Almost synchronous correlations defined within tracial von Neumann algebras

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This thesis concerns a class of non-local games known as synchronous games. In recent work, it was discovered independently by [Vid22] and [PP22] that, for any synchronous games, any near-optimal finite dimensional strategy is always near some convex combinations of projective strategies that employ a maximally entangled state. The main technical contribution of this thesis is a proposed proof for extending this result to a more general class of correlations known as the tracial embeddable strategies, which is a subset of the commuting operator strategies. Tracial embeddable strategies consist of the set of strategies which can be realized on the GNS representation of some tracial von Neumann algebra $(\mathscr{A}, \tau)$ using the state τ. In particular, we show that any near optimal tracial embeddable strategy is close to some averages of strategies that use projective measurements on tracial states, an infinite analogue of a result about maximally entangled states.

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Academia has always been a long and treacherous journey for me personally, and I was extremely fortunate to have many mentors and companions who had helped me along the way. It was still quite unbelievable about the progress I had made in the past four years throughout the latter half of my undergraduate degree and my master's degree. To put it quite frankly, I don't know how I made it this far, but I know I am forever in debt to those who helped to make this miracle possible.

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[1] The basement.

[2] and Babbage!

[3] Nice!

[4] Do not trust him under ANY scenario!

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Chapter 1

Introduction

Non-local games are R-E complete,
This means max tensors are not quite discrete,
Introspect the sampler,
P-C-P the player,
and an anchor to parallel repeat!

A limerick to the complexity of quantum non-local games

The study of two-player non-local games has been important to both the theoretical computer science and the operator algebra community. Roughly speaking, a two-player non-local game consists of two cooperating players who individually are given a question from a referee. Then, without communicating with each other, they are expected to output an answer from a set of predetermined answers. The players are then judged based on the questions that they were given and the answer they committed.

Non-local games have long been considered within the physics community as it could be used to model some of the advantages and limitations of entanglement. Although the players are assumed to be unable to communicate for the duration of the game, they are allowed to prepare a pair of entangled systems between them beforehand. The contrast between the model where the players are assumed to not be allowed to use quantum entanglement (known in the literature as *classical strategy*) and the model where the players are (known as *quantum strategies*) has been used to study the power of entanglement. Notably, in [Bel64], sharing an entangled quantum system between the players allows a richer set of strategies, which also allows the player to achieve a higher success probability in some instances of the non-local game[1]. The study of *quantum value* of a non-local game,

[1]This is the famous bell inequality within the literature

or the optimal success probability of a non-local game assuming the players are allowed to use quantum entanglement, turns out to have a more significant impact in areas outside of theoretical physics than initially expected.

The theoretical computer science community has also studied this type of cooperative game under a different name, a *two-prover, one-round, interactive proof system*. In the computer science literature, the system consists of a verifier with limited computation power, trying to determine whether some given string x is in a specific language (in the complexity theory sense) $\mathcal{L}$. To the verifier's relief, he is given access to two provers which he can query for some evidence for and against whether $x \in \mathcal{L}$. The two provers are assumed to be computationally unbounded but unable to communicate with each other during this process (they are allowed to perform some preprocessing). What makes this model interesting is that the provers are also assumed to act adversely, where they try to convince the verifier that $x \in \mathcal{L}$ even in the cases where this is not true. This assumption creates an interesting dynamic in which the verifier must find a clever way to query the prover to obtain sufficient information about the instance without being tricked. The class of language which can be decided, with high probability, in such a manner is roughly known as $\mathrm{MIP}(2,1)$. Interestingly enough, with access to these provers, the verifier can check a much larger class of languages as it was famously shown that $\mathrm{MIP}(2,1) = \mathrm{NEXP}$ [BFL91]. This line of work also gives rise to the study of *probabilistically checkable proof*, an interesting topic which will not be discussed in this thesis (see [AS98], [AS98], [Din07] for more references on this topic).

At the same time, the study of non-local games has been connected to one of the most important conjectures in operator algebra, the *Connes embedding conjecture*. We will briefly summarise this connection in the next two paragraphs.

This summary is written with the assumption of basic familiarity with the literature on von Neumann algebras. Note that this background is not assumed for the rest of this thesis. The study of hyperfinite von Neumann algebra has been one of the earliest topics in von Neumann algebra. It was famously shown by Murray and von Neumann that up to isomorphism; there is a unique tracial hyperfinite factor[2] [MN43]. The unique factor demonstrating these properties is commonly known as the *hyperfinite II_1 factor* and labelled as $\mathcal{R}$ in the literature. In essence, the conjecture asked whether every tracial von Neumann algebra can be represented on some ultraproduct of $\mathcal{R}$, or more plainly, can we see tracial von Neumann algebra as some tensor copy of $\mathcal{R}$. This conjecture is later shown to be equivalent to the famous Kirchberg QWEP conjecture [Kir93], which studies the *nuclear* properties, or the uniqueness of tensor product, between separable C*-

[2]See [Gol21] for more discussion for hyperfinite von Neumann algebras.

algebra. We will refer to [Oza04] for an excellent survey showing the equivalence of these two conjectures.

Finally, relating this problem to non-local games, it was shown in [JNP+11] that Kirchberg's conjecture is connected to *Tsirelson conjectures* about how entanglements between two quantum systems should be represented. The canonical way for quantum mechanics to represent an entangled system is usually in the form of a vector state within a tensor of two Hilbert spaces $\mathcal{H}_A \otimes \mathcal{B}_B$. Each of the two parties only has access to one of the Hilbert spaces and can only perform an observable on the state using one of the Hilbert spaces. This model is known as the *tensor product model*. Tsirelson conjectures[3] [Tsi93] that this is equivalent to a more general notion of Quantum Mechanics known as the *commuting operator model* of entanglement. In this model, instead of having two different Hilbert spaces, both parties interact in a single joint Hilbert space, assuming both players' observables commute with each other instead.

How does everything above connect? The connection between the study of non-local games and computational complexity was famously observed first on [CHT+10]. Roughly speaking, the role of the Referee in non-local games is being replaced by the role of a prover in an interactive proof system. Based on the language $\mathcal{L}$ and the instance string x, the Referee can design a non-local game in which the players, with the assumption of unbounded computation power, must provide answers without communication. The Referee will then either accept $x \in \mathcal{L}$ if the players win the instance of the game or reject it otherwise.

In this model, if we restrict the player to using only classical strategies, this is precisely the definition for $\mathrm{MIP}(2,1)$ that we have discussed above. However, we allowed both provers to prepare some arbitrary entanglement pairs under the tensor product model of entanglement before the protocol. This complexity class is known as $\mathrm{MIP}^*(2,1)$ (or commonly known as MIP^*) within the literature. Intuitively, a language is known to be in MIP^* if for every instance $x \in \{0,1\}^*$ with $|x| = n$, the Referee can always come up with some non-local games such that the players can win with high probability (usually $\frac{2}{3}$ in the literature) if $x \in \mathcal{L}$, or if $x \notin \mathcal{L}$, the players will have a low probability of winning (usually $\frac{1}{3}$ in the literature). This complexity class is often associated with the following problem:

Given a non-local game $\mathcal{G}$, estimate the success probability of the game $\mathcal{G}$ (up to some $\varepsilon > 0$) assuming the tensor product model of entanglement.

The connection between complexity theory and Connes embedding conjecture stems from finding an algorithm to solve the problem above. Since it is impossible to discretize a

[3] or rather, stated without proof, incorrectly

continuous space in which the quantum strategy is usually represented, it is impossible to compute the exact quantum value for a non-local game[4]. An estimation technique was first proposed in [NPA08], commonly known as the NPA hierarchy. This technique comprises two series, and each entry of the series could be approximated computationally. One of the two series converges from below to the optimal quantum value of the game. The other series converges, from above, to what is known as the *commuting operator value* of the game $\mathcal{G}$, the optimal success probability assuming the players are allowed to use the commuting operator model of entanglement. Hence, assuming Tsirelson's conjecture holds (i.e. the tensor product model can be used to approximate the commuting operator model), the quantum value of the game would be equal to the commuting operator value. Then we could compute the two series until the difference them is within $\frac{\varepsilon}{2}$ to approximate the game's quantum value. On the contrary, if this problem turns out to be uncomputable, then the two models cannot be equivalent; hence, by the chain of implications, this would refute Connes embedding conjecture.

It's not immediately clear what the complexity of MIP^* would be. On the one hand, giving the provers more tools could potentially give the provers more ways to cheat in some instances. However, this also allows the verifier to create a new and inventive protocol. It was first shown in [IV12] that MIP^* is at least as powerful as MIP. However, it was unclear back then whether the two complexity classes were equal. Numerous literatures were published in order to explore this area (See, for example [NV18],[FJV+18], [NV17], [BVY21]). Finally, the strict advantage (assuming $\text{P} \neq \text{NP}$) for MIP^* was shown in [NW19] as it was shown that even $\text{NEEXP} \subseteq \text{MIP}^*$.

Building off the technique from [NW19], [FJV+18] and [BVY21], [JNV+20a] was able to show that MIP^* is equivalent to the complexity class RE, a complexity class which is characterised by the halting problem. In essence, [JNV+20a] shows a procedure to convert an instance of determining whether a Turing Machine halts into an instance of a non-local game, such that if the game has a quantum value of 1 if the Turing machine halts, and less than $\frac{1}{2}$ if it doesn't. This also implies that it is impossible to approximating the quantum value of a non-local game in general, disproving Connes embedding conjecture.

The follow-up question remains, what is the complexity of MIP^{co}, the interactive proof system in which the provers are allowed to use the commuting operator model of entanglement instead? Computing a game's exact commuting operator value is already shown to be coRE-hard[5] [Slo19], or to be equivalent to the complexity class defined by the decision

[4] In fact, this is shown, recently by [MNY21], to be equivalent to Π_2, the second level of the arithmetic hierarchy.

[5] Note that co have a different connotation on both sides!

of a non-halting problem. However, the hardness of approximating this value, which is the complexity of MIP^{co}, is still unknown and it was conjectured to be equal to coRE. One of the primary challenges to this area is that it's still unclear whether certain techniques (for example, the parallel repetition step [BVY21]) in the tensor product model would carry over in the commuting operator model.

In this thesis, we will investigate a technique known as the *rounding lemma*. In the literature, the study of non-local games in the commuting operator model primarily focuses on a class of games known as *synchronous games*. Loosely speaking, a game is synchronous iff both players are given the same questions and answers set within the non-local game and an additional synchronous condition where the players are expected to output the same answers given the same questions. This class of games gave the notion of synchronous strategies, where both players will *always* give the same answer given the same question. Synchronous strategies have been studied extensively in the operator algebra community (see for example [DP16], [KPS18]) because, as proven in [PSS+16], it guarantees that the strategies will always be realized using a *tracial state*. It's been used recently in complexity theory (see [JNV+22], [MNY21], for example) because it simplifies a lot of the analysis required[6]. In principle, the rounding technique is designed to answer the following question: can we all approximate almost synchronous strategies with some averages of synchronous strategies? Where almost synchronous strategies, roughly speaking, are the set of strategies in which the two players could, with a small error probability, output different answers when given the same questions in a synchronous game. In the finite-dimensional case, this was shown to be true independently by [Vid22] and [PP22]. The rounding technique reduces the analysis of the complexity of the synchronous games to the case where the players only use synchronous strategies. Since roughly speaking, if we assigned a larger probability on the synchronous question pair, any optimal strategy with a success rate $1-\varepsilon$ must succeed on the synchronous question with probability $1-O(\varepsilon)$, which could then be approximated into a synchronous strategy. However, this result is unknown in the commuting operator framework.

This thesis is dedicated to studying almost synchronous strategies in a subset of the commuting operator strategies known as *tracial embeddable strategies*. Roughly speaking, tracial embeddable strategies correspond to the set of correlations which can be realized in the GNS representation of some tracial von Neumann algebra $(\mathscr{A}, \tau)$ on the trace τ (we will define this set of strategies in more detail in Section 3). Having the existence of the trace within the representation comes with two natural advantages. The first is the representation allows us to define a notion of symmetric strategies through a map we

[6] see page 7 of [JNV+22] for a more in-depth discussion.

denoted as the modular transpose map[7], which we will explain in more detail in Section 2. The other advantage is that we could apply the so-called "Connes distribution lemma", which is a crucial lemma within [Vid22] and [PP22]. In this thesis, we will replicate the argument made by [Vid22] and provide some insight into the cases where the strategies are defined within the general commuting operator model. Roughly speaking, we propose a proof for the following theorem:

Theorem 1.0.1 (Rounding lemma, informally). *Suppose there is a $\delta-$approximate synchronous, tracial embeddable commuting operator strategy defined within $\mathcal{H}$ on the tracial von Neumann algebra $(\mathcal{A}, \tau)$ with $\{P(x, y|a, b)\}$ being the corresponding correlation set. There exist some set of projectors $\{P_\lambda\}_{\lambda \in [0, \lambda_{max}]} \in \mathcal{A}$ in which we can find a synchronous strategy defined on the tracial von Neumann algebra $(P_\lambda \mathcal{A} P_\lambda)$ using the state $\frac{1}{\tau(P_\lambda)}\tau$, such that we can approximate the original strategy as an average of theses synchronous strategies.*

In Chapter 2, we will introduce some of the basic background material on C* and von Neumann algebras, as well as some of the tools that we invented which are used in this thesis. In Chapter 3, we will define, more precisely, the notion of quantum entanglement and the commuting operator model of entanglement. We will also define the notion of synchronous games and synchronous strategies and give a brief survey of the literature. Finally, we will introduce the notion of an approximate synchronous strategy and tracial embeddable strategies, two notions of commuting operator strategy which we will focus on in this thesis. In Chapter 4, we will introduce and prove the main theorem stated above and discuss some implications of the rounding theorem. We will also discuss some challenges of generalizing the rounding theorem to the general commuting operator model.

[7] This map is equivalent to the Tomita-Takesaki map [Tak70] composed with the adjoint in literature.

Chapter 2

Operator Theory and von Neumann algebras

In this chapter, we will introduce the operator theory background required for this thesis. We will assume basic knowledge of functional analysis for this introduction. We will also introduce the modular transpose map and the Connes joint distribution trick in this chapter, as this is a core component in showing the rounding lemma. For a more comprehensive introduction to this topic, we would recommend [KR97b] and [KR97a].

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the set of bounded linear operators acting on $\mathcal{H}$. Throughout this thesis, we will use $\mathcal{I}$ to represent the identity operator acting on $\mathcal{H}$ (we will often use a subscript to identify where the identity element lies if the context isn't clear). Instead of the standard notation used within the literature of operator algebras, we will adopt the braket notation to align our proof with [Vid22].

For readers not familiar with the braket notation, we will use $|\psi\rangle$ to denote a vector in $\mathcal{H}$ and $\langle\psi|$ to denote the dual function associated with $|\psi\rangle$. For any $A \in \mathcal{B}(\mathcal{H})$, we will use $\langle\psi|A|\psi\rangle$ to denote the conjugation by the vector $|\psi\rangle$ on A (i.e. $\langle A\,|\psi\rangle\,, |\psi\rangle\rangle$).

2.1 C*-algebra basics

In this thesis, we will primarily work with a class of operator algebras known as *von Neumann algebras*. Since many works on non-local games typically apply to a more general type of operator algebra known as C*-algebras, we will start by giving some fundamental theories related to C*-algebra which are required for this thesis.

2.1.1 Basic definitions

The definition from this subsection is taken from [Mar21]. We will first begin defining a notion of an (abstract) C*-algebra.

Definition 2.1.1 (C*-algebra)**.** *A algebra* M *associated with the field* $\mathbb{C}$ *is called a C*-algebra if it is equipped with a norm* $||\cdot|| : \mathsf{M} \to \mathbb{C}$*, and as well as an involution map* $* : \mathsf{M} \to \mathsf{M}$ *(by convention we denote this map as* $*(x) = x^*$*), which, for all* $x, y \in \mathsf{M}$ *and* $\alpha, \beta \in \mathbb{C}$*, satisfies the following relationships:*

- *Cauchy-Schwarz, or* $||x \cdot y|| \leq ||x|| \cdot ||y||$.
- $(x^*)^* = x$.
- $(\alpha x + \beta y)^* = \overline{\alpha} x^* + \overline{\beta} y^*$.
- $(xy)^* = y^* x^*$.
- $||x^*|| = ||x||$.
- *The* C^* *equation, or* $||x^* x|| = ||x||^2$.

An example of a C*-algebra is $\mathcal{B}(\mathcal{H})$, where the involution map corresponds to the adjoint map. For this thesis, we will assume all C*-algebras is *unital*, meaning the algebra M admits an algebraic identity operator $\mathcal{I}$ with the additional constraint that $\mathcal{I}^* = \mathcal{I}$.

C*-algebras can be seen as a natural abstraction of matrices, as the involution operator is designed to capture the characteristics of the conjugate transpose operator. Following this intuition, for every $x \in \mathsf{M}$, say that x is:

- *Self-adjoint* if $x^* = x$.
- *Positive* if there exist some $y \in \mathsf{M}$ such that $x = y^* y$.
- *Unitary* if $x^* x = x x^* = \mathcal{I}_{\mathsf{M}}$.
- A *projector* if $x^2 = x$, (we will often use p to denote a projector).
- A *partial isometry* if $x^* x$ and $x x^*$ are projectors.

For any positive element $x \in \mathsf{M}$, there will always exist some unique positive element $y \in \mathsf{M}$ such that $y^2 = x$ (we will use $\sqrt{x}$ to denote this element).

For every $x \in \mathsf{M}$, we define the *spectrum* of x as $\sigma(x) = \{c \in \mathbb{C}, x - c\mathcal{I}$ is not invertible in $\mathsf{M}\}$. The spectrum of an element intuitively corresponds to the set of eigenvalues for a matrix. However, we note that the spectrum could be continuous (meaning $\sigma(x)$ could be the set $[0, 1]$, unlike in the finite-dimensional case where the eigenvalues have to be discrete). The next theorem gives further intuition on this generalization:

Theorem 2.1.2.

- *If* $p \in \mathsf{M}$ *is a projector, then* $\sigma(p) \subseteq \{0, 1\}$.
- *If* $h \in \mathsf{M}$ *is self-adjoint, then* $\sigma(h) \subseteq \mathbb{R}$.
- *If* $p \in \mathsf{M}$ *is positive, then* $\sigma(p) \subseteq \mathbb{R}_+$.
- *If* $u \in \mathsf{M}$ *is unitary, then* $\sigma(u) \subseteq \mathcal{T}$, *where* $\mathcal{T} = \{x : x \in \mathbb{C}, |c| = 1\}$.

For a map $\tau : \mathsf{M} \to \mathbb{C}$, we call τ a linear functional if for all $a, b \in \mathsf{M}$, $\alpha, \beta \in \mathsf{M}$, we have

$$\tau(\alpha a + \beta b) = \alpha\tau(a) + \beta\tau(b).$$

Furthermore, we call a linear functional $\tau : \mathscr{A} \to \mathbb{C}$ to be:

- *Unital* if $\tau(\mathcal{I}_{\mathsf{M}}) = 1$.
- *Positive* if for every $A \in \mathscr{A}^+$, we have $\tau(A) \geq 0$.
- *Normal* if for every net $\{A_n\} \subseteq \mathscr{A}$ which converges to A, we have $\lim_n \tau(A_n) = \tau(A)$.
- A *state* if the functional is unital and positive.
- *Tracial* if for all $A, B \in \mathscr{A}$, we have $\tau(AB) = \tau(BA)$.

Note that for any unit vector in $|\psi\rangle \in \mathcal{H}$, we can naturally induce a state acting on $\mathcal{B}(\mathcal{H})$ via the linear functional $\pi_{|\psi\rangle}(a) = \langle\psi|a|\psi\rangle$. Note that for any C*-algebra $\mathcal{M}$ which is a subalgebra of $\mathcal{B}(\mathcal{H})$, the functional $\pi_{|\psi\rangle}$ would remain as a state for M. Throughout this thesis, we will always assume that the state we work with is normal.

For two C*-algebras M and N, we call an algebraic homomorphism π between M to N a $*$-*homomorphism* if π respects the involution operator ($\pi(x^*) = \pi(x)^*$), and we call the map to be a $*$-*isomorphism* if the the map is bijective. Furthermore, we say that the two C*-algebras are $*$-*isomorphic* to each other iff there exists a $*$-isomorphism between them. We call a $*$-homomorphism between two C*-algebras to be a representation iff the target space for the $*$-homomorphism is $\mathcal{B}(\mathcal{H})$. We call a $*$-representation *faithful* if π is injective, and *cyclic* if there exist some vector $|\psi\rangle \in \mathcal{H}$ in the representation such that $\overline{\pi(\mathsf{M})\,|\psi\rangle} = \mathcal{H}$ where $\bar{\cdot}$ denotes the norm closure of the Hilbert space. The following theorem shows that every C*-algebra is $*$-isomorphic to a subalgebra of some $\mathcal{B}(\mathcal{H})$.

Theorem 2.1.3 (GNS representation ([AP10], section 2.6.1))**.** *Given a C*-algebra M and a state ρ acting on M. Then there exist a cyclic representation $(\mathcal{H}_\rho, \pi_\rho, |\rho\rangle)$ of M, with $\overline{\pi_\rho(\mathsf{M})\,|\rho\rangle} = \mathcal{H}_\rho$. Such that $||\rho||^{\frac{1}{2}} = \langle\rho|\rho\rangle = 1$, and*

$$\rho(a) = \langle\rho|\pi_\rho(a)|\rho\rangle,$$

for all $a \in \mathsf{M}$.

We call a C*-algebra to be *concrete* iff it is faithfully represented as a subalgebra of some $\mathcal{B}(\mathcal{H})$ and we will call a C*-algebra *abstract* iff it is represented just as an algebra. Finally, for any vector in $|\psi\rangle \in \mathcal{H}$ and a concrete C*-algebra algebra $\mathsf{M} \subseteq \mathcal{B}(\mathcal{H})$, we say the vector is:

- *Cyclic* with respect to M if $\mathsf{M}\,|\psi\rangle = \overline{\mathsf{M}\,|\psi\rangle} = \mathcal{H}$.
- *Separating* with respect to M if $a\,|\psi\rangle = 0$, then $a = 0$ for all $a \in \mathsf{M}$.
- *Tracial* with respect to M if $\tau(A) = \langle\psi|A|\psi\rangle$ is a tracial state.

We note that for a C*-algebra M and a $*$-representation to $\mathcal{B}(\mathcal{H})$, we have the following connection between vectors with the properties listed above and its representation: if the algebra admits a cyclic vector, then by definition, the $*$-representation is cyclic, and if the algebra admits a separating vector $|\psi\rangle$, the $*$-representation has to be faithful (or else we will run into the scenario where $a - b \neq 0$ with $(a - b)\,|\psi\rangle = 0$).

2.1.2 Group C*-algebra

In this section, we will briefly summarise the group C*-algebra construction. The definitions below are taken from section 3.7/3.8 of [Gol21] and appendix C of [Fri12], and we will refer to these two works for a more in-depth introduction to this construction.

Let G be a group; we will define the vector space associated with group G, $l^2(G)$, as the set consists of a finite linear combination of the standard basis e_h for $h \in G$ over the field $\mathbb{C}$. This construction forms a (potentially infinite-dimensional) Hilbert space by taking the completion of the vector space above with respect to the following inner product $\langle \sum_h c_h e_h, \sum_h d_h e_h \rangle = \sum_h \overline{c_h} \cdot d_h$.

For each element $g \in G$, we can define an operator u_g acting on $l^2(G)$ by the action $u_g e_h = e_{hg}$. Note that by extending this construction linearly, this offers a natural homomorphism from the algebra $\mathbb{C}[G]$ to $\mathcal{B}(l^2(G))$. To extend this construction to a C*-algebra, we can further define an involution map on this algebra by the following mapping:

$$* : \sum_h c_h e_h \to \sum_h \overline{c_h} e_{h^{-1}}.$$

We note that the construction above actually induces an unital C*-algebra which is represented on $\mathcal{B}(l^2(G))$ with the identity being e, the operator generated from the identity element $e \in G$ (see [Fri12], proposition C.1 for more detail on how the norm is constructed). This construction is known as the *group C*-algebra*, and we will use $C^*(G)$ to denote this construction.

2.1.3 Tensor products between C*-algebras

In this section, we wish to define some notions of a tensor product between two abstract C*-algebras M and N. We will first present the analytic approach for such a construction. This construction is the common notion of tensor products used within quantum information.

For this construction, we will first associate each M and N with a concrete representation to $\mathcal{B}(\mathcal{H}_\mathsf{M})$ and $\mathcal{B}(\mathcal{H}_\mathsf{N})$. We can naturally extend the $*$-representation by mapping M to the larger Hilbert space $\mathcal{B}(\mathcal{H}_\mathsf{M} \otimes \mathcal{H}_\mathsf{N})$ by the map $x_\mathsf{M} \to x_\mathsf{M} \otimes \mathcal{I}_\mathsf{N}$ (and $x_\mathsf{N} \to \mathcal{I}_\mathsf{M} \otimes x_\mathsf{N}$). We note that for this construction, the representation only affects the norm of the resulting C*-algebra, but it doesn't change the underlying algebraic structure (see [Tak79], chapter IV.4 for more detail). Following this intuition, we define the *minimum tensor product* between two C*-algebra M and N to be the resulting C*-algebra described above, with the norm between the C*-algebra defined as,

$$\begin{aligned} ||m \otimes n||_{\min} = \sup\{||\pi_1(m) \otimes \pi_2(n)|| : \pi_1 \text{ is a } * \text{-representation for } \mathsf{M}, \\ \pi_2 \text{ is a } * \text{-representation for } \mathsf{N}\}, \end{aligned}$$

and we will use $\mathsf{M} \otimes_{\min} \mathsf{N}$ to denote the C*-algebra which arises from the above construction.

Another approach we could take is to use the algebraic approach in order to arrive with the same sets of algebra above. Let $\mathbb{F}(\mathsf{M} \times \mathsf{N})$ be the algebra generated from the elements from M and N respectively, with elements from M always commuting with N. In this regard, we can think of $\mathbb{F}(\mathsf{M} \times \mathsf{N})$ being represented by an order of pair of $(m, n) \in \mathsf{M} \times \mathsf{N}$. To define a tensor product, we will mod out the following properties in which a tensor product is supposed to follow. Namely, for all $m, m^{'} \in \mathsf{M}$, $n, n^{'} \in \mathsf{M}$, we will mod out the following set of elements from $\mathbb{F}(\mathsf{M} \times \mathsf{N})$:

- $(m + m^{'}, n) - (m, n) - (m^{'}, n)$ (this ensures that $(m + m^{'}, n) = (m, n) + (m^{'}, n)$)
- $(m, n + n^{'}) - (m, n) - (m, n^{'})$
- $(\alpha m, n) - \alpha(m, n)$
- $(m, \alpha n) - \alpha(m, n)$.

We will write the algebra generated by this construction as $\mathsf{M} \otimes_{\max} \mathsf{N}$. Note that this forms the same algebraic set as the construction above. However, since $\mathsf{M} \otimes_{\max} \mathsf{N}$ is defined to be a single $C*-$algebra, the norm will be defined according. Namely,

$$||m \otimes n||_{\max} = \sup\{||\pi(m \otimes_{\max} n)|| : \pi \text{ is a } * -\text{representation for } \mathsf{M} \otimes_{\max} \mathsf{N}\},$$

and we will define this the *maximum tensor product* between two C*-algebra and we will use $\mathsf{M} \otimes_{\max} \mathsf{N}$ to denote this construction. Note for any two $*$-representations π_1 for M and π_2 for N, $\pi_1 \otimes \pi_2$ is a valid representation for $m \otimes_{\max} n$. Hence we always have the case that $||m \otimes n||_{\min} \leq ||m \otimes n||_{\max}$ for all $m \in \mathsf{M}$ and $n \in \mathsf{N}$.

We note that the notation of a maximum and minimum tensor product comes from the fact that they correspond to the minimum and maximum C*-algebra norms the constructions above can incur. The study of whether these two tensor product norms are equivalent has been linked to Kirchberg's conjecture[1], which is an equivalent formulation of Connes embedding problem [Oza04]. However, since this is not the primary focus of this thesis, we will refer to [Gol21] chapter 3 for an excellent survey on this topic.

[1]specifically in the case between the tensor product of two groups C*-algebras of $\mathbb{F}_\infty$, the free group on a countable number of generators.

2.2 Von Neumann algebra

This section will provide some basic background related to von Neumann algebras which is required for this thesis. We note that since a von Neumann algebra is a specific class of C*-algebra, the theorem defined above will hold for von Neumann algebras as well.

2.2.1 Preliminary definition

For any concrete C*-algebra $\mathsf{M} \subseteq \mathcal{B}(\mathcal{H})$, we say the C*-algebra is closed under the weak $*$-topology if for every net[2] $\{A_\lambda\}_\lambda \subseteq \mathsf{M}$, there exists some $A \in \mathsf{M}$ such that, for all $|\psi\rangle, |\phi\rangle \in \mathcal{H}$, we have:

$$\lim_{\lambda} || \langle \psi | A_\lambda | \phi \rangle - \langle \psi | A | \phi \rangle || = 0.$$

A (concrete) von Neumann algebra $\mathscr{A}$ is a concrete unital C*-algebra which is also closed under the weak$-*$ topology. We note that it is possible to define a notion of abstract von Neumann algebra using the notion of being equal to it's double dual (see [Sak71] for more detail), and are commonly denoted as W^*-algebras within the literature to distinguish them. We will not go into detail since it will not be the primary focus of our thesis, and we will instead assume all von Neumann algebras used in this thesis are concrete.

A von Neumann has several defining characteristics that make it useful in analysing non-local games. One such property is the relationship between a von Neumann algebra and its commutant. For a von Neumann algebra $\mathscr{A}$, the commutant $\mathscr{A}^{'}$ is defined as

$$\mathscr{A}^{'} = \{B : B \in \mathcal{B}(\mathcal{H}), [A, B] = 0, \text{ for all } A \in \mathscr{A}\}.$$

We note that the commutant of a concrete von Neumann algebra is representation dependent. Two von Neumann algebras can be $*$-isomorphic and have different commutant. However, the commutant of a von Neumann algebra will always obey the following relationship:

Theorem 2.2.1 (Von Neumann bicommutant theorem ([Mar21], theorem 12.26)). *For any unital concrete C*-algebra* M*, we have* $\overline{\mathsf{M}}^{WOT} = (\mathsf{M})''$ *where* $\bar{\cdot}^{WOT}$ *denote the weak $*$-closure of the algebra. In particular, if* M *is a von Neumann algebra, then we have* $\mathsf{M} = (\mathsf{M})''$*. In other words, any von Neumann algebra is equal to its double commutant.*

[2] A net is a generalization of a limit.

The second property is that every element of a von Neumann algebra has a unique polar decomposition. Or a way to represent any element as a product of a positive element and a partial isometry. To be more precise:

Theorem 2.2.2 (Polar Decomposition ([KR97a], Theorem 6.1.2))**.** *For every $\sigma \in \mathscr{A}$, there exists a unique partial isometry $u \in \mathscr{A}$ and a positive element $\sigma^+ = \sqrt{\sigma^* \sigma} \in \mathscr{A}$ such that $\sigma = u\sigma^+$.*

We note that it is possible to define a notion of Borel functional calculus on the self-adjoint elements in a von Neumann algebra, as stated by the following theorem:

Theorem 2.2.3 (Functional Calculus for von Neumann algebras ([AP10], proposition 2.2.1))**.** *Let x be a self-adjoint element of a von Neumann algebra $\mathscr{A}$. Then, for every bounded Borel function f acting on $\sigma(x)$, we have $f(x) \in \mathscr{A}$. In particular, the spectral measure of x takes its value in $\mathscr{A}$.*

Taking any Borel function as any piecewise continuous function with finite domain and range is sufficient for this thesis.

This thesis will also consider cyclic and separating vectors for a given von Neumann algebra $\mathscr{A}$. We record the following theorem relating a vector's cyclic and separating properties. The proof mostly follows from [KR97b], Proposition 5.5.11.

Theorem 2.2.4. *Given a VNA $\mathscr{A} \subseteq \mathcal{B}(\mathcal{H})$, $|\psi\rangle \in \mathcal{H}$ is cyclic and separating for $\mathscr{A}$ if and only if it is cyclic and separating for $\mathscr{A}^{'}$.*

Proof. Note it is sufficient to show that $|\psi\rangle$ is cyclic for $\mathscr{A}$ if and only if $|\psi\rangle$ is separating for $\mathscr{A}^{'}$. Suppose $|\psi\rangle$ is cyclic for $\mathscr{A}$, and for some arbitrary $A^{'} \in \mathcal{H}$, we have $A^{'} |\psi\rangle = 0$. Then for any $|\phi\rangle \in \mathcal{H}$, since $|\psi\rangle$ is cyclic, we can write $|\phi\rangle = A |\psi\rangle$ for some $A \in \mathscr{A}$, note

$$A^{'} |\phi\rangle = A^{'} A |\psi\rangle = AA^{'} |\psi\rangle = 0,$$

which implies that $A^{'} = 0$, implying that $|\psi\rangle$ is separating for $\mathscr{A}^{'}$.

Now, suppose $|\psi\rangle$ is separating for $\mathscr{A}$, let P be the central projection from $\mathcal{H}$ onto $\overline{\mathscr{A}' |\psi\rangle}^{\perp}$. Note that $P \in \mathscr{A}$ (see [KR97b] proposition 5.5.2) and since $\mathbf{1}_{\mathcal{H}} \in \mathscr{A}^{'}$, we have $P |\psi\rangle = 0$, hence we have $P = 0$, and we have $\overline{\mathscr{A}' |\psi\rangle}^{\perp} = \mathcal{H}$, which implies that $|\psi\rangle$ is cyclic for $M^{'}$. □

Finally, this thesis will often consider von Neumann algebra of the form $P\mathscr{A}P$ for some projector $P \in \mathscr{A}$. The following result shows that $P\mathscr{A}P$ is indeed a von Neumann algebra with a structured commutant.

Theorem 2.2.5 ([KR97a], corollary 5.5.7)**.** *If $\mathscr{A}$ is a von Neumann algebra acting on $\mathcal{B}(\mathcal{H})$ and P is a projector in $\mathscr{A}$, then $P\mathscr{A}P$ acting on $P\mathcal{H}$ is a von Neumann algebra with commutant $\mathscr{A}'P$.*

2.2.2 Quantum measurements

In this section, we will define the notion of quantum measurement within the von Neumann algebra formalism. Given a von Neumann algebra $\mathscr{A}$ and a finite set $\mathcal{X}$, we call a set of positive elements $\{A_x\}_{x\in\mathcal{X}}$ a *positive operator valued measure* (POVM) with outcome $\mathcal{X}$ if $\sum_{x\in\mathcal{X}} A_x = \mathcal{I}_{\mathscr{A}}$. Intuitively, for every state π acting on $\mathscr{A}$, since

$$\sum_{x\in\mathcal{X}} \pi(A_x) = \pi\left(\sum_{x\in\mathcal{X}} A_x\right) = \pi(\mathcal{I}_{\mathscr{A}}) = 1.$$

Also note since π is positive, we can form a probability measure with outcome in $\mathcal{X}$ via $P(x) = \pi(A_x)$ for every state π action on $\mathscr{A}$. The following proposition shows that for every projector $P \in \mathscr{A}$, every POVM $\{A_x\}_{x\in\mathcal{X}}$ can be represented as a POVM in the subalgebra $P\mathscr{A}P$:

Proposition 2.2.6. *For any POVM $\{A_a^x\} \subseteq \mathscr{A}$ and a projector P, $\{PA_a^xP\}$ forms a POVM on $P\mathscr{A}P$.*

Proof. Note since conjugation by projectors does not change the positivity for the spectrum of A_a^x. Hence PA_a^xP is also positive. Furthermore,

$$\sum_a PA_a^xP = P\left(\sum_a A_a^x\right)P = P = \mathcal{I}_{P\mathscr{A}P},$$

proving the claim. □

If each of the A_x in a POVM is assumed to be a set of mutually orthogonal projective measurements, we called this a *projective valued measurement* (PVM). By representing the elements in $\mathcal{X}$ with $\mathbb{Z}_{|\mathcal{X}|}$, we can associate any set of PVM with a unitary element in $\mathscr{A}$ via the following transformation,

$$u = \sum_{x\in\mathbb{Z}_{|\mathcal{X}|}} e^{\frac{2x\pi i}{2\pi}} A_x, \tag{2.1}$$

and this is known as an *observable*.

We recall the following result shows that any set of POVM is always close to a PVM.

Theorem 2.2.7 (Orthogonalization lemma ([Sal22], Theorem 1.2))**.** *Let $\mathscr{A} \subseteq \mathcal{B}(\mathcal{H})$ be a VNA with a unit vector $|\sigma\rangle \in \mathcal{H}$, and let $\{A_x\} \subseteq \mathscr{A}$ be a POVM such that $\sum_x \langle\sigma|A_x^2|\sigma\rangle > 1-\epsilon$, Then there exist PVM $\{P_x\} \subseteq \mathscr{A}$ such that*

$$\sum_i \langle\sigma|(|A_x - P_x|^2)|\sigma\rangle < 9\varepsilon.$$

In this thesis, we will use the above lemma to convert POVMs to a PVMs. We note that the standard technique of converting POVMs to PVMs used within the study of quantum information is through the Stinespring dilation theorem (see [Fri12], appendix A for more detail). However, in this thesis, we often assume the existence of a tracial state acting on the von Neumann algebra generated by the measurement set, which does not necessarily hold after applying the Stinespring dilation theorem.

2.2.3 Tracial von Neumann algebra

For this thesis, we will mainly consider a type of von Neumann algebra known as a *tracial von Neumann algebra*, where the von Neumann algebra is assumed to admit a tracial state τ. We will often use $(\mathscr{A}, \tau)$ to denote such a von Neumann algebra to emphasize the existence of a tracial state. For the remainder of this thesis, we will also be assuming our tracial von Neumann algebra to be represented as the GNS representation on the tracial state $(\mathcal{H}_\tau, \pi_\tau, |\tau\rangle)$. This representation is known as the *standard representation* within the literature. In this thesis, we will often write $\tau(A)$ as $\langle\tau|A|\tau\rangle$.

Under the standard representation, the vector $|\tau\rangle$ is both tracial and cyclic with respect to the algebra $\mathscr{A}$. By the following theorem, the vector $|\tau\rangle$ is also separating:

Theorem 2.2.8. *Given a tracial von Neumann algebra $(\mathscr{A}, \tau)$ represented within the standard representation $(\mathcal{H}_\tau, \pi_\tau, |\tau\rangle)$, $|\tau\rangle$ is also separating for $\mathscr{A}$.*

Proof. Note by definition, we need show that for all $A \in \mathscr{A}$, $A|\tau\rangle = 0$ implies that $A = 0$. Hence take any $|\phi\rangle \in \mathcal{H}_\tau$, and note that, since $|\tau\rangle$ is cyclic for $\mathscr{A}$, we can write $|\phi\rangle = \sigma|\tau\rangle$ for some $\sigma \in \mathscr{A}$. Also note:

$$\begin{aligned}||A|\phi\rangle|| &= ||A\sigma|\tau\rangle|| \\ &= \langle\tau|\sigma^* A^* A\sigma|\tau\rangle \\ &= \langle\tau|\sigma\sigma^* A^* A|\tau\rangle = 0,\end{aligned}$$

where the third equality follows from $|\tau\rangle$ being a tracial state. Hence, since $|\phi\rangle$ is arbitrary, we have $A = 0$, and hence $|\tau\rangle$ is cyclic and separating vector for $\pi_\tau(\mathscr{A})$. $\square$

Note that the existence of the tracial state naturally induces an inner product within the von Neumann algebra using the definition

$$\langle A, B\rangle = \tau(A^* B)$$

and note, by Theorem 2.2.8, $\langle A, A\rangle = 0$ implies that $A = 0$. We note that this inner product obeys the Cauchy-Schwarz inequality, or

$$\langle A, B\rangle \leq \sqrt{\langle A^* A\rangle}\sqrt{\langle B^* B\rangle}.$$

Similarly, we can define a notion of the 2-norm for a tracial von Neumann algebra as

$$||A||_2 = \sqrt{\langle A, A\rangle}$$

and, by definition, we can write the above equation as

$$||A||_2 = \sqrt{\langle \tau|A^* A|\tau\rangle} = ||A\,|\tau\rangle\,||.$$

We will give a finite-dimensional example of the definition we described below.

Example 2.2.9 (The operator-vector correspondence ([Wat18], page 23))**.** Given the matrix algebra $\mathbf{M}_n$, the operator-vector correspondence to $\mathbb{C}^{n^2}$ can be described by the following linear map:

$$\begin{gathered} \mathrm{vec} : \mathbf{M}_n \to \mathbb{C}^{n^2} \\ \mathrm{vec}(|a\rangle\langle b|) = |a\rangle\,|b\rangle \end{gathered}$$

For example, for $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathbf{M}(C^2)$, we have $\mathrm{vec}(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) = |00\rangle - |11\rangle \in \mathbb{C}^4$.

Note that the above mapping induced a unique bijection between $\mathbf{M}_n$ to $\mathbb{C}^4$.

This map satisfies the identities:

$$\langle A, B\rangle = \langle \mathrm{vec}(A), \mathrm{vec}(B)\rangle \tag{2.2}$$

$$(A_0 \otimes A_1)\mathrm{vec}(B) = \mathrm{vec}(A_0 B A_1^T). \tag{2.3}$$

Also, for $\mathscr{A} = \mathbf{M}_2(\mathbb{C}) \otimes \mathbb{I}_2 \subseteq M_4(\mathbb{C})$ with $\mathscr{A}' = \mathbb{I}_2 \otimes \mathbf{M}_2(\mathbb{C})$, we note that $|EPR\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |EPR_2\rangle$ is a tracial state for $\mathscr{A}$. Furthermore, it is a cyclic and separating vector for $\mathbb{C}^4$ with respective to $\mathscr{A}$. To see this, note since $\text{vec}(\cdot)$ induces a unique bijection between $\mathbf{M}_2(\mathbb{C})$ and $\mathbb{C}^4$, by (2.3), we have

$$\text{vec}(A) = (A \otimes \mathbb{I}_2)\text{vec}(\mathbb{I}_4)$$

for all $A \in \mathbf{M}_2(\mathbb{C})$ and hence $\text{vec}(\mathbb{I}_4)$ is cyclic for $\mathscr{A}$. Similarly, since $\text{vec}(0) = \vec{0} \in \mathbb{C}^4$, we have $(A \otimes \mathbb{I}_2)\text{vec}(\mathbb{I}_4) = \text{vec}(A) = 0$ if and only if $A = 0$, and hence $\text{vec}(\mathbb{I}_4)$ is separating for $\mathscr{A}$. By a similar calculation, we can see that $\mathbb{I}$ is also cyclic and separating for $\mathscr{A}'$.

Finally, we wish to show the following utility lemma for tracial von Neumann algebras:

Lemma 2.2.10. *Let p, q be two projectors in a tracial VNA $(\mathscr{A}, \tau)$, and $\sigma \in \mathscr{A}^+$, then we have*

$$\langle \tau | \sigma p \sigma q | \tau \rangle = 0 \tag{2.4}$$

Proof. We can rewrite, using the fact that p, q are a projectors and $|\tau\rangle$ is tracial, (2.4) as

$$\langle \tau | q \sigma p p \sigma q | \tau \rangle$$

Note since $(p\sigma q)^* = q^* \sigma^* p^* = q \sigma p$, we have

$$\begin{aligned} 0 &\leq \langle \tau | q \sigma p p \sigma q | \tau \rangle \\ &\leq ||\sigma||_\infty^2 \langle \tau | p q q p | \tau \rangle = 0 \end{aligned}$$

Where the second inequality follows from $\sigma^+ \leq ||\sigma^+||_\infty \mathcal{I}$, hence proving the claim. □

2.2.4 Modular transpose map

In this section, we will define the modular transpose map for a tracial von Neumann algebra $(\mathscr{A}, \tau)$. We note that, by Theorem 2.2.4, for any tracial von Neumann $(\mathscr{A}, \tau)$ defined within the standard representation $(\mathcal{H}_\tau, \pi_\tau, |\tau\rangle)$, the vector $|\tau\rangle$ is both cyclic and separating with respect to $\mathscr{A}$ and $\mathscr{A}'$. This means that, for every $|\psi\rangle \in \mathcal{H}_\tau$, there exists a unique $A \in \mathscr{A}$ and $B \in \mathscr{A}'$ such that $A|\tau\rangle = B|\tau\rangle = |\psi\rangle$. Since $|\tau\rangle$ is also cyclic for both $\mathscr{A}$ and $\mathscr{A}'$, for every $A \in \mathscr{A}$, we can find some unique A' such that:

$$A|\tau\rangle = |\phi\rangle = A'|\tau\rangle.$$

This induced a natural bijection $\mathcal{T}_{|\tau\rangle} : \mathscr{A} \to \mathscr{A}^{'}$ via the map $\Phi(A) = A^{'}$. We will call this map the *modular transpose map*. Note that this map depends on the underlying tracial vector. Hence we will use $\mathcal{T}_{|\tau\rangle}$ to define this map with the underlying tracial state being $|\tau\rangle$. When the tracial state is clear from context, we will use $\mathcal{T}$ instead. We note that if we remove the tracial assumption of $|\tau\rangle$, the map we defined is equivalent to $S_{|\tau\rangle} \circ *$ where $S_{|\tau\rangle}$ is the anti-linear operator used within Tomita-Takasaki theory on the cyclic and separating vector $|\tau\rangle$ [Tak79] [3]. The naming of this map follows from the example given in the last section, where for a tracial vector $|EPR\rangle$, we have:

$$(A \otimes \mathcal{I}_2) |EPR_2\rangle = (\mathcal{I}_2 \otimes A^T) |EPR_2\rangle$$

for all $A \in \mathbf{M}_2(\mathbb{C})$ (in fact, this holds for any $A \in \mathbf{M}_n(\mathbb{C})$ and $|EPR_n\rangle = \frac{1}{\sqrt{n}} \sum_n |nn\rangle$). Similarly to the notion of a transpose, the modular transpose map has the following properties:

Theorem 2.2.11.

$$\mathcal{T}(\lambda A + B) = \lambda \mathcal{T}(A) + \mathcal{T}(B) \tag{2.5}$$
$$\mathcal{T}(AB) = \mathcal{T}(B)\mathcal{T}(A) \tag{2.6}$$
$$\mathcal{T}(A^*) = \mathcal{T}(A)^* \tag{2.7}$$

Proof. (2.5) follows from

$$\mathcal{T}(\lambda A + B) |\tau\rangle = (\lambda A + B) |\tau\rangle = \lambda A |\tau\rangle + B |\tau\rangle = \lambda \mathcal{T}(A) |\tau\rangle + \mathcal{T}(B) |\tau\rangle .$$

(2.6) follows from

$$\begin{aligned} \mathcal{T}(AB) |\tau\rangle = AB |\tau\rangle &= A\mathcal{T}(B) |\tau\rangle \\ &= \mathcal{T}(B)A |\tau\rangle = \mathcal{T}(B)\mathcal{T}(A) |\tau\rangle , \end{aligned}$$

where the second equality follows from $\mathcal{T}(B) \in \mathscr{A}^{'}$. For (2.7), since $|\tau\rangle$ is cyclic for $\mathscr{A}$, it is sufficient to show that for every $\sigma, \rho \in \mathscr{A}$, we have $\langle \tau|\sigma^* \mathcal{T}(A^*) \rho|\tau\rangle = \langle \tau|C^* \mathcal{T}(A)^* D|\tau\rangle$, hence

$$\begin{aligned} \langle \tau|\sigma^* \mathcal{T}(B^*) \rho|\tau\rangle &= \langle \tau|\rho\sigma^* \mathcal{T}(B^*)|\tau\rangle \\ &= \langle \tau|\rho\sigma^* B^*|\tau\rangle \\ &= \langle \tau|B^* \rho\sigma^*|\tau\rangle \\ &= \langle \tau|\mathcal{T}(B)^* \rho\sigma^*|\tau\rangle \\ &= \langle \tau|\sigma^* \mathcal{T}(B)^* \rho|\tau\rangle. \end{aligned}$$

[3] We note that the Tomita-Takasaki anti-linear operators inspire this map.

Where line 1,3,5 follows from $|\tau\rangle$ being a cyclic state, and line 2,4 follows from the definition of $\mathcal{T}$. $\square$

We note that if we only assume $|\tau\rangle$ is cyclic and separating, we can only conclude (2.5) and (2.6) for the above theorem. (2.7) turns out to be an important property within the study of Quantum Information, as we will see in the theorem below.

Theorem 2.2.12. *$\mathcal{T}$ has the following properties:*

1. *For every projector $P \in \mathscr{A}$, $\mathcal{T}(P)$ is also a projector in $\mathscr{A}'$*
2. *For every self-adjoint $A \in \mathscr{A}$, $\mathcal{T}(A)$ is also self-adjoint in $\mathscr{A}'$*
3. *For every unitary $U \in \mathscr{A}$, $\mathcal{T}(U)$ is also an unitary element in $\mathscr{A}'$*

Proof.

1. Note since
$$\mathcal{T}(P)^2 = \mathcal{T}(P^2) = \mathcal{T}(P),$$
we have $\mathcal{T}(P)^2 = \mathcal{T}(P)$ in $\mathscr{A}'$, which implies that $\mathcal{T}(P)$ is a projector.

2. Since every self-adjoint element A can be written as $A_+ - A_-$ for $A_+, A_- \in \mathscr{A}^+$, it is sufficient to show that $\mathcal{T}$ preserves positivity. To this end, note if $A \in \mathscr{A}^+$ then there exists some $B \in \mathscr{A}$ such that $A = B^*B$ and note $\mathcal{T}(A) = \mathcal{T}(B)\mathcal{T}(B)^*$, which implies that $\mathcal{T}(A) \in \mathscr{A}^+$.

3. Note $\mathcal{T}(\mathbf{1}) = \mathbf{1}$, hence we have
$$\mathbf{1} = \mathcal{T}(\mathbf{1}) = \mathcal{T}(\mathbf{UU^*}) = \mathcal{T}(\mathbf{U})^*\mathcal{T}(\mathbf{U})$$
and similarly, $\mathbf{1} = \mathcal{T}(\mathbf{U})^*\mathcal{T}(\mathbf{U})$ and hence $\mathcal{T}(U)$ is unitary.

$\square$

Since $\mathcal{T}(\mathcal{I}_{\mathscr{A}}) = \mathcal{I}_{\mathscr{A}'} = \mathcal{I}_{\mathscr{A}}$, this implies that the modular transpose map $\mathcal{T}$ will always map a PVM, POVM and unitary observable to its respective counterpart from $\mathscr{A}$ to $\mathscr{A}'$.

2.2.5 Connes joint distribution lemma

Lastly, we wish to introduce an important lemma which is used within both [Vid22] and [PP22]. For $\lambda \in \mathbb{R}$, we define the characteristic function $\chi_{\geq\lambda} : \mathbb{R} \to \mathbb{R}$ as

$$\chi_{\geq\lambda}(x) = \begin{cases} 1 & \text{if } x \geq \lambda \\ 0 & \text{otherwise} \end{cases}.$$

We note that, for all positive elements $A \in \mathscr{A}^+$,

$$\int_0^\infty \chi_{\geq\sqrt{\lambda}}(A) d\lambda = A^2, \tag{2.8}$$

and we will refer to [SV18] lemma 5.6 for more detail. Furthermore, since conjugating by a unitary element does not change the spectrum of the element, we have $U^*\chi_{\geq\lambda}(A)U = \chi_{\geq\lambda}(U^*AU)$ for all positive element $A \in \mathscr{A}$ and unitary $U \in \mathscr{A}$. Furthermore, since $\chi_{\geq\lambda}$ is a $\{0,1\}$-valued function, it will map any self-adjoint element to a projector in $\mathscr{A}$, note this fact combined with (2.8) implies that every positive element can be decomposed into an integral of projectors. With these properties in mind, we have the following lemma due to Connes:

Lemma 2.2.13 (Connes joint distribution lemma ([Con76], lemma 1.2.6))**.** *Let $(\mathscr{A}, \tau)$ be a tracial von Neumann algebra and ρ, σ be two positive element in $\mathscr{A}$. Then*

$$\int_0^\infty ||\chi_{\geq\sqrt{\lambda}}(\rho) - \chi_{\geq\sqrt{\lambda}}(\sigma)||_2^2 d\lambda \leq ||\rho - \sigma||_2 \cdot ||\rho + \sigma||_2.$$

Chapter 3

Quantum Non-local games

This section will be a brief survey on some recent literature on two-player non-local games, which is required for this Thesis. For a more comprehensive introduction to this subject, we would direct the readers to the following set of notes [Cle20].

3.1 Basic definitions

A *two player non-local game* consist of two question sets, $\mathcal{X}$ and $\mathcal{Y}$, and two answer sets $\mathcal{A}$ and $\mathcal{B}$. As well as an input distribution $\mu \sim \mathcal{X} \times \mathcal{Y}$ along with a *payoff function* $V : \mathcal{X} \times \mathcal{Y} \times \mathcal{A} \times \mathcal{B} \to \{0, 1\}$ where 0 is associate with the players losing the game and conversely 1 is associate with winning.

We can think of this game as a process between two cooperating players, Adam and Bill, and a referee Richard. In this process, Richard will first sample $(x, y) \in \mathcal{X} \times \mathcal{Y}$ according to the distribution μ and send x to Adam and y to Bill. The players must, without commuting with each other, come up with some answer output ($a \in \mathcal{A}$ and $b \in \mathcal{B}$) and return it to Richard. Adam and Bill is consider to win this process iff $V(x, y, a, b) = 1$. For ease of notation, we will often refer to a non-local game as the following sextuple of variables under $\mathcal{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, V)$.

With this description in mind, we can be more explicit about strategy for an arbitrary game $\mathcal{G}$. We can associate Adam and Bill's strategy based on a *correlation matrix* $\{P(x, y|a, b)\}_{(a,b)\in\mathcal{A}\times\mathcal{B}}^{(x,y)\in\mathcal{X}\times\mathcal{Y}} \subseteq \mathbb{R}_+^{|\mathcal{X}|} \times \mathbb{R}_+^{|\mathcal{Y}|} \times \mathbb{R}_+^{|\mathcal{A}|} \times \mathbb{R}_+^{|\mathcal{B}|}$ (we will often drop the subscript and the superscript if the context is clear), where $P(x, y, a, b)$ denotes the *probability* that Adam and Bill will output (a, b) based on receiving the question set (x, y). Note by this definition, for any $(x, y) \in \mathcal{X} \times \mathcal{Y}$, we must have

$$\sum_{(a,b)\in\mathcal{A}\times\mathcal{B}} P(x, y|a, b) = 1.$$

This intuitive corresponds to the total probability for Adam and Bill to output some answers to 1. Note depending on how we limit what Adam and Bill can do, the correlation matrix could take different values, and we will use $C_t(\mathcal{G})$ to denote the set possible correlation matrix which could arise from game $\mathcal{G}$ with the assumption t. Finally, we will denote the winning probability for game $\mathcal{G}$ under correlation set C_t, or the C_t-valued for game $\mathcal{G}$, as

$$\omega_{C_t}(G) = \sup_{\{P(x,y|a,b)\}\in C_t(\mathcal{G})} \{ \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \mu(x, y) \sum_{(a,b)\in\mathcal{A}\times\mathcal{B}} P(x, y|a, b)V(x, y, a, b)\}$$

Quantum correlation sets have been the interest of recent studies due to it's connection with operator algebra (see, for example [PSS+16], [DP16]). We will give some examples in the next section.

3.2 Quantum Correlations sets for Non-local games

The following definitions loosely follow the definition as presented within [PSS+16].

We will start with the most basic set of correlations that we will consider in this Thesis. Given two functions $f_A : \mathcal{X} \to \mathcal{A}$, and $f_B : \mathcal{Y} \to \mathcal{B}$, we can associate the correlation with f_A and f_B as

$$P(x, y|a, b) = \delta_{f_A(x),a}\delta_{f_B(y),b}.$$

Intuitively, this corresponds to Adam and Bill having a deterministic answer for any given question. This set of strategy is known as a *deterministic classical strategy*, and the set of possible correlations generated by this set of functions is known as *classical correlation* and denoted as $\mathcal{C}_{\text{loc}}(\mathcal{G})$.

We can generalize the above notion by allowing the players to use quantum mechanics. More specifically, let $\mathcal{H}_A$ and $\mathcal{H}_B$ be two finite-dimensional Hilbert space, and let $|\psi\rangle \in \mathcal{H}_A \otimes_{\min} \mathcal{H}_B$. For each question $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, we can associate it with a set of PVM with outcome listed as the potential set of output labelled as $\{A_a^x\}_{a\in\mathcal{A}} \subseteq \mathcal{B}(\mathcal{H}_A)$ (and similarly $\{B_b^y\}_{b\in\mathcal{B}} \subseteq \mathcal{B}(\mathcal{H}_B)$). We can associate the correlation with $|\psi\rangle$, $\{A_a^x\}_{a\in\mathcal{A}}$, and $\{B_b^y\}_{b\in\mathcal{B}}$ as:

$$P(x,y|a,b) = \langle\psi|A_a^x \otimes_{\min} B_b^y|\psi\rangle$$

Intuitively, this corresponds to Alice and Bob preparing an entangle state $|\psi\rangle$ at the beginning of the game and measuring their joint state during the game to obtain their answer. The set of possible correlations generated by such states and PVMs are known as the *quantum correlation* and denoted as $\mathcal{C}_q(\mathcal{G})$.

We can further generalize the above correlation by removing the assumption that Alice and Bob have different Hilbert space and assume both PVM lives in some joint Hilbert space $\mathcal{B}(\mathcal{H})$. In this case, we assume instead $A_a^x B_b^y = B_b^y A_a^x$, we associate the correlation generated by this set of correlation with

$$P(x,y|a,b) = \langle\psi|A_a^x B_b^y|\psi\rangle\,,$$

and we denoted the set of possible correlations generated this way as the *quantum commuting correlation* and denoted as $\mathcal{C}_{qc}(\mathcal{G})$.

It is easy to see that for any non-local game $\mathcal{G}$, we have the following set of containment:

$$\mathcal{C}_{\mathrm{loc}}(\mathcal{G}) \subseteq \mathcal{C}_q(\mathcal{G}) \subseteq \mathcal{C}_{qc}(\mathcal{G})$$

However, each of the above containment is strict, as the first one is shown by the famous CHSH game [CHS+69]. We will use this game as an example below. The second containment is recently demonstrated by the breakthrough result [JNV+20a].

Example 3.2.1 (CHSH game ([CHS+69]))**.** The CHSH game is perhaps the most famous non-local game in the literature as it experimentally proposed a way to validate the laws of quantum entanglement. In this specific game, Richard will randomly pick 2 bits $x, y \in \{0,1\}$ and send it to Adam and Bill, respectively, and they will independently send a bit, $a, b \in \{0,1\}$, back to the referee. Adam and Bill is considered to win this instance iff $a \wedge b = x \oplus y$.

To phrase it in the language consistent with this thesis, we have $\mathcal{X} = \mathcal{Y} = \mathcal{A} = \mathcal{B} = \{0,1\}$, since the referee is picking two random bits from $\mathcal{X} \times \mathcal{Y}$, we have $\mu(x,y) = \frac{1}{4}$ for $(x,y) \in \mathcal{X} \times \mathcal{Y}$. The verification function, in this case, would be

$$V(x,y|a,b) = \begin{cases} 1 & \textit{if } a \wedge b = x \oplus y \\ 0 & \text{Otherwise} \end{cases}.$$

In this case, a classical strategy for Adam and Bill would consist of a look-up table for the list of questions given, where they will always answer consistently according to the table. For example, we might have $f_A(x) = f_B(y) = 0$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, in other words, Adam and Bill will always ignore the input and always answer 0. In this case, their winning probability would be 75% since they can only lose on the question pair $(1, 1)$. In fact, this is the best they can do assuming classical correlation (or $\omega_{\mathcal{C}_{loc}} = 0.75$).

What about quantum correlations? in this case, the optimal strategy will consist of a joint state $|EPR\rangle = (\frac{|00\rangle+|11\rangle}{\sqrt{2}}) \in \mathbb{C}_2 \otimes \mathbb{C}_2$. In this case, Adam will measure his half of the joint state according to the following sets of observables:

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, A^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

and similarly, Bob will measure his half of the joint state according to the following set of observables:

$$B^0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, A^1 = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}.$$

Surprisingly, this strategy gives Adam and Bill a success rate of $\cos(\frac{\pi}{8})^2 \approx 0.854$, beating the classical strategy I have listed above (in fact, this is the optimal strategy if we are assuming q-strategy and it is known as Tsirelson's bound in the literature [Tsi06]).

Finally, we wish to draw the following connection between correlation sets and C*-algebra. Let $\mathbb{F}_n^m$ be a free group with n generator of order m, or

$$\mathbb{F}_n^m = \langle e_1, \cdots, e_n | e_i^m = \mathcal{I} \rangle.$$

Roughly speaking, we associate each of the generators of the free group above with one of the player's observables generated by their POVM. In this case, we have the following theorem relating states on a group C*-algebra and correlations which arise from non-local games:

Theorem 3.2.2 ([Fri12], Theorem 3.4). *Let $\mathcal{G}$ be a non-local games, given a correlation set $\{P(x, y|a, b)\}$, we say the correlation set is in...*

- *... $\mathcal{C}_{qa}(\mathcal{G})$ iff there exist a C^*-algebraic state $\rho \in \mathscr{S}(C^*(\mathbb{F}_{|\mathcal{X}|}^{|\mathcal{A}|}) \otimes_{min} C^*(\mathbb{F}_{|\mathcal{Y}|}^{|\mathcal{B}|}))$ such that*

$$P(a, b|x, y) = \rho(e_a^x \otimes e_b^y)$$

- ... $\mathcal{C}_{qc}(\mathcal{G})$ *iff there exist a* C^**-algebraic state* $\rho \in \mathscr{S}(C^*(\mathbb{F}_{|\mathcal{X}|}^{|\mathcal{A}|}) \otimes_{max} C^*(\mathbb{F}_{|\mathcal{Y}|}^{|\mathcal{B}|}))$ *such that*

$$P(a,b|x,y) = \rho(e_a^x e_b^y)$$

In this thesis, we will use the term *correlation* in order to refer to the correlation set $P(x,y|a,b)$, and the term *strategy* describing the set of POVM which achieved the given correlation. When the underlying strategy uses PVM, we will use the term *projective strategy* to refer to the strategy. For the consistency of notation, unless otherwise stated, we will always assume the corresponding strategy comes from a GNS representation of the given linear functional (in this case, we can represent each positive linear functional as a vector in some Hilbert space).

3.3 Synchronous games and synchronous correlations

In this Thesis, we will focus on a specific type of game known as synchronous games, which we will define in this section. A non-local game is *synchronous* iff the two players share the same question and answer set (i.e. $\mathcal{X} = \mathcal{Y}$ and $\mathcal{A} = \mathcal{B}$), and the players must answer the same response when they receive the same question, or

$$V(x,x|a,b) = \delta_{a,b}.$$

For the simplicity of notation, we will label any synchronous games with the following quadruple sets of elements $\mathcal{G} = (\mathcal{X}, \mathcal{A}, \mu, V)$.

For a synchronous game $\mathcal{G}$ and for any correlation type $t \in \{\text{loc}, \text{q}, \text{qa}, \text{qc}\}$, we say $\{P(x,y|a,b)\}$ in $\mathcal{C}_t(\mathcal{G})$ to be a *t-synchronous correlation* iff we have

$$P(x,x|a,a) = 1, \forall x \in \mathcal{X}, a \in \mathcal{A}.$$

We will denote this subset of $\mathcal{C}_t(\mathcal{G})$ as $\mathcal{C}_t^s(\mathcal{G})$. Intuitively, this corresponds to the case where the players will always strategize in a way such that they will always output the same answer given the same question. As shown within the following theorem, any t-synchronous correlation can always be realized utilizing strategies involving a tracial state acting on the algebra generated by Alice and Bob's observables:

Theorem 3.3.1 ([PSS+16], Theorem 5.5)**.** *For a non-local game* $\mathcal{G}$*, let* $(\{A_x^a\}_{x\in\mathcal{X}}^a,$ $\{B_x^a\}_{x\in\mathcal{X}}^a, \mathcal{H}, |\psi\rangle)$ *be a synchronous correlation for* $\mathcal{G}$*. Then*

- $A_x^a |\psi\rangle = B_x^a |\psi\rangle$, $\forall x \in \mathcal{X}, a \in \mathcal{A}$
- *Let* $\mathscr{A}^{||\cdot||} = \overline{\{A_x^a | a \in \mathcal{A}, x \in \mathcal{X}\}}^{||\cdot||}$ *be the* C^* *algebra generated by Alice's operator and likewise* $\mathscr{B}$ *be the* C^* *algebra generated by Bob's operator. The functional* $s : X \to \langle\psi|X|\psi\rangle$ *is a tracial state for both* $\mathscr{A}$ *and* $\mathscr{B}$*.*

Having a correlation set realized on some tracial state is a useful tool for analyzing properties related to non-local games (see [JNV+22] for more details). We note that it is possible to replace the C*-algebra assumption in the above theorem with a von Neumann algebra, as seen in the theorem below.

Theorem 3.3.2. *Condition 2 of the above theorem could be replace with* $\mathscr{A} = (\mathscr{A}^{||\cdot||})''$ *and* $\mathscr{B} = (\mathscr{B}^{||\cdot||})''$*, the von Neumann algebra generated by Alice's and Bob's operator respectively.*

Proof. Note $\mathscr{B} \subseteq (\mathscr{A}^{||\cdot||})''' = \mathscr{A}'$, which implies that Alice's algebra will always commute with Bob's algebra. Also note for all $a \in \mathscr{A}$, by Theorem 2.2.1, we have $a_n \xrightarrow{WOT} a$ for some $\{a_n\} \in A$. Note that for all $a, b \in \mathscr{A}$, we have

$$\langle\psi|ab|\psi\rangle = \lim_m \lim_n \langle\psi|b_n a_m|\psi\rangle = \lim_m \lim_n \langle\psi|a_m b_n|\psi\rangle = \langle\psi|ba|\psi\rangle .$$

Where the first and third equality follows from the definition of Weak-* topology and onond follows from $|\psi\rangle$ being tracial on $\mathscr{A}^{||\cdot||}$. □

Finally, we note that synchronous correlation cannot usually be used to substitute the regular correlation in the analysis of the synchronous non-local game. Although, as shown in [PSS+16], in the case where the game has a perfect strategy in any model (i.e. $\omega_{C_t}(\mathcal{G}) = 1$ for $t \in \{q, qa, qc\}$), then it has a perfect synchronous strategy under the same correlation set. However, this is not true when the game does not have a perfect strategy, as demonstrated by [HMN+21].

3.4 Approximately Synchronous correlation

Building off the notion of a synchronous correlation, we could also define a more relaxed notion which we will denote as *approximately synchronous correlation*. In essence, we consider the case where Adam and Bill have some margin of error δ outputting different answers when given the same questions. To be more precise, following [Vid22] line 2, we can define the synchronicity for a strategy as the following

Definition 3.4.1 (Synchronicity for a strategy)**.** *Let $\mathcal{G}$ be a synchronous game and let μ_x be the marginal distribution on the first question, and let $P(x,y|a,b)$ be a $\mathcal{C}_t(\mathcal{G})$ correlation for $t \in \{q, qa, qc\}$. Let $(\{A_a^x\}, \{B_a^x\}, |\phi\rangle)$ be the correspond strategy which realizes this correlation, the synchronicity of this correlation with respect to $\mathcal{G}$ is defined as*

$$\delta_{sync}(\mathcal{G}, \{A_a^x\}, \{B_a^x\}, |\phi\rangle) = \mathop{\mathbb{E}}_{x\sim\mu_x} \sum_{a\neq b} P(x,x|a,b).$$

Note that the above definition also works if the marginal distribution is assumed to be on the second question (i.e. marginal distribution on y instead). For the simplification of notation, we often use $x \sim \mu$ to denote the marginal distribution on the first question and $(x,y) \sim \mu$ to denote the question distribution for the game[1]. From this definition, we can define the set of *δ-approximately synchronous correlation* for game $\mathcal{G}$ as the set of correlations such that there exist some strategy $(\{A_a^x\}, \{B_a^x\}, |\phi\rangle)$ which realizes this correlation and

$$\delta_{\text{sync}}(\mathcal{G}, \{A_a^x\}, \{B_a^x\}, |\phi\rangle) \leq \delta,$$

and we will denote the set of correlation as $\mathcal{C}_t^{\delta-s}(\mathcal{G})$ for $t \in \{q, qa, qc\}$. Intuitively, this corresponds to the set of strategies in which Adam and Bill violate synchronicity with probability at most δ according to the marginal distribution of Adam's question.

We wish to provide two utility lemmas before we conclude this section, which will be used in the analysis of the next chapter. First, we wish to provide a distance bound between the two operators, δ-synchronous. This is similar to [JNV+20b] lemma 2.5.

Lemma 3.4.2. *Let $(\{A_a^x\}, \{B_a^x\}, |\psi\rangle)$ be a δ-synchronous strategy, then we have:*

$$\mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle\psi|(A_a^x - B_a^x)^2|\psi\rangle \leq 2\delta$$

Proof.

$$\begin{aligned}
\mathop{\mathbb{E}}_{x\sim\mu} \sum \langle\psi|(A_a^x - B_a^x)^2|\psi\rangle &\leq \mathop{\mathbb{E}}_{x\sim\mu} \sum \langle\psi|(A_a^x + B_a^x - 2A_a^x B_a^x)|\psi\rangle \\
&\leq \mathop{\mathbb{E}}_{x\sim\mu} (\sum \langle\psi|A_a^x|\psi\rangle + \langle\psi|B_a^x|\psi\rangle - 2\langle\psi|A_a^x B_a^x|\psi\rangle) \\
&\leq \mathop{\mathbb{E}}_{x\sim\mu} (2 - 2\langle\psi|A_a^x B_a^x|\psi\rangle) \\
&\leq 2\delta
\end{aligned}$$

[1] However, note that everything defined in this thesis works assuming μ is the marginal distribution on the second question instead

The first inequality follows because $A_a^x \leq \mathcal{I}$, hence $B_a^x \geq (B_a^x)^2$ (and similarly with B_a^x) $\square$

The second lemma gives a useful way to bound the synchronicity of any δ-synchronous strategy.

Lemma 3.4.3. *Let $(\{A_a^x\}, \{B_a^x\}, |\psi\rangle)$ be a δ-synchronous strategy for $\mathcal{G}$. Then:*

$$1 - 2\delta \leq \mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle\psi|(A_a^x)^2|\psi\rangle$$

and $\mathbb{E}_{x\sim\mu} \sum_a \langle\psi|(A_a^x)^2|\psi\rangle$, respectively.

Proof. Note, by Cauchy Schwartz and Jensen's inequality:

$$\begin{aligned}
1 - \delta &= \mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle\psi|A_a^x B_a^x|\psi\rangle \\
&\leq \sqrt{\mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle\psi|(A_a^x)^2|\psi\rangle} \sqrt{\mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle\psi|(B_a^x)^2|\psi\rangle} \\
&\leq \sqrt{\mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle\psi|(A_a^x)^2|\psi\rangle}
\end{aligned}$$

Where the last line follows since $\{B_a^x\}$ is a measurement with $\mathcal{I} \geq B_a^x \geq (B_a^x)^2$. Hence we obtain:

$$1 - 2\delta \leq 1 - 2\delta + \delta^2 \leq \mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle\psi|(A_a^x)^2|\psi\rangle$$

$\square$

The third lemma is a generalization of Theorem 2.2.7 to the case, which will be more helpful in analyzing non-local games.

Lemma 3.4.4. *Let $(\{A_a^x\}, \{B_a^x\}, |\phi\rangle)$ be a δ-synchronous strategy, then there exist a set of projective measurement $\{P_a^x\}$ such that*

$$\mathop{\mathbb{E}}_{x} \langle\psi|(A_a^x - P_i^x)^2|\psi\rangle \leq 9(\delta + \delta^2).$$

Proof. Note,

$$1-\delta = \mathop{\mathbb{E}}_{x\sim\mu}\sum_a \langle\psi|A_a^x\mathcal{T}(B_a^x)|\psi\rangle \le \sqrt{\mathop{\mathbb{E}}_{x\sim\mu}\sum_a \langle\psi|(A_a^x)^2|\psi\rangle}\sqrt{\mathop{\mathbb{E}}_{x\sim\mu}\sum_a \langle\psi|\mathcal{T}B_a^{x2}|\psi\rangle}$$

$$\le \sqrt{\mathop{\mathbb{E}}_{x\sim\mu}\sum_a \langle\psi|(A_a^x)^2|\psi\rangle}\cdot 1$$

Where the first inequality is by applying Cauchy Schwarz inequality. The second inequality follows because $\sum_a(\mathcal{T}(B_a^x))^2 \le \sum_a(\mathcal{T}(B_a^x))^x\mathcal{I}$ (since $\mathcal{T}(B_a^x)\le\mathcal{I}$). Hence we obtain $(1-\delta)^2 \le \mathbb{E}_{x\sim\mu}\sum_a\langle\psi|(A_a^x)^2|\psi\rangle$. By applying Theorem 2.2.7 to each set of $\{A_a^x\}$ for each x, we obtain a set of projectors $\{P_i^x\}$ such that

$$\mathop{\mathbb{E}}_x \langle\psi||A_a^x - P_i^x|^2|\psi\rangle \le 9(\delta+\delta^2).$$

Note since both A_a^x and P_i^x are self-adjoint, we can replace $|\cdot|$ in Theorem 2.2.7 with $(\cdot)$. □

3.5 Tracial embeddable correlations and Symmetric strategies

For any correlation sets $\{P(x,y|a,b)\}\in\mathcal{C}_{qc}(\mathcal{G})$ in the commuting operator framework, we called the set to be *tracial embeddable* if there exist some set of strategy $\{A_a^x\},\{B_a^x\},|\phi\rangle$ and a tracial von Neumann algebra $(\mathscr{A},\tau)$ with standard representation $(\mathcal{H}_\tau,\pi_\tau,|\tau\rangle)$ such that $\{A_a^x\}\subseteq\pi_\tau(\mathscr{A})$, $\{B_a^x\}\subseteq\pi_\tau(\mathscr{A}')$ and $|\phi\rangle\in\mathcal{H}_\tau$.

In other words, the tracial embeddable condition requires the correlation set to be realized on the standard representation of some tracial von Neumann algebra. We will denote this set of correlation as $\mathcal{C}_{qc}^{Tr}(\mathcal{G})$. One of the benefits of working in this representation is that, by Theorem 2.2.8, since $|\tau\rangle$ is both cyclic and separating for $\mathcal{H}$, we can write the underlying vector as $\sigma|\tau\rangle$ for some unique $\sigma\in\mathscr{A}$.

We note that $\mathcal{C}_{qc}^{s}(\mathcal{G})$ is contain within $\mathcal{C}_{qc}^{Tr}(\mathcal{G})$ by Theorem 3.3.2. Finally, we will refer to a correlation to be *tracial embeddable, δ-synchronous* with respect to game $\mathcal{G}$ iff the correlation is in $\mathcal{C}_{qc}^{Tr}(\mathcal{G})\bigcap\mathcal{C}_{qc}^{\varepsilon-s}(\mathcal{G})$.

We wish to define a similar notion of symmetric strategy for synchronous games in the case where the correlation set is in $\mathcal{C}_{qc}$. Intuitively, this is a generalization of the class of strategy such that $B_a^x = (A_a^x)^T$ within the $\mathcal{C}_q$ correlation set. We note that this set of strategies is a generalization of the one defined within [Vid22].

Definition 3.5.1 (Symmetric strategy for synchronous games)**.** *Let $\mathcal{G}$ be a synchronous game, and let $(\{A_a^x\}, \{B_a^x\}, |\phi\rangle)$ be a strategy defined within a tracial VNA $(\mathscr{A}, \tau)$. We call this strategy to be Symmetric iff $A_a^x = \mathcal{T}(B_a^x)$. In this case, we will write the strategy as $(|\psi\rangle, A_a^x)$ for the simplicity of notation.*

We note that this set of strategies can only be defined within the correlation set $\mathcal{C}_{qc}^{Tr}(\mathcal{G})$, as Theorem 2.2.11 guarantees $T_{|\tau\rangle}$ maps a set of POVM on $\mathscr{A}$ to another set of POVM on $\mathscr{A}'$, hence any Symmetric strategy is automatically tracial embeddable. Note that any synchronous strategy is tracial embeddable, in fact, since by definition, we have $A_a^x |\tau\rangle = B_a^x |\tau\rangle$, which means that $\mathcal{T}(A_a^x) = B_a^x$, and hence all synchronous strategy is symmetric by definition. Conversely, for any projective, symmetric strategy on using the trace $|\tau\rangle$, note for any $a \neq b$

$$P(x, x|a, b) = \langle\tau|A_a^x \mathcal{T}(A_b^x)|\tau\rangle = \langle\tau|A_a^x A_b^x|\tau\rangle = 0.$$

The above discussion implies that any projective, symmetric strategy on a tracial state is also a synchronous strategy.

Before we move on, we wish to show several utility lemmas related to these correlation sets which are tracial embeddable and δ-synchronous. The first lemma concerns an inequality to bound the synchronicity for a tracial-embeddable and δ-synchronous strategy. This lemma is similar to [Vid22] lemma 2.9.

Lemma 3.5.2. *Let $\sigma \in \mathscr{A}$ and let $(\{A_x^a\}, \{B_x^a\}, \sigma|\tau\rangle)$ be a tracial embeddable strategy defined on $(\mathscr{A}, \tau)$ and let $u\sigma^+ = \sigma$ be the polar decomposition of σ, we have*

$$1 - \delta_{sync}(\mathcal{G}, \{A_x^a\}, \{B_x^a\}, \sigma|\tau\rangle) \leq \sqrt{1 - \delta_{sync}(\mathcal{G}, \{A_x^a\}, \sigma|\tau\rangle)}\sqrt{1 - \delta_{sync}(\mathcal{G}, \{B_x^a\}, \sigma^+|\tau\rangle)}.$$

Proof. Note $\sigma^* = (u\sigma^+)^* = \sigma^+ u^*$, hence

$$\begin{aligned}
1 - \delta_{\text{sync}}(\mathcal{G}, \{A_x^a\}, \{B_x^a\}, \sigma|\tau\rangle) &= \mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle\tau|\sigma^+ u^* A_x^a B_x^a u\sigma|\tau\rangle \\
&\leq \mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle\tau|\sigma^+ u^* A_x^a u\sigma B_x^a|\tau\rangle \\
&= \mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle\tau|\sigma^+ u^* A_x^a u\sigma \mathcal{T}(B_x^a)|\tau\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle \tau | \sqrt{\sigma^+} u^* A_x^a u \sqrt{\sigma^+} \sqrt{\sigma^+} \mathcal{T}(B_x^a) \sqrt{\sigma^+} | \tau \rangle \\
&\leq \sqrt{\mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle \tau | \sqrt{\sigma^+} u^* A_x^a u \sqrt{\sigma^+} \sqrt{\sigma^+} u^* A_x^a u \sqrt{\sigma^+} | \tau \rangle} \sqrt{\mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle \tau | \sigma^+ \mathcal{T}(B_a^x) \sigma^+ \mathcal{T}(B_a^x) | \tau \rangle} \\
&= \sqrt{\mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle \tau | \sigma^* A_x^a \sigma u^* A_x^a u | \tau \rangle} \sqrt{\mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle \tau | \sigma^+ (B_a^x) \mathcal{T}(B_a^x) \sigma^+ | \tau \rangle} \\
&\leq \sqrt{\mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle \tau | \sigma^* A_x^a \sigma A_x^a | \tau \rangle} \sqrt{1 - \delta_{\text{sync}}(\mathcal{G}, \{B_x^a\}, \sigma^+ \, |\tau\rangle)} \\
&\leq \sqrt{1 - \delta_{\text{sync}}(\mathcal{G}, \{A_x^a\}, \sigma \, |\tau\rangle)} \sqrt{1 - \delta_{\text{sync}}(\mathcal{G}, \{B_x^a\}, \sigma^+ \, |\tau\rangle)}.
\end{aligned}
$$

Where the second line follows from $\{B_x^a\} \subseteq \mathscr{A}'$, the third line follows from $|\tau\rangle$ is tracial and σ^+ is positive. Fourth follows from Cauchy Schwartz. The sixth line follows from $|\tau\rangle$ is tracial and the definition of polar decomposition. The last line follows that $0 \leq u A_x^a u^* \leq A_x^a$ for a partial isometry. ☐

Notice in the calculation above that the measurement $\{A_x^a\}$ and $\{B_x^a\}$ are interchangeable within the lemma statement. Hence, we will be using this lemma to bound $\delta_{\text{sync}}(\mathcal{G}, \{A_x^a\}, \{B_x^a\}, \sigma \, |\tau\rangle)$ by both $\delta_{\text{sync}}(\mathcal{G}, \{A_x^a\}, \sigma \, |\tau\rangle)$ or $\delta_{\text{sync}}(\mathcal{G}, \{A_x^a\}, \sigma^+ \, |\tau\rangle)$ depending on the scenario. The second lemma relates the change in measurements to the change in correlation. The calculation to this lemma is similar to [Vid22], lemma 2.14.

Lemma 3.5.3. *Let $(\{A_x^a\}, \{B_x^a\}, |\psi\rangle)$ be a tracial embeddable strategy defined on $(\mathscr{A}, \tau)$ and let $\delta = \delta_{sync}(\mathcal{G}, \{A_x^a\}, |\psi\rangle)$ be the synchronicity of symmetric strategy $(\{A_x^a\}, |\psi\rangle)$. Let $\{C_x^a\} \subseteq \mathscr{A}$ be another POVM such that*

$$
\gamma = \mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle \psi | (A_x - C_x)^2 | \psi \rangle \tag{3.1}
$$

Let $P(x, y|a, b) = \langle \psi | A_x^a B_x^a | \psi \rangle$ and $P'(x, y|a, b) = \langle \psi | C_x^a B_x^a | \psi \rangle$, then we have

$$
\mathop{\mathbb{E}}_{(x,y)\sim\mu} \sum_{a,b} |P(x, y|a, b) - P'(x, y|a, b)| \leq O(poly(\delta, \gamma)).
$$

Proof. We wish to show this lemma using three inequality. We wish first to bound

$$
\mathop{\mathbb{E}}_{x,y\sim\mu} \sum_{a,b} | \langle \psi | A_a^x B_b^y | \psi \rangle - \langle \psi | (A_a^x)^2 B_b^y | \psi \rangle | \leq 2\delta \tag{3.2}
$$

and

$$\mathop{\mathbb{E}}_{x,y\sim\mu}\sum_{a,b}|\langle\psi|C^x_aB^y_b|\psi\rangle-\langle\psi|(C^x_a)^2B^y_b|\psi\rangle|\leq\delta+\sqrt{\gamma}. \tag{3.3}$$

Then, we will use (4.18) to show

$$\mathop{\mathbb{E}}_{x,y\sim\mu}\sum_{a,b}|\langle\psi|(A^x_a)^2B^y_a|\psi\rangle-\langle\psi|(C^x_a)^2B^y_a|\psi\rangle|\leq 2\sqrt{\gamma}. \tag{3.4}$$

Then we will use triangle inequality to conclude the bound. We wish first to show (3.2), note

$$\begin{aligned}
&\mathop{\mathbb{E}}_{(x,y)\sim\mu}\sum_{a,b}|\langle\psi|A^x_aB^y_a|\psi\rangle-\langle\psi|(A^x_a)^2B^y_b|\psi\rangle|\\
&=\mathop{\mathbb{E}}_{(x,y)\sim\mu}\sum_{a,b}\langle\psi|(A^x_a-(A^x_a)^2)B^y_b|\psi\rangle\\
&=\mathop{\mathbb{E}}_{(x,y)\sim\mu}\sum_{a}\langle\psi|(A^x_a-(A^x_a)^2)(\sum_b B^y_b)|\psi\rangle\\
&=\mathop{\mathbb{E}}_{x\sim\mu}\sum_{a}\langle\psi|(A^x_a-(A^x_a)^2)|\psi\rangle\\
&=1-\mathop{\mathbb{E}}_{x\sim\mu}\sum_{a}\langle\psi|(A^x_a)^2|\psi\rangle\leq 2\delta.
\end{aligned} \tag{3.5}$$

Where the second line follows from $(A^x_a)^2\leq(A^x_a)$, forth line follows from $\sum_b B^y_b=\mathcal{I}$, last line follows from Lemma 3.4.3.

Next, we wish to show (3.3), note by a similar calculation as (3.5), we have

$$\mathop{\mathbb{E}}_{(x,y)\sim\mu}\sum_{a,b}|\langle\psi|C^x_aB^y_a|\psi\rangle-\langle\psi|(C^x_a)^2B^y_b|\psi\rangle|=1-\mathop{\mathbb{E}}_{x\sim\mu}\sum_{a}\langle\psi|(C^x_a)^2|\psi\rangle\,. \tag{3.6}$$

We wish to bound the above quantity using δ and γ, first, note by a similar calculation as Lemma 3.4.3

$$\begin{aligned}
\mathop{\mathbb{E}}_{x\sim\mu}\sum_{a}\langle\psi|A^x_a\mathcal{T}(C^x_a)|\psi\rangle&\leq\sqrt{\mathop{\mathbb{E}}_{x\sim\mu}\sum_{a}\langle\psi|(A^x_a)^2|\psi\rangle}\sqrt{\mathop{\mathbb{E}}_{x\sim\mu}\sum_{a}\langle\psi|(C^x_a)^2|\psi\rangle}\\
&\leq\sqrt{\mathop{\mathbb{E}}_{x\sim\mu}\sum_{a}\langle\psi|(C^x_a)^2|\psi\rangle}
\end{aligned} \tag{3.7}$$

Also note

$$
\begin{aligned}
&1 - \mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle\psi| A_a^x \mathcal{T}(C_a^x) |\psi\rangle \\
&= 1 - \mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle\psi| A_a^x \mathcal{T}(A_a^x) |\psi\rangle + \mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle\psi| A_a^x \mathcal{T}(A_a^x - C_a^x) |\psi\rangle \\
&\le 2\delta + \sqrt{\mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle\psi| (A_a^x)^2 |\psi\rangle} \sqrt{\mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle\psi| \mathcal{T}(A_a^x - C_a^x) |\psi\rangle} \\
&\le 2\delta + \sqrt{\gamma}.
\end{aligned} \tag{3.8}
$$

Combing (3.7) and (3.8), we have

$$
1 - \sqrt{\mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle\psi| (C_a^x)^2 |\psi\rangle} \le 2\delta + \sqrt{\gamma}
$$

Since $\{C_a^x\}$ is a set of POVM, we have $\mathbb{E}_{x\sim\mu} \sum_a \langle\psi| (C_a^x)^2 |\psi\rangle \le \mathbb{E}_{x\sim\mu} \sum_a \langle\psi| C_a^x |\psi\rangle \le 1$ and hence $\mathbb{E}_{x\sim\mu} \sum_a \langle\psi| (C_a^x)^2 |\psi\rangle \ge \sqrt{\mathbb{E}_{x\sim\mu} \sum_a \langle\psi| (C_a^x)^2 |\psi\rangle}$, and hence

$$
1 - \mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle\psi| (C_a^x)^2 |\psi\rangle \le 1 - \sqrt{\mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle\psi| (C_a^x)^2 |\psi\rangle} \le \delta + \sqrt{\gamma}
$$

Hence showing (3.3). Finally, we wish to show (3.4). We will do this with two inequality

$$
\mathop{\mathbb{E}}_{x,y\sim\mu} \sum_{a,b} \langle\psi| ((A_a^x)^2 - A_a^x C_a^x) B_b^y |\psi\rangle \le \sqrt{\gamma} \tag{3.9}
$$

$$
\mathop{\mathbb{E}}_{x,y\sim\mu} \sum_{a,b} \langle\psi| ((C_a^x)^2 - A_a^x C_a^x) B_b^y |\psi\rangle \le \sqrt{\gamma}, \tag{3.10}
$$

then we can use the triangle inequality to conclude the proof; since the calculation is similar, we will only show the calculation for (3.9). Note

$$
\begin{aligned}
&\mathop{\mathbb{E}}_{x,y\sim\mu}\sum_{a,b}\langle\psi|((A_a^x)^2 - A_a^x C_a^x)B_b^y|\psi\rangle \\
&\leq \mathop{\mathbb{E}}_{x,y\sim\mu}\sum_{a,b}\langle\psi|((A_a^x)^2 - A_a^x C_a^x)B_b^y|\psi\rangle \\
&\leq \sqrt{\mathop{\mathbb{E}}_{x,y\sim\mu}\sum_{a,b}\langle\psi|(A_a^x)^2 B_b^y|\psi\rangle}\sqrt{\mathop{\mathbb{E}}_{x,y\sim\mu}\sum_{a,b}\langle\psi|(A_a^x - C_a^x)^2 B_b^y|\psi\rangle} \\
&\leq 1\cdot\sqrt{\mathop{\mathbb{E}}_{x,y\sim\mu}\sum_{a}\langle\psi|(A_a^x - C_a^x)^2(\sum_b B_b^y)|\psi\rangle} \\
&\leq 1\cdot\sqrt{\mathop{\mathbb{E}}_{x,y\sim\mu}\sum_{a}\langle\psi|(A_a^x - C_a^x)^2|\psi\rangle} \leq \sqrt{\gamma},
\end{aligned}
$$

showing (3.9), completing the claim. □

Following this lemma, we have the following corollary

Corollary 3.5.4 (Orthogonalization of δ-synchronous, tracial embeddable strategy)**.** *Let* $(\{A_a^x\},\{B_a^x\},|\psi\rangle)$ *be a* δ*-synchronous strategies for* $\mathcal{G}$ *and let* $P(x,y|a,b) = \langle\psi|A_a^x B_a^x\rangle$. *Then there exist a projective,* $O(\delta^{\frac{1}{4}})$*-synchronous strategies* $(\{P_a^x\},\{Q_a^x\},|\psi\rangle)$ *for* $\mathcal{G}$ *with correlation set* $P^{''}(x,y|a,b)$ *such that*

$$\mathop{\mathbb{E}}_{(x,y)\sim\mu}\sum_{a,b}|P(x,y|a,b) - P^{''}(x,y|a,b)| \leq O(poly(\delta)).$$

Proof. Note it is sufficient to show that there exist some projective measurement $\{P_a^x\}$ such that $(\{P_a^x\},\{B_a^x\},|\psi\rangle)$ forms a $O(\sqrt{\delta})$-synchronous strategy for $\mathcal{G}$, and for $P^{'}(x,y|a,b) = \langle\psi|P_a^x B_a^x\rangle$, we have

$$\mathop{\mathbb{E}}_{(x,y)\sim\mu}\sum_{a,b}|P(x,y|a,b) - P^{'}(x,y|a,b)| \leq O(poly(\delta)).$$

Applying this fact twice on $\{A_a^x\}$, then on $\{B_a^x\}$ on the resulting measurement obtained in the first part gives the desired result. Note by Lemma 3.4.3, we have

$$1 - O(\delta) \leq \sum_{x\sim\mu}\langle\psi|(A_a^x)^2|\psi\rangle\,, \tag{3.11}$$

and hence by Lemma 3.4.4, for each $x \in \mathcal{X}$, we can find some projective measurement $\{P_a^x\} \subseteq \mathscr{A}$ such that

$$\mathop{\mathbb{E}}_{x\sim\mu} \sum_{a\in\mathcal{A}} \langle\psi||A_a^x - P_a^x|^2|\psi\rangle \leq O(\delta).$$

Note by Lemma 3.5.2, since $\delta_{\text{sync}}(\mathcal{G}, \{A_a^x\}, |\psi\rangle) \leq O(\delta)$, we have

$$\mathop{\mathbb{E}}_{(x,y)\sim\mu} \sum_{a,b} |P(x,y|a,b) - P^{'}(x,y|a,b)| \leq O(poly(\delta)).$$

For the first part of the condition, let $\delta^{'} = \delta_{\text{sync}}(\mathcal{G}, \{P_a^x\}, \{B_a^x\}, |\psi\rangle)$, note:

$$\begin{aligned}
\delta - \delta^{'} &= (1-\delta) - (1-\delta^{'}) \\
&\leq \mathop{\mathbb{E}}_{x} \sum_{a} \langle\psi|A_a^x B_a^x|\psi\rangle - \langle\psi|P_a^x B_a^x|\psi\rangle \\
&\leq \mathop{\mathbb{E}}_{x} \sum_{a} \langle\psi|(A_a^x - P_a^x) B_a^x|\psi\rangle \\
&\leq \sqrt{\mathop{\mathbb{E}}_{x} \sum_{a} \langle\psi|(A_a^x - P_a^x)^2|\psi\rangle} \sqrt{(\mathop{\mathbb{E}}_{x} \sum_{a} \langle\psi|(B_x^a)^2|\psi\rangle} \\
&\leq O(\sqrt{\delta})
\end{aligned}$$

Where the last line follows from $\{B_a^x\}$, is a measurement. □

Hence, without the loss of generality, we will always assume our δ-synchronous, tracial embeddable correlations are realized using a projective strategy in this thesis.

Next, we wish to make a similar argument as [Vid22], lemma 3.4, in which we treat every δ-synchronous, symmetric strategy into a $O(\delta)$-synchronous, symmetric and projective strategy. From this, we have the following lemma.

Lemma 3.5.5. *For every symmetric δ-synchronous strategy $(\{A_a^x\}, |\psi\rangle)$ defined on $(\mathscr{A}, \tau)$, there exist some symmetric $O(\delta)$-synchronous and projective strategy $(\{B_a^x\}, |\psi\rangle)$ such that*

$$\mathop{\mathbb{E}}_{x\sim\mu} \sum_{a\in A} |\langle\psi|A_a^x \mathcal{T}(A_a^x)|\psi\rangle - \langle\psi|B_a^x \mathcal{T}(B_a^x)|\psi\rangle| \leq O(\delta).$$

The proof of this lemma is similar to [Vid22] lemma 3.4.

Proof. Note by Lemma 3.4.4, for each $x \in \mathcal{X}$, we can find some projective measurement $\{B_a^x\}$ such that

$$\mathop{\mathbb{E}}_{x \sim \mu} \sum_{a \in \mathcal{A}} \langle\psi||A_a^x - B_a^x|^2|\psi\rangle \leq O(\delta).$$

Note since A_a^x and B_a^x are self adjoint for all $a \in \mathcal{A}$, $x \in \mathcal{X}$, we can rewrite the above equation into

$$\mathop{\mathbb{E}}_{x \sim \mu} \sum_{a \in \mathcal{A}} \langle\psi|(A_a^x - B_a^x)^2|\psi\rangle \leq O(\delta).$$

Note by definition, $(\{B_a^x\}, |\psi\rangle)$ is a tracial embeddable strategy on $(\mathscr{A}, \tau)$. Now we have to show that it is also $O(\delta)$-synchronous. Let $\delta' = \delta_{\text{sync}}(\mathcal{G}, \{B_a^x\}, |\psi\rangle)$, note

$$\begin{aligned}
\delta - \delta' &= (1-\delta) - (1-\delta') \\
&\leq \mathop{\mathbb{E}}_x \sum_a \langle\psi|A_a^x \mathcal{T}(A_a^x)|\psi\rangle - \langle\psi|B_a^x \mathcal{T}(B_a^x)|\psi\rangle + \langle\psi|B_a^x \mathcal{T}(A_a^x)|\psi\rangle - \langle\psi|B_a^x \mathcal{T}(A_a^x)|\psi\rangle \\
&\leq \mathop{\mathbb{E}}_x \sum_a \langle\psi|(A_a^x - B_a^x)\mathcal{T}(A_a^x)|\psi\rangle - \langle\psi|B_a^x \mathcal{T}(A_a^x - B_a^x)|\psi\rangle \\
&\leq \mathop{\mathbb{E}}_x \sum_a |\langle\psi|(A_a^x - B_a^x)\mathcal{T}(A_a^x)|\psi\rangle - \langle\psi|B_a^x \mathcal{T}(A_a^x - B_a^x)|\psi\rangle| \\
&\leq \sqrt{\mathop{\mathbb{E}}_x \sum_a \langle\psi|(A_a^x - B_a^x)^2|\psi\rangle} \sqrt{\mathop{\mathbb{E}}_x \sum_a \langle\psi|\mathcal{T}(A_a^x)|\psi\rangle} \\
&\quad + \sqrt{\mathop{\mathbb{E}}_x \sum_a \langle\psi|B_a^x|\psi\rangle} \sqrt{\mathop{\mathbb{E}}_x \sum_a \langle\psi|\mathcal{T}(A_a^x - B_a^x)^2|\psi\rangle} \\
&\leq \sqrt{\mathop{\mathbb{E}}_x \sum_a \langle\psi|(A_a^x - B_a^x)^2|\psi\rangle} \Big(\sqrt{\mathop{\mathbb{E}}_x \sum_a \langle\psi|\mathcal{T}(A_a^x)|\psi\rangle} + \sqrt{\mathop{\mathbb{E}}_x \sum_a \langle\psi|B_a^x|\psi\rangle}\Big) \\
&\leq \sqrt{2} O(\delta) = O(\delta).
\end{aligned}$$

Where the fourth inequality follows from Cauchy Schwartz (and $\langle\psi|A^2|\psi\rangle = \langle\psi|\mathcal{T}(A)^2|\psi\rangle$). Hence showing that $(\{B_a^x\}, |\psi\rangle)$ is $O(\delta)$-synchronous. Lastly, note by (3.5) the fact that $(\{B_a^x\}, |\psi\rangle)$ is $O(\delta)$-synchronous, we can apply Lemma 3.5.3 twice in order to get the last part of the lemma. $\square$

The important part of the above construction is that the measurement would remain a symmetric strategy after the application of Lemma 3.4.4. Hence, in this thesis, we can always assume our δ-synchronous, symmetric strategies are also projective.

Chapter 4

Rounding

In this section, we will introduce and prove the rounding lemma. We will first prove the lemma in the case where we assume the underlying strategy is symmetric, and the state is in the form $a^+ |\tau\rangle$ for some positive element $a^+ \in \mathscr{A}$. Then we will see how this assumption can be lifted in the next section.

4.1 Main theorem

We will begin with the following main theorem.

Theorem 4.1.1 (Rounding in standard representation of von Neumann algebra)**.** *Let $\mathcal{G}$ be synchronous game and let $\sigma \in \mathscr{A}^+$ be a positive operator such that $||\sigma||_\tau = 1$, let μ be a distribution on $\mathcal{X}$. Let $(\{A_a^x\}, \sigma |\tau\rangle)$ be a symmetric and δ-synchronous strategy defined within a tracial VNA $(\mathscr{A}, \tau)$. Then there exist some set of projectors $\{P_\lambda, \lambda \geq 0\} \subseteq \mathscr{A}$ such that*

1.
$$\int_0^\infty P_\lambda d\lambda = \sigma^2 \tag{4.1}$$

2. *Let $\lambda_{max} = \sup\{\lambda : \tau(P_\lambda) \neq 0\}$, then $0 < \lambda_{max} < \infty$. For each $\lambda \in [0, \lambda_{max}]$, there exist some synchronous strategy $(A_x^{\lambda,x}, P_\lambda |\tau\rangle)$ defined within $(P_\lambda \mathscr{A} P_\lambda, \frac{1}{\tau(P_\lambda)}) \subseteq (\mathscr{A}, \tau)$, such that:*
$$\int_0^{\lambda_{max}} \mathop{\mathbb{E}}_{x\sim\mu} \sum_a ||(A_a^x - A_a^{\lambda,x})P_\lambda||_2^2 d\lambda \leq O(\sqrt{\delta}) \tag{4.2}$$

To show the above lemma, we wish first to show the following proposition.

Proposition 4.1.2. *Let $\mathcal{G}$ be synchronous game and let $\sigma \in \mathscr{A}^+$ be a positive operator such that $||\sigma||_\tau = 1$, let μ be a distribution on $\mathcal{X}$. Let $(\{A_a^x\}, \sigma\,|\tau\rangle)$ be a symmetric and δ-synchronous strategy defined within a tracial VNA $(\mathscr{A}, \tau)$. Then there exist some set of projectors $\{P_\lambda, \lambda \geq 0\}$ such that*

$$\int_0^\infty P_\lambda d\lambda = \sigma^2 \tag{4.3}$$

and,

$$\int_0^\infty \mathop{\mathbb{E}}_{x\sim\mu} \sum_a ||(A_a^x - P_\lambda A_a^x P_\lambda)P_\lambda||_2^2 d\lambda \leq 2\sqrt{\delta} \tag{4.4}$$

The proof follows the structure of [Vid22], lemma 3.5 and theorem 3.1.

Proof. We wish to first convert $\{A_a^x\}$ into observables, let $m = |\mathcal{A}|$ and associate the answer set with $\mathbb{Z}_n$, write

$$U_b^x = \sum_{a\in\mathbb{Z}_m} e^{\frac{2i\pi ab}{m}} A_a^x \tag{4.5}$$

and note since $\sum_{a\in\mathbb{Z}_m} A_a^x = \mathcal{I}_{\mathscr{A}}$ for all $x \in \mathcal{X}$, U_b^x defines a unitary in $\mathscr{A}$. Note by definition, for every $x \in \mathcal{X}$, we have

$$\begin{aligned}
&\mathop{\mathbb{E}}_{x\sim\mu}\mathop{\mathbb{E}}_{b}||U_b^x\sigma - \sigma U_b^x||_2^2 \\
&= \mathop{\mathbb{E}}_{x\sim\mu}\mathop{\mathbb{E}}_{b}(2\,\langle\tau|\sigma(U_b^x)^*U_b^x\sigma|\tau\rangle - \langle\tau|\sigma(U_b^x)^*\sigma U_b^x|\tau\rangle - \langle\tau|(U_b^x)^*\sigma U_b^x\sigma\rangle\,|\tau) \\
&= 2 - \mathop{\mathbb{E}}_{x\sim\mu}\mathop{\mathbb{E}}_{b} 2(\langle\tau|\sigma(U_b^x)^*\sigma U_b^x|\tau\rangle) \\
&= 2 - \mathop{\mathbb{E}}_{x\sim\mu}\mathop{\mathbb{E}}_{b}\sum_a\sum_c e^{\frac{2i\pi(a-c)b}{m}} 2(\langle\tau|\sigma A_a^x\sigma A_c^x|\tau\rangle) \\
&= 2 - 2\mathop{\mathbb{E}}_{x\sim\mu}\mathop{\mathbb{E}}_{a}\langle\tau|\sigma A_a^x\sigma A_a^x|\tau\rangle \\
&= 2\delta.
\end{aligned} \tag{4.6}$$

Where the second equality follows the definition of U_b^x and Lemma 2.2.10. Note that since U_b^x is a unitary, we have

$$(U_b^x)^*\chi_{\geq\lambda}(\sigma)U_b^x = \chi_{\geq\lambda}((U_b^x)^*\sigma U_b^x).$$

Also, since

$$(U_b^x)^*\sqrt{\sigma}U_b^x(U_b^x)^*\sqrt{\sigma}U_b^x = (U_b^x)^*\sigma U_b^x,$$

by uniqueness of square root, we have $(U_b^x)^*\sqrt{\sigma}U_b^x = \sqrt{(U_b^x)^*\sigma U_b^x}$. We defined $P_\lambda = \lambda_{\leq\sqrt{\lambda}}(\sigma)$ and note by (2.8) we have

$$\int_0^\infty \lambda_{\leq\sqrt{\lambda}}(P_\lambda) = \sigma^2.$$

For each λ, we have

$$\begin{aligned}\mathop{\mathbb{E}}_b ||U_b^x P_\lambda - P_\lambda U_b^x||_2^2 &= \mathop{\mathbb{E}}_{x\sim\mu} 2(\langle\tau|P_\lambda|\tau\rangle - \langle\tau|P_\lambda(U_b^x)^*P_\lambda U_b^x|\tau\rangle) \\ &= \sum_{a\in\mathbb{Z}_m} 2(\langle\tau|P_\lambda A_a^x|\tau\rangle - \langle\tau|P_\lambda A_a^x P_\lambda A_a^x|\tau\rangle) \\ &= \sum_{a\in\mathbb{Z}_m} ||A_a^x P_\lambda - P_\lambda A_a^x||_2^2. \end{aligned} \tag{4.7}$$

Where line 2 follows from U_b^x being unitary and P_λ being projector, line 3 follows from Lemma 2.2.10 and $\sum_a A_a^x = \mathcal{I}_{\mathcal{H}}$. Also, note by Lemma 2.2.13, we have

$$\mathop{\mathbb{E}}_{x\sim\mu}\mathop{\mathbb{E}}_b \int_0^\infty ||P_\lambda - (U_b^x)^*P_\lambda U_b^x||_2^2 d\lambda \leq \mathop{\mathbb{E}}_{x\sim\mu}\mathop{\mathbb{E}}_b ||\sigma - (U_b^x)^*\sigma U_b^x||_2||\sigma + (U_b^x)^*\sigma U_b^x||_2.$$

Now we have all the pieces to show the main theorem, note,

$$\begin{aligned}&\int_0^\infty \mathop{\mathbb{E}}_{x\sim\mu}\sum_a ||(A_a^x - P_\lambda A_a^x P_\lambda)P_\lambda||_2^2 d\lambda \\ &\leq \mathop{\mathbb{E}}_{x\sim\mu}\int_0^\infty \sum_a ||(A_a^x P_\lambda - P_\lambda A_a^x)||_2^2||P_\lambda||_2^2 d\lambda \\ &\leq \mathop{\mathbb{E}}_{x\sim\mu}\int_0^\infty \mathop{\mathbb{E}}_b ||U_b^x P_\lambda - P_\lambda U_b^x||_2^2 d\lambda \\ &= \mathop{\mathbb{E}}_{x\sim\mu}\mathop{\mathbb{E}}_b \int_0^\infty ||P_\lambda - (U_b^x)^*P_\lambda U_b^x||_2^2 d\lambda \\ &\leq \mathop{\mathbb{E}}_{x\sim\mu}\mathop{\mathbb{E}}_b ||\sigma - (U_b^x)^*\sigma U_b^x||_2||\sigma + (U_b^x)^*\sigma U_b^x||_2 \\ &\leq \mathop{\mathbb{E}}_{x\sim\mu}\mathop{\mathbb{E}}_b ||U_b^x\sigma - \sigma U_b^x||_2(||\sigma||_2 + ||(U_b^x)^*\sigma U_b^x||_2) \\ &= 2\sqrt{\delta}\end{aligned}$$

Where the second line follows from Holder's inequality, the third line follows from (4.7) and $\langle\tau|P_\lambda|\tau\rangle \leq \langle\tau|\mathcal{I}_{\mathcal{H}}|\tau\rangle = 1$. Fifth line follows from (4.7), last line follows from $||\sigma||_2 = 1$ and (4.6). Hence proving the claim. □

To see how the theorem above implies some set of symmetric strategies within $P_\lambda \mathscr{A} P_\lambda$, we have the following lemma.

Lemma 4.1.3. *Let P_λ be a projector in $\mathscr{A}$ such that $\tau(P_\lambda) \neq 0$. $\frac{1}{\tau(P_\lambda)}\tau$ forms a trace for the von Neumann algebra $P_\lambda \mathscr{A} P_\lambda$. Furthermore, $(\{P_\lambda A_a^x P_\lambda\}, \{\mathcal{T}_{|\tau\rangle}(P_\lambda A_a^x)P_\lambda\}, P_\lambda |\tau\rangle)$ forms a symmetric strategy on $(P_\lambda \mathscr{A} P_\lambda, \frac{1}{\tau(P_\lambda)}\tau)$.*

Proof. For the first part, note for $P_\lambda \mathcal{I} P_\lambda = P_\lambda \in P_\lambda \mathscr{A} P_\lambda$, clearly

$$\frac{1}{\tau(P_\lambda)}\tau(P_\lambda) = 1.$$

Secondly, for $P_\lambda A P_\lambda, P_\lambda B P_\lambda \in P_\lambda \mathscr{A} P_\lambda$, since both $P_\lambda A P_\lambda$ and $P_\lambda B P_\lambda \in \mathscr{A}$, since τ is tracial, we have

$$\frac{1}{\tau(P_\lambda)} \langle \tau | P_\lambda A P_\lambda P_\lambda B P_\lambda | \tau \rangle = \frac{1}{\tau(P_\lambda)} \langle \tau | P_\lambda B P_\lambda P_\lambda A P_\lambda | \tau \rangle . \tag{4.8}$$

Showing that $\frac{1}{\tau(P_\lambda)}\tau(P_\lambda)$ is a tracial state for the Von Neumann algebra $P_\lambda \mathscr{A} P_\lambda$.

For the "furthermore" part, note by Theorem 3.3.2, it is sufficient to show that $\mathcal{T}_{|\tau\rangle}(P_\lambda A_a^x)P_\lambda \in (P_\lambda \mathscr{A} P_\lambda)'$ and $(P_\lambda A_a^x P_\lambda)|\tau\rangle = \mathcal{T}_{|\tau\rangle}(P_\lambda A_a^x)P_\lambda |\tau\rangle$. For the first part, note by Theorem 2.2.5, we have $(P_\lambda \mathscr{A} P_\lambda)' = \mathscr{A}' P_\lambda$ and since $\mathcal{T}_{|\tau\rangle}(P_\lambda A_a^x) \in \mathscr{A}'$ by definition, we have $\mathcal{T}_{|\tau\rangle}(P_\lambda A_a^x)P_\lambda \in \mathscr{A}' P_\lambda$, proving that $\mathcal{T}_{|\tau\rangle}(P_\lambda A_a^x)P_\lambda \in (P_\lambda \mathscr{A} P_\lambda)'$. For the second claim, note

$$\begin{aligned}
\mathcal{T}_{|\tau\rangle}(P_\lambda A_a^x P_\lambda)|\tau\rangle &= P_\lambda A_a^x P_\lambda |\tau\rangle \\
&= P_\lambda \mathcal{T}_{|\tau\rangle}(P_\lambda A_a^x)|\tau\rangle \\
&= \mathcal{T}_{|\tau\rangle}(P_\lambda A_a^x)P_\lambda |\tau\rangle .
\end{aligned}$$

Hence, showing that $(\{P_\lambda A_a^x P_\lambda\}, \{\mathcal{T}_{|\tau\rangle}(P_\lambda A_a^x)P_\lambda\}, P_\lambda |\tau\rangle)$ forms a symmetric strategy on $(P_\lambda \mathscr{A} P_\lambda, \frac{1}{\tau(P_\lambda)}\tau)$. ☐

Now we are ready to proof Theorem 4.1.1.

Proof. Take the $\{P_\lambda\}$ we obtain from Proposition 4.1.2. Note for any λ such that $\tau(P_\lambda) = 0$, we have

$$0 \leq \langle \tau | P_\lambda (A_a^x - P_\lambda A_a^x P_\lambda)^2 P_\lambda | \tau \rangle \leq 2 \langle \tau | P_\lambda \mathcal{I} P_\lambda | \tau \rangle = 0. \tag{4.9}$$

Recall, from the proof of Proposition 4.1.2, we have $P_\lambda = \lambda_{\leq\sqrt{\lambda}}(\sigma)$. Note since σ is a positive element in $\mathcal{B}(\mathcal{H})$, this implies that $\lambda_{max} < \infty$, furthermore, since by assumption, we have $||\sigma||_2 = 1$, hence $\sigma \neq 0$ and hence $\lambda_{max} > 0$, completing the first part of claim 2. Note this also implies that we can rewrite (4.4) as

$$\int_0^{\lambda_{max}} \mathop{\mathbb{E}}_{x\sim\mu} \sum_a ||(A_a^x - P_\lambda A_a^x P_\lambda)P_\lambda||_2^2 d\lambda \leq 2\sqrt{\delta}. \tag{4.10}$$

For each $\lambda \in [0, \lambda_{max}]$, by applying Lemma 4.1.3, we obtained some symmetric strategy $\{P_\lambda A_a^x P_\lambda, \frac{1}{\sqrt{\tau(P_\lambda)}}|\tau\rangle\}$. Write $A_{a,\lambda}^{\tilde{x}} = P_\lambda A_a^x P_\lambda$. We wish to determine the synchronous of the above symmetric strategy in order to apply Lemma 3.5.5 to obtain some projective strategy instead (and hence we obtain a synchronous strategy by the remark after the definition of symmetric strategy), hence note

$$\begin{aligned}
\int_0^{\lambda_{max}} \mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle\tau|P_\lambda A_{a,\lambda}^{\tilde{x}} \mathcal{T}_{P_\lambda|\tau\rangle}(A_{a,\lambda}^{\tilde{x}})P_\lambda|\tau\rangle \, d\lambda &= \int_0^{\lambda_{max}} \mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle\tau|P_\lambda A_a^x P_\lambda A_a^x P_\lambda|\tau\rangle \\
&= \int_0^{\lambda_{max}} \mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle\tau|A_a^x P_\lambda A_a^x P_\lambda|\tau\rangle \\
&\leq \int_0^{\lambda_{max}} \mathop{\mathbb{E}}_{x\sim\mu} \sum_a \langle\tau|A_a^x P_\lambda A_a^x P_\lambda|\tau\rangle \\
&= 1 - \frac{1}{2}\int_0^{\lambda_{max}} \mathop{\mathbb{E}}_{x\sim\mu} \sum_a ||A_a^x P_\lambda - P_\lambda A_a^x||_2^2 \\
&= 1 - \frac{1}{2}\int_0^{\infty} \mathop{\mathbb{E}}_{x\sim\mu} \sum_a ||A_a^x P_\lambda - P_\lambda A_a^x||_2^2 = 1 - \delta.
\end{aligned}$$

The third last line follows a similar calculation as (4.7). The last line follows from the fact that $||P_\lambda||_2 = 0$ and (4.6). Hence, we can apply Lemma 3.5.5 to each $A_{a,\lambda}^{\tilde{x}}$ to obtain some measurement $A_{a,\lambda}^x$ such that

$$\int_0^{\lambda_{max}} \mathop{\mathbb{E}}_{x\sim\mu} \sum_a ||(A_{a,\lambda}^{\tilde{x}} - A_x^{\lambda,x})P_\lambda||_2^2 d\lambda \leq O(\delta). \tag{4.11}$$

Hence by applying the triangle inequality with (4.4) to obtain (4.2). □

4.2 Generalization for the main rounding theorem

4.2.1 Relaxing conditions required

We will briefly discuss the potential generalization to Theorem 4.1.1 by slowly relaxing the assumption required. The following generalizes the requirement to all tracial embeddable, δ-synchronous correlation for game G.

Corollary 4.2.1. *The condition of Theorem 4.1.1 generalize to all $\sigma \in \mathscr{A}$ with $||\sigma||_\tau = 1$, and all tracial embeddable, δ-synchronous strategy $(\{A_x^a\}, \{B_x^a\}, \sigma\,|\tau\rangle)$, with a symmetric strategies defined either on $\{A_x^a\}$ or $\{B_x^a\}$ on some state $\sigma^+\,|\tau\rangle$ where $\sigma^+ = \sqrt{\sigma^*\sigma}$*

Proof. For the purpose of this proof, we will perform the calculation on $\{A_x^a\}$, note by Lemma 3.5.2, we have

$$1 - \delta_{\text{sync}}(\mathcal{G}, \{A_x^a\}, \{B_x^a\}, \sigma\,|\tau\rangle) \leq \sqrt{1 - \delta_{\text{sync}}(\mathcal{G}, \{B_x^a\}, \sigma\,|\tau\rangle)} \cdot \sqrt{1 - \delta_{\text{sync}}(\mathcal{G}, \{A_x^a\}, \sigma^+\,|\tau\rangle)}. \tag{4.12}$$

Let $\delta_A = \delta_{\text{sync}}(\mathcal{G}, \{A_a^x\}, \sigma_A^+\,|\tau\rangle)$, note the above calculation implies that $1 - \delta_A \geq (1 - O(\delta))^2$ or $\delta_A \leq O(\delta)$. Hence by applying Theorem 4.1.1 using the symmetric strategy $(A_x^a, \sigma^+\,|\tau\rangle)$ yields the desire result.

□

The following corollary is designed to be used within the analysis of the non-local games, where the vector state could potentially have a different 2-norm after the transformation incurred.

Corollary 4.2.2. *The above proof works for any $\sigma \in \mathscr{A}$ with an error bound of $\frac{\sqrt{2}\delta_{sync}^{1/4}}{||\sigma||_2^2}$ instead.*

4.2.2 Connection to Correlation sets

We wish to relate the above theorem to correlation sets defined by strategies. To this end, we have the following theorem. Note if we let $B_b^{\lambda,y} = \mathcal{T}_{P_\lambda|\tau\rangle}(A_a^{\lambda,x})$, then the theorem below is equivalent to Theorem 1.0.1,

Theorem 4.2.3. *Let $\{\{A_a^x\}, \{B_a^x\}, \sigma\,|\tau\rangle\}$ be a tracial embeddable, δ-synchronous strategy. Let $\{P_\lambda\}_{\lambda\in[0,\lambda_{max}]}$ and $\{\{A_a^{\lambda,x}\}, P_\lambda\,|\tau\rangle\}$ be the projector and strategies generated by Corollary 4.2.1. Let $P^\lambda(x,y|a,b) = \frac{1}{\tau(P_\lambda)}\langle\tau|P_\lambda A_a^{\lambda,x}\mathcal{T}_{P_\lambda|\tau\rangle}(A_a^{\lambda,x})P_\lambda|\tau\rangle$, we have*

$$\mathop{\mathbb{E}}_{(x,y)\sim\mu}\sum_{a,b}|P(x,y|a,b) - \int_0^{\lambda_{max}}\tau(P_\lambda)\cdot P^\lambda(x,y|a,b)| \leq O(\delta^{\frac{1}{4}}). \tag{4.13}$$

Note by the definition of λ_{max}, we have

$$\int_0^{\lambda_{max}}\tau(P_\lambda) = \int_0^\infty \tau(P_\lambda) = \tau(\int_0^\infty P_\lambda) = \tau(\sigma^*\sigma) = 1$$

Hence $\int_0^{\lambda_{max}}\tau(P_\lambda)\cdot P^\lambda(x,y|a,b)$ can be seen as a convex combination of synchronous strategy. This corollary is similar to corollary 3.3 from [Vid22] and the proof follows a similar structure.

Proof. We wish first to relate the correlation set in the statement to the symmetric strategy on $\sigma^+|\tau\rangle$ obtained from Corollary 4.2.1. Let $P^{'}(x,y|a,b) = \langle\tau|\sigma^+A_x^a\mathcal{T}(A_y^b)\sigma^+|\tau\rangle$ be the correlation set generated by the symmetric strategy $(A_x^a, \sigma^+\,|\tau\rangle)$. We wish to show that

$$\mathop{\mathbb{E}}_{(x,y)\sim\mu}\sum_{a,b}|P(x,y|a,b) - P^{'}(x,y|a,b)| \leq O(\sqrt{\delta}). \tag{4.14}$$

To do this, defined the correlation set $P^{''}(x,y|a,b) = \langle\tau|\sigma A_x^a\mathcal{T}(A_y^b)\sigma|\tau\rangle$, we wish to show the following

$$\mathop{\mathbb{E}}_{(x,y)\sim\mu}\sum_{a,b}|P(x,y|a,b) - P^{''}(x,y|a,b)| \leq O(\delta), \tag{4.15}$$

$$\mathop{\mathbb{E}}_{(x,y)\sim\mu}\sum_{a,b}|P^{'}(x,y|a,b) - P^{''}(x,y|a,b)| \leq O(\sqrt{\delta}). \tag{4.16}$$

Combining the above two equations with a triangle inequality shows (4.14). In order to show (4.15), we will use Lemma 3.5.3. Note by Lemma 3.5.2, we have

$$1-\delta \leq \sqrt{1-\delta_{sync}(\mathcal{G}, \{B_x^a\}, \sigma\,|\tau\rangle)}. \tag{4.17}$$

Hence we have $\delta_{sync}(\mathcal{G}, \{B_x^a\}, \sigma\,|\tau\rangle) \leq O(\delta)$, which is the first condition required for Lemma 3.5.3. Next, we wish to show that

$$\mathop{\mathbb{E}}_{x\sim\mu}\sum_a\langle\psi|(B_x - \mathcal{T}(A_x))^2|\psi\rangle \leq O(\delta). \tag{4.18}$$

Note

$$\mathop{\mathbb{E}}_{x\sim\mu}\sum_a \langle\psi|(B_x-\mathcal{T}(A_x))^2|\psi\rangle = \mathop{\mathbb{E}}_{x\sim\mu}\sum_a ||(B_x-\mathcal{T}(A_x))\,|\psi\rangle\,||^2$$
$$\leq \mathop{\mathbb{E}}_{x\sim\mu}\sum_a ||(B_x-A_x)\,|\psi\rangle\,||^2 + ||(A_x-\mathcal{T}(A_x))\,|\psi\rangle\,||^2. \quad (4.19)$$

For the first part of (4.19), by Lemma 3.4.2, we have

$$\mathop{\mathbb{E}}_{x\sim\mu}\sum_a ||(B_x-A_x)\,|\psi\rangle\,||^2 \leq O(\delta)$$

For the second part of (4.19), note

$$\begin{aligned}\mathop{\mathbb{E}}_{x\sim\mu}\sum_a ||(A_x-\mathcal{T}(A_x))\,|\psi\rangle\,||^2 &= \mathop{\mathbb{E}}_{x\sim\mu}\sum_a \langle\psi|(A_x^a-\mathcal{T}(A_x^a))^2|\psi\rangle \\ &= \mathop{\mathbb{E}}_{x\sim\mu}\sum_a (2\,\langle\psi|(A_x^a)^2|\psi\rangle - 2\,\langle\psi|A_x^a\mathcal{T}(A_x^a)|\psi\rangle) \\ &\leq 2-2(1-\delta_{\text{sync}}(\mathcal{G},\{A_x^a\},|\psi\rangle)) \leq O(\delta). \end{aligned} \quad (4.20)$$

Where the last inequality follows from Lemma 3.5.2. Combining (4.2.2) and (4.20) gives us (4.18). Hence by Lemma 3.5.3, we have (4.14).

In order to show (4.16), we wish to show the following:

$$\mathop{\mathbb{E}}_{(x,y)\sim\mu}\sum_{a,b} |P'(x,y|a,b) - \tau(\mathcal{T}(A_a^x)\mathcal{T}(A_b^y)(\sigma^+)^2)| \leq O(\sqrt{\delta}), \quad (4.21)$$

$$\mathop{\mathbb{E}}_{(x,y)\sim\mu}\sum_{a,b} |P''(x,y|a,b) - \tau(\mathcal{T}(A_a^x)\mathcal{T}(A_b^y)\sigma^*\sigma)| \leq O(\sqrt{\delta}). \quad (4.22)$$

Then, by the definition for Corollary 4.2.1, we have $\sigma^*\sigma = (\sigma^+)^2$, hence we can use the triangle inequality to show (4.15). Since the inequality proof for both inequalities is the same, we will only show the bound for (4.21). Note

$$\begin{aligned} &\mathop{\mathbb{E}}_{(x,y)\sim\mu}\sum_{a,b} |P'(x,y|a,b) - \tau(\mathcal{T}(A_a^x)\mathcal{T}(A_b^y)(\sigma^+)^2)| \\ &= \mathop{\mathbb{E}}_{(x,y)\sim\mu}\sum_{a,b} |\,\langle\tau|\sigma^+A_a^x\mathcal{T}(A_b^y)\sigma^+|\tau\rangle - \langle\tau|\sigma^+\mathcal{T}(A_a^x)\mathcal{T}(A_b^y)\sigma^+|\tau\rangle\,| \\ &= \mathop{\mathbb{E}}_{(x,y)\sim\mu}\sum_{a,b} |\,\langle\tau|\sigma^+(A_a^x-\mathcal{T}(A_a^x))\mathcal{T}(A_b^y)\sigma^+|\tau\rangle\,| \end{aligned}$$

$$
\begin{aligned}
&\leq \sqrt{\mathop{\mathbb{E}}_{(x,y)\sim\mu}\sum_{a,b} |\langle\tau|\sigma^+(A_a^x - \mathcal{T}(A_a^x))^2\sigma^+|\tau\rangle|}\sqrt{\mathop{\mathbb{E}}_{(x,y)\sim\mu}\sum_{a,b} |\langle\tau|\sigma^+\mathcal{T}(A_b^y)^2\sigma^+|\tau\rangle|} \\
&\leq \sqrt{2\delta_{\text{sync}}(\mathcal{G},\{A_x^a\},\sigma^+|\tau\rangle)}\cdot 1 \\
&\leq \sqrt{O(\delta)} = O(\sqrt{\delta})
\end{aligned}
$$

Where the second line follows from $\mathcal{T}(A_a^x), \mathcal{T}(A_b^y) \in \mathscr{A}'$, fourth line follows from Cauchy Schwartz, the fifth line follows from $\{\mathcal{T}(A_b^y)\}$ being a measurement and a similar calculation as (4.20), and the last line follows from Lemma 3.5.2. Hence, by triangle inequality, on (4.21), (4.22) and (4.15), we arrive at (4.14).

Next, we wish to show the following equation,

$$
\mathop{\mathbb{E}}_{(x,y)\sim\mu}\sum_{a,b}|P^{'}(x,y|a,b) - \int_0^{\lambda_{max}} \tau(P^\lambda)\cdot P^\lambda(x,y|a,b)d\lambda| \leq O(\delta) \tag{4.23}
$$

then combine with (4.14) and (4.23) gives us (4.13). Define $(P^{'})^\lambda(x,y|a,b) = \langle\tau|P_\lambda A_x^a\mathcal{T}(A_x^a)P_\lambda|\tau\rangle$. We wish first to show that:

$$
\mathop{\mathbb{E}}_{(x,y)\sim\mu}\sum_{a,b}|P^{'}(x,y|a,b) - \int_0^{\infty}(P^{'})^\lambda(x,y|a,b)d\lambda| \leq O(\sqrt{\delta}). \tag{4.24}
$$

We will first the following equations:

$$
\mathop{\mathbb{E}}_{(x,y)\sim\mu}\sum_{a,b}|P^{'}(x,y|a,b) - \int_0^{\infty}\tau(\mathcal{T}(A_a^x)\mathcal{T}(A_b^y)P_\lambda)d\lambda| \leq O(\sqrt{\delta}), \tag{4.25}
$$

$$
\mathop{\mathbb{E}}_{(x,y)\sim\mu}\sum_{a,b}|P^{'}(x,y|a,b) - \int_0^{\infty}\tau(\mathcal{T}(A_a^x)\mathcal{T}(A_b^y)P_\lambda)d\lambda| \leq O(\sqrt{\delta}), \tag{4.26}
$$

and then use triangle inequality to show (4.24). To show (4.25), by (4.21) and (4.1), we have:

$$
\begin{aligned}
&\mathop{\mathbb{E}}_{(x,y)\sim\mu}\sum_{a,b}|P^{'}(x,y|a,b) - \int_0^{\infty}\tau(\mathcal{T}(A_a^x)\mathcal{T}(A_b^y)P_\lambda)d\lambda| \\
&= \mathop{\mathbb{E}}_{(x,y)\sim\mu}\sum_{a,b}|P^{'}(x,y|a,b) - \tau(\mathcal{T}(A_a^x)\mathcal{T}(A_b^y)(\sigma^+)^2)| \leq O(\sqrt{\delta}).
\end{aligned}
$$

Showing (4.25). To show (4.26), note:

$$\begin{aligned}
&\mathop{\mathbb{E}}_{(x,y)\sim\mu} |\sum_{a,b} |P^{'}(x,y|a,b) - \int_0^\infty \tau(\mathcal{T}(A_a^x)\mathcal{T}(A_b^y)P_\lambda)d\lambda| \\
&\leq \int_0^\infty \mathop{\mathbb{E}}_{(x,y)\sim\mu} \sum_{a,b} |(P^{'})^\lambda(x,y|a,b) - \tau(\mathcal{T}(A_a^x)\mathcal{T}(A_b^y)P_\lambda)|d\lambda \\
&\leq \int_0^\infty \mathop{\mathbb{E}}_{(x,y)\sim\mu} \sum_{a,b} |\langle\tau|P_\lambda(A_a^x - \mathcal{T}(A_a^x))\mathcal{T}(A_b^y)P_\lambda|\tau\rangle|d\lambda \\
&\leq \sqrt{\int_0^\infty \mathop{\mathbb{E}}_{(x,y)\sim\mu} \sum_{a,b} |\langle\tau|P_\lambda(A_a^x - \mathcal{T}(A_a^x))^2P_\lambda|\tau\rangle|d\lambda}\sqrt{\int_0^\infty \mathop{\mathbb{E}}_{(x,y)\sim\mu} \sum_{a,b} |\langle\tau|P_\lambda A_b^y P_\lambda|\tau\rangle|d\lambda} \\
&\leq \sqrt{2 - \int_0^\infty \mathop{\mathbb{E}}_{x\sim\mu} \sum_{a} \langle\tau|P_\lambda A_a^x \mathcal{T}(A_a^x)P_\lambda|\tau\rangle\, d\lambda}\sqrt{\langle\tau|\int_0^\infty P_\lambda d\lambda|\tau\rangle} \\
&\leq \sqrt{2 - \int_0^\infty \mathop{\mathbb{E}}_{x\sim\mu} \sum_{a} \langle\tau|P_\lambda A_a^x \mathcal{T}(A_a^x)|\tau\rangle d\lambda} \cdot 1 \\
&\leq \sqrt{2 - 2\mathop{\mathbb{E}}_{x\sim\mu} \sum_{a} \langle\tau|\sigma^+ A_a^x \sigma^+ A_a^x|\tau\rangle} \leq \sqrt{2\delta} = O(\sqrt{\delta}).
\end{aligned}$$

Where the second and the third line follows from Jensen's inequality and Cauchy Schwartz, the fourth line follows from expanding $(A_a^x - \mathcal{T}(A_a^x))^2$ and the tracial property. The fifth line follows from P_λ being a projector, and the sixth line follows from Lemma 2.2.10. This shows (4.26).

Finally, we wish to show that,

$$\mathop{\mathbb{E}}_{(x,y)\sim\mu} \sum_{a,b} |\int_0^{\lambda_{max}} \tau(P_\lambda)\cdot P^\lambda(x,y|a,b)d\lambda - \int_0^\infty (P^{'})^\lambda(x,y|a,b)|d\lambda \leq O(\sqrt{\delta}). \tag{4.27}$$

Note for all P_λ such that $\tau(P_\lambda) = 0$ (or $\lambda > \lambda_{max}$), since A_a^x is assume to be a positive projector and τ is a positive function, we have:

$$0 \leq \tau(\mathcal{T}(A_a^x)\mathcal{T}(A_b^y)P_\lambda) \leq \tau(\mathcal{I}_{\mathscr{A}} P_\lambda) \leq 0,$$

for all $x, y \in \mathcal{X}$ and $a, b \in \mathcal{A}$. Hence, we can rewrite (4.27) as:

$$\begin{aligned}
&\mathop{\mathbb{E}}_{(x,y)\sim\mu} \sum_{a,b} \Big| \int_0^{\lambda_{max}} \tau(P_\lambda) \cdot P^\lambda(x, y|a, b) d\lambda - \int_0^{\infty} (P^{'})^\lambda(x, y|a, b) d\lambda \Big| \\
&= \mathop{\mathbb{E}}_{(x,y)\sim\mu} \sum_{a,b} \Big| \int_0^{\lambda_{max}} \langle\tau| P_\lambda A_a^{\lambda,x} \mathcal{T}_{P_\lambda|\tau\rangle}(A_a^{\lambda,x}) P_\lambda |\tau\rangle \, d\lambda - \int_0^{\lambda_{max}} \langle\tau| P_\lambda A_x^a \mathcal{T}(A_x^a) P_\lambda |\tau\rangle \, d\lambda \Big| \\
&\leq \int_0^{\lambda_{max}} \mathop{\mathbb{E}}_{(x,y)\sim\mu} \sum_{a,b} \big| \langle\tau| P_\lambda A_a^{\lambda,x} \mathcal{T}_{P_\lambda|\tau\rangle}(A_a^{\lambda,x}) P_\lambda |\tau\rangle - \langle\tau| P_\lambda A_x^a \mathcal{T}(A_x^a) P_\lambda |\tau\rangle \big| d\lambda
\end{aligned} \tag{4.28}$$

To this end, note by (4.2), we have:

$$\int_0^{\lambda_{max}} \mathop{\mathbb{E}}_{x\sim\mu} \sum_a ||(A_a^x - A_x^{\lambda,x}) P_\lambda||_2^2 d\lambda \leq O(\sqrt{\delta}).$$

and hence, by applying a similar argument as (4.11) and (4.4), we can conclude:

$$\begin{aligned}
&\int_0^{\lambda_{max}} \mathop{\mathbb{E}}_{x\sim\mu} \sum_a ||(\mathcal{T}(A_a^x) - \mathcal{T}_{P_\lambda|\tau\rangle} A_x^{\lambda,x}) P_\lambda||_2^2 d\lambda \\
&\leq \int_0^{\lambda_{max}} \mathop{\mathbb{E}}_{x\sim\mu} \sum_a ||(\mathcal{T}(A_a^x) - \mathcal{T}(P_\lambda A_a^x P_\lambda)||_2^2 d\lambda \\
&\quad + \int_0^{\lambda_{max}} ||\mathcal{T}(P_\lambda A_a^x P_\lambda) - \mathcal{T}_{P_\lambda|\tau\rangle} A_x^{\lambda,x}) P_\lambda||_2^2 d\lambda \\
&\leq O(\sqrt{\delta}) + O(\delta) = O(\sqrt{\delta})
\end{aligned} \tag{4.29}$$

Hence, since each of the $\{\{A_a^{\lambda,x}\}, P_\lambda |\psi\rangle\}$ is a projective strategy and hence $\delta_{\text{sync}}(\mathcal{G}, \{A_x^{a,\lambda}\}, P_\lambda |\tau\rangle) = 0$, applying Lemma 3.5.3 twice, on each of the $\{A_a^x\}$ and $\{\mathcal{T}(A_a^x)\}$, we have

$$\int_0^{\lambda_{max}} \mathop{\mathbb{E}}_{(x,y)\sim\mu} \sum_{a,b} \big| \langle\tau| P_\lambda A_a^{\lambda,x} \mathcal{T}_{P_\lambda|\tau\rangle}(A_a^{\lambda,x}) P_\lambda |\tau\rangle - \langle\tau| P_\lambda A_x^a \mathcal{T}(A_x^a) P_\lambda |\tau\rangle \big| d\lambda \leq O(\sqrt{\delta}).$$

Completing the claim.

□

4.3 Open questions

4.3.1 Tracial embeddable correlations

In this thesis, we defined a notion of the tracial embeddable correlation. Notably, by Theorem 3.3.2, the set of synchronous correlations is contained within the set of tracial embeddable correlations. However, it is unclear whether the set of tracial embeddable correlations is equivalent to the set of commuting operator correlations. The following corollary gives some evidence against this fact.

Theorem 4.3.1. *For any $(a, b, x, y) \in \mathbb{Z}^4$, $\mathscr{S}(C^*(\mathbb{F}_x^a) \otimes_{max} C^*(\mathbb{F}_y^b))$ does not contain a faithful trace.*

Since the proof of the above theorem involves a chain of known results in the field, we will provide a sketch of the proof in this thesis.

Proof. By the ping-pong lemma (lemma D.1 [Fri12]), it is possible to show that every scenario group is isomorphic to $C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty)$ the free group generated by 2 elements (see lemma D.2 [Fri12] for more detail) and hence we can reduce the problem of determining a faithful trace on $\mathscr{S}(C^*(\mathbb{F}_x^a) \otimes_{\max} C^*(\mathbb{F}_y^b))$. This statement is equivalent to Kirchberg's conjecture (see [Oza04] proposition for more details), which is shown to be false by [JNV+20a] by its relationship to Connes embedding problem. $\square$

Note the above theorem shows that *there is no single tracial representation* which contains a tracial representation for *all possible* commuting operator correlation generated by the scenario algebra $\mathscr{S}(C^*(\mathbb{F}_x^a) \otimes_{\max} C^*(\mathbb{F}_y^b))$ for each $(a, b, x, y) \in \mathbb{Z}^4$. However, by Theorem 3.2.2, since the set of commuting operator correlations can be generated by some state ρ acting on the scenario algebra, it is possible that *each* such state could be tracial embeddable in a different von Neumann algebra. Hence, whether $\mathcal{C}_{qc}(\mathcal{G}) = \mathcal{C}_{qc}^{tr}(\mathcal{G})$ for all $\mathcal{G}$ still remains open.

In a similar vein, one could consider the *approximately tracial embeddable correlation*, which is defined as the closure of $\mathcal{C}_{qc}^{tr}(\mathcal{G})$ similar to the correlation qa to q (see [Slo19] for a more in-depth discussion between these two correlation sets). Note that Theorem 4.2.3 could easily be generalized to the approximately tracial embeddable correlation. To relax the condition above, if we can show that $\mathcal{C}_{qc}(\mathcal{G}) = \overline{\mathcal{C}_{qc}^{tr}(\mathcal{G})}$ for all $\mathcal{G}$. Then we could show that every δ-approximately synchronous correlation in the commuting operator correlation set could be written as a convex combination of synchronous correlations.

4.3.2 Complexity of the tracial embeddable correlation

Similar to [CHT+10], we can define a similar notion as the complexity class MIP^*. $\mathrm{MIP}^{\textit{Tr-emb}}$, which stand for *Multiprover interactive proof system with tracial embeddable strategies.* Roughly speaking, a computational language $\mathcal{L}$ is in $\mathrm{MIP}^{\textit{Tr-emb}}(2,1)$ if we can computationally reduce every $z \in \{0,1\}^*$ into a non-local game $\mathcal{G}_z$ such that $\omega_{\mathcal{C}_{qc}^{tr}(\mathcal{G}_z)} \geq \frac{2}{3}$ if $z \in \mathcal{L}$ and $\omega_{\mathcal{C}_{qc}^{tr}(\mathcal{G}_z)} \leq \frac{1}{3}$ otherwise. By the theorem shown within this thesis, we could potentially reduce analyzing the complexity of this set of language to $\mathrm{MIP}^{\textit{Sync}}$, which is defined similarly except using the synchronous value instead. This problem could offer further insight into the complexity of different operator algebra systems.

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