Numerical analysis of divergence-free discontinuous Galerkin methods for incompressible flow problems

by

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Author's Declarations

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

I am the sole author of all material in Chapters 3-4.

The material in Chapter 2 and Appendix A, and parts of the material in Chapters 1 and 5, is adapted from the preprint paper [4] (this paper has been accepted for publication in the Journal of Scientific Computing). All mathematical analysis that is presented in this paper was carried out solely by myself. I also wrote the majority of the computer code that was used to obtain the numerical results in this paper. Lastly, I am responsible for the large majority of the writing in this paper, with the co-authors (Sander Rhebergen and Garth Wells) having made only minor edits to the writing in various places.

Abstract

In the first major contribution of this thesis, we present analysis of two lowest-order hybridizable discontinuous Galerkin methods for the Stokes problem, while making only minimal regularity assumptions on the exact solution. The methods under consideration have previously been shown to produce $H(\text{div})$-conforming and divergence-free approximate velocities. Using these properties, we derive a priori error estimates for the velocity that are independent of the pressure. These error estimates, which assume only H^{1+s}-regularity of the exact velocity fields for any $s \in [0, 1]$, are optimal in a discrete energy norm. Error estimates for the velocity and pressure in the L^2-norm are also derived in this minimal regularity setting. In the second major contribution of this thesis, we extend this analysis to the setting of the steady Navier–Stokes problem. We begin by proposing a new divergence-free discontinuous Galerkin method for the steady Navier–Stokes problem, and we show that the resultant discretized problem admits a unique solution under a smallness condition on the problem data. We then present an error analysis of the method in the minimal regularity setting, and we take special care to properly estimate the nonlinear terms arising from convection. We show that it is possible to derive optimal a priori error estimates for the velocity in a discrete energy norm. Our velocity error estimates are independent of the pressure, and require only H^{1+s}-regularity of the exact velocity fields where $s \in (0, 1]$ in the two-dimensional case and $s \in (1/2, 1]$ in the three-dimensional case.

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Chapter 1

Introduction

Finite element methods for incompressible flows that are *pressure-robust* have become increasingly popular. Such methods produce approximate velocity fields for which the *a priori* velocity error estimates are independent of the pressure approximation. Numerous classical inf-sup stable finite elements, such as the MINI element [2] and Bernardi–Raugel elements [5], are not pressure-robust [26], with the velocity error polluted by the pressure approximation error scaled by the inverse of the viscosity, which can be large if the pressure is complicated or the viscosity is small.

One way of achieving pressure-robustness is by stable mixed methods with $\boldsymbol{H}(\mathrm{div})$-conforming and divergence-free approximate velocities [26]. Methods with these properties may relax $\boldsymbol{H}^1$-conformity and use discontinuous velocity approximations, as constructing $\boldsymbol{H}^1$-conforming and inf-sup stable schemes that are also divergence-free is difficult [26]. For this reason, discontinuous Galerkin (DG) methods seem to be a natural candidate for the construction of pressure-robust schemes. Several classes of pressure-robust DG methods that produce $\boldsymbol{H}(\mathrm{div})$-conforming and divergence-free approximate velocities were introduced in [13, 46].

A drawback of DG methods is that they are, on a given mesh, typically computationally more expensive than standard conforming methods. Hybridized discontinuous Galerkin (HDG) methods were introduced to improve upon the computational efficiency of DG methods while retaining their desirable properties [11]. This is accomplished by introducing extra degrees of freedom on cell facets which allow for local cell-wise variables to be eliminated by static condensation. Examples of $\boldsymbol{H}(\mathrm{div})$-conforming and divergence-free HDG methods are given in [10, 37, 39] for the Stokes problem and in [21, 31, 38] for the Navier–Stokes problem.

The most important contributions of this thesis are presented in Chapter 2, which is based on the preprint paper [4] (this paper has been accepted for publication in the Journal of Scientific Computing). In Chapter 2 we study two closely related lowest-order hybridizable DG methods for the velocity-pressure formulation of the Stokes problem, which is given by

$$\begin{aligned} -\nu\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 && \text{in } \Omega, \\ \boldsymbol{u} &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Both methods produce $\boldsymbol{H}(\text{div})$-conforming and divergence-free approximate velocities, and are therefore pressure-robust. The first method is the lowest-order HDG method analyzed in [37, 39]. The velocity finite element space for this method consists of discontinuous piecewise linear functions on cells and facets. As discussed in [39], the computational cost of this method can be reduced, while maintaining pressure-robustness, by using a continuous basis for the velocity facet space. Such an approach is reminiscent of embedded discontinuous Galerkin (EDG) methods [23]. This leads to the EDG–HDG method of [39], the lowest-order formulation of which is the second method considered in Chapter 2.

The standard error analysis of DG and HDG methods for eq. (1.1) assumes that the exact solution is sufficiently regular, i.e. it is assumed that $(\boldsymbol{u}, p) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)$. Under this assumption, it can be shown that (see e.g. [14, Chapter 6])

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,h} \le C\Big\{ h\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} + \frac{1}{\nu} h \|p\|_{H^1(\Omega)} \Big\},$$

where $\boldsymbol{u}_h$ is the discrete velocity solution and $\|\cdot\|_{1,h}$ is a discrete H^1-norm. For pressure-robust methods, the standard analysis can be straightforwardly modified to yield that if $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$, then (see e.g. [26])

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,h} \le Ch\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} . \tag{1.2}$$

Error bounds of the form in eq. (1.2), which predict optimal rates of convergence with respect to the mesh size h, were derived in [39] for the HDG and EDG–HDG methods. Moreover, numerical experiments in [39] suggest that even when $\boldsymbol{H}^2$-regularity of the exact velocity solution fails to hold, these methods remain convergent. Because of their computational efficiency, the lowest-order HDG and EDG–HDG methods are appealing for problems with minimal regularity. However, error analysis for this minimal regularity case has not been developed. The purpose of Chapter 2 is to close this gap by extending the analysis of [39] to the minimal regularity setting. In particular, we show in Chapter 2 that if $\boldsymbol{u} \in \boldsymbol{H}^{1+s}(\Omega)$ for some real number $s \in [0,1]$, then the lowest-order HDG and EDG–HDG methods satisfy

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,h} \leq C h^s \|\boldsymbol{u}\|_{\boldsymbol{H}^{1+s}(\Omega)},$$

plus additional higher-order terms on the right-hand side. In Chapter 2 we also derive L^2-error bounds for the velocity and pressure in the minimal regularity setting.

To put the error analysis of Chapter 2 into a broader context, we review briefly the literature relevant to our analysis. Numerous classes of non-pressure-robust nonconforming methods for the Stokes problem were analyzed under minimal regularity assumptions in [3, 32]. Key to the analysis of [3, 32] is a so-called *enrichment operator* that maps nonconforming discrete functions to $\boldsymbol{H}^1$-conforming functions. More recently, by using enrichment operators that map discretely divergence-free functions to exactly divergence-free ones, this minimal regularity analysis has been extended to pressure-robust schemes. This is done in the works of [28, 33, 34, 45], which establish quasi-optimal and pressure-robust error estimates for various finite element methods that achieve pressure-robustness by modifying the source term in the discrete formulation. In [34, 45] modified Crouzeix–Raviart methods are considered, while [33] focuses on modified conforming methods and [28] on modified DG methods. Finally, a variety of conforming and nonconforming pressure-robust methods based on an augmented Lagrangian formulation have been proposed and analyzed under minimal regularity assumptions in [29].

The main contributions of Chapter 2 are as follows. First, we derive a bound on the consistency error of the lowest-order HDG and EDG–HDG methods, by means of an enrichment operator of the type considered in [28]. A consequence of the hybridized formulation is that our consistency error bound contains a new term not found in previous works. However, we show that it is still possible to obtain optimal pressure-robust velocity error estimates in a discrete energy norm. Pressure-robust velocity error estimates in the L^2-norm are also derived, and we conclude our analysis in Chapter 2 by deriving L^2-error bounds for the pressure.

Chapters 3 and 4 of this thesis should be viewed as being complementary to the material presented in Chapter 2. Chapter 3 is dedicated to proving a technical but important inequality for functions belonging to the three-dimensional finite element space of Guzmán and Neilan [25]. This inequality is used to ensure that the enrichment operator considered in Chapter 2 satisfies some crucial stability and approximation properties.

Chapter 4 can be viewed as an extension of the results in Chapter 2 to the setting of the steady Navier–Stokes equations. We begin Chapter 4 by proposing a new divergence-free DG method for the steady Navier–Stokes problem. Our proposed method is essentially a combination of the lowest-order formulation of the methods considered in [13] and [28]. We prove that the resultant discretized steady Navier–Stokes problem admits a unique solution provided that a smallness condition on the problem data holds. Then, building on the ideas developed in Chapter 2, we derive optimal and pressure-robust a-priori velocity error estimates for the method in a discrete energy norm. The analysis holds under minimal regularity assumptions on the exact solution, and is similar to that of Chapter 2, but with the need to additionally estimate contributions arising from the nonlinear convective term.

An important remark on the use of boldface notation

In Chapter 2 and Appendix A, boldface notation is used solely to denote cell-facet function pairs, see eq. (2.9). Vectors, vector-valued functions and vector-valued function spaces *are not* written using boldface in Chapter 2 and Appendix A. This notational convention is used in other HDG papers [37, 39] and we are using it to be consistent with the literature. In contrast, HDG methods do not appear or play any role in Chapters 3 and 4. Therefore, in these chapters, we use the "standard" notational convention in which vectors, vector-valued functions and vector-valued function spaces *are* written using boldface.

Chapter 2

Analysis of pressure-robust embedded-hybridized discontinuous Galerkin methods for the Stokes problem under minimal regularity

This chapter is based on [4, Sections 2-4] and is organized as follows. In Section 2.1 we introduce the Stokes problem and the methods to be analyzed, and discuss some preliminary results. The main analysis is carried out in Section 2.2, where we derive our error estimates for the velocity and the pressure. In Section 2.3 our theoretical findings are illustrated by numerical examples.

2.1 Preliminaries

Let $\Omega \subset \mathbb{R}^d$ with $d \in \{2, 3\}$ be a connected and bounded domain with polyhedral boundary $\partial\Omega$. The codimension of $\partial\Omega$ is assumed to be one, but we do not require that Ω be a Lipschitz domain. In particular, domains with cracks are allowed. On a given set $D \subset \Omega$, we let $(\cdot, \cdot)_D$ denote the L^2-inner-product on D and $\|\cdot\|_D$ the L^2-norm on D. Given an integer $k \geq 0$, we let $\|\cdot\|_{k,D}$ and $|\cdot|_{k,D}$ denote the usual H^k-norm and H^k-semi-norm on D, respectively. If $k > 0$ is not an integer, we let $\|\cdot\|_{k,D}$ denote the fractional order H^k-norm on D as defined in [15, 16]. In the following we omit the subscript D in the case of $D = \Omega$.

2.1.1 Stokes problem

Let $f \in L^2(\Omega)^d$ be a prescribed body force and $\nu > 0$ a given constant kinematic viscosity. The Stokes problem seeks a velocity field $u \in H_0^1(\Omega)^d$ and kinematic pressure field $p \in L_0^2(\Omega) := \left\{q \in L^2(\Omega) : \int_\Omega q \,\mathrm{d}x = 0\right\}$ such that

$$\nu a(u, v) + b(v, p) = (f, v) \qquad \forall v \in H_0^1(\Omega)^d, \tag{2.1a}$$
$$b(u, q) = 0 \qquad \forall q \in L_0^2(\Omega), \tag{2.1b}$$

where $a : H_0^1(\Omega)^d \times H_0^1(\Omega)^d \to \mathbb{R}$ and $b : H_0^1(\Omega)^d \times L_0^2(\Omega) \to \mathbb{R}$ are the bilinear forms

$$a(w, v) := (\nabla w, \nabla v), \qquad b(v, q) := -(\nabla \cdot v, q).$$

It is known that eq. (2.1) is well-posed, see e.g. [19, Chapter 4]. Furthermore, within the reduced space $V := \{v \in H_0^1(\Omega)^d : \nabla \cdot v = 0\}$ of divergence-free functions, the velocity $u \in V$ equivalently satisfies the reduced problem

$$\nu a(u, v) = (f, v) \qquad \forall v \in V. \tag{2.2}$$

We introduce the space of weakly divergence-free vector fields with vanishing normal component on $\partial\Omega$,

$$L_\sigma^2(\Omega) := \{w \in L^2(\Omega)^d : (w, \nabla\psi) = 0 \ \forall \psi \in H^1(\Omega)\}, \tag{2.3}$$

and note that every vector field $f \in L^2(\Omega)^d$ admits a unique Helmholtz decomposition [34, Theorem 2.1]

$$f = \nabla\phi + \mathbb{P}f,$$

where $\phi \in H^1(\Omega)/\mathbb{R}$ and $\mathbb{P}f \in L_\sigma^2(\Omega)$.

The vector field $\mathbb{P}f$ is called the Helmholtz projection of f, see e.g. [26, Section 2]. We note that the reduced problem in eq. (2.2) is equivalent to

$$\nu a(u, v) = (\mathbb{P}f, v) \qquad \forall v \in V, \tag{2.4}$$

since for all $v \in V$ it holds that $(f, v) = (\mathbb{P}f, v)$. In particular, the velocity solution u is determined only by the Helmholtz projection $\mathbb{P}f$ of the body force. The presence of $\mathbb{P}f$ in eq. (2.4) will turn out to play an important role in the pressure-robustness of our error estimates in Section 2.2, and we discuss why this is the case in Remark 2.1.

2.1.2 Mesh-related notation

Let $\mathcal{T} = \{K\}$ be a conforming triangulation of Ω into simplices $\{K\}$. Let $K \in \mathcal{T}$. We use $\mathcal{F}_K$ to indicate the collection of $(d-1)$-dimensional faces of K. We set $h_K = \mathrm{diam}(K)$ and let n_K denote the outward unit normal on ∂K. The mesh size is defined as $h := \max_{K\in\mathcal{T}} h_K$ and the mesh skeleton is defined as $\Gamma_0 = \bigcup_{K\in\mathcal{T}} \partial K$.

Notice that, if cracks are present in the domain, it is possible for two distinct elements $K_1, K_2 \in \mathcal{T}$ to share a face $\sigma \in \mathcal{F}_{K_1} \cap \mathcal{F}_{K_2}$ that lies on the boundary, i.e. $\sigma \subset \partial\Omega$. In this case, it will not be convenient to view σ as a single mesh face, as is typically done for interior faces. Following [43], we therefore define the collection of mesh faces as the quotient set

$$\mathcal{F}_h := \big\{(\sigma, K) : K \in \mathcal{T},\ \sigma \in \mathcal{F}_K\big\}/\!\sim,$$
$$(\sigma_1, K_1) \sim (\sigma_2, K_2) \iff \big[(\sigma_1, K_1) = (\sigma_2, K_2)\big] \text{ or } \big[\sigma_1 = \sigma_2 \text{ and } \sigma_1 \not\subseteq \partial\Omega\big].$$

For $F = [(\sigma, K)] \in \mathcal{F}_h$ we set $h_F := \mathrm{diam}(\sigma)$. Also, surface integration on F is well-defined, with the understanding that $\int_F v \,\mathrm{d}s := \int_\sigma v \,\mathrm{d}s$ for all $v \in L^1(\sigma)$. The boundary faces $\mathcal{F}_b$ and interior faces $\mathcal{F}_i$ are naturally defined as

$$\mathcal{F}_b := \{[(\sigma, K)] \in \mathcal{F}_h : \sigma \subset \partial\Omega\}, \qquad \mathcal{F}_i := \mathcal{F}_h \setminus \mathcal{F}_b,$$

and we note that $\partial\Omega = \bigcup_{[(\sigma,K)]\in\mathcal{F}_b} \sigma$ since the codimension of $\partial\Omega$ is one.

If $F \in \mathcal{F}_i$ is an interior face belonging to two distinct elements $K_1, K_2 \in \mathcal{T}$, we let n_F denote the unit normal on F pointing from K_1 to K_2, and we define on F the jump operator $[\![\cdot]\!]$ and average operator $\{\!\{\cdot\}\!\}$ in the usual way:

$$[\![\phi]\!]|_F := \phi|_{K_1} - \phi|_{K_2}, \tag{2.5}$$

$$\{\!\{\phi\}\!\}|_F := \frac{1}{2}(\phi|_{K_1} + \phi|_{K_2}), \tag{2.6}$$

where ϕ is any function defined piecewise on $K_1 \cup K_2$. The ambiguity in the ordering of K_1, K_2 will be unimportant. If $F \in \mathcal{F}_b$ is a boundary face belonging to $K \in \mathcal{T}$, we let n_F denote the unit normal on F outward to K, and we define on F the jump and average operators as

$$[\![\phi]\!]|_F = \{\!\{\phi\}\!\}|_F := \phi|_K, \tag{2.7}$$

where ϕ is any function defined on K.

Finally, the following definition will be used in Appendix A. Let $K \in \mathcal{T}$. Observe that we do not have $\mathcal{F}_K \subset \mathcal{F}_h$ because of how $\mathcal{F}_h$ is defined using equivalence classes. We therefore define $\mathcal{F}_{K,h} := \{[(\sigma, K)] \in \mathcal{F}_h : \sigma \in \mathcal{F}_K\}$ so that $\mathcal{F}_{K,h} \subset \mathcal{F}_h$ holds. The sets $\mathcal{F}_K$ and $\mathcal{F}_{K,h}$ intuitively encode the same information; they both contain exactly $(d+1)$ elements which describe the faces of K. The only difference is that $\mathcal{F}_{K,h}$ is defined using equivalence classes in $\mathcal{F}_h$.

2.1.3 Discrete finite element spaces and norms

We introduce the following low-order discontinuous finite element spaces on Ω:

$$
\begin{aligned}
X_h &:= \{v_h \in L^2(\Omega)^d : v_h|_K \in [\mathcal{P}_1(K)]^d \ \forall K \in \mathcal{T}\},\\
Q_h &:= \{q_h \in L^2_0(\Omega) : q_h|_K \in \mathcal{P}_0(K) \ \forall K \in \mathcal{T}\},
\end{aligned}
$$

where $\mathcal{P}_k(D)$ is the space of polynomials with degree at most k on D. Also, let $\mathcal{P}_1(\mathcal{F}_h) := \prod_{F\in\mathcal{F}_h} \mathcal{P}_1(F)$. We introduce the low-order discontinuous facet finite element spaces

$$
\begin{aligned}
\bar{X}_h &:= \{\bar{v}_h \in [\mathcal{P}_1(\mathcal{F}_h)]^d : \bar{v}_h|_F = 0 \ \forall F \in \mathcal{F}_b\},\\
\bar{Q}_h &:= \mathcal{P}_1(\mathcal{F}_h).
\end{aligned}
$$

Notice that $\bar{X}_h$ can be viewed as the space of discontinuous piecewise-linear vector functions on Γ_0 that vanish on $\partial\Omega$. Likewise, $\bar{Q}_h$ can be viewed as the space of discontinuous piecewise-linear scalar functions on Γ_0, with the caveat that these functions are double-valued on boundary faces shared by two distinct cells.

It will also be convenient to introduce the extended velocity spaces

$$
X(h) := X_h + H^1_0(\Omega)^d, \tag{2.8a}
$$

$$
\bar{X}(h) := \bar{X}_h + H^{1/2}_0(\Gamma_0)^d, \tag{2.8b}
$$

where $H^{1/2}_0(\Gamma_0)^d$ is the trace space of functions in $H^1_0(\Omega)^d$ restricted to Γ_0. We use boldface notation for function pairs in $X(h) \times \bar{X}(h)$ and $Q_h \times \bar{Q}_h$, i.e.

$$
\boldsymbol{v} = (v, \bar{v}) \in X(h) \times \bar{X}(h) \quad \text{and} \quad \boldsymbol{q}_h = (q_h, \bar{q}_h) \in Q_h \times \bar{Q}_h. \tag{2.9}
$$

Throughout this chapter $\nabla_h : X(h) \to [L^2(\Omega)]^{d\times d}$ denotes the broken gradient $(\nabla_h v)|_K := \nabla(v|_K)$. On the space $X(h)$ we introduce the discrete H^1-norm

$$
\|v\|^2_{\mathrm{dg}} := \|\nabla_h v\|^2 + |v|^2_{\mathrm{J}},
$$

where $|\cdot|_{\mathrm{J}}$ is the following jump semi-norm on $X(h)$:

$$|v|_{\mathrm{J}}^2 := \sum_{F\in\mathcal{F}_h} \frac{1}{h_F} \|[\![v]\!]\|_F^2 .$$

Similarly, on the product space $X(h) \times \bar{X}(h)$ we introduce the discrete H^1-norm

$$|||\boldsymbol{v}|||_v^2 := \|\nabla_h v\|^2 + |\boldsymbol{v}|_{\mathrm{F}}^2 ,$$

where $|\cdot|_{\mathrm{F}}$ is the following facet semi-norm on $X(h) \times \bar{X}(h)$:

$$|\boldsymbol{v}|_{\mathrm{F}}^2 := \sum_{K\in\mathcal{T}} \frac{1}{h_K} \|v - \bar{v}\|_{\partial K}^2 .$$

Finally, on the space $Q_h \times \bar{Q}_h$ we introduce the norm

$$|||\boldsymbol{q}_h|||_p^2 := \|q_h\|^2 + \|\bar{q}_h\|_p^2 ,$$

where $\|\cdot\|_p$ is the following norm on $\bar{Q}_h$:

$$\|\bar{q}_h\|_p^2 := \sum_{K\in\mathcal{T}} h_K \|\bar{q}_h\|_{\partial K}^2 .$$

We use $a \lesssim b$ to indicate $a \le Cb$ where C is a positive constant depending only on d, Ω and the shape-regularity of $\mathcal{T}$. On occasion we will use inequalities of the form $a \le C(s)b$, where $C(s)$ is a positive constant depending only on d, Ω, shape-regularity of $\mathcal{T}$ and s, where $s \in [0, 1]$ corresponds to the order of the fractional Sobolev space $H^{1+s}(\Omega)^d$. In these cases, we will use the notation $a \lesssim_s b$.

We conclude this subsection with the observation that

$$|v|_{\mathrm{J}} \lesssim |\boldsymbol{v}|_{\mathrm{F}} , \tag{2.10}$$

which follows from the triangle inequality. Note that eq. (2.10) implies

$$\|v\|_{\mathrm{dg}} \lesssim |||\boldsymbol{v}|||_v .$$

These inequalities will be used frequently in Section 2.2.

2.1.4 The hybridized and embedded–hybridized discontinuous Galerkin methods

The lowest-order HDG and EDG–HDG methods analyzed in [39] utilize the following finite element spaces:

$$\boldsymbol{X}_h^v := \begin{cases} X_h \times \bar{X}_h & \text{(HDG method)},\\ X_h \times (\bar{X}_h \cap C^0(\Gamma_0)^d) & \text{(EDG–HDG method)}, \end{cases} \tag{2.11a}$$

$$\boldsymbol{Q}_h^p := Q_h \times \bar{Q}_h. \tag{2.11b}$$

The HDG and EDG–HDG methods differ only in their choice of velocity facet space, which is discontinuous for the HDG method and continuous for the EDG–HDG method. The remainder of the analysis is agnostic as to whether the HDG or EDG–HDG method is considered, with the presented analysis holding for both methods.

The discrete formulation of eq. (2.1) seeks $(\boldsymbol{u}_h, \boldsymbol{p}_h) \in \boldsymbol{X}_h^v \times \boldsymbol{Q}_h^p$ such that

$$\nu a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(v_h, \boldsymbol{p}_h) = (f, v_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h^v, \tag{2.12a}$$

$$b_h(u_h, \boldsymbol{q}_h) = 0 \qquad \forall \boldsymbol{q}_h \in \boldsymbol{Q}_h^p, \tag{2.12b}$$

where $a_h : \boldsymbol{X}_h^v \times \boldsymbol{X}_h^v \to \mathbb{R}$ and $b_h : X_h \times \boldsymbol{Q}_h^p \to \mathbb{R}$ are the bilinear forms

$$\begin{aligned} a_h(\boldsymbol{v}, \boldsymbol{w}) := {} & \sum_{K\in\mathcal{T}} \int_K \nabla v : \nabla w \,\mathrm{d}x + \sum_{K\in\mathcal{T}} \frac{\alpha}{h_K} \int_{\partial K} (v - \bar{v}) \cdot (w - \bar{w}) \,\mathrm{d}s \\ & - \sum_{K\in\mathcal{T}} \int_{\partial K} \left[(v - \bar{v}) \cdot \frac{\partial w}{\partial n_K} + (w - \bar{w}) \cdot \frac{\partial v}{\partial n_K} \right] \mathrm{d}s, \end{aligned} \tag{2.13}$$

$$b_h(v, \boldsymbol{q}) := - \sum_{K\in\mathcal{T}} \int_K (\nabla \cdot v) q \,\mathrm{d}x + \sum_{K\in\mathcal{T}} \int_{\partial K} (v \cdot n_K) \bar{q} \,\mathrm{d}s, \tag{2.14}$$

and $\alpha > 0$ is a penalty parameter. It was shown in [37, Lemma 4.2] that for sufficiently large α the following coercivity result holds:

$$|||\boldsymbol{v}_h|||_v^2 \lesssim a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h^v. \tag{2.15}$$

Let us also mention that inf-sup stability of b_h was established in [39, Lemma 8]:

$$|||\boldsymbol{q}_h|||_p \lesssim \sup_{\boldsymbol{v}_h \in \boldsymbol{X}_h^v \setminus \{\boldsymbol{0}\}} \frac{b_h(v_h, \boldsymbol{q}_h)}{|||\boldsymbol{v}_h|||_v} \qquad \forall \boldsymbol{q}_h \in \boldsymbol{Q}_h^p. \tag{2.16}$$

A consequence of the stability properties in eqs. (2.15) and (2.16) is that the discrete problem eq. (2.12) is well-posed, see e.g. [6, Chapter 4]. Furthermore, let us introduce the discrete reduced space

$$\begin{aligned}\boldsymbol{V}_h^v &:= \{\boldsymbol{v}_h \in \boldsymbol{X}_h^v : b_h(v_h, \boldsymbol{q}_h) = 0 \ \forall \boldsymbol{q}_h \in \boldsymbol{Q}_h^p\} \\ &= \{\boldsymbol{v}_h \in \boldsymbol{X}_h^v : v_h \in X_h^{\mathrm{BDM}} \text{ and } \nabla \cdot v_h = 0\},\end{aligned}$$

where X_h^{BDM} is the lowest-order Brezzi–Douglas–Marini (BDM) space [6],

$$X_h^{\mathrm{BDM}} = \{v_h \in X_h : [\![v_h]\!]|_F \cdot n_F = 0 \ \forall F \in \mathcal{F}_h\}.$$

Inf-sup stability of b_h implies the best approximation result [7, Section 12.5]

$$\inf_{\tilde{\boldsymbol{v}}_h \in \boldsymbol{V}_h^v} |||\boldsymbol{u} - \tilde{\boldsymbol{v}}_h|||_v \lesssim \inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h^v} |||\boldsymbol{u} - \boldsymbol{v}_h|||_v, \tag{2.17}$$

where $u \in H_0^1(\Omega)^d$ is the velocity solution to eq. (2.1) and $\boldsymbol{u} = (u, u) \in X(h) \times \bar{X}(h)$. Also, the discrete velocity solution $\boldsymbol{u}_h \in \boldsymbol{V}_h^v$ to eq. (2.12) satisfies the discrete reduced problem

$$\nu a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = (f, v_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^v. \tag{2.18}$$

However, for $\boldsymbol{v}_h \in \boldsymbol{V}_h^v$ it holds that $v_h \in L_\sigma^2(\Omega)$ (recall that $L_\sigma^2(\Omega)$ is defined in eq. (2.3)) and therefore $(f, v_h) = (\mathbb{P}f, v_h)$. Hence the reduced problem eq. (2.18) can equivalently be written as

$$\nu a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\mathbb{P}f, v_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^v. \tag{2.19}$$

Analogously to eq. (2.4), the presence of $\mathbb{P}f$ in eq. (2.19) will play an important role in the pressure-robustness of our error estimates in Section 2.2.

2.1.5 Enrichment and interpolation operators

Our minimal regularity error analysis of the lowest-order HDG and EDG–HDG methods will utilize an *enrichment operator* E_h with the following properties.

Lemma 2.1 (Enrichment operator). *There exists a linear operator $E_h : X_h^{\mathrm{BDM}} \to H_0^1(\Omega)^d$ such that for all $v_h \in X_h^{\mathrm{BDM}}$ we have*

(i) $\int_F \{\!\{v_h\}\!\} \,\mathrm{d}s = \int_F E_h v_h \,\mathrm{d}s$ *for all* $F \in \mathcal{F}_i$.

(ii) $\nabla \cdot v_h = \nabla \cdot E_h v_h$.

(iii) $\sum_{K\in\mathcal{T}} h_K^{2(k-1)} |v_h - E_h v_h|_{k,K}^2 \lesssim |v_h|_{\mathrm{J}}^2$ *for all* $k \in \{0,1\}$.

(iv) $\|\nabla E_h v_h\| = \|E_h v_h\|_{\mathrm{dg}} \lesssim \|v_h\|_{\mathrm{dg}}$.

An operator satisfying the statements in Lemma 2.1 was constructed in [28]. In [28] the construction is outlined in detail for the two-dimensional case, but sketched only briefly for the three-dimensional case. We present an alternative proof of Lemma 2.1 for the three-dimensional case in Appendix A. Our construction is based on the conforming and divergence-free finite element of [25] and is inspired by [34, Lemma 4.7], in which a similar result is established for Crouzeix–Raviart finite element functions. We mention in passing that our construction in Appendix A can also be adapted to the two-dimensional case by using the two-dimensional finite element of [24].

Let $X_h^c := \{v_h \in X_h \cap C^0(\overline{\Omega})^d : v_h|_{\partial\Omega} = 0\}$ be the conforming analogue of X_h. Aside from E_h, we will also use the following quasi-interpolation operator I_h to deduce optimal rates of convergence for the HDG and EDG–HDG methods.

Lemma 2.2 (Quasi-interpolation operator)**.** *There exists a linear operator* $I_h : H_0^1(\Omega)^d \to X_h^c$ *such that for all* $s \in [0,1]$ *and* $v \in H_0^1(\Omega)^d \cap H^{1+s}(\Omega)^d$ *we have*

$$\left\|\nabla_h(v - I_h v)\right\| \lesssim_s h^s \|v\|_{1+s}. \tag{2.20}$$

Proof. A proof of Lemma 2.2 can be found in [20, 42], although these works assume that Ω is a Lipschitz domain. For the sake of completeness, we now show that the quasi-interpolation operator of [20] still satisfies eq. (2.20) when it is not assumed that Ω is Lipschitz. The quasi-interpolation operator of [20] is given by $I_h = A_h \circ \Pi_h$, where $A_h : X_h \to X_h^c$ is the lowest-order averaging operator introduced in [27, Theorem 2.2] and $\Pi_h : H_0^1(\Omega)^d \to X_h$ the L^2-orthogonal projector onto X_h. Because we are assuming that $\partial\Omega$ has codimension one, every boundary vertex of the mesh is contained in some boundary face of the mesh. As a result, by the arguments used in [27, Theorem 2.2]:

$$\left\|\nabla_h(v_h - A_h v_h)\right\| \lesssim |v_h|_{\mathrm{J}}. \tag{2.21}$$

Let $v \in H_0^1(\Omega)^d \cap H^{1+s}(\Omega)^d$. Using the triangle inequality, eq. (2.21), and a continuous trace inequality [14, Lemma 1.49], we have

$$\begin{aligned}
\left\|\nabla_h(v - I_h v)\right\| &\le \left\|\nabla_h(v - \Pi_h v)\right\| + \left\|\nabla_h(\Pi_h v - A_h \Pi_h v)\right\| \\
&\lesssim \left\|\nabla_h(v - \Pi_h v)\right\| + |\Pi_h v|_{\mathrm{J}} \\
&= \left\|\nabla_h(v - \Pi_h v)\right\| + |v - \Pi_h v|_{\mathrm{J}} \\
&\lesssim \Big(\sum_{K\in\mathcal{T}} h_K^{-2}\|v - \Pi_h v\|_K^2 + |v - \Pi_h v|_{1,K}^2\Big)^{1/2}.
\end{aligned} \tag{2.22}$$

Finally, eq. (2.20) follows from eq. (2.22) and standard approximation properties of the L^2-orthogonal projector Π_h (see e.g. [14, Section 1.4.4]). □

2.2 Pressure-robust error analysis under minimal regularity

In [39], optimal and pressure-robust error estimates for the HDG and EDG–HDG methods were derived assuming $u \in H_0^1(\Omega)^d \cap H^2(\Omega)^d$. In this section, we carry out error analysis for the more general case of $u \in H_0^1(\Omega)^d \cap H^{1+s}(\Omega)^d$ for $s \in [0,1]$.

2.2.1 Velocity error estimates

Thus far we have considered a_h on the finite element space $\boldsymbol{X}_h^v$ (see eq. (2.11a)). The first step in our analysis is to extend a_h to the larger space $\boldsymbol{X}^v(h) := X(h) \times \bar{X}(h)$ (see eq. (2.8)). The main difficulty is that for $v \in X(h)$ and $K \in \mathcal{T}$ we have only $\nabla v \in [L^2(K)]^{d\times d}$ and therefore ∇v does not admit a well-defined trace on ∂K. To deal with this problem, let $\pi_K : [L^2(K)]^{d\times d} \to [\mathcal{P}_0(K)]^{d\times d}$ denote the L^2-orthogonal projector onto $[\mathcal{P}_0(K)]^{d\times d}$. Hence $(G - \pi_K G, H)_K = 0$ for all $G \in [L^2(K)]^{d\times d}$ and $H \in [\mathcal{P}_0(K)]^{d\times d}$. For any $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{X}^v(h)$ we now define

$$
\begin{aligned}
a_h(\boldsymbol{v}, \boldsymbol{w}) := &\sum_{K\in\mathcal{T}} \int_K \nabla v : \nabla w \,\mathrm{d}x + \sum_{K\in\mathcal{T}} \frac{\alpha}{h_K} \int_{\partial K} (v - \bar{v}) \cdot (w - \bar{w}) \,\mathrm{d}s \\
&- \sum_{K\in\mathcal{T}} \int_{\partial K} \Big[(v - \bar{v}) \cdot ([\pi_K \nabla w] n_K) + (w - \bar{w}) \cdot ([\pi_K \nabla v] n_K) \Big] \,\mathrm{d}s.
\end{aligned}
\tag{2.23}
$$

We will use this bilinear form in the following analysis. Observe that eq. (2.23) reduces to the previous definition of a_h (see eq. (2.13)) for $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{X}_h^v$. Moreover, the following boundedness result holds on the extended space $\boldsymbol{X}^v(h)$.

Lemma 2.3 (Boundedness of a_h). *For all $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{X}^v(h)$ there holds*

$$
a_h(\boldsymbol{v}, \boldsymbol{w}) \lesssim |||\boldsymbol{v}|||_v |||\boldsymbol{w}|||_v.
$$

Proof. By definition we have that

$$a_h(\boldsymbol{v},\boldsymbol{w}) = \underbrace{\sum_{K\in\mathcal{T}}\int_K \nabla v : \nabla w \,\mathrm{d}x}_{I_1} + \underbrace{\sum_{K\in\mathcal{T}}\frac{\alpha}{h_K}\int_{\partial K}(v-\bar{v})\cdot(w-\bar{w})\,\mathrm{d}s}_{I_2}$$
$$+\underbrace{\sum_{K\in\mathcal{T}} -\int_{\partial K}(v-\bar{v})\cdot([\pi_K\nabla w]n_K)\,\mathrm{d}s}_{I_3}$$
$$+\underbrace{\sum_{K\in\mathcal{T}} -\int_{\partial K}(w-\bar{w})\cdot([\pi_K\nabla v]n_K)\,\mathrm{d}s}_{I_4}.$$

An application of the Cauchy–Schwarz inequality yields $|I_1|+|I_2| \lesssim |||\boldsymbol{v}|||_v|||\boldsymbol{w}|||_v$. To bound $|I_3|$ we first apply Cauchy–Schwarz to get

$$\begin{aligned} |I_3| &\le |\boldsymbol{v}|_{\mathrm{F}}\Big(\sum_{K\in\mathcal{T}} h_K\|\pi_K\nabla w\|^2_{\partial K}\Big)^{1/2} \\ &\lesssim |\boldsymbol{v}|_{\mathrm{F}}\Big(\sum_{K\in\mathcal{T}}\|\pi_K\nabla w\|^2_K\Big)^{1/2} \\ &\le |\boldsymbol{v}|_{\mathrm{F}}\Big(\sum_{K\in\mathcal{T}}\|\nabla w\|^2_K\Big)^{1/2} \\ &\le |||\boldsymbol{v}|||_v|||\boldsymbol{w}|||_v. \end{aligned} \tag{2.24}$$

For the second inequality in eq. (2.24) we used a discrete trace inequality, and for the third inequality we used stability of π_K. Similar reasoning shows that $|I_4| \lesssim |||\boldsymbol{v}|||_v|||\boldsymbol{w}|||_v$. This completes the proof. □

The next ingredient in our analysis is to establish an upper bound on the consistency error for the velocity solution of the method in eq. (2.12).

Lemma 2.4 (Consistency error for a_h)**.** *Let $u \in H^1_0(\Omega)^d$ be the velocity solution of eq. (2.1), let $\boldsymbol{u} = (u,u)$, and let $\boldsymbol{u}_h \in \boldsymbol{X}^v_h$ be the discrete velocity solution of eq. (2.12). Then for all $\boldsymbol{v}_h \in \boldsymbol{X}^v_h$ and $\boldsymbol{w}_h \in \boldsymbol{V}^v_h$ it holds that*

$$a_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{w}_h) \lesssim \Big\{|||\boldsymbol{u}-\boldsymbol{v}_h|||_v + |v_h|_{\mathrm{G}} + \frac{1}{\nu}\mathrm{osc}(\mathbb{P}f)\Big\}|\boldsymbol{w}_h|_{\mathrm{F}}, \tag{2.25}$$

where we have introduced the notation

$$\operatorname{osc}(g)^2 := \sum_{K\in\mathcal{T}} h_K^2 \|g\|_K^2 \qquad \forall g \in L^2(\Omega)^d, \tag{2.26}$$

$$|t_h|_{\mathrm{G}}^2 := \sum_{F\in\mathcal{F}_i} h_F \left\| [\![\nabla_h t_h]\!] n_F \right\|_F^2 \qquad \forall t_h \in X_h. \tag{2.27}$$

Proof. Let $\boldsymbol{v}_h \in \boldsymbol{X}_h^v$ and $\boldsymbol{w}_h \in \boldsymbol{V}_h^v$. We set

$$\boldsymbol{z}_h = \boldsymbol{w}_h - (E_h w_h, E_h w_h) = (w_h - E_h w_h, \bar{w}_h - E_h w_h).$$

Then

$$\begin{aligned} a_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h) = & \left[a_h(\boldsymbol{u}, (E_h w_h, E_h w_h)) - a_h(\boldsymbol{u}_h, \boldsymbol{w}_h)\right] \\ & + a_h(\boldsymbol{u} - \boldsymbol{v}_h, \boldsymbol{z}_h) + a_h(\boldsymbol{v}_h, \boldsymbol{z}_h) \\ = & \underbrace{\left[a(u, E_h w_h) - a_h(\boldsymbol{u}_h, \boldsymbol{w}_h)\right]}_{I_1} + \underbrace{a_h(\boldsymbol{u} - \boldsymbol{v}_h, \boldsymbol{z}_h)}_{I_2} + \underbrace{a_h(\boldsymbol{v}_h, \boldsymbol{z}_h)}_{I_3}. \end{aligned} \tag{2.28}$$

We first bound I_1. Since $\boldsymbol{w}_h \in \boldsymbol{V}_h^v$ we have $w_h \in X_h^{\mathrm{BDM}}$ with $\nabla \cdot w_h = 0$. Thus $E_h w_h \in V$ by Item ii of Lemma 2.1. Using the reduced problems eq. (2.4) and eq. (2.19), the Cauchy–Schwarz inequality, and Item iii of Lemma 2.1 with $k = 0$,

$$\begin{aligned} I_1 &= \frac{1}{\nu}(\mathbb{P}f, E_h w_h - w_h) \\ &\le \frac{1}{\nu}\operatorname{osc}(\mathbb{P}f)\left[\sum_{K\in\mathcal{T}} h_K^{-2}\|E_h w_h - w_h\|_K^2\right]^{1/2} \\ &\lesssim \frac{1}{\nu}\operatorname{osc}(\mathbb{P}f)|w_h|_{\mathrm{J}} \\ &\lesssim \frac{1}{\nu}\operatorname{osc}(\mathbb{P}f)|\boldsymbol{w}_h|_{\mathrm{F}}. \end{aligned} \tag{2.29}$$

We now bound I_2. Using Item iii of Lemma 2.1 with $k = 1$ we have

$$|||\boldsymbol{z}_h|||_v^2 = \left\|\nabla_h(w_h - E_h w_h)\right\|^2 + |\boldsymbol{w}_h|_{\mathrm{F}}^2 \lesssim |\boldsymbol{w}_h|_{\mathrm{F}}^2$$

so that $|||\boldsymbol{z}_h|||_v \lesssim |\boldsymbol{w}_h|_{\mathrm{F}}$. Hence by Lemma 2.3 we have

$$I_2 \lesssim |||\boldsymbol{u} - \boldsymbol{v}_h|||_v |||\boldsymbol{z}_h|||_v \lesssim |||\boldsymbol{u} - \boldsymbol{v}_h|||_v |\boldsymbol{w}_h|_{\mathrm{F}}. \tag{2.30}$$

To bound I_3 we use the definition eq. (2.23) of a_h and integrate by parts element-wise. Using that $(\nabla^2 v_h)|_K = 0$ as v_h is piecewise linear, this results in

$$\begin{aligned} I_3 = &\underbrace{\sum_{K\in\mathcal{T}} \frac{\alpha}{h_K}\int_{\partial K}(v_h - \bar{v}_h)\cdot(z_h - \bar{z}_h)\,\mathrm{d}s - \int_{\partial K}(v_h - \bar{v}_h)\cdot([\pi_K\nabla z_h]n_K)\,\mathrm{d}s}_{I_{3,1}} \\ &+ \underbrace{\sum_{K\in\mathcal{T}}\int_{\partial K}\bar{z}_h\cdot([\nabla v_h]n_K)\,\mathrm{d}s}_{I_{3,2}}. \end{aligned} \tag{2.31}$$

Using the same arguments from Lemma 2.3 (namely those used in eq. (2.24)) one sees that

$$I_{3,1} \lesssim |\boldsymbol{v}_h|_{\mathrm{F}}\, |||\boldsymbol{z}_h|||_v = |\boldsymbol{u} - \boldsymbol{v}_h|_{\mathrm{F}}\, |||\boldsymbol{z}_h|||_v \lesssim |||\boldsymbol{u} - \boldsymbol{v}_h|||_v |\boldsymbol{w}_h|_{\mathrm{F}}\,. \tag{2.32}$$

Also, rewriting $I_{3,2}$ in terms of facet integrals, applying Item i of Lemma 2.1, and using the Cauchy–Schwarz inequality, we find

$$\begin{aligned} I_{3,2} &= \sum_{F\in\mathcal{F}_i}\int_F \bar{z}_h\cdot([\![\nabla v_h]\!]n_F)\,\mathrm{d}s \\ &= \sum_{F\in\mathcal{F}_i}\int_F (\bar{w}_h - E_h w_h)\cdot([\![\nabla v_h]\!]n_F)\,\mathrm{d}s \\ &= \sum_{F\in\mathcal{F}_i}\int_F (\bar{w}_h - \{\!\{w_h\}\!\})\cdot([\![\nabla v_h]\!]n_F)\,\mathrm{d}s \\ &\lesssim |\boldsymbol{w}_h|_{\mathrm{F}}|v_h|_{\mathrm{G}}\,. \end{aligned} \tag{2.33}$$

Using eq. (2.32) and eq. (2.33) in eq. (2.31) we obtain

$$I_3 \lesssim \Big[|||\boldsymbol{u} - \boldsymbol{v}_h|||_v + |v_h|_{\mathrm{G}}\Big]|\boldsymbol{w}_h|_{\mathrm{F}}\,. \tag{2.34}$$

Finally, using the bounds eqs. (2.29), (2.30) and (2.34) in eq. (2.28) yields the desired result eq. (2.25). □

With boundedness and consistency results established for the method eq. (2.12), we can now derive our main error estimate.

Theorem 2.1 (Velocity error)**.** *Let $u \in H_0^1(\Omega)^d$ be the velocity solution of eq. (2.1), let $\boldsymbol{u} = (u, u)$, and let $\boldsymbol{u}_h \in \boldsymbol{X}_h^v$ be the discrete velocity solution of eq. (2.12). Then*

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||_v \lesssim \inf_{\boldsymbol{v}_h\in\boldsymbol{X}_h^v}\Big[|||\boldsymbol{u} - \boldsymbol{v}_h|||_v + |v_h|_{\mathrm{G}}\Big] + \frac{1}{\nu}\mathrm{osc}(\mathbb{P}f). \tag{2.35}$$

Proof. Let $\boldsymbol{v}_h \in \boldsymbol{X}_h^v$. Owing to eq. (2.17) we can find $\tilde{\boldsymbol{v}}_h \in \boldsymbol{V}_h^v$ with $|||\boldsymbol{u}-\tilde{\boldsymbol{v}}_h|||_v \lesssim |||\boldsymbol{u}-\boldsymbol{v}_h|||_v$. Let $\boldsymbol{w}_h = (\boldsymbol{u}_h - \tilde{\boldsymbol{v}}_h) \in \boldsymbol{V}_h^v$. Using discrete coercivity eq. (2.15) along with the boundedness and consistency results Lemmas 2.3 to 2.4,

$$\begin{aligned} |||\boldsymbol{w}_h|||_v^2 &\lesssim a_h(\boldsymbol{w}_h, \boldsymbol{w}_h) \\ &= a_h(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{w}_h) + a_h(\boldsymbol{u} - \tilde{\boldsymbol{v}}_h, \boldsymbol{w}_h) \\ &\lesssim \Big\{ \Big[|||\boldsymbol{u} - \tilde{\boldsymbol{v}}_h|||_v + |\tilde{v}_h|_{\mathrm{G}} \Big] + \frac{1}{\nu}\mathrm{osc}(\mathbb{P}f) \Big\} |||\boldsymbol{w}_h|||_v. \end{aligned} \tag{2.36}$$

Therefore, dividing eq. (2.36) by $|||\boldsymbol{w}_h|||_v$ we arrive at

$$|||\boldsymbol{u}_h - \tilde{\boldsymbol{v}}_h|||_v \lesssim \Big[|||\boldsymbol{u} - \tilde{\boldsymbol{v}}_h|||_v + |\tilde{v}_h|_{\mathrm{G}} \Big] + \frac{1}{\nu}\mathrm{osc}(\mathbb{P}f). \tag{2.37}$$

Using the triangle inequality and eq. (2.37) we obtain

$$\begin{aligned} |||\boldsymbol{u} - \boldsymbol{u}_h|||_v &\le |||\boldsymbol{u} - \tilde{\boldsymbol{v}}_h|||_v + |||\boldsymbol{u}_h - \tilde{\boldsymbol{v}}_h|||_v \\ &\lesssim \Big[|||\boldsymbol{u} - \tilde{\boldsymbol{v}}_h|||_v + |\tilde{v}_h|_{\mathrm{G}} \Big] + \frac{1}{\nu}\mathrm{osc}(\mathbb{P}f). \end{aligned} \tag{2.38}$$

Also, by the triangle inequality and a discrete trace inequality we have

$$\begin{aligned} |\tilde{v}_h|_{\mathrm{G}} &\le |\tilde{v}_h - v_h|_{\mathrm{G}} + |v_h|_{\mathrm{G}} \\ &\lesssim \big\| \nabla_h(\tilde{v}_h - v_h) \big\| + |v_h|_{\mathrm{G}} \\ &\le |||\tilde{\boldsymbol{v}}_h - \boldsymbol{v}_h|||_v + |v_h|_{\mathrm{G}} \\ &\le |||\boldsymbol{u} - \tilde{\boldsymbol{v}}_h|||_v + |||\boldsymbol{u} - \boldsymbol{v}_h|||_v + |v_h|_{\mathrm{G}}. \end{aligned} \tag{2.39}$$

Combining eqs. (2.38) to (2.39) and using that $|||\boldsymbol{u} - \tilde{\boldsymbol{v}}_h|||_v \lesssim |||\boldsymbol{u} - \boldsymbol{v}_h|||_v$ we obtain

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||_v \lesssim \Big[|||\boldsymbol{u} - \boldsymbol{v}_h|||_v + |v_h|_{\mathrm{G}} \Big] + \frac{1}{\nu}\mathrm{osc}(\mathbb{P}f).$$

The desired result eq. (2.35) follows as $\boldsymbol{v}_h \in \boldsymbol{X}_h^v$ is arbitrary. □

Remark 2.1 (Pressure-robustness of the data oscillation term). As discussed in [34, Remark 5.9], the function $\frac{1}{\nu}\mathbb{P}f$ is independent of both the pressure p and the viscosity ν. This can be seen by extending the domain of the Helmholtz projector $\mathbb{P}$ to $[H^{-1}(\Omega)]^d$, and then utilizing the fact that $-\nu\Delta u + \nabla p = f$ holds in the distributional sense. One finds that

$$\frac{1}{\nu}\mathbb{P}f = \frac{1}{\nu}\mathbb{P}(-\nu\Delta u + \nabla p) = \mathbb{P}(-\Delta u) \in L^2(\Omega)^d, \tag{2.40}$$

since $\mathbb{P}(\nabla p) = 0$. We refer the reader to [33, Section 3] for a more detailed discussion of these ideas. A consequence of eq. (2.40) is that the data oscillation term appearing in eq. (2.35) can equivalently be written as $\frac{1}{\nu}\mathrm{osc}(\mathbb{P}f) = \mathrm{osc}(\mathbb{P}(-\Delta u))$. Because this quantity depends only on the velocity, the error estimate eq. (2.35) is pressure-robust. We emphasize that eq. (2.35) would not be a pressure-robust error estimate if it contained $\frac{1}{\nu}\mathrm{osc}(f)$ instead of $\frac{1}{\nu}\mathrm{osc}(\mathbb{P}f)$.

Our next step is to show that the interpolation error term appearing in Theorem 2.1 converges optimally with respect to the mesh size h.

Lemma 2.5 (Interpolation error)**.** *Let $\psi \in H_0^1(\Omega)^d \cap H^{1+s}(\Omega)^d$ with $s \in [0,1]$, and set $\boldsymbol{\psi} = (\psi, \psi)$. Then*

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h^v} \left[|||\boldsymbol{\psi} - \boldsymbol{v}_h|||_v + |v_h|_{\mathrm{G}} \right] \lesssim_s h^s \|\psi\|_{1+s}. \tag{2.41}$$

Proof. For each $F \in \mathcal{F}_i$ we introduce the patch ω_F of elements sharing F,

$$\omega_F := \bigcup \{K \in \mathcal{T} : F \text{ is a face of } K\}.$$

Note that ω_F is the union of exactly two elements. Now let $\boldsymbol{v}_h \in \boldsymbol{X}_h^v$ and $g_F \in [\mathcal{P}_0(\omega_F)]^{d\times d}$ be arbitrary. By a discrete trace inequality and the triangle inequality,

$$\begin{aligned} |v_h|_{\mathrm{G}}^2 &\leq \sum_{F\in\mathcal{F}_i} h_F \left\| [\![\nabla_h v_h]\!] \right\|_F^2 \\ &= \sum_{F\in\mathcal{F}_i} h_F \left\| [\![\nabla_h v_h - g_F]\!] \right\|_F^2 \\ &\lesssim \sum_{F\in\mathcal{F}_i} \|\nabla_h v_h - g_F\|_{\omega_F}^2 \\ &\lesssim \sum_{F\in\mathcal{F}_i} \left[\left\|\nabla_h(\psi - v_h)\right\|_{\omega_F}^2 + \|\nabla\psi - g_F\|_{\omega_F}^2 \right] \\ &\lesssim |||\boldsymbol{\psi} - \boldsymbol{v}_h|||_v^2 + \sum_{F\in\mathcal{F}_i} \|\nabla\psi - g_F\|_{\omega_F}^2 . \end{aligned} \tag{2.42}$$

Since $\boldsymbol{v}_h \in \boldsymbol{X}_h^v$ and $g_F \in [\mathcal{P}_0(\omega_F)]^{d\times d}$ are arbitrary, eq. (2.42) yields that

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h^v} \left[|||\boldsymbol{\psi} - \boldsymbol{v}_h|||_v + |v_h|_{\mathrm{G}} \right] \lesssim \underbrace{\left(\sum_{F\in\mathcal{F}_i} \inf_{g_F \in [\mathcal{P}_0(\omega_F)]^{d\times d}} \|\nabla\psi - g_F\|_{\omega_F}^2 \right)^{1/2}}_{I_1} + \underbrace{\inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h^v} |||\boldsymbol{\psi} - \boldsymbol{v}_h|||_v}_{I_2}. \tag{2.43}$$

A bound for I_1 follows from the fractional order Bramble–Hilbert lemma [17, Theorem 6.1] applied to the patches ω_F:

$$I_1 \lesssim_s \left(\sum_{F\in\mathcal{F}_i} h_F^{2s} \|\nabla\psi\|_{s,\omega_F}^2 \right)^{1/2} \lesssim h^s \|\psi\|_{1+s}. \tag{2.44}$$

To bound I_2 we take $\boldsymbol{v}_h = (I_h\psi, I_h\psi) \in \boldsymbol{X}_h^v$ where I_h is the quasi-interpolation operator introduced in Lemma 2.2. We find that

$$I_2 \le |||\boldsymbol{\psi} - \boldsymbol{v}_h|||_v = \left\| \nabla_h(\psi - I_h\psi) \right\| \lesssim_s h^s \|\psi\|_{1+s}. \tag{2.45}$$

Using the bounds eqs. (2.44) to (2.45) in eq. (2.43) yields the desired result. □

An immediate consequence of Theorem 2.1 and Lemma 2.5 is the following error estimate, which is pressure-robust and optimal in the discrete energy norm.

Corollary 2.1 (Pressure-robust error estimate)**.** *In addition to the assumptions of Theorem 2.1, assume that $u \in H^{1+s}(\Omega)^d$ with $s \in [0,1]$. Then*

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||_v \lesssim_s h^s \|u\|_{1+s} + \frac{1}{\nu}\mathrm{osc}(\mathbb{P}f).$$

Also, owing to eq. (2.26), the data oscillation term can be estimated as

$$\frac{1}{\nu}\mathrm{osc}(\mathbb{P}f) \le h \left\| \frac{1}{\nu}\mathbb{P}f \right\|.$$

Remark 2.2 (Convergence under H^1-regularity). In the case of $s = 0$, where only H^1-regularity of u is assumed, Corollary 2.1 does not predict that $|||\boldsymbol{u} - \boldsymbol{u}_h|||_v \to 0$ as $h \to 0$. This can still be proven, however, using Theorem 2.1 and a density argument. Indeed, let $\epsilon > 0$. By definition, $H_0^1(\Omega)^d$ is the closure of $C_0^\infty(\Omega)^d$ under the H^1-norm. As $u \in H_0^1(\Omega)^d$

we can therefore find $\phi \in C_0^\infty(\Omega)^d$ with $|u-\phi|_1 < \epsilon$. Setting $\boldsymbol{\phi} = (\phi, \phi)$, the triangle inequality and Lemma 2.5 then yield

$$\begin{aligned}\inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h^v}\left[|||\boldsymbol{u}-\boldsymbol{v}_h|||_v + |v_h|_{\mathrm{G}}\right] &\leq |||\boldsymbol{u}-\boldsymbol{\phi}|||_v + \inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h^v}\left[|||\boldsymbol{\phi}-\boldsymbol{v}_h|||_v + |v_h|_{\mathrm{G}}\right] \\ &= |u-\phi|_1 + \inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h^v}\left[|||\boldsymbol{\phi}-\boldsymbol{v}_h|||_v + |v_h|_{\mathrm{G}}\right] \\ &\lesssim \epsilon + h\|\phi\|_2 \\ &\leq 2\epsilon,\end{aligned} \tag{2.46}$$

where the last inequality in eq. (2.46) holds for h sufficiently small. But $\epsilon > 0$ is arbitrary, and therefore eq. (2.46) implies that

$$\lim_{h\to 0}\left\{\inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h^v}\left[|||\boldsymbol{u}-\boldsymbol{v}_h|||_v + |v_h|_{\mathrm{G}}\right]\right\} = 0. \tag{2.47}$$

Using eq. (2.47) in Theorem 2.1 we see that $|||\boldsymbol{u}-\boldsymbol{u}_h|||_v \to 0$ as $h \to 0$.

We now investigate convergence of the velocity in the L^2-norm, by means of the Aubin–Nitsche trick. In order to proceed we assume the domain Ω is such that the following regularity holds (see e.g. [16]).

Assumption 2.1 (Regularity of the reduced Stokes problem)**.** *Let $s_0 \in [0,1]$ be fixed. We assume that for all $g \in L^2(\Omega)^d$ there holds $\phi_g \in H^{1+s_0}(\Omega)^d$ and*

$$\left\|\phi_g\right\|_{1+s_0} \lesssim_{s_0} \|g\|$$

where $\phi_g \in V$ is the solution to the reduced Stokes problem

$$a(\phi_g, v) = (g, v) \qquad \forall v \in V.$$

Theorem 2.2 (Velocity error in the L^2-norm)**.** *In addition to the assumptions of Corollary 2.1 and under Assumption 2.1 we have*

$$\|u-u_h\| \lesssim_{s_0} h^{s_0}\left\{\inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h^v}\left[|||\boldsymbol{u}-\boldsymbol{v}_h|||_v + |v_h|_{\mathrm{G}}\right] + \frac{1}{\nu}\mathrm{osc}(\mathbb{P}f)\right\} \tag{2.48a}$$

$$\lesssim_s h^{s+s_0}\|u\|_{1+s} + h^{1+s_0}\left\|\frac{1}{\nu}\mathbb{P}f\right\|. \tag{2.48b}$$

Proof. Let $\phi \in V, \boldsymbol{\phi}_h \in \boldsymbol{V}_h^v$ solve the reduced problems

$$a(\phi, v) = (u-u_h, v) \qquad \forall v \in V, \tag{2.49a}$$

$$a_h(\boldsymbol{\phi}_h, \boldsymbol{v}_h) = (u-u_h, v_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^v. \tag{2.49b}$$

Set $\boldsymbol{\phi} = (\phi, \phi)$. By Assumption 2.1 we have $\phi \in H^{1+s_0}(\Omega)^d$ and $\|\phi\|_{1+s_0} \lesssim_{s_0} \|u - u_h\|$. Applying Corollary 2.1 to the reduced problems eq. (2.49) (for which the source term is $u - u_h$ and the viscosity is one), we find that

$$
\begin{aligned}
|||\boldsymbol{\phi} - \boldsymbol{\phi}_h|||_v &\lesssim_{s_0} h^{s_0}\|\phi\|_{1+s_0} + \operatorname{osc}(\mathbb{P}(u - u_h)) \\
&\leq h^{s_0}\|\phi\|_{1+s_0} + h\|u - u_h\| \\
&\lesssim_{s_0} h^{s_0}\|u - u_h\|\,.
\end{aligned} \tag{2.50}
$$

Using eqs. (2.49a) to (2.49b) and some algebraic manipulations, we have

$$
\begin{aligned}
\|u - u_h\|^2 &= (u - u_h, u) - (u - u_h, u_h) \\
&= a(\phi, u) - a_h(\boldsymbol{\phi_h}, \boldsymbol{u}_h) \\
&= a_h(\boldsymbol{\phi}, \boldsymbol{u}) - a_h(\boldsymbol{\phi_h}, \boldsymbol{u}_h) \\
&= \underbrace{a_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\phi} - \boldsymbol{\phi}_h)}_{I_1} + \underbrace{a_h(\boldsymbol{\phi} - \boldsymbol{\phi}_h, \boldsymbol{u}_h)}_{I_2} + \underbrace{a_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\phi}_h)}_{I_3}.
\end{aligned} \tag{2.51}
$$

To bound I_1 we use Lemma 2.3 and eq. (2.50):

$$
I_1 \lesssim |||\boldsymbol{u} - \boldsymbol{u}_h|||_v |||\boldsymbol{\phi} - \boldsymbol{\phi}_h|||_v \lesssim_{s_0} h^{s_0}\|u - u_h\|\; |||\boldsymbol{u} - \boldsymbol{u}_h|||_v. \tag{2.52}
$$

To bound I_2 we use Lemma 2.4, Lemma 2.5 and Assumption 2.1. This yields

$$
\begin{aligned}
I_2 &\lesssim \Big\{ \inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h^v} \Big[|||\boldsymbol{\phi} - \boldsymbol{v}_h|||_v + |v_h|_{\mathrm{G}} \Big] + \operatorname{osc}(\mathbb{P}(u - u_h)) \Big\} |\boldsymbol{u}_h|_{\mathrm{F}} \\
&\lesssim_{s_0} \Big\{ h^{s_0}\|\phi\|_{1+s_0} + \operatorname{osc}(\mathbb{P}(u - u_h)) \Big\} |\boldsymbol{u}_h|_{\mathrm{F}} \\
&\lesssim_{s_0} h^{s_0}\|u - u_h\| |\boldsymbol{u}_h|_{\mathrm{F}} \\
&\leq h^{s_0}\|u - u_h\|\; |||\boldsymbol{u} - \boldsymbol{u}_h|||_v.
\end{aligned} \tag{2.53}
$$

To bound I_3 we again use Lemma 2.4 along with eq. (2.50). We find

$$
\begin{aligned}
I_3 &\lesssim \Big\{ \inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h^v} \Big[|||\boldsymbol{u} - \boldsymbol{v}_h|||_v + |v_h|_{\mathrm{G}} \Big] + \frac{1}{\nu}\operatorname{osc}(\mathbb{P}f) \Big\} |\boldsymbol{\phi}_h|_{\mathrm{F}} \\
&\leq \Big\{ \inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h^v} \Big[|||\boldsymbol{u} - \boldsymbol{v}_h|||_v + |v_h|_{\mathrm{G}} \Big] + \frac{1}{\nu}\operatorname{osc}(\mathbb{P}f) \Big\} |||\boldsymbol{\phi} - \boldsymbol{\phi}_h|||_v \\
&\lesssim_{s_0} h^{s_0}\|u - u_h\| \Big\{ \inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h^v} \Big[|||\boldsymbol{u} - \boldsymbol{v}_h|||_v + |v_h|_{\mathrm{G}} \Big] + \frac{1}{\nu}\operatorname{osc}(\mathbb{P}f) \Big\}.
\end{aligned} \tag{2.54}
$$

Using the bounds eqs. (2.52) to (2.54) in eq. (2.51), and using Theorem 2.1 to bound $|||\boldsymbol{u} - \boldsymbol{u}_h|||_v$, we obtain

$$\|u - u_h\|^2 \lesssim_{s_0} h^{s_0}\|u - u_h\| \left\{ \inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h^v} \left[|||\boldsymbol{u} - \boldsymbol{v}_h|||_v + |v_h|_{\mathrm{G}} \right] + \frac{1}{\nu}\mathrm{osc}(\mathbb{P}f) \right\}. \tag{2.55}$$

Dividing eq. (2.55) by $\|u - u_h\|$ we obtain eq. (2.48a). Finally, eq. (2.48b) follows from eq. (2.48a) and Lemma 2.5. □

2.2.2 Pressure error estimate

Let $(u, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ be the solution of eq. (2.1) and $(\boldsymbol{u}_h, \boldsymbol{p}_h) \in \boldsymbol{X}_h^v \times \boldsymbol{Q}_h^p$ the solution of eq. (2.12). Let $\pi_h : L^2(\Omega) \to Q_h$ be the L^2-orthogonal projector onto Q_h. Note that $(\pi_h p - p_h) \in Q_h$ and therefore $(p - \pi_h p, \pi_h p - p_h) = 0$. Hence by the Pythagorean theorem, the pressure error can be decomposed as

$$\begin{aligned} \|p - p_h\|^2 &= \left\|(p - \pi_h p) + (\pi_h p - p_h)\right\|^2 \\ &= \|p - \pi_h p\|^2 + \|\pi_h p - p_h\|^2. \end{aligned} \tag{2.56}$$

The term $\|p - \pi_h p\|$ is the best approximation error of p by functions in the discrete space Q_h under the L^2-norm, and is unavoidably pressure-dependent. However, the following theorem shows that the second term $\|\pi_h p - p_h\|$ can be bounded above by an error that is dependent on the velocity only.

Theorem 2.3 (Pressure error)**.** *Let $(u, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ solve eq. (2.1) and set $\boldsymbol{u} = (u, u)$. Let $(\boldsymbol{u}_h, \boldsymbol{p}_h) \in \boldsymbol{X}_h^v \times \boldsymbol{Q}_h^p$ solve eq. (2.12). Then*

$$\|\pi_h p - p_h\| \lesssim \left\{ \nu \inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h^v} \left[|||\boldsymbol{u} - \boldsymbol{v}_h|||_v + |v_h|_{\mathrm{G}} \right] + \mathrm{osc}(\mathbb{P}f) \right\}. \tag{2.57}$$

Proof. Set $r_h := (\pi_h p - p_h) \in Q_h$. We utilize the auxiliary inf-sup condition established in [39, Lemma 5], which tells us that

$$\|r_h\| \lesssim \sup_{\boldsymbol{w}_h \in X_h^{\mathrm{BDM}} \times (\bar{X}_h \cap C^0(\Gamma_0)^d)} \frac{(r_h, \nabla \cdot w_h)}{|||\boldsymbol{w}_h|||_v}. \tag{2.58}$$

Consider $\boldsymbol{w}_h \in X_h^{\mathrm{BDM}} \times (\bar{X}_h \cap C^0(\Gamma_0)^d)$. Then $[\![w_h]\!]|_F \cdot n_F = 0$ for all $F \in \mathcal{F}_h$ so that

$$-(p_h, \nabla \cdot w_h) = b_h(w_h, \boldsymbol{p}_h) = (f, w_h) - \nu a_h(\boldsymbol{u}_h, \boldsymbol{w}_h). \tag{2.59}$$

On the other hand, since $\nabla \cdot w_h = \nabla \cdot E_h w_h \in Q_h$ we have

$$\begin{aligned}(\pi_h p, \nabla \cdot w_h) &= (p, \nabla \cdot E_h w_h) \\ &= -b(E_h w_h, p) \\ &= -(f, E_h w_h) + \nu a(u, E_h w_h) \\ &= -(f, E_h w_h) + \nu a_h(\boldsymbol{u}, (E_h w_h, E_h w_h)).\end{aligned} \tag{2.60}$$

Set $\boldsymbol{z}_h = \boldsymbol{w}_h - (E_h w_h, E_h w_h)$. Combining eq. (2.59) and eq. (2.60) we obtain

$$\begin{aligned}(r_h, \nabla \cdot w_h) &= (f, w_h - E_h w_h) \\ &\quad - \nu a_h(\boldsymbol{u}_h, \boldsymbol{w}_h) + \nu a_h(\boldsymbol{u}, (E_h w_h, E_h w_h)) \\ &= \underbrace{(f, w_h - E_h w_h)}_{I_1} \\ &\quad + \nu \underbrace{a_h(\boldsymbol{u} - \boldsymbol{u}_h, (E_h w_h, E_h w_h))}_{I_2} - \nu \underbrace{a_h(\boldsymbol{u}_h, \boldsymbol{z}_h)}_{I_3}.\end{aligned} \tag{2.61}$$

Since $w_h \in X_h^{\mathrm{BDM}}$ and $\nabla \cdot (w_h - E_h w_h) = 0$ we have that $(w_h - E_h w_h) \in L^2_\sigma(\Omega)$ (recall eq. (2.3)). As a result, $I_1 = (\mathbb{P}f, w_h - E_h w_h)$. Applying the Cauchy–Schwarz inequality and Item iii of Lemma 2.1 with $k = 0$ therefore yields

$$|I_1| \lesssim \operatorname{osc}(\mathbb{P}f) |w_h|_{\mathrm{J}} \lesssim \operatorname{osc}(\mathbb{P}f) |||\boldsymbol{w}_h|||_v. \tag{2.62}$$

A bound for $|I_2|$ follows from Lemma 2.3 and Item iv of Lemma 2.1:

$$\begin{aligned}|I_2| &\lesssim |||\boldsymbol{u} - \boldsymbol{u}_h|||_v |||(E_h w_h, E_h w_h)|||_v \\ &= |||\boldsymbol{u} - \boldsymbol{u}_h|||_v \|E_h w_h\|_{\mathrm{dg}} \\ &\lesssim |||\boldsymbol{u} - \boldsymbol{u}_h|||_v |||\boldsymbol{w}_h|||_v.\end{aligned} \tag{2.63}$$

Also, the same arguments used in Lemma 2.4 show that

$$|I_3| \lesssim \Big[|||\boldsymbol{u} - \boldsymbol{u}_h|||_v + |u_h|_{\mathrm{G}} \Big] |||\boldsymbol{w}_h|||_v. \tag{2.64}$$

But for any $\boldsymbol{v}_h \in \boldsymbol{X}_h^v$, the triangle inequality and a discrete trace inequality yields

$$\begin{aligned}|u_h|_{\mathrm{G}} &\le |u_h - v_h|_{\mathrm{G}} + |v_h|_{\mathrm{G}} \\ &\lesssim \big\|\nabla_h (u_h - v_h)\big\| + |v_h|_{\mathrm{G}} \\ &\le |||\boldsymbol{u}_h - \boldsymbol{v}_h|||_v + |v_h|_{\mathrm{G}} \\ &\le |||\boldsymbol{u} - \boldsymbol{u}_h|||_v + \Big[|||\boldsymbol{u} - \boldsymbol{v}_h|||_v + |v_h|_{\mathrm{G}} \Big].\end{aligned} \tag{2.65}$$

Combining eq. (2.64) and eq. (2.65) gives

$$|I_3| \lesssim \Big\{ |||\boldsymbol{u} - \boldsymbol{u}_h|||_v + \inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h^v} \Big[|||\boldsymbol{u} - \boldsymbol{v}_h|||_v + |v_h|_{\mathrm{G}} \Big] \Big\} |||\boldsymbol{w}_h|||_v. \tag{2.66}$$

Inserting the bounds eqs. (2.62), (2.63) and (2.66) into eq. (2.61), and using Theorem 2.1 to bound $|||\boldsymbol{u} - \boldsymbol{u}_h|||_v$, we obtain

$$(r_h, \nabla \cdot w_h) \lesssim \Big\{ \nu \inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h^v} \Big[|||\boldsymbol{u} - \boldsymbol{v}_h|||_v + |v_h|_{\mathrm{G}} \Big] + \mathrm{osc}(\mathbb{P}f) \Big\} |||\boldsymbol{w}_h|||_v. \tag{2.67}$$

Finally, combining eq. (2.67) and the inf-sup condition eq. (2.58) we get

$$\|r_h\| \lesssim \Big\{ \nu \inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h^v} \Big[|||\boldsymbol{u} - \boldsymbol{v}_h|||_v + |v_h|_{\mathrm{G}} \Big] + \mathrm{osc}(\mathbb{P}f) \Big\},$$

which is the desired result. □

Corollary 2.2 (Pressure convergence rate)**.** *In addition to the assumptions of Theorem 2.3, assume that $(u, p) \in H^{1+s}(\Omega)^d \times H^s(\Omega)$ for some $s \in [0, 1]$. Then*

$$\|p - p_h\| \lesssim_s h^s \|p\|_s + \nu h^s \|u\|_{1+s} + h \|\mathbb{P}f\|. \tag{2.68}$$

Proof. By standard approximation properties of the L^2-orthogonal projector,

$$\|p - \pi_h p\| \lesssim_s h^s \|p\|_s. \tag{2.69}$$

On the other hand, combining Theorem 2.3 and Lemma 2.5 we find that

$$\|\pi_h p - p_h\| \lesssim_s \nu h^s \|u\|_{1+s} + h \|\mathbb{P}f\|. \tag{2.70}$$

Using the bounds eq. (2.69) and eq. (2.70) in the decomposition eq. (2.56) yields the desired result in eq. (2.68). □

2.3 Numerical examples

In this section we support our theoretical findings with numerical examples. Strictly speaking, the examples that we consider are outside of the scope of our theory, because they involve inhomogenous Dirichlet boundary conditions. Nevertheless, we will see that our numerical observations agree with the theoretical predictions of Section 2.2.

All numerical examples have been implemented in NGSolve [41]. The penalty parameter is taken as $\alpha = 6k^2$ where k is the polynomial degree of the velocity finite element space. We discuss numerical results only for the EDG–HDG method; our findings for the HDG method are very similar in all cases.

2.3.1 Convergence under minimal regularity

We consider the Stokes problem on the unit square $\Omega = (0,1)^2$ with $f = 0$ and $\nu = 1$. We impose Dirichlet boundary conditions on the discrete solution by interpolating the exact solution. The exact solution is taken from [44, Example 4] and in polar coordinates is given by

$$u = \frac{3}{2}\sqrt{r}\begin{bmatrix} \cos\left(\frac{\theta}{2}\right) - \cos\left(\frac{3\theta}{2}\right) \\ 3\sin\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{2}\right) \end{bmatrix}, \quad p = -6r^{-1/2}\cos\left(\frac{\theta}{2}\right). \tag{2.71}$$

We note that $(u,p) \in H^{1+s}(\Omega)^d \times H^s(\Omega)$ for all $0 \le s < 1/2$.

The computed velocity and pressure errors for the EDG–HDG method using the lowest-order $P^1 - P^0$ discretization and the $P^2 - P^1$ discretization are shown in Table 2.1. Both discretizations are seen to converge at the same rate. The velocity error in the discrete H^1-norm and the pressure error in the L^2-norm are observed to converge as roughly $h^{1/2}$. This is consistent with the regularity of the exact solution and the predictions of Corollary 2.1 and Corollary 2.2. Finally, the velocity error in the L^2-norm is observed to converge as roughly $h^{3/2}$. Because Ω is convex and therefore Assumption 2.1 holds with $s_0 = 1$ (see e.g. [36]), this observed convergence rate is consistent with Theorem 2.2.

2.3.2 Pressure-robust velocity approximation

To demonstrate pressure-robustness in the minimal regularity setting, we consider a Stokes problem, taken from [44, Example 3], on the L-shaped domain $\Omega = (-1,1)^2 \setminus ([0,1]\times[-1,0])$ and we vary the viscosity ν. Consider

$$\begin{aligned}\psi(\theta) = \frac{1}{1+\lambda}&\sin((1+\lambda)\theta)\cos(\lambda\omega) - \cos((1+\lambda)\theta) \\ &- \frac{1}{1-\lambda}\sin((1-\lambda)\theta)\cos(\lambda\omega) + \cos((1-\lambda)\theta),\end{aligned}$$

and let $\lambda = 856399/1572864 \approx 0.54$ and $\omega = 3\pi/2$. Our exact solution (u,p) is given in polar coordinates by

$$u = r^{\lambda}\begin{bmatrix} (1+\lambda)\sin(\theta)\psi(\theta) + \cos(\theta)\psi'(\theta) \\ -(1+\lambda)\cos(\theta)\psi(\theta) + \sin(\theta)\psi'(\theta) \end{bmatrix}, \quad p = \nu p_1 + p_2,$$

where

$$p_1 = r^{\lambda-1}((1+\lambda)^2\psi'(\theta) + \psi'''(\theta))/(1-\lambda), \quad p_2 = x^3 + y^3.$$

Table 2.1: Computed errors for the minimal regularity test case of Section 2.3.1 using the EDG–HDG method with different polynomial orders. In all cases, the discrete velocity solution is divergence-free up to machine precision.

Degree	Cells	$\lVert u - u_h \rVert$	Rate	$\lvert\lvert\lvert \boldsymbol{u} - \boldsymbol{u}_h \rvert\rvert\rvert_v$	Rate	$\lVert p - p_h \rVert$	Rate
P^1–P^0	24	7.2e-02	-	1.5e+00	-	5.8e+00	-
	96	2.2e-02	1.7	8.1e-01	0.9	1.2e+00	2.3
	384	7.6e-03	1.5	5.9e-01	0.5	8.2e-01	0.5
	1536	2.8e-03	1.5	4.2e-01	0.5	5.8e-01	0.5
	6144	9.8e-04	1.5	3.0e-01	0.5	4.1e-01	0.5
P^2–P^1	24	2.8e-02	-	8.4e-01	-	1.4e+00	-
	96	7.6e-03	1.9	4.0e-01	1.1	5.2e-01	1.4
	384	2.7e-03	1.5	2.9e-01	0.5	3.7e-01	0.5
	1536	9.5e-04	1.5	2.0e-01	0.5	2.6e-01	0.5
	6144	3.4e-04	1.5	1.4e-01	0.5	1.8e-01	0.5

Note that $-\nabla^2 u + \nabla p_1 = 0$ and therefore $-\nu\nabla^2 u + \nabla p = f$ where $f = \nabla p_2$. Also, there holds $(u, p) \in H^{1+s}(\Omega)^d \times H^s(\Omega)$ for all $0 \le s < \lambda$.

We compare the lowest-order EDG–HDG method to the lowest-order EDG method of [39] (see also [30] on the EDG method). The EDG method, which uses a continuous facet finite element space for both the velocity and pressure, is not pressure-robust [39]. We set the viscosity to be either $\nu = 1$ or $\nu = 10^{-5}$. The computed velocity errors for this example are shown in Figure 2.1.

For the EDG–HDG method, the velocity error is observed to be independent of the viscosity, confirming pressure-robustness. The velocity error for this method converges in the discrete H^1-norm as roughly $h^{0.54}$. This is consistent with the regularity of u and Corollary 2.1. Furthermore, according to [9, Section 5], on this domain Assumption 2.1 holds with $s_0 \approx 0.54$. Therefore, Theorem 2.2 predicts the velocity error in the L^2-norm to converge as roughly $(h^{0.54})^2 = h^{1.08}$, which is consistent with the empirical convergence rates displayed in Figure 2.1.

When $\nu = 1$ the velocity error for the EDG method is comparable to that of the EDG–HDG method. However, when $\nu = 10^{-5}$ the velocity error for the EDG method increases substantially, at least in the regime of large h. In this regime, we hypothesize that the velocity error for the EDG method is dominated by the pressure best approximation error

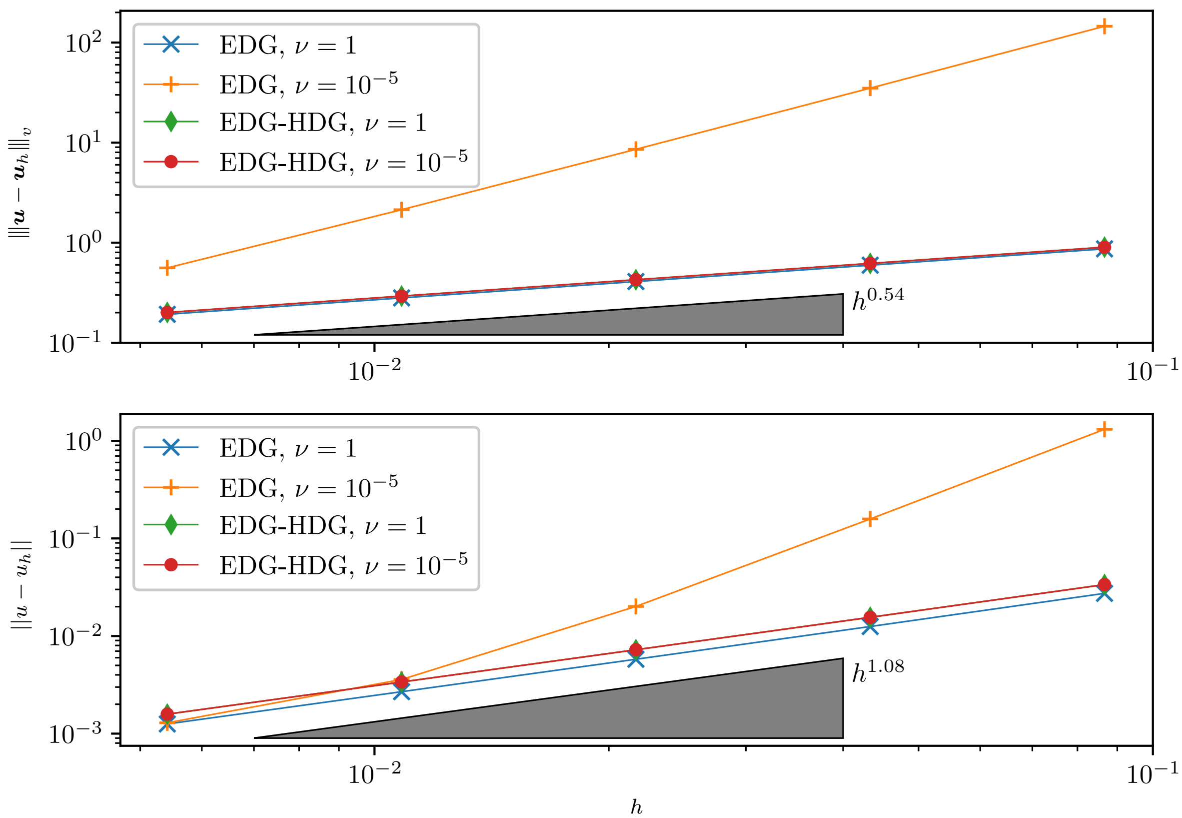

Figure 2.1: Computed velocity errors for the pressure-robustness test case of Section 2.3.2. We compare the lowest-order EDG and EDG–HDG methods with $\nu = 1$ and $\nu = 10^{-5}$.

Table 2.2: Computed errors for the cracked domain test case of Section 2.3.3 using the lowest-order EDG–HDG method. In all cases, the discrete velocity solution is divergence-free up to machine precision.

Cells	$\|u - u_h\|$	Rate	$\|\|\|\boldsymbol{u} - \boldsymbol{u}_h\|\|\|_v$	Rate	$\|p - p_h\|$	Rate
1680	2.0e-03	-	4.5e-01	-	5.8e-01	-
6720	1.0e-03	1.0	3.2e-01	0.5	3.6e-01	0.7
26880	5.0e-04	1.0	2.3e-01	0.5	2.4e-01	0.6
107520	2.5e-04	1.0	1.6e-01	0.5	1.6e-01	0.6
430080	1.2e-04	1.0	1.1e-01	0.5	1.1e-01	0.5

scaled by the inverse viscosity. Recalling that $p = \nu p_1 + p_2$, we can estimate the pressure best approximation error as

$$\begin{aligned}\inf_{q_h \in Q_h} \|p - q_h\| &\le \nu \inf_{q_h \in Q_h} \|p_1 - q_h\| + \inf_{q_h \in Q_h} \|p_2 - q_h\| \\ &\lesssim_s \nu h^s \|p_1\|_s + h^1 \|p_2\|_1 ,\end{aligned} \tag{2.72}$$

for any $0 \le s < \lambda$. For h sufficiently large eq. (2.72) converges pre-asymptotically at a rate of h^1, while the asymptotic convergence rate of eq. (2.72) is h^s. This behavior appears to be reflected in Figure 2.1, where for $\nu = 10^{-5}$ the velocity error of the EDG method pre-asymptotically converges at a faster rate than the EDG–HDG method.

2.3.3 Domain with a crack

We consider the Stokes problem on $\Omega = (-1/10, 1/10)^2 \setminus ([0, 1/10) \times \{0\})$ with $f = 0$ and $\nu = 1$. Notice that Ω has a crack along the positive x-axis. We use the same exact solution from eq. (2.71). The computed velocity and pressure errors for the lowest-order EDG–HDG method are shown in Table 2.2.

The velocity error in the discrete H^1-norm and the pressure error in the L^2-norm eventually both converge as roughly $h^{1/2}$. This is consistent with the regularity of the exact solution and the predictions of Corollary 2.1 and Corollary 2.2. Furthermore, according to [9, Section 5], on this domain Assumption 2.1 holds for any $s_0 < 1/2$. Therefore, Theorem 2.2 predicts the velocity error in the L^2-norm to converge as roughly h^1, which is consistent with the empirical convergence rate seen in Table 2.2.

Chapter 3

A technical inequality for three-dimensional Guzmán–Neilan finite element functions

An important tool in the analysis of Chapter 2 is the enrichment operator from Lemma 2.1. In Chapter 4 we will also make use of an enrichment operator satisfying the statements in Lemma 2.1. As discussed in Section 2.1.5, we have given a proof of Lemma 2.1 for the three-dimensional case in Appendix A. Our proof in Appendix A is based on the three-dimensional conforming and divergence-free finite element of Guzmán and Neilan [25]. The purpose of this chapter is to prove a technical inequality for functions in the three-dimensional Guzmán–Neilan finite element space. This inequality is used in Appendix A when proving Lemma 2.1.

To discuss the finite element space of [25], we must introduce some notation. Let $K \subset \mathbb{R}^3$ be a tetrahedron. We use $\mathcal{V}_K$ to denote the four vertices of K, while $\mathcal{E}_K$ denotes the six edges of K and $\mathcal{F}_K$ denotes the four faces of K. We let $\boldsymbol{V}(K)$ denote the local three-dimensional Guzmán–Neilan finite element space on K; this space is defined by [25, eq. (3.9)], and we note that functions in $\boldsymbol{V}(K)$ are $\mathbb{R}^3$-valued. We will delve into the precise definition of $\boldsymbol{V}(K)$ later on in this chapter, but for the time being we can ignore the precise details behind how $\boldsymbol{V}(K)$ is defined.

Our construction in Appendix A makes crucial use of the fact that any $\boldsymbol{v} \in \boldsymbol{V}(K)$ is uniquely determined by the unisolvent degrees of freedom [25, Theorem 3.5]

$$
\begin{aligned}
&\boldsymbol{v}(a) && \forall a \in \mathcal{V}_K, \\
&\langle \boldsymbol{v}, \boldsymbol{s} \rangle_{\boldsymbol{L}^2(e)} && \forall e \in \mathcal{E}_K, \boldsymbol{s} \in \boldsymbol{\mathcal{P}}_1(e), \\
&\langle \boldsymbol{v}, \boldsymbol{\kappa} \rangle_{\boldsymbol{L}^2(F)} && \forall F \in \mathcal{F}_K, \boldsymbol{\kappa} \in \boldsymbol{\mathcal{P}}_0(F),
\end{aligned} \tag{3.1}
$$

where $\boldsymbol{\mathcal{P}}_k(D)$ is the space of $\mathbb{R}^3$-valued polynomials of degree at most k on D. The degrees of freedom in eq. (3.1) naturally induce a norm $\|\cdot\|_{\boldsymbol{V},K}$ on $\boldsymbol{V}(K)$, which we define as follows. For $\boldsymbol{v} \in \boldsymbol{V}(K)$, we set

$$
\begin{aligned}
\|\boldsymbol{v}\|_{\boldsymbol{V},K}^2 := h_K \sum_{a \in \mathcal{V}_K} \left|\boldsymbol{v}(a)\right|^2 &+ \sum_{e \in \mathcal{E}_K} \sup_{\substack{\boldsymbol{s} \in \boldsymbol{\mathcal{P}}_1(e) \\ \|\boldsymbol{s}\|_{\boldsymbol{L}^2(e)} = 1}} \left|\langle \boldsymbol{v}, \boldsymbol{s} \rangle_{\boldsymbol{L}^2(e)}\right|^2 \\
&+ h_K^{-1} \sum_{F \in \mathcal{F}_K} \sup_{\substack{\boldsymbol{\kappa} \in \boldsymbol{\mathcal{P}}_0(F) \\ \|\boldsymbol{\kappa}\|_{\boldsymbol{L}^2(F)} = 1}} \left|\langle \boldsymbol{v}, \boldsymbol{\kappa} \rangle_{\boldsymbol{L}^2(F)}\right|^2,
\end{aligned} \tag{3.2}
$$

where $h_K := \operatorname{diam}(K)$. The factors of h_K and h_K^{-1} in eq. (3.2) are included for dimensional consistency. In a similar fashion we can define a dimensionally consistent $\boldsymbol{H}^1$-norm on $\boldsymbol{V}(K)$ as follows: For $\boldsymbol{v} \in \boldsymbol{V}(K)$, we set

$$
\|\boldsymbol{v}\|_{\boldsymbol{H},K}^2 := h_K^{-2} \int_K |\boldsymbol{v}|^2 \, \mathrm{d}x + \int_K |\nabla \boldsymbol{v}|^2 \, \mathrm{d}x. \tag{3.3}
$$

Notice that the norms $\|\cdot\|_{\boldsymbol{V},K}$ and $\|\cdot\|_{\boldsymbol{H},K}$ are dimensionally consistent with one another because both scale as $h_K^{1/2}$.

This entire thesis chapter is dedicated to proving the following inequality:

$$
\|\boldsymbol{v}\|_{\boldsymbol{H},K} \leq C \|\boldsymbol{v}\|_{\boldsymbol{V},K} \qquad \forall \boldsymbol{v} \in \boldsymbol{V}(K), \tag{3.4}
$$

where $C > 0$ is a constant depending on the shape-regularity of K only. To the best of the author's knowledge, a proof of eq. (3.4) is not available anywhere in the literature. However, the inequality in eq. (3.4) is quite useful because it allows one to estimate the $\boldsymbol{H}^1$-norm of any function $\boldsymbol{v} \in \boldsymbol{V}(K)$ in terms of its degrees of freedom in eq. (3.1). In the context of this thesis, we make use of eq. (3.4) in Appendix A (specifically in eq. (A.4)), when we prove that the enrichment operator of Lemma 2.1 satisfies Item iii.

The rest of this chapter is organized as follows. In Section 3.1 we introduce the necessary mathematical machinery that will allow us to give the precise definition of the space $\boldsymbol{V}(K)$, and we discuss some preliminary results concerning this space. In Section 3.2 we introduce some projection operators on $\boldsymbol{V}(K)$ which will prove to be useful in the main analysis. Finally, the main analysis is carried out in Section 3.3, where we give a proof of eq. (3.4).

3.1 The local Guzmán–Neilan finite element space

In what follows we use the standard notation for Lebesgue and Sobolev spaces and their norms. We frequently use the L^2-inner-product, which on a three-dimensional (resp. one- or two-dimensional) set $D \subset \mathbb{R}^3$ is denoted by $(\cdot,\cdot)_D$ (resp. $\langle\cdot,\cdot\rangle_D$). We denote by $\|\cdot\|_D$ the L^2-norm on D. We let $\mathcal{P}_k(D)$ denote the space of degree at most k polynomials on D, and spaces of $\mathbb{R}^3$-valued functions are written in boldface:

$$\boldsymbol{\mathcal{P}}_k(D) := [\mathcal{P}_k(D)]^3, \quad \boldsymbol{W}^{1,p}(D) := [W^{1,p}(D)]^3, \quad \boldsymbol{C}^0(D) := [C^0(D)]^3, \quad \text{etc.}$$

Let $K \subset \mathbb{R}^3$ be a tetrahedron. Following the notational conventions used in [25], the four vertices of K are denoted by $\mathcal{V}_K := \{x_i\}_{i=1}^4$. The four faces of K are denoted by $\mathcal{F}_K := \{F_i\}_{i=1}^4$ and are labeled such that the face F_i is opposite to the vertex x_i. The six edges of K are denoted by $\mathcal{E}_K := \{e_{i,j}\}_{1\le i<j\le 4}$ where $e_{i,j} := F_i \cap F_j$.

Remark 3.1 (Convention regarding the boundaries of these sets). We treat K as being an open set, so that K is disjoint from its boundary ∂K. In contrast, a face $F \in \mathcal{F}_K$ is treated as containing its constituent three edges (hence F is a closed set), and an edge $e \in \mathcal{E}_K$ is treated as containing its constituent two vertices (hence e is also a closed set).

Remark 3.2 (Traces of functions). Since we are assuming that K is open, we have $\partial K \not\subset K$. However, a function $f : K \to \mathbb{R}$ still admits a well-defined trace $f|_{\partial K} : \partial K \to \mathbb{R}$ provided that it is sufficiently regular. Given an integer $k \ge 0$, we adapt the usual definition

$$C^k(\overline{K}) := \{g \in C^k(K) \mid D^\alpha g \text{ is uniformly continuous for all } |\alpha| \le k\}.$$

Then any $f \in C^0(\overline{K})$ extends continuously to $\overline{K}$, which implies that $f|_{\partial K} \in C^0(\partial K)$. Similarly, in the context of Sobolev spaces, we have that if $f \in H^1(K)$ then $f|_{\partial K} \in H^{1/2}(\partial K)$. With this understood, in what follows we will often consider the restriction to ∂K of functions whose domain is K.

3.1.1 The bubble functions

Following [25], in this subsection we introduce some important bubble functions defined on the tetrahedron K. The barycentric coordinates on K are denoted by $\{\lambda_i\}_{i=1}^4 \subset \mathcal{P}_1(K)$, and are the unique linear functions satisfying $\lambda_i(x_j) = \delta_{ij}$ for all vertices $x_j \in \mathcal{V}_K$. Note in particular that λ_i vanishes on the face F_i. The *volume, face* and *edge bubble functions* are defined, respectively, as

$$b_K := \prod_{1\le k\le 4} \lambda_k \in \mathcal{P}_4(K), \quad b_i := \prod_{\substack{1\le k\le 4\\ k\ne i}} \lambda_k \in \mathcal{P}_3(K), \quad b_{i,j} := \prod_{\substack{1\le k\le 4\\ k\notin\{i,j\}}} \lambda_k \in \mathcal{P}_2(K).$$

It can be verified that these bubble functions have the following properties.

- b_K vanishes on ∂K. However, b_K is not identically zero on K.
- b_i vanishes on $\partial K \setminus F_i$. In particular, b_i vanishes on all vertices and edges of K. However, b_i is not identically zero on F_i.
- $b_{i,j}$ vanishes on $\partial K \setminus (F_i \cup F_j)$. In particular, $b_{i,j}$ vanishes on all vertices of K, and on all edges of K except for $e_{i,j}$. However, $b_{i,j}$ is not identically zero on $e_{i,j}$.

The *rational face bubble functions* are then defined, for $1 \leq i \leq 4$, as

$$B_i := (b_K b_i) / \prod_{k=1}^{3} (\lambda_i + \lambda_{i+k}),$$

where for $l > 4$ the expression λ_l should be interpreted as $\lambda_{1+(l-1 \bmod 4)}$. Hence, for example, $\lambda_5 := \lambda_1$ and $\lambda_6 := \lambda_2$. We also define $a_i := -|\nabla \lambda_i| < 0$, and we let $\boldsymbol{n}_i$ denote the outward unit normal to F_i. The following result is established in [25, Lemma 2.1].

Lemma 3.1 (Rational face bubbles)**.** *For all* $1 \leq i \leq 4$*, we have*

$$\begin{aligned} &B_i \in C^2(\overline{K}), \quad B_i|_{\partial K} = 0, \quad \nabla B_i(x_j) = 0 \ \forall 1 \leq j \leq 4, \\ &\nabla B_i|_{\partial K \setminus F_i} = 0, \quad \frac{\partial B_i}{\partial \boldsymbol{n}_i}|_{F_i} = a_i b_i, \quad \nabla B_i|_{F_i} \in \boldsymbol{\mathcal{P}}_3(F_i). \end{aligned}$$

Finally, to each edge $e_{i,j}$, where $1 \leq i < j \leq 4$, we define the corresponding *rational edge bubble function*

$$\boldsymbol{s}_{i,j} := \frac{b_K b_{i,j}}{2(\lambda_i \lambda_j + b_{i,j}(\lambda_i + \lambda_j))(\lambda_i + \lambda_j)} (\nabla(\lambda_j^2 - \lambda_i^2) + 4(\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i)).$$

The important properties of $\boldsymbol{s}_{i,j}$ are as follows.

Lemma 3.2 (Rational edge bubbles)**.** *For all* $1 \leq i < j \leq 4$*, we have*

$$\begin{aligned} \boldsymbol{s}_{i,j} &\in \boldsymbol{C}^0(\overline{K}) \cap \boldsymbol{W}^{1,\infty}(K), & \boldsymbol{s}_{i,j}|_{\partial K} &= 0, \\ \mathbf{curl}(\boldsymbol{s}_{i,j}) &\in \boldsymbol{C}^0(\overline{K}) \cap \boldsymbol{W}^{1,\infty}(K), & \mathbf{curl}(\boldsymbol{s}_{i,j})|_{\partial K} &= b_{i,j}(\nabla \lambda_i \times \nabla \lambda_j). \end{aligned}$$

Proof. The first property, $\boldsymbol{s}_{i,j} \in \boldsymbol{C}^0(\overline{K}) \cap \boldsymbol{W}^{1,\infty}(K)$, follows from the inequalities stated in [25, Lemma A.1]. The last three properties are stated in [25, Lemma 2.3]. □

3.1.2 The local spaces

We continue to follow the construction in [25], by now introducing the local spaces that are used to build $\boldsymbol{V}(K)$. For $m \geq 2$, we consider the local Nedelec space [35] of order $m-1$,

$$\begin{aligned}\boldsymbol{N}_{m-1}(K) &:= \boldsymbol{\mathcal{P}}_{m-2}(K) + \{\boldsymbol{w} \in \boldsymbol{\mathcal{P}}_{m-1}(K) \mid \boldsymbol{w} \cdot \boldsymbol{x} = 0\} \\ &= \{\boldsymbol{w} \in \boldsymbol{\mathcal{P}}_{m-1}(K) \mid \boldsymbol{w} \cdot \boldsymbol{x} \in \mathcal{P}_{m-1}(K)\}.\end{aligned} \tag{3.5}$$

We use this space to define the following local space of divergence-free polynomials:

$$\begin{aligned}\boldsymbol{Q}_m(K) := \ &\{\boldsymbol{v} \in \boldsymbol{\mathcal{P}}_m(K) \mid (\boldsymbol{v}, \boldsymbol{\rho})_K = 0 \ \ \forall \boldsymbol{\rho} \in \boldsymbol{N}_{m-1}(K) \text{ and} \\ &\langle \boldsymbol{v} \cdot \boldsymbol{n}_i, \kappa \rangle_{F_i} = 0 \ \ \forall \kappa \in \mathcal{P}_{m-1}(F_i), \ \forall 1 \leq i \leq 4\}.\end{aligned} \tag{3.6}$$

It is shown in [25, Section 3] that $\nabla \cdot \boldsymbol{Q}_m(K) = \{0\}$, so that these polynomials are indeed divergence-free. Next, we define the local $\boldsymbol{H}(\mathrm{div}; K)$-conforming space (cf. [25, eq. (3.1)]),

$$\boldsymbol{M}(K) := \boldsymbol{\mathcal{P}}_1(K) + \boldsymbol{Q}_2(K) + \boldsymbol{Q}_3(K). \tag{3.7}$$

The following result is proven in [25, Lemma 3.1].

Lemma 3.3 (Degrees of freedom for $\boldsymbol{M}(K)$)**.** *The degrees of freedom in eq. (3.8) are unisolvent on $\boldsymbol{M}(K)$:*

$$\boldsymbol{v}(x_i) \qquad \forall x_i \in \mathcal{V}_K, \tag{3.8a}$$

$$\langle \boldsymbol{v} \cdot \boldsymbol{n}_k, s \rangle_{e_{i,j}} \quad \forall s \in \mathcal{P}_1(e_{i,j}), \ 1 \leq i < j \leq 4, \ \ k \in \{i, j\}, \tag{3.8b}$$

$$\langle \boldsymbol{v} \cdot \boldsymbol{n}_i, \kappa \rangle_{F_i} \quad \forall \kappa \in \mathcal{P}_0(F_i), \ \ 1 \leq i \leq 4. \tag{3.8c}$$

The next step is to define a space whose associated degrees of freedom involve tangential moments on edges. For $1 \leq i < j \leq 4$, let $M^{(i,j)}(K) := \mathrm{span}(\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \setminus \{\lambda_i, \lambda_j\})$. Consider the local space (cf. [25, eq. (3.6)])

$$\boldsymbol{W}(K) := \left\{ \sum_{1 \leq i < j \leq 4} \mathbf{curl}(p_{i,j} \boldsymbol{s}_{i,j}) \mid p_{i,j} \in M^{(i,j)}(K) \right\}. \tag{3.9}$$

Using the product rule and Lemma 3.2, one sees that $\boldsymbol{W}(K) \subset \boldsymbol{C}^0(\overline{K}) \cap \boldsymbol{W}^{1,\infty}(K)$.

Lemma 3.4 (Degrees of freedom for $\boldsymbol{W}(K)$)**.** *The degrees of freedom in eq. (3.10) are unisolvent on $\boldsymbol{W}(K)$:*

$$\langle \boldsymbol{w} \cdot \boldsymbol{t}_{i,j}, s \rangle_{e_{i,j}} \quad \forall s \in \mathcal{P}_1(e_{i,j}), \ \ 1 \leq i < j \leq 4. \tag{3.10}$$

Here, $\boldsymbol{t}_{i,j}$ is a unit tangent vector to the edge $e_{i,j}$. Also, any function $\boldsymbol{w} \in \boldsymbol{W}(K)$ vanishes on the degrees of freedom in eq. (3.8).

Proof. Unisolvence of the degrees of freedom in eq. (3.10) is proven in [25, Lemma 3.3]. Next, let $\boldsymbol{w} \in \boldsymbol{W}(K)$, so that we may write $\boldsymbol{w} = \sum_{1 \le i < j \le 4} \mathbf{curl}(p_{i,j} \boldsymbol{s}_{i,j})$ where $p_{i,j} \in M^{(i,j)}(K)$. By [25, Lemma 3.3], we have

$$\boldsymbol{w}|_{e_{i,j}} = p_{i,j} b_{i,j} (\nabla \lambda_i \times \nabla \lambda_j) \quad \forall 1 \le i < j \le 4, \tag{3.11}$$

$$\boldsymbol{w} \cdot \boldsymbol{n}|_{\partial K} = 0. \tag{3.12}$$

Since edge bubbles vanish on all vertices, eq. (3.11) implies that $\boldsymbol{w}$ vanishes on the degrees of freedom in eq. (3.8a). Finally, an immediate consequence of eq. (3.12) is that $\boldsymbol{w}$ vanishes on the degrees of freedom in eqs. (3.8b) to (3.8c). ☐

The last step is to define a local space whose associated degrees of freedom involve tangential moments on faces. This space is taken to be (cf. [25, eq. (3.3)])

$$\boldsymbol{U}(K) := \sum_{i=1}^{4} \boldsymbol{U}^{(i)}(K), \quad \text{where} \quad \boldsymbol{U}^{(i)}(K) := \mathbf{curl}(B_i \boldsymbol{\mathcal{P}}_0(K) \times \boldsymbol{n}_i). \tag{3.13}$$

Using the product rule and Lemma 3.1, one sees that $\boldsymbol{U}(K) \subset \boldsymbol{C}^1(\overline{K})$.

Lemma 3.5 (Degrees of freedom for $\boldsymbol{U}(K)$)**.** *The degrees of freedom in eq. (3.14) are unisolvent on $\boldsymbol{U}(K)$:*

$$\langle \boldsymbol{z} \times \boldsymbol{n}_i, \boldsymbol{q} \times \boldsymbol{n}_i \rangle_{F_i} \quad \forall \boldsymbol{q} \in \boldsymbol{\mathcal{P}}_0(F_i), \ \ 1 \le i \le 4. \tag{3.14}$$

Also, any function $\boldsymbol{z} \in \boldsymbol{U}(K)$ vanishes on the degrees of freedom in eq. (3.8) and eq. (3.10).

Proof. Unisolvence of the degrees of freedom in eq. (3.14) is proven in [25, Lemma 3.2]. Next, for any $\boldsymbol{z} \in \boldsymbol{U}(K)$, it is also proven in [25, Lemma 3.2] that $\boldsymbol{z}$ vanishes on the degrees of freedom in eq. (3.8). To prove that $\boldsymbol{z}$ vanishes on the degrees of freedom in eq. (3.10), observe that by the product rule we can write

$$\boldsymbol{z} = \sum_{i=1}^{4} (\nabla B_i) \times (\boldsymbol{p}_i \times \boldsymbol{n}_i),$$

where $\boldsymbol{p}_i \in \boldsymbol{\mathcal{P}}_0(K)$. But ∇B_i vanishes on all edges (since $B_i \in C^2(\overline{K})$ and $\nabla B_i|_{\partial K \setminus F_i} = 0$ by Lemma 3.1), and it follows that $\boldsymbol{z}$ vanishes on the degrees of freedom in eq. (3.10). ☐

With the three auxiliary spaces $\boldsymbol{M}(K)$, $\boldsymbol{W}(K)$ and $\boldsymbol{U}(K)$ established, the local Guzmán–Neilan space $\boldsymbol{V}(K)$ is defined to be their sum (cf. [25, eq. (3.9)]):

$$\boldsymbol{V}(K) := \big[\boldsymbol{M}(K) + \boldsymbol{W}(K) + \boldsymbol{U}(K)\big] \subset \boldsymbol{C}^0(\overline{T}) \cap \boldsymbol{W}^{1,\infty}(K). \tag{3.15}$$

The degrees of freedom for $\boldsymbol{V}(K)$ are (recall eq. (3.1)) given in the following Theorem.

Theorem 3.1 (Degrees of freedom for $\boldsymbol{V}(K)$)**.** *The degrees of freedom in eq.* (3.16) *are unisolvent on* $\boldsymbol{V}(K)$*:*

$$\begin{aligned} &\boldsymbol{v}(x_i) && \forall x_i \in \mathcal{V}_K, && (3.16a)\\ &\langle \boldsymbol{v}, \boldsymbol{s}\rangle_{e_{i,j}} && \forall \boldsymbol{s} \in \boldsymbol{\mathcal{P}}_1(e_{i,j}),\ 1 \le i < j \le 4, && (3.16b)\\ &\langle \boldsymbol{v}, \boldsymbol{\kappa}\rangle_{F_i} && \forall \boldsymbol{\kappa} \in \boldsymbol{\mathcal{P}}_0(F_i),\ 1 \le i \le 4. && (3.16c) \end{aligned}$$

Proof. This is proven in [25, Theorem 3.5]. □

3.2 Projection operators

Motivated by the degrees of freedom considered in eqs. (3.8), (3.10) and (3.14), we introduce three linear *projection operators*

$$\boldsymbol{\Pi}_{\boldsymbol{M},K} : \boldsymbol{V}(K) \to \boldsymbol{M}(K), \quad \boldsymbol{\Pi}_{\boldsymbol{W},K} : \boldsymbol{V}(K) \to \boldsymbol{W}(K), \quad \boldsymbol{\Pi}_{\boldsymbol{U},K} : \boldsymbol{V}(K) \to \boldsymbol{U}(K),$$

in the following way. For all $\boldsymbol{v} \in \boldsymbol{V}(K)$, we require that

$$\begin{aligned} 0 &= (\boldsymbol{\Pi}_{\boldsymbol{M},K}\boldsymbol{v} - \boldsymbol{v})(x_i) && \forall x_i \in \mathcal{V}_K, && (3.17a)\\ 0 &= \langle (\boldsymbol{\Pi}_{\boldsymbol{M},K}\boldsymbol{v} - \boldsymbol{v}) \cdot \boldsymbol{n}_k, s\rangle_{e_{i,j}} && \forall s \in \mathcal{P}_1(e_{i,j}),\ 1 \le i < j \le 4,\ k \in \{i,j\}, && (3.17b)\\ 0 &= \langle (\boldsymbol{\Pi}_{\boldsymbol{M},K}\boldsymbol{v} - \boldsymbol{v}) \cdot \boldsymbol{n}_i, \kappa\rangle_{F_i} && \forall \kappa \in \mathcal{P}_0(F_i),\ 1 \le i \le 4, && (3.17c)\\ 0 &= \langle (\boldsymbol{\Pi}_{\boldsymbol{W},K}\boldsymbol{v} - \boldsymbol{v}) \cdot \boldsymbol{t}_{i,j}, s\rangle_{e_{i,j}} && \forall s \in \mathcal{P}_1(e_{i,j}),\ 1 \le i < j \le 4, && (3.17d)\\ 0 &= \langle (\boldsymbol{\Pi}_{\boldsymbol{U},K}\boldsymbol{v} - \boldsymbol{v}) \times \boldsymbol{n}_i, \boldsymbol{q} \times \boldsymbol{n}_i\rangle_{F_i} && \forall \boldsymbol{q} \in \boldsymbol{\mathcal{P}}_0(F_i),\ 1 \le i \le 4. && (3.17e) \end{aligned}$$

The unisolvence of the degrees of freedom in eqs. (3.8), (3.10) and (3.14) ensures that the projection operators $\boldsymbol{\Pi}_{\boldsymbol{M},K}, \boldsymbol{\Pi}_{\boldsymbol{W},K}$ and $\boldsymbol{\Pi}_{\boldsymbol{U},K}$ are well-defined.

Let $\boldsymbol{I} : \boldsymbol{V}(K) \to \boldsymbol{V}(K)$ be the identity operator on $\boldsymbol{V}(K)$. How can we decompose $\boldsymbol{I}$ in terms of the projection operators defined above? Since $\boldsymbol{V}(K) = \boldsymbol{M}(K) + \boldsymbol{W}(K) + \boldsymbol{U}(K)$, a reasonable guess would be that $\boldsymbol{I} = \boldsymbol{\Pi}_{\boldsymbol{M},K} + \boldsymbol{\Pi}_{\boldsymbol{W},K} + \boldsymbol{\Pi}_{\boldsymbol{U},K}$. However, this is not the case. The goal of this subsection is to figure out what the proper decomposition of $\boldsymbol{I}$ is.

Definition 3.1 (Vanishing on degrees of freedom)**.** Let $\boldsymbol{v} \in \boldsymbol{V}(K)$. We shall say that

(i) $\boldsymbol{v}$ *is* $\boldsymbol{M}$*-vanishing* if $\boldsymbol{v}$ vanishes on the degrees of freedom in eq. (3.8),

(ii) $\boldsymbol{v}$ *is* $\boldsymbol{W}$*-vanishing* if $\boldsymbol{v}$ vanishes on the degrees of freedom in eq. (3.10),

(iii) $\boldsymbol{v}$ *is* $\boldsymbol{U}$*-vanishing* if $\boldsymbol{v}$ vanishes on the degrees of freedom in eq. (3.14).

Let $\boldsymbol{v} \in \boldsymbol{V}(K)$. Notice that by construction, $(\boldsymbol{\Pi}_{\boldsymbol{M},K}\boldsymbol{v} - \boldsymbol{v})$ is $\boldsymbol{M}$-vanishing, $(\boldsymbol{\Pi}_{\boldsymbol{W},K}\boldsymbol{v} - \boldsymbol{v})$ is $\boldsymbol{W}$-vanishing and $(\boldsymbol{\Pi}_{\boldsymbol{U},K}\boldsymbol{v} - \boldsymbol{v})$ is $\boldsymbol{U}$-vanishing. Moreover, Lemma 3.4 states that any $\boldsymbol{w} \in \boldsymbol{W}(K)$ is $\boldsymbol{M}$-vanishing, and Lemma 3.5 states that any $\boldsymbol{z} \in \boldsymbol{U}(K)$ is both $\boldsymbol{M}$-vanishing and $\boldsymbol{W}$-vanishing.

Lemma 3.6 (Characterization of the zero function)**.** *If* $\boldsymbol{v} \in \boldsymbol{V}(K)$ *is simultaneously* $\boldsymbol{M}$*-vanishing,* $\boldsymbol{W}$*-vanishing and* $\boldsymbol{U}$*-vanishing, then* $\boldsymbol{v} = 0$*.*

Proof. Write $\boldsymbol{v} = \boldsymbol{v}_m + \boldsymbol{v}_w + \boldsymbol{v}_u$ for some $\boldsymbol{v}_m \in \boldsymbol{M}(K), \boldsymbol{v}_w \in \boldsymbol{W}(K)$ and $\boldsymbol{v}_u \in \boldsymbol{U}(K)$. Assume that $\boldsymbol{v}$ is simultaneously $\boldsymbol{M}$-vanishing, $\boldsymbol{W}$-vanishing and $\boldsymbol{U}$-vanishing. Then

$$\boldsymbol{v}_m = \boldsymbol{v} - \boldsymbol{v}_w - \boldsymbol{v}_u. \tag{3.18}$$

All terms on the right-hand side of eq. (3.18) are $\boldsymbol{M}$-vanishing, so $\boldsymbol{v}_m = 0$ by Lemma 3.3. Thus

$$\boldsymbol{v}_w = \boldsymbol{v} - \boldsymbol{v}_u. \tag{3.19}$$

Both terms on the right-hand side of eq. (3.19) are $\boldsymbol{W}$-vanishing, so $\boldsymbol{v}_w = 0$ by Lemma 3.4. Hence $\boldsymbol{v}_u = \boldsymbol{v}$ is $\boldsymbol{U}$-vanishing, so $\boldsymbol{v}_u = 0$ by Lemma 3.5. Therefore $\boldsymbol{v} = 0$. □

Corollary 3.1 (Factorization of the zero operator)**.** *There holds*

$$\boldsymbol{0} = (\boldsymbol{I} - \boldsymbol{\Pi}_{\boldsymbol{U},K})(\boldsymbol{I} - \boldsymbol{\Pi}_{\boldsymbol{W},K})(\boldsymbol{I} - \boldsymbol{\Pi}_{\boldsymbol{M},K}).$$

Proof. Let $\boldsymbol{v} \in \boldsymbol{V}(K)$. Define

$$\boldsymbol{v}_1 := (\boldsymbol{I} - \boldsymbol{\Pi}_{\boldsymbol{M},K})\boldsymbol{v}, \quad \boldsymbol{v}_2 := (\boldsymbol{I} - \boldsymbol{\Pi}_{\boldsymbol{W},K})\boldsymbol{v}_1, \quad \boldsymbol{v}_3 := (\boldsymbol{I} - \boldsymbol{\Pi}_{\boldsymbol{U},K})\boldsymbol{v}_2.$$

Our goal is to show that $\boldsymbol{v}_3 = 0$. Note that by definition $\boldsymbol{v}_1$ is $\boldsymbol{M}$-vanishing. But then

$$\boldsymbol{v}_2 = \boldsymbol{v}_1 - \boldsymbol{\Pi}_{\boldsymbol{W},K}\boldsymbol{v}_1, \tag{3.20}$$

and both terms on the right-hand side of eq. (3.20) are $\boldsymbol{M}$-vanishing, so that $\boldsymbol{v}_2$ is $\boldsymbol{M}$-vanishing. Also note that by definition $\boldsymbol{v}_2$ is $\boldsymbol{W}$-vanishing. Therefore $\boldsymbol{v}_2$ is both $\boldsymbol{M}$-vanishing and $\boldsymbol{W}$-vanishing. But then

$$\boldsymbol{v}_3 = \boldsymbol{v}_2 - \boldsymbol{\Pi}_{\boldsymbol{U},K}\boldsymbol{v}_2, \tag{3.21}$$

and both terms on the right-hand side of eq. (3.21) are $\boldsymbol{M}$-vanishing and $\boldsymbol{W}$-vanishing. Therefore $\boldsymbol{v}_3$ is $\boldsymbol{M}$-vanishing and $\boldsymbol{W}$-vanishing. But $\boldsymbol{v}_3$ is also $\boldsymbol{U}$-vanishing by definition. Hence by Lemma 3.6 we have that $\boldsymbol{v}_3 = 0$. This completes the proof. □

We are now ready to deduce the correct decomposition of the identity.

Theorem 3.2 (Decomposition of the identity)**.** *The identity $\boldsymbol{I}$ on $\boldsymbol{V}(K)$ is given by*

$$\boldsymbol{I} = \boldsymbol{\Pi}_{\boldsymbol{M},K} + \boldsymbol{\Pi}_{\boldsymbol{W},K}(\boldsymbol{I} - \boldsymbol{\Pi}_{\boldsymbol{M},K}) + \boldsymbol{\Pi}_{\boldsymbol{U},K}(\boldsymbol{I} - \boldsymbol{\Pi}_{\boldsymbol{W},K})(\boldsymbol{I} - \boldsymbol{\Pi}_{\boldsymbol{M},K}).$$

Proof. Proving this result is now just a matter of algebra. We have:

$$\begin{aligned}
\boldsymbol{I} &= \boldsymbol{\Pi}_{\boldsymbol{M},K} + (\boldsymbol{I} - \boldsymbol{\Pi}_{\boldsymbol{M},K}) \\
&= \boldsymbol{\Pi}_{\boldsymbol{M},K} + \boldsymbol{\Pi}_{\boldsymbol{W},K}(\boldsymbol{I} - \boldsymbol{\Pi}_{\boldsymbol{M},K}) + (\boldsymbol{I} - \boldsymbol{\Pi}_{\boldsymbol{W},K})(\boldsymbol{I} - \boldsymbol{\Pi}_{\boldsymbol{M},K}) \\
&= \boldsymbol{\Pi}_{\boldsymbol{M},K} + \boldsymbol{\Pi}_{\boldsymbol{W},K}(\boldsymbol{I} - \boldsymbol{\Pi}_{\boldsymbol{M},K}) + \boldsymbol{\Pi}_{\boldsymbol{U},K}(\boldsymbol{I} - \boldsymbol{\Pi}_{\boldsymbol{W},K})(\boldsymbol{I} - \boldsymbol{\Pi}_{\boldsymbol{M},K}) \\
&\quad + \underbrace{(\boldsymbol{I} - \boldsymbol{\Pi}_{\boldsymbol{U},K})(\boldsymbol{I} - \boldsymbol{\Pi}_{\boldsymbol{W},K})(\boldsymbol{I} - \boldsymbol{\Pi}_{\boldsymbol{M},K})}_{=\boldsymbol{0} \text{ by Corollary 3.1}}.
\end{aligned}$$

□

We will use Theorem 3.2 in the next section.

3.3 The main analysis

We are now ready to prove eq. (3.4). Since we want the constant C in eq. (3.4) to depend only on shape-regularity of the tetrahedron K, let us assume that K belongs to a shape-regular family of triangulations (see e.g. [18, Definition 11.2]). Hence there is a constant $\gamma > 0$ such that $h_K \leq \gamma \rho_K$, where ρ_K is the diameter of the largest ball inscribed in K and $h_K = \operatorname{diam}(K)$. In what follows, we will write $a \lesssim b$ if $a \leq Cb$ where C is any constant depending on γ only. We will write $a \sim b$ if $a \lesssim b$ and $b \lesssim a$.

3.3.1 The relevant norms and key estimates

We define on $\boldsymbol{V}(K)$ the scaled $\boldsymbol{H}^1$-norm $\|\cdot\|_{\boldsymbol{H},K}$ according to (recall eq. (3.3)):

$$\|\boldsymbol{v}\|_{\boldsymbol{H},K}^2 := h_K^{-2} \int_K |\boldsymbol{v}|^2 \,\mathrm{d}x + \int_K |\nabla \boldsymbol{v}|^2 \,\mathrm{d}x. \tag{3.22}$$

Motivated by the degrees of freedom in eq. (3.16) we also define on $\boldsymbol{V}(K)$ the norm $\|\cdot\|_{\boldsymbol{V},K}$ according to (recall eq. (3.2)):

$$\begin{aligned} \|\boldsymbol{v}\|_{\boldsymbol{V},K}^2 := h_K \sum_{i=1}^{4} \left|\boldsymbol{v}(x_i)\right|^2 + \sum_{1\le i<j\le 4} \sup_{\substack{\boldsymbol{s}\in\boldsymbol{\mathcal{P}}_1(e_{i,j})\\ \|\boldsymbol{s}\|_{e_{i,j}}=1}} \left|\langle \boldsymbol{v}, \boldsymbol{s}\rangle_{e_{i,j}}\right|^2 \\ + h_K^{-1} \sum_{i=1}^{4} \sup_{\substack{\boldsymbol{\kappa}\in\boldsymbol{\mathcal{P}}_0(F_i)\\ \|\boldsymbol{\kappa}\|_{F_i}=1}} \left|\langle \boldsymbol{v}, \boldsymbol{\kappa}\rangle_{F_i}\right|^2 . \end{aligned} \tag{3.23}$$

The degrees of freedom in eqs. (3.8), (3.10) and (3.14) also motivate us to define on $\boldsymbol{V}(K)$ the following semi-norms:

$$\begin{aligned} \|\boldsymbol{v}\|_{\boldsymbol{M},K}^2 := h_K \sum_{i=1}^{4} \left|\boldsymbol{v}(x_i)\right|^2 + \sum_{\substack{1\le i<j\le 4\\ k\in\{i,j\}}} \sup_{\substack{s\in\mathcal{P}_1(e_{i,j})\\ \|s\|_{e_{i,j}}=1}} \left|\langle \boldsymbol{v}\cdot\boldsymbol{n}_k, s\rangle_{e_{i,j}}\right|^2 \\ + h_K^{-1} \sum_{i=1}^{4} \sup_{\substack{\kappa\in\mathcal{P}_0(F_i)\\ \|\kappa\|_{F_i}=1}} \left|\langle \boldsymbol{v}\cdot\boldsymbol{n}_i, \kappa\rangle_{F_i}\right|^2 , \end{aligned} \tag{3.24a}$$

$$\|\boldsymbol{v}\|_{\boldsymbol{W},K}^2 := \sum_{1\le i<j\le 4} \sup_{\substack{s\in\mathcal{P}_1(e_{i,j})\\ \|s\|_{e_{i,j}}=1}} \left|\langle \boldsymbol{v}\cdot\boldsymbol{t}_{i,j}, s\rangle_{e_{i,j}}\right|^2 , \tag{3.24b}$$

$$\|\boldsymbol{v}\|_{\boldsymbol{U},K}^2 := h_K^{-1} \sum_{i=1}^{4} \sup_{\substack{\boldsymbol{q}\in\boldsymbol{\mathcal{P}}_0(F_i)\\ \|\boldsymbol{q}\|_{F_i}=1}} \left|\langle \boldsymbol{v}\times\boldsymbol{n}_i, \boldsymbol{q}\times\boldsymbol{n}_i\rangle_{F_i}\right|^2 . \tag{3.24c}$$

Note that for each $\boldsymbol{Y} \in \{\boldsymbol{M}, \boldsymbol{W}, \boldsymbol{U}\}$ the semi-norm $\|\cdot\|_{\boldsymbol{Y},K}$ is actually a norm on $\boldsymbol{Y}(K)$.

Lemma 3.7 (Properties of these norms)**.** *Let $\boldsymbol{Y} \in \{\boldsymbol{M}, \boldsymbol{W}, \boldsymbol{U}\}$. For all $\boldsymbol{v} \in \boldsymbol{V}(K)$ we have*

$$\left\|\boldsymbol{\Pi}_{\boldsymbol{Y},K}\boldsymbol{v}\right\|_{\boldsymbol{Y},K} = \|\boldsymbol{v}\|_{\boldsymbol{Y},K} \le C_{\boldsymbol{Y}} \|\boldsymbol{v}\|_{\boldsymbol{V},K} , \tag{3.25}$$

where $C_{\boldsymbol{M}} = \sqrt{2}$ and $C_{\boldsymbol{W}} = C_{\boldsymbol{U}} = 1$.

Proof. The equality in eq. (3.25) follows immediately from the definitions of $\mathbf{\Pi}_{\boldsymbol{Y},K}$ (see eq. (3.17)) and $\|\cdot\|_{\boldsymbol{Y},K}$ (see eq. (3.24)). The inequality in eq. (3.25) is a simple consequence of the definition of $\|\cdot\|_{\boldsymbol{V},K}$ and $\|\cdot\|_{\boldsymbol{Y},K}$ (see eq. (3.23) and eq. (3.24)). □

We now state three key estimates posed on the smaller spaces $\boldsymbol{M}(K)$, $\boldsymbol{W}(K)$ and $\boldsymbol{U}(K)$. We postpone the proof of these estimates to the upcoming subsections.

Proposition 3.1 (Estimate for $\boldsymbol{M}(K)$)**.** *For all $\boldsymbol{v} \in \boldsymbol{M}(K)$ we have*

$$\|\boldsymbol{v}\|_{\boldsymbol{H},K} + \|\boldsymbol{v}\|_{\boldsymbol{V},K} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{M},K}.$$

Proposition 3.2 (Estimate for $\boldsymbol{W}(K)$)**.** *For all $\boldsymbol{v} \in \boldsymbol{W}(K)$ we have*

$$\|\boldsymbol{v}\|_{\boldsymbol{H},K} + \|\boldsymbol{v}\|_{\boldsymbol{V},K} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{W},K}.$$

Proposition 3.3 (Estimate for $\boldsymbol{U}(K)$)**.** *For all $\boldsymbol{v} \in \boldsymbol{U}(K)$ we have*

$$\|\boldsymbol{v}\|_{\boldsymbol{H},K} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{U},K}.$$

We now show that these estimates, in combination with Theorem 3.2, yield the desired result of this chapter stated in eq. (3.4).

Theorem 3.3 (Main estimate)**.** *The inequality in eq. (3.4) holds. That is to say,*

$$\|\boldsymbol{v}\|_{\boldsymbol{H},K} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{V},K} \quad \forall \boldsymbol{v} \in \boldsymbol{V}(K). \tag{3.26}$$

Proof. Let $\boldsymbol{v} \in \boldsymbol{V}(K)$. By Theorem 3.2 we can write

$$\boldsymbol{v} = \mathbf{\Pi}_{\boldsymbol{M},K}\boldsymbol{v} + \mathbf{\Pi}_{\boldsymbol{W},K}\boldsymbol{v}_1 + \mathbf{\Pi}_{\boldsymbol{U},K}\boldsymbol{v}_2,$$

where $\boldsymbol{v}_1 := \boldsymbol{v} - \mathbf{\Pi}_{\boldsymbol{M},K}\boldsymbol{v}$ and $\boldsymbol{v}_2 := \boldsymbol{v}_1 - \mathbf{\Pi}_{\boldsymbol{W},K}\boldsymbol{v}_1$. Hence by the triangle inequality

$$\|\boldsymbol{v}\|_{\boldsymbol{H},K} \le \left\|\mathbf{\Pi}_{\boldsymbol{M},K}\boldsymbol{v}\right\|_{\boldsymbol{H},K} + \left\|\mathbf{\Pi}_{\boldsymbol{W},K}\boldsymbol{v}_1\right\|_{\boldsymbol{H},K} + \left\|\mathbf{\Pi}_{\boldsymbol{U},K}\boldsymbol{v}_2\right\|_{\boldsymbol{H},K}. \tag{3.27}$$

Now by Proposition 3.1 and Lemma 3.7 we have

$$\left\|\mathbf{\Pi}_{\boldsymbol{M},K}\boldsymbol{v}\right\|_{\boldsymbol{H},K} + \left\|\mathbf{\Pi}_{\boldsymbol{M},K}\boldsymbol{v}\right\|_{\boldsymbol{V},K} \lesssim \left\|\mathbf{\Pi}_{\boldsymbol{M},K}\boldsymbol{v}\right\|_{\boldsymbol{M},K} = \|\boldsymbol{v}\|_{\boldsymbol{M},K} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{V},K}. \tag{3.28}$$

Similarly we have by Proposition 3.2 and Lemma 3.7 that

$$\left\|\mathbf{\Pi}_{\boldsymbol{W},K}\boldsymbol{v}_1\right\|_{\boldsymbol{H},K} + \left\|\mathbf{\Pi}_{\boldsymbol{W},K}\boldsymbol{v}_1\right\|_{\boldsymbol{V},K} \lesssim \left\|\mathbf{\Pi}_{\boldsymbol{W},K}\boldsymbol{v}_1\right\|_{\boldsymbol{W},K} = \|\boldsymbol{v}_1\|_{\boldsymbol{W},K} \le \|\boldsymbol{v}_1\|_{\boldsymbol{V},K}. \tag{3.29}$$

But using the triangle inequality and eq. (3.28) we see that

$$\|\boldsymbol{v}_1\|_{\boldsymbol{V},K} \leq \|\boldsymbol{v}\|_{\boldsymbol{V},K} + \left\|\boldsymbol{\Pi}_{\boldsymbol{M},K}\boldsymbol{v}\right\|_{\boldsymbol{V},K} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{V},K}. \tag{3.30}$$

Therefore combining eq. (3.29) and eq. (3.30) we get

$$\left\|\boldsymbol{\Pi}_{\boldsymbol{W},K}\boldsymbol{v}_1\right\|_{\boldsymbol{H},K} + \left\|\boldsymbol{\Pi}_{\boldsymbol{W},K}\boldsymbol{v}_1\right\|_{\boldsymbol{V},K} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{V},K}. \tag{3.31}$$

Next, by Proposition 3.3 and Lemma 3.7 we have

$$\left\|\boldsymbol{\Pi}_{\boldsymbol{U},K}\boldsymbol{v}_2\right\|_{\boldsymbol{H},K} \lesssim \left\|\boldsymbol{\Pi}_{\boldsymbol{U},K}\boldsymbol{v}_2\right\|_{\boldsymbol{U},K} = \|\boldsymbol{v}_2\|_{\boldsymbol{U},K} \leq \|\boldsymbol{v}_2\|_{\boldsymbol{V},K}. \tag{3.32}$$

But by the triangle inequality, eq. (3.30) and eq. (3.31) we have

$$\|\boldsymbol{v}_2\|_{\boldsymbol{V},K} \leq \|\boldsymbol{v}_1\|_{\boldsymbol{V},K} + \left\|\boldsymbol{\Pi}_{\boldsymbol{W},K}\boldsymbol{v}_1\right\|_{\boldsymbol{V},K} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{V},K}. \tag{3.33}$$

Therefore combining eq. (3.32) and eq. (3.33) we get

$$\left\|\boldsymbol{\Pi}_{\boldsymbol{U},K}\boldsymbol{v}_2\right\|_{\boldsymbol{H},K} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{V},K}. \tag{3.34}$$

Finally, using the inequalities in eqs. (3.28), (3.31) and (3.34) to bound the terms on the right-hand side of eq. (3.27), we obtain the desired conclusion stated in eq. (3.26). □

Having proven Theorem 3.3, the remainder of this chapter is dedicated to proving the auxiliary inequalities stated in Propositions 3.1 to 3.3.

3.3.2 The reference tetrahedron

In what follows, it will be helpful to consider the so-called *reference tetrahedron*, which we denote by $\hat{K} \subset \mathbb{R}^3$. $\hat{K}$ is the unique tetrahedron with vertices $(1,0,0), (0,1,0), (0,0,1)$ and $(0,0,0)$. We can always find an affine map $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$, where $\boldsymbol{A} \in \mathbb{R}^{3\times 3}$ and $\boldsymbol{b} \in \mathbb{R}^3$, such that

$$\boldsymbol{F}|_{\hat{K}} : \hat{K} \to K$$

is a bijection. By shape-regularity it follows that (see e.g. [18, Lemma 11.1])

$$\|\boldsymbol{A}\| \sim h_K, \qquad \left\|\boldsymbol{A}^{-1}\right\| \sim h_K^{-1}, \qquad \left|\det(\boldsymbol{A})\right| \sim h_K^3, \tag{3.35}$$

where $\|\boldsymbol{A}\|$ is the operator norm of $\boldsymbol{A}$ induced by the Euclidean norm and likewise for $\left\|\boldsymbol{A}^{-1}\right\|$. The vertices of $\hat{K}$ are denoted by $\mathcal{V}_{\hat{K}} := \{\hat{x}_i\}_{i=1}^4$ and are labeled such that $x_i =$

$\boldsymbol{F}(\hat{x}_i)$ for all $x_i \in \mathcal{V}_K$. The faces of $\hat{K}$ are denoted by $\mathcal{F}_{\hat{K}} := \{\hat{F}_i\}_{i=1}^4$ while the edges of $\hat{K}$ are denoted by $\mathcal{E}_{\hat{K}} := \{\hat{e}_{i,j}\}_{1\le i<j\le 4}$. Hence $F_i = \boldsymbol{F}(\hat{F}_i)$ and $e_{i,j} = \boldsymbol{F}(\hat{e}_{i,j})$ for all $F_i \in \mathcal{F}_K$ and $e_{i,j} \in \mathcal{E}_K$.

We use hat notation for the barycentric coordinates $\{\hat{\lambda}_i\}_{i=1}^4 \subset \mathcal{P}_1(\hat{K})$ on $\hat{K}$, and likewise for the bubble functions $\hat{b}_{\hat{K}}, \hat{b}_i, \hat{b}_{i,j}, \hat{B}_i$ and $\hat{\boldsymbol{s}}_{i,j}$ on $\hat{K}$ (recall Section 3.1.1). Note that

$$\lambda_i = \hat{\lambda}_i \circ \boldsymbol{F}^{-1}, \tag{3.36}$$

and in particular this implies the relationships

$$\begin{aligned} &b_K = \hat{b}_{\hat{K}} \circ \boldsymbol{F}^{-1}, \quad b_i = \hat{b}_i \circ \boldsymbol{F}^{-1}, \quad b_{i,j} = \hat{b}_{i,j} \circ \boldsymbol{F}^{-1}, \\ &B_i = \hat{B}_i \circ \boldsymbol{F}^{-1}, \quad \boldsymbol{s}_{i,j} = \boldsymbol{A}^{-T}(\hat{\boldsymbol{s}}_{i,j} \circ \boldsymbol{F}^{-1}). \end{aligned} \tag{3.37}$$

Finally, the (contravariant) *Piola transform* of a vector field $\hat{\boldsymbol{v}} : \hat{K} \to \mathbb{R}^3$ is defined as (see e.g. [6, Section 2.1.3] or [18, Section 9.2]):

$$\boldsymbol{\mathcal{G}}(\hat{\boldsymbol{v}}) : K \to \mathbb{R}^3, \qquad \boldsymbol{\mathcal{G}}(\hat{\boldsymbol{v}}) := \frac{\boldsymbol{A}}{\det(\boldsymbol{A})}(\hat{\boldsymbol{v}} \circ \boldsymbol{F}^{-1}). \tag{3.38}$$

Note that $\boldsymbol{\mathcal{G}} : \{\hat{\boldsymbol{v}} : \hat{K} \to \mathbb{R}^3\} \to \{\boldsymbol{v} : K \to \mathbb{R}^3\}$ is a linear bijection, so we can also speak of the *inverse Piola transform* $\boldsymbol{\mathcal{G}}^{-1}$.

3.3.3 The proof of Proposition 3.1

We break down the proof of Proposition 3.1 into several lemmas. The idea is to transform via $\boldsymbol{\mathcal{G}}^{-1}$ onto the reference tetrahedron, utilize that Proposition 3.1 trivially holds on the reference tetrahedron by finite-dimensionality, and then transform back to the physical tetrahedron.

Lemma 3.8 (Piola transform of $\boldsymbol{M}(K)$)**.** *If $\boldsymbol{v} \in \boldsymbol{M}(K)$ and $\hat{\boldsymbol{v}} = \boldsymbol{\mathcal{G}}^{-1}(\boldsymbol{v})$ then $\hat{\boldsymbol{v}} \in \boldsymbol{M}(\hat{K})$.*

Proof. We first claim that, for all $m \ge 2$, there holds (recall eq. (3.6))

$$\boldsymbol{\mathcal{G}}^{-1}(\boldsymbol{Q}_m(K)) \subset \boldsymbol{Q}_m(\hat{K}). \tag{3.39}$$

Indeed, for any $\hat{\boldsymbol{\rho}} \in \boldsymbol{N}_{m-1}(\hat{K})$ (recall eq. (3.5)), one can verify that its so-called *covariant Piola transform* $\boldsymbol{\mathcal{H}}(\hat{\boldsymbol{\rho}}) := \boldsymbol{A}^{-T}(\hat{\boldsymbol{\rho}} \circ \boldsymbol{F}^{-1})$ satisfies $\boldsymbol{\mathcal{H}}(\hat{\boldsymbol{\rho}}) \in \boldsymbol{N}_{m-1}(K)$. Using this fact, along with the properties stated in [6, Lemma 2.1.6] and [6, Lemma 2.1.9], one then readily verifies that eq. (3.39) holds. To conclude, note that $\boldsymbol{\mathcal{G}}^{-1}(\boldsymbol{\mathcal{P}}_1(K)) \subset \boldsymbol{\mathcal{P}}_1(\hat{K})$. Recalling the definition of $\boldsymbol{M}(K)$ (see eq. (3.7)), we consequently obtain

$$\begin{aligned}\boldsymbol{\mathcal{G}}^{-1}(\boldsymbol{M}(K)) &= \boldsymbol{\mathcal{G}}^{-1}(\boldsymbol{\mathcal{P}}_1(K)) + \boldsymbol{\mathcal{G}}^{-1}(\boldsymbol{Q}_2(K)) + \boldsymbol{\mathcal{G}}^{-1}(\boldsymbol{Q}_3(K)) \\ &\subset \boldsymbol{\mathcal{P}}_1(\hat{K}) + \boldsymbol{Q}_2(\hat{K}) + \boldsymbol{Q}_3(\hat{K}) \\ &= \boldsymbol{M}(\hat{K}).\end{aligned}$$

This is the desired result. □

Lemma 3.9 (Norm estimate from K to $\hat{K}$)**.** *If $\boldsymbol{v} \in \boldsymbol{M}(K)$ and $\hat{\boldsymbol{v}} = \boldsymbol{\mathcal{G}}^{-1}(\boldsymbol{v})$ then*

$$\|\boldsymbol{v}\|_{\boldsymbol{H},K} + \|\boldsymbol{v}\|_{\boldsymbol{V},K} \lesssim h_K^{-3/2}\Big[\|\hat{\boldsymbol{v}}\|_{\boldsymbol{H},\hat{K}} + \|\hat{\boldsymbol{v}}\|_{\boldsymbol{V},\hat{K}}\Big].$$

Proof. First, using [6, Lemma 2.1.8] and eq. (3.35), we straightforwardly obtain

$$\|\boldsymbol{v}\|_{\boldsymbol{H},K} \lesssim h_K^{-3/2}\|\hat{\boldsymbol{v}}\|_{\boldsymbol{H},\hat{K}}\,.$$

To bound $\|\boldsymbol{v}\|_{\boldsymbol{V},K}$ let us write (recall eq. (3.23))

$$\begin{aligned}\|\boldsymbol{v}\|^2_{\boldsymbol{V},K} = &\underbrace{h_K\sum_{i=1}^{4}\big|\boldsymbol{v}(x_i)\big|^2}_{I_1} + \underbrace{\sum_{1\le i<j\le 4}\sup_{\substack{\boldsymbol{s}\in\boldsymbol{\mathcal{P}}_1(e_{i,j})\\ \|\boldsymbol{s}\|_{e_{i,j}}=1}}\big|\langle\boldsymbol{v},\boldsymbol{s}\rangle_{e_{i,j}}\big|^2}_{I_2} \\ &+ \underbrace{h_K^{-1}\sum_{i=1}^{4}\sup_{\substack{\boldsymbol{\kappa}\in\boldsymbol{\mathcal{P}}_0(F_i)\\ \|\boldsymbol{\kappa}\|_{F_i}=1}}\big|\langle\boldsymbol{v},\boldsymbol{\kappa}\rangle_{F_i}\big|^2}_{I_3}.\end{aligned} \tag{3.40}$$

We first bound I_1. Using eq. (3.35) we find

$$I_1 = h_K\sum_{i=1}^{4}\left|\frac{\boldsymbol{A}}{\det(\boldsymbol{A})}\hat{\boldsymbol{v}}(\hat{x}_i)\right|^2 \lesssim h_K^{-3}\sum_{i=1}^{4}\big|\hat{\boldsymbol{v}}(\hat{x}_i)\big|^2 \le h_K^{-3}\|\hat{\boldsymbol{v}}\|^2_{\boldsymbol{V},\hat{K}}\,.$$

Next we bound I_2. Fix $1 \le i < j \le 4$ and $\boldsymbol{s} \in \boldsymbol{\mathcal{P}}_1(e_{i,j})$ with $\|\boldsymbol{s}\|_{e_{i,j}} = 1$, and set $\hat{\boldsymbol{s}} := (\det(\boldsymbol{A})^{-1}\boldsymbol{A}^T(\boldsymbol{s} \circ \boldsymbol{F})) \in \boldsymbol{\mathcal{P}}_1(\hat{e}_{i,j})$. Using eq. (3.35), along with the fact that $\int_{e_{i,j}} 1\,\mathrm{d}l \sim h_K$

by shape-regularity, we find that

$$\begin{aligned}\left|\langle \boldsymbol{v}, \boldsymbol{s}\rangle_{e_{i,j}}\right| &\lesssim h_K \left|\langle \hat{\boldsymbol{v}}, \hat{\boldsymbol{s}}\rangle_{\hat{e}_{i,j}}\right| \\ &\le h_K \|\hat{\boldsymbol{v}}\|_{\boldsymbol{V},\hat{K}} \|\hat{\boldsymbol{s}}\|_{\hat{e}_{i,j}} \\ &\lesssim h_K^{1/2} \|\hat{\boldsymbol{v}}\|_{\boldsymbol{V},\hat{K}} \left\|\det(\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{s}\right\|_{e_{i,j}} \\ &\lesssim h_K^{-3/2} \|\hat{\boldsymbol{v}}\|_{\boldsymbol{V},\hat{K}} \|\boldsymbol{s}\|_{e_{i,j}} \\ &= h_K^{-3/2} \|\hat{\boldsymbol{v}}\|_{\boldsymbol{V},\hat{K}} \,.\end{aligned}$$

Since $\boldsymbol{s} \in \boldsymbol{\mathcal{P}}_1(e_{i,j})$ with $\|\boldsymbol{s}\|_{e_{i,j}} = 1$ was arbitrary, it then follows that $I_2 \lesssim h_K^{-3}\|\hat{\boldsymbol{v}}\|^2_{\boldsymbol{V},\hat{K}}$. Finally, we bound I_3. Fix $1 \le i \le 4$ and $\boldsymbol{\kappa} \in \boldsymbol{\mathcal{P}}_0(F_i)$ with $\|\boldsymbol{\kappa}\|_{F_i} = 1$, and set $\hat{\boldsymbol{\kappa}} := (\det(\boldsymbol{A})^{-1}\boldsymbol{A}^T(\boldsymbol{\kappa} \circ \boldsymbol{F})) \in \boldsymbol{\mathcal{P}}_0(\hat{F}_i)$. Using eq. (3.35), along with the fact that $\int_{F_i} 1 \, \mathrm{d}s \sim h_K^2$ by shape-regularity, we find that

$$\begin{aligned}\left|\langle \boldsymbol{v}, \boldsymbol{\kappa}\rangle_{F_i}\right| &\lesssim h_K^2 \left|\langle \hat{\boldsymbol{v}}, \hat{\boldsymbol{\kappa}}\rangle_{\hat{F}_i}\right| \\ &\le h_K^2 \|\hat{\boldsymbol{v}}\|_{\boldsymbol{V},\hat{K}} \|\hat{\boldsymbol{\kappa}}\|_{\hat{F}_i} \\ &\lesssim h_K \|\hat{\boldsymbol{v}}\|_{\boldsymbol{V},\hat{K}} \left\|\det(\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{\kappa}\right\|_{F_i} \\ &\lesssim h_K^{-1} \|\hat{\boldsymbol{v}}\|_{\boldsymbol{V},\hat{K}} \|\boldsymbol{\kappa}\|_{F_i} \\ &= h_K^{-1} \|\hat{\boldsymbol{v}}\|_{\boldsymbol{V},\hat{K}} \,.\end{aligned}$$

Since $\boldsymbol{\kappa} \in \boldsymbol{\mathcal{P}}_0(F_i)$ with $\|\boldsymbol{\kappa}\|_{F_i} = 1$ was arbitrary, it then follows that $I_3 \lesssim h_K^{-3}\|\hat{\boldsymbol{v}}\|^2_{\boldsymbol{V},\hat{K}}$. Using these bounds for I_1, I_2 and I_3 in eq. (3.40), we obtain

$$\|\boldsymbol{v}\|^2_{\boldsymbol{V},K} \lesssim h_K^{-3}\|\hat{\boldsymbol{v}}\|^2_{\boldsymbol{V},\hat{K}} \,.$$

This completes the proof. □

Lemma 3.10 (Norm estimate for $M(\hat{K})$)**.** *For all $\hat{\boldsymbol{v}} \in \boldsymbol{M}(\hat{K})$ we have*

$$\|\hat{\boldsymbol{v}}\|_{\boldsymbol{H},\hat{K}} + \|\hat{\boldsymbol{v}}\|_{\boldsymbol{V},\hat{K}} \lesssim \|\hat{\boldsymbol{v}}\|_{\boldsymbol{M},\hat{K}} \,.$$

Proof. This is a consequence of equivalence of norms on finite dimensional spaces. □

Lemma 3.11 (Norm estimate from $\hat{K}$ to K)**.** *If $\boldsymbol{v} \in \boldsymbol{M}(K)$ and $\hat{\boldsymbol{v}} = \boldsymbol{\mathcal{G}}^{-1}(\boldsymbol{v})$ then*

$$\|\hat{\boldsymbol{v}}\|_{\boldsymbol{M},\hat{K}} \lesssim h_K^{3/2}\|\boldsymbol{v}\|_{\boldsymbol{M},K} \,.$$

Proof. Let us write (recall eq. (3.24a))

$$\begin{aligned}\|\hat{\boldsymbol{v}}\|^2_{\boldsymbol{M},\hat{K}} = \underbrace{\sum_{i=1}^{4}\left|\hat{\boldsymbol{v}}(\hat{x}_i)\right|^2}_{I_1} + \underbrace{\sum_{\substack{1\le i<j\le 4\\ k\in\{i,j\}}}\sup_{\substack{\hat{s}\in\mathcal{P}_1(\hat{e}_{i,j})\\ \|\hat{s}\|_{\hat{e}_{i,j}}=1}}\left|\langle\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{n}}_k,\hat{s}\rangle_{\hat{e}_{i,j}}\right|^2}_{I_2} \\ + \underbrace{\sum_{i=1}^{4}\sup_{\substack{\hat{\kappa}\in\mathcal{P}_0(\hat{F}_i)\\ \|\hat{\kappa}\|_{\hat{F}_i}=1}}\left|\langle\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{n}}_i,\hat{\kappa}\rangle_{\hat{F}_i}\right|^2}_{I_3}.\end{aligned} \tag{3.41}$$

We first bound I_1. Using eq. (3.35) we have

$$I_1 = \sum_{i=1}^{4}\left|\det(\boldsymbol{A})\boldsymbol{A}^{-1}\boldsymbol{v}(x_i)\right|^2 \lesssim h_K^4\sum_{i=1}^{4}\left|\boldsymbol{v}(x_i)\right|^2 \le h_K^3\|\boldsymbol{v}\|^2_{\boldsymbol{M},K}.$$

Next we bound I_2. Fix $1\le i<j\le 4$ and $k\in\{i,j\}$. Note that by [6, eq. (2.1.94)] we have $\boldsymbol{n}_k = \alpha_k(\boldsymbol{A}^{-T}\hat{\boldsymbol{n}}_k)$ where $\alpha_k = \left|\boldsymbol{A}^{-T}\hat{\boldsymbol{n}}_k\right|^{-1}\sim h_K$. Now let $\hat{s}\in\mathcal{P}_1(\hat{e}_{i,j})$ with $\|\hat{s}\|_{\hat{e}_{i,j}}=1$ and set $s := (\hat{s}\circ\boldsymbol{F}^{-1})\in\mathcal{P}_1(e_{i,j})$. Using eq. (3.35), along with the fact that $\int_{e_{i,j}} 1\,\mathrm{d}l \sim h_K$ by shape-regularity, we find

$$\begin{aligned}\left|\langle\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{n}}_k,\hat{s}\rangle_{\hat{e}_{i,j}}\right| &\lesssim h_K^{-1}\left|\langle(\hat{\boldsymbol{v}}\circ\boldsymbol{F}^{-1})\cdot\hat{\boldsymbol{n}}_k,s\rangle_{e_{i,j}}\right| \\ &= h_K^{-1}\alpha_k^{-1}|\det\boldsymbol{A}|\left|\langle\boldsymbol{v}\cdot\boldsymbol{n}_k,s\rangle_{e_{i,j}}\right| \\ &\lesssim h_K\left|\langle\boldsymbol{v}\cdot\boldsymbol{n}_k,s\rangle_{e_{i,j}}\right| \\ &\le h_K\|\boldsymbol{v}\|_{\boldsymbol{M},K}\|s\|_{e_{i,j}} \\ &\lesssim h_K^{3/2}\|\boldsymbol{v}\|_{\boldsymbol{M},K}\|\hat{s}\|_{\hat{e}_{i,j}} \\ &= h_K^{3/2}\|\boldsymbol{v}\|_{\boldsymbol{M},K}.\end{aligned}$$

By arbitrariness of $\hat{s}$ it follows that $I_2\lesssim h_K^3\|\boldsymbol{v}\|^2_{\boldsymbol{M},K}$. Finally, we bound I_3. Fix $1\le i\le 4$ and $\hat{\kappa}\in\mathcal{P}_0(\hat{F}_i)$ with $\|\hat{\kappa}\|_{\hat{F}_i}=1$, and set $\kappa := (\hat{\kappa}\circ\boldsymbol{F}^{-1})\in\mathcal{P}_0(F_i)$. Using [6, Lemma 2.1.6], and $\int_{F_i}1\,\mathrm{d}s\sim h_K^2$ by shape-regularity, we find

$$\begin{aligned}\left|\langle\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{n}}_i,\hat{\kappa}\rangle_{\hat{F}_i}\right| &= \left|\langle\boldsymbol{v}\cdot\boldsymbol{n}_i,\kappa\rangle_{F_i}\right| \\ &\le h_K^{1/2}\|\boldsymbol{v}\|_{\boldsymbol{M},K}\|\kappa\|_{F_i} \\ &\lesssim h_K^{3/2}\|\boldsymbol{v}\|_{\boldsymbol{M},K}\|\hat{\kappa}\|_{\hat{F}_i} \\ &= h_K^{3/2}\|\boldsymbol{v}\|_{\boldsymbol{M},K}.\end{aligned}$$

By arbitrariness of $\hat{\kappa}$ it follows that $I_3 \lesssim h_K^3 \|\boldsymbol{v}\|_{\boldsymbol{M},K}^2$. Using these bounds for I_1, I_2 and I_3 in eq. (3.41) we obtain $\|\hat{\boldsymbol{v}}\|_{\boldsymbol{M},\hat{K}}^2 \lesssim h_K^3 \|\boldsymbol{v}\|_{\boldsymbol{M},K}^2$. This completes the proof. □

Proposition 3.1 now follows from the above lemmas. Indeed, let $\boldsymbol{v} \in \boldsymbol{M}(K)$ and set $\hat{\boldsymbol{v}} = \boldsymbol{\mathcal{G}}^{-1}(\boldsymbol{v})$. By Lemma 3.8 we have $\hat{\boldsymbol{v}} \in \boldsymbol{M}(\hat{K})$. Hence using Lemmas 3.9 to 3.11 we find

$$\|\boldsymbol{v}\|_{\boldsymbol{H},K} + \|\boldsymbol{v}\|_{\boldsymbol{V},K} \lesssim h_K^{-3/2} \Big[\|\hat{\boldsymbol{v}}\|_{\boldsymbol{H},\hat{K}} + \|\hat{\boldsymbol{v}}\|_{\boldsymbol{V},\hat{K}} \Big] \lesssim h_K^{-3/2} \|\hat{\boldsymbol{v}}\|_{\boldsymbol{M},\hat{K}} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{M},K} .$$

This completes the proof of Proposition 3.1. □

3.3.4 The proof of Proposition 3.2

The proof of Proposition 3.2 is broken into several lemmas. As in the previous subsection, the idea is to establish the result by transforming onto the reference tetrahedron.

Lemma 3.12 (Piola transform of $\boldsymbol{W}(K)$)**.** *If $\boldsymbol{v} \in \boldsymbol{W}(K)$ and $\hat{\boldsymbol{v}} = \boldsymbol{\mathcal{G}}^{-1}(\boldsymbol{v})$ then $\hat{\boldsymbol{v}} \in \boldsymbol{W}(\hat{K})$.*

Proof. Let $\boldsymbol{v} \in \boldsymbol{W}(K)$, so that by definition (recall eq. (3.9)) we can write

$$\boldsymbol{v} = \sum_{1 \le i < j \le 4} \mathbf{curl}(p_{i,j} \boldsymbol{s}_{i,j}),$$

where $p_{i,j} \in M^{(i,j)}(K)$. Using [6, eq. (2.1.92)] (see also [18, Lemma 9.6]), and eq. (3.37), we find that

$$\begin{aligned} \boldsymbol{\mathcal{G}}^{-1}(\boldsymbol{v}) &= \sum_{1 \le i < j \le 4} \boldsymbol{\mathcal{G}}^{-1}(\mathbf{curl}(p_{i,j} \boldsymbol{s}_{i,j})) \\ &= \sum_{1 \le i < j \le 4} \mathbf{curl}(\boldsymbol{A}^T (p_{i,j} \boldsymbol{s}_{i,j}) \circ \boldsymbol{F}) \\ &= \sum_{1 \le i < j \le 4} \mathbf{curl}((p_{i,j} \circ \boldsymbol{F}) \hat{\boldsymbol{s}}_{i,j}). \end{aligned}$$

Finally, because $p_{i,j} \in M^{(i,j)}(K)$ (recall that $M^{(i,j)}(K)$ is defined above eq. (3.9)), it follows from eq. (3.36) that $(p_{i,j} \circ \boldsymbol{F}) \in M^{(i,j)}(\hat{K})$. Therefore from the definition of $\boldsymbol{W}(\hat{K})$ we see that $\boldsymbol{\mathcal{G}}^{-1}(\boldsymbol{v}) \in \boldsymbol{W}(\hat{K})$. □

Lemma 3.13 (Norm estimate from K to $\hat{K}$)**.** *If $\boldsymbol{v} \in \boldsymbol{W}(K)$ and $\hat{\boldsymbol{v}} = \boldsymbol{\mathcal{G}}^{-1}(\boldsymbol{v})$ then*

$$\|\boldsymbol{v}\|_{\boldsymbol{H},K} + \|\boldsymbol{v}\|_{\boldsymbol{V},K} \lesssim h_K^{-3/2} \Big[\|\hat{\boldsymbol{v}}\|_{\boldsymbol{H},\hat{K}} + \|\hat{\boldsymbol{v}}\|_{\boldsymbol{V},\hat{K}} \Big].$$

Proof. The proof is identical to the proof of Lemma 3.9. □

Lemma 3.14 (Norm estimate for $W(\hat{K})$)**.** *For all* $\hat{\boldsymbol{v}} \in \boldsymbol{W}(\hat{K})$ *we have*

$$\|\hat{\boldsymbol{v}}\|_{\boldsymbol{H},\hat{K}} + \|\hat{\boldsymbol{v}}\|_{\boldsymbol{V},\hat{K}} \lesssim \|\hat{\boldsymbol{v}}\|_{\boldsymbol{W},\hat{K}}\,.$$

Proof. This is a consequence of equivalence of norms on finite dimensional spaces. □

Lemma 3.15 (Norm estimate from $\hat{K}$ to K)**.** *If* $\boldsymbol{v} \in \boldsymbol{W}(K)$ *and* $\hat{\boldsymbol{v}} = \boldsymbol{\mathcal{G}}^{-1}(\boldsymbol{v})$ *then*

$$\|\hat{\boldsymbol{v}}\|_{\boldsymbol{W},\hat{K}} \lesssim h_K^{3/2} \|\boldsymbol{v}\|_{\boldsymbol{W},K}\,.$$

Proof. Let $\boldsymbol{v} \in \boldsymbol{W}(K)$ and set $\hat{\boldsymbol{v}} = \boldsymbol{\mathcal{G}}^{-1}(\boldsymbol{v}) \in \boldsymbol{W}(\hat{K})$. By [25, Lemma 3.3] it holds that

$$(\boldsymbol{v}\cdot\boldsymbol{n})|_{\partial K} = 0, \qquad (\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{n}})|_{\partial\hat{K}} = 0.$$

This, in conjunction with the definition of $\|\cdot\|_{\boldsymbol{W},K}$ (recall eq. (3.24b)), implies that

$$\|\hat{\boldsymbol{v}}\|^2_{\boldsymbol{W},\hat{K}} = \sum_{1\le i<j\le 4} \sup_{\substack{\hat{\boldsymbol{s}}\in\boldsymbol{\mathcal{P}}_1(\hat{e}_{i,j})\\ \|\hat{\boldsymbol{s}}\|_{\hat{e}_{i,j}}=1}} \left|\langle\hat{\boldsymbol{v}},\hat{\boldsymbol{s}}\rangle_{\hat{e}_{i,j}}\right|^2, \tag{3.42a}$$

$$\|\boldsymbol{v}\|^2_{\boldsymbol{W},K} = \sum_{1\le i<j\le 4} \sup_{\substack{\boldsymbol{s}\in\boldsymbol{\mathcal{P}}_1(e_{i,j})\\ \|\boldsymbol{s}\|_{e_{i,j}}=1}} \left|\langle\boldsymbol{v},\boldsymbol{s}\rangle_{e_{i,j}}\right|^2. \tag{3.42b}$$

Fix $1 \le i < j \le 4$ and $\hat{\boldsymbol{s}} \in \boldsymbol{\mathcal{P}}_1(\hat{e}_{i,j})$ with $\|\hat{\boldsymbol{s}}\|_{\hat{e}_{i,j}} = 1$, and set $\boldsymbol{s} := (\det(\boldsymbol{A})\boldsymbol{A}^{-T}(\hat{\boldsymbol{s}}\circ\boldsymbol{F}^{-1})) \in \boldsymbol{\mathcal{P}}_1(e_{i,j})$. Using eq. (3.35), eq. (3.42b), and the fact that $\int_{e_{i,j}} 1\,\mathrm{d}l \sim h_K$ by shape-regularity, we find that

$$\begin{aligned}
\left|\langle\hat{\boldsymbol{v}},\hat{\boldsymbol{s}}\rangle_{\hat{e}_{i,j}}\right| &\lesssim h_K^{-1}\left|\langle\boldsymbol{v},\boldsymbol{s}\rangle_{e_{i,j}}\right| \\
&\le h_K^{-1}\|\boldsymbol{v}\|_{\boldsymbol{W},K}\|\boldsymbol{s}\|_{e_{i,j}} \\
&\lesssim h_K^{-1/2}\|\boldsymbol{v}\|_{\boldsymbol{W},K}\left\|\det(\boldsymbol{A})\boldsymbol{A}^{-T}\hat{\boldsymbol{s}}\right\|_{\hat{e}_{i,j}} \\
&\lesssim h_K^{3/2}\|\boldsymbol{v}\|_{\boldsymbol{W},K}\|\hat{\boldsymbol{s}}\|_{\hat{e}_{i,j}} \\
&= h_K^{3/2}\|\boldsymbol{v}\|_{\boldsymbol{W},K}\,.
\end{aligned} \tag{3.43}$$

Finally, combining eq. (3.42a) and eq. (3.43), we obtain $\|\hat{\boldsymbol{v}}\|_{\boldsymbol{W},\hat{K}} \lesssim h_K^{3/2}\|\boldsymbol{v}\|_{\boldsymbol{W},K}$. □

Proposition 3.2 now follows from the above lemmas. Indeed, let $\boldsymbol{v} \in \boldsymbol{W}(K)$ and set $\hat{\boldsymbol{v}} = \boldsymbol{\mathcal{G}}^{-1}(\boldsymbol{v})$. By Lemma 3.12 we have $\hat{\boldsymbol{v}} \in \boldsymbol{W}(\hat{K})$. Hence using Lemmas 3.13 to 3.15 we find

$$\|\boldsymbol{v}\|_{\boldsymbol{H},K} + \|\boldsymbol{v}\|_{\boldsymbol{V},K} \lesssim h_K^{-3/2}\left[\|\hat{\boldsymbol{v}}\|_{\boldsymbol{H},\hat{K}} + \|\hat{\boldsymbol{v}}\|_{\boldsymbol{V},\hat{K}}\right] \lesssim h_K^{-3/2}\|\hat{\boldsymbol{v}}\|_{\boldsymbol{W},\hat{K}} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{W},K}.$$

This completes the proof of Proposition 3.2. □

3.3.5 The proof of Proposition 3.3

The approach we take to prove Proposition 3.3 is different from that of Proposition 3.1 and Proposition 3.2. This is because for $\boldsymbol{v} \in \boldsymbol{U}(K)$, it does not appear to hold (as far as the author can tell) that $\boldsymbol{\mathcal{G}}^{-1}(\boldsymbol{v}) \in \boldsymbol{U}(\hat{K})$. We therefore prove Proposition 3.3 without transforming onto the reference tetrahedron.

Let $\boldsymbol{v} \in \boldsymbol{U}(K)$. By the definition of $\boldsymbol{U}(K)$ (see eq. (3.13)) and the product rule, we can write

$$\boldsymbol{v} = \sum_{i=1}^{4} (\nabla B_i) \times (\boldsymbol{p}_i \times \boldsymbol{n}_i),$$

where $\boldsymbol{p}_i \in \boldsymbol{\mathcal{P}}_0(K)$ and with no loss of generality $\boldsymbol{p}_i \cdot \boldsymbol{n}_i = 0$. By the triangle inequality,

$$\|\boldsymbol{v}\|_{\boldsymbol{H},K} \leq \sum_{i=1}^{4} \left\|(\nabla B_i) \times (\boldsymbol{p}_i \times \boldsymbol{n}_i)\right\|_{\boldsymbol{H},K}.$$

Using the definition of $\|\cdot\|_{\boldsymbol{H},K}$ (see eq. (3.22)), one verifies that for each $1 \leq i \leq 4$ we have

$$\left\|(\nabla B_i) \times (\boldsymbol{p}_i \times \boldsymbol{n}_i)\right\|_{\boldsymbol{H},K} \lesssim \left[h_K^{-1}|B_i|_{H^1(K)} + |B_i|_{H^2(K)}\right]|\boldsymbol{p}_i \times \boldsymbol{n}_i|.$$

Moreover, utilizing eq. (3.37), [6, Remark 2.1.8] and eq. (3.35), we can estimate

$$|B_i|_{H^1(K)} = \left|\hat{B}_i \circ \boldsymbol{F}^{-1}\right|_{H^1(K)} \lesssim h_K^{1/2}\left|\hat{B}_i\right|_{H^1(\hat{K})} \lesssim h_K^{1/2}.$$

Similar reasoning shows that $|B_i|_{H^2(K)} \lesssim h_K^{-1/2}$. We therefore obtain

$$\|\boldsymbol{v}\|_{\boldsymbol{H},K} \lesssim h_K^{-1/2} \sum_{i=1}^{4} |\boldsymbol{p}_i \times \boldsymbol{n}_i|. \tag{3.44}$$

Fix $1 \le i \le 4$; it remains to estimate $|\boldsymbol{p}_i \times \boldsymbol{n}_i|$. By shape-regularity $\int_{F_i} 1 \,\mathrm{d}s \sim h_K^2$, so that

$$|\boldsymbol{p}_i \times \boldsymbol{n}_i| \sim h_K^{-1} \|\boldsymbol{p}_i \times \boldsymbol{n}_i\|_{F_i} . \tag{3.45}$$

On the other hand, using Lemma 3.1 and the vector triple product formula, we see that

$$\begin{aligned}(\boldsymbol{v} \times \boldsymbol{n}_i)|_{F_i} &= ((\nabla B_i)|_{F_i} \times (\boldsymbol{p}_i \times \boldsymbol{n}_i)) \times \boldsymbol{n}_i \\ &= (\frac{\partial B_i}{\partial \boldsymbol{n}_i}|_{F_i})(\boldsymbol{p}_i \times \boldsymbol{n}_i) \\ &= a_i b_i|_{F_i}(\boldsymbol{p}_i \times \boldsymbol{n}_i).\end{aligned} \tag{3.46}$$

Here we recall that $a_i = -|\nabla \lambda_i|$, which by eq. (3.36) and eq. (3.35) satisfies

$$|a_i| = \left|\nabla(\hat{\lambda}_i \circ \boldsymbol{F}^{-1})\right| = \left|\boldsymbol{A}^{-T} \nabla \hat{\lambda}_i\right| \sim h_K^{-1}. \tag{3.47}$$

Let us also mention that shape-regularity implies

$$\left\|b_i^{1/2}(\boldsymbol{p}_i \times \boldsymbol{n}_i)\right\|_{F_i} \sim \|\boldsymbol{p}_i \times \boldsymbol{n}_i\|_{F_i} . \tag{3.48}$$

Indeed, eq. (3.48) is straightforwardly proven by a scaling argument where one transforms onto the reference face $\hat{F}_i$. Combining now eqs. (3.46) to (3.48), we find

$$\begin{aligned}\|\boldsymbol{p}_i \times \boldsymbol{n}_i\|_{F_i}^2 &\sim \langle b_i(\boldsymbol{p}_i \times \boldsymbol{n}_i), \boldsymbol{p}_i \times \boldsymbol{n}_i\rangle_{F_i} \\ &\sim h_K \langle -a_i b_i(\boldsymbol{p}_i \times \boldsymbol{n}_i), \boldsymbol{p}_i \times \boldsymbol{n}_i\rangle_{F_i} \\ &= h_K \langle \boldsymbol{v} \times \boldsymbol{n}_i, -\boldsymbol{p}_i \times \boldsymbol{n}_i\rangle_{F_i}.\end{aligned} \tag{3.49}$$

However, since $\boldsymbol{p}_i \in \boldsymbol{\mathcal{P}}_0(F_i)$, we see from the definition of $\|\cdot\|_{\boldsymbol{U},K}$ (see eq. (3.24c)) that

$$\langle \boldsymbol{v} \times \boldsymbol{n}_i, -\boldsymbol{p}_i \times \boldsymbol{n}_i\rangle_{F_i} \le h_K^{1/2} \|\boldsymbol{p}_i\|_{F_i} \|\boldsymbol{v}\|_{\boldsymbol{U},K} = h_K^{1/2} \|\boldsymbol{p}_i \times \boldsymbol{n}_i\|_{F_i} \|\boldsymbol{v}\|_{\boldsymbol{U},K} , \tag{3.50}$$

where $\|\boldsymbol{p}_i\|_{F_i} = \|\boldsymbol{p}_i \times \boldsymbol{n}_i\|_{F_i}$ follows from $\boldsymbol{p}_i \cdot \boldsymbol{n}_i = 0$. Combining eq. (3.49) and eq. (3.50) we get

$$\|\boldsymbol{p}_i \times \boldsymbol{n}_i\|_{F_i} \lesssim h_K^{3/2} \|\boldsymbol{v}\|_{\boldsymbol{U},K} . \tag{3.51}$$

Finally, combining eq. (3.44), eq. (3.45) and eq. (3.51) yields that $\|\boldsymbol{v}\|_{\boldsymbol{H},K} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{U},K}$. $\square$

Chapter 4

A new divergence-free discontinuous Galerkin method for the steady Navier–Stokes problem under minimal regularity

In Chapter 2 we derived optimal a-priori error estimates for some lowest-order hybridized DG methods in the context of the Stokes problem. The novelty of our analysis in Chapter 2 is that it is both pressure-robust (i.e. we are able to obtain velocity error estimates that are pressure independent) and is valid when the exact solution has low regularity. This chapter can be viewed as an extension of these results from Chapter 2 to the context of the steady Navier–Stokes problem. The material presented in this chapter is somewhat preliminary, and is part of an ongoing research project that is being undertaken by the author at the time of writing this thesis. Nevertheless, some interesting results are available at the present time, which is why this thesis chapter is included.

The first contribution of this chapter is to introduce a new lowest-order DG method for the steady Navier–Stokes problem. The proposed method is essentially a combination of the lowest-order formulation of the methods considered in [13] and [28]. Following [13] we use a (lowest-order) $\boldsymbol{H}(\mathrm{div})$-conforming finite element space for the velocity, while the pressure space consists of piecewise constant functions. The method consequently produces a discrete velocity solution that is exactly divergence-free. To ensure stability we discretize the viscous and convective terms in the same way as [13]. However, to discretize the source term we use the approach of [28], whereby a suitable divergence-preserving enrichment operator is utilized. The enrichment operator allows for the proposed method to be well-defined for source terms with low-regularity (see Remark 4.1), and the divergence-preserving property of the operator ensures that the method is pressure-robust.

After introducing the new DG method we proceed to analyze it. We first show that the discrete problem admits a unique solution under a smallness condition on the problem data. Building on the ideas developed in Chapter 2 for the Stokes problem, we then turn to the derivation of optimal and pressure-robust a-priori velocity error estimates in a discrete energy norm. We show that it is possible to derive such estimates under low regularity assumptions on the exact solution. In particular, our analysis requires only $\boldsymbol{H}^{1+s}$-regularity of the exact velocity solution for $s \in (0,1]$ in the two-dimensional case and $s \in (1/2,1]$ in the three-dimensional case. To the best of the author's knowledge, this is the first time that pressure-robust velocity a-priori error estimates have been derived for a DG method in the context of the steady Navier–Stokes problem under minimal regularity. We emphasize that error estimates of this nature can be obtained quite easily in the high regularity case of $s \geq 1$, by combining the ideas presented in [13, 26]. The novelty of our analysis is that it is valid in the minimal regularity setting where $s < 1$.

The rest of this chapter is organized as follows. In Section 4.1 we recall the weak formulation of the steady Navier–Stokes problem and discuss some of its basic properties. In Section 4.2 we introduce the proposed DG method and discuss some preliminary results concerning the discretization. In Section 4.3 we study well-posedness of the resultant discrete Navier–Stokes problem, and we show that it admits a unique solution under a discrete smallness condition on the problem data. Finally, in Section 4.4 we derive the pressure-robust velocity error estimates for the proposed method.

4.1 The steady Navier–Stokes problem

Let $\Omega \subset \mathbb{R}^d$ with $d \in \{2,3\}$ be a connected and bounded domain with polyhedral boundary. In this chapter we shall further assume for simplicity that Ω is a Lipschitz domain. The steady incompressible Navier–Stokes problem seeks a velocity field $\boldsymbol{u}$ and kinematic pressure field p such that

$$-\nu\Delta\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{4.1a}$$

$$\nabla\cdot\boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{4.1b}$$

$$\boldsymbol{u} = 0 \qquad \text{on } \partial\Omega, \tag{4.1c}$$

where $\nu > 0$ is a given constant kinematic viscosity and $\boldsymbol{f}$ is a given body force.

We use the standard notation for Lebesgue and Sobolev spaces and their norms. Boldface notation is used for $\mathbb{R}^d$-valued functions and function spaces; hence for example we write $\boldsymbol{H}_0^1(\Omega) := [H_0^1(\Omega)]^d$. To define the weak formulation of eq. (4.1), we introduce the following multilinear forms:

$$a : \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{H}_0^1(\Omega) \to \mathbb{R}, \qquad a(\boldsymbol{u}, \boldsymbol{v}) := (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}), \tag{4.2a}$$

$$b : \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega) \to \mathbb{R}, \qquad b(\boldsymbol{v}, q) := -(\nabla \cdot \boldsymbol{v}, q), \tag{4.2b}$$

$$c : \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{H}_0^1(\Omega) \to \mathbb{R}, \qquad c(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) := \int_\Omega (\boldsymbol{w} \cdot \nabla \boldsymbol{u}) \cdot \boldsymbol{v} \,\mathrm{d}x. \tag{4.2c}$$

Given $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$, the weak formulation of eq. (4.1) seeks $(\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\nu a(\boldsymbol{u}, \boldsymbol{v}) + c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \tag{4.3a}$$

$$b(\boldsymbol{u}, q) = 0 \qquad \forall q \in L_0^2(\Omega). \tag{4.3b}$$

To discuss the well-posedness of eq. (4.3), we mention the existence of constants C_p and C_τ which depend only on Ω, such that (see [14, eq. (6.6)] and [14, Lemma 6.32])

$$C_p \|\boldsymbol{v}\|_1^2 \le a(\boldsymbol{v}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \tag{4.4}$$

$$\left| c(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) \right| \le C_\tau \|\boldsymbol{w}\|_1 \|\boldsymbol{u}\|_1 \|\boldsymbol{v}\|_1 \qquad \forall \boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega). \tag{4.5}$$

It is known that under the smallness condition on the data

$$\frac{\|\boldsymbol{f}\|_{-1}}{\nu^2} < \frac{C_p^2}{C_\tau}, \tag{4.6}$$

the problem in eq. (4.3) admits a unique solution, see [14, Theorem 6.36] (strictly speaking [14, Theorem 6.36] only considers the case of $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, but the proof for the more general case of $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ is identical). Moreover, introducing the reduced space of divergence-free functions

$$\begin{aligned} \boldsymbol{V} &:= \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) : b(\boldsymbol{v}, q) = 0 \ \forall q \in L_0^2(\Omega)\} \\ &= \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) : \nabla \cdot \boldsymbol{v} = 0\}, \end{aligned}$$

the velocity solution $\boldsymbol{u} \in \boldsymbol{V}$ of eq. (4.3) equivalently satisfies the reduced problem

$$\nu a(\boldsymbol{u}, \boldsymbol{v}) + c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \qquad \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{4.7}$$

Finally, a useful feature of the trilinear form c is the skew-symmetric property

$$c(\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{v}) = 0 \qquad \forall \boldsymbol{w} \in \boldsymbol{V},\ \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \tag{4.8}$$

see [14, Lemma 6.33]. If we now take $\boldsymbol{v} = \boldsymbol{u} \in \boldsymbol{V}$ as a test function in eq. (4.7), we see by skew-symmetry that the convective term drops out, i.e. $c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}) = 0$, and we thus obtain

$$\nu a(\boldsymbol{u}, \boldsymbol{u}) = \langle \boldsymbol{f}, \boldsymbol{u} \rangle. \tag{4.9}$$

Combining eq. (4.9) with the coercivity result in eq. (4.4) yields the a-priori velocity energy estimate

$$\|\boldsymbol{u}\|_1 \leq (C_p \nu)^{-1} \|\boldsymbol{f}\|_{-1}, \tag{4.10}$$

which will prove to be useful later on in our error analysis (see Remark 4.4).

4.2 The divergence-free discontinuous Galerkin method

In this section we introduce the new divergence-free DG method for the steady Navier–Stokes problem in eq. (4.3).

4.2.1 Mesh related notation

Let $\mathcal{T} = \{K\}$ be a conforming triangulation of Ω into simplices $\{K\}$. Let $K \in \mathcal{T}$. We set $h_K := \operatorname{diam} K$, and the mesh size is defined as $h := \max_{K \in \mathcal{T}} h_K$. We let $\mathcal{F}_i$ and $\mathcal{F}_b$ denote the interior and boundary faces of $\mathcal{T}$ respectively, and we set $\mathcal{F}_h = \mathcal{F}_i \cup \mathcal{F}_b$. Note that we do not define the collection $\mathcal{F}_h$ of mesh faces using a quotient set, as we did in Section 2.1.2. This is because we are assuming that Ω is a Lipschitz domain, and therefore cracks are not present in the domain, so that the quotient set construction is unnecessary.

For $F \in \mathcal{F}_h$ we let $h_F := \operatorname{diam} F$. On an interior face $F = \partial K_1 \cap \partial K_2$ we let $\boldsymbol{n}_F$ denote the unit normal pointing from K_1 to K_2, while on a boundary face $F \subset \partial K \cap \partial \Omega$ we let $\boldsymbol{n}_F$ denote the outward unit normal to Ω. We shall also consider the jump operator $[\![\cdot]\!]$ and average operator $\{\!\{\cdot\}\!\}$, which are defined in the same way as Section 2.1.2. As always, the ambiguity in the ordering of K_1, K_2 in these definitions will be unimportant.

4.2.2 Discrete spaces and norms

The divergence-free DG method that we propose is based on the following lowest-order finite element spaces on Ω:

$$\boldsymbol{X}_h := \{\boldsymbol{v}_h \in \boldsymbol{H}(\mathrm{div};\Omega) : \boldsymbol{v}_h|_K \in \boldsymbol{\mathcal{P}}_1(K)\ \forall K \in \mathcal{T} \text{ and } \boldsymbol{v}_h \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}, \tag{4.11a}$$

$$Q_h := \{q_h \in L^2_0(\Omega) : q_h|_K \in \mathcal{P}_0(K)\ \forall K \in \mathcal{T}\}, \tag{4.11b}$$

where $\mathcal{P}_k(D)$ is the space of polynomials with degree at most k on D and $\boldsymbol{\mathcal{P}}_k(D) := [\mathcal{P}_k(D)]^d$. In other words, $\boldsymbol{X}_h$ is the lowest-order Brezzi–Douglas–Marini (BDM) space [6], which consists of all piecewise linear vector-valued functions whose normal component is continuous across interior faces and vanishes on boundary faces. Likewise, Q_h is the space of all piecewise constant functions with zero mean.

It will be convenient to introduce the extended velocity space

$$\boldsymbol{X}(h) := \boldsymbol{X}_h + \boldsymbol{H}^1_0(\Omega), \tag{4.12}$$

and we define on this space the discrete $\boldsymbol{H}^1$-norm $\|\cdot\|_{\mathrm{dg}}$ according to

$$\|\boldsymbol{v}\|^2_{\mathrm{dg}} := \|\nabla_h \boldsymbol{v}\|^2 + |\boldsymbol{v}|^2_{\mathrm{J}} \qquad \forall \boldsymbol{v} \in \boldsymbol{X}(h),$$

where ∇_h is the usual broken gradient operator [14, Section 1.2.5], and $|\cdot|_{\mathrm{J}}$ is the following jump semi-norm:

$$|\boldsymbol{v}|^2_{\mathrm{J}} := \sum_{F\in\mathcal{F}_h} \frac{1}{h_F} \left\|[\![\boldsymbol{v}]\!]\right\|^2_F \qquad \forall \boldsymbol{v} \in \boldsymbol{X}(h).$$

4.2.3 Discrete bilinear and convective forms

To discretize the bilinear form a defined in eq. (4.2a), we use the symmetric interior penalty method [1]. That is, we consider the discrete bilinear form $a_h : \boldsymbol{X}_h \times \boldsymbol{X}_h \to \mathbb{R}$ defined by

$$\begin{aligned} a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) := &\int_\Omega \nabla_h \boldsymbol{u}_h : \nabla_h \boldsymbol{v}_h \,\mathrm{d}x - \sum_{F\in\mathcal{F}_h} \int_F (\{\!\{\nabla_h \boldsymbol{u}_h\}\!\} \boldsymbol{n}_F) \cdot [\![\boldsymbol{v}_h]\!] \,\mathrm{d}s \\ &- \sum_{F\in\mathcal{F}_h} \int_F (\{\!\{\nabla_h \boldsymbol{v}_h\}\!\} \boldsymbol{n}_F) \cdot [\![\boldsymbol{u}_h]\!] \,\mathrm{d}s + \sum_{F\in\mathcal{F}_h} \frac{\alpha}{h_F} \int_F [\![\boldsymbol{u}_h]\!] \cdot [\![\boldsymbol{v}_h]\!] \,\mathrm{d}s, \end{aligned} \tag{4.13}$$

where $\alpha > 0$ is a penalty parameter. If α is taken sufficiently large we have the coercivity result (see [14, eq. (6.17)])

$$C_{pd} \|\boldsymbol{v}_h\|^2_{\mathrm{dg}} \le a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h, \tag{4.14}$$

with $C_{pd} > 0$ a constant depending on Ω and mesh shape-regularity only. Also, arguing similarly to that of Lemma 2.3, it is not hard to show that a_h is bounded on $\boldsymbol{X}_h$:

$$\left|a_h(\boldsymbol{v}_h, \boldsymbol{w}_h)\right| \le C_a \|\boldsymbol{v}_h\|_{\mathrm{dg}} \|\boldsymbol{w}_h\|_{\mathrm{dg}} \qquad \forall \boldsymbol{v}_h, \boldsymbol{w}_h \in \boldsymbol{X}_h, \tag{4.15}$$

with $C_a > 0$ a constant depending on Ω and mesh shape-regularity only.

Our discretization of the velocity-pressure coupling is based on the discrete bilinear form $b_h : \boldsymbol{X}_h \times Q_h \to \mathbb{R}$ given by

$$b_h(\boldsymbol{v}_h, q_h) := -\int_\Omega (\nabla \cdot \boldsymbol{v}_h) q_h \,\mathrm{d}x. \tag{4.16}$$

We then have the inf-sup stability result (see e.g. [46])

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \frac{b_h(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{\mathrm{dg}}} \ge \beta^* \|q_h\| \qquad \forall q_h \in Q_h, \tag{4.17}$$

with $\beta^* > 0$ a constant depending on Ω and mesh shape-regularity only. In what follows we will also need the discrete reduced space, which is defined as

$$\begin{aligned} \boldsymbol{V}_h &:= \{\boldsymbol{v}_h \in \boldsymbol{X}_h : b_h(\boldsymbol{v}_h, q_h) = 0 \ \forall q_h \in Q_h\} \\ &= \{\boldsymbol{v}_h \in \boldsymbol{X}_h : \nabla \cdot \boldsymbol{v}_h = 0\}. \end{aligned}$$

Finally, we define the discrete convective form $c_h : \boldsymbol{X}(h) \times \boldsymbol{X}(h) \times \boldsymbol{X}_h \to \mathbb{R}$ using a standard upwinding discretization [14, Section 6.2]:

$$\begin{aligned} c_h(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}_h) := &\int_\Omega (\boldsymbol{w} \cdot \nabla_h \boldsymbol{u}) \cdot \boldsymbol{v}_h \,\mathrm{d}x + \frac{1}{2} \int_\Omega (\nabla \cdot \boldsymbol{w})(\boldsymbol{u} \cdot \boldsymbol{v}_h) \,\mathrm{d}x \\ &+ \sum_{F \in \mathcal{F}_i} \int_F \Big(\frac{1}{2} |\boldsymbol{w} \cdot \boldsymbol{n}_F| \, [\![\boldsymbol{v}_h]\!] - (\boldsymbol{w} \cdot \boldsymbol{n}_F) \{\!\{\boldsymbol{v}_h\}\!\}\Big) \cdot [\![\boldsymbol{u}]\!] \,\mathrm{d}s. \end{aligned} \tag{4.18}$$

Note that $\boldsymbol{X}(h) \subset \boldsymbol{H}(\mathrm{div}; \Omega)$ and therefore it makes sense to speak of $\nabla \cdot \boldsymbol{w}$ for $\boldsymbol{w} \in \boldsymbol{X}(h)$. An elementwise integration by parts argument reveals that c_h has the stability property

$$c_h(\boldsymbol{w}, \boldsymbol{v}_h, \boldsymbol{v}_h) = \frac{1}{2} \sum_{F \in \mathcal{F}_i} \int_F |\boldsymbol{w} \cdot \boldsymbol{n}_F| \, [\![\boldsymbol{v}_h]\!]^2 \,\mathrm{d}s \ge 0 \qquad \forall \boldsymbol{w} \in \boldsymbol{X}(h), \ \forall \boldsymbol{v}_h \in \boldsymbol{X}_h, \tag{4.19}$$

which should be viewed as a discrete analogue of the skew-symmetric property in eq. (4.8). Lastly, we will require for the remainder of this chapter that the following assumption hold.

Assumption 4.1 (Lipschitz continuity property of c_h)**.** *We assume that c_h satisfies the following Lipschitz-continuity property. For all $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{u} \in \boldsymbol{X}(h)$ with $\nabla \cdot \boldsymbol{w}_1 = \nabla \cdot \boldsymbol{w}_2 = 0$ there holds*

$$\left| c_h(\boldsymbol{w}_1, \boldsymbol{u}, \boldsymbol{v}_h) - c_h(\boldsymbol{w}_2, \boldsymbol{u}, \boldsymbol{v}_h) \right| \leq C_t \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{\mathrm{dg}} \|\boldsymbol{u}\|_{\mathrm{dg}} \|\boldsymbol{v}_h\|_{\mathrm{dg}} \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h, \tag{4.20}$$

with $C_t > 0$ a constant depending on Ω and mesh shape-regularity only.

Assumption 4.1 *always* holds in the two-dimensional case ($d = 2$). This is proven in [12, Proposition 4.2], and the authors of [12] also claim that the results in their paper can be easily extended to the three-dimensional case. Note also that an inequality very similar to eq. (4.20), but in the context of hybridized DG methods, was proven for the three-dimensional case in [8, Proposition 3.4]. It is therefore expected that by following the ideas in [12, Proposition 4.2] and [8, Proposition 3.4], one should be able to prove that Assumption 4.1 *always* holds in three-dimensions. However, this has not been verified at the time of writing this thesis, and therefore eq. (4.20) is stated as an assumption.

4.2.4 Enrichment operator

Let $\boldsymbol{E}_h : \boldsymbol{X}_h \to \boldsymbol{H}_0^1(\Omega)$ be any operator satisfying the statements in Lemma 2.1. For example, $\boldsymbol{E}_h$ could be the enrichment operator constructed in Appendix A. Alternatively $\boldsymbol{E}_h$ could be the enrichment operator from [28]. Recall that we used the operator $\boldsymbol{E}_h$ in Chapter 2 as a theoretical tool for deriving a-priori error bounds. In contrast, in this chapter we will actually use the operator $\boldsymbol{E}_h$ in the very definition of our numerical scheme (see eq. (4.22a)).

Note that by Item iii with $k = 0$ and Item iv of Lemma 2.1, the operator $\boldsymbol{E}_h$ satisfies

$$\|\boldsymbol{E}_h \boldsymbol{v}_h - \boldsymbol{v}_h\| \leq C_L h \|\boldsymbol{v}_h\|_{\mathrm{dg}}, \qquad \|\boldsymbol{E}_h \boldsymbol{v}_h\|_1 \leq C_E \|\boldsymbol{v}_h\|_{\mathrm{dg}}, \tag{4.21}$$

for all $\boldsymbol{v}_h \in \boldsymbol{X}_h$, with $C_L, C_E > 0$ constants depending on Ω and mesh shape-regularity only. We will make use of the constants C_L and C_E in the following analysis.

4.2.5 The proposed discrete method

We are now ready to state the proposed divergence-free DG method for the problem in eq. (4.3). The proposed method reads: Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{X}_h \times Q_h$ such that

$$\nu a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + c_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) = \langle \boldsymbol{f}, \boldsymbol{E}_h \boldsymbol{v}_h \rangle \qquad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h, \tag{4.22a}$$

$$b_h(\boldsymbol{u}_h, q_h) = 0 \qquad \forall q_h \in Q_h. \tag{4.22b}$$

Several remarks about this method are in order.

Remark 4.1 (Duality pairing). Recall that we are assuming $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$. The duality pairing $\langle \boldsymbol{f}, \boldsymbol{E}_h \boldsymbol{v}_h \rangle$ in eq. (4.22a) is therefore well-defined since $\boldsymbol{E}_h \boldsymbol{v}_h \in \boldsymbol{H}_0^1(\Omega)$. We emphasize that, in contrast, the expression $\langle \boldsymbol{f}, \boldsymbol{v}_h \rangle$ does not make any sense because $\boldsymbol{v}_h \notin \boldsymbol{H}_0^1(\Omega)$ in general. This is one way of motivating why the operator $\boldsymbol{E}_h$ is present in eq. (4.22a). We refer the interested reader to [3, Section 6], [26, Section 5.2] and [28, 29, 33, 34, 45] for examples of other finite element methods that use an operator similar to $\boldsymbol{E}_h$ in the discretization of the source term. It is possible to avoid the use of $\boldsymbol{E}_h$ in eq. (4.22a), but in order to do so we would need some additional regularity assumptions on $\boldsymbol{f}$. If, for example, we assumed that $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, then we could use $\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h \, \mathrm{d}x$ in the right-hand side of eq. (4.22a).

Remark 4.2 (Divergence-free). The condition in eq. (4.22b) is equivalent to requiring that $\nabla \cdot \boldsymbol{u}_h = 0$. Therefore, any discrete velocity solution to eq. (4.22) is indeed divergence-free.

Remark 4.3 (Implementation). In order to implement the method in eq. (4.22) on a computer, it is necessary to be able to compute $\langle \boldsymbol{f}, \boldsymbol{E}_h \boldsymbol{\phi}_h \rangle$ where $\boldsymbol{\phi}_h$ belongs to some computational finite element basis for $\boldsymbol{X}_h$. Unfortunately, because all known constructions of $\boldsymbol{E}_h$ are very complicated, this is a highly non-trivial task. Consequently, at the time of writing this thesis, the method in eq. (4.22) remains unimplemented. However, in [28] the authors have successfully implemented a computer code that is able to compute quantities of the form $\langle \boldsymbol{f}, \boldsymbol{E}_h \boldsymbol{\phi}_h \rangle$, with the computation of $\boldsymbol{E}_h \boldsymbol{\phi}_h$ requiring only $O(1)$ operations. This suggests that it should be realistically possible (albeit difficult) to implement the method in eq. (4.22) on a computer. This is a task that we leave to future work.

4.3 Discrete well-posedness

Recall that we are able to guarantee the existence and uniqueness of a solution to the continuous steady Navier–Stokes problem in eq. (4.3) provided that the small data condition in eq. (4.6) holds. In a similar vein, the following Theorem ensures that the discrete problem in eq. (4.22) admits a unique solution, provided that a discrete small data condition holds. The argument is based on Banach's fixed point theorem, and is similar to previous arguments appearing in the literature (see e.g. [12, Theorem 4.7] and [14, Theorem 6.43]).

Theorem 4.1 (Discrete well-posedness). *Under the discrete small data condition*

$$\frac{\|\boldsymbol{f}\|_{-1}}{\nu^2} < \frac{C_{pd}^2}{C_t C_E}, \tag{4.23}$$

we have that eq. (4.22) admits a unique solution $(\boldsymbol{u}_h, p_h) \in \boldsymbol{X}_h \times Q_h$*. Moreover, we have the a-priori stability estimates*

$$\|\boldsymbol{u}_h\|_{\mathrm{dg}} \leq \frac{C_E \|\boldsymbol{f}\|_{-1}}{\nu C_{pd}}, \qquad \beta_* \|p_h\| \leq \nu C_a \|\boldsymbol{u}_h\|_{\mathrm{dg}} + C_t \|\boldsymbol{u}_h\|_{\mathrm{dg}}^2 + C_E \|\boldsymbol{f}\|_{-1} . \tag{4.24}$$

Here we recall that the constants $C_{pd}, C_a, \beta_*, C_t$ *and* C_E *are defined in eq. (4.14), eq. (4.15), eq. (4.17), eq. (4.20) and eq. (4.21) respectively.*

Proof. We first prove the existence and uniqueness of a velocity solution to the following reduced problem: Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$\nu a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + c_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) = \langle \boldsymbol{f}, \boldsymbol{E}_h \boldsymbol{v}_h \rangle \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{4.25}$$

We shall view the spaces $\boldsymbol{X}_h$ and $\boldsymbol{V}_h$ as being endowed with the norm $\|\cdot\|_{\mathrm{dg}}$. Let $\boldsymbol{V}_h'$ denote the dual space of $\boldsymbol{V}_h$. For fixed $\boldsymbol{w}_h \in \boldsymbol{V}_h$, let $S(\boldsymbol{w}_h) : \boldsymbol{V}_h \to \boldsymbol{V}_h'$ be the linear operator

$$\langle S(\boldsymbol{w}_h)\boldsymbol{u}_h, \boldsymbol{v}_h \rangle := \nu a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + c_h(\boldsymbol{w}_h, \boldsymbol{u}_h, \boldsymbol{v}_h).$$

Using eq. (4.14) and eq. (4.19), we observe that $S(\boldsymbol{w}_h)$ enjoys the coercivity property

$$\langle S(\boldsymbol{w}_h)\boldsymbol{v}_h, \boldsymbol{v}_h \rangle \geq \nu C_{pd} \|\boldsymbol{v}_h\|_{\mathrm{dg}}^2 \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

We can therefore apply the Lax–Milgram lemma (see e.g. [14, Lemma 1.4]) to deduce that $S(\boldsymbol{w}_h)$ is an isomorphism from $\boldsymbol{V}_h$ onto $\boldsymbol{V}_h'$, whose inverse satisfies the bound

$$\left\|S(\boldsymbol{w}_h)^{-1}\right\|_{op} \leq (\nu C_{pd})^{-1}, \tag{4.26}$$

where $\|\cdot\|_{op}$ denotes the operator norm. Moreover, letting $\boldsymbol{F} \in \boldsymbol{V}_h'$ be given according to $\langle \boldsymbol{F}, \boldsymbol{v}_h \rangle := \langle \boldsymbol{f}, \boldsymbol{E}_h \boldsymbol{v}_h \rangle$, we see that eq. (4.25) is equivalent to the operator equation $S(\boldsymbol{u}_h)\boldsymbol{u}_h = \boldsymbol{F}$, which in turn is equivalent to $\boldsymbol{u}_h = S(\boldsymbol{u}_h)^{-1}\boldsymbol{F}$. Therefore, to prove that eq. (4.25) admits exactly one solution, we just need to show that the map $\boldsymbol{v}_h \mapsto S(\boldsymbol{v}_h)^{-1}\boldsymbol{F}$ admits exactly one fixed point. By Banach's fixed point theorem, it suffices to show that the map $\boldsymbol{v}_h \mapsto S(\boldsymbol{v}_h)^{-1}\boldsymbol{F}$ is a contraction. Let us now show this. For $\boldsymbol{v}_h, \boldsymbol{w}_h \in \boldsymbol{V}_h$ we note the algebraic identity

$$S(\boldsymbol{v}_h)^{-1} - S(\boldsymbol{w}_h)^{-1} = S(\boldsymbol{v}_h)^{-1}\left[S(\boldsymbol{w}_h) - S(\boldsymbol{v}_h)\right]S(\boldsymbol{w}_h)^{-1}.$$

We can therefore estimate

$$\left\|S(\boldsymbol{v}_h)^{-1}\boldsymbol{F} - S(\boldsymbol{w}_h)^{-1}\boldsymbol{F}\right\|_{\mathrm{dg}} \leq \left\|S(\boldsymbol{v}_h)^{-1}\right\|_{op}\left\|S(\boldsymbol{w}_h) - S(\boldsymbol{v}_h)\right\|_{op}\left\|S(\boldsymbol{w}_h)^{-1}\right\|_{op}\|\boldsymbol{F}\|_{\boldsymbol{V}_h'} .$$

Using eq. (4.26) and the bound $\|\boldsymbol{F}\|_{\boldsymbol{V}_h'} \leq C_E \|\boldsymbol{f}\|_{-1}$, we then obtain

$$\left\| S(\boldsymbol{v}_h)^{-1}\boldsymbol{F} - S(\boldsymbol{w}_h)^{-1}\boldsymbol{F} \right\|_{\mathrm{dg}} \leq \frac{C_E \|\boldsymbol{f}\|_{-1}}{(\nu C_{pd})^2} \left\| S(\boldsymbol{w}_h) - S(\boldsymbol{v}_h) \right\|_{op}.$$

But using the Lipschitz continuity property in eq. (4.20), we can further estimate

$$\begin{aligned}
\left\| S(\boldsymbol{w}_h) - S(\boldsymbol{v}_h) \right\|_{op} &= \sup_{\boldsymbol{r}_h \in \boldsymbol{V}_h} \frac{\left\| \left[S(\boldsymbol{w}_h) - S(\boldsymbol{v}_h) \right] \boldsymbol{r}_h \right\|_{\boldsymbol{V}_h'}}{\|\boldsymbol{r}_h\|_{\mathrm{dg}}} \\
&= \sup_{\boldsymbol{r}_h \in \boldsymbol{V}_h} \sup_{\tilde{\boldsymbol{v}}_h \in \boldsymbol{V}_h} \frac{\left| \langle \left[S(\boldsymbol{w}_h) - S(\boldsymbol{v}_h) \right] \boldsymbol{r}_h, \tilde{\boldsymbol{v}}_h \rangle \right|}{\|\boldsymbol{r}_h\|_{\mathrm{dg}} \|\tilde{\boldsymbol{v}}_h\|_{\mathrm{dg}}} \\
&= \sup_{\boldsymbol{r}_h \in \boldsymbol{V}_h} \sup_{\tilde{\boldsymbol{v}}_h \in \boldsymbol{V}_h} \frac{\left| c_h(\boldsymbol{w}_h, \boldsymbol{r}_h, \tilde{\boldsymbol{v}}_h) - c_h(\boldsymbol{v}_h, \boldsymbol{r}_h, \tilde{\boldsymbol{v}}_h) \right|}{\|\boldsymbol{r}_h\|_{\mathrm{dg}} \|\tilde{\boldsymbol{v}}_h\|_{\mathrm{dg}}} \\
&\leq C_t \|\boldsymbol{w}_h - \boldsymbol{v}_h\|_{\mathrm{dg}}.
\end{aligned}$$

Combining the above inequalities we see that

$$\left\| S(\boldsymbol{v}_h)^{-1}\boldsymbol{F} - S(\boldsymbol{w}_h)^{-1}\boldsymbol{F} \right\|_{\mathrm{dg}} \leq \frac{C_t C_E \|\boldsymbol{f}\|_{-1}}{(\nu C_{pd})^2} \|\boldsymbol{w}_h - \boldsymbol{v}_h\|_{\mathrm{dg}},$$

which shows that $\boldsymbol{v}_h \mapsto S(\boldsymbol{v}_h)^{-1}\boldsymbol{F}$ is a contraction under the small data condition

$$\frac{C_t C_E \|\boldsymbol{f}\|_{-1}}{(\nu C_{pd})^2} < 1,$$

which is equivalent to eq. (4.23). In this case, we deduce that eq. (4.25) admits a unique solution $\boldsymbol{u}_h \in \boldsymbol{V}_h$. Moreover, taking $\boldsymbol{u}_h$ as a test function in eq. (4.25), one readily obtains (with the help of eq. (4.14) and eq. (4.19)) the desired a-priori stability estimate

$$\|\boldsymbol{u}_h\|_{\mathrm{dg}} \leq \frac{C_E \|\boldsymbol{f}\|_{-1}}{\nu C_{pd}}.$$

Lastly, to recover the pressure p_h, consider the linear functional $l : \boldsymbol{X}_h \to \mathbb{R}$ given by

$$l(\boldsymbol{v}_h) := \langle \boldsymbol{f}, \boldsymbol{E}_h \boldsymbol{v}_h \rangle - \nu a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) - c_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h).$$

By eq. (4.25) we have $l(\boldsymbol{v}_h) = 0$ for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$. Consequently, inf-sup stability of b_h yields (see e.g. [40, Lemma 6.4]) the existence of a unique pressure function $p_h \in Q_h$ such that

$$l(\boldsymbol{v}_h) = b_h(\boldsymbol{v}_h, p_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h.$$

It can now be deduced that $(\boldsymbol{u}_h, p_h)$ solves eq. (4.22) uniquely. Finally, to obtain the desired stability bound for $\|p_h\|$ we can again appeal to inf-sup stability:

$$\begin{aligned}
\beta_* \|p_h\| &\leq \sup_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \frac{b_h(\boldsymbol{v}_h, p_h)}{\|\boldsymbol{v}_h\|_{\mathrm{dg}}} \\
&= \sup_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \frac{l(\boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{\mathrm{dg}}} \\
&\leq \nu C_a \|\boldsymbol{u}_h\|_{\mathrm{dg}} + C_t \|\boldsymbol{u}_h\|_{\mathrm{dg}}^2 + C_E \|\boldsymbol{f}\|_{-1}.
\end{aligned}$$

The proof is finished. □

4.4 Error analysis of the method

Having studied well-posedness of the discrete method in eq. (4.22), we now turn to the derivation of error estimates.

4.4.1 Extension of the bilinear form a_h

Recall that the bilinear form a_h in eq. (4.13) is defined only on the finite element space $\boldsymbol{X}_h$ (see eq. (4.11a)). The first step in our analysis is to extend a_h to the larger space $\boldsymbol{X}(h)$ (see eq. (4.12)). The main difficulty is that for $\boldsymbol{v} \in \boldsymbol{X}(h)$ we have only $\nabla_h \boldsymbol{v} \in [L^2(\Omega)]^{d\times d}$ and therefore $\nabla_h \boldsymbol{v}$ does not admit a well-defined trace on the mesh faces. We will deal with this problem by mimicking the ideas used in Section 2.2.1. Let $\pi_h : [L^2(\Omega)]^{d\times d} \to [\mathcal{P}_0(\mathcal{T}_h)]^{d\times d}$ denote the L^2-orthogonal projector onto $[\mathcal{P}_0(\mathcal{T}_h)]^{d\times d}$, where we are introducing the space of piecewise constant matrices

$$[\mathcal{P}_0(\mathcal{T}_h)]^{d\times d} := \{G_h \in [L^2(\Omega)]^{d\times d} : G_h|_K \in [\mathcal{P}_0(K)]^{d\times d} \ \forall K \in \mathcal{T}\}.$$

For the sake of clarity, let us note that for any $G \in [L^2(\Omega)]^{d\times d}$ and $K \in \mathcal{T}$ we have

$$(\pi_h G)|_K = \frac{1}{|K|} \int_K G \, \mathrm{d}x,$$

i.e. $(\pi_h G)|_K$ is simply the component-wise mean of G on K. For any $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{X}(h)$ we now define

$$\begin{aligned} a_h(\boldsymbol{v}, \boldsymbol{w}) := \int_\Omega \nabla_h \boldsymbol{v} : \nabla_h \boldsymbol{w} \,\mathrm{d}x - \sum_{F \in \mathcal{F}_h} \int_F (\{\!\{\pi_h(\nabla_h \boldsymbol{v})\}\!\} \boldsymbol{n}_F) \cdot [\![\boldsymbol{w}]\!] \,\mathrm{d}s \\ - \sum_{F \in \mathcal{F}_h} \int_F (\{\!\{\pi_h(\nabla_h \boldsymbol{w})\}\!\} \boldsymbol{n}_F) \cdot [\![\boldsymbol{v}]\!] \,\mathrm{d}s + \sum_{F \in \mathcal{F}_h} \frac{\alpha}{h_F} \int_F [\![\boldsymbol{v}]\!] \cdot [\![\boldsymbol{w}]\!] \,\mathrm{d}s. \end{aligned} \tag{4.27}$$

We will use this bilinear form in the following analysis. Observe that eq. (4.27) reduces to the previous definition of a_h (see eq. (4.13)) for $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{X}_h$. Moreover, the following boundedness result holds on the extended space $\boldsymbol{X}(h)$.

Lemma 4.1 (Boundedness of a_h)**.** *For all $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{X}(h)$ there holds*

$$\left|a_h(\boldsymbol{v}, \boldsymbol{w})\right| \leq C_{\tilde{a}} \|\boldsymbol{v}\|_{\mathrm{dg}} \|\boldsymbol{w}\|_{\mathrm{dg}},$$

with $C_{\tilde{a}} > 0$ a constant depending on Ω and mesh shape-regularity only.

Proof. We omit the proof as it is entirely analogous to that of Lemma 2.3. □

4.4.2 A data-oscillation-type quantity

In what follows it will be convenient to introduce the data-oscillation-type quantity

$$\mathcal{R}(\boldsymbol{u}) := \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{1}{\|\boldsymbol{v}_h\|_{\mathrm{dg}}} \int_\Omega (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) \cdot (\boldsymbol{E}_h \boldsymbol{v}_h - \boldsymbol{v}_h) \,\mathrm{d}x, \tag{4.28}$$

where $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ is the velocity solution of eq. (4.3). Note that $\mathcal{R}(\boldsymbol{u}) \in \mathbb{R}$ is simply the dual norm of the linear functional

$$\boldsymbol{V}_h \ni \boldsymbol{v}_h \mapsto \left\{ \int_\Omega (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) \cdot (\boldsymbol{E}_h \boldsymbol{v}_h - \boldsymbol{v}_h) \,\mathrm{d}x \right\} \in \mathbb{R}.$$

Since $\boldsymbol{V}_h$ is finite dimensional, we therefore have that $\mathcal{R}(\boldsymbol{u}) < \infty$.

4.4.3 The main error estimate

Our main error estimate for the method in eq. (4.22) is the following.

Theorem 4.2 (Velocity error estimate)**.** *Suppose that the small data conditions in eq. (4.6) and eq. (4.23) hold. Let $\boldsymbol{u}$ and $\boldsymbol{u}_h$ be the unique velocity solutions of eq. (4.3) and eq. (4.22) respectively. Assume further the smallness condition on the solution norms,*

$$\frac{2C_t}{\nu C_{pd}}\left(\|\boldsymbol{u}\|_1 + \|\boldsymbol{u}_h\|_{\mathrm{dg}}\right) < \delta \tag{4.29}$$

where $\delta \in (0,1)$ is any fixed constant. Here we recall that the constants C_{pd} and C_t are defined in eq. (4.14) and eq. (4.20) respectively. Then we have the error estimate

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{dg}} \leq C_1 \inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathrm{dg}} + \frac{C_2}{\nu}\mathcal{R}(\boldsymbol{u}), \tag{4.30}$$

where the constants C_1 and C_2 depend only on Ω, mesh shape-regularity and δ.

Proof. Let $\boldsymbol{v}_h \in \boldsymbol{V}_h$ be arbitrary, and set $\boldsymbol{w}_h := \boldsymbol{u}_h - \boldsymbol{v}_h \in \boldsymbol{V}_h$. Using the discrete coercivity result in eq. (4.14) and the discrete problem in eq. (4.22), we have

$$\begin{aligned} \nu C_{pd}\|\boldsymbol{w}_h\|_{\mathrm{dg}}^2 &\leq \nu a_h(\boldsymbol{w}_h, \boldsymbol{w}_h) \\ &= \nu a_h(\boldsymbol{u}_h, \boldsymbol{w}_h) - \nu a_h(\boldsymbol{v}_h, \boldsymbol{w}_h) \\ &= \langle \boldsymbol{f}, \boldsymbol{E}_h\boldsymbol{w}_h\rangle - c_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{w}_h) - \nu a_h(\boldsymbol{v}_h, \boldsymbol{w}_h). \end{aligned} \tag{4.31}$$

However, note that $\boldsymbol{E}_h\boldsymbol{w}_h \in \boldsymbol{V}$ since $\nabla \cdot \boldsymbol{E}_h\boldsymbol{w}_h = \nabla \cdot \boldsymbol{w}_h = 0$ (recall Item ii of Lemma 2.1). Taking $\boldsymbol{E}_h\boldsymbol{w}_h$ as a test function in the reduced problem eq. (4.7), we then obtain

$$\nu a(\boldsymbol{u}, \boldsymbol{E}_h\boldsymbol{w}_h) + c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{E}_h\boldsymbol{w}_h) = \langle \boldsymbol{f}, \boldsymbol{E}_h\boldsymbol{w}_h\rangle. \tag{4.32}$$

Combination of eq. (4.31) and eq. (4.32) yields two terms I_1 and I_2 that we shall bound separately:

$$\nu C_{pd}\|\boldsymbol{w}_h\|_{\mathrm{dg}}^2 \leq \nu \underbrace{\left[a(\boldsymbol{u}, \boldsymbol{E}_h\boldsymbol{w}_h) - a_h(\boldsymbol{v}_h, \boldsymbol{w}_h)\right]}_{I_1} + \underbrace{\left[c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{E}_h\boldsymbol{w}_h) - c_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{w}_h)\right]}_{I_2}. \tag{4.33}$$

We first bound I_1. Note that $a(\boldsymbol{u}, \boldsymbol{E}_h\boldsymbol{w}_h) = a_h(\boldsymbol{u}, \boldsymbol{E}_h\boldsymbol{w}_h)$ since $[\![\boldsymbol{u}]\!]|_F = [\![\boldsymbol{E}_h\boldsymbol{w}_h]\!]|_F = 0$ for all $F \in \mathcal{F}_h$. Letting $\boldsymbol{z}_h := (\boldsymbol{E}_h\boldsymbol{w}_h - \boldsymbol{w}_h)$, we can therefore write

$$I_1 = a_h(\boldsymbol{u}, \boldsymbol{E}_h\boldsymbol{w}_h) - a_h(\boldsymbol{v}_h, \boldsymbol{w}_h) = \underbrace{a_h(\boldsymbol{u} - \boldsymbol{v}_h, \boldsymbol{E}_h\boldsymbol{w}_h)}_{I_{1,1}} + \underbrace{a_h(\boldsymbol{v}_h, \boldsymbol{z}_h)}_{I_{1,2}}.$$

By Lemma 4.1 and eq. (4.21) we can bound $I_{1,1}$ according to

$$\left|I_{1,1}\right| \leq C_{\tilde{a}} C_E \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathrm{dg}} \|\boldsymbol{w}_h\|_{\mathrm{dg}} . \tag{4.34}$$

On the other hand, using the definition eq. (4.27) of a_h and the fact that $\pi_h(\nabla_h \boldsymbol{v}_h) = \nabla_h \boldsymbol{v}_h$, we can decompose $I_{1,2}$ as

$$\begin{aligned} I_{1,2} = & \underbrace{\int_\Omega \nabla_h \boldsymbol{v}_h : \nabla_h \boldsymbol{z}_h \,\mathrm{d}x - \sum_{F \in \mathcal{F}_h} \int_F (\{\!\!\{\nabla_h \boldsymbol{v}_h\}\!\!\} \boldsymbol{n}_F) \cdot [\![\boldsymbol{z}_h]\!] \,\mathrm{d}s}_{I_{1,2,1}} \\ & \underbrace{- \sum_{F \in \mathcal{F}_h} \int_F (\{\!\!\{\pi_h(\nabla_h \boldsymbol{z}_h)\}\!\!\} \boldsymbol{n}_F) \cdot [\![\boldsymbol{v}_h]\!] \,\mathrm{d}s + \sum_{F \in \mathcal{F}_h} \frac{\alpha}{h_F} \int_F [\![\boldsymbol{v}_h]\!] \cdot [\![\boldsymbol{z}_h]\!] \,\mathrm{d}s}_{I_{1,2,2}} . \end{aligned}$$

We claim that $I_{1,2,1} = 0$. Indeed, integrating by parts elementwise, and using the fact that $(\nabla^2 \boldsymbol{v}_h)|_K = 0$ as $\boldsymbol{v}_h$ is piecewise linear, one can verify that

$$\begin{aligned} \int_\Omega \nabla_h \boldsymbol{v}_h : \nabla_h \boldsymbol{z}_h \,\mathrm{d}x = & \sum_{F \in \mathcal{F}_i} \int_F (\{\!\!\{\nabla_h \boldsymbol{v}_h\}\!\!\} \boldsymbol{n}_F) \cdot [\![\boldsymbol{z}_h]\!] + ([\![\nabla_h \boldsymbol{v}_h]\!] \boldsymbol{n}_F) \cdot \{\!\!\{\boldsymbol{z}_h\}\!\!\} \,\mathrm{d}s \\ & + \sum_{F \in \mathcal{F}_b} \int_F (\{\!\!\{\nabla_h \boldsymbol{v}_h\}\!\!\} \boldsymbol{n}_F) \cdot [\![\boldsymbol{z}_h]\!] \,\mathrm{d}s. \end{aligned} \tag{4.35}$$

Plugging eq. (4.35) into the definition of $I_{1,2,1}$, and using Item i of Lemma 2.1 along with the fact that $([\![\nabla_h \boldsymbol{v}_h]\!] \boldsymbol{n}_F)$ is a constant on any $F \in \mathcal{F}_i$, we obtain

$$\begin{aligned} I_{1,2,1} &= \sum_{F \in \mathcal{F}_i} \int_F ([\![\nabla_h \boldsymbol{v}_h]\!] \boldsymbol{n}_F) \cdot \{\!\!\{\boldsymbol{z}_h\}\!\!\} \,\mathrm{d}s \\ &= \sum_{F \in \mathcal{F}_i} \int_F ([\![\nabla_h \boldsymbol{v}_h]\!] \boldsymbol{n}_F) \cdot (\boldsymbol{E}_h \boldsymbol{w}_h - \{\!\!\{\boldsymbol{w}_h\}\!\!\}) \,\mathrm{d}s \\ &= 0. \end{aligned} \tag{4.36}$$

Next we consider $I_{1,2,2}$. To begin, note that the same arguments used to prove Lemma 4.1 straightforwardly show that $\left|I_{1,2,2}\right| \leq \tilde{C} |\boldsymbol{v}_h|_{\mathrm{J}} \|\boldsymbol{z}_h\|_{\mathrm{dg}}$ with $\tilde{C}$ a constant depending on Ω and mesh shape-regularity only. Note also that $|\boldsymbol{v}_h|_{\mathrm{J}} = |\boldsymbol{u} - \boldsymbol{v}_h|_{\mathrm{J}} \leq \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathrm{dg}}$. Moreover, by the triangle inequality and the second inequality in eq. (4.21) we have that $\|\boldsymbol{z}_h\|_{\mathrm{dg}} \leq (1 + C_E) \|\boldsymbol{w}_h\|_{\mathrm{dg}}$. Putting all of this together we get the bound

$$\left|I_{1,2,2}\right| \leq \tilde{C} (1 + C_E) \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathrm{dg}} \|\boldsymbol{w}_h\|_{\mathrm{dg}} . \tag{4.37}$$

Combining the results in eq. (4.34), eq. (4.36) and eq. (4.37) we obtain

$$|I_1| \le \hat{C}\|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathrm{dg}}\|\boldsymbol{w}_h\|_{\mathrm{dg}}\,, \tag{4.38}$$

with $\hat{C}$ a constant depending on Ω and mesh shape-regularity only.

Next we bound I_2. Using the definition of c (recall eq. (4.2c)) we can write

$$I_2 = \int_\Omega (\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{E}_h\boldsymbol{w}_h\,\mathrm{d}x - c_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{w}_h). \tag{4.39}$$

Since $\nabla\cdot\boldsymbol{u} = 0$ and $[\![\boldsymbol{u}]\!]|_F = 0$ for all $F \in \mathcal{F}_h$, we notice from the definition of c_h (recall eq. (4.18)) that

$$\int_\Omega (\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{w}_h\,\mathrm{d}x = c_h(\boldsymbol{u},\boldsymbol{u},\boldsymbol{w}_h). \tag{4.40}$$

Combining eq. (4.39) and eq. (4.40), and using the definition of $\mathcal{R}(\boldsymbol{u})$ (recall eq. (4.28)), we obtain

$$\begin{aligned} I_2 &= \int_\Omega (\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot(\boldsymbol{E}_h\boldsymbol{w}_h - \boldsymbol{w}_h)\,\mathrm{d}x + \big[c_h(\boldsymbol{u},\boldsymbol{u},\boldsymbol{w}_h) - c_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{w}_h)\big] \\ &\le \mathcal{R}(\boldsymbol{u})\|\boldsymbol{w}_h\|_{\mathrm{dg}} + \big[c_h(\boldsymbol{u},\boldsymbol{u},\boldsymbol{w}_h) - c_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{w}_h)\big]. \end{aligned} \tag{4.41}$$

Moreover, the triangle inequality and Lipschitz continuity of c_h (recall eq. (4.20)) results in

$$\begin{aligned} \big|c_h(\boldsymbol{u},\boldsymbol{u},\boldsymbol{w}_h) - c_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{w}_h)\big| &\le \big|c_h(\boldsymbol{u},\boldsymbol{u},\boldsymbol{w}_h) - c_h(\boldsymbol{u}_h,\boldsymbol{u},\boldsymbol{w}_h)\big| \\ &\quad + \big|c_h(\boldsymbol{u}_h,\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{w}_h)\big| \\ &\le 2C_t\big(\|\boldsymbol{u}\|_{\mathrm{dg}} + \|\boldsymbol{u}_h\|_{\mathrm{dg}}\big)\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathrm{dg}}\|\boldsymbol{w}_h\|_{\mathrm{dg}} \\ &\le 2C_t\big(\|\boldsymbol{u}\|_1 + \|\boldsymbol{u}_h\|_{\mathrm{dg}}\big)\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathrm{dg}}\|\boldsymbol{w}_h\|_{\mathrm{dg}}\,. \end{aligned} \tag{4.42}$$

Combining the bounds in eq. (4.41) and eq. (4.42) we obtain

$$I_2 \le \mathcal{R}(\boldsymbol{u})\|\boldsymbol{w}_h\|_{\mathrm{dg}} + 2C_t\big(\|\boldsymbol{u}\|_1 + \|\boldsymbol{u}_h\|_{\mathrm{dg}}\big)\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathrm{dg}}\|\boldsymbol{w}_h\|_{\mathrm{dg}}\,. \tag{4.43}$$

Having obtained suitable bounds for I_1 and I_2, we are now almost finished. Plugging the bounds eq. (4.38) and eq. (4.43) in to eq. (4.33), we get

$$\begin{aligned} \nu C_{pd}\|\boldsymbol{w}_h\|_{\mathrm{dg}}^2 &\le \nu\hat{C}\|\boldsymbol{u}-\boldsymbol{v}_h\|_{\mathrm{dg}}\|\boldsymbol{w}_h\|_{\mathrm{dg}} + \mathcal{R}(\boldsymbol{u})\|\boldsymbol{w}_h\|_{\mathrm{dg}} \\ &\quad + 2C_t\big(\|\boldsymbol{u}\|_1 + \|\boldsymbol{u}_h\|_{\mathrm{dg}}\big)\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathrm{dg}}\|\boldsymbol{w}_h\|_{\mathrm{dg}}\,. \end{aligned} \tag{4.44}$$

Dividing eq. (4.44) by $\nu C_{pd} \|\boldsymbol{w}_h\|_{\mathrm{dg}}$ yields

$$\|\boldsymbol{w}_h\|_{\mathrm{dg}} \leq \frac{\hat{C}}{C_{pd}} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathrm{dg}} + \frac{1}{\nu C_{pd}} \mathcal{R}(\boldsymbol{u}) + \frac{2C_t}{\nu C_{pd}} \big(\|\boldsymbol{u}\|_1 + \|\boldsymbol{u}_h\|_{\mathrm{dg}} \big) \|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{dg}} . \tag{4.45}$$

Using the triangle inequality, eq. (4.45), and the smallness condition eq. (4.29), we have

$$\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{dg}} &\leq \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathrm{dg}} + \|\boldsymbol{w}_h\|_{\mathrm{dg}} \\
&\leq (1 + \frac{\hat{C}}{C_{pd}}) \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathrm{dg}} + \frac{1}{\nu C_{pd}} \mathcal{R}(\boldsymbol{u}) \\
&\quad + \frac{2C_t}{\nu C_{pd}} \big(\|\boldsymbol{u}\|_1 + \|\boldsymbol{u}_h\|_{\mathrm{dg}} \big) \|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{dg}} \\
&\leq (1 + \frac{\hat{C}}{C_{pd}}) \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathrm{dg}} + \frac{1}{\nu C_{pd}} \mathcal{R}(\boldsymbol{u}) + \delta \|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{dg}} .
\end{aligned} \tag{4.46}$$

Since $\delta \in (0, 1)$ we can rearrange eq. (4.46) to get

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{dg}} \leq \frac{1}{1 - \delta} (1 + \frac{\hat{C}}{C_{pd}}) \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathrm{dg}} + \frac{1}{1 - \delta} \frac{1}{\nu C_{pd}} \mathcal{R}(\boldsymbol{u}). \tag{4.47}$$

The desired result in eq. (4.30) now follows from eq. (4.47) as $\boldsymbol{v}_h \in \boldsymbol{V}_h$ is arbitrary. □

Remark 4.4 (Smallness condition on the solution norms)*.* Owing to the stability estimates for $\|\boldsymbol{u}\|_1$ and $\|\boldsymbol{u}_h\|_{\mathrm{dg}}$ in eq. (4.10) and eq. (4.24), we see that the smallness condition on the solution norms in eq. (4.29) is guaranteed to hold whenever we have

$$\frac{\|\boldsymbol{f}\|_{-1}}{\nu^2} \cdot \frac{2C_t}{C_{pd}} \Big[\frac{1}{C_p} + \frac{C_E}{C_{pd}} \Big] < \delta.$$

In other words, eq. (4.29) will always hold provided that $\nu^{-2} \|\boldsymbol{f}\|_{-1}$ is sufficiently small. In this sense eq. (4.29) is similar to the small data conditions in eq. (4.6) and eq. (4.23).

Remark 4.5 (Pressure-robustness)*.* The error estimate in eq. (4.30) is pressure-robust in the sense that it does not contain any terms that involve the pressure (cf. for example the error estimate in [12, Theorem 4.8], which is not pressure-robust). The estimate in eq. (4.30) does however depend on the viscosity through a factor of ν^{-1}. While this might seem problematic, it is important to keep in mind that all of the theory in this chapter is based on smallness assumptions of the form $\nu^{-2} \|\boldsymbol{f}\|_{-1} \leq C$. In other words, for any of the theory in this chapter to be applicable, the viscosity cannot be too small. Because of this, the presence of ν^{-1} in eq. (4.30) is not particularly concerning. Also, it seems unlikely that the theory in this chapter could possibly be developed in a way that circumvents the need for assumptions of the form $\nu^{-2}\|\boldsymbol{f}\|_{-1} \leq C$, since for example the continuous problem in eq. (4.3) need not have a unique solution if the viscosity is sufficiently small.

The error estimate in eq. (4.30) is only useful if we understand the behavior of $\mathcal{R}(\boldsymbol{u})$. In particular, we would like to know whether $\mathcal{R}(\boldsymbol{u})$ goes to zero (and at what rate) as the mesh size h goes to zero. At the time of writing this thesis we have only a partial answer to this question, which goes as follows. Suppose we have the additional regularity $\boldsymbol{u} \in \boldsymbol{L}^\infty(\Omega)$. We can then use the Cauchy–Schwarz inequality and the first inequality in eq. (4.21) to estimate, for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$,

$$\int_\Omega (\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot(\boldsymbol{E}_h\boldsymbol{v}_h - \boldsymbol{v}_h)\,\mathrm{d}x \leq \|\boldsymbol{u}\|_{\boldsymbol{L}^\infty}\|\nabla\boldsymbol{u}\|\|\boldsymbol{E}_h\boldsymbol{v}_h - \boldsymbol{v}_h\| \leq \|\boldsymbol{u}\|_{\boldsymbol{L}^\infty}\|\nabla\boldsymbol{u}\|\, C_L h\|\boldsymbol{v}_h\|_{\mathrm{dg}}. \quad (4.48)$$

By the definition of $\mathcal{R}(\boldsymbol{u})$ (recall eq. (4.28)) we consequently obtain that

$$\mathcal{R}(\boldsymbol{u}) \leq C_L h\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty}\|\nabla\boldsymbol{u}\|. \quad (4.49)$$

When can we actually say that $\boldsymbol{u} \in \boldsymbol{L}^\infty(\Omega)$? Here is one possibility. When $d = 2$ we have the continuous Sobolev embedding (see e.g. [18, Theorem 2.31])

$$\boldsymbol{H}^{1+s}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$$

for any real number $s \in (0, 1]$. For $d = 3$ we have only (see e.g. [18, Theorem 2.31])

$$\boldsymbol{H}^{1+s}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$$

for $s \in (1/2, 1]$. This discussion, in combination with Theorem 4.2, yields the following.

Corollary 4.1 (Convergence rate). *Let the hypotheses of Theorem 4.2 hold. Suppose also that $\boldsymbol{u} \in \boldsymbol{H}^{1+s}(\Omega)$, where $s \in (0, 1]$ when $d = 2$, and $s \in (1/2, 1]$ when $d = 3$. Then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{dg}} \leq \tilde{C}_1 h^s\|\boldsymbol{u}\|_{1+s} + \frac{\tilde{C}_2}{\nu}h\|\boldsymbol{u}\|^2_{1+s},$$

where the constants $\tilde{C}_1$ and $\tilde{C}_2$ depend only on Ω, mesh shape-regularity, δ and s.

Proof. We estimate the two terms on the right-hand side of eq. (4.30). We use C to denote a generic constant that depends only on Ω, mesh shape-regularity and s. By inf-sup stability of b_h we have the best-approximation result (see e.g. [7, Section 12.5])

$$\inf_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathrm{dg}} \leq C \inf_{\boldsymbol{v}_h\in\boldsymbol{X}_h}\|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathrm{dg}}.$$

Also, by standard approximation properties of the BDM space (see e.g. [18, Chapter 22]), or alternatively using Lemma 2.2, we have

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathrm{dg}} \leq C h^s \|\boldsymbol{u}\|_{1+s} .$$

Moreover, by eq. (4.49) and the continuous embedding $\boldsymbol{H}^{1+s}(\Omega) \hookrightarrow \boldsymbol{L}^{\infty}(\Omega)$, we have

$$\mathcal{R}(\boldsymbol{u}) \leq C_L h \|\boldsymbol{u}\|_{\boldsymbol{L}^{\infty}} \|\nabla \boldsymbol{u}\| \leq C h \|\boldsymbol{u}\|_{1+s}^2 .$$

The desired conclusion follows from eq. (4.30) and these bounds. □

Corollary 4.1 predicts optimal rates of convergence in the discrete $\boldsymbol{H}^1$-norm for the method in eq. (4.22). When $d = 2$ Corollary 4.1 requires only that $\boldsymbol{u} \in \boldsymbol{H}^{1+s}(\Omega)$ for some $s > 0$. However, when $d = 3$ the applicability of Corollary 4.1 is limited to the higher regularity case of $\boldsymbol{u} \in \boldsymbol{H}^{1+s}(\Omega)$ for some $s > 1/2$. Addressing the case of $s \in (0, 1/2]$ when $d = 3$ is a topic for future work.

Remark 4.6 (Future work). How might we be able to generalize Corollary 4.1 to the case of $s \in (0, 1/2]$ when $d = 3$? Here is one possible approach. We can modify the argument used in eq. (4.48), by instead using Hölder's inequality and the Sobolev embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^4(\Omega)$ (which is valid in three-dimensions), to estimate

$$\begin{aligned} \int_{\Omega} (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) \cdot (\boldsymbol{E}_h \boldsymbol{v}_h - \boldsymbol{v}_h) \, \mathrm{d}x &\leq \|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)} \|\nabla \boldsymbol{u}\| \|\boldsymbol{E}_h \boldsymbol{v}_h - \boldsymbol{v}_h\|_{\boldsymbol{L}^4(\Omega)} \\ &\leq C \|\boldsymbol{u}\|_1^2 \|\boldsymbol{E}_h \boldsymbol{v}_h - \boldsymbol{v}_h\|_{\boldsymbol{L}^4(\Omega)} . \end{aligned}$$

By the definition of $\mathcal{R}(\boldsymbol{u})$ (recall eq. (4.28)) we consequently obtain that

$$\mathcal{R}(\boldsymbol{u}) \leq C \|\boldsymbol{u}\|_1^2 \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{\|\boldsymbol{E}_h \boldsymbol{v}_h - \boldsymbol{v}_h\|_{\boldsymbol{L}^4(\Omega)}}{\|\boldsymbol{v}_h\|_{\mathrm{dg}}}.$$

If we could then establish an inequality of the form

$$\|\boldsymbol{E}_h \boldsymbol{v}_h - \boldsymbol{v}_h\|_{\boldsymbol{L}^4(\Omega)} \leq C h^r \|\boldsymbol{v}_h\|_{\mathrm{dg}} \tag{4.50}$$

for some $r > 0$, the same arguments used in Corollary 4.1 would then yield that

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{dg}} \leq \tilde{C}_1 h^s \|\boldsymbol{u}\|_{1+s} + \frac{\tilde{C}_2}{\nu} h^r \|\boldsymbol{u}\|_1^2 ,$$

for any $s \in (0, 1]$ when $d = 3$. We see therefore that it suffices to investigate whether eq. (4.50) holds. Note also that eq. (4.50) is very similar to the first inequality in eq. (4.21), and it therefore seems plausible that eq. (4.50) could hold for some suitable choice of r.

Chapter 5

Conclusions and future work

In Chapter 2 we analyzed two lowest-order HDG methods for the Stokes problem, while requiring only $\boldsymbol{H}^{1+s}$-regularity of the exact velocity solution for any $s \in [0, 1]$. A salient feature of the analysis is that it allows for the case of a domain with cracks. The key ingredient in the analysis is a suitable upper bound on the consistency error of the HDG methods, which we have derived by means of a divergence-preserving enrichment operator. We give an explicit construction of this enrichment operator in Appendix A, and here we rely crucially on a technical inequality that is established in Chapter 3. The resultant error estimates in Chapter 2 for the velocity are pressure-robust and optimal in the discrete energy norm. We also obtained an error bound for the pressure that is dependent on the velocity only. Our theoretical findings are supported by various numerical examples.

In Chapter 4 we extended the ideas in Chapter 2 to the setting of the steady Navier–Stokes problem. We proposed a new divergence-free DG method for the steady Navier–Stokes problem and showed that the resultant discretized problem is well-posed under a smallness condition on the problem data. We also presented an error analysis of the method, where we obtained error estimates for the velocity that are pressure-robust and optimal in the discrete energy norm. The presented analysis requires only $\boldsymbol{H}^{1+s}$-regularity of the exact velocity solution for any $s \in (0, 1]$ in the two-dimensional case. However, an interesting consequence of the nonlinear convective term is that in three-dimensions the presented analysis only holds for $s \in (1/2, 1]$.

The research presented in Chapter 4 is still in a preliminary stage, and there is still much more that can be done in this area. To begin with, the method in eq. (4.22) has not yet even been implemented. As discussed in Remark 4.3, it is likely that the method could be implemented in practice, but it may be very difficult to do so because of the complicated definition of $\boldsymbol{E}_h$. An interesting question is whether the method in eq. (4.22) could somehow be redefined in such a way that it remains valid for low-regularity source terms $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$, and retains its other desirable properties (e.g. pressure-robustness, convergence behavior), but does not require $\boldsymbol{E}_h$ to be used explicitly in any computations. Avoiding $\boldsymbol{E}_h$ altogether in this sort of way would make implementation a much easier task. Another question is whether an HDG or EDG–HDG analogue of the method in eq. (4.22) could be proposed and analyzed. An EDG–HDG analogue would be particularly appealing as upon static condensation it would require fewer global unknowns in the discrete problem.

From a theoretical analysis standpoint, there are also still many tasks to carry out regarding the DG method introduced in Chapter 4. The most obvious task is to address whether the convergence result in Corollary 4.1 can be extended to $s \in (0, 1/2]$ in three-dimensions, and we discuss a possible route for this in Remark 4.6. Also, analogously to the discussion in Remark 2.2, another task is to investigate whether we have $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{dg}} \to 0$ as $h \to 0$ in the $s = 0$ case where only $\boldsymbol{H}^1$-regularity of the exact velocity solution is assumed. A final question to ask is whether error estimates in the L^2-norm for the velocity and the pressure can be derived, perhaps in a similar fashion to what we have for the Stokes problem in Theorem 2.2 and Theorem 2.3.

References

[1] D. N. Arnold. An interior penalty finite element method with discontinuous elements. *SIAM J. Numer. Anal.*, 19(4):742–760, 1982.

[2] D. N. Arnold, F. Brezzi, and M. Fortin. A stable finite element for the Stokes equations. *Calcolo*, 21(4):337–344, 1984.

[3] S. Badia, R. Codina, T. Gudi, and J. Guzmán. Error analysis of discontinuous Galerkin methods for the Stokes problem under minimal regularity. *IMA J. Numer. Anal.*, 34(2):800–819, 2013.

[4] A. Baier-Reinio, S. Rhebergen, and G. N. Wells. Analysis of pressure-robust embedded-hybridized discontinuous Galerkin methods for the Stokes problem under minimal regularity. *arXiv preprint arXiv:2110.10611*, 2021.

[5] C. Bernardi and G. Raugel. Analysis of some finite elements for the Stokes problem. *Math. Comput.*, 44(169):71–79, 1985.

[6] D. Boffi, F. Brezzi, and M. Fortin. *Mixed Finite Element Methods and Applications*, volume 44 of *Springer Series in Computational Mathematics*. Springer, Heidelberg, 2013.

[7] S. C. Brenner and L. R. Scott. *The Mathematical Theory of Finite Element Methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, 3rd edition, 2008.

[8] A. Cesmelioglu, B. Cockburn, and W. Qiu. Analysis of a hybridizable discontinuous Galerkin method for the steady-state incompressible Navier–Stokes equations. *Math. Comput.*, 86(306):1643–1670, 2017.

[9] N. Chorfi. Geometric singularities of the Stokes problem. *Abstr. Appl. Anal.*, 2014:Article ID 491326, 2014.

[10] B. Cockburn and F.-J. Sayas. Divergence-conforming HDG methods for Stokes flows. *Math. Comput.*, 83(288):1571–1598, 2014.

[11] B. Cockburn, J. Gopalakrishnan, and R. Lazarov. Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems. *SIAM J. Numer. Anal.*, 47(2):1319–1365, 2009.

[12] B. Cockburn, G. Kanschat, and D. Schötzau. A locally conservative LDG method for the incompressible Navier–Stokes equations. *Math. Comput.*, 74(251):1067–1095, 2005.

[13] B. Cockburn, G. Kanschat, and D. Schötzau. A note on discontinuous Galerkin divergence-free solutions of the Navier–Stokes equations. *J. Sci. Comput.*, 31(1-2):61–73, 2007.

[14] D.A. Di Pietro and A. Ern. *Mathematical Aspects of Discontinuous Galerkin Methods*, volume 69 of *Mathématiques et Applications*. Springer, Berlin, 2012.

[15] M. Dauge. *Elliptic Boundary Value Problems on Corner Domains*, volume 1341 of *Lecture Notes in Mathematics*. Springer, Berlin, 1988.

[16] M. Dauge. Stationary Stokes and Navier–Stokes systems on two- or three-dimensional domains with corners. Part I. Linearized equations. *SIAM J. Math. Anal.*, 20(1):74–97, 1989.

[17] T. Dupont and R. Scott. Polynomial approximation of functions in Sobolev spaces. *Math. Comput.*, 34(150):441–463, 1980.

[18] A. Ern and J.-L. Guermond. *Finite Elements I. Approximation and Interpolation*, volume 72 of *Texts in Applied Mathematics*. Springer Nature, Switzerland, 2021.

[19] A. Ern and J.-L. Guermond. *Theory and Practice of Finite Elements*, volume 159 of *Applied Mathematical Sciences*. Springer, New York, 2004.

[20] A. Ern and J.-L. Guermond. Finite element quasi-interpolation and best approximation. *ESAIM Math. Model. Numer. Anal.*, 51(4):1367–1385, 2017.

[21] G. Fu. An explicit divergence-free DG method for incompressible flow. *Comput. Methods Appl. Mech. Eng.*, 345:502–517, 2019.

[22] V. Girault and P.-A. Raviart. *Finite Element Methods for Navier–Stokes Equations: Theory and Algorithms*, volume 5 of *Springer Series in Computational Mathematics*. Springer, Berlin, 1986.

[23] S. Güzey, B. Cockburn, and H.K. Stolarski. The embedded discontinuous Galerkin method: application to linear shell problems. *Int. J. Numer. Methods Eng.*, 70(7):757–790, 2007.

[24] J. Guzmán and M. Neilan. Conforming and divergence-free Stokes elements on general triangular meshes. *Math. Comput.*, 83(285):15–36, 2014.

[25] J. Guzmán and M. Neilan. Conforming and divergence-free Stokes elements in three dimensions. *IMA J. Numer. Anal.*, 34(4):1489–1508, 2014.

[26] V. John, A. Linke, C. Merdon, M. Neilan, and L. G. Rebholz. On the divergence constraint in mixed finite element methods for incompressible flows. *SIAM Rev.*, 59(3):492–544, 2017.

[27] O. A. Karakashian and F. Pascal. A posteriori error estimates for a discontinuous Galerkin approximation of second-order elliptic problems. *SIAM J. Numer. Anal.*, 41(6):2374–2399, 2003.

[28] C. Kreuzer, R. Verfürth, and P. Zanotti. Quasi-optimal and pressure robust discretizations of the Stokes equations by moment- and divergence-preserving operators. *Comput. Methods Appl. Math.*, 21(2):423–443, 2021.

[29] C. Kreuzer and P. Zanotti. Quasi-optimal and pressure-robust discretizations of the Stokes equations by new augmented Lagrangian formulations. *IMA J. Numer. Anal.*, 40(4):2553–2583, 2019.

[30] R. J. Labeur and G. N. Wells. Energy stable and momentum conserving hybrid finite element method for the incompressible Navier–Stokes equations. *SIAM J. Sci. Comput.*, 34(2):A889–A913, 2012.

[31] C. Lehrenfeld and J. Schöberl. High order exactly divergence-free hybrid discontinuous Galerkin methods for unsteady incompressible flows. *Comput. Methods Appl. Mech. Eng.*, 307:339–361, 2016.

[32] M. Li, S. Mao, and S. Zhang. New error estimates of nonconforming mixed finite element methods for the Stokes problem. *Math. Methods Appl. Sci.*, 37(7):937–951, 2014.

[33] A. Linke, C. Merdon, and M. Neilan. Pressure-robustness in quasi-optimal a priori estimates for the Stokes problem. *Electron. Trans. Numer. Anal.*, 52:281–294, 2020.

[34] A. Linke, C. Merdon, M. Neilan, and F. Neumann. Quasi-optimality of a pressure-robust nonconforming finite element method for the Stokes-problem. *Math. Comput.*, 87(312):1543–1566, 2018.

[35] J.-C. Nédélec. Mixed finite elements in $\mathbb{R}^3$. *Numer. Math.*, 35(3):315–341, 1980.

[36] R.B. Kellogg and J.E. Osborn. A regularity result for the Stokes problem in a convex polygon. *J. Funct. Anal.*, 21(4):397–431, 1976.

[37] S. Rhebergen and G. N. Wells. Analysis of a hybridized/interface stabilized finite element method for the Stokes equations. *SIAM J. Numer. Anal.*, 55(4):1982–2003, 2017.

[38] S. Rhebergen and G. N. Wells. A hybridizable discontinuous Galerkin method for the Navier–Stokes equations with pointwise divergence-free velocity field. *J. Sci. Comput.*, 76(3):1484–1501, 2018.

[39] S. Rhebergen and G. N. Wells. An embedded–hybridized discontinuous Galerkin finite element method for the Stokes equations. *Comput. Methods Appl. Mech. Eng.*, 358:112619, 2020.

[40] B. Rivière. *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation*. Frontiers in Applied Mathematics. Society for Industrial and Applied Mathematics, Philadelphia, 2008.

[41] J. Schöberl. C++11 implementation of finite elements in NGSolve. Technical Report, ASC Report 30/2014, Institute for Analysis and Scientific Computing, Vienna University of Technology, 2014.

[42] L. R. Scott and S. Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comput.*, 54(190):483–493, 1990.

[43] A. Veeser and P. Zanotti. Quasi-optimal nonconforming methods for second-order problems on domains with non-Lipschitz boundary. In F. A. Radu, K. Kumar, I. Berre, J. M. Nordbotten, and I. S. Pop, editors, *Numerical Mathematics and Advanced Applications ENUMATH 2017*, pages 461–469, Cham, 2019. Springer.

[44] R. Verfürth. A posteriori error estimators for the Stokes equations. *Numer. Math.*, 55(3):309–325, 1989.

[45] R. Verfürth and P. Zanotti. A quasi-optimal Crouzeix–Raviart discretization of the Stokes equations. *SIAM J. Numer. Anal.*, 57(3):1082–1099, 2019.

[46] J. Wang and X. Ye. New finite element methods in computational fluid dynamics by H(div) elements. *SIAM J. Numer. Anal.*, 45(3):1269–1286, 2007.

APPENDICES

Appendix A

Enrichment operator

We prove here Lemma 2.1 for the three-dimensional case ($d = 3$). We shall use the angled bracket notation $\langle \cdot, \cdot \rangle_D$; this is simply used to denote the L^2-inner-product on a domain D with dimension strictly less than $d = 3$.

For $K \in \mathcal{T}$ we recall from Section 2.1.2 that $\mathcal{F}_{K,h} \subset \mathcal{F}_h$ denotes the four faces of K. Also, let $\mathcal{E}_K$ denote the six edges of K and $\mathcal{V}_K$ the four vertices of K. The collection of all mesh edges is written as $\mathcal{E}_h := \cup_{K \in \mathcal{T}} \mathcal{E}_K = \mathcal{E}_b \cup \mathcal{E}_i$ where $\mathcal{E}_b$ denotes the boundary edges and $\mathcal{E}_i$ the interior edges. Likewise, the collection of all mesh vertices is written as $\mathcal{V}_h := \cup_{K \in \mathcal{T}} \mathcal{V}_K = \mathcal{V}_b \cup \mathcal{V}_i$ where $\mathcal{V}_b$ denotes the boundary vertices and $\mathcal{V}_i$ the interior vertices. For an interior edge $e \in \mathcal{E}_i$, we define the average of a function v on e as

$$\{\!\{v\}\!\}_e := \frac{1}{|\mathcal{T}_e|} \sum_{K \in \mathcal{T}_e} v_K|_e,$$

where $\mathcal{T}_e := \{K \in \mathcal{T} : e \in \mathcal{E}_K\}$ denotes the collection of elements having e as an edge and $v_K := v|_K$. On boundary edges $e \in \mathcal{E}_b$, it will be convenient to define $\{\!\{v\}\!\}_e := 0$. Similarly, for an interior vertex $a \in \mathcal{V}_i$, the average of a function v on a is defined as

$$\{\!\{v\}\!\}_a := \frac{1}{|\mathcal{T}_a|} \sum_{K \in \mathcal{T}_a} v_K(a),$$

where $\mathcal{T}_a := \{K \in \mathcal{T} : a \in \mathcal{V}_K\}$ denotes the collection of elements having a as a vertex. On boundary vertices $a \in \mathcal{V}_b$ it will be convenient to define $\{\!\{v\}\!\}_a := 0$. Finally, throughout this proof we continue to use the definition eqs. (2.6) to (2.7) of the average operator on faces.

For $K \in \mathcal{T}$, let $V(K)$ denote the local three-dimensional Guzmán–Neilan finite element space defined by [25, eq. (3.9)]. This space has the properties [25, Lemma 3.4]

$$
\begin{aligned}
&[\mathcal{P}_1(K)]^3 \subset V(K), && V(K) \subset [W^{1,\infty}(K) \cap C^0(\bar{K})]^3, \\
&\nabla \cdot V(K) \subset \mathcal{P}_0(K), && V(K)|_{\partial K} \subset [\mathcal{P}_3(\partial K)]^3.
\end{aligned}
$$

Moreover, a set of unisolvent degrees of freedom for $v \in V(K)$ is given by [25, Theorem 3.5]

$$
\begin{aligned}
&v(a) && \forall a \in \mathcal{V}_K, && \text{(A.1a)} \\
&\langle v, w\rangle_e && \forall e \in \mathcal{E}_K, w \in [\mathcal{P}_1(e)]^3, && \text{(A.1b)} \\
&\langle v, w\rangle_F && \forall F \in \mathcal{F}_{K,h}, w \in [\mathcal{P}_0(F)]^3. && \text{(A.1c)}
\end{aligned}
$$

For $K \in \mathcal{T}$ we now define the local operator $E_K : X_h^{\mathrm{BDM}} \to V(K)$ as follows. For $v_h \in X_h^{\mathrm{BDM}}$ we require that

$$
\begin{aligned}
&(E_K v_h)(a) = \{\!\{v_h\}\!\}_a && \forall a \in \mathcal{V}_K, && \text{(A.2a)} \\
&\langle E_K v_h - \{\!\{v_h\}\!\}_e, w\rangle_e = 0 && \forall e \in \mathcal{E}_K, w \in [\mathcal{P}_1(e)]^3, && \text{(A.2b)} \\
&\langle E_K v_h - \{\!\{v_h\}\!\}, w\rangle_F = 0 && \forall F \in \mathcal{F}_{K,h} \cap \mathcal{F}_i, w \in [\mathcal{P}_0(F)]^3, && \text{(A.2c)} \\
&\langle E_K v_h, w\rangle_F = 0 && \forall F \in \mathcal{F}_{K,h} \cap \mathcal{F}_b, w \in [\mathcal{P}_0(F)]^3. && \text{(A.2d)}
\end{aligned}
$$

The degrees of freedom eq. (A.1) imply that the operator E_K is well-defined. We then define $E_h : X_h^{\mathrm{BDM}} \to H_0^1(\Omega)^d$ by $(E_h v_h)|_K = E_K v_h$ for all $v_h \in X_h^{\mathrm{BDM}}$. Utilizing the inclusion $V(K)|_{\partial K} \subset [\mathcal{P}_3(\partial K)]^3$ one can show that $[\![E_h v_h]\!]|_F = 0$ for all $F \in \mathcal{F}_h$, and thus $E_h v_h \in H_0^1(\Omega)^d$ holds. It remains to verify that E_h satisfies Items i to iv from Lemma 2.1.

That Item i holds is an immediate consequence of eq. (A.2c). To prove Item ii, consider the space $\mathcal{P}_{0,h} := \{q_h \in L^2(\Omega) : q_h|_K \in \mathcal{P}_0(K)\ \forall K \in \mathcal{T}\}$ of piecewise constant functions. Let $v_h \in X_h^{\mathrm{BDM}}$ and $q_h \in \mathcal{P}_{0,h}$. Then element-wise integration by parts and eq. (A.2c) shows that

$$
\begin{aligned}
\int_\Omega (\nabla \cdot E_h v_h) q_h \,\mathrm{d}x &= \sum_{F \in \mathcal{F}_i} \int_F (E_h v_h \cdot n_F)[\![q_h]\!] \,\mathrm{d}s \\
&= \sum_{F \in \mathcal{F}_i} \int_F (\{\!\{v_h\}\!\} \cdot n_F)[\![q_h]\!] \,\mathrm{d}s \qquad \text{(A.3)} \\
&= \int_\Omega (\nabla \cdot v_h) q_h \,\mathrm{d}x,
\end{aligned}
$$

where the last equality in eq. (A.3) follows from the fact that $[\![v_h]\!]|_F \cdot n_F = 0$ for all $F \in \mathcal{F}_h$. Item ii now follows from eq. (A.3) as $\nabla \cdot E_h v_h, \nabla \cdot v_h \in \mathcal{P}_{0,h}$ and $q_h \in \mathcal{P}_{0,h}$ is arbitrary.

To prove Item iii, fix $k \in \{0, 1\}$ and $v_h \in X_h^{\mathrm{BDM}}$. Consider $K \in \mathcal{T}$ and set $v_K := v_h|_K$ and $z_K := (E_K v_h) - v_K$. Since $v_K \in [\mathcal{P}_1(K)]^3 \subset V(K)$, there holds $z_K \in V(K)$. A scaling argument (see Chapter 3 – in particular we are making use of the inequality eq. (3.4)) utilizing the degrees of freedom eq. (A.1) then shows that

$$\begin{aligned}
h_K^{2(k-1)} |z_K|_{k,K}^2 &\lesssim \sum_{a \in \mathcal{V}_K} h_K \big|z_K(a)\big|^2 + \sum_{e \in \mathcal{E}_K} \sup_{\substack{\kappa_h \in [\mathcal{P}_1(e)]^3 \\ \|\kappa_h\|_e = 1}} \big|\langle z_K, \kappa_h \rangle_e\big|^2 \\
&\quad + \sum_{F \in \mathcal{F}_{K,h}} \frac{1}{h_K} \sup_{\substack{\kappa_h \in [\mathcal{P}_0(F)]^3 \\ \|\kappa_h\|_F = 1}} \big|\langle z_K, \kappa_h \rangle_F\big|^2 \\
&\leq \underbrace{\sum_{a \in \mathcal{V}_K} h_K \big|\{\!\{v_h\}\!\}_a - v_K(a)\big|^2}_{I_1} + \underbrace{\sum_{e \in \mathcal{E}_K} \big\|\{\!\{v_h\}\!\}_e - v_K\big\|_e^2}_{I_2} \\
&\quad \underbrace{+ \sum_{\substack{F \in \mathcal{F}_{K,h} \\ F \in \mathcal{F}_i}} \frac{1}{h_K} \big\|\{\!\{v_h\}\!\} - v_K\big\|_F^2 + \sum_{\substack{F \in \mathcal{F}_{K,h} \\ F \in \mathcal{F}_b}} \frac{1}{h_K} \|v_K\|_F^2}_{I_3}.
\end{aligned} \tag{A.4}$$

Because $\partial\Omega$ has codimension one, every boundary vertex of the mesh is contained in some boundary face of the mesh, and likewise for boundary edges. As a result, the same arguments used in [34, Lemma 4.7] show that

$$I_1 \lesssim \sum_{a \in \mathcal{V}_K} \sum_{F \in \mathcal{F}_a} \frac{1}{h_F} \big\|[\![v_h]\!]\big\|_F^2, \tag{A.5}$$

$$I_2 \lesssim \sum_{e \in \mathcal{E}_K} \sum_{F \in \mathcal{F}_e} \frac{1}{h_F} \big\|[\![v_h]\!]\big\|_F^2, \tag{A.6}$$

where $\mathcal{F}_a \subset \mathcal{F}_h$ denotes the collection of all mesh faces having a as a vertex, and $\mathcal{F}_e \subset \mathcal{F}_h$ denotes the collection of all mesh faces having e as an edge. We note that, due to midpoint continuity of Crouzeix–Raviart elements, there is no term analogous to I_3 in [34, Lemma 4.7]. Fortunately, it is easy to see that we can bound I_3 by means of

$$I_3 \lesssim \sum_{F \in \mathcal{F}_{K,h}} \frac{1}{h_F} \big\|[\![v_h]\!]\big\|_F^2. \tag{A.7}$$

Using the bounds eqs. (A.5) to (A.7) in eq. (A.4), and summing over $K \in \mathcal{T}$, one obtains

$$\sum_{K\in\mathcal{T}} h_K^{2(k-1)} |E_h v_h - v_h|_{k,K}^2 \lesssim \sum_{F\in\mathcal{F}_h} \frac{1}{h_F} \big\| [\![v_h]\!] \big\|_F^2 = |v_h|_{\mathrm{J}}^2 ,$$

so that Item iii holds. Lastly, Item iv follows from Item iii with $k = 1$ and the triangle inequality:

$$\begin{aligned} \|\nabla E_h v_h\| &\leq \Big(\sum_{K\in\mathcal{T}} |v_h - E_h v_h|_{1,K}^2 \Big)^{1/2} + \Big(\sum_{K\in\mathcal{T}} |v_h|_{1,K}^2 \Big)^{1/2} \\ &\lesssim |v_h|_{\mathrm{J}} + \Big(\sum_{K\in\mathcal{T}} |v_h|_{1,K}^2 \Big)^{1/2} \\ &\lesssim \|v_h\|_{\mathrm{dg}} . \end{aligned}$$

This completes the proof of Lemma 2.1. □