List 3-coloring P_t-free graphs with no induced 1-subdivision of $K_{1,s}$

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Abstract

Let s and t be positive integers. We use P_t to denote the path with t vertices and $K_{1,s}$ to denote the complete bipartite graph with parts of size 1 and s respectively. The one-subdivision of $K_{1,s}$ is obtained by replacing every edge $\{u, v\}$ of $K_{1,s}$ by two edges $\{u, w\}$ and $\{v, w\}$ with a new vertex w. In this paper, we give a polynomial-time algorithm for the list 3-coloring problem restricted to the class of P_t-free graph with no induced 1-subdivision of $K_{1,s}$.

1 Introduction

All graphs in this paper are finite and simple. We use $[k]$ to denote the set $\{1, \dots, k\}$. Let G be a graph. A *k-coloring* of G is a function $f : V(G) \to [k]$ such that for every edge $uv \in E(G)$, $f(u) \neq f(v)$, and G is *k-colorable* if G has a k-coloring. The k-COLORING PROBLEM is the problem of deciding, given a graph G, if G is k-colorable. This problem is well-known to be NP-hard for all $k \geq 3$.

A function $L : V(G) \to 2^{[k]}$ that assigns a subset of $[k]$ to each vertex of a graph G is a *k-list assignment* for G. For a k-list assignment L, a function $f : V(G) \to [k]$ is a *coloring of* (G, L) if f is a k-coloring of G and $f(v) \in L(v)$ for all $v \in V(G)$. We say that a graph G is *L-colorable*, and that the pair (G, L) is *colorable*, if (G, L) has a coloring. The LIST k-COLORING PROBLEM is the problem of deciding, given a graph G and a k-list assignment L, if (G, L) is colorable. Since this generalizes the k-coloring problem, it is also NP-hard for all $k \geq 3$.

We denote by P_t the path with t vertices and we use $K_{r,s}$ to denote the complete bipartite graph with parts of size r and s respectively. The one-subdivision of $K_{1,s}$ is obtained by replacing every edge $\{u, v\}$ of $K_{1,s}$ by two edges $\{u, w\}$ and $\{v, w\}$ with a new vertex w. For a set $\mathcal{H}$ of graphs, a graph G is $\mathcal{H}$-free if no element of $\mathcal{H}$ is an induced subgraph of G. If $\mathcal{H} = \{H\}$, we say that G is H-free. In this paper, we use the terms "polynomial time" and "polynomial size" to mean "polynomial in $|V(G)|$", where G is the input graph. Since the k-COLORING PROBLEM and the LIST-k COLORING PROBLEM are NP-hard for $k \geq 3$, their restrictions to H-free graphs, for various H, have been extensively studied. In particular, the following is known:

Theorem 1 ([7])**.** *Let H be a (fixed) graph, and let $k > 2$. Assume that $P \neq NP$. If the k-*COLORING PROBLEM *can be solved in polynomial time when restricted to the class of H-free graphs, then every connected component of H is a path.*

Thus if we assume that H is connected, then the question of determining the complexity of k-coloring H-free graph is reduced to studying the complexity of coloring graphs with certain induced paths excluded, and a significant body of work has been produced on this topic. Below we list a few such results.

Theorem 2 ([1]). *The* 3-COLORING PROBLEM *can be solved in polynomial time for the class of P_7-free graphs.*

Theorem 3 ([2]). *The* 4-COLORING PROBLEM *can be solved in polynomial time for the class of P_6-free graphs.*

Theorem 4 ([4]). *The* k-COLORING PROBLEM *can be solved in polynomial time for the class of P_5-free graphs.*

Theorem 5 ([5]). *The* 4-COLORING PROBLEM *is NP-complete for the class of P_7-free graphs.*

Theorem 6 ([5]). *For all $k \geq 5$, the* k-COLORING PROBLEM *is NP-complete for the class of P_6-free graphs.*

The only case for which the complexity of k-coloring P_t-free graphs is not known $k = 3$, $t \geq 8$. Then it is natural to consider forbidding another induced subgraph besides the path. The following are two known results when the other forbidden induced subgraph is a clique or a cycle.

Theorem 7 ([8]). *For all $k, r, s, t \geq 1$, the* LIST k-COLORING PROBLEM *can be solved in polynomial time for the class of $(K_{r,s}, P_t)$-free graphs.*

Theorem 8 ([6]). *The* k-COLORING PROBLEM *for the class of (C_s, P_t)-free graphs can be solved in polynomial time if $k \geq 5, s = 3$ and $t \leq k + 2$, and is NP-complete if*

1. $k = 4, s = 3$ *and* $t \geq 22$
2. $k = 4, s = 5$ *or* 6 *and* $t \geq 7$
3. $k = 4, s = 7$ *and* $t \geq 9$
4. $k = 4, s \geq 8$ *and* $t \geq 7$
5. $k \geq 5, s = 3$ *and* $t \geq t_k$ *where* t_k *is a constant only depends on* k
6. $k \geq 5, s = 5$ *and* $t \geq 7$
7. $k \geq 5, s \geq 6$ *and* $t \geq 6$.

In this paper, we consider the LIST 3-COLORING PROBLEM for P_t-free graphs with no induced 1-subdivision of $K_{1,s}$. We use SDK_s to denote the one-subdivision of $K_{1,s}$. The main result is the following:

Theorem 9. *For all positive integers s and t, the* LIST 3-COLORING PROBLEM *can be solved in polynomial time for the class of (SDK_s, P_t)-free graphs.*

2 Preliminaries

We need two theorems: the first one is the famous Ramsey Theorem [9], and the second is a result of Edwards [3]:

Theorem 10 ([9]). *For each pair of positive integers k and l, there exists an integer $R(k,l)$ such that every graph with at least $R(k,l)$ vertices contains a clique with at least k vertices or an independent set with at least l vertices.*

Theorem 11 ([3]). *Let G be a graph, and let L be a list assignment for G such that $|L(v)| \leq 2$ for all $v \in V(G)$. Then a coloring of (G, L), or a determination that none exists, can be obtained in time $O(|V(G)| + |E(G)|)$.*

Let G be a graph with list assignment L. For $X \subseteq V(G)$ we denote by $G|X$ the subgraph induced by G on X, by $G \setminus X$ the graph $G|(V(G) \setminus X)$ and by $(G|X, L)$ the list coloring problem where we restrict the domain of the list assignment L to X. For $v \in V(G)$ we write $N_G(v)$ (or $N(v)$ when there is no danger of confusion) to mean the set of vertices of G that are adjacent to v. For $X \subseteq V(G)$ we write $N_G(X)$ (or $N(X)$ when there is no danger of confusion) to mean $\bigcup_{v \in X} N(v)$. We say that $D \subseteq V(G)$ is a *dominating set* of G if for every vertex $v \in G \setminus D$, $N(v) \cap D \neq \emptyset$. By Theorem 11, the following corollary immediately follows.

Corollary 12. *Let G be a graph, L be a 3-list assignment for G and let D be a dominating set of G. Then a coloring of (G, L), or a determination that (G, L) is not colorable, can be obtained in time $O(3^{|D|}(|V(G)| + |E(G)|))$.*

Proof. For every coloring c of $(G|D, L)$, in time $O(|E(G)|)$ we can define a list assignment L_c of G as follows: if $v \in D$ we set $L_c(v) = \{c(v)\}$ and if $v \notin D$ we can pick $u \in N(v) \cap D$ by the definition of a dominating set and set $L_c(v) = L(v) \setminus c(u)$. Let $\mathcal{L} = \{L_c : c$ is a coloring of $(G|D, L)\}$, then clearly $|\mathcal{L}| \leq 3^{|D|}$ and (G, L) is colorable if and only if there exists a $L_c \in \mathcal{L}$ such that (G, L_c) is colorable. For every $L_c \in \mathcal{L}$, by construction $|L_c(v)| \leq 2$ for every $v \in G$ and hence by Theorem 11, a coloring of (G, L_c), or a determination that none exists, can be obtained in time $O(|V(G)| + |E(G)|)$. Therefore a coloring of (G, L), or a determination that (G, L) is not colorable, can be obtained in time $O(3^{|D|}(|V(G)| + |E(G)|))$. □

3 The Algorithm

Let s and t be positive integers, and let $G = (V, E)$ be a connected (P_t, SDK_s, K_4)-free graph. Pick an arbitrary vertex $a \in V$ and let $S_1 = \{a\}$. For $v \in V$, let $d(v)$ be the distance from v to a. For $i = 1, 2, \ldots, t-2$, we define the set S_{i+1} as follows:

- Let $B_i = N(S_i), W_i = V \setminus (B_i \cup S_i)$.
- Write $S_i = \{v_1, v_2, \ldots, v_{|S_i|}\}$ and define

$$B_i^j = \left\{ v \in \left(B_i \setminus \bigcup_{k=1}^{j-1} B_i^k \right) : v \text{ is adjacent to } v_j \right\}$$

  for $j = 1, 2, \ldots |S_i|$. Then $B_i = \bigcup_{j=1}^{|S_i|} B_i^j$.
- For $j = 1, 2, \ldots, |S_i|$, let $X_i^j \subseteq B_i^j$ be a minimal vertex set such that for every $w \in W_i$, if $N(w) \cap B_i^j \neq \emptyset$, then $N(w) \cap X_i^j \neq \emptyset$. Let $X_i = \bigcup_{j=1}^{|S_i|} X_i^j$.

- Let $S_{i+1} = S_i \cup X_i$.

It is clear that we can compute S_{t-1} in $O(t|V|^2)$ time. Next, we prove some properties of this construction.

Lemma 13. *For $i = 1, 2, \dots, t-2$, $|S_{i+1}| \leq |S_i|(1 + R(4, R(4, s)))$.*

Proof. It is sufficient to show that for each $\ell = 1, 2, \dots, |S_i|$, $|X_i^\ell| \leq R(4, R(4, s))$. Suppose not, $|X_i^\ell| = K > R(4, R(4, s))$ for some $\ell \in \{1, 2 \dots, |S_i|\}$. Let $X_i^\ell = \{x_1, x_2, \dots, x_K\}$. By the minimality of X_i^ℓ, for $j = 1, 2, \dots, K$, there exists $y_j \in W_i$ such that $N(y_j) \cap X_i^\ell = \{x_j\}$. Since G is K_4-free, by Theorem 10, there exists an independent set $X' \subseteq X_i^\ell$ of size $R(4, s)$. We may assume $X' = \{x_1, x_2, \dots, x_{R(4,s)}\}$. Let $Y' = \{y_1, y_2, \dots, y_{R(4,s)}\}$. Again by Theorem 10, there exists an independent set $Y'' \subseteq Y'$ of size s. We may assume $Y'' = \{y_1, y_2, \dots, y_s\}$ and let $X'' = \{x_1, x_2, \dots, x_s\}$. Then $G[\{v_\ell\} \cup X'' \cup Y'']$ is isomorphic to SDK_s, a contradiction. □

For convenience, we set $S_0 = \emptyset$, $B_0 = \{a\}$ and $B_{t-1} = N(S_{t-1})$. Then by construction it is clear that $S_i \subseteq \bigcup_{k=0}^{i-1} B_k$ for every $1 \leq i \leq t-1$. Moreover, the following property holds.

Lemma 14. *For $i = 0, 1, \dots, t-2$, $B_{i+1} \setminus (B_i \cup S_i) = \{v : d(v) = i+1\}$*

Proof. We use induction to prove this lemma. It is clear that for $i = 0$, $B_1 = N(a) = \{v : d(v) = 1\}$.

Now suppose this lemma holds for $i < k$, where $k \in \{1, 2 \dots, t-2\}$. First we show that for every $v \in B_{k+1} \setminus (B_k \cup S_k)$, $d(v) = k+1$. By construction $v \in W_k$, hence $d(v) > k$ by induction. Since $v \in B_{k+1} \setminus B_k$, v has a neighbor w in $S_{k+1} \setminus S_k \subseteq B_k$; and thus $d(v) \leq d(w) + 1 \leq k+1$.

Now let $v \in V$ with $d(v) = k+1$. It follows that $v \notin (B_k \cup S_k)$, and $v \in B_{k+1} \cup W_{k+1}$, and v has a neighbor $w \in V$ with $d(w) = k$. By induction, it follows that $v \in W_k$ and $w \in B_k$. Let $j \in \mathbb{N}$ such that $w \in B_k^j$. Since $v \in W_k$ and $N(w) \cap B_k^j \neq \emptyset$, it follows that v has a neighbor in $X_k^j \subseteq X_k \subseteq S_{k+1}$, and therefore $v \in B_{k+1}$, as required. This finishes the proof of Lemma 14. □

By applying Lemma 13 and Lemma 14, we deduce the following properties of S_{t-1}.

Lemma 15.
1. *There exists a constant $M_{s,t}$ which only depends on s and t such that $|S_{t-1}| \leq M_{s,t}$.*
2. *$W_{t-1} = V \setminus (S_{t-1} \cup N(S_{t-1})) = \emptyset$.*

Proof. Since we start with $|S_1| = 1$, by applying Lemma 13 $t-2$ times, it follows that $|S_{t-1}| \leq (1 + R(4, R(4, s)))^{t-2}$. Let $M_{s,t} = (1 + R(4, R(4, s)))^{t-2}$, then the first claim holds.

Suppose the second claim does not hold. From Lemma 14, it follows that $\{v : d(v) \leq t-1\} \subseteq S_{t-1} \cup N(S_{t-1})$. But if $w \in V$ satisfies $d(w) \geq t$, then a shortest w-a-path is an induced path of at least t vertices, a contradiction. Thus the second claim holds. □

We are now ready to prove our main result, which we rephrase here:

Theorem 16. *Let $M_{s,t} = (1 + R(4, R(4, s)))^{t-2}$. There exists an algorithm with running time $O(|V(G)|^4 + t|V(G)|^2 + 3^{M_{s,t}}(V(G) + E(G)))$ with the following specification.*

Input: *A (SDK_s, P_t)-free graph G and a 3-list assignment L for G.*

Output: *A coloring of (G, L), or a determination that (G, L) is not colorable.*

Proof. We may assume that G is connected, since otherwise we can run the algorithm for each component of G independently. In time $O(|V(G)|^4)$ we can determine that either (G, L) is not colorable, or G is K_4-free. If G is K_4-free, we can construct S_{t-1} in $O(tn^2)$ time as stated above. Then by Lemma 15, S_{t-1} is a dominating set of G and $|S_{t-1}| \leq M_{s,t}$. Now the theorem follows from Corollary 12. □

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