On the Dynamical Wilf-Zeilberger Problem

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

In this paper, we give an algorithmic solution to a dynamical analog of the problem of certifying combinatorial identities by Wilf-Zeilberger pairs. Given two sequences generated in a dynamical setting, we calculate an upper bound $N \geq 1$ such that whenever the first N terms of the two sequences agree pairwise, the two sequences agree term-by-term. Then, we give an algorithm that can be used to check whether two such sequences agree term-by-term. Our methods are mainly based on the theory of Chow rings of algebraic varieties.

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Table of Contents

Section 1

Introduction

In the early 1990s, Herbert S. Wilf and Doron Zeilberger introduced a method of proving certain combinatorial identities by automata [WZ90a, WZ90b, Ze06, Ze90, Ze91]. Specifically, they considered identities of the form

$$\sum_{k=0}^{n} F(n,k) = f(n),$$

where $F(n,k)$ is a function in n, k such that $\frac{F(n,k+1)}{F(n,k)}$ is a rational function in k for each $k \geq 0$, and $f(n)$ is a function in n. Such functions $F(n,k)$ are known as hypergeometric functions, so identities of the above form are called hypergeometric identities.

Definition 1.0.1. Let $F(n,k)$ be a function defined on $n \in \mathbb{Z}^{\geq 0}$, $k \in \mathbb{Z}$. We say that $F(n,k)$ is a hypergeometric function if

$$\frac{F(n,k+1)}{F(n,k)}$$

is a rational function in k for each $k \in \mathbb{Z}$ and for each $n \in \mathbb{Z}^{\geq 0}$.

Many interesting identities from combinatorial enumeration are hypergeometric identities. For example, summation identities involving binomial coefficients are hypergeometric identities. Specific examples include Vandermonde's identity

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r},$$

which is true for each $m, n, r \geq 0$, and Dixon's identity

$$\sum_{k \in \mathbb{Z}} (-1)^k \binom{n+b}{n+k}\binom{n+c}{c+k}\binom{b+c}{b+k} = \frac{(n+b+c)!}{n!b!c!},$$

which is true for each $n \geq 0$ [Wi94].

In general, hypergeometric identities of the above form from combinatorics terminate in the variable k at or before $k = n$, i.e., $F(n,k) = 0$ for all $k > n$. Under this assumption, we may write

$$\sum_{k} F(n,k) := \sum_{k=0}^{n} F(n,k).$$

Wilf and Zeilberger were primarily interested in hypergeometric identities of the above form whose summands terminate. In particular, they were interested in summands $F(n,k)$ of the form

$$F(n,k) = \frac{\prod_{i=1}^{A}(a_i n + a_i' k + a_i'')!}{\prod_{i=1}^{B}(b_i n + b_i' k + b_i'')!} z^k$$

where the coefficients of n and k are constant coefficients (say rational numbers), and A, B, z are also constants [WZ90b, Ze06, Ze90, Ze91]. By using the theory of holonomic functions, they were able to show that summands of the above form always satisfy some recurrence relations of the form

$$s_0(n)F(n,k) + s_1(n)F(n+1,k) + \cdots + s_L(n)F(n+L,k) = G(n,k+1) - G(n,k)$$

where L is a positive integer, $s_0(n), \ldots, s_L(n)$ are polynomials in n and $G(n,k)$ is a function that terminates in the variable k, i.e.,

$$\lim_{k \to \pm\infty} G(n,k) = 0$$

for each $n \in \mathbb{Z}^{\geq 0}$ [WZ90b, Ze90]. Hence, the sum

$$S(n) := \sum_k F(n,k)$$

satisfies the recurrence relation

$$s_0(n)S(n) + \cdots + s_L(n)S(n+L) = \sum_k (G(n,k+1) - G(n,k)) = 0.$$

Thus, to prove the identity

$$S(n) = \sum_k F(n,k) = f(n),$$

it is enough to check that the function $f(n)$ is a solution to the above recurrence relation, and that $S(n+i) = f(n)$ for $i = 0, \ldots, L$. This can be done on any computer algebra software in finitely many number of steps. Further, by using the algorithm of Gosper [Go78], Wilf and Zeilberger were able to compute the polynomials $s_0(n), \ldots, s_L(n)$ and the function $G(n,k)$ for each $F(n,k)$ of the above form [WZ90a, Ze06, Ze90, Ze91]. Therefore, the method of Wilf and Zeilberger gives an algorithmic proof for every hypergeometric identity of the above form.

In this paper, we will consider the following dynamical analog of Wilf and Zeilberger:

Problem 1.0.2. Let (X, ϕ, x, f), (Y, ψ, y, g) be quadruples where

1. X, Y are quasi-projective varieties over a field k of characteristic zero;
2. $\phi : X \to X$ and $\psi : Y \to Y$ are dominant rational maps;
3. x, y are points on X and Y respectively;

4. f, g are rational functions from X and Y to $\mathbb{P}^1_k$ respectively such that the images of f and g are both dense in $\mathbb{P}^1_k$.

Does there exist an integer $N \geq 1$ such that $f(\phi^i(x)) = g(\psi^i(y))$ for all $i = 1, \ldots, N$ implies that $f(\phi^i(x)) = g(\psi^i(y))$ for all $i \geq 1$?

We shall see that the answer to this question is yes – there exists such a positive integer $N \geq 1$. This will be proved by Noetherian induction in Section 3. Thus, the problem becomes computing this positive integer $N \geq 1$ given the data (X, ϕ, x, f), (Y, ψ, y, g), and this will constitute the main problem to be addressed in this paper. In particular, assuming that the varieties X and Y are defined over an algebraically closed field k of characteristic zero, we shall compute an upper bound on the positive integer $N \geq 1$ that is dependent only on:

- The dimensions of X and Y;
- The degree of X and the degree of Y, where X and Y are considered as subvarieties of some $\mathbb{P}^m$ and $\mathbb{P}^n$ with the embeddings being fixed;
- The dimensions of the projective spaces $\mathbb{P}^m$ and $\mathbb{P}^n$;
- The degrees of the maps ϕ, ψ, f and g.

Specifically, we shall prove the following theorem giving a general upper bound on the positive integer $N \geq 1$.

Theorem 1.0.3. *Let (X, ϕ, x, f), (Y, ψ, y, g) be data given in Problem 1.0.2, and suppose that X and Y are defined over an algebraically closed field of characteristic zero. Fix the following:*

- $r := \dim X$, $s := \dim Y$*;*
- $\iota_X : X \hookrightarrow \mathbb{P}^m$, $\iota_Y : Y \hookrightarrow \mathbb{P}^n$ *embeddings for some* $m \geq r$, $n \geq s$*;*
- $\deg X = d_1$, $\deg Y = d_2$ *under the above embeddings;*
- $\deg f \circ \phi = e_1$, $\deg g \circ \psi = e_2$ *under the embeddings* ι_X *and* ι_Y *respectively;*
- $e := \max\{e_1, e_2\}$.

Also, denote

$$\widetilde{\gamma_{r,s,1}} := \binom{r+s}{r}\binom{m+n}{n}^2 d_1 d_2 e^2.$$

and define, inductively, for each $j = 2, \ldots, r+s$ that

$$\widetilde{\gamma_{r,s,j}} := \prod_{i=1}^{\widetilde{\lambda_{r,s,j-1}}} \binom{r+s}{r}\binom{m+n}{n}^2 d_1 d_2 e^{2i}$$

where

$$\widetilde{\lambda_{r,s,j}} := \widetilde{\gamma_{r,s,j}} + \widetilde{\gamma_{r,s,j-1}} + \cdots + \widetilde{\gamma_{r,s,1}} + j.$$

If

$$f(\phi^i(x)) = g(\psi^i(y))$$

for each $i = 1, \ldots, \widetilde{\lambda_{r,s,r+s}}$, *then*

$$f(\phi^i(x)) = g(\psi^i(y))$$

for every $i \geq 1$.

In Section 2, we will first give some background information on the method of Wilf and Zeilberger for proving hypergeometric identities by automata. Then, we will give an overview of intersection theory via the theory of Chow rings, and this will form the basis for finding an upper bound on the positive integer $N \geq 1$ in Problem 1.0.2.

Next, in Section 3, we will prove some general results concerning the intersection of algebraic varieties, using the theory of Chow rings as stated in Section 2. These results will be the main tool for solving Problem 1.0.2.

Then, in Section 4, we will present the solution of Problem 1.0.2. Before presenting the solution of the general case, we will give an upper bound on the positive integer $N \geq 1$ in the case where $X = Y$ and the maps ϕ, ψ, f and g are all regular and surjective. This simplification is relevant, as we would like to consider orbits of points given on a fixed algebraic variety. We then present the solution of the general case, by noticing that the proof technique of the general case is the same as that of the special case. Based on our results, we will present an algorithm that can be carried out by any computer algebra software to solve Problem 1.0.2 given the data (X, ϕ, x, f), (Y, ψ, y, g). The algorithm will attempt to find an upper bound on the positive integer $N \geq 1$ that is less than the general upper bound given in Theorem 1.0.3, so as to reduce the computational complexity for checking the N initial conditions for solving Problem 1.0.2.

Finally, in Section 5, we will present an application of our method for elliptic divisibility sequences.

Section 2

Background

2.1 Algorithms for proving combinatorial identities

Recall that Wilf and Zeilberger were interested in finding algorithms for proving identities of the form

$$\sum_{k=0}^{n} L(n,k) = f(n),$$

where $L(n,k)$ is a hypergeometric function in k and $f(n)$ is a function. Equivalently, the above may be written as

$$\sum_{k=0}^{n} F(n,k) := \sum_{k=0}^{n} \frac{L(n,k)}{f(n)} = const,$$

assuming that $f(n) \neq 0$ for each n.

Most combinatorial identities of the above form are such that the summand $F(n,k)$ terminate in the variable k at $k \leq n$, i.e.,

$$F(n,k) = 0 \text{ for all } k > n.$$

Thus, we will assume that hypergeometric functions $F(n,k)$ terminate in the variable k at $k \leq n$ throughout this section, and denote

$$\sum_{k} F(n,k) := \sum_{k=0}^{n} F(n,k)$$

for convenience.

In particular, summands of the form

$$F(n,k) = \frac{\prod_{i=1}^{A}(a_i n + a_i' k + a_i'')!}{\prod_{i=1}^{B}(b_i n + b_i' k + b_i'')!} z^k \tag{2.1}$$

are of primary interest to Wilf and Zeilberger [PWZ96, WZ90a, WZ90b, WZ92a, WZ92b, Ze90, Ze91]. Here, the coefficients of n and k are constants for each i and for each j, and z

is also a constant. It is easily seen that $F(n, k)$ is a hypergeometric function in k. Moreover, summands of this form include most functions involving binomial coefficients, so most combinatorial identities of interest are covered under this form.

Notice that the summand $F(n, k)$ in the above form is doubly hypergeometric, i.e., hypergeometric in both n and k. Under this assumption, Sister Mary Celine Fasenmyer discovered an algorithm that can check the identity [Fa47, Fa49]

$$\sum_k F(n, k) = const.$$

Her method is based on finding a linear recurrence on the function $F(n, k)$ in the variable k, and a summary of her method may be found in [PWZ96]. In [WZ92a], Wilf and Zeilberger were able to verify that the method of Fasenmyer always succeeds for summands of the form 2.1. However, her method is often computationally unfeasible, given that the process of finding the required linear recurrence requires a large number of trials and errors.

To come up with a computationally feasible algorithm, Wilf and Zeilberger employed the theory of holonomic functions [WZ90b, Ze90]. Heuristically, holonomic functions are solutions to homogeneous partial differential equations with polynomial coefficients, and one may show that summands of the form 2.1 are holonomic [Ze90]. In the following, we will give the definition of holonomic functions and holonomic sequences by building up the definition step by step, starting with the definition of C-finite functions.

Definition 2.1.1 ([Ze90]).

1. Let $D := \frac{\mathrm{d}}{\mathrm{d}x}$ be the differential operator defined on the space of k-valued functions in the variable x. Then, a k-valued function f is C-finite if there exists a non-zero polynomial P with coefficients in k such that

$$P(D)f = 0$$

identically, where it is assumed that the derivatives of f exist for each x.

2. Let E be the shift operator defined on the space of k-valued sequences, i.e., for a sequence $\{a_n\}_{n=0}^{\infty} \subset k$, $Ea_n = a_{n+1}$. Then, a k-valued sequence $a := \{a_n\}_{n=0}^{\infty}$ is C-finite if there exists a non-zero polynomial P with coefficients in k such that

$$P(E)a = 0.$$

Now, let $C[D] \cong k[D]$ be the ring of polynomials in D with coefficients in k, so that $C[D]$ is a principle ideal domain. It follows that for a k-valued function f, the ideal

$$I_f := \{P \in C[D] \mid P(D)f = 0\}$$

is a principle ideal generated by $P_0(D)$ of the minimal order annihilating f. Thus, f is C-finite if and only if the vector space

$$C[D]f := \mathrm{span}\{D^i f \mid i \geq 0\}$$

is finite dimensional over k. Equivalently, f is C-finite if and only if

$$\dim_k C[D]/I_f < \infty.$$

We may also obtain an equivalent definition of C-finite sequences by replacing the differential operator D in the above with the difference operator E. Let $C[E] \cong k[E]$ be the ring of polynomials in E with coefficients in k. Then, for a k-valued sequence a, $I_a := \{P \in C[E] \mid P(E)a = 0\} \subseteq C[E]$ is also a principal ideal. It follows that a is C-finite if and only if

$$\dim_k C[E]/I_a < \infty.$$

Notice that C-finite functions are solutions to linear ordinary differential equations with constant coefficients in k, so it would be natural to generalize the above to solutions to linear ODEs with polynomial coefficients in $k[x]$. This leads to the so-called P-finite functions [Ze90], which is the next step towards the definition of holonomic functions.

Definition 2.1.2 ([Ze90])**.** Let D and E be the differential and difference operators as defined in Definition 2.1.1.

1. A k-valued function f is P-finite if there exists a non-zero polynomial P with coefficients in $k[x]$ such that
$$P(D)f = 0$$
identically.

2. A k-valued sequence $a := \{a_n\}_{n=0}^{\infty}$ is P-finite if there exists a non-zero polynomial P with coefficients in $k[n]$ such that
$$P(E)a = 0.$$

Moreover, we may extend C-finiteness and P-finiteness to functions of several variables, or sequences with multi-indices. Let $D_1 = \frac{\partial}{\partial x_1}, D_2 = \frac{\partial}{\partial x_2}, \ldots, D_n = \frac{\partial}{\partial x_n}$ be differential operators defined on the space of k-valued functions in $x_1, \ldots, x_n$. As before, let $C[D_1, \ldots, D_n] \cong k[D_1, \ldots, D_n]$ be the polynomial ring in $D_1, \ldots, D_n$ with coefficients in k. Although this is no longer a principle ideal domain, we may still define the ideal

$$I_f = \{P \in C[D_1, \ldots, D_n] \mid P(D_1, \ldots, D_n)f = 0\}$$

where f is a k-valued function in $x_1, \ldots, x_n$ whose partial derivatives exist in each variable $x_1, \ldots, x_n$. We then have the following definition.

Definition 2.1.3 ([Ze90])**.** Let f be a k-valued function in variables $x_1, \ldots, x_n$. Then, f is multi-C-finite if

$$\dim_k C[D_1, \ldots, D_n]/I_f < \infty,$$

where the $C[D_1, \ldots, D_n]$-module $C[D_1, \ldots, D_n]/I_f$ is seen as a k-vector space.

Again, we may define multi-C-finiteness for sequences analogously. Let a be a k-valued sequence with multi-indices, i.e.,

$$a = \{a_{m_1, m_2, \ldots, m_n}\}_{m_1, \ldots, m_n \geq 0} := a(m_1, m_2, \ldots, m_n)$$

as a discrete-valued function mapping $\mathbb{N}^n$ to k. As before, let $E_1, \ldots, E_n$ be shift operators in variables $m_1, m_2, \ldots, m_n$ respectively, i.e.,

$$E_i a(m_1, m_2, \ldots, m_n) = a(m_1, \ldots, m_{i-1}, m_i + 1, m_{i+1}, \ldots, m_n).$$

Then, let $C[E_1, \dots, E_n] \cong k[E_1, \dots, E_n]$ be the polynomial ring in $E_1, \dots, E_n$ with coefficients in k, and define the ideal

$$I_a = \{P \in C[E_1, \dots, E_n] \mid P(E_1, \dots, E_n)a = 0\} \subseteq C[E_1, \dots, E_n].$$

We say that a is multi-C-finite if $\dim_k C[E_1, \dots, E_n]/I_a < \infty$.

The next step is to define multi-P-finite functions, which will become the correct notion for holonomic functions. As in the single variable case, we would like to replace linear PDEs with constant coefficients in k by linear PDEs with polynomial coefficients in $k[x_1, \dots, x_n]$. To this end, define the Weyl-algebra in variables $x_1, \dots, x_n$ [Ze90]:

$$A_n(C) := C\langle D_1, \dots, D_n, x_1, \dots, x_n\rangle := C[D_1, \dots, D_n, x_1, \dots, x_n]/S$$

where x_i is interpreted as the multiplication operator $f \mapsto x_i f$ for each i, and S is the set of commutation relations given by

- $x_i x_j = x_j x_i$ for every i, j;
- $D_i D_j = D_j D_i$ for every i, j;
- $[D_i, x_j] := D_i x_j - x_j D_i = \delta_{i,j}$ for every i, j, where $\delta_{i,j}$ is the Kronecker delta function.

As before, for a k-valued function f that is differentiable in each variable, we may define

$$I_f := \{P \in A_n(C) \mid Pf = 0\},$$

which is a left ideal of $A_n(C)$. The ring $A_n(C)$ is left- and right-Noetherian, and we may define the dimension of $A_n(C)/I_f$ as an $A_n(C)$-module as follows [Ze90]. Let $F := \{F_\nu\}$ be a filtration on $A_n(C)$, so that $\{F_\nu/(F_\nu \cap I_f)\}$ is an induced filtration on $A_n(C)/I_f$. Denote the dimension of each $F_\nu/(F_\nu \cap I_f)$ by $H(\nu)$. One may show that $H(\nu)$ is a polynomial function in ν with rational coefficients when $\nu \gg 0$ [Ze90]. We then define the F-dimension of $A_n(C)/I_f$ as a $A_n(C)$-module to be

$$d_F(A_n(C)/I_f) := \deg H(\nu).$$

Under certain assumptions on the filtration F, $d_F(A_n(C)/I_f)$ is an invariant for every F [Ze90]. Then, we may define the dimension of $A_n(C)/I_f$ as a $A_n(C)$-module to be

$$d(A_n(C)/I_f) = d_F(A_n(C)/I_f).$$

When a is not identically zero, Bernstein's inequality implies that [Ze90]:

$$d(A_n(C)/I_f) \geq n.$$

Definition 2.1.4 ([Ze90]). Let f be a k-valued function in variables $x_1, \dots, x_n$, and suppose f is not identically zero. Then, f is holonomic if

$$d(A_n(C)/I_f) = n$$

as an $A_n(C)$-module.

Thus, holonomic sequences f are such that $A_n(C)/I_f$ has the smallest possible dimension as an $A_n(C)$-module. For non-zero multi-C-finite functions f, the left ideal $I_f \subset A_n(C)$ is the extension of the ideal $I'_f \subset C[D_1, \ldots, D_n]$ under the inclusion map $C[D_1, \ldots, D_n] \hookrightarrow A_n(C)$, since we are simply adding the variables $x_1, \ldots, x_n$ into the ideal I'_f. Then, it can be shown that $d(A_n(C)/I_f) = n$ [Ze90], so the above definition is indeed a natural extension of multi-C-finiteness.

It is possible to define an action of the Weyl algebra $A_n(C)$ on the space of k-valued discrete functions in n variables, i.e., sequences with multi-indices in n variables. Let $a : \mathbb{N}^n \to k$ be a discrete function, and write

$$a = \{a_{m_1,m_2,\ldots,m_n}\}_{m_1,\ldots,m_n \geq 0} := a(m_1, m_2, \ldots, m_n),$$

where $m_1, \ldots, m_n \geq 0$ are the n mutually independent variables (indices) of a. Then, we let the action of $A_n(C)$ on a be the following ([Ze90]):

$$\begin{aligned} x_i \cdot a &= E_i a = a(m_1, \ldots, m_{i-1}, m_i + 1, m_{i+1}, \ldots, m_n), i = 1, \ldots, n, \\ D_i \cdot a &= m_i E_i^{-1} a = m_i a(m_1, \ldots, m_{i-1}, m_i - 1, m_{i+1}, \ldots, m_n), i = 1, \ldots, n. \end{aligned}$$

In particular, notice that the above action preserves the commutation relations of the generators of $A_n(C)$, so the action is well-defined. Thus, analogously as above, we may define a left-ideal

$$I_a := \{P \in A_n(C) \mid Pa = 0\},$$

and we say that the sequence a is holonomic if

$$d(A_n(C)/I_a) = n$$

as an $A_n(C)$-module. Similarly as above, one can show that P-finite sequences are holonomic.

The class of all holonomic k-valued functions in n variables forms an algebra, and this class is closed under integration in the continuous case or summation in the discrete case [Ze90]. In this way, this is a very large class, and it already contains the class of all multi-C-finite functions and the class of all P-finite functions.

Many examples of holonomic functions and operations that preserve holonomicity may be found in [Ze90]. Now, using the theory of holonomic functions discovered by I. N. Bernstein as described in [Ze90], Wilf and Zeilberger were able to generate linear recurrences for holonomic functions $F(n, k)$. This will become a key step in Wilf and Zeilberger's method for proving identities of the above form automatically.

Theorem 2.1.5 ([WZ90b, Ze90])**.** *Let $F(n, k)$ be a holonomic function in n and k. Then, there exists polynomials $s_0(n), \ldots, s_L(n)$ with L a positive integer and a holonomic function $G(n, k)$ such that*

$$s_0(n)F(n, k) + s_1(n)F(n + 1, k) + \cdots + s_L(n)F(n + L, k) = G(n, k + 1) - G(n, k)$$

with the LHS being indefinitely summable with respect to k.

Hence, once we find the polynomials $s_0(n), \ldots, s_L(n)$ and the holonomic function $G(n,k)$ for the function $F(n,k)$ as in the above theorem, we may sum both sides of the equation in k to obtain

$$s_0(n)S(n) + s_1(n)S(n+1) + \cdots + s_L(n)S(n+L) = G(n,+\infty) - G(n,-\infty)$$

where $S(n) = \sum_k F(n,k)$.

Now, suppose that we would like to check the identity

$$S(n) = \sum_k F(n,k) = C$$

corresponding to some holonomic functions $F(n,k)$, where C is a constant. The above theorem guarantees that $S(n)$ satisfies a recurrence relation of the form

$$s_0(n)S(n) + s_1(n)S(n+1) + \cdots + s_L(n)S(n+L) = G(n,+\infty) - G(n,-\infty),$$

where $s_i(n)$ is a polynomial in n for each i and $G(n,k)$ is holonomic as above. Thus, it suffices to check that $S(n) = C$ for each $n = 0, 1, \ldots, L-1$, and that C is a solution to the above recurrence, i.e.,

$$Cs_0(n) + Cs_1(n) + \cdots + Cs_L(n) = G(n,+\infty) - G(n,-\infty).$$

Further, it is shown in [Ze90] that whenever $F(n,k)$ is of the form 2.1 and that the holonomic function $F(n,k)$ terminates in the variable k, the function $G(n,k)$ is always of the form $R(n,k)F(n,k)$ for some rational function $R(n,k)$. Since $R(n,k)F(n,k)$ would also have to terminate in the variable k, i.e., $R(n,k)F(n,k) = 0$ for each n whenever k is large enough, we have that

$$\lim_{k \to \pm\infty} G(n,k) = 0.$$

Thus, the above linear recurrence becomes

$$s_0(n)S(n) + s_1(n)S(n+1) + \cdots + s_L(n)S(n+L) = 0.$$

The last step of Wilf and Zeilberger's method is an algorithm for computing the polynomials $s_0(n), \ldots, s_L(n)$ and the holonomic function $G(n,k)$. By using elimination theory on the Weyl algebra $A_n(C)$ to find operators $P \in A_n(C)$ that annihilate the function $F(n,k)$, Zeilberger came up with a ten-step algorithm that allows one to compute the functions $s_0(n), \ldots, s_L(n)$ and $G(n,k)$ [Ze90]. However, this algorithm is very time consuming [WZ90b, Ze90, Ze91]. A better algorithm known as creative telescoping is given in [WZ90b, Ze90, Ze91, Ze06], and the theory of holonomic functions from [Ze90] guarantees that this algorithm works for verifying identities involving summands of the form 2.1.

The method of creative telescoping may be summarized as follows. Let N, K be the difference operators of the variables n and k respectively, i.e., $Nn = n+1, Kk = k+1$. Notice that Theorem 2.1.5 guarantees a linear difference operator $s(N,K)$ such that

$$s(N,K)F(n,k) = G(n+1,k) - G(n,k).$$

Although finding $s(N, K)$ explicitly via the slow algorithm in [Ze90] is time consuming, one can make some guesses for what $s(N, K)$ might be. In the simplest case, $s(N, K)$ would be a first-order difference operator, as the linear recurrence relation which the sum $S(n) = \sum_k F(n, k)$ satisfies is non-trivial. Using the fact that $\lim_{k\to\pm\infty} G(n, k) = 0$ from [Ze90] and assuming $s(N, K)$ is of first-order, we would get

$$s_0(n)S(n) + s_1(n)S(n + 1) = 0.$$

For the identity

$$S(n) = C$$

to hold, we would necessarily have $s_0(n) + s_1(n) = 0$ assuming that $S(n) \neq 0$ for each n. Thus, normalizing the operator $s(N, K)$ by letting $s_1(n) = 1$, we would obtain $s_0(n) = -1$ and that

$$F(n + 1, k) - F(n, k) = G(n, k + 1) - G(n, k).$$

In [Go78], Gosper found an algorithm for determining whether there exists a solution $g(k)$ to the difference equation $g(k + 1) - g(k) = a(k)$ for any hypergeometric function $a(k)$, and the output of the algorithm would be the solution $g(k)$ whenever it exists. Thus, by setting $a(k) = F(n + 1, k) - F(n, k)$ as the input of Gosper's algorithm, we may find the required function $G(n, k)$ whenever $G(n, k + 1) - G(n, k) = a(k)$ has a solution. If a solution does not exist, the next simplest case for $s(N, K)$ would be that it is of second-order, so that

$$(s_0(n) + s_1(n)N + s_2(n)N^2)F(n, k) = G(n, k + 1) - G(n, k).$$

Gosper's algorithm allows for additional unknowns in the input hypergeometric function $a(k)$, so we may set $a(k) = s_0(n) + s_1(n)F(n + 1, k) + s_2(n)F(n + 2, k)$ as the input and look for the solution $G(n, k)$. Since Theorem 2.1.5 guarantees the existence of the holonomic function $G(n, k)$, we will be able to find the appropriate $G(n, k)$ once we have tried each input $a(k) = s(N, K)F(n, k)$ with the order of $s(N, K)$ large enough.

This method of trial and error is fast, since $s(N, K)$ turns out to be of first order for the vast majority of interesting examples from combinatorics [WZ90b, Ze91, Ze06]. In this case, the proof of the original identity $\sum_k F(n, k) = C$ is immediate via the below theorem, which is based on the above discussion.

Theorem 2.1.6 ([WZ90a]). *Suppose that $G(n, k)$ is a function in n, k defined for each $n \in \mathbb{Z}^{\geq 0}$ and each $k \in \mathbb{Z}$ such that*

$$F(n + 1, k) - F(n, k) = G(n, k + 1) - G(n, k)$$

for each n and each k. If

$$\lim_{k\to\pm\infty} G(n, k) = 0$$

for every integer $n \geq 0$, then

$$\sum_k F(n, k) = \sum_k F(n = 0, k) = \mathit{const.}$$

Definition 2.1.7 ([Wi94])**.** Given a hypergeometric identity

$$\sum_k F(n,k) = f(n),$$

if there exists a function $G(n,k)$ satisfying the conditions as outlined in Theorem 2.1.6, then the pair $(F(n,k), G(n,k))$ is called a Wilf-Zeilberger (WZ) pair. We also say that the pair $(F(n,k), G(n,k))$ certify the original hypergeometric identity.

In the case of $F(n,k)$ being holonomic, the WZ complement $G(n,k)$ of $F(n,k)$ is also guaranteed to be of the form $R(n,k)F(n,k)$, where $R(n,k)$ is some rational function in n and k [Ze90]. In this way, the method of creative telescoping for identities given by holonomic summands is also called the method of rational function certification, and the rational function $R(n,k)$ is called the proof certificate of the identity [WZ90a].

Finally, we remark that the method of rational function certification extends to multi-sum identities in the discrete variable case, and to integral identities in the continuous variable case [WZ92a, WZ92b]. The method also applies to the "q"-analog of every integral or summation identity to which it applies [WZ92a, WZ92b].

2.2 Intersection theory

In this section, we will give an overview of intersection theory for schemes via the theory of Chow rings. The results presented in this section are mainly based on the ones found in [Fu84] and [EH16]. Although the main focus of this paper is on quasi-projective algebraic varieties, we will establish the formalism of Chow rings for separated schemes of finite type. This will be relevant should we need to consider an analog of Problem 1.0.2 for schemes in general. Also, as we shall see, computing rational equivalence classes in Chow rings of projective spaces will be a main tool for solving Problem 1.0.2.

For the following, we will assume that all schemes are separated and of finite type over an algebraically closed field k of characteristic 0. This includes all algebraic varieties when considered as schemes over the field k.

Definition 2.2.1. Let X be a separated scheme of finite type over an algebraically closed field k of characteristic 0.

1. The group of k-cycles on X, $Z_k(X)$, is the free abelian group generated by all reduced irreducible k-dimensional subschemes of X.

2. The group of cycles on X, $Z(X)$, is the free abelian group generated by all reduced irreducible subschemes of X, i.e.,

$$Z(X) = \bigoplus_{k\geq 0} Z_k(X).$$

3. A cycle $Z = \sum_i n_i Y_i \in Z(X)$ is effective if $n_i \geq 0$ for all i.

Remark 2.2.2. 1. For all $k > \dim X$, $Z_k(X) = 0$ since subschemes of X must be of dimension less than or equal to $\dim X$.

2. Let X_{red} be the reduced scheme of X. Then $Z(X) = Z(X_{red})$ by definition.

Definition 2.2.3. Let X be a scheme, and let Y be a closed subscheme of X. Suppose that the reduced scheme of Y, Y_{red}, has irreducible components $Y_1, Y_2, \ldots, Y_s$. Then the effective cycle associated to Y is the cycle

$$\langle Y \rangle := \sum_{i=1}^{s} l_i Y_i \in Z(X)$$

where $l_i := \mathrm{len}(\mathcal{O}_{Y,Y_i})$ is the length of the local ring $\mathcal{O}_{Y,Y_i}$ for each $i = 1, \ldots, s$.

Thus, we may view the effective cycle associated to a closed subscheme as a coarse approximation to that subscheme. Our next goal is to define the notion of degree of subschemes of projective spaces. To this end, we will form equivalence classes in the group of cycles.

Definition 2.2.4. Let X be a scheme, and let $V \subseteq X$ be a reduced and irreducible subscheme of codimension one. Denote the local ring of X at V by $A := \mathcal{O}_{X,V}$. Let $r \in k(X)^*$ be a non-zero rational function over X, and write $r = \frac{a}{b}$ for $a, b \in A$ regular functions on V. The order of vanishing of r along V is given by

$$\mathrm{ord}_V(r) := \mathrm{ord}_V(a) - \mathrm{ord}_V(b) := \mathrm{len}_A(A/(a)) - \mathrm{len}_A(A/(b))$$

where $\mathrm{len}_A(A/(a))$, $\mathrm{len}_A(A/(b))$ are the lengths of the A-modules $A/(a)$, $A/(b)$ over A respectively.

Fulton [Fu84] showed that when viewed as a homomorphism from $k(X)^*$ to $\mathbb{Z}$, $\mathrm{ord}_V(-)$ is a well-defined group homomorphism. Moreover, for any non-zero rational function r over X, there are only finitely many codimension one reduced and irreducible subschemes V such that $\mathrm{ord}_V(r) \neq 0$. This allows us to associate cycles to every non-zero rational functions.

Definition 2.2.5. Let X be a scheme, and let r be a non-zero rational function over a $(k+1)$-dimensional subscheme W that is reduced and irreducible. The divisor of r is the k-cycle given by

$$[\mathrm{div}(r)] := \sum_V \mathrm{ord}_V(r)[V]$$

where the sum is taken over all reduced and irreducible k-dimensional subschemes $V \subset W$, and $\mathrm{ord}_V(r)$ is defined over the local ring $\mathcal{O}_{W,V}$.

In particular, notice that $[\mathrm{div}(r)]$ is well-defined since it is always a finite linear combination of subschemes of W of codimension one. Then, equivalence classes in $Z(X)$ are formed by cycles modulo divisors of rational functions. This gives rise to the Chow group of X.

Definition 2.2.6. Let $Z(X)$ be the group of cycles of a scheme X. The subgroup of rational equivalence on k-cycles, $\mathrm{Rat}_k(X)$, is the free abelian group generated by all $[\mathrm{div}(r)]$ where r is a non-zero rational function over a $(k+1)$-dimensional reduced and irreducible subscheme of X.
The Chow group of k-cycles of X is the quotient group

$$A_k(X) := Z_k(X)/\mathrm{Rat}_k(X).$$

The Chow group of X is given by the direct sum of the Chow groups of k-cycles:

$$A(X) := \bigoplus_{k\geq 0} A_k(X).$$

Notice that $\mathrm{Rat}_k(X) \subseteq Z_k(X)$, and it is indeed a subgroup since the sum of divisors of rational functions is still a divisor of a rational function: if $[\mathrm{div}(r)], [\mathrm{div}(s)] \in \mathrm{Rat}_k(X)$, then we can write

$$[\mathrm{div}(r)] = \sum \mathrm{ord}_V(r)[V], [\mathrm{div}(s)] = \sum \mathrm{ord}_V(s)[V]$$

where the sums run over k-dimensional reduced and irreducible subschemes of X. Then, we have that

$$[\mathrm{div}(r)] + [\mathrm{div}(s)] = \sum(\mathrm{ord}_V(r) + \mathrm{ord}_V(s))[V] = \sum \mathrm{ord}_V(r+s)[V] = [\mathrm{div}(r+s)]$$

since $\mathrm{ord}_V(-)$ is a group homomorphism by [Fu84].

Next, we will introduce a product structure on the Chow group, so that it becomes a ring. This product structure will allow us to interpret multiplicities of points of intersection.

Definition 2.2.7. Let X be a reduced and irreducible scheme, and let $A, B \subseteq X$ be reduced and irreducible subschemes. A and B intersect transversely at a point $p \in A \cap B$ if A, B, X are all smooth at p and

$$T_pA + T_pB = T_pX,$$

where T_pA, T_pB, T_pX are the tangent spaces of A, B and X at p respectively. Equivalently:

$$\mathrm{codim}(T_pA \cap T_pB) = \mathrm{codim}\, T_pA + \mathrm{codim}\, T_pB.$$

We say that A and B are generically transverse if they meet transversely at a general point of each irreducible component $C \subseteq A \cap B$.

Then, the product structure on $A(X)$ is given by intersections of generically transverse subschemes of X.

Theorem 2.2.8 ([EH16, Fu84])**.** *Let X be a smooth, reduced and irreducible scheme. Then, there exists a unique product structure on $A(X)$ such that for any two generically transverse subschemes $A, B \subseteq X$, we have $[A][B] := [A \cap B]$.*

We would also like to extend this result to subschemes that do not intersect generically transversely, given that the conditions imposed in Definition 2.2.7 are very strict. When two reduced and irreducible subschemes $A, B \subseteq X$ are dimensionally transverse, i.e., $\mathrm{codim}(A \cap B) = \mathrm{codim}\, A + \mathrm{codim}\, B$, we may relate the classes $[A]$ and $[B]$ with $[A \cap B]$ by taking into account of intersection multiplicities of A and B along components of $A \cap B$.

Theorem 2.2.9 ([EH16])**.** *Let X be a smooth, reduced and irreducible scheme, and let $A, B \subseteq X$ be reduced and irreducible subschemes. Suppose that every irreducible component $C \subseteq A \cap B$ is such that $\mathrm{codim}\, C = \mathrm{codim}\, A + \mathrm{codim}\, B$. Then, for each component $C \subseteq A \cap B$, there exists a positive integer $m_C(A, B)$ such that:*

1. *$[A][B] = \sum_{C \subseteq A\cap B} m_C(A, B)[C]$, where the sum is taken over all irreducible components of $A \cap B$.*

2. *$m_C(A, B) = 1$ if and only if A and B intersect generically transversely on C.*

3. *If A and B are Cohen-Macaulay at a general point of C, then $m_C(A, B)$ is the multiplicity of the component of the scheme $A \cap B$ supported on C. In particular, if A and B are everywhere Cohen-Macaulay, then*

$$[A][B] = [A \cap B].$$

4. *$m_C(A, B)$ depends only on the local structure of A and B at a general point of C.*

The following criterion allows us to check generic transverseness when two subvarieties intersect dimensionally transversely.

Proposition 2.2.10 ([EH16]). *Let X be a smooth, reduced and irreducible scheme. Two reduced and irreducible subschemes $A, B \subseteq X$ are generically transverse if and only if they are dimensionally transverse, and each irreducible component of $A \cap B$ contains a point where X is smooth and $A \cap B$ is reduced.*

Finally, we may define the notion of degree of subschemes of projective spaces, and show that this is a generalization of the notion for algebraic varieties. To this end, we need the fact that the pushforward of a rational equivalence class under a proper morphism of schemes is well-defined. This is demonstrated in [EH16].

Definition 2.2.11 ([EH16]). Let $f : Y \to X$ be a proper map of schemes, and let $A \subseteq Y$ be a subvariety, i.e., a reduced and irreducible subscheme of Y. Notice that $f(A)$ is a subvariety of X since f is proper. Denote the effective cycle associated with A by $\langle A \rangle$, and the one associated with $f(A)$ by $\langle f(A) \rangle$.

1. If $\dim f(A) < \dim A$, then we set $f_*\langle A \rangle = 0$, where f_* denotes the pushforward of f on $Z(Y)$.

2. If $\dim f(A) = \dim A$ and $f|_A$ has degree n, i.e., $[k(A) : k(f(A))] = n$ where $k(A)$ and $k(f(A))$ are the respective function fields of A and $f(A)$, then we set $f_*\langle A \rangle = n\langle f(A) \rangle$.

3. We extend f_* to all of $Z(Y)$ by linearity, i.e., for any collection of subvarieties $A_i \subseteq Y$,

$$f_* \left(\sum_i m_i \langle A_i \rangle \right) := \sum_i m_i f_* \langle A_i \rangle.$$

Rational equivalence is preserved under pushforward of cycles by proper morphisms:

Theorem 2.2.12 ([EH16]). *Suppose that $f : Y \to X$ is a proper morphism of schemes. Then, the map $f_* : Z(Y) \to Z(X)$ as defined above induces a group homomorphism $f_* : A_k(Y) \to A_k(X)$ for each k.*

A proof of this theorem can be found in [Fu84]. In fact, this theorem also gives rise to a degree map on $A(X)$ whenever X is proper over $\mathrm{Spec}(k)$.

Proposition 2.2.13 ([EH16]). *Let k be an algebraically closed field of characteristic zero, and let X be separated scheme of finite type over k. Suppose that X is proper over* $\operatorname{Spec}(k)$*. Then, there exists a unique map $\delta : A(X) \to \mathbb{Z}$ sending the class of a closed point $p \in X$ to 1, and vanishing on the class of any cycle of positive pure dimension.*

In particular, the above proposition applies when $X = \mathbb{P}^n$ for any n, since projective spaces are proper over $\operatorname{Spec}(k)$ when viewed as schemes over k.

Definition 2.2.14 ([EH16]). Let $A \subseteq \mathbb{P}^n$ be a subvariety of dimension k, and let $L \subseteq \mathbb{P}^n$ be an intersection of k hyperplanes of $\mathbb{P}^n$ such that A and L intersect generically transversely. (Notice that one can always find such an L.) Then, we define the degree of $A \subseteq \mathbb{P}^n$ to be

$$\deg A := \delta([A][L]),$$

where $\delta : \mathbb{P}^n \to \mathbb{Z}$ is the unique map given by Proposition 2.2.13.

Thus, the above definition is an extension of the definition of degree of algebraic varieties, since

$$\delta([A][L]) = \delta([A \cap L]) = \#|A \cap L|$$

by the definition of δ from Proposition 2.2.13. This allows us to treat the two different definitions of degrees of subvarieties of $\mathbb{P}^n$ as equivalent notions.

For the rest of this paper, Bézout's theorem for subvarieties of $\mathbb{P}^n$ will be a main tool for solving Problem 1.0.2. The theorem follows from the formula of the Chow ring of $\mathbb{P}^n$.

Theorem 2.2.15 ([EH16]). *The Chow ring of $\mathbb{P}^n$ is given by*

$$A(\mathbb{P}^n) \cong \mathbb{Z}[\zeta]/(\zeta^{n+1})$$

where $\zeta \in A_{n-1}(\mathbb{P}^n)$ is the rational equivalence class of a hyperplane. In general, the class of a k-dimensional irreducible subvariety of degree d is $d\zeta^{n-k}$.

Corollary 2.2.16 (Bézout, [EH16]). *Let $X_1, \ldots, X_k \subseteq \mathbb{P}^n$ be irreducible subvarieties of codimensions $c_1, \ldots, c_k$ respectively. Suppose that the X_i's intersect generically transversely. Then,*

$$\deg(X_1 \cap X_2 \cap \cdots \cap X_k) = \prod_{i=1}^{k} \deg(X_i).$$

Proof. By Theorem 2.2.8,

$$[X_1 \cap X_2 \cap \cdots \cap X_k] = \prod_{i=1}^{k} [X_i]$$

in $A(\mathbb{P}^n)$ since the X_i's intersect generically transversely. The degree of $X_1 \cap X_2 \cap \cdots \cap X_k$ is computed by intersecting the set with $\sum c_i$ general hyperplanes. Since $[X_i] = \deg(X_i)\zeta^{c_i}$

for all i's, we have that

$$\begin{aligned}\deg(X_1 \cap X_2 \cap \cdots \cap X_k) &= \deg([X_1 \cap X_2 \cap \cdots \cap X_k]\zeta^{n-\sum c_i}) \\ &= \deg\left(\prod_{i=1}^{k}[X_i]\zeta^{n-\sum c_i}\right) \\ &= \deg\left(\prod_{i=1}^{k}\deg(X_i)\zeta^{n}\right) \\ &= \prod_{i=1}^{k}\deg(X_i).\end{aligned}$$

□

The case where irreducible varieties do not intersect generically transversely may be handled as follows.

Corollary 2.2.17 (Bézout, [EH16]). *Let $X_1, \ldots, X_k \subseteq \mathbb{P}^n$ be irreducible subvarieties of codimensions $c_1, \ldots, c_k$ respectively. Suppose that the X_i's intersect dimensionally transversely, i.e., each irreducible component $Z_1, Z_2, \ldots, Z_s \subseteq X_1 \cap X_2 \cap \cdots \cap X_k$ is such that* $\operatorname{codim} Z_j = \sum_{i=1}^{k} c_i$ *for $j = 1, \ldots, s$. If the X_i's are Cohen-Macaulay at a general point of Z_j for each $j = 1, \ldots, s$, then*

$$\prod_{i=1}^{k}[X_i] = \sum_{j=1}^{s} m_{Z_j}(X_1, \ldots, X_k)[Z_j],$$

where $m_{Z_j}(X_1, \ldots, X_k)$ is equal to the multiplicity of $X_1 \cap X_2 \cap \cdots \cap X_k$ at a general point of Z_j. It follows that

$$\prod_{i=1}^{k}\deg X_i = \sum_{j=1}^{s} m_{Z_j}(X_1, \ldots, X_k)\deg Z_j.$$

Proof. Denote $Z = X_1 \cap X_2 \cap \cdots \cap X_k$. By Theorem 2.2.9, we have that

$$\prod_{i=1}^{k}[X_i] = \sum_{j=1}^{s} m_{Z_j}(X_1, \ldots, X_k)[Z_j]$$

where $m_{Z_j}(X_1, \ldots, X_k)$ is equal to the multiplicity of Z at a general point of Z_j by $X_1, \ldots, X_k$ being Cohen-Macaulay at a general point of Z_j. The equality of degrees follows similarly as in Theorem 2.2.16: we may intersect each subvariety with $\sum_{i=1}^{k} c_i$ general hyperplanes in $\mathbb{P}^n$ and take degrees of the intersections, which are all equal. □

We remark that when $X_1, \ldots, X_k$ are all hypersurfaces, the assumption that $X_1, \ldots, X_k$ are Cohen-Macaulay at a general point of each component of the intersection is automatically satisfied [EH16].

To conclude this subsection, we present a version of the generalized Bézout's theorem, which gives a bound on the degree of the intersection $X_1 \cap \cdots \cap X_k \subseteq \mathbb{P}^n$ whenever the intersection is not dimensionally transverse. This will be very useful for steps involved in solving Problem 1.0.2, as we shall see in the next section. The below theorem is due to Example 8.4.6 found in Chapter 8 of [Fu84].

Theorem 2.2.18 (Bézout, [Fu84])**.** *Let $X_1, \ldots, X_k \subseteq \mathbb{P}^n$ be irreducible subvarieties of $\mathbb{P}^n$, and let $Z_1, \ldots, Z_s$ be the irreducible components of $X_1 \cap X_2 \cap \cdots \cap X_k$. Then,*

$$\deg X_1 \cap X_2 \cap \cdots \cap X_k \leq \sum_{i=1}^{s} \deg Z_i \leq \prod_{i=1}^{k} \deg X_i.$$

Section 3

Preliminary results

For this section, we will assume that all algebraic varieties are defined over some field k of characteristic zero.

Let $(X, \phi, x, f), (Y, \psi, y, g)$ be as given in Problem 1.0.2. Firstly, notice that without the loss of generality, we may assume that X and Y are both irreducible. This is because if X and Y are reducible, we may restrict ϕ and ψ to some irreducible components of X and Y respectively and consider Problem 1.0.2 on those components. By solving Problem 1.0.2 for each pair of irreducible components of X and Y, we will have solved the problem for X and Y themselves.

Now, the answer to Problem 1.0.2 is yes – there exists a positive integer $N \geq 1$ such that for any pair of points $x, y \in X$, $f(\phi^i(x)) = g(\psi^i(y))$ for all $i = 1, \dots, N$ implies that $f(\phi^i(x)) = g(\psi^i(y))$ for all $i \in \mathbb{N}$. This may be shown via Noetherian induction. Denote $F_i := (f, g) \circ (\phi^i, \psi^i)$, and let

$$Y_i := F_i^{-1}(\Delta)$$

be the pre-image of the diagonal subvariety Δ in $\mathbb{P}^1 \times \mathbb{P}^1$ under F_i. By definition, Y_i does not intersect the locus of indeterminacy of F_i for each i, and

$$Y_i = \{(x, y) \in X \times Y \mid f(\phi^i(x)) = g(\psi^i(y))\}.$$

Thus, the condition that $f(\phi^i(x)) = g(\psi^i(y))$ for each $i = 1, \dots, N$ translates to

$$(x, y) \in Y_1 \cap Y_2 \cap \cdots \cap Y_N.$$

Since we have a descending chain of algebraic sets in $X \times Y$ given by

$$Y_1 \supseteq Y_1 \cap Y_2 \supseteq \cdots \supseteq Y_1 \cap Y_2 \cap \cdots \cap Y_n \supseteq \dots$$

it follows that this descending chain must eventually stabilise by $X \times Y$ being a Noetherian topological space.

Moreover, we have the following lemma giving a characterization of when the above chain stabilises.

Lemma 3.0.1. *Let $(X, \phi, x, f), (Y, \psi, y, g)$ be quadruples satisfying the conditions as outlined in Problem 1.0.2, and set $F_i := (f, g) \circ (\phi^i, \psi^i)$. For each $i \geq 1$, let $Y_i := F_i^{-1}(\Delta) \subseteq X \times Y$ be the closed subset as defined above. If*

$$Y_1 \cap Y_2 \cap \cdots \cap Y_N = Y_1 \cap Y_2 \cap \cdots \cap Y_{N+1}$$

for some positive integer N, then

$$Y_1 \cap Y_2 \cap \cdots \cap Y_N = Y_1 \cap Y_2 \cap \cdots \cap Y_j$$

for all $j > N$.

Proof. We will prove this by induction on j. Set $l = j - N$ with $j > N$, and suppose that

$$Y_1 \cap Y_2 \cap \cdots \cap Y_N = Y_1 \cap Y_2 \cap \cdots \cap Y_{N+1}$$

for some positive integer N. Then, the base case $l = 1$ is true by assumption. Now, suppose that

$$Y_1 \cap Y_2 \cap \cdots \cap Y_N = Y_1 \cap Y_2 \cap \cdots \cap Y_{N+1} = \cdots = Y_1 \cap Y_2 \cap \cdots \cap Y_{N+l}$$

for some $l \geq 1$. Since

$$Y_1 \cap Y_2 \cap \cdots \cap Y_{N+l} \supseteq Y_1 \cap Y_2 \cap \cdots \cap Y_{N+l+1},$$

it suffices to prove the inclusion of the LHS to the RHS, i.e.,

$$Y_1 \cap Y_2 \cap \cdots \cap Y_{N+l} \subseteq Y_{N+l+1}.$$

Notice that for each $l \geq 0$,

$$Y_{N+l+1} := F_{N+l+1}^{-1}(\Delta) = (\phi^{N+l+1}, \psi^{N+l+1})^{-1} \circ (f^{-1}, g^{-1})(\Delta) = (\phi, \psi)^{-1}(Y_{N+l})$$

as subsets of $X \times Y$. By assumption, $Y_N \subseteq Y_{N+1} = (\phi, \psi)^{-1}(Y_N)$, so we have that

$$Y_{N+l} = (\phi, \psi)^{-l}(Y_N) \subseteq (\phi, \psi)^{-l-1}(Y_N) = Y_{N+l+1}.$$

By the inductive hypothesis:

$$Y_1 \cap Y_2 \cap \cdots \cap Y_{N+l-1} \subseteq Y_{N+l}$$

for each $l \geq 1$. Therefore,

$$Y_1 \cap Y_2 \cap \cdots \cap Y_{N+l} \subseteq Y_{N+l} \subseteq Y_{N+l+1},$$

as required. $\square$

Thus, finding the positive integer $N \geq 1$ in Problem 1.0.2 is equivalent to finding the smallest number $N \geq 1$ such that

$$Y_1 \cap Y_2 \cap \cdots \cap Y_N = Y_1 \cap Y_2 \cap \cdots \cap Y_N \cap Y_{N+1}.$$

Remark 3.0.2. Problem 1.0.2 may be reduced to the case where X and Y are projective varieties. Notice that for X and Y quasi-projective and irreducible, the closures $\overline{X}$, $\overline{Y}$ are irreducible projective varieties. We may extend ϕ and ψ to dominant rational self-maps $\tilde{\phi}$, $\tilde{\psi}$ on $\overline{X}$ and $\overline{Y}$ respectively. Similarly, f and g may be extended to rational functions $\tilde{f}$, $\tilde{g}$ on $\overline{X}$ and $\overline{Y}$ with dense images on $\mathbb{P}^1$. Denote $\tilde{F}_i = (\tilde{f}, \tilde{g}) \circ (\tilde{\phi}^i, \tilde{\psi}^i)$. Thus,

$$\tilde{F_i}^{-1}(\Delta) = \overline{F_i^{-1}(\Delta)} \subseteq \overline{X \times Y}$$

by ϕ, f and ψ, g being continuous on a dense subset of X and Y respectively.

Hence, the descending chain

$$Y_1 \supseteq Y_1 \cap Y_2 \supseteq Y_1 \cap Y_2 \cap Y_3 \supseteq \dots$$

stabilises at $N \geq 1$ if and only if the descending chain

$$\overline{Y_1} \supseteq \overline{Y_1} \cap \overline{Y_2} \supseteq \overline{Y_1} \cap \overline{Y_2} \cap \overline{Y_3} \supseteq \dots$$

stabilises at $N \geq 1$, where $\overline{Y_i} = \tilde{F}_i^{-1}(\Delta) = \overline{F_i^{-1}(\Delta)}$ for each i. Indeed, we have that

$$Y_1 \cap Y_2 \cap \cdots \cap Y_N = Y_1 \cap Y_2 \cap \cdots \cap Y_{N+1}$$

implies that

$$\overline{Y_1} \cap \overline{Y_2} \cap \cdots \cap \overline{Y_N} = \overline{Y_1 \cap Y_2 \cap \cdots \cap Y_N} = \overline{Y_1 \cap Y_2 \cap \cdots \cap Y_{N+1}} = \overline{Y_1} \cap \overline{Y_2} \cap \cdots \cap \overline{Y_{N+1}},$$

so the latter chain stabilises whenever the former does. For the other direction, suppose that

$$\overline{Y_1} \cap \overline{Y_2} \cap \cdots \cap \overline{Y_N} = \overline{Y_1} \cap \overline{Y_2} \cap \cdots \cap \overline{Y_{N+1}},$$

and let $(x, y) \in Y_1 \cap Y_2 \cap \cdots \cap Y_N$. Thus,

$$\tilde{F}_{N+1}(x, y) = (\tilde{f}, \tilde{g}) \circ (\tilde{\phi}^{N+1}, \tilde{\psi}^{N+1})(x, y) \in \Delta.$$

By definition, we have that $\tilde{F}_i|_{X \times Y} = F_i$, so

$$F_{N+1}(x, y) = \tilde{F}_{N+1}(x, y) \in \Delta,$$

implying that $(x, y) \in Y_{N+1}$. This completes the proof of our claim.

To give the positive integer $N \geq 1$ as described in Problem 1.0.2, we will fix embeddings $\iota_X : X \hookrightarrow \mathbb{P}^m$ and $\iota_Y : Y \hookrightarrow \mathbb{P}^n$. By Lemma 3.0.1, finding $N \geq 1$ is equivalent to finding the smallest number $N \geq 1$ such that

$$Y_1 \cap Y_2 \cap \cdots \cap Y_N = Y_1 \cap Y_2 \cap \cdots \cap Y_{N+1},$$

where

$$Y_i = F_i^{-1}(\Delta) = (f \circ \phi^i, g \circ \psi^i)^{-1}(\Delta)$$

for each i, with Δ being the diagonal subvariety of $\mathbb{P}^1 \times \mathbb{P}^1$.

Notice that if $Y_1 \cap Y_2 \cap \cdots \cap Y_j$ is a zero-dimensional set for some $j \geq 1$, then we may embed $Y_1 \cap Y_2 \cap \cdots \cap Y_j$ into some projective space and use Theorem 2.2.18 to bound its size from the above, assuming that the field k is algebraically closed:

$$|Y_1 \cap Y_2 \cap \cdots \cap Y_j| \leq \prod_{i=1}^{j} \deg Y_i.$$

This suggests the following approach for giving an upper bound on the positive integer N.

Idea 3.0.3. 1. Find a positive integer $N_1 \geq 1$ such that for every positive dimensional component
$$C \subseteq Y_1 \cap Y_2 \cap \cdots \cap Y_{N_1},$$
$C \subseteq Y_i$ for each $i \geq 1$.

2. By Theorem 2.2.18, the size of the zero-dimensional components of $Y_1 \cap Y_2 \cap \cdots \cap Y_{N_1}$ is bounded above by
$$|(Y_1 \cap Y_2 \cap \cdots \cap Y_{N_1})^0| \leq \prod_{i=1}^{N_1} \deg Y_i := N_2$$
whenever the field k is algebraically closed. Using the characterization given in Lemma 3.0.1, we have that
$$N \leq N_1 + N_2.$$

Hence, we would need to bound the degree of each Y_i from the above. Embedding each Y_i into $\mathbb{P}^m \times \mathbb{P}^n$ under (ι_X, ι_Y), we may write
$$(\iota_X, \iota_Y)(Y_i) = (\iota_X, \iota_Y)(F_i^{-1}(\Delta_X)) = (\iota_X, \iota_Y)(X \times Y) \cap (\iota_X, \iota_Y)(F_i^{-1}(\Delta_X)).$$
When composed with ι_X, $f \circ \phi^i$ can be written locally as homogeneous polynomials in the coordinate ring of $\mathbb{P}^m$:
$$f \circ \phi^i(x) = (f_0 \circ \phi^i(x) : f_1 \circ \phi^i(x)) \in \mathbb{P}^1, x \in U \text{ for some } U \subseteq X \text{ open.}$$
Similarly, $g \circ \psi^i$ can be written as homogeneous polynomials in the coordinate ring of $\mathbb{P}^n$ on an open subset of Y. Thus, considering Y_i and $X \times Y$ as subsets of $\mathbb{P}^m \times \mathbb{P}^n$ under (ι_X, ι_Y), we have the following lemma.

Lemma 3.0.4. *Let k be a field of characteristic zero, but not necessarily algebraically closed. Let (X, ϕ, x, f), (Y, ψ, y, g) be as in Problem 1.0.2, where X and Y are varieties defined over k. Also, let Y_i be defined over k as above for each $i \geq 1$. Then,*
$$Y_i = (X \times Y) \cap V(f_0 \circ \phi^i(x) = g_0 \circ \psi^i(y), f_1 \circ \phi^i(x) = g_1 \circ \psi^i(y)) \subseteq \mathbb{P}^m \times \mathbb{P}^n.$$

Proof. Notice that the inclusion of the RHS to Y_i follows from the definition of Y_i. For the other inclusion, suppose that $(x, y) = ((x_0 : \cdots : x_n), (y_0 : \cdots : y_n)) \in Y_i$. Then, $(f \circ \phi^i, g \circ \psi^i)(x, y) \in \Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$, so
$$(f_0 \circ \phi^i(x) : f_1 \circ \phi^i(x)) = \lambda(g_0 \circ \psi^i(y) : g_1 \circ \psi^i(y))$$
for some $\lambda \in k^\times$ and for each $i = 0, 1, \ldots, n$. Thus, by taking $(x, y) = (x, \lambda y) \in \mathbb{P}^m \times \mathbb{P}^n$, we recover the equations $f_0 \circ \phi^i(x) = g_0 \circ \psi^i(y)$ and $f_1 \circ \phi^i(x) = g_1 \circ \psi^i(y)$, so each Y_i can be written as the above expression as claimed. □

Remark 3.0.5. Notice that in the above expression of Y_i, the polynomials
$$f_0 \circ \phi^i(x) = g_0 \circ \psi^i(y), f_1 \circ \phi^i(x) = g_1 \circ \psi^i(y)$$

are not bihomogeneous in the coordinates of $\mathbb{P}^m \times \mathbb{P}^n$ in general. This is problematic in practical computations, since we would like subvarieties of $\mathbb{P}^m \times \mathbb{P}^n$ to be defined by bihomogeneous polynomials in general. However, in practical applications, we often know that one of the above polynomials is never zero by assumption. For example, if we suppose that $f_0 \circ \phi^i(x), g_0 \circ \psi^i(y)$ are never zero for any $x \in X, y \in Y$, then

$$(f_0 \circ \phi^i(x) : f_1 \circ \phi^i(x)) = \left(1 : \frac{f_1 \circ \phi^i(x)}{f_0 \circ \phi^i(x)}\right), (g_0 \circ \psi^i(y) : g_1 \circ \psi^i(y)) = \left(1 : \frac{g_1 \circ \psi^i(y)}{g_0 \circ \psi^i(y)}\right).$$

Thus, using the argument of Lemma 3.0.4, we also have that

$$\begin{aligned} Y_i &= (X \times Y) \cap V\left(\frac{f_1 \circ \phi^i(x)}{f_0 \circ \phi^i(x)} = \frac{g_1 \circ \psi^i(y)}{g_0 \circ \psi^i(y)}\right) \\ &= (X \times Y) \cap V\left(f_1(\phi^i(x))g_0(\psi^i(y)) = g_1(\psi^i(y))f_0(\phi^i(x))\right) \subseteq \mathbb{P}^m \times \mathbb{P}^n \end{aligned}$$

provided that $f_0 \circ \phi^i(x), g_0 \circ \psi^i(y)$ are never zero. This gives a bihomogeneous generator for the ideal of Y_i.

The above suggests that we need to bound the degrees of the closed subsets $X \times Y$ and $V(f_0 \circ \phi^i(x) = g_0 \circ \psi^i(y), f_1 \circ \phi^i(x) = g_1 \circ \psi^i(y))$ of $\mathbb{P}^m \times \mathbb{P}^n$. More precisely, we will need to bound the degrees of the images of the above under the Segre embedding

$$\sigma_{m,n} : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{(m+1)(n+1)-1}, ((x_0 : \cdots : x_m), (y_0 : \cdots : y_n)) \mapsto (x_0y_0 : x_0y_1 : \cdots : x_my_n).$$

The Segre embedding is necessary, because we will need to apply Theorem 2.2.18 in some projective space to get

$$\deg Y_i \leq \deg(X \times Y) \deg V(f_0 \circ \phi^i(x) = g_0 \circ \psi^i(y), f_1 \circ \phi^i(x) = g_1 \circ \psi^i(y))$$

for each i, where each of the above varieties are considered as subvarieties of the projective space.

We shall see that the degree of $X \times Y$ can be calculated by finding the class $[X \times Y]$ in the Chow ring of $\mathbb{P}^m \times \mathbb{P}^n$, while the degree of the latter closed algebraic set is related to the degrees of the maps ϕ, ψ, f and g.

Notation 3.0.6. Let X, Y be quasi-projective varieties, and let $\iota_X : X \hookrightarrow \mathbb{P}^m$, $\iota_Y : Y \hookrightarrow \mathbb{P}^n$ be closed embeddings. Let $\phi : X \to Y$ be a rational map, and suppose that $\phi|_U : U \to Y$ is a morphism defined on an open subset $U \subseteq X$. Identify each $x \in X$ with its image $\iota_X(x) \in \mathbb{P}^m$, so that x may be written in coordinates of $\mathbb{P}^m$. Then, ϕ may be written in terms of homogeneous polynomials of the same degree on $U \subseteq X$:

$$\iota_Y \circ \phi(x) = (\phi_0(x) : \phi_1(x) : \cdots : \phi_n(x)) \in \mathbb{P}^n,$$

where each ϕ_i is an element of the homogeneous coordinate ring of $\mathbb{P}^m$. We denote

$$\deg \phi := \deg \phi_i,$$

and we call this the degree of ϕ under the embedding ι_Y.

Remark 3.0.7. In the above, $\deg \phi$ under the embedding ι_Y is not the same as the usual definition of the degree of ϕ. In Chapter 1, Section 3 of [EH16], the degree of a surjective morphism $f : W \to Z$ is given as follows. For a subvariety $A \subseteq W$ such that $\dim A = \dim f(A)$, the field of rational functions $k(A)$ is a finite extension of the field $k(f(A))$. Then, the degree of the extension $[k(A) : k(f(A))]$ is said to be the degree of the covering of $f(A)$ by A.

3.1 Degree of $X \times Y \subseteq \mathbb{P}^m \times \mathbb{P}^n$

For this subsection, we assume that all varieties are defined over an algebraically closed field k of characteristic zero.

The method for calculating the class of $X \times Y$ in the Chow ring of $\mathbb{P}^m \times \mathbb{P}^n$ can be summarized as follows. Given the product of projective spaces $\mathbb{P}^m \times \mathbb{P}^n$, let π_1, π_2 be the projections onto the first and second factors of the product space. We would like to find the pullback of the classes $[X] \in A(\mathbb{P}^m)$ and $[Y] \in A(\mathbb{P}^n)$ via the maps π_1, π_2 respectively. Then, we will show that the intersection of the two pullback classes in $A(\mathbb{P}^m \times \mathbb{P}^n)$ agrees with the class $[X \times Y] \in A(\mathbb{P}^m \times \mathbb{P}^n)$.

Hence, we will need to define the pullback of cycles via morphisms of algebraic varieties, and show that pullbacks preserve rational equivalence classes.

Definition 3.1.1. Let $f : Y \to X$ be a morphism of smooth, reduced and irreducible schemes. A subvariety $A \subseteq X$ is generically transverse to f if:

1. The pre-image $f^{-1}(A) \subset Y$ is a generically reduced scheme, and
2. $\operatorname{codim}_Y(f^{-1}(A)) = \operatorname{codim}_X(A)$.

In particular, notice that the projections π_1, π_2 of $\mathbb{P}^m \times \mathbb{P}^n$ onto its first and second factors are morphisms of a smooth variety. For any subvariety $X \subseteq \mathbb{P}^m$, $\pi_1^{-1}(X) \subseteq \mathbb{P}^m \times \mathbb{P}^n$ is an irreducible subvariety, so it is reduced and irreducible by X being reduced and irreducible. Also, the codimension of $X \subseteq \mathbb{P}^m$ is preserved under pullback by π_1. Thus, any subvariety of $\mathbb{P}^m$ is generically transverse to π_1, and similarly for any subvariety of $\mathbb{P}^n$ to π_2.

The following fundamental theorem from [EH16] guarantees that pullbacks preserve rational equivalence classes of subvarieties generically transverse to the corresponding morphism.

Theorem 3.1.2 ([EH16])**.** *Let $f : Y \to X$ be a map of smooth quasi-projective varieties. Denote $A^c(X) := A_{r-c}(X)$, where $r = \dim X$. Then, for each $c = 0, 1, \dots, r$, there exists a unique group homomorphism $f^* : A^c(X) \to A^c(Y)$ such that for each subvariety $A \subseteq X$ generically transverse to f,*

$$f^*([A]) = [f^{-1}(A)].$$

The map f^* extends to a ring homomorphism on $A(X)$, and makes the operation $A(-)$ a contravariant functor from the category of smooth projective varieties to the category of graded rings.

Thus, for each subvariety $X \subseteq \mathbb{P}^m$, we have that

$$\pi_1^*([X]) = [\pi_1^{-1}(X)]$$

Similarly, $\pi_2^*([Y]) = [\pi_2^{-1}(Y)]$ for each subvariety $Y \subseteq \mathbb{P}^n$.

Next, we will need to compute the Chow ring of $\mathbb{P}^m \times \mathbb{P}^n$. This is due to the following theorem from [EH16].

Theorem 3.1.3 ([EH16])**.** *Fix $m, n \geq 1$. Let $\alpha, \beta \in A^1(\mathbb{P}^m \times \mathbb{P}^n)$ denote the pullbacks, via the projection maps, of the hyperplane classes in $A(\mathbb{P}^m), A(\mathbb{P}^n)$ respectively. Then, the Chow ring of $\mathbb{P}^m \times \mathbb{P}^n$ is given by*

$$A(\mathbb{P}^m \times \mathbb{P}^n) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^{m+1}, \beta^{n+1}).$$

Moreover, if $V(f) \subset \mathbb{P}^m \times \mathbb{P}^n$ is a hypersurface with f being a bi-homogeneous polynomial of bi-degree (d, e), then $[V(f)] = d\alpha + e\beta \in A(\mathbb{P}^m \times \mathbb{P}^n)$.

As an immediate application, we will calculate the class $[X \times Y] \in A(\mathbb{P}^m \times \mathbb{P}^n)$ where $X \subseteq \mathbb{P}^m$, $Y \subseteq \mathbb{P}^n$ are subvarieties of dimensions r and s respectively.

Lemma 3.1.4. *Let $X \subseteq \mathbb{P}^m$, $Y \subseteq \mathbb{P}^n$ be irreducible subvarieties of dimensions r and s respectively, and suppose that their degrees are d and e under the respective embeddings. Then the class $[X \times Y]$ in the Chow ring of $\mathbb{P}^m \times \mathbb{P}^n$ is given by*

$$[X \times Y] = de\alpha^{m-r}\beta^{n-s},$$

where $\alpha, \beta \in \mathbb{P}^m \times \mathbb{P}^n$ are the pullbacks, via projection maps, of the hyperplane classes of $\mathbb{P}^m, \mathbb{P}^n$ respectively.

Proof. Let $\pi_1 : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^m$, $\pi_2 : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^n$ be the projection maps onto the first and second factors of $\mathbb{P}^m \times \mathbb{P}^n$. By Theorem 2.2.15, $[X] = d\zeta^{m-r}$ where $\zeta \in A(\mathbb{P}^m)$ is the hyperplane class. Thus, by Theorem 3.1.2,

$$[\pi_1^{-1}(X)] = \pi_1^*([X]) = \pi_1^*(d\zeta^{m-r}) = d\alpha^{m-r}.$$

Similarly,

$$[\pi_2^{-1}(Y)] = \pi_2^*([Y]) = \pi_2^*(e\xi^{n-s}) = e\beta^{n-s},$$

where $\xi \in A(\mathbb{P}^n)$ is the hyperplane class.

Now, $X \times Y = (X \times \mathbb{P}^n) \cap (\mathbb{P}^m \times Y) = \pi_1^{-1}(X) \cap \pi_2^{-1}(Y)$ with $\pi_1^{-1}(X)$ and $\pi_2^{-1}(Y)$ intersecting dimensionally transversely. Since $\mathbb{P}^m \times \mathbb{P}^n$ is smooth, and the intersection $X \times Y$ is reduced, we have that $\pi_1^{-1}(X)$ and $\pi_2^{-1}(Y)$ intersect generically transversely by Proposition 2.2.10. Therefore, by Theorem 2.2.8:

$$[X \times Y] = [\pi_1^{-1}(X)][\pi_2^{-1}(Y)] = de\alpha^{m-r}\beta^{n-s}.$$

□

As a Corollary, we obtain the degree of $\sigma_{m,n}(X \times Y) \subset \mathbb{P}^{(m+1)(n+1)-1}$, where $\sigma_{m,n}$ is the Segre embedding of $\mathbb{P}^m \times \mathbb{P}^n$ into $\mathbb{P}^{(m+1)(n+1)-1}$.

Corollary 3.1.5. *Let $X \subseteq \mathbb{P}^m$, $Y \subseteq \mathbb{P}^n$ be irreducible subvarieties of dimensions r and s respectively, and suppose that their degrees are d and e under the respective embeddings. Then*

$$\deg \sigma_{m,n}(X \times Y) = de\binom{r+s}{r}$$

where $\sigma_{m,n}(X \times Y)$ is considered as a subvariety of $\mathbb{P}^{(m+1)(n+1)-1}$.

Proof. Let ζ be the hyperplane class in the Chow ring of $\mathbb{P}^{(m+1)(n+1)-1}$. Since $\dim \sigma_{m,n}(X \times Y) = \dim X \times Y = r+s$, the degree of $\sigma_{m,n}(X \times Y) \subset \mathbb{P}^{(m+1)(n+1)-1}$ can be computed by intersecting $\sigma_{m,n}(X \times Y)$ with $r+s$ general hyperplanes. Let L be the intersection of $r+s$ general hyperplanes of $\mathbb{P}^{(m+1)(n+1)-1}$, so that $[L] = \zeta^{r+s}$. Since $\sigma_{m,n}$ is injective, we have that

$$|L \cap \sigma_{m,n}(X \times Y)| = |\sigma_{m,n}^{-1}(L) \cap (X \times Y)|.$$

The pullback of a general hyperplane under $\sigma_{m,n}$ is a general hypersurface of bidegree $(1,1)$, so $[\sigma_{m,n}^{-1}(L)] = (\alpha+\beta)^{r+s}$ by Theorem 3.1.3. Since $\sigma_{m,n}^{-1}(L)$ intersects with $X \times Y$ generically transversely,

$$\begin{aligned}
[\sigma_{m,n}^{-1}(L) \cap (X \times Y)] &= [\sigma_{m,n}^{-1}(L)][X \times Y] \\
&= de(\alpha+\beta)^{r+s}\alpha^{m-r}\beta^{n-s} \\
&= de\alpha^{m-r}\beta^{n-s}\sum_{i=0}^{r+s}\binom{r+s}{i}\alpha^i\beta^{r+s-i} \\
&= de\binom{r+s}{r}\alpha^m\beta^n
\end{aligned}$$

by Theorem 2.2.8 and Lemma 3.1.4. Hence,

$$\deg \sigma_{m,n}(X \times Y) := \deg(L \cap \sigma_{m,n}(X \times Y)) = de\binom{r+s}{r}.$$

□

3.2 Bounding the degree of intersections of hypersurfaces in $\mathbb{P}^m \times \mathbb{P}^n$

For this subsection, we will assume that all varieties are defined over an algebraically closed field k of characteristic zero.

Lemma 3.2.1 (cf. IV.§2, [Sh74])**.** *Let $F_1, F_2, \ldots, F_s$ be bihomogeneous polynomials in the coordinate ring of $\mathbb{P}^m \times \mathbb{P}^n$, and suppose that F_i is of bidegree (d_i, e_i) for each $i = 1, \ldots, s$. Then,*

$$\deg \sigma_{m,n}(V(F_1, F_2, \ldots, F_s)) \leq \binom{m+n}{n}^s \prod_{i=1}^{s} D_i$$

where $\sigma_{m,n} : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{(m+1)(n+1)-1}$ is the Segre embedding.

Proof. Firstly, we will calculate $\deg \sigma_{m,n}(V(F_i))$ for each $i = 1, \ldots, s$. Write the Chow ring of $\mathbb{P}^m \times \mathbb{P}^n$ as

$$A(\mathbb{P}^m \times \mathbb{P}^n) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^{m+1}, \beta^{n+1})$$

as in Theorem 3.1.3. We have that

$$[V(F_i)] = d_i\alpha + e_i\beta$$

for each i. Since $\dim V(F_i) = m+n-1$, the degree of $\sigma_{m,n}(V(F_i))$ is found by intersecting it with $m+n-1$ general hyperplanes in $\mathbb{P}^{(m+1)(n+1)-1}$. Let L be the intersection of $m+n-1$ general hyperplanes of $\mathbb{P}^{(m+1)(n+1)-1}$. Then,

$$\#|L \cap \sigma_{m,n}(V(F_i))| = \#|\sigma_{m,n}^{-1}(L) \cap V(F_i)|$$

since $\sigma_{m,n}$ is injective. Because the intersection is generically transverse, we have that

$$[\sigma_{m,n}^{-1}(L) \cap V(F_i)] = [\sigma_{m,n}^{-1}(L)][V(F_i)] = (\alpha+\beta)^{m+n-1}(d_i\alpha + e_i\beta)$$

by Theorem 2.2.8. Thus,

$$\begin{aligned} \deg \sigma_{m,n}(V(F_i)) &= \deg(\alpha+\beta)^{m+n-1}(d_i\alpha + e_i\beta) \\ &= \deg\left(\binom{m+n-1}{m-1} d_i\alpha^m\beta^n + \binom{m+n-1}{m} e_i\alpha^m\beta^n\right) \\ &= \binom{m+n-1}{m-1} d_i + \binom{m+n-1}{m} e_i. \end{aligned}$$

If we let $D_i = \max\{d_i, e_i\}$, then

$$\deg \sigma_{m,n}(V(F_i)) \leq \binom{m+n}{n} D_i.$$

Hence, by the generalized Bézout's theorem:

$$\deg \sigma_{m,n}(V(F_1, F_2, \ldots, F_s)) \leq \prod_{i=1}^{s} \deg \sigma_{m,n}(V(F_i)) \leq \binom{m+n}{n}^s \prod_{i=1}^{s} D_i.$$

□

3.3 The degree of the diagonal subvariety of $\mathbb{P}^n \times \mathbb{P}^n$

For this subsection, we will again assume that all varieties are defined over an algebraically closed field k of characteristic zero.

Finally, we will calculate the degree of the image of the diagonal $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$ under the Segre embedding $\sigma_{n,n}$. This will be helpful in later proofs. The following expression for the class of $[\Delta] \in A(\mathbb{P}^n \times \mathbb{P}^n)$ is due to [EH16].

Lemma 3.3.1 ([EH16]). *Let $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$ be the diagonal subvariety. Then*

$$[\Delta] = \sum_{i=0}^{n} \alpha^i \beta^{n-i} \in A(\mathbb{P}^n \times \mathbb{P}^n),$$

where α, β are the pullbacks, via the projection maps, of the hyperplane classes on the first and second factors of $\mathbb{P}^n \times \mathbb{P}^n$ respectively.

Corollary 3.3.2. *Let $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$ be the diagonal subvariety, and let $\sigma_{n,n} : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^{(n+1)^2-1}$ be the Segre embedding. Then,*

$$\deg \sigma_{n,n}(\Delta) = 2^n.$$

Proof. Since $\dim \sigma_{n,n}(\Delta) = \dim \Delta = n$, the degree of $\sigma_{n,n}(\Delta)$ can be computed by intersecting it with n general hyperplanes in $\mathbb{P}^{(n+1)^2-1}$. Let $L \subseteq \mathbb{P}^{(n+1)^2-1}$ be the intersection of n general hyperplanes. As in previous proofs,

$$|L \cap \sigma_{n,n}(\Delta)| = |\sigma_{n,n}^{-1}(L) \cap \Delta|$$

since the Segre embedding is injective. This also means that the pre-image of L intersects generically transversely with Δ, so

$$[\sigma_{n,n}^{-1}(L) \cap \Delta] = [\sigma_{n,n}^{-1}(L)][\Delta] = (\alpha + \beta)^n \sum_{i=0}^{n} \alpha^i \beta^{n-i}$$

by Lemma 3.3.1. Thus, taking degrees of both sides:

$$\begin{aligned}
\deg \sigma_{n,n}(\Delta) &= \deg(\alpha + \beta)^n \sum_{i=0}^{n} \alpha^i \beta^{n-i} \\
&= \deg \sum_{j=0}^{n} \binom{n}{j} \alpha^{n-j} \beta^j \sum_{i=0}^{n} \alpha^i \beta^{n-i} \\
&= \deg \sum_{j=0}^{n} \binom{n}{j} \alpha^n \beta^n \\
&= \sum_{j=0}^{n} \binom{n}{j} \\
&= 2^n,
\end{aligned}$$

as required. □

Section 4

Results

Throughout this section, we will assume that all varieties are defined over an algebraically closed field k of characteristic zero unless otherwise stated.

Recall from Idea 3.0.3 that for the descending chain of closed algebraic sets

$$Y_1 \supseteq Y_1 \cap Y_2 \supseteq Y_1 \cap Y_2 \cap Y_3 \supseteq \dots, \text{ where } Y_i := F_i^{-1}(\Delta) := (f \circ \phi^i, g \circ \psi^i)^{-1}(\Delta)$$

in Problem 1.0.2, our first goal is to find an $N_1 \geq 1$ such that

$$Y_1 \cap Y_2 \cap \dots \cap Y_{N_1} = (Y_1 \cap Y_2 \cap \dots \cap Y_{N_1})^0 \cup C_1 \cup C_2 \cup \dots \cup C_s,$$

where $(Y_1 \cap Y_2 \cap \dots \cap Y_{N_1})^0$ are the zero-dimensional components and $C_1, \dots, C_s$ are the positive dimensional components such that $C_j \subseteq Y_i$ for each $i \geq 1$, $j = 1, \dots, s$.

Our approach will be based on the following. For each irreducible component $C \subseteq Y_1$, notice that $\dim C \leq \dim Y_1$. Suppose that $\dim C = \dim Y_1$, and denote $r = \dim Y_1$. Also, denote

$$C_i := \overline{F^i(C)}$$

for each $i \geq 1$.

We will show that for some positive integer $\gamma_1 \geq 1$, if

$$C \subseteq Y_1 \cap Y_2 \cap \dots \cap Y_{\gamma_1},$$

then either $C \subseteq Y_i$ for all $i \geq 1$, or

$$\dim C_{\gamma_1+1} \cap Y_1 < \dim C = r.$$

Hence, we may write

$$Y_1 \cap Y_2 \cap \dots \cap Y_{\gamma_1} = C_{1,1} \cup C_{1,2} \cup \dots \cup C_{1,s_1} \cup Y'_{\gamma_1},$$

where $C_{1,j}$, $j = 1, \dots, s_1$, are the irreducible components of dimension r such that $C_{1,j} \subseteq Y_i$ for all $i \geq 1$, and Y'_{γ_1} is the union of all the other components.

Then, we may take further intersections of $Y_1 \cap Y_2 \cap \dots \cap Y_{\gamma_1}$ with Y_i's, $i > \gamma_1$, and repeat the above process for the irreducible components of $Y_1 \cap Y_2 \cap \dots \cap Y_{\gamma_1}$ of dimension

$\geq r-1$. Proceeding inductively, we will show that there exists a sequence of positive integers $1 \leq \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_r$, where γ_j is such that for each irreducible component

$$C \subseteq Y_1 \cap Y_2 \cap \cdots \cap Y_{\gamma_j}$$

with $\dim C \geq r-j$, either $C \subseteq Y_i$ for each $i \geq 1$ or

$$\dim C_{\gamma_j+1} \cap Y_1 < r-j.$$

Thus, the required $N \geq 1$ for Problem 1.0.2 is bounded above by

$$N \leq \gamma_1 + \gamma_2 + \cdots + \gamma_r.$$

Hence, we will first calculate $\dim Y_1$.

Lemma 4.0.1. *Let X and Y be irreducible quasi-projective varieties of positive dimensions, and let $\phi : X \to \mathbb{P}^1$, $\psi : Y \to \mathbb{P}^1$ be dominant rational maps. Set $f := (\phi, \psi) : X \times Y \to \mathbb{P}^1 \times \mathbb{P}^1$, and denote the diagonal subvariety of $\mathbb{P}^1 \times \mathbb{P}^1$ by Δ. Then,*

$$\dim f^{-1}(\Delta) \leq \dim(X \times Y) - 1.$$

Proof. Suppose, for contradiction, that $\dim f^{-1}(\Delta) = \dim X \times Y$. Since both X and Y are irreducible varieties, $X \times Y$ is also irreducible, so $f^{-1}(\Delta)$ must be a dense subset of $X \times Y$. However, $f(f^{-1}(\Delta)) \subseteq \Delta$ with $\dim \Delta < \dim(\mathbb{P}^1 \times \mathbb{P}^1)$, contradicting the fact that f is a dominant map. Thus, we must have that

$$\dim f^{-1}(\Delta) \leq \dim(X \times Y) - 1.$$

□

If the dominant rational maps ϕ, ψ, f and g in Problem 1.0.2 are morphisms, then we may calculate the dimension of each irreducible component of Y_1.

Lemma 4.0.2. *Let X, Y and Z be irreducible quasi-projective varieties, and let $\phi : X \to Z$, $\psi : Y \to Z$ be surjective morphisms. Set $f := (\phi, \psi) : X \times Y \to Z \times Z$, and denote the diagonal subvariety of $Z \times Z$ by Δ_Z. Then, each component of $f^{-1}(\Delta_Z) \subseteq X \times Y$ has codimension equaling* $\dim Z$*.*

Proof. By writing X, Y and Z as finite unions of open affine subsets, we may reduce this problem to the case where X, Y and Z are all affine. Let $\varphi : k[Z \times Z] \to k[X \times Y]$ be the map of coordinate rings induced by f, so that $\varphi(s) = s \circ f$ for each $s \in k[Z \times Z]$. Since f is surjective, φ must be an injective ring homomorphism. Now, $\Delta_Z \subset Z \times Z$ is an irreducible subvariety, so $\Delta_Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \subset k[Z \times Z]$.

Notice that

$$f^{-1}(\Delta_Z) = V(\varphi(\mathfrak{p})) = V(\mathfrak{p}^e).$$

where $\mathfrak{p}^e$ is the extension of $\mathfrak{p}$ under φ. Write $\mathfrak{p}^e$ as the intersection of its minimal prime ideals:

$$\mathfrak{p}^e = \bigcap_{i=1}^{s} \mathfrak{q}_i.$$

Thus, each irreducible component of $f^{-1}(\Delta_Z)$ is the vanishing locus of $\mathfrak{q}_i$ for some $i = 1, \ldots, s$. Denote $Z_i = V(\mathfrak{q}_i)$. We claim that the contraction of each $\mathfrak{q}_i$ via φ is $\mathfrak{p}$, so that $f(Z_i)$ is dense in Δ_Z for each $i = 1, \ldots, s$.

To prove this claim, first notice that $V(\mathfrak{p}) = V(\mathfrak{p}^{ec})$, where $\mathfrak{p}^{ec}$ is the contraction of $\mathfrak{p}^e$ under φ. The inclusion $\mathfrak{p} \subseteq \mathfrak{p}^{ec}$ follows from the definition of extension and contraction of ideals, so $V(\mathfrak{p}) \supseteq V(\mathfrak{p}^{ec})$. For the other inclusion, let $(x, x) \in V(\mathfrak{p}) = \Delta_Z$, and let $r \in \mathfrak{p}^{ec}$. Then, $\varphi(r) := r \circ f \in \mathfrak{p}^e$ and $f^{-1}(x, x) \subseteq V(\mathfrak{p}^e)$. Hence, $0 = \varphi(r)(f^{-1}(x, x)) = r \circ f(f^{-1}(x, x)) = r(x, x)$, implying that $(x, x) \in V(\mathfrak{p}^{ec})$.

Hence, $\mathfrak{p} = rad(\mathfrak{p}) = rad(\mathfrak{p}^{ec}) \supseteq \mathfrak{p}^{ec}$, so $\mathfrak{p} = \mathfrak{p}^{ec}$. By Proposition 3.16 of [AM69], $\mathfrak{p}$ is the contraction of a prime ideal. Since

$$\mathfrak{p} = \bigcap_{i=1}^{s} \mathfrak{q_i}^c,$$

we must have that $\mathfrak{p} = \mathfrak{q_i}^c$ for each $\mathfrak{q}_i$, $i = 1, \ldots, s$. Hence, $f(Z_i) = f(V(\mathfrak{q}_i))$ is dense in $\Delta_Z = V(\mathfrak{p})$ for each $i = 1, \ldots, s$.

Finally, by Theorem 3 of [Mu99, I.§8], there exists a non-empty open subset $U \subseteq Z \times Z$ such that for each irreducible closed subset $W \subseteq Z \times Z$ with $W \cap U \neq \varnothing$ and for all irreducible components $C \subseteq f^{-1}(W)$ such that $C \cap f^{-1}(U) \neq \varnothing$, $\operatorname{codim}_{X\times Y} C = \operatorname{codim}_{Z\times Z} W$. Now, $\Delta_Z \cap U \neq \varnothing$, since $U = U_1 \times U_2$ for $U_1, U_2 \subset X$ open subsets and $U_1 \cap U_2 \neq \varnothing$. Also, each irreducible component $Z_i \subseteq f^{-1}(\Delta_Z)$ has non-empty intersection with $f^{-1}(U)$, since $f(Z_i)$ is dense in Δ_X. Therefore,

$$\operatorname{codim}_{X\times Y} Z_i = \operatorname{codim}_{Z\times Z} \Delta_Z = \dim Z,$$

as required. □

4.1 Problem 1.0.2 with $X = Y$ and ϕ, ψ, f, g surjective morphisms

When $X = Y$ in Problem 1.0.2, we may further simplify the problem of stabilising the descending chain of closed subsets as described at the start of this section. In this case, we will omit the maps f and g from our consideration, since we may determine if the points $\phi^i(x), \psi^i(y) \in X$ are the same given $x, y \in X$. For the following, we will denote

$$X_i := \{(x, y) \in X \times X \mid \phi^i(x) = \psi^i(y)\}.$$

As before, solving Problem 1.0.2 in this case is equivalent to finding the smallest integer $N \geq 1$ such that

$$X_1 \cap X_2 \cap \cdots \cap X_N = X_1 \cap X_2 \cap \cdots \cap X_{N+1} \tag{4.1}$$

given that Lemma 3.0.1 also applies to this case.

To simplify the problem further, we will assume that both ϕ and ψ are morphisms, so that ϕ and ψ become surjective morphisms given that they are dominant maps. Then, we may follow the general approach as in Idea 3.0.3 to give an upper bound on N. In fact, we will apply the general approach to the chain of descending subsets

$$\Delta_X \supseteq \Delta_X \cap X_1 \supseteq \Delta_X \cap X_1 \cap X_2 \supseteq \ldots \tag{4.2}$$

where $\Delta_X \subset X \times X$ is the diagonal subvariety. Given that ϕ and ψ are surjective morphisms, $(\phi, \psi)(X_1) = \Delta_X$ and $(\phi, \psi)(X_i) = X_{i-1}$ for each $i > 1$. Thus, equality 4.1 is true if and only if the chain 4.2 is such that

$$\Delta_X \cap X_1 \cap X_2 \cap \cdots \cap X_{N-1} = \Delta_X \cap X_1 \cap X_2 \cap \cdots \cap X_N.$$

When X is a curve, the upper bound on N has a particularly simple expression. This is because there are no positive dimensional components of $\Delta_X \cap X_1$ unless $\Delta_X \subseteq X_1$.

Lemma 4.1.1. *Let X be a one-dimensional projective variety, and let $\phi : X \to X, \psi : X \to X$ be surjective morphisms. Let $\Delta_X \subset X \times X$ be the diagonal subvariety, and set*

$$X_i := (\phi^{-i}, \psi^{-i})(\Delta_X) := \{(x, y) \in X \times X \mid \phi^i(x) = \psi^i(y)\} \subseteq X \times X.$$

If there exists a component $C \subseteq \Delta_X \cap X_1$ of positive dimension, then $\Delta_X \subseteq X_1$.

Proof. Denote $F := (\phi, \psi)$. Suppose that $C \subseteq \Delta_X \cap X_1$ is a positive dimensional irreducible component. Then, $\dim C = 1 = \dim \Delta_X$. Since $\Delta_X \subset X \times X$ is irreducible and $C \subseteq \Delta_X \cap X_1 \subseteq \Delta_X$, we must have that $C = \Delta_X$. Hence, $\Delta_X \subseteq X_1$. □

Proposition 4.1.2. *Let X, ϕ, ψ and X_i be as in Lemma 4.1.1. Fix an embedding $X \subseteq \mathbb{P}^n$ for some $n \geq 1$, and suppose that $\deg X = d$, $\deg \phi = e_1$ and $\deg \psi = e_2$ under this embedding. Denote $e = \max\{e_1, e_2\}$. Then,*

$$\Delta_X \cap X_1 \cap X_2 \cap \cdots \cap X_{\gamma_{1,1}} = \Delta_X \cap X_1 \cap X_2 \cap \cdots \cap X_{\gamma_{1,1}+1}$$

where

$$\gamma_{1,1} = 2^{n+1} \binom{2n}{n}^{n+1} d^2 e^{n+1}.$$

Proof. Denote $F := (\phi, \psi)$. Firstly, if $\Delta_X \cap X_1$ contains a positive dimensional component, then $F^{-1}(\Delta_X) = X_1 = \Delta_X$ by Lemma 4.1.1. By simple induction, $X_i = F^{-i}(\Delta_X) = \Delta_X$ for all $i \geq 1$, so

$$\Delta_X = \Delta_X \cap X_1 = \Delta_X \cap X_1 \cap X_2 = \ldots$$

Since $\gamma_{1,1} > 1$ for any $n \geq 1$, WLOG we may assume that $\Delta_X \neq X_1$. Thus, $\dim(\Delta_X \cap X_1) = 0$. By Bézout's theorem,

$$|\Delta_X \cap X_1| \leq \deg \Delta_X \deg X_1.$$

By Corollary 3.3.2, $\deg \sigma_{n,n}(\Delta) = 2^n$, where Δ is the diagonal subvariety of $\mathbb{P}^n \times \mathbb{P}^n$. Also, using the same argument as in Lemma 3.0.4,

$$X_1 = (X \times X) \cap V(\phi_i(x) = \psi_i(y) \mid i = 0, 1, \ldots, n)$$

as a subset of $\mathbb{P}^n \times \mathbb{P}^n$, where ϕ_i, ψ_i are coordinate expressions of ϕ and ψ respectively as homogeneous polynomials in the coordinate ring of $\mathbb{P}^n$. Thus, by Corollary 3.1.5,

$$\deg \sigma_{n,n}(X \times X) = \binom{2}{1} d^2 = 2d^2.$$

Also,

$$\deg \sigma_{n,n}(V(\phi_i(x) = \psi_i(y) \mid i = 0, 1, \dots, n)) \leq \binom{2n}{n}^{n+1} e^{n+1},$$

where the bound is calculated by applying Lemma 3.2.1 to homogenizations of the polynomials $\phi_i(x) = \psi_i(y)$ for each i (notice that the bidegree of the homogenization would still be (e_1, e_2), so the bound from Lemma 3.2.1 still applies).

Hence, using Theorem 2.2.18 and the fact that $\sigma_{n,n}$ is injective:

$$\deg \sigma_{n,n}(X_1) \leq 2\binom{2n}{n}^{n+1} d^2 e^{n+1}.$$

Since $\sigma_{n,n}(\Delta_X \cap X_1) = \sigma_{n,n}(\Delta) \cap \sigma_{n,n}(X_1)$, we have that

$$\begin{aligned} |\Delta_X \cap X_1| &= |\sigma_{n,n}(\Delta_X \cap X_1)| \\ &\leq \deg \sigma_{n,n}(\Delta) \deg \sigma_{n,n}(X_1) \\ &\leq 2^{n+1}\binom{2n}{n}^{n+1} d^2 e^{n+1} \\ &= \gamma_{1,1}. \end{aligned}$$

Therefore, the result follows from Lemma 3.0.1. □

When X is of arbitrary dimension, we will characterize when irreducible components $C \subseteq \Delta_X \cap X_1$ stabilise in the chain 4.2 by considering the dimension of $\overline{(\phi, \psi)^i(C)}$.

Lemma 4.1.3. *Let X be an r-dimensional irreducible projective variety, and let $\phi, \psi : X \to X$ be surjective morphisms. Fix an embedding $\iota_X : X \hookrightarrow \mathbb{P}^n$ under which $\deg X = d$, $\deg \phi = e_1$ and $\deg \psi = e_2$, and denote $e = \max\{e_1, e_2\} \geq 1$. Let $\Delta_X \subset X \times X$ be the diagonal subvariety, and set $X_i := (\phi^{-i}, \psi^{-i})(\Delta_X) \subseteq X \times X$. If there exists an irreducible subvariety $C \subseteq \Delta_X$ of dimension $\geq r - 1$ such that $C \subseteq \Delta_X \cap X_1 \cap X_2 \cap \cdots \cap X_{\gamma_{r,1}}$ where*

$$\gamma_{r,1} = 2^n \binom{2r}{r}\binom{2n}{n}^{n+1} d^2 e^{n+1},$$

then either

1. *$C \subseteq X_i$ for all $i \geq 1$, or*
2. *$\dim \overline{(\phi, \psi)^{\gamma_{r,1}+1}(C)} \cap \Delta_X < r - 1$.*

Proof. Firstly, if $\dim C = r = \dim \Delta_X$, then $X_1 \supseteq C = \Delta_X$ since both C and Δ_X are irreducible. In this case, $X_1 = X_1 \cap X_2 \cap \cdots \cap X_i$ for all i's.

Thus, WLOG we may assume that $X_1 \not\supseteq \Delta_X$. Suppose that $C \subseteq \Delta_X$ is an irreducible subvariety of dimension $\geq r-1$ such that $C \subseteq \Delta_X \cap X_1 \cap X_2 \cap \cdots \cap X_{\gamma_{r,1}}$. By $C \subseteq \Delta_X \cap X_1$, $\dim C \leq \dim \Delta_X \cap X_1 \leq r - 1$, so $\dim C = r - 1$.

Denote $F := (\phi, \psi)$, $C_i = \overline{F^i(C)}$ and $X_0 := \Delta_X$. Since F is surjective, $F(X_i) = X_{i-1}$ for each $i \geq 1$. Then,

$$F^i(C) \subseteq C_i \subseteq X_0 \cap X_1 \cap X_2 \cdots \cap X_{\gamma_{r,1}-i}.$$

By Theorem 2 of [Mu99, I.§8],

$$\dim \overline{F^i(C)} \leq \dim C = r - 1$$

since the image of C under F^i is dense in $\overline{F^i(C)}$ for each i. Thus, if $\dim C_i < \dim C$ for any $i = 1, \ldots, \gamma_{r,1}$, then $\dim C_{\gamma_{r,1}+1} \cap X_0 < \dim C = r - 1$.

Now, suppose that $\dim C_i = \dim C = r - 1$ for each $i = 0, 1, \ldots, \gamma_{r,1}$, so each C_i is an irreducible $(r-1)$-dimensional component of $X_0 \cap X_1$ given that $\dim X_0 \cap X_1 \leq r - 1$. We may calculate the maximum number of unique irreducible $(r-1)$-dimensional components of $X_0 \cap X_1$ as follows. By Corollary 3.3.2, $\deg \sigma_{n,n}(\Delta) = 2^n$. Also, by Corollary 3.1.5, Lemma 3.2.1 and Theorem 2.2.18:

$$\deg \sigma_{n,n}(X_1) \leq \binom{2r}{r}\binom{2n}{n}^{n+1} d^2 e^{n+1}.$$

Thus, by applying Theorem 2.2.18 on $\sigma_{n,n}(X_0 \cap X_1) = \sigma_{n,n}(\Delta \cap X_1)$ in $\mathbb{P}^{(n+1)^2-1}$:

$$\deg \sigma_{n,n}(\Delta \cap X_1) \leq \deg \sigma_{n,n}(\Delta) \deg \sigma_{n,n}(X_1) \leq 2^n \binom{2r}{r}\binom{2n}{n}^{n+1} d^2 e^{n+1} = \gamma_{r,1}.$$

Hence, $X_0 \cap X_1$ can contain at most $\gamma_{r,1}$ unique irreducible $(r-1)$-dimensional components. In this way, we have the following cases:

1. If $C_i = C_j$ for some $i \neq j$, $i, j = 0, 1, \ldots, \gamma_{r,1}$, then $C_i \subseteq X_0$ for all $i \geq 1$.
2. Otherwise, $C_i \neq C_j$ for any $i \neq j$, $i, j = 0, 1, \ldots, \gamma_{r,1}$. By assumption, $C_i \subseteq X_0 \cap X_1$ for each $i = 0, 1, \ldots, \gamma_{r,1} - 1$. We know that $X_0 \cap X_1$ contains $\gamma_{r,1}$ unique irreducible $(r-1)$-dimensional components, so $C_{\gamma_{r,1}} \nsubseteq X_0 \cap X_1$. Since $C_{\gamma_{r,1}} \subseteq X_0$ by assumption, we must have $C_{\gamma_{r,1}} \nsubseteq X_1$. Thus, $C_{\gamma_{r,1+1}} \nsubseteq X_0$ as $X_1 = F^{-1}(X_0)$ and F is surjective on $X \times X$. This implies that

$$\dim C_{\gamma_{r,1}+1} \cap X_0 < \dim C = r - 1,$$

as required.

□

Notation 4.1.4. Denote

$$\gamma_{r,1} := 2^n \binom{2r}{r}\binom{2n}{n}^{n+1} d^2 e^{n+1},$$

as in Lemma 4.1.3. For each $j = 2, \ldots, r$, define $\gamma_{r,j}$ inductively as follows:

$$\gamma_{r,j} := 2^n \prod_{i=1}^{\lambda_{r,j-1}} \binom{2r}{r}\binom{2n}{n}^{n+1} d^2 e^{i(n+1)}.$$

where

$$\lambda_{r,j} := \gamma_{r,j} + \gamma_{r,j-1} + \cdots + \gamma_{r,1} + 1.$$

Further, we will adopt the following conventions:

$$\gamma_{r,0} := 0, \lambda_{r,0} := \gamma_{r,0} + 1 = 1.$$

Proposition 4.1.5. *Let X, ϕ, ψ and X_i be as defined in Lemma 4.1.3. Suppose that $\deg X = d$, $\deg \phi = e_1$ and $\deg \psi = e_2$ under a fixed embedding $\iota_X : X \hookrightarrow \mathbb{P}^n$, and set $e = \max\{e_1, e_2\}$. Let $C \subseteq \Delta_X$ be an irreducible subvariety of dimension $\geq r - k$ with $k = 0, 1, \ldots, r$. Suppose that*

$$C \subseteq \Delta_X \cap X_1 \cap X_2 \cap \cdots \cap X_{\lambda_{r,k}-1}.$$

Denote $F := (\phi, \psi)$, $C_i := \overline{F^i(C)}$ for each $i \geq 0$, and $X_0 := \Delta_X$. Then, either

1. *$C \subseteq X_i$ for each $i \geq 0$, or*
2. *$\dim C_{\lambda_{r,k}} \cap X_0 < r - k$.*

Proof. The proof follows by induction on k. Denote $F := (\phi, \psi)$. The case $k = 1$ is covered by Lemma 4.1.3. For the inductive case, suppose that the statement is true for any irreducible subvariety of $X_0 := \Delta_X$ of dimension $\geq r - (k-1)$ for $k > 1$. Let $C \subseteq X_0$ be an irreducible subvariety of dimension $\geq r - k$, and suppose that

$$C \subseteq X_0 \cap X_1 \cap \cdots \cap X_{\lambda_{r,k}-1}.$$

Claim: Either $C \subseteq X_i$ for each $i \geq 0$ or

$$\dim C_{\lambda_{r,k-1}+i} \leq r - k$$

for each $i \geq 0$.

This claim follows from the inductive hypothesis. When $\dim C \geq r - (k-1)$, we know by the inductive hypothesis that either $C \subseteq X_i$ for each $i \geq 0$ or

$$\dim C_{\lambda_{r,k-1}} \cap X_0 \leq r - k,$$

Since $C_{\lambda_{r,k-1}} \subseteq X_0$ by assumption, this implies that

$$\dim C_{\lambda_{r,k-1}+i} \leq \dim C_{\lambda_{r,k-1}} \leq r - k$$

for each $i \geq 0$.

When $\dim C = r - k$, $\dim C_i \leq r - k$ for every $i \geq 0$. Thus, in this case,

$$\dim C_{\lambda_{r,k-1}+i} \leq r - k$$

for each $i \geq 0$ as well.

For the rest of the proof, we will suppose that $\dim C_{\lambda_{r,k-1}+i} \leq r - k$ for each $i \geq 0$, since $C \subseteq X_i$ for each $i \geq 0$ otherwise. Let $S \subseteq X_0 \cap X_1 \cap \cdots \cap X_{\lambda_{r,k-1}}$ be the union of all irreducible components $Z \subseteq X_0 \cap X_1 \cap \cdots \cap X_{\lambda_{r,k-1}}$ of dimension $\geq (r-k)+1$ such that

$$\dim Z_{\lambda_{r,k-1}} \cap X_0 \geq (r-k)+1.$$

By the inductive hypothesis, $Z \subseteq X_i$ for each $i \geq 0$, implying that $S \subseteq X_i$ for each $i \geq 0$.

Let E be the union of components of $X_0 \cap X_1 \cap \cdots \cap X_{\lambda_{r,k-1}}$ of dimension $\geq (r-k)+1$ which are not in S, and let L be the union of the lower dimensional components, i.e., $\dim L \leq r-k$. Hence, we may write

$$X_0 \cap X_1 \cap X_2 \cap \cdots \cap X_{\lambda_{r,k-1}} = S \bigsqcup E \bigsqcup L$$

as a disjoint union. Thus, whenever an irreducible subvariety Z of the above set is such that $Z \nsubseteq S$, we must have that $Z \cap S = \varnothing$ and similarly for $Z \cap E$ as well as $Z \cap L$.

Notice that

$$C_i \subseteq X_0 \cap X_1 \cap X_2 \cap \cdots \cap X_{\lambda_{r,k-1}}$$

for every $i = 0, 1, \ldots, \gamma_{r,k} - 1$. In the following, we will divide the cases according to $C_i \subseteq S$, E or L for $i = 0, 1, \ldots, \gamma_{r,k} - 1$.

1. If $C_{i_0} \subseteq S$ for any $i_0 = 0, 1, \ldots, \lambda_{r,k}$, then $C_i \subseteq X_0 = \Delta_X$ for every $i \geq i_0$ since

$$C_i = \overline{F^{i-i_0}(C_{i_0})} \subseteq \overline{F^{i-i_0}(S)} \subseteq \Delta_X.$$

2. Thus, suppose that $C_i \nsubseteq S$ for any $i = 0, 1, \ldots, \lambda_{r,k}$, so that $C_i \cap S = \varnothing$ for $i = 0, 1, \ldots, \lambda_{r,k}$. Then, we can divide up the cases as follows.

   (a) Suppose that there does not exist any irreducible component $E_l \subseteq E$ such that $C_j, C_{j'} \subseteq E_l$ for $j \neq j'$ and $j, j' = 0, 1, \ldots, \gamma_{r,k}$. WLOG, suppose that $C_j \neq C_{j'}$ for any $j \neq j'$, $j, j' = 0, 1, \ldots, \gamma_{r,k}$, as we would have $C_i \subseteq X_0 = \Delta_X$ for each $i \geq 1$ otherwise.

   Since $C_i \cap S = \varnothing$ for any $i \leq \gamma_{r,k} - 1 < \lambda_{r,k}$, $C_i \subseteq E \sqcup L$ for any $i \leq \gamma_{r,k} - 1$. The number of irreducible components of $E \sqcup L$ is bounded by its degree, which is turn bounded by the degree of $X_0 \cap X_1 \cap \cdots \cap X_{\lambda_{r,k-1}}$:

$$\begin{aligned}\deg X_0 \cap X_1 \cap \cdots \cap X_{\lambda_{r,k-1}} &\leq \deg \Delta \prod_{i=1}^{\lambda_{r,k-1}} \deg X_i \\ &\leq 2^n \prod_{i=1}^{\lambda_{r,k-1}} \binom{2r}{r} \binom{2n}{n}^{n+1} d^2 e^{i(n+1)}, \\ &= \gamma_{r,k}.\end{aligned}$$

   where $\deg \Delta = 2^n$ by Corollary 3.3.2, and the upper bound on the product of the degrees of the X_i's follows from Corollary 3.1.5, Lemma 3.2.1 and Theorem 2.2.18.

   Now, recall that

$$\dim C_{\lambda_{r,k-1}+i} \leq r-k$$

   for each $i \geq 0$ and that $\dim L \leq r-k$. We have the following two sub-cases.

   - $\dim C_i \geq r-k$ for each $i = 0, 1, \ldots, \gamma_{r,k}$. By assumption, no irreducible component of E can contain more than one of such C_i. It follows that each

C_i, $i = 0, 1, \ldots, \gamma_{r,k} - 1$, must be on a unique component of $E \sqcup L$, so $C_{\gamma_{r,k}}$ cannot be contained in any component of E or L. Thus,

$$C_{\gamma_{r,k}} \nsubseteq S \sqcup E \sqcup L = X_0 \cap X_1 \cap X_2 \cap \cdots \cap X_{\lambda_{r,k-1}}.$$

By assumption,

$$C_{\gamma_{r,k}} \subseteq X_0 \cap X_1 \cap X_2 \cap \cdots \cap X_{\lambda_{r,k-1}-1},$$

so we must have that $C_{\gamma_{r,k}} \nsubseteq X_{\lambda_{r,k-1}}$. Hence, $C_{\lambda_{r,k}} = C_{\gamma_{r,k}+\lambda_{r,k-1}} \nsubseteq X_0$, so

$$\dim C_{\lambda_{r,k}} \cap X_0 < \dim C_{\lambda_{r,k}} \leq r - k$$

as required.

- $\dim C_{i_0} < r - k$ for some $i_0 = 0, 1, \ldots, \gamma_{r,k}$. Then,

$$\dim C_{\lambda_{r,k}} \cap X_0 \leq \dim C_{\lambda_{r,k}} \leq \dim C_{i_0} < r - k,$$

as required.

(b) Otherwise, there exists an irreducible component $E_l \subseteq E$ such that $C_j, C_{j'} \subseteq E_l$ for $j \neq j'$ and $j, j' = 0, 1, \ldots, \gamma_{r,k}$. WLOG, let $j < j'$. Then,

$$C_{\lambda_{r,k-1}+j} = \overline{F^{\lambda_{r,k-1}}(C_j)}, C_{\lambda_{r,k-1}+j'} = \overline{F^{\lambda_{r,k-1}}(C_j')} \subseteq \overline{F^{\lambda_{r,k-1}}(E_l)}.$$

By the definition of E, $E_l \cap S = \varnothing$, so

$$\dim \overline{F^{\lambda_{r,k-1}}(E_l)} \cap X_0 < r - (k - 1).$$

Since $\overline{F^{\lambda_{r,k-1}}(E_l)} \subseteq X_0$ by

$$E_l \subseteq E \subseteq X_0 \cap X_1 \cap \cdots \cap X_{\lambda_{r,k-1}}$$

it follows that $\dim \overline{F^{\lambda_{r,k-1}}(E_l)} < r - (k - 1)$ by the inductive hypothesis. Hence, we have the following two sub-cases:

- $\dim C_{\lambda_{r,k-1}+j} = \dim C_{\lambda_{r,k-1}+j'} = \dim \overline{F^{\lambda_{r,k-1}}(E_l)}$, implying

$$C_{\lambda_{r,k-1}+j} = C_{\lambda_{r,k-1}+j'} = \overline{F^{\lambda_{r,k-1}}(E_l)}$$

because all these sets are irreducible.
Since $\overline{F^{\lambda_{r,k-1}}(E_l)} \subseteq X_0$, we must have that

$$C_{\lambda_{r,k-1}+j} = C_{\lambda_{r,k-1}+j'} \subseteq X_0 = \Delta_X.$$

Thus, $C_i \subseteq \Delta_X$ for each $i \geq 1$. It follows that $C \subseteq X_i$ for each $i \geq 1$ in this case.

- $\dim C_{\lambda_{r,k-1}+j'} < \dim \overline{F^{\lambda_{r,k-1}}(E_l)} < r-(k-1)$. (Notice that the case $\dim C_{\lambda_{r,k-1}+j} < \dim \overline{F^{\lambda_{r,k-1}}(E_l)}$ implies the above case as $j' > j$, so it suffices to consider the above case.) Hence,

$$\dim C_i \leq \dim C_{\lambda_{r,k-1}+j'} < r-k$$

for all $i \geq \lambda_{r,k-1}+j'$, which implies that

$$\dim C_{\lambda_{r,k}+i} \cap X_0 < r-k$$

for all $i \geq 0$ given that $\lambda_{r,k-1}+j' \leq \lambda_{r,k}$.

□

If a subvariety $C \subseteq X_0 = \Delta_X$ is such that $C \subseteq X_i$ for all $i \geq 0$, then clearly

$$\dim \overline{(\phi^i, \psi^i)}(C) \cap X_0 \geq 0$$

for each $i \geq 0$. Hence, Proposition 4.1.5 leads us to the following solution to the case $X = Y$ of arbitrary dimensions.

Theorem 4.1.6. *Let X, ϕ, ψ and X_i be as defined in Lemma 4.1.3. Suppose that* $\deg X = d$, $\deg \phi = e_1$ *and* $\deg \psi = e_2$ *under a fixed embedding $\iota_X : X \hookrightarrow \mathbb{P}^n$, and set $e = \max\{e_1, e_2\}$. Let $C \subseteq \Delta_X$ be an irreducible subvariety, and suppose that*

$$C \subseteq \Delta_X \cap X_1 \cap X_2 \cap \cdots \cap X_{\lambda_{r,r}-1}.$$

Denote $F := (\phi, \psi)$, $C_i := \overline{F^i(C)}$ for each $i \geq 0$, and $X_0 := \Delta_X$. Then, either

1. *$C \subseteq X_i$ for each $i \geq 0$, or*
2. *$C_{\lambda_{r,r}} \cap X_0 = \varnothing$.*

Proof. The theorem follows from applying Proposition 4.1.5 with $k = r$. Since $\dim C \geq 0$, it follows that either $C \subseteq X_i$ for each $i \geq 0$ or

$$\dim C_{\lambda_{r,r}} \cap X_0 < 0,$$

i.e., $C_{\lambda_{r,r}} \cap X_0 = \varnothing$. □

Therefore, the descending chain of closed subsets

$$\Delta_X \supseteq \Delta_X \cap X_1 \supseteq \Delta_X \cap X_1 \cap X_2 \supseteq \dots$$

stabilises at some

$$N \leq \lambda_{r,r}.$$

It follows that the descending chain of closed subsets

$$X_1 \supseteq X_1 \cap X_2 \supseteq \cdots \supseteq X_1 \cap X_2 \cap \cdots \cap X_N \supseteq \dots$$

is such that
$$X_1 \cap X_2 \cap \cdots \cap X_N = X_1 \cap X_2 \cap \cdots \cap X_{N+1}$$
for each $m \geq \lambda_{r,r} + 1$. By Lemma 3.0.1, this gives a solution to Problem 1.0.2 where $X = Y$ and all dominant rational maps are surjective morphisms.

In practice, we would expect the above descending chain of closed subsets to stabilize at some $N \geq 1$ much less than $\lambda_{r,r}$, given that the upper bound of the number of components of the intersection
$$X_1 \cap X_2 \cap \cdots \cap X_{\lambda_{r,k}}$$
is very pessimistic.

In fact, if we view $X_1 \cap X_2$ as a subset of $X \times X$, we would expect the codimension of $X_1 \cap X_2$ in $X \times X$ to be
$$\dim X + \dim X = 2 \dim X,$$
given that X_1, X_2 are of pure dimension $\dim X$ by Lemma 4.0.2. This implies that $X_1 \cap X_2$ is a zero-dimensional algebraic set in the generic case. Even in the non-generic case, we would expect the intersection $X_1 \cap X_2 \cap \cdots \cap X_i$ to have all but its zero-dimensional components stabilise for some small i. Hence, we reformulate Theorem 4.1.6 below, which will be helpful for practical computations.

Theorem 4.1.7. *Let X, ϕ, ψ and X_i be as defined in Lemma 4.1.3, with $\dim X = r$. Suppose that $\deg X = d$, $\deg \phi = e_1$ and $\deg \psi = e_2$ under a fixed embedding $\iota_X : X \hookrightarrow \mathbb{P}^n$, and set $e = \max\{e_1, e_2\}$. Denote $X_0 := \Delta_X$, and write*
$$X_0 \cap X_1 \cap \cdots \cap X_i = (X_0 \cap X_1 \cap \cdots \cap X_i)^+ \sqcup (X_0 \cap X_1 \cap \cdots \cap X_i)^0$$
as a disjoint union where $(X_0 \cap X_1 \cap \cdots \cap X_i)^+$ is the union of the positive-dimensional components and $(X_0 \cap X_1 \cap \cdots \cap X_i)^0$ is the union of the zero-dimensional ones. If
$$(X_0 \cap X_1 \cap \cdots \cap X_{N_1})^+ = \varnothing$$
for some $N_1 \geq 1$, then
$$X_0 \cap X_1 \cap \cdots \cap X_N = X_0 \cap X_1 \cap \cdots \cap X_{N+1}$$
for some
$$N \leq 2^n \prod_{i=1}^{N_1} \binom{2r}{r} \binom{2n}{n}^{n+1} d^2 e^{i(n+1)} + N_1.$$

Proof. Denote $Z_{N_1} := (X_0 \cap X_1 \cap \cdots \cap X_{N_1})^+$. If $Z_{N_1} = \varnothing$ for some $N_1 \geq 1$, then the set $X_0 \cap X_1 \cap \cdots \cap X_{N_1}$ is zero-dimensional, and its size can be bounded using Theorem 2.2.18:
$$\begin{aligned} \#|X_0 \cap X_1 \cap \cdots \cap X_{N_1}| &\leq \deg \Delta \prod_{i=1}^{N_1} \deg X_i \\ &\leq 2^n \prod_{i=1}^{N_1} \binom{2r}{r} \binom{2n}{n}^{n+1} d^2 e^{i(n+1)}, \end{aligned}$$
where $\deg \Delta = 2^n$ by Corollary 3.3.2, and the bounds on the degrees of the X_i's follow from Corollary 3.1.5, Lemma 3.2.1 and Theorem 2.2.18. The result follows from Lemma 3.0.1. $\square$

Remark 4.1.8. We would like to point out that, while the bounds given by Theorem 4.1.6 and Theorem 4.1.7 are calculated with the assumption that the base field k is algebraically closed, those bounds also apply for the case where k is of characteristic zero but not algebraically closed. Indeed, suppose that k is of characteristic zero but not necessarily algebraically closed, and denote the algebraic closure of k by $\overline{k}$. Then, for each $i \geq 1$:

$$X_i(k) \subseteq X_i\left(\overline{k}\right)$$

where $X_i(k)$ denotes X_i as an algebraic variety defined over k. Thus, if

$$X_1\left(\overline{k}\right) \cap X_2\left(\overline{k}\right) \cap \cdots \cap X_N\left(\overline{k}\right) = X_1\left(\overline{k}\right) \cap X_2\left(\overline{k}\right) \cap \cdots \cap X_{N+1}\left(\overline{k}\right)$$

for some $N \geq 1$, and $(x, y) \in X(k) \times X(k)$ is such that

$$f \circ \phi^i(x) = g \circ \psi^i(y)$$

for each $i = 1, \ldots, N+1$, then

$$f \circ \phi^i(x) = g \circ \psi^i(y)$$

for each $i \geq 1$ given that $(x, y) \in X\left(\overline{k}\right) \times X\left(\overline{k}\right)$.

4.2 Solving Problem 1.0.2

In this subsection, we will give a complete proof of Theorem 1.0.3 and give a solution to Problem 1.0.2. The steps taken in the solution will be very similar to the simplified case where $X = Y$ and ϕ, ψ, f, g are morphisms. Here, we would only need to account for the fact that the dominant rational maps may have non-empty locus of indeterminacy. Recall that we would like to consider the descending chain

$$Y_1 \supseteq Y_1 \cap Y_2 \supseteq \cdots \supseteq Y_1 \cap Y_2 \cap \cdots \cap Y_n \supseteq \ldots$$

where

$$Y_i := \{(x, y) \in X \times Y \mid f(\phi^i(x)) = g(\psi^i(y)) \in \mathbb{P}^1\} \subseteq X \times Y$$

for each $i \geq 0$. By fixing embeddings $\iota_X : X \hookrightarrow \mathbb{P}^m$, $\iota_Y : X \hookrightarrow \mathbb{P}^n$, we may express Y_i as an intersection of two hypersurfaces in $\mathbb{P}^m \times \mathbb{P}^n$ using Lemma 3.0.4:

$$Y_i = (X \times Y) \cap V(f_0 \circ \phi^i(x) = g_0 \circ \psi^i(y), f_1 \circ \phi^i(x) = g_1 \circ \psi^i(y)) \subseteq \mathbb{P}^m \times \mathbb{P}^n,$$

where f_j, g_j, $j = 0, 1$ are the coordinate expressions of f and g under the embeddings ι_X, ι_Y respectively. In particular, each Y_i is still a closed algebraic set in $\mathbb{P}^m \times \mathbb{P}^n$. Hence, we may apply the exact same procedure as in the previous subsection to the descending chain of intersection of Y_i's.

The following lemma will allow us to generalize the proof technique from the last subsection.

Lemma 4.2.1. *Let (X, ϕ, x, f), (Y, ψ, y, g) be as in Problem 1.0.2, and let Y_i be as defined above. Denote $F := (\phi, \psi)$ and $F_i := (f \circ \phi^i, g \circ \psi^i)$ for each $i \geq 0$. Then, for each $i \geq 1$,*

$$\overline{F(Y_i)} = Y_{i-1}.$$

Proof. Let $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the diagonal subvariety. Firstly, notice that $Y_i = F^{-1}(Y_{i-1})$ for each $i \geq 1$. This is because if $(x, y) \in Y_i$, then $F_i(x, y) = F_{i-1} \circ F(x, y) \in \Delta$, so $(x, y) \in F^{-1}(Y_{i-1})$. For the other inclusion, if $(x, y) \in F^{-1}(Y_{i-1})$, then $F(x, y) \in Y_{i-1}$ implies that $F_{i-1} \circ F(x, y) = F_i(x, y) \in \Delta$, so $(x, y) \in Y_i$.

Hence,

$$\overline{F(Y_i)} = \overline{F(F^{-1}(Y_{i-1}))} = Y_{i-1}$$

since F is a dominant rational map on $X \times X$. $\square$

The proof technique for solving Problem 1.0.2 in general is very similar to the one given in the last section: we will still apply a proof by induction on the dimension of subvarieties of Y_1. In the following, we will only give a proof sketch of the solution, and we will refer to Lemma 4.1.3 and Proposition 4.1.5 where the same proof technique is applied.

Proposition 4.2.2. *Let (X, ϕ, x, f), (Y, ψ, y, g) be as in Problem 1.0.2, and let Y_i be as defined above. Denote $\dim X = r, \dim Y = s$, and fix embeddings $X \subseteq \mathbb{P}^m$, $Y \subseteq \mathbb{P}^n$ for some $m \geq r$, $n \geq s$. Suppose that $\deg X = d_1, \deg Y = d_2$, $\deg f \circ \phi = e_1$ and $\deg g \circ \psi = e_2$ under these embeddings, and set $e = \max\{e_1, e_2\}$. If $C \subseteq Y_1$ is an irreducible subvariety of dimension $\geq r + s - 1$ such that*

$$C \subseteq Y_1 \cap Y_2 \cap \cdots \cap Y_{\widetilde{\gamma_{r,s,1}}-1}$$

where

$$\widetilde{\gamma_{r,s,1}} := \binom{r+s}{r}\binom{m+n}{n}^2 d_1 d_2 e^2,$$

then either

1. *$C \subseteq Y_i$ for each $i \geq 1$, or*
2. *$\dim C_{\widetilde{\gamma_{r,s,1}}} \cap Y_1 < \dim C$, where $C_{\widetilde{\gamma_{r,s,1}}} = \overline{(\phi^{\widetilde{\gamma_{r,s,1}}}, \psi^{\widetilde{\gamma_{r,s,1}}})(C)}$.*

Proof. Denote $F := (\phi, \psi)$ and $C_i = \overline{F^i(C)}$ for each $i \geq 0$. Firstly, $\dim Y_1 \leq \dim X \times Y - 1 = r + s - 1$ by Lemma 4.0.1, so $\dim C = r + s - 1$. Thus, $\dim C_i \leq r + s - 1$ for each $i \geq 0$ since F^i is dominant for each $i \geq 1$.

Also, notice that for each $i \geq 1$,

$$C_{i+1} := \overline{F^{i+1}(C)} = \overline{F(F^i(C))} = \overline{F(C_i)}$$

since F is dominant. Thus, if $\dim C_i < r + s - 1$ for any $i = 1, \ldots, \widetilde{\gamma_{r,s,1}}$, we have

$$\dim C_{\widetilde{\gamma_{r,s,1}}} \cap Y_1 \leq \dim C_{\widetilde{\gamma_{r,s,1}}} \leq \dim C_i < r + s - 1$$

as required.

Now, suppose that $\dim C_i = \dim C = r + s - 1$ for each $i = 0, 1, \ldots, \widetilde{\gamma_{r,s,1}}$. Since $C_i = \overline{F^i(C)} \subseteq \overline{F^{i-1}(Y_i)} = Y_1$ for every $i \leq \widetilde{\gamma_{r,s,1}} - 1$, each C_i is an irreducible $(r + s - 1)$-dimensional component of Y_1. Thus, the maximum number of unique irreducible $(r + s - 1)$-dimensional components of Y_1 is given by Lemma 3.2.1:

$$\deg \sigma_{n,n}(Y_1) \leq \binom{r+s}{r}\binom{m+n}{n}^2 d_1 d_2 e^2 = \widetilde{\gamma_{r,s,1}}.$$

Hence, Y_1 can contain at most $\widetilde{\gamma_{r,s,1}}$ unique irreducible $(r+s-1)$-dimensional components. In this way, we have the following cases:

1. If $C_i = C_j$ for some $i \neq j$, $i, j = 0, 1, \dots, \widetilde{\gamma_{r,s,1}}$, then $C_i \subseteq Y_1$ for all $i \geq 1$.

2. Otherwise, $C_i \neq C_j$ for any $i \neq j$, $i, j = 0, 1, \dots, \widetilde{\gamma_{r,s,1}}$. By assumption, $C_i \subseteq Y_1$ for each $i = 0, 1, \dots, \widetilde{\gamma_{r,s,1}} - 1$. We know that Y_1 contains at most $\widetilde{\gamma_{r,s,1}}$ unique irreducible $(r+s-1)$-dimensional components, so $C_{\widetilde{\gamma_{r,s,1}}} \nsubseteq Y_1$. Thus,

$$\dim C_{\widetilde{\gamma_{r,s,1}}} \cap Y_1 < \dim C = r + s - 1,$$

as required.

□

Notation 4.2.3. Denote

$$\widetilde{\gamma_{r,s,1}} := \binom{r+s}{r}\binom{m+n}{n}^2 d_1 d_2 e^2,$$

as in Lemma 4.2.2. For each $j = 2, \dots, r+s$, define $\widetilde{\gamma_{r,s,j}}$ inductively as follows:

$$\widetilde{\gamma_{r,s,j}} := \prod_{i=1}^{\widetilde{\lambda_{r,s,j-1}}} \binom{r+s}{r}\binom{m+n}{n}^2 d_1 d_2 e^{2i}.$$

where

$$\widetilde{\lambda_{r,s,j}} := \widetilde{\gamma_{r,s,j}} + \widetilde{\gamma_{r,s,j-1}} + \dots + \widetilde{\gamma_{r,s,1}} + j.$$

Further, we will adopt the following conventions:

$$\widetilde{\gamma_{r,s,0}} = 0, \widetilde{\lambda_{r,s,0}} = \widetilde{\gamma_{r,s,0}} + 1 = 1.$$

Proposition 4.2.4. *Let (X, ϕ, x, f), (Y, ψ, y, g) be as in Problem 1.0.2, and let Y_i be as defined above. Denote $\dim X = r, \dim Y = s$, and fix embeddings $X \subseteq \mathbb{P}^m$, $Y \subseteq \mathbb{P}^n$ for some $m \geq r$, $n \geq s$. Suppose that $\deg X = d_1, \deg Y = d_2$, $\deg f \circ \phi = e_1$ and $\deg g \circ \psi = e_2$ under these embeddings, and set $e = \max\{e_1, e_2\}$. Also, suppose that $C \subseteq Y_1$ is an irreducible subvariety of dimension $\geq r + s - k$ such that*

$$C \subseteq Y_1 \cap Y_2 \cap \dots \cap Y_{\widetilde{\lambda_{r,s,k}}-2}.$$

Denote $F := (\phi, \psi)$ and $C_i := \overline{F^i(C)}$ for each $i \geq 0$. Then, either

1. *$C \subseteq Y_i$ for each $i \geq 1$, or*

2. *$\dim C_{\widetilde{\lambda_{r,s,k}}-1} \cap Y_1 < r + s - k$.*

Proof. The proof follows by induction on k. The case $k = 1$ is covered by Lemma 4.2.2. For the inductive case, suppose that the statement is true for any irreducible subvariety of Y_1 of dimension $\geq r + s - (k-1)$ for $k > 1$. Let $C \subseteq Y_1$ be an irreducible subvariety of dimension $\geq r + s - k$, and suppose that

$$C \subseteq Y_1 \cap Y_2 \cap \cdots \cap Y_{\widetilde{\lambda_{r,s,k}}-2}.$$

Then, using the inductive hypothesis and the fact that

$$C_i = \overline{F^i(C)} \subseteq \overline{F^i(Y_{i+1})} \subseteq Y_1$$

for each $i = 0, 1, \ldots, \widetilde{\lambda_{r,s,k}} - 2$, we have the following claim.

Claim: Either $C \subseteq Y_i$ for each $i \geq 0$ or

$$\dim C_{\widetilde{\lambda_{r,s,k-1}}-1+i} \leq r + s - k$$

for each $i \geq 0$.

The proof of this claim can be carried out in exactly the same way as in Proposition 4.1.5. For the rest of the proof, we will suppose that $\dim C_{\widetilde{\lambda_{r,s,k-1}}-1+i} \leq r + s - k$ for each $i \geq 0$, since $C \subseteq Y_i$ for each $i \geq 1$ otherwise. Let

$$S \subseteq Y_1 \cap Y_2 \cap \cdots \cap Y_{\widetilde{\lambda_{r,s,k-1}}}$$

be the union of all irreducible components Z of the RHS of dimension $\geq (r + s - k) + 1$ such that

$$\dim Z_{\widetilde{\lambda_{r,s,k-1}}} \cap Y_1 \geq (r + s - k) + 1.$$

By the inductive hypothesis, $Z \subseteq Y_i$ for each $i \geq 1$, implying that $S \subseteq Y_i$ for each $i \geq 1$.

Let E be the union of components of $Y_1 \cap Y_2 \cap \cdots \cap Y_{\widetilde{\lambda_{r,s,k-1}}}$ of dimension $\geq (r+s-k)+1$ which are not in S, and let L be the union of the lower dimensional components, i.e., $\dim L \leq r + s - k$. Hence, similarly as in Proposition 4.1.5, we may write

$$Y_1 \cap Y_2 \cap \cdots \cap Y_{\widetilde{\lambda_{r,s,k-1}}} = S \bigsqcup E \bigsqcup L$$

as a disjoint union. Thus, whenever an irreducible subvariety Z of the above set is such that $Z \nsubseteq S$, we must have that $Z \cap S = \varnothing$ and similarly for $Z \cap E$ as well as $Z \cap L$.

Notice that

$$C_i \subseteq Y_1 \cap Y_2 \cap \cdots \cap Y_{\widetilde{\lambda_{r,s,k-1}}}$$

for every $i = 0, 1, \ldots, \widetilde{\gamma_{r,s,k}} - 1$. In the following, we will divide the cases according to $C_i \subseteq S$, E or L for $i = 0, 1, \ldots, \widetilde{\gamma_{r,s,k}} - 1$, as in Proposition 4.1.5.

1. If $C_{i_0} \subseteq S$ for any $i_0 = 0, 1, \ldots, \widetilde{\lambda_{r,s,k}} - 1$, then $C_i \subseteq Y_1$ for every $i \geq i_0$ since

$$C_i = \overline{F^{i-i_0}(C_{i_0})} \subseteq \overline{F^{i-i_0}(S)} \subseteq Y_1.$$

Hence, $C \subseteq Y_i$ for each $i \geq 1$ in this case.

2. Thus, suppose that $C_i \nsubseteq S$ for any $i = 0, 1, \ldots, \widetilde{\lambda_{r,s,k}} - 1$, so that $C_i \cap S = \varnothing$ for $i = 0, 1, \ldots, \widetilde{\lambda_{r,s,k}} - 1$. Then, we can divide up the cases as follows.

   (a) Suppose that there does not exist any irreducible component $E_l \subseteq E$ such that $C_j, C_{j'} \subseteq E_l$ for $j \neq j'$ and $j, j' = 0, 1, \ldots, \widetilde{\gamma_{r,s,k}}$. WLOG, suppose that $C_j \neq C_{j'}$ for any $j \neq j'$, $j, j' = 0, 1, \ldots, \widetilde{\gamma_{r,s,k}}$, as we would have $C_i \subseteq Y_1$ for each $i \geq 1$ otherwise.
   Since $C_i \cap S = \varnothing$ for any $i \leq \widetilde{\gamma_{r,s,k}} - 1 < \widetilde{\lambda_{r,s,k}} - 1$, $C_i \subseteq E \sqcup L$ for any $i \leq \widetilde{\gamma_{r,s,k}} - 1$. Then, calculating the number of irreducible components of $E \sqcup L$, we find that it is bounded above by:

   $$\deg Y_1 \cap Y_2 \cap \cdots \cap Y_{\widetilde{\lambda_{r,s,k-1}}} \leq \widetilde{\gamma_{r,s,k}}.$$

   Now, recall that

   $$\dim C_{\widetilde{\lambda_{r,s,k-1}} - 1 + i} \leq r + s - k$$

   for each $i \geq 0$ and that $\dim L \leq r + s - k$. We have the following two sub-cases.

   - $\dim C_i \geq r + s - k$ for each $i = 0, 1, \ldots, \widetilde{\gamma_{r,s,k}}$. By assumption, no irreducible component of E can contain more than one of such C_i. It follows that each C_i, $i = 0, 1, \ldots, \widetilde{\gamma_{r,s,k}} - 1$, must be on a unique component of $E \sqcup L$, so $C_{\widetilde{\gamma_{r,s,k}}}$ cannot be contained in any component of E or L. Moreover, $C_{\widetilde{\gamma_{r,s,k}}} \nsubseteq S$ by assumption. Thus, using the same argument as in Proposition 4.1.5:

     $$C_{\widetilde{\lambda_{r,s,k}} - 1} = C_{\widetilde{\gamma_{r,s,k}} + \widetilde{\lambda_{r,s,k-1}}} \nsubseteq Y_1,$$

     so

     $$\dim C_{\widetilde{\lambda_{r,s,k}} - 1} \cap Y_1 < \dim C_{\widetilde{\lambda_{r,s,k}} - 1} \leq r + s - k,$$

     as required.
   - $\dim C_{i_0} < r + s - k$ for some $i_0 = 0, 1, \ldots, \widetilde{\gamma_{r,s,k}}$. Then,

     $$\dim C_{\widetilde{\lambda_{r,s,k}} - 1} \cap Y_1 \leq \dim C_{\widetilde{\lambda_{r,s,k}} - 1} \leq \dim C_{i_0} < r + s - k,$$

     as required.

   (b) Otherwise, there exists an irreducible component $E_l \subseteq E$ such that $C_j, C_{j'} \subseteq E_l$ for $j \neq j'$ and $j, j' = 0, 1, \ldots, \widetilde{\gamma_{r,s,k}}$. WLOG, let $j < j'$. Then,

   $$C_{\widetilde{\lambda_{r,s,k-1}} - 1 + j}, C_{\widetilde{\lambda_{r,s,k-1}} - 1 + j'} \subseteq \overline{F^{\widetilde{\lambda_{r,s,k-1}} - 1}(E_l)}.$$

   By the definition of E, $E_l \cap S = \varnothing$, so

   $$\dim \overline{F^{\widetilde{\lambda_{r,s,k-1}} - 1}(E_l)} \cap Y_1 < r + s - (k - 1)$$

   using the inductive hypothesis.
   Since

   $$E_l \subseteq E \subseteq Y_1 \cap Y_2 \cap \cdots \cap Y_{\widetilde{\lambda_{r,k-1}}},$$

   we know that $\overline{F^{\widetilde{\lambda_{r,s,k-1}} - 1}(E_l)} \subseteq Y_1$. It follows that $\dim \overline{F^{\widetilde{\lambda_{r,s,k-1}} - 1}(E_l)} < r + s - (k - 1)$. Hence, either

- $\dim C_{\widetilde{\lambda_{r,s,k-1}}-1+j} = \dim C_{\widetilde{\lambda_{r,s,k-1}}-1+j'} = \dim \overline{F^{\widetilde{\lambda_{r,s,k-1}}-1}(E_l)}$, implying that these sets are the same set as they are all irreducible. Thus,

$$C_{\widetilde{\lambda_{r,s,k-1}}-1+j} = C_{\widetilde{\lambda_{r,s,k-1}}-1+j'} \subseteq Y_1,$$

  which implies that $C_i \subseteq Y_1$ for each $i \geq 1$. Thus, $C \subseteq Y_i$ for each $i \geq 1$ in this case. Or,

- $\dim C_{\widetilde{\lambda_{r,s,k-1}}-1+j'} < \dim \overline{F^{\widetilde{\lambda_{r,s,k-1}}-1}(E_l)} < r+s-(k-1)$. Hence, using the same argument as in Proposition 4.1.5,

$$\dim C_{\widetilde{\lambda_{r,s,k}}-1} \cap Y_1 < r+s-k.$$

□

Then, Theorem 1.0.3 follows from the below corollary.

Corollary 4.2.5. *Let (X, ϕ, x, f), (Y, ψ, y, g) be as in Problem 1.0.2, and let Y_i be defined as previously. Denote* $\dim X = r, \dim Y = s$, *and fix embeddings $X \subseteq \mathbb{P}^m$, $Y \subseteq \mathbb{P}^n$ for some $m \geq r$, $n \geq s$. Suppose that* $\deg X = d_1, \deg Y = d_2$, $\deg f \circ \phi = e_1$ *and* $\deg g \circ \psi = e_2$ *under these embeddings, and set $e =$* $\max\{e_1, e_2\}$. *Also, suppose that $C \subseteq Y_1$ is an irreducible subvariety such that*

$$C \subseteq Y_1 \cap Y_2 \cap \cdots \cap Y_{\widetilde{\lambda_{r,s,r+s}}-2}.$$

Denote $F := (\phi, \psi)$ and $C_i := \overline{F^i(C)}$ for each $i \geq 0$. Then, either

1. *$C \subseteq Y_i$ for each $i \geq 1$, or*
2. *$C_{\widetilde{\lambda_{r,s,r+s}}-1} \cap Y_1 = \varnothing$, implying that $C \cap Y_{\widetilde{\lambda_{r,s,r+s}}-1} = \varnothing$.*

Proof. Apply Proposition 4.2.4 to C with $\dim C \geq 0 = (r+s)-(r+s)$. □

Therefore, the descending chain of closed subsets

$$Y_1 \supseteq Y_1 \cap Y_2 \supseteq \cdots \supseteq Y_1 \cap Y_2 \cap \cdots \cap Y_N \supseteq \ldots$$

from Problem 1.0.2 stabilises at some

$$N \leq \widetilde{\lambda_{r,s,r+s}} - 1$$

whenever $\dim X = r, \dim Y = s$. This completes the proof of Theorem 1.0.3.

Finally, as in the last subsection, we have the following reformulation of Corollary 4.2.5 for practical computations.

Theorem 4.2.6. *Let (X, ϕ, x, f), (Y, ψ, y, g) be as in Problem 1.0.2, and let Y_i be defined as previously. Denote* $\dim X = r, \dim Y = s$, *and fix embeddings $X \subseteq \mathbb{P}^m$, $Y \subseteq \mathbb{P}^n$ for some $m \geq r$, $n \geq s$. Suppose that* $\deg X = d_1, \deg Y = d_2$, $\deg f \circ \phi = e_1$ *and* $\deg g \circ \psi = e_2$ *under these embeddings, and set $e = \max\{e_1, e_2\}$. Write*

$$Y_1 \cap Y_2 \cap \cdots \cap Y_i = (Y_1 \cap Y_2 \cap \cdots \cap Y_i)^+ \sqcup (Y_1 \cap Y_2 \cap \cdots \cap Y_i)^0$$

as a disjoint union where $(Y_1 \cap Y_2 \cap \cdots \cap Y_i)^+$ *is the union of the positive-dimensional components and* $(Y_1 \cap Y_2 \cap \cdots \cap Y_i)^0$ *is the union of the zero-dimensional ones. If*

$$(Y_1 \cap Y_2 \cap \cdots \cap Y_{N_1})^+ = \varnothing$$

for some $N_1 \geq 1$*, then*

$$Y_1 \cap Y_2 \cap \cdots \cap Y_N = Y_1 \cap Y_2 \cap \cdots \cap Y_{N+1}$$

for some

$$N \leq \prod_{i=1}^{N_1} \binom{r+s}{r} \binom{m+n}{n}^2 d_1 d_2 e^{2i} + N_1.$$

Proof. Denote $Z_{N_1} = (Y_1 \cap Y_2 \cap \cdots \cap Y_{N_1})^+$. Notice that if $Z_{N_1} = \varnothing$, then $Y_1 \cap Y_2 \cap \cdots \cap Y_{N_1}$ is zero-dimensional, and its size is bounded above by its degree. The degrees of each Y_i may be bounded from the above via Corollary 3.1.5, Lemma 3.2.1 and Theorem 2.2.18. The theorem then follows from Lemma 3.0.1. □

Remark 4.2.7. As in the last subsection, we remark that the bounds obtained in Corollary 4.2.5 and Theorem 4.2.6 also apply for the case where the base field k is of chracteristic zero but not algebraically closed. In other words, Theorem 1.0.3 applies for the case where k is not algebraically closed as well.

4.3 An Algorithm for Solving Problem 1.0.2 in Practice

Let (X, ϕ, x, f) and (Y, ψ, y, g) be data given in Problem 1.0.2. As in Theorem 1.0.3, we will fix the following:

- $r := \dim X$, $s := \dim Y$;
- $\iota_X : X \hookrightarrow \mathbb{P}^m$, $\iota_Y : Y \hookrightarrow \mathbb{P}^n$ embeddings for some $m \geq r$, $n \geq s$;
- $\deg X = d_1$, $\deg Y = d_2$ under the above embeddings;
- $\deg f \circ \phi = e_1$, $\deg g \circ \psi = e_2$ under the embeddings ι_X and ι_Y respectively;
- $e := \max\{e_1, e_2\}$.

To solve Problem 1.0.2 with the above given data, we need to check

$$f(\phi^i(x)) = g(\psi^i(y))$$

for each $i = 1, \ldots, N$, where

$$N = \widetilde{\lambda_{r,s,r+s}} - 1$$

using the upper bound given by Theorem 1.0.3. Notice that $\widetilde{\lambda_{r,s,r+s}}$ is a very large number in general. This is because

$$\widetilde{\gamma_{r,s,1}} := \binom{r+s}{r} \binom{m+n}{n}^2 d_1 d_2 e^2$$

and

$$\begin{aligned}
\widetilde{\lambda_{r,s,2}} &:= \prod_{i=1}^{\widetilde{\lambda_{r,s,1}}} \binom{r+s}{r}\binom{m+n}{n}^2 d_1 d_2 e^{2i} \\
&\gg \left(\binom{r+s}{r}\binom{m+n}{n}^2 d_1 d_2 e^2\right)^{\widetilde{\lambda_{r,s,1}}} \\
&> \left(\binom{r+s}{r}\binom{m+n}{n}^2 d_1 d_2 e^2\right)^{d_1 d_2 e^2} \\
&> \left(\binom{r+s}{r}\binom{m+n}{n}^2 d_1 d_2 e^2\right)^{D},
\end{aligned}$$

where $D = \max\{d_1, d_2, e^2\}$.

Thus, even in the case where both X and Y are one-dimensional, the upper bound on the number $N \geq 1$ given by Theorem 1.0.3 grows much faster than an exponential function in the variable D. This means that the upper bound given by Theorem 1.0.3 is not computationally useful in general.

In the spirit of Theorem 4.1.7 and Theorem 4.2.6, we propose the following algorithm for solving Problem 1.0.2. The general idea is that, given the data (X, ϕ, x, f) and (Y, ψ, y, g), it is easy to compute the ideal generating Y_i for each $i \geq 1$ using any computer algebra software: indeed, Lemma 3.0.4 allows one to compute generators of the ideal of Y_i given the maps ϕ, ψ, f and g. Then, one may compute Y_i by computing the vanishing locus of the ideal of each Y_i, which is also not difficult when the ideals are generated by polynomials of low orders. Thus, as soon as one finds

$$Y_1 \cap Y_2 \cap \cdots \cap Y_N = Y_1 \cap Y_2 \cap \cdots \cap Y_{N+1}$$

for some $N \geq 1$, Problem 1.0.2 may be solved by checking

$$f(\phi^i(x)) = g(\psi^i(y))$$

for each $i = 1, \ldots, N+1$.

Algorithm 4.3.1. Let $(X, \phi, x, f), (Y, \psi, y, g)$ be data as given in Problem 1.0.2.

1. If the pairs of maps (ϕ, f) and (ψ, g) are not given explicitly, write them down on some affine covers of X and Y respectively.

2. Choose embeddings $\iota_X : X \hookrightarrow \mathbb{P}^m$ and $\iota_Y : Y \hookrightarrow \mathbb{P}^n$, so that X and Y may be viewed as subvarieties of $\mathbb{P}^m$ and $\mathbb{P}^n$ respectively. Homogenize the pairs of maps (ϕ, f) and (ψ, g) and write down their coordinate expressions on $\mathbb{P}^m$ and $\mathbb{P}^n$ respectively. Thus, for each $i \geq 1$, the coordinate expressions of ϕ^i and ψ^i may be computed iteratively with those of ϕ and ψ.

3. Check that both ϕ and ψ are dominant rational maps when considered as self-maps on $\mathbb{P}^m$ and $\mathbb{P}^n$ respectively. If applicable, check that they are regular and surjective on the respective projective spaces. In addition, check that f and g are both dominant rational maps.

4. Compute the ideal generating Y_1 via Lemma 3.0.4.
5. Compute the ideal generating Y_{i+1}, $i = 1$, via Lemma 3.0.4.
6. Check if

   $$Y_1 \cap Y_2 \cap \cdots \cap Y_i = Y_1 \cap Y_2 \cap \cdots \cap Y_{i+1},$$

   $i = 1$, by computing $Y_1, \ldots, Y_{i+1}$ based on their ideals. Since inclusion of the RHS to the LHS is trivial, it suffices to check the inclusion of the LHS to the RHS in practice. If the above equality is true, then proceed to step 7(a). If the above equality is not true, then check if

   $$\dim Y_1 \cap Y_2 \cap \cdots \cap Y_{i+1} = 0.$$

   If this is true, then proceed to step 7(b). If this is not true either, then update i with $i + 1$ and repeat steps 5 and 6.
7. (a) Check if

   $$f(\phi^j(x)) = g(\psi^j(y))$$

   for each $j = 1, \ldots, i + 1$. By Lemma 3.0.1, if this is true, then

   $$f(\phi^j(x)) = g(\psi^j(y))$$

   for every $j \geq 1$. If this is false, then there exists some positive integer j such that

   $$f(\phi^j(x)) \neq g(\psi^j(y)).$$

   (b) Compute $d_1 := \deg X$, $d_2 := \deg Y$, $e_1 := \deg f \circ \phi$ and $e_2 := \deg g \circ \psi$ under the embeddings ι_X and ι_Y as defined in step 2. Denote $r := \dim X$, $s := \dim Y$ and $e := \max\{e_1, e_2\}$. Check if

   $$f(\phi^j(x)) = g(\psi^j(y))$$

   for each $j = 1, \ldots, N + 1$, where

   $$N = \prod_{k=1}^{i} \binom{r+s}{r} \binom{m+n}{n}^2 d_1 d_2 e^{2k} + i.$$

   By Theorem 4.2.6, if the above is true, then

   $$f(\phi^j(x)) = g(\psi^j(y))$$

   for every $j \geq 1$. If the above is false, then there exists some positive integer j such that

   $$f(\phi^j(x)) \neq g(\psi^j(y)).$$

In particular, the above algorithm is guaranteed to terminate by Theorem 1.0.3.

We remark that the bounds given by Theorem 4.2.6 may be replaced by those given by Theorem 4.1.7 in the case of $X = Y$ and ϕ, ψ are both regular and surjective. The algorithm may also be re-written accordingly to adapt to this case, as in general, Theorem 4.1.7 gives a better bound when $X = Y$.

Moreover, by Remark 4.1.8 and Remark 4.2.7, the above algorithm may also be applied to solve Problem 1.0.2 when the base field is of characteristic zero but not algebraically closed.

Section 5

Applications

5.1 Elliptic Divisibility Sequences

In this section, we will apply Algorithm 4.3.1 to show that an elliptic divisibility sequence, as defined below, is equal term-by-term to the even terms of the Fibonacci sequence.

An elliptic divisibility sequence (EDS) is a sequence $\{w_n\}_{n\geq 1}$ satisfying the recurrence relation

$$w_{n+m}w_{n-m} = w_{n+1}w_{n-1}w_m^2 - w_{m+1}w_{m-1}w_n^2$$

for every $n > m$, with $w_1 = 1$ (cf. [Wa48]). Such sequences were first defined by Morgan Ward in the late 1940s, and their basic arithmetic properties are studied by Ward in [Wa48].

In particular, an EDS $\{w_n\}$ is uniquely determined by its first four terms, $w_1 = 1, w_2, w_3$ and w_4 [Wa48]. Ward also proved that an EDS may be associated with an elliptic curve, provided that the first four terms of the EDS satisfy some arithmetic relations. More precisely, one may define the discriminant of an EDS based on its first four terms, and the EDS may be associated with an elliptic curve as long as its discriminant is non-zero [Wa48].

Some applications of EDS include the work of Katherine E. Stange [St07], in which higher rank generalizations of EDS are used to compute the Tate pairing of an elliptic curve over a finite field. This is significant since such pairings have applications in pairing-based cryptography. For this paper, EDS are interesting since they are defined by nonlinear recurrrence relations. Thus, EDS are not holonomic sequences in general, so one cannot apply the method of Wilf and Zeilberger [WZ90a, WZ90b, Ze06, Ze90, Ze91] directly to show that an EDS is the same as another sequence. As we shall see in the following example, Algorithm 4.3.1 can be applied directly to show that an EDS is equal to another sequence. In this way, our technique is indeed an analog of the method of Wilf and Zeilberger in the dynamical setting.

For our specific example, we will set the initial values of the EDS $\{w_n\}$ to be $w_1 = 1, w_2 = 3, w_3 = 8$ and $w_4 = 21$. These are the first four even terms of the Fibonacci sequence, and we will indeed prove that $\{w_n\}$ consists of the even terms of the Fibonacci sequence.

Proposition 5.1.1. *Let $\{w_n\}_{n\geq 1}$ be the EDS given by the initial values $w_1 = 1, w_2 = 3, w_3 = 8$ and $w_4 = 21$. Denote the Fibonacci sequence by $\{F_n\}_{n\geq 1}$, i.e., $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Then, $w_n = F_{2n}$ for each $n \geq 1$.*

To prove this proposition, we apply Algorithm 4.3.1 as follows.

Step 1: Write down maps associated with the recurrences.

By definition, the Fibonacci numbers F_n satisfy the recurrence $F_{n+2} = F_{n+1} + F_n$. We may generate $\{F_n\}$ as a sequence on the affine plane $\mathbb{A}^2$ as follows. Let $\phi : \mathbb{A}^2 \to \mathbb{A}^2$ be the map given by $\phi(y_1, y_2) = (y_2, y_1 + y_2)$. Then, it is straightforward to show that

$$\phi^n(0,1) = (F_n, F_{n+1})$$

for each $n \geq 1$. Thus, $(\phi^2)^n(0,1) = (F_{2n}, F_{2n+1})$.

As for the defining recurrence relation for the EDS $\{w_n\}$, notice that for $m = 2$,

$$w_{n+2}w_{n-2} = w_{n+1}w_{n-1}w_2^2 - w_3w_1w_n^2$$

for each $n > 2$. Using the initial values $w_1 = 1, w_2 = 3, w_3 = 8$, we may write

$$w_{n+2} = \frac{9w_{n+1}w_{n-1} - 8w_n^2}{w_{n-2}}$$

for each $n > 2$. This gives us a means of computing w_n recursively for each $n > 4$. Thus, similarly as above, let $\psi : \mathbb{A}^4 \to \mathbb{A}^4$ be the map given by

$$\psi(x_1, x_2, x_3, x_4) = \left(x_2, x_3, x_4, \frac{9x_4x_2 - 8x_3^2}{x_1}\right).$$

Then, $\psi(w_1, w_2, w_3, w_4) = (w_2, w_3, w_4, w_5)$, and it is straightforward to show that

$$\psi^n(w_1, w_2, w_3, w_4) = (w_{n+1}, w_{n+2}, w_{n+3}, w_{n+4})$$

for each $n \geq 1$.

Step 2: Write down homogenizations of ϕ^2 and ψ on projective spaces.

We will homogenize ϕ^2 and ψ by embedding the affine spaces $\mathbb{A}^2$ and $\mathbb{A}^4$ into $\mathbb{P}^2$ and $\mathbb{P}^4$ respectively. Let $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$ be the embedding given by $(y_1, y_2) \mapsto (1 : y_1 : y_2)$, so that

$$\phi^2(y_1, y_2) = (1 : y_1 + y_2 : y_1 + 2y_2)$$

under this embedding. Thus,

$$\Phi : \mathbb{P}^2 \to \mathbb{P}^2, (y_0 : y_1 : y_2) \mapsto (y_0 : y_1 + y_2 : y_1 + 2y_2)$$

is the homogenization of ϕ^2. Note that we will be working over the open subset $\mathbb{P}^2 \setminus V(y_0 = 0) \subset \mathbb{P}^2$ by assumption.

Similarly, let $\mathbb{A}^4 \hookrightarrow \mathbb{P}^4$ be the embedding given by $(x_1, x_2, x_3, x_4) \mapsto (1 : x_1 : x_2 : x_3 : x_4)$. Thus,

$$\begin{aligned}\Psi : \mathbb{P}^4 \to \mathbb{P}^4, (x_0 : x_1 : x_2 : x_3 : x_4) &\mapsto \left(x_0 : x_2 : x_3 : x_4 : \frac{9x_4x_2 - 8x_3^2}{x_1}\right) \\ &= \left(x_0x_1 : x_1x_2 : x_1x_3 : x_1x_4 : 9x_4x_2 - 8x_3^2\right)\end{aligned}$$

is the homogenization of ψ. Note that we will be working over the open subset $\mathbb{P}^4 \setminus V(x_0x_1 = 0) \subset \mathbb{P}^4$ by assumption.

Now, let $f : \mathbb{P}^2 \to \mathbb{P}^1$, $g : \mathbb{P}^4 \to \mathbb{P}^1$ be the projections onto the first two homogeneous coordinates of $\mathbb{P}^2$ and $\mathbb{P}^4$ respectively. Then,

$$f \circ \Phi^i(1 : 1 : 2) = (1 : F_{2(i+1)}),$$
$$g \circ \Psi^i(1 : w_1 : w_2 : w_3 : w_4) = (1 : w_{i+1}).$$

Step 3: Check that the maps Φ and Ψ are dominant rational maps on $\mathbb{P}^2$ and $\mathbb{P}^4$ respectively.

It is clear that Φ and Ψ are morphisms on $\mathbb{P}^2 \setminus V(y_0 = 0) \subset \mathbb{P}^2$ and $\mathbb{P}^4 \setminus V(x_0x_1 = 0) \subset \mathbb{P}^4$ respectively. If we can show that they are surjective maps on $\mathbb{P}^2$ and $\mathbb{P}^4$ respectively, we may conclude that they are surjective morphisms on the respective projective spaces.

We would like to use computer algebra softwares for computations. In order to ensure the accuracy of computations, we will let $\mathbb{Q}$ be the base field over which all varieties are defined for the following calculation. Using Macaulay2, we may check that Φ and Ψ are indeed surjective on $\mathbb{P}^2(\mathbb{Q})$ and $\mathbb{P}^4(\mathbb{Q})$ respectively:

```
i1 : R = QQ[x_0..x_4];

i2 : S = QQ[y_0,y_1,y_2];

i3 : needsPackage "RationalMaps";

i4 : psi = rationalMapping(Proj(R),Proj(R),
{x_0*x_1,x_1*x_2,x_1*x_3,x_1*x_4,9*x_2*x_4-8*x_3^2});

o4 : RationalMapping

i5 : phi = rationalMapping(Proj(S),Proj(S),{y_0,y_1+y_2,y_1+2*y_2});

i6 : idealOfImageOfMap(psi)

o6 = ideal 0

o6 : Ideal of R

i7 : idealOfImageOfMap(phi)

o7 = ideal 0

o7 : Ideal of S
```

Thus, Φ and Ψ are surjective endomorphisms on $\mathbb{P}^2(\mathbb{Q})$ and $\mathbb{P}^4(\mathbb{Q})$ respectively, so they are dominant rational self-maps on $\mathbb{P}^2(\overline{\mathbb{Q}})$ and $\mathbb{P}^4(\overline{\mathbb{Q}})$ respectively, where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$.

Step 4: Compute the ideal of Y_1.

For illustration purposes, we will compute the ideal of each Y_i by hand instead of using Macaulay2. We have that

$$f \circ \Phi(y_0 : y_1 : y_2) = (y_0 : y_1 + y_2) = \left(1 : \frac{y_1 + y_2}{y_0}\right),$$
$$g \circ \Psi(x_0 : x_1 : x_2 : x_3 : x_4) = (x_0x_1 : x_1x_2) = \left(1 : \frac{x_2}{x_0}\right),$$

since $y_0, x_0, x_1 \neq 0$ by assumption. Thus,

$$Y_1 = V\left(\frac{y_1 + y_2}{y_0} = \frac{x_2}{x_0}\right) = V(y_0x_2 - x_0(y_1 + y_2)),$$

which is a hypersurface of bidegree $(1,1)$ in variables x_i, y_j.

Step 5: Compute the ideal of Y_{i+1}, $i = 1$.

Similarly, by computing $f \circ \Phi^2(y_0 : y_1 : y_2)$ and $g \circ \Psi^2(x_0 : x_1 : x_2 : x_3 : x_4)$, we may find:

$$Y_2 = V(y_0x_3 - x_0(2y_1 + 3y_2)).$$

In particular, by Lemma 3.0.4 and Remark 3.0.5, note that the above expressions are the ideals generating Y_n over both the field $\mathbb{Q}$ and the field $\overline{\mathbb{Q}}$.

Step 6: Check if $Y_1 = Y_1 \cap Y_2$, and then if $\dim Y_1 \cap Y_2 = 0$.

Firstly, we check if $Y_1 = Y_1 \cap Y_2$ using Macaulay2. Again, this is done over the field $\mathbb{Q}$ for accuracy of the computations:

```
i1 : R = QQ[x_0..x_4,y_0..y_2,Degrees=>{5:{1,0},3:{0,1}}];

i2 : needsPackage "MultiprojectiveVarieties";

i3 : I1 = ideal(x_2*y_0-x_0*(y_1+y_2));

o3 : Ideal of R

i4 : I2 = ideal(y_0*x_3-x_0*(2*y_1+3*y_2));

o4 : Ideal of R

i5 : Z1 = projectiveVariety I1;

i6 : Z2 = projectiveVariety (I1 + I2);

i7 : isSubset(Z1, Z2)

o7 = false
```

Since this is false, we proceed to check if $\dim Y_1 \cap Y_2 = 0$:

```
i8 : dim Z2

o8 = 4
```

It turns out that $\dim Y_1 \cap Y_2 > 0$, so we must repeat steps 5 and 6 with $i = 2$.

Repeats of Steps 5 and 6: Similarly as above,

$$Y_3 = V(y_0 x_4 - x_0(5y_1 + 8y_2)).$$

In fact, it is straightforward to show that

$$Y_n = V\left(y_0 \Psi_4^{n-3}(x_0 : x_1 : x_2 : x_3 : x_4) - x_0 x_1^{2^{n-4}} x_2^{\lfloor 2^{n-5} \rfloor} x_3^{\lfloor 2^{n-6} \rfloor} x_4^{\lfloor 2^{n-7} \rfloor} \Phi_1^n(y_0 : y_1 : y_2)\right)$$

for each $n \geq 4$, where we have denoted

$$\Phi(y) := \Phi(y_0 : y_1 : y_2) = (\Phi_0(y) : \Phi_1(y) : \Phi_2(y)),$$
$$\Psi(x) := \Psi(x_0 : x_1 : x_2 : x_3 : x_4) = (\Psi_0(x) : \Psi_1(x) : \cdots : \Psi_4(x)).$$

This may also be checked using computer algebra softwares.

Then, using Macaulay2, we proceed to check if $Y_1 \cap Y_2 = Y_1 \cap Y_2 \cap Y_3$ and if $\dim Y_1 \cap Y_2 \cap Y_3 = 0$, then update i with $i + 1$ and repeat these steps until one of these two statements is true. In the following, we continue to compute over the field $\mathbb{Q}$:

```
i9 : I3 = ideal(y_0*x_4-x_0*(5*y_1+8*y_2));

o9 : Ideal of R

i10 : Z3 = projectiveVariety (I1 + I2 + I3);

i11 : isSubset(Z2, Z3)

o11 = false

i12 : dim Z3

o12 = 4

i13 : I4 = ideal(y_0*(9*x_2*x_4-8*x_3^2)-x_0*x_1*(13*y_1+21*y_2));

o13 : Ideal of R

i14 : Z4 = projectiveVariety (I1 + I2 + I3 + I4);
```

```
i15 : isSubset(Z3, Z4)

o15 = false

i16 : dim Z4

o16 = 4

i17 : I5 = ideal(y_0*(9*9*x_1*x_3*x_4*x_2-9*8*x_1*x_3^3-8*(x_1^2)*(x_4^2))
-x_0*(x_1^2)*x_2*(34*y_1+55*y_2));

o17 : Ideal of R

i18 : Z5 = projectiveVariety (I1 + I2 + I3 + I4 + I5);

i19 : isSubset(Z4, Z5)

o19 = true
```

Thus, the iterations terminate at $i = 4$, since $Y_1(\mathbb{Q}) \cap \cdots \cap Y_4(\mathbb{Q}) = Y_1(\mathbb{Q}) \cap \cdots \cap Y_5(\mathbb{Q})$. Denote $N = 4$.

Step 7: Check that the first $N + 1 = 5$ terms of $\{F_{2(n+1)}\}_{n\geq 1}$ and $\{w_{n+1}\}_{n\geq 1}$ are the same.

Notice that both the EDS $\{w_n\}$ and the even Fibonacci sequence $\{F_{2n}\}$ are in $\mathbb{Q}$ by their defining recurrence relations. Thus, having worked over the field $\mathbb{Q}$ in the above steps, we find that the iterative steps in Algorithm 4.3.1 terminate at $N = 4$. Also, recall that

$$\begin{aligned} &f \circ \Phi^n(1 : 1 : 2) = (1 : F_{2(n+1)}), \\ &g \circ \Psi^n(1 : w_1 : w_2 : w_3 : w_4) = (1 : w_{n+1}). \end{aligned}$$

Thus, $F_{2(n+1)} = w_{n+1}$ for each $n \geq 1$ if $f \circ \Phi^n(1 : 1 : 2) = g \circ \Psi^n(1 : w_1 : w_2 : w_3 : w_4)$ for $n = 1, 2, 3, 4, 5$. Computing the first five terms of $\{F_{2(n+1)}\}_{n\geq 1}$ and $\{w_{n+1}\}_{n\geq 1}$ respectively, we see that this is indeed true!

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