Connectivity Properties of the Flip Graph After Forbidding Triangulation Edges

by

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Author’s Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.
Statement of Contributions

Reza Bigdeli was the sole author of the Sections 1, 2, 3.1, 3.2, 3.3.1, 3.3.2, and 4.1 which were written under the supervision of Prof. Anna Lubiw and were not written for publication.

The rest of the chapters are joint work with my supervisor Prof. Anna Lubiw and appeared in the Young Researchers Forum of the Symposium on Computational Geometry and on arXiv [5].
Abstract

The flip graph for a set $P$ of points in the plane has a vertex for every triangulation of $P$, and an edge when two triangulations differ by one flip that replaces one triangulation edge by another. The flip graph is known to have some connectivity properties: (1) the flip graph is connected; (2) connectivity still holds when restricted to triangulations containing some constrained edges between the points; (3) for $P$ in general position of size $n$, the flip graph is $\lceil \frac{n}{2} - 2 \rceil$-connected, a recent result of Wagner and Welzl (SODA 2020).

We introduce the study of connectivity properties of the flip graph when some edges between points are forbidden. An edge $e$ between two points is a flip cut edge if eliminating triangulations containing $e$ results in a disconnected flip graph. More generally, a set $X$ of edges between points of $P$ is a flip cut set if eliminating all triangulations that contain edges of $X$ results in a disconnected flip graph. The flip cut number of $P$ is the minimum size of a flip cut set.

We give a characterization of flip cut edges that leads to an $O(n \log n)$ time algorithm to test if an edge is a flip cut edge and, with that as preprocessing, an $O(n)$ time algorithm to test if two triangulations are in the same connected component of the flip graph. For a set of $n$ points in convex position (whose flip graph is the 1-skeleton of the associahedron) we prove that the flip cut number is $n - 3$. 

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Chapter 1

Introduction

1.1 Triangulations

Given a set $P$ of $n$ points in the plane, which may include collinear points, an edge of $P$ is a line segment $pq$ that intersects $P$ in exactly the two endpoints $p$ and $q$. A triangulation of $P$ is a maximal set of non-crossing edges. To give an intuition of what a triangulation of a point set would look like, first we introduce the convex hull of a point set. The convex hull of a point set $P$ consists of the edges $pq$ such that one of the two closed half-planes determined by the line through $pq$ contains all the points of $P$. This means that the Convex Hull can be seen as the smallest container of the points in the point set. Any triangulation of the point set contains all the edges of the Convex Hull of that point set. Then, edges are added to the points inside the hull in a way that finally the area inside the hull is divided into empty triangles. A triangle is empty if there do not exist any points of $P$ in the triangle. This is why it is called a triangulation; we add edges until the whole area is divided into triangle regions. Figure 1.1 shows some point sets and a valid triangulation of each point set. Triangulations have important applications in graphics and mesh generation [4, 13] and are of significant mathematical interest [12].
A fundamental approach to understanding triangulations is by means of flips. A flip operates on a triangulation by removing one edge $pq$ and adding another edge $uv$ to obtain a new triangulation—of necessity, the edges $pq$ and $uv$ will cross and their four endpoints will form a convex quadrilateral with no other points of $P$ inside it. For example, in Figure 1.4, edge $a_1b_1$ can be flipped to $uv$. We say we can transform or reconfigure one triangulation into another one, if there exists a sequence of flips after which the first triangulation is converted into the other one. In 1972, Lawson [23, 24] proved that any triangulation of point set $P$ can be reconfigured to any other triangulation of $P$ by a sequence of flips.
1.2 Reconfiguration and Flip Graph

A graph $G$ is a mathematical tool used for expressing the relation between some objects. It has a wide range of applications in networks, combinatorial optimization, machine learning, and so on. Each graph consists of some nodes or vertices and some edges. Each vertex represents an object in the environment we are studying. There exists an edge between vertices of the graph if the objects those vertices are representing satisfy a specific relation. For example, consider the people in a university. We want to construct a graph that shows the friendship relation between people at the university. So, each person would be represented with a node and we place an edge between nodes that correspond to two people who are friends. This would be the friendship graph of the mentioned university.

We denote a graph by $G = (V, E)$ where set $V = \{v_1, v_2, ..., v_n\}$ is the set of the graph’s vertices and $E$ denotes the set of graph edges. We call a sequence of vertices $v_1, v_2, ..., v_{l+1}$ a path of length $l$ between $v_1$ and $v_{l+1}$ if there exists an edge between every two consecutive vertices $v_i$ and $v_{i+1}$ for $i = 1, ..., l$. The length of this path is the number of edges along this path. A shortest path between two vertices $u$ and $v$ is defined as a path with the smallest length and the length of this path is called the distance between the two vertices and is denoted by $d(u, v)$. The maximum distance between any pair of vertices is called the diameter of the graph. That is, diameter $= \max\{d(v, u) : v, u \in V\}$. We say a graph is connected if there exists a path between every arbitrary pair of vertices.

We can represent flips and triangulations using graphs. A flip graph has a vertex for every triangulation of $P$ and an edge when two triangulations differ by a flip. In this way we can define triangulation flips as a reconfiguration problem. Reconfiguration is the study of the relationship between the feasible solutions of a problem [31]. Each feasible solution is considered as a valid configuration and reconfiguration consists of some steps to transform one feasible solution into another one in a way that each intermediary step is also a feasible solution to the problem (in other words a valid configuration). Some sensible examples of reconfiguration problems are Sliding Puzzles, Rubik’s Cube, etc. In the reconfiguration graph, each vertex represents a configuration and each edge shows the reconfiguration operation which is the flip in triangulations. As mentioned, Lawson proved that any two triangulations can be reconfigured into each other. This means there exists a path between any two vertices in the flip graph. Therefore, the flip graph is connected. Reconfiguring triangulations via flips is well studied, see the survey by Bose and Hurtado [6]. However, there are some very interesting open questions, and many properties of flip graphs remain to be discovered.

In addition to properties of the regular flip graph, properties of some specific flip graphs have been studied too. Especially, the flip graph of triangulations that must contain a
specific set of edges has been studied widely. In this case, the flip graph would be a subgraph of the regular flip graph. One intriguing thing about flip graphs of triangulations is that many properties carry over when we restrict to triangulations containing some specified non-crossing edges—so-called constrained triangulations. The subgraph of the flip graph consisting of triangulations that contain all the constrained edges is connected, as proved by Chew [10]. This encourages people working on combinatorial reconfiguration to consider special cases of the reconfiguration graph to find more properties (especially connectivity properties) for different reconfiguration problems. As a matter of fact, that is one of the main reasons for our contribution to the subject of this thesis.

### 1.3 Points in Convex Position

A well-known and important class of points sets is the class of convex point sets. A point set $P$ is a **convex point set** if all the points of $P$ are extreme points of the convex hull of $P$, i.e., for every point $p \in P$ there is a line through $p$ with all other points of $P$ strictly to one side of the line.

The case of points in convex position is especially interesting because there is a bijection between flips in triangulations of a convex point set and rotations in binary trees [35], so that flip distance becomes rotation distance between binary trees. Figure 1.2 demonstrates the relationship between flips in convex point sets and rotation trees.
(a) Two triangulations that can be reconfigured into each other using a flip. Their corresponding tree and how it can be built has been shown in red.

(b) Two corresponding trees and how they can be reconfigured into each other using a rotation.

Figure 1.2: The relationship between triangulations of a convex point set and binary trees.

Finding the rotation distance between two binary trees is of great interest in biology for phylogenetic trees [11], and in data structures for splay trees [35]. Furthermore, the flip graph for $n$ points in convex position is the 1-skeleton (the graph of vertices and edges) of an $(n - 3)$-dimensional polytope called the associahedron [25]. See Figure 1.3.

### 1.4 Our results

In this thesis, we study connectivity properties of the flip graph when—instead of constraining certain edges between points to be present as in constrained triangulations—we forbid certain edges between points. To be precise, if a set $X$ of edges between points is forbidden, we eliminate all triangulations that contain an edge of $X$, and examine whether
the flip graph on the remaining triangulations is connected. We say that $X$ is a **flip cut set** if the resulting flip graph is disconnected; in the special case where $X$ is a single edge, we say that the edge is a **flip cut edge**. For example the edge $uv$ in Figure 1.4 is a flip cut edge, but the two red edges in Figure 1.3 do not form a flip cut set. We define the **flip cut number** of a set of points to be the minimum size of a flip cut set. This is analogous to the **connectivity** of a graph—the minimum number of vertices whose removal disconnects the graph.

Recently, Wagner and Welzl [38] have investigated some connectivity properties of the flip graph. They have shown that the flip graph is $\lceil \frac{n}{2} - 2 \rceil$-connected for $P$ in general position. Since the structure of the flip graph depends on the edges between the points, it seems more natural to study connectivity of the flip graph after deleting some of these edges, rather than deleting some vertices of the flip graph, as standard graph connectivity does, and as the result of Wagner and Welzl [38] does.

As our main result, in Chapter 3.2 we characterize when an edge $e$ is a flip cut edge in terms of connectivity (in the usual graph sense) of the edges that cross $e$. Observe that a triangulation that does not contain $e$ must contain an edge that crosses $e$. We then use the characterization to give an $O(n \log n)$ time algorithm to test if a given edge $e$ in a point
Figure 1.4: The smallest point set that has a flip cut edge. The edge \( e = uv \) is a flip cut edge since forbidding \( e \) leaves two possible triangulations (as shown) and neither one allows a flip.

set of size \( n \) is a flip cut edge in Section 3.3. With that algorithm as preprocessing, we give a linear-time algorithm to test if two triangulations are still connected after we eliminate from the flip graph all triangulations containing edge \( e \).

For the case of \( n \) points in convex position, there are no flip cut edges, and we show that the flip cut number is \( n - 3 \). For example, in Figure 1.3 the leftmost and rightmost triangulations become disconnected if we forbid one more edge, which yields a flip cut set of size 3 for \( n = 6 \). More details can be found in Chapter 4.

Figure 1.5: The “channel”, and a triangulation that becomes \textbf{frozen} (an isolated vertex in the flip graph) if we forbid the edge \( b_2, t_{n-1} \) (in red). In fact, every edge \( b_i t_j, i, j \not\in \{1, 5\} \) is a flip cut edge.

In Section 3.4.2, we show that a point set of size \( n \) may have \( \Theta(n^2) \) flip cut edges (see Figure 1.5), and in Section 3.4.4, we show that a flip cut edge may result in \( \Theta(n) \) disconnected components in the flip graph. We also examine various special point sets whose flip graphs have been previously studied in Section 3.4, such as points on an integer grid [8] and, more generally, point sets without empty convex pentagons [14]. Our characterization of flip cut edges becomes simpler in the absence of empty convex pentagons. Point sets without empty convex pentagons must have collinear points; our results do not assume
points in general position.

1.4.1 Examples

In this chapter we are going to provide some examples of flip cut edges and flip cut sets. However the details for each example are provided in later related chapters.

Figure 1.4 shows the smallest example of a point set that has a flip cut edge—in fact, by forbidding one edge, we get a triangulation from which no flips are possible. The triangulation associated with such an isolated vertex in the flip graph is called a **frozen** triangulation.

The point set shown in Figure 1.5, which is called a **channel**, is created in the following way: first, put four points on four corners of an axis-aligned rectangle. Then, add $n$ points from a flat convex curve connecting two points on the upper corners. These $n$ points are called the upper chain. Finally, add $n$ points of a concave flat curve that connects the two points on the lower corners. This is called the lower chain. The triangulation of this point set shown in Figure 1.5 becomes frozen (modulo triangulating the upper and lower convex subpolygons) when we forbid one flip cut edge. In fact, any edge joining an interior vertex of the top curve and an interior vertex of the bottom curve is a flip cut edge. So there are $\Theta(n^2)$ flip cut edges. We justify this more carefully in Section 3.4.2.

Grid points are another well-studied case for triangulation flips [8], in part because of physics applications. For $n$ points lying on a $\sqrt{n} \times \sqrt{n}$ grid, there are again $\Theta(n^2)$ flip cut edges. In fact, for an infinite grid, every edge is a flip cut edge, though boundary effects interfere in finite grids. See Section 3.4.2. Points in a grid have the special property that there are no empty convex pentagons. Flips for point sets without empty convex pentagons were studied by Eppstein [14]. Such point sets must have collinear points [1]. See Section 3.4.1.

The “hourglass” shown in Figure 1.6 has a flip cut edge $e = uv$ such that forbidding $e$ results in $\Theta(n)$ disconnected components in the flip graph, which is the most possible. See Section 3.4.4.

Some point sets have $\Theta(n^2)$ flip cut edges, but at the other extreme, some point sets have no flip cut edges. For example, there are no flip cut edges for points in convex position. The standard way to flip between two triangulations of a convex polygon is via a **star triangulation**. A **star triangulation** is a triangulation with all edges (except the edges of the convex hull) incident to one point. If there is a point not incident to any forbidden edge, we can still flip to a star centered on that point. See Figure 1.7. A more
Figure 1.6: In this “hourglass” $uv$ is a flip cut edge that creates $n$ disconnected components in the flip graph, one for each $a_i b_i$.

detailed version of this argument is given in Lemma 24 when we investigate flip cut sets for points in convex position.

Figure 1.7: A convex hexagon with a set $X$ of two forbidden edges (in red), and a flip sequence to connect one triangulation in $\mathcal{T}_X$ to the triangulation that is a star at $v$, without using any forbidden edges.

Figure 1.8 shows some more examples of flip cut edges in point sets.

1.5 Notation and Definitions

We denote the flip graph of point set $P$ by $\mathcal{F}(P)$, or just $\mathcal{F}$, when $P$ is clear from context.
Figure 1.8: Some point sets and their flip cut edges (in red).

For a subset $E$ of the edges of $P$, let $\mathcal{T}_+(P)$ be the set of triangulations of $P$ that include all the edges of $E$. These are known as constrained triangulations. Let $\mathcal{F}_+(P)$ be the subgraph of the flip graph induced on the vertex set $\mathcal{T}_+(P)$. It is known that $\mathcal{F}_+(P)$ is connected [10].

Let $\mathcal{T}_-(P)$ be the set of triangulations of $P$ that include none of the edges of $E$, and let $\mathcal{F}_-(P)$ be the subgraph of the flip graph induced on $\mathcal{T}_-(P)$. When $E$ consists of a single edge $e$, we will write $\mathcal{T}_-(P)$, and so on. Also, we will omit $P$ when the point set is clear from the context.

A subset $E$ of edges of $P$ is a flip cut set if the flip graph $\mathcal{F}_-(P)$ is disconnected. The smallest size of a flip cut set is called the flip cut number of $P$. If $\{e\}$ is a flip cut set of size one, we call $e$ a flip cut edge.

An empty convex quadrilateral or EC4 is a set of 4 points in $P$ that form a convex
quadrilateral with no other points of $P$ inside or on the boundary. We also use $EC3$ for empty triangles, $EC5$ for empty convex pentagons, and so on.

Point set $P$ is a **convex point set** if all the points of $P$ are extreme points of the convex hull of $P$, i.e., for every point $p \in P$, there is a line through $p$ with all other points of $P$ strictly to one side of the line. Point set $P$ is in **general position** if no three points are collinear.
Chapter 2

Related Work

In this section, we briefly discuss related work on general reconfiguration problems, flip graphs, and questions related to these topics. Then, we give some background on flip graphs of triangulations, some variations, and the problems that arise in this specific setting of reconfiguration.

Reconfiguration and Flip Graphs

As described in the previous section, reconfiguration is the study of the relationship between the feasible solutions of a problem. The reconfiguration, or “flip” graph has a vertex for each feasible solution of the problem, and an edge between two solutions if they can be transformed into each other via an operation called a flip. There are many famous reconfiguration problems including “15-puzzle”, “independent set reconfiguration”, “vertex cover reconfiguration”, “k-coloring reconfiguration”, “dominating set reconfiguration”, “sliding puzzle”, etc. People interested in reconfiguration problems study three important question for each problem: (1) Is the flip graph for the reconfiguration problem connected? (2) What is the asymptotic (or exact) diameter of the flip graph? (3) What is the complexity of finding the shortest path between two configuration in the flip graph? These are the three well-known questions that people studying reconfiguration try to answer. Further information on these questions and different reconfiguration problems can be found in two surveys about reconfiguration by Nishimura [31] and van den Heuvel [36].
Reconfiguring Triangulations

Connectivity of the flip graph. In the previous chapter we discussed that Lawson proved that any two arbitrary triangulations $T_1, T_2$ of a point set $P$ can be transformed into each other using a sequence of flips. Lawson proved this using two approaches. Both of the proofs use a canonical triangulation and prove that any triangulation can be transformed into this canonical triangulation. In the first approach [23], the canonical triangulation can be constructed in the following way: Sort the points by their $x$-coordinates. First, create a triangle with the first three points. Then, add other points in order one by one and add the edge between the added point and each of the previous ones if it does not cross the previous edges. Lawson [23] proved that any triangulation can be reconfigured into this canonical triangulation which shows that any two triangulations can be transformed into each other via a sequence of flips. The second approach [24] used Delaunay triangulations (which were introduced by Lee and Lin [26]) as canonical triangulations. A triangulation of a point set $P$ is a Delaunay triangulation if for any triangle $\triangle abc$, we have that the circle through $a, b$ and $c$ does not contain any other point of $P$ inside it. Again, Lawson showed that any triangulation can be reconfigured to the Delaunay triangulation, hence any two triangulations can be transformed into each other by a sequence of flips. Note that in both of these proofs, in the sequence of the flips, first we reconfigure one triangulation to the canonical one and then reconfigure the canonical triangulation to the second one.

Diameter of the Flip Graph. There is considerable work on distances and diameter of flip graphs. Lawson [24], for example, showed that for both canonical triangulations, $O(n^2)$ flips are sufficient to reconfigure any triangulation into the canonical one. This means $O(n^2)$ is an upper bound for the number of flips needed for transforming any triangulation into another one. Hurtado et al. [21] proved that this upper bound is tight. In other words, they found a point set and two specific triangulations of that point set which need $\Omega(n^2)$ flips in order to reconfigure one into the other. This point set is known as a “channel” and the two triangulations are shown in Figure 2.1. Note that the upper and lower polygons need to be triangulated in order to complete the triangulations. Also, it can be seen that the only possible flip in the middle for each of the two triangulations, are $b_1t_5$ for the left channel and $t_1b_5$ for the right channel. In order to prove the $\Omega(n^2)$ bound we can consider a binary encoding of the triangulations. We encode triangles with two vertices from the upper chain and one vertex from the lower chain by 1 and triangles with two vertices from the lower chain and one vertex from the upper chain by 0. Then the right triangulation is encoded by 11110000 and the left one is encoded by 00001111. Each flip here is equal to swapping a neighbouring 0 and 1. So, it can be shown that $\Omega(n^2)$ swaps or flips are needed.
to transform one encoding (triangulation) to another one.

Figure 2.1: Two triangulations of a channel that require $\Omega(n^2)$ flips in order to transform one into the other.

Hurtado et al. [21] showed that for a point set $P$ with $k$ convex layers (the convex layers of a point set are the sequence of convex hulls of the point set, each obtained after removing the vertices of the previous layer), each triangulation of $P$ can be transformed to another one with at most $O(nk)$ flips. They also showed that at most $O(n + k^2)$ flips are needed to reconfigure a triangulation of an $n$-gon with $k$ reflex vertices to another one. A reflex vertex is a vertex that has a concave internal angle inside the polygon.

Bounds on Flip Distance Between Triangulations. As can be seen from the above discussion, most of the proofs for connectivity of the flip graph use some canonical triangulation to reconfigure two triangulations. However, usually the shortest way to transform one configuration into another does not include a canonical configuration in the intermediary steps. This observation led to finding some bounds on the distance between two triangulations $T_1$ and $T_2$. Hanke et al. [18] provided an upper bound on the minimum number of flips needed to transform one triangulation into another one. They showed that the number of intersections between the edges of the two triangulations is an upper bound on the distance between those two triangulations. Eppstein [14] provided a lower bound on the flip distance between two triangulations of the point set using quadrilateral graphs. In a quadrilateral graph of point set $P$, there is a vertex for each edge between two points of the point set and there exists a graph edge between those two vertices if the four endpoints of the corresponding edges create a convex empty quadrilateral (EC4). We denote the quadrilateral graph by $QG$. Eppstein showed for two triangulations $T_1$ and $T_2$ on point set $P$ if we define a complete bipartite graph between the edges of $T_1$ and $T_2$ and assign the weight of an edge to be the distance between the corresponding edges in $QG$, then the weight of the minimum perfect matching in this bipartite graph is a lower bound on the flip distance between $T_1$ and $T_2$. 

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Complexity of the Flip Distance Between Two Triangulations. Kanj et al. [22] showed that the problem of finding the flip distance between two triangulations is fixed-parameter tractable, that is, there is an \(O(n + k^c)\) time algorithm that identifies whether two triangulations can be reconfigured into each other using at most \(k\) flips. For a general point set, finding the distance in the flip graph between two triangulations (the “flip distance”) is known to be NP-hard [28], and even APX-hard [32]. NP-hardness of the flip distance problem is proved using channels as the building block for the reduction.

Variations: Constrained Triangulations. As mentioned in the previous chapter, sometimes some special families and types of triangulations (and flip graphs as a result) are studied for their properties and constrained triangulations is the most famous one among them. In 1987, Chew [10] introduced the notion of constrained triangulations and constrained Delaunay triangulations. Given a set \(X\) of constrained edges, the constrained triangulations are the ones that contain all the edges of \(X\), i.e., \(T_{+X}\) in our notation. The constrained Delaunay triangulation (CDT) is a triangulation in \(T_{+X}\) that is as close as possible to the Delaunay triangulation. That is, CDT contains all the edges of \(X\) and for each edge \(e = uv\) of the CDT there exists a circle through \(u\) and \(v\), and for any point \(p \in P\) inside the circle there is an edge \(f \in X\) that crosses one of the segments \(pv\) or \(pu\). Chew [10] showed that any constrained triangulation can be flipped to a CDT while keeping all the constrained edges, which implies connectivity of the flip graph for constrained triangulations (which we denote by \(F_{+X}\)). Houle et al. [20] introduced another family of triangulations and another subgraph of the flip graph. This new family, which consists of only triangulations that admit a perfect matching, was also proven to have connected flip graphs [20].

Variations: Simultaneous Flips The next variation, introduced by Hurtado et al. [21], is the notion of simultaneous flips in triangulations. There are some flips in triangulations that can be done at the same time. A set of flips can be done simultaneously if no pair of the flipping edges are sides of the same triangle. In that sense, these are independent from each other. Based on this observation, for a triangulation \(T\), if we flip a set \(E\) of triangulation edges in \(T\) and obtain a new triangulation \(T'\), this would be called a simultaneous flip. Hurato et al. also showed that there are always \(\lceil \frac{n-4}{2} \rceil\) edges that can be flipped. Galtier et al. [16] showed that \(O(n)\) simultaneous flips is sufficient to reconfigure one triangulation into another one. They provided an example of two triangulations that needed at least \(\Omega(n)\) simultaneous flips in order to transform one into the other one. They also showed that there are always \(\lceil \frac{n-4}{6} \rceil\) edges that can be flipped simultaneously, and provided an example where the maximum number of edges that can be flipped simultaneously is \(\lceil \frac{n-4}{5} \rceil\).
Variations: Edge-Labeled Flips  Another variation on flips in triangulations is to consider edge-labelled triangulations, as introduced by Bose et al. [7]. Here the edges of a triangulation have distinct labels and a flip transfers the label from the edge that is removed to the edge that is added. Bose et al. proved a connectivity and diameter result for points in convex position—any labelled triangulation can be flipped to any other with $O(n \log n)$ flips. For general point sets, the labelled flip graph is not connected, but Lubiw et al. [27] characterized when one labelled triangulation can be flipped to another. The proof uses the flip complex, which is an abstract complex (i.e., a complex which is defined combinatorially rather than geometrically) that generalizes the associahedron. Some properties of the associahedron carry over to the flip complex, e.g., the two dimensional faces have size 4 or 5.

Connectivity and Expander Properties  Recently, Wagner and Welzl [38] showed that for $n$ points in general position in the plane, the flip graph is $\lceil \frac{n}{2} - 2 \rceil$-connected. More generally, researchers who study connectivity properties of flip graphs are interested in mixing rate results. Wagner and Welzl also proposed the question of whether their connectivity results are going to be helpful for rapid mixing of triangulations and expander properties of the triangulation flip graph. An open frontier in the study of flip graphs and triangulation mixing rate has to do with expander properties. The reason for these studies is that expander properties would potentially be helpful for rapid mixing via random flips. Mixing time of triangulations has been studied in [30, 29]. Molloy et al. [30] gave a lower bound of $\Omega(n^{3/2})$ and an upper bound of $O(n^{23} \log(n/\epsilon))$. McShine and Tetali [29] improved the upper bound to $O(n^5 \log(n/\epsilon))$. Eppstein and Frishberg [15] showed that mixing time for the flip walk on a convex point set is $O(n^{1.75})$. Caputo et al. [8] studied rapid mixing of lattice triangulations, which has important applications in quantum physics. Lattice triangulations are triangulations on a grid point set. It has recently been shown that the reconfiguration graph of bases of a matroid is an expander [2], and it would be interesting to know if similar results hold for triangulation flip graphs.

Convex Point Sets and Associahedra

As mentioned in the previous chapter, the flip graph for $n$ points in convex position can be realized geometrically as the 1-skeleton of an $(n-3)$-dimensional polytope called the associahedron. This means that each vertex of the associahedron represents one triangulation of the point set and each edge of the associahedron represents a flip. The $(n-3)$-dimensional associahedron is denoted $A_{n-1}$. Using the fact that the flip graph of convex point sets is the
1-skeleton of associahedra, and applying Balinski’s theorem [3] to this 1-skeleton Wagner and Welzl [38] showed that for points in convex position, the flip graph is \((n - 3)\)-connected.

Another interesting fact about associahedra is that each vertex has \(n - 3\) adjacent edges, which follows from the fact that each triangulation of a convex point set has \(n - 3\) possible flips.

We showed that there exists a bijection between binary trees and triangulations in Section 1.3, which means there exists a bijection between rotations in trees and flips in triangulations. As a result, associahedra are the realizations of the reconfiguration graphs for rotations in binary trees. Also, it is known that there is a correspondence between each vertex of the associahedra and each different way of inserting parentheses in an expression of length \(n\); then, each edge represents applying the associativity rule once. Many other realizations of associahedra have been discussed in [9].

The 3-dimensional associahedron \(A_5\) is the one that can be understood better than the higher dimensional ones because of its simplicity. As shown in Figure 1.3 it has 14 vertices, 21 edges, and 9 faces (2D facets). Also, it is interesting that each face corresponds to one of the inner diagonals of the convex hexagon, in the sense that the face consists of all the triangulations that contain that diagonal. More generally, each \((n - 4)\)-dimensional face of the \((n - 3)\)-dimensional associahedron consists of the triangulations that contain one specific inner diagonal of the convex point set of size \(n\).

The diameter of associahedra has been studied widely in the past. Sleator et al. [35] showed that the diameter of \(A_{n-1}\) is \(2n - 6\) for large enough \(n\). Equivalently, maximum rotation distance for trees of size \(n\) is \(2n - 6\) for large \(n\)’s. Later, Pournin [33] proved this for any \(n\) larger than 9. This means that for a convex point set with \(n\) points, the diameter of the flip graph is \(2n - 10\) for \(n > 12\). However, a main open question is the complexity of the flip distance problem for convex point sets (is it NP-hard or in P?).
Chapter 3

Flip Cut Edges

In this chapter, we explain the initial steps in identifying flip cut edges (Section 3.1), then we characterize flip cut edges (Section 3.2) and give an $O(n \log n)$ time algorithm to test if a given edge is a flip cut edge (Section 3.3). In Section 3.4 we examine some special point sets and establish bounds on the number of flip cut edges and the number of connected components.

3.1 Initial Steps

This section contains an analysis of flip cut edges in small point sets, and a discussion of the role of EC4’s (empty convex quadrilaterals) and EC5’s in characterizing flip cut edges. This material adds intuition but is not necessary for the following sections.

3.1.1 The Simplest Example of a Flip Cut Edge

The main problem at the beginning is whether it is possible to disconnect the flip graph by forbidding exactly one of the edges between a pair of points (which we call a flip cut edge). In order to solve this problem, one should begin with some simple examples and test whether an edge with the described property can be found. Any point set containing three points only has one valid triangulation, which can be constructed by creating a triangle that contains the points as its vertices. Therefore, the flip graph contains only one vertex and cannot become disconnected in any way. Remember that if there exists a valid flip for a triangulation, we need a convex empty quadrilateral where the flip is exchanging its
The flip graph of a convex point set of size four contains only one frozen triangulation. First, note that we cannot forbid any edge of the convex hull since every valid triangulation contains all of them. Thus, in the previous case, we only can forbid one of the diagonals of the convex quadrilateral. This leads to a new flip graph with exactly one vertex which is connected. In point set $P$, $|P| = 4$ where the convex hull is a triangle and there exists a point inside this triangle, no flips are possible, so we have a frozen triangulation and an isolated vertex in the flip graph which again is not possible to become disconnected (Figure 3.1b). This means no point set of size less than five can have a flip cut edge. So, we are going to consider point sets of size five.

The first option is a convex point set of size five. Remember that a point set is in convex position if all the points are vertices of the convex hull of that point set. The flip graph for this point set contains a cycle of size five (Figure 3.2). By forbidding any single edge of this point set, two triangulations become invalid and a path of size three will be the new flip graph. So, again, it is not possible to disconnect the flip graph. Now, consider in our point set $P$, $|P| = 5$, the convex hull is a convex quadrilateral, and the other point is inside this quadrilateral. In this scenario, if you have colinear points, then again, the flip graph only contains an isolated vertex, and if it does not, there exists only one valid flip which means disconnecting it is not possible (Figure 3.3a). The last possibility for $|P| = 5$ is a point set with a triangle convex hull and two points inside the hull. In this case, again, there exists only one valid flip (Figure 3.3b) and disconnecting the flip graph is impossible. As a result, we move on to point sets of size six.

In the previous chapters, we saw that the flip graph for a convex point set of size six is a 3D associahedron (Figure 1.3). We also saw that by forbidding one edge of this point set, we remove triangulations on one of the 2D faces of this associahedron. Thus, forbidding one edge does not disconnect the flip graph. The second possibility is having a pentagon convex hull and a point inside. For this case, the flip graph is shown in Figure 3.4. Again, there is no edge such that forbidding the edge disconnects the flip graph. Finally, consider
a point set with a quadrilateral as its convex hull which contains two points inside the hull. The flip graph is shown in Figure 3.5; by forbidding the edge between the two points inside the hull, the middle triangulations in the flip graph become invalid and we are left with two isolated vertices in the resulting flip graph. This means that we have a flip cut edge in this example, and since we have shown there exist no flip cut edges in simpler examples, this is the smallest possible example.

### 3.1.2 The role of EC4’s and EC5’s

In this section we investigate the relationship between flip cut edges and EC4’s (empty convex quadrilaterals) and EC5’s. We make a conjecture about EC4’s which we disprove, and a conjecture about EC5’s which we later show is correct.
Observation 1. In Figure 3.5, each of the two disconnected triangulations contains one of the possible convex quadrilaterals that includes the forbidden edge as its diagonal.

This observation leads us to the thinking that maybe the \textit{quadrilateral graph} introduced by Eppstein [14] can help us in identifying flip cut edges. Recall that in a \textit{quadrilateral graph (QG)} of point set $P$, there is a vertex for each edge between two points of the point set and there exists a graph edge between those two vertices if the four endpoints of the corresponding edges create a convex empty quadrilateral (EC4). In other words, suppose by removing the forbidden edge in QG we disconnect two edges in QG that cross the forbidden edge. Then, does it mean that forbidding that edge disconnects the flip graph? The point set in Figure 3.5 is an example that gives a positive answer to this question. The QG for this point set is shown in Figure 3.6. After removing vertex $uv$, $a_1b_1$ and $a_2b_2$ become disconnected in the QG.
Figure 3.5: The flip graph for a point set which gives us the smallest example that contains a flip cut edge: by forbidding the edge between two points inside the quadrilateral, the resulting flip graph only contains the leftmost and rightmost triangulations, which are disconnected from each other.

Figure 3.6: The QG for the smallest example in Figure 3.5: forbidding $uv$ disconnects $a_1b_1$ and $a_2b_2$

**Conjecture 2.** Suppose that the forbidden edge $e = uv$ is contained in two EC4’s, $Q_1$ and $Q_2$ with diagonals $ab$ and $cd$ respectively. Suppose that removing $uv$ from the QG disconnects $ab$ from $cd$. Then, in the flip graph $F_e$, every triangulation containing $Q_1 \cup ab$ is disconnected from every triangulation containing $Q_2 \cup cd$.

Unfortunately, this conjecture is FALSE. The point set shown in Figure 3.7a is a counter-example for the conjecture. The QG of this point set becomes disconnected after removing edge $uv$ (After forbidding $uv$, we need to remove vertex $uv$ and red edges in the QG that corresponds to invalid flips as shown in Figure 3.7b). But, the flip graph is not disconnected as we can transform one of the triangulations containing one of the quadrilaterals to the triangulation that contains the other quadrilateral. In more detail, forbidding $uv$ in the QG disconnects $a_1b_1$ from $a_3b_2$, so the intuition was that we might be unable to flip from a triangulation containing $a_1b_1$ to a triangulation containing $a_3b_2$. However,
although we cannot flip directly between those two edges, the example flip sequence shows that \(a_3b_2\) flips to position \(va_2\), and \(a_1b_1\) flips from position \(ua_2\).

(a) The flip graph of the simplest point set with flip cut edge after adding point \(a_2\). Blue edges are added after adding \(a_2\). Green edges are the edges that are created during the flip operations.

(b) The QG of the point set in Figure 3.7a. Red edges correspond to invalid flips after forbidding \(uv\).

Figure 3.7: An example where removing \(uv\) disconnects \(a_1b_1\) and \(a_3b_2\) in the QG (b) but triangulations containing those diagonals are still connected in the flip graph (a).

We need a more subtle way to analyze how one EC4 is transformed to another one. To do this, we will examine EC5’s. We first revisit the flip sequence of Figure 3.7a and then formulate several refined conjectures. Figure 3.7a shows that in the path from transforming the first quadrilateral to the second one, we use flips inside three intermediary pentagons: \(ua_2a_3vb_2\), \(ua_2vb_1b_2\), \(ua_1a_2vb_1\). So, it shows adding \(a_2\) creates some EC5’s inside of which we can flip in a way that enables us to transform EC4 \(ua_3vb_2\) in Figure 1.4 to EC4 \(ua_1vb_1\). So, we conjectured that EC4’s containing the forbidden edge are fundamental elements for identifying whether forbidding that edge disconnects the flip graph and EC5’s are the bridges that help us transform each of the EC4’s to the other ones using a sequence of flips occurring inside the EC5’s. The formal description of the conjecture is as follows:
Conjecture 3. Edge $e = uv$ is a flip cut edge iff there exist two EC4’s $Q_1$ and $Q_2$ containing $e$ in a way that there does not exist any other point $p$ where $p \cup Q_1$ and $p \cup Q_2$ are both EC5’s.

We are going to DISPROVE this conjecture, though it seems promising based on Figures 3.5 and 3.8. Adding a new point to the point set in Figure 3.5 in different areas acknowledges our conjecture; we draw the extension of the edges of the two EC4’s (Figure 3.8a). After adding one point to each of areas 1, 2, 3, and 4, we see that only adding a point to area 4, prevents edge $uv$ from being a flip cut edge. And, that is because only a point in area 4 can create EC5’s with the vertices of both $ua_1vb_1$ and $ua_3vb_2$. Note that since the picture is symmetric, the same areas are created on the bottom half of the point set. However, checking only area 4 for an additional point is not enough for identifying whether $uv$ is a flip cut edge.

In Figure 3.8a some points are added but not to the area 4. So, this point set satisfies the condition in Conjecture 3. However, Figure 3.8b shows how we can transform EC4 $ua_1vb_1$ to EC4 $ua_3vb_3$ using flips inside the pentagons shown step by step. More specifically, first we use EC5 $ua_1vb_1b_2$ to transform EC4 $ua_1vb_1$ to EC4 $ua_1vb_2$; then, use flips inside $ua_1a_2vb_2$ to transform $ua_1vb_2$ to $ua_2vb_2$. We continue by using $ua_2vb_2b_3$ to transform $ua_2vb_2$ to $ua_2vb_3$. $ua_1a_2vb_2$ is used in the same way to transform $ua_1vb_2$ to $ua_2vb_2$. Finally, we flip inside $ua_3vb_3b_4$ to transform $ua_3vb_3$ to $ua_3vb_4$. Figure 3.9 shows how to use an EC5 to flip one EC4 to another one even with having a forbidden edge. A careful analysis of the flip graph of the point set in Figure 3.8a shows that it is connected. So, instead of the previous conjecture, we propose a new one that supports our observation on Figure 3.8 to characterize flip cut edges. In order to do that, first, we introduce a new graph called pentagon graph.

Definition 4. For an edge $e = uv$ in the point set $P$, the pentagon graph (PG) of $e$ can be constructed in the following way: every EC4 containing $e$ as a diagonal is a vertex. There exists an edge between two vertices if there exists an EC5 that contains both corresponding EC4’s inside it.

The next conjecture relates flip cut edges to the PG. Later, in Section 3.3.2, we will prove this conjecture to be CORRECT (Lemma 13).

Conjecture 5. Edge $e$ is a flip cut edge iff its pentagon graph is disconnected.
(a) Some points added to areas other than 4

(b) How to use EC5’s to transform EC4 $ua_1vb_1$ to EC4 $ua_3vb_4$

Figure 3.8: Different areas for adding an additional point to point set in Figure 3.5 to see the effect on the forbidden edge remaining a flip cut edge

Figure 3.9: Transforming EC4 $a_1a_2a_3a_4$ to $a_2a_3a_4a_5$ using EC5 $a_1a_2a_3a_4a_5$
3.2 Characterizing Flip Cut Edges

In this section we give three characterizations of flip cut edges. The characterizations are in terms of connectivity of associated graphs that we call \( G_Y \), \( G_Z \), and PG. The pentagon graph (PG) was introduced in Section 3.1.2, and the other two will be defined below. The first characterization helps identifying connected components better, the second one is helpful for having an efficient algorithm, and the characterization based on PG more precisely demonstrates the transition between triangulations using flips.

3.2.1 Main Characterization, \( G_Y \)

Consider an edge \( e = uv \) of \( P \). Let us orient \( e \) horizontally so we can use the terms “above” and “below” to refer to the two sides of \( e \). In the horizontal orientation, suppose that \( u \) lies to the left of \( v \).

Let \( Y \) be the set of edges \( f \) of \( P \) such that \( f \) crosses \( e \). Let \( G_Y \) be the line graph of \( Y \), i.e., the vertex set of \( G_Y \) is \( Y \), and two edges of \( Y \) are adjacent in \( G_Y \) if they meet at a point. See Figure 3.10.

We will prove that \( e \) is a flip cut edge if and only if \( G_Y \) is disconnected. In fact, we will be able to identify the connected components of \( F_e \) from \( G_Y \).

Observation 6. \( T_e \) is the union of the sets of triangulations \( T_f \) for \( f \in Y \). Each flip graph \( F_f \) is connected.

Proof. For any edge \( f \), \( F_f \) is connected by properties of constrained triangulations [10]. For \( f \in Y \), \( T_f \) is a subset of \( T_e \) because triangulations that contain \( f \) cannot contain \( e \). Finally, since every triangulation of \( T_e \) contains some edge of \( Y \), \( T_e \) is the union of \( T_f \) for \( f \in Y \).

The observation says that every vertex of the graph \( F_e \) appears in some \( F_f \) for \( f \in Y \). In fact, every edge of the graph \( F_e \) appears in some \( F_f \) for \( f \in Y \), as we will prove as part of Theorem 8.

Based on Observation 6, in order to identify connected components of the flip graph \( F_e \), it suffices to figure out which subgraphs \( F_f \) are connected to which other ones in \( F_e \), i.e., to know when there is a path in \( F_e \) from an element of \( T_f \) to an element of \( T_g \).

Before giving the main theorem, we give one more observation.
Figure 3.10: The edges $Y$ that cross $e = uv$. Here, $G_Y$ is connected, so $e$ is not a flip cut edge.

**Observation 7.** For any triangulation $T$ in $\mathcal{T}_{-e}$, the edges of $Y$ in $T$, which we denote $Y \cap T$, are connected in $G_Y$.

*Proof.* The triangles of $T$ that cross the segment $uv$ form a path in the planar dual of the triangulation, and so the edges of $Y \cap T$ which are the side edges of these triangles form a connected subgraph of $G_Y$. See Figure 3.11. \qed

**Theorem 8.** Edge $e$ is a flip cut edge if and only if $G_Y$ is disconnected. More specifically, edges $f$ and $g$ in $Y$ are connected in $G_Y$ if and only if $\mathcal{T}_{+f}$ and $\mathcal{T}_{+g}$ are connected in $\mathcal{F}_{-e}$.

*Proof.* If $fg$ is an edge of $G_Y$, then, in the point set, the edges $f$ and $g$ are incident at a common endpoint so they do not cross, which implies that there is a triangulation containing $f$ and $g$, so $\mathcal{F}_{+f}$ and $\mathcal{F}_{+g}$ are connected (they have a triangulation in common). Thus, by transitivity, if $f$ and $g$ are connected in $G_Y$ then $\mathcal{F}_{+f}$ and $\mathcal{F}_{+g}$ are connected.

For the converse, if $\mathcal{F}_{+f}$ and $\mathcal{F}_{+g}$ are connected, this means that there exists a sequence of flips that transforms triangulation $T_1 \in \mathcal{T}_{+f}$ to $T_2 \in \mathcal{T}_{+g}$ where intermediary triangulations are also in $\mathcal{T}_{-e}$. So, it suffices to show that if we flip from $T_1$ in $\mathcal{T}_{-e}$ to $T_2$ in $\mathcal{T}_{-e}$,
Figure 3.11: The edges of $Y \cap T$ (in thick black) are connected in $G_Y$. A flip of $f_1$ to $f_2$ requires another edge $h \in Y \cap T_1 \cap T_2$.

then $Y \cap T_1$ and $Y \cap T_2$ are connected in $G_Y$. This implies that if there is a sequence of flips transforming $T_1$ in $\mathcal{T}_e$ to $T_2$ in $\mathcal{T}_-e$, then there is a path connecting $f$ to $g$ in $G_Y$. First note that by Observation 7, $Y \cap T_1$ is a connected set in $G_Y$ and the same is true for $Y \cap T_2$. We will show that $Y \cap T_1$ and $Y \cap T_2$ have an element in common. Let $f_i \in T_i \cap Y$, $i = 1, 2$. If either edge is in the other triangulation, we are done. Otherwise, the flip from $T_1$ to $T_2$ must flip $f_1$ to $f_2$. Then the EC4 formed by the endpoints of $f_1$ and $f_2$ has a side edge $h$ that is an edge of $Y$ in both $T_1$ and $T_2$. See Figure 3.11. (Note that this argument shows that every edge of the flip graph $\mathcal{F}_-e$ lies in $\mathcal{F}_{+h}$ for some $h \in Y$.)

Algorithmic implications. The theorem gives an immediate polynomial time algorithm to test if $e$ is a flip cut edge: construct $G_Y$ (a graph on $O(n^2)$ vertices) and test connectivity. The details of this algorithm can be found in Algorithm 1 in Section 3.3.1. In Section 3.3.3 we give a faster $O(n \log n)$ time algorithm.

Our other algorithmic goal is to “identify” the connected components of $\mathcal{F}_-e$. Although the flip graph is exponentially large, we can identify connected components by grouping them into $\mathcal{T}_+f$’s for $f \in Y$, and identify which groups of triangulations are connected to which other ones. That being said, we focus on the problem of testing whether two triangulations $T_1$ and $T_2$ in $\mathcal{T}_-e$ are in the same connected component of $\mathcal{F}_-e$. Using
Theorem 8, we can do that as follows. Pick $f_1$ in $Y \cap T_1$ and $f_2$ in $Y \cap T_2$ (note that such edges exist). Then $T_1 \in T_{+f_1}$ and $T_2 \in T_{+f_2}$. So $T_1$ and $T_2$ are connected in $\mathcal{F}_{-e}$ iff $T_{+f_1}$ and $T_{+f_2}$ are connected in $\mathcal{F}_{-e}$ iff (by Theorem 8) $f_1$ and $f_2$ are connected in $G_Y$, which we can test in polynomial time. In Subsection 3.3 we give a faster $O(n)$ time algorithm.

3.2.2 Second Characterization, $G_Z$.

For the efficient algorithms in Section 3.3.3, we need an alternative characterization of flip cut edges in terms of a subgraph of $G_Y$.

For any $f \in Y$, let $Q(f)$ be the convex quadrilateral formed by the endpoints of $f$ and $e = uv$. Let $Z$ be the set of edges $f \in Y$ such that $Q(f)$ is an EC4. Let $G_Z$ be the line graph of $Z$. Then $G_Z$ is an induced subgraph of $G_Y$ with a smaller vertex set.

Note that any $f \in Z$ has one point above $e$ and one point below $e$, and these points make empty triangles with $e$. Let $A$ be the set of points $a$ above $e$ such that $auv$ is an empty triangle (an EC3). Let $B$ be the set of points $b$ below $e$ such that $bvu$ is an empty triangle. Thus $Z$ consist of the edges from $A$ to $B$ that cross $e = uv$, i.e., $Z = Y \cap (A \times B)$. See Figure 3.13.

We will prove an analogue of Theorem 8:

**Theorem 9.** Edge $e$ is a flip cut edge if and only if $G_Z$ is disconnected.

We will also be able to characterize connected components of $\mathcal{F}_{-e}$ in terms of $G_Z$, but this cannot be analogous to Theorem 8, because not every triangulation contains an edge of $Z$. We begin with some preliminary results.

**Lemma 10.** Edges $f, g \in Z$ are connected in $G_Z$ if and only if they are connected in $G_Y$.

**Proof.** The forward direction is clear since $G_Z$ is an induced subgraph of $G_Y$.

For the other direction, suppose $f$ and $g$ are connected in $G_Y$. Suppose, $f = f_1, f_2, ..., f_m = g$ is a shortest path in $G_Y$ between $f$ and $g$. If all the $f_i$'s are in $Z$, then $f$ and $g$ are connected in $G_Z$. Otherwise, we will modify the path to replace edges of $Y$ by edges of $Z$. Suppose $f_i = p_i q_i$ with $p_i$ above $uv$ and $q_i$ below $uv$. Because the path is shortest, the common points between successive edges are alternately above and below the line $L$ through $uv$. In particular, suppose without loss of generality that $p_1 = p_2, q_2 = q_3, ..., p_{2i-1} = p_{2i}, q_{2i} = q_{2i+1}, ...$.
Figure 3.12: Proof of Lemma 10. The original path in $G_Y$ from $f_1 \in Z$ to $f_6 \in Z$ is $f_1, \ldots, f_6$ (shown in black), and the modified path replaces the edges of $Y - Z$, which are $f_3, f_4, f_5$ by the edges $f'_3, f'_4, f'_5 \in Z$ (shown in red).

The plan is to replace $p_i$ by some point $p'_i$ and replace $q_i$ by some point $q'_i$ so that the segments $f'_i = p'_i q'_i$ are edges in $Z$ and form a path connecting $f$ to $g$.

If triangle $p_iuv$ is empty, then $p'_i = p_i$; otherwise $p'_i$ is a point inside triangle $p_iuv$ that is closest to line $L$. Define $q'_i$ similarly with respect to triangle $q_iuv$. Note that $f'_1 = f_1$ and $f'_m = f_m$.

We claim that each $f'_i$ is in $Z$. We must show that the line segment $p'_i q'_i$ is an edge, that it crosses $e = uv$, and that the quadrilateral $Q(f'_i)$ is empty.

First note that triangles $p'_i uv$ and $q'_i uv$ are empty because we picked $p'_i$ and $q'_i$ closest to line $L$. Next we claim that these two empty triangles together form a convex quadrilateral. This is because $f_i$ crosses $uv$, so $Q(f_i)$ is convex, and when we move $p_i$ to $p'_i$ and $q_i$ to $q'_i$, convexity is preserved. Thus $f'_i$ is an edge of $Z$.

Finally, note that $p'_{2i-1} = p'_{2i}$ and $q'_{2i} = q'_{2i+1}$ because that was true of the original points. Thus the edges $f'_i$ form a path in $G_Z$ connecting $f$ and $g$. \hfill \qed

**Lemma 11.** Every connected component of $G_Y$ contains an edge of $Z$.

This lemma can be proved in several different ways. One possibility is to take an edge $f \in Y$ and construct the edge $f' \in Z$ inside $Q(f)$ as in the above proof. However, showing
that \( f \) and \( f' \) are connected in \( G_Y \) runs into some complications due to the possibility of collinear points.

Instead we take an edge \( f \in Y \) and consider a triangulation \( T \) containing \( f \) and show that \( T \) can flip in \( \mathcal{F}_{-e} \) to a triangulation containing an edge \( g \in Z \). Below we give a short proof based on this idea, and in Section 3.3 we give an alternative algorithmic proof that efficiently finds an edge of \( Z \) connected to \( f \) in \( G_Y \).

**Proof.** Let \( f \) be an edge of \( Y \) and let \( T \) be a triangulation that contains \( f \). We will prove there is an edge \( g \in Z \) that is connected to \( f \) in \( G_Y \). There is a flip sequence from \( T \) to a triangulation that contains \( e = uv \). The last flip in this sequence must involve an edge \( g \) of \( Z \) flipping to \( e \). Until the last flip, we are in one connected component of \( \mathcal{F}_{-e} \), so, by Theorem 8, all the edges of \( Y \) in all the triangulations in the sequence are in the same connected component of \( G_Y \). Thus \( f \) and \( g \) are connected in \( G_Y \). \hfill \Box

**Proof of Theorem 9.** If \( e \) is a flip cut edge, then by Theorem 8, \( G_Y \) is disconnected. By Lemma 11, every connected component of \( G_Y \) contains an edge of \( G_Z \). Thus \( G_Z \) is disconnected.

In the other direction, if \( G_Z \) is disconnected, then by Lemma 10, \( G_Y \) is disconnected, so \( e \) is a flip cut edge by Theorem 8. \hfill \Box

In Section 3.3.3 we give efficient algorithm that uses \( G_Z \) to test if edge \( e \) is a flip cut edge, and, if so, to identify when two given triangulations are in different components of \( \mathcal{F}_{-e} \).

### 3.2.3 Third Characterization, PG.

The third characterization says that \( e \) is a flip cut edge if and only if \( PG \) is disconnected. This was stated as Conjecture 5 and is proved below as Lemma 13 using Theorem 9.

Recall from Definition 4 that \( PG \) has a vertex for every EC4 that has \( e \) as a diagonal and an edge when two EC4’s are contained in an EC5.

**Claim 12.** \( PG \) is a subgraph of \( G_Z \) with the same vertex set.

**Proof.** A vertex of \( PG \) corresponds to an EC4 that has \( e \) as a diagonal, and the other diagonal of such an EC4 is an edge of \( Z \). This gives a one-to-one correspondence between the vertices of \( PG \) and the set \( Z \).
Figure 3.13: The points that form empty triangles with $uv$ are $A = a_1, \ldots, a_5$ and $B = b_1, b_2, b_3$ ordered cyclically around $u$. The edges of $Z$ are $f_1, \ldots, f_5$ and they form two connected components in $G_Z$, $\{f_1\}$ and $\{f_2, \ldots, f_5\}$. For any point $b \in B$, the points $a \in A$ such that $ab \in Z$ form a subinterval of $A$.

To show that any edge of $PG$ is an edge of $G_Z$, consider two vertices of $PG$ corresponding to edges $f, g \in Z$. Then $Q(f)$ and $Q(g)$ are EC4’s that are contained in a common EC5. This implies that $f$ and $g$ share a common endpoint, so they are joined by an edge in $G_Z$.

Lemma 13. Edge $e = uv$ is a flip cut edge iff its pentagon graph $PG$ is disconnected.

Proof. Based on Theorem 9, $e$ is a flip cut edge iff $G_Z$ is disconnected. So, we need to prove $PG$ is disconnected iff $G_Z$ is disconnected, i.e., $PG$ is connected iff $G_Z$ is connected.

Based on the above claim, $PG$ is a subgraph of $G_Z$ on the same vertex set. Thus, the forward direction is trivial.

For the other direction, it suffices to show that if $f$ and $g$ are joined by an edge in $G_Z$ then their corresponding vertices ($Q(f)$ and $Q(g)$) are connected in $PG$. The existence of an edge between $f$ and $g$ means they have a common endpoint. Suppose, without loss of generality, this endpoint is $b$ below $e = uv$, and $f = ab$ and $g = a'b$. Suppose $a = a_0, a_1, \ldots, a' = a_l$ are the points above $uv$ that create an empty triangle with $u$ and
$v$ in cyclic order around $u$. Note that $a$ and $a'$ create an empty triangle with $u$ and $v$ because $f = ab$ and $g = a'b$ are in $Z$. Because of the conditions about order and having empty triangles, we can conclude that $a_iubv$, $a_{i+1}ubv$, and $ua_ia_{i+1}vb$ are all empty for $i = 0, ..., l - 1$. Hence, there is an edge between every $a_iubv$, $a_{i+1}ubv$ in $PG$. Thus, $Q(f)$ and $Q(g)$ are connected in $PG$. 

### 3.3 Algorithms for Flip Cut Edges

In this section we give three algorithms to test if an edge is a flip cut edge using the three characterizations from the previous section. One is easy to describe, one led to our implementation, and one is efficient. In more detail:

1. Algorithm 1 is straight-forward. It tests if $G_Y$ is connected, as justified by Theorem 8. The running time is $O(n^4)$.

2. Algorithm 2 uses the simplest graph which is $PG$ (it contains fewer vertices and edges in comparison with $G_Y$ and $G_Z$) and checks if it is connected. It also is a better representation of flips for transforming triangulations containing different quadrilaterals. Algorithm 2 is the one we used for the examples in Figure 1.8. The reason we implemented this one at first was that Conjecture 5 was our first idea of characterizing flip cut edges and it also is a better demonstration of the way we can transform between triangulations containing EC4s created with $u$ and $v$. The running time is $O(n^4)$.

3. Algorithm 3 tests if $G_Z$ is connected (justified by Theorem 9) with an efficient $O(n \log n)$ time implementation.

Algorithms 1, 2, and 3 are in the first three subsections. In Section 3.3.4, we show how to find all flip cut edges in time $O(n^3)$ (note that there may be $\Theta(n^2)$ flip cut edges). In Section 3.3.5, with Algorithm 3 as a preprocessing step, we give an $O(n)$ time algorithm to test if two triangulations in $T_\omega$ are connected in $F_\omega$.

#### 3.3.1 Testing for a Flip Cut Edge Using $G_Y$

Recall that Theorem 8 states:
Theorem 8. Edge $e$ is a flip cut edge if and only if $G_Y$ is disconnected. More specifically, edges $f$ and $g$ in $Y$ are connected in $G_Y$ if and only if $T_{+f}$ and $T_{+g}$ are connected in $F_{-e}$.

In order to use this characteristic in implementing an algorithm, we build the graph $G_Y$ and check whether it is connected or not. Note that the vertices of $G_Y$ are edges in point set $P$ that cross the edge $e = uv$ (the one we are testing). So, we can have $\Theta(n^2)$ vertices and $\Theta(n^4)$ edges. So, running DFS or BFS to check whether $G_Y$ is connected takes $O(n^4)$ time. Also, note that this algorithm implicitly identifies the connected components of the flip graph. Triangulations containing edges represented by vertices in connected components of $G_Y$ are in the same connected component in the flip graph $F_{-e}$. Algorithm 1 describes the detailed implementation of the explained algorithm.
Algorithm 1 Connected components of $G_Y$

**Input:** $u, v, p_1, \ldots, p_n$

**Output:** Connected components of $G_Y$ in the form of $C_1, \ldots, C_k$

1: procedure TestFlipCutEdge1
2:    flipCutEdge ← False
3:    $k ← 0$
4:    // Find $Y$
5:    $Y ← \emptyset$
6:    for $i ← 1$ to $n$ do
7:        for $j ← i + 1$ to $n$ do
8:            if $p_ip_j$ crosses $uv$ then add $p_ip_j$ to $Y$
9:        end for
10:    end for
11:    // Find the edges of $G_Y$
12:    for all $v$ in $Y$, adjacencyList[$v$] = $\emptyset$, visited[$v$] ← False
13:    for $v_1$ in $Y$ do
14:        for $v_2$ in $Y$ do
15:            if $v_1 \neq v_2$ and $v_1, v_2$ share a common endpoint then
16:                insert $v_2$ into adjacencyList[$v_1$]
17:            end if
18:        end for
19:    end for
20:    // Run DFS on $G_Y$ to find connected components
21:    for $i ← 1$ to $Y.size()$ do
22:        if not visited[$v_i$] then
23:            $k ← k + 1$
24:            $C_k ← \emptyset$
25:            run DFS on graph with $Y$ as its set of vertices and adjacencyList representing the edges
26:            for every $v$ visited during DFS, insert $v$ into $C_k$
27:            for every $v$ visited during DFS, visited[$v$] ← True
28:        end if
29:    end for
30:    if $k > 1$ then flipCutEdge ← True
31: end procedure
3.3.2 Testing for a Flip Cut Edge Using PG

Since, in the beginning, we were looking for characterization based on Conjecture 5 and PG this was the first algorithm that we devised, thus this was the one we implemented. Also, note that the graph built based on this algorithm better represents the sequence of flips that transforms one triangulation into other ones. Algorithm 2 builds PG for \( e \) by finding EC5’s and EC4’s that contain diagonal \( e \). Then, based on the connectivity of this graph, it decides whether \( e \) is a flip cut edge. We check every two points with \( u \) and \( v \) to see if they create a convex quadrilateral (this takes \( O(n^2) \) time) and then we need a linear-time check to see whether that quadrilateral is empty. Similarly, for every set of three points, we check if they can create an empty convex pentagon with \( u \) and \( v \), which takes \( O(n^4) \) time. This algorithm works correctly as a result of Lemma 13. However, \( G_Z \) can be used for finding flip cut edges much more efficiently. This will be shown in the following section.
Algorithm 2 Connected components of PG

Input: $u, v, p_1, \ldots, p_n$

Output: Connected components of PG in the form of $C_1, \ldots, C_k$

1: procedure $\text{TestFlipCutEdge2}$
2: \hspace{1em} flipCutEdge $\leftarrow$ False
3: \hspace{1em} $k \leftarrow 0$
4: \hspace{1em} // Find $V$, the vertex set of PG
5: \hspace{1em} $V \leftarrow \emptyset$
6: \hspace{1em} for $i \leftarrow 1$ to $n$ do
7: \hspace{2em} for $j \leftarrow i + 1$ to $n$ do
8: \hspace{3em} if $u, v, p_i, p_j$ create a convex quadrilateral containing $uv$ as a diagonal then
9: \hspace{3em} \hspace{1em} isEmpty $\leftarrow$ True
10: \hspace{3em} \hspace{1em} for $l \leftarrow 1$ to $n$ do
11: \hspace{3em} \hspace{2em} if $p_l$ is inside quadrilateral created by $u, v, p_i, p_j$ then
12: \hspace{3em} \hspace{3em} \hspace{1em} isEmpty $\leftarrow$ False
13: \hspace{3em} \hspace{3em} break
14: \hspace{3em} \hspace{1em} end if
15: \hspace{3em} \hspace{1em} end for
16: \hspace{3em} \hspace{1em} if isEmpty then
17: \hspace{3em} \hspace{2em} insert $E_{4u vp_i vp_j}$ to $V$
18: \hspace{3em} \hspace{2em} adjacencyList[$u vp_i vp_j$] $= \emptyset$, visited[$u vp_i vp_j$] $\leftarrow$ False
19: \hspace{3em} \hspace{1em} end if
20: \hspace{2em} end if
21: \hspace{2em} end for
22: \hspace{1em} end for
23: \hspace{1em} // Find the edges of PG
24: \hspace{1em} for $i \leftarrow 1$ to $n$ do
25: \hspace{2em} for $j \leftarrow i + 1$ to $n$ do
26: \hspace{3em} if $u v p_i p_j$ $\in V$ then
27: \hspace{4em} for $l \leftarrow j + 1$ to $n$ do
28: \hspace{5em} if $u, v, p_i, p_j, p_l$ create a convex pentagon containing $uv$ as a diagonal then
29: \hspace{6em} \hspace{1em} isEmpty $\leftarrow$ True

for $t \leftarrow 1$ to $n$ do
  if $p_t$ is inside pentagon created by $u, v, p_i, p_j, p_l$ then
    isEmpty $\leftarrow$ False
    break
  end if
end for
if isEmpty then
  if $up_i vp_j \in V$ then
    insert $up_i vp_j$ to $\text{adjacencyList}[up_i vp_j]$
    insert $up_i vp_l$ to $\text{adjacencyList}[up_i vp_l]$
  end if
  if $up_j vp_l \in V$ then
    insert $up_i vp_j$ to $\text{adjacencyList}[up_i vp_j]$
    insert $up_j vp_l$ to $\text{adjacencyList}[up_i vp_l]$
  end if
end if
end if
end for
end for
end for
// Run DFS to find connected components of PG
for all $v$ in $V$, $\text{visited}[v] \leftarrow$ False
for $i \leftarrow 1$ to $V$.size() do
  if not $\text{visited}[V_i]$ then
    $k \leftarrow k + 1$
    $C_k \leftarrow \emptyset$
    run DFS on graph with $V$ as its set of vertices and $\text{adjacencyList}$ representing the edges
    for every $v$ visited during DFS, insert $v$ into $C_k$
    for every $v$ visited during DFS, $\text{visited}[v] \leftarrow$ True
  end if
end for
if $k > 1$ then
  $\text{FlipCutEdge} \leftarrow$ True
end if
end procedure
3.3.3 Efficiently Testing for a Flip Cut Edge Using $G_Z$

To test if edge $e$ is a flip cut edge, by Theorem 9 it suffices to test if $G_Z$ is connected. The algorithm also identifies the connected components of $G_Z$ (though without explicitly listing the elements of $Z$, since there can be $\Theta(n^2)$ of them). In particular, we find disjoint subsets $A_1, \ldots, A_c$ of $A$, and $B_1, \ldots, B_c$ of $B$ such that the $i$th connected component of $G_Z$ consists of the edges of $Z$ between $A_i$ and $B_i$. Note that $Z$ will not in general contain all pairs from $A_i \times B_i$.

We first need some more properties of the sets $A$ and $B$ and the graph $G_Z$. Because no empty triangle is contained in another, the ordering of $a \in A$ by decreasing (convex) angle $\angle auv$ is the same as the ordering by increasing (convex) angle $\angle avu$. Let $a_1, \ldots, a_k$ be this ordering of $A$. Similarly, let $b_1, \ldots, b_l$ be the ordering of $B$ by decreasing (convex) angle $\angle bvu$, or equivalently, by increasing angle $\angle buv$. Thus the cyclic order of $A \cup B$ around $u$ is $a_1, \ldots, a_k, b_1, \ldots, b_l$. See Figure 3.13.

Observation 15. For any point $b_j$, the set of points $a_i$ such that the edge $a_ib_j$ is in $Z(e)$ form a subinterval of the ordering of $A$, and similarly for $a_i$.

Later on, we will find it useful to have an even stronger property:

Observation 16. Let $L$ be the line through $uv$.

1. If $a_ib_j$ crosses $L$ to the right of $v$, then the same is true for all $a_ib_{j'}$, $i' \geq i, j' \leq j$.

2. If $a_ib_j$ crosses $L$ to the left of $u$, then the same is true for all $a_ib_{j'}$, $i' \leq i, j' \geq j$.

3. If $a_i b_{i_2}$ and $a_j b_{j_2}$ are in $Z$ for some $i_1 \leq i_2$ and $j_1 \leq j_2$, then $a_i b_j$ is in $Z$ for all $i_1 \leq i \leq i_2$ and all $j_1 \leq j \leq j_2$.

Proof. Since (2) is symmetric to (1), it suffices to prove (1). To prove (1), suppose $a_ib_j$ crosses $L$ to the right of $v$. Then the angle $\angle a_{i}vb_j$ is convex to the right of $v$, so (by the ordering) the same is true of $\angle a_{i+1}vb_j$ and $\angle a_i b_{j-1}$. The result follows by induction.

Also, note that (3) follows from (1) and (2). Suppose that for some $i, i_1 \leq i \leq i_2$ and $j, j_1 \leq j \leq j_2$, $a_ib_j$ is not in $Z$, then it either crosses $L$ to the right of $v$ or to the left of $u$. If $a_ib_j$ crosses $L$ to the right of $v$, then based on (1), $a_i b_{j_1}$ is not in $Z$ either, which is a contradiction. If $a_ib_j$ crosses $L$ to the left of $u$, then based on (2) $a_i b_{j_2}$ is not in $Z$, which is a contradiction again. So, for any $i, i_1 \leq i \leq i_2$ and $j, j_1 \leq j \leq j_2$, $a_ib_j$ is in $Z$. \qed

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Finding the ordered sets $A$ and $B$. We show how to find the ordered set $A$ in $O(n \log n)$ time. Sort all the points $p \in P$ that lie above $e$ by decreasing angle $\angle puv$, breaking ties by distance from $u$. Call this the $u$-order, $<_u$. Also, sort the same points by increasing angle $\angle puv$, and call this the $v$-order, $<_v$. Define $a_1$ to be the first point in the $v$-order. Observe that $a_1$ is a member of $A$ (the first member of $A$), and that any points $p$ with $p <_u a_1$ do NOT belong to $A$ and can be discarded. In general, we proceed through the $v$-order. Define $a_i$ to be the next un-discarded element of the $v$-order, and discard all points $p$ with $p <_u a_i$. To prove that this is correct, observe that when we discard $p$, we have $p <_u a_i$ and $p >_v a_i$. Thus, the triangle $puv$ contains $a_i$, so discarding $p$ is correct. Observe that when we choose $a_i$, we have $a_{i-1} <_v a_i$ because we follow the $v$-order, and we have $a_{i-1} <_u a_i$ because we discarded all points $p <_u a_{i-1}$ in the previous step.

We can find the ordered set $B$ similarly. The time to find the ordered sets $A$ and $B$ is $O(n \log n)$, and in fact, it is $O(n)$ after the preliminary sorting steps.

Finding the connected components of $G_Z$. We do not explicitly list the vertices of each connected component (there are too many). Rather, we find a partition of $A$ into $A_1, \ldots, A_c$ and a partition of $B$ into $B_1, \ldots, B_c$ such that the $i$th connected component of $G_Z$ consists of the edges of $G_Z$ contained in $A_i \times B_i$. Given the ordered lists $A$ and $B$, we find the connected components of $G_Z$ in linear time as follows. The algorithm has two phases. In Phase 1, we find the first edge of the next component, and in Phase 2, we complete the component. Phase 1 initially begins with $a_1$ and $b_1$, but more generally, we will find the first edge of the next component $A_c, B_c$ among the “active” points $a_1, \ldots, a_k$ and $b_1, \ldots, b_\ell$, maintaining the invariant that there are no edges of $Z$ from an inactive point to an active point. We search for an edge that crosses $uv$. Let $L$ be the line through $uv$. If $a_ib_j$ crosses $L$ to the right of $v$, then by Observation 16(1), there are no edges of $Z$ from $b_j$ to any active point. So we increment $j$ and the invariant is maintained. If $a_ib_j$ crosses $L$ to the left of $u$, then, with similar justification, we increment $i$. Continue until $a_ib_j$ crosses $uv$. This completes Phase 1—we have found the first edge of the connected component. Add $a_i$ to $A_c$ and add $b_j$ to $B_c$.

For Phase 2, we complete the connected component by alternately “pivoting” on $a_i$ and $b_j$. To pivot on $a_i$, update $j$ to the maximum index such that $a_ib_j$ crosses $uv$, putting all the $b$ points we find into $B_c$. To pivot on $b_j$, update $i$ to the maximum index such that $a_ib_j$ crosses $uv$, putting all the $a$ points we find into $A_c$. When no more pivots are possible, we are done with this connected component and done with Phase 2. We increment $i$ and $j$ by 1, and go back to Phase 1 to find the next connected component.

More details are given as Algorithm 3. The algorithm runs in linear time since it only
performs a linear scan through each of the ordered lists $A$ and $B$, doing constant work for each point.

We now justify correctness of the algorithm. We already argued that during Phase 1 we maintain the invariant that there are no edges of $Z$ between an inactive point and an active point, i.e., there are no edges of $Z$ of the form $a_i b_{j'}$ with $i' \leq i$ and $j' > j$ or with $i' > i$ and $j' \leq j$. Now consider Phase 2. Observe that when we add points to $A_c$ and $B_c$, they are part of the same connected component of $G_Z$. This is because when we pivot on $a_i$, all the $b$ points that we add to $B_c$ are adjacent to $a_i$ in $Z$, and similarly, when we pivot on $b_j$, all the $a$ points that we add to $A_c$ are adjacent to $b_j$ in $Z$. It remains to show that when we declare a connected component “done” because we can no longer pivot at $a_i$ or $b_j$, then the invariant holds for active points $a_{i+1}, \ldots, a_k$ and $b_{j+1}, \ldots, b_{\ell}$, i.e., there are no edges of $Z$ of the form $a_i b_{j'}$ with $i' \leq i$ and $j' > j$ or with $i' > i$ and $j' \leq j$. First suppose $a_i b_{j'} \in Z$ for some $i' \leq i$ and $j' > j$. Then by Observation 16(3), $a_i b_{j+1}$ is in $Z$, so we would not have been done pivoting at $a_i$. The case of $i' > i$ and $j' \leq j$ follows by symmetry.

### 3.3.4 Finding All Flip Cut Edges

To find all the flip cut edges, we run the above test on each of the $O(n^2)$ edges. We preprocess by sorting the points cyclically around each point $p$ in a total of $O(n^2 \log n)$ time. Then, to test a particular edge $e = uv$, we find the ordered sets $A$ and $B$ around $uv$, and apply Algorithm 3, which takes linear time apart from the sorting. Thus the total time to find all flip cut edges is $O(n^3)$.

### 3.3.5 Testing Connectivity of Two Triangulations

We now show how to test if two triangulations $T_1$ and $T_2$ in $T_{-e}$ are connected in $F_{-e}$. We assume that we have the output of the above algorithm, i.e., the sets $A_1, \ldots, A_c$ and $B_1, \ldots, B_c$ such that the $i$th connected component of $G_Z$ consists of the edges of $Z$ between $A_i$ and $B_i$. We can then find the component $i$ of a given edge $e \in Z$ in constant time.

As mentioned in Section 3.2 one approach is to pick one edge $f_1$ from $Y \cap T_1$, and one edge $f_2$ from $Y \cap T_2$, and test if $f_1$ and $f_2$ are in the same connected component of $G_Y$. However, we only have $G_Z$ available to us, and there are triangulations in $T_{-e}$ that contain no edges of $Z$. 

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Algorithm 3 Connected components of $G_Z$

Input: $u, v, a_1, \ldots, a_k, b_1, \ldots, b_\ell$
Output: Connected components of $G_Z$ in the form of $A_1, \ldots, A_c, B_1, \ldots, B_c$

1: procedure ConnectedComponents
2: for all $i$, done[$a_i$] $\leftarrow$ False; for all $j$, done[$b_j$] $\leftarrow$ False
3: $i \leftarrow 1; j \leftarrow 1; c \leftarrow 0$
4: while $i \leq k$ and $j \leq \ell$ do
5:     // Phase 1. Find the first edge of component $c$
6:     while $a_i b_j$ does not cross $uv$ do
7:         if $a_i b_j$ crosses $L$ to the right of $v$ then
8:             $j \leftarrow j + 1; $ if $j > \ell$ then Halt
9:         end if
10:        if $a_i b_j$ crosses $L$ to the left of $u$ then
11:           $i \leftarrow i + 1; $ if $i > k$ then Halt
12:      end if
13:     end while
14:     $c \leftarrow c + 1; $ Insert $a_i$ into $A_c$, and $b_j$ into $B_c$
15:     // Phase 2. Find all edges in component $c$ by alternately pivoting on $a_i, b_j$ until no further pivot is possible
16:     repeat
17:         // Pivot on $a_i$
18:         while $a_i b_j$ crosses $uv$ and $j \leq \ell$ do
19:             insert $b_j$ into $B_c; $ $j \leftarrow j + 1$
20:         end while
21:         $j \leftarrow j - 1; $ done[$a_i$] $\leftarrow$ True
22:     // Pivot on $b_j$
23:     while $a_i b_j$ crosses $uv$ and $i \leq k$ do
24:         insert $a_i$ into $A_c; $ $i \leftarrow i + 1$
25:     end while
26:     $i \leftarrow i - 1; $ done[$b_j$] $\leftarrow$ True
27:     until done[$a_i$] and done[$b_j$]
28:     $i \leftarrow i + 1; $ $j \leftarrow j + 1$
29: end while
30: end procedure
Instead, we give an algorithmic version of Lemma 11 that finds an edge $g_1 \in Z$ in the same component of $G_Y$ as the edges $Y \cap T_1$, and an edge $g_2 \in Z$ in the same component of $G_Y$ as the edges $Y \cap T_2$. Then we simply test if $g_1$ and $g_2$ are in the same set $A_i \times B_i$.

We first establish correctness and then give the details of finding $g_1$ and $g_2$ in linear time. We must prove that $g_1$ and $g_2$ are connected in $G_Z$ if and only if $T_1$ and $T_2$ are connected in $F - e$. Let $f_i$ be an edge of $Y \cap T_i$. By Observation 7, the edges of $Y \cap T_i$ are all connected in $G_Y$, so the choice of $f_i$ is arbitrary. Now $T_1$ and $T_2$ are connected in $F - e$ iff (by Theorem 8) $f_1$ and $f_2$ are connected in $G_Y$ iff (by choice of $g_1$, $g_2$) $g_1$ and $g_2$ are connected in $G_Y$ iff (by Lemma 10) $g_1$ and $g_2$ are connected in $G_Z$.

We now give the details of finding $g_1$ (finding $g_2$ is similar). Triangulation $T_1$ has a sequence $C$ of triangles that intersect $uv$. See Figure 3.14. Each triangle in $C$ shares an edge of $Y \cap T_1$ with the previous triangle in $C$, and all the edges of $Y \cap T_1$ are in one connected component of $G_Y$ (by Observation 7). Among the vertices of triangles of $C$, let $a$ be a vertex above the line $L$ through $uv$ that is closest to $L$ and let $b$ be a vertex below the line $L$ that is closest to $L$. Then $ab$ is an edge of $Z$ and is in the same connected component as $Y \cap T_1$. This can be done in linear time.

![Figure 3.14: The triangles $C$ of $T_1$ that cross edge $e = uv$, and an edge $a_3b_3$ of $Z$ (in blue) that is connected in $G_Y$ to the edges of $Y \cap T_1$.](image)

### 3.4 Further Results on Flip Cut Edges

In this section we establish some bounds on the number of flip cut edges, and on the number of disconnected components caused by forbidding one flip cut edge. To do this,
we fill in the details of the claims made in Section 1.4.1 (Examples) about some special point sets: channels, hourglasses, and grid point sets. We begin by relating flip cut edges to empty convex pentagons. Throughout the section we refer to a forbidden edge $e$, and to $Z, G_Z, A, B$, and so on, as defined above.

### 3.4.1 Empty Convex Pentagons (EC5’s)

Empty convex pentagons (EC5’s) play a significant role in the study of flip graphs. In Section 3.1.2 we showed how EC5’s are used to flip between two EC4’s without using the forbidden edge. Later, we proved how PG illustrates the role of EC5’s in transforming different triangulations of $F_{-e}$. EC5’s have important roles in other works too. Eppstein [14] studied the flip graph of points without EC5’s, and gave a polynomial time algorithm to find the minimum number of flips between two given triangulations. Points on an $n \times m$ grid have no EC5’s, and in the other direction, a point set without EC5’s must have collinear points. In particular, any $n \geq 10$ points in general position contain an EC5 [19], and sufficiently large point sets with no EC5 have arbitrarily large subsets of collinear points [1]. Empty convex pentagons also play a significant role in analyzing flips for coloured edges and proving the Orbit Theorem [27].

We show that the absence of EC5’s leads to a simple condition for flip cut edges. In particular, if a point set has no EC5’s, then edge $e$ is a flip cut edge if and only if $e$ is in (i.e., is a diagonal of) at least two EC4’s. In fact, it suffices to exclude local EC5’s:

**Lemma 17.** An edge $e$ that is not a diagonal of an EC5 is a flip cut edge if and only if it is a diagonal of at least two EC4’s, i.e., $|Z| \geq 2$.

**Proof.** If $e$ is a flip cut edge, then by Theorem 9, $G_Z$ is disconnected, so we must have $|Z| \geq 2$. (For this direction we did not use the assumption about EC5’s.)

For the other direction, suppose $e$ is not a flip cut edge and $|Z| \geq 2$. We will show that $e$ is a diagonal of an EC5. By Theorem 9, $G_Z$ is connected. Because neighbours in $G_Z$ are consecutive in the orderings of $A$ and $B$ (by Observation 15), there must be two edges $f, g$ in $Z$ that are incident at one end and consecutive in the ordering of $A$ (or $B$) at the other end.

Without loss of generality, suppose $f = a_i b_j$ and $g = a_{i+1} b_j$. See Figure 3.15. The five points $a_i, a_{i+1}, b_j, u, v$ form a convex pentagon. (Observe that the angle at $u$ is convex because of the EC4 $Q(f)$, and the angle at $a_i$ is convex because $a_i$ and $a_{i+1}$ make empty triangles with $uv$ and $a_i$ precedes $a_{i+1}$ in $A$, and similar arguments show that the other
angles are convex.) If this pentagon is not empty, then it contains a point \( p \) outside \( Q(f) \cup Q(g) \) and such a point of minimum \( y \) coordinate is a point of \( A \) between \( a_i \) and \( a_{i+1} \), a contradiction. Thus the pentagon is empty, so \( e \) is a diagonal of an EC5.

3.4.2 Grid Point Sets

The points of a \( k \times \ell \) grid have no empty convex pentagons, so by Lemma 17, an edge is a flip cut edge if and only if \(|Z| \geq 2\). Edges “near” the boundary of the grid may fail to be flip cut edges, but we show that edges farther from the boundary are flip cut edges, and we show that in an infinite grid, all edges are flip cut edges. We first deal with horizontal/vertical edges.

Claim 18. Let \( e \) be a horizontal/vertical edge of a grid point set. Then \( e \) is a flip cut edge if and only if it does not lie on the boundary of the grid.

Proof. Note that \( e \) must have unit length since no edge goes through intermediate points. We may assume without loss of generality that \( e \) goes from \((x, y)\) to \((x+1, y)\). If \( e \) lies on the boundary, then \( Z \) is empty. For the converse, suppose \( e \) is not on the boundary. Then the points just above \( e \), \( a_1 = (x, y+1) \), \( a_2 = (x+1, y+1) \), and the points just below \( e \), \( b_1 = (x+1, y-1) \), \( b_2 = (x, y-1) \), lie in the grid, and the two edges \( a_1, b_1 \) and \( a_2, b_2 \) lie in \( Z \), so \( e \) is a flip cut edge by Lemma 17.

Next consider an edge \( e \) that is not horizontal or vertical. Reflect so that \( e \) goes from the point \((x, y)\) to the point \((x+\Delta x, y+\Delta y)\) with \( \Delta x \geq \Delta y > 1 \). Since an edge cannot
go through intermediate points, gcd(Δx, Δy) = 1. Let L be the line through e. On the infinite grid, translate L upward and parallel to itself until it hits grid points and call the resulting line $L_U$. Similarly, translate L downward and parallel to itself until it hits grid points, and call the resulting line $L_D$.

**Claim 19.** A grid point makes an empty triangle with e iff it lies on $L_U$ or $L_D$.

**Proof.** The “if” direction is clear. For the other direction, consider a triangle $T$ determined by $e$ and an apex grid point strictly above $L_U$. The area of $T$ is strictly larger than the area of a triangle with apex on $L_U$. By Pick’s theorem [17] (the area of a triangle on the grid is the number of grid points strictly inside the triangle plus half the number on the boundary of the triangle), triangle $T$ cannot be empty.

By the easy direction of this claim, a grid point $a \in L_U$ and a grid point $b \in L_D$ provide an edge $ab \in Z$ iff $ab$ crosses $e$. The other direction of the claim is used in Figure 3.16b to demonstrate that some edges are not flip cut edges.

We now analyze the points on $L_U$ and $L_D$. We claim that there is a point $a = (a_x, a_y)$ on $L_U$ with $a_x \in [x, x + \Delta x)$ and $a_y \in (y, y + \Delta y]$. To justify this, note that if $a = (a_x, a_y)$ is a point on $L_U$, then so is $(a_x + \Delta x, a_y + \Delta y)$ and $(a_x - \Delta x, a_y - \Delta y)$. This implies that there is a point $a = (a_x, a_y)$ on $L_U$ with $a_x \in [x, x + \Delta x)$. Then we must have $a_y \in (y, y + \Delta y]$, as otherwise $(a_x, a_y - 1)$ would be between $L$ and $L_U$. By symmetry, the point $b = (b_x, b_y) := ((x + \Delta x) - (a_x - x), (y + \Delta y) - (a_y - y)) = (2x + \Delta x - a_x, 2y + \Delta y - a_y)$ lies on $L_D$ and has $b_x \in (x, x + \Delta x]$ and $b_y \in [y, y + \Delta y]$. Then, the segment $ab$ crosses $e$ at their midpoints, so $ab \in Z$. The same is true for the segment from $(a_x + i \Delta x, a_y + i \Delta y)$ to $(b_x - i \Delta x, b_y - i \Delta y)$ for any $i = 0, 1, \ldots$. This implies that on an infinite grid, $Z$ has infinite size. Thus we have proved:

**Lemma 20.** For the points of the infinite integer grid, every edge is a flip cut edge.

In a finite grid, the boundary edges are not flip cut edges, but they are not the only exceptions, see Figure 3.16. It is possible to characterize flip cut edge in terms of integer solutions to equations, but for now we simply note that a large enough grid of $n$ points has $\Theta(n^2)$ flip cut edges, as justified by the following.

**Proposition 21.** Let $G$ be a $k \times \ell$ grid and let $G'$ be the middle one-third grid (the grid which contains points that have $x$-coordinates between $\frac{k}{3}$ and $\frac{2k}{3}$ and $y$-coordinates between $\frac{l}{3}$ and $\frac{2l}{3}$). Then every edge of $G'$ is a flip cut edge of $G$.  

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Proof. Let \( e \) be an edge of \( G' \). We will show that \( Z \) has at least two elements. As above, we may assume that \( e \) goes from \((x, y)\) to \((x + \Delta x, y + \Delta y)\) with \( \Delta x \geq \Delta y > 1 \) and \( \gcd(\Delta x, \Delta y) = 1 \). Since \( e \) lies in \( G' \) we have \( \Delta x \leq k/3 \), \( \Delta y \leq \ell/3 \).

As noted above, there is a point \( a = (a_x, a_y) \) on \( L_U \) with \( a_x \in [x, x + \Delta x) \) and \( a_y \in (y, y + \Delta y] \) and a point \( b = (b_x, b_y) = (2x + \Delta x - a_x, 2y + \Delta y - a_y) \) on \( L_D \) with \( b_x \in (x, x + \Delta x] \) and \( b_y \in [y, y + \Delta y) \). Both \( a \) and \( b \) lie in the grid \( G \) and \( ab \in Z \). For our second element of \( Z \), we use the edge from \((a_x + \Delta x, a_y + \Delta y)\) on \( L_U \) to \((b_x - \Delta x, b_y - \Delta y)\) on \( L_D \). Note that these points lie in \( G \).

\( \square \)

![Diagram showing example edges in a 7 \times 7 grid.](image)

Figure 3.16: Example edges in a 7 × 7 grid.

3.4.3 The Number of Flip Cut Edges

In Section 1.4.1, we claimed that channels, as shown in Figure 1.5, are point sets with \( \Theta(n^2) \) flip cut edges. Here we justify that claim.

**Proposition 22.** For a channel, every edge between an interior point on the upper reflex chain and an interior point on the lower reflex chain is a flip cut edge.

**Proof.** Let the points of the upper reflex chain be \( t_1, \ldots, t_n \) and the points of the lower reflex chain be \( b_1, \ldots, b_n \), as shown in Figure 1.5. Consider the edge \( b_it_j \) where \( i, j \notin \{1, n\} \). We
use Lemma 17 to show that \( b_it_j \) is a flip cut edge. There is no EC5 with \( b_it_j \) as a diagonal because such an EC5 would have to have at least 3 points of one reflex chain, say the upper one, and those points cannot be part of a convex polygon with \( b_i \). However, both the edges \( b_{i-1}t_{j+1} \) and \( b_{i+1}t_{j-1} \) form EC4’s together with \( b_it_j \). Thus by Lemma 17, \( b_it_j \) is a flip cut edge.

\[\square\]

### 3.4.4 The Number of Components from a Flip Cut Edge

In Section 1.4.1, we claimed that one flip cut edge \( e \) can cause \( \Theta(n) \) disconnected components in \( F_{-e} \). Here we justify that claim.

**Theorem 23.** A flip cut edge can create \( O(n) \) disconnected components in the flip graph, and this is the most possible.

**Proof.** First of all, connected components in the flip graph \( F_{-e} \) correspond to connected components in \( G_Y \) (by Theorem 8) and therefore correspond to disjoint sets of points, so there are at most \( n \) of them.

The hourglass in Figure 1.6 is an example of a point set of size \( 2n + 2 \) with a flip cut edge \( e \) that results in \( n \) components in \( F_{-e} \). To construct the hourglass, place points \( a_1, \ldots, a_n \) along the top half of a circle, and let \( b_1, \ldots, b_n \) be the diametrically opposite points. For each \( i \) construct the wedge from \( a_i \) to \( b_{i-1} \) and \( b_{i+1} \). Place points \( u \) and \( v \) on the horizontal diameter of the circle, so close to the center of the circle that they are inside all the wedges. Then \( G_Y \) consists of the \( n \) edges \( a_ib_i \), and no two of these are connected in \( G_Y \). Thus (by Theorem 8) \( F_{-e} \) has \( n \) disconnected components. \( \square \)
Chapter 4

Flip Cut Set for Points in Convex Position

In this chapter, we are going to give some examples of flip cut sets, and then prove that flip cut number for points in convex position is $n - 3$.

4.1 Some Examples of Flip Cut Sets

As mentioned in the introduction, a point set in convex position has no flip cut edge, i.e., its flip cut number is greater than 1. We are going to prove this in the next section of this chapter. In this section, we will show all the possible flip cut sets for convex point sets with sizes six, seven, and eight. We have not yet been able to characterize flip cut sets for convex point sets. That being said, examining these examples should give some clue on where to start.

First, note that for a convex point set of size five, there exists no flip cut set, since forbidding two edges fails to disconnect the flip graph and forbidding more edges than that leaves us with no valid triangulation. For a convex point set of size six, there are two minimal flip cut sets with size three, which are shown in Figures 4.1 and 4.2.

The connected components of the flip cut sets in Figures 4.1 and 4.2 contain only frozen triangulations. In Figure 4.3, a flip cut set for a convex point set of size seven is shown. Each of the connected components of the flip graph for this set of forbidden edges contains 3 vertices (triangulations). Hence, both components are non-frozen components.
Figure 4.4 shows all the other possible minimal flip cut sets of a point set with size seven. We can see that all of these flip cut sets have size 4. Figure 4.5a shows some minimal flip cut sets for a convex point set of size eight. The size of all these sets is five. However, Figure 4.5b shows that not all sets of size five are flip cut sets.

Although we have not been able to characterize flip cut sets for points in convex position, in the next section of this chapter, we show that the flip cut number for convex point sets is $n - 3$ where $n$ is the size of the point set.

![Figure 4.1: An example of flip cut set for $n = 6$](image1)

(a) A flip cut set $X$ for a convex point set of size six

(b) The components of $F_X$ are frozen

![Figure 4.2: Another example of flip cut set for $n = 6$ whose components are frozen](image2)

(a) Another flip cut set $X$ for a convex point set of size six

(b) The frozen components of $F_X$
4.2 Flip Cut Number for Points in Convex Position

As mentioned in the introduction, a point set in convex position has no flip cut edge, i.e., its flip cut number is greater than 1. In this section we show that the flip cut number of $n$ points in convex position is $n - 3$.

We give a direct proof, but first we discuss what the result means in terms of associahedra. The flip graph of $n$ points in convex position is the 1-skeleton of the $(n - 3)$-dimensional associahedron $A_{n-1}$ (see Figure 1.3), so by Balinski’s theorem [3], the 1-skeleton is $(n - 3)$-connected. The dual polytope $\bar{A}_{n-1}$ is also an $(n - 3)$-dimensional polytope with an $(n - 3)$-connected 1-skeleton. The face lattice of $A_{n-1}$ is the inverted face lattice (in which the order of layers has been reversed) of $A_{n-1}$, so the vertices of $\bar{A}_{n-1}$
correspond to the \((n - 4)\)-dimensional faces (the facets) of \(A_{n-1}\) and the edges of \(\overline{A}_{n-1}\) correspond to \((n - 5)\)-dimensional faces of \(A_{n-1}\). Forbidding a chord of the original polygon corresponds to deleting a facet of \(A_{n-1}\), i.e., a vertex of \(\overline{A}_{n-1}\). Deleting fewer than \(n - 3\) facets of \(A_{n-1}\) leaves the remaining facets connected via \((n - 5)\)-dimensional faces. What our result shows is that the remaining vertices (0-dimensional faces) of \(A_{n-1}\) are connected via the remaining edges (1-dimensional faces) of \(A_{n-1}\). We do not see how to prove our result using this polyhedral interpretation. Instead, we give a direct proof that the flip cut number of \(n\) points in convex position is \(n - 3\).

For points in convex position, we number the points \(p_1, \ldots, p_n\) in cyclic order around the convex hull. The edges \(p_ip_{i+1}\) (addition modulo \(n\)) must be present in every triangulation. A chord is a line segment joining points that are not consecutive in the cyclic ordering. An ear is a chord of the form \(p_{i-1}p_{i+1}\) (addition modulo \(n\)), and we say that this ear cuts off point \(p_i\). Every triangulation of a convex point set on \(n \geq 4\) points has at least two ears (because the dual tree has at least two leaves). A star triangulation is a triangulation all of whose chords are incident to the same point.

For a set \(X\) of chords, we study connectivity of \(\mathcal{F}_X\). We define the degree of a point \(p\) in \(X\) to be the number of chords of \(X\) incident to \(p\).

We begin with a necessary condition for a flip cut set. This provides a rigorous proof that the flip cut number is greater than 1, and will also be needed for our main result.

**Lemma 24.** If \(X\) is a flip cut set for a set of points in convex position, then every point is incident to an edge of \(X\), i.e., the degree in \(X\) of every point is at least 1.
Proof. We prove the contrapositive: if there is a point \( p \in P \) with no incident edges of \( X \), then \( X \) is not a flip cut set. Specifically, we prove that any triangulation \( T \) in \( \mathcal{T}_{-X} \) is connected to the star triangulation centered at \( p \). This is in fact the standard way to show connectivity of the (full) flip graph of a convex point set. In our situation, we must show that the required flips do not use forbidden edges.

So, suppose triangulation \( T \) contains a triangle incident to \( p \) whose other two points, \( q \) and \( r \), are not consecutive on the convex hull. Now \( T \) contains a triangle, say \( qrs \), on the other side of \( qr \). Then \( pqsr \) forms a convex quadrilateral, and we can flip the chord \( qr \) to the chord \( ps \) (which is not forbidden). This increases the number of chords incident to \( p \), so, by induction, \( T \) can be flipped to the star triangulation centered at \( p \) without using forbidden edges.

Corollary 25. There is no flip cut edge in a convex point set.

In the rest of this section we prove that the flip cut number for \( n \) points in convex position is \( n - 3 \). We begin with the following two lemmas.

Lemma 26. A convex point set of size \( n \) has a flip cut set of size \( n - 3 \).

Proof. Consider a “zigzag” triangulation \( T \) of a convex \( n \)-gon as shown in Figure 4.6a. This triangulation has \( n - 3 \) chords that form a path \( p_0, p_{n-2}, p_1, p_{n-3}, \ldots, p_{\lceil \frac{n}{2} \rceil} \). Each chord \( e \) of \( T \) can flip to a unique other chord \( f(e) \). If we forbid the \( n - 3 \) chords \( X := \{ f(e) : e \) a chord of \( T \} \) then \( T \) becomes a frozen triangulation, i.e., it becomes an isolated node in the flip graph \( \mathcal{F}_{-X} \). In order to complete the proof, we only need to show that there
exists another triangulation in $F$ different from $T$. (This is where we use the fact that $T$ is a zig-zag triangulation—for example, a star triangulation would not work.) We create another zigzag triangulation $T'$ in the following way. The path of chords for $T'$ starts at the same vertex $p_0 = p_n$, but the first chord from $p_0$ goes in the other direction, i.e., to $p_2$. See Figure 4.6b. The new zigzag path is $p_0, p_2, p_{n-1}, p_3, p_{n-2}, ..., p_{\frac{n}{2}}$. We prove that $T'$ does not use any forbidden edges. Note that each forbidden edge crosses only one chord of $T$. However, each chord of $T'$, crosses at least two chords of $T$. This is because each chord in $T'$ of the form $p_i p_{i+2}$, for $i = 0, ..., \frac{n}{2}$, crosses $p_{i+1} p_{n-i-2}$ and $p_{i+1} p_{n-i-3}$ from $T$ and any chord of the form $p_{i+1} p_{n-i}$, for $i = 1, ..., \frac{n}{2}$, in $T'$ crosses $p_{i} p_{n-i-1}$ and $p_{i-1} p_{n-i-1}$ from $T$. So, since each forbidden edge $e \in X$ crosses exactly one chord of $T$, but each chord of $T'$ crosses at least two chords of $T$; therefore $T'$ does not contain any forbidden edges.

**Lemma 27.** Consider a convex point set $P$ with $n$ points and a set $X$ of forbidden edges with $|X| \leq n - 3$. Then there is a triangulation of $P$ that uses no forbidden edges, i.e., $T_{-X}$ is non-empty.

**Proof.** The proof is by induction on $n$ with base case $n = 3$. If there is a point $p$ that is incident to no forbidden edges, then take the star triangulation at $p$. Otherwise, suppose every point is incident to at least one forbidden edge. There are $n$ chords of the form $p_{i-1} p_{i+1}$ (addition modulo $n$). So, there exists at least one of them that is not forbidden. Suppose it is $p_{i-1} p_{i+1}$. We build a triangulation containing the ear $p_{i-1} p_{i+1}$. Let $P'$ be $P - \{p_i\}$ of size $n' = n - 1$. Since $p_i$ is incident to at least one forbidden edge, there are at most $n - 4 = n' - 3$ forbidden edges of $P'$. Thus, by induction, there is a triangulation of $P'$ that uses no forbidden edges. Together with chord $p_{i-1} p_{i+1}$ this provides the desired triangulation of $P$. Thus $T_{-X}$ is non-empty. 

**Theorem 28.** The flip cut number of a convex point set $P$ with $n$ points is $n - 3$.

**Proof.** By Lemma 26, we can disconnect $P$’s flip graph by forbidding $n - 3$ edges. So, now we only need to show that if $X$ is a set of forbidden edges and $|X| \leq n - 4$, then $F_{-X}(P)$ is connected. By Lemma 24, $F_{-X}(P)$ is connected if there is a point $p \in P$ not incident to any edge of $X$. Thus we may assume that every point $p \in P$ is incident to some edge of $X$. Let $S$ and $T$ be two triangulations of $P$ that do not contain any edges of $X$. We will prove that $S$ is connected to $T$ in $F_{-X}(P)$. We will prove this by induction on $n$, with the base case $n = 3$ where the statement is vacuously true.

We know that each triangulation contains at least two ears. We consider two cases.
**Case 1.** $S$ and $T$ have a shared ear that cuts off point $p_i$. Consider sub-polygon $P'$ obtained by removing point $p_i$. By assumption there is a forbidden edge incident to $p_i$. Let $X'$ be the forbidden edges of $X$ not incident to $p_i$. Then $|X'| \leq |X| - 1 \leq |P| - 5 = |P'| - 4$, so by induction, the triangulations of $P'$ induced from $S$ and $T$ are connected in $\mathcal{F}_{-X'}(P')$. Thus $S$ and $T$ are connected in $\mathcal{F}_{-X}(P)$.

**Case 2.** $S$ and $T$ have no shared ear. We claim that there is an ear $e_1$ in $S$ and an ear $e_2$ in $T$ such that $e_1$ and $e_2$ do not cross, i.e., the points that are cut off by $e_1$ and by $e_2$ are not adjacent on the convex hull. Suppose $S$ has an ear $e_1$ that cuts off point $p_i$. $T$ has at least two ears, and if they cross $e_1$, they must cut off points $p_{i-1}$ and $p_{i+1}$. But then the second ear of $S$ cannot cross both these ears of $T$ unless $n = 4$ (and $X = \emptyset$), in which case, a single flip converts $S$ to $T$.

Thus we may assume a pair of non-crossing ears $e_1$ of $S$ and $e_2$ of $T$. Suppose $e_1$ cuts off point $q_1$ and $e_2$ cuts off point $q_2$. Let $P_1$ be $P - \{q_1\}$. $P_1$ has $n_1 = n - 1$ points. Let $X_1$ be the forbidden edges of $X$ induced on $P_1$. By assumption there is at least one edge of $X$ incident to $q_1$, so $|X_1| \leq |X| - 1 \leq n - 5 = n_1 - 4$. By induction, the flip graph $\mathcal{F}_{-X_1}(P_1)$ is connected. Let $S_1$ be the triangulation of $P_1$ formed by cutting the ear $e_1$ off $S$. The plan is to apply induction on $P_1$ to connect triangulation $S_1$ to a new triangulation $R_1$ of $P_1$ that includes the chord $e_2$.

We construct $R_1$ as follows. Let $P_2$ be $P - \{q_1, q_2\}$. Then $P_2$ has size $n_2 = n - 2$. Let $X_2$ be the forbidden edges of $X$ induced on $P_2$. Then $|X_2| \leq |X_1| \leq n_1 - 4 = n_2 - 3$. By Lemma 27, $P_2$ has a triangulation that uses no chords of $X_2$. Adding chord $e_2$ yields the triangulation $R_1$.

By induction on $P_1$ and $X_1$ the triangulations $S_1$ and $R_1$ are connected in $\mathcal{F}_{-X_1}(P_1)$. Finally, $R_1$ and $T$ share the ear $e_2$, so by Case 1, they are connected in $\mathcal{F}_{-X}(P)$. Altogether, we have connected $S$ to $T$ in $\mathcal{F}_{-X}(P)$, as required. □
Chapter 5

Conclusions and Open Problems

We examined connectivity of the flip graph of triangulations when some edges between points are forbidden, and introduced the concepts of flip cut edges, flip cut sets, and the flip cut number. We gave an $O(n \log n)$ time algorithm to identify flip cut edges and test connectivity after forbidding a flip cut edge, and we proved that the flip cut number of a convex $n$-gon is $n - 3$. We conclude with some open questions.

1. Is there a polynomial-time algorithm to test if a set of edges is a flip cut set? To compute the flip cut number?

2. What happens if we specify both a set of forbidden edges and a set of forced (constrained) edges. Can we characterize when the resulting flip graph is disconnected? A better understanding of this situation might be helpful for solving the previous open question.

3. The asymptotic diameter of the flip graph of a convex $n$-gon is $2n - 10$—a famous result of Sleater, Tarjan and Turston [35], improved to all $n > 12$ by Pournin [33]. For a convex point set and a set $X$ of forbidden edges with $|X| < n - 3$ what is the diameter of $\mathcal{F}_X$ in terms of $n$ and $|X|$?

4. It is open whether there is a polynomial-time algorithm for the flip distance problem for a convex polygon. Is the following generalization NP-complete: Given $n$ points in convex position, a set $X$ of forbidden chords, two triangulations $T_1$ and $T_2$, and a number $k$, is the flip distance from $T_1$ to $T_2$ in $\mathcal{F}_X$ less than or equal to $k$?
5. A flip cut edge is a bottleneck to connecting triangulations via flips. One might guess that point sets with no flip cut edges provide better mixing properties. More generally, how does the flip cut number affect mixing properties?
References


