Dilation methods in semigroup dynamics and noncommutative convexity

by

Adam Humeniuk

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Pure Mathematics

Waterloo, Ontario, Canada, 2022

Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

External Examiner:	Ivan Todorov Professor, Dept. of Mathematical Sciences University of Delaware
Supervisor(s):	Kenneth Davidson Professor Emeritus, Dept. of Pure Mathematics University of Waterloo
	Matthew Kennedy Associate Professor, Dept. of Pure Mathematics University of Waterloo
Internal Member:	Laurent Marcoux Professor, Dept. of Pure Mathematics University of Waterloo
Internal Member:	Alexandru Nica Professor, Dept. of Pure Mathematics University of Waterloo
Internal-External Member:	Richard Cleve Professor, Cheriton School of Computer Science University of Waterloo

Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

I am the sole author of Chapters 1, 2, and 3. Chapters 2 and 3 have appeared in print as [46] and [47], respectively. Chapter 4 is joint work with my advisor Matthew Kennedy and fellow student Nicholas Manor.

Abstract

Since seminal work of Stinespring, Arveson, and others, dilation theory has been an indispensable tool for understanding operator algebras. Dilations are fundamental to the representation theory of operator systems and (non-selfadjoint) operator algebras. This thesis is a compilation of three research papers in operator algebras and noncommutative convexity linked by their use of dilations and operator systems.

A semicrossed product is a non-selfadjoint operator algebra encoding the action of a semigroup on an operator or C*-algebra. In Chapter 2, we describe the C*-envelopes of a large class of semicrossed products. We prove that, when the positive cone of a discrete lattice ordered abelian group acts on a C*-algebra, the C*-envelope of the associated semicrossed product is a full corner of a crossed product by the whole group. After dilating the semigroup action to an automorphic action of the whole group using a direct product construction, we explicitly compute the Shilov ideal and therefore compute the C*-envelope. This generalizes a result of Davidson, Fuller, and Kakariadis from $\mathbb{Z}_+^n$ to the class of all discrete lattice ordered abelian groups.

Chapters 3 and 4 present results in noncommutative (or "matrix") convexity. By the noncommutative Kadison duality of Webster-Winkler and Davidson-Kennedy in the unital setting, and Kennedy-Kim-Manor in the nonunital setting, the category of compact noncommutative (nc) convex sets is dual to the category of operator systems. Thus nc convexity allows a new avenue to study operator systems geometrically. In Chapter 3, we prove a noncommutative generalization of the classical Jensen's Inequality for multivariable nc functions which are convex in each variable separately. The proof involves a sequence of dilations resembling a noncommutative analogue of Fubini's Theorem. This extends a single-variable nc version of Jensen's Inequality of Davidson and Kennedy. We demonstrate an application of the multivariable separate nc Jensen's Inequality to free semicircular systems in free probability.

In Chapter 4, we discuss duals of operator systems. Recently, C.K. Ng obtained a nice duality theory for operator systems. Call a (possibly nonunital) operator system S *dualizable* if its dual S^* embeds into $B(H)$ via a complete order embedding and complete norm equivalence. Through the nonunital noncommutative Kadison duality of Kennedy, Kim, and Manor, we characterize dualizability of S using geometric conditions on its associated nc convex quasistate space K in two ways. Firstly, in terms of an nc affine embedding of K into the nc unit ball of a Hilbert space satisfying a certain extension property. Secondly, we show that Ng's characterization is dual to a normality condition between K and the nc cone $\mathbb{R}_+K$. As applications, we obtain some permanence properties for dualizability, and give a new nc convex-geometric proof of Choi's Theorem.

Acknowledgements

Firstly, I must thank both my advisors Ken Davidson and Matt Kennedy for their guidance, advice, and humour. I am happy to have had two supervisors with vastly different, but complementary, styles. I give special thanks to Ken in particular for continuing to supervise me even after his retirement. It was an honour to be your student.

There are too many people in the Pure Mathematics department at UWaterloo to thank. I have had helpful conversations about mathematics with (at least) Ben Anderson-Sackaney, Eric Boulter, Zack Cramer, Adina Goldberg, Sam Kim, Nicole Kitt, Nick Manor, Alexandru Nica, Vern Paulsen, Daniel Perales Anaya, Vern Paulsen, Pawel Sarkowicz, John Sawatzky, Nico Spronk, Zsolt Tanko, and Dan Ursu. Beyond math, the graduate student culture at UWaterloo has been fantastically welcoming. In addition to those grad students listed so far, I have been blessed to have great friendships with Paul Lawrence, Max Levit, Luke MacLean, Wilson Poulter, Joaco Prandi, Hayley Reid, Nolan Shaw, Carlos Valero, and others I am surely forgetting.

Completing a PhD during a global pandemic has been at times challenging, sad, fascinating, boring, and freeing. In the end, I am grateful to have been able to work remotely and spend nearly a full year's worth of time with family and friends at home.

Lastly, I want to thank the Canada geese of Kitchener-Waterloo for their courage, tenacity, and beautiful birdsong.

Dedication

This thesis is dedicated to my parents Arline and Wade Humeniuk. I wouldn't be who I am without their support, guidance, and sense of humour.

Table of Contents

"It often happens that you have no success at all with a problem; you work very hard yet without finding anything. But when you come back to the problem after a night's rest, or a few days' interruption, a bright idea appears and you solve the problem easily.

Such happenings give the impression of *subconscious work*. The fact is that a problem, after prolonged absence, may return into consciousness essentially clarified, much nearer to its solution than it was when it dropped out of consciousness. Who clarified it, who brought it nearer to the solution? Obviously, oneself, working at it *subconsciously*. It is difficult to give any other answer.

Past ages regarded a sudden good idea as an inspiration, a gift of the gods. You must deserve such a gift by work, or at least a fervent wish."

George Pólya - *How to Solve It*

Chapter 1

Introduction

This thesis contains three complete research projects in operator algebras tied together in their use of dilation theory. The ubiquitous players in our story are Hilbert spaces H, the algebra $B(H)$ of bounded operators $H \to H$, and **C*-algebras** $A \subseteq B(H)$, which are norm closed $*$-subalgebras. C*-algebras have multiplicative, order, and norm structure, and relaxing the requirement for all three leads to larger categories of well-behaved subobjects. **Operator systems** are unital $*$-closed subspaces of $B(H)$. (Non-selfadjoint) **Operator algebras** are simply subalgebras of $B(H)$. Most generally, **operator spaces** are closed $*$-subspaces of $B(H)$. While these are "concrete" descriptions, all of these objects have pleasing "abstract" descriptions due respectively to Gelfand-Naimark [39], Blecher-Ruan-Sinclair [9], Choi-Effros [15], and Ruan [71]. The key theme is *matrix structure*, where operator systems, algebras, and spaces are convincingly thought of as matricial or "quantized" versions of functions systems, Banach algebras, and Banach spaces, respectively.

Broadly, the ethos of dilation theory, which dates back to Halmos [41], is that one can study an operator $T \in B(H)$ by viewing it as a corner of an operator on a larger space. We say an operator $S \in B(K)$ **dilates** T if $K \supseteq H$ and with respect to the orthogonal decomposition $S = H \oplus H^\perp$, the block matrix of S is

$$S = \begin{pmatrix} T & * \\ * & * \end{pmatrix},$$

where each $*$ may be any operator. That is, the **compression** $P_H S|_H$ of S to H is T, where P_H is the orthogonal projection to H.

It is helpful to think of dilation as a "give-and-take" process. Given T, we usually wish to dilate T to an operator S which has much nicer properties, but at the cost of enlarging the ambient Hilbert space. The prototypical result in this direction is Sz.-Nagy's Dilation Theorem [75], which states that any contraction T can be dilated to an S which is *unitary*. In fact, one can arrange that S^n dilates T^n for every power $n \geq 1$, and Sz.-Nagy used this to give a beautiful proof of von Neumann's inequality. Put another way, the diagram

$$\begin{tikzcd} & B(K) \arrow[d] \\ \mathbb{Z}_+ \arrow[ur, "S"] \arrow[r, "T"'] & B(H) \end{tikzcd}$$

commutes, where the vertical arrow is the compression map, and the other arrows are **representations** of the semigroup $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. That is, every contractive representation of $\mathbb{Z}_+$ extends to a unitary representation of the whole group $\mathbb{Z}$. For the purposes of operator algebras, this illustrates the general principle that we are most interested in dilating *whole representations at once*. An operator algebraist might translate the diagram above as follows: Let $\mathbb{D}$ and $\mathbb{T}$ be the closed unit disk and circle in $\mathbb{C}$. Every contractive representation of the universal operator algebra

$$A(\mathbb{D}) = \overline{\operatorname{span}}\{1, z, z^2, \dots\} \subseteq C(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} \text{ continuous}\}$$

that is generated by a single contraction–the coordinate function $z : \mathbb{D} \to \mathbb{D}$, dilates to a representation of the universal C*-algebra $C(\mathbb{T})$ that is generated by a single unitary $z|_{\mathbb{T}}$. Since dilations increase norm, this implies that the restriction map $A(\mathbb{D}) \to C(\mathbb{T})$ is isometric on the **disk algebra** $A(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} \text{ continuous and holomorphic}\}$. In particular, Sz.-Nagy's dilation theoretic approach recovers the maximum modulus principle for polynomials on $\mathbb{D}$ with a proof that uses no complex analysis at all.

The operator algebraic approach to dilation theory in its modern form is owed to the seminal papers [3] and [4] by Arveson. The most fundamental result is Stinespring's Dilation Theorem [74], which shows that any unital and **completely positive** (u.c.p.) map $A \to B(H)$ on a C*-algebra A dilates to a $*$-representation. Arveson showed that any u.c.p. map $S \to B(H)$ on any operator system $S \subseteq A$ extends in Hahn-Banach fashion to all of A, a kind of injectivity result for $B(H)$. This result, now called Arveson's Extension Theorem, combines with Stinespring's Dilation Theorem to assert that any completely positive map on an operator system dilates to a restriction of a $*$-representation. In this way, operator systems and their u.c.p. representations are the most fundamental objects of dilation theory in operator algebras.

Every operator algebra $A \subseteq B(H)$ generates a C*-algebra $C^*(A)$, but the C*-algebra $C^*(A)$ is not an invariant for A. It is common that isomorphic copies of A generate non-$*$-isomorphic C*-algebras. To what extent is the structure of $C^*(A)$ determined intrinsically by A? In [3], Arveson answered this questions by defining the *C^*-envelope* of A , denoted $C_e^*(A)$ or $C_{\min}^*(A)$, as the unique universal quotient among all C*-algebras generated by A. That is, for every operator algebra isomorphism $\varphi : A \to \varphi(A)$, there is a unique C*-algebra quotient map $C^*(\varphi(A)) \to C_e^*(A)$ fixing the copy of A. The C*-envelope is a C*-algebra intrinsic to A, and one defines the C*-envelope for operator systems in the same way. Arveson was initially unable to show the C*-envelope exists in all cases, but proposed a proof by showing A has enough so-called **boundary representations**, which are representations of A on Hilbert space that must lift automatically to irreducible representations of the C*-envelope. Proving existence of the C*-envelope became a 40-year journey. Its existence was first shown 10 years later by Hamana [42] using *injective* envelopes instead of boundary representations, then by Dritschel and McCullough [31] using maximal dilations. Using this new dilation-theoretic approach, Arveson [5] (in the separable case) and Davidson and Kennedy [25] (in general) showed how to construct enough boundary representations to yield the C*-envelope. Since even showing the existence of $C_e^*(A)$ was challenging and non-constructive, concretely describing $C_e^*(A)$ in specific cases is a constant goal when studying operator algebras.

In Chapter 2, we concretely describe the structure of the C*-envelope for a large class of non-selfadjoint operator algebras arising from semigroup dynamics. In a **C*-dynamical system**, a discrete group G acts on a unital C*-algebra B by $*$-automorphisms, and there is a natural associated (full) **crossed product** C*-algebra $B \rtimes G$. The crossed product contains a copy of B, and a copy of G in its unitary group such that unitary conjugation implements the G-action on B. The $*$-operation unifies the $*$-operation in B with inversion in G. In a **semigroup dynamical system**, a *semigroup* P acts on an *operator algebra* A (often a C^*-algebra) by *endomorphisms*. The associated **semicrossed product** $A \times P$ is naturally a non-selfadjoint operator algebra encoding the P-action on A. In fact, a wrinkle in the theory is that there are multiple semicrossed products depending on what representation theory for P one permits, but we will suppress that here.

It is natural to hope that the C*-envelope of a semicrossed product is a crossed product. This is sensible in the following situation. Suppose P is a subsemigroup of a group G, and P acts on an operator algebra A. Can the P-action be extended to a G-action on a larger C*-algebra $\tilde{A} = C^*(A)$, such that the C*-envelope of $A \times P$ is $\tilde{A} \rtimes G$? This is impossible in general, but in specific cases $C_e^*(A \times P)$ is at least a **full corner** in $\tilde{A} \rtimes G$, a projection of it that does not sit in any proper ideal. The main result of Chapter 2 concerns the case where (G, P) is a **lattice ordered abelian group**. Here G is abelian, and P induces a partial order

$$g \le h \iff h - g \in P$$

that makes G into a lattice. In this situation, one considers only **Nica-covariant** representations of P [66], and the associated **Nica-covariant** semicrossed product $A \times^{\mathrm{nc}} P$ [18]. Nica-covariant representations of P respect the lattice structure, and have a tractable dilation theory. We prove that the C*-envelope $C_e^*(A \times^{\mathrm{nc}} P)$ is a full corner of a crossed product $\tilde{A} \rtimes G$, where $\tilde{A}$ is a so-called **Nica-covariant automorphic dilation** of A. This means that the G action on $\tilde{A}$ extends the P action on A in a way that respects the lattice structure of G. We give an explicit description of the envelope via a new construction, where we build such an $\tilde{A}$ via a direct product construction, and then compute the Shilov ideal. This extends a result of Davidson, Fuller, and Kakariadis [18] in the special case $(G, P) = (\mathbb{Z}^n, \mathbb{Z}_+^n)$ to the class of all lattice ordered abelian groups. As applications, we obtain some simplicity results for the C*-envelope and permanence properties under direct limits.

Chapters 3 and 4 are filled with the language of noncommutative–or "matrix", convexity. Classical convexity plays a key role in functional analysis, and in the study of operator algebras. Familiar contexts include the state or trace space of a C*-algebra, or less generally the set of Radon probability measures on a compact space. Wittstock [79] argued convincingly that ordinary convexity is not innately compatible with operator algebra theory, because it doesn't encode matricial information. A classical convex set C is closed under convex combinations

$$\sum_i t_i x_i \in C$$

of $x_i \in C$ with normalized scalar coefficients $t_i \geq 0$. The matricial version is a **noncommutative (nc or matrix) convex set** K which is closed under **nc convex combinations**

$$\sum_i \alpha_i^* x_i \alpha_i \in K,$$

where $x_i \in K$ are matrices, and α_i are rectangular matrices which are positively "matrix normalized" to the identity matrix $\sum_i \alpha_i^* \alpha_i = I$. This definition requires that K contains matrices of all sizes. Formally, K must be graded into levels

$$K = \coprod_{n \geq 1} K_n,$$

where each $K_n \subseteq M_n(V)$ consists of $n \times n$ matrices over a common vector space V.

A matrix convex set K is classically convex at each level, with two fundamental additional features. It is closed under taking direct sums

$$x \oplus y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

and closed under compression $x \mapsto \alpha^* x \alpha$ by any isometry matrix α, including any unitary. Once one is comfortable with the cognitive dissonance of allowing differently-sized matrices to form a single geometric object, nc convex sets begin to appear fundamental to the study of operator spaces and operator systems. For instance, Ruan's axioms for an abstract operator space E [71] just assert that the "noncommutative unit ball"

$$\coprod_{n \geq 1} \{x \in M_n(E) \mid \|x\|_{M_n(E)} \leq 1\}$$

is an nc convex set. Likewise, the Choi-Effros axioms for an abstract operator system S [15] require that the "noncommutative positive cone" $\coprod_{n \geq 1} M_n(S)^+$ is nc convex. The superior replacement for the state space of an operator system S is the **nc state space**

$$\mathcal{S}(S) = \coprod_{n \geq 1} \{\varphi : S \to M_n \mid \varphi \text{ is unital and completely positive}\},$$

which is nc convex when one identifies $B(S, M_n) \cong M_n(S^*)$.

Noncommutative convexity is a matricial or "quantized" version of ordinary convexity. So, a major theme is the search for matricial or "quantized" versions of the major theorems of classical convexity, and there is much success in this direction. In a seminal paper, Effros and Winkler [32] proved noncommutative versions of the Hahn-Banach Separation and Extension Theorems, and the Bipolar Theorem. Subsequently, Webster and Winkler [77] considered the problem of extreme points and obtained a noncommutative Krein-Milman Theorem. Along the way, they showed that there is a noncommutative form of Kadison duality. The functor which sends an operator system S to its nc state space $\mathcal{S}(S)$ is a contravariant equivalence between the category of operator systems and the category of levelwise compact nc convex sets. So, an operator system is remembered by its nc state space, and the study of nc convex sets completely captures all features of operator system theory. Recently, Davidson and Kennedy [26] developed a noncommutative version of classical Choquet theory, and with a more restrictive definition of extreme point obtained a stronger nc Krein-Milman theorem. A key innovation was the view that the disjoint union $K = \coprod_{n \geq 1} K_n$ should rightly include *infinite* cardinals n, because the nc extreme points may necessarily be infinite matrices.

Along the way, Davidson and Kennedy observed that there is a natural notion of noncommutative convex (nc) function, and that nc functions satisfy a noncommutative version of Jensen's inequality. Classically, Jensen's inequality states that if $f : X \to \mathbb{R}$ is a convex function on a compact convex set X, then for any probability measure μ on X,

$$\int_X f \, d\mu \geq f(\mathrm{bar}(\mu)),$$

where the **barycenter** $\mathrm{bar}(\mu) \in X$ is the unique (weakly) average point of X with respect to the measure μ. Probability measures on X are just states on $C(X)$, so the noncommutative Jensen's Inequality is concerned with nc states. If f is an nc convex function on a compact nc convex set K, then

$$\mu(f) \geq f(\mathrm{bar}(\mu))$$

for every nc state $\mu : C(K) \to M_n$. Here $C(K)$ denotes the C*-algebra of continuous *nc functions* on K, and if $\mu : C(X) \to M_n$ is an nc state, it has a barycenter in the nth level K_n of K.

In Chapter 3, we prove a noncommutative Jensen's Inequality for multivariable nc functions which are nc convex in each variable separately. Such a function has a domain which is the levelwise free product $K_1 \times \cdots \times K_d$ for compact nc convex sets $K_1, \ldots, K_d$. We show that

$$C(K_1 \times \cdots \times K_d) \cong C(K_1) * \cdots * C(K_d)$$

is the unital free product of $C(K_1), \ldots, C(K_d)$. Here the same nc Jensen Inequality holds for a large but restricted class of nc states μ on $C(K_1) * \cdots * C(K_d)$ which manifest as certain "free products" of nc states μ_i on $C(K_i)$. The proof involves writing a chain of dilations for μ which are trivial in all but one variable, resembling a noncommutative version of Fubini's Theorem. In fact, this Jensen Inequality holds for a larger class of ucp maps satisfying a noncommutative verson of Fubini's Theorem. The connection to free products suggests a connection to free probability, and we show that nc states μ which are **conditionally free** in the operator-valued free probabilistic sense of Młotkowski [62] satisfy this multivariable nc Jensen Inequality. We give a sample application to free probability by deriving some operator inequalities for conditionally free nc states on an algebra of free semicircular elements, suggesting a connection between noncommutative convexity and free probability.

In Chapter 4, we discuss duality for operator systems from an nc convex geometric perspective. If S is an operator system, the dual S^* is an operator space equipped with an involution and matrix ordering. It is a *matrix ordered operator space* in the sense of Werner [78]. We say S is **dualizable** if its dual S^* embeds into $B(H)$ via a complete order isomorphism which is completely bounded below. That is, S^* can be re-normed into a **nonunital operator system** $S^d \hookrightarrow B(H)$. Recently, C.K. Ng [65] characterized dualizability for (possibly nonunital) S in terms of a bounded positive decomposition property.

Kennedy, Kim, and Manor [56] extended noncommutative Kadison duality to the nonunital setting, and showed that nonunital operator systems are categorically dual to *pointed* compact nc convex sets (K, z). Here, a nonunital operator system S is dual to its pointed nc quasistate space

$$K = \mathcal{QS}(S) = \coprod_{n\geq 1}\{\varphi : S \to M_n \mid \varphi \text{ is completely contractive and positive}\}$$

with the zero map 0 as the basepoint. We find two equivalent geometric conditions on K for dualizability of S. The first condition is extrinsic, and requires that $(K, 0)$ embeds pointedly into the pointed nc quasistate space of the operator system of trace class operators $\mathcal{T}(H) = B(H)_*$ with a bounded and positive extension property for nc affine functions. The second equivalent condition is intrinsic to K. By dualizing Ng's characterization, we require that the nc convex set

$$(K - \mathbb{R}_+K) \cap \mathbb{R}_+K$$

of positive elements order-dominated by nc quasistates in K is norm-bounded in $\coprod_{n\geq 1} M_n(S^*)$. The geometric condition lets us prove permanence properties for dualizability, showing that quotients, coproducts, and pushouts of dualizable operator systems are again dualizable. Using Ng's framework, we also give an nc convex-geometric proof of Choi's Theorem [14].

Chapters 2 and 3 have appeared in publication as [46] and [47], respectively. Throughout this thesis, we will generally assume the reader has a general comfort with the basics of C*-algebra theory, such as in [16]. For more detail on operator spaces, systems, algebras, and dilation theory, we refer the reader to [68].

Chapter 2

C*-envelopes of semicrossed products by lattice ordered abelian semigroups

2.1 Introduction

2.1.1 Preliminaries

A semicrossed product is a non-selfadjoint generalization of the crossed product of a C*-algebra by a group. A crossed product $B \rtimes G$ encodes the action of a group G on a C*-algebra B, by embedding both into a larger C*-algebra in which the G-action is by unitaries. Built similarly, a semicrossed product of a (possibly non-selfadjoint) operator algebra A by an abelian semigroup P encodes a given action of P on A by completely contractive endomorphisms. First introduced by Arveson in [2], and first formally studied by Peters in [69] in the case $P = \mathbb{Z}_+$, subsequent work on semicrossed products has focused on conjugacy problems [6, 20, 22, 24, 27, 40, 52] and their C*-envelopes [18, 23, 49, 51, 54, 63]. For a complete survey of the history of semicrossed products, and a thorough discussion of the conjugacy problem, we recommend Davidson, Fuller, and Kakariadis' treatment in [19]. For a given action of P on A, there are multiple associated semicrossed products $A \times^{\mathcal{F}} P$, depending on what family of admissible representations $\mathcal{F}$ of P one considers. Generally, we have distinct unitary, isometric, and contractive semicrossed products $A \times^{\mathrm{un}} P$, $A \times^{\mathrm{is}} P$, and $A \times P$, which satisfy universal properties for "covariant" contractive/isometric/unitary representations of P with respect to A.

Following the programme outlined in [18, Page 1], our main question of interest is: If P is a generating subsemigroup of an abelian group G, can the C*-envelope of a semicrossed product $A \times^{\mathcal{F}} P$ be realized as a full corner of a crossed product $B \rtimes G$ by G, for some G-C*-algebra $B \supseteq A$? If the action of P on A is by automorphisms, then $A \times^{\mathrm{is}} P = A \times^{\mathrm{un}} P$, and the P action extends to $*$-automorphisms of the C*-envelope $C_e^*(A)$. It follows that

$$C_e^*(A \times^{\mathrm{is}} P) \cong C_e^*(A) \rtimes G$$

is a crossed product [18, Theorem 3.3.1]. If $G = P - P$, and P acts on a C*-algebra A by $*$-monomorphisms, then

$$C_e^*(A \times^{\mathrm{un}} P) \cong \tilde{A} \rtimes G \tag{2.1}$$

is a crossed product for a certain unique minimal C*-algebra $\tilde{A} \supseteq A$ whose G-action extends the action of P, called the minimal automorphic extension of A. Kakariadis and Katsoulis [51, Theorem 2.6] established (2.1) in the case $P = \mathbb{Z}_+$. Laca [57] showed how to build the automorphic dilation $\tilde{A}$ in general, and from this Davidson, Fuller, and Kakariadis establish (2.1) in [18, Theorem 3.2.3].

Parrott's example [68, Chapter 7] of three commuting contractions without a simultaneous isometric dilation, shows that the dilation theory of representations of any semigroup at least as complicated as $\mathbb{Z}_+^3$ is intractable. To make progress, we need to restrict our class of representations $\mathcal{F}$ if we wish a nice dilation theory for $A \times^{\mathcal{F}} P$. Of interest are lattice ordered abelian groups (G, P). These are pairs consisting of a subsemigroup P of a group G, where the induced ordering

$$g \leq h \iff h - g \in P$$

makes G a lattice. In the lattice ordered setting, one studies the more tractable class of Nica-covariant representations, first introduced by Nica in [66]. Nica-covariance is a $*$-commutation type condition which ensures a nice dilation theory. For instance, Li [60, 59] showed that every Nica-covariant representation of P has an isometric dilation.

In the Nica-covariant setting, for injective C*-systems (2.1) holds with $A \times^{\mathrm{nc}} P$ in place of $A \times^{\mathrm{un}} P$. For non-injective systems, it is not possible to embed $A \times^{\mathrm{nc}} P$ into any crossed product $B \rtimes G$ via inclusions $A \subseteq B$ and $P \subseteq G$, because such a system has no faithful unitary covariant pairs. The best one can do is embed $A \times^{\mathrm{nc}} P$ into a *full corner* of a crossed product. For a lattice ordered abelian group (G, P) and an action of P on a C*-algebra A, one expects to prove

$$C_e^*(A \times^{\mathrm{nc}} P) \cong p_A(B \rtimes G)p_A, \tag{2.2}$$

is a full corner of a crossed product of some G-C*-algebra B. Here A embeds into B non-unitally, and $p_A := 1_A$ is the projection coming from the unit in A. In the case

$(G,P) = (\mathbb{Z}^n, \mathbb{Z}_+^n)$, the result (2.2) was established in the case $n = 1$ by Kakariadis and Katsoulis [49, 51], and extended to general $n \geq 1$ by Davidson, Fuller, and Kakariadis [18, Theorem 4.3.7]. Their construction of the G-C*-algebra B was in two stages. First, one builds a bigger C*-algebra $B_0 \supseteq A$ which has an injective P-action dilating the P-action on A. This is accomplished by a tail-adding technique. Then one takes the minimal automorphic dilation $B := \tilde{B}_0$.

2.1.2 Main results

We establish that (2.2) holds for any discrete lattice ordered abelian group (G,P), when A is a C*-algebra (Corollary 2.3.16). Our approach differs from Davidson, Fuller, and Kakariadis' construction for $P = \mathbb{Z}_+^n$. First, we define a notion of a *Nica-covariant automorphic dilation* of A, which is a certain G-C*-algebra B with a non-unital embedding $A \subseteq B$. This definition is meant to capture a sufficient set of conditions to get a completely isometric embedding

$$A \times^{\mathrm{nc}} P \subseteq p_A (B \rtimes G) p_A,$$

with $p_A := 1_A$. When the dilation B is *minimal*, this is a C*-cover. Then, we show that the Shilov ideal in such a cover has the form $p(I \rtimes G)p$, for a unique maximal G-invariant ideal $I \lhd B$ not intersecting A. Upon taking a quotient by the Shilov ideal,

$$C_e^*(A \times^{\mathrm{nc}} P) \cong (p_A + I) \left(\frac{B}{I} \rtimes G \right) (p_A + I)$$

is a full corner of a crossed product. Then it suffices to show that any C*-algebra A with P-action has at least one minimal Nica-covariant automorphic dilation. We build one via a direct product construction (Proposition 2.3.5).

A semicrossed product is a special instance of the tensor algebra of a C*-correspondence [28, 36, 50, 63] (when $P = \mathbb{Z}_+$) or a product system [29, 33, 34, 35, 73]. Katsoulis and Kribs [54] showed that the C*-envelope of the tensor algebra of a C*-correspondence X is the associated Cuntz-Pimsner algebra $\mathcal{O}_X$, a generalization of the usual crossed product. In [30], Dor-On and Katsoulis extend this result and show that the C*-envelope of the Nica tensor algebra $\mathcal{NT}_X^+$ associated to a product system X over P coincides with the associated Cuntz-Nica-Pimsner algebra $\mathcal{NO}_X$ considered by Carlsen, Larsen, Sims, and Vittadello [13], and also coincides with an associated covariance algebra $A \times_X P$ defined by Sehnem [72]. Our result shows further that, when this product system arises from a C*-dynamical system, this same C*-envelope has the structure of a corner of a crossed product, and so is Morita equivalent to a crossed product.

Before proceeding, we should also direct the reader to the extensive literature on C*-algebras associated to semigroups and semigroup dynamical systems, including [45, 57, 58, 61, 64, 66, 80, 81]. Following Nica [66], Laca and Raeburn [58] demonstrated for quasi-lattice ordered (G, P) that the universal C*-algebra $C^*(G, P)$ for Nica-covariant representations of P has the structure of a semigroup crossed product. Interestingly, we will see (Remark 2.3.8) that our direct product construction of an automorphic dilation reduces to Laca-Raeburn's in the case where P acts on $\mathbb{C}$ trivially.

2.1.3 Structure of this chapter

Throughout this section, (G, P) is a (discrete) lattice ordered abelian group, and P acts on a C*-algebra A by $*$-endomorphisms. In Section 4.2, we review the construction of the semicrossed product, and necessary background on ordered groups and C*-envelopes. Section 2.3 contains our main results. We define the notion of a *minimal Nica-covariant automorphic dilation*, construct such a canonical dilation which we call the *product dilation*, and show that any such dilation always yields a C*-cover of the Nica-covariant semicrossed product $A \times^{\mathrm{nc}} P$ via full corner of a crossed product (Proposition 2.3.9). We show the Shilov ideal arises from a unique maximal G-invariant A-boundary ideal in any such C*-cover in Theorem 2.3.14, and hence show that the C*-envelope of $A \times^{\mathrm{nc}} P$ is a full corner of a crossed product (Corollary 2.3.16). In two immediate applications, we show that Theorem 2.3.14 reduces to the known result (2.1) for $A \times^{\mathrm{nc}} P$ in the injective case (Proposition 2.3.19), and we compute the unique maximal boundary ideal in the product dilation in the case $P = \mathbb{Z}_+$ (Proposition 2.3.20).

Section 2.4 is devoted to explicitly computing the Shilov ideal in the C*-cover arising from the product dilation for any Nica-covariant semicrossed product $A \times^{\mathrm{nc}} P$. We do so by describing a unique maximal G-invariant boundary ideal I in the product dilation B. Then

$$C_e^*(A \times^{\mathrm{nc}} P) \cong p_A \left(\frac{B}{I} \rtimes G \right) p_A$$

is a full corner by $p_A := 1_A + I$. Using the explicit construction of I from Section 2.4, in Section 2.5 we show that the G-C*-algebra B/I in the case $P = \mathbb{Z}_+^n$ is equivariantly $*$-isomorphic to the construction given by Davidson, Fuller, and Kakariadis in [18, Section 4.3]. So, our description of the C*-envelope reduces to the known result when $P = \mathbb{Z}_+^n$. In Section 2.6, we give some applications both of Theorem 2.3.14 and the explicit description of I from Section 2.4. In Section 6.1, we establish a simplicity result for the C*-envelope in the commutative case analogous to [18, Corollary 4.4.9]. In Section 6.2, we show that for totally ordered groups (G,P) which are direct limits of ordered subgroups $(G,P) = \bigcup_\lambda (G_\lambda, P_\lambda)$, such as $\mathbb{Q} = \bigcup_n \mathbb{Z}/n!$, we have

$$C_e^*(A \times^{\mathrm{nc}} P) = \varinjlim_{\lambda} C_e^*(A \times^{\mathrm{nc}} P_\lambda)$$

naturally, as long as P acts on A by surjections. This result is sharp and fails for non-totally ordered groups and non-surjective actions.

2.2 Background

In this chapter, a (discrete, unital) **semigroup** P is a set equipped with an associative binary operation, and we require that P contains a two-sided identity element. We are primarily interested in abelian semigroups. In the abelian setting, we will always denote the semigroup operation by + and the identity element by 0. A semigroup homomorphism is a function between semigroups preserving the semigroup operations and the identity.

If A is a C*-algebra, an **ideal** $I \lhd A$ always means a closed, two-sided ideal. We make frequent use of the following two inductivity properties of ideals in C*-algebras. Firstly, if

$$A = \overline{\bigcup_{\lambda\in\Lambda} A_\lambda} \cong \varinjlim_{\lambda\in\Lambda} A_\lambda$$

is an internal direct limit of C*-subalgebras A_λ, and $I \lhd A$ is an ideal, then

$$I = \overline{\bigcup_{\lambda\in\Lambda} I \cap A_\lambda}.$$

In particular, $I = \{0\}$ if and only if $I \cap A_\lambda = \{0\}$ for all $\lambda \in \Lambda$. Secondly, if $\{I_\lambda \mid \lambda \in \Lambda\}$ is a family of ideals in A that is directed under inclusion, then $I := \overline{\bigcup_{\lambda\in\Lambda} I_\lambda}$ is also an ideal in A.

Let P be a semigroup. An **(operator algebra) dynamical system** (A, α, P) over P consists of an operator algebra A and a semigroup action α of P on A by completely contractive algebra endomorphisms. That is, there is a distinguished (unital) semigroup homomorphism

$$p \mapsto \alpha_p : P \to \mathrm{End}(A).$$

We do not require the α_p to be automorphisms. We will say that (A, α, P) is **injective/surjective/automorphic** if each α_p is injective/surjective/automorphic. When A has an identity 1_A and each α_p is unital, we call (A, α, P) a **unital** dynamical system. If A is a C*-algebra, and hence each α_p is an $*$-endomorphism, then (A, α, P) is a **C*-dynamical system**.

Let G be an abelian group. A subsemigroup $P \subseteq G$ is a **positive cone** if $P \cap (-P) = \{0\}$, and a **spanning cone** if in addition $G = P - P$. Any positive cone $P \subseteq G$ induces a partial order on G by defining

$$g \leq h \iff h - g \in P.$$

This ordering respects the group operation $+$. A **lattice ordered abelian group** (G, P) consists of an abelian group G and a spanning cone $P \subseteq G$ such that the partial order $\leq$ induced by P on G makes G into a lattice. That is, for any $g, h \in G$, the $\{g, h\}$ has a least upper bound $g \vee h$ and a greatest lower bound $g \wedge h$. If (G, P) is a lattice ordered abelian group, we also refer to P as a **lattice ordered abelian semigroup**.

Example 2.2.1. The pair $(\mathbb{Z}^n, \mathbb{Z}_+^n)$ forms a lattice ordered abelian group. Here, a dynamical system $(A, \alpha, \mathbb{Z}_+^n)$ consists of a choice of n commuting completely contractive endomorphisms of A, which we usually just write as $\alpha_1, \ldots, \alpha_n \in \operatorname{End}(A)$.

Example 2.2.2. Any **totally ordered** group (G, P) is automatically lattice ordered. For instance, $(\mathbb{Q}, \mathbb{Q}_+)$ and $(\mathbb{R}, \mathbb{R}_+)$ are both totally ordered groups. If $P \subseteq \mathbb{Z}^n$ is the set of elements larger than $(0, \ldots, 0)$ in the lexicographic ordering of $\mathbb{Z}^n$, then $(\mathbb{Z}^n, P)$ is totally ordered, and the induced ordering is lexicographic.

A representation $T : P \to B(H)$ is a (unital) semigroup homomorphism, and we usually write $T(p) = T_p$. The representation T is **contractive/isometric/unitary** whenever each T_p is contractive/isometric/unitary. If (G, P) is a lattice ordered group, a contractive representation $T : P \to B(H)$ is **Nica-covariant** if whenever $p, q \in P$ satisfy $p \wedge q = 0$, we have $T_p T_q^* = T_q^* T_p$, so T_p and T_q not only commute, but $*$-commute [66]. If $V : P \to B(H)$ is an isometric representation, V is Nica-covariant if and only if

$$V_p V_p^* V_q V_q^* = V_{p \vee q} V_{p \vee q}^*.$$

That is, the range projections of the V_p's give a lattice homomorphism $P \to \operatorname{proj}(H)$. A representation T of $\mathbb{Z}_+^n$ is Nica-covariant if and only if the generators $T_1, \ldots, T_n$ $*$-commute, and in this case we can find a simultaneous dilation to isometries $V_1, \ldots, V_n$, which yield an isometric Nica-covariant representation V that dilates T [18, Theorem 2.5.10]. More generally, for any lattice ordered abelian semigroup P, any contractive Nica-covariant representation $T : P \to B(H)$ has an isometric Nica-covariant co-extension. For any lattice ordered abelian semigroup P, Li [60] showed that any Nica-covariant representation $T : P \to B(H)$ extends to a completely positive definite function on the whole group, and so T co-extends to an isometric Nica-covariant representation of P by [18, Theorem 2.5.10]

Let (A, α, P) be a dynamical system over an abelian semigroup P. A **covariant pair**

$$(\pi, T) : (A, P) \to B(H)$$

consists of a $*$-homomorphism $\pi : A \to B(H)$, and a representation $T : P \to B(H)$, such that, if $a \in A$ and $p \in P$,

$$\pi(a)T_p = T_p\pi(\alpha_p(a)). \tag{2.3}$$

We say (π, T) is **unitary/isometric/contractive/Nica-covariant** when T is so. Let $\mathcal{F}$ be a sufficiently well behaved family of representations of P on Hilbert space (cf. [18, Definition 2.1.1]). For our purposes, $\mathcal{F}$ is always one of "un" (unitary representations), "is" (isometric representations), "c" (contractive representations), or when P is a lattice ordered abelian semigroup, "nc" (Nica-covariant representations).

The **semicrossed product**

$$A \times_\alpha^{\mathcal{F}} P$$

is an operator algebra defined by the following universal property [18, Section 3.1]. There is a covariant pair $(i, v) : (A, P) \to A \times_\alpha^{\mathcal{F}} P$ such that whenever $(\pi, T) : (A, P) \to B(H)$ is a covariant pair with $T \in \mathcal{F}$, there is a unique completely contractive homomorphism

$$\pi \times T : A \times_\alpha^{\mathcal{F}} P \to B(H) \tag{2.4}$$

with $(\pi \times T) \circ i = \pi$ and $(\pi \times T) \circ v = T$. Concretely, $A \times_\alpha^{\mathcal{F}} P$ is densely spanned by formal monomials $v_p a$, for $p \in P$ and $a \in A$, which satisfy the relation

$$(v_p a) \cdot (v_q b) = v_{p+q}\alpha_q(a)b.$$

Indeed, one can construct $A \times_\alpha^{\mathcal{F}} P$ by starting with the algebraic tensor product

$$\mathbb{C}[P] \odot A,$$

defining a multiplication relation

$$(\delta_p \otimes a) \cdot (\delta_q \otimes b) = \delta_{p+q} \otimes (\alpha_q(a)b),$$

and completing in the universal operator algebra norm defined by

$$\|X\| := \sup\left\{\left\|(\pi \times T)^{(n)}(X)\right\| \;\middle|\; (\pi, T) \text{ is a covariant pair with } T \in \mathcal{F}\right\},$$

for any $X \in M_n(\mathbb{C}[P] \odot A)$. Here $\pi \times T : \delta_p \otimes a \mapsto T_p\pi(a)$ defines a homomorphism on $\mathbb{C}[P] \odot A$. When the action α is clear, we usually just write $A \times^{\mathcal{F}} P$. We do not omit $\mathcal{F}$, because it is standard that

$$A \times_\alpha P := A \times_\alpha^{\mathrm{c}} P$$

always denotes the contractive semicrossed product. Note that commutativity of the operation in P is used to prove the multiplication on $\mathbb{C}[P]\odot A$ is associative.

We are primarily interested in the Nica-covariant semicrossed product $A\times^{\mathrm{nc}} P$, when (G,P) is a lattice ordered abelian group. By [18, Proposition 4.2.1] and [60], $A\times^{\mathrm{nc}} P$ is also universal for *isometric* Nica-covariant pairs, and these completely norm the algebra $A\times^{\mathrm{nc}} P$. In fact, there is a distinguished isometric Nica-covariant pair for any pair (A,P). Let $\pi : A\to B(H)$ be any completely isometric representation. Define a pair $(\tilde{\pi},V) : (A,P)\to B(H\otimes\ell^2(P))$ by

$$\tilde{\pi}(a)(x\otimes\delta_p)=\alpha_p(a)x\otimes\delta_p,\quad V_q(x\otimes\delta_p)=x\otimes\delta_{p+q}.$$

Then $(\tilde{\pi},V)$ is an isometric Nica-covariant pair, and we call $\tilde{\pi}\times V : A\times^{\mathrm{nc}} P\to B(H)$ the **Fock representation** (induced by π) for $A\times^{\mathrm{nc}} P$. By [18, Theorem 4.2.9], any Fock representation is completely isometric. This is a key tool which makes it easy to prove that $A\times^{\mathrm{nc}} P$ embeds completely isometrically into a crossed product.

Let G be an abelian group and (B,β,G) a C*-dynamical system over G. In this chapter, we use the nonstandard convention that the crossed product $B\rtimes_\beta G$ is the universal C*-algebra generated by monomials

$$u_g a,\quad g\in G, a\in B$$

satisfying $u_g a\cdot u_h b = u_{g+h}\beta_h(a)b$, or when B is unital,

$$u_g^* a u_g=\beta_g(a).$$

Usually one takes the convention that $u_g a u_g^*=\beta_g(a)$. This backwards convention is only valid because G is abelian, so $g\mapsto u_g^*$ defines a unitary representation of G. Clearly this construction is isomorphic to the usual crossed product, so we lose no generality. What we gain is an alignment with the semicrossed covariance relations (2.3) and (2.4). Indeed

$$B\rtimes_\beta G\cong B\times_\beta^{\mathrm{un}} G\cong B\times_\beta^{\mathrm{is}} G$$

is also a semicrossed product, and a C*-algebra with the obvious $*$-structure. As with semicrossed products, we usually write $B\rtimes G$ when the action β is clear.

Generally, for any dynamical system (A,α,P), the semicrossed product $A\times_\alpha^{\mathcal{F}} P$ is a (non-selfadjoint) operator algebra, even when A is a C*-algebra. Let A be any operator algebra. A **C*-cover** $\varphi : A\to B$ for A consists of a C*-algebra B, and a unital completely isometric homomorphism φ such that $B = C^*(\varphi(A))$. The **C*-envelope** $C^*_e(A)$ is a co-universal or terminal C*-cover $\iota : A \to C^*_e(A)$. That is, whenever $\varphi : A \to B$ is a C*-cover, there is a $*$-homomorphism $\pi : B \to C^*_e(A)$ such that

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[dr, "\iota"'] & B \arrow[d, "\pi"] \\ & C^*_e(A) \end{tikzcd}$$

commutes. The homomorphism π is necessarily unique and surjective. The C*-envelope exists and is unique up to a $*$-homomorphism fixing A [42]. In fact, it can be produced from any C*-cover. If $\varphi : A \to B$ is a C*-cover, a **boundary ideal** $I \lhd B$ is an ideal such that the quotient map $q : B \to B/I$ is completely isometric on $\varphi(A)$. Existence of the C*-envelope implies that there is a unique maximal boundary ideal in B for A called the **Shilov ideal**, and $q\varphi : A \to B \to B/I$ is a C*-envelope for A [3].

2.3 Main results

Let (G, P) be a lattice ordered abelian group, and let (A, α, P) be a unital C*-dynamical system over P. Our goal is to embed the Nica-covariant semicrossed product $A \times^{\mathrm{nc}}_\alpha P$ into a crossed product $B \rtimes_\beta G$. Here, A should be a C*-subalgebra of B and the action β of G on A should extend or dilate the action α of P. Write

$$B \rtimes_\beta G = \overline{\operatorname{span}}\{u_g b \mid g \in G,\, b \in B\}.$$

We might hope to embed $A \times^{\mathrm{nc}} P$ in $B \rtimes G$ via a map of the form $\iota \times u$, where $\iota : A \to B$ is some unital $*$-monomorphism that intertwines α and β. However, this is impossible whenever any α_p has kernel. Indeed, if $a \in \ker \alpha_p \subseteq A$ is nonzero, then we would require $\iota(a) \neq 0$, but

$$0 = \iota(\alpha_p(a)) = u_p^* \iota(a) u_p.$$

This is impossible when u_p is unitary.

In the non-injective case, the best we can do is embed $A \times^{\mathrm{nc}} P$ into a corner of $B \rtimes G$. We do this by taking a nonunital embedding $\iota : A \to B$. Then, $p_A := \iota(1_A)$ is a projection in B. Consequently,

$$up_A : p \mapsto u_p p_A$$

defines an isometric representation of P in the corner $p_A(B \rtimes G)p_A$. The following definition is meant to capture a set of sufficient conditions for (ι, up_A) to be a Nica-covariant covariant pair, and give an embedding $\iota \times up_A : A \times^{\mathrm{nc}} P \to p_A(B \rtimes G)p_A$ (Proposition 2.3.9).

Definition 2.3.1. Let (G, P) be a lattice ordered abelian group. Suppose (A, α, P) is a C*-dynamical system over P. An **automorphic dilation** (B, β, G) is a C*-dynamical system (B, β, G) together with

(1) a $*$-monomorphism $\iota : A \to B$, such that

(2) for all $a, b \in A$ and $p \in P$,
$$\iota(a)\beta_p(\iota(b)) = \iota(a\alpha_p(b)).$$

By taking adjoints we have that $\beta_p(\iota(A))\iota(b) = \iota(\alpha_p(a)b)$. Moreover we say that (B, β, G) is a **Nica-covariant** automorphic dilation if in addition

(3) for all $a, b \in A$ and $g, h \in G$, we have
$$\beta_g(\iota(a))\,\beta_h(\iota(b)) = \beta_{g\wedge h}(\iota(\alpha_{g-g\wedge h}(a)\alpha_{h-g\wedge h}(b))).$$

We call an automorphic dilation (B, β, G) **minimal** if

(4) $\iota(A)$ generates B as a G-C*-algebra, i.e.
$$B = C^*\left(\bigcup_{g\in G}\beta_g\iota(A)\right).$$

We are primarily concerned with minimal Nica-covariant automorphic dilations, which satisfy all of (1)-(4). Note that if the automorphic dilation (B, β, G) is both minimal and Nica-covariant then property (3) above implies that
$$\sum_{g\in G}\beta_g\iota(A)$$
is a $*$-subalgebra, and hence
$$B = \overline{\sum_{g\in G}\beta_g\iota(A)}.$$
We are also mostly concerned with the unital case.

Remark 2.3.2. Let (G, P) be a lattice ordered abelian group, and let (A, α, P) be a unital C*-dynamical system. Suppose (B, β, G) is an automorphic C*-dynamical system, with a (possibly nonunital) $*$-monomorphism $\iota : A \to B$. Setting $p_A := \iota(1_A)$, it is straightforward to check that properties (2) and (3) in Definition 2.3.1 are equivalent to:

(2') For all $a \in A$ and $p \in P$,

$$p_A \beta_p(\iota(a)) = \iota(\alpha_p(a)) \quad (= \beta_p(\iota(a)) p_A),$$

and

(3') for all $g, h \in G$,

$$\beta_g(p_A)\beta_h(p_A) = \beta_{g \wedge h}(p_A).$$

Clearly if (2) and (3) hold, so do (2') and (3'). Conversely, if both (2') and (3') hold, then (2) holds because $\iota(a) = \iota(a)p_A$. Given $a, b \in A$ and $g, h \in G$, using both (2') and (3') we find

$$\begin{aligned} \beta_g(\iota(a))\beta_h\iota(b)) &= \beta_g(\iota(a))(\beta_g(p_A)\beta_h(p_A))\beta_h\iota(b)) \\ &= \beta_g(\iota(a))\beta_{g\wedge h}(p_A)\beta_h(\iota(b)) \\ &= \beta_{g\wedge h}(\beta_{g-g\wedge h}(a)p_A\beta_{h-g\wedge h}(b)) \\ &= \beta_{g\wedge h}(\iota(\alpha_{g-g\wedge h}(a)\alpha_{h-g\wedge h}(b))), \end{aligned}$$

showing (3) holds.

The reason we assign property (3) the name "Nica-covariant" is because in the unital case, the identity $\beta_g(p_A)\beta_h(p_A) = \beta_{g\wedge h}(p_A)$ ensures that the isometric semigroup representation $p \mapsto u_p p_A \in p_A(B \rtimes G)p_A$ is Nica-covariant. Indeed,

$$(u_p p_A)(u_p p_A)^* = u_p p_A u_p^* = \beta_{-p}(p_A).$$

So if (3) holds, the element $(u_p p_A)(u_p p_A)^* \cdot (u_q p_A)(u_q p_A)^*$ equals

$$\beta_{-p}(p_A)\beta_{-q}(p_A) = \beta_{(-p)\wedge(-q)}(p_A) = \beta_{-p\vee q}(p_A) = (u_{p\vee q}p_A)(u_{p\vee q}p_A)^*.$$

In a minimal Nica-covariant automorphic dilation, the projections $\beta_p(p_A)$ for $p \in P$ are central and in fact form an approximate identity.

Lemma 2.3.3. *Let (G, P) be a lattice ordered abelian group. Suppose (A, α, P) is a unital C*-dynamical system, with (B, β, G) a minimal Nica-covariant automorphic dilation. Considering P (a lattice) as a directed set, the net*

$$(\beta_p(p_A))_{p\in P}$$

is an increasing approximate identity for B, consisting of central projections.

Proof. Suppose $p \leq q$ in P. Then

$$\beta_p(p_A)\beta_q(p_A) = \beta_{p\wedge q}(p_A) = \beta_p(p_A).$$

Therefore $\beta_p(p_A) \leq \beta_q(p_A)$, since both are projections. These projections are central, because for an element of the form $\beta_g\iota(a)$, where $a \in A$ and $g \in G$, we have

$$\beta_p(p_A)\beta_g(\iota(a)) = \beta_{p\wedge g}\iota(\alpha_{g-p\wedge g}(a)) = \beta_g(\iota(a))\beta_p(p_A).$$

Here, we have used property (3) in Definition 2.3.1 and the fact that each α_p fixes 1_A. Since (G, P) is lattice ordered, any element $g \in G$ is dominated by an element $p = g \vee 0 \in P$. Further, when $p \geq g$, we have $p \wedge g = g$ and the same computation shows

$$\beta_p(p_A)\beta_g\iota(a) = \beta_g\iota(a).$$

Thus $(\beta_p(p_A))_{p\in P}$ is an approximate identity for $\beta_g\iota(A)$, which commutes with $\beta_g\iota(A)$. Since the automorphic dilation (B, β, G) is minimal,

$$B = \overline{\sum_{g\in G} \beta_g\iota(A))}.$$

Thus each $\beta_p(p_A)$ is central. Since the net $(\beta_p(p_A))_p$ is norm-bounded, a standard $\varepsilon/3$ argument shows it is an approximate identity on all of B. $\square$

The key observation is that any C*-dynamical system over a lattice ordered abelian semigroup admits a minimal Nica-covariant automorphic dilation. In fact, we can build one with an infinite product construction.

Definition 2.3.4. Let (G, P) be a lattice ordered abelian group, and (A, α, P) a C*-dynamical system. We construct a minimal Nica-covariant automorphic dilation as follows. Define the $*$-monomorphism

$$\iota : A \to \prod_G A$$

by

$$\iota(a)_g = \begin{cases} \alpha_g(a) & g \in P, \\ 0 & g \notin P. \end{cases}$$

Throughout, $[x]_g$ or simply x_g always denotes the g'th element of a tuple $x \in \prod_G A$. Then, G acts on $\prod_G A$ by the "left-shift" $\beta : G \to \mathrm{End}(\prod_G A)$, where

$$[\beta_g(x)]_h = x_{h+g}.$$

Set
$$B := C^*\left(\bigcup_{g\in G}\beta_g\iota(A)\right) = \overline{\sum_{g\in G}\beta_g\iota(A)}. \tag{2.5}$$
Then, (B,β,G) is a minimal Nica-covariant automorphic dilation of (A,α,P), which we call the **product dilation** of (A,α,P).

Proposition 2.3.5. *The product dilation (B,β,G) is a minimal Nica-covariant automorphic dilation of (A,α,P).*

Proof. Given $a,b\in A$, $p\in P$, and $g\in G$, we compute
$$[\iota(a)\beta_p\iota(b)]_g = \begin{cases}\alpha_g(a)\alpha_{g+p}(b) & g\ge 0,\\ 0 & \text{else,}\end{cases}$$
which equals $[\iota(a\alpha_p(b))]_g$. Thus $\iota(a)\beta_p\iota(b) = \iota(a\alpha_p(b))$. So, (B,β,G) is an automorphic dilation of (A,α,P). By (2.5), this dilation is minimal. Let $a,b\in A$ and $g,h,k\in G$. Then
$$[\beta_g\iota(a)\beta_h\iota(b)]_k = \begin{cases}\alpha_{g+k}(a)\alpha_{h+k}(b) & k\ge -g \text{ and } k\ge -h,\\ 0 & \text{else.}\end{cases}$$
Because $k\ge -g$ and $k\ge -h$ if and only if $k\ge(-g)\vee(-h) = -(g\wedge h)$, it follows that
$$\beta_g\iota(a)\beta_h\iota(b) = \beta_{g\wedge h}\iota(\alpha_{g-g\wedge h}(a)\alpha_{h-g\wedge h}(b)),$$
so the dilation is Nica-covariant. □

Remark 2.3.6. While finalizing this chapter of the thesis, the author was made aware that the product dilation defined here was defined first by Zahmatkesh for totally ordered abelian groups in [80], and for general lattice ordered abelian groups in [81]. In Proposition 2.3.9, we prove that a full corner of the crossed product associated to the product dilation is a C*-cover of the semicrossed product $A\times^{\mathrm{nc}}P$. Zahmatkesh proves in [81] that this same full corner is the universal C*-algebra associated to Nica-Toeplitz covariant representations of (A,α,P).

Example 2.3.7. It is most instructive to consider the product dilation of a unital system in the case $(G,P) = (\mathbb{Z},\mathbb{Z}_+)$. Here, we embed A in $\prod_{\mathbb{Z}} A$ via
$$\iota(a) := (\dots,0,0,a,\alpha(a),\alpha^2(a),\dots),$$
the "a" occurring at index 0. Then, $\mathbb{Z}$ acts on $\prod_{\mathbb{Z}} A$ by the backwards bilateral shift β. This is an automorphic dilation, because $p_A = (\dots,0,0,1,1,1,\dots)$ and
$$p_A\beta\iota(a) = (\dots,0,0,\alpha(a),\alpha^2(a),\dots) = \iota\alpha(a).$$

Remark 2.3.8. When $A = \mathbb{C}$ and P acts trivially, the product dilation (B, β, G) is the C*-algebra B_P that Laca and Raeburn define in [58, Section 2]. In terms of their notation, $1_0 = p_A$, and for $p \in P$, $1_p = \beta_{-p}(p_A)$. Nica-covariance of the dilation B_P is seen in Equation (1.2) in [58].

As promised, the Nica-covariant semicrossed product $A \times^{\mathrm{nc}} P$ embeds into the crossed product of any Nica-covariant automorphic dilation.

Proposition 2.3.9. *Let (G, P) be a lattice ordered abelian group. Let (A, α, P) be a unital C*-dynamical system. Suppose (B, β, G) is a Nica-covariant automorphic dilation of (A, α, P), with $*$-embedding $\iota : A \to B$. With $p_A = \iota(1_A)$, there is a completely isometric homomorphism*

$$\varphi = \iota \times up_A : A \times_\alpha^{nc} P \to B \rtimes_\beta G,$$

where $(up_A)_p = u_p p_A$. Moreover, if (B, β, P) is a minimal automorphic dilation, then

$$C^*(\varphi(A \times_\alpha^{nc} P)) = p_A(B \rtimes_\beta G)p_A$$

is a full corner of $B \rtimes_\beta G$.

Proof. As shown after Remark 2.3.2, Nica-covariance of the dilation (B, β, G) ensures that $up_A : P \to p_A(B \rtimes G)p_A$ is an isometric Nica-covariant representation of P. Further, because $p_A = \iota(1_A)$, ι maps A into $p_A(B \rtimes G)p_A$. The pair (ι, up_A) is covariant, as for $a \in A$ and $p \in P$,

$$\iota(a)u_p p_A = u_p \beta_p \iota(a) p_A = u_p \iota(\alpha_p(a)) = u_p p_A \iota(\alpha_p(a)).$$

By the universal property, there exists a completely contractive homomorphism

$$\varphi = \iota \times up_A : A \times^{\mathrm{nc}} P \to p_A(B \rtimes G)p_A \subseteq B \rtimes G.$$

We have to show that φ is completely isometric. Fix any faithful nondegenerate representation $\pi : B \to B(H)$. As G is abelian, the left regular representation

$$U \times \tilde{\pi} : B \rtimes G \to B(H \otimes \ell^2(G))$$

is faithful. Then $H \otimes \ell^2(P) \subseteq H \otimes \ell^2(G)$ is a $\tilde{\pi}(B)$ and $U(P)$-invariant subspace. Let

$$\sigma := \tilde{\pi} \circ \iota|_{H \otimes \ell^2(P)} : A \to B(H \otimes \ell^2(P)), \quad \text{and}$$
$$V := U|_{H \otimes \ell^2(P)} : P \to B(H \otimes \ell^2(P)).$$

Then, it is immediate that (σ, V) is a Nica-covariant covariant pair for (A, P), and by definition, $\sigma \times V$ is the Fock representation of $A \times^{\mathrm{nc}} P$ on $H \otimes \ell^2(P)$. By [18, Theorem 4.2.9], the Fock representation is completely isometric. Let $\kappa : B(H \otimes \ell^2(G)) \to B(H \otimes \ell^2(P))$ be the compression map. The diagram

$$\begin{tikzcd} A \times^{\mathrm{nc}} P \arrow[r, "\varphi"] \arrow[d, "\sigma\times V"] & B \rtimes G \arrow[d, "\pi\times U"] \\ B(H \otimes \ell^2(P)) & B(H \otimes \ell^2(G)) \arrow[l, "\kappa"'] \end{tikzcd}$$

commutes. As the vertical maps are complete isometries, and κ is a complete contraction, it follows that φ is completely isometric, as claimed.

Now suppose that (B, β, G) is minimal. We claim that the corner $p_A(B \rtimes G)p_A$ is full and generated by $\varphi(A \times^{\mathrm{nc}} P)$. This is a full corner, because $p_A(B \rtimes G)p_A$ contains $A \subseteq p_A B p_A$, and as A generates B as a G-C*-algebra, the ideal that A generates in $B \rtimes G$ is everything. By minimality,

$$B = \overline{\sum_{g\in G} \beta_g \iota(A)},$$

so $B \rtimes G$ is densely spanned by monomials $x = u_g \beta_h \iota(a))$ for $a \in A$ and $g, h \in G$. Given such a monomial, as p_A is central in B,

$$\begin{aligned} p_A x p_A &= p_A u_g p_A \beta_h \iota(a) p_A \\ &= p_A u_g p_A u_h^* \iota(a) u_h p_A \\ &= (u_{g_-} p_A)^* (u_{g_+} p_A)(u_h p_A)^* \iota(a)(u_h p_A). \end{aligned}$$

Here, since P is a spanning cone we have written $g = g_+ - g_-$, where $g_\pm \in P$. Thus, $x \in C^*(\iota, up_A)$ and $p_A(B \rtimes G)p_A \subseteq C^*(\iota, up_A)$. Conversely, since (ι, up_A) is a Nica-covariant isometric pair, by [18, Proposition 4.2.3], $C^*(\iota, up_A)$ is densely spanned by monomials $y = (u_p p_A)\iota(a)(u_q p_A)^*$, for $a \in A$ and $p, q \in P$. Given $p \in P$, we have

$$p_A u_p p_A = u_p \beta_p(p_A) p_A = u_p \beta_{p\wedge 0}(p_A) = u_p p_A,$$

and by taking adjoints $p_A u_p^* = p_A u_p^* p_A$. Then, for such a monomial y, we find

$$\begin{aligned} y &= u_p p_A \iota(a) p_A u_q^* \\ &= p_A u_p p_A \iota(a) p_A u_q^* p_A = p_A y p_A. \end{aligned}$$

This proves $C^*(\iota, up_A) = p_A(B \rtimes G)p_A$, as desired. ∎

Proposition 2.3.9 asserts that $p_A(B \rtimes G)p_A$ is a C*-cover of $A \times^{\mathrm{nc}} P$. To find the C*-envelope $C^*_e(A \times^{\mathrm{nc}} P)$, it suffices to describe the Shilov ideal. In Theorem 2.3.14 we will show that the Shilov ideal arises as a corner of a crossed product $I \rtimes G$, where $I \lhd B$ is some G-invariant ideal in B.

Definition 2.3.10. Let A be an operator algebra, B a C*-algebra, and suppose there is a completely isometric homomorphism $\iota : A \to B$. A (closed) ideal $I \lhd B$ is called an **A-boundary ideal** (with respect to ι) if the quotient map $B \to B/I$ restricts to be completely isometric on A.

Note that if A is a C*-algebra as in Definition 2.3.10, then I is a boundary ideal if and only if the quotient map $B \to B/I$ is faithful on A. This occurs if and only if $\iota(A) \cap I = \{0\}$.

Remark 2.3.11. It is routine to check that if (B, β, G) is a C*-dynamical system, and $I \lhd B$ is a β-invariant ideal, then $(B/I, \tilde{\beta}, G)$ is also a C*-dynamical system. Here $\tilde{\beta}_g(b+I) := \beta_g(b) + I$ is well defined, by invariance of I. The quotient map $q : B \to B/I$ is G-equivariant.

Further, suppose that (G, P) is a lattice ordered abelian group, and (B, β, G) is an automorphic dilation of (A, α, P) with inclusion $\iota : A \to B$. If $I \lhd B$ is a β-invariant A-boundary ideal (meaning $\iota(A) \cap I = \{0\}$), then $(B/I, \tilde{\beta}, G)$ is also an automorphic dilation of (A, α, P), because $q\iota$ is faithful on A. Moreover, if (B, β, G) is Nica-covariant or minimal, then so too is $(B/I, \tilde{\beta}, G)$, which easily follows from equivariance of q.

The following lemma summarizes that under reasonable hypotheses we can "commute" taking quotients with either taking corners or crossed products.

Lemma 2.3.12. *(i) Suppose C is a C*-algebra, and $p \in C$ is a projection. Let $J \lhd pCp$ be an ideal. If $K = \langle J \rangle_C = \overline{CJC}$ is the ideal J generates in C, then $J = pKp$. Moreover, there is a canonical isomorphism*

$$\frac{pCp}{J} \cong (p + K)\left(\frac{C}{K}\right)(p + K).$$

(ii) Suppose (B, β, G) is an automorphic C-dynamical system over an abelian group G. Let $I \lhd B$ be a G-invariant ideal. Then the natural map $B \rtimes G \to (B/I) \rtimes G$ induces an isomorphism*

$$\frac{B \rtimes_\beta G}{I \rtimes_\beta G} \cong \frac{B}{I} \rtimes_{\tilde{\beta}} G.$$

Proof. (i) Since $J \subseteq K$, certainly $J = pJp \subseteq pKp$. Conversely, for any term ajb, for $a, b \in C$ and $j \in J \subseteq pAp$, the product

$$p(ajb)p = pa(pjp)bp = (pap)j(pbp)$$

lies in J, since $J \lhd pCp$. Thus $pKp = J$. Restricting the quotient map $C \to C/K$ gives a $*$-homomorphism with range $(p+K)(C/K)(p+K)$ and kernel $K \cap pCp = pKp = J$, so the stated isomorphism follows.

(ii) This follows because G is abelian, and hence an exact group [12, Theorem 5.1.10]. $\square$

Recall that when G is an abelian group, the compact dual group $\widehat{G}$ has a natural gauge action γ on any crossed product $B \rtimes G$, which satisfies

$$\gamma_\chi(u_g b) = \chi(g) u_g b.$$

Consequently, there is a faithful expectation

$$E_{\widehat{G}} : B \rtimes G \to B \rtimes G$$

with range B, defined by the formula

$$E_{\widehat{G}}(x) = \int_{\widehat{G}} \gamma_\chi(x)\, d\chi.$$

Here $d\chi$ denotes integration against Haar measure.

Lemma 2.3.13. *Suppose (B, β, G) is an automorphic C^*-dynamical system over an abelian group G. Let $J \lhd B \rtimes_\beta G$ be an ideal. Then J is invariant under the gauge action of $\widehat{G}$ if and only if $J = I \rtimes_\beta G$, where $I = J \cap B \lhd B$ is a β-invariant ideal in B.*

Proof. Since $\widehat{G}$ acts diagonally on the spanning monomials $u_g b$ in $B \rtimes G$, any ideal of the form $I \rtimes_\beta G$ is $\widehat{G}$-invariant. Conversely, let $J \lhd B$ be $\widehat{G}$-invariant. Then $I := J \cap B \lhd B$ is a G-invariant ideal, since the action β is implemented by unitaries in $B \rtimes_\beta G$. Then, $I \subseteq J$ implies $I \rtimes_\beta G \subseteq J$.

For the reverse inclusion, as in Lemma 2.3.12.(ii), there is a canonical onto $*$-homomorphism $\pi : B \rtimes_\beta G \to (B/I) \rtimes_{\tilde{\beta}} G$ with kernel $I \rtimes_\beta G$. Given $x \in J$, because J is closed and $\widehat{G}$-invariant $E_{\widehat{G}}(x^* x) \in J \cap B = I$ and hence $\pi(E_{\widehat{G}}(x^* x)) = 0$. Since π is $\widehat{G}$-equivariant, we find

$$0 = \pi(E_{\widehat{G}}(x^* x)) = E_{\widehat{G}}(\pi(x^* x)).$$

As the expectation $E_{\widehat{G}}$ is faithful, $\pi(x) = 0$ and $x \in \ker \pi = I \rtimes_\beta G$. Therefore $J = I \rtimes_\beta G$. $\square$

We can now identify the Shilov ideal in $C^*(\varphi(A\times^{\mathrm{nc}} P)) = p_A(B\rtimes G)p_A$, for any minimal Nica-covariant automorphic dilation (B,β,G).

Theorem 2.3.14. *Let (G,P) be a lattice ordered abelian group, and let (A,α,P) be a unital C^*-dynamical system over P. Suppose (B,β,G) is any minimal Nica-covariant automorphic dilation of (A,α,P), with $*$-embedding $\iota : A \to B$. Then, there is a unique maximal β-invariant A-boundary ideal $I \lhd B$. Further, if $p_A = \iota(1_A) \in B$ and $\varphi = \iota \times up_A : A \times_\alpha^{nc} P \to B \rtimes_\beta G$ is the completely isometric embedding from Proposition 2.3.9, then*

$$p_A(I \rtimes_\beta G)p_A \lhd p_A(B \rtimes_\beta G)p_A = C^*(\varphi(A \times_\alpha^{nc} P))$$

is the Shilov ideal for $A \times_\alpha^{nc} P$. Consequently

$$C_e^*(A \times_\alpha^{nc} P) \cong (p_A + I)\left(\frac{B}{I} \rtimes_{\tilde{\beta}} G\right)(p_A + I)$$

is a full corner of a crossed product.

Proof. Let $\varphi = \iota \times up_A : A \times_\alpha^{\mathrm{nc}} P \to B \rtimes_\beta G$ be the completely isometric representation from Proposition 2.3.9. Let $J \lhd p_A(B \rtimes_\beta G)p_A$ be the Shilov ideal for $A \times_\alpha^{\mathrm{nc}} P$. Since $\varphi(A \times_\alpha^{\mathrm{nc}} P) = \overline{\operatorname{span}}\{u_p\iota(a) \mid p \in P, a \in A\}$ is invariant under the gauge action of $\widehat{G}$, it follows that J is also $\widehat{G}$-invariant. Let $K = \overline{(B \rtimes_\beta G)J(B \rtimes_\beta G)}$ be the ideal J generates in the entire crossed product $B \rtimes_\beta G$. Since J is $\widehat{G}$-invariant, so too is K. By Lemma 2.3.13, we have $K = I \rtimes_\beta G$ for some β-invariant $I \lhd B$. By Lemma 2.3.12.(i), we find

$$J = p_A K p_A = p_A(I \rtimes_\beta G)p_A.$$

Because $\iota(A) \subseteq p_A(B \rtimes_\beta G)p_A$, we also find

$$I \cap \iota(A) = K \cap \iota(A) = p_A(K \cap \iota(A))p_A = J \cap \iota(A) = \{0\},$$

since J does not intersect $\varphi(A \times_\alpha^{\mathrm{nc}} P) \supseteq \iota(A)$. Therefore I is a β-invariant boundary ideal. By Lemma 2.3.12, we have a canonical isomorphism

$$C_e^*(A \times_\alpha^{\mathrm{nc}} P) \cong \frac{p_A(B \rtimes_\beta G)p_A}{p_A(I \rtimes_\beta G)p_A} \cong (p_A + I)\left(\frac{B}{I} \rtimes_{\tilde{\beta}} G\right)(p_A + I).$$

To see that I is the unique maximal such ideal, suppose that $R \lhd B$ is any β-invariant A-boundary ideal. Then $p_A(R \rtimes_\beta G)p_A \lhd p_A(B \rtimes_\beta G)p_A$. By Lemma 2.3.12 again,

$$\frac{p_A(B \rtimes_\beta G)p_A}{p_A(R \rtimes_\beta G)p_A} \cong (p_A + R)\left(\frac{B}{R} \rtimes_{\tilde{\beta}} G\right)(p_A + R).$$

Then by Remark 2.3.11, $(B/R, \tilde{\beta}, G)$ is a minimal Nica covariant automorphic dilation. By Proposition 2.3.9, $(p_A + R)((B/R) \rtimes_\beta G)(p_A + R)$ is a C*-cover for $A \times^{\mathrm{nc}} P$. By definition of the C*-envelope, there is an onto $*$-homomorphism

$$\frac{p_A(B \rtimes_\beta G)p_A}{p_A(R \rtimes_\beta G)p_A} \cong (p_A + R)\left(\frac{B}{R} \rtimes_{\tilde{\beta}} G\right)(p_A + R) \to C_e^*(A \times_\alpha^{\mathrm{nc}} P) \cong \frac{p_A(B \rtimes_\beta G)p_A}{p_A(I \rtimes_\beta G)p_A},$$

which fixes $A \times^{\mathrm{nc}} P$. It follows that $p_A(R \rtimes_\beta G)p_A \subseteq p_A(I \rtimes_\beta G)p_A$. Upon intersecting with B, in which p_A is central, we find $p_A R \subseteq p_A I$. Since R and I are β-invariant, and $(\beta_g(p_A))_{g \in G}$ is an approximate identity in B, by Lemma 2.3.3, it follows that $R \subseteq I$. Indeed, for $x \in R$,

$$x\beta_g(p_A) = \beta_g(\beta_{-g}(x)p_A)$$

lies in $\beta_g(Rp_A) \subseteq \beta_g(I) \subseteq I$, and converges as a net indexed by $g \in G$ to $x \in R$. □

Corollary 2.3.15. *Suppose (A, α, P) is a unital C*-dynamical system over a lattice ordered abelian group (G, P). If (B, β, G) is a minimal Nica-covariant automorphic dilation of (A, α, P), then the C*-cover*

$$\varphi : A \times_\alpha^{nc} P \to p_A(B \rtimes_\beta G)p_A$$

is a C-envelope if and only if B contains no nontrivial β-invariant A-boundary ideals.*

Corollary 2.3.16. *Suppose that (A, α, P) is a unital C*-dynamical system, where (G, P) is a lattice ordered abelian group. The C*-envelope $C_e^*(A \times_\alpha^{nc} P)$ is a full corner of a crossed product of a minimal Nica-covariant automorphic dilation of (A, α, P).*

Proof. To apply Theorem 2.3.14, it is enough to note that (A, α, P) has at least one minimal Nica-covariant automorphic dilation. The product dilation (B, β, G) from Definition 2.3.4 suffices. Then

$$C_e^*(A \times_\alpha^{\mathrm{nc}} P) \cong (p_A + I)\left(\frac{B}{I} \rtimes G\right)(p_A + I),$$

and by Remark 2.3.11, $(B/I, \tilde{\beta}, G)$ is itself a minimal Nica-covariant automorphic dilation. □

Remark 2.3.17. Note that when $A \cong C(X)$ is a commutative C*-algebra, the product dilation $B \subseteq \prod_{g \in G} A$ is also commutative. Consequently the minimal Nica-covariant automorphic dilation in Corollary 2.3.16 is a quotient of the product dilation, and hence also commutative.

Corollary 2.3.16 extends even to nonunital systems. To show this, we use essentially the same unitization technique as in [18, Section 4.3].

Corollary 2.3.18. *Let (G, P) be a lattice ordered abelian group, and (A, α, P) a (possibly nonunital) C*-dynamical system. The C*-envelope of $A \times_{\alpha}^{nc} P$ is a full corner of a crossed product associated to a minimal Nica-covariant automorphic dilation of (A, α, P).*

Proof. Form the unitization $\tilde{A} := A \oplus \mathbb{C}1_{\tilde{A}}$, even if A is unital. Then we get a unital C*-dynamical system $(\tilde{A}, \tilde{\alpha}, P)$ by setting $\tilde{\alpha}(a + \lambda 1_{\tilde{A}}) := \alpha(a) + \lambda 1_{\tilde{A}}$ for $a \in A$ and $\lambda \in \mathbb{C}$. Let $(\tilde{B}, \tilde{\beta}, G)$ be the product dilation of $(\tilde{A}, \tilde{\alpha}, P)$, with inclusion $\iota : \tilde{A} \to \tilde{B}$.

Now define
$$B := \overline{\bigcup_{g \in G} \tilde{\beta}\iota(A)} = \overline{\sum_{g \in G} \tilde{\beta}_g \iota(A)} \subseteq \tilde{B},$$
and set $\beta := \tilde{\beta}|_B$. Since A is an $\tilde{\alpha}$-invariant ideal in $\tilde{A}$, it follows that B is a $\tilde{\beta}$-invariant ideal in $\tilde{B}$. By definition, (B, β, G) is just the product dilation for (A, α, P). Using the faithfulness of the associated Fock or left regular representations, one can prove that $A \times_{\alpha}^{\mathrm{nc}} P$ embeds completely isometrically into $\tilde{A} \times_{\tilde{\alpha}}^{\mathrm{nc}} P$, and that $B \rtimes_{\beta} G$ embeds into $\tilde{B} \rtimes_{\tilde{\beta}} G$. Let $p_{\tilde{A}} := \iota(1_{\tilde{A}})$, and let $\varphi = \iota \times u p_{\tilde{A}} : \tilde{A} \times^{\mathrm{nc}} P \to p_{\tilde{A}}(\tilde{B} \rtimes G) p_{\tilde{A}}$ be the completely isometric embedding from Proposition 2.3.9. A similar argument as in the proof of Proposition 2.3.9 proves that
$$C^*(\varphi(A \times_{\alpha}^{\mathrm{nc}} P)) = p_{\tilde{A}}(B \rtimes_{\beta} G) p_{\tilde{A}}.$$

Thus, the corner $p_{\tilde{A}}(B \rtimes_{\beta} G) p_{\tilde{A}}$ is a C*-cover for $A \times^{\mathrm{nc}} P$. This is a full corner of $B \rtimes_{\beta} G$, because it contains $\iota(A) \subseteq B$, which generates B as a G-C*-algebra. Let J be the Shilov ideal for $A \times^{\mathrm{nc}} P$ in $p_{\tilde{A}}(B \rtimes G) p_{\tilde{A}}$. Observe that Lemma 2.3.12.(i) holds even in the setting where $C \lhd \tilde{C}$ is an ideal in some larger C*-algebra $\tilde{C}$, and the projection p lies in $\tilde{C}$. In particular, using $B \rtimes G \lhd \tilde{B} \rtimes G$, and the projection $p_A \in \tilde{B} \rtimes G$, the proof given for Theorem 2.3.14 applies verbatim to show that $J = p_{\tilde{A}}(I \rtimes G) p_{\tilde{A}}$ for some unique maximal β-invariant A-boundary ideal I in B.

Let $\tilde{I}$ be the unique maximal $\tilde{\beta}$-invariant $\tilde{A}$-boundary ideal in $\tilde{B}$. By construction of the product dilation, we have $B \cap \iota(\tilde{A}) = \iota(A)$. It follows that a β-invariant ideal in B is an A-boundary ideal if and only if it is also an $\tilde{A}$-boundary ideal. Therefore $I = \tilde{I} \cap B$. Identifying $B/I = B/(\tilde{I} \cap B) \subseteq \tilde{B}/\tilde{I}$, applying Remark 2.3.11 and Lemma 2.3.12.(ii) shows that
$$C_e^*(A \times^{\mathrm{nc}} P) \cong (p_{\tilde{A}} + \tilde{I}) \left(\frac{B}{I} \rtimes G \right) (p_{\tilde{A}} + \tilde{I})$$
is a full corner of a crossed product associated to a minimal Nica-covariant automorphic dilation. ∎

When (A, α, P) is an injective C*-dynamical system, we recover a known result that the C*-envelope of $A \times^{\mathrm{nc}} P$ is a crossed product of a certain minimal automorphic extension of A.

Proposition 2.3.19. [18, Theorem 4.2.12] *Let (G, P) be a lattice ordered abelian group, and (A, α, P) an injective unital C*-dynamical system. Then*

$$C_e^*(A \times_\alpha^{nc} P) \cong \tilde{A} \rtimes_{\tilde{\alpha}} G,$$

where $(\tilde{A}, \tilde{\alpha}, G)$ is an automorphic C-dynamical system (unique up to equivariant $*$-isomorphism) satisfying $A \subseteq \tilde{A}$ and $\tilde{\alpha}_p|_A = \alpha_p$ for $p \in P$.*

Proof. Let (B, β, G) be the product dilation for (A, α, P). Then $B \subseteq \prod_G A$. Let

$$c_0(G, A) \coloneqq \{x \in \prod_G A \mid \lim_g \|x_g\| = 0\} \lhd \prod_G A.$$

Here, by writing "$\lim_{g \in G}$", we are considering G as a directed set in its ordering induced by P, and thinking of G-tuples as nets. We will show that

$$I \coloneqq B \cap c_0(G, A) \subset B$$

is the unique maximal β-invariant A-boundary ideal in B. It is easy to check it is a β-invariant ideal. Because the action α is injective, each α_p is isometric. So, if $a \in \iota^{-1}(I)$, we have

$$0 = \lim_{g \in G} \|\iota(a)_g\| = \lim_{p \in P} \|\alpha_p(a)\| = \lim_{p \in P} \|a\| = \|a\|,$$

hence $a = 0$. Note that the second equality holds because P is a cofinal subset of G. This proves $\iota(A) \cap I = \{0\}$, so I is a β-invariant A-boundary ideal.

Suppose $J \lhd B$ is any other β-invariant A-boundary ideal. Let $x \in J \subseteq \prod_G A$. Let $\varepsilon > 0$. Because B is a minimal dilation, we can choose an element of the form

$$y = \sum_{g \in F} \beta_{-g} \iota(a_g),$$

where $F \subseteq G$ is finite, and $a_g \in A$, and $\|y - x\| < \varepsilon$. Since J is a β-invariant ideal, $p_A \beta_{\vee F}(x)$ is in J. However, since (B, β, G) is a Nica-covariant automorphic dilation,

$$\begin{aligned} p_A \beta_{\vee F}(y) &= \sum_{g \in F} p_A \beta_{\vee F - g} \iota(a_g) \\ &= \sum_{g \in F} \iota \alpha_{\vee F - g}(a_g) \\ &= \iota\left(\sum_{g \in F} \alpha_{\vee F - g}(a_g)\right) \end{aligned}$$

is in $\iota(A)$. Since J is an A-boundary ideal, the projection $A \to B \to B/J$ is injective, and so isometric. Therefore

$$\begin{aligned}\|p_A\beta_{\vee F}(y)\| &= \|p_A\beta_{\vee F}(y) + J\| \\ &\leq \|p_A\beta_{\vee F}(y) - p_A\beta_{\vee F}(x)\| \\ &\leq \|y - x\| < \varepsilon.\end{aligned}$$

Since $[p_A\beta_{\vee F}(y)]_p = y_{p+\vee F}$ for $p \in P$, it follows that $g \geq \vee F$ implies $\|y_g\| < \varepsilon$, and also $\|x_g\| \leq \|y_g\| + \|x - y\| < 2\varepsilon$. This proves that $x \in c_0(G, A)$, so $J \subseteq I$, and I is the unique maximal β-invariant A-boundary ideal in B.

By Theorem 2.3.14,

$$C_e^*(A \times_\alpha^{\mathrm{nc}} P) \cong (p_A + I)\left(\frac{B}{I} \rtimes_{\tilde{\beta}} G\right)(p_A + I).$$

However, $p_A = 1_{\prod_G A}$ modulo $c_0(G, A)$, because $p \geq 0$ implies $[p_A]_p = 1$. It follows that $p_A + I$ is a two-sided identity $1_{B/I}$, and the C*-envelope is just the crossed product $(B/I) \rtimes_{\tilde{\beta}} G$. By Remark 2.3.11, Nica-covariance of the dilation $(B/I, \tilde{\beta}, G)$, with unital embedding $\eta = q_I\iota : A \to B \to B/I$, implies that, for $p \in P$ and $a \in A$,

$$\beta_p\eta(a) = (p_A + I)\beta_p(\eta(a)) = \eta(\alpha_p(a)).$$

So, $\tilde{\beta}_p\eta = \eta\alpha_p$, which when we identify $A \cong \eta(A) \subseteq B/I$, reads $\tilde{\beta}_p|_A = \alpha_p$. Since the automorphic dilation $(B/I, \tilde{\beta}, G)$ is minimal, it also follows easily that

$$\frac{B}{I} = \overline{\bigcup_{p \in P} \tilde{\beta}_{-p}\eta(A)}.$$

Thus $(B/I, \tilde{\beta}, G)$ is a minimal automorphic extension of (A, α, P). Such an extension is unique up to an equivariant isomorphism fixing A, since if

$$\tilde{A} = \overline{\bigcup_{p \in P} \tilde{\alpha}_{-p}(A)} \supseteq A,$$

with G-action $\tilde{\alpha}$ extending α, the map $\tilde{\beta}_{-p}\eta(a) \mapsto \tilde{\alpha}_{-p}(a)$ extends to an equivariant $*$-isomorphism $B/I \cong \tilde{A}$. □

In the proof of Proposition 2.3.19, we showed the maximal β-invariant A-boundary ideal was $B \cap c_0(G, A)$. In the case $(G, P) = (\mathbb{Z}, \mathbb{Z}_+)$, this result generalizes readily to the non-injective case.

Recall that if A is a C*-algebra and $I \lhd A$ is an ideal, then

$$I^{\perp} := \{a \in A \mid b \in I \implies ab = 0\} \subseteq A$$

is also an ideal, and satisfies $I \cap I^{\perp} = \{0\}$. If $\pi : A \to B$ is a $*$-homomorphism and $a \in (\ker \pi)^{\perp}$, then

$$\|\pi(a)\| = \|a + \ker \pi\| = \|a + (\ker \pi)^{\perp} \cap \ker \pi\| = \|a\|.$$

This shows $\pi|_{(\ker \pi)^{\perp}}$ is always isometric.

Proposition 2.3.20. *Let $(A, \alpha, \mathbb{Z}_+)$ be a unital C*-dynamical system over $\mathbb{Z}_+$, and let $(B, \beta, \mathbb{Z})$ be its product dilation. The unique maximal β-invariant A-boundary ideal for $\iota(A)$ in B is*

$$I = B \cap c_0(\mathbb{Z}, (\ker \alpha)^{\perp}).$$

Consequently,

$$C_e^*(A \times_{\alpha}^{nc} P) \cong (p_A + I)\left(\frac{B}{I} \rtimes_{\tilde{\beta}} G\right)(p_A + I).$$

Proof. Because $(\ker \alpha)^{\perp}$ is an ideal in A, it follows easily that I is a β-invariant ideal in B. Suppose $a \in A$ with $\iota(a) \in I \subseteq c_0(\mathbb{Z}, (\ker \alpha)^{\perp})$. Then, each $\alpha^n(a) \in (\ker \alpha)^{\perp}$. Because α is isometric on $(\ker \alpha)^{\perp}$, one sees that $\|\alpha^n(a)\| = \|a\|$ by an easy induction on n, and so

$$0 = \lim_{n \to \infty} \|\iota(a)_n\| = \lim_{n \to \infty} \|\alpha^n(a)\| = \lim_{n \to \infty} \|a\| = \|a\|.$$

Thus $\iota(A) \cap I = \{0\}$.

Suppose $J \lhd B$ is any β-invariant boundary ideal for A. The same argument as in the proof of Proposition 2.3.19 shows that all tuples in J vanish at $+\infty$. So, it suffices to let $x \in J$ and prove each $x_g \in (\ker \alpha)^{\perp}$. If $b \in \ker \alpha$, then

$$\iota(b) = (\ldots, 0, 0, b, 0, 0, \ldots).$$

So,

$$\beta_g(x)\iota(b) = (\ldots, 0, 0, x_g b, 0, 0 \ldots) = \iota(x_g b) \in \iota(A) \cap J = \{0\}.$$

Since ι is injective, $x_g b = 0$, and this proves each $x_g \in (\ker \alpha)^{\perp}$. So, $J \subseteq I$. Therefore I is the unique maximal β-invariant A-boundary ideal in B, and Theorem 2.3.14 applies. □

Example 2.3.21. Proposition 2.3.20 does not generalize so readily to the case $P = \mathbb{Z}_+^n$. Consider the unital C*-dynamical system $(A, \alpha, \mathbb{Z}_+^2)$, where $A = \mathbb{C}^3$ and the action is determined by generators by

$$\alpha_1(a, b, c) = (a, c, c),$$
$$\alpha_2(a, b, c) = (c, b, c).$$

This is the unitization of the nonunital system $(\mathbb{C} \oplus \mathbb{C}, \alpha^0, \mathbb{Z}_+^2)$, where $\alpha_1^0(a, b) = (a, 0)$ and $\alpha_2^0(a, b) = (0, b)$. Reviewing Proposition 2.3.20, we might expect

$$B \cap c_0(\mathbb{Z}, R_\alpha^\perp) = \{b \in B \mid b_g \in R_\alpha^\perp \text{ and } \lim_{g \in \mathbb{Z}^2} b_g = 0\},$$

for $R_\alpha = \ker \alpha_1 \cap \ker \alpha_2$, to be the unique maximal β-invariant A-boundary ideal. However, this fails to even be a boundary ideal, since here $R_\alpha = \{0\}$, and for any element $x = (a, b, 0) \in A$, the tuple

$$\iota(x) = \begin{pmatrix} \ddots & & & & & \\ & 0 & 0 & 0 & 0 & \cdots \\ & 0 & (a,b,0) & (0,b,0) & (0,b,0) & \cdots \\ & 0 & (a,0,0) & 0 & 0 & \cdots \\ & 0 & (a,0,0) & 0 & \ddots & \\ & \vdots & \vdots & \vdots & & \end{pmatrix}$$

lies in $A \cap c_0(\mathbb{Z}, R_\alpha^\perp) = \iota(A) \cap c_0(\mathbb{Z}, A)$. A correct description of the Shilov ideal in the case $P = \mathbb{Z}_+^n$ is more complicated, and follows as described in Section 2.4. See Section 2.5 for more discussion in the case $P = \mathbb{Z}_+^n$.

2.4 Explicit computation of the Shilov Ideal

Throughout this section, let (G, P) be a lattice ordered abelian group, and let (A, α, P) be a unital C*-dynamical system. Also, let (B, β, G) be the product dilation for (A, α, P), with inclusion $\iota : A \to B \subseteq \prod_G A$, as in Definition 2.3.4. By Theorem 2.3.14, B contains a unique maximal ideal I which is both β-invariant and an A-boundary ideal (does not intersect $\iota(A)$). In this section, we will explicitly describe I. The following construction of I was inspired both by the construction in [18, Section 4.3], and the construction of Sehnem's covariance algebra in [72, Section 3.1].

Definition 2.4.1. Define the following ideals.

(1) Given a finite subset $F \subseteq G$, let

$$K_F := \bigcap_{\substack{g \in F \\ g \nleq 0}} \ker \alpha_{g \vee 0} \lhd A$$

be the ideal of elements vanishing under the action of any strictly positive part of an element in F. Here we take the convention that the empty intersection yields $K_F = A$.

(2) For $F \subseteq G$ finite, let

$$J_F := K_F^{\perp} \lhd A$$

be the annihilator of K_F.

(3) For $F \subseteq G$ finite, define

$$I_F := \{b \in B \mid b_g \in J_{F-g} \text{ for all } g \in G\} \lhd B.$$

(4) Finally, set

$$I := \overline{\bigcup_{F \subseteq G \text{ finite}} I_F} \lhd B.$$

It is straightforward to check that if $F \subseteq F'$ are finite subsets of G, then $K_F \supseteq K_{F'}$, and hence $J_F \subseteq J_{F'}$. Consequently $I_F \subseteq I_{F'}$, so $\{I_F \mid F \subseteq G \text{ finite}\}$ is a directed system of ideals, and so I is indeed an ideal in B. Further, it's just as straightforward to show that for any $g \in G$, and any finite $F \subseteq G$, that

$$\beta_g(I_F) = I_{F-g}.$$

It follows that $I = \overline{\bigcup_F I_F}$ is a β-invariant ideal.

Theorem 2.4.2. *Let (A, α, P) be a unital C^*-dynamical system over a lattice ordered abelian group (G, P), with product dilation (B, β, G). The ideal $I \lhd B$ from Definition 2.4.1 is the unique maximal β-invariant A-boundary ideal in the product dilation (B, β, G). Consequently,*

$$C_e^*(A \times_\alpha^{nc} P) \cong (p_A + I)\left(\frac{B}{I} \rtimes_{\tilde{\beta}} G\right)(p_A + I)$$

is a full corner of a crossed product.

For clarity, we break the proof of Theorem 2.4.2 into lemmas. Our first lemma is a verification that I is indeed a β-invariant A-boundary ideal.

Lemma 2.4.3. *The ideal I satisfies $\iota(A) \cap I = \{0\}$.*

Proof. Since $I = \overline{\bigcup_F I_F}$ is an inductive union of ideals, it suffices to prove $I_F \cap \iota(A) = \{0\}$ for every finite $F \subseteq G$. Suppose for a contradiction that there is some finite $F_0 \subseteq G$, and some nonzero $a_0 \in A \setminus \{0\}$ with $\iota(a_0) \in I_{F_0}$. By definition of I,

$$0 \neq a_0 = \iota(a_0)_0 \in J_{F_0} = K_{F_0}^{\perp} = \left(\bigcap_{\substack{g \in F_0 \\ g \nleq 0}} \ker \alpha_{g \vee 0} \right)^{\perp}.$$

Since $a_0 \neq 0$ and $K_{F_0} \cap K_{F_0}^{\perp} = \{0\}$, the element a_0 is not in K_{F_0}. So, there is a $g \in F_0$ with $g \vee 0 > 0$ and $\alpha_{g \vee 0}(a_0) \neq 0$.

Set $a_1 = \alpha_{g\vee 0}(a_0) \neq 0$. Since $\iota(a_0) \in I_{F_0}$, it will follow that $\iota(a_1) = \iota\alpha_{g \vee 0}(a_0)$ lies in I_{F_1}, where

$$F_1 \coloneqq \{h - g \vee 0 \mid h \in F_0, h \nleq g\} \subset F_0 - g \vee 0.$$

Because $g - g \vee 0$ lies in $(F_0 - g \vee 0) \setminus F_1$, we also have $|F_1| < |F_0|$ strictly. But then, because $a_1 \neq 0$ and $\iota(a_1) \in I_{F_1}$, we may repeat the same argument to find a nonzero $a_2 \in A$ and an $F_2 \subseteq G$, with $\iota(a_2) \in I_{F_2}$, and $|F_2| < |F_1|$. Continuing recursively, we find an infinite sequence

$$|F_0| > |F_1| > |F_2| > \cdots$$

of finite subsets of G, and each $I_{F_n} \cap \iota(A) \neq \{0\}$. This is absurd, since eventually such a sequence must terminate at $\varnothing$, and $I_\varnothing = \{0\}$. This proves $I_F \cap \iota(A) = \{0\}$.

To prove that $\iota(a_1) \in I_{F_1}$ as needed in the paragraph above, it suffices to note that for any $p \in P$, that

$$\begin{aligned} K_{F_0 - g\vee 0 - p} &= \bigcap_{\substack{h \in F_0 \\ h - g\vee 0 - p \nleq 0}} \ker \alpha_{(h - g \vee 0 - p)\vee 0} \\ &\supseteq \bigcap_{\substack{k \in F_1 \\ k - p \nleq 0}} \ker \alpha_{(k-p)\vee 0} = K_{F_1 - p}. \end{aligned}$$

This is because if $h \in F_0$ with $h - g \vee 0 - p \nleq 0$, then

$$\begin{aligned} 0 &< (h - g \vee 0 - p) \vee 0 \\ &\leq (h - g \vee 0) \vee 0 \\ &= h \vee g \vee 0 - g \vee 0. \end{aligned}$$

Therefore $h \vee g \neq g$ and $h \nleq g$, so in fact $h - g \vee 0 \in F_1$. Knowing this, for any $p \in P$, we have

$$\iota(a_1)_p = \alpha_{g \vee 0 + p}(a_0) \in J_{F_0 - g\vee 0 - p} \subseteq J_{F_1 - p},$$

proving $\iota(a_1) \in I_{F_1}$. $\square$

To prove Theorem 2.4.2, it will be very helpful to identify B as a direct limit over certain finite subsets of G.

Definition 2.4.4. [18, Section 4.2] Let (G,P) be a lattice ordered abelian group. A subset $F \subseteq G$ is a **grid** if F is finite and closed under $\vee$.

Since any finite subset F of G is contained in a grid, found by appending all joins of finite subsets of F, the set of all grids in G is directed under inclusion and $G = \bigcup\{F \subseteq G \text{ grid}\}$.

Lemma 2.4.5. *The product dilation B is an internal direct limit*

$$B = \overline{\bigcup_{F\subseteq G \text{ grid}} B_F}$$

of C^*-subalgebras

$$B_F := \sum_{g\in F} \beta_{-g}\iota(A).$$

Proof. In fact, for any minimal Nica-covariant automorphic dilation (B,β,G) (Definition 2.3.1), we have

$$B = \overline{\sum_{g\in G} \beta_g \iota(A)} = \overline{\bigcup_{F\subseteq G \text{ grid}} B_F}.$$

The subspaces B_F are always $*$-subalgebras, because all maps involved are $*$-linear, and the multiplication formula

$$\beta_{-g}\iota(a)\beta_{-h}\iota(b) = \beta_{-(g\vee h)}\iota(\alpha_{g\vee h-g}(a)\alpha_{h-g\vee h}(b)),$$

for $g,h \in G$ and $a,b \in A$, implies that B_F is multiplicatively closed when F is $\vee$-closed.

So we need only show each B_F is norm closed, and this is where we use the construction of the product dilation. We will use induction on $|F|$. Certainly $B_\varnothing = \{0\}$ is closed. Fix a nonempty grid $F \subseteq G$, and suppose whenever $F' \subseteq G$ is a grid with $|F'| < |F|$, that $B_{F'} \subseteq B$ is closed. Choose a convergent sequence $x_n \in B_F$, and write

$$x_n = \sum_{g\in F} \beta_{-g}\iota(a_n^g), \quad a_n^g \in A.$$

Since F is finite, F contains a minimal element g_0. By minimality of g_0, we have $[x_n]_{g_0} = a_n^{g_0}$. Then,

$$\|a_n^{g_0} - a_m^{g_0}\| \le \|x_n - x_m\|,$$

so the sequence $a_n^{g_0}$ is Cauchy, and has a limit $a^{g_0} \in A$. Then,

$$y_n := x_n - \beta_{-g_0}\iota(a_n^{g_0}) = \sum_{g \in F \setminus \{g_0\}} \beta_{-g}\iota(a_n^g)$$

is a Cauchy sequence in $B_{F\setminus\{g_0\}}$. As $F \setminus \{g_0\}$ is a grid of smaller size then F, y_n has a limit $y \in B_{F\setminus\{g_0\}} \subseteq B_F$. But then $x_n = \beta_{-g_0}\iota(a_n^{g_0}) + y_n$ converges to

$$\beta_{-g_0}\iota(a^{g_0}) + y \in B_F.$$

So, B_F is closed, finishing the induction. $\square$

The next lemma offloads a technical step in the proof of Theorem 2.4.2. The point is that, when $F \subseteq G$ is any grid, and $a \in A$, the entries of the tuple $\iota(a) \in B \subseteq \prod_G A$ are realized by an element of B_F for "large enough" $g \in G$.

Lemma 2.4.6. *Let $F \subseteq G$ be a grid. Then there are integers $c_g \in \mathbb{Z}$, such that whenever $a \in A$, and $h \geq g$ for at least one element $g \in F$, we have*

$$\iota(a)_h = \alpha_h(a) = \left[\sum_{g \in F} c_g \cdot \beta_{-g}\iota\alpha_g(a)\right]_h.$$

Proof. It will be enough to find integers c_g such that, for any $g \in F$,

$$c_g = 1 - \sum_{\substack{h \in F \\ h < g}} c_h.$$

We can build such c_g recursively. Choose some minimal element $g_0 \in F$, and set $c_{g_0} := 1$. Assuming inductively that $c_{g_0}, \dots, c_{g_n}$ have been defined, so that each g_k is minimal in $F \setminus \{g_0, \dots, g_{k-1}\}$, we can set

$$c_{g_{n+1}} := 1 - \sum_{\substack{h \in F \\ h < g_{n+1}}} c_h.$$

Note that if $h \in F$ and $h < g_{n+1}$, then minimality of g_{n+1} implies that h appears in the list $\{g_0, \dots, g_n\}$, so c_h is defined.

Completing the inductive construction, we find integers c_g, $g \in F$, with

$$\sum_{\substack{h \in F \\ h \leq g}} c_h = 1$$

for any $g \in F$. Now, let

$$x := \sum_{g \in F} c_g \cdot \beta_{-g} \iota \alpha_g(a) \in B_F.$$

Then, whenever h dominates at least one element of F,

$$\begin{aligned} [x]_h &= \sum_{g \in F} c_g \, [\iota \alpha_g(a)]_{h-g} \\ &= \Bigg(\sum_{\substack{g \in F \\ g \leq h}} c_g \Bigg) \alpha_h(a). \end{aligned}$$

Because F is $\vee$-closed, $U := \{g \in F \mid g \leq h\}$ is equal to $\{g \in F \mid g \leq \vee U\}$. So,

$$\sum_{\substack{g \in F \\ g \leq h}} c_g = \sum_{\substack{g \in F \\ g \leq \vee U}} c_g = 1.$$

This shows that $[x]_h = \alpha_h(a) = \iota(a)_h$. Otherwise, h dominates no element of F and $[\beta_{-g} \iota \alpha_g(a)]_h = [\iota \alpha_g(a)]_{h-g} = 0$ for each $g \in F$, so $[x]_h = 0$. $\square$

Proof of Theorem 2.4.2. From Lemma 2.4.3, we already know that I is a β-invariant A-boundary ideal. So, it remains to prove I is maximal among all such ideals. Suppose $R \lhd B$ is the unique maximal β-invariant boundary ideal for A, from Theorem 2.3.14. Then $I \subseteq R$, but we wish to prove $I = R$. Since $B = \overline{\bigcup \{B_F \mid F \text{ grid}\}}$ is a direct limit (Lemma 2.4.5), and ideals in a C*-algebra are inductive, $R \subseteq I$ if and only if $R \cap B_F \subseteq I \cap B_F$ for every grid $F \subseteq G$.

We will prove $R \cap B_F \subseteq I \cap B_F$ for every grid F by induction on $|F|$. This is immediate when $|F| = 0$, since $B_\varnothing = \{0\}$. Suppose now that $|F| > 0$ and that if F' is any grid with $|F'| < |F|$, then $R \cap B_{F'} \subseteq I \cap B_{F'}$. Choose any element

$$x = \sum_{g \in F} \beta_{-g} \iota(a_g) \in R \cap B_F.$$

Pick a minimal element $g_0 \in F$. In fact, since R is β-invariant we are free to translate so that $g_0 = 0$ is minimal in F. By Lemma 2.4.6 applied for a_0 and the grid $F \smallsetminus \{0\}$, we can find an element

$$y = \sum_{g \in F \smallsetminus \{0\}} c_g \cdot \beta_{-g} \iota \alpha_g(a_0)$$

such that if $h \geq g$ for any $g \in F \smallsetminus \{0\}$, then $y_h = \iota(a_0)_h$. Let

$$\begin{aligned} z &:= y + \sum_{g \in F \smallsetminus \{0\}} \beta_{-g} \iota(a_g) \\ &= \sum_{g \in F \smallsetminus \{0\}} \beta_{-g} \iota(a_g + c_g \, \alpha_g(a_0)), \end{aligned}$$

so that $z \in B_{F \smallsetminus \{0\}}$. Whenever $h \in G$ dominates a nonzero element of F, we have $[y]_h = \iota(a_0)_h$ and so

$$[z]_h = \sum_{g \in F} [\beta_{-g} \iota(a_g)]_h = [x]_h.$$

Otherwise, if $h \ngeq g$ for all $g \in F \smallsetminus \{0\}$, then any element $w \in B_{F \smallsetminus \{0\}}$ satisfies $w_h = 0$. Indeed if $g \in F \smallsetminus \{0\}$ and $d \in A$, then $[\beta_g \iota(d)]_h = [\iota(d)]_{h-g} = 0$ because $h - g \ngeq 0$, and any $w \in B_{F \smallsetminus \{0\}}$ is a sum of such terms. So, in this case $[x]_h = \iota(a_0)_h$ and $[z]_h = 0$.

We will show that $x - z$ lies in $I_F \subseteq I \subseteq R$. We have

$$[x - z]_h = \begin{cases} \alpha_h(a_0) & h \geq 0 \text{ and } h \ngeq g \text{ for all } g \in F \smallsetminus \{0\}, \\ 0 & \text{else.} \end{cases} \tag{2.6}$$

So, suppose $p \in P$, with $p \ngeq g$ for all $g \in F \smallsetminus \{0\}$. Let $b \in K_{F-p}$, so that

$$b \in \bigcap_{h \in (F-p) \smallsetminus \{0\}} \ker \alpha_{h \vee 0} = \bigcap_{g \in F \smallsetminus \{p\}} \ker \alpha_{(g-p) \vee 0}.$$

Then it follows from (2.6) that

$$\beta_p(x - z)\iota(b) = \beta_p(x)\iota(b) = \iota(\alpha_p(a_0)b),$$

which, since $x \in R$, and R is β-invariant, lies in $\iota(A) \cap R = \{0\}$. Since ι is injective, $\alpha_p(a_0)b = 0$. This proves $\alpha_p(a_0) = [x - z]_p \in J_{F-p} = K_{F-p}^{\perp}$, so indeed $x - z \in I_F \subseteq I \subseteq R$.

As x and $x - z$ are in R,

$$z = x - (x - z) \in R \cap B_{F \smallsetminus \{0\}}.$$

By inductive hypothesis, since $|F \smallsetminus \{0\}| < |F|$, we conclude $z \in I \cap B_{F \smallsetminus \{0\}}$. Since $x - z \in I$, we find $x = z + (x - z)$ lies in $I \cap B_F$, completing the induction. □

Recall that an ideal I in a C*-algebra A is **essential** if it intersects every nonzero ideal of A, or equivalently if $I^{\perp} = \{0\}$.

Corollary 2.4.7. *Let (G, P) be a lattice ordered abelian group. Let (A, α, P) be a unital C^*-dynamical system, with product dilation (B, β, G). Then the C^*-cover $p_A(B \rtimes_\beta G)p_A$ is the C^*-envelope of $A \times_\alpha^{nc} P$ if and only if for every finite subset $F \subseteq P \setminus \{0\}$,*

$$K_F = \bigcap_{p \in F} \ker \alpha_p$$

is an essential ideal in A.

Proof. With $I \lhd B$ as in Theorem 2.4.2, the product dilation yields the C*-envelope if and only if $I = \{0\}$. But by construction, this occurs if and only if each $I_F = \{0\}$, which occurs if and only if each $K_F^\perp = \{0\}$ for any finite subset $F \subseteq G$, or equivalently any finite $F \subseteq P$. $\square$

2.5 The case $P = \mathbb{Z}_+^n$.

In [18, Theorem 4.3.7], Davidson, Fuller, and Kakariadis identify the C*-envelope of a semicrossed product $A \times_\alpha^{\mathrm{nc}} \mathbb{Z}_+^n$ by $\mathbb{Z}_+^n$ as a full corner of a crossed product by $\mathbb{Z}^n$, when $(A, \alpha, \mathbb{Z}_n^+)$ is a C*-dynamical system. In this section, we show that the C*-dynamical system $(B/I, \tilde{\beta}, \mathbb{Z}^n)$ from Theorem 2.4.2 (in the case $(G, P) = (\mathbb{Z}^n, \mathbb{Z}_+^n)$) is $\mathbb{Z}^n$-equivariantly $*$-isomorphic to the C*-dynamical system constructed in [18, Section 4.3]. It follows that the latter system is a minimal automorphic Nica-covariant dilation of $(A, \alpha, \mathbb{Z}_+^n)$ without nontrivial $\mathbb{Z}^n$-invariant A-boundary ideals, and we recover [18, Theorem 4.3.7] from Corollary 2.3.15.

We now recall the construction in [18, Section 4.3]. Since our notation clashes with the notation in that paper, we must introduce new notation. We write the standard generators in $\mathbb{Z}_n^+$ as $\mathbf{1}, \dots, \mathbf{n}$. Given $x = (x_1, \dots, x_n) \in \mathbb{Z}_+^n$,

$$\operatorname{supp}(x) := \{\mathbf{k} \in \{\mathbf{1}, \dots, \mathbf{n}\} \mid x_k > 0\}.$$

If $x, y \in \mathbb{Z}_+^n$, we write $x \perp y$ if $x \wedge y = 0$, or equivalently $\operatorname{supp}(x) \cap \operatorname{supp}(y) = \varnothing$. Moreover, let $x^\perp := \{y \in \mathbb{Z}_+^n \mid y \perp x\}$. Let $(A, \alpha, \mathbb{Z}_n^+)$ be a C*-dynamical system. For $x \in \mathbb{Z}_n^+$, define ideals

$$Q_x^0 := \left(\bigcap_{\mathbf{i} \in \operatorname{supp}(x)} \ker \alpha_i \right)^\perp \lhd A,$$

and

$$Q_x := \bigcap_{y \in x^\perp} \alpha_y^{-1}(Q_x^0) \subseteq Q_x^0.$$

Form the C*-algebra

$$C := \bigoplus_{x \in \mathbb{Z}_+^n} \frac{A}{Q_x}.$$

Let $q_x : A \to A/Q_x$ be the quotient map. Since $Q_0 = \{0\}$, $\eta := q_0$ is a $*$-monomorphism $A \to C$. For convenience, we notationally identify

$$C = \overline{\sum_{x \in \mathbb{Z}_+^n} \frac{A}{Q_x} \otimes e_x},$$

where e_x are formal generators, as in [18, Section 4.3]. Then $(C, \gamma, \mathbb{Z}_+^n)$ is an injective C*-dynamical system, where the action γ is determined on generators by

$$\gamma_i(q_x(a) \otimes e_x) = \begin{cases} q_x(\alpha_{\mathbf{i}}(a)) \otimes e_x + q_{x+\mathbf{i}}(a) \otimes e_{x+\mathbf{i}} & \mathbf{i} \perp x, \\ q_{x+\mathbf{i}}(a) \otimes e_{x+\mathbf{i}} & \mathbf{i} \in \operatorname{supp}(x). \end{cases}$$

Since $\gamma_i(q_0(a) \otimes e_0) = q_0(\alpha_i(a) \otimes e_0 + q_{\mathbf{i}}(a) \otimes e_{\mathbf{i}}$ has 0'th entry $\alpha_i(a)$, the system $(C, \gamma, \mathbb{Z}_+^n)$ dilates $(A, \alpha, \mathbb{Z}_+^n)$ in the same sense as Definition 2.3.1.

Let $(\tilde{C}, \tilde{\gamma}, \mathbb{Z}^n)$ be the minimal automorphic extension of $(C, \gamma, \mathbb{Z}_+^n)$, from [18, Theorem 4.2.12]. This C*-dynamical system satisfies

$$C \subseteq \tilde{C}, \quad \tilde{\gamma}|_C = \gamma \quad \text{and} \quad C = \overline{\bigcup_{x \in \mathbb{Z}_+^n} \tilde{\gamma}_{-x}(C)}.$$

Then, $(\tilde{C}, \tilde{\gamma}, \mathbb{Z}^n)$ is a minimal Nica-covariant automorphic dilation of $(A, \alpha, \mathbb{Z}_+^n)$. Nica-covariance of this dilation is found in [18, Lemma 4.3.8]. The content of [18, Theorem 4.3.7] is that the natural map $A \times_\alpha^{\mathrm{nc}} \mathbb{Z}_+^n \to \tilde{C} \rtimes_{\tilde{\gamma}} \mathbb{Z}^n$ is completely isometric, and via this map

$$C_e^*(A \times^{\mathrm{nc}} \mathbb{Z}_+^n) = p_0(\tilde{C} \rtimes \mathbb{Z}^n) p_0$$

is a full corner by the projection $p_0 = 1_A \otimes e_0 = \eta(1_A)$.

Proposition 2.5.1. *Let $(A, \alpha, \mathbb{Z}_+^n)$ be a unital C*-dynamical system. Let $(B, \beta, \mathbb{Z}^n)$ be the product dilation (Definition 2.3.4), and $(\tilde{C}, \tilde{\gamma}, \mathbb{Z}^n)$ be the automorphic dilation defined above. Let $I \lhd B$ be the unique maximal β-invariant A-boundary ideal as in Theorem 2.4.2. Then there is a $\mathbb{Z}^n$-equivariant $*$-isomorphism $B/I \cong \tilde{C}$ that fixes A.*

Proof. Define

$$\pi : \sum_{x \in \mathbb{Z}^n} \beta_x \iota(A) \to \tilde{C}$$

to be the unique $*$-linear map satisfying

$$\pi(\beta_x \iota(a)) = \tilde{\gamma}_x \eta(a).$$

Note that π is well defined, because if $F \subseteq G$ is finite, and $a_x \in A$ satisfy

$$b := \sum_{x \in F} \beta_x \iota(a_x) = 0,$$

then each $a_x = 0$, so such a representation is well defined. Indeed, if $x_0 \in F$ is minimal, then $b_{x_0} = a_{x_0} = 0$. By replacing F with $F \setminus \{x_0\}$ and recursing, we eventually find each $a_x = 0$. Since both $(B, \beta, \mathbb{Z}^n)$ and $(\tilde{C}, \tilde{\gamma}, \mathbb{Z}^n)$ are minimal automorphic dilations, π is a $*$-linear map defined on a dense subalgebra with dense range. By construction π is $\mathbb{Z}^n$-equivariant. Nica-covariance of both dilations imply that, if $x, y \in \mathbb{Z}^n$ and $a, b \in A$,

$$\beta_x \iota(a) \beta_y \iota(b) = \beta_{x \wedge y} \iota(\alpha_{x - x \wedge y}(a) \alpha_{y - x \wedge y}(b)),$$

and identically

$$\tilde{\gamma}_x \eta(a) \tilde{\gamma}_y \eta(b) = \tilde{\gamma}_{x \wedge y} \eta(\alpha_{x - x \wedge y}(a) \alpha_{y - x \wedge y}(b)).$$

Extending linearly, it follows that π is a $*$-homomorphism.

We claim π is bounded. Given an element $b = \sum_{x \in F} \beta_g \iota(a_x) \in \sum_g \beta_g \iota(A)$ as above, using [18, Lemma 4.3.6], we find

$$\begin{aligned}
\|\pi(b)\| &= \left\| \sum_{x \in F} \sum_{0 \le y \le x} q_y(\alpha_{x-y}(a_x)) \otimes e_y \right\| \\
&\le \sup_{y \in \mathbb{Z}_+^n} \left\| q_y\Big(\sum_{\substack{x \in F \\ x \ge y}} \alpha_{x-y}(a_x) \Big) \right\| \\
&\le \sup_{y \in \mathbb{Z}^n} \left\| \sum_{\substack{x \in F \\ x \ge y}} \alpha_{x-y}(a_x) \right\|,
\end{aligned}$$

or upon swapping y with $-y$,

$$\begin{aligned}
\|\pi(b)\| &\le \sup_{y \in \mathbb{Z}^n} \left\| \sum_{\substack{x \in F \\ x+y \ge 0}} \alpha_{x+y}(a_x) \right\| \\
&= \left\| \sum_{x \in F} \beta_x \iota(a_x) \right\| = \|b\|.
\end{aligned}$$

So, π is contractive. Therefore, the map π extends uniquely to an equivariant surjective $*$-homomorphism $B \to \tilde{C}$, which we denote by the same symbol. Note also that $\pi\iota = \eta$, so π fixes the respective copies of A.

The result follows if we can prove $\ker\pi = I$. Since π is equivariant and isometric on $\iota(A) \cong \eta(A) \cong A$, $\ker\pi$ is a β-invariant boundary ideal and so $\ker\pi \subseteq I$, by maximality of I.

To show $I \subseteq \ker\pi$, by inductivity of ideals it suffices to prove

$$B_F \cap I_H \subseteq \ker\pi,$$

for any grid $F \subseteq \mathbb{Z}^n$ (Lemma 2.4.5) and any finite subset $H \subseteq \mathbb{Z}^n$. So, it suffices to assume we have an element

$$b = \sum_{x \in F} \beta_x \iota(a_x) \in I_H,$$

where $F \subseteq G$ is finite, and prove $\pi(b) = 0$. In fact, since π is $\mathbb{Z}^n$-equivariant, and $\beta_g(I_H) = I_{H-g}$, we are free to apply β_g for any $g \geq (-\wedge F) \vee (\vee H)$ and so assume $F \subseteq \mathbb{Z}^n_+$ and $H \subseteq -\mathbb{Z}^n_+$. Now, compute

$$\begin{aligned}
\pi(b) &= \sum_{x \in F} \gamma_x \eta(a_x) \\
&= \sum_{y \in \mathbb{Z}^n_+} q_y \Big(\sum_{\substack{x \in F \\ x \geq y}} \alpha_{x-y}(a_x) \Big) \otimes e_y \\
&= \sum_{y \in \mathbb{Z}^n_+} q_y(b_{-y}) \otimes e_y.
\end{aligned}$$

So, we must show $b_{-y} \in Q_y$ for all $y \in \mathbb{Z}^n_+$. Suppose that $z \in \mathbb{Z}^n_+$ with $z \perp y$. Then

$$\begin{aligned}
b_{z-y} &= \sum_{\substack{x \in F \\ x+z \geq y}} \alpha_{x+z-y}(a_x) \\
&= \sum_{\substack{x \in F \\ x \geq y}} \alpha_{x+z-y}(a_x) = \alpha_z(b_{-y}),
\end{aligned}$$

since $z \perp y$ implies that $x + z \geq y$ if and only if $x \geq y$. Because $b \in I_H$, we have

$$\alpha_z(b_{-y}) = b_{z-y} \in J_{H-z+y} = \Bigg(\bigcap_{\substack{h \in H \\ h \nleq z-y}} \ker\alpha_{(h-z+y)\vee 0} \Bigg)^{\perp}.$$

However, we also have

$$\bigcap_{\mathbf{i}\in\operatorname{supp}(y)} \ker\alpha_{\mathbf{i}} \subseteq \bigcap_{\substack{h\in H\\ h\nleq z-y}} \ker\alpha_{(h-z+y)\vee 0}.$$

Indeed, if $h\in H$ with $h-z+y\nleq 0$, then since $h,-z\le 0$,

$$\varnothing \ne \operatorname{supp}((h-z+y)\vee 0) \subseteq \operatorname{supp}(y),$$

so $\ker\alpha_{(h-z+y)\vee 0} \supseteq \ker\alpha_{\mathbf{i}}$ for at least one $\mathbf{i}\in\operatorname{supp}(y)$. Upon taking annihilators, which reverses containment,

$$\alpha_z(b_{-y}) \in \left(\bigcap_{\mathbf{i}\in\operatorname{supp}(y)} \ker\alpha_i\right)^{\perp} = Q_y^0.$$

Therefore $b_{-y}\in Q_y$ for all $y\ge 0$. So, $\pi(b)=0$, proving $I=\ker\pi$. □

Proposition 2.5.1 implies that there is a $*$-isomorphism

$$C_e^*(A\times^{\mathrm{nc}}\mathbb{Z}_+^n) = (p_A+I)\left(\frac{B}{I}\rtimes\mathbb{Z}^n\right)(p_A+I) \cong p_0\left(\tilde{C}\rtimes\mathbb{Z}^n\right)p_0$$

which fixes the respective completely isometric copies of $A\times^{\mathrm{nc}}P$.

2.6 Applications and examples

2.6.1 Simplicity of the C*-envelope

In the commutative case, we can give a dynamical characterization of when the C*-envelope of a Nica-covariant semicrossed product is simple. The following definition is standard.

Definition 2.6.1. A C*-dynamical system (A,α,P) is **minimal** if A contains no nontrivial α-invariant ideals.

Throughout Section 2.6.1, let (G,P) be a lattice-ordered abelian group, and let (A,α,P) be a unital C*-dynamical system. Let (B,β,G) be the associated product dilation, with inclusion $\iota: A\to B$ and unique maximal β-invariant A-boundary ideal I, as in Corollary 2.3.16. The C*-envelope of $A\times_\alpha^{\mathrm{nc}}P$ is a full corner of $(B/I)\rtimes_{\tilde{\beta}}G$. The following result is an analogue of [18, Corollary 4.4.4].

Proposition 2.6.2. *The C*-dynamical system (A, α, P) is minimal if and only if the automorphic C*-dynamical system $(B/I, \tilde{\beta}, G)$ is minimal.*

Proof. Suppose (A, α, P) is minimal. Since P is abelian, for any $p \in P$ the ideal $\ker \alpha_p \lhd A$ is α-invariant and doesn't contain the unit 1_A. By minimality, we must have each $\ker \alpha_p = \{0\}$. Therefore the system (A, α, P) is injective. As in Proposition 2.3.19, the dilation B/I is a minimal automorphic extension of A. By [18, Proposition 4.4.3], it follows that $(B/I, \tilde{\beta}, G)$ is minimal.

Conversely, suppose $(B/I, \tilde{\beta}, G)$ is minimal. Suppose that $J \lhd A$ is a nonzero α-invariant ideal. Let

$$K := \overline{\bigcup_{g \in G} \beta_g \iota(J)} = \overline{\sum_{g \in G} \beta_g \iota(J)}$$

be the β-invariant ideal in the product dilation B generated by $\iota(J)$. Because $K \cap \iota(A) = \iota(J)$ is nonzero, and I is an A-boundary ideal, we must have $K \notin I$, so the ideal

$$\frac{K + I}{I} \lhd B/I$$

is nonzero and $\tilde{\beta}$-invariant. By assumption, we must have $(K + I)/I = B/I$, and therefore $K + I = B$. Because $\iota(J) \subseteq K \subseteq K + I$, the injection ι induces a $*$-monomorphism

$$\frac{A}{J} \to \frac{B}{K + I} \cong \{0\}.$$

Therefore $A/J \cong \{0\}$, so $J = A$. □

Definition 2.6.3. Let φ be an action of a semigroup P by continuous maps on a locally compact Hausdorff space X. Then (X, φ, P) is a **classical system**. The system (X, φ, P) is **minimal** if X contains no proper nonempty closed φ-invariant subsets.

Definitions 2.6.1 and 2.6.3 are equivalent in the commutative setting $A = C_0(X)$, since ideals correspond to closed subsets. In the classical setting, the following dynamical notion is related to simplicity for crossed products.

Definition 2.6.4. Let (X, φ, P) be a classical system. The action φ is **topologically free** if for any $p, q \in P$ with $p \neq q$, the set $\{x \in X \mid \varphi_p(x) = \varphi_q(x)\}$ has empty interior.

Now suppose $A = C(X)$ is a commutative unital C*-algebra. Here X is a compact Hausdorff space. Let (B, β, G) be the associated product dilation, with inclusion $\iota : A \to B$, and unique maximal β-invariant A-boundary ideal I. By Remark 2.3.17, B and B/I are commutative. The C*-dynamical systems (A, α, P) and $(B/I, \tilde{\beta}, G)$ arise from classical systems $(C(X), \varphi, P)$ and $(C_0(Y), \psi, G)$ via the usual duality for commutative C*-algebras. The author is grateful to Evgenios Kakariadis and to the referee for suggesting the following variant of [18, Corollary 4.4.9].

Proposition 2.6.5. *With notation as above, the following are equivalent.*

(i) The system (X, φ, P) is minimal and $\varphi_p \neq \varphi_q$ for all $p, q \in P$ with $p \neq q$.

(ii) The system (Y, ψ, G) is minimal and topologically free.

(iii) The crossed product $C_0(Y) \rtimes_\psi G$ is simple.

(iv) The C-envelope $C_e^*(C(X) \times_\varphi^{nc} P)$ is simple.*

If any of the above hold, then Y is compact, $C(Y)$ is a minimal automorphic extension of $C(X)$, and

$$C_e^*(C(X) \times_\varphi^{nc} P) \cong C(Y) \rtimes_\psi G$$

is a crossed product.

Proof. The equivalence of (ii) and (iii) for an amenable group G is a standard result of Archbold and Spielberg [1, Corollary]. Because $C_e^*(C(\times_\varphi^{\mathrm{nc}} P)$ is a full corner of $C(Y) \rtimes_\psi G$, items (iii) and (iv) are equivalent.

It therefore suffices to prove (i) and (ii) are equivalent. If the system (A, α, P) is injective, then Proposition 2.3.19 implies that $(B/I, \beta, G)$ is a minimal automorphic extension of (A, α, P), and that the C*-envelope is the associated crossed product. In this case, $C_0(Y) \cong B/I$ is unital, so Y is compact. When (A, α, P) is injective, (i) and (ii) are shown to be equivalent in [18, Theorem 4.4.8]. However, minimality of (A, α, P) implies that this system is injective as in Proposition 2.6.2 above. Using Proposition 2.6.2, either (i) or (ii) implies (A, α, P) is minimal, so the result follows. □

If Proposition 2.6.5 holds, then the simplicity of the C*-envelope implies that any proper ideal in a C*-cover of $C(X) \times_\varphi^{\mathrm{nc}} P$ is a $(C(X) \times_\varphi^{\mathrm{nc}} P)$-boundary ideal.

2.6.2 Direct limits of subgroups

Given a lattice ordered abelian group (G, P), we call $H \subseteq G$ a **sub-lattice ordered group** of G if H is a subgroup closed under $\vee$ and $\wedge$. (In fact, the identity $g + h = g \vee h + g \wedge h$ shows that it is enough to assume closure under at least one of $\vee$ or $\wedge$.) For any sub-lattice ordered group, $(H, H \cap P)$ is itself a lattice ordered abelian group. Suppose (A, α, P) is a C*-dynamical system, and set $Q := H \cap P$. By the universal property, there is a natural homomorphism $A \times_{\alpha|_Q}^{\mathrm{nc}} Q \to A \times_{\alpha}^{\mathrm{nc}} P$ induced by the inclusion $P \subseteq Q$. By [18, Theorem 4.2.9], the Fock representation is completely isometric on any Nica-covariant semicrossed product. Suppose A acts faithfully on a Hilbert space K, then $A \times_{\alpha|_Q}^{\mathrm{nc}} Q$ acts faithfully on $K \otimes \ell^2(Q)$. Then the diagram

$$\begin{array}{ccc} A \times_{\alpha}^{\mathrm{nc}} P & \longrightarrow & B(K \otimes \ell^2(P)) \\ \uparrow & & \downarrow \\ A \times_{\alpha|_Q}^{\mathrm{nc}} Q & \longrightarrow & B(K \otimes \ell^2(Q)) \end{array}$$

commutes, where the right-hand map is compression to $K \otimes \ell^2(Q) \subseteq K \otimes \ell^2(P)$. As the bottom map is completely isometric, it follows that the natural map

$$A \times_{\alpha|_Q}^{\mathrm{nc}} Q \to A \times_{\alpha}^{\mathrm{nc}} P$$

is completely isometric. Moreover, if $G = \bigcup_{\lambda \in \Lambda} G_\lambda$ is an internal direct limit of sub-lattice ordered groups $G_\lambda \subseteq G$, then it follows that

$$A \times_{\alpha}^{\mathrm{nc}} P \cong \varinjlim_{\lambda \in \Lambda} A \times_{\alpha|_{P_\lambda}}^{\mathrm{nc}} P_\lambda,$$

is a direct limit. Here, $P_\lambda := G_\lambda \cap P$. Upon identification, we think of

$$A \times_{\alpha}^{\mathrm{nc}} P = \overline{\bigcup_{\lambda \in \Lambda} A \times_{\alpha|_{P_\lambda}}^{\mathrm{nc}} P_\lambda}$$

as an internal direct limit. The next result is that the respective product dilations (Definition 2.3.4) over P_λ embed just as nicely.

Proposition 2.6.6. *Let (G, P) be a lattice ordered abelian group. Let (A, α, P) be a C*-dynamical system, with product dilation (B, β, G).*

(1) Suppose $H \subseteq G$ is a sub-lattice ordered group. Setting $Q = H \cap P$, let (C, γ, H) be the product dilation for $(A, \alpha|_Q, Q)$. Then C embeds into B via an equivariant $*$-monomorphism fixing A.

(2) If $G = \bigcup_{\lambda\in\Lambda} G_\lambda$, for sub-lattice ordered groups G_λ, let $(B_\lambda, \beta_\lambda, G_\lambda)$ be the product dilation for $(A, \alpha_\lambda, P_\lambda)$, where $P_\lambda := G_\lambda \cap P$ and $\alpha_\lambda := \alpha|_{P_\lambda}$. Then up to identification, we have

$$B \cong \overline{\bigcup_{\lambda\in\Lambda} B_\lambda} \cong \varinjlim_{\lambda\in\Lambda} B_\lambda.$$

Proof. As in the proof of Proposition 2.5.1, there is a well defined $*$-homomorphism $\pi : \sum_{g\in G} \beta_g \eta(A) \to B$ with $(\beta_H)_g \eta(a) \mapsto \beta_g \iota(a)$. Here $\iota : A \to B$ and $\eta : A \to C$ are the usual inclusions. Then, (1) follows if we can prove π is isometric. Let

$$b = \sum_{g\in F} \beta_{-g} \iota(a_g),$$

where $F \subseteq H$ is finite and $a_g \in A$. Then

$$\begin{aligned}
\|b\| &= \Big\| \sum_{g\in F} \gamma_{-g} \iota_H(a_g) \Big\| \\
&= \sup_{h\in H} \Big\| \sum_{\substack{g\in F \\ g\le h}} \alpha_{h-g}(a_g) \Big\| \\
&\le \sup_{k\in G} \Big\| \sum_{\substack{g\in F \\ g\le k}} \alpha_{k-g}(a_g) \Big\| = \|\pi(b)\|.
\end{aligned}$$

Conversely, given $k \in G$, since H is $\vee$-closed we have

$$\{g \in F \mid g \le k\} = \{g \in F \mid g \le h\},$$

where $h := \vee\{g \in F \mid g \le k\} \in H$. Then,

$$\begin{aligned}
\|[\pi(b)]_k\| &= \Big\| \sum_{\substack{g\in F \\ g\le k}} \alpha_{k-g}(a_g) \Big\| \\
&= \Big\| \alpha_{k-h} \Big(\sum_{\substack{g\in F \\ g\le h}} \alpha_{h-g}(a_g) \Big) \Big\| \\
&= \|\alpha_{k-h}(b_h)\| \le \|b_h\| \le \|b\|.
\end{aligned}$$

So, $\|\pi(b)\| = \|b\|$ and π extends to a $*$-monomorphism.

For claim (2), it follows from (1) that each B_λ embeds in B. By minimality,

$$B = \overline{\sum_{g\in G} \beta_g \iota(A)} = \overline{\bigcup_{\lambda\in\Lambda} \sum_{g\in G_\lambda} \beta_g \iota(A)} = \overline{\bigcup_{\lambda\in\Lambda} B_\lambda},$$

as claimed. $\square$

Since the embedding in Proposition 2.6.6 is equivariant and fixes the copy of A, and since all groups involved are abelian and so exact, we also get a $*$-embedding

$$B_\lambda \rtimes_{\beta_\lambda} G_\lambda \subseteq B \rtimes_\beta G.$$

This embedding restricts to the natural embedding $A \times^{\mathrm{nc}} P_\lambda \subseteq A \times^{\mathrm{nc}} P$. Moreover

$$B \rtimes_\beta G \cong \overline{\bigcup_{\lambda\in\Lambda} B_\lambda \rtimes_{\beta_\lambda} G_\lambda}$$

is again a direct product. It's then tempting to ask when this result still holds after passing to quotients by Shilov ideals. That is, when is

$$C_e^*(A \times_\alpha^{\mathrm{nc}} P) \cong \varinjlim_{\lambda\in\Lambda} C_e^*(A \times_{\alpha_\lambda}^{\mathrm{nc}} P_\lambda)?$$

This does occur for surjective systems over totally ordered groups.

Proposition 2.6.7. *Let (G, P) be a totally ordered abelian group, and suppose that (A, α, P) is a unital surjective C^*-dynamical system.*

(1) Let $H \subseteq G$ be a subgroup, and set $Q := H \cap P$. Let (B, β, G) (resp. (C, γ, H)) be the product dilation for (A, α, P) (resp. $(A, \alpha|_Q, Q)$). Let I (resp. J) be the unique maximal G-invariant (resp. H-invariant) A-boundary ideal in B (resp. C). After identifying $C \subseteq B$, we have that

$$J = I \cap C.$$

(2) Suppose $G = \bigcup_{\lambda\in\Lambda} G_\lambda$ is a directed limit of subgroups. If (B, β, G) (respectively $(B_\lambda, \beta_\lambda, G_\lambda)$) is the product dilation for (A, α, P) (resp. $(A, \alpha_\lambda, P_\lambda) = (A, \alpha_\lambda, G_\lambda \cap P)$), and $I \lhd B$ and $I_\lambda \lhd B_\lambda$ are the respective unique maximal β or β_λ-invariant A-boundary ideals, then $I_\lambda = I \cap B_\lambda$ and

$$I = \overline{\bigcup_{\lambda\in\Lambda} I_\lambda}.$$

Proof. To prove (1), we use Proposition 2.6.6 to identify $C \subseteq B$. Since I is a G-invariant A-boundary ideal, $I \cap C$ is an H-invariant boundary ideal in C. So, $I \cap B_H \subseteq J$. Conversely, suppose $x \in J$. By Lemma 2.4.5, and inductivity of ideals, it suffices to assume x has the form

$$x = \sum_{g\in F} \beta_{-g}\iota(a_g) \in I_S^{(H)} := \{y \in C \mid y_h \in J_{S-h} \lhd A \text{ for all } h \in H\}$$

for some grid $F \subseteq H \subseteq G$, and some finite subset $S \subseteq H$. We will prove

$$x \in I_S = \{y \in B \mid y_g \in J_{S-g} \text{ for all } g \in G\} \subseteq I.$$

Let $g \in G$, and

$$b \in K_{S-g} = \bigcap_{\substack{s \in S \\ s \nleq g}} \ker \alpha_{(s-g) \vee 0} = \bigcap_{\substack{s \in S \\ s > g}} \ker \alpha_{s-g}.$$

The second equality is where we use the assumption that G is totally ordered. As in the proof of Proposition 2.6.6.(1), we find

$$x_g = \alpha_{g-h}(x_h),$$

where

$$h := \bigvee \{k \in F \cup S \mid k \leq g\} \in H.$$

Since the action α is by surjections, we can write $b = \alpha_{g-h}(c)$ for some $c \in A$. Then because $b \in K_{S-g}$, it follows that

$$c \in \bigcap_{\substack{s \in S \\ s > h}} \ker \alpha_{s-h} = K_{S-h}.$$

Because $x \in I_S^{(H)}$, we conclude $x_h c = 0$, so $x_g b = \alpha_{g-h}(x_h c) = 0$. Thus $x \in I_S \subseteq I$, as needed.

Claim (2) follows because from (1) and the identification $B = \overline{\bigcup_\lambda B_\lambda}$ (Proposition 2.6.6.(2)), because in this case inductivity of ideals implies

$$I = \overline{\bigcup_{\lambda \in \Lambda} I \cap B_\lambda}.$$

But by (1), $I \cap B_\lambda = I_\lambda$. □

Corollary 2.6.8. *Suppose (G, P) is a totally ordered group with $G = \bigcup_{\lambda \in \Lambda} G_\lambda$, for subgroups G_λ. If (A, α, P) is a surjective unital C*-dynamical system, then*

$$C_e^*(A \times_\alpha^{nc} P) \cong \varinjlim_{\lambda \in \Lambda} C_e^*(A \times_{\alpha|_{P_\lambda}}^{nc} P_\lambda),$$

where $P_\lambda = G_\lambda \cap P$.

Corollary 2.6.8 applies to the totally ordered group $(\mathbb{Q}, \mathbb{Q}_+)$, where we can decompose

$$\mathbb{Q} = \bigcup_{n \geq 1} \frac{\mathbb{Z}}{n!}$$

as a direct limit of an increasing sequence of totally ordered subgroups. More generally, it applies to any subgroup of $\mathbb{R}$ which is built as a union of an increasing sequence of cyclic subgroups, such as the dyadic rationals. It is not clear that one can obtain Corollary 2.6.8 in vacuo without the explicit description of the Shilov ideal from Theorem 2.4.2.

The following examples show that the hypotheses of surjectivity or total ordering of G cannot be dropped from Proposition 2.6.7.

Example 2.6.9. Define an action φ of $\mathbb{R}_+$ on $[-1,1]$ by the continuous maps

$$\varphi_x(t) = \begin{cases} t, & x = 0, \\ e^{-x}|t|, & x > 0. \end{cases}$$

Then φ is a semigroup action, which is jointly continuous away from $x = 0 \in \mathbb{R}_+$. This induces an action α of $\mathbb{R}_+$ on $A = C([-1,1])$ by $*$-homomorphisms

$$\alpha_t(f) = f \circ \varphi_t.$$

For any $x > 0$, φ_x is not injective, and so α_x is not surjective. Indeed, for any $f \in C([-1,1])$, $\alpha_x(f)$ is an even function.

Restrict α to get C*-dynamical systems $(A, \alpha, \mathbb{Z}_+)$ and $(A, \alpha, \mathbb{Z}_+/2)$. Build the product dilation $(B, \beta, \mathbb{Z}/2)$ for $(A, \alpha, \mathbb{Z}_+/2)$. By Proposition 2.6.6.(1), we can identify the product dilation for $(A, \alpha, \mathbb{Z}_+)$ as the C*-subalgebra

$$B_1 = \overline{\sum_{n\in\mathbb{Z}} \beta_n \iota(A)}.$$

We will show that the unique maximal β-invariant boundary ideal I_1 for A in $(B_1, \beta, \mathbb{Z})$ is not a subset of the unique maximal boundary ideal $I \lhd B$ in $(B, \beta, \mathbb{Z}/2)$. By Proposition 2.3.20, we have

$$I = \left\{ x \in B \,\middle|\, x_{n/2} \in (\ker \alpha_{1/2})^\perp \text{ for all } n \in \mathbb{Z}, \text{ and } \lim_{n\to\infty} x_n = 0 \right\},$$

and

$$I_1 = \left\{ x \in B_1 \,\middle|\, x_n \in (\ker \alpha_1)^\perp \text{ for all } n \in \mathbb{Z}, \text{ and } \lim_{n\to\infty} x_n = 0 \right\}.$$

Suppose that we had $I_1 \subseteq I$. Then it would follow that

$$\alpha_{1/2}\left((\ker \alpha_1)^\perp\right) \subseteq (\ker \alpha_{1/2})^\perp.$$

To prove this, suppose $a \in (\ker \alpha_1)^\perp$. Then

$$a - \beta_{-1}\iota(\alpha_1(a)) \in I_1,$$

so by assumption $a - \beta_{-1}\iota(\alpha_1(a)) \in I$. Then

$$[a - \beta_{-1}\iota(\alpha_1(a))]_{1/2} = \alpha_{1/2}(a) \in (\ker \alpha_{1/2})^\perp.$$

However, in our case, for $x > 0$,

$$\ker \alpha_x = \left\{ f \in A \mid f|_{[0,e^{-x}]} = 0 \right\}.$$

So,

$$(\ker \alpha_x)^\perp = C_0((0, e^{-x})) = \{ f \in A \mid \operatorname{supp}(f) \subseteq [0, e^{-x}] \}.$$

We certainly cannot have

$$\alpha_{1/2}\left(C_0((0, e^{-1})) \right) \subseteq C_0(0, e^{-1/2}),$$

because $\alpha_{1/2}(f)$ is always an even function and $\alpha_{1/2} \neq 0$. For instance, $f(x) = \max\{x(1 - ex), 0\}$ satisfies

$$f \in C_0(0, e^{-1}) \quad \text{and} \quad \alpha_{1/2}(f) \notin C_0(0, e^{-1/2}),$$

because $\alpha_{1/2}(f)(-e^{-1/2}/2) = f(e^{-1}/2) > 0$. So, we cannot have $I_1 \subseteq I$ and the conclusion in Proposition 2.6.7.(1) fails for the sub-lattice ordered group $\mathbb{Z} \subseteq \mathbb{Z}/2$ when α is not surjective.

Example 2.6.10. Proposition 2.6.7.(1) fails in the case $H = \mathbb{Z} \oplus \{0\} \subseteq \mathbb{Z} \oplus \mathbb{Z} = G$, even for surjective actions. Take any C*-dynamical system $(A, \alpha, \mathbb{Z}_+^2)$. Using the same notation as Proposition 2.6.7, let C and B be the respective product dilations for $(A, \alpha, \mathbb{Z}_+ \oplus \{0\})$ and $(A, \alpha, \mathbb{Z}_+^2)$. Let J and I be the respective unique maximal invariant A-boundary ideals in C and B. As in Proposition 2.6.6.(1), identify $C \subseteq B$. Then, suppose for a contradiction that $J \subseteq I$.

As $H \cong \mathbb{Z}$, Proposition 2.3.20 gives

$$J = \left\{ x \in B \subseteq \prod_{\mathbb{Z}^2} A \;\middle|\; x_{(n,0)} \in (\ker \alpha_1)^\perp \text{ for all } n \in \mathbb{Z}, \text{ and } \lim_{n \to \infty} x_{(n,0)} = 0 \right\}.$$

Therefore, if $a \in (\ker \alpha_1)^\perp$, we have

$$x = \iota(a) - \beta_1^{-1}\iota\alpha_1(a) \in J.$$

Given $\varepsilon > 0$, there is a finite subset $F \subseteq \mathbb{Z}^2$ and an element $y \in I_F$ with $\|x - y\| < \varepsilon$. Since $\{I_F \mid F \subseteq G \text{ finite}\}$ is directed, we are free to enlarge F so that $(1,1) \in F$. Set

$$k = \max\{m \mid (n,m) \in F\}.$$

Then for $j \geq k$, we have

$$\begin{aligned} y_{(0,j)} &\in \Bigg(\bigcap_{\substack{(n,m)\in F \\ (n,m-j)\nleq 0}} \ker \alpha_{(n,m-j)\vee 0} \Bigg)^{\perp} \\ &= \Bigg(\bigcap_{\substack{(n,m)\in F \\ n>0}} \ker \alpha_1^n \Bigg)^{\perp} \subseteq (\ker \alpha_1)^{\perp}, \end{aligned}$$

so

$$\operatorname{dist}(\alpha_2^j(a), (\ker \alpha_1)^{\perp}) \leq \|x - y\| < \varepsilon.$$

This proves that for any commuting unital endomorphisms $\alpha_1, \alpha_2 \in \operatorname{End}(A)$, and any $a \in (\ker \alpha_1)^{\perp}$, that

$$\lim_{j\to\infty} \operatorname{dist}(\alpha_2^j(a), (\ker \alpha_1)^{\perp}) = \lim_{j\to\infty} \|\alpha_2^j(a) + (\ker \alpha_1)^{\perp}\| = 0. \tag{2.7}$$

However, the identity (2.7) fails in general. Let $X = [0,1] \times [0,1]$ and $A = C(X)$. The two injective continuous maps $\varphi_1, \varphi_2 : X \to X$ defined by

$$\varphi_1(s,t) = \left(\frac{s}{2}, t\right), \qquad \varphi_2(s,t) = \left(\frac{s}{2}, \frac{t}{2}\right)$$

commute and define surjective $*$-endomorphisms $\alpha_i \in \operatorname{End}(A)$, where $\alpha_i(f) = f \circ \varphi_i$, for $i = 1, 2$. Then

$$\begin{aligned} &(\ker \alpha_1)^{\perp} = C_0([0,1/2) \times [0,1]), \text{ and} \\ &(\ker \alpha_2)^{\perp} = C_0([0,1/2) \times [0,1/2)). \end{aligned}$$

Pick any $f \in (\ker \alpha_1)^{\perp}$ with $f(s,t) = 1$ whenever $s \in [0,3/8]$. Then we have $\alpha_2^j(f)(3/4,0) = 1$ for any $j \geq 1$. So,

$$\|\alpha_2^j(f) + (\ker \alpha_1)^{\perp}\| = \left\|\alpha_2^j(f)|_{[1/2,1]\times[0,1]}\right\| \geq 1$$

for all j, and (2.7) does not hold. We conclude that Proposition 2.6.7.(1) fails for the surjective system $(A, \alpha, \mathbb{Z}_+^2) = (C(X), \alpha, \mathbb{Z}_+^2)$, with the sub-lattice ordered group $H = \mathbb{Z} \oplus \{0\} \subseteq \mathbb{Z}^2$.

Chapter 3

Jensen's Inequality for separately convex noncommutative functions

3.1 Introduction

Noncommutative convexity is now an exciting and developing toolbox for use in operator algebras and functional analysis. Wittstock [79] introduced the central notion of a matrix convex set. The main idea is that matrix convex sets are graded by matrix levels, and include points at each level. Here, the classical notion of "convex combination" $\sum_i t_i x_i$ is replaced with a "matrix convex combination" $\sum_i \alpha_i^* x_i \alpha_i$, where the "points" x_i are matrices of possibly different sizes, and α_i are rectangular matrices satisfying $\sum_i \alpha_i^* \alpha_i = I$. Matrix convexity is a more natural notion for the study of operator algebras, where the study of structure at all matrix levels via completely positive or completely bounded maps is a central part of the theory. In fact, Webster and Winkler [77] showed that the category of compact matrix convex sets is contravariantly equivalent to the category of operator systems, so matrix convex sets faithfully encode the information of any operator system.

The theory of matrix convex sets contains many noncommutative analogues of classical facts in convexity and Choquet theory. For instance, Effros and Winkler [32] gave noncommutative analogues of the Hahn-Banach Separation Theorem and Bipolar Theorem. A persistent difficulty in matrix convexity was the search for the right notion of extreme point and a working Krein-Milman theorem. A fully realized version of a Krein-Milman type theorem in matrix convexity was given by Webster and Winkler in [77]. Recently, Davidson and Kennedy [26] obtained new results by working in a framework of noncommutative–or "nc", convex sets, obtaining a Krein-Milman theorem and even a noncommutative Choquet-Bishop-De Leeuw Integral Representation Theorem. Their Krein-Milman theorem is stronger in the sense that it requires fewer extreme points, but finding any and all of their "nc extreme points" requires considering infinite matrices.

A key ingredient in Davidson and Kennedy's framework is that one needs to include infinite matrix levels, and we refer to such convex sets as "noncommutative" or "nc" convex sets as opposed to "matrix" convex sets. Closed nc convex sets are determined by their finite levels [26, Proposition 2.2.10], so in a sense the two theories contain the same information. Using the dual equivalence to the category of operator systems, the nc extreme points in the nc convex set of nc states on an operator system S correspond exactly to boundary representations in the sense of Arveson [3]. Through this lens, the long search for sufficiently many boundary representations of S dualizes to a search for sufficiently many nc extreme points. Dritschel and McCullough [31] used maximal dilations to obtain enough boundary representations to produce the C*-envelope. Building on their techniques, Arveson [5] (for separable S) and Davidson and Kennedy [25] (for general S) showed that the boundary representations completely norm S. Then, having the language of nc convexity in place, Davidson and Kennedy's Krein-Milman theorem [26, Theorem 6.4.2] can be viewed as a dual version of [25, Theorem 3.4].

Sufficiently many boundary representations may not exist at finite levels, and so a compact nc convex set may not have enough nc extreme points at finite matrix levels to generate the whole set. For instance, if S is a C*-algebra, then boundary representations are just irreducible representations, and S may have no finite dimensional representations whatsoever. So, in Davidson and Kennedy's framework one is forced to include infinite matrix levels for large classes of duals of operator systems to be able to recover all information.

On matrix or nc convex sets, classical functions are more naturally replaced by noncommutative functions. Usually, one requires an nc function to be graded along matrix levels, to preserve direct sums, and respect similarities either by arbitrary invertible matrices, or just unitary equivalences. The theory of similarity invariant functions parallels complex analysis, because similarity invariant nc functions turn out to be automatically analytic. See [53] for a detailed treatment. Studying the notion of convex nc functions requires selfadjoint-valued functions, so that there is an ordering on the codomain. Because similarities don't preserve selfadjointness, we instead only require our nc functions in this context to be unitarily invariant.

If X is a (classical) compact convex set, any convex function $f : X \to \mathbb{R}$ satisfies *Jensen's inequality*

$$f(\mathrm{bar}(\mu)) \le \int_X f \, d\mu$$

for any probability measure $\mu \in \mathrm{Prob}(X)$. The *barycenter* $\mathrm{bar}(\mu)$ is the unique point in X that satisfies $\varphi(\mathrm{bar}(\mu)) = \int \varphi \, d\mu$ for every affine function $\varphi : X \to \mathbb{C}$. In fact, Jensen's inequality characterizes convexity, because we can take μ to be a convex combination of point masses.

In [26, Section 7], Davidson and Kennedy show that a noncommutative convex function

$$f : K = \bigcup_n K(n) \to \mathcal{M} = \bigcup_n M_n(\mathbb{C})$$

satisfies the Jensen inequality

$$f(\mathrm{bar}(\mu)) \leq \mu(f)$$

whenever μ is a ucp map $C(K) \to M_k$ defined on the C*-algebra $C(K)$ of continuous nc functions on K. Here, the *barycenter* $\mathrm{bar}(\mu)$ of μ is the unique point in the kth matrix level $K(k)$ of K that satisfies $a(\mathrm{bar}(\mu)) = \mu(a)$ for all nc affine functions a on K.

This noncommutative Jensen Inequality sheds some light on classical operator convexity. A function $f : I \to \mathbb{R}$ defined on some interval $I \subseteq \mathbb{R}$ is operator convex if its associated functional calculus defines a convex function, i.e. if

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y)$$

for all $t \in [0,1]$ and all *selfadjoint matrices* x, y with spectrum in I. Hansen and Pedersen [43] demonstrated that operator convexity is equivalent to a noncommutative Jensen inequality. Hansen and Pedersen's characterization can be obtained as a special case of Davidson and Kennedy's nc Jensen Inequality, by restricting to the case where

$$K = \mathrm{MIN}(I) = \{x \in \bigcup_n M_n^{\mathrm{sa}} \mid \sigma(x) \subseteq I\}$$

is the unique minimal compact nc convex set with first level $\mathrm{MIN}(I)(1) = I$, see [17, Section 4]. Operator convexity for multivariate functions is more delicate. For instance, Hansen [44] established a multivariate nc Jensen inequality for an operator convex function f of two variables, but this inequality is more technical and can't be simply obtained from Davidson and Kennedy's inequality by working in $\mathrm{MIN}(I \times J)$, because on this domain an nc function may not be determined entirely by its first level.

3.1.1 Main results

Our main result is a fully noncommutative analogue of the following classical fact (see Proposition 3.4.2). Let $X_1, \ldots, X_d$ be compact convex sets, and let $f : X_1 \times \cdots \times X_d \to \mathbb{R}$

be a *separately convex* function, meaning f is convex as a function in any one variable as long as all other variables are fixed. This is a much weaker assumption than convexity of f. For instance, the function $f(x,y) = xy$ on $\mathbb{R}^2$ is linear and hence convex in each variable separately, but fails to be "jointly" convex in both variables (cf. Example 3.4.1 and its noncommutative version in Example 3.4.5). Then the function f satisfies the Jensen inequality

$$f(\mathrm{bar}(\mu)) \leq \int_{X_1 \times \cdots \times X_d} f \, d\mu$$

for any *product* measure of the form $\mu = \mu_1 \times \cdots \times \mu_d$, where $\mu_i \in \mathrm{Prob}(X_i)$. In fact, this characterizes separate convexity of f.

We are interested in convex nc functions of multiple variables. Therefore, in Section 3.3 we first study nc convex sets of the form $K_1 \times \cdots \times K_d$, where K_i are compact nc convex sets, and the product is taken separately at each matrix level. By the categorical duality between compact nc convex sets, each compact nc convex set K_i corresponds to the operator system $A(K_i)$ of continuous nc affine functions on K_i. Because $K_1 \times \cdots \times K_d$ is the categorical product, it follows from the equivalence of categories that (Proposition 3.3.1)

$$A(K_1 \times \cdots \times K_d) \cong A(K_1) \oplus_1 \cdots \oplus_1 A(K_d)$$

is the categorical coproduct of the associated operator systems. This is the "unital direct sum"

$$S \oplus_1 T = \frac{S \oplus T}{\mathbb{C}((1_S, 0) - (0, 1_T))}$$

constructed by Fritz in [37].

Davidson and Kennedy [26, Section 4.4] showed that if K is a compact nc convex set, the operator system $A(K)$ of continuous nc affine functions generates the C*-algebra $C(K)$ of (point-ultrastrong-$*$) continuous nc functions on K, and that

$$C(K) = C^*_{\max}(A(K))$$

is in fact the maximal C*-algebra. By comparing the right universal properties, it follows (Corollary 3.3.3) that

$$C(K_1 \times \cdots \times K_d) \cong C(K_1) * \cdots * C(K_d)$$

is a free product of the associated maximal C*-algebras, with amalgamation over $\mathbb{C}$. This is evidently a noncommutative analogue of the classical result that $C(X_1 \times \cdots \times X_d) \cong C(X_1) \otimes \cdots \otimes C(X_d)$ for compact Hausdorff spaces $X_1, \ldots, X_d$.

So, in analogy to the classical case, we should expect that a selfadjoint nc function $f : K_1 \times \cdots \times K_d \to \mathcal{M}^{\mathrm{sa}}$ which is *separately nc convex* (Definition 3.4.3) should satisfy an nc Jensen inequality for any ucp map

$$\mu : C(K_1 \times \cdots \times K_d) \cong C(K_1) * \cdots * C(K_d) \to M_k$$

which is a "free product" of ucp maps $\mu_i : C(K_i) \to M_k$. The central difficulty is that the notion of "free product" for ucp maps is not uniquely defined. If A_i, $i \in I$, are unital C*-algebras, then Boca's theorem [10], or its generalized version in [21], gives a standard recipe for how to glue a collection of ucp maps $\mu_i : A_i \to M_k$ to a ucp map

$$\mu : *_{i \in I} A_i \to M_k$$

with $\mu|_{A_i} = \mu_i$. However, such a map is not unique, and many such gluings might exist. Nonetheless, we show f satisfies an nc Jensen inequality for *any* ucp map built from Boca's theorem. We call any ucp map glued together as in the proof of Boca's theorem or the more general construction in [21, Theorem 3.1] a *free product ucp map* (see Definition 3.4.13), and get the following result.

Theorem 3.1.1. *Let $K_1, \ldots, K_d$ be compact nc convex sets, and suppose*

$$f : K_1 \times \cdots \times K_d \to \mathcal{M}^{sa}$$

is a continuous separately nc convex nc function. Suppose

$$\mu : C(K_1 \times \cdots \times K_d) \cong C(K_1) * \cdots * C(K_d) \to M_k$$

is a free product ucp map of any ucp maps $\mu_i : C(K_i) \to M_k$, $i = 1, \ldots, d$. Then f satisfies the Jensen inequality

$$f(bar(\mu)) \leq \mu(f).$$

In fact, free product ucp maps are not the most general class of ucp maps on $C(K_1) * \cdots * C(K_d)$ for which we get a Jensen inequality. Our strongest version of Theorem 3.1.1 is Theorem 3.4.10, which shows that f satisfies the Jensen inequality for any ucp map which satisfies a certain dilation-theoretic analogue of Fubini's theorem. We call such ucp maps "Fubini type" (Definition 3.4.8), and they form a larger family than just maps coming from Boca's theorem. This class is large enough that the Jensen inequality characterizes separate nc convexity of f.

3.1.2 Connection to free probability

In [11], Bożejko, Leinert, and Speicher introduced the notion of a *conditionally free* or *c-free* product of states on a free product $A_1 * \cdots * A_d$ of C*-algebras. Młotowski [62] generalized this definition to include a conditionally free product of ucp maps as follows. Suppose we have an index set I, unital C*-algebras A_i, $i \in I$, and prescribed ucp maps

$$\mu_i : A_i \to M_k, \quad \text{and states}$$
$$\varphi_i : A_i \to \mathbb{C},$$

for $i \in I$. The $(\varphi_i)_{i\in I}$*-conditionally free product* of the ucp maps μ_i is a ucp map

$$\mu : *_{i\in I} A_i \to M_k$$

which satisfies $\mu|_{A_i} = \mu_i$ for each $i \in I$, and whenever $a_1 \cdots a_m \in *_{i\in I} A_i$ is a reduced word (meaning $a_\ell \in A_{j_\ell}$ with $j_1 \neq j_2 \neq \cdots \neq j_m$) that satisfies

$$\varphi_{j_\ell}(a_\ell) = 0$$

for each $\ell = 1, \ldots, m$, then one has the independence rule

$$\mu(a_1 \cdots a_m) = 0.$$

If μ is a conditionally free product for any tuple of states $(\varphi_i)_{i\in I}$, we will simply say μ is a conditionally free or c-free ucp map. We can decompose each A_i as a direct sum $A_i = \ker \varphi_i \oplus \mathbb{C}1_{A_i}$, and the value of μ on any reduced word is recursively determined by μ_i and φ_i for $i \in I$. Thus the $(\varphi_i)_{i\in I}$-c-free product is uniquely determined.

Existence and complete positivity of the $(\varphi_i)_{i\in I}$-c-free product μ follows from Boca's theorem. Indeed, examining [10, Theorem 3.1] or [21, Theorem 3.4] in the case of amalgamation over $\mathbb{C}$ shows that the constructed ucp map on the free product is the unique c-free product ucp map on the unital free product. Since c-free products can be built from Boca's theorem, they are product ucp maps in our definition and so Theorem 3.1.1 in this context gives

Corollary 3.1.2. *Suppose $K_1, \ldots, K_d$ are compact nc convex sets and let $A_i = C(K_i)$, $i = 1, \ldots, d$. Then for any continuous separately nc convex function*

$$f : K_1 \times \cdots \times K_d \to \mathcal{M}^{sa},$$

and any conditional free ucp map

$$\mu : A_1 * \cdots * A_d \to M_k,$$

the Jensen inequality

$$f(bar(\mu)) \leq \mu(f)$$

holds.

Note that Corollary 3.1.2 applies exactly to those unital C*-algebras A_i which are of the form $A_i = C^*_{\max}(S_i)$ for any operator systems S_i. In this case, we may assume $K_i = \mathcal{S}(S_i)$ is the nc state space $\bigcup_n \mathrm{UCP}(S_i, M_n)$. For example, the result applies to commutative C*-algebras of the form $C(X_i)$, where $X_i \subseteq \mathbb{R}^m$ are simplices [38, Theorem 4.7].

As an application of Corollary 3.1.2, we obtain some operator inequalities for conditionally free ucp maps on free semicircular families. For instance, let a and b be free semicircular elements in a C*-probability space (A, φ), where φ is faithful and tracial. Let $S = \mathrm{span}\,\{1_A, a\}$ and $T = \mathrm{span}\,\{1_A, b\}$ be the operator systems they generate. Then because the spectra $\sigma(a)$ and $\sigma(b)$ are closed intervals, the continuous functional calculus implies that

$$C^*(a) \cong C^*_{\max}(S) \quad \text{and} \quad C^*(b) \cong C^*_{\max}(T).$$

Therefore Corollary 3.1.2 applies to

$$C^*(a, b) \cong C^*(a) * C^*(b).$$

With this identification, elements such as $ab+ba$ or ab^2a correspond to separately nc convex functions. Consequently, if $\mu : C^*(a) * C^*(b) \to B(H)$ is a c-free ucp map, or a ucp map built from Boca's theorem, in Example 3.5.2 we obtain the operator inequalities

$$\begin{aligned} \mu(a)\mu(b) + \mu(b)\mu(a) \leq \mu(ab + ba) \quad \text{and} \\ \mu(a)\mu(b)^2\mu(a) \leq \mu(ab^2a). \end{aligned}$$

More generally, if $a_1, \ldots, a_k$ is any free semicircular family in (A, φ), and μ is a c-free ucp map, we show (Corollary 3.5.3) that

$$\begin{aligned} \mu(a_1)\cdots\mu(a_k) + \mu(a_k)\cdots\mu(a_1) \leq \mu(a_1\cdots a_k + a_k\cdots a_1), \quad \text{and} \\ \mu(a_1)\cdots\mu(a_{k-1})\mu(a_k)^2\mu(a_{k-1})\cdots\mu(a_1) \leq \mu(a_1\cdots a_{k-1}a_k^2a_{k-1}\cdots a_1). \end{aligned}$$

The appearance of conditional freeness in Corollary 3.1.2 suggests that some analogue of free independence for ucp maps may play a role in our main Theorem 3.4.10.

Question 3.1.3. *Is the class of ucp maps $C(K_1) * \cdots * C(K_d) \to M_k$ for which a noncommutative Jensen inequality for separately nc convex functions holds described by some free independence condition? In the language of Section 3.4.2, are ucp maps of Fubini type, or free product ucp maps, characterized by some generalized free independence condition?*

Question 3.1.3 has a positive answer for states, in which case $k = 1$ and $M_k = \mathbb{C}$. Indeed it is straightforward to check that if

$$\varphi : *_{i\in I} A_i \to \mathbb{C}$$

is a state which is a free product ucp map (Definition 3.4.13), then the C*-subalgebras A_i are freely independent, and this occurs if and only if φ is the unique free product of the states $\varphi_i := \varphi|_{A_i}$ built from Boca's theorem or by [8, Proposition 1.1].

3.2 Background

3.2.1 Noncommutative convexity

Throughout, we work in the framework of nc convexity developed by Davidson and Kennedy in [26]. Because their results are still fairly novel, we devote a larger-than-normal portion of this section to an exposition of the main results of their paper that play a role here.

Given an operator system E, we let

$$\mathcal{M}(E) = \coprod_{n\le\kappa} M_n(E),$$

where κ is any fixed sufficiently large cardinal greater than the density character of E. In the special case where E is separable, usually $\kappa = \aleph_0$. When $E = \mathbb{C}$, we write

$$\mathcal{M} := \mathcal{M}(E)$$

for simplicity. Moreover, we define

$$\mathcal{M}^{\mathrm{sa}} := \bigcup_{n\le\kappa} M_n^{\mathrm{sa}} \quad \text{and} \quad \mathcal{M}^d := \mathcal{M}(\mathbb{C}^d),$$

and in the latter we freely identify an element of $M_n(\mathbb{C}^d)$ with a d-tuple of matrices in M_n in the natural way. The key difference from the existing theory of matrix convex sets is that we allow for infinite matrix levels, i.e. we consider all cardinals $n \le \kappa$, with the convention that $M_n(E) \cong M_n \otimes_{\min} E$. Note that when $E = \mathbb{C}$, by convention we have $M_n := M_n(\mathbb{C}) = B(H_n)$ for any Hilbert space H_n of dimension n.

If E is an operator system, we call a subset

$$K \subseteq \mathcal{M}(E)$$

an *nc convex set* if it is closed under direct sums and compression by isometries. Equivalently, given a bounded collection of $x_i \in K(n_i)$ for some index set $i \in I$, and matrices $\alpha_i \in M_{n_i,n}$ satisfying

$$\sum_{i \in I} \alpha_i^* \alpha_i = I_n,$$

where the series converges weak-$*$, the "nc convex combination"

$$\sum_{i \in I} \alpha_i^* x_i \alpha_i$$

is also in K. The nth *matrix level* of K is

$$K(n) \coloneqq K \cap M_n(E).$$

If K is nonempty and nc convex, then each level $K(n)$ is nonempty. Note that we require closure under infinite sums to ensure, for instance, that if $x \in K(n)$ for some finite level $n < \infty$, then the infinite amplification

$$1_{\aleph_0} \otimes x = \begin{pmatrix} x & 0 & \\ 0 & x & \\ & & \ddots \end{pmatrix}$$

(as a block matrix with $n \times n$ blocks) lies in $K(\aleph_0)$. If $E = (E_*)^*$ is a dual operator system, then we may identify

$$M_n(E) = M_n(\mathrm{CB}(E_*, \mathbb{C})) \cong \mathrm{CB}(E_*, M_n) \subseteq B(E_*, M_n),$$

which has a standard weak-$*$ topology. Hence $M_n(E)$ has an induced weak-$*$ topology, with agrees with the topology of pointwise-weak-$*$ convergence on $B(E_*, M_n)$ on bounded subsets. When $E = \mathbb{C}$, this is just the usual weak-$*$ topology on each level $\mathcal{M}(n) = M_n = B(H_n)$ of $\mathcal{M}$. If E is a dual operator space, we say that K is a *compact nc convex set* if it is nc convex and each level $K(n) \subseteq M_n(E)$ is compact in the weak-$*$ topology.

Given nc convex sets K and L, a function $f : K \to L$ is called an *nc function* if f is graded (i.e. $f(K(n)) \subseteq L(n)$), preserves unitary equivalence, and preserves direct sums. If K and L are compact nc convex sets, we say f is continuous if it is weak-$*$ continuous on each level of K. Moreover f is *nc affine* if it also preserves compressions, so if $x \in K(n)$, and $\alpha \in M_{n,k}$ is an isometry, then

$$f(\alpha^* x \alpha) = \alpha^* f(x) \alpha.$$

Compact nc convex sets are a noncommutative analogue of classical compact convex sets (in locally convex spaces). In the classical case, given a compact convex set X, the space $A(X)$ of continuous affine functions $X \to \mathbb{C}$ forms a function system. Kadison's Representation Theorem [48] shows that the functor $A : x \mapsto A(X)$ is an equivalence of categories between the category of compact convex sets and the category of function systems. The essential inverse functor is $F \mapsto \mathcal{S}(F)$, where $\mathcal{S}(F)$ is the state space of F, and so the map $C \to \mathcal{S}(A(C))$ which embeds C as point evaluations is a natural isomorphism.

In the noncommutative setting, operator systems are the correct analogue of function systems. Given a compact nc convex sets K and L, we form the space

$$A(K, L) = \{a : K \to L \mid a \text{ nc affine}\}$$

of nc affine functions with values in $L = \bigcup_n L_n$. When $L = \mathcal{M}$, we set $A(K) := A(K, \mathcal{M})$. The space $A(K)$ is an operator system, with $*$-structure

$$a^*(x) := a(x)^*.$$

The matrix order unit is the "constant function"

$$1_{A(K)}(x) = 1_{M_n}, \quad x \in K(n).$$

The matrix order structure on $M_n(A(K)) \cong A(K, \mathcal{M}(M_n))$ is pointwise. The functor $K \mapsto A(K)$ implements an equivalence of categories between the category of compact nc convex sets, with continuous nc affine maps as morphisms, and the category of operator systems, with ucp maps as morphisms [26, Theorem 3.2.5]. The essential inverse takes an operator system S to the nc state space

$$\mathcal{S}(S) \subseteq \mathcal{M}(S^*)$$

whose nth level is

$$\mathcal{S}(S)(n) = \{\mu : S \to M_n \mid \mu \text{ ucp}\}.$$

In particular $\mathcal{S}(A(K)) \cong K$ naturally, and the isomorphism means that every ucp map $A(K) \to M_n$ is of the form $f \mapsto f(x)$, for some $x \in K(n)$.

Given an operator system S, the maximal C*-algebra $C^*_{\max}(S)$ satisfies the following universal property. We have an embedding

$$S \subseteq C^*_{\max}(S) = C^*(S),$$

and for any unital complete order embedding $\iota : S \to B(H)$, there is a unique $*$-homomorphism $\pi : C^*_{\max}(S) \to B(H)$ with $\pi|_S = \iota$. In the classical setting, for a compact convex set X, one has $C^*_{\max}(A(X)) \cong C(X)$, via the usual inclusion $A(X) \subseteq C(X)$. Let K be a compact nc convex set. Let $B(K)$ denote the C*-algebra of bounded nc functions $K \to \mathcal{M}$, where the C*-operations are pointwise. Then $A(K) \subseteq B(K)$, and we set

$$C(K) := C^*(A(K)) \subseteq B(K).$$

Davidson and Kennedy demonstrated a noncommutative analogue of the classical result $C^*_{\max}(A(X)) \cong C(X)$ in [26, Theorem 4.4.3]. The C*-algebra $C(K)$ is both

- the C*-algebra of all bounded nc functions $K \to \mathcal{M}$ which are continuous levelwise in the point-ultrastrong-$*$ topology on each $K(n) \subseteq B(E_*, M_n)$, and
- the maximal C*-algebra $C^*_{\max}(A(K))$, with the usual inclusion $A(K) \subseteq C(K)$.

By the universal property and categorical duality, any $*$-homomorphism $\pi : C(K) \to M_n$ is of the form $\pi = \delta_x : f \mapsto f(x)$, for some $x \in K(n)$. Then, Stinespring's theorem implies that any ucp map $\mu : C(K) \to M_n$ has the form

$$\mu = \alpha^* \delta_x \alpha : f \mapsto \alpha^* f(x) \alpha,$$

for some $x \in K_k$ and isometry $\alpha \in M_{k,n}$.

On a classical compact convex set X, a function $f : X \to \mathbb{R}$ is *convex* if for all $x_1, \dots, x_n \in X$ and $t_1, \dots, t_n \in [0,1]$ with $\sum_{k=1}^n t_k = 1$, we have

$$f\Big(\sum_{k=1}^n t_k x_k\Big) \leq \sum_{k=1}^n t_k f(x_k).$$

If f is continuous, then f is convex if and only if f satisfies *Jensen's inequality*

$$f(\mathrm{bar}(\mu)) \leq \int f \, d\mu$$

for all Radon probability measures $\mu \in \mathrm{Prob}(X)$. Here, the *barycenter* $\mathrm{bar}(\mu)$ is the unique point in X such that $a(\mathrm{bar}(\mu)) = \int a \, d\mu$, which exists by Kadison duality.

In the noncommutative case, a selfadjoint nc function $f : K \to \mathcal{M}^{\mathrm{sa}}$ on a compact nc convex set K is *nc convex* if whenever $x_i \in K(n_i)$ and $\alpha_i \in M_{n_i,n}$ with $\sum_i \alpha_i^* \alpha_i = I_n$, we have

$$f\left(\sum_{i \in I} \alpha_i^* x_i \alpha_i\right) \leq \sum_{i \in I} \alpha_i^* f(x_i) \alpha_i.$$

Since f is a continuous nc function, it automatically preserves direct sums, and so it is equivalent to simply require

$$f(\alpha^* x\alpha) \leq \alpha^* f(x)\alpha \tag{3.1}$$

whenever $x \in K(k)$ and $\alpha \in M_{k,n}$ is an isometry. Consequently, a continuous bounded nc function $f \in C(K)^{\mathrm{sa}}$ is nc convex if and only if it satisfies the *nc Jensen inequality*

$$f(\mathrm{bar}(\mu)) \leq \mu(f) \tag{3.2}$$

for all ucp maps $\mu : C(K) \to M_n$. Here the *barycenter* of a ucp map $\mu : C(K) \to M_n$ is the unique point $\mathrm{bar}(\mu) \in K(n)$ such that $\mu(a) = a(\mathrm{bar}(\mu))$ for all $a \in A(K) \subseteq C(K)$. The nc Jensen inequality above follows directly from (3.1) together with the observation that any ucp map $\mu : C(K) \to M_n$ must be of the form $\mu = \alpha^* \delta_x \alpha$ for a point $x \in K$ and isometry α, and in this case $\mathrm{bar}(\mu) = \alpha^* x\alpha \in K(n)$. In fact [26, Theorem 7.6.1] applies even to matrix-valued bounded nc functions $f : K \to \mathcal{M}(M_k)^{\mathrm{sa}}$ which are *lower semicontinuous* in the sense that their *nc epigraph*

$$\mathrm{epi}(f) = \bigcup_{n \leq \kappa} \{(x, y) \in K(n) \times M_k(M_n) \mid y \geq f(x)\}$$

is levelwise weak-$*$ closed, see [26, Theorem 7.6.1].

3.2.2 Minimal nc convex sets

If $X \subseteq \mathbb{C}^d$ is a compact convex set, there is a minimal compact nc convex set $K = \mathrm{MIN}(X) \subseteq \mathcal{M}^d$ with $K(1) = X$ [17, Definition 4.1]. A tuple $x = (x_1, \ldots, x_d) \in \mathcal{M}^d(n) = M_n^d$ lies in $\mathrm{MIN}(X)(n)$ if and only if there is a normal tuple $n = (n_1, \ldots, n_d) \in \mathcal{M}^d$ which dilates x and has joint spectrum

$$\sigma(n) \subseteq X.$$

In particular, if $d = 1$ and $X = [a, b]$ is an interval, then

$$\mathrm{MIN}([a, b]) = \{x \in \mathcal{M}^{\mathrm{sa}} \mid \sigma(x) \subseteq [a, b]\}.$$

For general $X \subseteq \mathbb{C}^d$, if $a \in A(K)$ with first level $f := a|_{K(1)} = a|_X$, because a preserves unitaries and direct sums, an application of the spectral theorem shows that

$$a(n_1, \ldots, n_d) = f(n_1, \ldots, n_d),$$

in the sense of the functional calculus, for every normal tuple $n = (n_1, \ldots, n_d)$ with joint spectrum $\sigma(n) \subseteq X$. Because a preserves compressions, and every $x \in \mathrm{MIN}(I)$ is a compression of a normal tuple, the nc affine function a is determined by its restriction to the first level. Consequently $A(\mathrm{MIN}(X)) \cong A(X)$ via the isomorphism that restricts to the first level.

3.2.3 Dilations and notation

If E is an operator space, $x \in M_n(E)$, and $y \in M_k(E)$, we say that y *dilates* x if there is an isometry $\alpha \in M_{k,n}$ such that $\alpha^* y \alpha = x$. In this case we write, $x \prec_\alpha y$, or just $x \prec y$ if the associated isometry is clear.

If S is an operator system and $\mu : S \to B(H)$, $\nu : S \to B(K)$ are ucp maps, then we say ν dilates μ if $\alpha^* \nu \alpha = \mu$, for some isometry $\alpha : H \to K$. Usually up to a unitary we assume α is just an inclusion map $H \subseteq K$. This is really the same perspective as above, where $E = S^*$ is the dual operator space and there is a standard identification

$$M_n(E) \cong \mathrm{CB}(S, M_n),$$

where for infinite cardinals n we take $M_n = B(H_n)$ for an n-dimensional Hilbert space H_n, and identify

$$H \cong H_{\dim H} \quad \text{and} \quad K \cong H_{\dim K}$$

up to some fixed hidden unitary. Since our nc functions are always unitarily equivariant, these hidden unitaries are harmless, so we may freely switch between working with M_n and $B(H)$ as long as $\dim H = n$.

A dilation $x \prec y$ is *trivial* if $y \cong x \oplus z$ with respect to the range of α, or equivalently $\alpha\alpha^* y = y \alpha\alpha^*$. For ucp maps $\mu : S \to B(H)$ and $\nu : S \to B(K)$, with $H \supseteq K$ and $\mu \prec \nu$, the dilation $\mu \prec \nu$ is trivial if and only if H is invariant/reducing for $\nu(S)$.

3.2.4 Free products

If A_i, $i \in I$ are unital C*-algebras, we denote their unital free product C*-algebra by

$$\star_{i \in I} A_i,$$

or, if $I = \{1, \ldots, d\}$, by $A_1 \star \cdots \star A_d$. Here, we amalgamate only over the subalgebras $\mathbb{C} \cong \mathbb{C}1_{A_i} \subseteq A_i$. For convenience, we freely identify each C*-algebra A_j as a literal subalgebra $A_j \subseteq \star_{i \in I} A_i$ of the free product.

3.3 Products of nc convex sets

Suppose $K_1 \subseteq \mathcal{M}(E_1)$ and $K_2 \subseteq \mathcal{M}(E_2)$ are compact nc convex sets, where $E_i = (E_{i,*})^*$ are dual operator systems. As in [26], compactness is meant levelwise in the weak-$*$ topology.

The Cartesian product

$$K_1 \times K_2 := \coprod_n (K_1)(n) \times (K_2)(n) \subseteq \mathcal{M}(E_1 \times E_2)$$

is also an nc convex set. By convention $E_1 \times E_2$ is the usual ℓ^∞-product of operator spaces. We have the standard operator space duality [70, Section 2.6]

$$E_1 \times E_2 = ((E_1)_* \times_1 (E_2)_*)^*$$

and the corresponding weak-$*$ topology agrees with the product topology on $K_1 \times K_2$. Hence $K_1 \times K_2$ is a compact nc convex set when given the product topology. It is straightforward to verify that $E_1 \times E_2$ is the categorical product of E_1 and E_2 in the category of compact nc convex sets with continuous nc affine maps as morphisms.

Davidson and Kennedy [26, Theorem 3.2.5] showed that the functor $K \mapsto A(K)$ implements an equivalence of categories between this category of compact nc convex sets and the category of operator systems with ucp maps as morphisms. Fritz [37, Proposition 3.3] showed that the categorical coproduct in the category of operator systems S, T is the *unital direct sum*

$$S \oplus_1 T := \frac{S \times T}{\mathbb{C}((1_S, 0) - (0, 1_T))}.$$

Here, we naturally identify

$$M_n(S \oplus_1 T) \cong \frac{M_n(S) \times M_n(T)}{M_n(\mathbb{C}((1_S, 0) - (0, 1_T)))}.$$

Write the coset of a pair $(s, t) \in M_n(S) \times M_n(T)$ in $M_n(S \oplus_1 T)$ as $s \oplus_1 t$. The matrix order structure is determined by declaring

$$s \oplus_1 t \geq 0 \iff s - \lambda \geq 0 \text{ and } t + \lambda \geq 0 \text{ for some } \lambda \in M_n(\mathbb{C}).$$

Proposition 3.3.1. *Let $K_1 \subseteq \mathcal{M}(E_1)$ and $K_2 \subseteq \mathcal{M}(E_2)$ be compact nc convex sets. Then there is a natural complete order isomorphism*

$$A(K_1 \times K_2) \cong A(K_1) \oplus_1 A(K_2).$$

Here $a \oplus_1 b \in A(K_1) \oplus_1 A(K_2)$ corresponds to the continuous nc affine function

$$(a \oplus_1 b)(x, y) = a(x) + b(y), \quad (x, y) \in K_1 \times K_2.$$

Proof. By [26, Theorem 3.2.5], the functor $K \mapsto A(K)$ is a contravariant equivalence of categories. Let $\pi_i : K_1 \times K_2 \to K_i$ be the usual projection. Since the diagram

$$\begin{tikzcd}[row sep=small] & K_1 \\ K_1 \times K_2 \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ & K_2 \end{tikzcd}$$

is a categorical product in the category of compact nc convex sets, the diagram

$$\begin{tikzcd}[row sep=small] A(K_1) \arrow[dr, "\epsilon_1"] & \\ & A(K_1 \times K_2) \\ A(K_2) \arrow[ur, "\epsilon_2"'] & \end{tikzcd}$$

is a coproduct of operator systems, where $\epsilon_i(a) = a \circ \pi_i$. Since the coproduct $A(K_1) \oplus_1 A(K_2)$ is unique up to isomorphism, the induced map

$$\epsilon_1 \oplus_1 \epsilon_2 : A(K_1) \oplus_1 A(K_2) \to A(K_1 \times K_2)$$

given by $(\epsilon_1 \oplus \epsilon_2)(a, b) = \epsilon_1(a) + \epsilon_2(b)$ is an isomorphism. $\square$

For a compact nc convex set K, recall that $C(K) = C^*_{\max}(A(K))$ is the maximal C*-algebra generated by the operator system $A(K)$. Since $A(K_1 \times K_2) = A(K_1) \oplus_1 A(K_2)$ is a coproduct of operator systems, it is natural to expect that

$$C(K_1 \times K_2) \cong C^*_{\max}(A(K_1) \oplus_1 A(K_2))$$

is itself a coproduct in the category of unital C*-algebras, with unital $*$-homomorphisms as morphisms. Indeed this is the case. Here, the coproduct of unital C*-algebras A and B is the unital free product $A * B$ with amalgamation over $\mathbb{C} \cong \mathbb{C}1_A \cong \mathbb{C}1_B$.

Proposition 3.3.2. *Let S and T be operator systems. Then*

$$C^*_{max}(S \oplus_1 T) \cong C^*_{max}(S) * C^*_{max}(T)$$

naturally. The $*$-isomorphism $C^*_{max}(S) * C^*_{max}(T) \to C^*_{max}(S \oplus_1 T)$ is induced by the $*$-monomorphisms

$$\iota_S : C^*_{max}(S) \to C^*_{max}(S \oplus_1 T), \quad \iota_T : C^*_{max}(T) \to C^*_{max}(S \oplus_1 T),$$

which are themselves induced by the natural complete order embeddings $\iota_S : S \to S \oplus_1 T$ and $\iota_T : T \to S \oplus_1 T$.

Proof. It suffices to show that

$$\begin{tikzcd} C^*_{\max}(S) \arrow[dr, "\iota_S"] & \\ & C^*_{\max}(S \oplus_1 T) \\ C^*_{\max}(T) \arrow[ur, "\iota_T"'] & \end{tikzcd}$$

is a coproduct in the category of unital C*-algebras, and then the natural isomorphism $C^*_{\max}(S \oplus_1 T) \cong C^*_{\max}(S) * C^*_{\max}(T)$ follows by uniqueness of coproducts up to isomorphism.

Suppose $A \subseteq B(H)$ is a C*-algebra and we have $*$-homomorphisms $\pi_S : C^*_{\max}(S) \to A$ and $\pi_T : C^*_{\max}(T) \to A$. Set $\varphi_S = \pi_S|_S$ and $\varphi_T = \pi_T|_T$, which are ucp maps $S \to A$ and $T \to A$, respectively. By the universal property, these induce a ucp map $\varphi : S \oplus_1 T \to A$ with $\varphi\iota_S = \varphi_S$ and $\varphi\iota_T = \varphi_T$. The ucp map φ induces a $*$-homomorphism

$$\pi : C^*_{\max}(S \oplus_1 T) \to B(H)$$

with $\pi|_{S\oplus_1 T} = \varphi$. Because $S \oplus_1 T$ generates $C^*_{\max}(S \oplus_1 T)$, and $\varphi(S \oplus_1 T) \subseteq A$, we in fact have

$$\pi(C^*_{\max}(S \oplus_1 T)) = C^*(\varphi(S \oplus_1 T)) \subseteq A.$$

Since $\iota_S(C^*_{\max}(S)) = C^*(\iota_S(S))$ is generated by $\iota_S(S) = S \oplus_1 0$, and $\pi\iota_S|_S = \varphi\iota_S = \varphi_S = \pi_S|_S$, we have $\pi\iota_S = \pi_S$. Identically, we find $\pi\iota_T = \pi_T$. Since the $*$-homomorphism π is determined by its action on the generating set $S \oplus_1 T = \iota_S(S) + \iota_T(T)$, it follows that π is unique. □

Corollary 3.3.3. *Let $K_1 \subseteq \mathcal{M}(E_1)$ and $K_2 \subseteq \mathcal{M}(E_2)$ be compact nc convex sets. Then*

$$C(K_1 \times K_2) \cong C(K_1) * C(K_2)$$

naturally via the isomorphism $\epsilon_1 * \epsilon_2 : C(K_1) * C(K_2) \to C(K_1 \times K_2)$ which satisfies

$$\epsilon_1(f)(x,y) = f(x) \text{ and } \epsilon_2(g)(x,y) = g(y)$$

for $f \in C(K_1)$ and $g \in C(K_2)$.

Proof. This follows from combining Proposition 3.3.1 with Proposition 3.3.2. Note that each inclusion

$$A(K_i) \to A(K_1) \oplus_1 A(K_2) \cong A(K_1 \times K_2)$$

is exactly the map ϵ_i, which clearly extends to a $*$-homomorphism $C(K_i) \to C(K_1 \times K_2)$. Thus when identifying $C^*_{\max}(A(K_i)) \cong C(K_i)$, in the notation of Proposition 3.3.2 we must have $\iota_{A(K_i)} = \epsilon_i$, so the isomorphism is implemented by

$$\iota_{A(K_1)} * \iota_{A(K_2)} = \epsilon_1 * \epsilon_2.$$

□

Remark 3.3.4. Propositions 3.3.1 and 3.3.2, and Corollary 3.3.3 all extend immediately to finitely many variables. Since categorical products and coproducts such as $\times$, $\oplus_1$, and $*$ are all associative up to natural isomorphism, a straightforward induction shows that for any $d \in \mathbb{N}$, we have natural isomorphisms

$$\begin{aligned} A(K_1 \times \cdots \times K_d), &\cong A(K_1) \oplus_1 \cdots \oplus_1 A(K_d) \\ C^*_{\max}(S_1 \oplus_1 \cdots \oplus_1 S_d) &\cong C^*_{\max}(S_1) * \cdots * C^*_{\max}(S_d), \text{ and} \\ C(K_1 \times \cdots \times K_d) &\cong C(K_1) * \cdots * C(K_d), \end{aligned}$$

whenever $K_1, \ldots, K_d$ are compact nc convex sets and $S_1, \ldots, S_d$ are operator systems.

3.4 Jensen's Inequality for separately nc convex functions

3.4.1 The commutative case

Let $X_1, \ldots, X_d$ be (classical) compact convex sets. A function $f : X_1 \times \cdots \times X_d \to \mathbb{R}$ is separately convex if it is convex in each variable separately. That is, if

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y)$$

whenever the points $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ in $X_1 \times \cdots \times X_d$ differ in at most one coordinate.

Example 3.4.1. The function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = xy$ is separately convex but not convex. Indeed, it's affine in each variable, yet for example we find

$$f(1/2, -1/2) = -1/4 > -1/2 = \frac{f(1,-1)}{2} + \frac{f(0,0)}{2}.$$

Separately convex functions satisfy Jensen's inequality for product measures. The following result is classical, but we include a proof for completeness, and because the use of Fubini's theorem motivates our approach in the noncommutative case.

Proposition 3.4.2. *Let $X_1, \ldots, X_d$ be compact convex sets, and let $f : X_1 \times \cdots \times X_d \to \mathbb{R}$ be a continuous function. Then f is separately convex if and only if*

$$f(\mathit{bar}(\mu)) \leq \int_{X_1 \times \cdots \times X_d} f \, d\mu$$

for all product measures $\mu = \mu_1 \times \cdots \times \mu_d$, where $\mu_k \in \mathit{Prob}(X_k)$ for each $1 \leq k \leq d$.

Proof. For simplicity we will prove only the case $d = 2$. Suppose f satisfies $f(\mathrm{bar}(\mu)) \le \mu(f)$ for any product of probability measures. Suppose $x, y \in X_1$, $t \in [0,1]$, and $z \in X_2$. Define the product measure

$$\mu := ((1-t)\delta_x + t\delta_y) \times \delta_z$$

Then

$$\begin{aligned} f((1-t)x + ty, z) &= f(\mathrm{bar}(\mu)) \le \mu(f) \\ &= (1-t)f(x,z) + tf(y,z). \end{aligned}$$

Therefore the function f is convex in its first argument, and a symmetrical argument works in the second argument.

Conversely, suppose f is separately convex, and let $\mu = \mu_1 \times \mu_2$ be a product of probability measures. By Fubini's theorem, we find

$$\begin{aligned} \mu(f) &= \int_{X_1} \int_{X_2} f(x,y)\, d\mu_2(y)\, d\mu_1(x) \\ &\ge \int_{X_1} f(x, \mathrm{bar}(\mu_2))\, d\mu_1(x) \\ &\ge f(\mathrm{bar}(\mu_1), \mathrm{bar}(\mu_2)) = f(\mathrm{bar}(\mu)). \end{aligned}$$

Here, in the first inequality, we had

$$f(x, \mathrm{bar}(\mu_2)) \le \int_{X_1} f(x,y)\, d\mu_2(y)$$

by the one-variable version of Jensen's inequality applied to the convex function $y \mapsto f(x,y)$, and similarly in the second. □

Taking a more algebraic perspective, we have the standard identification

$$C(X_1 \times \cdots \times X_d) \cong C(X_1) \otimes \cdots \otimes C(X_d)$$

as C*-algebras. Product measures, as states on $C(X_1 \times \cdots \times X_d)$, are exactly those states of the form $\mu_1 \otimes \cdots \otimes \mu_d$, where each $\mu_i \in S(C(X_i))$ is a state.

3.4.2 Noncommutative analogue

In the setting of noncommutative convexity, we've seen in Corollary 3.3.3 that

$$C(K_1 \times \cdots \times K_d) \cong C(K_1) * \cdots * C(K_d)$$

for compact nc convex sets $K_1, \dots, K_d$. The expected noncommutative analogue of Proposition 3.4.2 should be that a "separately nc convex" nc function $f : K_1 \times \cdots \times K_d \to \mathcal{M}^{\mathrm{sa}}$ satisfies a Jensen-type inequality for any nc state/ucp map

$$\mu : C(K_1 \times \cdots \times K_d) \cong C(K_1) * \cdots * C(K_d) \to M_m$$

that arises as a "free product" of ucp maps $\mu_i : C(K_i) \to M_m$. Defining a noncommutative version of separate convexity is straightforward, but the main difficulty is identifying which ucp maps should "count" as free products. Boca's theorem [10] and its more general version given by Davidson and Kakariadis [21] provide a standard recipe for taking free products for ucp maps. We will show ucp maps built from Boca's theorem satisfy a Jensen inequality for separately nc convex functions, but this is not the biggest class that works.

Definition 3.4.3. Suppose K_i, $i \in I$, are compact nc convex sets, for $i \in I$. Let $f : K = \prod_{i \in I} K_i \to \mathcal{M}^{\mathrm{sa}}$ be an nc function. We say f is *separately nc convex* if whenever $x, y \in K$, and $x \prec_\alpha y$ is a dilation in K such that at most one of the dilations

$$x_i \prec_\alpha y_i, \quad i \in I,$$

is not trivial, we have

$$f(x) \leq \alpha^* f(y) \alpha.$$

The following observation justifies the terminology "separately nc convex".

Proposition 3.4.4. *Let $f : K = \prod_{i \in I} K_i \to \mathcal{M}^{sa}$ be an nc function. Then f is separately nc convex if and only if the restriction $f|_{K(n)}$ is a separately convex function for each matrix level $K(n)$ of K.*

Proof. The proof is essentially the same as for [26, Proposition 7.2.3], so we only provide a sketch in the two-variable case $I = \{1, 2\}$.

Suppose $f : K_1 \times K_2 \to \mathcal{M}^{\mathrm{sa}}$ is separately nc convex. Given $x, y \in K_1(n)$, $\lambda \in [0, 1]$, and $z \in K_2(n)$, the dilation

$$((1-\lambda)x + \lambda y, z) = \begin{pmatrix} \sqrt{1-\lambda} & \sqrt{\lambda} \end{pmatrix} \left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \right) \begin{pmatrix} \sqrt{1-\lambda} \\ \sqrt{\lambda} \end{pmatrix}$$

is trivial in the second entry. So, separate nc convexity gives

$$\begin{aligned} f((1-\lambda)x + \lambda y, z) &\leq \begin{pmatrix} \sqrt{1-\lambda} & \sqrt{\lambda} \end{pmatrix} \begin{pmatrix} f(x,z) & 0 \\ 0 & f(y,z) \end{pmatrix} \begin{pmatrix} \sqrt{1-\lambda} \\ \sqrt{\lambda} \end{pmatrix} \\ &= (1-\lambda) f(x,z) + \lambda f(y,z). \end{aligned}$$

Thus f is convex in its first argument, and a symmetrical argument works for the second argument

Conversely, suppose f is separately convex at each level. Take any dilation of the form

$$(x,z) \prec_\alpha (y,w)$$

in $K_1 \times K_2$ where $w \cong z \oplus z'$ with respect to $\mathrm{ran}(\alpha)$. Up to a unitary, which the nc function f respects, we may write

$$y = \begin{pmatrix} x & * \\ * & * \end{pmatrix}, \quad w = \begin{pmatrix} z & 0 \\ 0 & z' \end{pmatrix}, \quad f(y,w) = \begin{pmatrix} a & * \\ * & * \end{pmatrix}, \quad \text{and} \quad \alpha = \begin{pmatrix} I \\ 0 \end{pmatrix}$$

as operator matrices. Define a selfadjoint unitary

$$U := \alpha\alpha^* - (I - \alpha\alpha^*) = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Then by assumption

$$f\left(\frac{y + UyU}{2}, w\right) = \begin{pmatrix} f(x,z) & 0 \\ 0 & f(*,z') \end{pmatrix} \le \frac{f(y,w) + Uf(y,w)U}{2} = \begin{pmatrix} a & 0 \\ 0 & * \end{pmatrix}.$$

Cutting down to the (1,1) corner shows

$$f(x,z) \le \alpha^* f(y,w)\alpha.$$

Hence f respects dilations which are trivial in the second argument, and a symmetric argument works when the dilation is trivial in its first argument. This shows that f is separately nc convex. □

Example 3.4.5. Let $K_1, K_2 \subseteq \mathcal{M}^{\mathrm{sa}}$ be compact nc convex sets. Proposition 3.4.4 implies that the nc function $f : K_1 \times K_2 \to \mathcal{M}^{\mathrm{sa}}$ defined by

$$f(x,y) = \frac{xy + yx}{2}$$

is separately nc convex. It is not an nc convex function, because its restriction to the first level $(K_1 \times K_2)_1$ is the non-convex function $f(x,y) = xy$ from Example 3.4.1.

Example 3.4.6. More generally, Proposition 3.4.4 shows that any function of the form

$$a_1 \cdots a_k + a_k^* \cdots a_1^* \in C(K_1 \times \cdots \times K_d)$$

for nc affine functions $a_i \in A(K_{j_i})$ in separate variables, i.e. with $i \ne i'$ implying $j_i \ne j_{i'}$, is separately nc convex. Moreover, sums of such functions are also separately nc convex.

Example 3.4.7. If I is a finite set and $J \subseteq I$, we identify

$$C\left(\prod_{j\in J} K_j\right) \cong *_{j\in J} C(K_j) \subseteq *_{i\in I} C(K_i) \cong C\left(\prod_{i\in I} K_i\right)$$

as a C*-subalgebra. If $f \in C(\prod_{j\in J} K_j)_+$ is a positive separately nc convex nc function, $i \in I \setminus J$, and $a \in A(K_i) \subseteq C(K_i) \subseteq C(\prod_i K_i)$ is an nc affine function, then the function $F := a^* f a \in C(\prod_i K_i)$ is separately nc convex.

Suppose $x = (x_i)_{i\in I} \prec_\alpha y = (y_i)_{i\in I}$ is a dilation that is nontrivial in at most one coordinate. Indeed, if $\pi_J : \prod_{i\in I} K_i \to \prod_{j\in J} K_J$ is the natural projection, then by separate nc convexity of f,

$$\begin{aligned} F(x) &= \alpha^* a(y_i)^* \alpha f(\alpha^* \pi_J(y)\alpha)\alpha^* a(y_i)\alpha \\ &\le \alpha^* a(y_i)^* (\alpha\alpha^*) f(\pi_J(y))(\alpha\alpha^*) a(y_i)\alpha. \end{aligned}$$

If the dilation $x_i \prec_\alpha y_i$ is trivial, then $a(y_i)$ commutes with $\alpha\alpha^*$ and the right hand side becomes

$$\alpha^* a(y_i)^* f(\pi_J(y)) a(y_i)\alpha = \alpha^* F(y)\alpha.$$

Otherwise, the dilation $x_j \prec_\alpha y_j$ is trivial for each $j \in J$, so $f(\pi_J(y))$ commutes with $\alpha\alpha^*$. Since $f(\pi_J(y)) \ge 0$, the right hand side is

$$\alpha^* a(y_i)^* f(\pi_J(y))^{1/2} a a^* f(\pi_J(y))^{1/2} a(y_i)\alpha \le \alpha^* a(y_i)^* f(\pi_J(y)) a(y_i)\alpha = \alpha^* F(y)\alpha.$$

In either case, $F(x) \le \alpha^* F(y)\alpha$.

For example, if $a \in A(K_i)$ and $b \in A(K_j)$ are nc affine functions in separate variables $i \ne j$, then

$$F = a^* b^* b a$$

is separately nc convex. An easy induction shows that if $a_1 \in A(K_{i_1}), \dots, a_k \in A(K_{i_k})$ with $i_1, \dots, i_k$ distinct indices, then the function

$$F = a_1^* \cdots a_k^* a_k \cdots a_1$$

is separately nc convex.

The following definition is designed to exactly capture the largest class of ucp maps for which our approach can prove a Jensen-type inequality that characterizes separately nc convex functions. The name and involved chain of dilations are meant as a noncommutative analogue of the role that Fubini's theorem plays in the proof of Proposition 3.4.2.

Definition 3.4.8. Let A_i, $i \in I$, be unital C*-algebras. A ucp map $\mu : *_{i\in I} A_i \to B(H)$ is of *Fubini type* if there exists an ordinal α and a chain of dilations

$$\{\mu_\lambda : *_{i\in I} A_i \to B(H_\lambda) \mid \lambda \le \alpha\}$$

such that

(i) $\mu_0 = \mu$ and $H_0 = H$,

(ii) μ_α is a $*$-homomorphism,

(iii) $\lambda \le \rho \le \alpha$ implies that $H_\lambda \subseteq H_\rho$ and $\mu_\lambda \prec \mu_\rho$,

(iii) each dilation

$$\mu_\lambda \prec \mu_{\lambda+1}$$

is nontrivial in at most one of the algebras A_i, $i \in I$,

(iv) if $\beta \le \alpha$ is a limit ordinal, then $H_\beta = \overline{\bigcup_{\lambda\le\beta} H_\lambda}$.

In most examples, when I is finite we will take α to be a finite ordinal with $|\alpha| = d := |I|$. Usually, we can arrange that

$$\mu = \mu_0 \prec \mu_1 \prec \cdots \prec \mu_d = \pi$$

where π is a $*$-homomorphism and each dilation $\mu_{k-1} \prec \mu_k$ is nontrivial only in the kth coordinate.

Example 3.4.9. Suppose $\mu : *_{i\in I} A_i \to B(H)$ is a ucp map such that all but one of the ucp maps $\mu|_{A_i}$ is a $*$-homomorphism. Then μ is of Fubini type. Let $\pi : *_{i\in I} A_i \to B(L)$ be the minimal Stinespring dilation of μ, with $L \supseteq H$. Then since $\mu = P_H \pi|_H$ is a $*$-homomorphism on all but one algebra A_i, it follows that H is reducing for $\pi(A_i)$ for all but possibly one $i \in I$. Hence the dilation

$$\mu =: \mu_0 \prec \mu_1 := \pi$$

is trivial in all but one algebra A_i.

In the following theorem, we freely identify $C(\prod_i K_i)$ with $*_i C(K_i)$.

Theorem 3.4.10. *Let $K_1, \dots, K_d$ be compact nc convex sets, let $I = \{1, \dots, d\}$, and let*

$$f : K_1 \times \cdots \times K_d \to \mathcal{M}^{sa}$$

be a bounded and (weak-$*$ to weak-$*$ or ultrastrong-$*$ to ultrastrong-$*$) upper semicontinuous nc function. If f is separately nc convex, and

$$\mu : *_{i \in I} C(K_i) \to M_m$$

is a ucp map of Fubini type, then the Jensen inequality

$$f(bar(\mu)) \leq \mu(f)$$

holds.

Proof. Suppose f is continuous and separately nc convex. Let

$$\mu : C(\prod_{i \in I} K_i) \cong *_{i \in I} C(K_i) \to M_m = B(H)$$

be a ucp map of Fubini type, with associated dilation chain $\{\mu_\lambda \mid \lambda \leq \alpha\}$, where $\mu_0 = \mu$ and $\pi := \mu_\alpha$ is a $*$-homomorphism. Set $x_\lambda := \mathrm{bar}(\mu_\lambda)$ for all $\lambda \leq \alpha$. Because π is a $*$-homomorphism, we have $\pi = \delta_{x_\alpha}$. Since the barycenter map is continuous and nc affine, then $\lambda \leq \rho \leq \alpha$ implies $x_\lambda \prec x_\rho$, and whenever $\lambda + 1 \leq \alpha$, the dilation $x_\lambda \prec x_{\lambda+1}$ in $K_1 \times \cdots \times K_d$ is trivial in all but at most one variable. That is, the chain of dilations $\{x_\lambda \mid \lambda \leq \alpha\}$ shows that $x_\alpha \in K \cong S(A(K))$ is itself of Fubini type when viewed as a ucp map on $A(K)$.

We will show that

$$f(x_0) \leq P_{H_0} f(x_\lambda)|_{H_0}$$

for any λ by transfinite induction on λ. This is a tautology when $\lambda = 0$. Suppose for $\lambda \leq \alpha$ that $f(x_0) \leq P_{H_0} f(x_\lambda)|_{H_0}$. Because the dilation $x_\lambda \prec x_{\lambda+1}$ is trivial in all but one variable, and f is separately nc convex, we have

$$f(x_\lambda) \leq P_{H_\lambda} f(x_{\lambda+1})|_{H_\lambda}.$$

Compressing to H_0 and using the inductive hypothesis yields

$$f(x_0) \leq P_{H_0} f(x_\lambda)|_{H_0} \leq P_{H_0} f(x_{\lambda+1})|_{H_0}.$$

Finally, suppose $\beta \leq \alpha$ is a limit ordinal, and $f(x_0) \leq P_{H_0} f(x_\lambda)|_{H_0}$ for all $\lambda < \beta$. Fix any constant vector $c \in (K_1 \times \cdots \times K_d)(1)$. Define a net

$$z_\lambda = x_\lambda \oplus c 1_{d_\lambda}$$

with respect to the decomposition $H_\beta = H_\lambda \oplus (H_\beta \ominus H_\lambda)$, where $d_\lambda = \dim(H_\beta \ominus H_\lambda)$. Then z_λ converges to x_β ultrastrong-$*$. For $\rho \le \lambda < \beta$ we have

$$P_{H_\rho} f(z_\lambda)|_{H_\rho} = P_{H_\rho} f(x_\lambda)|_{H_\rho}$$

as f respects direct sums. Compressing to H_0 gives

$$P_{H_0} f(z_\lambda)|_{H_0} = P_{H_0} f(x_\lambda)|_{H_0} \ge f(x_0)$$

by inductive hypothesis. Since $z_\lambda \to x_\beta$ and f is upper semicontinuous, we have

$$P_{H_0} f(x_\beta)|_{H_0} \ge P_{H_0}(\limsup_{\lambda<\beta} f(z_\lambda))|_{H_0} \ge \limsup_{\lambda<\beta} P_{H_0} f(z_\lambda)|_{H_0} \ge f(x_0).$$

This completes the induction, and by taking $\lambda = \alpha$ we conclude

$$f(\mathrm{bar}(\mu)) = f(x_0) \le P_{H_0} f(x_\alpha)|_{H_0} = P_{H_0}\pi(f)|_{H_0} = \mu(f). \qquad \square.$$

Remark 3.4.11. Theorem 3.4.10 applies to weak-$*$ or ultrastrong-$*$ continuous nc functions, including all functions in $C(K_1 \times \cdots \times K_d)^{\mathrm{sa}}$. The same proof also shows that a non-continuous separately nc convex function still satisfies a Jensen inequality $f(\mathrm{bar}(\mu)) \le \alpha^*\pi(f)\alpha$ whenever μ is a Fubini type ucp map with an associated dilation chain of finite length.

Remark 3.4.12. In fact, the Jensen inequality in Theorem 3.4.10 completely characterizes separate nc convexity of f. Suppose f satisfies the claimed Jensen inequality for ucp maps of Fubini type. Let

$$x \prec_\alpha y$$

be a dilation in $\prod_{i\in I} K_i$ which is trivial in all but at most one coordinate. Let $\mu = \alpha^*\delta_y\alpha$. Because the dilation $\mu \prec \delta_y$ is trivial in all but possibly one algebra $C(K_i)$, the ucp map μ is of Fubini type. Therefore

$$f(x) = f(\mathrm{bar}(\mu)) \le \mu(f) = \alpha^* f(y)\alpha,$$

so f is separately nc convex.

Theorem 3.4.10 applies to the following wide class of ucp maps which we might consider "free products" of ucp maps. This upcoming definition is just a reorganizing of the construction given by Davidson and Kakariadis [21]. Given an index set I, let S_I denote the set of finite words in I without repeated letters. We include the empty word $\varnothing$ in S_I. In what follows we are usually interested in the case where I is a (von Neumann) ordinal, i.e. a certain form of well ordered set. In practice, usually $I = \{1, \dots, d\}$ for some $d \in \mathbb{N}$, but what follows works in generality for infinite sets I.

Definition 3.4.13. Let A_i, $i \in I$, be unital C*-algebras with unital free product $A = *_{i \in I} A_i$. Let $\mu : A \to B(H)$ be a ucp map with minimal Stinespring dilation $\pi : A \to B(L)$, where $L \supseteq H$. Define subspaces L_w for $w \in S_I$ by setting

$$L_\varnothing = H,$$

and inductively

$$L_{iw} := \overline{\pi(A_i)L_w} \ominus L_w$$

whenever $w = w_1 \cdots w_m \in S_I$ with $w_1 \neq i \in I$. We call μ a *free product ucp map* (of the ucp maps $\mu_i = \mu|_{A_i}$, $i \in I$) if the spaces L_w are pairwise orthogonal for $w \in S_I$.

Minimality of the Stinespring dilation in Definition 3.4.13 implies that

$$L = \overline{\sum_{w \in S_I} L_w},$$

so the definition of "free product" is just that this sum is direct.

Remark 3.4.14. By definition, any ucp map $\mu : A \to B(H)$ which is built from ucp maps $\mu_i : A_i \to B(H)$ via the proof of [21, Theorem 3.1] is a free product ucp map. Examining the proof of [21, Theorem 3.4] also shows that any ucp map built from Boca's theorem, which produces a unique product map μ given the additional data of prescribed states $\varphi_i : A_i \to \mathbb{C}$, is also a free product ucp map.

Proposition 3.4.15. *Let A_i, $i \in I$ be unital C*-algebras with unital free product $A = *_{i \in I} A$. Then any free product ucp map $\mu : A \to B(H)$ is a ucp map of Fubini type.*

Proof. Suppose μ is a free product ucp map. Let $\pi : A \to B(L)$ be its minimal Stinespring dilation and define the spaces L_w, $w \in S_I$, exactly as in Definition 3.4.13, so that

$$L = \bigoplus_{w \in S_I} L_w$$

as an orthogonal direct sum. Up to a fixed bijection we may assume I is just some ordinal α. For any ordinal $\lambda \leq \alpha$, let

$$H_\lambda = \bigoplus_{w \in S_\lambda} L_w.$$

Note that as above we take the von Neumann definition of "ordinal", so λ is understood as an actual set and not some equivalence class. With this convention the notation S_λ is well defined. (In the case where I is a finite set, this reduces to identifying $\alpha = \{1, \ldots, n\}$ and

$\lambda = \{1, \ldots, k\}$ for some natural numbers $k \le n$.) Then $H_0 = H$, $H_\alpha = L$, and the sequence $(H_\lambda)_{\lambda \le \alpha}$ is increasing with

$$H_\beta = \overline{\bigcup_{\lambda < \beta} L_\lambda}$$

for any limit ordinal β.

Set $\mu_\lambda := P_{H_\lambda} \pi|_{H_\lambda}$. It suffices to show for any λ with $\lambda + 1 \le \alpha$ that the dilation $\mu_\lambda \prec \mu_{\lambda+1}$ is trivial in all variables except $\lambda + 1$. Since the sum $L = \bigoplus_w L_w$ is direct, we find

$$H_{\lambda+1} \ominus H_\lambda = \bigoplus \{L_w \mid w = w_1 \cdots w_m \in S_{\lambda+1} \text{ with some } w_j = \lambda + 1\}.$$

Given $i \in I$ and $w = w_1 \cdots w_m \in S_I$, by definition $\pi(A_i)$ maps L_w into

$$\begin{cases} L_w \oplus L_{iw} & i \ne w_1, \\ L_w \oplus L_{w_2 \cdots w_m} & i = w_1. \end{cases}$$

Knowing this, it follows that for any $i \in I$ with $i \ne \lambda + 1$ that $\pi(A_i)$ maps $H_{\lambda+1} \ominus H_\lambda$ into

$$\begin{aligned} &\bigoplus \{L_w \mid w = w_1 \cdots w_m \in S_{(\lambda+1) \cup \{i\}} \text{ with some } w_j = \lambda + 1\} \\ &\qquad \subseteq \bigoplus \{L_w \mid w = w_1 \cdots w_m \in S_I \text{ with some } w_j \ne i\} = (H_\lambda)^\perp. \end{aligned}$$

Compressing to $H_{\lambda+1}$ shows that $H_{\lambda+1} \ominus H_\lambda$ is invariant for $\mu_{\lambda+1}(A_i) = P_{H_{\lambda+1}} \pi(A_i)|_{H_{\lambda+1}}$. So, the dilation $\mu_\lambda \prec \mu_{\lambda+1}$ is trivial in any variable $i \ne \lambda + 1$. The chain of dilations $\{\mu_\lambda \mid \lambda \le \alpha\}$ shows that μ is of Fubini type. $\square$

While every free product ucp map is of Fubini type, the following example demonstrates that a ucp map of Fubini type need not be a free product ucp map.

Example 3.4.16. Fix some large $M \ge 2$, and let $I = [-M, M] \subseteq \mathbb{R}$ be an interval, so we have a compact nc convex set

$$\mathrm{MIN}(I) = \{x \in \mathcal{M}^{\mathrm{sa}} \mid \sigma(x) \subseteq I\}.$$

Set $A_1 = A_2 = C(\mathrm{MIN}(I))$. By Corollary 3.3.3 we may identify

$$A_1 * A_2 \cong C(\mathrm{MIN}(I) \times \mathrm{MIN}(I)).$$

Let

$$y = (y_1, y_2) := \left(\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right) \in (\mathrm{MIN}(I) \times \mathrm{MIN}(I))(3) \subseteq M_3(\mathbb{C})^2.$$

Setting $L := \mathbb{C}^3$, the evaluation map

$$\pi := \delta_y : A_1 * A_2 \cong C(\mathrm{MIN}(I) \times \mathrm{MIN}(I)) \to M_3 = B(K)$$

is a $*$-homomorphism. Define subspaces

$$H := \mathbb{C} \oplus 0 \oplus 0 \quad \text{and}$$
$$H_1 := \mathbb{C} \oplus \mathbb{C} \oplus 0.$$

Compress to get ucp maps

$$\mu := \mu_0 := P_H \pi|_H,$$
$$\mu_1 := P_{H_1} \pi|_{H_1}.$$

Because H is reducing for $\mu_1(A_2)$, and H_1 is reducing for $\pi(A_1)$, the chain of dilations

$$\mu \prec \mu_1 \prec \pi$$

demonstrates that μ is a ucp map of Fubini type.

However, μ is not a free product ucp map. For $I = \{1,2\}$, define the subspaces $L_w \subseteq L$ for $w \in S_I$ as in Definition 3.4.13. Because

$$C(\mathrm{MIN}(I)) \cong C(I)$$

is generated by the polynomials, it is straightforward to see that

$$\pi(A_1)H = C^*(y_1)H = \mathbb{C} \oplus \mathbb{C} \oplus 0 = H_1.$$

Hence

$$L_1 = \overline{\pi(A_1)H} \ominus H = 0 \oplus \mathbb{C} \oplus 0.$$

Then, we have

$$\pi(A_2)L_1 = C^*(y_2)L_1 = \mathbb{C}^3 = K.$$

For instance, if $\{e_1, e_2, e_3\}$ is the standard basis for $\mathbb{C}^3$, then $e_1 = (y_1^2 - 2y_1)e_2$ and $e_3 = (y_1 - I)e_2$ both lie in $C^*(y_2)L_1$. Thus

$$L_{21} = \overline{\pi(A_2)L_1} \ominus L_1 = \mathbb{C} \oplus 0 \oplus \mathbb{C},$$

which is not orthogonal to $L_\varnothing = L = \mathbb{C} \oplus 0 \oplus 0$. Therefore

$$L = H + L_1 + L_{21},$$

but this sum is not direct, and so μ isn't a free product ucp map.

3.5 Connection to free probability

Given C*-algebras A_i, $i \in I$, and prescribed states $\varphi_i : A_i \to \mathbb{C}$, a ucp map

$$\mu : *_{i\in I} A_i \to B(H)$$

is *conditionally free* (with respect to the family $(\varphi_i)_{i\in I}$) if for every reduced word

$$a_1 \cdots a_m \in *_{i\in I} A_i$$

(reduced meaning that $a_k \in A_{j(k)}$ with $j(1) \neq \cdots \neq j(m)$) that satisfies

$$\varphi_{j(k)}(a_k) = 0 \text{ for all } k = 1, \ldots, m,$$

the multiplication rule

$$\mu(a_1 \cdots a_m) = \mu_{j(1)}(a_1) \cdots \mu_{j(m)}(a_m)$$

holds [11, 62]. Boca's theorem ([10] or see [21, Theorem 3.4]) shows that if $\mu_i : A_i \to B(H)$ are any ucp maps and $\varphi_i \in S(A_i)$ are prescribed states, there exists a unique ucp map

$$\mu : *_{i\in I} A_i \to B(H)$$

which is conditionally free with respect to $(\varphi_i)_{i\in I}$.

If μ is any ucp map which is conditionally free with respect to some family of states $(\varphi_i : A_i \to \mathbb{C})_{i\in I}$, then μ is a free product ucp map in the sense of Definition 3.4.13. This follows from the uniqueness in Boca's theorem. Any such conditionally free map agrees with the one constructed by Boca's theorem, and so Remark 3.4.14 applies. Or, examining the proof of [21, Theorem 3.2] in the case of amalgamation over $\mathbb{C}$ shows how to build the Stinespring dilation. The minimal Stinespring dilation

$$\pi : *_{i\in I} A_i \to B(L)$$

lives on a direct sum

$$L = \bigoplus_{w\in S_I} L_w,$$

where S_I is the set of words with letters in I without repeated letters. The dilation is constructed recursively so that whenever $i \in I$ and $w = w_1 \cdots w_m \in S_I$ with $i \neq w_1$, the sum $L_w \oplus L_{iw}$ is reducing for $\pi(A_i)$, and the compression $P_{L_w} \pi|_{L_w}$ to L_w satisfies

$$(P_{L_w} \pi|_{L_w})|_{A_i} = \varphi_i \otimes \mathrm{id}_{L_w}.$$

Applying Theorem 3.4.10 in this context yields the following.

Corollary 3.5.1. *Suppose $A_1, \dots, A_d$ are unital C*-algebras such that $A_i = C^*_{max}(S_i)$ for some operator systems $S_1 = A(K_1), \dots, S_d = A(K_d)$, where $K_i \cong \mathcal{S}(S_i)$ are compact nc convex sets. If*

$$\mu : A_1 * \cdots * A_d \to B(H)$$

is a conditionally free ucp map, and

$$f \in (A_1 * \cdots * A_d)^{sa} \cong C(K_1 \times \cdots \times K_d)^{sa}$$

is a continuous separately nc convex function, then the Jensen inequality

$$f(bar(\mu)) \leq \mu(f)$$

holds.

As in Remark 3.4.6, Corollary 3.5.1 applies e.g. to any function f which is a (limit of) sum(s) of the form

$$a_1 \cdots a_k + a_k^* \cdots a_1^*,$$

where each $a_i \in A(K_{j_i}) \cong S_i$ is continuous and nc affine, and no two a_i's depend on the same variable, i.e. $i \neq i'$ implies $j_i \neq j_{i'}$.

In the commutative case, if $K_i \subseteq \mathbb{R}^{m_i}$ are (classical) Choquet simplices, then

$$C(\mathrm{MIN}(K_i)) \cong C(K_i)$$

is a commutative C*-algebra, where the isomorphism is implemented by restriction to the first level [38, Theorem 4.7]. In particular, the theorem applies to commutative C*-algebras of the form $A_i = C(I_i)$, where $I_i \subseteq \mathbb{R}$ are intervals. Any selfadjoint element a_i with spectrum $\sigma(a_i) = I_i$ generates such a C*-algebra. What follows is an application of Corollary 3.5.1 in the special case where a_i are free semicircular elements.

Example 3.5.2. Suppose (A, φ) is a C*-probability space, where A is a unital C*-algebra, and φ is a faithful tracial state on A. Suppose $a, b \in A^{\mathrm{sa}}$ are free semicircular elements. This means that a and b have the semicircular $*$-distribution with some radii $r > 0$ and $s > 0$, respectively, and that the C*-algebras $C^*(A)$ and $C^*(b)$ are freely independent with respect to φ. Then $\sigma(a) = I := [-r, r]$ and $\sigma(b) = J = [-s, s]$. By [67, Theorem 7.9], faithfulness and traciality of φ implies that

$$C^*(a, b) \cong C^*(a) * C^*(b)$$

via the natural map.

Consider the operator systems $S = \text{span}\,\{1_A, a\}$ and $T = \text{span}\,\{1_A, b\}$ generated by a and by b. Because $\sigma(a) = I$ and $\sigma(b) = J$ are closed intervals, the functional calculus gives standard isomorphisms

$$C^*(a) \cong C(I) \cong C(\text{MIN}(I)) \quad \text{and} \quad C^*(b) \cong C(J) \cong C(\text{MIN}(J)).$$

We then identify

$$C^*(a,b) \cong C(\text{MIN}(I)) * C(\text{MIN}(J)) \cong C(\text{MIN}(I) \times \text{MIN}(J)).$$

Suppose

$$\mu : C^*(a) * C^*(b) \cong C(\text{MIN}(I) \times \text{MIN}(J)) \to B(H)$$

is a conditionally free ucp map, or a ucp map built from Boca's theorem. The barycenter $\text{bar}(\mu) \in \text{MIN}(I) \times \text{MIN}(J)$ corresponds to the point evaluation

$$\sigma = \delta_{\text{bar}(\mu)} : C^*(a) * C^*(b) \to B(H)$$

that is the unique $*$-homomorphism determined by $\sigma(a) = \mu(a)$ and $\sigma(b) = \mu(b)$. Note that such a $*$-homomorphism exists because $\sigma(\mu(a)) \subseteq I$ and $\sigma(\mu(b)) \subseteq J$.

Theorem 3.4.10 implies that for every element $x \in C^*(a)*C^*(b) \cong C(\text{MIN}(I)\times\text{MIN}(J))$ which corresponds to a separately nc convex function on $\text{MIN}(I) \times \text{MIN}(J)$, the operator inequality $\sigma(x) \leq \mu(x)$ holds. By Examples 3.4.6 and 3.4.7, elements of the form $x = ab+ba$, or $x = ab^2a$ correspond to separately convex functions. Therefore if a and b are free semicircular elements and μ is a conditionally free ucp map, we get the inequalities

$$\mu(a)\mu(b) + \mu(b)\mu(a) \leq \mu(ab + ba), \tag{3.3}$$

$$\mu(a)\mu(b)^2\mu(a) \leq \mu(ab^2a). \tag{3.4}$$

In this same context, the "one-variable" nc Jensen inequality of Davidson and Kennedy [26, Theorem 7.6.1] implies that if $y \in S + T = \text{span}\,\{1_A, a, b\}$, we have

$$\mu(y)^*\mu(y) \leq \mu(y^*y)$$

because the element $x = y^*y$ corresponds to a (jointly) nc convex function. In this case, such an inequality trivially reduces to the usual Schwarz inequality for ucp maps. In contrast, the inequalities (3.3) and (3.4) do not reduce to some trivial application of the Schwarz inequality because they do not hold for general ucp maps μ. For instance, one could take $A = M_2$,

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and let $\mu : M_2 \to \mathbb{C}$ be the normalized trace. (In this example, $(A, \varphi) := (M_2, \operatorname{tr}/2)$ is a faithful tracial C*-probability space, but a and b are not freely independent and not semicircular.)

The reasoning of Example 3.5.2 generalizes readily to free semicircular families of arbitrary size as follows.

Corollary 3.5.3. *Let (A, φ) be a C*-probability space with faithful tracial state $\varphi \in \mathcal{S}(A)$. Suppose $a_1, \ldots, a_d \in A$ is a free family of semicircular elements. If*

$$\mu : C^*(a_1, \ldots, a_d) \cong C^*(a_1) * \cdots * C^*(a_d) \to B(H)$$

is a conditionally free ucp map, or a ucp map built from Boca's theorem, then for every list of distinct indices $i_1, \ldots, i_k \in \{1, \ldots, d\}$, the operator inequalities

$$\mu(a_{i_1}) \cdots \mu(a_{i_k}) + \mu(a_{i_k}) \cdots \mu(a_{i_1}) \le \mu(a_{i_1} \cdots a_{i_k} + a_{i_k} \cdots a_{i_1})$$

and

$$\mu(a_{i_1}) \cdots \mu(a_{i_{k-1}}) \mu(a_{i_k})^2 \mu(a_{i_{k-1}}) \cdots \mu(a_{i_1}) \le \mu(a_{i_1} \cdots a_{i_k}^2 \cdots a_{i_1})$$

hold.

Note that in Corollary 3.5.3, the ucp map μ need not be conditionally free with respect to the states $\varphi_i := \varphi|_{C^*(a_i)}$. Conditional freeness with respect to *any* family of states is enough.

Chapter 4

Operator system duals and noncommutative convexity

4.1 Introduction

A unital operator system S is a $*$-closed unital subpsace of the bounded operators $B(H)$ on a Hilbert space H. In this chapter, we assume that all operator spaces and operator systems are norm-complete. Choi and Effros [15] gave an abstract characterization of unital operator systems as matrix ordered $*$-vector spaces which contain an archimedean matrix order unit. Using this characterization, it is natural to ask if the dual space S^* is itself a unital operator system.

The dual S^* is at least a complete operator space, and inherits a $*$-operation and matrix ordering from S. One says that S^* is a *matrix ordered operator space*. However, S^* typically fails to have an order unit in infinite dimensions. For instance, if $S = C(X)$ is a commutative C*-algebra, then the dual $C(X)^*$ is the space of Radon measures on the compact Hausdorff space X, which never has an order unit if X is uncountable. So, one requires a theory of *nonunital* operator systems if S^* is to be an operator system. Werner [78] defined nonunital operator systems–which we hereafter refer to as simply "operator systems", as matrix ordered operator spaces which embed completely isometrically and completely order isomorphically into $B(H)$. Werner gave an abstract characterization that extends the Choi-Effros axioms in the unital setting. One would hope that S^* is such an operator system, but it turns out that this is too much to ask. For instance, we have the standard duality $M_n^* \cong M_n$, but this duality is not completely isometric. In fact, an embedding $M_n^* \hookrightarrow B(H)$ cannot be completely isometric and completely order isomorphic at the same time. However, the isomorphism $M_n^* \cong M_n$ is a *complete isomorphism*, inducing completely equivalent matrix norms.

Call an operator system S *dualizable* if the dual matrix ordered operator space S^* embeds into $B(H)$ via a map which is a complete order isomorphism and is *completely bounded below*. That is, S^* can be re-normed with completely equivalent matrix norms in a way that makes it an operator system. Recently, C.K. Ng [65] obtained an intrinsic characterization of dualizability. The operator system S is dualizable if and only if it satisfies the following completely bounded positive decomposition property: There is a constant $C > 0$, such that for every $n \geq 1$ and every selfadjoint $x \in M_n(S)^{\mathrm{sa}}$, there are positives $y, z \in M_n(S)^+$ with $x = y - z$ and $\|y\| + \|z\| \leq C\|x\|$. Using the order unit, every unital operator system S is dualizable with $C = 2$. Similarly, the continuous functional calculus implies that every (possibly nonunital) C*-algebra is dualizable with $C = 1$. So, the dualizable systems form a large class. However, not every operator system is dualizable. For instance, the operator systems

$$S = \{a : [-1,1] \to \mathbb{R} \mid a \text{ is affine and } a(0) = 0\} \subseteq C([-1,1]) \quad \text{and}$$
$$T = \mathrm{span}\{E_{12}, E_{21}\} \subseteq M_2$$

contain no nonzero positive elements. So, the matrix cones in S^* and T^* are not proper, and these cannot be re-normed into operator systems.

Recently, the Kennedy, Kim, and Manor [56] showed that the study of nonunital operator systems is categorically dual to studying *pointed* noncommutative compact convex sets. This is a nonunital, quantized version of classical Kadison duality for function systems [48]. A noncommutative (nc) convex set is graded into matrix levels

$$K = \coprod_{n\geq 1} K_n \subseteq \coprod_{n\geq 1} M_n(E)$$

over an operator space E, which is closed under direct sums and compression by scalar isometries [26]. Nc convex sets are essentially equivalent to the *matrix* convex sets of Wittstock [79], with the distinction that an nc convex set contains infinite matrix levels up to some infinite cardinal α depending on E. (In separable settings, usually one takes $\alpha = \aleph_0$.) While nc convex sets are determined by their finite levels, one needs the infinite levels to find all nc extreme points. Here when n is an infinite cardinal, we use the convention $M_n = B(H)$, where $\dim H = n$. We say that K is closed/compact if each level K_n is closed/compact.

The canonical example of an nc convex set is the nc state space

$$\mathcal{S}(S) = \coprod_{n\geq 1} \{\varphi : S \to M_n \mid \varphi \text{ is unital and completely positive}\}$$

of a unital operator system S. The unital noncommutative Kadison duality of Webster-Winkler [77] and Davidson-Kennedy [26] asserts that $\mathcal{S}(S)$ completely determines S. If S is a *nonunital* operator system, the appropriate replacement for the nc state space is the *nc quasistate space*

$$\mathcal{QS}(S) = \coprod_{n\geq 1}\{\varphi : S \to M_n \mid \varphi \text{ is completely contractive and positive}\}.$$

If K is a compact nc convex set, and $z \in K_1$ is a prescribed basepoint, we form the nonunital operator system $A(K,z)$ of *pointed* nc affine functions

$$a : (K,z) \to (\mathcal{M},0),$$

where $\mathcal{M} = \coprod_{n\geq 1} M_n$. The pair (K,z) is a *pointed* nc convex set if every nc quasistate on $A(K,z)$ is a point evaluation in K. In [56], it was shown that the functor

$$S \mapsto (\mathcal{QS}(S),0)$$

is a contravariant equivalence of categories between the category of operator systems and the category of pointed compact nc convex sets, with essential inverse $(K,z) \mapsto A(K,z)$.

Via this equivalence, operator systems can be completely described by the nc convex geometry of the nc quasistate space $(K,z) = (\mathcal{QS}(S),0)$. Our main question is: What geometric condition on $(K,0)$ detects dualizability of S? We obtain two geometric answers to this question. The first is extrinsic, and the second is intrinsic to K.

Firstly, in Theorem 4.4.9, we show that S is dualizable if and only if there is a Hilbert space H and a pointed embedding

$$(K,0) \hookrightarrow (L,0)$$

into the positive nc unit ball

$$\coprod_{n\geq 1} B_1(M_n(B(H)))^+$$

satisfying the following extension property: Every nc affine function on K extends to an nc affine on L with a complete norm bound, and every positive nc affine function $a \in M_n(A(K,0))^+$ extends to a positive nc affine on L. Equivalently, the restriction map $A(L,0) \to A(K,0)$ is an operator space quotient map that maps the positives onto the positives at all matrix levels.

Secondly, in Theorem 4.5.7, we show that Ng's bounded positive decomposition property for S is equivalent to a complete normality condition for S^* in the sense of [7, Section 2.1]. This is equivalent to the geometric condition

$$(K - \mathbb{R}_+ K) \cap \mathbb{R}_+ K \subseteq CK$$

in $\coprod_{n \geq 1} M_n(S^*)$, for some constant $C > 0$. It is equivalent to simply require that the set $(K - \mathbb{R}_+ K) \cap \mathbb{R}_+ K$ is norm-bounded.

The structure of this chapter is as follows. After some preliminaries in Section 4.2, we discuss quotients of matrix ordered operator spaces in Section 4.3. In Section 4.4, given an inclusion $0 \in K \subseteq L$ of pointed compact nc convex sets, we discuss the problem of extending nc affine functions from K to L with norm bounds or while preserving positivity, which characterizes when the restriction map $A(L,0) \to A(K,0)$ is a quotient. In Section 4.5, we prove our main results, characterizing dualizability of S via geometric conditions on the nc quasistate space $\mathcal{QS}(S)$. In Section 4.6, we discuss positive generation for a nonunital system S, and show that–in contrast to the classical case, a matrix ordered operator space may be positively generated but not satisfy Ng's condition of bounded positive generation. In Section 4.7, we give some examples and applications. We obtain some permanence properties, showing that quotients and pushouts of dualizable operator systems are again dualizable. Using the nc quasistate space, we obtain a new proof of Choi's Theorem.

4.2 Background

4.2.1 Nonunital operator systems

All vector spaces in this chapter are over $\mathbb{C}$, unless stated otherwise. If V is a vector space and $n \in \mathbb{N}$, we let $M_n(V)$ be the vector space of $n \times n$-matrices with entries in V. Frequently we naturally identify $M_n(V)$ with $M_n \otimes V$, and write for instance

$$1_2 \otimes x = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$

where $x \in V$ and $1_2 \in M_2$ is the identity matrix. We will also use the notation

$$\mathcal{M}(V) := \coprod_{n \geq 1} M_n(V)$$

to denote the matrix universe over V.

If V is any normed vector space and $r \geq 0$, we will frequently use $B_r(V)$ to denote the closed ball in V with radius r and center $0 \in V$.

Following Ng [65], we fix the following definitions. An **operator space** E is a vector space equipped with a complete family of L^∞-matrix norms, which we will denote either by $\|\cdot\|$, $\|\cdot\|_E$, or $\|\cdot\|_{M_n(E)}$ as appropriate.

Definition 4.2.1. A **semi-matrix ordered operator space** (X, P) consists of an operator space X equipped with a conjugate-linear completely isometric involution $x \mapsto x^*$, and a distinguished selfadjoint matrix convex cone $P = \coprod_{n \geq 1} P_n \subseteq \coprod_n M_n(X)^{\mathrm{sa}}$ such that each P_n is norm-closed in $M_n(X)$. Usually we omit the symbol P and write $M_n(X)^+ := P_n$. If in addition each cone $M_n(X)^+$ satisfies $M_n(X)^+ \cap (-M_n(X)^+) = \{0\}$, then we say X is a **matrix ordered operator space**. If X is in addition a dual space $X = (X_*)^*$, we say X is a **dual matrix ordered operator space** if the positive cones $M_n(X)^+$ are weak-$*$ closed.

Definition 4.2.2. A semi-matrix ordered operator space X is **positively generated** if

$$M_n(X)^{\mathrm{sa}} = M_n(X)^+ - M_n(X)^+$$

for all $n \geq 1$.

Example 4.2.3. If X is a positively generated matrix ordered operator space, then X^* is naturally a dual matrix ordered operator space with the standard norm and order structure that identifies

$$\begin{aligned} M_n(X^*) &\cong \mathrm{CB}(X, M_n) \quad \text{isometrically, and} \\ M_n(X^*)^+ &\cong \mathrm{CP}(X, M_n). \end{aligned}$$

Definition 4.2.4. Let X and Y be matrix ordered operator spaces, and let $\varphi : X \to Y$ be a linear map. For any $n \geq 1$, φ induces a linear map $\varphi_n : M_n(X) \to M_n(Y)$. We say that φ is completely bounded, contractive, bounded below, isometric, positive, or a complete order isomorphism when each induced map φ_n satisfies the same property uniformly in n. If φ is completely bounded below and positive, we say φ is a **complete embedding**. If φ is completely isometric and positive, we say φ is a **completely isometric embedding**. If φ is also a linear isomorphism, we call φ a **complete isomorphism** or **completely isometric isomorphism** as appropriate.

The class of all matrix ordered operator spaces forms a category, where one usually chooses the morphisms to be completely contractive and completely positive (ccp) maps, or completely bounded and completely positive (cbp) maps. In the interest of readability, we hereafter adopt the convention that "**completely contractive and positive**" always means "completely bounded and completely positive", and similarly for "completely bounded and positive". That is, "completely" modifies both the words "contractive" and "positive". Since we have no need to consider maps which are positive but not completely positive, there is hopefully no risk of confusion.

Example 4.2.5. Let S be a unital operator system, i.e. an $*$-matrix ordered space with archimedian matrix order unit 1_S. Then S is a matrix ordered operator space with norm

$$\|x\|_{M_n(S)} = \inf\left\{t \geq 0 \;\middle|\; \begin{pmatrix} t(1_n \otimes 1_s) & x \\ x^* & t(1_n \otimes 1_s) \end{pmatrix} \geq 0 \text{ in } M_{2n}(S)^{\mathrm{sa}}\right\}.$$

This norm agrees with the induced norm from any unital complete order embedding $S \subseteq B(H)$. In particular, for any Hilbert space H, the space $B(H)$ is a unital operator system.

Definition 4.2.6. Let S be a matrix ordered operator space. We say that S is a **quasi-operator system** if there is a complete embedding $S \to B(H)$ for some Hilbert space H, and that S is a **operator space** if there is a completely isometric embedding $S \to B(H)$. If S is in addition a *dual* matrix ordered operator space, then we say S is a dual (quasi-)operator system if there is a weak-$*$ homeomorphic (complete embedding) completely isometric embedding into some $B(H)$.

That is, a quasi-operator system S is a matrix ordered operator space which is completely isomorphic to an operator system. Put another way, one can choose a completely equivalent system of norms on S, for which S embeds completely isometrically and order isomorphically into $B(H)$.

4.2.2 Pointed noncommutative convex sets

Suppose that $E = (E_*)^*$ is a dual operator space. Let

$$\mathcal{M}(E) := \coprod_{n \geq 1} M_n(E),$$

where the union is taken over all cardinals $n \geq 1$ up to some fixed cardinal α at least as large as the density character of E. (In practice we suppress α.) When n is infinite, we take the convention $M_n := B(H_n)$, where H_n is a Hilbert space of dimension n. By naturally identifying

$$M_n(E) = \mathrm{CB}(E_*, E),$$

we may equip each $M_n(E)$ with its corresponding point-weak-$*$ topology. Note that if $E = M_k$, this is the just the usual weak-$*$ topology on $M_n(M_k) \cong M_{nk}$.

Definition 4.2.7. We say that a graded subset

$$K = \coprod_{n\geq 1} K_n \subseteq \mathcal{M}(E)$$

is an **nc convex set** if for every norm-bounded family $(x_i) \in K_{n_i}$ and every family of matrices $\alpha_i \in M_{n_i,n}$ which satisfies

$$\sum_i \alpha_i^* \alpha_i = 1_n, \tag{4.1}$$

we have

$$\sum_i \alpha_i^* x_i \alpha_i \in K_n. \tag{4.2}$$

Here the sums (4.1) and (4.2) are required to converge in the point-weak-$*$ topologies on M_n and $M_n(E)$, respectively. We say in addition that K is a **compact nc convex set** if each matrix level K_n is point-weak-$*$ compact in $M_n(E)$.

Usually we refer to the sum in (4.2) as an **nc convex combination** of the points x_i. Succinctly, an nc convex set is one that is *closed under nc convex combinations*. It is equivalent to require only that K is closed under direct sums (4.1) in which the α_i's are co-isometries with orthogonal domain projections, and *compressions* (4.2) when there is only one α_i, which must be an isometry.

Definition 4.2.8. Let K and L be nc convex sets. A function $a : K \to L$ is an **nc affine function** if it is graded

$$a(K_n) \subseteq L_n, \qquad \text{for all } n \geq 1,$$

and respects nc convex combinations, i.e. whenever $x_i \in K$ are bounded and α_i are scalar matrices of appropriate sizes such that $\sum_i \alpha_i^* x_i \alpha_i$, then

$$a\Big(\sum_i \alpha_i^* x_i \alpha_i\Big) = \sum_i \alpha_i^* a(x_i) \alpha_i.$$

We say a is **continuous** if each restriction $a|_{K_n}$ is point-weak-$*$ continuous.

Classical Kadison duality [48] asserts that the category of **function systems**– partially ordered Banach spaces with an archimedean order unit, is equivalent to the category of compact convex sets with continuous affine functions as morphisms. Noncommutative Kadison duality asserts a similar equivalence for unital operator systems.

Theorem 4.2.9. [77, Proposition 3.5][26, Theorem 3.2.5] *The category of unital operator systems with ucp maps as morphisms is contravariantly equivalent to the category of compact nc convex sets with continuous nc affine functions as morphisms. On objects, the essential inverse functors send an operator system S to its nc state space*

$$\mathcal{S}(S) = \coprod_{n\geq 1}\{\varphi : S \to M_n \mid \varphi \text{ is unital and completely positive}\},$$

and send a compact nc convex set K to the operator system

$$A(K) = \{a : K \to \mathcal{M} = \mathcal{M}(\mathbb{C}) \mid a \text{ continuous nc affine}\}.$$

The operator system structure and norm on $A(K)$ is pointwise, i.e. one identifies $M_n(A(K)) \cong A(K, \mathcal{M}(M_n))$, and declares a matrix valued nc affine function if it takes positive values at every point. The order unit is the "constant function" $x \in K_n \mapsto 1_n \in M_n$. Both essential inverse functors act on morphisms by precomposition. That is, if $\pi : S \to T$ is a ucp map between operator systems, then the corresponding map on nc state spaces sends $\rho : T \to M_n$ to $\rho\pi : S \to M_n$. Likewise, if $a : K \to L$ is nc affine, then $f \mapsto f \circ a : A(L) \to A(K)$ is nc affine.

Recently, the Kennedy, Kim, and Manor [56] settled the question of Kadison duality for nonunital operator systems. The key challenge is that in the absence of order units, if S is a nonunital operator system then one must remember the whole **nc quasistate space**

$$\mathcal{QS}(S) = \coprod_{n\geq 1}\{\varphi : S \to M_n \mid \varphi \text{ is contractive and completely positive}\}$$

and consider **pointed** nc affine functions which fix the zero quasistates.

Definition 4.2.10. Let K be a compact nc convex set and fix a distinguished point z. We let

$$A(K, z) = \{a \in A(K) \mid a(z) = 0\}$$

denote the operator system of nc affine functions which vanish at z. We say that the pair (K, z) is a **pointed nc convex set** if the natural evaluation map

$$\begin{aligned} K &\to \mathcal{QS}(A(K,z)) \\ x &\mapsto (a \mapsto a(x)) \end{aligned}$$

is surjective (and hence bijective).

The main subtlety in nonunital Kadison duality is that while the correspondence $S \mapsto (\mathcal{QS}(S), 0)$ is a full and faithful functor, it is only essentially surjective onto the *pointed* compact nc convex sets.

Theorem 4.2.11. [56, Theorem 4.9] *The category of operator systems with ccp maps as morphisms is contravariantly equivalent to the category of pointed compact nc convex sets with pointed continuous nc affine functions as morphism. On objects, the essential inverse functors send an operator system S to is pointed nc quasistate space $(\mathcal{QS}(S), 0)$, and send a pointed compact nc convex set K to the operator system $A(K, z)$ of pointed continuous nc affine functions on (K, z).*

Again, on morphisms the essential inverse functors in Theorem 4.2.11 act in the natural way by precomposition on either nc affine functions or on nc quasistates.

4.3 Quotients of matrix ordered spaces

4.3.1 Operator space quotients

Here, we recall the basic theory of quotients for operator spaces. If E is an operator space, and $F \subseteq E$ is a closed subspace, then the quotient vector space E/F is an operator space where the matrix norms isometrically identify $M_n(E/F)$ with the standard Banach space quotient $M_n(E)/M_n(F)$.

Definition 4.3.1. Let $\varphi : E \to F$ be a completely bounded map between operator spaces E and F. We will say that $\varphi : E \to F$ is a **operator space quotient map with constant** $C > 0$ if any of the following equivalent conditions hold

(1) $B_1(M_n(F)) \subseteq \overline{\varphi_n(B_C(M_n(E)))} = C\overline{\varphi_n(B_1(M_n(E)))}$ for all $n \in \mathbb{N}$.

(2) $B_1(M_n(F)) \subseteq (C + \epsilon) \cdot \varphi_n(B_1(M_n(E)))$ for all $n \in \mathbb{N}$ and every $\epsilon > 0$.

(3) The induced map $\tilde{\varphi} : E/\ker\varphi \to F$ is an isomorphism and satisfies $\|\tilde{\varphi}^{-1}\|_{\mathrm{cb}} \leq C$.

The equivalence of (1) and (2) follows from a standard series argument using completeness of E. We will simply say **operator space quotient map** if we have no need to refer to C explicitly.

The following fact is standard in operator space theory, but we provide a proof for completeness.

Proposition 4.3.2. *Let $\varphi : E \to F$ be a completely bounded map between operator spaces E and F. The map φ is a quotient map with constant $C > 0$ if and only if the dual map $\varphi^* : F^* \to E^*$ is completely bounded below by $1/C$. Moreover, in this case, φ^* is weak-$*$ homeomorphism onto its range.*

Proof. Suppose that $C\varphi_n(B_1(M_n(E)))$ is dense in $B_1(M_n(F))$ for every n. Given $f \in M_m(F^*) \cong \mathrm{CB}(F, M_m)$, approximating unit vector $y \in B_1(M_n(F))$ with vectors of the form $\varphi(x)$ for $x \in B_C(E)$ shows that $\|f\|_{\mathrm{cb}} \le C\|\varphi_m^*(f)\|_{\mathrm{cb}}$.

Conversely, suppose that

$$\coprod_{n\geq 1} B_1(M_n(F)) \notin C \coprod_{n\geq 1} \varphi_n(B_1(M_n(E))).$$

By the Effros-Winkler nc Bipolar theorem [32], there are $m, n \ge 1$, an $x \in CB_1(M_n(E))$, and an $f \in M_m(F^*) \cong \mathrm{CB}(F, M_m)$, such that

$$\operatorname{Re} f_k(y) \le 1_{mk} \quad \text{for all } k \ge 1, y \in B_1(M_k(F)),$$

and yet $\operatorname{Re} f_n(x) \nleq 1_{mn}$. It follows that $\|f\| \le 1$, but $\|x\| \le C$ and $\|f_n(\varphi_n(x))\| > 1$, so $\|\varphi_m^*(f)\| > \|f\|_{\mathrm{cb}}/C$. This shows φ^* is not completely bounded below by $1/C$.

Finally, if φ is an operator space quotient map, it is bounded and surjective, and so its dual map φ^* is weak-$*$ homeomorphic onto its range. □

4.3.2 Matrix ordered operator space quotients

Definition 4.3.3. Let X be a matrix ordered operator space. We call a closed subspace $J \subseteq X$ a **kernel** if it is the kernel of a ccp map $\varphi : X \to Y$ for some matrix ordered operator space Y. In this case, we define an matrix ordered operator space structure on the operator space X/J with involution

$$(x + J)^* := x^* + J$$

and matrix order

$$M_n(X/J)^+ := \overline{\{x + M_n(J) \mid x \in M_n(X)^+\}},$$

where the closure is taken in the quotient norm topology on $M_n(X/J) \cong M_n(X)/M_n(J)$.

Proposition 4.3.4. *If X is a matrix ordered operator space, and $J = \ker \varphi$ is a kernel, then X/J is a matrix ordered operator space.*

Proof. Since the involution on X is completely isometric and J is selfadjoint, it follows that the involution on $M_n(X/J)$ is completely isometric. It is straightforward to check that X/J is a matrix ordered operator space. To prove that it is a matrix ordered operator space, suppose $x+J \in M_n(X/J)^+ \cap (-M_n(X/J)^+)$. Then for any ϵ, there are $y, z \in M_n(X)^+$ with $\|x - y + M_n(J)\|, \|x + z + M_n(J)\| < \epsilon$. Hence

$$\|\varphi_n(x) - \varphi_n(y)\| \leq \|x - y + M_n(J)\| < \epsilon$$

and similarly $\|\varphi_n(x) + \varphi_n(z)\| < \epsilon$. Since φ is cp, $\varphi_n(y), \varphi_n(z) \geq 0$. As ϵ is arbitrary and Y is a matrix ordered operator space, this shows

$$\varphi_n(x) \in \overline{M_n(Y)^+} \cap (-\overline{M_n(Y)^+}) = M_n(Y)^+ \cap (-M_n(Y)^+) = \{0\}.$$

Therefore $x \in M_n(\ker\varphi) = M_n(J)$, and so $x + M_n(J) = 0$. This shows

$$M_n(X/J)^+ \cap (-M_n(X/J)^+) = \{0\},$$

so X/J is a matrix ordered operator space. □

One can form a category of matrix ordered operator spaces with morphisms as either completely contractive and positive (ccp) or completely bounded and positive (cbp) maps. If V is a normed vector space and $r > 0$, then we let $B_r(V)$ denote the closed ball in V with radius r and center 0.

Definition 4.3.5. Let X and Y be matrix ordered operator spaces, and let $\varphi : X \to Y$ be a cbp map. We say that φ is a **matrix ordered operator space quotient map with constant** $C > 0$ if for all $n \in \mathbb{N}$ we have both

(1) $B_1(M_n(Y)) \subseteq C\overline{\varphi_n(B_1(M_n(X)))}$, and

(2) $M_n(Y)^+ = \overline{\varphi_n(M_n(X))^+}$.

For brevity, we will usually simply refer to φ as a **quotient map**, whenever it is clear that we are speaking only in the context of matrix ordered operator spaces.

That is, a matrix ordered operator space quotient map is just an operator space quotient map that maps the positives (densely) onto the positives at each matrix level. Comparing to Definition 4.3.1.(2), a quotient map is surjective. Each map $\varphi_n : M_n(X) \to M_n(Y)$ is therefore open and closed, and since the positive cones $M_n(X)^+$ and $M_n(Y)^+$ are norm-closed, it follows that $\varphi_n(M_n(X)^+)$ is closed and $\varphi_n(M_n(X)^+) = M_n(Y)^+$ for all n. That is, the closure in condition (2) is redundant. The first thing to show is that such maps are in fact categorical quotients in the category of matrix ordered operator spaces.

Proposition 4.3.6. *Let $\varphi : X \to Y$ be a cbp map between matrix ordered operator spaces. The following are equivalent.*

(1) The map φ is a quotient map with constant $C > 0$.

(2) The dual map $\varphi^* : Y^* \to X^*$ is completely bounded below by $1/C$ and a complete order injection.

(3) With $J = \ker\varphi$, the induced map $\tilde{\varphi} : X/J \to Y$ such that

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "q"] & Y \\ X/J \arrow[ur, dashed, "\tilde{\varphi}"'] & \end{tikzcd}$$

commutes is an isomorphism with cbp inverse satisfying $\|\tilde{\varphi}^{-1}\|_{cb} \le C$.

(4) For every matrix ordered operator space Z and cbp map $\psi : X \to Z$ with $\ker\varphi \subseteq \ker\psi$, there is a unique cbp map $\tilde{\psi} : Y \to Z$ making the diagram

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "\varphi"] & Z \\ Y \arrow[ur, dashed, "\tilde{\psi}"'] & \end{tikzcd}$$

commute, with $\|\tilde{\psi}\|_{cb} \le C\|\psi\|_{cb}$.

In this case, φ^* is weak-$*$ homeomorphic onto its range.

Proof. To prove (1) and (2) are equivalent, after invoking Proposition 4.3.2, it suffices to show that φ^* is a complete order injection if and only if Condition (2) in Definition 4.3.5 holds. Note that because φ is completely positive, so is φ^*. Suppose $\varphi_n(M_n(X)^+)$ is dense in $M_n(Y)^+$ for every $n \ge 0$. Let $f \in M_m(Y^*)$ with $\varphi_m^*(f) \ge 0$. Given $n \ge 1$ and $y \in M_n(Y)^+$, approximating y with a net of points of the form $\varphi_n(x_i)$ for $x_i \in M_n(X)^+$ shows that

$$f_n(y) = \lim_i f_n(\varphi_n(x_i)) = \lim_i (\varphi_m^*(f))_n(x_i) \ge 0.$$

This shows $f \ge 0$.

Conversely, suppose that $\varphi_n(M_n(X)^+)$ is not dense in $M_n(Y)^+$ for some $n \geq 1$. By the Effros-Winkler nc Bipolar Theorem [32] applied to the closed nc convex sets

$$\coprod_{k\geq 1} \overline{\varphi_k(M_n(X)^+)} \quad \not\supseteq \quad \coprod_{k\geq 1} M_k(Y)^+,$$

there is a selfadjoint matrix functional $f \in M_m(Y^*)^{\mathrm{sa}}$ such that $f_k(y) \geq -1_{mk}$ for every k and every $y \in M_k(Y)^+$, but

$$f_n(z) \not\geq -1_{mn}$$

for some $z \in \overline{\varphi_n(M_n(X)^+)} \setminus M_k(Y)^+$. A rescaling argument shows that $f \geq 0$ in $M_k(Y)$. However, approximating x by points of the form $\varphi_n(x)$, $x \in M_n(X)^+$ shows that $\varphi_m^*(f)$ cannot be positive. Hence, φ^* is not a complete order isomorphism.

If φ is a quotient map with constant $C > 0$, then it follows immediately from the definition of the matrix order and matrix norms on X/J that $\tilde{\varphi} : X/J \to Y$ is a complete order and norm isomorphism with $\|\tilde{\varphi}^{-1}\|_{\mathrm{cb}} \leq C$. Conversely, note that by definition the quotient map $q : X \to X/J$ is a quotient map with constant 1. Hence, if $\tilde{\varphi}$ is a complete order isomorphism with $\|\tilde{\varphi}^{-1}\|_{\mathrm{cb}} \leq C$, it follows that $\varphi = \tilde{\varphi} \circ q$ is a quotient map with constant C. This proves (1) and (3) are equivalent.

To show (3) and (4) are equivalent, it is enough to note that the quotient map $q : X \to X/J$ satisfies the universal property (4) with constant $C = 1$. In detail, if (3) holds, composing the universal map from (4) applied to $q : X \to X/J$ with $\tilde{\varphi}^{-1}$ shows that (4) holds for φ with constant C. Conversely, if (4) holds, then it holds for both φ and q, and there are induced maps $\tilde{\varphi} : X/J \to Y$ and $\tilde{q} : Y \to X/J$ with $\|\tilde{q}\| \leq \|\tilde{\varphi}\|_{\mathrm{cb}}$ and $\|\tilde{q}\| \leq C\|q\|_{\mathrm{cb}} = C$. Comparing diagrams shows $\tilde{q} = \tilde{\varphi}^{-1}$, and $\tilde{\varphi}$ is an isomorphism. □

Condition (4) in Proposition 4.3.6 shows that a matrix ordered operator space quotient map is a categorical quotient in the category of matrix ordered operator spaces with cbp maps as morphisms. Moreover, the norm bound shows that a quotient map with constant $C = 1$ is a categorical quotient in the subcategory of matrix ordered operator spaces with ccp maps as morphisms.

Remark 4.3.7. Every unital operator system is a matrix ordered operator space, and so if $\varphi : S \to T$ is a ucp map between operator systems with $J = \ker S$, we may form the quotient matrix ordered operator space $S/\ker\varphi$, but there is no a priori guarantee that this quotient is again an operator system. The matrix ordered operator space quotient is generally *not* isomorphic to the unital operator system quotient defined by Kavruk, Paulsen, Todorov, and Tomforde [55]. For example, they show in [55, Example 4.4] that the order norm on the unital operator system quotient need not be completely equivalent to the quotient operator space norm.

4.4 Extension property for compact nc convex sets

If $K = \coprod_n K_n$ is a compact nc convex set, we will define

$$\operatorname{span}_{\mathbb{R}} K := \coprod_{n \geq 1} \operatorname{span}_{\mathbb{R}} K_n \subseteq \mathcal{M}(E).$$

The set $\operatorname{span}_{\mathbb{R}} K$ is also nc convex, but need not be closed in E.

Lemma 4.4.1. *Let $0 \in K \subseteq \mathcal{M}(E)$ be a compact nc convex set containing 0. Let $K - K$ denote the levelwise Minkowski difference of K with itself. Then we have inclusions*

$$\frac{K - K}{2} \subseteq \overline{\operatorname{ncconv}}(K \cup (-K)) \subseteq K - K.$$

Consequently, $\overline{\operatorname{ncconv}}(K \cup (-K)) \subseteq \operatorname{span}_{\mathbb{R}} K$.

Proof. It is immediate that $(K - K)/2 \subseteq \operatorname{ncconv}(K \cup (-K)) \subseteq \overline{\operatorname{ncconv}}(K \cup (-K))$. Given $z \in \operatorname{ncconv}(K \cup (-K))_n$, we can write

$$z = \sum_i \alpha_i^* x_i \alpha_i - \sum_j \beta_j^* y_j \beta_j$$

for uniformly bounded families $\{x_i\}, \{y_i\}$ in K and matrix coefficients satisfying $\sum_i \alpha_i^* \alpha_i + \sum_j \beta_j^* \beta_j = 1_n$. Since $0 \in K$ and $\sum_i \alpha_i^* \alpha_i \leq 1$, we have $x := \sum_i \alpha_i^* x_i \alpha_i \in K_n$. Similarly $y := \sum_j \beta_j^* y_j \beta_j \in K$, and so $z = x - y$ is in $(K - K)_n = K_n - K_n$. Therefore

$$\operatorname{ncconv}(K \cup (-K)) \subseteq K - K,$$

and since the latter is compact, $\overline{\operatorname{ncconv}}(K \cup (-K)) \subseteq K - K$. □

When $0 \in K$, by extending the inclusion map $K \subseteq \coprod_n M_n(A(K, 0)^*)$ linearly at each level, we will think of elements in $(\operatorname{span}_{\mathbb{R}} K)_n$ as nc functionals in

$$M_n(A(K, 0)^*) = \operatorname{CB}(A(K, 0), M_n).$$

Proposition 4.4.2. *Let $0 \in K \subseteq \mathcal{M}(E)$ be a compact nc convex set in a dual operator space $E = (E_*)^*$. For each $n \in \mathbb{N}$, the inclusion $K \to QS(A(K,0))$ extends uniquely to a well-defined nc affine isomorphism*

$$\eta : \coprod_{n\geq 1} \operatorname{span}_{\mathbb{R}} K_n \to \coprod_{n\geq 1} M_n(A(K,0)^*)^{sa}$$

which is levelwise linear. The norm unit ball in $M_n(A(K,0)^*)^{sa}$ is

$$B_1(M_n(A(K,0)^*)) = CC(A(K,0), M_n) = \overline{\operatorname{ncconv}}(\eta(K) \cup (-\eta(K)))_n,$$

and for each n, η is homeomorphic on $K_n - K_n$.

Proof. Since K_n is convex, we have $\operatorname{span}_{\mathbb{R}} K_n = \{sx - ty \mid x, y \in K_n, s, t \geq 0\}$. Given $sx - ty \in \operatorname{span}_{\mathbb{R}} K_n$, we define

$$\eta(sx - ty)(a) = sa(x) - ta(y)$$

for $a \in A(K,0)$. Since such functions a are affine and satisfy $a(0) = 0$, it follows that $\eta|_{K_n}$ is well-defined and linear, and that η is nc affine. Since E_* contains a separating family of functionals, which restrict to nc affine functions in $A(K,0)$, the map η is injective.

Next we will show the closed unit ball is

$$B_1(M_n(A(K,0)^*)^{\mathrm{sa}}) = \overline{\operatorname{ncconv}}(\eta(K) \cup (-\eta(K)))_n$$

for every n. That is, if L is the compact nc convex set

$$L = \coprod_{n\geq 1} L_n = \coprod_{n\geq 1} B_1(M_n(A(K,0)^*)^{\mathrm{sa}}),$$

we want to show $L = \overline{\operatorname{ncconv}}(\eta(K) \cup (-\eta(K)))$. Since $\eta(K)$ consists of nc quasistates on $A(K,0)$, it is clear that $L \supseteq \overline{\operatorname{ncconv}}(\eta(K) \cup (-\eta(K)))$. To prove the reverse inclusion, by the nc Bipolar theorem of Effros and Winkler [32], it suffices to suppose that for some $n \in \mathbb{N}$ and $a \in M_n(A(K,0))^{\mathrm{sa}}$ that we have

$$\varphi_n(a) \leq 1_k \otimes 1_n = 1_{kn}$$

for all $k \in \mathbb{N}$ and all $\varphi \in \overline{\operatorname{ncconv}}(\eta(K) \cup (-\eta(K)))$, and then show that $\psi_n(a) \leq 1_k \otimes 1_n$ for all k and all $\psi \in L_k$. Because $\overline{\operatorname{ncconv}}(\eta(K) \cup (-\eta(K)))$ contains both $\eta(K)$ and $-\eta(K)$, we have

$$-1_{kn} \leq a(x) \leq 1_{kn}$$

for all k and all $x \in K_k$. Hence $\|a\|_{M_n(A(K,0))} \leq 1$, and so $\psi_n(a) \leq \|a\| 1_{kn} \leq 1_{kn}$ for every $\psi \in L$. This proves $L = \overline{\text{ncconv}}(\eta(K) \cup (-\eta(K)))$, and consequently η is also surjective. Since η is homeomorphic on K and $K - K$ is (levelwise) compact, it is easy to check that η is continuous on each $K_n - K_n$. Being a continuous injection on a compact Hausdorff space, the map $\eta|_{K_n - K_n}$ is automatically a homeomorphism onto its range. $\square$

Recall that the pair $(K, 0)$ in Proposition 4.4.2 is a **pointed nc convex set** exactly when we have

$$\coprod_{n\geq 1} B_1(M_n(A(K,0)^*)^+) = \mathrm{QS}(A(K,0)) = \eta(K).$$

In practice, we will often identify $M_n(A(K,0)^*)^{\mathrm{sa}}$ with $\mathrm{span}_{\mathbb{R}} K_n$ and so omit the symbol η. Note that since η is homeomorphic on $K - K \supseteq \overline{\text{ncconv}}(K \cup (-K))$ (Lemma 4.4.1), we are free to identify

$$\overline{\text{ncconv}}(\eta(K) \cup (-\eta(K))) = \eta(\overline{\text{ncconv}}(K \cup (-K))).$$

That is, when we identify $M_n(A(K,0)^*)^{\mathrm{sa}} = \mathrm{span}_{\mathbb{R}} K_n$, the unit ball of $M_n(A(K,0)^*)^{\mathrm{sa}}$ is $\overline{\text{ncconv}}(K \cup (-K))_n$.

For a closed convex set X in a vector space V containing 0, we use the usual Minkowski functional

$$\gamma_X(v) := \inf\{t \geq 0 \mid v \in tX\}, \qquad v \in V.$$

If $0 \in K = \bigcup_n K_n$ is a compact nc convex set over a dual operator space E, we will use the shorthand

$$\gamma_K(x) = \gamma_{K_n}(x)$$

when $x \in M_n(E)$.

Definition 4.4.3. [76] If X is a closed convex set in some vector space V, then for $d \in V$, we define the **width of V (with respect to d)** or the **d-width of V** as

$$\begin{aligned} |X|_d &:= \sup\{t \geq 0 \mid td \in X - X\} \\ &= \frac{1}{\gamma_{X-X}(d)}. \end{aligned}$$

Definition 4.4.4. If $K = \coprod_n K_n \subseteq \mathcal{M}(E)$ is a closed nc convex set over a dual operator space E, then for any n and any $d \in M_n(E)$ we define the **width**

$$|K|_d := |K_n|_d = \frac{1}{\gamma_{K-K}(d)},$$

Lemma 4.4.5. *If $0 \in K \subseteq \mathcal{M}(E)$ is a compact nc convex set containing 0, then for $d \in M_n(E)$, we have $|K|_d > 0$ if and only if $d \in \operatorname{span}_{\mathbb{R}} K$. Moreover, for $d \in \operatorname{span}_{\mathbb{R}} K$, we have*

$$\frac{1}{|K|_d} \le \|\eta(d)\|_{M_n(A(K,0)^*)} \le \frac{2}{|K|_d}.$$

That is, $d \mapsto 1/|K|_d = 1/|K_n|_d$ defines a norm on $\operatorname{span}_{\mathbb{R}} K_n$ that is equivalent to the norm induced by the isomorphism $\eta : \operatorname{span}_{\mathbb{R}} K_n \to M_n(A(K,0)^*)^{sa}$.

Proof. By Lemma 4.4.1, we have inclusions

$$\frac{K-K}{2} \subseteq \overline{\operatorname{ncconv}}(K \cup (-K)) \subseteq K - K.$$

It follows that for $d \in \operatorname{span}_{\mathbb{R}} K$, we have

$$2\gamma_{K-K}(d) \ge \gamma_{\overline{\operatorname{ncconv}}(K\cup(-K))}(d) \ge \gamma_{K-K}(d).$$

By definition, $\gamma_{K-K} = 1/|K|_d$. By Proposition 4.4.2, the norm unit ball of $M_n(A(K,0)^*)^{\mathrm{sa}}$ is

$$\overline{\operatorname{ncconv}}(\eta(K) \cup (-\eta(K))) = \eta(\overline{\operatorname{ncconv}}(K \cup (-K))),$$

and hence $\gamma_{\overline{\operatorname{ncconv}}(K\cup(-K))}(d) = \gamma_{\overline{\operatorname{ncconv}}(\eta(K)\cup(-\eta(K)))}(\eta(d)) = \|\eta(d)\|$. ☐

Given compact nc convex sets $0 \in L \subseteq K$. The restriction map $\rho : A(K,0) \to A(L,0)$ is always completely contractive and positive, and has dense range. When is this map an operator space quotient map? Equivalently, this means there is a constant $C > 0$ so that any nc affine function $g \in M_n(A(L,0))$ extends to an nc affine function f on all of K with

$$f|_L = g \quad \text{and} \quad \|f\|_{M_n(A(K,0))} \le C\|g\|_{M_n(A(L,0))}.$$

Here is a noncommutative version of [76, Theorem 1].

Proposition 4.4.6. *Let $0 \in L \subseteq K \subseteq \mathcal{M}(E)$ be compact nc convex sets containing 0. The following are equivalent*

(1) The restriction map $A(K) \to A(L)$ is an operator space quotient map.

(2) The restriction map $\rho : A(K,0) \to A(L,0)$ is an operator space quotient map.

(3) The dual map $\rho^* : A(L,0)^* \to A(K,0)^*$ is completely bounded below.

(4) *There is a constant $c > 0$ such that for all $n \geq 1$ and all $d \in M_n(E)$ with $|L|_d > 0$, we have*

$$|L|_d \geq c|K|_d.$$

(5) *There is a constant $C > 0$ such that*

$$(K - K) \cap \operatorname{span}_{\mathbb{R}} L \subseteq C(L - L).$$

Proof. Clearly (1) implies (2). Suppose $\rho : A(K, 0) \to A(L, 0)$ is an operator space quotient map with constant $C \geq 0$. Given $a \in A(L)$, we have $a - a(0) \otimes 1_{A(L)} \in A(L, 0)$. Thus there is a $b \in A(K, 0)$ with $b|_L = a - a(0) \otimes 1_{A(L)}$ and $\|b\| \leq C\|a - a(0) \otimes 1_{A(L)}\| \leq 2C\|a\|$. Then, $b + a(0) \otimes 1_{A(K)} \in A(K)$ restricts to a on L and satisfies $\|b + a(0) \otimes 1_{A(K)}\| \leq \|b\| + \|a\| \leq (2C + 1)\|a\|$. This proves $A(K) \to A(L)$ is an operator space quotient map with constant $2C + 1$, so (2) implies (1).

The equivalence of (2) and (3) is Proposition 4.3.2. To prove (3) is equivalent to (4), first note by taking real and imaginary parts that (3) occurs if and only if the restrictions $\rho_n^* : M_n(A(L, 0)^*)^{\text{sa}} \to M_n(A(K, 0)^*)^{\text{sa}}$ are bounded below by a universal constant. By Proposition 4.4.2, we may identify

$$\operatorname{span}_{\mathbb{R}} L_n = M_n(A(L, 0)^*)^{\text{sa}} \quad \text{and} \quad \operatorname{span}_{\mathbb{R}} K_n = M_n(A(K, 0)^*)^{\text{sa}}.$$

With this identification, ρ^* is just the inclusion map $\operatorname{span}_{\mathbb{R}} L_n \to \operatorname{span}_{\mathbb{R}} K_n$. By Lemma 4.4.5, the induced norms on $\operatorname{span}_{\mathbb{R}} L$ and $\operatorname{span}_{\mathbb{R}} K$ are completely equivalent to $d \mapsto 1/|L|_d$ and $d \mapsto 1/|K|_d$. Thus the dual map ρ^* is completely bounded below if and only if for some constant $c > 0$, we have

$$\frac{1}{c|K|_d} \leq \frac{1}{|L|_d} \iff |L|_d \geq c|K|_d$$

whenever $d \in \operatorname{span}_{\mathbb{R}} L = \{d \in \mathcal{M}(E) \mid |L|_d > 0\}$, by Lemma 4.4.5.

For $d \in \mathcal{M}(E)$, recall that $|K|_d = \frac{1}{\gamma_{K-K}(d)}$ and $|L|_d = \frac{1}{\gamma_{L-L}(d)}$. Hence condition (3) holds if and only if

$$\gamma_{L-L}|_{\operatorname{span}_{\mathbb{R}} L} \leq \frac{1}{c}\gamma_{K-K}|_{\operatorname{span}_{\mathbb{R}} L} = \gamma_{c(K-K)}|_{\operatorname{span}_{\mathbb{R}} L}.$$

Using only the definition of the Minkowski gauges γ_{K-K} and γ_{L-L}, this holds if and only if

$$c(K - K) \cap \operatorname{span}_{\mathbb{R}} L \subseteq L - L.$$

Hence condition (4) holds with constant $c > 0$ if and only if condition (5) holds with constant $C = 1/c > 0$. □

Note that for any general inclusion $L \subseteq K$ of compact nc convex sets, we can freely translate to assume $0 \in L$ and apply Proposition 4.4.6. Thus conditions (1), (4), and (5) are equivalent in total generality. Note also that we do not require in 4.4.6 that $(L, 0)$ and $(K, 0)$ are *pointed* nc convex sets.

Example 4.4.7. It is possible that the restriction map $A(K, 0) \to A(L, 0)$ in Proposition 4.4.6 is surjective but not an operator space quotient. For instance, let E be an infinite dimensional Banach space. Let $\max(E)$ and $\min(E)$ denote E equipped with its maximal and minimal operator space norms which restrict to the usual norm on E [32, Section 3.3]. There are standard operator space dualities $\max(E)^* \cong \min(E^*)$ and $\min(E)^* \cong \max(E^*)$. As E is infinite dimensional, the maximal and minimal matrix norms on E are not completely equivalent [68, Theorem 14.3]. So, the identity map $\max(E) \to \min(E)$ is surjective and not an operator space quotient map. Consider the minimal and maximal nc unit balls

$$K = \coprod_{n \geq 1} B_1(M_n(\min(E^*))) \quad \text{and} \quad L = \coprod_{n \geq 1} B_1(M_n(\max(E^*)))$$

in $\mathcal{M}(E^*)$. By the dualities $\max(E)^* \cong \min(E^*)$ and $\min(E)^* \cong \max(E^*)$, we have

$$A(K, 0) \cong \max(E) \quad \text{and} \quad A(L, 0) \cong \min(E)$$

completely isometrically. The restriction map $A(K, 0) \to A(L, 0)$ is just the identity map $\max(E) \to \min(E)$, which is surjective, but not an operator space quotient map.

Proposition 4.4.6 provides a guarantee that every matrix-valued nc affine function on L lifts to an nc affine function on K with a complete norm bound. However, there is no guarantee that we can lift a *positive* affine function to one that is positive. For instance, the restriction map of function systems

$$A([-1, 1], 0) \to A([0, 1], 0)$$

is an operator space quotient map with constant $c = 1$, but does not map the positives onto the positives because $A([-1, 1], 0)^+ = \{0\}$.

Proposition 4.4.8. *Let $0 \in L \subseteq K \subseteq \mathcal{M}(E)$ be compact nc convex sets such that $(L, 0)$ and $(K, 0)$ are pointed compact nc convex sets. Let $\rho : A(K, 0) \to A(L, 0)$ be the restriction map. The following are equivalent*

(1) For all $n \geq 1$, $\overline{\rho_n(M_n(A(K, 0)^+))} = M_n(A(L, 0))^+$.

(2) The dual map $\rho^* : A(L, 0)^* \to A(K, 0)^*$ is a complete order embedding.

(3) $K \cap \operatorname{span}_{\mathbb{R}} L \subseteq \mathbb{R}_+ L$.

(4) $K \cap \overline{\operatorname{ncconv}}(L \cup (-L)) = L$.

Proof. To prove (1) $\iff$ (2), consider the closed nc convex sets

$$P = \coprod_{n \geq 1} M_n(A(L,0))^+ \quad \text{and} \quad Q = \coprod_{n \geq 1} \overline{\rho_n(M_n(A(K,0))^+)}.$$

By the nc Bipolar theorem of Effros and Winkler [32], we have $Q = P$ if and only if their nc polars Q^π and P^π are equal. But by scaling, we have

$$\begin{aligned} P^\pi &= \{\varphi \in M_k(A(L,0)^*) \mid k \in \mathbb{N}, \operatorname{Re} \varphi_n(b) \leq 1_{nk} \text{ for all } n \geq 1, b \in P_n\} \\ &= \{\varphi \in M_k(A(L,0)^*) \mid k \in \mathbb{N}, \operatorname{Re} \varphi \leq 0\} \end{aligned}$$

and similarly

$$\begin{aligned} Q^\pi &= \{\varphi \in M_k(A(L,0)^*) \mid k \in \mathbb{N}, \operatorname{Re} \varphi_n(\rho_n(a)) \leq 1_{nk} \text{ for all } n \geq 1, a \in M_n(A(K,0))^+\} \\ &= \{\varphi \in M_k(A(L,0)^*) \mid k \in \mathbb{N}, \operatorname{Re} \rho_k^*(\varphi) \leq 0\}. \end{aligned}$$

Thus $P = Q$ if and only if ρ^* is a complete order injection.

When we identify $A(K,0)^* = \operatorname{span}_{\mathbb{R}} K_1$ and $A(L,0)^* = \operatorname{span}_{\mathbb{R}} L_1$ as in Proposition 4.4.2, the dual map $\rho^* : \operatorname{span}_{\mathbb{R}} L \to \operatorname{span}_{\mathbb{R}} K$ is just the inclusion map. Since $(K,0)$ and $(L,0)$ are pointed, the positive cones in $M_n(A(K,0)^*) = \operatorname{span}_{\mathbb{R}} K_n$ and $M_n(A(L,0)^*) = \operatorname{span}_{\mathbb{R}} L_n$ are just $\mathbb{R}_+ K_n$ and $\mathbb{R}_+ L_n$, respectively. Hence the inclusion map is a complete order injection if and only if we have

$$\mathbb{R}_+ K \cap \operatorname{span}_{\mathbb{R}} L = \mathbb{R}_+ L.$$

A rescaling argument shows that this is equivalent to

$$K \cap \operatorname{span}_{\mathbb{R}} L \subseteq \mathbb{R}_+ L,$$

and so (2) and (3) are equivalent.

If $K \cap \overline{\operatorname{ncconv}}(L \cup (-L)) = L$, then scaling gives

$$\mathbb{R}_+(K \cap \operatorname{span}_{\mathbb{R}} L) = \mathbb{R}_+ K \cap \operatorname{span}_{\mathbb{R}} L = \mathbb{R}_+ L,$$

which is again equivalent to (3), so (4) implies (3). Now suppose that $K \cap \operatorname{span}_{\mathbb{R}} L \subseteq \mathbb{R}_+ L$. Clearly $L \subseteq K \cap \overline{\operatorname{ncconv}}(L \cup (-L))$. Conversely, if $x \in K \cap \overline{\operatorname{ncconv}}(L \cup (-L))$, then by Lemma 4.4.1, we also have $x \in K \cap \operatorname{span}_{\mathbb{R}} L = \mathbb{R}_+ L$. Hence

$$x \in \overline{\operatorname{ncconv}}(L \cup (-L)) \cap \mathbb{R}_+ L.$$

Because $(L,0)$ is pointed, this implies $x \in L$, proving that (3) implies (4). $\square$

Combining Propositions 4.4.6 and 4.4.8 yields

Theorem 4.4.9. *Let* $(L, 0)$ *and* $(K, 0)$ *be pointed compact nc convex sets with* $L \subseteq K \subseteq \mathcal{M}(E)$. *The following are equivalent.*

(1) The restriction map $A(K, 0) \to A(L, 0)$ *is a matrix ordered operator space quotient map.*

(2) There is a constant $C > 0$ *such that*

(i) $(K - K) \cap \operatorname{span}_{\mathbb{R}} L \subseteq C(L - L)$, *and*

(ii) $K \cap \operatorname{span}_{\mathbb{R}} L \subseteq \mathbb{R}_+ L$.

4.5 Dualizability via nc quasistate spaces

Recall that the trace class operators $\mathcal{T}(H) = B(H)_*$ inherit a matrix ordered operator space structure via the embedding $\mathcal{T}(H) = B(H)_* \subseteq B(H)^*$, where $B(H) \cong (B(H)_*)^*$ completely isometrically and order isomorphically. By Ng's [65] results, since $B(H)$ is a C*-algebra, $B(H)^*$ is an operator system, and so $\mathcal{T}(H) = B(H)_* \subseteq B(H)^*$ is also an operator system. The nc quasistate space of $\mathcal{T}(H)$ is the compact nc convex set

$$\mathcal{P}(H) := \coprod_n M_n(B(H))_1^+ = \coprod_n \{x \in M_n(B(H)) | x \geq 0, \|x\| \leq 1\}.$$

Applying Theorem 4.4.9 and Proposition 4.3.6 yields the following extrinsic geometric characterization of dualizability for an operator system.

Corollary 4.5.1. *Let* S *be an operator system with pointed nc quasistate space* $(K, 0)$, *and let* H *be a Hilbert space. The following are equivalent.*

(1) There is a weak-$*$ homeorphic complete embedding $S^* \to B(H)$.

(2) There is a matrix ordered operator space quotient map $\mathcal{T}(H) \to S$.

(3) There is a pointed continuous nc affine injection $\varphi : (K, 0) \to \mathcal{P}(H)$ *such that*

(i) $(\mathcal{P}(H) - \mathcal{P}(H)) \cap \operatorname{span}_{\mathbb{R}} \varphi(K) \subseteq C(\varphi(K) - \varphi(K))$ *for some constant* $C > 0$, *and*

(ii) $\mathcal{P}(H) \cap \operatorname{span}_{\mathbb{R}} \varphi(K) \subseteq \mathbb{R}_+ \varphi(K)$.

Definition 4.5.2. Let E be an ordered $*$-Banach space with closed positive cone E^+. We say E is **α-generated** for a constant $\alpha > 0$ if for each $x \in E^{\mathrm{sa}}$, we can write

$$x = y - z$$

for $y, z \in E^+$ satisfying $\|y\| + \|z\| \le \alpha\|x\|$. Or, equivalently,

$$B_1(E) = \alpha \operatorname{conv}(B_1(E^+) \cup (-B_1(E^+))).$$

If X is a matrix ordered operator space, then we say X is **completely α-generated** if each matrix level $M_n(X)$ is α-generated.

In [65, Theorem 3.9], Ng proved that an operator system S is dualizable if and only if it is completely α-generated for some $\alpha > 0$. The following definition is the dual property of α-generation.

Definition 4.5.3. An ordered $*$-Banach space E is **α-normal** for some $\alpha > 0$ if for all $x, y, z \in E^{\mathrm{sa}}$,

$$x \le y \le z \implies \|y\| \le \alpha \max\{\|x\|, \|z\|\}. \tag{4.3}$$

If X is a matrix ordered operator space, then X is **completely α-normal** if each matrix level $M_n(X)$ is α-normal.

The condition of α-normality can be viewed as a strict requirement about how the norm and order structure on E interact. Normality means that "order bounds" $x \le y \le z$ should imply "norm bounds" $\|y\| \le \alpha \max\{\|x\|, \|z\|\}$. If one does not care about the exact value of α, it is enough to check the normality identity (4.3) on positive elements in the special case $x = 0$.

Proposition 4.5.4. *If E is an ordered $*$-Banach space, then E is α-normal for some $\alpha > 0$ if and only if there is a constant $\beta > 0$ such that*

$$0 \le x \le y \implies \|x\| \le \beta\|y\| \tag{4.4}$$

for $x, y \in E^+$.

Proof. If E is α-normal, then (4.4) holds with $\beta = \alpha$. Conversely, suppose (4.4) holds, and let $x \le y \le z$ in E^{sa}. Then $0 \le y - x \le z - x$, and so $\|y - x\| \le \beta\|z - x\|$. Then, we get the bound

$$\begin{aligned}\|y\| &\le \|y - x\| + \|x\| \\ &\le \beta\|z - x\| + \|x\| \\ &\le \beta(\|z\| + \|x\|) + \|x\| \\ &\le (2\beta + 1)\max\{\|x\|, \|z\|\},\end{aligned}$$

proving E is $(2\beta + 1)$-normal. $\square$

Proposition 4.5.5. *Let X be a matrix ordered operator space, with dual matrix ordered operator space X^*, and let $\alpha > 0$. If X is completely α-generated, then X^* is completely 2α-normal. Conversely, if X^* is completely α-normal, then X is completely 2α-generated.*

Proof. Suppose that X is completely α-generated. Let $k \in \mathbb{N}$ and suppose $x, y, z \in M_k(X^*)^{\mathrm{sa}}$ satisfy $x \le y \le z$ in the dual matrix ordering on X^*. By definition of the dual norm, we have

$$\|y\|_{M_k(X^*)} = \sup\{\|\langle\!\langle a, x\rangle\!\rangle\| \mid n \ge 1, a \in M_n(X)^{\mathrm{sa}}\},$$

where $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denotes the usual matrix pairing between $\mathcal{M}(X)$ and $\mathcal{M}(X^*)$. Given $n \in \mathbb{N}$ and $a \in M_n(X)^{\mathrm{sa}}$, we can write $a = b - c$ where $b, c \in M_n(X)^+$ satisfy $\|b\| + \|c\| \le \alpha\|a\|$. Then, we have the operator inequality

$$\begin{aligned}\langle\!\langle a, y\rangle\!\rangle &= \langle\!\langle b, y\rangle\!\rangle - \langle\!\langle c, y\rangle\!\rangle \\ &\le \langle\!\langle b, z\rangle\!\rangle - \langle\!\langle c, x\rangle\!\rangle \\ &\le (\|z\|\|b\| + \|x\|\|c\|)1_{nk} \\ &\le (\|x\| + \|z\|)\alpha\|a\|1_{nk}.\end{aligned}$$

Symmetrically,

$$\begin{aligned}\langle\!\langle a, y\rangle\!\rangle &\ge \langle\!\langle b, x\rangle\!\rangle - \langle\!\langle c, z\rangle\!\rangle \\ &\ge -(\|x\|\|b\| + \|z\|\|c\|)1_{nk} \\ &\ge -(\|x\| + \|z\|)\alpha\|a\|1_{nk}.\end{aligned}$$

It follows that

$$\|\langle\!\langle a, y\rangle\!\rangle\| \le (\|x\| + \|z\|)\alpha\|a\|.$$

Since a was arbitrary, this shows $\|y\| \le \alpha(\|x\| + \|z\|) \le 2\alpha \max\{\|x\|, \|z\|\}$, proving X^* is completely 2α-normal.

Now suppose X^* is completely 2α-normal. Consider the closed matrix convex subsets

$$\begin{aligned}K &:= \coprod_{n\ge1} B_1(M_n(X)^{\mathrm{sa}}) = B_1(\mathcal{M}(X)^{\mathrm{sa}}), \\ K^+ &:= \coprod_{n\ge1} B_1(M_n(X)^+) = K \cap \mathcal{M}(X)^+, \\ L &:= \overline{\mathrm{ncconv}}\,(K^+ \cup (-K^+))\end{aligned}$$

of $\mathcal{M}(X)$. We will show that $K \subseteq \alpha L$.

To prove $K \subseteq \alpha L$, by the selfadjoint version of the nc separation Theorem of Effros and Winkler [26, Theorem 2.4.1], it suffices to show that the selfadjoint nc polars

$$K^\rho := \coprod_{n\geq 1}\{x \in M_n(X)^{\mathrm{sa}} \mid \langle\!\langle a, x\rangle\!\rangle \leq 1_{nk} \text{ for all } k \geq 1, x \in K_k\}$$

and L^ρ (defined similarly) satisfy $L^\rho \subseteq \alpha K^\rho$. The relevant selfadjoint polars are

$$\begin{aligned}
K^\rho &= \coprod_{k\geq 1} B_1(M_k(X^*)),\\
(K^+)^\rho &= K^\rho - \mathcal{M}(X^*)^+\\
&= \coprod_{k\geq 1}\{x \in M_k(X^*)^{\mathrm{sa}} \mid x \leq y \text{ for some } y \in K^\rho\}, \qquad \text{and}\\
L^\rho &= (K^+)^\rho \cap (-K^+)^\rho\\
&= (K^\rho - \mathcal{M}(X^*)^+) \cap (K^\rho + \mathcal{M}(X^*)^+)\\
&= \coprod_{k\geq 1}\{y \in M_k(X^*)^{\mathrm{sa}} \mid x \leq y \leq z \text{ for some } x, z \in K^\rho\}.
\end{aligned}$$

Hence, if $y \in L^\rho_k$, then y satisfies $x \leq y \leq z$ for some $x, z \in M_k(X^*)^+$ with $\|x\|, \|z\| \leq 1$. By complete α-normality, this implies $\|y\| \leq \alpha$, so $y \in \alpha K^\rho$. This proves $L^\rho \subseteq \alpha K^\rho$, so $K \subseteq \alpha L$.

Hence $K \subseteq \alpha L = \alpha\overline{\mathrm{ncconv}}(K^+ \cup (-K^+))$. Using Lemma 4.4.1, we have

$$\overline{\mathrm{ncconv}}(K^+ \cup (-K^+)) \subseteq K^+ - K^+.$$

Hence $K \subseteq \alpha(K^+ - K^+)$, and by rescaling every element $x \in \mathcal{M}(X)^{\mathrm{sa}}$ can be decomposed as $x = y - z$ with $y, z \geq 0$ and $\|y\|, \|z\| \leq \alpha\|x\|$, and so $\|y\| + \|z\| \leq 2\alpha\|x\|$. This shows X is completely 2α-normal. □

Remark 4.5.6. If H is a Hilbert space, then $B(H)$ is completely 1-normal. Consequently, if X is a matrix ordered operator space which is completely norm and order isomorphic to a subspace of $B(H)$ (a quasi-operator system), then X must be α-normal for some $\alpha > 0$.

Because complete α-normality is dual to complete α-generation, [65, Theorem 3.9] can be viewed as a partial converse to Remark 4.5.6. If $X = S^*$ is the dual of an operator space, then if X is completely α-normal, it is a dual quasi-operator system. Translating the normality condition into a condition on the nc quasistate space gives the following intrinsic characterization of dualizability.

Theorem 4.5.7. *Let $(K, 0)$ be a pointed compact nc convex set, with associated operator space $S = A(K, 0)$. The following are equivalent.*

(1) S^* is a dual quasi-operator system.

(2) S is completely α-generated for some $\alpha > 0$.

(3) S^* is completely α-normal for some $\alpha > 0$.

(4) There is a constant $C > 0$ such that

$$(K - \mathbb{R}_+ K) \cap \mathbb{R}_+ K \subseteq CK,$$

where $K - \mathbb{R}_+ K$ denotes the levelwise Minkowski difference.

(5) The closed nc convex set $(K - \mathbb{R}_+ K) \cap \mathbb{R}_+ K$ is bounded.

Proof. The equivalence of (1) and (2) was proved by Ng in [65, Theorem 3.9]. Proposition 4.5.5 shows that (2) and (3) are equivalent. To prove that (3) and (4) are equivalent, we may use Proposition 4.4.2 to identify $\coprod_{n\geq 1} M_n(S^*)^{\mathrm{sa}} = \operatorname{span}_{\mathbb{R}} K$. After doing so, the positive elements in $\mathcal{M}(S^*)$ correspond to the closed nc convex set $\mathbb{R}_+ K$, and for $d \in \mathbb{R}_+ K_n$, we have $\|d\|_{M_n(S^*)} = \gamma_K(d)$. Consequently,

$$\begin{aligned}(K - \mathbb{R}_+ K) \cap \mathbb{R}_+ K &= \{d \in \operatorname{span}_{\mathbb{R}} K \mid 0 \leq d \leq x \text{ for some } x \in K\} \\ &= \{d \in \coprod_n M_n(S^*)^{\mathrm{sa}} \mid 0 \leq d \leq x \text{ for some } x > 0 \text{ in } K_n \text{ with } \|x\| \leq 1\}.\end{aligned}$$

Thus (4) holds if and only if

$$0 \leq x \leq y \text{ and } \|y\| \leq 1 \implies \|x\| \leq C,$$

in $M_n(S^*)^{\mathrm{sa}}$ for all $n \in \mathbb{N}$. By rescaling, this is equivalent to asserting that

$$0 \leq x \leq y \implies \|x\| \leq C\|y\|$$

in $M_n(S^*)^{\mathrm{sa}}$. Then, Proposition 4.5.4 shows that if (3) holds, then (4) holds with $C = \alpha$, and if (4) holds, then (3) holds with $\alpha = 2C + 1$. Finally, because $(K - \mathbb{R}_+ K) \cap \mathbb{R}_+ K$ is a subset of $\mathbb{R}_+ K$, on which the matrix norms from S^* agree with the Minkowski gauge γ_K, (4) holds if and only if $(K - \mathbb{R}_+ K) \cap \mathbb{R}_+ K$ is bounded by $C > 0$, i.e. if and only if (5) holds. □

Remark 4.5.8. The analogous version of Theorem 4.5.7 holds in the classical case: If $(K, 0)$ is a pointed compact convex set, then the nonunital function system $A(K, 0)$ is α-generated for some $\alpha > 0$ if and only if $(K - \mathbb{R}_+) \cap \mathbb{R}_+ K$ is bounded.

Corollary 4.5.9. *Let $z \in K \subseteq L$ be compact nc convex sets such that (K, z) and (L, z) are pointed. If $A(L, z)$ is dualizable, then so is $A(K, z)$.*

Proof. By translating, it suffices to consider this when $z = 0$. This follows by noting that

$$(K - \mathbb{R}_+ K) \cap \mathbb{R}_+ K \subseteq (L - \mathbb{R}_+ L) \cap \mathbb{R}_+ L,$$

and using condition (5) in Theorem 4.5.7. □

In [56, Section 8], quotients of (nonunital) operator systems were defined. There, a quotient of operator systems $S \to S/J$ corresponds dually to a restriction map $A(K, z) \to A(M, z)$ between pointed compact nc convex sets, where $M \subseteq K$ is the annihilator of the kernel $J \subseteq K$. Applying Corollary 4.5.9 gives

Corollary 4.5.10. *If S is a dualizable operator system, then every quotient of S is dualizable.*

4.6 Positive generation versus completely bounded positive generation

Classically, if an ordered Banach space E is **positively generated** in the sense that $E^{\mathrm{sa}} = E^+ - E^+$, then E is in fact α-generated for some $\alpha > 0$. This is a consequence of the Baire category theorem [7, Theorem 2.1.2]. In the special case where $E = A(K, 0)$ is the nonunital function system of continuous affine functions on a pointed compact convex set K containing 0 which vanish at 0, the following classical analogue of Theorem 4.5.7 holds: The function system $A(K, 0)$ is α-generated if and only if the classical convex set $(K - \mathbb{R}_+ K) \cap \mathbb{R}_+ K \subseteq A(K, 0)^*$ is bounded. If $A(K, 0)$ is positively generated, it is a consequence of the Banach-Steinhaus Principle of Uniform Boundedness that $(K - \mathbb{R}_+ K) \cap \mathbb{R}_+ K$ is bounded, and so $A(K, 0)$ automatically has bounded positive generation. This proof is essentially the dual version of the proof of [7, Theorem 2.1.2].

In this section, we discuss the noncommutative situation. First, we show that an operator system S has **complete positive generation**, meaning $M_n(S)^{\mathrm{sa}} = M_n(S)^+ - M_n(S)^+$ for all $n \geq 1$, if and only if S is positively generated at the first level. In contrast to the classical situation, complete positive generation need not imply complete α-generation. In Example 4.6.6, we give an example of a matrix ordered operator space which is positively generated but not completely α-generated for any $\alpha > 0$.

One might also consider the following weaker property. Call an ordered Banach space E **approximately positively generated** if $E^+ - E^+$ is dense in E. Note that even though the postiive cone E^+ is closed, it need not be the case that $E^+ - E^+$ is closed, even when E is an operator space, as the following example shows.

Example 4.6.1. Let $S = C([0,1])$, and define S^+ to be the closed cone of functions which are both positive and convex. Then $S^+ - S^+$ is dense in $S = C([0,1])$, because it contains all C^2 functions, but $S^+ - S^+ \neq S$, because the convex functions in S^+ are automatically differentiable almost everywhere on the interior $(0,1)$. So, S is an ordered Banach space which is approximately positively generated, but not positively generated. In fact, S is an operator system. Indeed, if we let

$$K = \{\varphi \in S^* \mid \|\varphi\| \leq 1 \text{ and } \varphi(S^+) \subseteq [0,\infty)\}$$

be the classical quasistate space of K, then since every probability measure on $[0,1]$ lies in K, the natural map

$$S \to A(K)$$

into the continuous affine functions on K is isometric and order isomorphic. That is, S is isometrically order isomorphic to a nonunital function system, and so inherits an operator system structure.

There are many examples of the same kind as Example 4.6.1. It suffices to take any function system S, and equip it with a new closed positive cone $P \subseteq S^+$ for which $P - P$ is not closed. In a private correspondence, Ken Davidson suggested another example in which $S = \mathbb{C} \oplus c_0$ is equipped with the new positive cone

$$P = \Big\{(t,(x_n)_{n\geq 1}) \in \mathbb{C} \oplus c_0 \mid t \geq 0, (x_n)_{n\geq 1} \geq 0, \text{ and } \sum_{n=1}^{\infty} x_n \leq t\Big\}.$$

Here, again $P - P$ is dense and not closed in S.

Proposition 4.6.2. *Let S be an operator system with quasistate space $K \subseteq S^*$. Then S is approximately positively generated if and only if S^+ separates points in K.*

Proof. If S is densely spanned by its positives, then the positives must separate points in K. Conversely, suppose that S is not positively generated. Then there exists an element $x \in S^{sa} \setminus \overline{(S^+ - S^+)}$. By the Hahn-Banach Separation Theorem, there is a self-adjoint linear functional $\varphi \in S^*$ so that for all $y \in S^+ - S^+$ we have

$$\varphi(x) < \varphi(y).$$

But since $S^+ - S^+$ is a real vector space, this implies that φ is identically zero on $S^+ - S^+$. Moreover, by the Hahn-Jordan decomposition theorem there are positive functionals $\varphi^+, \varphi^- \in E^d$ with $\varphi = \varphi^+ - \varphi^-$. Since $\varphi(x) < 0$, the functionals φ^+ and φ^- are necessarily distinct, but they are equal on $S^+ - S^+$ and hence on S^+. Normalizing $\varphi^{\pm}$ to obtain quasistates shows that S^+ does not separate quasistates. □

Remark 4.6.3. The Hahn-Jordan decomposition theorem ensures that, as an ordered vector space, the dual space S^* is always positively generated.

By the following result, if S is positively generated then so are each of its matrix levels $M_n(S)$. Again by [7, Chapter 2, Theorem 1.2], each $M_n(S)$ is α_n-generated for some α_n. In order for S to be dualizable, we would need the sequence (α_n) to be bounded.

Proposition 4.6.4. *If S is positively generated, then so is $M_n(S)$ for each n.*

Before proving this, we will need a technical lemma which proves a much stronger statement in the finite dimensional setting.

Lemma 4.6.5. *If S is a finite dimensional and positively generated operator system, then it contains a matrix order unit.*

Proof. Since S is positively generated, then it admits a basis $B = \{p_1, \dots, p_m\}$ consisting of positive elements. We claim that $e := \sum_{i=1}^m p_i$ is an order unit. For any x in S^{sa}, we can write x uniquely as a real linear combination

$$x = \sum_{i=1}^m \alpha_i p_i,$$

and we define $\lambda_x := \max\{1, |\alpha_1|, \dots, |\alpha_m|\}$. It is clear that $\lambda_x e \pm x$ are positive in S, so e is an order unit.

Next we let $n \geq 0$ and show that $e_n := e \otimes I_n$ is an order unit for $M_n(S)$, so fix an $X = (x_{ij})_{i,j=1}^n \in M_n(S)^{\text{sa}}$. Since E is positively generated, for every $i \leq j$ we can decompose the corresponding entries of X as

$$x_{ij} = \text{Re } x_{ij}^+ - \text{Re } x_{ij}^- + i(\text{Im} x_{ij}^+ - \text{Im} x_{ij}^-).$$

To find a large enough coefficient of e_n to dominate X, we let

$$\lambda_X := \lambda_d + \lambda_{\text{Re}} + \lambda_{\text{Im}}.$$

Where $\lambda_d \coloneqq \max\{\lambda_{x_{ii}}\}_{i=1}^n$, $\lambda_{\mathrm{Re}} \coloneqq \sum_{i<j} \lambda_{\mathrm{Re}\, x_{ij}^+ + \mathrm{Re}\, x_{ij}^-}$, and $\lambda_{\mathrm{Im}} \coloneqq \sum_{i<j} \lambda_{\mathrm{Im} x_{ij}^+ + \mathrm{Im} x_{ij}^-}$. Note that it makes sense to write $x_{ii}^{\pm}$ since the x_{ii} must all be self-adjoint, as they lie on the diagonal of $X = X^*$.

Fix a concrete representation $S \hookrightarrow B(H)$ of S as a norm closed and $*$-closed subspace of the bounded operators on a Hilbert space. We'll show that $\lambda_X e_n + X \geq 0$ concretely using inner products. Take an arbitrary vector $a = (a_i)_{i=1}^n \in H^n = \bigoplus_{i=1}^n H$, and compute

$$\begin{aligned}
\langle(\lambda_X e_n + X)a, a\rangle =& \lambda_X \langle e_n a, a\rangle + \langle X a, a\rangle \\
=& \lambda_X \sum_{i=1}^n \langle e a_i, a_i\rangle + \sum_{i=1}^n \langle x_{ii} a_i, a_i\rangle + \sum_{i<j} \langle x_{ij} a_j, a_i\rangle + \langle x_{ji} a_i, a_j\rangle \\
=& \lambda_X \sum_{i=1}^n \langle e a_i, a_i\rangle + \sum_{i=1}^n \langle x_{ii} a_i, a_i\rangle + \sum_{i<j} 2\mathrm{Re}\, \langle x_{ij} a_j, a_i\rangle \\
=& \left(\lambda_d \sum_{i=1}^n \langle e a_i, a_i\rangle + \sum_{i=1}^n \langle x_{ii} a_i, a_i\rangle\right) \\
&+ \left((\lambda_{\mathrm{Re}} + \lambda_{\mathrm{Im}}) \sum_{i=1}^n \langle e a_i, a_i\rangle + \sum_{i<j} 2\mathrm{Re}\, \langle x_{ij} a_j, a_i\rangle\right) \\
=& \left(\lambda_d \sum_{i=1}^n \langle e a_i, a_i\rangle + \sum_{i=1}^n \langle x_{ii} a_i, a_i\rangle\right) \\
&+ \left(\lambda_{\mathrm{Re}} \sum_{i=1}^n \langle e a_i, a_i\rangle + 2\sum_{i<j} \mathrm{Re}\, \langle \mathrm{Re}\, x_{ij} a_j, a_i\rangle\right) \\
&+ \left(\lambda_{\mathrm{Im}} \sum_{i=1}^n \langle e a_i, a_i\rangle - 2\sum_{i<j} \mathrm{Im}\langle \mathrm{Im} x_{ij} a_j, a_i\rangle\right).
\end{aligned}$$

For the remainder of the proof, we will show that each of the three terms above is non-negative. Starting with the first term,

$$\begin{aligned}
\lambda_d \sum_{i=1}^n \langle e a_i, a_i\rangle + \sum_{i=1}^n \langle x_{ii} a_i, a_i\rangle &= \sum_{i=1}^n \langle(\lambda_d e + x_{ii}) a_i, a_i\rangle \\
&\geq \sum_{i=1}^n \langle(\lambda_{x_{ii}} e + x_{ii}) a_i, a_i\rangle \\
&\geq 0,
\end{aligned}$$

where the last inequality follows from the first paragraph of the proof.

To prove that the second term is non-negative, note

$$\begin{aligned}
&\lambda_{\mathrm{Re}} \sum_{k=1}^{n} \langle e a_k, a_k\rangle + 2\sum_{i<j} \mathrm{Re}\ \langle \mathrm{Re}\ x_{ij} a_j, a_i\rangle \\
=&\sum_{i<j} (\lambda_{\mathrm{Re}\ x_{ij}^+ + \mathrm{Re}\ x_{ij}^-}) \sum_{k=1}^{n} \langle e a_k, a_k\rangle + 2\sum_{i<j} \mathrm{Re}\ \langle \mathrm{Re}\ x_{ij} a_j, a_i\rangle \\
=&\sum_{i<j} (\lambda_{\mathrm{Re}\ x_{ij}^+ + \mathrm{Re}\ x_{ij}^-}) \sum_{k=1}^{n} \langle e a_k, a_k\rangle + 2\mathrm{Re}\ \langle \mathrm{Re}\ x_{ij} a_j, a_i\rangle.
\end{aligned}$$

We now show that for each pair $i < j$, the corresponding summand is non-negative:

$$\begin{aligned}
&(\lambda_{\mathrm{Re}\ x_{ij}^+ + \mathrm{Re}\ x_{ij}^-}) \sum_{k=1}^{n} \langle e a_k, a_k\rangle + 2\mathrm{Re}\ \langle \mathrm{Re}\ x_{ij} a_j, a_i\rangle \\
=&(\lambda_{\mathrm{Re}\ x_{ij}^+ + \mathrm{Re}\ x_{ij}^-}) \sum_{k=1}^{n} \langle e a_k, a_k\rangle + 2\mathrm{Re}\ \langle (\mathrm{Re}\ x_{ij}^+ - \mathrm{Re}\ x_{ij}^-) a_j, a_i\rangle \\
\geq&(\lambda_{\mathrm{Re}\ x_{ij}^+ + \mathrm{Re}\ x_{ij}^-})\langle e a_i, a_i\rangle + (\lambda_{\mathrm{Re}\ x_{ij}^+ + \mathrm{Re}\ x_{ij}^-})\langle e a_j, a_j\rangle + 2\mathrm{Re}\ \langle (\mathrm{Re}\ x_{ij}^+ - \mathrm{Re}\ x_{ij}^-) a_j, a_i\rangle \\
\geq&\left(\langle \mathrm{Re}\ x_{ij}^+ a_i, a_i\rangle + \langle \mathrm{Re}\ x_{ij}^+ a_j, a_j\rangle + 2\mathrm{Re}\ \langle \mathrm{Re}\ x_{ij}^+ a_j, a_i\rangle\right) \\
+&\left(\langle \mathrm{Re}\ x_{ij}^- a_i, a_i\rangle + \langle \mathrm{Re}\ x_{ij}^- a_j, a_j\rangle - 2\mathrm{Re}\ \langle \mathrm{Re}\ x_{ij}^- a_j, a_i\rangle\right) \\
=&\langle \mathrm{Re}\ x_{ij}^+ (a_i + a_j), a_i + a_j\rangle \\
+&\langle \mathrm{Re}\ x_{ij}^- (a_i - a_j), a_i - a_j\rangle \\
\geq&0.
\end{aligned}$$

The last inequality follows since each $\mathrm{Re}\ x_{ij}^{\pm}$ is a positive operator. The proof that the third term is non-negative is similar. □

We now prove Proposition 4.6.4

Proof of Proposition 4.6.4. To show $M_n(S)$ is positively generated, fix $X = (x_{ij})_{i,j=1}^n \in M_n(S)^{\mathrm{sa}}$. Since S is positively generated, each x_{ij} can be written as a linear combination of four positives $\mathrm{Re}\ x_{ij}^+$, $\mathrm{Re}\ x_{ij}^-$, $\mathrm{Im} x_{ij}^+$, and $\mathrm{Im} x_{ij}^-$. Let S_X denote the linear span of these positives, as i and j range from 1 to n. Since S_X is a finite dimensional operator system, by the previous lemma there is a matrix order unit $e_X \in S_X$ and in particular there is a constant $\lambda > 0$ so that both $\lambda 1_n \otimes e_X \pm X \geq 0$. Since $X = (\lambda 1_n \otimes e_X + X)/2 - (\lambda 1_n \otimes e_X - X)/2$ and all entries are ultimately in S, this shows $M_n(S)$ is positively generated. □

So, complete positive generation coincides with positive generation at the first level. However, the following example shows that for matrix ordered operator spaces, positive generation at all matrix levels does not imply complete α-generation for any α.

Example 4.6.6. Any Banach space E has a unique maximal and minimal system of L^∞-matrix norms which give E an operator space structure and restrict to the norm on E at the first matrix level. We denote the resultant operator spaces by $\max(E)$ and $\min(E)$, respectively. There are natural operator space dualities $\max(E)^* = \min(E^*)$ and $\min(E)^* = \max(E)^*$ [32, Section 3.3].

We will consider the Banach space ℓ^1 and its dual ℓ^∞. Because ℓ^∞ is a commutative C^*-algebra, we have $\ell^\infty = \min(\ell^\infty)$ [32, Proposition 3.3.1]. The embedding $\ell^1 \subseteq (\ell^\infty)^*$ gives a matrix ordered operator space structure on ℓ^1, which coincides with the max norm $\ell^1 = \max(\ell^1)$. Using the natural linear identifications

$$M_n(\ell^\infty) = \ell^\infty(\mathbb{N}, M_n) \quad \text{and} \quad M_n(\ell^1) = \ell^1(\mathbb{N}, M_n),$$

the resultant positive cones in ℓ^∞ and ℓ^1 consist of those sequences of matrices which are positive in each entry.

We will consider the minimal operator space $\min(\ell^1)$ equipped with the same matrix ordering as $\ell^1 = \max(\ell^1)$. Because the matrix cones $M_n(\ell^1)^+ = \ell^1(\mathbb{N}, M_n^+)$ are closed in the topology of pointwise weak-$*$ convergence, which is weaker than the topology induced by either the minimal or maximal norms on $M_n(\ell^1)$, the matrix cones $M_n(\ell^1)^+$ are closed in the minimal norm topology. Thus $\min(\ell^1)$ has the structure of a matrix ordered operator space. Because M_n is 1-generated, it follows that each $M_n(\min(\ell^1)) = \ell^1(\mathbb{N}, M_n)$ is positively generated, so $\min(\ell^1)$ is completely positively generated.

However, we will show that $\min(\ell^1)$ is not completely α-generated for any $\alpha > 0$. We will do so using Proposition 4.5.5, by proving the dual matrix ordered operator space $\min(\ell^1)^* = \max(\ell^\infty)$ (equipped with the usual matrix ordering on ℓ^∞) is not completely α-normal for any $\alpha > 0$. Since ℓ^∞ is infinite dimensional, the minimal and maximal matrix norms on ℓ^∞ are not completely equivalent [68, Theorem 14.3]. Thus there is a sequence $x_k \in M_{n_k}(\ell^\infty)$ for which

$$\|x_k\|_{\min} \le 1 \quad \text{and} \quad \|x_k\|_{\max} \ge k.$$

In the C*-algebras $M_{n_k}(\ell^\infty)$, we can write each x_k as a linear combination

$$x_k = (\operatorname{Re} x_k)^+ - (\operatorname{Re} x_k)^- + i(\operatorname{Im} x_k)^+ - i(\operatorname{Im} x_k)^-$$

of positive elements $(\operatorname{Re} x_k)^\pm, (\operatorname{Im} x_k)^\pm$ of min-norm at most 1. Since $\|x_k\|_{\max} > k$, by suitably choosing $y_k \in \{(\operatorname{Re} x_k)^\pm, (\operatorname{Im} x_k)^\pm\}$, we can obtain a sequence of positive elements $y_k \in M_{n_k}(\ell^\infty)^+$ with

$$\|y_k\|_{\min} \le 1 \quad \text{and} \quad \|y_k\|_{\max} > k/4.$$

Since the minimal norm on $M_{n_k}(\ell^\infty)$ is just the usual C*-algebra norm, we have $0 \le y_k \le 1_{M_{n_k}(\ell^\infty)}$. Because the maximal norms satisfy the L^∞-matrix identity, we have $\|1_{M_{n_k}(\ell^\infty)}\|_{\max} = 1$. Thus

$$0 \le y_k \le 1_{M_{n_k}(\ell^\infty)}, \quad \|1_{M_{n_k}(\ell^\infty)}\|_{\max} \le 1, \quad \text{and} \quad \|y_k\|_{\max} > k/4$$

for all $k \in \mathbb{N}$. So, ℓ^∞ is not completely $k/4$-normal, and taking $k \to \infty$ shows that ℓ^∞ cannot be completely α-normal for any $\alpha > 0$.

Example 4.6.6 is a minimal example of this kind. One cannot restrict to the finite dimensional spaces ℓ^1_d and $\ell^\infty_d = (\ell^1_d)^*$ because the maximal and minimal norms on a finite dimensional Banach space are completely equivalent [68, Theorem 14.3], and so $\max(\ell^1_d) \cong \min(\ell^1_d)$ is a dualizable quasi-operator system.

4.7 Examples and applications

4.7.1 Nonunital operator system pushouts and coproducts

If $K = \coprod_{n\ge1} K_n$ and $L = \coprod_{n\ge1} L_n$ are compact nc convex sets, we denote by

$$K \times L := \coprod_{n\ge1} K_n \times L_n$$

their levelwise cartesian product. In [47], it was shown that $A(K \times L)$ is the categorical coproduct of the unital operator systems $A(K)$ and $A(L)$ in the category of unital operator systems with ucp maps as morphisms. The following result will let us assert a similar result in the pointed context, for nonunital operator systems.

Proposition 4.7.1. *Let (K, z) and (L, w) be pointed compact nc convex sets. Then $(K \times L, (z, w))$ is pointed, and there is a vector space isomorphism*

$$A(K \times L, (z, w)) \cong A(K, z) \oplus A(L, w).$$

Proof. We will prove the result in the special case when $z = 0$ and $w = 0$ in the ambient spaces containing K and L. The general case follows by translation. Define a linear map $A(K, z) \oplus A(L, w) \to A(K \times L, (z, w))$ by $(a, b) \mapsto a \oplus b$, where $(a \oplus b)(x, y) := a(x) + b(y)$ for $x \in K$, $y \in L$. Since $a(z) = 0 = b(w)$, it is easy to see that this map is injective. Given

$c \in A(K \times L, (0,0))$, let $a(x) = c(x,0)$ and $b(y) = c(0,y)$ for $x \in K$, $y \in L$. Then since $c(0,0) = 0$,

$$\begin{aligned} c(x,y) &= 2c\left(\frac{x}{2}, \frac{y}{2}\right) \\ &= 2\left(\frac{c(x,0)}{2} + \frac{c(0,y)}{2}\right) \\ &= a(x) + b(y) = (a \oplus b)(x,y). \end{aligned}$$

This proves that $A(K,0) \oplus A(L,0) \to A(K \times L, (0,0))$ is a linear isomorphism.

Now, it will follow from this isomorphism that $(K \times L, (z,w))$ is pointed. Let $\rho : A(K \times L, (z,w)) \to M_n$ be any nc quasistate. Then

$$\varphi(a) = \rho(a \oplus 0) \quad \text{and} \quad \psi(b) = \rho(0 \oplus b)$$

define nc quasistates on $A(K,0)$ and $A(L,0)$, respectively. Because $(K,0)$ and $(L,0)$ are pointed, all nc quasistates are point evaluations, so we have $\varphi(a) = a(x)$ and $\varphi(b) = b(y)$ for some $(x,y) \in (K \times L)_n$ and all $a \in A(K,0)$, $b \in A(L,0)$. From linearity, it follows that ρ is just point evaluation at (x,y), so $(K \times L, (0,0))$ is pointed. $\square$

Definition 4.7.2. Let S and T be operator systems with respective nc quasistate spaces $(K,0)$ and $(L,0)$. We define the **operator system coproduct** to be the vector space $S \oplus T$ equipped with the operator system structure such that

$$S \oplus T \cong A(K,0) \oplus A(L,0) \cong A(K \times L, (0,0))$$

is a completely isometric complete order isomorphism.

Explicitly, the matrix norms on $S \oplus T$ satisfy

$$\|(x,y)\|_{M_n(S \oplus T)} = \sup\{\|\varphi_n(x) + \psi_n(y)\| \mid \varphi \in K, \psi \in L\}$$

for $(x,y) \in M_n(S \oplus T) = M_n(S) \oplus M_n(T)$. The matrix cones just identify $M_n(S \oplus T)^+ = M_n(S)^+ \oplus M_n(T)^+$.

Proposition 4.7.3. *The bifunctor $(S,T) \mapsto S \oplus T$ is the categorical coproduct in the category of operator systems with ccp maps as morphisms. That is, given any operator system R and ccp maps $\varphi : S \to R$ and $\psi : T \to R$, the linear map $\varphi \oplus \psi : S \oplus T \to R$ is ccp.*

Proof. This follows either by the explicit description of the matrix norms and order on $S \oplus T$, or by showing that $(K \times L, (0,0))$ is the categorical *product* of $(K,0)$ and $(L,0)$ in the category of pointed compact nc convex sets, and using Theorem 4.2.11. $\square$

Remark 4.7.4. The operator space norm on $S \oplus T$ is neither the usual ℓ^∞-product nor the ℓ^1-product of the operator spaces S and T. For example, if

$$K = L = \coprod_{n \geq 1} \{x \in M_n^+ \mid 0 \leq x \leq 1_n\}$$

is the nc simplex generated by $[0,1]$, and $a \in A(K,0)$ is the coordinate function $a(x) = x$, then

$$\begin{aligned} \|a \oplus a\|_{A(K^2,(0,0))} &= 2 > \|a \oplus a\|_\infty \quad \text{and} \\ \|a \oplus (-a)\|_{A(K^2,(0,0))} &= 1 < \|a \oplus a\|_1. \end{aligned}$$

Proposition 4.7.5. *Let S and T be operator systems. If S and T are dualizable, then $S \oplus T$ is dualizable.*

Proof 1. We will use Theorem 4.5.7. Let the nc quasistate spaces of S and T be $(K,0)$ and $(L,0)$, respectively. Then $(K - \mathbb{R}_+ K) \cap \mathbb{R}_+ K$ and $(L - \mathbb{R}_+ L) \cap \mathbb{R}_+ L$ are norm bounded. Checking that

$$(K \times L - \mathbb{R}_+(K \times L)) \cap \mathbb{R}_+(K \times L) \subseteq ((K - \mathbb{R}_+ K) \cap \mathbb{R}_+ K) \times ((L - \mathbb{R}_+ L) \cap \mathbb{R}_+ L)$$

shows that $(K \times L - \mathbb{R}_+(K \times L)) \cap \mathbb{R}_+(K \times L)$ is bounded, so $S \oplus T \cong A(K \times L, (0,0))$ is dualizable. □

It is also possible to give a proof of Proposition 4.7.5 using only Ng's bounded decomposition property, which appears in 4.5.7.(2).

More generally, we can form finite pushouts in the operator system category by taking pullbacks in the category of pointed compact nc convex sets.

Definition 4.7.6. Let

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \downarrow{\scriptstyle \psi} & & \\ T & & \end{array}$$

be a diagram of operator systems with ccp maps as morphisms. Let S, T, and R, have respective quasistate spaces $(K,0)$, $(L,0)$, and $(M,0)$. We define the **pushout** $S \oplus_{R,\varphi,\psi} T$ as the operator system

$$A(K \times_{M,\varphi^*,\psi^*} L, (0,0)),$$

where

$$K \times_{M,\varphi^*,\psi^*} L = \{(x,y) \in K \times L \mid \varphi^*(x) = \psi^*(y)\} \subseteq K \times L,$$

equipped with the natural maps

$$\begin{aligned} \iota_S &: S \to S \oplus T \to S \oplus_{R,\varphi,\psi} T \quad \text{and} \\ \iota_T &: T \to S \oplus T \to S \oplus_{R,\varphi,\psi} T \end{aligned}$$

which make the diagram

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "\psi"'] & S \arrow[d, "\iota_S"] \\ T \arrow[r, "\iota_T"] & S \oplus_{R,\varphi,\psi} T \end{tikzcd} \tag{4.5}$$

commute.

When the morphisms φ and ψ are understood, we will usually just write $S \oplus_R T$ and $K \times_M L$. Note that the coproduct $S \oplus T$ coincides with the pushout $S \oplus_0 T$ of the diagram

$$\begin{tikzcd} 0 \arrow[r, "0"] \arrow[d, "0"'] & S \\ T & \end{tikzcd}$$

as expected, where 0 denotes the 0 operator system.

To verify that $A(K \times_M L, (0,0))$ is an operator system, we need to show that:

Proposition 4.7.7. *$(K \times_M L, (0,0))$ is pointed.*

Proof. Let $\rho : A(K \times_M L, (0,0)) \to M_n$ be an nc quasistate. Pulling ρ back to $A(K \times L, (0,0))$ gives a point evaluation at some point $(x,y) \in K \times L$. It will suffice to show that $(x,y) \in K \times_M L$, in which case ρ must be point evaluation at (x,y).

We must show that $\varphi^*(x) = \psi^*(y)$ in M. Given $a \in R \cong A(M,0)$. Since the diagram (4.5) commutes, upon pulling back to $S \oplus T$, we have

$$\rho(\iota_S\varphi(a)) = (\varphi(a) \oplus 0)(x,y) = (0 \oplus \psi(a))(x,y) = \rho(\iota_T\psi(a)),$$

that is, $\varphi(a)(x) = a(\varphi^*(x)) = \psi(a)(y) = a(\psi^*(y))$. Since $a \in R = A(M,0)$ was arbitrary, this proves $\varphi^*(x) = \psi^*(y)$, so $(x,y) \in K \times_M L$. $\square$

Proposition 4.7.8. *The diagram 4.5 is a pushout in the category of operator systems with ccp maps as morphisms.*

Proof. It is easiest to verify that the diagram

$$\begin{tikzcd} (K \times_M L, (0,0)) \arrow[r] \arrow[d] & (K, 0) \arrow[d, "\varphi^*"] \\ (L, 0) \arrow[r, "\psi^*"] & (M, 0) \end{tikzcd}$$

is a pullback in the category of pointed compact nc convex sets with pointed continuous nc affine functions as morphisms, where the unlabeled maps are just the coordinate projections. Checking this is fairly immediate, using the fact that the point-weak-$*$ topology on $K \times_M L \subseteq K \times L$ coincides with the restriction of the product topology. By the contravariant equivalence of categories Theorem 4.2.11, it follows that (4.5) is a pushout. □

Proposition 4.7.9. *If S and T are dualizable operator systems, then any pushout $S \oplus_{R,\varphi,\psi} T$ is also dualizable.*

Proof. This follows from Proposition 4.7.5 combined with Corollary 4.5.9 used with the inclusion $(0,0) \subseteq K \times_M L \subseteq K \times L$. □

It follows by induction that any finite pushout of dualizable operator systems is again dualizable.

4.7.2 A new proof of Choi's theorem

In [65], Ng showed that if S is a dualizable operator system, then there is a canonical choice of completely equivalent matrix norm on the dual S^* for which S^d is an operator system, embedding completely isometrically into some $B(H)$. This canonical dual matrix norm is

$$\|f\|_d = \sup\{\|f_n(x)\| \mid n \geq 1, x \in M_n(S)^+, \|x\| \leq 1\}, \quad m \geq 1, f \in M_m(S^*)$$

where the key difference is that the supremum is taken only over *positive* elements x. Ng denotes by S^d the operator system S^* renormed with the matrix norms $\|\cdot\|_d$.

Theorem 4.7.10. *The nc quasistate space (K, z) of M_n is pointedly affinely homeomorphic to $(\coprod_{k=1}^{\infty} M_k(M_n)_1^+, 0)$, and its nc extreme points consist of unitary conjugates of the Choi matrix $\sum_{i,j=1}^{n} E_{ij} \otimes E_{ij}$ together with the zero scalar.*

Proof. Note that the canonical map $\Phi : M_n \to M_n^d$ given by $\Phi(E_{ij}) = \delta_{ij}$ is a complete order isomorphism, where $\delta_{ij}(E_{kl}) = 1$ when $(i,j) = (k,l)$ and 0 otherwise.

In particular, we can write $M_n^d = A(K,z)$ and view K as lying in the ambient space $\coprod_{k=1}^\infty M_k(M_n^{dd}) = \coprod_{k=1}^\infty M_k(M_n)$. The last equality follows from M_n being finite dimensional. Under this identification, and with the matrix norms $\|\cdot\|_d$ on M_n^d, it is clear that the completely contractive and completely positive maps on M_n^d are precisely the elements of $(\coprod_{k=1}^\infty M_k(M_n)_1^+, 0)$.

This proves that M_n is isomorphic as an operator system to $A(\coprod_{k=1}^\infty M_k(M_n)_1^+, 0)$. To describe the boundary of the nc quasistate space we note that M_n is a C*-algebra, and so the boundary consists of its irreducible representations. These are precisely the unitary conjugates of the identity map on M_n together with the zero map.

Using the same notation as above, the identity map on M_n can be written as $\sum_{i,j=1}^n E_{ij} \otimes \delta_{ij}$. Indeed, if $(x_{kl}) \in M_n$ then

$$\begin{aligned}\left(\sum_{i,j=1}^n E_{ij} \otimes \delta_{ij}\right)(x_{kl}) &= \sum_{i,j=1}^n E_{ij} \otimes \delta_{ij}((x_{kl})) \\ &= \sum_{i,j=1}^n E_{ij} \otimes x_{ij} \\ &= \sum_{i,j=1}^n x_{ij} E_{ij} \\ &= (x_{kl}).\end{aligned}$$

This shows that $\Phi_n^{-1}(\mathrm{id}_{M_n}) = \sum_{i,j=1}^n E_{ij} \otimes E_{ij}$, where Φ_n denotes the n^{th} amplification of Φ. Hence, as the unitary orbit of id_{M_n} together with the zero map are the extreme boundary of K, the Choi matrix together with the zero scalar are the extreme boundary of $\coprod_{k=1}^\infty M_k(M_n)_1^+$ under the identification $\coprod_{k=1}^\infty M_k(M_n)_1^+ = K$ given above. □

As a corollary, we obtain a celebrated result of Choi [14].

Corollary 4.7.11. *A map $\Phi : M_n \to M_k$ is completely positive if and only if $\sum_{i,j=1}^n E_{ij} \otimes \Phi(E_{ij})$ is positive in $M_n(M_k)$.*

Proof. By identifying M_k to its vector space dual, and applying the standard operation of uncurrying Φ, we obtain a new map $\tilde{\Phi} : M_k(M_n) \to \mathbb{C}$ defined by

$$\tilde{\Phi}(E_{ij} \otimes E_{kl}) = \langle \Phi(E_{ij}), E_{kl} \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the Hilbert-Schmidt inner product on M_k. It is a well known fact that $\tilde{\Phi}$ is a positive functional if and only if Φ is completely positive.

In this way, we may view $\tilde{\Phi}$ as an element of the dual $M_k(M_n)^d = M_k(M_n^d)$. Using the identification $M_k(M_n^d) = M_k(A(\coprod_{m=1}^{\infty} M_m(M_n)_1^+, 0))$, and the further identification that $M_k(A(\coprod_{m=1}^{\infty} M_m(M_n)_1^+, 0)) = A((\coprod_{m=1}^{\infty} M_m(M_n)_1^+, 0), (\mathcal{M}_k, 0))$, we obtain that $\tilde{\Phi}$ is a positive functional if and only if it takes positive values on the extreme boundary of $\coprod_{m=1}^{\infty} M_m(M_n)_1^+$ when viewed as an element of $A((\coprod_{m=1}^{\infty} M_m(M_n)_1^+, 0), (\mathcal{M}_k, 0))$. By the previous result, this happens precisely when its evaluation at the Choi matrix is positive. □

Corollary 4.7.12. *For any contractive positive matrix $A \in M_n$, there are k matrices $X_1, \ldots, X_k \in M_{n,n^2}$ with $X_1 X_1^* + \cdots + X_k X_k^* = 1_n$ so that*

$$A = X_1 C X_1 + \cdots + X_k C X_k,$$

where $C = \sum_{i,j=1}^{n} E_{ij} \otimes E_{ij}$ denotes the Choi matrix. Moreover, k is a polynomial in n.

References

[1] Robert J Archbold and Jack S Spielberg. Topologically free actions and ideals in discrete C*-dynamical systems. *Proceedings of the Edinburgh Mathematical Society*, 37:119–124, 1994.

[2] William B. Arveson. Operator algebras and measure preserving automorphisms. *Acta Mathematica*, 118:95–109, 1967.

[3] William B. Arveson. Subalgebras of C*-algebras. *Acta Mathematica*, 123:141–224, 1969.

[4] William B. Arveson. Subalgebras of C*-algebras II. *Acta Mathematica*, 128:271–308, 1972.

[5] William B. Arveson. The noncommutative Choquet boundary. *Journal of the American Mathematical Society*, 21(4):1065–1084, 2008.

[6] William B. Arveson and Keith B. Josephson. Operator algebras and measure preserving automorphisms II. *Journal of Functional Analysis*, 4:100–134, 1969.

[7] L. Asimow and A. J. Ellis. *Convexity theory and its applications in functional analysis*, volume 16. Academic Press (London), 1983.

[8] Daniel Avitzour. Free products of C*-algebras. *Transactions of the American Mathematical Society*, 271(2):423–435, 1982.

[9] David P. Blecher, Zhong-Jin Ruan, and Allan M. Sinclair. A characterization of operator algebras. *Journal of Functional Analysis*, 89(1):188–201, 1990.

[10] Florin Boca. Free products of completely positive maps and spectral sets. *Journal of Functional Analysis*, 97(2):251–263, 1991.

[11] Marek Bożejko, Michael Leinert, and Roland Speicher. Convolution and limit theorems for conditionally free random variables. *Pacific Journal of Mathematics*, 175(2):357–388, 1996.

[12] Nathanial P. Brown and Narutaka Ozawa. *C*-Algebras and Finite-Dimensional Approximations*. American Mathematical Society, 2008.

[13] Toke M. Carlsen, Nadia S. Larsen, Aidan Sims, and Sean T. Vittadello. Co-universal algebras associated to product systems, and gauge-invariant uniqueness theorems. *Proceedings of the London Mathematical Society*, 103:563–600, 2011.

[14] Man-Duen Choi. Completely positive linear maps on complex matrices. *Linear algebra and its applications*, 10(3):285–290, 1975.

[15] Man-Duen Choi and Edward G. Effros. Injectivity and operator spaces. *Journal of Functional Analysis*, 24(2):156–209, 1977.

[16] Kenneth R. Davidson. *C*-algebras by Example*, volume 6. American Mathematical Society, 1996.

[17] Kenneth R. Davidson, Adam Dor-On, Orr Moshe Shalit, and Baruch Solel. Dilations, inclusions of matrix convex sets, and completely positive maps. *International Mathematics Research Notices*, 2017(13):4069–4130, 2017.

[18] Kenneth R. Davidson, Adam Fuller, and Evgenios T.A. Kakariadis. Semicrossed products of operator algebras by semigroups. *Memoirs of the American Mathematical Society*, 247(1168), 2017.

[19] Kenneth R. Davidson, Adam H. Fuller, and Evgenios T.A. Kakariadis. Semicrossed products of operator algebras: a survey. *New York Journal of Mathematics*, 24:56–86, 2018.

[20] Kenneth R. Davidson and Evgenios T. A. Kakariadis. Conjugate dynamical systems on C*-algebras. *International Mathematics Research Notices*, 2014:1289–1311, 2014.

[21] Kenneth R. Davidson and Evgenios T.A. Kakariadis. A proof of Boca's Theorem. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 149(4):869–876, 2019.

[22] Kenneth R. Davidson and Elias G. Katsoulis. Isomorphisms between topological conjugacy algebras. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2008(621):29–51, 2008.

[23] Kenneth R. Davidson and Elias G. Katsoulis. Dilating covariant representations of the non-commutative disc algebras. *Journal of Functional Analysis*, 259:817–831, 2010.

[24] Kenneth R. Davidson and Elias G. Katsoulis. *Operator Algebras for Multivariable Dynamics*. American Mathematical Society, 2011.

[25] Kenneth R. Davidson and Matthew Kennedy. The Choquet boundary of an operator system. *Duke Mathematical Journal*, 164(15):2989–3004, 2015.

[26] Kenneth R. Davidson and Matthew Kennedy. Noncommutative Choquet theory. *arXiv preprint arXiv:1905.08436*, 2019.

[27] Kenneth R. Davidson and Gelu Popescu. Noncommutative Disc Algebras for Semigroups. *Canadian Journal of Mathematics*, 50:290–311, 1998.

[28] Kenneth R. Davidson and Jean Roydor. C*-envelopes of tensor algebras for multivariable dynamics. *Proceedings of the Edinburgh Mathematical Society*, 53:333–351, 2010. *Contains an error, corrected in corrigendum (2011) or in [50].*

[29] Hung T. Dinh. Discrete product systems and their C*-algebras. *Journal of Functional Analysis*, 102:1–34, 1991.

[30] Adam Dor-On and Elias G. Katsoulis. Tensor algebras of product systems and their C*-envelopes. *Journal of Functional Analysis*, page 108416, 2019.

[31] Michael A. Dritschel and Scott A. McCullough. Boundary representations for families of representations of operator algebras and spaces. *Journal of Operator Theory*, pages 159–167, 2005.

[32] Edward G Effros and Soren Winkler. Matrix convexity: operator analogues of the Bipolar and Hahn–Banach Theorems. *Journal of Functional Analysis*, 144(1):117–152, 1997.

[33] Neal J. Fowler. Compactly-aligned discrete product systems, and generalizations of $\mathcal{O}_\infty$. *International Journal of Mathematics*, 10:721–738, 1999.

[34] Neal J. Fowler. Discrete product systems of Hilbert bimodules. *Pacific Journal of Mathematics*, 204:335–375, 2002.

[35] Neal J. Fowler, Paul S. Muhly, and Iain Raeburn. Representations of Cuntz-Pimsner algebras. *Indiana University Mathematics Journal*, 52:569–606, 2003.

[36] Neal J. Fowler and Iain Raeburn. Discrete product systems and twisted crossed products by semigroups. *Journal of Functional Analysis*, 155:171–204, 1998.

[37] Tobias Fritz. Operator system structures on the unital direct sum of C*-algebras. *Rocky Mountain Journal of Mathematics*, 44(3):913–936, 2014.

[38] Tobias Fritz, Tim Netzer, and Andreas Thom. Spectrahedral containment and operator systems with finite-dimensional realization. *SIAM Journal on Applied Algebra and Geometry*, 1(1):556–574, 2017.

[39] I. Gelfand and M. Naimark. On the imbedding of normed rings into the ring of operators in Hilbert space. *Contemporary Mathematics*, 167:3–19, 1994.

[40] Donald W. Hadwin and Thomas B. Hoover. Operator algebras and the conjugacy of transformations. *Journal of Functional Analysis*, 77:112–122, 1988.

[41] Paul R. Halmos. Normal dilations and extensions of operators. *Summa Brasil. Math*, 2(12):134, 1950.

[42] Masamichi Hamana. Injective envelopes of operator systems. *Publications of the Research Institute for Mathematical Sciences*, 15(3):773–785, 1979.

[43] Frank Hansen and Gert K. Pedersen. Jensen's operator inequality. *Bulletin of the London Mathematical Society*, 35(4):553–564, 2003.

[44] Frank Hansen and Gert K. Pedersen. Jensen's trace inequality in several variables. *International Journal of Mathematics*, 14(06):667–681, 2003.

[45] Astrid an Huef, Brita Nucinkis, Camila F. Sehnem, and Dilian Yang. Nuclearity of semigroup C*-algebras. *Journal of Functional Analysis*, 280(2):108793, 2021.

[46] Adam Humeniuk. C*-envelopes of semicrossed products by lattice ordered abelian semigroups. *Journal of Functional Analysis*, 279(9):108731, 2020.

[47] Adam Humeniuk. Jensens Inequality for Separately Convex Noncommutative Functions. *International Mathematics Research Notices*, 10 2021. rnab282.

[48] Richard Vincent Kadison. *A representation theory for commutative topological algebra*. American Mathematical Society, 1951.

[49] Evgenios T. A. Kakariadis. Semicrossed products of C*-algebras and their C*-envelopes. *arXiv preprint arXiv:1102.2252*, 2011.

[50] Evgenios T. A. Kakariadis and Elias G. Katsoulis. Contributions to the theory of C*-correspondences with applications to multivariable dynamics. *Transactions of the American Mathematical Society*, 364(12):6605–6630, 2012.

[51] Evgenios T. A. Kakariadis and Elias G. Katsoulis. Semicrossed products of operator algebras and their C*-envelopes. *Journal of Functional Analysis*, 262:3108–3124, 2012.

[52] Evgenios T. A. Kakariadis and Justin R. Peters. Representations of C*-dynamical systems implemented by Cuntz families. *Muenster Journal of Mathematics*, 6:383–411, 2013.

[53] Dmitry S. Kaliuzhnyi-Verbovetskyi and Victor Vinnikov. *Foundations of free noncommutative function theory*, volume 199. American Mathematical Society, 2014.

[54] Elias G. Katsoulis and David W. Kribs. Tensor algebras of C*-correspondences and their C*-envelopes. *Journal of Functional Analysis*, 234:226–233, 2006.

[55] Ali S. Kavruk, Vern I. Paulsen, Ivan G. Todorov, and Mark Tomforde. Quotients, exactness, and nuclearity in the operator system category. *Advances in Mathematics*, 235:321–360, 2013.

[56] Matthew Kennedy, Se-Jin Kim, and Nicholas Manor. Nonunital operator systems and noncommutative convexity. *arXiv preprint arXiv:2101.02622*, 2021.

[57] Marcelo Laca. From endomorphisms to automorphisms and back: dilations and full corners. *Journal of the London Mathematical Society*, 61:893–904, 2000.

[58] Marcelo Laca and Iain Raeburn. Semigroup Crossed Products and the Toeplitz Algebras of Nonabelian Groups. *Journal of Functional Analysis*, 139:415–440, 1996.

[59] Boyu Li. Regular dilation and Nica-covariant representation on righ. *Integral Equations and Operator Theory*.

[60] Boyu Li. Regular representations of lattice ordered semigroups. *Journal of Operator Theory*, 76:33–56, 2016.

[61] Xin Li. Semigroup C*-algebras and amenability of semigroups. *Journal of Functional Analysis*, 262:4302–4340, 2012.

[62] Wojciech Młotkowski. Operator-valued version of conditionally free product. *Studia Math.*, 153(1):13–30, 2002.

[63] Paul S. Muhly and Baruch Solel. Tensor algebras over C*-correspondences: Representations, dilations, and C*-envelopes. *Journal of Functional Analysis*, 158:389–457, 1998.

[64] Gerard J. Murphy. Crossed products of C*-algebras by semigroups of automorphisms. *Proceedings of the London Mathematical Society*, 3:423–448, 1994.

[65] Chi-Keung Ng. Dual spaces of operator systems. *Journal of Mathematical Analysis and Applications*, 508(2):125890, 2022.

[66] Alexandru Nica. C*-algebras generated by isometries and Weiner-Hopf operators. *Journal of Operator Theory*, 27:17–52, 1992.

[67] Alexandru Nica and Roland Speicher. *Lectures on the combinatorics of free probability.* Number 335 in London Mathematical Society Lecture Note Series. Cambridge University Press, 2006.

[68] Vern Paulsen. *Completely bounded maps and operator algebras.* Number 78. Cambridge University Press, 2002.

[69] Justin Peters. Semi-crossed products of C*-algebras. *Journal of Functional Analysis*, 59:498–534, 1984.

[70] Gilles Pisier. *Introduction to operator space theory.* Number 294 in London Mathematical Society Lecture Note Series. Cambridge University Press, 2003.

[71] Zhong-Jin Ruan. Subspaces of C*-algebras. *Journal of functional analysis*, 76(1):217–230, 1988.

[72] Camila F. Sehnem. On C*-algebras associated to product systems. *Journal of Functional Analysis*, 277:558–593, 2019.

[73] Aidan Sims and Trent Yeend. C*-algebras associated to product systems of Hilbert bimodules. *Journal of Operator Theory*, pages 349–376, 2010.

[74] W. Forrest Stinespring. Positive functions on C*-algebras. *Proceedings of the American Mathematical Society*, 6(2):211–216, 1955.

[75] B. Sz.-Nagy. Sur les contractions de l'espace de Hilbert. *Acta Sci. Math. (Szeged)*, 15:87–92, 1953.

[76] Peter D. Taylor. The extension property for compact convex sets. *Israel Journal of Mathematics*, 11(2):159–163, 1972.

[77] Corran Webster and Soren Winkler. The Krein-Milman theorem in operator convexity. *Transactions of the American Mathematical Society*, 351(1):307–322, 1999.

[78] Wend Werner. Subspaces of L(H) that are *-invariant. *Journal of Functional Analysis*, 193(2):207–223, 2002.

[79] Gerd Wittstock. On matrix order and convexity. In *North-Holland Mathematics Studies*, volume 90, pages 175–188. Elsevier, 1984.

[80] Saeid Zahmatkesh. The partial-isometric crossed products by semigroups of endomorphisms are Morita equivalent to crossed products by groups. *New Zealand Journal of Mathematics*, 47, 2017.

[81] Saeid Zahmatkesh. The Nica-Toeplitz algebras of dynamical systems over abelian lattice-ordered groups as full corners. *arXiv preprint arXiv:1912.09682*, 2019.