conservation laws as desired. We now discuss how its components, which are interpreted as Carrollian fluid momenta, can be expressed in terms of the extrinsic geometry of the surface. As a tensor tangent to the surface $H$, the energy-momentum tensor decomposes as

$$T_a{}^b := \mathcal{W}_a{}^b - \mathcal{W}\Pi_a{}^b = -\left( \mathcal{E}\ell^b + \mathcal{J}^b \right) k_a + \pi_a \ell^b + \left( \mathcal{K}_a{}^b + \mathcal{P} q_a{}^b \right), \tag{6.31}$$

which is precisely the form of the zeroth order Carrollian fluid energy-momentum tensor (5.67b). The rigged Weingarten tensor $W_a{}^b$ (sometimes called the shape operator) captures essential elements of extrinsic geometry of the surface $H$. It has been established that components of the extrinsic geometry serve as the conjugate momenta to the intrinsic geometry of the surface, in the gravitational phase space (see [111, 155] for the case of null boundaries). In our construction, the intrinsic geometry is encoded in the Carroll structure and the extrinsic geometry is the Carrollian fluid momenta. We will come back and have an in-depth discussion on this aspect at the level of phase space in the upcoming Chapter.

The rigged Weingarten tensor by definition is the covariant derivative of the vector $n^a$ projected, using the rigged projector, on to $H$,

$$\mathcal{W}_a{}^b = \Pi_a{}^c \Pi_d{}^b (\nabla_c n^d) = \Pi_a{}^c \nabla_c n^b, \tag{6.32}$$

where the last equality holds because we have chosen the norm of the 1-form $\boldsymbol{n}$ to be constant on $H$, that is $n_d \Pi_a{}^c \nabla_c n^d = D_a \rho = 0$. On the null hypersurface $N$, the tensor $W_a{}^b = \Pi_a{}^c \nabla_c \ell^b$ is called the null Weingarten tensor [111, 149, 157]. Following from the definition of the energy-momentum tensor, the Weingarten tensor (6.61), which is a tensor field on $H$, can be parameterized in terms of Carrollian fluid momenta as

$$\mathcal{W}_a{}^b = \mathcal{K}_a{}^b + \frac{1}{2}\mathcal{E} q_a{}^b + \pi_a \ell^b - \mathcal{J}^b k_a - \left(\mathcal{P} + \frac{1}{2}\mathcal{E}\right) k_a \ell^b, \tag{6.33}$$

where we emphasize again that the tensors $\mathcal{K}_a{}^b$, $\pi_a$, and $\mathcal{J}^a$ are horizontal. Since the vector $n^a$ is a linear combination of the tangential vector $\ell^a$ and the transverse vector $k^a$, the Weingarten tensor then decomposes as follows

$$\mathcal{W}_a{}^b = \mathcal{W}_{(\boldsymbol{\ell})a}{}^b + 2\rho \mathcal{W}_{(\boldsymbol{k})a}{}^b, \tag{6.34}$$

where we defined

$$\mathcal{W}_{(\boldsymbol{\ell})a}{}^b := D_a \ell^b = \theta_a{}^b + \pi_a \ell^b + A^b k_a + \kappa k_a \ell^b \tag{6.35}$$

$$\mathcal{W}_{(\boldsymbol{k})a}{}^b := D_a k^b = \overline{\theta}_a{}^b - (\pi^b + \varphi^b) k_a. \tag{6.36}$$

Note that the absence of the $\ell^b$ terms in $\mathcal{W}_{(\boldsymbol{k})a}{}^b$ is due to the fact that the vector $k$ is null. Let us now elaborate the connection between fluid momenta and the components of the tensor $\mathcal{W}_{(\boldsymbol{\ell})a}{}^b$ and $\mathcal{W}_{(\boldsymbol{k})a}{}^b$.

### Viscous stress tensor and Energy density

Let us first consider the spin-2 components of the rigged Weingarten tensor, which are the extrinsic curvature tensor, $q_a{}^c q_{bd} \mathcal{W}_c{}^d = q_a{}^c q_b{}^d \nabla_c n_d$. Observe that this object is symmetric in its two indices which follows from the fact that the normal form $\boldsymbol{n}$ defines foliation, $\nabla_{[a}n_{b]} = a_{[a}n_{b]}$. Its trace corresponds to the Carrollian fluid energy density $\mathcal{E}$,

$$\mathcal{E} := q_a{}^b \nabla_b n^a \qquad \text{or equivalently,} \qquad \mathcal{E} := q^{AB} g(e_B, \nabla_{e_A} n), \tag{6.37}$$

and the traceless part corresponds to the viscous stress tensor, $\mathcal{K}_{ab} = \mathcal{K}_{AB} e^A{}_a e^B{}_b$, of Carrollian fluids,

$$\mathcal{K}_{ab} := q_{\langle a}{}^c q_{b\rangle}{}^d \nabla_c n_d, \qquad \text{or,} \qquad \mathcal{K}_{AB} := g(e_B, \nabla_{e_A} n) - \frac{1}{2} q^{CD} g(e_D, \nabla_{e_C} n) q_{AB}. \tag{6.38}$$

We can also define the extrinsic curvature tensor[2] associated with the tangential vector $\ell$ to be $\theta_{ab} := q_a{}^c q_{bd} \mathcal{W}_{(\boldsymbol{\ell})c}{}^d = q_a{}^c q_b{}^d \nabla_c \ell_d$. Components of this extrinsic curvature tensor can be expressed in the horizontal basis as

$$\theta_{AB} = g(e_B, \nabla_{e_A} \ell) = \frac{1}{2} \ell[q_{AB}] + \rho w_{AB}. \tag{6.39}$$

Notice that this tensor is symmetric only on the boundary $N$ and its antisymmetric part is given by the Carrollian vorticity. The trace and the symmetric traceless components of tensor $\theta_{AB}$ are the expansion and the shear tensor associated with the tangential vector $\ell$,

$$\theta := q^{AB} \theta_{AB} = \ell[\ln \sqrt{q}], \qquad \text{and} \qquad \sigma_{AB} := \theta_{(AB)} - \frac{1}{2} \theta q_{AB}. \tag{6.40}$$

In a similar manner, we define the extrinsic curvature tensor associated with the transverse direction $k$ as $\overline{\theta}_{ab} := q_a{}^c q_{bd} \mathcal{W}_{(\boldsymbol{k})c}{}^d = q_a{}^c q_b{}^d \nabla_c k_d$, and its components can be expressed as

$$\overline{\theta}_{AB} = g(e_B, \nabla_{e_A} k) = \frac{1}{2} k[q_{AB}] - \frac{1}{2} w_{AB}. \tag{6.41}$$

Observe that $\overline{\theta}_{AB}$ is not symmetric even on the null surface. Its trace and its symmetric traceless components are respectively the expansion and the shear associated with $k$ and they are given by

$$\overline{\theta} := q^{AB} \overline{\theta}_{AB} = k[\ln \sqrt{q}], \qquad \text{and} \qquad \overline{\sigma}_{AB} := \overline{\theta}_{(AB)} - \frac{1}{2} \overline{\theta} q_{AB}. \tag{6.42}$$

[2]Note that despite the terminology, the tensor $D_a \ell^b$ does not truly describe the extrinsic geometry of the space $H$ as $\ell$ is tangent to $H$. Its values are completely determined by the intrinsic geometry, i.e. the Carrollian structure of the surface.

Let us also note that the combination

$$g(e_B, \nabla_{e_A} n) = \theta_{AB} + 2\rho\overline{\theta}_{AB} = \frac{1}{2} n[q_{AB}] \tag{6.43}$$

is symmetric as we have already stated. The fluid energy density and the viscous stress tensor are given in terms of expansions and shear tensors by

$$\mathcal{E} = \theta + 2\rho\overline{\theta}, \qquad \text{and} \qquad \mathcal{K}_{AB} = \sigma_{AB} + 2\rho\overline{\sigma}_{AB}. \tag{6.44}$$

It is important to appreciate that geometrically, the internal energy $\mathcal{E}$ computes the expansion of the area element of the sphere $S$ along the vector $n$. On the null surface $N$, it therefore computes the expansion of the area element along null vector $\ell$, while the traceless part $\mathcal{K}_{ab}$ corresponds to the shear tensor [111, 149, 155].

**Momentum density**

There are two spin-1 components of the energy-momentum tensor $T_a{}^b$. The first one corresponds to the Carrollian fluid momentum density, $\pi_a = \pi_A e^A{}_a$, which is defined as

$$\pi_a := q_a{}^c k_b \nabla_c n^b, \qquad \text{or in the horizontal basis,} \qquad \pi_A := g(k, \nabla_{e_A} n). \tag{6.45}$$

It then follows from the null rigged condition, $k^a k_a = 0$, that $\pi_a = q_a{}^c k_b \nabla_c \ell^b$ is the Hájíček field computed with the basis vector $(\ell, k, e_A)$. The expression of the fluid momentum can be derived starting from the commutators 6.13 as follows

$$\begin{aligned} e_A[\overline{\alpha}] = g(\ell, [k, e_A]) &= g(\ell, \nabla_k e_A) - g(\ell, \nabla_{e_A} k) \\ &= g(k, \nabla_\ell e_A) + g(k, \nabla_{e_A} \ell) \\ &= g(k, [\ell, e_A]) + 2g(k, \nabla_{e_A} \ell) \\ &= \varphi_A + 2\pi_A, \end{aligned} \tag{6.46}$$

where to get from the first line to the second line, we repeatedly applied Leibniz rule and used that $g([k, \ell], e_A) = 0$. We therefore arrive at the expression for the fluid momentum

$$\pi_A = \frac{1}{2} \left( e_A[\overline{\alpha}] - \varphi_A \right). \tag{6.47}$$

### Carrollian heat current

Another spin–1 quantity is the Carrollian heat current, $\mathcal{J}^a = \mathcal{J}^A e_A{}^a$, defined as

$$\mathcal{J}^a := -q_b{}^a \nabla_\ell n^b, \qquad \text{or in the horizontal basis,} \qquad \mathcal{J}^A := -q^{AB} g(e_B, \nabla_\ell n) \tag{6.48}$$

This object is related to the tangential acceleration $A^a = q_b{}^a \nabla_\ell \ell^b$ of the vector $\ell$ and the Carrollian momentum density as one can check using (6.5) and repeatedly applying Leibniz rule and the commutators (6.13),

$$\begin{aligned} \mathcal{J}_A &= -g(e_A, \nabla_\ell \ell) - 2\rho g(e_A, \nabla_\ell k) \\ &= -A_A + 2\rho g(e_A, [k, \ell]) + 2\rho g(\ell, [k, e_A]) - 2\rho g(k, \nabla_{e_A} \ell) \\ &= -A_A + 2\rho \left( \mathrm{e}_A[\overline{\alpha}] - \pi_A \right). \end{aligned} \tag{6.49}$$

We can evaluate the tangential acceleration as follows

$$A_A = g(e_A, \nabla_\ell \ell) = -g(\ell, [\ell, e_A]) - g(\ell, \nabla_{e_A} \ell) = e_A[\rho] + 2\rho\varphi_A. \tag{6.50}$$

Observe that the acceleration vanishes on the null surface $N$. Overall, the Carrollian heat current $\mathcal{J}^a$ thus becomes

$$\mathcal{J}_A = 2\rho\pi_A - e_A[\rho], \tag{6.51}$$

and it vanishes on the null boundary $N$.

### Surface gravity and Pressure

The last spin-0 component of the energy-momentum tensor is the fluid pressure $\mathcal{P}$ defined as the combination

$$\mathcal{P} = -\mu \qquad \text{where we define} \qquad \mu := \kappa + \frac{1}{2}(\theta + 2\rho\overline{\theta}). \tag{6.52}$$

Note that our combination $\mu$ on the stretched horizon is the generalization of what is called the gravitational pressure in [111] defined for the case of null boundary. The surface gravity $\kappa$ is defined as

$$\kappa = k_a \nabla_\ell \ell^a = g(k, \nabla_\ell \ell). \tag{6.53}$$

It also measures the vertical acceleration of the vector $\ell$. Its value is non-zero even on the null boundary $N$. Let us also comment that we write the directional derivative of the Carrollian vector field $\ell$ along itself as

$$\nabla_\ell \ell = \kappa \ell + A^A e_A + (2\rho\kappa - \ell[\rho])k \stackrel{N}{=} \kappa \ell, \tag{6.54}$$

Recalling that $A^A \stackrel{N}{=} 0$, this means $\nabla_\ell \ell = \kappa \ell$ which clearly dictates that on the null boundary $N$, the Carrollian vector $\ell$ generates non-affine null geodesics, and the in-affinity is measured by the surface gravity $\kappa$. We can show that the surface gravity is given by

$$\kappa = g(k, \nabla_\ell \ell) = -g(\ell, [\ell, k]) - g(\ell, \nabla_k \ell) = \ell[\overline{\alpha}] + k[\rho]. \tag{6.55}$$

Let us summarize below the dictionary between Carrollian fluid quantities and the components of the Weingarten tensors $\mathcal{W}_{(\boldsymbol{\ell})a}{}^b$ and $\mathcal{W}_{(\boldsymbol{k})a}{}^b$,

| | | |
|---|---|---|
| **Energy density:** | $\mathcal{E} = \theta + 2\rho\overline{\theta}$ | (6.56a) |
| **Pressure:** | $\mathcal{P} = -\mu$ | (6.56b) |
| **Momentum density:** | $\pi_A = \frac{1}{2}\left(e_A[\overline{\alpha}] - \varphi_A\right),$ | (6.56c) |
| **Carrollian heat current:** | $\mathcal{J}^A = 2\rho\pi_A - e_A[\rho],$ | (6.56d) |
| **Viscous stress tensor:** | $\mathcal{K}_{AB} = \sigma_{AB} + 2\rho\overline{\sigma}_{AB}.$ | (6.56e) |

Lastly, let us provide the general form of the covariant derivative of the tangential vector $\ell$, the transverse vector $k$, an their combination $n = \ell + 2\rho k$ which will become handy in further computations,

$$\nabla_a \ell^b = \mathcal{W}_{(\boldsymbol{\ell})a}{}^b + \left(2\rho(\pi_a + \kappa k_a) - D_a\rho\right) k^b - n_a \left(k[\rho]k^b + (\pi^b + \varphi^b)\right) \tag{6.57}$$

$$\nabla_a k^b = \mathcal{W}_{(\boldsymbol{k})a}{}^b - (\pi_a + \kappa k_a)k^b \tag{6.58}$$

$$\nabla_a n^b = \mathcal{W}_a{}^b + (D_a\rho)k^b + n_a \left(k[\rho]k^b - (\pi^b + \varphi^b)\right). \tag{6.59}$$

### 6.2.3 Comment on the energy-momentum tensor

As we have explained, the condition $D_a\rho$ is necessary to have conservations of the energy-momentum tensor (6.61) and that this condition can always be chosen by properly rescaling the normal form $\boldsymbol{n}$. Let us now demonstrate how this is done. Suppose that we start from the norm $\rho$ that is not constant on the surface, $D_a\rho \neq 0$, and consequently the energy-momentum tensor $T_a{}^b$ naively defined as in (6.61) is no longer conserved. In close vicinity of the null boundary $N$, we can always express the norm as $\rho = r\eta$, where $\eta$ is a function on $M$. We can now define the new normal form as

$$\widehat{\boldsymbol{n}} := \frac{1}{\sqrt{\eta}}\boldsymbol{n}, \qquad \text{with its norm being} \qquad \widehat{n}_a\widehat{n}^a = 2r, \tag{6.60}$$

which is now constant on the surface $H$. Notice that this corresponds to the change in the scale factor $\overline{\alpha}$. One can then define the new energy momentum tensor,

$$\widehat{T}_a{}^b := \widehat{\mathcal{W}}_a{}^b - \widehat{\mathcal{W}}\Pi_a{}^b = -\left(\widehat{\mathcal{E}}\ell^b + \widehat{\mathcal{J}}^b\right)k_a + \widehat{\pi}_a\ell^b + \left(\widehat{\mathcal{K}}_a{}^b + \widehat{\mathcal{P}}q_a{}^b\right), \tag{6.61}$$

where $\widehat{\mathcal{W}}_a{}^b$ is the Weingarten tensor now defined with the rescaled vector $\widehat{n}^a$. This new energy-momentum tensor is now conserved, $D_b\widehat{T}_a{}^b = 0$. One can check that this new conserved tensor is related to the naive, non-conserved, one by

$$\widehat{T}_a{}^b = \frac{1}{\sqrt{\eta}}\left(T_a{}^b - q_a{}^b\partial_b(\ln\sqrt{\eta})\ell^b + \ell[\ln\sqrt{\eta}]q_a{}^b\right). \tag{6.62}$$

Note that when working with the closed normal form, $\overline{\alpha} = 0$, the function $\eta$ coincides with the surface gravity $\kappa$ on the null boundary. In such case, this particular form of the conserved energy-momentum $\widehat{T}_a{}^b$, with the presence of the derivatives $D_a \ln\sqrt{\kappa}$ terms, has been proposed in [121]. In our previous construction, we have already bypassed this construction by assuming a priori the condition $D_a\rho = 0$.

### 6.2.4 Einstein equation of the stretched horizons

We have already proved the the Einstein equations corresponds to the conservation laws of energy-momentum tensor (6.61). With the extrinsic geometry of the stretched horizon $H$ defined, we now finally explicitly write the Einstein equations on $H$ in terms of the Carrollian fluid momenta.

Recalling the conservation equation (6.30), the component $G_{n\ell}$ of the Einstein tensor can be written as

$$\begin{aligned} G_{n\ell} &= \ell^a D_b T_a{}^b \\ &= D_a(\ell^b T_b{}^a) - T_a{}^b D_b\ell^a \\ &= -D_a\left(\mathcal{E}\ell^a + \mathcal{J}^a\right) - T_a{}^b \mathcal{W}_{(\boldsymbol{\ell})b}{}^a \\ &= -(\ell+\theta)[\mathcal{E}] - \mathcal{P}\theta - (\mathcal{D}_A + 2\varphi_A)\mathcal{J}^A - \mathcal{K}_A{}^B\theta_B{}^A, \end{aligned} \tag{6.63}$$

where we used that $D_a \mathcal{J}^a = \mathcal{D}_A \mathcal{J}^A + (\pi_A + \varphi_A)\mathcal{J}^A$. This is precisely the form of the energy equation (5.70) of Carrollian hydrodynamics. It is necessary to note that in this Chapter, we have not expanded the sphere metric $q_{AB}$ as in (5.46). This is the reason why we will not see the term $\lambda_{AB}$ and the viscous tensor $\Sigma_{AB}$ in the present derivation. A similar expansion to those in hydrodynamics will be discussed in the next chapter.

The remaining components of the Einstein tensor are $G_{nA}$. In a similar manner, we can show that

$$\begin{aligned} G_{nA} &= e_A{}^a D_b T_a{}^b \\ &= D_a(e_A{}^b T_b{}^a) - T_a{}^b D_b e_A{}^a \\ &= D_a \left(\mathcal{K}_A{}^a + \mathcal{P} q_A{}^a + \pi_A \ell^a\right) - T_a{}^b D_b e_A{}^a \\ &= (\ell + \theta)[\pi_A] + \mathcal{E}\varphi_A - w_{AB}\mathcal{J}^B + (\mathcal{D}_B + \varphi_B)(\mathcal{K}_A{}^B + \mathcal{P}\delta_A^B), \end{aligned} \tag{6.64}$$

where we recalled that $\mathcal{J}_A = 2\rho\pi_A$ and $\mathcal{P} = -\kappa - \frac{1}{2}\mathcal{E}$. We observe that $G_{nA}$ has the same form as the Carrollian momentum equations (5.73).

# Chapter 7

# Symmetries and Einstein Equations

In this final chapter, we explore the gravitational phase space and the Noether charges associated with the so-called *near-horizon diffeomorphism*. The main plots of this Chapter are the following:

*i*) The pre-symplectic potential capturing information about gravitational phase space on the stretched horizon $H$ are given in terms of the Carrollian conjugate pairs as in (5.76)

*ii*) The complete set of the Einstein equations, governing the evolution of the geometry of the null boundary $N$ can be derived from the near-horizon symmetries, and we compute the Noether charges associated with these symmetries.

To accomplish the point *ii*), we need to forsake the preconception that the pre-symplectic potential strictly evaluated at the null boundary fully captures all information, including the dynamics, of the null boundary. Instead, we will show that a wealth of information of the null boundary can be gained by considering the stretched horizon as the near-horizon deviation from the boundary. This gives us access to the *sub-leading* contributions to the null boundary phase space, and from these we can ultimately derived the Einstein equations, specifically the components $\mathring{G}_{\ell k} = 0$ and $\mathring{G}_{AB} = 0$. The latter equations are associated with the diffeomorphism-derived spin-2 symmetries of the surfaces. Let us now discuss these plot points in details.

## 7.1 Pre-Symplectic Potential

Gravitational phase space of the stretched horizon $H$ can be constructed using the covariant phase space formalism (see an introductory review given in Chapter 2). The main object that encodes phase space information of the theory is the *pre-symplectic potential*. In our study, we consider the 4-dimensional Einstein-Hilbert Lagrangian without the cosmological constant and without matter degrees of freedom, $\boldsymbol{L} = \frac{1}{2}R\boldsymbol{\epsilon}_M$ where $R$ stands for the spacetime Ricci scalar and $\boldsymbol{\epsilon}_M$ denotes the spacetime volume form. The standard pre-symplectic potential of the Einstein-Hilbert gravity pulling back to the stretched horizon $H$ is given by

$$\boldsymbol{\Theta}_H = -\Theta^a n_a \boldsymbol{\epsilon}_H, \qquad \text{where} \qquad \Theta^a = \frac{1}{2}\left(g^{ac}\nabla^b \delta g_{bc} - \nabla^a \delta g\right), \tag{7.1}$$

where we recalled the volume form on the surface $\boldsymbol{\epsilon}_H := -\iota_k \boldsymbol{\epsilon}_M$ and we also denote the trace of the metric variation with $\delta g := g^{ab}\delta g_{ab}$. In order to evaluate the pre-symplectic potential $\boldsymbol{\Theta}_H$, one starts with the variation of the spacetime metric, whose components are expressed in terms of the co-frame fields as,

$$\delta g_{ab} = \delta q_{ab} + 2k_{(a}\delta n_{b)} + 2\ell_{(a}\delta k_{b)} - 2(\delta\rho)k_a k_b. \tag{7.2}$$

Computations of the variation $\delta g_{ab}$ then boil down to the computation of variations of the co-frame $\boldsymbol{n}$ and $\boldsymbol{k}$ and the null metric $q_{ab}$. These variations are given by

$$\delta\boldsymbol{n} = \delta\overline{\alpha}\boldsymbol{n}, \quad \delta\boldsymbol{k} = \mathbb{\delta}\alpha\boldsymbol{k} - \mathrm{e}^{\alpha}\mathbb{\delta}\beta_A\boldsymbol{e}^A, \quad \delta q = -2\mathrm{e}^{\alpha}q_{AB}\mathbb{\delta}V^B\boldsymbol{k}\circ\boldsymbol{e}^A + \mathbb{\delta}q_{AB}\boldsymbol{e}^A\circ\boldsymbol{e}^B, \tag{7.3}$$

where we define the variation $\mathbb{\delta}$ as follows

$$\begin{aligned}
\mathbb{\delta}\alpha &:= \delta\alpha + \beta_A\mathbb{\delta}V^A, && (7.4)\\
\mathbb{\delta}\beta_A &:= (J^{-1})_A{}^C\delta\left(J_C{}^B\beta_B\right) - (\beta\cdot\mathbb{\delta}V)\beta_A, && (7.5)\\
\mathbb{\delta}q_{AB} &:= (J^{-1})_A{}^C(J^{-1})_B{}^D\delta\left(J_C{}^E J_D{}^F q_{EF}\right) - 2q_{C(A}\beta_{B)}\mathbb{\delta}V^C, && (7.6)\\
\mathbb{\delta}V^A &:= \left(\delta V^B\right)J_B{}^A. && (7.7)
\end{aligned}$$

One can then compute the trace of the metric variations and it is given by

$$\delta g = 2(\delta\overline{\alpha} + \delta\alpha + \delta\ln\sqrt{q}) = 2(\delta\overline{\alpha} + \mathbb{\delta}\alpha + \mathbb{\delta}\ln\sqrt{q}). \tag{7.8}$$

After tedious but straightforward computations, we finally obtain the expression for the pre-symplectic potential on the stretched horizon,

$$\begin{aligned}
\Theta_H = &\int_H \left(-\mathcal{E}\mathbb{\delta}\alpha + \mathrm{e}^{\alpha}\mathcal{J}^A\mathbb{\delta}\beta_A - \pi_A\mathrm{e}^{-\alpha}\mathbb{\delta}V^A + \frac{1}{2}\left(\mathcal{K}^{AB} + \mathcal{P}q^{AB}\right)\mathbb{\delta}q_{AB} - \overline{\theta}\delta\rho\right)\boldsymbol{\epsilon}_H \\
&+ \delta\int_H \left(\kappa + 2\rho\overline{\theta}\right)\boldsymbol{\epsilon}_H + \int_S \left(\frac{1}{2}(\delta\alpha - \delta\overline{\alpha}) + \delta\ln\sqrt{q}\right)\boldsymbol{\epsilon}_S
\end{aligned} \tag{7.9}$$

We observe that the bulk piece of the pre-symplectic potential contains the same conjugate pairs as in the Carrollian hydrodynamics (5.76) with the addition of the term $\overline{\theta}\delta\rho$ that vanishes on the null boundary $N$. We also notice that the scale $\overline{\alpha}$ of the normal form only appear in the corner term, in agreement with the one presented in [111, 155] for the case of null boundary. This suggests that we can safely set $\overline{\alpha} = 0$ without losing any phase space data. We will do so for the rest of this chapter.

## 7.2 Near-Horizon Expansion

As we have promoted, we shall focus our attention on the study of spacetime geometry in extremely close vicinity of the null boundary $N$ located at the coordinate $r = 0$. To this end, we are considering the stretched horizon $H$ located at an infinitesimally small value of the coordinate $r$. This means that variables on $H$ that are functions of $r$ admit power series expansions in the coordinate $r$. This picture indeed in the same spirit as to when studying spacetime geometry near (null) horizon of black holes (see for example [121, 219, 220] and [112] for general null surfaces at finite distances). We will hence adopt in this thesis the terminology *near-horizon* to refer to this particular scenario.

Let us recall that the variables contained in the metric (6.9) fall into two categories: those that are independent of the radial coordinate $r$ and those that depend on $r$. The former comprises $(\alpha, \beta_A, V^A, J_A{}^B)$ while the latter contains $(\rho, q_{AB})$ (we recall here again that we will set $\overline{\alpha} = 0$ from this point on). Since the variables $\rho$ and $q_{AB}$ depend on the radial coordinate, they hence admit the power series in $r$ around the null boundary $(r = 0)$. Let us first focus on the norm $\rho$ which we now take to be a function on $M$[1]. Let us recall the expression (6.55) for the surface gravity, $\kappa = k[\rho]$ infers that the norm $\rho$ completely determines the surface gravity, or vice versa. Let us now express the surface gravity as

$$\kappa = \mathring{\kappa} + r\kappa_{(1)} + \mathcal{O}(r^2), \tag{7.10}$$

where $\mathring{\kappa}$ and $\kappa_{(1)}$ are functions on the surface. The leading-order[2] $\mathring{\kappa}$ is the surface gravity evaluated on the null boundary, $\kappa \stackrel{N}{=} \mathring{\kappa}$, and the sub-leading order is given by $\partial_r\kappa \stackrel{N}{=} \kappa_{(1)}$.

[1] This seemingly contradicts the condition $D_a\rho = 0$ required for conservation laws of energy-momentum tensor we have placed previously. However, as we have explained in the previous chapter, when focusing on the null boundary, this condition can be relaxed. We will see that, when setting $\overline{\alpha} = 0$, the norm $\rho$ completely determines the surface gravity. Imposing $\rho$ to be constant on the surface means the surface gravity on the surface is constant, and this scenario corresponds to the case when the Carrollian fluid on the null surface is in global equilibrium. This condition in a sense is too restrictive in this case.

[2] We will use the symbol $\mathring{-}$ to denote the zeroth (or leading) order of quantities.

Given the form of $\kappa$, we can then integrate it to obtain the norm $\rho$ as

$$\rho = r\mathring{\kappa} + \frac{1}{2}r^2\kappa_{(1)} + \mathcal{O}(r^3). \tag{7.11}$$

Another element of geometry that can depend on the radial coordinate $r$ is the sphere metric $q_{AB}$. In a much similar manner, we express the sphere metric as a power series in $r$ as follows

$$q_{AB} = \mathring{q}_{AB} + 2r\lambda_{AB} + \mathcal{O}(r^2), \tag{7.12}$$

where again $q_{AB} \overset{N}{=} \mathring{q}_{AB}$ and $\partial_r q_{AB} \overset{N}{=} 2\lambda_{AB}$, and both $\mathring{q}_{AB}$ and $\lambda_{AB}$ are functions on the surface. To avoid mixing different orders of the $r$-expansion, we will use the metric $\mathring{q}_{AB}$ and its inverse $\mathring{q}^{AB}$ to lower and raise horizontal indices. In this setup, the inverse of the sphere metric is then $q^{AB} = \mathring{q}^{AB} - 2r\lambda^{AB} + O(r^2)$. As a consequence of the near-horizon expansion of $\rho$ and $q_{AB}$, some elements of extrinsic geometry derived from them also admit the near-horizon expansion.

Let us first look at the extrinsic curvature associated with the vector $k$, that is $\overline{\theta}_{AB} = \frac{1}{2}k[q_{AB}]$. One can then check that $\lambda_{AB}$ coincides with the leading-order in $r$ of the symmetric part $\overline{\theta}_{(AB)} \overset{N}{=} \lambda_{AB}$. We then have the trace-traceless split[3],

$$\lambda_{AB} = \overline{\theta}_{(AB)} = \overline{\sigma}_{AB} + \frac{1}{2}\overline{\theta}\mathring{q}_{AB}, \qquad \text{and} \qquad \overline{\theta} := \mathring{q}^{AB}\overline{\theta}_{AB} = \lambda, \tag{7.13}$$

where we denote the trace $\lambda = \mathring{q}^{AB}\lambda_{AB}$. In the same vein, the extrinsic curvature associated to the tangential vector $\ell$ is then written as $\theta_{AB} = \mathring{\theta}_{AB} + r\theta^{(1)}_{AB} + \mathcal{O}(r^2)$ where we have

$$\mathring{\theta}_{AB} = \frac{1}{2}\ell[\mathring{q}_{AB}] \qquad \text{and} \qquad \theta^{(1)}_{(AB)} = \ell[\overline{\theta}_{(AB)}]. \tag{7.14}$$

The expansion $\theta := q^{AB}\theta_{AB} = \mathring{\theta} + r\theta^{(1)} + \mathcal{O}(r^2)$ is then

$$\mathring{\theta} = \ell[\ln\sqrt{\mathring{q}}], \qquad \text{and} \qquad \theta^{(1)} = \ell[\overline{\theta}]. \tag{7.15}$$

We can also define the leading-order shear tensor, which is traceless, as

$$\mathring{\sigma}_{AB} := \mathring{\theta}_{AB} - \frac{1}{2}\mathring{\theta}\mathring{q}_{AB}. \tag{7.16}$$

[3]The extrinsic curvature $\overline{\theta}_{AB}$ and its offsprings will always appear at the sub-leading order in our analysis and hence only its value at the zeroth order $\mathring{\overline{\theta}}_{AB}$ will become relevant in our computations. To prevent cumbersome notations, we will omit the symbol $\mathring{-}$ in $\overline{\theta}_{AB}$, $\overline{\theta}$, and $\overline{\sigma}_{AB}$, while keeping in mind that these notations already represent their values on the null boundary.

Lastly, we comment that the volume forms on the stretched horizon $H$ and on the sphere $S$ also admit a near-horizon expansion as

$$\boldsymbol{\epsilon}_H = (1 + r\overline{\theta})\boldsymbol{\epsilon}_N + \mathcal{O}(r^2), \qquad \text{and} \qquad \boldsymbol{\epsilon}_S = (1 + r\overline{\theta})\mathring{\boldsymbol{\epsilon}}_S + \mathcal{O}(r^2), \tag{7.17}$$

where the volume form on the null boundary is $\boldsymbol{\epsilon}_N := \mathring{\boldsymbol{\epsilon}}_H = \boldsymbol{\epsilon}_H|_{r=0}$.

### 7.2.1 Near-horizon expansion of pre-symplectic potential

We now write the near-horizon expansion of the pre-symplectic potential on the stretched horizon, $\Theta_H$, as

$$\Theta_H = \Theta_N + r\Theta_N^{(1)} + \mathcal{O}(r^2) \tag{7.18}$$

The leading-order term represents the pre-symplectic potential when strictly evaluated on the null boundary. It is given by

$$\Theta_N = \int_N \left( -\mathring{\mathcal{E}}\delta\alpha - \pi_A \mathrm{e}^{-\alpha}\delta V^A + \frac{1}{2}\left(\mathring{\mathcal{K}}^{AB} + \mathring{\mathcal{P}}\mathring{q}^{AB}\right)\delta\mathring{q}_{AB}\right)\boldsymbol{\epsilon}_N + \delta\int_N \mathring{\kappa}\boldsymbol{\epsilon}_N + \mathring{\Theta}_S \tag{7.19}$$

where the conjugate momenta are the Carrollian fluid quantities living on the null boundary $N$. They are given by

$$\mathring{\mathcal{E}} = \mathring{\theta}, \qquad \pi_A = -\frac{1}{2}\varphi_A, \qquad \mathring{\mathcal{K}}^{AB} = \mathring{\sigma}^{AB}, \qquad \mathring{\mathcal{P}} = -\left(\mathring{\kappa} + \frac{1}{2}\mathring{\theta}\right). \tag{7.20}$$

Notice that the momentum $\pi_A = -\frac{1}{2}\varphi_A$ does not depend on the radial coordinate $r$ because its value is determined from the $r$-independence sector of the metric components $(\alpha, \beta_A, V^A, J_A{}^B)$. The corner term in the pre-symplectic potential is

$$\mathring{\Theta}_S = \int_S \left(\frac{1}{2}\delta\alpha + \delta\ln\sqrt{\mathring{q}}\right)\mathring{\boldsymbol{\epsilon}}_S. \tag{7.21}$$

The sub-leading contributions to the null boundary phase space are encoded in the sub-leading pre-symplectic potential,

$$\begin{aligned}\Theta_N^{(1)} = \int_N \Big( &\mathbb{P}_\alpha\delta\alpha + \mathbb{P}_\kappa\delta\kappa + \delta\kappa_{(1)} + \mathbb{P}_{\overline{\theta}}\delta\overline{\theta} + \mathrm{e}^{-\alpha}(\mathbb{P}_V)_A\delta V^A + \mathrm{e}^{\alpha}(\mathbb{P}_\beta)^A\delta\beta_A \\ &+ \frac{1}{2}(\mathbb{P}_q)^{AB}\delta\mathring{q}_{AB} + (\mathbb{P}_\lambda)^{AB}\delta\lambda_{AB}\Big)\boldsymbol{\epsilon}_N + \Theta_S^{(1)},\end{aligned} \tag{7.22}$$

where we defined the spin-0 sub-leading conjugate momenta as

$$\mathbb{P}_\alpha = -\ell[\overline{\theta}] + (\mathring{\kappa} - \mathring{\theta})\overline{\theta} + \kappa_{(1)}, \qquad \mathbb{P}_\kappa = 2\overline{\theta}, \qquad \text{and} \qquad \mathbb{P}_{\overline{\theta}} = 2\mathring{\kappa} - \frac{1}{2}\mathring{\theta}. \tag{7.23}$$

For spin-1 sector, we have that

$$(\mathbb{P}_V)_A = -\overline{\theta}\pi_A, \qquad \text{and} \qquad (\mathbb{P}_\beta)^A = -\mathring{q}^{AB}e_B[\kappa] - \kappa\varphi^A. \tag{7.24}$$

Lastly, the spin-2 momenta are given by

$$(\mathbb{P}_q)^{AB} = \mathring{q}^{AC}\mathring{q}^{BD}\ell[\overline{\sigma}_{CD}] + (2\mathring{\kappa} - \mathring{\theta})\overline{\sigma}^{AB} - 4\overline{\sigma}^{C(A}\mathring{\sigma}^{B)}{}_C - \left(\frac{1}{2}(\ell + \mathring{\theta})[\overline{\theta}] - \mathring{\kappa}\overline{\theta}\right)\mathring{q}^{AB} \tag{7.25}$$

$$(\mathbb{P}_\lambda)^{AB} = \mathring{\sigma}^{AB}. \tag{7.26}$$

The sub-leading corner piece is then

$$\Theta_S^{(1)} = \int_S \left(\overline{\theta}\left(\frac{1}{2}\delta\alpha + \delta\ln\sqrt{\mathring{q}}\right) + \delta\overline{\theta}\right)\mathring{\boldsymbol{\epsilon}}_S. \tag{7.27}$$

## 7.3 Near-Horizon Symmetries

We are now looking at the diffeomorphism that preserves our geometrical structures near the null boundary, the so-called near-horizon symmetries. To derive the form of near-horizon diffeomorphism and how the metric variables transform under such symmetries, we first write the diffeomorphism vector field $\xi$ as a power series in the radial coordinate,

$$\xi = (\tau + r\psi)\ell + (W + \frac{1}{2}rU)rk + (Y^A + rZ^A + \frac{1}{2}r^2X^A)e_A + ... \tag{7.28}$$

where ... denotes possible higher-order terms. The components $(\tau, \psi, W, U, Y^A, Z^A, X^A)$ are a prior independent functions on the null boundary. They are however not completely independent as some of them are determined from the others, stemming from the fact that there exist fixed background structures in our constructions.

In order to derive constraints on near-horizon symmetries, let us recall that the spacetime metric is covariant under diffeomorphisms, meaning that its anomaly vanishes, $\Delta_\xi g_{ab} = 0$. Using the decomposition (6.9), we write the anomaly of the metric as

$$\Delta_\xi g = -2(\Delta_\xi\rho)\boldsymbol{k}\circ\boldsymbol{k} - 4\rho(\Delta_\xi\boldsymbol{k})\circ\boldsymbol{k} + 2(\Delta_\xi\boldsymbol{n})\circ\boldsymbol{k} + 2(\Delta_\xi\boldsymbol{k})\circ\boldsymbol{n} + \Delta_\xi q = 0. \tag{7.29}$$

Components of this equation dictate the form of the vector field $\xi$ and how the metric variables change under diffeomorphisms generated by $\xi$. First, let us observe that the component $\Delta_\xi g(k,k) = 0$ imposes the condition

$$-\iota_k \Delta_\xi \boldsymbol{k} = \iota_k \mathcal{L}_\xi \boldsymbol{k} = k[\iota_\xi \boldsymbol{k}] = 0 \tag{7.30}$$

where we used the variation (7.3) (for the case $\overline{\alpha} = 0$) and that the vector $k$ is null and geodesic. The condition $k[\iota_\xi \boldsymbol{k}] = 0$ dictates that the vertical component of the near-horizon diffeomorphism is independent of the radial coordinate $r$. In other words, the transformation parameter $\psi$ and the higher-order terms vanish for near-horizon diffeomorphism. Other constraints on $\xi$ are derived following from that the variables $(\alpha, \beta_A, V^A)$ are independent of $r$. For instance, the component $\Delta_\xi g(\ell, k) = 0$ yields

$$\delta_\xi \alpha = \big(\ell[\tau] + Y^A \varphi_A + W\big) + r\left(U + Z^A \varphi_A\right) + \mathcal{O}(r^2), \tag{7.31}$$

which fixes the transformation $\delta_\xi \alpha$ and imposes the condition $U = -Z^A \varphi_A$. The equations $\Delta_\xi g(k, e_A) = 0$ gives

$$-\mathrm{e}^{\alpha} \delta_\xi \beta_A = \big((e_A - \varphi_A)[\tau] + w_{AB} Y^B + Z_A\big) + r\left(X_A + 2\overline{\theta}_{BA} Z^B\right) + \mathcal{O}(r^2), \tag{7.32}$$

thereby fixing the form of $\delta \beta_A$ and imposing the relation $X^A = -2Z^B \overline{\theta}_B{}^A$. Furthermore, the components $\Delta_\xi g(e, e_A) = 0$ of the metric anomaly can be expressed as

$$-\mathrm{e}^{-\alpha} \delta V^A = \ell[Y^A] + r\left((\ell + 2\mathring{\kappa})[Z^A] + \mathring{\mathcal{D}}^A W\right) + \mathcal{O}(r^2). \tag{7.33}$$

Since $\delta V^A$ is $r$-independent, the above equation places the condition

$$(\ell + 2\mathring{\kappa})[Z^A] + \mathring{\mathcal{D}}^A W = 0. \tag{7.34}$$

The remaining components of the anomaly $\Delta_\xi g = 0$ only impose transformations of the remaining metric variables, namely $(\mathring{\kappa}, \kappa_{(1)}, \mathring{q}_{AB}, \lambda_{AB})$, under the near-horizon diffeomorphism, and they do not impose additional constraints on the diffeomorphism vector field $\xi$, at least up to the sub-leading order in $r$.

To summarize, the near-horizon symmetries are labelled by the diffeomorphism vector field of the form

$$\xi = \tau \ell + \left(W + r Z^A \pi_A\right) r\partial_r + \left(Y^A + r Z^A - r^2 Z^B \overline{\theta}_B{}^A\right) e_A + \mathcal{O}(r^3), \tag{7.35}$$

where $(\tau, W, Y^A, Z^A)$ are generic functions on $H$. We emphasize again that the functions $W$ and $Z^A$ are related via the equation (7.34). Transformations of leading-order metric components under this near horizon-diffeomorphism are

$$\delta_\xi \alpha = \ell[\tau] + Y^A \varphi_A + W, \tag{7.36a}$$

$$\delta_\xi \mathring{\kappa} = \left(\tau\ell + Y^A e_A - W\right)[\mathring{\kappa}] - \ell[W], \tag{7.36b}$$

$$-\mathrm{e}^{\alpha}\delta_\xi \beta_A = (e_A - \varphi_A)[\tau] + w_{AB}Y^B + Z_A, \tag{7.36c}$$

$$\delta_\xi V^A = -D_u Y^A, \tag{7.36d}$$

$$\delta_\xi \mathring{q}_{AB} = 2\left(\tau \mathring{\theta}_{AB} + \mathring{\mathcal{D}}_{(A}Y_{B)}\right). \tag{7.36e}$$

For the sub-leading components, we have that

$$\delta_\xi \kappa_{(1)} = \left(\tau\ell + Y^A e_A\right)[\kappa_{(1)}] + 2Z^A e_A[\mathring{\kappa}] - 2(\ell + 3\mathring{\kappa})[Z\cdot\pi], \tag{7.37a}$$

$$\delta_\xi \overline{\theta} = \left(\tau\ell + Y^A e_A + W\right)[\overline{\theta}] + \mathring{\mathcal{D}}_A Z^A, \tag{7.37b}$$

$$\delta_\xi \lambda_{AB} = (\tau\ell + W)[\lambda_{AB}] + \mathring{\mathcal{D}}_{(A}Z_{B)} + Y^C \mathring{\mathcal{D}}_C \lambda_{AB} + 2\lambda_{C(A}\mathring{\mathcal{D}}_{B)}Y^C. \tag{7.37c}$$

Let us comment on some special cases. The first special case, which we will assume in the upcoming section, is the case where we set the Carrollian connection $\beta_A = 0$. In this particular case, we have that the Carrollian acceleration is a total derivative $\varphi_A = e_A[\alpha]$ and the Carrollian vorticity vanishes $w_{AB} = 0$. It also follows from (7.36c) that the parameter $Z^A$ is now given by

$$Z_A = -(e_A - \varphi_A)[\tau], \qquad \text{for the case} \qquad \beta_A = 0. \tag{7.38}$$

In this case, the near-horizon diffeomorphism (7.35) agrees with [112, 221].

The second special case one can consider is when the velocity field vanishes, $V^A = 0$. In this case, the near-horizon diffeomorphism reduces to the Carrollian diffeomorphism,

$$\ell[Y^A] = 0, \qquad \text{for the case} \qquad V^A = 0. \tag{7.39}$$

In the upcoming section, we will also set the sub-leading horizontal diffeomorphism to be zero, i.e., $Z^A = 0$. The transformation labelled by $Z^A$ does not however contribute to the leading-order symplectic potential, and in a sense can be regarded as gauge on the null boundary $N$. Requiring $Z^A$ also imposes, following from (7.34) that

$$\mathring{\mathcal{D}}_A W = 0. \tag{7.40}$$

## 7.4 Einstein Equations and Noether Charges

We demonstrate in this section how the Einstein equations on the null boundary $N$ can be derived from the near-horizon diffeomorphism (7.35) and compute the Noether charges associated with these symmetries.

As we have mentioned, in the following analysis, we will consider the case where the Carrollian connection vanishes, $\beta_A = 0$. In this particular case, the Carrollian acceleration becomes the total derivative, $\varphi_A = e_A[\alpha]$, while the Carrollian vorticity now vanishes, $w_{AB} = 0$. Furthermore, in the 4-dimensional spacetime, the 2-dimensional curvature tensor ${}^{(2)}R_{AB}$ is diagonal,

$$ {}^{(2)}\mathring{R}_{AB} = \frac{1}{2}{}^{(2)}\mathring{R}\mathring{q}_{AB}. \tag{7.41} $$

where $\mathring{R} := \mathring{q}^{AB}\mathring{R}_{AB}$.

The components of the Einstein tensor whose corresponding Einstein equations governing dynamics of the null boundary are (see derivations in Appendix C)

$$ -\mathring{G}_{\ell\ell} = (\ell + \mathring{\theta})[\mathring{\mathcal{E}}] + \mathring{\mathcal{P}}\mathring{\theta} + \mathring{\mathcal{K}}^{AB}\mathring{\sigma}_{AB} \tag{7.42a} $$

$$ \mathring{G}_{\ell A} = (\ell + \mathring{\theta})[\pi_A] + \mathring{\mathcal{E}}\varphi_A + (\mathring{\mathcal{D}}_B + \varphi_B)\left(\mathring{\mathcal{K}}_A{}^B + \mathring{\mathcal{P}}\delta_A^B\right) \tag{7.42b} $$

$$ \mathring{G}_{\ell k} = (\ell + \mathring{\theta} + \mathring{\kappa})[\overline{\theta}] - (\mathring{\mathcal{D}}_A - \pi_A)\pi^A - \frac{1}{2}{}^{(2)}R \tag{7.42c} $$

$$ -\mathring{G}_{\langle AB\rangle} = 2\ell[\overline{\sigma}_{AB}] + (2\mathring{\kappa} - \mathring{\theta})\overline{\sigma}_{AB} + \overline{\theta}\mathring{\sigma}_{AB} - 4\overline{\sigma}_{C(A}\mathring{\sigma}_{B)}{}^C - 2(\mathring{\mathcal{D}} - \pi)_{\langle A}\pi_{B\rangle} \tag{7.42d} $$

The first two equations are kwon as the null Raychaudhuri equation and the Damour equations, respectively. Let us also remark here that in general, there will be the curvature term ${}^{(2)}\mathring{R}_{\langle AB\rangle}$ contained in the expression of the components $\mathring{G}_{\langle AB\rangle}$, although this term vanishes in the four dimensional case. In addition, the trace part of the components $\mathring{G}_{AB}$ is determined by the Ricci tensor,

$$ \frac{1}{2}\mathring{q}^{AB}\mathring{G}_{AB} = -R_{\ell k} = (\ell + \mathring{\mu})[\overline{\theta}] + \kappa_{(1)} - (\mathring{\mathcal{D}}_A + \varphi_A)\pi^A + \mathring{\sigma} : \overline{\sigma}. \tag{7.43} $$

Others components of the Einstein equations do not describe the time-evolution dynamics of the null boundary. Instead, they are viewed as constraints. The components that we will encounter during the analysis below are

$$ \mathring{G}_{kA} = -\overline{\theta}\pi_A + \mathring{\mathcal{D}}_B(\overline{\sigma}_A{}^B - \frac{1}{2}\overline{\theta}.\delta_A^B) \tag{7.44} $$

### 7.4.1 Noether Charges for tangential symmetries

To start with, we consider the tangential near-horizon diffeomorphism, $\xi = \tau\ell + Y^A e_A$ and analyze how it acts on the null boundary phase space. Let us recap that the Noether charges $Q_\xi$ associated with the symmetries $\xi$ are derived from the pre-symplectic potential $\Theta_H$ by evaluating the field space contraction,

$$I_\xi \Theta_H - \int_H \iota_\xi \boldsymbol{L} = C_\xi + Q_\xi \tag{7.45}$$

where $\boldsymbol{L}$ is the Lagrangian of the theory and $C_\xi$ is the constraint that vanishes once enforcing equations of motion.

Following the procedure that was put forward by the authors of [111], we can define for tangential diffeomorphism the canonical pre-symplectic potential on the boundary $N$ by exploiting the ambiguities of the pre-symplectic potential, known as the JKM ambiguities [162]. In our consideration, we define the canonical pre-symplectic potential on $N$ as

$$\Theta_N^{\text{can}} := \Theta_N - \delta\left(\int_N \mathring{\kappa}\boldsymbol{\epsilon}_N\right) - \mathring{\Theta}_S \tag{7.46}$$

$$= \int_N \left(-\mathring{\mathcal{E}}\delta\alpha - \pi_A \mathrm{e}^{-\alpha}\delta V^A + \frac{1}{2}\left(\mathring{\mathcal{K}}^{AB} + \mathring{\mathcal{P}}\mathring{q}^{AB}\right)\delta\mathring{q}_{AB}\right)\boldsymbol{\epsilon}_N, \tag{7.47}$$

which are given only in terms of the Carrollian conjugate pairs. We can then show that

$$I_{(\tau,Y)}\Theta_N^{\text{can}} = -\int_N \left(\tau\mathring{G}_{\ell\ell} + Y^A\mathring{G}_{\ell A}\right)\boldsymbol{\epsilon}_N + \mathring{Q}_{(\tau,Y)}. \tag{7.48}$$

We now see that the null Raychaudhuri equation $\mathring{G}_{\ell\ell} = 0$ and the Damour equations $\mathring{G}_{\ell A}$ are associated with the tangential diffeomorphism on $N$. This result has been already well-established in the literature (see [111]). The Noether charges are given (for non-zero $\beta_A$ by) by

$$\mathring{Q}_{(\tau,Y)} = \int_S \left(-\tau\mathring{\mathcal{E}} + Y^A\left(\pi_A + (\mathring{\mathcal{K}}_A{}^B + \mathring{\mathcal{P}}\delta_A^B)\mathrm{e}^\alpha\beta_B\right)\right)\mathring{\boldsymbol{\epsilon}}_S. \tag{7.49}$$

They are precisely the charges for Carrollian hydrodynamics on the null boundary.

### 7.4.2 The Einstein equation $\mathring{G}_{\ell k} = 0$ from symmetries

We next demonstrate that the component $\mathring{G}_{\ell k} = 0$ of the Einstein equations can be derived from symmetries. To accomplish this, we consider the rescaling symmetry generated by the diffeomorphism vector field $\xi = Wr\partial_r$. Recalling the transformations (7.36), we first perform the field space contraction of the leading-order pre-symplectic potential. One can show that it simply is the corner term,

$$I_W\Theta_N = -\int_S \frac{1}{2}W\mathring{\boldsymbol{\epsilon}}_S. \tag{7.50}$$

We next evaluate the sub-leading pre-symplectic potential with this symmetry,

$$\begin{aligned} I_W\Theta_N^{(1)} &= \int_N W\left(\ell[\overline{\theta}] + (\kappa + \frac{1}{2}\theta)\overline{\theta} + \kappa_{(1)} + \sigma : \overline{\sigma}\right)\boldsymbol{\epsilon}_N - \int_S \frac{1}{2}W\overline{\theta}\mathring{\boldsymbol{\epsilon}}_S \\ &= \int_H W\left(\ell[\overline{\theta}] + (\kappa + \frac{1}{2}\theta)\overline{\theta} + \kappa_{(1)} + \sigma : \overline{\sigma} - (\mathring{\mathcal{D}}_A + \varphi_A)\pi^A\right)\mathring{\boldsymbol{\epsilon}}_H - \int_S \frac{1}{2}W\overline{\theta}\mathring{\boldsymbol{\epsilon}}_S \\ &= -\int_H W\mathring{R}_{\ell k}\mathring{\boldsymbol{\epsilon}}_H - \int_S \frac{1}{2}W\overline{\theta}\mathring{\boldsymbol{\epsilon}}_S \end{aligned} \tag{7.51}$$

where to obtain the second equality, we added to the intregrand the term $(\mathring{\mathcal{D}}_A + \varphi_A)(W\pi^A)$ which integrated to zero and recall the condition $\mathring{\mathcal{D}}_A W = 0$.

When considering transverse diffeomorphisms, unlike the tangential one, the Lagrangian term $\iota_\xi\boldsymbol{L}$ now becomes relevant. For the Einstein-Hilbert Lagrangian $\boldsymbol{L} = \frac{1}{2}R\boldsymbol{\epsilon}_M$, we use the relation between spacetime curvature tensors and the Einstein tensor, $\frac{1}{2}R = R_{\ell k} - G_{\ell k}$, to show that the contribution from the Lagrangian term appears at the sub-leading order,

$$\iota_\xi\boldsymbol{L} \overset{H}{=} -rW(\mathring{R}_{\ell k} - \mathring{G}_{\ell k})\mathring{\boldsymbol{\epsilon}}_H + \mathcal{O}(r^2). \tag{7.52}$$

Gathering all the results, we can finally show that

$$\boxed{I_W\Theta_H - \int_H \iota_\xi\boldsymbol{L} = -r\int_N W\mathring{G}_{\ell k}\mathring{\boldsymbol{\epsilon}}_N + Q_W.} \tag{7.53}$$

It is important to appreciate the result we have just derived. We have shown that the Einstein equation $\mathring{G}_{\ell k} = 0$ on the null boundary $N$ actually corresponds to the rescaling transformation $\xi = Wr\partial_r$. This feature however is manifest only when considering the sub-leading terms in the pre-symplectic potential. The Noether charges associated with this symmetry, up to sub-leading order in powers of $r$, are given by

$$Q_W = \mathring{Q}_W + rQ_W^{(1)} = -\int_S \frac{1}{2}W\left(1 + r\overline{\theta}\right)\mathring{\boldsymbol{\epsilon}}_S. \tag{7.54}$$

Observe also that while the conservation laws $\mathring{G}_{\ell k} = 0$ only appear at sub-leading order, the Noether charges nonetheless have non-zero values on the null boundary $N$.

### 7.4.3 Deriving $G_{AB} = 0$ from symmetries

The remaining evolution equations on the null boundary are the horizontal components of the Einstein equations, $\mathring{G}_{AB} = 0$. Similar to the component $\mathring{G}_{\ell k} = 0$ the equations $\mathring{G}_{AB} = 0$ are unveiled at the sub-leading order of the pre-symplectic potential. The spin-2 symmetries that generate these Einstein equations are labelled by $D^{(A}Y^{B)}$. In the same vein as the previous computations, we consider the field space contraction of the sub-leading pre-symplectic potential, $I_Y\Theta_H^{(1)}$. Let us consider each sector of the pre-symplectic potential separately.

For the spin-0 sector of the pre-symplectic potential, we have that

$$\begin{aligned}
&\mathbb{P}_\alpha \delta_Y \alpha + \mathbb{P}_\kappa \delta_Y \kappa + \delta_Y \kappa_{(1)} + \mathbb{P}_{\overline{\theta}} \delta_Y \overline{\theta} \\
&= Y^A(\mathring{\mathcal{D}}_A + \varphi_A)(\kappa_{(1)} + 2\mathring{\kappa}\overline{\theta}) - (Y\cdot\varphi)(\ell[\overline{\theta}] + (\mathring{\kappa} + \mathring{\theta})\overline{\theta}) - \frac{1}{2}\mathring{\theta}Y^A\mathring{\mathcal{D}}_A\overline{\theta} \\
&= -(\mathring{\mathcal{D}}\cdot Y)\left(\kappa_{(1)} + 2\mathring{\kappa}\overline{\theta} - \frac{1}{2}\mathring{\theta}\overline{\theta}\right) - (Y\cdot\varphi)\left(\ell[\overline{\theta}] + (\mathring{\kappa}+\mathring{\theta})\overline{\theta} - \frac{1}{2}\mathring{\theta}\overline{\theta}\right) + \frac{1}{2}\overline{\theta}\mathcal{L}_Y\mathring{\theta},
\end{aligned} \tag{7.55}$$

where we dropped the divergence term as its integration on $H$ vanishes for the particular case when $\beta_A = 0$. The last term can be expressed as

$$\mathcal{L}_Y\mathring{\theta} = -(\mathring{\mathcal{D}}_A + \varphi_A)(\ell[Y^A]) - (Y\cdot\varphi)\mathring{\theta} + \ell[\mathring{\mathcal{D}}\cdot Y]. \tag{7.56}$$

We therefore obtain,

$$\begin{aligned}
&\left(\mathbb{P}_\alpha \delta_Y \alpha + \mathbb{P}_\kappa \delta_Y \kappa + \delta_Y \kappa_{(1)} + \mathbb{P}_{\overline{\theta}} \delta_Y \overline{\theta}\right)\mathring{\boldsymbol{\epsilon}}_H \\
&= -(\mathring{\mathcal{D}}\cdot Y)\left(\frac{1}{2}\ell[\overline{\theta}] + \kappa_{(1)} + 2\mathring{\kappa}\overline{\theta}\right)\mathring{\boldsymbol{\epsilon}}_H - (Y\cdot\varphi)\left(\ell[\overline{\theta}] + (\mathring{\kappa}+\mathring{\theta})\overline{\theta}\right)\mathring{\boldsymbol{\epsilon}}_H \\
&\quad + \frac{1}{2}\ell[Y^A]\mathring{\mathcal{D}}_A\overline{\theta}\mathring{\boldsymbol{\epsilon}}_H + \mathbf{d}\left(\frac{1}{2}(\mathring{\mathcal{D}}\cdot Y)\overline{\theta}\mathring{\boldsymbol{\epsilon}}_S\right).
\end{aligned} \tag{7.57}$$

The spin-1 sector only contains the velocity field term, which is simply $-e^{-\alpha}(\mathbb{P}_V)_A\delta_Y V^A = \ell[Y^A](\overline{\theta}\pi_A)$. There are two spin-2 components in the sub-leading pre-symplectic potential, namely the terms $(\mathbb{P}_q)^{AB}\delta_Y\mathring{q}_{AB}$ and $(\mathbb{P}_\lambda)^{AB}\delta_Y\lambda_{AB}$. The former is straightforwardly evaluated,

$$\begin{aligned}
\frac{1}{2}(\mathbb{P}_q)^{AB}\delta_Y\mathring{q}_{AB} = \mathring{\mathcal{D}}^{\langle A}Y^{B\rangle}&\left(\ell[\overline{\sigma}_{AB}] + (2\mathring{\kappa} - \mathring{\theta})\overline{\sigma}_{AB} - 4\overline{\sigma}_{C\langle A}\mathring{\sigma}_{B\rangle}{}^C\right) \\
&- (\mathring{\mathcal{D}}\cdot Y)\left(\frac{1}{2}(\ell+\mathring{\theta})[\overline{\theta}] - \mathring{\kappa}\overline{\theta} + \mathring{\sigma} : \overline{\sigma}\right).
\end{aligned} \tag{7.58}$$

The latter however requires more careful analysis. We first employ the Leibniz rule and write

$$(\mathbb{P}_\lambda)^{AB}\delta_Y\lambda_{AB} = \mathring{\sigma}^{ab}\mathcal{L}_Y\lambda_{ab} = Y^A\mathring{\mathcal{D}}_A\left(\mathring{\sigma}:\overline{\sigma}\right) - \lambda_{ab}\mathcal{L}_Y\left(\mathring{\theta}^{ab} - \frac{1}{2}\mathring{\theta}\mathring{q}^{ab}\right) \tag{7.59}$$

Let us focus on the term $\lambda_{ab}\mathcal{L}_Y\mathring{\theta}^{ab}$, which can be manipulated as follows

$$\begin{aligned}\lambda_{ab}\mathcal{L}_Y\mathring{\theta}^{ab} &= \frac{1}{2}\lambda_{ab}\mathcal{L}_Y\left(\mathring{q}^{ac}\mathring{q}^{bd}\mathcal{L}_\ell\mathring{q}_{cd}\right)\\ &= \lambda_a{}^b(\mathcal{L}_Y q^{ac})(\mathcal{L}_\ell q_{cb}) + \frac{1}{2}\lambda^{ab}\mathcal{L}_{[Y,\ell]}\mathring{q}_{ab} + \frac{1}{2}\lambda^{ab}\mathcal{L}_\ell\mathcal{L}_Y\mathring{q}_{ab}\\ &= \lambda_{AB}\ell\left[\mathring{\mathcal{D}}^{(A}Y^{B)}\right] - \lambda_A{}^B(\mathring{\mathcal{D}}_B + \varphi_B)\left(\ell[Y^A]\right) - (Y\cdot\varphi)\left(\mathring{\sigma}:\overline{\sigma} + \frac{1}{2}\mathring{\theta}\overline{\theta}\right).\end{aligned} \tag{7.60}$$

Recalling the result of (7.56), we arrive at the expression

$$\begin{aligned}(\mathbb{P}_\lambda)^{AB}\delta_Y\lambda_{AB}\mathring{\boldsymbol{\epsilon}}_H = {}&\mathring{\mathcal{D}}^{\langle A}Y^{B\rangle}\left(\ell[\overline{\sigma}_{AB}] + \overline{\theta}\mathring{\sigma}_{AB}\right)\mathring{\boldsymbol{\epsilon}}_H - \ell[Y^A]\mathring{\mathcal{D}}_B\overline{\sigma}_A{}^B\mathring{\boldsymbol{\epsilon}}_H\\ &- \mathbf{d}\left(\mathring{\mathcal{D}}^{\langle A}Y^{B\rangle}\overline{\sigma}_{AB}\mathring{\boldsymbol{\epsilon}}_S\right)\end{aligned} \tag{7.61}$$

Upon including the contribution from the corner term $\Theta_S^{(1)}$ in the sub-leading pre-symplectic potential, we overall obtain the contraction

$$\begin{aligned}I_Y\Theta_N^{(1)} = &\int_N\Big[\mathring{\mathcal{D}}^{\langle A}Y^{B\rangle}\left(2\ell[\overline{\sigma}_{AB}] + (2\mathring{\kappa} - \mathring{\theta})\overline{\sigma}_{AB} + \overline{\theta}\mathring{\sigma}_{AB} - 4\overline{\sigma}_{C(A}\mathring{\sigma}_{B)}{}^C\right)\\ &\qquad - (\mathring{\mathcal{D}}\cdot Y)\left(\ell[\overline{\theta}] + (\mathring{\kappa} + \frac{1}{2}\mathring{\theta})\overline{\theta} + \kappa_{(1)} + \mathring{\sigma}:\overline{\sigma}\right)\\ &\qquad - (Y\cdot\varphi)\left(\ell[\overline{\theta}] + (\mathring{\kappa} + \mathring{\theta})\overline{\theta}\right)\\ &\qquad - \ell[Y^A]\left(-\overline{\theta}\pi_A + \mathring{\mathcal{D}}_B(\overline{\sigma}_A{}^B - \frac{1}{2}\overline{\theta}\delta_A^B)\right)\Big]\boldsymbol{\epsilon}_N\\ &+ \int_S\left(-\mathring{\mathcal{D}}^{\langle A}Y^{B\rangle}\overline{\sigma}_{AB} + \frac{1}{2}(\mathring{\mathcal{D}}\cdot Y)\overline{\theta} + \frac{1}{2}(Y\cdot\varphi)\overline{\theta}\right)\mathring{\boldsymbol{\epsilon}}_S.\end{aligned} \tag{7.62}$$

Let us observe that the first three terms contained in the bulk piece almost have the form of the Einstein tensors $\mathring{G}_{\langle AB\rangle}$, the Ricci tensor $\mathring{R}_{\ell k}$ and the Einstein tensor $\mathring{G}_{\ell k}$, respectively. The last term however already has the form of the components $\mathring{G}_{kA}$ of the Einstein tensors. To write all these terms in the form of the Einstein tensors, we use that

$$\int_H 2\mathring{\mathcal{D}}^{\langle A}Y^{B\rangle}\left(-\mathring{\mathcal{D}}_{\langle A}\pi_{B\rangle} + \pi_{\langle A}\pi_{B\rangle}\right) + (\mathring{\mathcal{D}}\cdot Y)(\mathring{\mathcal{D}}_A + \varphi_A)\pi^A + (Y\cdot\varphi)\left((\mathring{\mathcal{D}}_A - \pi_A)\pi^A + \frac{1}{2}{}^{(2)}R\right) = 0, \tag{7.63}$$

up to the divergence terms of the form $(\mathring{\mathcal{D}}_A + \varphi_A)X^A$ that vanish once integrated on the stretched horizon $H$. We therefore arrive at the final result

$$\boxed{\begin{aligned} I_Y \Theta_H^{(1)} = &\int_H \left( -\mathring{\mathcal{D}}^{(A} Y^{B)} \mathring{G}_{AB} - (Y \cdot \varphi) \mathring{G}_{\ell k} - \ell[Y^A] \mathring{G}_{kA} \right) \mathring{\boldsymbol{\epsilon}}_H \\ &+ \int_S \left( -\mathring{\mathcal{D}}^{\langle A} Y^{B\rangle} \overline{\sigma}_{AB} + \frac{1}{2} (\mathring{\mathcal{D}} \cdot Y) \overline{\theta} + \frac{1}{2} (Y \cdot \varphi) \overline{\theta} \right) \mathring{\boldsymbol{\epsilon}}_S . \end{aligned}} \tag{7.64}$$

Note that we used the fact that in 4-dimensional spacetime, the trace of the horizontal components of the Einstein tensor coincides with the Ricci tensor $\mathring{q}^{AB} \mathring{G}_{AB} = -2\mathring{R}_{\ell k}$.

We have discovered that the remaining Einstein equations $\mathring{G}_{AB} = 0$ on the finite distance null boundary are associated with the spin-2 symmetries. These spin-2 symmetries are related to the diffeomorphism $\xi = Y^A e_A$ and the parameters labelling these transformations are the specific symmetric tensor $\mathring{\mathcal{D}}^{(A} Y^{B)}$. We also recover the Einstein equation $\mathring{G}_{\ell k} = 0$ with the symmetry parameter being $Y \cdot \varphi$ in this case. The Einstein equation $\mathring{G}_{kA} = 0$, although it is not the evolution equation on the boundary $N$, is also obtained. Note however that if one restricts to a diffeomorphism that is Carrollian, such that $\ell[Y^A] = 0$, the term $\mathring{G}_{kA}$ disappears from (7.64).

### 7.4.4 Bianchi Identity

We have shown that the Einstein equations governing gravitational dynamics on the null boundary are indeed consequences of near-horizon symmetries. Interestingly for some components of the Einstein equations, namely $\mathring{G}_{\ell k} = 0$ and $\mathring{G}_{AB} = 0$, this appealing correspondence is only manifest when carefully analyzing phase space of the stretched horizon located near the null boundary.

We will now show that the result (7.64), revealing the special (diffeomorphism-related) spin-2 symmetries that generate the spin-2 equations $\mathring{G}_{AB} = 0$, follows from the Bianchi identity of the Einstein tensor,

$$\nabla^a G_{ab} = 0. \tag{7.65}$$

In what follows, we will focus on the horizontal component $e_A{}^a \nabla^b G_{ab}$ of the Bianchi identity. Using the expression of the metric in terms of the frame fields, we can then write the divergence of the Einstein tensor as

$$\begin{aligned} e_A{}^a \nabla^b G_{ab} &= e_A{}^a (n^b k^c + k^b \ell^c + q^{BC} e_B{}^b e_C{}^c) \nabla_c G_{ab} \\ &= e_A{}^a n^b \nabla_k G_{ab} + e_A{}^a k^b \nabla_\ell G_{ab} + q^{BC} e_A{}^a e_B{}^b \nabla_{e_C} G_{ab}. \end{aligned} \tag{7.66}$$

Let us consider each terms separately. We begin with the first term which, by using the Leibniz rule, can be written as

$$\begin{aligned} e_A{}^a n^b \nabla_k G_{ab} &= k[G_{nA}] - G_{Aa}\nabla_k n^a - G_{na}\nabla_k e_A{}^a \\ &\overset{N}{=} \partial_r G_{nA} + (\omega^B + \varphi^B)\mathring{G}_{AB} - \mathring{\kappa}\mathring{G}_{kA} - \overline{\theta}_A{}^B \mathring{G}_{\ell B} + \omega_A \mathring{G}_{\ell k}, \end{aligned} \tag{7.67}$$

where we used that $[k, e_A] = 0$. Similarly for the second term, we can evaluate it on the null boundary $N$ as

$$\begin{aligned} e_A{}^a k^b \nabla_\ell G_{ab} &= \ell[G_{kA}] - G_{Aa}\nabla_\ell k^a - G_{ka}\nabla_\ell e_A{}^a \\ &\overset{N}{=} (\ell + \mathring{\kappa})[\mathring{G}_{kA}] + (\omega^B + \varphi^B)\mathring{G}_{AB} - (\varphi_A + \omega_A)\mathring{G}_{\ell k} - \mathring{\theta}_A{}^B \mathring{G}_{kB}. \end{aligned} \tag{7.68}$$

The last term can be expressed as follows

$$\begin{aligned} e_A{}^a e_B{}^b \nabla_{e_C} G_{ab} &= e_C[G_{AB}] - G_{Aa}\nabla_{e_C} e_B{}^a - G_{Ba}\nabla_{e_C} e_A{}^a \\ &\overset{N}{=} e_C[\mathring{G}_{AB}] - {}^{(2)}\mathring{\Gamma}^D_{CB}\mathring{G}_{AD} - {}^{(2)}\mathring{\Gamma}^D_{CA}\mathring{G}_{BD} + \overline{\theta}_{CB}\mathring{G}_{A\ell} + \mathring{\theta}_{CB}\mathring{G}_{kA} \\ &\quad + \overline{\theta}_{CA}\mathring{G}_{B\ell} + \mathring{\theta}_{CA}\mathring{G}_{kB} \\ &= \mathring{\mathcal{D}}_C \mathring{G}_{AB} + \overline{\theta}_{CB}\mathring{G}_{A\ell} + \mathring{\theta}_{CB}\mathring{G}_{kA} + \overline{\theta}_{CA}\mathring{G}_{B\ell} + \mathring{\theta}_{CA}\mathring{G}_{kB}. \end{aligned} \tag{7.69}$$

Putting these results together, we therefore express the Bianchi identity $e_A{}^a \nabla^b G_{ab} = 0$ as

$$\boxed{-\partial_r G_{nA} \overset{N}{=} \left(\mathring{\mathcal{D}}^B + \varphi^B\right)\mathring{G}_{AB} + (\ell + \mathring{\theta})\mathring{G}_{kA} + \overline{\theta}\mathring{G}_{\ell A} - \varphi_A \mathring{G}_{\ell k}.} \tag{7.70}$$

Having derived the desired Bianchi identity, we now finally explain how it connects to the result (7.64) provided in the previous section. The key point of this connection lies in the constraint term when computing the Noether charges associated to the near-horizon diffeomorphism on the surface $H$. For the Einstein-Hilbert theory, this constraint is the Einstein tensor on $H$,

$$C_\xi = -\int_H G_{n\xi}\boldsymbol{\epsilon}_H. \tag{7.71}$$

When considering the diffeomorphism $\xi = Y^A e_A$, the constraint only imposes the Einstein equations $G_{nA} = 0$ on $H$. As we have been trying to emphasize the central plot of this Chapter, more underlying information of the null boundary $N$ can only be accessed by considering the surface $H$, not as its own surface, but instead as a near-horizon expansion around $N$. This means that the constraint $C_\xi$ can be expand as a power series in $r$ around the boundary located at $r=0$ as

$$C_\xi = \mathring{C}_\xi + r C_\xi^{(1)} + \mathcal{O}(r^2), \tag{7.72}$$

where the leading-order imposes the Damour equation on $N$,

$$\mathring{C}_\xi = \int_N Y^A \mathring{G}_{\ell A} \boldsymbol{\epsilon}_N = 0. \tag{7.73}$$

The sub-leading order of the constraint is given by

$$\begin{aligned} C_\xi^{(1)} &= -\int_N Y^A \left(\partial_r G_{nA} + \overline{\theta}\mathring{G}_{\ell A}\right) \boldsymbol{\epsilon}_N \\ &= \int_N Y^A \left(\left(\mathring{\mathcal{D}}^B + \varphi^B\right) \mathring{G}_{AB} + (\ell + \mathring{\theta})[\mathring{G}_{kA}] - \varphi_A \mathring{G}_{\ell k}\right) \boldsymbol{\epsilon}_N, \end{aligned} \tag{7.74}$$

where we used the previously derived Bianchi identity (7.70). We hence obtain, up to the boundary term, the sub-leading constraint

$$C_\xi^{(1)} = \int_H \left(-\mathring{\mathcal{D}}^{(A} Y^{B)} \mathring{G}_{AB} - (Y \cdot \varphi)\mathring{G}_{\ell k} - \ell[Y^A]\mathring{G}_{kA}\right). \tag{7.75}$$

This is precisely the constraint obtained in the equation (7.64).

# Chapter 8

# Concluding Remarks

Let us summarize what has been done in this thesis and mention some possible future directions.

**Part I: Edge Modes: Dynamics and Duality**

The first part of this thesis was dedicated to the investigation of emergent boundary degrees of freedoms called *edge modes* and their associated *corner symmetries*. Our objective was to unravel the role of these edge modes in theories with internal gauge symmetries.

In Chapter 3, we proposed an extended variational principle which supplements the bulk symplectic structure with a boundary symplectic structure including the edge mode fields and descending from a boundary action. With the addition of a boundary action with edge modes, we then demonstrated that edge modes are necessary in order to factorize the Hilbert space, phase space, or path integral of a theory (this property is summarized schematically on figure 3.1). Most importantly, once a theory has been split between two subregions by introducing edge modes on the boundary, the bulk of a subregion can be evaluated on-shell and a residual dynamics gets imprinted on the boundary. This agrees with the proposal made in [173], which we have now therefore connected with the extended phase space constructions originally proposed in [32] and later expanded in [34, 36] (for the cases concerning internal gauge symmetries). We put our proposal for deriving the boundary dynamics of edge modes to the test in various examples of theories with Abelian internal gauge symmetries, including Chern-Simons theory, Maxwell theory, Maxwell-Chern-Simons theory, and the topological BF theory.

In Chapter 4, we explored a surprising role of corner symmetries in the understanding of electromagnetic duality, the existence of dual boundary magnetic charges and their centrally-extended algebra with electric charges which has been studied in [82, 108, 109]. We argued in this thesis that the existence of dual charges is naturally described in the first order formulation. We validated this proposal by studying the first order formulation of Maxwell's theory and we showed that in this picture, the magnetic charges are the zero-modes of the translational symmetry of the first order theory. Moreover, the electric and magnetic charges inherit the centrally-extended charge algebra also descending from the first order formulation.

Many directions however remain to be explored. Let us list some of them below.

*i) Relationship between boundary conditions, Hamiltonian, and dynamics.* It is now important to study more precisely the nature of the boundary theory obtained for non-topological theories ( e.g., Maxwell theory) and investigate in particular its dependency on the choice of boundary Hamiltonian.

*ii) Extension to other theories*: There are essentially two types of theories to which the present work should be extended — non-Abelian gauge theories, and theories with diffeomorphism symmetry. Here we have presented the construction of the extended action and symplectic structure for non-Abelian gauge theories in appendix A.4. This does not present any conceptual difference with the Abelian case. However, the study of the boundary dynamics of non-Abelian theories is for the most part unknown (although preliminary steps have been taken in [173]). In the case of diffeomorphism symmetry however, already the introduction of the edge modes in the extended phase space or the boundary action is conceptually different from what we have presented in this work, since it requires embedding variables [32]. Inclusion of gravitational edge modes through the boundary action has been studied extensively in [38–40]. It would then be interesting to apply or generalize our proposal to study dynamics and entanglement of edge modes in gravity.

Concerning the dualities, it would also be interesting to study how this construction can be extended to the non-Abelian case. A new complication in this case is that the reducibility condition (4.22) now involves a gauge covariant derivative $\mathrm{d}_A\phi = 0$, and cannot naively be solved without imposing a condition on the boundary field strength.

*iii) Link with soft modes at asymptotic infinity.* A major open question is to relate the edge modes presented in this work (and in all the references) to soft modes which appear in the infrared regime of massless theories [21]. Some steps in this direction have already been taken in [53, 56], and there have also been proposals for the description of the infrared dynamics itself [227, 228]. It is therefore natural to ask whether such proposals can be recovered (or corrected) from the boundary dynamics of Maxwell theory presented in this work, provided we can choose appropriate boundary conditions and push the boundary to infinity.

It would also be interesting to properly study the appearance of magnetic charges from the first order BF theory asymptotically, following e.g. [110] and [214].

**Part II: Carrollian Hydrodynamics**

In the second part of this thesis, we turned our attention to the study of gravitational dynamics of spacetime around null boundary and explored the emerging connection with the so-called Carroll geometries and Carrollian hydrodynamics (which was first proposed in [121]).

Chapter 5 was devoted to the construction of Carroll geometries starting from the most fundamental building blocks, namely Carroll structures. We then proceeded to examine Carrollian fluid and their corresponding hydrodynamics. We presented two methods to derive Carrollian hydrodynamic equations. In the first (and rather old-school) method, we started from conservation laws of the relativistic energy-momentum tensor, then properly took the Carrollian limit ($c \to 0$) of the standard relativistic conservation laws. Our derivations could be viewed as a generalization of the one originally presented in [124] due to the fact that we now have in our construction the fluid velocity $V^A$ and the sub-leading sphere metric $\lambda_{AB}$. These two quantities are indeed important parts of the phase space of Carrollian hydrodynamics. The second route, which was the highlight of this chapter, was to viewed Carrollian hydrodynamics as the consequence of symmetries. We argued that Carrollian diffeomorphisms are not sufficient to derive the full set of Carrollian fluid equations (this has already been done in [144]) and that we need to go beyond Carrollian diffeomorphisms. To this end, we introduced the notion of near-Carrollian symmetries (5.89) and showed, once and for all, that it leads to the complete Carrollian conservation laws.

In Chapter 6, we considered a subregion of spacetime bounded by a null boundary and located at a finite distance. We expressed this spacetime region as a series of timelike hypersurfaces, called stretched horizons. We then studied the geometry of theses stretched horizons using the so-called rigging technique and showed that there is a natural Carroll structure induced from this rigged structure, enabling us to talk about Carroll geometries and Carrollian hydrodynamics beyond null surfaces. We then proposed the Carrollian energy-momentum tensor of the stretched horizon and showed that its conservation laws infer the Einstein equations on the surfaces (or vice versa).

Gravitational phase space of stretched horizons was finally investigated in Chapter 7. We showed that the pre-symplectic potential, capturing phase space information, is given by the Carrollian conjugate pairs. We additionally put forward the perspective that pre-symplectic potential of stretched horizons located extremely close to the null boundary actually encodes *sub-leading* information of the null boundary phase space. In this thesis, we unraveled for the first time the symmetries associated with the Einstein equations governing the evolutions of the radial expansion $\overline{\theta}$ and the shear $\overline{\sigma}_{AB}$ on a finite distance null boundary

Some possible future avenues of investigation include:

*i*) *Thermodynamics of Carrollian fluids*: Having this new type of fluid, the Carrollian fluid, a natural question therefore emerges — what are thermodynamical properties of Carrollian fluids? Although this question may not garner much interest in the field of fluid mechanics due to the sole fact that everyday life fluids are Galilean in nature, we believe that this question will provide unprecedented insights to the realm of black hole physics. One possible direction we would like to investigate in the future is the notion of *thermodynamical horizons*, the types of horizons that obey all laws of thermodynamics, and also the universal notion of equilibrium in any surface.

*ii*) *Understanding better the sub-leading charges*: We believe that the results presented in this thesis may open new investigation windows to probe hidden (sub-leading) symmetries and charges on null surfaces at finite distances. However, we only managed to scratch the surface, laid down the idea, and presented the symmetries and Noether charges in some limiting cases. Needless to say that numerous studies are required in order to deepen our understanding on this topic. One question, for example, concerns the algebras of these charges, especially how to compute the algebra of the charges appearing at different orders of the pre-symplectic potential.

*iii*) *Connection with null infinity*: It would be interesting to connect our idea at finite distances with asymptotic null infinity. The understanding of sub-leading (and sub-sub-leading) symmetries and charges at null infinity is in a sense far more developed than the finite distance cases. For example, the recent works [225, 229] have suggested the existence of infinite tower of higher-spins symmetries, charges, and associated conservations at null infinity. This characteristic, we believe, should also persist at finite distances. In-depth investigations however need to be done.

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# APPENDICES

# Appendix A

# Edge Modes

## A.1 Maxwell theory in radial gauge

Inspired by holography, let us work in the radial Hamiltonian formulation, where the radial coordinate $r$ is treated as the Hamiltonian time. The natural gauge fixing corresponding to this choice is then the radial gauge where we set $A_r = 0$. This condition, together with the Lorenz gauge $\partial^\mu A_\mu = 0$, are the gauge choices used in [173] and in section 3.3.2.

Let us focus on the 3-dimensional case, and decompose the spacetime coordinates $x^\mu = (t, r, \phi)$ as $x^\mu = (r, y^i)$, with $y^i = (t, \phi)$ the coordinates on the $r =$ constant hypersurfaces. With this, the spacetime metric can be decomposed as

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}r^2 + q_{ij}(r)\mathrm{d}y^i \mathrm{d}y^j, \tag{A.1}$$

where in cylindrical coordinates

$$q_{ij}(r)\mathrm{d}y^i \mathrm{d}y^j = -\mathrm{d}t^2 + r^2 \mathrm{d}\phi^2. \tag{A.2}$$

Placing the time-like boundary $\partial M$ at $r = \ell$, we have that the induced metric at the boundary is $g_{ij}\big|_{\partial M} = q_{ij}(\ell) =: q_{ij}$. With this radial decomposition, all the total derivatives are along the directions $y^i$, and can therefore be discarded because of our choice of cylindrical topology $M = \mathbb{R} \times D$. We will therefore freely integrate by parts over $t$ and $\phi$.

Similarly to the standard Hamiltonian analysis with respect to time $t$, the bulk Maxwell action in (3.33) can be written in radial Hamiltonian form as

$$S_M = \int_M \sqrt{|g|}\big(\Pi^i \partial_r A_i - H\big), \tag{A.3}$$

where the conjugate momentum to $A_i$ is $\Pi^i \coloneqq -F^{ri}$, and the Hamiltonian is

$$H = \frac{1}{4} F^{ij} F_{ij} - \frac{1}{2} \Pi^i \Pi_i. \tag{A.4}$$

We now use a Hodge decomposition of the gauge field and the momenta by writing

$$A_i = \partial_i \alpha + \varepsilon_{ij} \partial^j \beta \coloneqq \partial_i \alpha + \beta_i, \qquad \Pi^i = \partial^i \xi + \varepsilon^{ij} \partial_j \lambda \coloneqq \partial^i \xi + \lambda^i, \tag{A.5}$$

where $\beta_i$ and $\lambda^i$ are the divergence-free parts $\partial^i \beta_i = 0 = \partial_i \lambda^i$. The constraint enforced by the radial gauge fixing is that

$$\partial_i \Pi^i = \partial^2 \xi = 0, \tag{A.6}$$

where we define $\partial^2 \coloneqq \partial_i \partial^i$ to be the Laplace operator on the slices of constant radius $r$. In terms of this Hodge decomposition, the canonical term of the action and the terms of the Hamiltonian can be decomposed as

$$\Pi^i \partial_r A_i = \partial^i \xi \partial_i \partial_r \alpha + \lambda^i \partial_i \partial_r \alpha + \partial^i \xi \partial_r \beta_i + \lambda^i \partial_r \beta_i = -\partial^2 \xi \partial_r \alpha + \lambda^i \partial_r \beta_i, \tag{A.7a}$$

$$\Pi^i \Pi_i = \partial^i \xi \partial_i \xi + 2 \partial^i \xi \lambda_i + \lambda^i \lambda_i = -\xi \partial^2 \xi + \lambda^i \lambda_i, \tag{A.7b}$$

$$\frac{1}{2} F^{ij} F_{ij} = \partial^i \beta^j \partial_i \beta_j - \partial^i \beta^j \partial_j \beta_i = -\beta^i \partial^2 \beta_i. \tag{A.7c}$$

With this, the bulk Maxwell action becomes

$$S_M = \int_M \sqrt{|g|} \left( -\left( \partial_r \alpha + \frac{1}{2} \xi \right) \partial^2 \xi + \lambda^i \partial_r \beta_i + \frac{1}{2} \beta^i \partial^2 \beta_i + \frac{1}{2} \lambda^i \lambda_i \right). \tag{A.8}$$

The first term vanishes once the constraint (A.6) is imposed. Path integrating over the momentum variable $\lambda$ then yields

$$S_M = \frac{1}{2} \int_M \sqrt{|g|} \left( \beta^i \partial^2 \beta_i - \partial_r \beta^i \partial_r \beta_i \right), \tag{A.9}$$

so one can see that the bulk contribution is determined by the divergence-free part of the Hodge decomposition. Notice that if we write explicitly the path integral with the Hodge decomposition, taking into account the change of measure

$$\mathcal{D}[A, \Pi] = \mathcal{D}[\alpha, \beta, -\partial^2 \xi, \lambda] = \det(-\partial^2) \mathcal{D}[\alpha, \beta, \xi, \lambda] \tag{A.10}$$

and the change of variables in the Gauss constraint using

$$\delta\big(\partial^2 \xi\big) = \det(-\partial^2)^{-1} \delta(\xi), \tag{A.11}$$

the two determinant factors cancel out. Furthermore, there is no determinant factor produced when integrating over $\lambda$, and we are therefore left with the path integral over $\beta$ of (A.9). This is a crucial difference with the Hodge decomposition of the path integral in the temporal gauge, which produces determinant factors as explained in appendix A.2.

We can now add to this the boundary action containing the edge mode field $a$ to obtain the full bulk + boundary action (3.33). More precisely, using the radial gauge with the Hodge decomposition, imposing the constraint (A.6) and integrating out the momenta $\lambda$, we obtain the extended Maxwell action[1]

$$S = \frac{1}{2}\int_M \sqrt{|g|}\big(\beta^i\partial^2\beta_i - \partial_r\beta^i\partial_r\beta_i\big) + \int_{\partial M}\sqrt{|q|}\,\varepsilon^{im} j_i(\beta_m + \partial_m\varphi), \tag{A.12}$$

where we have introduced the gauge-invariant scalar $\varphi \coloneqq a + \alpha$. Variation with respect to $\beta_i$ gives in the bulk the flat massless Klein–Gordon equation in cylindrical coordinates,

$$\frac{1}{\sqrt{|g|}}\partial_\mu\big(\sqrt{|g|}\partial^\mu\beta^i\big) = -\partial_t^2\beta^i + \frac{1}{r}\partial_r(r\partial_r\beta^i) + \frac{1}{r^2}\partial_\phi^2\beta^i = 0, \tag{A.13}$$

and on the boundary the boundary condition

$$\partial_r\beta^i(\ell, y) = -\varepsilon^{im} j_m(y), \tag{A.14}$$

where $\beta^i(\ell, y) = \beta^i(r, y)\big|_{\partial M}$. Similarly to the calculation (3.38), path integrating over $\beta$ in the bulk produces an operator determinant for the massless scalar, and, recalling that on-shell the bulk action is a boundary term, the total boundary action (i.e. the initial one in (3.33) plus the piece coming on-shell from the bulk) becomes

$$S_{\text{edge}} = \int_{\partial M}\sqrt{|q|}\left(\varepsilon^{im} j_i(\beta_m + \partial_m\varphi) - \frac{1}{2}\beta_i\partial_r\beta^i\right) = \int_{\partial M}\sqrt{|q|}\varepsilon^{im} j_i\left(\frac{1}{2}\beta_m[j] + \partial_m\varphi\right), \tag{A.15}$$

where we have used the boundary condition. This expression is the effective boundary action in (3.38).

The boundary value $\beta_i[j]$ is now determined by solving the bulk equation of motion with respect to a Neumann-type boundary condition, i.e.

$$\Box\beta^i = 0, \qquad \partial_r\beta^i(\ell, y) = -\varepsilon^{im} j_m(y), \tag{A.16}$$

[1] We deliberately include the boundary metric into the boundary volume form. This does not alter the analysis of the extended phase space symplectic structure, as it can be viewed as the redefinition of $j$.

which is the 3-dimensional version of (3.39) in the radial gauge. The bulk equation is solved by going to momentum space, with $k_i = (k_t, k_\phi)$, as

$$\beta^i(r,y) = \int \frac{\mathrm{d}^2k}{2\pi} J_{k_\phi}(rk_t)\tilde{\beta}^i(k)e^{\mathrm{i}k\cdot y}, \tag{A.17}$$

where $k \cdot y = k_i y^i$, and $J_n$ denotes the Bessel function of integer order. Note that $k_\phi$ has a discrete spectrum due to the compactness of the $\phi$ direction. Now, writing

$$j_i(y) = \int \frac{\mathrm{d}^2k}{2\pi}\tilde{j}_i(k)e^{\mathrm{i}k\cdot y}, \qquad \varphi(y) = \int \frac{\mathrm{d}^2k}{2\pi}\tilde{\varphi}(k)e^{\mathrm{i}k\cdot y}, \tag{A.18}$$

the boundary condition translates into

$$\tilde{\beta}^i(k) = -\frac{1}{k_t\partial_r J_{k_\phi}(\ell k_t)}\varepsilon^{im}\tilde{j}_m(k). \tag{A.19}$$

We therefore obtain $\beta$ at the boundary in the form

$$\beta^i(\ell,y) = -\varepsilon^{im}\int \frac{\mathrm{d}^2k}{2\pi}\tilde{G}(k)\tilde{j}_m(k)e^{\mathrm{i}k\cdot y}, \tag{A.20}$$

where $\tilde{G}(k)$ represents the momentum space Green function. In the present case, we have

$$\tilde{G}(k) = \frac{J_{k_\phi}(\ell k_t)}{k_t\partial_r J_{k_\phi}(\ell k_t)}. \tag{A.21}$$

Putting this together, the effective edge mode action becomes

$$S_{\text{edge}} = \int \mathrm{d}^2k \left(\frac{1}{2}\tilde{j}^i(k)\tilde{G}(k)\tilde{j}_i(-k) + \varepsilon^{im}\tilde{j}_i(k)(\mathrm{i}k_m)\tilde{\varphi}(-k)\right). \tag{A.22}$$

As expected, this action is quadratic in $\tilde{j}$. We can now choose to integrate over $\tilde{\varphi}$ to obtain the condition

$$\varepsilon^{im}\tilde{j}_i(k)k_m = 0 \qquad \Rightarrow \qquad \tilde{j}_i(k) \sim k_i, \tag{A.23}$$

which is equivalent to the condition that boundary current is conserved, i.e. $\mathrm{d}j = 0$. The path integral then reduces to an integral over all conserved currents of the quadratic action written above. Alternatively, we can integrate out the current $\tilde{j}_i$ in order to get the quadratic action for $\tilde{\varphi}$ which is

$$S_{\text{edge}} = \frac{1}{2}\int \mathrm{d}^2k\, k^2\tilde{\varphi}(k)\tilde{G}(k)^{-1}\tilde{\varphi}(-k). \tag{A.24}$$

This is the effective boundary action in momentum space.

## A.2 Maxwell–Chern–Simons theory in temporal gauge

Here we present the detailed calculations of the Hodge decomposition of MCS theory, keeping carefully all the boundary terms, together with the manipulations of the path integral with and without boundaries. We recall that the spacetime has the topology $M=\mathbb{R}\times D$ of an infinite cylinder with $x^\mu=(t,r,\phi)$ such that $\varepsilon^{tr\phi}=1$, so total derivatives $\partial_t$ and $\partial_\phi$ can be ignored. We will constantly use this fact to freely move these derivatives around. For simplicity, even though we are using cylindrical-looking coordinates, our choice of metric will be $g_{\mu\nu}=\mathrm{diag}(-1,1,1)$, so we will allow ourselves to write the expressions below with all spatial indices downstairs for simplicity, and drop the determinant $\sqrt{|g|}$ in the integrals. This does not affect the results of this appendix. We denote the spatial Levi–Civita tensor by $\varepsilon_{ab}\coloneqq\varepsilon^t{}_{ab}$, with $\varepsilon_{ab}\varepsilon_{ac}=\delta_{bc}$ and $\varepsilon_{r\phi}=1$. Finally, the spatial Laplacian will be denoted by $\Delta\coloneqq\partial^a\partial_a$, the wave operator by $\Box=-\partial_t^2+\Delta$, and the time derivative $\partial_t\alpha$ by a dot $\dot{\alpha}$.

With a $2+1$ decomposition identifying the momentum, the Gauss constraint, and the Hamiltonian, the bulk part of the action in (3.45) can be written as

$$S_M=\int_M \Pi_a\dot{A}_a+A_tG-H-\int_{\partial M}A_t\Pi_r. \tag{A.25}$$

The momentum conjugated to the gauge field is

$$\Pi_a=-F^{ta}+m\varepsilon_{ab}A_b=F_{ta}+m\varepsilon_{ab}A_b, \tag{A.26}$$

where one should notice that the first term has picked up a sign because we have lowered the indices. The Gauss constraint is

$$G=\partial_a(\Pi_a+m\varepsilon_{ab}A_b), \tag{A.27}$$

and the Hamiltonian is

$$H=\frac{1}{4}(F_{ab})^2+\frac{1}{2}(\Pi_a-m\varepsilon_{ab}A_b)^2\eqqcolon\frac{1}{4}(F_{ab})^2+\frac{1}{2}(E_a)^2. \tag{A.28}$$

We are now going to rewrite these quantities using the Hodge decomposition

$$A_a=\partial_a\alpha+\varepsilon_{ab}\partial_b\beta,\qquad\qquad \Pi_a=\partial_a\xi+\varepsilon_{ab}\partial_b\lambda. \tag{A.29}$$

In various expressions, we will only keep total derivatives in $r$ since the ones in $\phi$ vanish when going to the boundary. In these total derivative, which will give boundary terms, we will furthermore use (A.26) and the temporal gauge $A_t=0$ to rewrite

$$\xi=\dot{\alpha}-m\beta,\qquad\qquad \lambda=\dot{\beta}+m\alpha. \tag{A.30}$$

This is justified since on the boundary it is $A$ (and therefore $\alpha$ and $\beta$) which is conjugated to the edge mode field $a$. The Hodge decomposition gives

$$\begin{aligned}
\Pi_a\dot{A}_a &= -\xi\Delta\dot{\alpha}-\lambda\Delta\dot{\beta}+\partial_r(\xi\partial_r\dot{\alpha}+\lambda\partial_r\dot{\beta}+\xi\partial_\phi\dot{\beta}-\lambda\partial_\phi\dot{\alpha})\\
&= -\xi\Delta\dot{\alpha}-\lambda\Delta\dot{\beta}+\partial_r\big(\dot{\alpha}\partial_r\dot{\alpha}+\dot{\beta}\partial_r\dot{\beta}+2\dot{\alpha}\partial_\phi\dot{\beta}-m(\alpha\partial_\phi\dot{\alpha}+\beta\partial_\phi\dot{\beta})-m\partial_r(\dot{\alpha}\beta)\big),
\end{aligned} \tag{A.31a}$$

$$G=\Delta(\xi-m\beta)=:\Delta\tilde{G}, \tag{A.31b}$$

$$\frac{1}{2}(F_{ab})^2=\beta\Delta^2\beta+\partial_r^2(\partial_r\beta\partial_r\beta-\beta\partial_r^2\beta), \tag{A.31c}$$

$$\begin{aligned}
(E_a)^2 &= -\xi\Delta\xi-\lambda\Delta\lambda-2m(\xi\Delta\beta-\lambda\Delta\alpha)-m^2(\alpha\Delta\alpha+\beta\Delta\beta)\\
&\quad+\partial_r\big(\xi\partial_r\xi+\lambda\partial_r\lambda+2\xi\partial_\phi\lambda-2m(\lambda\partial_r\alpha-\xi\partial_r\beta+\xi\partial_\phi\alpha+\lambda\partial_\phi\beta)+m^2(\alpha\partial_r\alpha+\beta\partial_r\beta+2\alpha\partial_\phi\beta)\big)\\
&= -\xi\Delta\xi-\lambda\Delta\lambda-2m(\xi\Delta\beta-\lambda\Delta\alpha)-m^2(\alpha\Delta\alpha+\beta\Delta\beta)+\partial_r(\dot{\alpha}\partial_r\dot{\alpha}+\dot{\beta}\partial_r\dot{\beta}+2\dot{\alpha}\partial_\phi\dot{\beta}).
\end{aligned} \tag{A.31d}$$

From this, we can now derive several results.

First, let us focus on the bulk contributions to explain how the path integral for the massive scalar field arises. We will perform manipulations at the level of the Lorentzian path integral, but simply write the actions alone in order to avoid unnecessary notational cluttering. Working in the temporal gauge $A_t=0$ we have to impose the Gauss constraint, which we write here in the form $\tilde{G}=\xi-m\beta=0$. This transforms all the bulk terms above according to

$$\begin{aligned}
S_M &= \frac{1}{2}\int_M -2\xi\Delta\dot{\alpha}-2\lambda\Delta\dot{\beta}-\beta\Delta^2\beta+\xi\Delta\xi+\lambda\Delta\lambda+2m(\xi\Delta\beta-\lambda\Delta\alpha)+m^2(\alpha\Delta\alpha+\beta\Delta\beta)\\
&= \frac{1}{2}\int_M -2m\beta\Delta\dot{\alpha}-2\lambda\Delta\dot{\beta}-\beta\Delta^2\beta+\lambda\Delta\lambda-2m\lambda\Delta\alpha+m^2\alpha\Delta\alpha+4m^2\beta\Delta\beta,
\end{aligned} \tag{A.32}$$

where for the second equality we have used the constraint enforced by the temporal gauge. Performing now the Gaussian integral over $\lambda$ leads to

$$S_M=\frac{1}{2}\int_M\beta(-\Delta)(\Box-4m^2)\beta-\frac{m}{2}\int_{\partial M}\alpha\partial_r\dot{\beta}+\beta\partial_r\dot{\alpha}. \tag{A.33}$$

Let us now keep track of the various determinants which have been produced by these manipulations in the path integral. First, when using the Hodge decomposition, the measure on phase space changes as

$$\mathcal{D}[A,\Pi]=\mathcal{D}[-\Delta\alpha,-\Delta\beta,\xi,\lambda]=(\det\Delta)^2\mathcal{D}[\alpha,\beta,\xi,\lambda]. \tag{A.34}$$

Using the identity

$$\delta(G) = \delta(\Delta\tilde{G}) = (\det\Delta)^{-1}\delta(\tilde{G}), \tag{A.35}$$

we have also picked up a factor $(\det\Delta)^{-1}$ when imposing the Gauss law with a delta function in the path integral. Then, the Gaussian integral over $\lambda$ has produced a factor of $(\det\Delta)^{-1/2}$. Putting all these factors together, assuming that there is no boundary, and performing a final Gaussian integral over $\beta$ in (A.33), we finally get that

$$\begin{aligned}
\mathcal{Z}_M &= \int \mathcal{D}[A,\Pi]\delta(G)\exp(\mathrm{i}S_M) \\
&= (\det\Delta)^2 \int \mathcal{D}[\alpha,\beta,\xi,\lambda]\delta(G)\exp(\mathrm{i}S_M) \\
&= \det\Delta \int \mathcal{D}[\alpha,\beta,\xi,\lambda]\delta(\tilde{G})\exp(\mathrm{i}S_M) \\
&= (\det\Delta)^{1/2} \int \mathcal{D}[\beta]\exp\left(\frac{\mathrm{i}}{2}\int_M \beta(-\Delta)(\Box - 4m^2)\beta\right) \\
&= \big(\det(\Box - 4m^2)\big)^{-1/2},
\end{aligned} \tag{A.36}$$

where we have dropped gauge volume factors (which can be absorbed by properly normalizing the path integral). As expected, we recover the evaluation of the path integral for a massive scalar field, and all the factors of $\det\Delta$ have cancelled out.

We can now look more carefully at all the boundary contributions coming from the bulk action $S_M$, we will will denote by $\partial S_M$. More precisely, these contributions come from the decomposition (A.31) and from (A.33). On the boundary, we will use the relation (A.30) to write the Gauss law as

$$\tilde{G}\big|_{\partial M} = \dot{\alpha} - 2m\beta = 0. \tag{A.37}$$

With this the boundary term in (A.33) is actually vanishing. More precisely, the boundary contributions are

$$\begin{aligned}
\partial S_M &= \frac{1}{2}\int_{\partial M} \dot{\alpha}\partial_r\dot{\alpha} + \dot{\beta}\partial_r\dot{\beta} + 2\dot{\alpha}\partial_\phi\dot{\beta} - 2m(\alpha\partial_\phi\dot{\alpha} + \beta\partial_\phi\dot{\beta}) - 2m\partial_r(\dot{\alpha}\beta) - \partial_r(\partial_r\beta\partial_r\beta - \beta\partial_r^2\beta) \\
&\qquad - m(\alpha\partial_r\dot{\beta} + \beta\partial_r\dot{\alpha}) \\
&= \frac{1}{2}\int_{\partial M} \dot{\beta}\partial_r\dot{\beta} + 2m\beta\partial_\phi\dot{\beta} - 4m^2\beta\partial_r\beta - 4m^2\alpha\partial_\phi\beta - \partial_r(\partial_r\beta\partial_r\beta - \beta\partial_r^2\beta).
\end{aligned} \tag{A.38}$$

In order to get the dynamics of the edge modes, we have to compute the path integral for this boundary theory coupled to the boundary action in (3.45), in which we have to take into account the constraint

$$j_\phi = \Pi_r + m\partial_\phi a = \partial_r\dot{\alpha} + \partial_\phi\dot{\beta} + m\big(\partial_\phi(a+\alpha) - \partial_r\beta\big) = m\partial_r\beta + \partial_\phi\dot{\beta} + m\partial_\phi(a+\alpha) \tag{A.39}$$

imposed by the temporal gauge $A_t = 0$. With this the boundary action (3.45) becomes

$$\begin{aligned} S_{\partial M} &= \int_{\partial M} (j_\phi + mA_\phi)\dot{a} - j_t(\partial_\phi a + A_\phi) + h \\ &= \int_{\partial M} \big(j_\phi + m(\partial_\phi\alpha - \partial_r\beta)\big)\dot{a} - j_t\big(\partial_\phi(a+\alpha) - \partial_r\beta\big) + h \\ &= \int_{\partial M} \big(\partial_\phi\dot{\beta} + m\partial_\phi\varphi + m\partial_\phi\alpha\big)\dot{a} - j_t(\partial_\phi\varphi - \partial_r\beta) + h, \end{aligned} \tag{A.40}$$

where we have introduced $\varphi := a + \alpha$. Combining the two boundary actions (A.38) and (A.40) into a total boundary action

$$S_{\text{edge}} := \partial S_M + S_{\partial M} \tag{A.41}$$

now leads to

$$\begin{aligned} S_{\text{edge}} = \int_{\partial M} &\frac{1}{2}\dot{\beta}\partial_r\dot{\beta} + m\beta\partial_\phi\dot{\beta} + 2m^2\beta\partial_r\beta - \frac{1}{2}\partial_r(\partial_r\beta\partial_r\beta - \beta\partial_r^2\beta) \\ &+ \dot{\varphi}\partial_\phi\dot{\beta} + m\dot{\varphi}\partial_\phi\varphi - j_t(\partial_\phi\varphi - \partial_r\beta) + h. \end{aligned} \tag{A.42}$$

With a further change of variables $\chi := \varphi + \dot{\beta}/(2m)$ we finally get

$$S_{\text{edge}} = \int_{\partial M} B[\beta] + m\dot{\chi}\partial_\phi\chi - j_t\left(\partial_\phi\chi - \frac{1}{2m}\partial_\phi\dot{\beta} - \partial_r\beta\right) + h, \tag{A.43}$$

where

$$B[\beta] := \frac{1}{2}\dot{\beta}\partial_r\dot{\beta} + m\beta\partial_\phi\dot{\beta} + 2m^2\beta\partial_r\beta - \frac{1}{2}\partial_r(\partial_r\beta\partial_r\beta - \beta\partial_r^2\beta) - \frac{1}{4m}\ddot{\beta}\partial_\phi\dot{\beta}. \tag{A.44}$$

We then have to choose a boundary Hamiltonian $h$ and integrate over $j_t$ in order to get the final form of the effective boundary action. In the main text we use the Hamiltonian

$$h = \frac{1}{m}(j_t \mp j_\phi)j_\phi. \tag{A.45}$$

Noticing that the expression (A.39) for $j_\phi$ can be written in terms of $\chi$ as

$$j_\phi = m\partial_r\beta + \frac{1}{2}\partial_\phi\dot{\beta} + m\partial_\phi\chi, \tag{A.46}$$

with this Hamiltonian we get

$$S_{\text{edge}} = \int_{\partial M} B[\beta] + m\dot{\chi}\partial_\phi\chi + j_t\left(\frac{1}{m}\partial_\phi\dot{\beta} + 2\partial_r\beta\right) \mp \frac{1}{m}\left(m\partial_r\beta + \frac{1}{2}\partial_\phi\dot{\beta} + m\partial_\phi\chi\right)^2, \tag{A.47}$$

which can be path integrated over $j_t$ to finally obtain

$$S_{\text{edge}} = m\int_{\partial M} \dot{\chi}\partial_\phi\chi \mp (\partial_\phi\chi)^2. \tag{A.48}$$

Note that this last step involves the fact that, under the constraint imposed by $j_t$, we have

$$B[\beta]\Big|_{\left(2m\partial_r\beta + \partial_\phi\dot{\beta} = 0\right)} = 0, \tag{A.49}$$

as one can easily check.

## A.3 Boundary conditions as boundary sources

In this appendix we show how the boundary conditions obtained from the extended boundary + boundary action can equivalently be treated as boundary sources. To see this in a simpler setting, we will use the 3-dimensional radial gauge formulation of Maxwell theory introduced in appendix A.1, where the coordinates are $x^\mu = (t, r, \phi) = (r, y^i)$ with $y^i = (t, \phi)$. We have shown in (A.12) that in this case the extended action is given by

$$S = \frac{1}{2}\int_M \sqrt{|g|}\big(\beta^i\partial^2\beta_i - \partial_r\beta^i\partial_r\beta_i\big) + \int_{\partial M}\sqrt{|q|}\,\varepsilon^{im}j_i(\beta_m + \partial_m\varphi), \tag{A.50}$$

where $A_i = \partial_i\alpha + \beta_i$ and $\varphi = a + \alpha$. Variation with respect to $\beta_i$ leads to the bulk and boundary equations of motion

$$\Box\beta^i = 0, \qquad\qquad \partial_r\beta^i(\ell, y) = -\varepsilon^{im}j_m(y), \tag{A.51}$$

where $r = \ell$ is the location of the boundary $\partial M$. Alternatively, we can use a Dirac delta to rewrite this action as

$$S = \frac{1}{2}\int_M \sqrt{|g|}\big(\beta^i\partial^2\beta_i - \partial_r\beta^i\partial_r\beta_i + 2\varepsilon^{im}j_i\beta_m\delta(r-\ell)\big) + \int_{\partial M}\sqrt{|q|}\,\varepsilon^{im}j_i\partial_m\varphi, \tag{A.52}$$

which leads to the massless Klein–Gordon equation with boundary sources

$$\Box\beta^i(r,y) = \varepsilon^{im}j_m(y)\delta(r-\ell), \qquad \partial_r\beta^i(\ell,y) = 0. \tag{A.53}$$

To solve this boundary problem we use the decomposition

$$\beta^i(r,y) = \beta_0^i(r,y) + \int \mathrm{d}^2y'\sqrt{|g|}\,G(r-\ell,y-y')\varepsilon^{im}j_m(y'), \tag{A.54}$$

where $\beta_0^i$ is the homogeneous solution with Dirichlet boundary condition, i.e.

$$\Box\beta_0^i(r,y) = 0, \qquad \beta_0^i(\ell,y) = 0, \qquad \partial_r\beta_0^i(\ell,y) = 0, \tag{A.55}$$

and the Green function satisfies

$$\Box G(x-x') = \delta(x-x'), \qquad \partial_r G(r-\ell,y-y')\big|_{r=\ell} = 0. \tag{A.56}$$

Using this ansatz, the extended action can be written as

$$S = -\frac{1}{2}\int_M \sqrt{|g|}\,\beta_0^i\Box\beta_{i0} + \int_{\partial M}\mathrm{d}^2y\,\sqrt{|q|}\left(\varepsilon^{im}j_i\partial_m\varphi + \frac{1}{2}\int_{\partial M}\mathrm{d}^2y'\,\sqrt{|q|}\,j^i(y)G(0,y-y')j_i(y')\right). \tag{A.57}$$

Finally, focusing on the boundary piece (the bulk gives a factor of $(\det\Box)^{-1/2}$ computed with Dirichlet boundary conditions) and path integrating over $j_i$ leads to the effective action

$$S_{\text{edge}} = -\frac{1}{2}\int_{\partial M}\mathrm{d}^2y\,\sqrt{|q|}\int_{\partial M}\mathrm{d}^2y'\,\sqrt{|q|}\,\partial^i\varphi(y)G(0,y-y')^{-1}\partial_i\varphi(y'). \tag{A.58}$$

One can verify that in momentum space this effective edge mode action coincides with (A.24).

We have shown that the boundary current $j$ can be treated either as a boundary condition or as a boundary source. In general, this is valid for any theory whose equations of motion involve a Laplace-type operator. In order to see this, let us consider two functions $\varphi$ and $\psi$ defined over $M$ and satisfying

$$\Box\varphi = 0, \qquad \Box\psi = -j\delta(x - x|_{\partial M}), \tag{A.59}$$

with the boundary conditions

$$n[\varphi] = j, \qquad n[\psi] = 0, \tag{A.60}$$

where $n = n^\mu\partial_\mu$ is a unit normal vector to the boundary. From Green's second identity

$$\int_M \big(\varphi\Box\psi - \psi\Box\varphi\big)\mathrm{d}V = \int_{\partial M} \big(\varphi n[\psi] - \psi n[\varphi]\big)\mathrm{d}S \tag{A.61}$$

we get

$$\int_{\partial M} j(\varphi - \psi)\mathrm{d}S = 0, \tag{A.62}$$

which therefore means that $\varphi|_{\partial M} = \psi|_{\partial M}$.

## A.4 Extended action and phase space for non-Abelian theories

In this appendix we present the extended actions for non-Abelian Chern–Simons, Yang–Mills and BF theories, and show that they lead as expected to the extended phase space structures which have been derived in [32, 34]. We postpone the study of the effective boundary dynamics to future work.

Throughout this appendix, the gauge fields are 1-forms with values in the Lie algebra $\mathfrak{g}$, whose bracket is denoted by $[\cdot\,,\cdot]$. The non-Abelian covariant derivative and field strength are given by

$$\mathrm{d}_A P = \mathrm{d}P + [A \wedge P], \qquad F = \mathrm{d}A + \frac{1}{2}[A \wedge A]. \tag{A.63}$$

For forms $P$ and $Q$ of respective degree $p$ and $q$ the bracket satisfies $[P \wedge Q] = (-1)^{pq+1}[Q \wedge P]$. For a group element $g$ we denote the finite gauge transformations by

$$g^*A = g^{-1}(A + \mathrm{d})g. \tag{A.64}$$

Finally, recall that all the expressions below should be understood with an implicit pairing between the Lie algebra elements, which we choose to drop for notational clarity, and which is furthermore invariant under the adjoint action of the group on its algebra.

### A.4.1 Chern–Simons theory

In non-Abelian CS theory, the edge mode field that we need to introduce is now a group element. We will denote it by $u$. Under the action of finite gauge transformations, this edge mode field transforms as $g^*u = g^{-1}u$, and the current $j$ transform as $g^*j = g^{-1}jg$. With this, the extended bulk + boundary action naturally takes the form

$$S = \int_M A \wedge \left(F - \frac{1}{6}[A \wedge A]\right) - \frac{1}{6}\mathrm{d}uu^{-1} \wedge [\mathrm{d}uu^{-1} \wedge \mathrm{d}uu^{-1}] + \int_{\partial M} A \wedge \mathrm{d}uu^{-1} + j \wedge (A + \mathrm{d}uu^{-1}), \tag{A.65}$$

where by comparison with the Abelian case we have now included the bulk NWZW term. The variation of this action can be written as

$$\begin{aligned} \delta S = 2&\int_M \delta A \wedge F \\ &+ \int_{\partial M} \delta A \wedge (A + \mathrm{d}uu^{-1} - j) + \delta j \wedge (A + \mathrm{d}uu^{-1}) + u^{-1}\delta u \mathrm{d}\big(u^*(A + j)\big) - \mathrm{d}\big(\delta uu^{-1}(A + j)\big), \end{aligned} \tag{A.66}$$

where

$$u^*(A + j) = u^{-1}(A + j)u + u^{-1}\mathrm{d}u. \tag{A.67}$$

To obtain this form of the variation of the action, we have used several identities. The first one is the variation of the bulk WZNW term, which gives a boundary term according to

$$\frac{1}{6}\delta\big(\mathrm{d}uu^{-1} \wedge [\mathrm{d}uu^{-1} \wedge \mathrm{d}uu^{-1}]\big) = \frac{1}{2}\mathrm{d}\big(\delta uu^{-1}[\mathrm{d}uu^{-1} \wedge \mathrm{d}uu^{-1}]\big). \tag{A.68}$$

The second one is $\delta(\mathrm{d}uu^{-1}) = u\mathrm{d}(u^{-1}\delta u)u^{-1}$. Finally, we have also used the fact that

$$u^{-1}\delta u\mathrm{d}(u^{-1}\mathrm{d}u) = u^{-1}\delta u\mathrm{d}u^{-1}u \wedge u^{-1}\mathrm{d}u = -\frac{1}{2}\delta uu^{-1}[\mathrm{d}uu^{-1} \wedge \mathrm{d}uu^{-1}], \tag{A.69}$$

which comes from the invariance of the (implicit) pairing under the adjoint action of $u$.

To obtain the extended potential, we have to remember that the bulk NWZW term also brings a contribution. The total extended potential is therefore given by

$$\begin{aligned} \theta_\mathrm{e} &= \delta A \wedge A - \frac{1}{2}\delta uu^{-1}[\mathrm{d}uu^{-1} \wedge \mathrm{d}uu^{-1}] + \mathrm{d}\big(\delta uu^{-1}(A + j)\big) \\ &\approx \delta A \wedge A - \frac{1}{2}\delta uu^{-1}[\mathrm{d}uu^{-1} \wedge \mathrm{d}uu^{-1}] + \mathrm{d}\big(\delta uu^{-1}(2A + \mathrm{d}uu^{-1})\big) \\ &= \delta A \wedge A + \mathrm{d}(\delta uu^{-1}) \wedge \mathrm{d}uu^{-1} + 2\mathrm{d}(A\delta uu^{-1}). \end{aligned} \tag{A.70}$$

Upon taking a further variation the NWZW term gets pushed to the corner using

$$\delta\big(\mathrm{d}(\delta u u^{-1}) \wedge \mathrm{d}u u^{-1}\big) = \mathrm{d}\big(\mathrm{d}(\delta u u^{-1})\delta u u^{-1}\big), \tag{A.71}$$

and one finally obtains the extended symplectic structure

$$\Omega = -\int_\Sigma \delta A \wedge A + \int_S \big(2\delta A + \mathrm{d}_A(\delta u u^{-1})\big)\delta u u^{-1}, \tag{A.72}$$

in agreement with [36]. As explained in this reference, similarly to what happens in Abelian Chern–Simons theory, we then have that the generators of $\delta_\alpha$ are integrable and vanishing on-shell, while the boundary symmetries acting as $\Delta_\alpha A = 0$ and $\Delta_\alpha u = u\alpha$ have a generator $I_{\Delta_\alpha}\Omega$ which satisfies a non-Abelian current algebra.

### A.4.2 Yang–Mills theory

To treat the case of 4-dimensional Yang–Mills theory, we need once again a group element $u$ and a current 2-form $j$, transforming respectively under gauge transformations as $g^*u = g^{-1}u$ and $g^*j = g^{-1}jg$. With this we can then form the extended action

$$S = -\frac{1}{2}\int_M \star F \wedge F + \int_{\partial M} j \wedge (A + \mathrm{d}u u^{-1}). \tag{A.73}$$

Its variation is given by

$$\delta S = -\int_M \delta A \wedge \mathrm{d}_A \star F + \int_{\partial M} \delta A \wedge (j - \star F) + \delta j \wedge (A + \mathrm{d}u u^{-1}) - u^{-1}\delta u\, \mathrm{d}(u^* j) + \mathrm{d}(j\delta u u^{-1}), \tag{A.74}$$

where the third term on the boundary can actually be rewritten using

$$u^{-1}\delta u\, \mathrm{d}(u^* j) = \delta u u^{-1}(\mathrm{d}j - [\mathrm{d}u u^{-1} \wedge j]). \tag{A.75}$$

The two boundary equations of motion imposed by $\delta j$ and $\delta u u^{-1}$ imply that the boundary current is conserved, i.e. $\mathrm{d}_A j = 0$. The extended potential is given by

$$\theta_\mathrm{e} = -\delta A \wedge \star F - \mathrm{d}(j\delta u u^{-1}) \approx -\delta A \wedge \star F - \mathrm{d}(\star F \delta u u^{-1}), \tag{A.76}$$

in agreement with [32] and with the Abelian limit (3.35).

### A.4.3 BF theory

For 3-dimensional non-Abelian BF theory, which is actually 3-dimensional first order gravity (here with a vanishing cosmological constant), the edge mode fields are a group element $u$ and a Lie algebra element $b$, transforming respectively as $g^*u = g^{-1}u$ and $g^*b = g^{-1}bg$. The extended action is

$$S = \int_M B \wedge F + \int_{\partial M} bF + j \wedge (A + \mathrm{d}uu^{-1}), \tag{A.77}$$

and is of course invariant under the shift symmetry $\delta_\phi$ and the non-Abelian gauge transformation $\delta_\alpha$. The variation of this action is

$$\begin{aligned}\delta S = &\int_M \delta B \wedge F + \delta A \wedge \mathrm{d}_A B \\ &+ \int_{\partial M} \delta A \wedge (B + \mathrm{d}_A b - j) + \delta j \wedge (A + \mathrm{d}uu^{-1}) + u^{-1}\delta u \, \mathrm{d}(u^* j) + \delta bF - \mathrm{d}(j\delta uu^{-1} - b\delta A).\end{aligned} \tag{A.78}$$

From this we can read once again the bulk and boundary equations of motions, and the extended potential becomes

$$\theta_\mathrm{e} = \delta A \wedge B + \mathrm{d}(j\delta uu^{-1} - b\delta A) \approx \delta A \wedge B + \mathrm{d}\big((B + \mathrm{d}_A b)\delta uu^{-1} - b\delta A\big), \tag{A.79}$$

in agreement with [34]. The computation of the extended symplectic structure, the boundary observables, and their algebra, then follows the results of this reference.

# Appendix B

# More details on Carroll geometries

## B.1 Coordinate expressions for $\varphi_A$ and $w_{AB}$

Expressions for the Carrollian acceleration $\varphi_A$ and the Carrollian vorticity in coordinates are straightforwardly computed from the Carrollian commutators. Let us start with the acceleration, we evaluate

$$\begin{aligned}\varphi_A \ell &= [\ell, e_A] \\ &= [\mathrm{e}^{-\alpha} D_u, e_A] \\ &= e_A[\alpha]\ell + \mathrm{e}^{-\alpha}[D_u, (J^{-1})_A{}^B \partial_B + \beta_A D_u] \\ &= (D_u \beta_A + e_A[\alpha])\, \ell + \mathrm{e}^{-\alpha} \left( D_u (J^{-1})_A{}^B - (J^{-1})_A{}^C \partial_C V^B \right) \partial_B.\end{aligned} \tag{B.1}$$

The last term vanishes due to the condition (5.25). We therefore obtain the expression

$$\varphi_A = D_u \beta_A + e_A[\alpha]. \tag{B.2}$$

In a similar vein, the Carrollian vorticity can be evaluated as follows,

$$\begin{aligned}w_{AB}\ell &= [e_A, e_B] \\ &= [(J^{-1})_A{}^C \partial_C + \beta_A D_u, (J^{-1})_B{}^D \partial_D + \beta_B D_u] \\ &= [(J^{-1})_A{}^C \partial_C, (J^{-1})_B{}^D \partial_D] + [(J^{-1})_A{}^C \partial_C, \beta_B D_u] + [\beta_A D_u, (J^{-1})_B{}^D \partial_D] \\ &\quad + [\beta_A D_u, \beta_B D_u] \\ &= \mathrm{e}^{\alpha} \left( e_A[\beta_B] - e_B[\beta_A] \right) \ell + \left( e_A[J_B{}^C] - \beta_A (J^{-1})_B{}^D \partial_D V^C - (A \leftrightarrow B) \right) \partial_C\end{aligned} \tag{B.3}$$

The last term, again, computes to zero by means of (5.25). The Carrollian vorticity is then given by

$$w_{AB} = \mathrm{e}^{\alpha} \left( e_A[\beta_B] - e_B[\beta_A] \right). \tag{B.4}$$

One can alternatively check by computing the curvature of $\boldsymbol{k} = \mathrm{e}^{\alpha}(\mathbf{d}u - \beta_A \boldsymbol{e}^A)$, which is

$$\begin{aligned} \mathbf{d}\boldsymbol{k} &= \mathbf{d}\alpha \wedge \boldsymbol{k} - \mathrm{e}^{\alpha}\mathbf{d}\beta_A \wedge \boldsymbol{e}^A \\ &= -\left(D_u\beta_A + e_A[\alpha]\right)\boldsymbol{k} \wedge \boldsymbol{e}^A - \frac{1}{2}\mathrm{e}^{\alpha}\left(e_A[\beta_B] - e_B[\beta_A]\right)\boldsymbol{e}^A \wedge \boldsymbol{e}^B \\ &= -\varphi_A \boldsymbol{k} \wedge \boldsymbol{e}^A - \frac{1}{2} w_{AB} \boldsymbol{e}^A \wedge \boldsymbol{e}^B. \end{aligned} \tag{B.5}$$

## B.2 Horizontal covariant derivative

One property of the horizontal covariant derivative $\mathcal{D}_A$ is that we can define the analog of the Riemann tensor with this connection and it is called the Riemann-Carroll tensor, ${}^{(2)}R^A{}_{BCD}$. Its components are determined from the commutator,

$$[\mathcal{D}_C, \mathcal{D}_D]X^A = {}^{(2)}R^A{}_{BCD}X^B + w_{CD}\ell[X^A], \tag{B.6}$$

where the vertical derivative term $\ell[X^A]$ appeared due to the non-integrability of the horizontal subspace. We can then define corresponding the Ricci-Carroll tensor, ${}^{(2)}R_{AB} := {}^{(2)}R_{CADB}q^{CD}$, and the Ricci-Carroll scalar, ${}^{(2)}R := {}^{(2)}R_{AB}q^{AB}$. Let us also note that the Ricci-Carroll tensor is not symmetric, ${}^{(2)}R_{AB} \neq {}^{(2)}R_{BA}$, in general.

Since we are dealing with the expansion in $c^2$ of the sphere metric, $q_{AB} = \mathring{q}_{AB} + 2c^2\lambda_{AB}$, it then becomes essential to define the similar expansion for the connection ${}^{(2)}\Gamma^A_{BC}$. With this in mind, let us define the following connection,

$${}^{(2)}\mathring{\Gamma}^A_{BC} := \frac{1}{2}\mathring{q}^{AD}\left(e_B[\mathring{q}_{DC}] + e_C[\mathring{q}_{BD}] - e_D[\mathring{q}_{BC}]\right), \tag{B.7}$$

and the new horizontal covariant derivative $\mathring{\mathcal{D}}_A$ compatible with the zeroth-order of the sphere metric $\mathring{q}_{AB}$, that is $\mathring{\mathcal{D}}_A\mathring{q}_{BC} = 0$. This operator $\mathring{\mathcal{D}}_A$ acts on a horizontal tensor the same way as $\mathcal{D}_A$ but with the new connection ${}^{(2)}\mathring{\Gamma}^A_{BC}$ instead of ${}^{(2)}\Gamma^A_{BC}$. One can therefore show that ${}^{(2)}\Gamma^A_{BC}$ admits the following expansion in $c^2$,

$${}^{(2)}\Gamma^A_{BC} = {}^{(2)}\mathring{\Gamma}^C_{AB} + c^2\left(\mathring{\mathcal{D}}_A\lambda_B{}^C + \mathring{\mathcal{D}}_B\lambda_A{}^C - \mathring{\mathcal{D}}^C\lambda_{AB}\right) + \mathcal{O}(c^4). \tag{B.8}$$

## B.3 Covariant derivatives

In the main text, we already presented the derivation of the covariant derivative $\nabla_\ell \ell$. Here we complete the detailed derivations of the remaining covariant derivatives, which are $\nabla_{e_A}\ell, \nabla_\ell e_A$, and $\nabla_{e_A} e_B$.

- **Derivation of $\nabla_\ell \ell$:** For completeness, let us quote the result derived in the main text,

$$\nabla_\ell \ell = c^2 \varphi^A e_A + \mathcal{O}(c^4). \tag{B.9}$$

- **Derivation of $\nabla_{e_A}\ell$:** We begin by writing the vector $\nabla_{e_A}\ell$ in the Carrollian basis $(\ell, e_A)$,

$$\nabla_{e_A}\ell = \left(k_i \nabla_{e_A}\ell^i\right)\ell + \left(e^B{}_i \nabla_{e_A}\ell^i\right) e_B, \tag{B.10}$$

then consider each component separately. The vertical component is identically zero as one can easily see from

$$k_i \nabla_{e_A}\ell^i = -\frac{1}{c^2} h(\ell, \nabla_{e_A}\ell) = -\frac{1}{2c^2} e_A \left[h(\ell,\ell)\right] = 0. \tag{B.11}$$

The horizontal components are computed using repeatedly the Leibniz rule and the commutators (5.14),

$$\begin{aligned} e^B{}_i \nabla_{e_A}\ell^i &= q^{BC} h(e_C, \nabla_{e_A}\ell) \\ &= \frac{1}{2} q^{BC}\left(h(e_C, \nabla_{e_A}\ell) + h(e_C, \nabla_{e_A}\ell)\right) \\ &= \frac{1}{2} q^{BC}\left(-h(\nabla_{e_A} e_C, \ell) + h(e_C, \nabla_{e_A}\ell)\right) \\ &= \frac{1}{2} q^{BC}\left(-h([e_A, e_C], \ell) - h(\nabla_{e_C} e_A, \ell) + h(e_C, \nabla_{e_A}\ell)\right) \\ &= \frac{1}{2} q^{BC}\left(c^2 w_{AC} + h(e_A, \nabla_{e_C}\ell) + h(e_C, \nabla_{e_A}\ell)\right) \\ &= \frac{1}{2} q^{BC}\left(c^2 w_{AC} + h(e_A, \nabla_\ell e_C) + h(e_C, \nabla_\ell e_A)\right) \\ &= \frac{1}{2} q^{BC}\left(c^2 w_{AC} + 2\theta_{AC}\right), \end{aligned} \tag{B.12}$$

where we recalled $2\theta_{AB} = \ell[q_{AB}]$. Expanding the metric $q_{AB}$ in $c^2$, we therefore obtain

$$\nabla_{e_A}\ell = \left(\mathring{\theta}_A{}^B + c^2\left(\frac{1}{2} w_A{}^B + \mathring{q}^{BC}\ell[\lambda_{AC}] - 2\lambda^{BC}\mathring{\theta}_{AC}\right)\right) e_B + \mathcal{O}(c^4). \tag{B.13}$$

- **Derivation of $\nabla_\ell e_A$:** This term decomposes as

$$\nabla_\ell e_A = \left(k_i \nabla_\ell e_A{}^i\right) \ell + \left(e^B{}_i \nabla_\ell e_A{}^i\right) e_B. \tag{B.14}$$

Its components are already determined by the components of $\nabla_\ell \ell$ and $\nabla_{e_A} \ell$. For the vertical component, we have

$$k_i \nabla_\ell e_A{}^i = -\frac{1}{c^2} h(\ell, \nabla_\ell e_A) = \frac{1}{c^2} h(\nabla_\ell \ell, e_A) = \varphi_A, \tag{B.15}$$

and for the horizontal components, we have

$$\begin{aligned} e^B{}_i \nabla_\ell e_A{}^i &= q^{BC} h(e_C, \nabla_\ell e_A) \\ &= q^{BC} \left(h(e_C, \nabla_{e_A} \ell) + h(e_C, [\ell, e_A])\right) \\ &= \mathring{\theta}_A{}^B + c^2 \left(\frac{1}{2} w_A{}^B + \mathring{q}^{BC} \ell[\lambda_{AC}] - 2\lambda^{BC} \mathring{\theta}_{AC}\right) + \mathcal{O}(c^4). \end{aligned} \tag{B.16}$$

Together, they give

$$\nabla_\ell e_A = \varphi_A \ell + \left(\mathring{\theta}_A{}^B + c^2 \left(\frac{1}{2} w_A{}^B + \mathring{q}^{BC} \ell[\lambda_{AC}] - 2\lambda^{BC} \mathring{\theta}_{AC}\right)\right) e_B + \mathcal{O}(c^4). \tag{B.17}$$

- **Derivation of $\nabla_{e_A} e_B$:** For this covariant derivative, we write its decomposition in the Carrollian basis as

$$\nabla_{e_A} e_B = \left(k_i \nabla_{e_A} e_B{}^i\right) \ell + \left(e^C{}_i \nabla_{e_A} e_B{}^i\right) e_C, \tag{B.18}$$

where the vertical component is

$$k_i \nabla_{e_A} e_B{}^i = -\frac{1}{c^2} h(\ell, \nabla_{e_A} e_B) = \frac{1}{c^2} h(\nabla_{e_A} \ell, e_B) = \frac{1}{c^2} \mathring{\theta}_{AB} + \left(\frac{1}{2} w_{AB} + \ell[\lambda_{AB}]\right). \tag{B.19}$$

The horizontal components, $e^C{}_i \nabla_{e_A} e_B{}^i = q^{CD} h(e_D, \nabla_{e_A} e_B)$, can be evaluated using the following trick. First, the covariant derivative is metric compatible, which following from this yields the obvious identity, $e_A[q_{DB}] = h(e_D, \nabla_{e_A} e_B) + h(e_B, \nabla_{e_A} e_D)$. It then become a straightforward computation to show that

$$\begin{aligned} e_A[q_{DB}] + e_B[q_{AD}] - e_D[q_{AB}] = 2h(e_D, \nabla_{e_A} e_B) + h(e_A, [e_B, e_D]) \\ + h(e_B, [e_A, e_D]) + h(e_D, [e_B, e_A]). \end{aligned} \tag{B.20}$$

Using the commutator $[e_A, e_B] = w_{AB}\ell$ and that $h(e_A, \ell) = 0$, we arrive at the expression for the horizontal components,

$$e^C{}_i \nabla_{e_A} e_B{}^i = \frac{1}{2} q^{CD} \left( e_A[q_{DB}] + e_B[q_{AD}] - e_D[q_{AB}] \right) = {}^{(2)}\Gamma^C_{AB}. \tag{B.21}$$

We finally obtain the covariant derivative $\nabla_{e_A} e_B$ expanded in $c^2$ as

$$\begin{aligned} \nabla_{e_A} e_B = & \left( \frac{1}{c^2} \mathring{\theta}_{AB} + \left( \frac{1}{2} w_{AB} + \ell[\lambda_{AB}] \right) \right) \ell + {}^{(2)}\mathring{\Gamma}^C_{AB} e_C \\ & + c^2 \left( \mathcal{D}_A \lambda_B{}^C + \mathcal{D}_B \lambda_A{}^C - D^C \lambda_{AB} \right) e_C. \end{aligned} \tag{B.22}$$

## B.4 Anomaly computations

To evaluate the anomaly of the Ehresmann connection, $\Delta_\xi \boldsymbol{k} = \delta_\xi \boldsymbol{k} - \mathcal{L}_\xi \boldsymbol{k}$, one first computes its variation under the near-Carrollian diffeomorphism. Using the fact that the the coordinates $x^i = (u, y^A)$ are field-independent and thus $\delta \mathbf{d} x^i = 0$, we can straightforwardly write the variation of the Ehresmann connection as

$$\delta_\xi \boldsymbol{k} = \mathbb{\delta}_\xi \alpha \boldsymbol{k} - \mathrm{e}^\alpha \mathbb{\delta}_\xi \beta_A \boldsymbol{e}^A. \tag{B.23}$$

Next, we need to compute the Lie derivative of the Ehresmann connection. Using the Cartan formula and recalling the curvature of the Ehresmann connection (5.10), one can proof that

$$\begin{aligned} \mathcal{L}_\xi \boldsymbol{k} &= \mathbf{d}(\iota_\xi \boldsymbol{k}) + \iota_\xi \mathbf{d} \boldsymbol{k} \\ &= \mathbf{d} f + (X \cdot \varphi) \boldsymbol{k} + \left( -f \varphi_A + w_{AB} X^B \right) \boldsymbol{e}^A \\ &= (\ell[f] + X \cdot \varphi) \boldsymbol{k} + \left( (e_A - \varphi_A)[f] + w_{AB} X^B \right) . \boldsymbol{e}^A \end{aligned} \tag{B.24}$$

Expanding the transformation parameter $f = \tau + c^2 \psi$ and $X^A = Y^A + c^2 Z^A$, the anomaly of the Ehresmann connection $\Delta_\xi \boldsymbol{k}$ decomposes as

$$\Delta_\xi \boldsymbol{k} = (\iota_\ell \Delta_\xi \boldsymbol{k}) \, \boldsymbol{k} + (\iota_{e_A} \Delta_\xi \boldsymbol{k}) \, \boldsymbol{e}^A, \tag{B.25}$$

where the components are

$$\begin{aligned} \iota_\ell \Delta_\xi \boldsymbol{k} &= \mathbb{\delta}_\xi \alpha - \mathbb{\delta}^{\text{Carr}}_{(\tau, Y)} \alpha + \mathcal{O}(c^2), \\ \iota_{e_A} \Delta_\xi \boldsymbol{k} &= -\mathrm{e}^\alpha \left( \mathbb{\delta}_\xi \beta_A - \mathbb{\delta}^{\text{Carr}}_{(\tau, Y)} \beta_A \right) + \mathcal{O}(c^2). \end{aligned} \tag{B.26}$$

Next, we compute the anomaly of the null Carrollian metric, $q = q_{AB}\boldsymbol{e}^A \circ \boldsymbol{e}^B$. We begin by considering its variation under the near-Carrollian diffeomorphism and show that

$$\delta_\xi q = -2\mathrm{e}^{-\alpha}\left(q_{AB}\delta\!\!\!\delta_\xi V^B\right)\boldsymbol{k}\circ\boldsymbol{e}^A + \delta\!\!\!\delta_\xi q_{AB}\boldsymbol{e}^A\circ\boldsymbol{e}^B. \tag{B.27}$$

Using the Cartan formula and the fact that $\mathbf{d}\boldsymbol{e}^A = 0$, the Lie derivative of the null Carrollian metric thus given by

$$\begin{aligned}\mathcal{L}_\xi q &= \xi[q_{AB}]\boldsymbol{e}^A\circ\boldsymbol{e}^B + 2q_{AB}(\mathcal{L}_\xi\boldsymbol{e}^{\boldsymbol{A}})\circ\boldsymbol{e}^B \\ &= 2q_{AB}\ell[X^B]\boldsymbol{k}\circ\boldsymbol{e}^B + \left(\xi[q_{AB}] + q_{C(A}e_{B)}[X^C]\right)\boldsymbol{e}^A\circ\boldsymbol{e}^B.\end{aligned} \tag{B.28}$$

The anomaly of the null Carrollian metric is

$$\Delta_\xi q = \delta_\xi q - \mathcal{L}_\xi q = 2\Delta_\xi q(\ell, e_A)\boldsymbol{k}\circ\boldsymbol{e}^A + \Delta_\xi q(e_A, e_B)\boldsymbol{e}^A\circ\boldsymbol{e}^B, \tag{B.29}$$

where its components, in the $c^2$–expansion, are

$$\Delta_\xi q(\ell, e_A) = -\mathrm{e}^{-\alpha}(\mathring{q}_{AB} + 2c^2\lambda_{AB})\left(\delta\!\!\!\delta_\xi V^B + D_u Y^B\right) - c^2\mathring{q}_{AB}\ell[Z^B], \tag{B.30}$$

and

$$\begin{aligned}\Delta_\xi q(e_A, e_B) = &\left(\delta\!\!\!\delta\mathring{q}_{AB} - \delta\!\!\!\delta^{\text{Carr}}_{(\tau,Y)}\mathring{q}_{AB}\right) \\ &+ 2c^2\left(\delta\!\!\!\delta_\xi\lambda_{AB} - \frac{1}{2}\delta\!\!\!\delta^{\text{Carr}}_{(\psi,Z)}\mathring{q}_{AB} - \tau\ell[\lambda_{AB}] - Y^C\mathring{\mathcal{D}}_C\lambda_{AB} - 2\lambda_{C(A}\mathring{\mathcal{D}}_{B)}Y^C\right).\end{aligned} \tag{B.31}$$

# Appendix C

# Einstein equations on the null boundary

## C.1 Gauss-Codazzi equation

In this section, we give a derivation of the Gauss-Codazzi equation which will be used to write the Einstein equations on the null boundary.

On the null boundary $N$, the covariant derivative of the horizontal basis $e_A$ along another horizontal basis is given by

$$\nabla_{e_A} e_B = {}^{(2)}\mathring{\Gamma}^C_{AB} e_C - \overline{\theta}_{AB}\ell - \mathring{\theta}_{AB} k \tag{C.1}$$

Using the decomposition of the metric (6.9) we express the divergence of the horizontal basis as

$$\nabla_a e_A{}^a = \left(n_a k^b + k_a \ell^b + q^{BC} e_{Ba} e_C{}^b\right) \nabla_b e_A{}^a \stackrel{N}{=} {}^{(2)}\mathring{\Gamma}^B_{BA} + \varphi_A. \tag{C.2}$$

where we recall our choice that we set the scale factor $\overline{\alpha} = 0$. With these, we show that the covariant derivative of a generic horizontal vector fields $X^a := X^A e_A{}^a$ projected onto the horizontal subspace is

$$e^B{}_a \nabla_{e_A} X^a = e_A[X^B] + X^C e^B{}_b \nabla_{e_A} e_C{}^b = \mathring{\mathcal{D}}_A X^B. \tag{C.3}$$

Furthermore, the spacetime divergence of the horizontal vector is

$$\nabla_a \left(X^A e_A{}^a\right) = e_A[X^A] + X^A \nabla_a e_A{}^a = \left(\mathring{\mathcal{D}}_A + \varphi_A\right) X^A. \tag{C.4}$$

Armed with these tools, we are ready to derive the Gauss-Codazzi equation on $N$ which is the relation between the Riemann tensor $\mathring{R}_{ABCD} := e_A{}^a e_B{}^b e_C{}^c e_D{}^d \mathring{R}_{abcd}$ and the

Riemann-Carroll tensor ${}^{(2)}\mathring{R}_{ABCD}$ defined in (B.6)

$$[\mathring{\mathcal{D}}_C, \mathring{\mathcal{D}}_D]X^A = {}^{(2)}\mathring{R}^A{}_{BCD}X^B + w_{CD}\ell[X^A]. \tag{C.5}$$

To derive the relation, let us start by considering the following projection of the covariant derivative,

$$e^A{}_a e_C{}^c \nabla_{e_B}\nabla_c X^a = e_B[e^A{}_a \nabla_{e_C} X^a] - (\nabla_{e_C}X^a)(\nabla_{e_B}e^A{}_a) - e^A{}_a(\nabla_b X^a)(\nabla_{e_B}e_C{}^b). \tag{C.6}$$

The second term can be written as

$$(\nabla_{e_C}X^a)(\nabla_{e_B}e^A{}_a) = -{}^{(2)}\mathring{\Gamma}^A_{BD}\mathring{\mathcal{D}}_C X^D + (\overline{\theta}_B{}^A\mathring{\theta}_{CD} + \mathring{\theta}_B{}^A\overline{\theta}_{CD})X^D. \tag{C.7}$$

The third term can also be written as

$$e^A{}_a(\nabla_b X^a)(\nabla_{e_B}e_C{}^b) = {}^{(2)}\mathring{\Gamma}^D_{BC}\mathring{\mathcal{D}}_D X^A - (e^A{}_a\nabla_\ell X^a)\overline{\theta}_{BC} - (e^A{}_a\nabla_k X^a)\mathring{\theta}_{BC}. \tag{C.8}$$

We therefore obtain

$$\begin{aligned} e^A{}_a e_C{}^c \nabla_{e_B}\nabla_c X^a = {}& \mathring{\mathcal{D}}_B\mathring{\mathcal{D}}_C X^A - (\overline{\theta}_B{}^A\mathring{\theta}_{CD} + \mathring{\theta}_B{}^A\overline{\theta}_{CD})X^D \\ &+ (e^A{}_a\nabla_\ell X^a)\overline{\theta}_{BC} + (e^A{}_a\nabla_k X^a)\mathring{\theta}_{BC}. \end{aligned} \tag{C.9}$$

Using that $[\nabla_b, \nabla_c]X^a = R^a{}_{dbc}X^d$ and recalling the definition of the Riemann-Carroll tensor, we obtain the Carrollian analog of the Gauss-Codazzi equation on the null boundary,

$$\mathring{R}_{ADBC} = {}^{(2)}\mathring{R}_{ADBC} - \left(\overline{\theta}_{BA}\mathring{\theta}_{CD} + \mathring{\theta}_{BA}\overline{\theta}_{CD} - \overline{\theta}_{CA}\mathring{\theta}_{BD} - \mathring{\theta}_{CA}\overline{\theta}_{BD}\right) - w_{BC}\mathring{\theta}_{DA}. \tag{C.10}$$

Observe that if the Carrollian vorticity is zero, such as when one considers the case $\mathring{\beta}_A = 0$, we recover the standard Gauss-Codazzi equation. The trace of the Riemann tensor $\mathring{R}_{ABCD}$ is related to the symmetric part of the Ricci-Caroll tensor,

$$\mathring{q}^{CD}\mathring{R}_{CADB} = {}^{(2)}\mathring{R}_{(AB)} - \left(\overline{\theta}\mathring{\theta}_{AB} + \mathring{\theta}\overline{\theta}_{(AB)}\right) + \left(\mathring{\theta}_{C(A}\overline{\theta}_{B)}{}^C + \overline{\theta}_{C(A}\mathring{\theta}_{B)}{}^C\right) - w_{C(A}\mathring{\theta}_{B)}{}^C. \tag{C.11}$$

Taking the trace of the Riemann tensor, we obtain the scalar relation

$$\mathring{q}^{AB}\mathring{q}^{CD}\mathring{R}_{CADB} = {}^{(2)}\mathring{R} - 2\overline{\theta}\mathring{\theta} + 2\overline{\theta}^{AB}\mathring{\theta}_{AB}. \tag{C.12}$$

## C.2 Derivation of the Einstein tensors on the null hypersurface

We present in this section the derivation of the complete set of the Einstein equations on the null hypersurface $N$. We provide the general form of the covariant derivative of the tangential vector $\ell$, the transverse vector $k$, an their combination $n = \ell + 2\rho k$ which will become handy in further computations,

$$\nabla_a \ell^b = \mathcal{W}_{(\boldsymbol{\ell})a}{}^b + (2\rho(\pi_a + \kappa k_a) - D_a\rho)\, k^b - n_a \left(\kappa k^b + (\pi^b + \varphi^b)\right) \tag{C.13}$$

$$\nabla_a k^b = \mathcal{W}_{(\boldsymbol{k})a}{}^b - (\pi_a + \kappa k_a) k^b \tag{C.14}$$

### The $(\ell\ell)$-component

Let us first consider the $(\ell\ell)$-component of the Einstein tensor. For this component, the Einstein tensor on the null boundary $N$ is simply given by the corresponding Ricci tensor,

$$\mathring{G}_{\ell\ell} = \mathring{R}_{\ell\ell} = \ell^a \nabla_b \nabla_a \ell^b - \ell^a \nabla_a \nabla_b \ell^b. \tag{C.15}$$

Using the decomposition (C.13), we can write the Einstein tensor as

$$\begin{aligned} \mathring{G}_{\ell\ell} &= \nabla_a \left(\nabla_\ell \ell^a\right) - (\nabla_a \ell^b)(\nabla_b \ell^a) - \ell[\nabla_a \ell^a] \\ &= \nabla_a(\mathring{\kappa}\ell^a + 2r\mathring{\kappa}k^a) - \mathring{\theta}^{AB}\mathring{\theta}_{AB} - 2\mathring{\kappa}^2 - \ell[\mathring{\theta}] \\ &\overset{N}{=} -\ell[\mathring{\theta}] + \mathring{\kappa}\mathring{\theta} - \mathring{\theta}^{AB}\mathring{\theta}_{AB}. \end{aligned} \tag{C.16}$$

Recalling the definition $\mathring{\mu} = \mathring{\kappa} + \frac{1}{2}\mathring{\theta}$, we can then write the Einstein tensor as

$$-\mathring{G}_{\ell\ell} = (\ell + \mathring{\theta})[\mathring{\theta}] - \mathring{\mu}\mathring{\theta} + \mathring{\sigma}^{AB}\mathring{\sigma}_{AB}. \tag{C.17}$$

### The $(\ell A)$-components

Similarly, these components of the Einstein tensors are simply given by the corresponding Ricci tensor,

$$\mathring{G}_{\ell A} = \mathring{R}_{\ell A} = e_A{}^a \nabla_b \nabla_a \ell^b - e_A{}^a \nabla_a \nabla_b \ell^b. \tag{C.18}$$

Then by using the decomposition (C.13), one can express the Einstein tensors on the null hypersurface $N$ as

$$
\begin{aligned}
\mathring{G}_{\ell A} &= e_A{}^a \nabla_b \left( \mathring{\theta}_a{}^b + \pi_a \ell^b + \mathring{\kappa} k_a \ell^b + \left(2r\mathring{\kappa}(\pi_a + \mathring{\kappa} k_a) - r D_a \mathring{\kappa}\right) k^b - \ell_a (\mathring{\kappa} k^b + \pi^b + \varphi^b) \right) - e_A[\mathring{\theta}] \\
&= (\mathring{\mathcal{D}}_B + \varphi_B)\mathring{\theta}_A{}^B + \ell[\pi_A] + \mathring{\theta}\pi_A - (2\pi_B + \varphi_B)\mathring{\theta}_A{}^B + 2\mathring{\kappa}\pi_A - e_A[\mathring{\theta} + \mathring{\kappa}]
\end{aligned}
\tag{C.19}
$$

Recalling that $2\pi_A + \varphi_A = 0$, we arrive at the following expression of the Einstein tensor

$$
\mathring{G}_{\ell A} \overset{N}{=} (\ell + \mathring{\theta})[\pi_A] + \mathring{\theta}\varphi_A + (\mathring{\mathcal{D}}_B + \varphi_B)\left(\mathring{\sigma}_A{}^B - \mathring{\mu}\delta_A^B\right). \tag{C.20}
$$

### The $(\ell k)$-component

Expressing the $(\ell k)$–component of the Einstein tensor in terms of the extrinsic geometry of the null hypersurface requires the Gauss-Codazzi relation. Let us start by writing the Einstein equation as

$$
G_{\ell k} = R_{\ell k} - \frac{1}{2} R. \tag{C.21}
$$

By using the decomposition of the spacetime metric, the spacetime Ricci scalar can be expressed on the null boundary $N$ as

$$
\begin{aligned}
R = g^{ab} R_{ab} &\overset{N}{=} (\ell^a k^b + k^a \ell^b + \mathring{q}^{AB} e_A{}^a e_B{}^b)\mathring{R}_{ab} \\
&= 2\mathring{R}_{\ell k} + \mathring{q}^{AB}\mathring{R}_{AB} \\
&= 2\mathring{R}_{\ell k} + 2\mathring{q}^{AB}\mathring{R}_{\ell A k B} + \mathring{q}^{AB}\mathring{q}^{CD}\mathring{R}_{CADB}.
\end{aligned}
\tag{C.22}
$$

This allows us to express the Einstein tensor as

$$
\mathring{G}_{\ell k} = -\mathring{q}^{AB}\mathring{R}_{\ell A k B} - \frac{1}{2}\mathring{q}^{AB}\mathring{q}^{CD}\mathring{R}_{CADB}, \tag{C.23}
$$

The second term is the scalar Gauss-Codazzi equation we have already derived. Let us focus on the first term which can be expressed as the commutator as

$$
\mathring{q}^{AB}\mathring{R}_{\ell A k B} = \mathring{q}^{AB} e_{Aa} e_B{}^b \ell^c [\nabla_b, \nabla_c] k^a = \mathring{q}^{AB} e_{Aa} e_B{}^b \ell^c (\nabla_b \nabla_c k^a - \nabla_c \nabla_b k^a). \tag{C.24}
$$

Using the decomposition (C.13) and (C.14), we are able to write the following terms as

$$
\begin{aligned}
e_{Aa}\ell^c \nabla_{e_B} \nabla_c k^a &= e_{Aa}\nabla_{e_B}(\nabla_\ell k^a) - e_{Aa}(\nabla_c k^a)(\nabla_{e_B}\ell^c) \\
&= -e_{Aa}\nabla_{e_B}\left(\pi^a + \varphi^a + \mathring{\kappa} k^a\right) - e_{Aa}(\nabla_c k^a)\left(\theta_B{}^c + \pi_B \ell^c\right) \\
&= -(\mathring{\mathcal{D}}_B - \pi_B)(\pi_A + \varphi_A) - \mathring{\kappa}\overline{\theta}_{BA} - \mathring{\theta}_B{}^C \overline{\theta}_{CA},
\end{aligned}
\tag{C.25}
$$

and for the second term, we have

$$
\begin{aligned}
-e_{Aa}e_B{}^b\nabla_\ell\nabla_b k^a &= -\ell[e_{Aa}\nabla_{e_B}k^a] + e_{Aa}(\nabla_b k^a)(\nabla_\ell e_B{}^b) + (\nabla_{e_B}k^a)(\nabla_\ell e_{Aa}) \\
&= -\ell[\overline{\theta}_{BA}] + \big(\overline{\theta}_{bA} - (\pi_A + \varphi_A)k_b\big)\,\nabla_\ell e_B{}^b + \big(\overline{\theta}_{Ba} - \pi_B k_a\big)\,\nabla_\ell e_A{}^a \\
&= -\ell[\overline{\theta}_{BA}] + \overline{\theta}_{CA}\mathring{\theta}_B{}^C + \mathring{\theta}_A{}^C\overline{\theta}_{BC}
\end{aligned} \tag{C.26}
$$

where we used again $2\pi_A + \varphi_A = 0$. We thus obtain

$$
\mathring{R}_{\ell AkB} = -(\ell + \mathring{\kappa})[\overline{\theta}_{BA}] + \mathring{\mathcal{D}}_B\pi_A - \pi_A\pi_B + \mathring{\theta}_A{}^C\overline{\theta}_{BC}. \tag{C.27}
$$

After taking the trace, we arrive at the expression

$$
\mathring{q}^{AB}\mathring{R}_{\ell AkB} = -(\ell + \mathring{\kappa})[\overline{\theta}] + (\mathring{\mathcal{D}}_A - \pi_A)\pi^A - \theta^{AB}\overline{\theta}_{AB}, \tag{C.28}
$$

where we note that on $N$ the tensor $\theta_{AB}$ is symmetric. Using the Gauss-Codazzi equation (C.12), we then arrive at the expression for the Einstein tensor

$$
\mathring{G}_{\ell k} = (\ell + \mathring{\theta} + \mathring{\kappa})[\overline{\theta}] - (\mathring{\mathcal{D}}_A - \pi_A)\pi^A - \frac{1}{2}{}^{(2)}R. \tag{C.29}
$$

For completeness, let us also consider the component $\mathring{R}_{\ell k}$ of the spacetime Ricci tensor. This can be written as

$$
-\mathring{R}_{\ell k} = \ell^a[\nabla_a, \nabla_b]k^b = \ell[\nabla_a k^a] - \nabla_a\left(\nabla_\ell k^a\right) + (\nabla_b\ell^a)(\nabla_a k^b). \tag{C.30}
$$

Now using the decomposition (C.13) and (C.14), we finally arrive at

$$
-\mathring{R}_{\ell k} = (\ell + \mathring{\mu})[\overline{\theta}] + \kappa_{(1)} - (\mathring{\mathcal{D}}_A + \varphi_A)\pi^A + \mathring{\sigma} : \overline{\sigma}, \tag{C.31}
$$

where we recall the notation $\mathring{\sigma} : \overline{\sigma} = \mathring{\sigma}_{AB}\overline{\sigma}^{AB}$

### The $(AB)$-components

Next, we consider the fully horizontal components of the Einstein tensor,

$$
\mathring{G}_{AB} = \mathring{R}_{AB} - \frac{1}{2}\mathring{R}\mathring{q}_{AB} = \mathring{R}_{AB} - \left(\mathring{R}_{\ell k}\mathring{q}_{AB} + \frac{1}{2}\mathring{q}^{CD}\mathring{R}_{CD}\right)\mathring{q}_{AB}. \tag{C.32}
$$

These components can be split into the traceless part and the trace part as[1]

$$
\mathring{G}_{\langle AB\rangle} = \mathring{R}_{\langle AB\rangle}, \qquad \text{and} \qquad \mathring{q}^{AB}\mathring{G}_{AB} = -2\mathring{R}_{\ell k}. \tag{C.33}
$$

[1] In $D$-dimensional spacetime, the relation is $\mathring{q}^{AB}\mathring{G}_{AB} = (D-4)\mathring{G}_{\ell k} - (D-2)\mathring{R}_{\ell k}$.

Let us first consider the horizontal components of the spacetime Ricci tensor, $R_{AB} = e_A{}^a e_B{}^b R_{ab}$. Using the previous result and the Gauss-Codazzi equation, we have

$$\begin{aligned}
-\mathring{R}_{AB} &= -\left(\mathring{R}_{\ell AkB} + \mathring{R}_{\ell BkA}\right) - \mathring{q}^{CD}\mathring{R}_{CADB} \\
&= 2(\ell + \mathring{\kappa})[\overline{\theta}_{(AB)}] - 2\mathring{\mathcal{D}}_{(A}\pi_{B)} + 2\pi_{(A}\pi_{B)} - 4\mathring{\theta}_{(A}{}^C\overline{\theta}_{B)C} + \overline{\theta}\mathring{\theta}_{AB} + \mathring{\theta}\overline{\theta}_{(AB)} - {}^{(2)}\mathring{R}_{(AB)}.
\end{aligned} \tag{C.34}$$

One can verify that $-\mathring{q}^{AB}\mathring{R}_{AB} = 2\mathring{G}_{\ell k}$. We can therefore show that the symmetric traceless components of the Einstein tensor are given by

$$\begin{aligned}
-\mathring{G}_{\langle AB\rangle} = {} & 2\ell[\overline{\sigma}_{AB}] - 2\mathring{\mathcal{D}}_{\langle A}\pi_{B\rangle} + 2\pi_{\langle A}\pi_{B\rangle} + \overline{\theta}\mathring{\sigma}_{AB} + (2\mathring{\kappa} - \mathring{\theta})\overline{\sigma}_{AB} \\
& - 4\mathring{\sigma}_{(A}{}^C\overline{\sigma}_{B)C} + 2\mathring{\sigma}_{(A}{}^C w_{B)C} - {}^{(2)}\mathring{R}_{\langle AB\rangle}.
\end{aligned} \tag{C.35}$$

**The $(kA)$-components**

For these components, we can use the decomposition (C.14) to write the Einstein tensor as

$$\mathring{G}_{kA} = \mathring{R}_{kA} = e_A{}^a\nabla_b\nabla_a k^b - e_A{}^a\nabla_a\nabla_b k^b. \tag{C.36}$$

Using the decomposition (C.14), we can derive the following expression

$$\begin{aligned}
\mathring{G}_{kA} = \mathring{R}_{kA} &= e_A{}^a\nabla_b\nabla_a k^b - e_A{}^a\nabla_a\nabla_b k^b \\
&= -\overline{\theta}\pi_A + \mathring{\mathcal{D}}_B\left(\overline{\theta}_A{}^B - \overline{\theta}\delta_A{}^B\right) + w_{AB}\pi^B.
\end{aligned} \tag{C.37}$$

The last term vanishes when the vorticity vanishes, $w_{AB} = 0$.

**The $(kk)$–component**

For the last component, we have that $\mathring{G}_{kk} = \mathring{R}_{kk}$ and we use (C.14) to write

$$\mathring{G}_{kk} = k^a\nabla_b\nabla_a k^b - k^a\nabla_a\nabla_b k^b = -\nabla_a k^b\nabla_b k^a - \partial_r(\nabla_a k^a). \tag{C.38}$$

The equation $\mathring{G}_{kk} = 0$ fixes the higher order terms in the near-horizon expansion of $q_{AB}$. Let us now write $q_{AB} = \mathring{q}_{AB} + 2r\lambda_{AB} + r^2 K_{AB} + \ldots$ with the symmetric tensor $K_{AB}$ being a function of $(u, \sigma^A)$. One has that

$$\nabla_a k^a = \frac{1}{2}q^{AB}\partial_r q_{AB} = \overline{\theta} + r\left(-2\overline{\theta}^{(AB)}\overline{\theta}_{AB} + \mathring{q}^{AB}K_{AB}\right), \tag{C.39}$$

and thus the Einstein tensor can be expressed as

$$\mathring{G}_{kk} = \overline{\theta}^{AB}\overline{\theta}_{AB} - \mathring{q}^{AB} K_{AB}. \tag{C.40}$$

Demanding $\mathring{G}_{kk} = 0$ determines the trace of the sub-sub-leading order in the sphere metric.