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List 3-Coloring Graphs with No Induced $P_6 + rP_3$

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Abstract

For an integer t, we let P_t denote the t-vertex path. We write $H + G$ for the disjoint union of two graphs H and G, and for an integer r and a graph H, we write rH for the disjoint union of r copies of H. We say that a graph G is *H-free* if no induced subgraph of G is isomorphic to the graph H. In this paper, we study the complexity of k-coloring, for a fixed integer k, when restricted to the class of H-free graphs with a fixed graph H. We provide a polynomial-time algorithm to test if, for fixed r, a $(P_6 + rP_3)$-free is three-colorable, and find a coloring if one exists. We also solve the list version of this problem, where each vertex is assigned a list of possible colors, which is a subset of $\{1, 2, 3\}$. This generalizes results of Broersma, Golovach, Paulusma, and Song, and results of Klimošová, Malik, Masařík, Novotná, Paulusma, and Slívová. Our proof uses a result of Ding, Seymour, and Winkler relating matchings and hitting sets in hypergraphs. We also prove that the problem of deciding if a $(P_5 + P_2)$-free graph has a k-coloring is NP-hard for every fixed $k \geq 5$.

Keywords Graph coloring · Forbidden induced subgraph · Polynomial algorithm

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1 Introduction

All graphs in this paper are finite and simple. We use $[k]$ to denote the set $\{1, \dots, k\}$. Let G be a graph. A *k-coloring* of G is a function $f : V(G) \to [k]$ such that for every edge $uv \in E(G), f(u) \neq f(v)$, and G is *k-colorable* if G has a k-coloring. The k -COLORING PROBLEM is the problem of deciding, given a graph G, if G is k-colorable. This problem is well-known to be NP-hard for all $k \geq 3$.

A function $L : V(G) \to 2^{[k]}$ that assigns a subset of $[k]$ to each vertex of a graph G is a *k-list assignment* for G. For a k-list assignment L, a function $f : V(G) \to [k]$ is a *coloring of (G, L)* if f is a k-coloring of G and $f(v) \in L(v)$ for all $v \in V(G)$. We say that a graph G is *L-colorable*, and that the pair (G, L) is *colorable*, if (G, L) has a coloring. The LIST k -COLORING PROBLEM is the problem of deciding, given a graph G and a k-list assignment L, if (G, L) is colorable. Since this generalizes the k-coloring problem, it is also NP-hard for all $k \geq 3$.

Let G be a graph, and let $X \subseteq V(G)$. We denote by $G|X$ the subgraph of G induced by X. For a k-list assignment L for G, a *k-precoloring* (G, L, X, f) of (G, L) is a function $f : X \to \mathbb{N}$ for a set $X \subseteq V(G)$ such that $f(v) \in L(v)$ for every $v \in X$, and f is a k-coloring of $G|X$. A *k-precoloring extension* for (G, L, X, f) is a k-coloring g of (G, L) such that $g|_X = f|_X$. The k -PRECOLORING EXTENSION PROBLEM is the problem of deciding if a given k-precoloring (G, L, X, f) of (G, L) extends to a coloring of (G, L).

We denote by P_t the path with t vertices. Given a path P, its *interior* is the set of vertices that have degree 2 in P. A *P_t in a graph G* is a sequence $v_1 - \dots - v_t$ of pairwise distinct vertices where for $i, j \in [t]$, v_i is adjacent to v_j if and only if $|i - j| = 1$. We denote by $V(P_t)$ the set $\{v_1, \dots, v_t\}$, and if $a, b \in V(P)$, say $a = v_i$ and $b = v_j$ and $i < j$, then $a - P - b$ is the path $v_i - v_{i+1} - \dots - v_j$. We denote by $P_6 + rP_3$ the graph with $r + 1$ components, one of which is a P_6, and each of the others is a P_3.

For two graphs H, G we say that G is *H-free* if no induced subgraph of G is isomorphic to H. In this paper, we use the terms "polynomial time" and "polynomial size" to mean "polynomial in $|V(G)|$", where G is the input graph. Since the k -COLORING PROBLEM and the k -PRECOLORING EXTENSION PROBLEM are NP-hard for $k \geq 3$, their restrictions to H-free graphs, for various H, have been extensively studied. In particular, the following is known:

Theorem 1 ([1]) *Let H be a (fixed) graph, and let $k > 2$. If the k -COLORING PROBLEM can be solved in polynomial time when restricted to the class of H-free graphs, then every connected component of His a path.*

In this paper we focus on the case when $k = 3$. In this case, the converse of Theorem 1 may be true (it is known to be false for $k \geq 4$ unless P=NP: for example, 4-coloring P_7-free graphs was shown to be NP-complete in[2]), since the following question is still open:

Question 1 Is it true that for every (fixed) integer $t > 0$, the 3-COLORING PROBLEM can be solved in polynomial time when restricted to the class of P_t-free graphs?

Below are the positive results we know in this direction:

Theorem 2 ([3]) *The* LIST 3-COLORING PROBLEM*can be solved in polynomial time for the class of P_7-free graphs.*

Theorem 3 ([4]) *If the* LIST 3-COLORING PROBLEM *can be solved in polynomial time for the class of H-free graphs, then the* LIST 3-COLORING PROBLEM *can be solved in polynomial time for the class of* $(H + sP_1)$*-free graphs for every* $s \geq 0$.

Theorem 4 ([5]) *The* LIST 3-COLORING PROBLEM *can be solved in polynomial time for the class of* $(P_5 + P_2)$*-free graphs, and for the class of* $(P_4 + P_3)$*-free graphs.*

Theorem 5 ([6]) *The* 3-PRECOLORING EXTENSION PROBLEM*can be solved in polynomial time for the class of* $(rP3)$*-free graphs for every* $r \geq 1$.

Theorem 6 ([7]) *The* LIST 3-COLORING PROBLEM *can be solved in subexponential time* $2^{O(\sqrt{t|V(G)|\log(|V(G)|)})}$ *where the input graph G is* P_t*-free.*

Our main result is the following, which simultaneously generalizes Theorem 5 and both parts of Theorem 4:

Theorem 7 *The* LIST 3-COLORING PROBLEM*can be solved in polynomial time for the class of* $(P_6 + rP_3)$*-free graphs for every* $r \geq 0$.

This immediately implies that the 3-COLORING PROBLEM can be solved in polynomial time for the class of $(P_6 + rP_3)$-free graphs. It also gives a classification for the complexity of the 3-COLORING PROBLEM and LIST-3-COLORING PROBLEM for the class of H-free graphs for all graphs H on at most eight vertices except for P_8 and $P_4 + P_4$ (note that the result for $P_7 + P_1$ follows from Theorem 2 and Theorem 7 in[8]).

In contrast, we also show that

Theorem 8 *The* k -COLORING PROBLEM *restricted to* $(P_5 + P_2)$*-free graphs is NP-hard for* $k \geq 5$.

This paper is organized as follows. Section 2 is a collection of tools for general P_t -free graphs that we use in the proof. Section 3 describes the main object we work with, an "r-seeded precoloring". An r-seeded precoloring consists of a graph G, a precolored subset S of vertices, and a list of allowed colors for every vertex of $V(G)$. Sections 4, 5 and 6 contain a sequence of theorems that start with a general r-seeded precoloring, and, by "guessing" (by exhaustive enumeration) the coloring of a certain bounded-size set of vertices, transform it into a precoloring that is "tractable". Here by tractable we mean a precoloring for which the precoloring extension problem can be solved in polynomial time. Section 7 combines the results of the previous three sections to obtain a proof of Theorem 7. Finally Sect. 8 is devoted to the proof of Theorem 8.

2 Tools

In this section we discuss several tools that we repeatedly use in this paper. The first is a result of [9]:

Lemma 1 ([9]) *Let G be a graph, and let L be a list assignment for G such that $|L(v)| \leq 2$ for all $v \in V(G)$. Then a coloring of (G, L), or a determination that none exists, can be obtained in time $O(|V(G)| + |E(G)|)$.*

We also need a modification of Lemma 1. For a graph G, a coloring c of G, and a set $X \subseteq V(G)$, we say that X is *monochromatic in c* if $c(u) = c(v)$ for all $u, v \in X$. Let L be a list assignment for G, and $\mathcal{X}$ a set of subsets of $V(G)$. We say that the triple $(G, L, \mathcal{X})$ is *colorable* if there is a coloring c of (G, L) such that X is monochromatic in c for all $X \in \mathcal{X}$. We need the following.

Lemma 2 ([10]) *Let G be a graph, and let L be a list assignment for G such that $|L(v)| \leq 2$ for all $v \in V(G)$. Let $\mathcal{X}$ be a set of subsets $V(G)$ where $|\mathcal{X}|$ is polynomial. Then a coloring of $(G, L, \mathcal{X})$, or a determination that none exists, can be obtained in polynomial time.*

Note that if two sets X and X' with $X \cap X' \neq \emptyset$ are monochromatic in a coloring c, then $X \cup X'$ is also monochromatic in c. Thus, given a triple $(G, L, \mathcal{X})$ as in Lemma 2 we can compute in polynomial time a triple $(G, L, \mathcal{X}')$ where the sets in $\mathcal{X}'$ are pairwise disjoint and $(G, L, \mathcal{X})$ has a coloring if and only if $(G, L, \mathcal{X}')$ does. Thus Lemma 2 follows from

Lemma 3 ([3, 10]) *Let G be a graph, and let L be a list assignment for G such that $|L(v)| \leq 2$ for all $v \in V(G)$. Let $\mathcal{X}$ be a set of pairwise disjoint subsets of $V(G)$. Then a coloring of $(G, L, \mathcal{X})$, or a determination that none exists, can be obtained in time $O(|V(G)| + |E(G)|)$.*

Next we present a result from[11]. A *hypergraph H* consists of a finite set $V(H)$ of vertices and a set $E(H)$ of non-empty subsets of $V(H)$ called *hyperedges*. A *matching* in H is a set of pairwise disjoint hyperedges, and a *hitting set* in H is a set of vertices meeting every hyperedge. We denote by $\nu(H)$ the maximum size of a matching in H, and by $\tau(H)$ the minimum size of a hitting set in H. The parameters $\nu(H)$ and $\tau(H)$ are well-known, but we need one more. We denote by $\lambda(H)$ the maximum $k \geq 2$ such that there are edges $e_1, \dots, e_k \in E(H)$ with the property that for every i, j with $1 \leq i < j \leq k$ there exist $v_{i,j} \in V(H)$ satisfying $\{h : 1 \leq h \leq k \text{ such that } v_{i,j} \in e_h\} = \{i, j\}$. If there is no such k, we set $\lambda(H) = 2$. We need the following:

Lemma 4 ([11]) *For every hypergraph H, $\tau(H) \leq 11\lambda(H)^2(\lambda(H) + \nu(H) + 3)\binom{\lambda(H)+\nu(H)}{\nu(H)}^2$.*

Let us discuss how we apply Lemma 4. Let G be a graph. A set $X \subseteq V(G)$ is *stable* if no edge of G has both its ends in X. Let $A \subseteq V(G)$. An *attachment* of A is a vertex of $V(G) \setminus A$ with a neighbor in A. For $B \subseteq V(G) \setminus A$ we denote by $B(A)$ the set of attachments of A in B. If $F = G|A$, we sometimes write $B(F)$ to mean $B(V(F))$. Note that the following result is for general P_t-free graphs, and thus we expect that it will have further applications in the context of Question 1.

Lemma 5 *Let t be an integer and let G be a P_t-free graph, and let $X, Y \subseteq V(G)$ be disjoint, where X is stable and every component of Y has size at most p. Let $\mathcal{Z}$ be a set of connected subsets of size q of Y, each of which has an attachment in X. Let H be a hypergraph with vertex set X and hyperedge set $\{X(Z) : Z \in \mathcal{Z}\}$. Then $\lambda(H) \leq \binom{p}{q}\lceil\frac{t+1}{2}\rceil$.*

Proof Let $\lambda = \lambda(H)$ and let $e_1, \dots, e_\lambda$, and $\{v_{i,j}\}_{1\leq i<j\leq \lambda}$ be as in the definition of $\lambda(H)$. Suppose $\lambda > \binom{p}{q}\lceil\frac{t+1}{2}\rceil$. For $i \in \{1, \dots, \lambda\}$, let $Y_i \subseteq \mathcal{Z}$ be such that $e_i = X(Y_i)$. Define a graph F with vertex set $\{e_1, \dots, e_\lambda\}$ and such that e_i is adjacent to e_j in F if either $Y_i \cap Y_j \neq \emptyset$, or in G there is an edge with one end in Y_i and the other end in Y_j. Then $deg_F(e_i) \leq \binom{p}{q} - 1$ for every $e_i \in V(F)$. It follows that F is $\binom{p}{q}$-colorable, and so F has a stable set S with $|S| \geq \frac{\lambda}{\binom{p}{q}} > \lceil\frac{t+1}{2}\rceil$. Write $m = \lceil\frac{t+1}{2}\rceil$. Renumbering if necessary, we may assume that $e_1, \dots, e_m \in S$. Let q_1 be a neighbor of $v_{1,2}$ in Y_1, and let q_m be a neighbor of $v_{m-1,m}$ in Y_m. For $i \in \{2, \dots, m-2\}$ let Q_i be a path from $v_{i-1,i}$ to $v_{i,i+1}$ with interior in Y_i (such a path exists by the definition of H). Now, since X is stable, $q_1 - v_{1,2} - Q_2 - v_{2,3} - \cdots - v_{m-2,m-1} - Q_{m-1} - v_{m-1,m} - q_m$ is a path of length at least t in G, a contradiction. This proves Lemma 5. □

We deduce

Lemma 6 *Let r be an integer, and let G be a $(P_6 + rP_3)$-free graph. Let H be a hypergraph as in Lemma 5. Let $C = \binom{p}{q}(2r+4)$. Then $\tau(H) \leq 11C^2(C + \nu(H) + 3)\binom{C+\nu(H)}{\nu(H)}^2$. In particular, there is a non-decreasing function $f_{r,p,q} : \mathbb{N} \to \mathbb{N}$ such that $\tau(H) \leq f_{r,p,q}(\nu(H))$.*

Proof Since $P_6 + rP_3$ is contained in P_{4r+6}, and since G is $(P_6 + rP_3)$-free, it follows that G is P_{4r+6}-free. By Lemma 5, $\lambda(H) \leq C$. But now Lemma 6 follows directly from Lemma 4. □

We finish this section with some terminology. Let G be a graph. For $X \subseteq V(G)$ we denote by $G \setminus X$ the graph $G|(V(G) \setminus X)$. If $X = \{x\}$, we write $G \setminus x$ to mean $G \setminus \{x\}$. For disjoint subsets $A, B \subset V(G)$ we say that A is *complete* to B if every vertex of A is adjacent to every vertex of B, and that A is *anticomplete* to B if every vertex of A is non-adjacent to every vertex of B. If $A = \{a\}$ we write a is complete (or anticomplete) to B to mean $\{a\}$ that is complete (or anticomplete) to B. If $a \notin B$ is not complete and not anticomplete to B, we say that a is *mixed* on B. Finally, if H is an induced subgraph of G and $a \in V(G) \setminus V(H)$, we say that a is *complete to,* *anticomplete to*, or *mixed on H* if a is complete to, anticomplete to, or mixed on $V(H)$, respectively. For $v \in V(G)$ we write $N_G(v)$ (or $N(v)$ when there is no danger of confusion) to mean the set of vertices of G that are adjacent to v. Observe that since G is simple, $v \notin N(v)$. For $X \subseteq V(G)$ a *component* of X (or of $G|X$) is the vertex set of a maximal connected subgraph of $G|X$.

Let L be a list assignment for G. We denote by $X^0(L)$ the set of all vertices v with $|L(v)| = 1$. For $X \subseteq V(G)$, we write $(G|X, L)$ to mean the list coloring problem where we restrict the domain of the list assignment L to X. Let $X \subset X^0(L)$, and let $Y \subset V(G)$. We say that a list assignment M is *obtained from L by updating Y from X* if $M(v) = L(v)$ for every $v \notin Y$, and $M(v) = L(v) \setminus \bigcup_{x \in N(v) \cap X} L(x)$ for every $v \in Y$. If $Y = V(G)$, we say that M is *obtained from L by updating from X*. If M is obtained from L by updating from $X^0(L)$, we say that M is *obtained from L by updating*. For $v \in X^0(L)$ we will not distinguish between the set $L(v)$ and its unique element. For $X \subseteq X^0(L)$, we will regard L as a coloring of $G|X$. Let $L_0 = L$, and for $i \geq 1$ let L_i be obtained from L_{i-1} by updating. If $L_i = L_{i-1}$, we say that L_i is *obtained from L by updating exhaustively*. Since $0 \leq \sum_{v \in V(G)} |L_j(v)| < \sum_{v \in V(G)} |L_{j-1}(v)| \leq 3|V(G)|$ for all $j < i$, it follows that $i \leq 3|V(G)|$ and thus L_i can be computed from L in polynomial time. This observation allows us to set the following convention.

Lemma 7 *If G is a graph, L a list assignment for G, and $v \in V(G)$, then there is no $u \in N(v) \cap X^0(L)$ with $L(u) \subseteq L(v)$.*

A *seagull S* in G is a P_3 $a - b - c$ in G. We write $V(S) = \{a, b, c\}$. The vertices a and c are called the *wings* of the seagull, and b is the *body* of the seagull. For $X, Y \subseteq V(G), S$ is an X-seagull if $V(S) \in X$, and S is an (X, Y)-seagull if S has one wing in X, and the body and the other wing in Y. A *flock* is a set of pairwise disjoint seagulls that are pairwise anticomplete to each other. The *size* of a flock is its cardinality.

3 Seeded Precolorings

Given a $(P_6 + rP_3)$-free graph G with a 3-list assignment L, our strategy for checking if (G, L) is colorable involves several steps, each of which consists of choosing a small subset $S \subseteq V(G)$, precoloring S and updating the lists. In view of Lemma 1, if we arrive at a situation where every vertex has list of size at most two, then we are done. We show, roughly, that this can always be achieved. To keep track of the precoloring and updating process, we define the following object.

An *r-seeded precoloring* of the pair (G, L') is a triple $P = (G, L, S)$ such that

1. L is a list assignment for G, and $L(v) \subseteq L'(v)$ for every $v \in V(G)$, and $|L(v)| = 1$ for every $v \in S$,
2. $(G, L', S, L|_S)$ is a precoloring of (G, L');
3. $G|S$ contains an induced $P_6 + (r-1)P_3$.

We call S the *seed* of P, and write $S(P)$ to mean S. Similarly, we use the notation $G(P)$ and $L(P)$. The *boundary* $B(P)$ of P is the set of all vertices $v \in V(G)$ with $|L(v)| = 2$ and such that v has a neighbor $s \in S$ with $L(s) = \{1, 2, 3\} \setminus L(v)$. We denote by $B(P, i)$ the set of vertices $b \in B(P)$ with $i \in L(b)$, and by $B(P)_{ij}$ the set of vertices $b \in B(P)$ with $L(b) = \{i, j\}$. Finally, the *wilderness* $W(P)$ of P is the set $V(G) \setminus (X^0(L) \cup B(P))$. The reason for the name "wilderness" is that every $v \in V(G)$ with $|L(v)| = 3$ belongs to $W(P)$, and so by Lemma 1 $W(P)$ is the set where the algorithmic difficulty lies.

We observe the following.

Lemma 8 *Let r be an integer, G a $(P_6 + rP_3)$-free graph, L' a 3-list assignment for G, and let $P = (G, L, S)$ be an r-seeded precoloring of (G, L'). Assume that G does not contain a clique of size four. Then each component of $W(P)$ is a clique of size at most three.*

Proof Since $G|S$ contains an induced $P_6 + (r-1)P_3$, it follows that $G|W$ is P_3-free. Consequently every component of W is a clique, and since G has no clique of size four, Lemma 8 follows. □

Let $P = (G, L, S)$ be an r-seeded precoloring. Let $U \subseteq V(G)$ and a let c be a coloring of $G|U$. We say that the seeded precoloring $P' = (G, L', S')$ is obtained from P by *moving U to the seed with c* if $S' = S \cup U$, and L' is obtained by updating exhaustively from the list assignment L'', defined as follows: $L''(u) = c(u)$ for every $u \in U$, and $L''(v) = L(v)$ for every $v \in V(G) \setminus U$.

For an r-seeded precoloring P and a collection $\mathcal{L}$ of r-seeded precolorings, we say that $\mathcal{L}$ is an *equivalent collection* for P (or that P is *equivalent* to $\mathcal{L}$) if P has a precoloring extension if and only if at least one of the precolorings in $\mathcal{L}$ has a precoloring extension, and a precoloring extension of P can be constructed from a precoloring extension of a member of $\mathcal{L}$ in polynomial time.

Let $P = (G, L, S)$ be an r-seeded precoloring. A *type* is a non-empty monochromatic subset of S. Thus for every $b \in B(P), N(b) \cap S$ is a type; we call $N(b) \cap S$ the *type of b*. For $S' \subseteq S$ we denote by $B(P, S')$ the set of all vertices of $B(P)$ whose type is a superset of S'. In what follows we will often need to handle each type of S separately, and so it is important that we keep track of the size of the seed in every precoloring we consider.

4 Nice and Easy Precolorings

An r-seeded precoloring P is *nice* if no vertex of $B(P)$ is mixed on an edge of $W(P)$, and it is *easy* if $G|(B(P) \cup W(P))$ is P_6-free. Our first goal is to show that an r-seeded precoloring P can be replaced by an equivalent collection of precolorings each of which is either nice or easy, such that the size of the collection is polynomial, and the size of the seed of each of its members is bounded by a function of $|S(P)|$. For a precoloring extension c of P, we will define several "characteristics" of c. While we cannot (in polynomial time) "try" all possible ways to extend P to a coloring of G, we can exhaustively enumerate all possible characteristics of such extensions, and that turns out to be enough for our purposes.

Thus let $P = (G, L, S)$ be an r-seeded precoloring and let c be a precoloring extension of P. For every $i \in \{1, 2, 3\}$ we define the hypergraph $H(P, i, c)$ as follows. The vertex set $V(H(P, i, c)) = \{b \in B(P, i) : c(b) = i\}$. Next we construct the hyperedges. Let K be the set of all edges w_1w_2 of G with both ends in $W(P)$ such that some vertex of $V(H(P, i, c))$ is mixed on $\{w_1, w_2\}$. For every $e = w_1w_2 \in K$, let $h(e)$ be the set of attachments of $\{w_1, w_2\}$ in $V(H(P, i, c))$. Then $\{h(e) : e \in K\}$ is the set of the hyperedges of $H(P, i, c)$.

Lemma 9 *For every integer $r > 0$there is a function $f : \mathbb{N} \to \mathbb{N}$ (that depends on r) with the following properties. Let G be a $(P_6 + rP_3)$-free graph with no clique of size four, $P = (G, L, S)$ an r-seeded precoloring, and let c be a coloring of (G, L). Write $M = 2^{|S|}(r + 6)$. Then for every $i \in \{1, 2, 3\}$ either*

1. *there exists $X \subseteq V(H(P, i, c))$ with $|X| \leq f(3M)$ where for every edge w_1w_2 of $G|W(P)$ such that some vertex of $V(H(P, i, c))$ is mixed on $\{w_1, w_2\}$, at least one of w_1, w_2 has a neighbor in X, or*
2. *there exists a flock $F = \{a_1 - b_1 - c_1, \dots, a_M - b_M - c_M\}$ where for every $i, a_i \in V(H(P, i, c))$ and $b_i, c_i \in W(P)$, and such that every vertex of $V(H(P, i, c))$ has a neighbor in $\{b_i, c_i\}$ for at most one value of i.*

Proof By Lemma 8 every component of W is a clique of size at most three. Let $f = f_{r,3,2}$ be as in Lemma 6. Applying Lemma 6 to $H = H(P, i, c)$ with $p = 3$ and $q = 2$, we deduce that either $\nu(H) \geq 3M$ or $\tau(H) \leq f(3M)$.

Suppose first that $\nu(H) \geq 3M$. Let $b_1c_1, \dots, b_{3M}c_{3M}$ be edges of $G|W$ such that $M = \{h(b_1c_1), \dots, h(b_{3M}c_{3M})\}$ is a matching of H. Since every component of $G|W$ has size at most three, we may assume that the edges $b_1c_1, \dots, b_Mc_M$ are all in distinct components of $G|W$. It follows from the definition of $h(b_jc_j)$ (using symmetry) that for every j there exists $a_j \in V(H(P, i, c))$ such that $a_j - b_j - c_j$ is a seagull. Moreover, since M is a matching of H, no $v \in V(H(P, i, c)$ belongs to more than one $h(b_jc_j)$, and therefore every vertex of $V(H(P, i, c))$ has a neighbor in $\{b_j, c_j\}$ for at most one value of j. This proves that if $\nu(H) \geq 3M$, then Lemma 9.2 holds.

Thus we may assume that $\tau(H) \leq f(3M)$. Letting $X \subseteq V(H(P, i, c))$ be a hitting set for H, we immediately see that Lemma 9.1 holds. □

Given an r-seeded precoloring $P, i \in \{1, 2, 3\}$ and a precoloring extension c of P, we say that an *M-characteristic* of P, i, c (denoted by $char_M(P, i, c)$) is X if Lemma 9.1 holds for P, i and c, and F if Lemma 9.2 holds for P, i and c and Lemma 9.1 does not hold for P, i and c. We denote by $V(char_M(P, i, c))$ the set of all the vertices involved in $char_M(P, i, c)$.

We also need a version of the hypergraph above for each type, as follows. For every type $T \subseteq S$ and every $i \in \{1, 2, 3\}$ we define the hypergraph $H(P, T, i, c)$. The vertex set $V(H(P, T, i, c)) = \{b \in B(P, T) : c(b) = i\}$. Next we construct the hyperedges. Let K be set of all edges w_1w_2 of G with both ends in $W(P)$ such that some vertex of $V(H(P, T, i, c))$ is mixed on $\{w_1, w_2\}$. For every $e = w_1w_2 \in K$, let $h(e)$ be the set of attachments of $\{w_1, w_2\}$ in $V(H(P, T, i, c))$. Then $\{h(e) : e \in K\}$ is the set of the hyperedges of $H(P, T, i, c)$.

Lemma 10 *For every integer $r > 0$ there is a function $f : \mathbb{N} \to \mathbb{N}$ (that depends on r) with the following properties. Let G a $(P_6 + rP_3)$-free graph with no clique of size four, $P = (G, L, S)$ an r-seeded precoloring and let c be a coloring of (G, L). Then for every type T of S either*

1. *there exists $X \subseteq V(H(P, T, i, c))$ with $|X| \leq f(6)$ where for every edge w_1w_2 of $G|W(P)$ such that some vertex of $V(H(P, T, i, c))$ is mixed on $\{w_1, w_2\}$, at least one of w_1, w_2 has a neighbor in X,*
2. *there exists a flock $F = \{a_1 - b_1 - c_1, a_2 - b_2 - c_2\}$ with $a_1, a_2 \in V(H(P, T, i, c))$, and $b_1, b_2, c_1, c_2 \in W(P)$.*

Proof By Lemma 8 every component of W is a clique of size at most three. Let $f = f_{r,3,2}$ be as in Lemma 6. Applying Lemma 6 to $H = H(P, T, i, c)$ with $p = 3$ and $q = 2$, we deduce that either $\nu(H) \geq 6$ or $\tau(H) \leq f(6)$. Suppose first that $\nu(H) \geq 6$. Let $b_1c_1, \dots, b_6c_6$ be edges of $G|W$ such that $M = \{h(b_1c_1), ..., h(b_6c_6)\}$ is a matching of H. Since every component of $G|W$ has size at most three, we may assume that b_1c_1 and b_2c_2 are in different components of $G|W$. It follows from the definition of $h(b_ic_i)$ (using symmetry) that for every i there exists $a_i \in V(H(P, T, i, c))$ such that $a_i - b_i - c_i$ is a seagull, and Lemma 10.2 holds. Thus we may assume that $\tau(H) \leq f(6)$. Letting $X \subseteq V(H(P, T, i, c))$ be a hitting set for H, we immediately see that Lemma 10.1 holds. □

Given an r-seeded precoloring P, a type T of $S(P), i \in \{1, 2, 3\}$ and a precoloring extension c of P, we say that an 2-*characteristic* of P, T, i, c (denoted by $char_2(P, T, i, c)$) is X if Lemma 10.1 holds for P, T, i and c, and F if Lemma 10.2 holds for P, T, i and c and Lemma 10.1 does not hold for P, T, i and c. We denote by $V(char_2(P, T, i, c))$ the set of all the vertices involved in $char_2(P, T, i, c)$.

We can now prove the main result of the section.

Lemma 11 *There exists a function $g_1 : \mathbb{N} \to \mathbb{N}$ with the following properties. Let G be a $(P_6 + rP_3)$-free graph with no clique of size four, and let $P = (G, L, S)$ be an r-seeded precoloring of G. There is a collection $\mathcal{L}$ of r-seeded precolorings such that*

1. *$\mathcal{L}$ is equivalent to P*
2. *every $P' \in \mathcal{L}$ is either nice or easy*
3. *$|S(P')| \leq g_1(|S(P)|)$ for every $P' \in \mathcal{L}$*
4. *$|\mathcal{L}| \leq |V(G)|^{g_1(|S|)}$*

Moreover, given P, the collection $\mathcal{L}$ can be constructed in time $O(|V(G)|^{g_1(|S|)})$.

Proof First we define two kinds of constructions, both of which start with an arbitrary r-seeded precoloring P' and construct a new r-seeded precoloring by moving some vertices to the seed and modifying the lists. Let f be as in Lemma 9 and let $M = 2^{|S|}(r+6)$. For an r-seeded precoloring P' of (G, L) and for $i \in \{1,2,3\}$ let $smallguess(P', i)$ be the set of all subsets of $B(P', i)$ of size at most $f(3M)$, and $bigguess(P', i)$ be the set of all flocks of size M such that every seagull of the flock is $(B(P', i), W(P'))$-seagull. Let $guess(P', i) = smallguess(P', i) \cup bigguess(P', i)$. Thus $guess(P', i)$ is the set of all possible M-characteristics of a precoloring extension of P'. We say that $X_i \in guess(P', i)$ is *small* if $X_i \in smallguess(P', i)$ and that X_i is *big* if $X_i \in bigguess(P', i)$. If X_i is big, we denote by U_i the set of the wings of the flock that are contained in $B(P', i)$, by W_i the vertices of the set of the bodies and the wings of the flock that are contained in $W(P')$, and write $V(X_i) = V_i = U_i \cup W_i$. If X_i is small, we write $V(X_i) = V_i = U_i = X_i$, and $W_i = \emptyset$. Thus in both cases $W_i = V(X_i) \cap W(P')$ and $U_i = V(X_i) \cap B(P')$. A coloring c of $(G|V(X_i), L)$ is *i-consistent* if $c(v) = i$ for every $v \in U_i$.

Let $X_i \in guess(P', i)$, and let c be an i-consistent coloring of $(G|V_i, L(P'))$. Now we define the r-seeded precoloring $P'(X_i, c)$, which is the first kind of construction we introduce. Let $\tilde{P} = (G, \tilde{L}, \tilde{S})$ be obtained from P' by moving X_i to the seed with c. Next we modify $\tilde{L}$ further.

- Assume first that X_i is small. If $b \in B(\tilde{P}, i) \cap B(P')$ and b is mixed on an edge of $W(\tilde{P})$, remove i from $\tilde{L}(b)$.
- Next assume that X_i is big. Let $X_i = \{a_1 - b_1 - c_1, \dots, a_M - b_M - c_M\}$ where $U_i = \{a_1, \dots, a_M\}$. If $b \in B(\tilde{P}, i) \cap B(P')$ and b has a neighbor in $\{b_q, c_q\}$ for more than one value of q, remove i from $\tilde{L}(b)$.

Let $P'(X_i, c)$ be the r-seeded precoloring thus obtained. Note that given P', the r-seeded precoloring $P'(X_i, c)$ can be constructed in polynomial time.

For an r-seeded precoloring P' of (G, L), every type T of $S(P)$ and every $i \in \{1,2,3\} \setminus L(T)$, let $smallguess(P', T, i)$ be the set of all subsets of $B(P', T)$ of size at most $f(6)$ (here f is as in Lemma 10), and $bigguess(P', T, i)$ be the set of all flocks of size 2 such that every seagull of the flock is a $(B(P', T), W(P'))$-seagull. Please note that here we are referring to types of P, and not of P'. Let $guess(P', T, i) = smallguess(P', T, i) \cup bigguess(P', T, i)$. For $i \in \{1,2,3\}$, let $\mathcal{T}_i$ be the set of every type T of $S(P)$ such that $i \in \{1,2,3\} \setminus L(T)$ and assume $|\mathcal{T}_i| = t_i$. Let $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ and assume $|\mathcal{T}| = t$. Now, let $\mathcal{C}_i(P')$ be the set of all $2t_i$-tuples $X^i = (X_{T,i})$ where $T \in \mathcal{T}_i$ and $X_{T,i} \in guess(P', T, i)$. We say that $X_{T,i}$ is *small* if $X_{T,i} \in smallguess(P', T, i)$ and that $X_{T,i}$ is *big* if $X_{T,i} \in bigguess(P', T, i)$. If $X_{T,i}$ is big, we denote by $U_{T,i}$ the set of the wings of the flock that are contained in $B(P', T)$, by $W_{T,i}$ the vertices of the set of the bodies and the wings of the flock that are contained in $W(P')$, and write $V_{T,i} = U_{T,i} \cup W_{T,i}$. If $X_{T,i}$ is small, we write $V_{T,i} = U_{T,i} = X_{T,i}$, and $W_{T,i} = \emptyset$. Finally, let $V(X^i) = \bigcup_{T \in \mathcal{T}_i} V_{T,i}$.

A coloring c of $(G|V(X^i), L)$ is *i′-consistent* if $c(v) = i$ for every $v \in U_{T,i}$. Let $X^i \in \mathcal{C}_i(P')$ and let c be a i'-consistent coloring of $(G|V(X^i), L(P'))$. Now we define the r-seeded precoloring $P'_{X^i,c}$, which is the second kind of construction we introduce. Let $P'' = (G, L'', S'')$ be obtained from P' by moving $V(X^i)$ to the seed with c. Next we modify L'' further. For every $T \in \mathcal{T}_i$, proceed as follows.

- Assume that $X_{T,i}$ is small. If $b \in B(P'') \cap X_{T,i}$ and $T \subseteq N(b) \cap S$, and b is mixed on an edge of $W(P'')$, remove i from $L''(b)$.

Let $P'_{X^i,c}$ be the r-seeded precoloring thus obtained. Note that given P', the r-seeded precoloring $P'_{X^i,c}$ can be constructed in polynomial time.

Now we have defined two kinds of constructions and can proceed as follows.

- For every $X_1 \in guess(P, 1)$ and every 1-consistent coloring c_1 of $(G|V(X_1), L(P))$ construct $P'(X_1, c_1)$ as the first kind of construction described above. Let $\mathcal{L}_1$ be the set of precolorings thus constructed.
- For every r-seeded precoloring $P^1 \in \mathcal{L}_1$, every $X^1 \in \mathcal{C}_1(P^1)$, and every $1'$-consistent coloring c'_1 of $(G|V(X^1), L(P^1))$ construct P'_{X^1,c'_1} as the second kind of construction described above. Let $\mathcal{L}'_1$ be the set of precolorings thus constructed.
- For every r-seeded precoloring $P'^1 \in \mathcal{L}'_1$, every $X_2 \in guess(P'^1, 2)$ and every 2-consistent coloring c_2 of $(G|V(X_2), L(P'^1))$ construct $P'(X_2, c_2)$ as the first kind of construction described above. Let $\mathcal{L}_2$ be the set of precolorings thus constructed.
- For every r-seeded precoloring $P^2 \in \mathcal{L}_2$, every $X^2 \in \mathcal{C}_2(P^2)$, and every $2'$-consistent coloring c'_2 of $(G|V(X^2), L(P^2))$ construct P'_{X^2,c'_2} as the second kind of construction described above. Let $\mathcal{L}'_2$ be the set of precolorings thus constructed.
- For every r-seeded precoloring $P'^2 \in \mathcal{L}'_2$, every $X_3 \in guess(P'^2, 3)$ and every 3-consistent coloring c_3 of $(G|V(X_3), L(P'^2))$ construct $P'(X_3, c_3)$ as the first kind of construction described above. Let $\mathcal{L}_3$ be the set of precolorings thus constructed.
- For every r-seeded precoloring $P^3 \in \mathcal{L}_3$, every $X^3 \in \mathcal{C}_3(P^3)$, and every $3'$-consistent coloring c'_3 of $(G|V(X^3), L(P^3))$ construct P'_{X^3,c'_3} as the second kind of construction described above. Let $\mathcal{L}'_3$ be the set of precolorings thus constructed.

It is clear that for every r-seeded precoloring $P' \in \mathcal{L}'_3$, there exist $X_1, X_2, X_3, X^1, X^2, X^3$ chosen as above. Let $\mathcal{Q}$ be the set of all such 6-tuples $X = (X_1, X_2, X_3, X^1, X^2, X^3)$. Write $V(X) = V_1 \cup V_2 \cup V_3 \cup V(X^1) \cup V(X^2) \cup V(X^3)$. A coloring c of $(G|V(X), L)$ is *consistent* if $c_i = c|X_i$ is i-consistent and $c'_i = c|X^i$ is i'-consistent. For every $X \in \mathcal{Q}$ and every consistent coloring c of $G|V(X)$, let $Q^1_{X,c}, Q^{1'}_{X,c}, Q^2_{X,c}, Q^{2'}_{X,c}, Q^3_{X,c}$ and $Q^{3'}_{X,c}$ be the precolorings we obtain in the first, second, third, fourth, fifth and sixth bullets above respectively with the choices of X_i, X^i the same as the corresponding element in X and the choices of c_i, c'_i as a partial coloring of c. For simplicity, we denote $Q_{X,c} = Q^4_{X,c} = Q^{3'}_{X,c}$. Let $\mathcal{L}$ be the collection of all precolorings $Q_{X,c}$.

We now list several properties of $Q^i_{X,c}$ and $Q^{i'}_{X,c}$.

$$\text{Let } i \in \{1,2,3\}, j \in \{1,2,3,4\}$$
$$\begin{aligned}&1.\quad W(Q^j_{X,c}) \text{ is anticomplete to } V(\bigcup_{i\leq j} X_i).\\ &2.\quad \text{if } b \in B(Q^j_{X,c}, i) \text{ with } j \geq i \text{ then } b \text{ is anticomplete to } U_i.\\ &3.\quad \text{if } b \in B(Q^j_{X,c}, i) \text{ with } j \geq i \text{ and } X_i \text{ is big, then } b\\ &\qquad \text{has neighbors in at most one of the seagulls of } X_i.\end{aligned} \tag{1}$$

The first two statements of (1) follow from the fact that $V(\bigcup_{i\leq j} X_j) \subseteq S(Q^j_{X,c})$. By the second bullet of the first kind of construction, we deduce that if X_i is big and b has neighbors in more than one of the seagulls of X_i, then i is removed from the list of b; thus the third statement of (1) follows. This proves (1).

$$\begin{aligned}&\text{Let } i,j \in \{1,2,3\} \text{ such that } i < j. \text{Then for every } X = (X_1, X_2, X_3, X^1, X^2, X^3) \in\\ &\mathcal{Q} \text{ where } X_j \text{ is big, the following hold.}\\ &-\quad W_j \text{ is anticomplete to } V_i,\\ &-\quad \text{if } u \in U_j \text{ and } i \in L(P)(u), \text{ then } u \text{ is anticomplete to } U_i, \text{ and}\\ &-\quad \text{if } u \in U_j \text{ and } i \in L(P)(u) \text{ and } X_i \text{ is big, then}\\ &\quad u \text{ has neighbors in at most one of the seagulls of } X_i.\end{aligned} \tag{2}$$

Recall that $X_j \in guess(Q^{j-1'}_{X,c}, j)$. Since $j-1 \geq i$, $V_i \in S(Q^{j-1'}_{X,c})$, and therefore W_j is anticomplete to V_i, thus the first statement of (2) holds. Next we prove the second and third statements. Let $u \in U_j$, then $u \in B(Q^{j-1'}_{X,c}, j)$. Since $Q^{j-1'}_{X,c}$ was obtained from P by moving vertices to the seed, it follows that $u \in B(P) \cup W(P)$. Recall that by Lemma 8 each component of $W(P)$ is a clique. Since X_j is big, it follows that u is mixed on an edge of $W(P)$, and therefore $u \in B(P)$. Since $u \in B(Q^{j-1'}_{X,c}, j) \cap B(P)$, it follows that $L(P)(u) = L(Q^{j-1'}_{X,c})(u)$. Assume that $i \in L(P)(u)$. Then $L(P)(u) = L(Q^{j-1'}_{X,c})(u) = \{i,j\}$.
It follows that u has no neighbor in $S(Q^{j-1'}_{X,c})$ with color i. Since $U_i \subseteq S(Q^{j-1'}_{X,c})$ and $L(Q^{j-1'}_{X,c})(U_i) = i$, the second statement of (2) follows. By the second bullet of the first kind of construction, we deduce that if X_i is big and u has neighbors in more than one of the seagulls of X_i, then i is removed from the list of u during the process of constructing $Q^{j-1'}_{X,c}$; thus the third statement of (2) follows. This proves (2).

$$\begin{aligned}&\text{Let } i \in \{1,2,3\}, j \in \{1,2,3,4\}. \text{If } X_i \text{ is small, then no vertex } b \in B(Q^j_{X,c}, i)\\ &\text{is mixed on an edge of } W(Q^j_{X,c}) \text{ for } j \geq i. \text{ Moreover,}\\ &X_{T,i} \text{ is small for every element } X_{T,i} \text{ of } X^i.\end{aligned} \tag{3}$$

Suppose $b \in B(Q^j_{X,c}, i)$ is mixed on an edge of $W(Q^j_{X,c})$. Since $Q^j_{X,c}$ is obtained from P by moving a set of vertices to the seed, it follows that $B(Q^j_{X,c}) \subseteq B(P) \cup W(P)$. From Lemma 8 we can deduce that no vertex of $B(Q^j_{X,c}) \setminus B(P)$ is mixed on an edge of $W(P)$, and therefore $b \in B(P)$. Consequently, $L(P)(b) = L(Q^j_{X,c})(b)$. However, i

would be removed from the list of b in the construction process of $L(Q^i_{X,c})$ (according to the first bullet of the first kind of construction), thus $i \notin L(Q^i_{X,c})(b)$, a contradiction. This proves the first part of (3). Recall that $X_{T,i} \in guess(Q^i_{X,c}, T, i)$, and $bigguess(Q^i_{X,c}, T, i) = \emptyset$ by what we just proved. It follows that $X_{T,i}$ is small.

$$\text{Let } T \in \mathcal{T}, i \in \{1,2,3\} \setminus L(T) \text{ and } j \geq i. \text{If } X_{T,i} \text{ is small, then no vertex } b \in B(Q^{j'}_{X,c}) \text{ such that } T \subseteq N(b) \cap S(Q^{j'}_{X,c}) \text{ is mixed on an edge of } W(Q^{j'}_{X,c}). \tag{4}$$

Suppose $b \in B(Q^{j'}_{X,c})$ with $T \subseteq N(b) \cap S(Q^{j'}_{X,c})$ is mixed on an edge of $W(Q^{j'}_{X,c})$. Since $Q^{j'}_{X,c}$ is obtained from P by moving a set of vertices to the seed, it follows that $B(Q^{j'}_{X,c}) \subseteq B(P) \cup W(P)$. By Lemma 8, no vertex of $B(Q^{j'}_{X,c}) \setminus B(P)$ is mixed on an edge of $W(P)$, and therefore $b \in B(P)$ and $L(Q^{j'}_{X,c})(b) = L(b) = \{1,2,3\} \setminus L(T)$. In particular $i \in L(Q^{j'}_{X,c}(b))$. However, in the construction process of $L(Q^{j'}_{X,c})$, i would be removed from the list of b (according to the second kind of construction), a contradiction. This proves (4).

$$Q_{X,c} \text{ is nice or easy.} \tag{5}$$

Write $B = B(Q_{X,c})$ and $W = W(Q_{X,c})$. For $i,j \in \{1,2,3\}$ let $B_{i,j} = \{b \in B\colon L(Q_{X,c})(b) = \{i,j\}\}$. Suppose first that for every $T \in \mathcal{T}$ there exists $i \in \{1,2,3\} \setminus L(T)$ such that $X_{T,i}$ is small. Then by (4) no vertex of $B(P_{X,c})$ is mixed on an edge of W, and therefore $Q_{X,c}$ is nice.

Thus we may assume that there exist $\{i,j,k\} = \{1,2,3\}$ and $D \in \mathcal{T}$ such that $L(D) = k$ and both $X_{D,i}$ and $X_{D,j}$ are big. By (3) both X_i and X_j are big. Since $M = 2^{|S|}(r+6)$ and there are at most $2^{|S|}$ types in S, it follows that there exist $T_i, T_j \subseteq S(P)$ such that $|U_i \cap B(P,T_i)| \geq r+6$, and $|U_j \cap B(P,T_j)| \geq r+6$. We now show that $Q_{X,c}$ is easy. We may assume that $i = 1$ and $j = 2$, and that $G|(B \cup W)$ contains an induced six-vertex path $R = p_1 - p_2 - p_3 - p_4 - p_5 - p_6$.

Let $X_1 = \{a_1 - b_1 - c_1, \dots, a_M - b_M - c_M\}$ and $X_2 = \{x_1 - y_1 - z_1, \dots, x_M - y_M - z_M\}$. We may assume that $U_1 \cap B(P,T_1) = \{a_1, \dots, a_{r+6}\}$ and $U_2 \cap B(P,T_2) = \{x_1, \dots, x_{r+6}\}$. Let $Y'_1 = \{a_1 - b_1 - c_1, \dots, a_{r+6} - b_{r+6} - c_{r+6}\}$ and let $Y'_2 = \{x_1 - y_1 - z_1, \dots, x_{r+6} - y_{r+6} - z_{r+6}\}$.

First we show that for $i = 1,2$, if $D' \in \mathcal{T}$ and $L(D') \neq i$, and $X_{D',i}$ is big, then $D' \subseteq T_i$. Suppose that there exists $s \in D' \setminus T_i$. Let $X_{D',i} = \{s_1 - t_1 - r_1, s_2 - t_2 - r_2\}$ where $U_{D',i} = \{s_1, s_2\}$. Then $R' = t_2 - s_2 - s - s_1 - t_1 - r_1$ is a P_6. Since $s \in S(P) \setminus T_i$, it follows that s has no neighbors in the seagulls of Y'_i. Recall that $X_{D',i} \in guess(Q^i_{X,c}, T, i)$. Thus $t_1, t_2, r_1, r_2 \in W(Q^i_{X,c})$ and $s_1, s_2 \in B(Q^i_{X,c}, i)$. By (1).1 $\{t_1, t_2, r_1\}$ is anticomplete to V_i. By (1).2 $\{s_1, s_2\}$ is anticomplete to U_i. By (1).3 each of s_1, s_2 has neighbors in at most one seagull of Y'_i, and thus $V(R')$ is anticomplete to at least $r+4$ seagulls of Y'_i, contrary to the fact that G is $(P_6 + rP_3)$-free. This proves that $D' \subseteq T_1$.

By the claim of the previous paragraph with $D' = D$ and $i = 1,2$, we deduce that $D \subseteq T_1 \cap T_2$. Consequently, $L(T_1) = L(T_2) = L(D) = 3$.

Recall that $V(X_1) \cup V_{D,1} \cup V_{D,2} \subseteq S(Q_{X,c})$. Therefore $V(R) \cap W$ is anticomplete to $V(X_1) \cup V_{D,1} \cup V_{D,2}$, $V(R) \cap (B_{12} \cup B_{13})$ is anticomplete to $U_1 \cup U_{T,1}$ and $V(R) \cap (B_{12} \cup B_{23})$ is anticomplete to $U_2 \cup U_{T,2}$.

By (1).3, every vertex of $B_{12} \cup B_{13}$ has neighbors in at most one seagull of Y'_1, and every vertex of $B_{12} \cup B_{23}$ has neighbors in at most one seagull of Y'_2. If every vertex of $V(R)$ has neighbors in at most one of the seagulls of Y'_2, then at least $|Y'_2| - 6 \geq r$ of the seagulls in Y'_2 are anticomplete to $V(R)$, contrary to the fact that G is $(P_6 + rP_3)$-free. This proves that some vertex $p_q \in V(R)$ has neighbors in at least two of the seagulls of Y'_2. It follows that $V(R) \cap B_{13} \neq \emptyset$, and we may assume $p_q \in V(R) \cap B_{13}$ has a neighbor in $x_1 - y_1 - z_1$ and in $x_2 - y_2 - z_2$. Since $c(x_1) = c(x_2) = 2$, it follows that $c(y_1) \neq 2$ and $c(y_2) \neq 2$, and so since $p_q \in B_{13}$, we deduce that p_q is anticomplete to $\{y_1, y_2\}$.

We claim that p_q is not mixed on either of the the sets $\{y_1, z_1\}, \{y_2, z_2\}$. If $p_q \notin B(P)$, this follows immediately from Lemma 8. Thus we may assume that $p_q \in B(P)$. Let $T' = N(p_q) \cap S(P)$. Since $c(T') \neq 3$, it follows that $T' \nsubseteq T_1$, and therefore $X_{T',1}$ is small. Recall that $X_2 \in guess(Q^{1'}_{X,c}, 2)$. Then $\{y_1, z_1, y_2, z_2\} \in W(Q^{1'}_{X,c})$ and $p_q \in B(Q^{1'}_{X,c})$. Hence the claim follows from (4).

We deduce that p_q is adjacent to x_1, x_2 and anticomplete to $\{y_1, z_1, y_2, z_2\}$. Now $R' = z_1 - y_1 - x_1 - p_q - x_2 - y_2$ is a six-vertex path. Since $x_1, x_2 \in U_2 \cap B(P, T_2)$ and $L(T_2) = 3, L(P)(x_1) = L(P)(x_2) = \{1, 2\}$. By (2) $\{x_1, y_1, z_1, x_2, y_2\}$ is anticomplete to U_1, $\{y_1, z_1, y_2\}$ is anticomplete to $V(Y_1)$, and each of x_1, x_2 has neighbors in at most one seagull of Y'_1. Since $p_q \in B_{13}$, (1) implies that p_q is anticomplete to U_1, and p_q has neighbors in at most one seagull in Y'_1. But now R' is anticomplete to at least $|Y'_1| - 3 > r$ of the seagulls of Y'_1, contrary to the fact that G is $(P_6 + rP_3)$-free. This proves (5).

$$|S(Q_{X,c})| \leq |S| + 3 \times \max(3M, f(3M)) + 2^{|S|+1} \times \max(6, f(6)) \tag{6}$$

Recall that we define f be as in Lemma 9 and it is also the same f in Lemma 10. Observe that for every $i, |V(X_i)| \leq \max(3M, f(3M))$. The number of possible pairs (T, i) where $T \in \mathcal{T}$ and $i \in \{1, 2, 3\} \setminus L(T)$ is $2t \leq 2^{|S|+1}$. For every such $(T, i), |V_{T,i}| \leq \max(6, f(6))$, and therefore $|V(X^1) + V(X^2) + V(X^3)| \leq 2^{|S|+1} \times \max(6, f(6))$. Since $S(Q_{X,c}) = S \cup V(X) \leq |S| + V(X) \leq |S| + \sum_{i=1}^{3} V(X_i) + \sum_{i=1}^{3} V(X^i)$, (6) follows.

$$|\mathcal{L}| \text{ is polynomial} \tag{7}$$

Let $E = 3 \times \max(3M, f(3M)) + 2^{|S|+1} \times \max(6, f(6))$, then $|V(X)| \leq E$ as shown above. It follows that the number of possible choices of $V(X)$ is at most $|V(G)|^E$ and the number of consistent colorings for of a given X is at most $3^{|V(X)|} \leq 3^E$. It follows that $\mathcal{L} \leq (3|V(G)|)^E$, as required. This proves (7).

By (5), (6) and (7), it remains to show that $\mathcal{L}$ is equivalent to P. Since for every $P_{X,c} \in \mathcal{L}, c$ is a coloring of $(G|V(X), L)$, from the construction process it is clear that if some $P_{X,c}$ has a precoloring extension, then so does P. It remains to show that if d is a precoloring extension of P, then some $R \in \mathcal{L}$ has a precoloring extension.

Let d be a precoloring extension of P. First we construct $P(X_1, d) \in \mathcal{L}_1$ that has a precoloring extension. Let $X_1 = char_M(P, 1, d)$. Then $X_1 \in guess(P, 1)$ and d is a

1-consistent coloring of $V(X_1)$. Define the r-seeded precoloring $P(X_1, d)$ as follows. Let $\tilde{P} = (G, \tilde{L}, \tilde{S})$ be obtained from P by moving $V(X_1)$ to the seed with d. Next we modify $\tilde{L}$ further (the same way as we obtain $P'(X_1, c)$ earlier in the proof).

- Assume first that X_1 is small. If $b \in B(\tilde{P}, 1) \cap B(P)$ and b is mixed on an edge of $W(\tilde{P})$, remove 1 from $\tilde{L}(b)$.
- Next assume that X_1 is big. Let $X_1 = \{a_1 - b_1 - c_1, \dots, a_M - b_M - c_M\}$ where $U_1 = \{a_1, \dots, a_M\} \subseteq B(P, 1)$. If $b \in B(\tilde{P}, 1) \cap B(P)$ has a neighbor in $\{b_q, c_q\}$ for more than one value of q, remove 1 from $\tilde{L}(b)$.

Denote the precoloring we have constructed so far by $P(X_1, d)$. It follows from the construction that $P(X_1, d) \in \mathcal{L}_1$

We claim that $d(v) \in L(P(X_1, d))(v)$ for every $v \in V(G)$. Suppose not. Since $\tilde{P}$ is obtained from P by moving a set of vertices to the seed with d, it follows that $d(v) \in \tilde{L}(v)$ for every $v \in V(G)$. Thus we may assume that for some $v \in V(G), d(v) \in \tilde{L}(v) \setminus L(P(X_1, d))(v)$. Suppose first that X_1 is small. Then $v \in B(\tilde{P}, 1) \cap B(P), v$ is mixed on an edge w_1w_2 of $G|W(\tilde{P})$, and $d(v) = 1$. Then $w_1, w_2 \in W(P)$ and by Lemma 9.1, at least one of w_1, w_2 has a neighbor in X_1. It follows that not both w_1, w_2 are in $W(\tilde{P})$, a contradiction. Thus we may assume that X_1 is big, $v \in B(\tilde{P}, 1) \cap B(P), v$ has a neighbor in $\{b_q, c_q\}$ for more than one value of q, and $d(v) = 1$. But this immediately contradicts Lemma 9.2. This proves that $d(v) \in L(P(X_1, d))(v)$ for every $v \in V(G)$.

Next we construct $P(X_1, X^1, d) \in \mathcal{L}'_1$ that has a precoloring extension. For every $T \in \mathcal{T}_1$, let $X_{T,1} = char_2(P(X_1, d), T, 1, d)$. Let $X^1 = (X_{T,1})$, then d is a $1'$-consistent coloring of X^1. Let Q' be obtained from $P(X_1, d)$ by moving $V(X^1) = \bigcup_T V(X_{T,1})$ to the seed with d; write $L' = L(Q')$. We modify L' further. For every $T \in \mathcal{T}$ we proceed as follows (the same way as we obtain $P'_{X^1,c}$ earlier in the proof).

- Assume $X_{T,1}$ is small. If $b \in B(Q') \cap B(P(X_1, d), T)$ is mixed on an edge of $W(Q')$, remove 1 from $L'(b)$.

Denote the precoloring thus obtained by $P(X_1, X^1, d)$. It follows from the construction that $P(X_1, X^1, d) \in \mathcal{L}'_1$.

We claim that $d(v) \in L(P(X_1, X^1, d))(v)$ for every $v \in V(G)$. Suppose not. Since Q' is obtained from $P(X_1, d)$ by moving a set of vertices to the seed with d, it follows that $d(v) \in L'(v)$ for every $v \in V(G)$. Thus we may assume that for some $v \in V(G), d(v) \in L'(v) \setminus L(P(X_1, X^1, d))(v)$. Then $v \in B(Q') \cap B(P(X_1, d), T), X_{T,i}$ is small, v is mixed on an edge w_1w_2 of $G|W(Q')$, and $d(v) = i$. Then $w_1, w_2 \in W(P)$ and by Lemma 10.1, at least one of w_1, w_2 has a neighbor in $X_{T,i}$. It follows that not both w_1, w_2 are in $W(Q')$, a contradiction. This proves that $d(v) \in L(P(X_1, X^1, d))(v)$ for every $v \in V(G)$.

By applying the above argument three times, we can deduce that there exists $P(X_1, X^1, X_2, X^2, X_3, X^3, d) \in \mathcal{L}'_3$ that has a precoloring extension with d being an i-consistent coloring of X_i and an i'-consistent coloring of X^i.

Consequently, let $X = (X_1, X_2, X_3, X^1, X^2, X^3)$, then $d|X$ is consistent. It follows that $P(X_1, X^1, X_2, X^2, X_3, X^3, d) = Q_{X,d|X} \in \mathcal{L}$. This proves Lemma 11. □

5 From Nice to Stable

In this section we show that in order to be able to test if a nice r-seeded precoloring has a precoloring extension, it is enough to be able to answer the same question for a more restricted kind of r-seeded precoloring, that we call "stable".

We start with a lemma. Let G be a graph and L a list assignment for G. We say that $v \in V(G)$ is *connected* if $G|N(v)$ is connected. Let $v \in V(G)$ be connected such that $G|N(v)$ is bipartite. Let (A_1, A_2) be the (unique) bipartition of $G|N(v)$. We say that G' is obtained from G by *reducing* v if $V(G') = (V(G) \setminus (\{v\} \cup N(v))) \cup \{a_1, a_2\}, G' \setminus \{a_1, a_2\} = G \setminus (\{v\} \cup N(v)), a_1a_2 \in E(G')$, and for $u \in V(G) \cap V(G')$ and $i \in \{1, 2\}, a_iu \in E(G')$ if and only if (in G) u has a neighbor in A_i. We say that (G', L') is obtained from (G, L) by *reducing* v if G' is obtained from G by reducing $v, L'(u) = L(u)$ for every $u \in V(G') \setminus \{a_1, a_2\}$, and for $i = 1, 2, L(a_i) = \bigcap_{a \in A_i} L(a)$.

Lemma 12 *Let r be an integer and let G be a $(P_6 + rP_3)$-free graph. Let $v \in V(G)$ be connected such that $G|N(v)$ is bipartite with (unique) bipartition (A_1, A_2), and let G' be obtained from G by reducing v. Then G'* is *$(P_6 + rP_3)$-free.*

Proof Suppose Q is an induced subgraph of G' isomorphic to $P_6 + rP_3$. Recall that v is anticomplete to $V(G') \setminus \{a_1, a_2\}$ (in G). Then $V(Q) \cap \{a_1, a_2\} \neq \emptyset$.

If only one vertex of $V(Q) \setminus \{a_1, a_2\}$, say q, has a neighbor in $V(Q) \cap \{a_1, a_2\}$, say, a_1, then we get a $P_6 + rP_3$ in G by replacing a_1 with a vertex of $N_G(q) \cap A_1$, and, if $a_2 \in V(Q)$, replacing a_2 with v. Thus we may assume that two vertices q, q' of $V(Q) \setminus \{a_1, a_2\}$ have a neighbor in $V(Q) \cap \{a_1, a_2\}$. If q and q' have a common neighbor $u \in A_1 \cup A_2$, then $G|((V(Q) \setminus \{a_1, a_2\}) \cup \{u\})$ is a $P_6 + rP_3$, a contradiction. So no such u exists. Let Q' be an induced path from q to q' with $V(Q') \setminus \{q, q'\} \subseteq A_1 \cup A_2 \cup \{v\}$, meeting only one of A_1, A_2 if possible. Then $V(Q')$ is anticomplete to $V(Q) \setminus \{a_1, a_2, q, q'\}$ and $G|((V(Q) \setminus \{a_1, a_2\}) \cup V(Q'))$ contains an induced $P_6 + rP_3$, a contradiction. This proves Lemma 12. □

An r-seeded precoloring $P = (G, L, S)$ is *stable* if

1. P is nice.
2. Every component C of $W(P)$ such that some $w \in C$ has $|L(w)| = 3$ satisfies $C = \{w\}$
3. Let $\{i, j, k\} = \{1, 2, 3\}$. Then for every $b \in B(P)_{ij}$ the set $N(b) \cap B(P)_{ik}$ is stable.
4. Let $\{i, j, k\} = \{1, 2, 3\}$, let $w \in W(P)$ with $|L(w)| = 3$ and let $n \in B(P)_{ij}$ and $n' \in B(P)_{jk}$ be adjacent to w. Then no $u \in B(P)_{ik}$ is complete to $\{n, n'\}$.
5. No $w \in W(P)$ with $|L(w)| = 3$ is connected, and

6. $deg(v) > 2$ for every $v \in V(G)$ with $|L(v)| = 3$.

We can now prove the main result of this section.

Lemma 13 *For every integer $r > 0$ there exists $b \in \mathbb{N}$ with the following properties. Let G be a $(P_6 + rP_3)$-free graph. Let $P = (G, L, S)$ be a nice r-seeded precoloring of G. Assume that for every $X \subseteq V(G)$ with $|X| \le 4r + 8$, the pair $(G|X, L)$ is colorable. Then there exists a collection $\mathcal{L}$ of stable r-seeded precolorings such that*

1. $|V(G(P'))| \le |V(G)|$ *for every* $P' \in \mathcal{L}$,
2. $S(P') = S(P)$ *for every* $P' \in \mathcal{L}$
3. $\mathcal{L} \le |V(G)|$, *and*
4. *P has a precoloring extension if and only if every $P' \in \mathcal{L}$ has a precoloring extension. Moreover, if it exists, we can construct a precoloring extension of P from the precoloring extensions of every $P' \in \mathcal{L}$ in polynomial time.*

Moreover, $\mathcal{L}$ can be constructed in time $O(|V(G)|^b)$.

Proof In the proof we describe several modifications that can be made to P (in polynomial time) without changing the existence of a precoloring extension.

$$\begin{gathered}\text{Let } v \in V(G) \text{ with } |L(v)| = 3 \text{ such that either } deg_G(v) \le 2, \text{ or } v \text{ is connected.}\\ \text{Then we can construct in polynomial time a } (P_6 + rP_3)\text{-free graph}\\ G' \text{ and an } r\text{-seeded precoloring } P' = (G', L', S) \text{ such that } |V(G')| < |V(G)|,\\ \text{and } \{P'\} \text{is equivalent to } P.\end{gathered} \tag{8}$$

If $deg_G(v) \le 2$ we can set $(G', L', S) = (G \setminus v, L, S)$; thus we may assume that v is connected. If $G|N(v)$ is not bipartite (which can be checked in polynomial time), then $G|(V(C) \cup \{v\})$ is not 3-colorable, and therefore is not L-colorable. So we may assume that $G|N(v)$ is bipartite with bipartition (A_1, A_2). Since $G|N(v)$ is connected, it follows that the bipartition is unique. Let (G', L') be obtained from (G, L) by reducing v. Since $|L(v)| = 3$, it follows that $v \in W(P)$, and therefore $(A_1 \cup A_2) \cap S = \emptyset$. Thus $P' = (G', L', S)$ is an r-seeded precoloring of G'. By Lemma 12 G' is $(P_6 + rP_3)$ -free. The uniqueness of the bipartition (A_1, A_2) implies that in every coloring of (G, L) each of the sets A_1, A_2 is monochromatic. This in turn implies that if c is a precoloring extension of P, then a precoloring extension c' of P' can be obtained by setting $c'(u) = c(u)$ for every $u \in V(G') \setminus \{a_1, a_2\}$, and $c'(a_i) = c(A_i)$ for $i = 1, 2$. Conversely, if c' is a precoloring extension of P', then a precoloring extension c of P can be obtained by setting $c(u) = c'(u)$ for every $u \in V(G) \cap V(G'), c(a) = c'(a_i)$ for every $a \in A_i$, and $c(v) = \{1, 2, 3\} \setminus \{c'(a_1), c'(a_2)\}$. This proves (8).

$$\begin{gathered}\text{Let } \{i, j, k\} = \{1, 2, 3\}. \text{ Suppose that } b \in B(P)_{ij} \text{ has neighbors } n,\\ n' \in B(P)_{ik} \text{ such that } n \text{ is adjacent to } n'.\\ \text{Let } P' = (G', L', S) \text{ be the } r\text{-seeded precoloring obtained by setting } L'(b) = \{j\},\\ \text{and } L'(v) = L(v) \text{ for every } v \in V(G) \setminus \{b\}.\text{Then } \{P'\} \text{ is equivalent to } P.\end{gathered} \tag{9}$$

(9) follows from the fact that in every precoloring extension of P one of n, n' receives color i.

Let $\{i,j,k\} = \{1,2,3\}$, let $w \in W(P)$ with $|L(w)| = 3$ and let $n \in B(P)_{ij}$ and $n' \in B(P)_{jk}$ be adjacent to w. Suppose that some $u \in B(P)_{ik}$ is complete to $\{n, n'\}$. Let P' be the seeded precoloring (G, L', S) obtained by setting $L'(w) = \{1,2,3\} \setminus \{j\}$. Then $\{P'\}$ is equivalent to P. (10)

Clearly a precoloring extension of P' is also a precoloring extension of P. To see the converse, let c be a precoloring extension of P. We may assume by symmetry that $c(u) = i$. Then $c(n) = j$, and so $c(w) \neq j$, and c is a precoloring extension of P'. This proves (10).

Repeatedly applying (8), (9) and (10) we may assume that

- No $w \in W$ with $|L(w)| = 3$ is connected,
- $deg(v) > 2$ for every $v \in V(G)$ with $|L(v)| = 3$.
- For every $\{i,j,k\} = \{1,2,3\}$ and for every $b \in B(P)_{ij}$, the set $N(b) \cap B(P)_{ik}$ is stable.
- Let $\{i,j,k\} = \{1,2,3\}$, let $w \in W(P)$ with $|L(w)| = 3$ and let $n \in B(P)_{ij}$ and $n' \in B(P)_{jk}$ be adjacent to w. Then no $u \in B(P)_{ik}$ is complete to $\{n, n'\}$. (11)

We may assume that $G \setminus X^0(L)$ is connected. (12)

Suppose not. Let $C_1, \dots, C_m$ be the components of $G \setminus X^0(L)$. For $i \in \{1, \dots, m\}$ let $G_i = G|(X^0(L) \cup C_i)$. Then $S \subseteq V(G_i)$ for every i. Moreover, $P_i = (G_i, S, L)$ is an r-seeded precoloring of G_i, where $B(P_i) = B(P) \cap C_i$ and $W(P_i) = W(P) \cap C_i$. It follows that each of P_i is nice, and satisfies (11). Clearly if P has a precoloring extension, then each P_i does. Conversely, if each P_i has a precoloring extension c_i, then setting $c(v) = c_i(v)$ for $v \in V(G_i)$, we obtain a precoloring extension of P. Now it is enough to prove the theorem for each P_i separately, and (12) follows.

In view of (12) from now on we assume that $G \setminus X^0(L)$ is connected. It remains to show that:

If C is a component of $W(P)$ and $w \in C$ has $|L(w)| = 3$, then $C = \{w\}$. (13)

Let C and w be as above. Since $G \setminus X^0(L)$ is connected, it follows that $B(P)(C)$ (this is the set of attachments of C in $B(P)$) is non-empty and complete to C. Since $|L(w)| = 3$, it follows from the definition of a seeded precoloring that w is anti-complete to $X^0(L)$, and consequently $N(w) = (B(P)(C)) \cup (N(w) \cap C)$. If $C \neq \{w\}$, then $N(w) \cap C \neq \emptyset$, and therefore w is connected, a contradiction to (11). This proves (13).

Now Lemma 13 follows from (11) and (13). $\square$

6 Reducing Lists

The goal of this section is to deal with stable precolorings. Similarly to Sect. 4, we will define several "characteristics" of a precoloring extension of an r-seeded precoloring, and then, in the algorithm, given an r-seeded precoloring, enumerate all possible characteristics of its precoloring extensions.

First we need a few more definitions. Let $P(G, L, S)$ be an r-seeded precoloring. We write $\tilde{W}(P) = \{w \in W(G) \ : \ |L(w)| = 3\}$ and denote by $\tilde{B}(P)$ the set of attachments of $\tilde{W}(P)$ in $B(P)$. We write $\tilde{B}(P,i) = B(P,i) \cap \tilde{B}(P), \tilde{B}(P)_{ij} = B(P)_{ij} \cap \tilde{B}(P)$ and $\tilde{B}(P,T) = B(P,T) \cap \tilde{B}(P)$.

Let $P = (G,S,L)$ be an r-seeded precoloring of a $(P_6 + rP_3)$-free graph G, and let c be a precoloring extension of P. We define several hypergraphs associated with P and c. For every $i \in \{1,2,3\}$, let $V_i = \{b \in \tilde{B}(P,i) : c(b) = i\}$.

For every distinct $i,j \in \{1,2,3\}$ we define the hypergraph $R(P, i, j, c)$ with vertex set V_i as follows. Let K be the set of all vertices $w \in \tilde{W}(P)$ such that w has two neighbors $n,n' \in \tilde{B}(P)_{ij}$ with $c(n) = c(n') = i$. Let $h(w) = N(w) \cap V_i$. Then $\{h(w) : w \in K\}$ is the set of the hyperedges of $R(P, i, j, c)$.

We prove:

Lemma 14 *For every integer $r > 0$ there is a function $f : \mathbb{N} \to \mathbb{N}$ (that depends on r) with the following properties. Let G be a $(P_6 + rP_3)$-free graph with no clique of size four, $P = (G,L,S)$ a stable r-seeded precoloring, and let c be a coloring of (G, L). Then for every integer M and every distinct $i,j \in \{1,2,3\}$ either*

1. *there exists $X \subseteq V(R(P,i,j,c))$ with $|X| \leq f(M)$ such that if $w \in \tilde{W}(P)$ is anticomplete to X, then w has at most one neighbor $n \in \tilde{B}(P)_{ij}$ with $c(n) = i$, or*
2. *there exists a flock $F = \{a_1 - b_1 - c_1, \dots, a_M - b_M - c_M\}$ where for every $l, a_l, c_l \in V_i \cap \tilde{B}(P)_{ij}$ and $b_l \in \tilde{W}(P)$, and such that every vertex of V_i is adjacent to b_l for at most one value of l.*

Proof Let $f = f_{r,1,1}$ be as in Lemma 6. Since P is stable, $\tilde{W}(P)$ is a stable set. Applying Lemma 6 to $H = R(P,i,j,c)$ with $p = q = 1$, we deduce that either $\nu(H) \geq M$ or $\tau(H) \leq f(M)$.

Suppose first that $\nu(H) \geq M$. Let $b_1, \dots, b_M \in \tilde{W}(P)$ be such that $M = \{h(b_1), \dots, h(b_M)\}$ is a matching of H. It follows from the definition of $h(b_l)$ that for every l there exists $a_l, c_l \in \tilde{B}(P)_{ij}$ such that $c(a_l) = c(c_l) = i$, and consequently $a_l - b_l - c_l$ is a seagull. Moreover, since M is a matching of H, no $v \in V_i$ belongs to more than one $h(b_l)$, and therefore every vertex of V_i is adjacent to b_l for at most one value of l. This proves that if $\nu(H) \geq M$, then Lemma 14.2 holds. Thus we may assume that $\tau(H) \leq f(M)$. Letting $X \subseteq V_i$ be a hitting set for H, we immediately see that Lemma 14.1 holds. □

Given an r-seeded precoloring P, distinct $i,j \in \{1,2,3\}$ and a precoloring extension c of P, we say that an *R, M-characteristic* of P, i, j, c (denoted by $char_{R,M}(P,i,j,c)$) is X if Lemma 14.1 holds for P, i, j and c, and F if Lemma 14.2

holds for P, i, j and c and Lemma 14.1 does not hold for P, i, j and c. We denote by $V(char_{R,M}(P, i, j, c))$ the set of all the vertices involved in $char_{R,M}(P, i, j, c)$.

Next, for every $i \in \{1, 2, 3\}$, we define another hypergraph, $S(P, i, c)$, with vertex set V_i. Let K be the set of all vertices $w \in \tilde{W}(P)$ such that w has a neighbor $n \in \tilde{B}(P)_{ij}$ and $n' \in \tilde{B}(P)_{ik}$ with $c(n) = c(n') = i$. Let $h(w) = N(w) \cap V_i$. Then $\{h(w) : w \in K\}$ is the set of the hyperedges of $S(P, i, c)$.

We prove an analogue of Lemma 14.

Lemma 15 *For every integer $r > 0$ there is a function $f : \mathbb{N} \to \mathbb{N}$ (that depends on r) with the following properties. Let G be a $(P_6 + rP_3)$-free graph with no clique of size four, $P = (G, L, S)$ a stable r-seeded precoloring, and let c be a coloring of (G, L). Then for every integer M and every $i \in \{1, 2, 3\}$ either*

1. *there exists $X \subseteq V_i$ with $|X| \le f(M)$ such that if $w \in \tilde{W}(P)$ is anticomplete to X, then either $c(N(w) \cap \tilde{B}(P)_{ij}) = j$ or $c(N(w) \cap \tilde{B}(P)_{ik}) = k$, or*
2. *there exists a flock $F = \{a_1 - b_1 - c_1, \dots, a_M - b_M - c_M\}$ where for every $l, a_l \in \tilde{B}(P)_{ij}, b_l \in \tilde{W}(P)$ and $c_l \in \tilde{B}(P)_{ik}, c(a_l) = c(c_l) = i$, and such that every vertex of V_i is adjacent to b_l for at most one value of l.*

Proof Since P is stable, $\tilde{W}(P)$ is a stable set. Let $f = f_{r,1,1}$ be as in Lemma 6. Applying Lemma 6 to $H = S(P, i, c)$ with $p = q = 1$, we deduce that either $\nu(H) \ge M$ or $\tau(H) \le f(M)$.

Suppose first that $\nu(H) \ge M$. Let $b_1, \dots, b_M \in \tilde{W}(P)$ be such that $M = \{h(b_1), \dots, h(b_M)\}$ is a matching of H. It follows from the definition of $h(b_l)$ that for every l there exists $a_l \in \tilde{B}(P)_{ij}$ and $c_l \in \tilde{B}(P)_{ik}$ with $c(a_l) = c(c_l) = i$, and consequently $a_l - b_l - c_l$ is a seagull. Moreover, since M is a matching of H, no $v \in V_i$ belongs to more than one $h(b_l)$, and therefore every vertex of V_i is adjacent to b_l for at most one value of l. This proves that if $\nu(H) \ge M$, then Lemma 15.2 holds. Thus we may assume that $\tau(H) \le f(M)$. Letting $X \subseteq V_i$ be a hitting set for H, we immediately see that Lemma 15.1 holds. □

Given an r-seeded precoloring $P, i \in \{1, 2, 3\}$ and a precoloring extension c of P, we say that an *S, M-characteristic* of P, i, c (denoted by $char_{S,M}(P, i, c)$) is X if Lemma 15.1 holds for P, i and c, and F if Lemma 15.2 holds for P, i and c and Lemma 15.1 does not hold for P, i and c. We denote by $V(char_{S,M}(P, i, c))$ the set of all the vertices involved in $char_{S,M}(P, i, c)$.

We also need a version of the hypergraph above for types, as follows. Let $\{i, j, k\} = \{1, 2, 3\}$. For every pair of types $T_1, T_2 \subseteq S$ with $c(T_1) = k$ and $c(T_2) = j$, we define the hypergraph $H(P, T_1, T_2, c)$. The vertex set $V(H(P, T_1, T_2, c)) = \{b \in \tilde{B}(P, T_1) \cup \tilde{B}(P, T_2) : c(b) = i\}$. Next we construct the hyperedges. Let K be set of all $w \in \tilde{W}$ such that w has neighbors $n \in \tilde{B}(P, T_1)$ and $n' \in \tilde{B}(P, T_2)$ with $c(n) = c(n')$ (and therefore $c(n) = c(n') = i$). For every $w \in K$, let $h(w) = N(w) \cap V(H(P, T_1, T_2, c))$. Then $\{h(w) : w \in K\}$ is the set of the hyperedges of $H(P, T_1, T_2, c)$.

Lemma 16 *For every integer $r > 0$ there is a function $f : \mathbb{N} \to \mathbb{N}$ with the following properties. Let G be a $(P_6 + rP_3)$-free graph with no clique of size four, $P = (G, L, S)$ a stable r-seeded precoloring, and let c be a coloring of (G, L). Let $\{i, j, k\} = \{1, 2, 3\}$. Then for every integer M and for every pair of types T_1, T_2 of S with $c(T_1) = k$ and $c(T_2) = j$ either*

1. *there exists $X \subseteq V(H(P, T_1, T_2, c))$ with $|X| \leq f(M)$ such that if $w \in \tilde{W}(P)$ is anticomplete to X, then either $c(N(w) \cap \tilde{B}(P, T_1)) = j$ or $c(N(w) \cap \tilde{B}(P, T_2)) = k$, or*
2. *there exists a flock $F = \{a_1 - b_1 - c_1, \dots, a_M - b_M - c_M\}$ with $a_1, \dots, a_M \in V(H(P, T_1, T_2, c)) \cap \tilde{B}(P, T_1)$, $c_1, \dots, c_M \in V(H(P, T_1, T_2, c)) \cap \tilde{B}(P, T_2)$, and $b_1, \dots, b_M \in \tilde{W}(P)$.*

Proof Since P is stable, $\tilde{W}(P)$ is a stable set. Let $f = f_{r,1,1}$ be as in Lemma 6. Applying Lemma 6 to $H = H(P, T_1, T_2, c)$ with $p = q = 1$, we deduce that either $\nu(H) \geq M$ or $\tau(H) \leq f(M)$. Suppose first that $\nu(H) \geq M$. Let $b_1, \dots, b_M \in \tilde{W}(P)$ be such that $M = \{h(b_1), \dots, h(b_M)\}$ is a matching of H. It follows from the definition of $h(b_l)$ that there exists a flock as in Lemma 16.2. Thus we may assume that $\tau(H) \leq f(M)$. Letting $X \subseteq V(H(P, T_1, T_2, c))$ be a hitting set for H, we immediately see that Lemma 16.1 holds. □

Given an r-seeded precoloring P and types T_1, T_2 of $S(P)$ with $c(T_1) \neq c(T_2)$, and a precoloring extension c of P, we say that an *M-characteristic* of P, T_1, T_2, c (denoted by $char_M(P, T_1, T_2, c)$) is X if Lemma 16.1 holds for P, T_1, T_2 and c, and F if Lemma 16.2 holds for P, T_1, T_2 and c and Lemma 16.1 does not hold for P, T_1, T_2 and c. We denote by $V(char_M(P, T_1, T_2, c))$ the set of all the vertices involved in $char_M(P, T_1, T_2, c)$.

In contrast to Sect. 4 here we will need another type of characteristic, that is not related to Lemma 6. Let $\{i, j, k\} = \{1, 2, 3\}$. A seagull $a - b - d$ is an *ij-typed seagull* if $a \in \tilde{B}(P)_{ik}, b \in \tilde{W}(P)$ and $d \in \tilde{B}(P)_{jk}$. An ij-typed seagull is *ij-colored* if $c(a) = i$ and $c(d) = j$ (and therefore $c(b) = k$). Let $width_{ij}(c)$ be the maximum size of a flock F of ij-colored seagulls. We say that two ij-typed seagulls $a - b - d$ and $a' - b' - d'$ are *related* a is adjacent to d', d is adjacent to a', and there are no other edges between $\{a, b, d\}$ and $\{a', b', d'\}$. A(P, i, j, c) *-key* is a pair (X_1, X_2) such that

- X_1 is a maximal flock of ij-colored seagulls. Let $X_1 = \{x_1 - y_1 - z_1, \dots, x_m - y_m - z_m\}$. Let $P' = (G, L', S')$ be obtained from P by moving $V(X_1)$ to the seed with c.
- For every $l \in \{1, \dots, m\}$ let S_l be a flock of size at most one, such that if $S_l \neq \emptyset$, then the member of S_l is an ij-typed seagull of P' related to $x_l - y_l - z_l$. Let $X_2 = \bigcup_{l=1}^{m} S_l$.
- For every $s_2 \in X_2$, at least one wing of s_2 has color k (in c).
- For every $l \in \{1, \dots, m\}$, if $S_l = \emptyset$, then no ij-typed seagull of P' that is related to s_l has a wing u with $c(u) = k$.

The *order* of the key is $|X_1|$.

Lemma 17 *Let $r > 0$ be an integer, let G be a $(P_6 + rP_3)$-free graph with no clique of size four, and let $P = (G, L, S)$ be a stable r-seeded precoloring. Assume that $G|(\tilde{B}(P)_{ik} \cup \tilde{B}(P)_{jk} \cup \tilde{W}(P))$ is P_6-free. Let F be a flock of ij-typed seagulls in P. Let c be a coloring of $G|V(F)$ where $c(V(F) \cap \tilde{B}(P)_{ik}) = i$, $c(V(F) \cap \tilde{B}(P)_{jk}) = j, c(V(F) \cap \tilde{W}(P)) = k$, and let P' be the precoloring obtained from P by moving $V(F)$ to the seed with c. Then*

1. *For every ij-typed seagull s of P', either $F \cup \{s\}$ is a flock, or s is related to a seagull of F.*
2. *If $s \in F$, and $s_1 = x_1 - y_1 - z_1$ and $s_2 = x_2 - y_2 - z_2$ are ij-typed seagulls of P' such that both s_1 and s_2 are related to s, and y_1 is anticomplete to $\{x_2, z_2\}$, and y_2 is anticomplete to $\{x_1, z_1\}$, then s_1 is related to s_2.*

Proof Let $F = \{a_1 - b_1 - c_1, \dots, a_m - b_m - c_m\}$. Let $s = x - y - z$ be an ij-typed seagull of P'. We may assume that $F \cup \{s\}$ is not a flock. Since every seagull of F is an ij-colored seagull in c, and s is an ij-typed seagull of P', and $V(F) \subseteq S(P')$, it follows that for every $l \in \{1, \dots, m\}$, the only possible edges between $\{a_l, b_l, c_l\}$ and $\{x, y, z\}$ are $a_l z$ and $c_l x$. By symmetry we may assume that for some $l \in \{1, \dots, m\}$ x is adjacent to c_l. Since $a_l - b_l - c_l - x - y - z$ is not a P_6 in $G|(\tilde{B}(P)_{ik} \cup \tilde{B}(P)_{jk} \cup \tilde{W}(P))$, it follows that a_l is adjacent to z, and thus s is related to $a_l - b_l - c_l$. This proves the first assertion of Lemma 17.

We now prove the second assertion. Assume that $s_1 = x_1 - y_1 - z_1$ and $s_2 = x_2 - y_2 - z_2$ are both ij-typed seagulls of P' that are related to $a_1 - b_1 - c_1$, and y_1 is anticomplete to $\{x_2, z_2\}$, and y_2 is anticomplete to $\{x_1, z_1\}$. Then $b_1, y_1, y_2 \in \tilde{W}(P)$. Since P is stable, it follows that $x_1, z_1, x_2, z_2, a_1, c_1 \in \tilde{B}(P)$. This implies that y_1 is not adjacent to y_2. The fact that s_1 and s_2 are related to $a_1 - b_1 - c_1$ implies that c_1 is complete to $\{x_1, x_2\}$ and anticomplete to $\{z_1, z_2\}$. Since P is stable and c_1 is complete to $\{x_1, x_2\}$, it follows that x_1 is non-adjacent to x_2, and similarly and z_1 is non-adjacent to z_2. Since $y_1 - x_1 - c_1 - x_2 - y_2 - z_2$ is not a P_6, it follows that x_1 is adjacent to z_2, and similarly x_2 is adjacent to z_1. This proves the second assertion of Lemma 17 and and completes the proof. □

Lemma 18 *Let $r > 0$ be an integer, G a $(P_6 + rP_3)$-free graph with no clique of size four, $P = (G, L, S)$ a stable r-seeded precoloring, and let c be a coloring of (G, L). Assume that $G|(\tilde{B}(P)_{ik} \cup \tilde{B}(P)_{jk} \cup \tilde{W}(P)$ is P_6-free. Then for every integer M and every $\{i, j, k\} = \{1, 2, 3\}$ either*

1. *$width_{i,j}(c) \geq M$, or*
2. *There exists a (P, i, j, c)-key (X_1, X_2) of order less than M.*

Proof We may assume that Lemma 18.1 does not hold. Let X_1 be a maximal flock of ij-colored seagulls; then $|X_1| = m < M$. Write $X_1 = \{a_1 - b_1 - c_1, \dots, a_m - b_m - c_m\}$. Note that for every $v \in V(X_1) \cap \tilde{W}(P)$ we have $c(v) = k$. Let P' be the precoloring obtained from P by moving $V(X_1)$ to the seed with c. For every l, let S_l contain an ij-typed seagull $x - w - y$ of P' that is related to s_l and that has $c(x) = k$ or $c(y) = k$.

If no such $x-w-y$ exists, let $S_l = \emptyset$. Let $X_2 = \bigcup_{l=1}^{m} S_l$. Clearly $X = X_1 \cup X_2$ is a (P, i, j, c)-key. □

Given an r-seeded precoloring P, distinct $i,j \in \{1,2,3\}$ and a precoloring extension c of P, we say that a *heterogeneous M-characteristic* of P, i, j, c (denoted by $char_{h,M}(P,i,j,c)$) is a flock of size M of ij-colored seagulls if Lemma 18.1 holds for P, i, j and c, and a (P, i, j, c)-key (X_1, X_2) of order $< M$ if Lemma 18.2 holds for P, i, j and c. We denote by $V(char_{h,M}(P,i,j,c))$ the set of all the vertices involved in $char_{h,M}(P,i,j,c)$.

Next we generalize the notion of an r-seeded precoloring in order to be able to use Lemma 2. For a graph G, the pair $(P, \mathcal{X})$ is an *augmented r -seeded precoloring* of G if P is an r-seeded precoloring of an induced subgraph G' of G and $\mathcal{X}$ is a set of subsets of $V(G)$ where $|\mathcal{X}|$ is polynomial. A precoloring extension of P is a *coloring* of $(P, \mathcal{X})$ if every $X \in \mathcal{X}$ is monochromatic in c. We say that $(P, \mathcal{X})$ is *tractable* if $|L(P)(v)| \leq 2$ for every $v \in V(G')$.

For an r-seeded precoloring P, a collection $\mathcal{L}$ of augmented r-seeded precolorings is *equivalent* to P if P has a precoloring extension if and only if some member of $\mathcal{L}$ has a coloring, and given a coloring of a member of $\mathcal{L}$, a precoloring extension of P can be constructed in polynomial time.

We can now prove the main result of this section. The strategy of the proof is similar to Lemma 11, but it is technically more involved since we need to consider more characteristics.

Lemma 19 *There exists a function $g : \mathbb{N} \to \mathbb{N}$ with the following properties. Let $r > 0$ be an integer, G a $(P_6 + rP_3)$-free graph with no clique of size four, and $P = (G, L, S)$ a stable r-seeded precoloring of G. Then there exists an equivalent collection $\mathcal{L}$ of tractable augmented r-seeded precolorings such that $|\mathcal{L}| \leq |V(G)|^{g(|S|)}$. Moreover, $\mathcal{L}$ can be constructed in time $O(|V(G)|^{g(|S|)})$.*

Proof We start with an observation.

$$\text{Let } w \in \tilde{W}(P). \text{ Then } N(w) \subseteq \tilde{B}(P). \tag{14}$$

(14) follows immediately from the fact that P is stable.

Let $i,j \in \{1,2,3\}$ be distinct, f_R as in Lemma 14 and $M = r + 6$. First we enumerate R, M-characteristics of P. Let *smallguess*(P, i, j) be the set of all subsets of $\tilde{B}(P, i)$ of size at most $f_R(M)$, and *bigguess*(P, i, j) the set of all flocks of size M such that every seagull of the flock has body in $\tilde{W}(P)$ and both its wings in $\tilde{B}(P)_{ij}$. Let *guess*(P,i,j) = *smallguess*$(P,i,j) \cup$ *bigguess*(P,i,j). Then *guess*(P, i, j) is the set of all possible objects that can be an R, M-characteristic of a precoloring extension of P. We say that $X_{ij} \in$ *guess*(P,i,j) is *small* if $X_{ij} \in$ *smallguess*(P,i,j) and that X_{ij} is *big* if $X_{ij} \in$ *bigguess*(P,i,j). If X_{ij} is big, we denote by U_{ij} the set of the wings of the flock, by W_{ij} the the bodies of the flock, and write $V_{ij} = U_{ij} \cup W_{ij}$. If X_{ij} is small, we write $V_{ij} = U_{ij} = X_{ij}$, and $W_{ij} = \emptyset$. In both cases we set $V(X_{ij}) = V_{ij}$. A coloring c of $(G|V_{ij}, L)$ is *i, j-consistent* if $c(v) = i$ for every $v \in U_{ij}$.

Let $\mathcal{Q}$ be the set of all 6-tuples $X = (X_{12}, X_{21}, X_{13}, X_{31}, X_{23}, X_{32})$ such that $X_{ij} \in guess(P, i, j)$. Let $V(X) = \bigcup_{i \neq j \in \{1,2,3\}} V(X_{ij})$. We say that a coloring c of $(G|V(X), L)$ is *consistent* if $c|V_{ij}$ is i, j-consistent for all i, j.

Let $X \in \mathcal{Q}$ and let c be a consistent coloring of X, and let $P' = (G, L', S')$ be the precoloring obtained from P by moving $V(X)$ to the seed with c. We modify L' further as follows. Let $i, j \in \{1, 2, 3\}$.

- Assume first that X_{ij} is small. If $w \in \tilde{W}(P')$ and $|N(w) \cap \tilde{B}(P')_{ij}| > 1$, remove j from $L'(w)$.
- Next assume that X_{ij} is big. Let $X_{ij} = \{a_1 - b_1 - c_1, \dots, a_M - b_M - c_M\}$. If $b \in \tilde{B}(P', i) \cap \tilde{B}(P)$ and b is adjacent to b_q for more than one value of q, remove i from $L'(b)$.

Let $P_{X,c}$ be the r-seeded precoloring thus obtained. We list several properties of $P_{X,c}$.

Let $i, j \in \{1, 2, 3\}$.
- $W(P_{X,c})$ is anticomplete to $V(X)$.
- If $b \in B(P_{X,c}, i)$ then b is anticomplete to $U_{i,j}$. (15)
- If $b \in \tilde{B}(P_{X,c}, i)$ and X_{ij} is big, then b has neighbors in at most one of the seagulls of X_{ij}.

The first two statements of (15) follow from that fact that $V(X) \subseteq S(P_{X,c})$. By the second bullet of the construction process of $P_{X,c}$, we deduce that if X_{ij} is big and b has neighbors in more than one of the seagulls of X_{ij}, then i is removed from the list of b; thus the third statement of (1) follows. This proves (15).

Let $i, j \in \{1, 2, 3\}$. If X_{ij} is small, then no vertex $w \in \tilde{W}(P_{X,c})$ has two neighbors in $B(P_{X,c})_{ij}$. (16)

Suppose that such w has two neighbors n, n' in $B(P_{X,c})_{ij}$. Since $P_{X,c}$ is obtained from P by moving a set of vertices to the seed, it follows that $B(P_{X,c}) \subseteq B(P) \cup W(P)$. By (14) no vertex of $W(P)$ is adjacent to w, and therefore $n, n' \in \tilde{B}(P)$. It follows that in the construction process of $L(P_{X,c}), j$ was removed from the list of w, contrary to the fact that $w \in \tilde{W}(P_{X,c})$. This proves (16).

Let $\mathcal{L}_R$ be the collection of all precolorings $P_{X,c}$ as above where $X \in \mathcal{Q}$ and c is a consistent coloring of $V(X)$.

- $|\mathcal{L}_R| \leq (3|V(G)|)^{6 \max(f_R(M), 3M)}$.
- $|S(Q)| \leq |S(P)| + 6 \max(f_R(M), 3M)$ for every $Q \in \mathcal{L}_R$. (17)

For every $X \in \mathcal{Q}, V(X) = \bigcup_{i \neq j \in \{1,2,3\}} V(X_{ij})$, and $|V_{ij}| \leq \max(f_R(M), 3M)$. Thus $|V(X)| \leq 6 \max(f_R(M), 3M)$. It follows that $|S(Q)| \leq |S(P)| + 6 \max(f_R(M), 3M)$ for every $Q \in \mathcal{L}_R$. Moreover, there are at most $|V(G)|^{6 \max(f_R(M), 3M)}$ possible choices for

X. Since there are at most $3^{|V(X)|} \leq 3^{6\max(f_R(M),3M)}$ possible colorings of $G|X$, it follows that $|\mathcal{L}_R| \leq (3|V(G)|)^{6\max(f_R(M),3M)}$. This proves (17).

Let $Q \in \mathcal{L}_R$ and $\{i,j,k\} = \{1,2,3\}$. Let f_S be as in Lemma 15 and let $N = \max(2^{|S(Q)|+1}, r+6)$. We enumerate all S, N-characteristics of Q. Let *smallguess*(W, k) be the set of all subsets of $\tilde{B}(Q,k)$ of size at most $f_S(N)$, and *bigguess*(W, k) be the set of all flocks of size N such that every seagull of the flock has one wing in $\tilde{B}(Q)_{ik}$, body $w \in \tilde{W}(Q)$, and the other wing in $\tilde{B}(Q)_{jk}$. Let *guess*(Q,k) = *smallguess*$(Q,k) \cup$ *bigguess*(Q,k). We say that $Y_k \in$ *guess*(Q,k) is *small* if $Y_k \in$ *smallguess*(Q,k) and that Y_k is *big* if $Y_k \in$ *bigguess*(Q,k). If Y_k is big, we denote by U_k the set of the wings of the flock, by W_k the set of the bodies of the flock, and write $V_k = U_k \cup W_k$. If Y_k is small, we write $V_k = U_k = Y_k$, and $W_k = \emptyset$. In both cases $V(Y_k) = V_k$. A coloring c of $(G|V_k, L(Q))$ is *k-consistent* if $c(v) = k$ for every $v \in U_k$.

Let $\mathcal{S}(Q)$ be the set of all triples $Y = (Y_1, Y_2, Y_3)$ such that $Y_k \in$ *guess*(Q,k). Let $V(Y) = V(Y_1) \cup V(Y_2) \cup V(Y_3)$. We say that a coloring c of $(G|V(Y), L(Q))$ is *consistent* if $c|V_i$ is i-consistent for all i.

Let $Y \in \mathcal{S}(Q)$ and let c be a consistent coloring of $(G|V(Y), L(Q))$. Denote by $P' = (G, L', S')$ the precoloring obtained from Q by moving $V(Y)$ to the seed with c. We modify L' further. For every $i,j \in \{1,2,3\}$ with $i \neq j$, proceed as follows.

- Assume first that both Y_i and Y_j are small. If $w \in \tilde{W}(P')$ has a neighbor in all three of the sets $\tilde{B}(P')_{ij}, \tilde{B}(P')_{ik}, \tilde{B}(P')_{jk}$, remove k from $L'(w)$.
- Repeat the following for $l = i,j$. Assume that Y_l is big. Let $Y_l = \{a_1 - b_1 - c_1, \dots, a_N - b_N - c_N\}$. If $b \in \tilde{B}(P', i) \cap \tilde{B}(Q)$ and b is adjacent to b_q for more than one value of q, remove l from $L'(b)$.

Let $Q_{Y,c}$ be the r-seeded precoloring thus obtained. We list several properties of $Q_{Y,c}$.

Let $i,j \in \{1,2,3\}$.
- $W(Q_{Y,c})$ is anticomplete to $V(Y)$.
- If $b \in B(Q_{Y,c}, i)$, then b is anticomplete to U_i.
- If $b \in \tilde{B}(Q_{Y,c}, i)$ and Y_i is big, then u has neighbors in at most one of the seagulls of Y_i.

(18)

The first two statements of (18) follow from that fact that $V(Y) \subseteq S(Q_{Y,c})$. We now prove the third bullet. Let $b \in \tilde{B}(Q_{Y,c}, i)$ and suppose that b has neighbors in more that one of the seagulls of Y_i. Since $Q_{Y,c}$ is obtained from Q by moving vertices to the seed, it follows that $B(Q_{Y,c}) \subseteq B(Q) \cup W(Q)$, and thus $b \in B(Q) \cup W(Q)$. Moreover, $N(b) \cap V(Y_i) \subseteq W_i$. Since $W_i \subseteq \tilde{W}(Q)$, it follows that no two vertices of W_i belong to the same component of $W(Q)$, and thus $b \notin W(Q)$. Consequently, $b \in \tilde{B}(Q)$. By the second bullet of the construction process of $Q_{Y,c}$, we deduce that if b has neighbors in more than one of the seagulls of Y_i, then i is removed from the list of b; thus the third statement of (18) follows. This proves (18).

Let $i,j \in \{1,2,3\}$ be distinct. If Y_i and Y_j are both small, then $|L(Q_{Y,c})(w)| < 3$ for every $w \in W(Q_{Y,c})$ with a neighbor in all three of the sets $B(Q_{Y,c})_{ij}, B(Q_{Y,c})_{ik}, B(Q_{Y,c})_{jk}$.

(19)

(19) follows immediately from the first bullet of the description of the modification of L'.

$$\text{Let } i \in \{1,2,3\}. \text{ If } Y_i \text{ is big, then } G|(\tilde{B}(Q_{Y,c}, i) \cup \tilde{W}(Q_{Y,c})) \text{ is } P_6\text{-free.} \tag{20}$$

Suppose that R is a P_6 in $G|(\tilde{B}(Q_{Y,c}, i) \cup \tilde{W}(Q_{Y,c}))$. By (18) every vertex of R has neighbors in at most one seagull of Y_i. Consequently at least $N - 6 \geq r$ seagulls of Y_i are anticomplete to R, contrary to the the fact that G is $(P_6 + rP_3)$-free. This proves (20).

$$\text{Write } Q = P_{X,d} \text{ and let } \{i,j,k\} = \{1,2,3\}. \text{ If } Y_i \text{ is big, then } X_{ij} \text{ and } X_{ik} \text{ are small.} \tag{21}$$

Suppose that (21) is false; by symmetry we may assume that X_{ij} is big. Since Y_i is big, there is a type T of $S(Q)$ with $L(T) = j$ such that at least $\frac{N}{2^{|S(Q)|}} \geq 2$ of the seagulls of Y_i have a wing in $\tilde{B}(Q, T)$. Let $a_1 - b_1 - c_1$ and $a_2 - b_2 - c_2$ be such seagulls, where $a_1, a_2 \in \tilde{B}(Q, T)$. Let $s \in T$. Then $R = b_1 - a_1 - s - a_2 - b_2 - c_2$ is a P_6, and by (15) every vertex of R has neighbors in at most one of the seagulls of X_{ij}. Therefore $V(R)$ is anticomplete to at least $M - 6 \geq r$ of the seagulls of X_{ij}, contrary to the fact that G is $(P_6 + rP_3)$-free. This proves (21).

Let $\mathcal{L}_S(Q)$ be the list of all precolorings $Q_{Y,c}$ where $Y \in \mathcal{S}(Q)$ and c is a consistent coloring of $(G|V(Y), L(Q))$. We claim the following:

$$\begin{aligned} &- \quad |\mathcal{L}_S(Q)| \leq (3|V(G)|)^{3\max(f_S(N),3N)}. \\ &- \quad |S(Z)| \leq |S(Q)| + 3\max(f_S(N), 3N) \text{ for every } Z \in \mathcal{L}_S(Q). \end{aligned} \tag{22}$$

For every $Y \in \mathcal{S}, V(Y) = V_1 \cup V_2 \cup V_3$, and $|V_i| \leq \max(f_S(N), 3N)$. Thus $|V(Y)| \leq 3\max(f_S(N), 3N)$. It follows that $|S(Z)| \leq |S(Q)| + 3\max(f_S(N), 3N)$ for every $Z \in \mathcal{L}_S(Q)$. Moreover, there are at most $|V(G)|^{3\max(f_S(N),3N)}$ possible choices for Y. Since there are at most $3^{|V(Y)|} \leq 3^{3\max(f_S(N),3N)}$ possible colorings of $G|V(Y)$, it follows that $|\mathcal{L}_S(Q)| \leq (3|V(G)|)^{3\max(f_R(N),3N)}$. This proves (22).

Let $S \in \mathcal{L}_S(Q)$. Let $T = r + 6$ and let $\{i,j,k\} = \{1,2,3\}$. Our next step is to enumerate all heterogeneous T-characteristics of S. A *potential* (S, i, j) *-key* is a pair (X_1, X_2) such that

- X_1 is a flock of *ij*-typed seagulls of S with $|X_1| < T$. Write $X_1 = \{x_1 - y_1 - z_1, \dots, x_m - y_m - z_m\}$ where $x_1, \dots, x_m \in \tilde{B}(S)_{ik}, y_1, \dots, y_m \in \tilde{W}(S)$, and $z_1, \dots, z_m \in \tilde{B}(S)_{jk}$.
- Let c be a coloring of $G|V(X_1)$ such that for every $l \in \{1, \dots, m\}, c(x_l) = i, c(z_l) = j$ and $c(y_l) = k$. Let $P' = (G, L', S')$ be obtained from S by moving $V(X_1)$ to the seed with c. For every $l \in \{1, \dots, m\}$ let S_l be a flock of size at most one, such that if $S_l \neq \emptyset$, then the member of S_l is an *ij*-typed seagull of P' related to $x_l - y_l - z_l$. Let $X_2 = \bigcup_{l=1}^{m} S_l$.

Let *smallguess*(S, i, j) be the set of all potential (S, i, j)-keys and *bigguess*(S, i, j) be the set of all flocks of size T such that every seagull of the flock is an *ij*-typed seagull of S. Let *guess*(S,i,j) = *smallguess*$(S,i,j) \cup$ *bigguess*(S,i,j). We say that

$Z_{ij} \in guess(S, i, j)$ is *small* if $Z_{ij} \in smallguess(S, i, j)$ and that Z_{ij} is *big* if $Z_{ij} \in bigguess(S, i, j)$. If Z_{ij} is big, we denote by U^i_{ij} the set of the wings of the flock that are contained in $\tilde{B}(S)_{ik}$, by U^j_{ij} the set of the wings of the flock that are contained in $\tilde{B}(S)_{jk}$, and by W_{ij} the set of the bodies of the flock, and write $V_{ij} = U^i_{ij} \cup U^j_{ij} \cup W_{ij}$. Next assume that $Z_{ij} = (Z^{ij}_1, Z^{ij}_2)$ is small. Denote by U^i_{ij} the set of the wings of Z^{ij}_1 that are contained in $\tilde{B}(S)_{ik}$, by U^j_{ij} the set of the wings of Z^{ij}_1 that are contained in $\tilde{B}(S)_{jk}$, and by W_{ij} the set of the bodies of Z^{ij}_1. We write $V_{ij} = U^i_{ij} \cup U^j_{ij} \cup W_{ij} \cup V(Z^{ij}_2)$. In both cases $V(Z_{ij}) = V_{ij}$. A coloring c of $(G|V_{ij}, L(S))$ is *ij-consistent* if

- $c(v) = i$ for every $v \in U^i_{ij}$, $c(v) = j$ for every $v \in U^j_{ij}$, $c(v) = k$ for every $v \in W_{ij}$, and
- if Z_{ij} is small, then every seagull of Z^{ij}_2 has at least one wing v with $c(v) = k$.

Let $\mathcal{T}(S)$ be the set of all triples $Z = (Z_{12}, Z_{13}, Z_{23})$ such that $Z_{ij} \in guess(S, i, j)$. Let $V(Z) = V(Z_{12}) \cup V(Z_{13}) \cup V(Z_{23})$. We say that a coloring c of $(G|V(Z), L)$ is *consistent* if $c|V_{ij}$ is ij-consistent for all $i, j \in \{1, 2, 3\}$ with $i < j$.

Let $Z \in \mathcal{T}(S)$ and let c be a consistent coloring of $G|V(Z)$. Let $P' = (G, L', S')$ be the precoloring obtained from S by moving $V(Z)$ to the seed with c. If Z_{ij} is small, we modify L' further, as follows. Let $i, j \in \{1, 2, 3\}$. Write $Z^{ij}_1 = \{x_1 - y_1 - z_1, \dots, x_m - y_m - z_m\}$ and $Z^{ij}_2 = \{S^{ij}_1, \dots, S^{ij}_m\}$ as in the definition of a potential key.

- Let $l \in \{1, \dots, m\}$. If $S^{ij}_l = \emptyset$, and $p - q - r$ is an ij-typed seagull of P' related to $x_l - y_l - z_l$, remove k from $L'(p)$ and from $L'(r)$.
- If $p - q - r$ is an ij-typed seagull of P' and $Z^{ij}_1 \cup \{p - q - r\}$ is a flock, remove k form $L'(q)$.

Let $S_{Z,c}$ be the r-seeded precoloring thus obtained. We list several properties of $S_{Z,c}$.

Let $\{i, j, k\} \in \{1, 2, 3\}$.

- $W(S_{Z,c})$ is anticomplete to $V(Z)$.
- If $b \in B(S_{Z,c}, i)$ then b is anticomplete to every $v \in V(Z)$ with $c(v) = i$.
- $B(S_{Z,c}, k)$ is anticomplete to W_{ij}.

(23)

All three statements of (23) follow from that fact that $V(Z) \subseteq S(S_{Z,c})$.

Let $\{i, j, k\} = \{1, 2, 3\}$ and assume that Z_{ij} is small. Write

$Z^{ij}_1 = \{x_1 - y_1 - z_1, \dots, x_m - y_m - z_m\}$ as in the definition of a potential key.

1. Let $l \in \{1, \dots, m\}$. If $S^{ij}_l = \emptyset$, then no ij-typed seagull of $S_{Z,c}$ is related to $x_l - y_l - z_l$.
2. There is no ij-typed seagull $p - q - r$ of $S_{Z,c}$ such that $Z^{ij}_1 \cup \{s\}$ is a flock.

(24)

(24) follows immediately from the the description of the modification of L', and the fact that, by Lemma 7, an ij-typed seagull as in (24).2 is also an ij-typed seagull of P' (with the notation as in the description of the modification of L').

Recall that there exists an r-seeded precoloring Q such that $S \in \mathcal{L}_S(Q)$ and $Q \in \mathcal{L}_R(P)$. Let Y and d be such that $S = Q_{Y,d}$.

$$\text{Let } \{i,j,k\} = \{1,2,3\} \text{ and assume that } Z_{ij} \text{ is small and } Y_k \text{ is big. Then there is no } ij\text{-typed seagull in } S_{Z,c}. \tag{25}$$

Since Y_k is big, (20) implies that $G|(\tilde{B}(S,k) \cup \tilde{W}(S))$ is P_6-free. Suppose $p-q-r$ is ij-typed seagull of $S_{Z,c}$. Then $p-q-r$ is an ij-typed seagull of P' (with the notation as in the description of the modification of L'). Then $k \in L(S_{Z,c})(p) \cap L(S_{Z,c})(r)$. It follows from (24).2 that $Z_1^{ij} \cup \{p-q-r\}$ is not a flock, and so by Lemma 17 $p-q-r$ is related to some seagull of Z_1^{ij}, say to $a_1 - b_1 - c_1$. We claim that $S_1 = \emptyset$. Suppose not; write $S_1 = \{s_1\}$. Then, by (23) and Lemma 17, $p-q-r$ is related to s_1. But $V(s_1) \subseteq S(S_{Z,c})$, and at least one wing of s_1 has color k, a contradiction. This proves that $S_1 = \emptyset$, and we get a contradiction to (24).1 This proves (25).

Let $\mathcal{L}_T(S)$ be the list of all precolorings $S_{Z,c}$ where $Z \in \mathcal{T}(S)$ and c is a consistent coloring of $V(Z)$.

$$\begin{aligned} &- \quad |\mathcal{L}_T(S)| \leq (3|V(G)|)^{18T}. \\ &- \quad |S(T)| \leq |S(S)| + 18T \text{ for every } T \in \mathcal{L}_T(S). \end{aligned} \tag{26}$$

For every $Z \in \mathcal{T}, Z$ consists of three parts, each of which is a set of at most $2T$ seagulls. Therefore $|V(Z)| \leq 18T$. It follows that $|S(T)| \leq |S(S)| + 18T$ for every $T \in \mathcal{L}_T(S)$. Moreover, there are at most $|V(G)|^{18T}$ possible choices for Z. Since there are at most $3^{|Z|} \leq 3^{18T}$ possible colorings of $G|V(Z)$, it follows that $|\mathcal{L}_T(S)| \leq (3|V(G)|)^{18T}$. This proves (26).

Let $R \in \mathcal{L}_T(S)$ and let $\{i,j,k\} = \{1,2,3\}$. Let $B = r + 2$. Let f be as in Lemma 16. Finally we enumerate all possible B-characteristics of pairs of types of R. For every pair of types T_i, T_j of $S(R)$ with $L(R)(T_i) = i$ and $L(R)(T_j) = j$, we define the following sets. Assume first that Y_k is small (here Y_k is in the same notation as in (25)). Let $smallguess(T_i, T_j) = \{\emptyset\}$ and $bigguess(T_i, T_j) = \emptyset$. Next assume that Y_k is big. Let $smallguess(T_i, T_j)$ be the set of all subsets of $\tilde{B}(R, T_i) \cup \tilde{B}(R, T_j)$ of size at most $f(B)$, and $bigguess(T_i, T_j)$ be the set of all flocks of size B such that every seagull of the flock has a wing in $\tilde{B}(R, T_i)$, a wing in $\tilde{B}(R, T_j)$ and body in $w \in \tilde{W}(R)$.

In all cases, let $guess(T_i, T_j) = smallguess(T_i, T_j) \cup bigguess(T_i, T_j)$. Let $\mathcal{T}$ be the set of all types of $S(R)$, say $|\mathcal{T}| = t$. Now let $\mathcal{C}(R)$ be the set of all vectors (A_{T_i,T_j}) where $T_i, T_j \in \mathcal{T}$ with $L(R)(T_i) = i$ and $L(R)(T_j) = j$, and $A_{T_i,T_j} \in guess(T_i, T_j)$. Then every $A \in \mathcal{C}(R)$ has at most t^2 components.

We say that A_{T_i,T_j} is *small* if $A_{T_i,T_j} \in smallguess(T_i, T_j)$ and that A_{T_i,T_j} is *big* if $A_{T_i,T_j} \in bigguess(T_i, T_j)$. If A_{T_i,T_j} is big, we denote by U_{T_i,T_j} the set of the wings of the flock, and by W_{T_i,T_j} the set of the bodies of the flock, and write $V_{T_i,T_j} = U_{T_i,T_j} \cup W_{T_i,T_j}$. If A_{T_i,T_j} is small, we write $V_{T_i,T_j} = U_{T_i,T_j} = A_{T_i,T_j}$ and $W_{T_i,T_j} = \emptyset$. Finally, let $V(A) = \bigcup_{T_i,T_j} V_{T_i,T_j}$.

A coloring c of $G|V(A)$ is *consistent* if $c(v) = k$ for every $v \in U_{T_i,T_j}$. Let $A \in \mathcal{C}(R)$ and let c be a consistent coloring of $G|V(A)$. We construct the r-seeded precoloring $R_{A,c}$ as follows. Let $P' = (G, L', S')$ be obtained from R by moving $V(A)$ to the seed with c. Next we modify L' further. Please note that we are still dealing with types of R, and not with types of P'.

– Let $\{i,j,k\} = \{1,2,3\}$, let T_i, T_j, T_k be such that $L(R)(T_l) = l$, and assume that A_{T_i,T_j} and A_{T_i,T_k} are both small. If $w \in \tilde{W}(P')$ is both in a seagull with wings in $\tilde{B}(R, T_i)$ and $\tilde{B}(R, T_j)$, and in a seagull with wings in $\tilde{B}(R, T_i)$ and $\tilde{B}(R, T_k)$, remove i from $L'(w)$.
– Let $\{i,j,k\} = \{1,2,3\}$, let T_i, T_j, T_k be such that $L(R)(T_l) = l$, and assume that A_{T_i,T_j} is small. If $w \in \tilde{W}(P')$ is in a seagull with wings in $\tilde{B}(R, T_i)$ and $\tilde{B}(R, T_j)$, and $N(w) \cap \tilde{B}(R, T_i)$ is complete to $N(w) \cap \tilde{B}(R, T_k)$, then remove i from $L'(w)$.

Denote the precoloring thus obtained by $R_{A,c}$.

Let $w \in \tilde{W}(R_{A,c})$. Then either

1. w only has neighbors in at most two of $B(R_{A,c})_{ij}$, or
2. there exist T_1, T_2, T_2', T_3 such that $L(R)(T_1), L(R)(T_2), L(R)(T_3)$ are all distinct, $L(R)(T_2) = L(R)(T_2')$, and w is both in a seagull with wings in $\tilde{B}(R_{A,c}, T_1)$ and $\tilde{B}(R_{A,c}, T_2)$, and in a seagull with wings in $\tilde{B}(R_{A,c}, T_2')$ and $\tilde{B}(R_{A,c}, T_3)$. Moreover, if T_2 and T_2' can be chosen with $T_2 = T_2'$, then A_{T_1,T_2} and A_{T_2,T_3} are both big.

(27)

Let $\{i,j,k\} = \{1,2,3\}$ and let $N_k = N(w) \cap B(R_{A,c})_{ij}$. We may assume that all three of the sets N_1, N_2, N_3 are non-empty, for otherwise (27).1 holds. Since $w \in \tilde{W}(R_{A,c})$, it follows that $w \in \tilde{W}(P)$. By (14) $N(w) \subseteq \tilde{B}(P)$. Since P is stable, we deduce that $N(w)$ is not connected. Consequently, N_1 is not complete to $N_2 \cup N_3, N_2$ is not complete to $N_1 \cup N_3$, and N_3 is not complete to $N_1 \cup N_2$. If follows that there exist types T_1, T_2, T_2', T_3 of R such that $L(R)(T_1), L(R)(T_2), L(R)(T_3)$ are all distinct, $L(R)(T_2) = L(R)(T_2')$, and w is both in a seagull with wings in $\tilde{B}(R_{A,c}, T_1)$ and $\tilde{B}(R_{A,c}, T_2)$, and in a seagull with wings in $\tilde{B}(R_{A,c}, T_2')$ and $\tilde{B}(R_{A,c}, T_3)$. Now suppose T_2 and T_2' can be chosen with $T_2 = T_2'$. By the first bullet of the construction process of $L(R_{A,c})$ we may assume that A_{T_1,T_2} is big. We may also assume that A_{T_2,T_3} is small, for otherwise (27).2 holds. Now by the second bullet point of the construction of $L(R_{A,c})$, we deduce that w is also in a seagull with wings in $\tilde{B}(R_{A,c}, T_1)$ and $\tilde{B}(R_{A,c}, T_3)$. But now A_{T_1,T_3} is big (by the first bullet point of the construction of $L(R_{A,c})$), and again (27).2 holds. This proves (27).

Write $Q = P_{X,d}, S = Q_{Y,f}$ and $R = S_{Z,g}$

$|L(R_{A,c})(w)| < 3$ for every $w \in W(R_{A,c})$ with neighbors in all three of the sets $B(R_{A,c})_{ij}, B(R_{A,c})_{ik}, B(R_{A,c})_{jk}$ (here $\{i,j,k\} = \{1,2,3\}$).

(28)

We may assume that there exists $w \in \tilde{W}(R_{A,c})$ with neighbors in all three of the sets $B(R_{A,c})_{ij}, B(R_{A,c})_{ik}, B(R_{A,c})_{jk}$. Then $w \in \tilde{W}(Q_{Y,f})$ with a neighbor in all three of the sets $B(Q_{Y,f})_{ij}, B(Q_{Y,f})_{ik}, B(Q_{Y,f})_{jk}$. By (19) there exist distinct $i,j \in \{1,2,3\}$

such that both Y_i and Y_j are big. It follows from (21) that X_{12}, X_{13} and X_{23} are all small, and so by (16) every $w' \in \tilde{W}(Q)$ has at most one neighbor in $B(Q)_{ij}$ for every $i,j \in \{1,2,3\}$. Using (14) we deduce that every $w' \in \tilde{W}(R_{A,c})$ has at most one neighbor in $B(R_{A,c})_{ij}$ for every $i,j \in \{1,2,3\}$. So (27).2 holds for w and $T_2 = T_2'$. With the notation of (27).2, we may assume that $L(T_i) = i$; consequently w is both in a 23-seagull, and in a 12-seagull. Using symmetry, we may assume that Y_1 is big. Now by (25), it follows that Z_{23} is big. Let $Z_{23}' \subseteq Z_{23}$ be a flock of size exactly r. Let $A_{T_1,T_2} = \{p_1 - q_1 - r_1, \dots, p_B - q_B - r_B\}$, where $p_1, \dots, p_B \in \tilde{B}(R, T_1)$ and $r_1, \dots, r_B \in \tilde{B}(R, T_2)$ (recall that A_{T_1,T_2} is big by (27)). Since each of the bodies of the seagulls of Z_{23}' has at most one neighbor in $\tilde{B}(Q)_{23}$ (note that the bodies of Z_{23}' are in $\tilde{W}(Q)$), it follows that the set of bodies of Z_{23}' is anticomplete to at least $B - r = 2$ of $p_1, \dots, p_B$. We may assume that the set of bodies of Z_{23}' is anticomplete to $\{p_1, p_2\}$. Let $s \in T_1$. Then $M = q_2 - p_2 - s - p_1 - q_1 - r_1$ and $M' = q_1 - p_1 - s - p_2 - q_2 - r_2$ are copies of P_6 in G. If every vertex of M has neighbors in at most one seagull of Y_1, then $V(M)$ is anticomplete to at least $N - 6 \geq r$ seagulls of Y_1, contrary to the fact that G is $(P_6 + rP_3)$-free. By (18) we may assume that p_1 has neighbors in at least two seagulls of Y_1, say $a_1 - b_1 - c_1$ and $a_2 - b_2 - c_2$. It follows that $q_1, b_1, b_2 \in \tilde{W}(P)$. Now (14) implies that $p_1, a_1, c_1, a_2, c_2 \in \tilde{B}(P)$, and since P is stable, p_1 is not complete to either of $\{a_1, c_1\}, \{a_2, c_2\}$. Also, since $L(R_{A,c})(a_1) = L(R_{A,c})(c_1) = L(R_{A,c})(a_2) = L(R_{A,c})(c_2) = \{1\}$, it follows that $L(R_{A,c})(b_1) \neq \{1\}$ and $L(R_{A,c})(b_2) \neq \{1\}$, and since $p_1 \in \tilde{B}(R_{A,c})_{23}$, we have that p_1 is anticomplete to $\{b_1, b_2\}$. We deduce that $G|\{p_1, a_1, b_1, c_1, a_2, b_2, c_2\}$ contains an induced P_6, say K. Recall that each of b_1, b_2 has at most one neighbor in each of $B(Q)_{12}$ and$B(Q)_{13}$ (namely a_1, c_1 and a_2, c_2, respectively) from the construction of $Q_{Y,f}$. Since $V(Y_1) \subseteq S(Q_{Y,f})$, it follows that $\{a_1, b_1, c_1, a_2, b_2, c_2\}$ is anticomplete to $V(Z_{23})$. Now, since Z_{23}' are all 23-typed seagulls of $S = Q_{Y,f}$ with bodies anticomplete to p_1, and since $p_1 \in \tilde{B}(R)_{23} \subseteq \tilde{B}(S)_{23} \cup \tilde{W}(S)$, it follows that $\{p_1, a_1, b_1, c_1, a_2, b_2, c_2\}$ is anticomplete to $V(Z_{23}')$. Since Z_{23}' is a flock of size r, this contradicts the fact that G is $(P_6 + rP_3)$-free. This proves (28).

$$\begin{aligned}&\text{If } w \in \tilde{W}(R_{A,c}) \text{ has neighbors in } B(R_{A,c})_{ij} \text{ and in } B(R_{A,c})_{ik},\\ &\text{then for every pair of types } T_1, T_2 \text{ with } L(R)(T_1) = k \text{ and } L(R)(T_2) = j,\\ &\text{we have that } A(T_1, T_2) \text{ is small.}\end{aligned} \tag{29}$$

By (28) we may assume that w is anticomplete to $\tilde{B}(R_{A,c})_{jk}$. Suppose that there exist T_1, T_2 as above with $A(T_1, T_2)$ big. It follows that Z_i is big (since if Z_i is small, we would set $guess(T_i, T_j) = \emptyset$ and then $A(T_1, T_2) = \emptyset$ is small). But now by (21) both Y_{ij} and Y_{ik} are small, and so by (16) $deg(w) < 2$. By (14) we get a contradiction to the fact that P is stable. This proves (29).

Now we construct an augmented r-seeded precoloring $M_{A,c}$ as follows. If $|L(R_{A,c})(w)| = 2$ for every $w \in W(R_{A,c})$, let $M_{A,c} = (R_{A,c}, \emptyset)$. Now we may assume that $\tilde{W}(R_{A,c}) \neq \emptyset$. Let W_l be the set of $w \in \tilde{W}(R_{A,c})$ with neighbors in exactly l of the sets $B(R_{A,c})_{ij}$. By (28), $W_3 = \emptyset$. Let $G' = G \setminus (W_1 \cup W_2)$, and let $\mathcal{X}$ be the set of all the non-empty sets $N(w) \cap B(R_{A,c})_{ij}$ with $w \in W_2$. Let $M_{A,c} = ((G', L(R_{A,c}), S(R_{A,c}), \mathcal{X})$.

Let $\mathcal{M}(R)$ be the set of all the augmented r-seeded precolorings $M_{A,c}$ where $A \in \mathcal{C}(R)$ and c is a consistent coloring of A.

$$|\mathcal{M}(R)| \text{ is polynomial.} \tag{30}$$

The number of possible pairs (T_i, T_j) where $T_i, T_j \in \mathcal{T}$ and $L(R)(T_i) \neq L(R)(T_j)$ is at most $t^2 \leq 2^{2|S(R)|}$. By (17), (22), (26), there is a constant D that depends on r but not on G, such that $|S(R)| \leq D$. Since $|V_{T_i,T_j}| \leq max(3B, f(B))$ for every T_i, T_j, it follows that for every $A \in \mathcal{C}(R)$, $|V(A)| \leq 2^{2D} max(3B, f(B))$. Let $K = 2^{2D} max(3B, f(B))$. Then there are at most $|V(G)|^K$ choices for the members of $\mathcal{C}$, and so $|\mathcal{C}(R)| \leq |V(G)|^K$. Moreover, for every $A \in \mathcal{C}(R)$, the number of colorings of $G|V(A)$ is at most $3^{|V(A)|} \leq 3^K$. Since $|\mathcal{M}(R)|$ is at most the total number of pairs (A, c) where $A \in \mathcal{C}(R)$ and c is a coloring of $G|V(A)$, we deduce that $|\mathcal{M}(R)| \leq (3|V(G)|)^K$. This proves (30).

Finally, let $\mathcal{L} = \bigcup_{Q \in \mathcal{L}_R} \bigcup_{S \in \mathcal{L}_S(Q)} \bigcup_{R \in \mathcal{L}_T(S)} \mathcal{M}(R)$.

It follows from (17), (22), (26) and (30) that $|\mathcal{L}|$ is polynomial. Moreover, for every $(M, \mathcal{X}) \in \mathcal{L}$, $|L(M)(v)| \leq 2$ for every $v \in V(G(M))$, and $|\mathcal{X}| \leq 2|V(G)|$; thus $(M, \mathcal{X})$ is tractable. It remains to show that $\mathcal{L}$ is equivalent to P. Suppose $(M, \mathcal{X}) \in \mathcal{L}$ has a precoloring extension d. We observe that M is an r-seeded precoloring (G', L', S'), where G' is an induced subgraph of G, and $L'(v) \subseteq L(v)$ for every $v \in V(G')$. Thus a precoloring extension c of M is also a precoloring extension of (G', L, S). Next we observe that if $v \in V(G) \setminus V(G')$, then $|L(w)| = 3$, and either

- there exist $i, j \in \{1, 2, 3\}$ such that $L'(n) = \{i, j\}$ for every $n \in N(v)$, or
- there exist $X_1, X_2 \in \mathcal{X}$ such that $N(v) = X_1 \cup X_2$.

In both cases we have $L(v) \setminus c(N(v)) \neq \emptyset$, and so c can be extended to a precoloring extension of P.

Now we show the converse. Let c be a precoloring extension of P. We will construct $(M, \mathcal{X}) \in \mathcal{L}$ that has a precoloring extension. For every $i, j \in \{1, 2, 3\}$, let $X_{ij} = char_{R,M}(P, i, j, c)$, and let $X = (X_{12}, X_{21}, X_{13}, X_{31}, X_{23}, X_{32})$. Then c is a consistent coloring of $V(X)$, and so $Q = Q_{X,c} \in \mathcal{L}_R$. We claim that c is a precoloring extension of Q. Suppose that $c(v) \notin L(Q)(v)$ for some $v \in V(G)$. Let P' be the precoloring obtained from P by moving $V(X)$ to the seed with $c|V(X)$. Then there exist $i, j \in \{1, 2, 3\}$ such that either

- X_{ij} is small, $v \in \tilde{W}(P')$, $|N(v) \cap B(P')_{ij}| > 1$, and $c(v) = j$, or
- X_{ij} is big, $v \in \tilde{B}(P', i) \cap \tilde{B}(P)$, and v is adjacent to the bodies of at least two seagulls of X_{ij}, and $c(v) = i$.

In the former case Lemma 14.1 implies that v has a neighbor in $B(P')_{ij}$ of color j in c, and in the latter case Lemma 14.2 immediately implies that $c(v) \neq i$, in both cases a contradiction. This proves that c is a precoloring extension of Q.

Let $Y_i = char_{S,N}(Q, i, c)$, and let $Y = (Y_1, Y_2, Y_3)$. Then c is a consistent coloring of $V(Y)$, and so $S = Q_{Y,c} \in \mathcal{L}_S(Q)$. We claim that c is a precoloring extension of S. Suppose that $c(v) \notin L(S)(v)$ for some $v \in V(G)$. Let Q' be the precoloring obtained from Q by moving $V(Y)$ to the seed with $c|V(Y)$. Then there exist $\{i, j, k\} = \{1, 2, 3\}$ such that

– X_i and X_j are small, $v \in \tilde{W}(Q')$ has a neighbor in each of the sets $\tilde{B}(Q')_{ij}, \tilde{B}(Q')_{ik}, \tilde{B}(Q')_{jk}$, and $c(v) = k$, or
– for $l \in \{i,j\}, X_l$ is big, $v \in \tilde{B}(Q', l) \cap \tilde{B}(Q)$ and v is adjacent to the body of at least two seagulls of X_l, and $c(v) = l$

We deal with the first bullet first. In the case of the first bullet Lemma 15.1 implies that

1. v does not have neighbors $n \in B(Q')_{ij}$ and $n' \in B(Q')_{ik}$ with $c(n) = c(n') = i$, and
2. v does not have neighbors $m \in B(Q')_{ij}$ and $m' \in B(Q')_{jk}$ with $c(n) = c(n') = j$.

We claim that v has a neighbor $n'' \in B(Q')$ with $c(n'') = k$. Suppose not. Then $c(N(v) \cap B(Q')_{ik}) = i$, and therefore $c(N(v) \cap B(Q')_{ij}) = j$. But also $c(N(v) \cap B(Q')_{jk}) = j$, and therefore $c(N(v) \cap B(Q')_{ij}) = i$, a contradiction. This proves that the first bullet above does not happen. If the case of the second bullet Lemma 15.2 immediately implies that $c(v) \neq i$, and in the case of the third bullet Lemma 15.2 immediately implies that $c(v) \neq j$. Thus we get a contradiction in all cases. This proves that c is a precoloring extension of S.

Let $i,j \in \{1,2,3\}$. Let $Z_{ij} = char_{h,T}(S,i,j,c)$. Let $Z = (Z_{12}, Z_{13}, Z_{23})$. Then c is a consistent coloring of $V(Z)$. Let $R = S_{Z,c}$, then $R \in \mathcal{L}_T(S)$. We claim that c is a precoloring extension of R. Suppose that $c(v) \notin L(R)(v)$ for some $v \in V(G)$. Let S' be the precoloring obtained from S by moving $V(X)$ to the seed with $c|V(X)$. Then there exists $\{i,j,k\} = \{1,2,3\}$ such that Z_{ij} is small, $Z_1^{ij} = \{x_1 - y_1 - z_1, \dots, x_m - y_m - z_m\}$ and, with the notation of the definition of an (S, i, j, c)-key, either

– there exists $l \in \{1, \dots, m\}$, $v - q - r$ is an ij-typed seagull of S' related to $x_l - y_l - z_l$, and $c(v) = k$, or
– $p - v - r$ is an ij-typed seagull of $S', Z_1^{ij} \cup \{p - v - r\}$ is a flock, and $c(v) = k$.

Suppose first that the first case happens. Since $S_l = \emptyset$, it follows from the definition of a key that no seagull related to $x_l - y_l - z_l$ has a wing colored k in c, a contradiction. Thus the second bullet holds, and we get a contradiction to the fact that Z_1^{ij} is a maximal flock of ij-colored seagulls. This proves that c is a precoloring extension of R.

For every pair of types T_i, T_j of $S(R)$ with $L(R)(T_i) = i$ and $L(R)(T_j) = j$, let $A_{T_i,T_j} = char_B(R, T_i, T_j, c)$. Note that, by Lemma 15.1, if Y_k is small, then no vertex of $\tilde{W}(R)$ has neighbors $n \in B(R, T_i)$ and $n' \in B(R, T_j)$ with $c(n) = c(n') = k$, and therefore $A_{T_i,T_j} = \emptyset$. Let $A = (A_{T_i,T_j})$ (so A is a vector indexed by pairs T_i, T_j and $A \in \mathcal{C}(R)$). Then c is a consistent coloring of $V(A)$. Let $D = R_{A,c}$. Let R' be the seeded precoloring obtained from R by moving $V(A)$ to the seed with c. We claim that c is a precoloring extension of D.

Suppose $c(v) \notin L(D)(v)$ for some $v \in V(G)$. Then there exist types T_i, T_j, T_k of R where $\{i,j,k\} = \{1,2,3\}$ and $L(R)(T_l) = l$ for every $l \in \{1,2,3\}$, and such that (please note that we are still dealing with types of R, and not with types of R')

– A_{T_i,T_j} and A_{T_j,T_k} are both small, $v \in \tilde{W}(R')$ is in both in a seagull with wings in $B(R,T_i)$ and $B(R,T_j)$, in a seagull with wings in $B(R,T_i)$ and $B(R,T_k)$, and $c(v)=i$, or
– A_{T_i,T_j} is small, and $N(v) \cap \tilde{B}(R,T_i)$ is complete to $N(v) \cap \tilde{B}(R,T_k)$, and $v \in \tilde{W}(R')$ is in a seagull with wings in $\tilde{B}(R,T_i)$ and $\tilde{B}(R,T_j)$, and $c(v)=i$.

Now Lemma 16.1 implies that (note that $\tilde{W}(R') \subseteq \tilde{W}(R)$)

1. v does not have neighbors $n \in B(R',T_i)$ and $n' \in B(R',T_j)$ with $c(n)=c(n')=k$, and
2. v does not have neighbors $m \in B(R',T_i)$ and $m' \in B(R',T_k)$ with $c(m)=c(m')=j$.

We claim that v has a neighbor in $n'' \in B(R')$ with $c(n'')=i$. Suppose not. Then $c(N(v) \cap B(R',T_j))=k$, and therefore $c(N(v) \cap B(R',T_i))=j$. Also, $c(N(v) \cap B(R',T_k))=j$, and therefore $c(N(v) \cap B(R',T_i))=k$, a contradiction. This proves that c is a precoloring extension of D.

Finally, we construct an augmented seeded precoloring $M_{A,c}$. If $|L(D)(w)|=2$ for every $w \in W(D)$, let $M_{A,c}=(D,\emptyset)$. Then c is a coloring of $M_{A,c}$ and $M_{A,c} \in \mathcal{M}(R)$, as required. Thus we may assume that there exists $w \in \tilde{W}(D)$. It follows from (28), (29) and

Lemma 16.1 that for every $\{i,j,k\}=\{1,2,3\}$, and every pair T_i, T_j with $L(T_i)=i$ and $L(T_j)=j, w$ does not have neighbors $n \in B(R',T_i)$ and $n' \in B(R',T_j)$ with $c(n)=c(n')=k$. For $l \in \{1,2,3\}$ let W_l be the set of vertices $w \in \tilde{W}(D)$ with neighbors in exactly l of the sets $B(D)_{ij}$. By (28), $W_3=\emptyset$.

$$\text{Let } w \in W_2. \text{ We claim that each of the sets } N(w) \cap B(D)_{ij} \text{ is monochromatic in } c. \tag{31}$$

Suppose not; we may assume that $1,2 \in c(N(w) \cap B(D)_{12})$. It follows that $c(N(w) \cap (B(D)_{13} \cup B(D)_{23}))=3$, and so all three colors appear in $N(w)$, contrary to the fact that c is a precoloring extension of D. This proves (31).

Let $G'=G \setminus (W_1 \cup W_2)$, and let $\mathcal{X}$ be the set of all the non-empty sets $N(w) \cap B(D)_{ij}$ with $w \in W_2$. Let $M_{A,c}=((G',L(D),S(D)),\mathcal{X})$. Then $M_{A,c} \in \mathcal{M}(D)$, and by (31) c is a precoloring extension of $M_{U,c}$, as required. This proves Lemma 19. □

7 The Complete Algorithm

We can now prove Theorem 7 which we restate.

Theorem 9 *The* LIST 3-COLORING PROBLEM *can be solved in polynomial time for the class of* (P_6+rP_3)*-free graphs.*

Proof The proof is by induction on r. For $r = 0$, the result follows from Theorem 2, so we may assume that $r \geq 1$. Let G be a $(P_6 + rP_3)$-free graph and let $\tilde{L}$ be a 3-list assignment for G. We can test (by enumeration) if there exists $X \subseteq V(G)$ with $|X| \leq 4r + 8$ such that $(G|X, \tilde{L})$ is not colorable. If such X exists, stop and output that $(G, \tilde{L})$ is not colorable.

We may assume that G contains an induced $P_6 + (r-1)P_3$. Let $S \subseteq V(G)$ be such that $G|S = P_6 + (r-1)P_3$. For every precoloring $(G, \tilde{L}, S, L)$ of $(G, \tilde{L})$, $P_L = (G, L, S)$ is an r-seeded precoloring. Since $|S| = 3r + 6$, it follows that the number of such r-seeded precolorings is at most 3^{3r+6}. For each P_L as above, let $\mathcal{L}_1(P_L)$ be be as in Lemma 11, and let $\mathcal{L}_1 = \bigcup_{P_L} \mathcal{L}_1(P_L)$. Then $|\mathcal{L}_1| \leq 3^{3r+6}|V(G)|^{g_1(3r+6)}$ and every member of $\mathcal{L}_1$ is nice or easy and has seed of size at most $g_1(3r+6)$.

For every $P' \in \mathcal{L}_1$ proceed as follows. If P' is easy, set $\mathcal{L}(P') = \{P'\}$. Next assume that P' is nice. Let $\mathcal{L}_2(P')$ be as in Lemma 13. Then $|\mathcal{L}_2(P')| \leq |V(G)|$, every member of $\mathcal{L}_2(P')$ is stable and has seed of size at most $g_1(3r+6)$. Now for every $P'' \in \mathcal{L}_2(P')$ let $\mathcal{L}_3(P'')$ be as in Lemma 19. Then $|\mathcal{L}_3(P'')| \leq |V(G)|^{g(g_1(3r+6))}$. Let $\mathcal{L}(P') = \bigcup_{P'' \in \mathcal{L}_2(P')} \mathcal{L}_3(P'')$. Finally, let $\mathcal{L} = \bigcup_{P' \in \mathcal{L}_1} \mathcal{L}(P')$. It follows that $|\mathcal{L}|$ is polynomial.

It is now enough to test in polynomial time if each member of $\mathcal{L}$ has a precoloring extension. Let $Q \in \mathcal{L}$. It follows from the construction of $\mathcal{L}$ that Q is either a tractable augmented r-seeded precoloring, or an easy r-seeded precoloring. If Q is a tractable augmented r-seeded precoloring, then a coloring of Q, or a determination that none exists, can be found by Lemma 2. Thus we may assume that Q is an easy r-seeded precoloring. It is now enough to test if $(G \setminus X^0(L(Q)), L(Q))$ is colorable, and find a coloring if one exists. Since $V(G) \setminus X^0(L(Q)) \subseteq B(Q) \cup W(Q)$, this can be done by Theorem 2. This proves Theorem 9. □

8 A Hardness Result

A graph $G = (V, E)$ is said to be *k-critical* if $\chi(G) = k$ and $\chi(G - v) < k$ for any vertex $v \in V$. A k-critical graph G is *nice* if G contains three pairwise non-adjacent vertices c_1, c_2 and c_3 such that $\omega(G - \{c_1, c_2, c_3\}) = \omega(G) = k - 1$. For instance, any odd cycle of length at least seven with any 3-vertex stable contained in it is a nice 3-critical graph. The graph H^* with its vertices c_1, c_2 and c_3 (see Fig. 1) is a nice 4-critical graph.

In[2], the following generic framework of showing *NP*-completeness of the k-coloring problem was proposed. Let I be a 3-SAT instance with variables $x_1, x_2, \dots, x_n$ and clauses $C_1, C_2, \dots, C_m$. Let H be a nice k-critical graph. We construct a graph $G_{H,I}$ as follows.

- For each variable x_i there is a *variable component* T_i consisting of two adjacent vertices x_i and $\overline{x_i}$. Call these vertices *X-type*.
- For each variable x_i there is a vertex d_i. Call these vertices *D-type*.
- For each clause $C_j = y_{i_1} \vee y_{i_2} \vee y_{i_3}$ where y_{i_t} is either x_{i_t} or $\overline{x_{i_t}}$ there is a *clause component* H_j that is isomorphic to H. Denote the three specified pair-wise

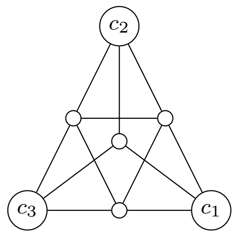

Fig. 1 A nice 4-critical graph H^*

non-adjacent vertices in H_j by $c_{i,j}$ for $t = 1, 2, 3$. Vertices $c_{i,j}$ are referred to as *C-type* and all remaining vertices in H_j are referred to as *U-type*.
- Add an edge between every vertex of U-type and every vertex of X-type or D-type.
- For each C-type vertex c_{ij} we say that x_i or $\overline{x_i}$ is its *literal vertex* depending on whether $x_i \in C_j$ or $\overline{x_i} \in C_j$. Add an edge between c_{ij} and its literal vertex.
- For each C-type vertex c_{ij} add an edge between c_{ij} and d_i.

Lemma 20 ([2]) *A* 3-SAT *instance I is satisfiable if and only if $G_{H,I}$ is $(k+1)$-colorable.*

Now we use the generic framework to prove Theorem 8 which we restate.

Theorem 10 *The k* -COLORING PROBLEM *restricted to $(P_5 + P_2)$-free graphs is NP-hard for $k \geq 5$.*

Proof First we show:

$$\begin{aligned}&\text{Let } I \text{ be a 3-SAT instance and } H \text{ be a nice } k\text{-critical graph.}\\ &\text{If } H \text{ is } P_5\text{-free, then } G_{H,I} \text{ is } (P_5+P_2)\text{-free.}\end{aligned} \tag{32}$$

Suppose that $G_{H,I}$ contains an induced $Q = Q_1 + Q_2$ where Q_1 and Q_2 are isomorphic to a P_5 and a P_2, respectively. Let C_i (respectively $\overline{C_i}$) be the set of C-type vertices that connect to x_i (respectively $\overline{x_i}$). We observe that each connected component of $G - U$ has a specific structure, namely it is the result of substituting stable sets into a 5-cycle (and possibly removing some vertices). Specifically, the 5 stable sets are, in the cyclical order, $X_0 = \{x_i\}, X_1 = C_i, X_2 = \{d_i\}, X_3 = \overline{C_i}$, and finally $X_4 = \{\overline{x_i}\}$. This subgraph does not contain an induced P_5, since the 5-cycle does not and substituting stable sets cannot create a P_5. This implies that $Q_1 \cap U \neq \emptyset$. Since U is complete to $X \cup D, Q_2 \subseteq U \cup C$. Since C is an stable set, this implies that $Q_2 \cap U \neq \emptyset$ and thus $Q_1 \subseteq U \cup C$. This means that Q_1 is entirely contained in some clause component. This, however, contradicts the assumption that H is P_5-free. This proves (32).

Now observe that the graph H^* (Fig. 1) is P_5-free. It follows then from Lemma 20 and (32) that 5-COLORING $(P_5 + P_2)$-free graphs is NP-hard. □

9 Conclusions

In this paper, we introduced a new proof technique, which involves applying a hypergraph theorem in [11], and used it to solve the list 3-coloring problem restricted to the class of $(P_6 + rP_3)$-free graphs. This technique (or its extensions) may be applied to other coloring problems with forbidden induced subgraphs, such as the list 3-coloring problem on the family of $(P_4 + P_6)$-free graphs. We have not explored any of these directions as of yet. Besides the positive results, we have also proved that the k-coloring problem on $(P_5 + P_2)$-free graphs is NP-hard for every fixed $k \geq 5$.

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