MATRIX ALGEBRAS WITH A CERTAIN COMPRESSION PROPERTY I

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Abstract. An algebra $\mathcal{A}$ of $n \times n$ complex matrices is said to be *projection compressible* if $P\mathcal{A}P$ is an algebra for all orthogonal projections $P \in \mathbb{M}_n(\mathbb{C})$. Analogously, $\mathcal{A}$ is said to be *idempotent compressible* if $E\mathcal{A}E$ is an algebra for all idempotents E in $\mathbb{M}_n(\mathbb{C})$. In this paper we construct several examples of unital algebras that admit these properties. In addition, a complete classification of the unital idempotent compressible subalgebras of $\mathbb{M}_3(\mathbb{C})$ is obtained up to similarity and transposition. It is shown that in this setting, the two notions of compressibility agree: a unital subalgebra of $\mathbb{M}_3(\mathbb{C})$ is projection compressible if and only if it is idempotent compressible. Our findings are extended to algebras of arbitrary size in [2].

§1 Introduction

In this paper we examine the following question: *Which unital subalgebras $\mathcal{A}$ of $\mathbb{M}_n(\mathbb{C})$ have the property that $P\mathcal{A}P$ is an algebra for all orthogonal projections $P \in \mathbb{M}_n(\mathbb{C})$?*

Since for every orthogonal projection P one may decompose $\mathcal{A}$ as an algebra of block 2×2 matrices with respect to the orthogonal decomposition $\mathbb{C}^n = ran(P) \oplus \ker(P)$, this question may be restated as follows: *Which unital subalgebras $\mathcal{A}$ of $\mathbb{M}_n(\mathbb{C})$ have the property that with respect to every orthogonal direct sum decomposition $\mathbb{C}^n = ran(P) \oplus \ker(P)$, the compression of $\mathcal{A}$ to the $(1,1)$-corner is an algebra of linear maps acting on $ran(P)$?* This condition will be known as the *projection compression property.* If $\mathcal{A}$ is a subalgebra of $\mathbb{M}_n(\mathbb{C})$ for which this property holds, we say that $\mathcal{A}$ is *projection compressible.*

Of course, one's attention need not be restricted to just the *orthogonal* direct sum decompositions of $\mathbb{C}^n$. If $\mathcal{A}$ is a subalgebra of $\mathbb{M}_n(\mathbb{C})$ such that $E\mathcal{A}E$ is an algebra for all idempotents $E \in \mathbb{M}_n(\mathbb{C})$, we shall say that $\mathcal{A}$ exhibits the *idempotent compression property* or that $\mathcal{A}$ is *idempotent compressible.* As in the case of projections, the idempotent compression property can be stated in terms of the compressions of an algebra to the $(1,1)$-corner with respect to each (potentially non-orthogonal) decomposition $\mathbb{C}^n = ran(E) \dot{+} \ker(E)$. It is clear that any algebra possessing the idempotent compression property must also be projection compressible.

If $E \in \mathbb{M}_n(\mathbb{C})$ is an idempotent, then the corner $E\mathcal{A}E$ is always a linear space. This means that $E\mathcal{A}E$ is an algebra if and only if it is multiplicatively closed. It is easy to see that this holds trivially for any idempotent from the algebra $\mathcal{A}$ itself. Furthermore, dimension considerations imply that this is also true for any idempotent of rank 1. It follows that any subalgebra of $\mathbb{M}_2(\mathbb{C})$ is trivially idempotent compressible, and hence projection compressible as well. Our study will therefore only concern subalgebras of $\mathbb{M}_n(\mathbb{C})$ for integers $n \geq 3$.

While it is immediate from the definitions that every idempotent compressible algebra is also projection compressible, the converse is much less clear. As will be shown in §2 and §3, all of our preliminary examples indicate either the presence of the idempotent compression property or the absence of the projection compression property, thus providing evidence to the affirmative. Despite this evidence, our attempts at obtaining an intrinsic proof of the equivalence of these notions have been unsuccessful. Instead, a systematic case-by-case analysis is used to investigate whether or not such an equivalence exists. Our analysis reveals that the techniques for studying the compression properties for subalgebras of $\mathbb{M}_3(\mathbb{C})$ differed significantly from those used for subalgebras of $\mathbb{M}_n(\mathbb{C})$ when $n \geq 4$. For this reason, our study has been divided into two parts.

Our examination begins in §2 by introducing the notation and basic theory surrounding these notions of compressibility. This is followed by §3 in which we investigate these properties in various concrete examples. As we shall see, the unital idempotent compressible algebras constructed in this section form an exhaustive

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list in $\mathbb{M}_3(\mathbb{C})$ up to similarity and transposition. In order to show that this is the case, we will require certain results on the structure theory for matrix algebras outlined in §4. We then devote §5 to the classification of unital idempotent compressible subalgebras of $\mathbb{M}_3(\mathbb{C})$, ultimately proving that in this setting, the notions of projection compressibility and idempotent compressibility coincide.

In [2], the sequel to this paper, our attention is devoted to the unital subalgebras of $\mathbb{M}_n(\mathbb{C})$ when $n \geq 4$. The main result, [2, Theorem 6.1.1], states that the two notions of compressibility agree in this setting as well. In fact, it is shown that up to similarity and transposition, the unital algebras admitting one (and hence both) of the compression properties are exactly those outlined in §3 of this paper.

§2 Preliminaries

In this section we will introduce some basic results on the algebras admitting the one or both of the compression properties. Our first task is to establish the notation and terminology that will be used throughout.

Since we will only be concerned with algebras of $n \times n$ matrices over $\mathbb{C}$, we will write $\mathbb{M}_n$ in place of $\mathbb{M}_n(\mathbb{C})$ from here on.

Notation. Given vectors $x, y \in \mathbb{C}^n$, define $x \otimes y^* : \mathbb{C}^n \to \mathbb{C}^n$ to be the rank-one operator $z \mapsto \langle z, y \rangle x$.

Definition 2.0.1. If A is an $n \times n$ matrix written with respect to a fixed orthonormal basis for $\mathbb{C}^n$, then the *anti-transpose* of A is the matrix

$$A^{aT} = JA^TJ,$$

where $J = J^*$ is the unitary matrix whose (i,j)-entry is $\delta_{j,n-i+1}$. If $\mathcal{A}$ is a subset of $\mathbb{M}_n$, then we will define the *transpose* and *anti-transpose* of $\mathcal{A}$ as

$$\mathcal{A}^T = \left\{A^T : A \in \mathcal{A}\right\} \quad \text{and} \quad \mathcal{A}^{aT} = \left\{A^{aT} : A \in \mathcal{A}\right\},$$

respectively.

While transposition has the affect of reflecting a matrix about its main diagonal, anti-transposition has the affect of reflecting a matrix about its anti-diagonal (i.e., the diagonal from the $(n,1)$-entry to the $(1,n)$-entry). It is clear that if $\mathcal{A}$ is an algebra, then so too are $\mathcal{A}^T$ and $\mathcal{A}^{aT}$.

Definition 2.0.2. If $\mathcal{A}$ and $\mathcal{B}$ are subsets of $\mathbb{M}_n$ such that $\mathcal{A}$ or $\mathcal{A}^T$ is similar to $\mathcal{B}$, we will say that $\mathcal{A}$ and $\mathcal{B}$ are *transpose similar*. If $\mathcal{A}$ or $\mathcal{A}^T$ is unitarily equivalent to $\mathcal{B}$, we will say that $\mathcal{A}$ and $\mathcal{B}$ are *transpose equivalent*.

It is easy to verify that transpose similarity and transpose equivalence are equivalence relations that generalize the notions of similarity and unitary equivalence, respectively. Any algebra $\mathcal{A}$ and its transpose $\mathcal{A}^T$ are obviously transpose equivalent. Furthermore, since $\mathcal{A}^T = J\mathcal{A}^{aT}J$, we have that $\mathcal{A}$ and $\mathcal{A}^{aT}$ are transpose equivalent as well.

Proposition 2.0.3. *Let n be a positive integer, and let $\mathcal{A}$ and $\mathcal{B}$ be subalgebras of $\mathbb{M}_n$.*

(i) If $\mathcal{A}$ and $\mathcal{B}$ are transpose equivalent, then $\mathcal{A}$ is projection compressible if and only if $\mathcal{B}$ is projection compressible.

(ii) If $\mathcal{A}$ and $\mathcal{B}$ are transpose similar, then $\mathcal{A}$ is idempotent compressible if and only if $\mathcal{B}$ is idempotent compressible.

Proof. Part (i) follows from the observation that the set of projections in $\mathbb{M}_n$ is invariant under transposition and unitary equivalence. In a similar fashion, one may prove (ii) by noting that the set of idempotents in $\mathbb{M}_n$ is invariant under transposition and similarity. ∎

The following proposition states that if $\mathcal{A}$ is projection (resp. idempotent) compressible, then so too is its unitization $\mathcal{A} + \mathbb{C}I$. A counterexample following the proof of Corollary 2.0.9 demonstrates that the converse is false.

Proposition 2.0.4. *If $\mathcal{A}$ is a projection (resp. idempotent) compressible subalgebra of $\mathbb{M}_n$, then its unitization*

$$\tilde{\mathcal{A}} := \mathcal{A} + \mathbb{C}I$$

is projection (resp. idempotent) compressible.

Proof. Assume that $\mathcal{A}$ is projection compressible, and let P be an arbitrary projection in $\mathbb{M}_n$. Let $A, B \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$, so that $A + \alpha I$ and $B + \beta I$ define elements of $\tilde{\mathcal{A}}$. Since $PAP \cdot PBP$ belongs to $P\mathcal{A}P$, we can write $PAP \cdot PBP = PCP$ for some $C \in \mathcal{A}$. As a result,

$$\begin{aligned} P(A+\alpha I)P \cdot P(B+\beta I)P &= PAP \cdot PBP + \beta PAP + \alpha PBP + (\alpha\beta)I \\ &= P(C + \beta A + \alpha B)P + (\alpha\beta)I. \end{aligned}$$

Therefore $P\tilde{\mathcal{A}}P$ is an algebra, so $\tilde{\mathcal{A}}$ is projection compressible.

One may obtain a proof for the case of idempotent compressibility by replacing the projection P in the above argument with a general idempotent E. ■

The following proposition describes an obvious sufficient condition for an algebra to exhibit the projection or idempotent compression property, and will be useful in building our first class of examples.

Proposition 2.0.5. *Let n be a positive integer, and let $\mathcal{A}$ be a subalgebra of $\mathbb{M}_n$. If $AEB \in \mathcal{A}$ for all A, $B \in \mathcal{A}$, and all idempotents (resp. projections) $E \in \mathbb{M}_n$, then $\mathcal{A}$ is idempotent (resp. projection) compressible.*

Proof. Let E be an idempotent (resp. projection) in $\mathbb{M}_n$. Given $A, B \in \mathcal{A}$, we have that $AEB \in \mathcal{A}$, and hence

$$(EAE)(EBE) = E(AEB)E$$

belongs to $E\mathcal{A}E$. Thus, $E\mathcal{A}E$ is an algebra. ■

The condition described in the above result strongly resembles the multiplicative absorption property satisfied by ideals. In particular, Proposition 2.0.5 implies that any (one- or two-sided) ideal of $\mathbb{M}_n$ exhibits the idempotent compression property. It will be shown in Corollary 2.0.9 that this property also holds for the intersection of one-sided ideals, or equivalently, the intersection of a single left ideal with a single right ideal. Thus, we make following definition.

Definition 2.0.6. If $\mathcal{A}$ is a subalgebra of $\mathbb{M}_n$ given by an intersection of a left ideal and a right ideal in $\mathbb{M}_n$, then $\mathcal{A}$ is said to be an *$\mathcal{LR}$-algebra*.

It is straightforward to show that any algebra that is transpose similar to an $\mathcal{LR}$-algebra $\mathcal{A}$ is again an $\mathcal{LR}$-algebra. Indeed, if $\mathcal{A} = \mathcal{L} \cap \mathcal{R}$ for some left ideal $\mathcal{L}$ and right ideal $\mathcal{R}$ of $\mathbb{M}_n$, then $\mathcal{R}^T$ is a left ideal, $\mathcal{L}^T$ is a right ideal, and $\mathcal{A}^T = \mathcal{R}^T \cap \mathcal{L}^T$. Hence, $\mathcal{A}^T$ is also an $\mathcal{LR}$-algebra. If $\mathcal{B}$ is transpose similar to $\mathcal{A}$, then by replacing $\mathcal{A}$ with $\mathcal{A}^T$ if necessary, we may assume that

$$\mathcal{B} = S^{-1}\mathcal{A}S = \left(S^{-1}\mathcal{L}S\right) \cap \left(S^{-1}\mathcal{R}S\right)$$

for some invertible $S \in \mathbb{M}_n$. Since $S^{-1}\mathcal{L}S$ and $S^{-1}\mathcal{R}S$ are left and right ideals of $\mathbb{M}_n$, respectively, $\mathcal{B}$ is again an $\mathcal{LR}$-algebra.

It is well known that the one-sided ideals in $\mathbb{M}_n$ can be described entirely in terms of projections. In particular, each left ideal of $\mathbb{M}_n$ has the form $\mathbb{M}_nQ$ for some orthogonal projection Q, while each right ideal has the form $P\mathbb{M}_n$ for some orthogonal projection P. More generally, we have the following classical ring-theoretic result concerning the $\mathbb{M}_n$-submodules of the space of complex $n \times p$ matrices (see [3, Theorem 3.3]). This result will be used in §5 and invoked extensively throughout the classification in [2].

Theorem 2.0.7. *Let n and p be positive integers.*

(i) If $\mathcal{S} \subseteq \mathbb{M}_{n\times p}$ is a left $\mathbb{M}_n$-module, then there is a projection $Q \in \mathbb{M}_p$ such that $\mathcal{S} = \mathbb{M}_{n\times p}Q$.
(ii) If $\mathcal{S} \subseteq \mathbb{M}_{p\times n}$ is a right $\mathbb{M}_n$-module, then there is a projection $Q \in \mathbb{M}_p$ such that $\mathcal{S} = Q\mathbb{M}_{p\times n}$.

Corollary 2.0.8. *A subalgebra $\mathcal{A}$ of $\mathbb{M}_n$ is an $\mathcal{LR}$-algebra if and only if there are projections P and Q in $\mathbb{M}_n$ such that $\mathcal{A} = P\mathbb{M}_nQ$.*

The description of $\mathcal{LR}$-algebras presented in Corollary 2.0.8 allows us to quickly verify that these algebras admit the idempotent compression property.

Corollary 2.0.9. *Every $\mathcal{LR}$-algebra is idempotent compressible.*

Proof. Let $\mathcal{A}$ be an $\mathcal{LR}$-algebra, so $\mathcal{A} = P\mathbb{M}_n Q$ for some projections P and Q. If E is an idempotent in $\mathbb{M}_n$, then for any $A, B \in \mathcal{A}$,

$$(PAQ)E(PBQ) = P(AQEPB)Q \in P\mathbb{M}_n Q = \mathcal{A}.$$

Thus, $\mathcal{A}$ satisfies the assumptions of Proposition 2.0.5 in the case of idempotents. We conclude that $\mathcal{A}$ is idempotent compressible.

■

The fact that $\mathcal{LR}$-algebras admit the idempotent compression property gives us a means to disprove the converse to Proposition 2.0.4. We will exhibit a subalgebra of $\mathbb{M}_3$ that is not projection compressible, but whose unitization is. Indeed, let $\{e_1, e_2, e_3\}$ denote the standard basis for $\mathbb{C}^3$ and for each i, let Q_i denote the orthogonal projection onto the span of $\{e_i\}$.

Consider the algebra $\mathcal{A} = \mathbb{C}(Q_1 + Q_2)$. Note that the unitization of $\mathcal{A}$ is also the unitization of the $\mathcal{LR}$-algebra

$$\mathcal{B} \coloneqq \mathbb{C}Q_3 = Q_3\mathbb{M}_3 Q_3.$$

By Corollary 2.0.9 and Proposition 2.0.4, $\tilde{A}$ is idempotent compressible, a fortiori, projection compressible.

To see that $\mathcal{A}$ is not projection compressible, consider the matrix

$$P = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix},$$

and note that $\frac{1}{3}P$ is a projection in $\mathbb{M}_3$. We claim that $\left(\frac{1}{3}P\right)\mathcal{A}\left(\frac{1}{3}P\right)$ is *not* an algebra. Of course, since $(\frac{1}{3}P)\mathcal{A}(\frac{1}{3}P)$ is an algebra if and only if $P\mathcal{A}P$ is an algebra, it suffices to prove that $P\mathcal{A}P$ is not multiplicatively closed.

One may verify that every element $B = (b_{ij})$ in $P\mathcal{A}P$ satisfies the equation $b_{22} + 5b_{23} = 0$. With $B = e_1 \otimes e_1^* + e_2 \otimes e_2^*$, however, we have that

$$(PBP)^2 = \begin{bmatrix} 42 & -39 & -3 \\ -39 & 42 & -3 \\ -3 & -3 & 6 \end{bmatrix}.$$

This matrix clearly does not satisfy the above equation, and hence $(PBP)^2$ does not belong to $P\mathcal{A}P$. Thus, $P\mathcal{A}P$ is not an algebra, so $\mathcal{A}$ is not projection compressible.

Remark 2.0.10. When determining whether or not a corner $P\mathcal{A}P$ is an algebra, it is often more computationally convenient to consider a multiple of the projection P rather than P itself. This simplification will frequently be used without mention.

§3 Examples

While $\mathcal{LR}$-algebras comprise a large collection of algebras that admit the idempotent compression property, they are not the only examples. The purpose of §3 is to expand our library of matrix algebras that admit one or both of the compression properties.

We begin with §3.1, which showcases three distinct families of idempotent compressible algebras that occur as subalgebras of $\mathbb{M}_n$ for each $n \geq 3$. The algebras outlined in this section will be important for the classification in [2]. In §3.2, we present three additional examples of idempotent compressible algebras that occur uniquely in the setting of 3×3 matrices.

§3.1 Subalgebras of $\mathbb{M}_n$, $n \geq 3$.

Example 3.1.1. *Let $n \geq 3$ be an integer. If Q_1, Q_2, and Q_3 are projections in $\mathbb{M}_n$ which sum to I, then*

$$\mathcal{A} \coloneqq \mathbb{C}Q_1 + (Q_1 + Q_2)\mathbb{M}_n(Q_2 + Q_3)$$

has the idempotent compression property. Consequently, its unitization

$$\widetilde{\mathcal{A}} = \mathbb{C}Q_1 + \mathbb{C}Q_3 + (Q_1 + Q_2)\mathbb{M}_n(Q_2 + Q_3)$$

has the idempotent compression property.

Proof. Define

$$\mathcal{A}_1 \coloneqq \mathbb{C}Q_1 \quad \text{and} \quad \mathcal{A}_2 \coloneqq (Q_1+Q_2)\mathbb{M}_n(Q_2+Q_3),$$

so that $\mathcal{A} = \mathcal{A}_1 \dotplus \mathcal{A}_2$. Let E be an idempotent in $\mathbb{M}_n$. We will show that $E\mathcal{A}E$ contains the product $E\mathcal{A}_iE \cdot E\mathcal{A}_jE$ for each choice of i and j.

Since $\mathcal{A}_2$ is an $\mathcal{LR}$-algebra, it is easy to see that $(E\mathcal{A}_2E)^2$ is contained in $E\mathcal{A}E$. What's more, the equation $Q_1 = (Q_1+Q_2)Q_1$ shows that $E\mathcal{A}_1E \cdot E\mathcal{A}_2E$ is contained in $E\mathcal{A}_2E$, and hence in $E\mathcal{A}E$. To see that $(E\mathcal{A}_1E)^2$ is contained in $E\mathcal{A}E$, write

$$(EQ_1E)^2 = EQ_1E - E(Q_1+Q_2)Q_1E \cdot E(Q_2+Q_3)E.$$

Finally, if $T \in \mathbb{M}_n$, then the equation

$$\begin{aligned} E(Q_1+Q_2)T(Q_2+Q_3)E \cdot EQ_1E = {}& E(Q_1+Q_2)T(Q_2+Q_3)E \\ & - E(Q_1+Q_2)T(Q_2+Q_3)E \cdot E(Q_2+Q_3)E, \end{aligned}$$

proves that $E\mathcal{A}_2E \cdot E\mathcal{A}_1E$ is contained in $E\mathcal{A}E$.

■

Remark 3.1.2. *If $\{e_1, e_2, e_3\}$ denotes the standard basis for $\mathbb{C}^3$, and for each i, Q_i denotes the orthogonal projection onto the span of $\{e_i\}$, then*

$$\mathcal{T}_3 \coloneqq \mathbb{C}Q_1 + \mathbb{C}Q_3 + (Q_1+Q_2)\mathbb{M}_n(Q_2+Q_3),$$

is the algebra of all 3×3 upper triangular matrices. By Example 3.1.1, this algebra is idempotent compressible.

Example 3.1.3. *Let $n \geq 3$ be an integer. If Q_1 and Q_2 are mutually orthogonal rank-one projections in $\mathbb{M}_n$, and $Q_3 = I - Q_1 - Q_2$, then*

$$\mathcal{A} \coloneqq \mathbb{C}Q_1 + \mathbb{C}Q_2 + (Q_1+Q_2)\mathbb{M}_nQ_3$$

has the idempotent compression property. Consequently, its unitization

$$\widetilde{\mathcal{A}} = \mathbb{C}Q_1 + \mathbb{C}Q_2 + \mathbb{C}Q_3 + (Q_1+Q_2)\mathbb{M}_nQ_3$$

has the idempotent compression property as well.

Proof. Define

$$\mathcal{A}_1 \coloneqq \mathbb{C}Q_1, \quad \mathcal{A}_2 \coloneqq \mathbb{C}Q_2, \quad \text{and} \quad \mathcal{A}_3 \coloneqq (Q_1+Q_2)\mathbb{M}_nQ_3,$$

so that $\mathcal{A} = \mathcal{A}_1 \dotplus \mathcal{A}_2 \dotplus \mathcal{A}_3$. Let E be an idempotent in $\mathbb{M}_n$. As in the previous proof, we will show that $E\mathcal{A}E$ contains the product $E\mathcal{A}_iE \cdot E\mathcal{A}_jE$ for all choices of i and j.

Note that $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ are $\mathcal{LR}$-algebras, so $E\mathcal{A}E$ contains $(E\mathcal{A}_iE)^2$ for all i. Moreover, it can be shown that $E\mathcal{A}_1E \cdot E\mathcal{A}_3E$ and $E\mathcal{A}_2E \cdot E\mathcal{A}_3E$ are contained in $E\mathcal{A}E$ by writing $Q_1 = (Q_1+Q_2)Q_1$ and $Q_2 = (Q_1+Q_2)Q_2$. From these inclusions it follows that $E\mathcal{A}_1E \cdot E\mathcal{A}_2E$ and $E\mathcal{A}_2E \cdot E\mathcal{A}_1E$ are contained in $E\mathcal{A}E$, as

$$EQ_1E \cdot EQ_2E = EQ_1E - EQ_1E \cdot EQ_1E - E(Q_1+Q_2)Q_1E \cdot EQ_3E,$$

and

$$EQ_2E \cdot EQ_1E = EQ_2E - EQ_2E \cdot EQ_2E - E(Q_1+Q_2)Q_2E \cdot EQ_3E.$$

The proof will be complete upon showing that $E\mathcal{A}_3E \cdot E\mathcal{A}_1E$ and $E\mathcal{A}_3E \cdot E\mathcal{A}_2E$ are contained in $E\mathcal{A}E$. To achieve this, observe that for any $T \in \mathbb{M}_n$, one has

$$E(Q_1+Q_2)TQ_3E \cdot EQ_1E = EQ_1TQ_3E \cdot EQ_1E - EQ_2TQ_3E \cdot EQ_2E + EQ_2T(I-Q_3E)Q_3E.$$

The summands on the right-hand side of this equation belong to $E\mathcal{A}_1E$, $E\mathcal{A}_2E$, and $E\mathcal{A}_3E$, respectively. Consequently, $E\mathcal{A}_3E \cdot E\mathcal{A}_1E$ is contained in $E\mathcal{A}E$. The inclusion $E\mathcal{A}_3E \cdot E\mathcal{A}_2E \subseteq E\mathcal{A}E$ can be deduced in a similar fashion.

■

It was fairly routine to verify that the algebras presented in Examples 3.1.1 and 3.1.3 admit the idempotent compression property. Showing that this condition holds for the algebra $\mathcal{A}$ in our next example is not so straightforward. We will first present two lemmas that showcase sufficient conditions for an arbitrary corner of this algebra to be an algebra itself. It will be shown in Example 3.1.6 that every such corner of $\mathcal{A}$ must satisfy one of these conditions. This will prove that the algebra is indeed idempotent compressible.

Lemma 3.1.4. *Let $n \geq 3$ be an integer, let Q_1 and Q_2 be mutually orthogonal rank-one projections in $\mathbb{M}_n$, and define $Q_3 \coloneqq I - Q_1 - Q_2$. Let $\mathcal{A}$ denote the subalgebra of $\mathbb{M}_n$ given by*

$$\mathcal{A} \coloneqq \mathbb{C}(Q_1 + Q_2) + Q_1\mathbb{M}_nQ_2 + (Q_1 + Q_2)\mathbb{M}_nQ_3.$$

If E is an idempotent in $\mathbb{M}_n$ and $E\mathcal{A}E$ contains EQ_2E, then $E\mathcal{A}E$ is an algebra.

Proof. Let E be a fixed idempotent in $\mathbb{M}_n$ and suppose that $EQ_2E \in E\mathcal{A}E$. If $\mathcal{A}_0$ denotes the algebra

$$\mathcal{A}_0 \coloneqq \mathbb{C}Q_1 + \mathbb{C}Q_2 + Q_1\mathbb{M}_nQ_2 + (Q_1 + Q_2)\mathbb{M}_nQ_3,$$

then as seen in Example 3.1.1, $\mathcal{A}_0$ is idempotent compressible. Consequently,

$$\begin{aligned} E\mathcal{A}E &= \mathbb{C}E(Q_1 + Q_2)E + EQ_1\mathbb{M}_nQ_2E + E(Q_1 + Q_2)\mathbb{M}_nQ_3E \\ &= \mathbb{C}EQ_1E + \mathbb{C}EQ_2E + EQ_1\mathbb{M}_nQ_2E + E(Q_1 + Q_2)\mathbb{M}_nQ_3E \\ &= E\mathcal{A}_0E \end{aligned}$$

is an algebra.

■

Lemma 3.1.5. *Let $n \geq 3$ be an integer, let Q_1 and Q_2 be mutually orthogonal rank-one projections in $\mathbb{M}_n$, and define $Q_3 \coloneqq I - Q_1 - Q_2$. Let $\mathcal{A}$ denote the subalgebra of $\mathbb{M}_n$ given by*

$$\mathcal{A} \coloneqq \mathbb{C}(Q_1 + Q_2) + Q_1\mathbb{M}_nQ_2 + (Q_1 + Q_2)\mathbb{M}_nQ_3.$$

If E is an idempotent in $\mathbb{M}_n$ such that $EQ_1 = Q_1$, then $E\mathcal{A}E$ is an algebra.

Proof. Let E be an idempotent in $\mathbb{M}_n$ such that $EQ_1 = Q_1$. Define

$$\mathcal{A}_1 \coloneqq \mathbb{C}(Q_1 + Q_2), \quad \mathcal{A}_2 \coloneqq Q_1\mathbb{M}_nQ_2, \quad \text{and} \quad \mathcal{A}_3 \coloneqq (Q_1 + Q_2)\mathbb{M}_nQ_3,$$

so that $\mathcal{A} = \mathcal{A}_1 \dotplus \mathcal{A}_2 \dotplus \mathcal{A}_3$. As in the previous examples, we will show that $E\mathcal{A}E$ contains the product $E\mathcal{A}_iE \cdot E\mathcal{A}_jE$ for all i and j.

Since $\mathcal{A}_2$ and $\mathcal{A}_3$ are $\mathcal{LR}$-algebras, it is easy to see that $E\mathcal{A}E$ contains $(E\mathcal{A}_2E)^2$ and $(E\mathcal{A}_3E)^2$. Moreover, it is clear that $E\mathcal{A}_2E \cdot E\mathcal{A}_3E$ is contained in $E\mathcal{A}_3E$, and hence in $E\mathcal{A}E$. Observe that since the algebra

$$\mathcal{A}_0 \coloneqq \mathcal{A}_1 \dotplus \mathcal{A}_3$$

was shown to be idempotent compressible in Example 3.1.1, we have that $E\mathcal{A}_1E \cdot E\mathcal{A}_3E$, $E\mathcal{A}_3E \cdot E\mathcal{A}_1E$, and $(E\mathcal{A}_1E)^2$ are contained in $E\mathcal{A}_0E \subseteq E\mathcal{A}E$. Proving these inclusions directly is also straightforward.

The equation $EQ_1 = Q_1$ will now be used to obtain the remaining inclusions. We have that for all S and T in $\mathbb{M}_n$,

$$E(Q_1 + Q_2)SQ_3E \cdot EQ_1TQ_2E = 0,$$

$$E(Q_1 + Q_2)E \cdot EQ_1TQ_2E = EQ_1TQ_2E,$$

and

$$EQ_1TQ_2E \cdot E(Q_1 + Q_2)E = EQ_1(TQ_2E)Q_2E.$$

The right-hand side of each expression above is easily seen to belong to $E\mathcal{A}E$. As a result, $E\mathcal{A}E$ contains $E\mathcal{A}_3E \cdot E\mathcal{A}_2E$, $E\mathcal{A}_1E \cdot E\mathcal{A}_2E$, and $E\mathcal{A}_2E \cdot E\mathcal{A}_1E$, as claimed.

■

Example 3.1.6. *Let $n \geq 3$ be a positive integer, let Q_1 and Q_2 be mutually orthogonal rank-one projections in $\mathbb{M}_n$, and define $Q_3 \coloneqq I - Q_1 - Q_2$. If $\mathcal{A}$ is the subalgebra of $\mathbb{M}_n$ given by*

$$\mathcal{A} \coloneqq \mathbb{C}(Q_1 + Q_2) + Q_1\mathbb{M}_nQ_2 + (Q_1 + Q_2)\mathbb{M}_nQ_3,$$

then $\mathcal{A}$ is idempotent compressible. Consequently, its unitization

$$\tilde{\mathcal{A}} = \mathbb{C}(Q_1 + Q_2) + Q_1\mathbb{M}_nQ_2 + (Q_1 + Q_2)\mathbb{M}_nQ_3 + \mathbb{C}Q_3$$

is also idempotent compressible.

Proof. In light of Lemmas 3.1.4 and 3.1.5, it suffices to prove that if $r \in \{2, 3, \ldots, n-1\}$, and E is an idempotent in $\mathbb{M}_n$ of rank r, then either $EQ_2E \in E\mathcal{A}E$ or $EQ_1 = Q_1$.

Fix such an integer r and idempotent E. Choose an orthonormal basis $\mathcal{B} = \{e_1, e_2, \ldots, e_n\}$ for $\mathbb{C}^n$ so that $e_1 \in ran(Q_1)$ and $e_2 \in ran(Q_2)$, and consider the rank-r idempotent

$$E_0 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

expressed with respect to this basis. Rank considerations imply that there is an invertible matrix $S = (s_{ij})$ in $\mathbb{M}_n$ such that $E = SE_0S^{-1}$.

The product EQ_2E belongs $E\mathcal{A}E$ if and only if there is an $A \in \mathcal{A}$ such that

$$E_0S^{-1}(A - Q_2)SE_0 = 0.$$

In showing this equality, it is clearly sufficient to exhibit an $A \in \mathcal{A}$ such that $(A - Q_2)SE_0 = 0$. To this end, observe that for any $A \in \mathcal{A}$, the operator $B := A - Q_2$ admits the following matrix representation with respect to the basis $\mathcal{B}$:

$$B = \begin{bmatrix} \alpha & w_2 & w_3 & \cdots & w_n \\ 0 & \alpha - 1 & v_3 & \cdots & v_n \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Since the last $n-2$ rows of B and the last $n-r$ columns of E_0 are zero, one may verify that the product BSE_0 is zero whenever $(BS)_{ij} = 0$ for all $i \in \{1, 2\}$ and $j \in \{1, 2, \ldots, r\}$. That is, such a B exists if there is a solution to the following non-homogeneous $2r \times 2(n-1)$ system of linear equations:

$$\begin{array}{c} \begin{matrix} w_2 & w_3 & \cdots & w_n & \alpha & v_3 & \cdots & v_n & \phantom{s_{21}} \end{matrix} \\ \left[\begin{array}{cccccccc|c} s_{21} & s_{31} & \cdots & s_{n1} & s_{11} & 0 & \cdots & 0 & 0 \\ s_{22} & s_{32} & \cdots & s_{n2} & s_{12} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{2r} & s_{3r} & \cdots & s_{nr} & s_{1r} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & s_{21} & s_{31} & \cdots & s_{n1} & s_{21} \\ 0 & 0 & \cdots & 0 & s_{22} & s_{32} & \cdots & s_{n2} & s_{22} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & s_{2r} & s_{3r} & \cdots & s_{nr} & s_{2r} \end{array}\right] \end{array}.$$

If the rank of the above (non-augmented) matrix is $2r$, then its columns span $\mathbb{C}^{2r}$ and a solution exists. In this case, EQ_2E belongs to $E\mathcal{A}E$, so $E\mathcal{A}E$ is an algebra by Lemma 3.1.4.

Suppose that this is not the case, so the above (non-augmented) matrix has rank $< 2r$. It is then apparent that

$$S_0 := \begin{bmatrix} s_{21} & s_{31} & \cdots & s_{n1} \\ s_{22} & s_{32} & \cdots & s_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ s_{2r} & s_{3r} & \cdots & s_{nr} \end{bmatrix}$$

has rank $< r$. From here we will demonstrate that $EQ_1 = Q_1$, or equivalently, that $E_0S^{-1}Q_1 = S^{-1}Q_1$.

To see this, note that if $S^{-1} = (t_{ij})$, then $t_{i1} = 0$ for all $i > r$. Indeed,

$$t_{i1} = \frac{C_{1i}}{\det(S)}$$

where C_{ij} denotes the (i,j)-cofactor of S. When $i > r$, C_{1i} is equal to $(-1)^{i+1}\det(M)$, where M is an $(n-1)\times(n-1)$ matrix of the form

$$M = \begin{bmatrix} s_{21} & s_{22} & \cdots & s_{2r} & * & \cdots & * \\ s_{31} & s_{32} & \cdots & s_{3r} & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \cdots & s_{nr} & * & \cdots & * \end{bmatrix}.$$

Since the $(n-1)\times r$ matrix obtained by keeping only the first r columns of M is exactly S_0^T and $\operatorname{rank}(S_0) < r$, one has

$$\operatorname{rank}(M) < r + (n-1-r) = n-1.$$

Consequently, $t_{i1} = 0$ for all $i > r$. A straightforward computation now shows that $E_0 S^{-1} Q_1 = S^{-1} Q_1$.

■

§3.2 Subalgebras of $\mathbb{M}_3$. The families of idempotent compressible algebras presented in §3.1 include subalgebras of $\mathbb{M}_n$ for each integer $n \geq 3$. It turns out that up to similarity and transposition, these are the only examples of unital idempotent compressible algebras that exist in $\mathbb{M}_n$ when $n \geq 4$. In fact, up to unitary equivalence and transposition, this list also represents all unital *projection* compressible subalgebras of $\mathbb{M}_n$ when $n \geq 4$. Obtaining a proof of this result is the focus of [2].

Unfortunately, the story for unital subalgebras of $\mathbb{M}_3$ is not so sweet. As we will see in this section, there exist several examples of unital idempotent compressible subalgebras of $\mathbb{M}_3$ that are not accounted for in §3.1. A partial explanation as to why these pathological examples arise is due to dimension. Just as $\mathbb{M}_2$ is simply "too small" to contain the projections required to disprove the existence of the compression properties for any of its subalgebras, certain subalgebras of $\mathbb{M}_3$ acquire the compression properties because $\mathbb{M}_3$ does not contain projections of large enough rank. Support for this explanation is given by [2, Theorem 2.0.5], where it is shown that in the case of $\mathbb{M}_n$, $n \geq 4$, one can very often prove that an algebra lacks the compression properties using projections of rank 3.

Before introducing these examples, it will be important to recall the following facts concerning matrices of rank-one.

Proposition 3.2.1. *Let A be an element of $\mathbb{M}_n$ with $\operatorname{rank}(A) = 1$. The linear space $\mathbb{C}A$ is an algebra, and $A\mathbb{M}_n A$ is contained in $\mathbb{C}A$.*

Proof. As a rank-one operator, A is either nilpotent or a scalar multiple of an idempotent. Hence, $\mathbb{C}A$ is closed under multiplication. Writing $A = x \otimes y^*$ for some vectors x and y in $\mathbb{C}^n$, we have that for an arbitrary $R \in \mathbb{M}_n$,

$$ARA = (x\otimes y^*)R(x\otimes y^*) = \langle Rx, y^*\rangle (x\otimes y^*) = \langle Rx, y\rangle A \in \mathbb{C}A.$$

Thus, $A\mathbb{M}_n A \subseteq \mathbb{C}A$.

■

Example 3.2.2. *Let Q_1, Q_2, and Q_3 be rank-one projections in $\mathbb{M}_3$ that sum to I. If $\mathcal{A}$ is the subalgebra of $\mathbb{M}_3$ defined by*

$$\mathcal{A} := \mathbb{C}Q_1 + \mathbb{C}Q_2 + (Q_2+Q_3)\mathbb{M}_3 Q_3,$$

then $\mathcal{A}$ is idempotent compressible.

Proof. Define

$$\mathcal{A}_1 := \mathbb{C}Q_1, \quad \mathcal{A}_2 := \mathbb{C}Q_2 \quad \text{and} \quad \mathcal{A}_3 := (Q_2+Q_3)\mathbb{M}_3 Q_3,$$

so that $\mathcal{A} = \mathcal{A}_1 \dotplus \mathcal{A}_2 \dotplus \mathcal{A}_3$. Let E be a fixed idempotent in $\mathbb{M}_3$. We will show that $E\mathcal{A}E$ contains the product $E\mathcal{A}_i E \cdot E\mathcal{A}_j E$ for all i and j.

For each $i \in \{1,2,3\}$, $\mathcal{A}_i$ is an $\mathcal{LR}$-algebra; hence, we have that $(E\mathcal{A}_i E)^2 \subseteq E\mathcal{A}_i E \subseteq E\mathcal{A}E$. Upon writing $Q_2 = (Q_2+Q_3)Q_2$, one can show that $EQ_2\mathbb{M}_3 Q_3 E$ is contained in $E\mathcal{A}_3 E$, and hence $E\mathcal{A}_2 E \cdot E\mathcal{A}_3 E \subseteq E\mathcal{A}E$ as well. These inclusions, together with the identities

$$\begin{aligned} EQ_1E\cdot EQ_2E &= EQ_1E - EQ_1E\cdot EQ_1E - EQ_3E + EQ_2E\cdot EQ_3E + EQ_3E\cdot EQ_3E \quad \text{and} \\ EQ_2E\cdot EQ_1E &= EQ_2E - EQ_2E\cdot EQ_2E - EQ_2E\cdot EQ_3E, \end{aligned}$$

demonstrate that $E\mathcal{A}_1E\cdot E\mathcal{A}_2E$ and $E\mathcal{A}_2E\cdot E\mathcal{A}_1E$ are contained in $E\mathcal{A}E$. Furthermore, if T is an arbitrary element of $\mathbb{M}_3$, then by writing

$$EQ_1E\cdot E(Q_2+Q_3)TQ_3E = E(Q_2+Q_3)TQ_3E - E(Q_2+Q_3)E\cdot E(Q_2+Q_3)TQ_3E,$$

it becomes apparent that $EQ_1E\cdot E(Q_2+Q_3)TQ_3E\in E\mathcal{A}E$. Consequently, $E\mathcal{A}_1E\cdot E\mathcal{A}_3E\subseteq E\mathcal{A}E$.

For the final inclusions, it will be helpful to first prove that $EQ_3E\cdot EQ_2E\in E\mathcal{A}E$. Indeed, this is a consequence of the identity

$$EQ_3E\cdot EQ_2E = EQ_3E - EQ_3E\cdot EQ_3E - EQ_1E + EQ_1E\cdot EQ_1E + EQ_2E\cdot EQ_1E$$

and the inclusions established above. One may then apply Proposition 3.2.1 to the rank-one operator Q_3 to deduce that $EQ_3\mathbb{M}_3Q_3E\cdot EQ_2E$ is contained in $E\mathcal{A}E$ as well. From here, Proposition 3.2.1, together with the identities

$$E(Q_2+Q_3)TQ_3E\cdot EQ_2E \quad = \quad EQ_2TQ_3E\cdot EQ_2E + EQ_3TQ_3E\cdot EQ_2E$$

and

$$\begin{aligned} E(Q_2+Q_3)TQ_3E\cdot EQ_1E \quad = \quad & E(Q_2+Q_3)TQ_3E - E(Q_2+Q_3)TQ_3E\cdot EQ_3E \\ & -EQ_2TQ_3E\cdot EQ_2E - EQ_3TQ_3E\cdot EQ_2E, \end{aligned}$$

shows that $E\mathcal{A}E$ contains $E\mathcal{A}_3E\cdot E\mathcal{A}_2E$ and $E\mathcal{A}_3E\cdot E\mathcal{A}_1E$. Therefore, $E\mathcal{A}E$ is an algebra.

■

Proving the existence of the idempotent compression property for our next two examples will be somewhat more challenging. In the same spirit of the proof of Example 3.1.6, Examples 3.2.5 and 3.2.8 will each be preceded by two lemmas that highlight sufficient conditions for a corner of the algebra to be an algebra itself. We will then prove that all corners of these algebras must satisfy one of these two conditions.

Lemma 3.2.3. *Let Q_1, Q_2, and Q_3 be rank-one projections in $\mathbb{M}_3$ that sum to I. Let $\mathcal{A}$ be the subalgebra of $\mathbb{M}_3$ defined by*

$$\mathcal{A} \coloneqq \mathbb{C}(Q_1+Q_2)+\mathbb{C}Q_3+Q_1\mathbb{M}_3(Q_2+Q_3).$$

If E is an idempotent in $\mathbb{M}_3$ such that $EQ_2E\in E\mathcal{A}E$, then $E\mathcal{A}E$ is an algebra.

Proof. Suppose that E is an idempotent such that $EQ_2E\in E\mathcal{A}E$, and define

$$\mathcal{A}_0 \coloneqq \mathbb{C}Q_1+\mathbb{C}Q_2+\mathbb{C}Q_3+Q_1\mathbb{M}_3(Q_2+Q_3).$$

We have that

$$\begin{aligned} E\mathcal{A}E &= \mathbb{C}E(Q_1+Q_2)E+\mathbb{C}EQ_3E+EQ_1\mathbb{M}_3(Q_2+Q_3)E \\ &= \mathbb{C}EQ_1E+\mathbb{C}EQ_2E+\mathbb{C}EQ_3E+EQ_1\mathbb{M}_3(Q_2+Q_3)E \\ &= E\mathcal{A}_0E. \end{aligned}$$

Since $\mathcal{A}_0^{aT}$ is the unital algebra from Example 3.1.3, $\mathcal{A}_0$ is idempotent compressible. Thus, $E\mathcal{A}_0E=E\mathcal{A}E$ is an algebra.

■

Lemma 3.2.4. *Let Q_1, Q_2, and Q_3 be rank-one projections in $\mathbb{M}_3$ that sum to I. Let $\mathcal{A}$ be the subalgebra of $\mathbb{M}_3$ defined by*

$$\mathcal{A} \coloneqq \mathbb{C}(Q_1+Q_2)+\mathbb{C}Q_3+Q_1\mathbb{M}_3(Q_2+Q_3).$$

If E is an idempotent in $\mathbb{M}_3$ such that $EQ_1=Q_1$, then $E\mathcal{A}E$ is an algebra.

Proof. Let E be an idempotent such that $EQ_1=Q_1$. Define

$$\mathcal{A}_1 \coloneqq \mathbb{C}(Q_1+Q_2), \quad \mathcal{A}_2 \coloneqq \mathbb{C}Q_3, \quad \text{and} \quad \mathcal{A}_3 \coloneqq Q_1\mathbb{M}_3(Q_2+Q_3),$$

so that $\mathcal{A}=\mathcal{A}_1\dotplus\mathcal{A}_2\dotplus\mathcal{A}_3$. To show that $E\mathcal{A}E$ is an algebra, we will verify that the product $E\mathcal{A}_iE\cdot E\mathcal{A}_jE$ is contained in $E\mathcal{A}E$ if all i and j.

Observe that $\mathcal{A}_2$ and $\mathcal{A}_3$ are $\mathcal{LR}$-algebras. Thus, $(E\mathcal{A}_iE)^2 \subseteq E\mathcal{A}_iE \subseteq E\mathcal{A}E$ for each $i \in \{2,3\}$. Moreover, since

$$\begin{aligned} E(Q_1+Q_2)E \cdot EQ_3E &= EQ_3E - EQ_3E \cdot EQ_3E, \\ EQ_3E \cdot E(Q_1+Q_2)E &= EQ_3E - EQ_3E \cdot EQ_3E, \end{aligned}$$

and

$$E(Q_1+Q_2)E \cdot E(Q_1+Q_2)E = E - 2EQ_3E + EQ_3E \cdot EQ_3E,$$

it follows that $E\mathcal{A}_1E \cdot E\mathcal{A}_2E$, $E\mathcal{A}_2E \cdot E\mathcal{A}_1E$, and $(E\mathcal{A}_1E)^2$ are all contained in $E\mathcal{A}E$.

For the remaining inclusions, note that for any $T \in \mathbb{M}_3$,

$$EQ_1T(Q_2+Q_3)E \cdot E(Q_1+Q_2)E = EQ_1T(Q_2+Q_3)E - EQ_1T(Q_2+Q_3)E \cdot EQ_3(Q_2+Q_3)E$$

and

$$EQ_1T(Q_2+Q_3)E \cdot EQ_3E = EQ_1T(Q_2+Q_3)E \cdot EQ_3(Q_2+Q_3)E.$$

Consequently, $E\mathcal{A}_3E \cdot E\mathcal{A}_1E$ and $E\mathcal{A}_3E \cdot E\mathcal{A}_2E$ are contained in $E\mathcal{A}_3E \subseteq E\mathcal{A}E$. Finally, since $EQ_1 = Q_1$ by hypothesis, we have that

$$E(Q_1+Q_2)E \cdot EQ_1T(Q_2+Q_3)E = EQ_1T(Q_2+Q_3)E$$

and

$$EQ_3E \cdot EQ_1T(Q_2+Q_3)E = 0.$$

This implies that $E\mathcal{A}E$ contains $E\mathcal{A}_1E \cdot E\mathcal{A}_3E$ and $E\mathcal{A}_2E \cdot E\mathcal{A}_3E$.

■

Example 3.2.5. *Let Q_1, Q_2, and Q_3 be rank-one projections in $\mathbb{M}_3$ that sum to I. If $\mathcal{A}$ is the subalgebra of $\mathbb{M}_3$ defined by*

$$\mathcal{A} := \mathbb{C}(Q_1+Q_2) + \mathbb{C}Q_3 + Q_1\mathbb{M}_3(Q_2+Q_3),$$

then $\mathcal{A}$ is idempotent compressible.

Proof. It is obvious that $E\mathcal{A}E$ is an algebra whenever E is an idempotent of rank 1 or 3. In light of Lemmas 3.2.3 and 3.2.4, it suffices to show that for every rank-two idempotent E in $\mathbb{M}_3$, either EQ_2E belongs to $E\mathcal{A}E$ or $EQ_1 = Q_1$.

To this end, suppose that E is a rank-two idempotent in $\mathbb{M}_3$ such that EQ_2E does not belong to $E\mathcal{A}E$, and consider the projection $E_0 := (Q_1+Q_2)$. By rank considerations, there is an invertible matrix $S = (s_{ij})$ with inverse $S^{-1} = (t_{ij})$ such that $E = SE_0S^{-1}$.

Since EQ_2E is not contained in $E\mathcal{A}E$, then there is no $A \in \mathcal{A}$ that satisfies the equation

$$SE_0S^{-1}(A - Q_2)SE_0S^{-1} = 0.$$

In particular, there is no $A \in \mathcal{A}$ such that $(A - Q_2)SE_0 = 0$. Since every $A \in \mathcal{A}$ can be expressed as a matrix of the form

$$A = \begin{bmatrix} \alpha & x & y \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix}$$

with respect to $\mathbb{C}^3 = ran(Q_1) \oplus ran(Q_2) \oplus ran(Q_3)$, it follows that there do not exist constants $\alpha, \beta, x, y \in \mathbb{C}$ that solve the following system of equations:

$$\begin{cases} \alpha s_{11} + x s_{21} + y s_{31} & = 0 \\ \alpha s_{12} + x s_{22} + y s_{32} & = 0 \\ \alpha s_{21} & = s_{21} \\ \alpha s_{22} & = s_{22} \\ \beta s_{31} & = 0 \\ \beta s_{32} & = 0 \end{cases}.$$

Note that if the determinant of $S_0 := \begin{bmatrix} s_{21} & s_{31} \\ s_{22} & s_{32} \end{bmatrix}$ were zero, then a solution to the above system could be obtained by taking $\alpha = 1$, $\beta = 0$, and x and y such that

$$x \begin{bmatrix} s_{21} \\ s_{22} \end{bmatrix} + y \begin{bmatrix} s_{31} \\ s_{32} \end{bmatrix} = \begin{bmatrix} -s_{11} \\ -s_{12} \end{bmatrix}.$$

It must therefore be the case that $\det S_0 = 0$.

We conclude the proof by showing that $EQ_1 = Q_1$, or equivalently, that $E_0 S^{-1} Q_1 = S^{-1} Q_1$. It is easy to see that this equation holds when $t_{31} = 0$. But if C_{ij} denotes the (i, j)-cofactor of S, then indeed,

$$t_{31} = \frac{C_{13}}{\det(S)} = \frac{\det(S_0^T)}{\det(S)} = 0.$$

Thus, the proof is complete.

■

Lemma 3.2.6. *Let Q_1, Q_2, and Q_3 be rank-one projections in $\mathbb{M}_3$ that sum to I. Let $\mathcal{A}$ be the subalgebra of $\mathbb{M}_3$ defined by*

$$\mathcal{A} := Q_1 \mathbb{M}_3 (Q_2 + Q_3) + Q_2 \mathbb{M}_3 Q_3 + \mathbb{C} I.$$

If E is an idempotent in $\mathbb{M}_3$ such that $EQ_1E \in E\mathcal{A}E$, then $E\mathcal{A}E$ is an algebra.

Proof. Suppose that E is an idempotent such that $EQ_1E \in E\mathcal{A}E$, and define

$$\mathcal{A}_0 := \mathbb{C} Q_1 + \mathbb{C}(Q_2 + Q_3) + Q_1 \mathbb{M}_3 (Q_2 + Q_3) + Q_2 \mathbb{M}_3 Q_3.$$

We have that

$$\begin{aligned} E\mathcal{A}E &= EQ_1 \mathbb{M}_3 (Q_2 + Q_3) E + EQ_2 \mathbb{M}_3 Q_3 E + \mathbb{C} E \\ &= EQ_1 \mathbb{M}_3 (Q_2 + Q_3) E + EQ_2 \mathbb{M}_3 Q_3 E + \mathbb{C} EQ_1 E + \mathbb{C} E (Q_2 + Q_3) E \\ &= E\mathcal{A}_0 E. \end{aligned}$$

Since $\mathcal{A}_0^{aT}$ is the unital algebra from Example 3.1.6, $\mathcal{A}_0$ is idempotent compressible. Thus, $E\mathcal{A}_0E = E\mathcal{A}E$ is an algebra.

■

Lemma 3.2.7. *Let Q_1, Q_2, and Q_3 be rank-one projections in $\mathbb{M}_3$ that sum to I. Let $\mathcal{A}$ be the subalgebra of $\mathbb{M}_3$ defined by*

$$\mathcal{A} := Q_1 \mathbb{M}_3 (Q_2 + Q_3) + Q_2 \mathbb{M}_3 Q_3 + \mathbb{C} I.$$

If E is an idempotent in $\mathbb{M}_3$ such that $EQ_1 = Q_1$, then $E\mathcal{A}E$ is an algebra.

Proof. Let E be an idempotent such that $EQ_1 = Q_1$. Define

$$\mathcal{A}_1 := Q_1 \mathbb{M}_3 (Q_2 + Q_3) \quad \mathcal{A}_2 := Q_2 \mathbb{M}_3 Q_3 \quad \text{and} \quad \mathcal{A}_3 := \mathbb{C} I,$$

so that $\mathcal{A} = \mathcal{A}_1 \dotplus \mathcal{A}_2 \dotplus \mathcal{A}_3$. Yet again, to show that $E\mathcal{A}E$ is an algebra, we will prove that the product $E\mathcal{A}_iE \cdot E\mathcal{A}_jE$ is contained in $E\mathcal{A}E$ for all i and j.

Observe that $E\mathcal{A}_iE \cdot E\mathcal{A}_jE$ is clearly contained in $E\mathcal{A}E$ when $i = 3$ or $j = 3$. Moreover, it is easy to see that $(E\mathcal{A}_1E)^2$ and $(E\mathcal{A}_2E)^2$ are contained in $E\mathcal{A}E$, as $\mathcal{A}_1$ and $\mathcal{A}_2$ are $\mathcal{LR}$-algebras.

Given $T, S \in \mathbb{M}_3$, we have that

$$EQ_1 S (Q_2 + Q_3) E \cdot EQ_2 T Q_3 E = EQ_1 S (Q_2 + Q_3) E \cdot EQ_2 T Q_3 (Q_2 + Q_3) E,$$

so $E\mathcal{A}_1E \cdot E\mathcal{A}_2E$ is contained in $E\mathcal{A}_1E$, and hence in $E\mathcal{A}E$. Finally, we may use the equation $EQ_1 = Q_1$ to see that

$$EQ_2 S Q_3 E \cdot EQ_1 T (Q_2 + Q_3) E = 0,$$

so $E\mathcal{A}_2E \cdot E\mathcal{A}_1E = \{0\}$.

■

Example 3.2.8. *Let Q_1, Q_2, and Q_3 be rank-one projections in $\mathbb{M}_3$ that sum to I. If $\mathcal{A}$ is the subalgebra of $\mathbb{M}_3$ defined by*

$$\mathcal{A} := Q_1 \mathbb{M}_3 (Q_2 + Q_3) + Q_2 \mathbb{M}_3 Q_3 + \mathbb{C} I,$$

then $\mathcal{A}$ is idempotent compressible.

Proof. It is obvious that $E\mathcal{A}E$ is an algebra whenever E is an idempotent of rank 1 or 3. In light of Lemmas 3.2.6 and 3.2.7, it suffices to show that for every rank-two idempotent E in $\mathbb{M}_3$, either EQ_1E belongs to $E\mathcal{A}E$, or $EQ_1 = Q_1$.

To this end, suppose that E is a rank-two idempotent in $\mathbb{M}_3$ such that EQ_1E does not belong to $E\mathcal{A}E$. Define $E_0 := (Q_1 + Q_2)$, and let $S = (s_{ij})$ be an invertible matrix with inverse $S^{-1} = (t_{ij})$ satisfying $E = SE_0S^{-1}$.

Since EQ_1E is not contained in $E\mathcal{A}E$, then there is no $A \in \mathcal{A}$ satisfying the equation

$$SE_0S^{-1}(A - Q_1)SE_0S^{-1} = 0.$$

In particular, there is no $A \in \mathcal{A}$ such that $(A-Q_1)SE_0 = 0$. Since every $A \in \mathcal{A}$ can be expressed as a matrix of the form

$$A = \begin{bmatrix} \alpha & x & y \\ 0 & \alpha & z \\ 0 & 0 & \alpha \end{bmatrix}$$

with respect to the decomposition $\mathbb{C}^3 = ran(Q_1) \oplus ran(Q_2) \oplus ran(Q_3)$, it follows that there do not exist constants $\alpha, \beta, x, y,$ and z in $\mathbb{C}$ that solve the following system of equations :

$$\begin{cases} \alpha s_{11} + x s_{21} + y s_{31} & & = & s_{11} \\ \alpha s_{12} + x s_{22} + y s_{32} & & = & s_{12} \\ \alpha s_{21} & + z s_{31} & = & 0 \\ \alpha s_{22} & + z s_{32} & = & 0 \\ \alpha s_{31} & & = & 0 \\ \alpha s_{32} & & = & 0 \end{cases}.$$

Observe, however, that if the determinant of $S_0 := \begin{bmatrix} s_{21} & s_{31} \\ s_{22} & s_{32} \end{bmatrix}$ were non-zero, then a solution could be obtained by taking $\alpha = z = 0$, and x and y such that

$$x \begin{bmatrix} s_{21} \\ s_{22} \end{bmatrix} + y \begin{bmatrix} s_{31} \\ s_{32} \end{bmatrix} = \begin{bmatrix} s_{11} \\ s_{12} \end{bmatrix}.$$

It must therefore be the case that $\det S_0 = 0$.

We are now prepared to show that $EQ_1 = Q_1$, or equivalently, that $E_0S^{-1}Q_1 = S^{-1}Q_1$. This will be accomplished by proving that $t_{31} = 0$. Indeed, if C_{ij} denotes the (i, j)-cofactor of S,

$$t_{31} = \frac{C_{13}}{\det(S)} = \frac{\det(S_0)^T}{\det(S)} = 0,$$

as claimed. ∎

§4 Structure Theory for Matrix Algebras

In §2 and §3, we introduced several families of unital algebras admitting the idempotent compression property. By Proposition 2.0.3, any algebra obtained by applying a transposition or similarity to one of these algebras also enjoys the idempotent compression property. It becomes interesting to ask whether or not this list is exhaustive. That is, is every unital idempotent compressible subalgebra of $\mathbb{M}_n$ transpose similar to one of the algebras from §2 or §3? In order to decide whether or not additional examples exist, it will be necessary to establish a systematic approach to listing the unital subalgebras of $\mathbb{M}_n$. Thus, this section will be devoted to recording a few key results on the structure theory for matrix algebras over $\mathbb{C}$. The primary reference for this section is [4].

Perhaps the most important result in this vein is the following theorem of Burnside [1], which states that the only irreducible subalgebra of $\mathbb{M}_n$ is the entire matrix algebra $\mathbb{M}_n$ itself.

Theorem 4.0.1 (Burnside's Theorem). *If $\mathcal{A}$ is an irreducible algebra of linear transformations on a finite-dimensional vector space $\mathcal{V}$ over an algebraically closed field, then $\mathcal{A}$ is the algebra of all linear transformations on $\mathcal{V}$.*

As a consequence of Burnside's Theorem, every proper subalgebra of $\mathbb{M}_n$ can be block upper triangularized with respect to some basis for $\mathbb{C}^n$. The diagonal blocks of this decomposition are themselves algebras. Thus, Burnside's Theorem may be applied to these blocks successively to obtain a maximal block upper triangularization of the algebra.

Definition 4.0.2. [4, Definition 9] A subalgebra $\mathcal{A}$ of $\mathbb{M}_n$ is said to have a *reduced block upper triangular form* with respect to a decomposition $\mathbb{C}^n = \mathcal{V}_1 \dotplus \mathcal{V}_2 \dotplus \cdots \dotplus \mathcal{V}_m$ if

(i) when expressed as a matrix, each A in $\mathcal{A}$ has the form

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1m} \\ 0 & A_{22} & A_{23} & \cdots & A_{2m} \\ 0 & 0 & A_{33} & \cdots & A_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{mm} \end{bmatrix}$$

with respect to this decomposition, and

(ii) for each i, the subalgebra $\mathcal{A}_{ii} := \{A_{ii} : A \in \mathcal{A}\}$ is irreducible. That is, either $\mathcal{A}_{ii} = \{0\}$ and $\dim \mathcal{V}_i = 1$, or $\mathcal{A}_{ii} = \mathbb{M}_{\dim \mathcal{V}_i}$.

If $\mathcal{A}$ is a reduced block upper triangular algebra and $A \in \mathcal{A}$, define the *block-diagonal* of A to be the matrix $BD(A)$ obtained by replacing the block-'off-diagonal' entries of A with zeros. In addition, define the *block-diagonal* of $\mathcal{A}$ to be the algebra

$$BD(\mathcal{A}) = \{BD(A) : A \in \mathcal{A}\}.$$

By definition, the non-zero diagonal blocks of a reduced block upper triangular matrix algebra $\mathcal{A}$ are full matrix algebras. There may, however, exist dependencies among different diagonal blocks. That is, while it may be the case that any matrix of suitable size can be realized as a diagonal block for some element of $\mathcal{A}$, there is no guarantee that matrices for different blocks can be chosen at will simultaneously. The following result states that any dependencies that occur among the diagonal blocks of $\mathcal{A}$ can be described in terms of dimension and similarity.

Theorem 4.0.3. [4, Corollary 14] *If a subalgebra $\mathcal{A}$ of $\mathbb{M}_n$ has a reduced block upper triangular form with respect to a decomposition $\mathbb{C}^n = \mathcal{V}_1 \dotplus \mathcal{V}_2 \dotplus \cdots \dotplus \mathcal{V}_m$, then the set $\{1, 2, \ldots, m\}$ can be partitioned into disjoint sets $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ such that*

(i) If $i \in \Gamma_s$ and $\mathcal{A}_{ii} \neq \{0\}$, then there exists $G^{<i>}$ in $\mathcal{A}$ such that $G^{<i>}_{jj} = I_{\mathcal{V}_j}$ for all $j \in \Gamma_s$, and $G^{<i>}_{jj} = 0$ for all $j \notin \Gamma_s$.

(ii) If i and j belong to the same Γ_s, then $\dim \mathcal{V}_i = \dim \mathcal{V}_j$, and there is an invertible linear map $S_{ij} : \mathcal{V}_i \to \mathcal{V}_j$ such that

$$A_{ii} = S_{ij}^{-1} A_{jj} S_{ij}$$

for all $A \in \mathcal{A}$.

(iii) If i and j do not belong to the same Γ_s, then

$$\{(A_{ii}, A_{jj}) : A \in \mathcal{A}\} = \{A_{ii} : A \in \mathcal{A}\} \times \{A_{jj} : A \in \mathcal{A}\}.$$

Definition 4.0.4. Let $\mathcal{A}$ be an algebra of the form described in Theorem 4.0.3. Indices i and j are said to be *linked* if they belong to the same Γ_s, and are said to be *unlinked* otherwise.

It should be noted that if $\mathcal{A}$ is an algebra in reduced block upper triangular form, and S is an invertible matrix that is block upper triangular with respect to the same decomposition as that of $\mathcal{A}$, then $S^{-1}\mathcal{A}S$ has a reduced block upper triangular form with respect to this decomposition, and indices i and j are linked in $S^{-1}\mathcal{A}S$ if and only if they are linked in $\mathcal{A}$. From this it follows that every subalgebra of $\mathbb{M}_n$ has a reduced block upper triangular form with respect to some *orthogonal* decomposition of $\mathbb{C}^n$.

The theorems presented above provide insight into the structure of the block-diagonal of a reduced block upper triangular matrix algebra $\mathcal{A}$. It will now be important to develop an understanding of blocks that are located above the block-diagonal.

As described in [4, Corollary 28], every subalgebra $\mathcal{A}$ of $\mathbb{M}_n$ can be written as $\mathcal{A} = \mathcal{S} \dotplus Rad(\mathcal{A})$, where $\mathcal{S}$ is semi-simple and $Rad(\mathcal{A})$ is the nil radical of $\mathcal{A}$. If $\mathcal{A}$ is a reduced block upper triangular algebra, then $\mathcal{S}$ is block upper triangular and $Rad(\mathcal{A})$ consists of all strictly upper triangular elements of $\mathcal{A}$ [4, Proposition 19]. Thus, the blocks above the block-diagonal are, in general, comprised of blocks from $\mathcal{S}$ and blocks from $Rad(\mathcal{A})$. Of course in the simplest scenario, $\mathcal{S}$ is equal to $BD(\mathcal{A})$.

Definition 4.0.5. Let $\mathcal{A}$ be a subalgebra of $\mathbb{M}_n$ that has a reduced block upper triangular form with respect to some decomposition of $\mathbb{C}^n$. The algebra $\mathcal{A}$ is said to be *unhinged* with respect to this decomposition if

$$\mathcal{A} = BD(\mathcal{A}) \dotplus Rad(\mathcal{A}).$$

The following result indicates that if $\mathcal{A}$ is an algebra in reduced block upper triangular form with respect to some decomposition of $\mathbb{C}^n$, then $\mathcal{A}$ can be unhinged with respect to this decomposition via conjugation by a block upper triangular similarity.

Theorem 4.0.6 ([4], Corollary 30)**.** *If a subalgebra $\mathcal{A}$ of $\mathbb{M}_n$ has a reduced block upper triangular form with respect to a decomposition of $\mathbb{C}^n$, then after a block upper triangular similarity, $\mathcal{A}$ has an unhinged reduced block upper triangular form with respect to this decomposition.*

We end this section with the following lemma concerning the independence of the blocks in the radical of an algebra $\mathcal{A}$ in reduced block upper triangular form. This result will be used extensively in §5 and throughout the classification in [2].

Lemma 4.0.7. *Let n be a positive integer, and let $\mathcal{A}$ be a unital subalgebra of $\mathbb{M}_n$ expressed in reduced block upper triangular form with respect to a decomposition $\bigoplus_{i=1}^m \mathcal{V}_i$ of $\mathbb{C}^n$. Suppose that there is an index $1 < k < m$ that is unlinked from all indices $i \neq k$, and let Q_1, Q_2, and Q_3 denote the orthogonal projections onto $\bigoplus_{i<k} \mathcal{V}_i$, $\mathcal{V}_k$, and $\bigoplus_{i>k} \mathcal{V}_i$, respectively.*

(i) For every $R \in Rad(\mathcal{A})$, there are elements $R' = Q_1 R'$ and $R'' = R'' Q_3$ in $Rad(\mathcal{A})$ such that

$$R' Q_2 = Q_1 R Q_2 \quad \textit{and} \quad Q_2 R'' = Q_2 R Q_3.$$

(ii) If there exist projections $Q_1' \leq Q_1$ and $Q_3' \leq Q_3$ such that

$$Q_1 Rad(\mathcal{A}) Q_2 = Q_1' \mathbb{M}_n Q_2, \quad Q_2 Rad(\mathcal{A}) Q_3 = Q_2 \mathbb{M}_n Q_3',$$

and

$$Q_1 Rad(\mathcal{A}) Q_3 = Q_1' Rad(\mathcal{A}) Q_3',$$

then

$$Rad(\mathcal{A}) = Q_1' \mathbb{M}_n Q_2 \dotplus Q_1' \mathbb{M}_n Q_3' \dotplus Q_2 \mathbb{M}_n Q_3'.$$

Proof. Let R belong to $Rad(\mathcal{A})$. Since $\mathcal{V}_k$ is unlinked from all other spaces $\mathcal{V}_i$, there is an element $A \in \mathcal{A}$ such that $Q_1 A Q_1 = Q_3 A Q_3 = 0$ and $Q_2 A Q_2 = Q_2$. Thus, with respect to the decomposition $\mathbb{C}^n = ran(Q_1) \oplus ran(Q_2) \oplus ran(Q_3)$, A and R may be expressed as

$$A = \begin{bmatrix} 0 & A_{12} & A_{13} \\ 0 & I & A_{23} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 0 & R_{12} & R_{13} \\ 0 & 0 & R_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

for some A_{ij} and R_{ij}. It is then easy to see that $R' := RA$ and $R'' := AR$ define elements of $Rad(\mathcal{A})$ that satisfy the requirements of (i).

For (ii), let M_1 and M_2 denote arbitrary elements of $Q_1' \mathbb{M}_n Q_2$ and $Q_2 \mathbb{M}_n Q_3'$, respectively. By (i), there are elements S_1 and S_2 in $Q_1 \mathbb{M}_n Q_3$ such that $M_1 + S_1$ and $M_2 + S_2$ belong to $Rad(\mathcal{A})$. Moreover, since $Q_1 Rad(\mathcal{A}) Q_3 = Q_1' Rad(\mathcal{A}) Q_3'$, we have that S_1 and S_2 are contained in $Q_1' \mathbb{M}_n Q_3'$.

Observe that

$$R := (M_1 + S_1)(M_2 + S_2)$$

belongs to $Rad(\mathcal{A})$. With respect to the decomposition described above, this element can be expressed as

$$R = \begin{bmatrix} 0 & M_1 & S_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & S_2 \\ 0 & 0 & M_2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & M_1M_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

But since M_1 and M_2 were arbitrary, this implies that $Q_1'\mathbb{M}_nQ_3' \subseteq Rad(\mathcal{A})$. In particular, $Rad(\mathcal{A})$ contains S_1 and S_2. It then follows that M_1 and M_2 belong to $Rad(\mathcal{A})$ as well. We conclude that $Rad(\mathcal{A})$ contains $Q_1'\mathbb{M}_nQ_2$ and $Q_2\mathbb{M}_nQ_3'$, as M_1 and M_2 were arbitrary.

■

§5 Compressibility in $\mathbb{M}_3$

We now turn our attention to assessing the completeness of the list of idempotent compressible algebras established in §2 and §3. That is, we wish to determine whether or not there exist additional examples of unital idempotent compressible algebras up to transpose similarity.

Our findings in §3.2 suggest that there may be pathological examples of such algebras that exist in $\mathbb{M}_3$. For this reason, we devote this section to classifying the unital subalgebras in $\mathbb{M}_3$ that admit the idempotent compression property, and reserve the classification of such subalgebras of $\mathbb{M}_n$, $n \geq 4$, for [2].

Using the structure theory established in §4, we show in §5.1 that up to transposition and similarity, the only unital idempotent compressible subalgebras of $\mathbb{M}_3$ are those constructed in §2 and §3. As a consequence of this analysis, we will observe that a unital subalgebra $\mathcal{A}$ of $\mathbb{M}_3$ that lacks the idempotent compression property is necessarily transpose similar to one of the following algebras:

$$\mathcal{B} := \left\{ \begin{bmatrix} \alpha & x & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix} : \alpha, \beta, x \in \mathbb{C} \right\},$$

$$\mathcal{C} := \left\{ \begin{bmatrix} \alpha & x & y \\ 0 & \alpha & x \\ 0 & 0 & \alpha \end{bmatrix} : \alpha, x, y \in \mathbb{C} \right\}, \quad \text{or}$$

$$\mathcal{D} := \left\{ \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{C} \right\}.$$

This observation has interesting implications for *projection* compressibility $\mathbb{M}_3$. In particular, it leads to an avenue for proving that in the case of unital subalgebras of $\mathbb{M}_3$, the notions of projection compressibility and idempotent compressibility coincide. Indeed, note that if there were a unital projection compressible subalgebra $\mathcal{A}$ of $\mathbb{M}_3$ that did not exhibit the idempotent compression property, then $\mathcal{A}$ must be similar to $\mathcal{B}$, $\mathcal{C}$, or $\mathcal{D}$. Thus, one could establish the above equivalence by proving that no algebra similar to $\mathcal{B}$, $\mathcal{C}$, or $\mathcal{D}$ is projection compressible. We follow this approach in §5.2 to show that the notions do in fact agree.

§5.1 Classification of Idempotent Compressibility. Here we begin the classification of unital idempotent compressible subalgebras of $\mathbb{M}_3$, up to transposition and similarity. Note that from the structure theory developed in §4, we may assume that all algebras $\mathcal{A}$ are expressed in reduced block upper triangular form with respect to an orthogonal decomposition of $\mathbb{C}^3$, and that $\mathcal{A}$ is unhinged with respect to this decomposition. That is, we will assume that

$$\mathcal{A} = BD(\mathcal{A}) \dotplus Rad(\mathcal{A}),$$

where $Rad(\mathcal{A})$ consists of all strictly block upper triangular elements of $\mathcal{A}$. With this in mind, the algebras in this list will be organized according to the configuration of their block-diagonal and the dimension of their radical.

Let $\mathcal{A} = BD(\mathcal{A}) \dotplus Rad(\mathcal{A})$ be a unital subalgebra of $\mathbb{M}_3$ that is in reduced block upper triangular form with respect to a decomposition $\mathbb{C}^3 = \bigoplus_{i=1}^m \mathcal{V}_i$. If $\mathcal{A} = \mathbb{M}_3$, then $\mathcal{A}$ is clearly idempotent compressible. Furthermore, if some $\mathcal{V}_i$ has dimension 2, Theorem 2.0.7 implies that $\mathcal{A}$ is transpose equivalent to $\mathbb{C} \oplus \mathbb{M}_2$ or

$$\left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} : a_{ij} \in \mathbb{C} \right\}.$$

In either case, $\mathcal{A}$ is the unitization of an $\mathcal{LR}$-algebra, and hence is idempotent compressible.

Thus, we may assume from here on that all spaces $\mathcal{V}_i$ have dimension 1. For each i, let e_i be a unit vector in $\mathcal{V}_i$, and let Q_i denote the orthogonal projection onto $\mathcal{V}_i$.

Case I: $\dim BD(\mathcal{A}) = 3$. If $\dim BD(\mathcal{A}) = 3$, then the spaces $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}_3$ are mutually unlinked. An application of Lemma 4.0.7 then shows that

$$Rad(\mathcal{A}) = Q_1 Rad(\mathcal{A}) Q_2 \dot{+} Q_1 Rad(\mathcal{A}) Q_3 \dot{+} Q_2 Rad(\mathcal{A}) Q_3.$$

(i) If $Rad(\mathcal{A}) = \{0\}$, then $\mathcal{A} = \mathcal{D}$, one of the three algebras presented at the outset of §5. It will be shown in Theorem 5.2.6 that no algebra similar to $\mathcal{D}$ is projection compressible. In particular, $\mathcal{A}$ is not idempotent compressible.

(ii) If $\dim Rad(\mathcal{A}) = 1$, then there is exactly one pair of indices (i, j) such that $i < j$ and $Q_i Rad(\mathcal{A}) Q_j$ is non-zero. In this case, $\mathcal{A}$ is unitarily equivalent to

$$\mathbb{C}Q_1 + \mathbb{C}Q_2 + (Q_2 + Q_3)\mathbb{M}_3 Q_3,$$

the algebra described in Example 3.2.2. Consequently, $\mathcal{A}$ is idempotent compressible.

(iii) If $\dim Rad(\mathcal{A}) = 2$, then $Q_i Rad(\mathcal{A}) Q_j = \{0\}$ for exactly one pair of indices (i, j) with $i < j$. By considering products in $Rad(\mathcal{A})$, one can show that $Q_1 Rad(\mathcal{A}) Q_3$ is non-zero whenever both $Q_1 Rad(\mathcal{A}) Q_2$ and $Q_2 Rad(\mathcal{A}) Q_3$ are non-zero. This means that either $Q_1 Rad(\mathcal{A}) Q_2 = \{0\}$ or $Q_2 Rad(\mathcal{A}) Q_3 = \{0\}$; hence $\mathcal{A}$ is transpose equivalent to

$$\mathbb{C}Q_1 + \mathbb{C}Q_2 + \mathbb{C}Q_3 + (Q_1 + Q_2)\mathbb{M}_3 Q_3.$$

This algebra was shown to admit the idempotent compression property in 3.1.3. Therefore, $\mathcal{A}$ is idempotent compressible.

(iv) If $\dim Rad(\mathcal{A}) = 3$, then $\mathcal{A}$ is equal to

$$\mathbb{C}Q_1 + \mathbb{C}Q_3 + (Q_1 + Q_2)\mathbb{M}_3(Q_2 + Q_3),$$

the unital algebra from Example 3.1.1. Consequently, $\mathcal{A}$ is idempotent compressible.

Case II: $\dim BD(\mathcal{A}) = 2$. If $\dim BD(\mathcal{A}) = 2$, then exactly two of the spaces $\mathcal{V}_i$ and $\mathcal{V}_j$ are linked. By replacing $\mathcal{A}$ with $\mathcal{A}^{aT}$ if necessary, we may assume that $\mathcal{V}_1$ is one of the linked spaces.

(i) If $Rad(\mathcal{A}) = \{0\}$, then $\mathcal{A}$ is unitarily equivalent to $\mathbb{C}(Q_1 + Q_2) + \mathbb{C}Q_3$, and hence $\mathcal{A}$ is the unitization of the $\mathcal{LR}$-algebra $\mathbb{C}Q_3$. Consequently, $\mathcal{A}$ is idempotent compressible.

(ii) If $\dim Rad(\mathcal{A}) = 1$, then $Rad(\mathcal{A}) = \mathbb{C}R$ for some strictly upper triangular element

$$R = \begin{bmatrix} 0 & r_{12} & r_{13} \\ 0 & 0 & r_{23} \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $R^2 \in Rad(\mathcal{A})$, we have that $R^2 = \alpha R$ for some $\alpha \in \mathbb{C}$. From this it follows that at least one of r_{12} or r_{23} is equal to zero.

First consider the case in which $\mathcal{V}_2$ is not linked to $\mathcal{V}_1$. By Lemma 4.0.7,

$$Rad(\mathcal{A}) = Q_1 Rad(\mathcal{A}) Q_2 \dot{+} Q_1 Rad(\mathcal{A}) Q_3 \dot{+} Q_2 Rad(\mathcal{A}) Q_3.$$

If $r_{12} = r_{13} = 0$ or $r_{13} = r_{23} = 0$, then $\mathcal{A}$ or $\mathcal{A}^{aT}$ is equal to

$$\mathcal{A} = Q_2 \mathbb{M}_3 (Q_2 + Q_3) + \mathbb{C}I.$$

In this case, $\mathcal{A}$ is idempotent compressible as it is the unitization of an $\mathcal{LR}$-algebra. If instead $r_{12} = r_{23} = 0$, then $\mathcal{A}$ is unitarily equivalent to $\mathcal{B}$, one of the three algebras described at the beginning of §5. It will be shown in Theorem 5.2.2 that no algebra similar to $\mathcal{B}$ is projection compressible. In particular, $\mathcal{A}$ is not idempotent compressible.

Now consider the case in which $\mathcal{V}_1$ is linked to $\mathcal{V}_2$. This means that $\mathcal{V}_3$ is unlinked from $\mathcal{V}_1$ and $\mathcal{V}_2$, and therefore

$$Rad(\mathcal{A}) = Q_1 Rad(\mathcal{A}) Q_2 \dotplus (Q_1 + Q_2) Rad(\mathcal{A}) Q_3.$$

If $r_{12} = 0$, then $\mathcal{A}$ is unitarily equivalent to

$$(Q_2 + Q_3)\mathbb{M}_3 Q_3 + \mathbb{C}I.$$

In this case, $\mathcal{A}$ is idempotent compressible as it is the unitization of an $\mathcal{LR}$-algebra. If instead $r_{12} \neq 0$, then $r_{13} = r_{23} = 0$ and hence $\mathcal{A}$ is equal to $\mathcal{B}$.

(iii) Suppose now that $\dim Rad(\mathcal{A}) = 2$. If $\mathcal{V}_2$ is the unlinked space, then

$$Rad(\mathcal{A}) = Q_1 Rad(\mathcal{A}) Q_2 \dotplus Q_1 Rad(\mathcal{A}) Q_3 \dotplus Q_2 Rad(\mathcal{A}) Q_3.$$

It then follows that either $Q_1 Rad(\mathcal{A}) Q_2 = \{0\}$ or $Q_2 Rad(\mathcal{A}) Q_3 = \{0\}$, so $\mathcal{A}$ is transpose equivalent to

$$\mathbb{C}(Q_1 + Q_2) + \mathbb{C}Q_3 + Q_1 \mathbb{M}_3 (Q_2 + Q_3).$$

This algebra was shown to admit the idempotent compression property in Example 3.2.5, so $\mathcal{A}$ is idempotent compressible as well.

Now consider the case where $\mathcal{V}_2$ is linked to $\mathcal{V}_1$, so that

$$Rad(\mathcal{A}) = Q_1 Rad(\mathcal{A}) Q_2 \dotplus (Q_1 + Q_2) Rad(\mathcal{A}) Q_3.$$

If $Q_1 Rad(\mathcal{A}) Q_2 = \{0\}$, then

$$\mathcal{A} = \mathbb{M}_3 Q_3 + \mathbb{C}I.$$

Consequently, $\mathcal{A}$ is idempotent compressible as it is the unitization of an $\mathcal{LR}$-algebra. If instead $Q_1 Rad(\mathcal{A}) Q_2 = Q_1 \mathbb{M}_3 Q_2$, then $(Q_1 + Q_2) Rad(\mathcal{A}) Q_3$ is 1-dimensional. Thus, there is a non-zero matrix $R \in (Q_1 + Q_2)\mathbb{M}_3 Q_3$ such that

$$Rad(\mathcal{A}) = Q_1 \mathbb{M}_3 Q_2 \dotplus \mathbb{C}R.$$

It is then easy to see that $\langle Re_3, e_2 \rangle = 0$. For if not, $Rad(\mathcal{A})$ would contain an element of the form $e_2 \otimes e_3^* + te_1 \otimes e_3^*$ for some $t \in \mathbb{C}$; hence $Rad(\mathcal{A})$ also contains

$$(e_1 \otimes e_2^*)\,(e_2 \otimes e_3^* + te_1 \otimes e_3^*) = e_1 \otimes e_3^*$$

This would then imply that $Rad(\mathcal{A})$ is 3-dimensional—a contradiction.

Thus, $\langle Re_3, e_2 \rangle = 0$, so $\mathcal{A}$ is equal to

$$\mathbb{C}(Q_1 + Q_2) + \mathbb{C}Q_3 + Q_1 \mathbb{M}_3 (Q_2 + Q_3),$$

the idempotent compressible algebra from Example 3.2.5. In all cases, $\mathcal{A}$ is idempotent compressible.

(iv) Suppose that $\dim Rad(\mathcal{A}) = 3$. If $\mathcal{V}_2$ is the unlinked space, then $\mathcal{A}$ is equal to

$$(Q_1 + Q_2)\mathbb{M}_3 (Q_2 + Q_3) + \mathbb{C}I.$$

In this case $\mathcal{A}$ is the unitization of an $\mathcal{LR}$-algebra, and hence is idempotent compressible. If instead $\mathcal{V}_2$ is linked to $\mathcal{V}_1$, then $\mathcal{A}$ is equal to

$$\mathbb{C}(Q_1 + Q_2) + \mathbb{C}Q_3 + Q_1 \mathbb{M}_3 Q_2 + (Q_1 + Q_2)\mathbb{M}_3 Q_3,$$

the unital algebra described in Example 3.1.6. Consequently, $\mathcal{A}$ is idempotent compressible.

Case III: $\dim BD(\mathcal{A}) = 1$. Suppose now that $\dim BD(\mathcal{A}) = 1$, so that all spaces $\mathcal{V}_i$ are mutually linked. That is, $BD(\mathcal{A}) = \mathbb{C}I$.

(i) If $Rad(\mathcal{A}) = \{0\}$, then $\mathcal{A} = \mathbb{C}I$. Clearly $\mathcal{A}$ is idempotent compressible.

(ii) If $\dim Rad(\mathcal{A}) = 1$, then $Rad(\mathcal{A}) = \mathbb{C}R$ for some strictly upper triangular matrix

$$R = \begin{bmatrix} 0 & r_{12} & r_{13} \\ 0 & 0 & r_{23} \\ 0 & 0 & 0 \end{bmatrix}.$$

As in part (ii) of the previous case, one can show that $r_{12} = 0$ or $r_{23} = 0$, so R necessarily has rank 1. By replacing $\mathcal{A}$ with $\mathcal{A}^{aT}$ if necessary, we may assume that $r_{12} = 0$. But then $\mathcal{A}$ is unitarily equivalent to

$$Q_2 \mathbb{M}_3 Q_3 + \mathbb{C}I,$$

the unitization of an $\mathcal{LR}$-algebra. Thus, $\mathcal{A}$ is idempotent compressible.

(iii) Suppose that $\dim Rad(\mathcal{A}) = 2$. If $Q_1 Rad(\mathcal{A}) Q_2 = \{0\}$ or $Q_2 Rad(\mathcal{A}) Q_3 = \{0\}$, then $\mathcal{A}$ or $\mathcal{A}^{aT}$ is equal to

$$Q_1 \mathbb{M}_3 (Q_2 + Q_3) + \mathbb{C}I.$$

Thus, $\mathcal{A}$ is idempotent compressible as it is the unitization of an $\mathcal{LR}$-algebra.

Now consider the case in which $Rad(\mathcal{A})$ contains an element

$$R = \begin{bmatrix} 0 & r_{12} & r_{13} \\ 0 & 0 & r_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

with $r_{12} \neq 0$ and $r_{23} \neq 0$. In this case, $Rad(\mathcal{A})$ contains $\frac{1}{r_{12} r_{23}} R^2 = e_1 \otimes e_3^*$; hence

$$Rad(\mathcal{A}) = \operatorname{span}\{e_1 \otimes e_2^* + r e_2 \otimes e_3^*, e_1 \otimes e_3^*\}$$

where $r := r_{23}/r_{12}$. Consequently,

$$\mathcal{A} = \left\{ \begin{bmatrix} \alpha & x & y \\ 0 & \alpha & rx \\ 0 & 0 & \alpha \end{bmatrix} : \alpha, x, y \in \mathbb{C} \right\},$$

which is easily seen to be similar to the algebra $\mathcal{C}$ described at the outset of §5. It will be shown in Theorem 5.2.4 that no algebra similar to $\mathcal{C}$ is projection compressible. In particular, $\mathcal{A}$ is not idempotent compressible.

(iv) If $\dim Rad(\mathcal{A}) = 3$, then $\mathcal{A}$ is equal to

$$Q_1 \mathbb{M}_3 (Q_2 + Q_3) + Q_2 \mathbb{M}_3 Q_3 + \mathbb{C}I,$$

the idempotent compressible algebra described in Example 3.2.8.

Let us quickly summarize the analysis from this section. We have shown that if $\mathcal{A}$ is a unital subalgebra of $\mathbb{M}_3$ that affords the idempotent compression property, then either $\mathcal{A}$ is the unitization of an $\mathcal{LR}$-algebra, or $\mathcal{A}$ is transpose similar to one of the algebras described in Example 3.1.1, 3.1.3, 3.1.6, 3.2.2, 3.2.5, or 3.2.8. If instead $\mathcal{A}$ is a unital subalgebra of $\mathbb{M}_3$ that lacks the idempotent compression property, then $\mathcal{A}$ must be transpose similar to one of the algebras $\mathcal{B}$, $\mathcal{C}$, or $\mathcal{D}$ described at the outset of §5.

§5.2 Projection Compressibility = Idempotent Compressibility. Our final goal of this manuscript is to show that no unital subalgebra of $\mathbb{M}_3$ can possess the projection compression property without also possessing the idempotent compression property. If such an algebra did exist, it would necessarily be transpose similar to $\mathcal{B}$, $\mathcal{C}$, or $\mathcal{D}$ by the analysis in §5.1. Thus, to show that the notions of projection compressibility and idempotent compressibility agree for unital subalgebras of $\mathbb{M}_3$, it suffices to prove that no algebra similar to $\mathcal{B}$, $\mathcal{C}$, or $\mathcal{D}$ is projection compressible. This goal will be accomplished by first characterizing the algebras similar to $\mathcal{B}$, $\mathcal{C}$, or $\mathcal{D}$ up to *unitary equivalence*.

Lemma 5.2.1. *Let $\mathcal{A}$ be a subalgebra of of $\mathbb{M}_3$. If $\mathcal{A}$ is similar to*

$$\mathcal{B} = \left\{ \begin{bmatrix} \alpha & x & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix} : \alpha, \beta, x \in \mathbb{C} \right\},$$

then there are constants $s, t \in \mathbb{C}$ such that $\mathcal{A}$ is unitarily equivalent to

$$\mathcal{B}_{st} := \left\{ \begin{bmatrix} \alpha & s(\alpha-\beta) & x \\ 0 & \beta & t(\alpha-\beta) \\ 0 & 0 & \alpha \end{bmatrix} : \alpha, \beta, x \in \mathbb{C} \right\}.$$

Proof. If the matrices in $\mathcal{B}$ are expressed with respect to the standard basis $\{e_1, e_2, e_3\}$ for $\mathbb{C}^3$, then $\mathcal{B}$ is spanned by $\{E_{11}+E_{22}, E_{12}, E_{33}\}$, where $E_{ij} := e_i \otimes e_j^*$. Thus, if S is an invertible matrix in $\mathbb{M}_3$ such that $\mathcal{A} = S^{-1}\mathcal{B}S$, then $\mathcal{A}$ is spanned by $\{E'_{11}+E'_{22}, E'_{12}, E'_{33}\}$, where $E'_{ij} := S^{-1}E_{ij}S$.

Since E'_{12} is a rank-one nilpotent of order 2, there is a unitary $U \in \mathbb{M}_3$ and a non-zero $y_0 \in \mathbb{C}$ such that

$$U^* E'_{12} U = \begin{bmatrix} 0 & 0 & y_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let x_{ij} be complex constants such that $U^*(E'_{11}+E'_{22})U = (x_{ij})$. Using the fact that

$$(E'_{11}+E'_{22})E'_{12} = E'_{12}(E'_{11}+E'_{22}) = E'_{12},$$

one can show that $x_{21} = x_{31} = x_{32} = 0$ and $x_{11} = x_{33} = 1$. Moreover, since $U^*(E'_{11}+E'_{22})U$ is an idempotent of trace 2, it follows that $x_{22} = 0$ and $x_{13} = -x_{12}x_{23}$. Thus,

$$U^*(E'_{11}+E'_{22})U = \begin{bmatrix} 1 & x_{12} & -x_{12}x_{23} \\ 0 & 0 & x_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

Finally, we have that

$$U^* E'_{33} U = I - U^*(E'_{11}+E'_{22})U = \begin{bmatrix} 0 & -x_{12} & x_{12}x_{23} \\ 0 & 1 & -x_{23} \\ 0 & 0 & 0 \end{bmatrix}.$$

As a result,

$$U^*\mathcal{A}U = \operatorname{span}\left\{ \begin{bmatrix} 1 & x_{12} & -x_{12}x_{23} \\ 0 & 0 & x_{23} \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -x_{12} & x_{12}x_{23} \\ 0 & 1 & -x_{23} \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & y_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} = \mathcal{B}_{st},$$

where $s := x_{12}$ and $t := x_{23}$. ∎

Theorem 5.2.2. *If s and t are complex constants, then the algebra $\mathcal{B}_{st}$ as in Lemma 5.2.1 is not projection compressible. Consequently, no algebra similar to $\mathcal{B}$ is projection compressible.*

Proof. Consider the elements A and B of $\mathcal{B}_{st}$ given by

$$A = \begin{bmatrix} 1 & s & 0 \\ 0 & 0 & t \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We will construct a matrix P that is a multiple of a projection in $\mathbb{M}_3$, and such that $(PAP)(PBP)$ does not belong to $P\mathcal{B}_{st}P$. To do this, let k be any element of $\mathbb{R} \setminus \{0, s, t\}$, and define

$$P := \begin{bmatrix} k^2+1 & -k & -1 \\ -k & 2 & -k \\ -1 & -k & k^2+1 \end{bmatrix}.$$

Note that $\frac{1}{k^2+2}P$ is a projection in $\mathbb{M}_3$.

If $(PAP)(PBP)$ were an element of $P\mathcal{B}_{st}P$, there would exist a matrix

$$C = \begin{bmatrix} \alpha_0 & s(\alpha_0-\beta_0) & x_0 \\ 0 & \beta_0 & t(\alpha_0-\beta_0) \\ 0 & 0 & \alpha_0 \end{bmatrix} \in \mathcal{B}_{st}$$

such that $G := PAPBP - PCP = (g_{ij})$ is equal to 0. By examining the value of g_{31}, one can show that x_0 must be given by

$$k(\alpha_0-\beta_0+1)(2k-s-t) + 2(\alpha_0+1) + k^2\beta_0.$$

Direct computations then show that

$$(k-s)g_{11} - (k-t)g_{33} = k(k^2+2)(k-s)(k-t).$$

Since $g_{11} = g_{33} = 0$, but the right-hand side is non-zero by construction, we have reached a contradiction. Thus, there does not exist a C as above, so $P\mathcal{B}_{st}P$ is not an algebra. The final claim is now a consequence of Lemma 5.2.1.

■

Lemma 5.2.3. *Let $\mathcal{A}$ be a subalgebra of $\mathbb{M}_3$. If $\mathcal{A}$ is similar to*

$$\mathcal{C} := \left\{ \begin{bmatrix} \alpha & x & y \\ 0 & \alpha & x \\ 0 & 0 & \alpha \end{bmatrix} : \alpha, x, y \in \mathbb{C} \right\},$$

then there is a non-zero constant $r \in \mathbb{C}$ such that $\mathcal{A}$ is unitarily equivalent to

$$\mathcal{C}_r := \left\{ \begin{bmatrix} \alpha & x & y \\ 0 & \alpha & rx \\ 0 & 0 & \alpha \end{bmatrix} : \alpha, x, y \in \mathbb{C} \right\}.$$

Proof. Observe that $\mathcal{C}$ is spanned by $\{I, N_1, N_2\}$, where

$$N_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad N_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, if $S \in \mathbb{M}_3$ is an invertible matrix such that $\mathcal{A} = S^{-1}\mathcal{C}S$, then $\mathcal{A}$ is spanned by $\{I, N_1', N_2'\}$, where $N_i' = S^{-1}N_iS$ for $i \in \{1, 2\}$.

It is evident that N_1' is a rank-one nilpotent, N_2' is a rank-two nilpotent, and $N_1'N_2' = N_2'N_1' = 0$. In particular, since N_1' and N_2' commute, there is a unitary $U \in \mathbb{M}_3$ such that $U^*N_1'U$ and $U^*N_2'U$ are upper triangular. If a_{ij} and b_{ij} are such that

$$U^*N_1'U = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad U^*N_2'U = \begin{bmatrix} 0 & b_{12} & b_{13} \\ 0 & 0 & b_{23} \\ 0 & 0 & 0 \end{bmatrix},$$

then rank considerations imply that neither b_{12} nor b_{23} is equal to 0. But

$$(U^*N_1'U)(U^*N_2'U) = \begin{bmatrix} 0 & 0 & a_{12}b_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad (U^*N_2'U)(U^*N_1'U) = \begin{bmatrix} 0 & 0 & a_{23}b_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so it must be that $a_{12} = a_{23} = 0$. By setting $r = b_{23}/b_{12}$, it follows that

$$U^*\mathcal{A}U = \operatorname{span}\left\{ I, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & b_{13}/b_{12} \\ 0 & 0 & r \\ 0 & 0 & 0 \end{bmatrix} \right\} = \mathcal{C}_r.$$

■

Theorem 5.2.4. *If r is a non-zero element of $\mathbb{C}$, then the algebra $\mathcal{C}_r$ as in Lemma 5.2.3 is not projection compressible. Consequently, no algebra similar to $\mathcal{C}$ is projection compressible.*

Proof. Consider element $A, B \in \mathcal{C}_r$ given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & r \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Furthermore, define the matrix

$$P := \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix},$$

so $\frac{1}{3}P$ is a projection in $\mathbb{M}_3$.

We claim that $(PAP)(PBP)$ does not belong to $P\mathcal{C}_rP$. For if it did, there would exist an element

$$C = \begin{bmatrix} \alpha_0 & x_0 & y_0 \\ 0 & \alpha_0 & rx_0 \\ 0 & 0 & \alpha_0 \end{bmatrix}$$

in $\mathcal{C}_r$ such that $G = PAPBP - PCP = (g_{ij})$ is equal to 0. Direct computations show that

$$0 = g_{31} = 3\alpha_0 - (x_0+1)(r+1) - y_0,$$

hence $y_0 = 3\alpha_0 - (x_0+1)(r+1)$. From here, further calculations reveal that

$$g_{21} - rg_{32} = 3r.$$

Since $g_{21} = g_{32} = 0$ but $r \neq 0$, we have reached a contradiction. Thus, there does not exist an element $C \in \mathcal{C}_r$ as described above. This shows that $(PAP)(PBP) \notin P\mathcal{C}_rP$, so $\mathcal{C}_r$ is not projection compressible. The final claim is now immediate from Lemma 5.2.3.

■

Lemma 5.2.5. *Let $\mathcal{A}$ be a subalgebra of $\mathbb{M}_3$. If $\mathcal{A}$ is similar to*

$$\mathcal{D} = \left\{ \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{C} \right\},$$

then there are complex constants r, s, and t such that $\mathcal{A}$ is unitarily equivalent to

$$\mathcal{D}_{rst} := \left\{ \begin{bmatrix} \alpha & r(\alpha-\beta) & s(\alpha-\gamma) - rt(\gamma-\beta) \\ 0 & \beta & t(\gamma-\beta) \\ 0 & 0 & \gamma \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{C} \right\}.$$

Proof. If $\mathcal{D}$ is written with respect to the standard basis $\{e_1, e_2, e_3\}$ for $\mathbb{C}^3$, then $\mathcal{D}$ is spanned by $\{E_{11}, E_{22}, E_{33}\}$ where $E_{jj} = e_j \otimes e_j^*$. Let S be an invertible element of $\mathbb{M}_3$ such that $\mathcal{A} = S^{-1}\mathcal{D}S$. Clearly $\mathcal{A}$ is spanned by $\{E'_{11}, E'_{22}, E'_{33}\}$ where $E'_{jj} = S^{-1}E_{jj}S$.

Observe that the matrices E'_{jj} commute, so there is a unitary $U \in \mathbb{M}_3$ such that $U^*E'_{jj}U$ is upper triangular for each $j \in \{1,2,3\}$. Further, since each $U^*E'_{jj}U$ is an idempotent of rank 1, and

$$(U^*E'_{ii}U)(U^*E'_{jj}U) = \delta_{ij}U^*E'_{jj}U$$

for all i and j, one may re-index the matrices E'_{jj} if necessary to write

$$U^*E'_{11}U = \begin{bmatrix} 1 & x_{12} & x_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad U^*E'_{22}U = \begin{bmatrix} 0 & y_{12} & y_{12}y_{23} \\ 0 & 1 & y_{23} \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad U^*E'_{33}U = \begin{bmatrix} 0 & 0 & z_{13} \\ 0 & 0 & z_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

for some x_{ij}, y_{ij}, and z_{ij} in $\mathbb{C}$. The fact that these matrices add to I implies that

$$y_{12} = -x_{12}, \quad y_{23} = -z_{23}, \quad \text{and} \quad z_{13} = -x_{13} - x_{12}z_{23} \ .$$

As a result,

$$U^*\mathcal{A}U = \operatorname{span}\left\{ \begin{bmatrix} 1 & x_{12} & x_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -x_{12} & x_{12}z_{23} \\ 0 & 1 & -z_{23} \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -x_{13} - x_{12}z_{23} \\ 0 & 0 & z_{23} \\ 0 & 0 & 1 \end{bmatrix} \right\} = \mathcal{D}_{rst},$$

where $r := x_{12}$, $s := x_{13}$, and $t := z_{23}$.

■

Theorem 5.2.6. *If $r, s,$ and t are complex constants, then the algebra $\mathcal{D}_{rst}$ as in Lemma 5.2.5 is not projection compressible. Consequently, no algebra similar to $\mathcal{D}$ is projection compressible.*

Proof. Consider the elements A and B of $\mathcal{D}_{rst}$ given by

$$A = \begin{bmatrix} 1 & r & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -r & rt \\ 0 & 1 & -t \\ 0 & 0 & 0 \end{bmatrix}.$$

We wish to construct a matrix P that is a multiple of a projection in $\mathbb{M}_3$, and such that $(PAP)(PBP)$ does not belong to $P\mathcal{D}_{rst}P$. To do this, choose elements $k, m \in \mathbb{R} \setminus \{0\}$ subject to the following constraints:

$$\begin{array}{rcrcrl} tk & & & \neq & 1, & \\ & & rm & \neq & 1, & \\ sk & + & m & \neq & -r, & \text{and} \\ k & - & (rt+s)m & \neq & -t. & \end{array}$$

Of course, such k and m always exist. Using these values, define

$$P = \begin{bmatrix} k^2+1 & -m & -mk \\ -m & k^2+m^2 & -k \\ -mk & -k & m^2+1 \end{bmatrix}.$$

It is straightforward to check that $\frac{1}{k^2+m^2+1}P$ is a projection in $\mathbb{M}_3$.

Suppose to the contrary that $(PAP)(PBP)$ were an element of $P\mathcal{D}_{rst}P$. In this case, there is a matrix

$$C = \begin{bmatrix} \alpha_0 & r(\alpha_0-\beta_0) & s(\alpha_0-\gamma_0)-rt(\gamma_0-\beta_0) \\ 0 & \beta_0 & t(\gamma_0-\beta_0) \\ 0 & 0 & \gamma_0 \end{bmatrix} \in \mathcal{D}_{rst}$$

such that $G := PAPBP - PCP$ is equal to 0. Our goal is to obtain a contradiction by examining specific entries of $G = (g_{ij})$.

Firstly, one may check that

$$0 = g_{31} - kg_{21} = km(k^2+m^2+1)(tk-1)(\beta_0-\gamma_0).$$

By construction, the product on the right-hand side is zero if and only if $\beta_0 = \gamma_0$. But if this is the case, then

$$kg_{23} - g_{33} = \beta_0(k^2+m^2+1),$$

so we must have $\beta_0 = \gamma_0 = 0$. Direct computation then show that

$$\begin{aligned} &(r(k^2+m^2)-sk-m)g_{21} - (k^2-skm-rm+1)g_{22} \\ &\qquad = km(k^2+m^2+1)(rm-1)(sk+m+r)(k-(rt+s)m+t). \end{aligned}$$

Since $g_{21} = g_{22} = 0$ while the right-hand side of this equation is non-zero by construction, we obtain the required contradiction.

Thus, $(PAP)(PBP)$ does not belong to $P\mathcal{D}_{rst}P$, so $\mathcal{D}_{rst}$ is not projection compressible. The final claim now follows from Lemma 5.2.5. ■

§6 Conclusion

Our analysis from §5 leads to the following classification of unital subalgebras of $\mathbb{M}_3$ that admit one, and hence both of the compression properties.

Theorem 6.0.1. *If $\mathcal{A}$ is a unital subalgebra of $\mathbb{M}_3$, then the following are equivalent:*

(i) $\mathcal{A}$ is projection compressible;
(ii) $\mathcal{A}$ is idempotent compressible;
(iii) $\mathcal{A}$ is the unitization of an $\mathcal{LR}$-algebra, or $\mathcal{A}$ is transpose similar to one of the unital algebras from Example 3.1.1, 3.1.3, 3.1.6, 3.2.2, 3.2.5, or 3.2.8.

The fact that the set of projection compressible and idempotent compressible subalgebras of $\mathbb{M}_3$ (and as will be shown in [2], of $\mathbb{M}_n$ for all $n \geq 4$) coincide is rather surprising. As mentioned in the introduction, despite a considerable amount of effort, we have been unable to provide a direct proof of this fact that does not involve characterizing each class of algebras. Such a proof might shed further light on why these algebras have the particular structures described above.

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