

Para-Holomorphic Algebroids and Para-Complex Connections

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners. I understand that my thesis may be made electronically available to the public.

Abstract

The goal of this paper is to develop the theory of Courant algebroids with integrable para-Hermitian vector bundle structures by invoking the theory of Lie bialgebroids. We consider the case where the underlying manifold has an almost para-complex structure, and use this to define a notion of para-holomorphic algebroid. We investigate connections on para-holomorphic algebroids and determine an appropriate sense in which they can be para-complex. Finally, we show through a series of examples how the theory of exact para-holomorphic algebroids with a para-complex connection is a generalization of both para-Kähler geometry and the theory of Poisson-Lie groups.

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1 Introduction

There has been a growing interest in generalized geometry since Hitchin introduced the concept in his paper *Generalized Calabi-Yau Manifolds* in 2002 [17]. Given a smooth manifold M , the natural setting for generalized geometry is the vector bundle $\mathbb{T}M := TM \oplus T^*M$. This vector bundle comes equipped with a natural anti-symmetric bracket, $[\cdot, \cdot]$, introduced by Courant in his 1990 paper *Dirac Manifolds* [8], a symmetric bilinear form $\langle \cdot, \cdot \rangle$ and a natural anchor map $\rho : TM \oplus T^*M \rightarrow TM$ given by projection onto the first factor. For $X \oplus \xi, Y \oplus \eta \in \Gamma(TM \oplus T^*M)$, the bracket and symmetric pairing are given explicitly by:

$$\begin{aligned} [X \oplus \xi, Y \oplus \eta] &= [X, Y] \oplus \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2}d(\iota_X \eta - \iota_Y \xi), \\ \langle X \oplus \xi, Y \oplus \eta \rangle &= \xi(Y) + \eta(X). \end{aligned} \tag{1}$$

Courant also introduced the notion of a Dirac subbundle in [8].

Definition 1.1. *Let M be a smooth manifold. A Dirac structure on $\mathbb{T}M$ is a subbundle $L \subset \mathbb{T}M$ that is maximally isotropic and whose space of sections, $\Gamma(L)$, is closed under the Courant bracket.*

Dirac structures correspond directly to Poisson bivectors in the case where $L \neq TM$ or T^*M (for a good explanation of why this is the case, see [32]), and so the search for Dirac subbundles is equivalent to the search for Poisson bivectors on M .

On the other hand, research in para-complex geometry has been expanding as well. The fundamental object here is the *almost product manifold*.

Definition 1.2. *Let M be a smooth manifold, and $J \in \Gamma(\text{End}(TM))$ with $J^2 = \text{Id}_{TM}$. The pair (M, J) is called an almost product manifold.*

Since $J^2 = Id_{TM}$, J has two eigenvalues, ± 1 , and so TM admits a decomposition into the direct sum of the ± 1 -eigenbundles $TM = T^+M \oplus T^-M$. If the eigenbundles have the same rank, then J is called an *almost para-complex structure*. The pair (M, J) is called a *half integrable para-complex manifold* if one of the eigenbundles is closed under the Lie bracket, and a *para-complex manifold* if both eigenbundles are integrable. For an interesting survey of the historical context, and some basic results in the field leading up to the mid 1990's, see [9]. Just as in the complex case, one can ask what it means for an almost para-complex structure to be compatible with a metric on M . This gives the following definition.

Definition 1.3. *Let (M, J) be an almost para-complex manifold and g be a non-degenerate symmetric bilinear form on M . Then we say that J is compatible with g if $g(J\cdot, J\cdot) = -g(\cdot, \cdot)$ and in this case, we refer to (M, g, J) as an almost para-Hermitian manifold.*

It was Bejan who introduced the concepts of para-complex and para-Hermitian vector bundles in her paper *The Existence Problem of Hyperbolic Structures on Vector Bundles* [4]. These structures lack a concept of integrability however, and so these constructions are done basically at the level of vector spaces. Importantly, Erdem introduced the concept of a para-holomorphic map between para-complex manifolds. In the context of vector bundles, we understand para-holomorphic maps as vector bundle morphisms $T : (E, J_E) \rightarrow (F, J_F)$ such that $J_F \circ T = T \circ J_E$. Erdem also introduced the idea of “para-complexifying” a vector bundle using the para-complex numbers C to obtain $\pm j$ -eigenbundles, similar to the complex case, but with the caveat that $j^2 = 1$.

Where the concept of Courant algebroids and para-Hermitian vector bundles meet is in Svoboda's paper *Algebroid Structures on Para-Hermitian Manifolds* [36]. Motivated by the physics of Double Field Theory, Svoboda identifies para-Hermitian structures on Courant algebroids as corresponding to Lie bialgebroids, which are thoroughly studied

in [34]. The condition of integrability of the eigenbundles of the para-complex structure becomes closure under the Courant bracket, and so one obtains a pair of transverse Dirac structures corresponding to the para-Hermitian structure.

In this paper, we are interested in exploring para-Hermitian algebroids $E \rightarrow M$ over para-Hermitian manifolds (M, g, J) in order to show that an even wider class of objects can be realized in the para-complex setting than previously thought. In this setting, it is natural to ask questions like: “Is the obstruction to integrability of the eigenbundles of E related to structures on TM ?”, “What happens if the anchor map $\rho : E \rightarrow TM$ is para-holomorphic?”, and “Given a connection $A : TM \rightarrow E$, what does it mean for A to be para-holomorphic?”. The contribution that this paper aims to make is to introduce the concept of para-holomorphic algebroids, and para-complex connections. The interplay between para-complex connections and para-Hermitian/holomorphic algebroids is particularly interesting. For instance, we find that

Theorem 1.1. *If (E, J) is an exact para-Hermitian algebroid over a para-Hermitian manifold, and admits a flat para-complex connection A , then E is a para-holomorphic algebroid.*

A consequence of considering para-complex connections is that we arrive naturally at the concept of a *split-para-complex structure*. In short, a split-para-complex structure is a vector bundle E together with two para-complex structures J, K , such that $JK = KJ$. For an exact para-Hermitian algebroid (E, J) with a connection A and an anchor map $\rho : E \rightarrow TM$, we can define the para-complex structure K to take the value $+1$ on $A(TM)$ and -1 on $\rho^*(T^*M)$. In general, we can take any splitting $E = \rho^*(T^*M) \oplus H$, and define the para-complex structure K similarly. This identification of connections and almost para-complex algebroids leads us to the observation:

Theorem 1.2. *If (E, J) is an exact para-holomorphic algebroid, then E admits a flat*

para-complex connection if and only if (E, J, K) is split-para-complex.

Finally, we will present a class of para-Hermitian algebroids over the exact Courant algebroid $\mathbb{T}G$ for a Lie group G with a quadratic Lie algebra \mathfrak{g} , as well as a class of exact para-holomorphic algebroids with flat para-complex connections over $\mathbb{T}G$ when \mathfrak{g} is also semi-simple. In the second class of examples, we realize a compact real form of the Lie algebra \mathfrak{g} as corresponding to a special para-holomorphic structure on $\mathbb{T}G$, where G is the Drinfeld double of K , endowed with a compatible para-complex connection induced by the diagonal inclusion $G \hookrightarrow G \times G$ (referred to as the Cartan-Dirac structure). We will also construct a separate para-holomorphic structure with flat para-complex connection on $\mathbb{T}G$ in the case where G is the Drinfeld double of a complete simply-connected Poisson-Lie group (K, Π_K) . The goal of this section is to concretely establish that exact para-holomorphic algebroids with flat para-complex connections are not only common, but comprise some well known objects. The impression we try to give is that para-Hermitian geometry is general enough to include interesting objects and provides a direction for future generalizations.

Roytenberg studied Lie bialgebroids using the framework of supermanifolds in [34], and so in the future it would be interesting to generalize para-holomorphic algebroids using this formalism. In particular, it would be interesting to try and cast the para-complex structure as a function on $T^*\Pi E_+$, as they do with other important objects like the Courant bracket. It would also be interesting to try and find examples of para-holomorphic structures with compatible para-complex connections that are not simply the diagonal embedding, and are perhaps not flat as well.

2 Para-Complex Vector Bundles

Para-complex manifolds and para-complex vector bundles are straightforward generalizations of complex manifolds and vector bundles. The utility of the para-complex setting is that the splitting of the vector bundle into eigenbundles occurs at the level of the vector bundle, and does not require you to “para-complexify”. One can generalize basically any fact about complex manifolds and vector bundles to the para-complex setting to obtain analogous results and fundamental objects, which is what we will summarize in this chapter.

Of special importance is the so called para-Hermitian structure. The main interest in this structure comes from the fact that the eigenbundles of the para-complex structure become maximally isotropic. In the context of quadratic Lie algebras, orthogonal maximally isotropic subalgebras arise naturally in the study of Poisson-Lie groups. With our goal of finding a single object that corresponds naturally to para-Kähler manifolds and also Poisson-Lie groups, we present the theory of para-complex vector bundles.

2.1 Basic Properties

Throughout this paper, let M be a smooth manifold of dimension n and $E \rightarrow M$ be a smooth real vector bundle over M .

Definition 2.1. *A product structure on E is a section $J \in \Gamma(\text{End}(E))$ such that $J^2 = Id_E$.*

On each fiber $E_p := \pi^{-1}(p)$, $p \in M$, we find that J has the two possible eigenvalues 1 and -1 . This induces a decomposition of E_p into the direct sum of the eigenspaces $E_p = E_p^+ \oplus E_p^-$, which in turn induces an eigenbundle decomposition $E = E^+ \oplus E^-$.

When the subbundles E^+ and E^- have the same rank (hence the rank of E must be even), we say that J is a *para-complex structure*. This decomposition of the vector bundle E induces a decomposition of the dual bundle as follows:

Let $E_+^* := \{\omega \in E^* \mid \omega(Z) = 0 \text{ for all } Z \in E^-\}$ and $E_-^* := \{\omega \in E^* \mid \omega(Z) = 0 \text{ for all } Z \in E^+\}$. We have the decomposition $E^* = E_+^* \oplus E_-^*$. This splitting induces a decomposition of the k^{th} -exterior bundle of E given by

$$\bigwedge^k E^* = \bigoplus_{p+q=k} E_{(p,q)}^*,$$

where in this case $E_{(p,q)}^* := \Lambda^p E_+^* \wedge \Lambda^q E_-^*$. An alternative definition involves the ring of para-complex numbers $C = \{a + jb \mid j^2 = 1, a, b \in \mathbb{R}\}$.

Definition 2.2. *A para-complex vector bundle is a vector bundle whose fibers are isomorphic to the free module C^n , with the multiplication by j as the para-complex structure.*

The transition functions of such a bundle will then be smooth (not necessarily para-holomorphic) maps $g_{\alpha\beta} : (U_\alpha \cap U_\beta) \times C^n \rightarrow (U_\alpha \cap U_\beta) \times C^n$ satisfying the relations $g_{\alpha\alpha} = Id_{E|_{U_\alpha}}$ and $g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = Id_{E|_{U_\alpha \cap U_\beta \cap U_\gamma}}$. Just as in the complex case, placing restrictions on the transition functions allows us to introduce the concept of para-holomorphic vector bundles, which we discuss later. We now consider the case of the tangent bundle and the interaction of these eigenbundles with the natural Lie algebroid structure on TM that comes from the Lie bracket.

2.2 The Case of the Tangent Bundle

An important case to consider is when $E = TM$. In this case, the pair (M, J) is referred to as an *almost para-complex manifold*. The famous Newlander-Nirenberg theorem in this context relates the almost para-complex structure J on TM to the integrability of the eigenbundles T^+M and T^-M .

Theorem 2.1. (Newlander-Nirenberg [33]): *Let (M, J) be an almost para-complex manifold. Then J is induced by a para-holomorphic atlas if and only if the distributions T^+M and T^-M are integrable.*

We will discuss the concept of para-holomorphicity in the next section. The integrability of the eigenbundles induces coordinates on M of the form $(\mathbf{z}_+, \mathbf{z}_-)$. The partial derivatives with respect to the \mathbf{z}_\pm coordinates span eigenbundles $T^\pm M$. As in the complex case, there is a $(2, 1)$ -tensor N^J , called the *Nijenhuis tensor*, that measures the integrability of these eigenbundles:

$$4N^J(X, Y) = ([X, Y] - J[X, JY] - J[JX, Y] + [JX, JY]). \quad (2)$$

One can easily see that $N^J = 0$ if and only if T^+M and T^-M are integrable. In the case where M is itself a para-complex manifold (meaning that it has an integrable para-complex structure), we can define the notion of a para-holomorphic vector bundle over M using the concept of (p, q) -forms with values in E . If M is a para-complex manifold, then we have a decomposition of the differential forms on M into types and can give an expression for the image of the deRham differential with respect to this decomposition:

$$\begin{aligned} \Omega^k M &= \bigoplus_{p+q=k} \Omega^{(p,q)} M, \\ d\Gamma(\Omega^{(p,q)} M) &\subset \Gamma(\Omega^{(p+1,q)} M) \oplus \Gamma(\Omega^{(p,q+1)} M). \end{aligned}$$

This is summed up in the following theorem.

Theorem 2.2. ([22], [33]) *Let J be an almost para-complex structure on M . Then the following are equivalent:*

1. *J is a para-complex structure.*
2. *T^+M and T^-M are integrable.*

3. $d\Gamma(\Omega^{(1,0)}M) \subset \Gamma(\Omega^{(2,0)}M \oplus \Omega^{(1,1)}M)$ and $d\Gamma(\Omega^{(0,1)}M) \subset \Gamma(\Omega^{(1,1)}M \oplus \Omega^{(0,2)}M)$.
4. $d\Gamma(\Omega^{(p,q)}M) \subset \Gamma(\Omega^{(p+1,q)}M \oplus \Omega^{(p,q+1)}M)$.
5. $N^J = 0$.

The fourth condition allows us to decompose the exterior derivative into the sum of two operators, $d = \partial_+ + \partial_-$, with the property that $\partial_+ : \Omega^{(p,q)}M \rightarrow \Omega^{(p+1,q)}M$ and $\partial_- : \Omega^{(p,q)}M \rightarrow \Omega^{(p,q+1)}M$. From the fact that $d^2 = 0$, we derive the identities $\partial_+^2 = 0$, $\partial_-^2 = 0$ and $\partial_- \partial_+ + \partial_+ \partial_- = 0$ from the fact that the type decomposition of forms is disjoint. In keeping with the complex setting, we obtain a local ∂_\pm -Poincaré lemma and a local $\partial_+ \partial_-$ -lemma.

Lemma 2.1. (*Local ∂_\pm -Poincaré Lemma [22]*): Any ∂_+ -closed form $\omega \in \Omega^{(p,q)}U$, $p \geq 1$, is locally ∂_+ -exact, and similarly for ∂_- when $q \geq 1$.

Lemma 2.2. (*Local $\partial_+ \partial_-$ -Lemma [22]*): Let $\omega \in \Omega^{(1,1)}M$ be a 2-form. Then ω is closed if and only if for every point $x \in M$, there exists an open neighborhood U containing x such that $\omega|_U = \partial_+ \partial_- u$ for some real function u on U .

2.2.1 Para-Holomorphic Bundles

Let (M, J) be a para-complex manifold and $f : M \rightarrow C$ be a smooth function. Following [11], we say that f is *para-holomorphic* when $df \circ J = j \circ df$. We note that any function $f : M \rightarrow C$ can be expressed as $f = \frac{1+j}{2} f_1 + \frac{1-j}{2} f_2$ for some real-valued functions $f_1, f_2 \in C^\infty(M)$. Now, since M is a para-complex manifold, we have local coordinates $(\mathbf{z}_+, \mathbf{z}_-)$, and in these coordinates, the para-Cauchy-Riemann equations read

$$\partial_- f_1 = 0, \quad \partial_+ f_2 = 0. \quad (3)$$

Alternatively, if we model M locally as C^n , then using the $+j$ - and $-j$ -eigencoordinates of j extended to $TM \otimes_{\mathbb{R}} C$, \mathbf{z} and $\bar{\mathbf{z}}$, the para-Cauchy-Riemann equations take on the

familiar form of $\bar{\partial}f = 0$, where $\bar{\partial}$ corresponds to partial derivatives with respect to the \bar{z} coordinates. We desire a simple expression for para-holomorphicity similar to the one in the complex setting. In order to do this, we consider the operators

$$\partial = \frac{1+j}{2}\partial_+ + \frac{1-j}{2}\partial_-, \quad \bar{\partial} = \frac{1+j}{2}\partial_- + \frac{1-j}{2}\partial_+. \quad (4)$$

We can say that $f : M \rightarrow C$ is para-holomorphic if and only if $\bar{\partial}f = 0$, which is useful because it allows us to avoid para-complexifying the tangent bundle in order to obtain $\pm j$ -eigenbundles. Further, we retain the identity $d = \partial + \bar{\partial}$. As was hinted earlier, $E \rightarrow M$ is a para-holomorphic vector bundle over a para-complex manifold if there exists an atlas with para-holomorphic transition functions. With this in mind, we can now examine what it means for the transition functions of a para-complex bundle E over a para-complex manifold M to be para-holomorphic. Given a trivialization $\Psi_{U_\alpha} : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^{2n}$, we can construct the transition functions $\Psi_{U_\alpha} \circ \Psi_{U_\beta}^{-1}(x, v) = (x, \tau_{\alpha\beta}(x)v)$. The component functions of the map $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL_{2n}(\mathbb{R})$ are of interest here when determining what it means for this map to be para-holomorphic. To understand what this means, we first consider that these transition functions should be compatible with the para-complex structure on \mathbb{R}^{2n} . If we choose eigenvectors as the basis, we can express $\tau_{\alpha\beta}$ as a block matrix in the z_+ and z_- coordinates. we find that when

$$J \circ \begin{bmatrix} \rho_{\alpha\beta}^+ & \rho_{\alpha\beta}^- \\ \sigma_{\alpha\beta}^+ & \sigma_{\alpha\beta}^- \end{bmatrix} = \begin{bmatrix} \rho_{\alpha\beta}^+ & \rho_{\alpha\beta}^- \\ \sigma_{\alpha\beta}^+ & \sigma_{\alpha\beta}^- \end{bmatrix} \circ J,$$

$$\begin{bmatrix} \rho_{\alpha\beta}^+ & \rho_{\alpha\beta}^- \\ -\sigma_{\alpha\beta}^+ & -\sigma_{\alpha\beta}^- \end{bmatrix} = \begin{bmatrix} \rho_{\alpha\beta}^+ & -\rho_{\alpha\beta}^- \\ \sigma_{\alpha\beta}^+ & -\sigma_{\alpha\beta}^- \end{bmatrix}$$

and so

$$\tau_{\alpha\beta} = \begin{bmatrix} \rho_{\alpha\beta} & 0 \\ 0 & \sigma_{\alpha\beta} \end{bmatrix}.$$

Further, we will see that in light of Proposition 2.2 and the previous discussion on para-holomorphic functions, and the fact that under the decomposition $E_+ \oplus E_-$ the transition

functions are in the form $\rho_{\alpha\beta} \oplus \sigma_{\alpha\beta}$, the transition functions of the para-complexified bundle would be $\frac{1+j}{2}\rho_{\alpha\beta} + \frac{1-j}{2}\sigma_{\alpha\beta}$. For this reason, we present the following.

Proposition 2.1. [4] *Let $E \rightarrow M$ be a rank $2n$ real vector bundle over a para-complex manifold. Then E is para-complex if and only if the structure sheaf (I.e. the collection of transition functions) admits a decomposition*

$$g_{\alpha\beta} = \begin{bmatrix} \rho_{\alpha\beta} & 0 \\ 0 & \sigma_{\alpha\beta} \end{bmatrix}$$

where $\rho_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ both have rank n . Moreover E is para-holomorphic if and only if $\partial_-(\rho_{\alpha\beta})_{ij} = 0$ and $\partial_+(\sigma_{\alpha\beta})_{ij} = 0$.

We now consider the bundle of forms. Beginning on the manifold M , we can construct the bundle $\Lambda^{(p,q)}E = \Lambda^{(p,q)}M \otimes E$ of E -valued forms on M of type (p, q) . At this point, it is pertinent to address the difference between the real bundle $E \rightarrow M$ with the para-complex structure J and its para-complexification. Recall that the para-complexification of the bundle E is simply the vector bundle E_C whose fibers are $(E_C)_x = E_x \otimes_{\mathbb{R}} C$, $x \in M$. If we replace the para-complex bundle E with E_C , then one can decompose E_C into its $\pm j$ -eigenbundles, and much of the complex theory can be replicated, including the Dolbeault sequence. In order to extend the operator $\bar{\partial}$ defined in equation (4) to one compatible with our understanding of the complexification, we consider the following proposition from [22]:

Proposition 2.2. *For a para-complex manifold M , let $\Lambda^{(p,q)}M_C$ denote the forms of type (p, q) with respect to the para-complexified tangent bundle TM_C . Then there is an (\mathbb{R} -linear) isomorphism*

$$\begin{aligned} \varphi : \Lambda^{(p,q)}M \times \Lambda^{(q,p)}M &\rightarrow \Lambda^{(p,q)}M_C \\ (\eta, \eta') &\mapsto \frac{1+j}{2}\eta + \frac{1-j}{2}\eta' \end{aligned}$$

such that the following diagram commutes:

$$\begin{array}{ccc}
\Omega^{(p,q)}M \times \Omega^{(q,p)}M & \xrightarrow{\varphi} & \Omega^{(p,q)}M_C \\
\downarrow \partial_- \times \partial_+ & & \downarrow \bar{\partial} \\
\Omega^{(p,q+1)}M \times \Omega^{(q+1,p)}M & \xrightarrow{\varphi} & \Omega^{(p,q+1)}M_C
\end{array}
.$$

Proof. Suppose that J is a para-complex structure on M . Then there are coordinates z_+^α on T^+M and z_-^α on T^-M , which allow us to define para-holomorphic coordinates z^α in the usual way, $z^\alpha := \frac{1+j}{2}z_+^\alpha + \frac{1-j}{2}z_-^\alpha$. From the fact that $(\frac{1\pm j}{2})^2 = \frac{1\pm j}{2}$ and $\frac{1+j}{2}\frac{1-j}{2} = 0$, we can see that $\frac{1\pm j}{2}dz^\alpha = \frac{1\pm j}{2}dz_\pm^\alpha$ and $\frac{1\pm j}{2}dz^\alpha = \frac{1\pm j}{2}dz_\mp^\alpha$. This means that if $\eta \in \Omega^{(p,q)}M$, then $\frac{1+j}{2}\eta \in \Omega^{(p,q)}M_C$, and if $\eta' \in \Omega^{(q,p)}M$, then $\frac{1-j}{2}\eta' \in \Omega^{(p,q)}M_C$. Therefore, the image of φ is what we claim it to be. It is also easy to see that φ is \mathbb{R} -linear as

$$\begin{aligned}
\varphi((\eta, \eta') + \alpha(\omega, \omega')) &= \frac{1+j}{2}(\eta + \alpha\omega) + \frac{1-j}{2}(\eta' + \alpha\omega') \\
&= \frac{1+j}{2}\eta + \frac{1-j}{2}\eta' + \alpha\left(\frac{1+j}{2}\omega + \frac{1-j}{2}\omega'\right) \\
&= \varphi(\eta, \eta') + \alpha\varphi(\omega, \omega'),
\end{aligned}$$

for any $\alpha \in \mathbb{R}$. Furthermore, $\ker \varphi = \{0\}$ as $\phi(\eta, \eta') = 0$ implies that $\frac{1+j}{2}\eta = -\frac{1-j}{2}\eta'$, which by multiplying through by $\frac{1+j}{2}$ (or $\frac{1-j}{2}$) gives $\frac{1+j}{2}\eta = 0$ (or $\frac{1-j}{2}\eta' = 0$). Since η and η' are real-valued forms, we can conclude that they are the zero form on each of their respective spaces, thus φ is an isomorphism. For reference, the inverse of φ is given by $(\eta, \eta') = (Re(\omega) + Im(\omega), Re(\omega) - Im(\omega))$. Finally, we can show that the diagram

commutes. Recall that $\frac{1+j}{2} \frac{\partial}{\partial z_{\mp}^{\alpha}} = \frac{1+j}{2} \frac{\partial}{\partial \bar{z}^{\alpha}}$, implying

$$\begin{aligned} \varphi(\partial_- \eta, \partial_+ \eta') &= \frac{1+j}{2} \partial_- \eta + \frac{1-j}{2} \partial_+ \eta' \\ &= \frac{1+j}{2} \bar{\partial} \eta + \frac{1-j}{2} \bar{\partial} \eta' \\ &= \bar{\partial} \left(\frac{1+j}{2} \eta + \frac{1-j}{2} \eta' \right) \\ &= \bar{\partial} \varphi(\eta, \eta'). \end{aligned}$$

□

The interesting part of Proposition 2.2 is that it appears as though the operator $\partial_- \oplus \partial_+$ plays the same role as $\bar{\partial}$ does in the complex setting. In the complex setting, the operator $\bar{\partial}$ sets up the Dolbeault sequence, which is the cohomological sequence of spaces of differential forms, and maps type (p, q) -forms to type $(p, q+1)$ -forms. For para-complex bundles, the stand-in for this operator is $\partial_- \oplus \partial_+$ acting on $(p, q) \oplus (q, p)$ -forms, and so we have an analogous Dolbeault sequence in the para-complex setting:

$$\dots \xrightarrow{\partial_- \oplus \partial_+} \Omega^{(p,q)} M \oplus \Omega^{(q,p)} M \xrightarrow{\partial_- \oplus \partial_+} \Omega^{(p,q+1)} M \oplus \Omega^{(q+1,p)} M \xrightarrow{\partial_- \oplus \partial_+} \dots$$

The choice to swap (p, q) for (q, p) comes directly from Proposition 2.2 and ensures that, as in the complex case, $\Omega^{(p,0)} M_C \cong \Omega^{(p,0)} M \oplus \Omega^{(0,p)} M$ is a para-holomorphic vector bundle (Where in the complex setting, $\Omega^{(0,p)}(M)$ is holomorphic [33]). We can extend both ∂_+ and ∂_- to $\Omega^{(p,q)} E$ in the same way as in the complex case:

$$\partial_{\pm}(\omega_1, \dots, \omega_{2n}) = (\partial_{\pm} \omega_1, \dots, \partial_{\pm} \omega_{2n}), \quad (5)$$

where $\omega_i \in \Omega^{(p,q)}(M)$. Additionally, we can consider the bundle

$$\Omega^{(p,q)}(E) := \Omega^{(p,q)}(M) \otimes E_+ \oplus \Omega^{(q,p)}(M) \otimes E_-. \quad (6)$$

In the case where E is para-holomorphic, we can extend the operator $\partial_- \oplus \partial_+$ to the local sections $\omega = (\omega_1^+, \dots, \omega_n^+) \oplus (\omega_1^-, \dots, \omega_n^-) \in \Omega^{(p,q)}(E)$, where ω_i^+ are local (p, q) -forms and the ω_i^- are local (q, p) -forms, in a way that does not depend on the chosen trivialization, as the E_{\pm} trivialization is invariant under ∂_{\mp} . Both ∂_+ and ∂_- satisfy the Leibniz rule, and therefore so does $\partial_- \oplus \partial_+$. Using this convention, we retain the sequence

$$\dots \xrightarrow{\partial_- \oplus \partial_+} \Omega^{(p,q)}(E) \xrightarrow{\partial_- \oplus \partial_+} \Omega^{(p,q+1)}(E) \xrightarrow{\partial_- \oplus \partial_+} \dots,$$

with the special property that $(\partial_- \oplus \partial_+)^2 = 0$.

Definition 2.3. *An operator $\partial_- \oplus \partial_+ : \Omega^{(p,q)}(E) \rightarrow \Omega^{(p,q+1)}(E)$ for all $0 \leq p, q \leq n$ on a para-complex vector bundle E that satisfies the Leibniz rule is called a pseudo-paraholomorphic structure. If, moreover $(\partial_- \oplus \partial_+)^2 = 0$, then $\partial_- \oplus \partial_+$ is called a para-holomorphic structure.*

Just as in the holomorphic case, we have the following theorem.

Theorem 2.3. *A para-complex vector bundle (E, J) is para-holomorphic if and only if it has a para-holomorphic structure.*

2.2.2 Para-Hermitian Manifolds

In contrast to the theory of complex manifolds, where the appropriate objects of study in the Hermitian context are Riemannian metrics, para-complex geometry is concerned with pseudo-Riemannian metrics of split signature. To begin:

Definition 2.4. *A pseudo-Riemannian metric g is a smooth, symmetric, non-degenerate section of $E^* \otimes E^*$. Given a para-complex vector bundle (E, J) , g is said to be compatible with J if $g(JX, Y) = -g(X, JY)$ for any sections $X, Y \in \Gamma(E)$. In this case, we refer*

to g as a para-Hermitian metric. We define the induced fundamental form $F \in \Lambda^2 E^*$ by $F(X, Y) = g(X, JY)$.

We now quote some useful results from Bejan [4].

Proposition 2.3. *Let (J, g) be a para-Hermitian structure on E . Then:*

1. J is a para-complex structure on E and $\text{rank}(E) = 2n$.
2. g is a pseudo-Riemannian structure of signature (n, n) .
3. The eigenbundles E^+ and E^- are maximally isotropic with respect to g .

Proof. Recall that g is non-degenerate, so there exists $s_1 \in \Gamma(E)$ such that $g(s_1, s_1) \neq 0$. We may also assume that $g(s_1, s_1) > 0$, with the negative case being similar. By the definition of compatibility and the fact that g is symmetric, we have that s_1 is orthogonal to Js_1 with respect to g . If $\text{rank}(E) > 2$, then we let $s_2 \in \Gamma(E)$ be orthogonal to both s_1 and Js_1 with respect to g . It follows that Js_2 is also orthogonal to both s_1 and Js_1 , as well as s_2 , and so by continuing this process, we get a local orthogonal basis $\mathcal{B} = \{s_1, Js_1, \dots, s_n, Js_n\}$, and so $\text{rank}(E) = 2n$.

The signature of g is (n, n) as $g(s_i, s_i) > 0$, and so $g(Js_i, Js_i) = -g(s_i, s_i) < 0$. Let us denote $u_i = s_i + Js_i$ and $v_i = s_i - Js_i$. We can see that, as before, $\{u_i\}$ and $\{v_i\}$ are local bases for the eigenbundles E^+ and E^- respectively. Finally, we simply have to show that g is identically zero when restricted to either bundle. The maximality condition is satisfied as $\text{rank}(E^\pm) = n$, and $E = E^+ \oplus E^-$. To this end, we see that for

the basis vectors $\{u_i\}$, we have

$$\begin{aligned} g(s_i + Js_i, s_j + Js_j) &= g(s_i, s_j) + g(Js_i, Js_j) + g(Js_i, s_j) + g(s_i, Js_j) \\ &= g(s_i, s_j) - g(s_i, s_j) + g(s_i, Js_j) - g(s_i, Js_j) \\ &= 0. \end{aligned}$$

And a similar calculation shows that g also vanishes on E^- . \square

In keeping with the previous section, the addition of a compatible pseudo-Riemannian metric induces a further reduction in the structure sheaf of the para-Hermitian bundle, consisting of the collection of transition functions over some choice of open cover of M . We have the well known result due to Bejan [4]:

Theorem 2.4. *A vector bundle E over M admits a para-Hermitian structure if and only if the structure group acting on its frame bundle can be reduced to the group of matrices of the form:*

$$\left\{ g_{\alpha\beta}(p) = \begin{pmatrix} S & 0 \\ 0 & (S^t)^{-1} \end{pmatrix} : S \in GL_k(\mathbb{R}), p \in M \right\}.$$

Proof. Suppose that E admits a para-Hermitian structure (J, g) . The first thing to note is that we require a compatible pseudo-Riemannian metric, whereas in the complex case, the complex structure is enough to provide a reduction of $GL_{2n}(\mathbb{C}) \hookrightarrow GL_n(\mathbb{R})$. In the complex case, it is possible to construct local linearly independent frames just using the complex structure, but the same method does not work in the para-complex case unless a compatible metric is introduced. This is a consequence of the classification of almost vector cross-product structures. With this in mind, we proceed as follows.

Let (E, π, M) be a vector bundle of rank $2n$, and let J be an endomorphism of E such that $J^2 = Id_E$. Let $\varphi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^{2n}$, and $\varphi_\beta : \pi^{-1}(U_\beta) \rightarrow U_\beta \times \mathbb{R}^{2n}$ be two

local trivializations such that $U_\alpha \cap U_\beta \neq \emptyset$. Consider the action of J on \mathbb{R}^{2n} , given by $J_\alpha = \varphi_\alpha \circ J \circ \varphi_\alpha^{-1}$ and $J_\beta = \varphi_\beta \circ J \circ \varphi_\beta^{-1}$. Furthermore, we let $\varphi_\beta \circ \varphi_\alpha^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)(v))$ for $(p, v) \in U_\alpha \cap U_\beta \times \mathbb{R}^{2n}$. Recall that g is non-degenerate, so there exists $s_1 \in \Gamma(E)$ such that $g(s_1, s_1) \neq 0$. We may also assume that $g(s_1, s_1) > 0$, with the negative case being similar. By the definition of compatibility and the fact that g is symmetric, we have that s_1 is orthogonal to $J s_1$ with respect to g . If $\text{rank}(E) > 2$, then we let $s_2 \in \Gamma(E)$ be orthogonal to both s_1 and $J s_1$ with respect to g . It follows that $J s_2$ is also orthogonal to both s_1 and $J s_1$, as well as s_2 , and so by continuing this process, we get a local orthogonal frame $\mathcal{B}_\alpha = \{s_1, J_\alpha s_1, \dots, s_n, J_\alpha s_n\}$ on the image of φ_α . Following as in the complex case, we get a reduction into matrices of the form

$$\tau_{\alpha\beta} = \begin{pmatrix} a_{11} & b_{11} & \cdots & a_{n1} & b_{n1} \\ b_{11} & a_{11} & \cdots & b_{n1} & a_{n1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{1n} & b_{1n} & \cdots & a_{nn} & b_{nn} \\ b_{1n} & a_{1n} & \cdots & b_{nn} & a_{nn} \end{pmatrix}$$

In the para-complex setting, we may use the natural ± 1 -eigenbasis, which is given by $\mathcal{B}_\alpha = \{\frac{1}{\sqrt{2}}(s^1 + J_\alpha s^1), \dots, \frac{1}{\sqrt{2}}(s^n + J_\alpha s^n), \frac{1}{\sqrt{2}}(s^1 - J_\alpha s^1), \dots, \frac{1}{\sqrt{2}}(s^n - J_\alpha s^n)\}$. In this basis, the transformation takes the form

$$\tau_{\alpha\beta} = \begin{pmatrix} a_{11} + b_{11} & a_{21} + b_{21} & \cdots & 0 & 0 \\ a_{12} + b_{12} & a_{22} + b_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{(n-1)(n-1)} - b_{(n-1)(n-1)} & a_{n(n-1)} - b_{n(n-1)} \\ 0 & 0 & \cdots & a_{(n-1)n} - b_{(n-1)n} & a_{nn} - b_{nn} \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & V \end{pmatrix} \quad (7)$$

Now, we make an observation: $\tau_{\alpha\beta}$ maps the ± 1 -eigenspace of J_α into the ± 1 -eigenspace of J_β . With this in mind, we construct a basis $\mathcal{B}_\beta = \{\frac{1}{\sqrt{2}}(s^1 + J_\beta s^1), \dots, \frac{1}{\sqrt{2}}(s^n +$

$J_\beta s^n), \frac{1}{\sqrt{2}}(s^1 - J_\beta s^1), \dots, \frac{1}{\sqrt{2}}(s^n - J_\beta s^n)\}$. Using this basis, we remark that

$$J_\beta = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}, \quad g_\beta = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

Knowing that $\tau_{\alpha\beta}$ vanishes on complimentary eigenspaces allows us to keep the form that we arrived at in equation (7.2.1). We also note that by definition $g_\alpha = \tau_{\alpha\beta}^t \circ g_\beta \circ \tau_{\beta\alpha}$, and so

$$\begin{aligned} \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} &= \begin{pmatrix} S^t & 0 \\ 0 & V^t \end{pmatrix} \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & V \end{pmatrix} \\ \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} &= \begin{pmatrix} 0 & S^t V \\ V^t S & 0 \end{pmatrix}. \end{aligned}$$

And thus we have $S^t V = 0$, or $V = (S^t)^{-1}$. Therefore, the structure group has a reduction into matrices of the form

$$\tau_{\alpha\beta}(p) = \begin{pmatrix} S(p) & 0 \\ 0 & (S^t)^{-1}(p) \end{pmatrix}, \quad S(p) \in GL_k(\mathbb{R}), \quad p \in M.$$

Conversely, suppose that $\tau_{\alpha\beta}$ has the desired form. Simply picking

$$g|_{U_\alpha} = g_\alpha = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

and then extending g_α to all of E as

$$g_\beta = \tau_{\alpha\beta}^t \circ g_\alpha \circ \tau_{\alpha\beta} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

gives a well defined metric. As well, we get two sub-bundles of E as the invariant spaces of $\tau_{\alpha\beta}$. Simply defining J to be the identity on one of them and minus the identity on

the other gives a para-complex structure, and one can see that on the overlap,

$$\begin{aligned}
J_\beta &= \tau_{\alpha\beta}^{-1} \circ J_\alpha \circ \tau_{\alpha\beta} \\
&= \begin{pmatrix} S^{-1} & 0 \\ 0 & S^t \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & (S^t)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}.
\end{aligned}$$

Thus, from the reduction of the structure group, we obtain a para-Hermitian structure. \square

In this case, if we assume that E is also para-holomorphic, then it is quite easy to see that the transition functions have to be constant. This is because $\partial_-(S)_{ij} = 0$, and so S can only be a function of the z^+ variables. But then we could say that the transition functions $(S^T)^{-1}$ can only be functions of the z_- variables. These matrices are inverse transposes of each other, and so one can see that they must be constant. It is quite easy to show that when g is compatible with the almost para-complex structure, $F \in \Omega^{(1,1)}M$. If the para-complex structure is integrable and F is closed, then we know by the local $\partial_+\partial_+$ -lemma that locally $F = \partial_+\partial_-u$ for some real function u , called the *local para-Kähler potential*.

Definition 2.5. *Let (M, J, g, F) be an almost para-Hermitian manifold. We say that this structure is para-Kähler and refer to g as the para-Kähler metric if and only if J is integrable and $dF = 0$. In summary:*

$$g \text{ is para-Kähler} \Leftrightarrow \begin{cases} N^J & = 0 \\ dF & = 0 \end{cases}. \quad (8)$$

The utility of this definition is that since the associated 2-form is closed, the para-Kähler metric g can be written locally in term of the local para-Kähler potential,

giving

$$g_{i\bar{j}} = \frac{\partial^2 u}{\partial z_+^i \partial z_-^{\bar{j}}}.$$

Many interesting examples of para-complex, para-Hermitian, and para-Kähler manifolds exist. For an overview of some historical examples as well as a survey of the development of the field of para-complex geometry, see [9], in particular section 3. We will now present some illustrative and interesting examples.

Example 2.1. *The natural almost para-Kähler structure on T^*M :*

This example is due Bejan in [3], and will be informative to have in mind when we talk about para-complex connections in section 4. Let M be a smooth manifold and let ∇ be a torsion free connection on M . The connection ∇ induces a decomposition $T_\xi(T^*M) = H_\xi(T^*M) \oplus V_\xi(T^*M)$, where $V_\xi(T^*M)$ is the vertical bundle with respect to the projection $\pi : T^*M \rightarrow M$, and $H_\xi(T^*M)$ is the horizontal bundle induced by the connection. In local coordinates $\{x^i, p^i\}$ for T^*M , we have that

$$\begin{aligned} V_{(x,p)}(T^*M) &= \left\{ v^i \frac{\partial}{\partial p^i} \mid v^i \in C^\infty(T^*M) \right\}, \\ H_{(x,p)}(T^*M) &= \left\{ v^i \frac{\partial}{\partial x^i} + w^i \frac{\partial}{\partial p^i} \mid v^i, w^i \in C^\infty(T^*M), \text{ and } w^i + \Gamma_{jk}^i(x) v^j p^k = 0 \right\}, \end{aligned}$$

where Γ_{jk}^i are the Christoffel symbols of the connection ∇ . Under this decomposition, we can define an almost para-complex structure J with +1-eigenbundle equal to the horizontal bundle, and -1-eigenbundle equal to the vertical bundle. The natural para-Hermitian metric in the coordinates $\{x^i, p^i\}$ is then given by

$$g = \begin{pmatrix} -2p^k \Gamma_{jk}^i(x) & \delta_j^i \\ \delta_j^i & 0 \end{pmatrix}.$$

It is easy to see that both $H(T^*M)$ and $V(T^*M)$ are maximally isotropic with respect to this metric. Further, Bejan gives us the following Theorem:

Theorem 2.5. *With the above notation, (M, g, J) is an almost para-Kähler structure on the total space of the cotangent bundle T^*M , whose fundamental 2-form F satisfies $F = d\theta$ (where θ denotes the Liouville form), and thus coincides with the canonical symplectic structure on T^*M . Moreover if ∇ has vanishing curvature, then the structure (M, g, J) is para-Kähler.*

Example 2.2. *The Pseudosphere $S^{(3,3)}$.*

In complex geometry, there is the famous question of whether the sphere S^6 admits an integrable complex structure. There are many known almost complex structures, but none so far discovered are integrable. In this vein of exploration, Libermann in [26] constructs an almost para-Hermitian structure on the pseudosphere $S^{(3,3)}$ that is not integrable, where $S^{(3,3)}$ is the submanifold of \mathbb{R}^7 determined by the polynomial

$$-x_1^2 - x_2^2 - x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 = 1.$$

In [35], Smolentsev reconstructs this almost para-Hermitian structure by considering the quotient $G_2^*/SL(3; \mathbb{R})$, where G_2^* is the non-compact real form of the exceptional Lie group G_2 , and shows that its Nijenhuis tensor does not vanish. Interestingly, in section 5.1 of [35], Smolentsev finds an integrable para-complex structure on $S^{(3,3)}$ by showing that $S^{(3,3)}$ is diffeomorphic to the cylinder $D^3 \times S^3$, and so the question of whether integrable para-complex structures exist on $S^{(3,3)}$ is closed.

3 Lie Algebroids, Courant Algebroids and Lie-Bialgebroids

We now move towards a discussion of algebroids. Roughly, algebroids allow us to combine the generality of para-Hermitian vector bundles with the integrability results that only appear in the context of the tangent bundle. By endowing the vector bundle with a bracket that has certain properties, we can ask questions about integrability of the eigenbundles that we would be unable to do otherwise. The two main objects of study here are Lie algebroids and Courant algebroids. A Lie algebroid is the most straightforward generalization of the tangent bundle, and a Courant algebroid is a generalization of the bundle $TM \oplus T^*M$, as we discussed in the introduction.

The objects that we intend to retain are the non-degenerate symmetric bracket $\langle \cdot, \cdot \rangle$ on the vector bundle $E \rightarrow M$ and the almost para-complex structure J . Recall that the eigenbundles of J are maximally isotropic with respect to $\langle \cdot, \cdot \rangle$. The fundamental take-away from the concept of a para-Hermitian manifold is that when the subbundles $T^\pm M$ are integrable, they form subalgebroids of the Lie algebroid TM . Therefore, the natural object to consider is the Lie bialgebroid (E_+, E_-) consisting of dual pairs of Lie algebroids. As we will see, Lie bialgebroids are related to Courant algebroids in precisely the way that makes the Courant algebroid viable for a generalization of para-Hermitian geometry, which we then discuss in the next section.

3.1 Lie Algebroids

In the context of vector bundles, we lack a concept of integrability for the eigenbundles of a para-complex structure. Further, another weakness is that there is no accompanying derivative operator to decompose that acts on functions to produce sections of the dual bundle (as opposed to sections of $\Omega(M) \otimes E$). It's clear that the two objects that are

missing are an anchor map $\rho : E \rightarrow TM$, which will allow us to pull back the exterior derivative on M , and a bracket on the space of sections of E . For this reason, we introduce the notion of Lie algebroid using the formalism in [34].

Definition 3.1. *A Lie algebroid is a vector bundle $A \rightarrow M$ together with an anti-symmetric bracket $[\cdot, \cdot]_A$ on the space of sections $\Gamma(A)$ and a bundle map $a : A \rightarrow TM$ called the anchor. The bracket and the anchor satisfy the following conditions:*

1. For all $\sigma, \tau \in \Gamma(A)$, $a([\sigma, \tau]_A) = [a(\sigma), a(\tau)]_{TM}$.
2. For all $\sigma, \tau \in \Gamma(A)$ and $f \in C^\infty(M)$, $[\sigma, f\tau]_A = f[\sigma, \tau]_A + (a(\sigma)f)\tau$.

One can use this bracket and anchor to produce a derivation of degree 1, d_A , on the space of sections $\Gamma(\bigwedge^* A^*)$ by using an identical form of the Cartan formula. For any $\omega \in \Gamma(\bigwedge^p A^*)$, we can define $d_A\omega \in \Gamma(\bigwedge^{p+1} A^*)$ by

$$\begin{aligned} d_A\omega(\sigma_0, \dots, \sigma_p) &= \sum_{i=0}^p (-1)^i (a(\sigma_i))\omega(\sigma_1, \dots, \hat{\sigma}_i, \dots, \sigma_p) \\ &\quad + \sum_{0 \leq i < j \leq p} (-1)^{i+j} \omega([\sigma_i, \sigma_j]_A, \sigma_0, \dots, \hat{\sigma}_i, \dots, \hat{\sigma}_j, \dots, \sigma_p) \end{aligned} \quad (9)$$

Note that $d_A^2 = 0$. Moreover, the Lie bracket on A is recoverable from d_A since for any $\omega \in \Gamma(\bigwedge^1 A^*)$,

$$\omega([\sigma, \tau]_A) = a(\sigma)\omega(\tau) - a(\tau)\omega(\sigma) - d_A\omega(\sigma, \tau).$$

All of the structural identities of the Lie bracket are induced by the fact that $d_A^2 = 0$ and can be discovered by considering $d_A^2 g(\sigma, f\tau)$ for arbitrary $f, g \in C^\infty(M)$. Further, it is possible to extend the Lie bracket to multi-vector fields (sections of $\bigwedge^* A$) via the following rules. For $\sigma \in \Gamma(\bigwedge^p A)$, we let the degree of σ be $|\sigma| = p$.

1. $[\sigma, \tau]_A = -(-1)^{(|\sigma|-1)(|\tau|-1)}[\tau, \sigma]$.

2. For $\sigma \in \Gamma(A)$ and $f \in C^\infty(M)$, $[\sigma, f] = a(\sigma)f$.

3. $[\sigma, \cdot]_A$ is a derivation of degree $|\sigma| - 1$, meaning

$$[\sigma, \tau \wedge \xi]_A = [\sigma, \tau]_A \wedge \xi + (-1)^{(|\sigma|-1)(|\tau|)} \tau \wedge [\sigma, \xi]_A, \quad |[\sigma, \tau]| = |\sigma| + |\tau| - 1.$$

The derivation d_A and interior derivative ι , defined by contracting sections of $\bigwedge A^*$ with sections of A , can be used to define the Lie derivative operator $\mathcal{L}_X^A = [d_A, \iota_X]$. All of the usual commutation relations satisfied by these operators continue to hold in this setting.

Example 3.1. *The tangent bundle, TM , to a smooth manifold M .*

As we remarked in the introduction to this section, the simplest Lie algebroid is the tangent bundle TM with the Lie bracket as the bracket on the space of sections and the anchor given by the identity map.

Example 3.2. *The action Lie algebroid $\mathfrak{g} \times M \rightarrow M$.*

A Lie algebra action of \mathfrak{g} on M is a map $\sigma : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ such that $\sigma([X, Y]_{\mathfrak{g}}) = [\sigma(X), \sigma(Y)]_{TM}$ for $X, Y \in \mathfrak{g}$. If one considers the trivial bundle $\mathfrak{g} \times M$, we have an anchor given by $\rho(X, p) = \sigma(X)|_p$ and can extend the bracket on \mathfrak{g} to $\mathfrak{g} \times M$ by derivations over smooth functions on M to obtain a Lie algebroid called the *action Lie algebroid* of \mathfrak{g} on M . Given an element of the dual Lie algebra $\xi \in \mathfrak{g}^*$, one can compute the derivative of ξ with respect to the induced derivation $d_{\mathfrak{g}}$ as

$$d_{\mathfrak{g}}\xi(X, Y) = \sigma(X)\xi(Y) - \sigma(Y)\xi(X) - \xi([X, Y]_{\mathfrak{g}}),$$

which when X and Y are constant sections, meaning elements of \mathfrak{g} , reduces to $d_{\mathfrak{g}}\xi(X, Y) = -\xi([X, Y]_{\mathfrak{g}})$, which is the standard Poisson structure on \mathfrak{g}^* [10].

3.2 Courant Algebroids

Now, the Lie algebroid structure may be sufficient to define a para-complex structure by simply requiring the eigenbundles to be closed under the bracket structure, but this definition does not extend easily to the para-Hermitian setting. The more natural objects in this setting are the Courant algebroid and Lie bialgebroid, which we also draw from [34].

Definition 3.2. *Consider the vector bundle $E \rightarrow M$ equipped with a symmetric bilinear form $\langle \cdot, \cdot \rangle$, a skew-symmetric bracket $[\cdot, \cdot]_E : \wedge^2 E \rightarrow E$, and an anchor map $\rho : E \rightarrow TM$. We define $d_E : C^\infty(M) \rightarrow \Gamma(E^*)$ by $d_E = \rho^*d$, meaning for a section $e \in \Gamma(E)$ and a smooth function $f \in C^\infty(M)$, $d_E f(e) = \rho(e)f$, and the map T given by $T(e_1, e_2, e_3) = \frac{1}{6}\langle [e_1, e_2], e_3 \rangle + c.p.$. Then E is a Courant algebroid if the following conditions are satisfied. For all $e, e_1, e_2, e_3, h_1, h_2 \in \Gamma(E)$ and $f \in C^\infty(M)$,*

1. $J(e_1, e_2, e_3) = \beta^{-1}d_E T(e_1, e_2, e_3)$,
2. $\rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)]$,
3. $[e_1, f e_2] = f[e_1, e_2] + (\rho(e_1)f)e_2 - \frac{1}{2}\langle e_1, e_2 \rangle \beta^{-1}d_E f$,
4. $\rho \circ \beta^{-1} \circ d_E = 0$,
5. $\rho(e)\langle h_1, h_2 \rangle = \langle [e, h_1], h_2 \rangle + \langle h_1, [e, h_2] \rangle + \frac{1}{2}d_E \langle e, h_1 \rangle(h_2) + \frac{1}{2}d_E \langle e, h_2 \rangle(h_1)$,

where $J(e_1, e_2, e_3) = [[e_1, e_2], e_3] + [[e_2, e_3], e_1] + [[e_3, e_1], e_2]$ and $\beta : E \rightarrow E^*$ is the non-degenerate bundle isomorphism induced by $\langle \cdot, \cdot \rangle$.

A special kind of Courant algebroid that will be useful for consideration later are exact Courant algebroids. An exact Courant algebroid is a Courant algebroid that

admits a fiber-wise decomposition $E_x = T_x M \oplus T_x^* M$, $x \in M$. This is equivalent to the following sequence being exact:

$$0 \longrightarrow T^*M \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \longrightarrow 0,$$

where in this case, ρ^* is the dual map to ρ under the pairing $\langle \cdot, \cdot \rangle$, meaning $\langle \rho^* \xi, e \rangle = \xi(\rho(e))$. Exactness allows us to say that $\ker(\rho) = \text{Im}(\rho^*)$ and $\ker(\rho^*) = 0$.

Given an exact Courant algebroid E , one can deform the Courant bracket on E by any closed 3-form in the following way.

Definition 3.3. *Let $E \rightarrow M$ be an exact Courant algebroid and let $\eta \in \Omega^3(M)$. We define the η -twist of the Courant bracket $[\cdot, \cdot]$ by*

$$[e_1, e_2]_\eta = [e_1, e_2] + \iota_{e_1} \iota_{e_2} \rho^* \eta. \quad (10)$$

Then, $[\cdot, \cdot]_\eta$ is a Courant bracket precisely when η is closed [34].

In the definition of the Courant algebroid, we have introduced a derivative operator d_E that will play a special role. Note that d_E can be extended to all of $\bigwedge^* E^*$ via the Cartan formula, as in equation (9). There remains the question of what morphisms between Courant algebroids should look like. At the bare minimum, we should ask that a map $\Psi : E \rightarrow F$ between Courant algebroids preserves the anchor and the symmetric bilinear form (meaning that Ψ should be an isometry). The way that Ψ should interact with the bracket is informed by the following lemma.

Lemma 3.1. *Let E, F be Courant algebroids over M and $\Psi : E \rightarrow F$ be a vector bundle map that preserves the metric and anchor ($\rho_F \circ \Psi = \rho_E$). Then $\Psi \circ d_E = d_F$ if and only if Ψ preserves the Courant brackets up to an element of $\rho_E^*(\Omega^2(M)) \otimes \rho_F^*(\Omega^1(M))$.*

Proof. All that we have to show is that if $\Psi \circ d_E = d_F$, then $\Psi([e_1, e_2]_E) - [\Psi(e_1), \Psi(e_2)]_F$ is tensoral in both arguments. Since the brackets are anti-symmetric, we need only show this for one entry. For $e_1, e_2 \in \Gamma(E)$, $f \in C^\infty(M)$, we have

$$\begin{aligned} \Psi([e_1, fe_2]_E) - [\Psi(e_1), \Psi(fe_2)]_F &= \Psi \left(f[e_1, e_2]_E + \rho_E(e_1)fe_2 - \frac{1}{2}\langle e_1, e_2 \rangle d_E f \right) \\ &\quad - \left(f[\Psi(e_1), \Psi(e_2)]_F + \rho_F(\Psi(e_1))fe_2 - \frac{1}{2}\langle \Psi(e_1), \Psi(e_2) \rangle d_F f \right) \\ &= f(\Psi([e_1, e_2]_E) - [\Psi(e_1), \Psi(e_2)]_F) \\ &\quad + \frac{1}{2}\langle \Psi(e_1), \Psi(e_2) \rangle (\Psi(d_E f) - d_F f). \end{aligned}$$

And so we can see that $\phi = \Psi([e_1, e_2]_E) - [\Psi(e_1), \Psi(e_2)]_F$ is tensoral if and only if $\Psi \circ d_E = d_F$. By the fact that Ψ preserves the anchor, we find that $\rho_F(\phi(e_1, e_2)) = 0$, and so by exactness $\phi(e_1, e_2) = \rho_F^*(\omega_{e_1, e_2})$, for $\omega \in \Omega^1(M)$. Finally, since E and F are exact, we can dualize the expression $\rho_E = \rho_F \circ \Psi$ to obtain $\rho_F^* = \Psi \circ \rho_E^*$, and hence $\phi(\rho_E^*(\xi), \rho_E^*(\zeta)) = 0$ for all $\eta, \zeta \in \Omega^1(M)$, meaning $\phi \in \rho_E^*(\Omega^2(M)) \otimes \rho_F^*(\Omega^1(M))$, as desired. \square

Based on this understanding of the way that isometric vector bundle morphisms that preserve the anchor interact with the bracket, we introduce the following definition.

Definition 3.4. *Let $E, F \rightarrow M$ be two exact Courant algebroids and let $\Psi : E \rightarrow F$ be an isometry that preserves the anchors of E and F . Then we call Ψ an almost Courant algebroid morphism if $\Psi \circ d_E = d_F$. We call Ψ Courant algebroid morphism if it preserves the bracket.*

If $\Psi : E \rightarrow E$ is an almost Courant algebroid automorphism, then $\Psi([\cdot, \cdot])$ is the Courant bracket on E up to a ϕ -twist by a closed form $\phi \in \Omega^3(M)$. We mention these maps in lieu of Example 5.2, wherein it makes more sense to represent the double tangent bundle to a Lie group $\mathbb{T}G$ as $G \times (\mathfrak{g} \oplus \bar{\mathfrak{g}})$, but a Courant algebroid morphism is needed

to make this identification in a way that allows us to extract useful information about the natural Courant algebroid on $\mathbb{T}G$. The goal now is to examine the situation where E admits a decomposition into dual subbundles for which the restriction of the Courant bracket corresponds to a Lie bracket on the subbundles.

3.3 Lie Bialgebroids

There are specific subbundles of Courant algebroids that are of interest in the classical theory, namely, Dirac subbundles. A *Dirac subbundle* is a maximally isotropic subbundle L whose sections are closed under the bracket on E . In this case, we see that the function T from the definition of a Courant algebroid has the property that $T|_L = 0$ by the fact that L is Lagrangian and closed under the bracket, and so the Jacobiator J is identically zero by condition 1 for Courant algebroids, meaning the bracket $[\cdot, \cdot]_E$ is a Lie bracket on L . Further, for any $l \in \Gamma(L)$, $[l, \cdot]_E$ acts by derivations on sections of L , and so $L \rightarrow M$ is a Lie algebroid. We now consider the special case where $E = A \oplus A^*$ for two Dirac subbundles A and A^* of E . Our choice of notation here comes from the fact that A and A^* must be maximally isotropic and hence disjoint, and so we can identify the second subspace as the dual to A under the inner-product. In this case, we have the following definition.

Definition 3.5. *Suppose that A and A^* are both Lie algebroids over M . We say that the pair (A, A^*) is a Lie bialgebroid if d_A is a derivation of the Schouten bracket on A^* . I.e., $d_A[\xi, \eta]_{A^*} = [d_A\xi, \eta]_{A^*} + [\xi, d_A\eta]_{A^*}$ for $\xi, \eta \in \Gamma(\wedge A^*)$.*

It is possible to cast the theory of Lie bialgebroids in terms of Courant algebroids using the following theorems from [27].

Theorem 3.1. *Let (A, A^*) be a Lie bialgebroid with corresponding anchors a and a_* . Then $(E = A \oplus A^*, \rho, \langle \cdot, \cdot \rangle_+, [\cdot, \cdot])$ is a Courant algebroid, where $\rho = a + a_*$, $\langle X + \xi, Y + \eta \rangle_+$*

$\eta\rangle_{\pm} = \langle \xi, Y \rangle \pm \langle \eta, X \rangle$, and for $e_1 = X + \xi$ and $e_2 = Y + \eta$,

$$\begin{aligned} [e_1, e_2] &= ([X, Y]_A + \mathcal{L}_{\xi}^{A^*} Y - \mathcal{L}_{\eta}^{A^*} X - \frac{1}{2} d_{A^*} \langle e_1, e_2 \rangle_-) \\ &\quad + ([\xi, \eta]_{A^*} + \mathcal{L}_X^A \eta - \mathcal{L}_Y^A \xi + \frac{1}{2} d_A \langle e_1, e_2 \rangle_-) \end{aligned} \quad (11)$$

We will return to this choice of $\langle \cdot, \cdot \rangle_-$ later as it will have significance in the para-Hermitian setting. It is easy to see that, in this case, we have a splitting of the derivative operator $d_E = d_A + d_{A^*}$. This construction is dual in the sense that if one begins with a Courant algebroid, we have the following:

Theorem 3.2. *Let $(E, \rho, \langle \cdot, \cdot \rangle, [\cdot, \cdot])$ be a Courant algebroid, and suppose that L_1 and L_2 are two maximally isotropic subbundles of E that are both closed under the Courant bracket. Then $E = L_1 \oplus L_2$ and (L_1, L_2) is a Lie bialgebroid, where L_2 is considered the dual to L_1 under $\langle \cdot, \cdot \rangle$.*

Definition 3.6. *A Courant algebroid E that admits a direct sum decomposition $E = A \oplus A^*$ into isotropic subbundles is called a proto-bialgebroid. If one of A or A^* is closed under the bracket, then E is called a quasi-Lie bialgebroid.*

The obstruction to any Courant algebroid E being a proto-bialgebroid is simply the existence of a dual pair of Lagrangian subbundles. Given a proto-bialgebroid $E = A \oplus A_*$, we can ask what the obstruction to closure with respect to the bracket on E for each of the subbundles is. Given sections $\sigma_1, \sigma_2 \in \Gamma(A)$ and $\tau_1, \tau_2 \in \Gamma(A^*)$, we define the sections $\phi \in \Gamma(\wedge^3(A^*))$ and $\psi \in \Gamma(\wedge^3(A))$ by

$$\begin{aligned} \iota_{\sigma_1} \iota_{\sigma_2} \phi &= \pi_{A^*} [\sigma_1, \sigma_2]_E, \\ \iota_{\tau_1} \iota_{\tau_2} \psi &= \pi_A [\tau_1, \tau_2]_E. \end{aligned} \quad (12)$$

We can be sure that ϕ is a 3-form on A , as the bracket is skew-symmetric, and for any

$f \in C^\infty(M)$,

$$\begin{aligned} [\sigma_1, f\sigma_2]_E &= f[\sigma_1, \sigma_2]_E + \rho(\sigma_1)f\sigma_2 - \frac{1}{2}\langle \sigma_1, \sigma_2 \rangle d_E f \\ &= (f\pi_A([\sigma_1, \sigma_2]_E) + \rho(\sigma_1)f\sigma_2) \oplus f\pi_{A^*}([\sigma_1, \sigma_2]_E), \end{aligned}$$

where at the end we have split the sum into its A and A^* part. We can easily see that $\pi_{A^*}[\cdot, \cdot]$ is tensoral from the final expression. We also obtain a proto-Lie bracket $[\cdot, \cdot]_A$ on A which fails to satisfy the Jacobi identity defined as $[\cdot, \cdot]_A := \pi_A[\cdot, \cdot]|_A$. We can do the same thing for A^* , and obtain $\psi \in \Gamma(\bigwedge^3(A))$ that measures the failure of A^* to be closed and a proto-Lie bracket $[\cdot, \cdot]_{A^*} = \pi_{A^*}[\cdot, \cdot]|_{A^*}$. From the discussion on quasi-bialgebroids in [34], in particular equation 3.18, we arrive at the following theorem.

Theorem 3.3. *Let $E = A \oplus A^*$ be a proto-bialgebroid. The obstructions to the closure of the subbundles A and A^* under the Courant bracket are the 3-forms $\phi \in \Gamma(\bigwedge^3(A^*))$ and $\psi \in \Gamma(\bigwedge^3(A))$. Further $d_A\phi = d_{A^*}\psi = 0$ for any proto-bialgebroid, where d_A and d_{A^*} are defined in equation (9) using $\pi_A[\cdot, \cdot]_E$ and $\pi_{A^*}[\cdot, \cdot]_E$ as the stand-in for the Lie bracket respectively.*

This final condition that $d_A\phi = d_{A^*}\psi = 0$ will have more significance in the para-Hermitian setting. We are now in a position to define our main object of study.

4 Para-Hermitian Algebroids

We now have all of the ingredients necessary to define a para-Hermitian algebroid. The goal will be to cast all of the previous facts about Courant and Lie algebroids in terms of the para-Hermitian structure. To begin, we have

Definition 4.1. *Let $(E, \rho, \langle \cdot, \cdot \rangle, [\cdot, \cdot])$ be a Courant algebroid over M . Then E is called an almost para-Hermitian algebroid if it is equipped with a para-complex structure J that is compatible with the inner product (i.e., $\langle J\cdot, J\cdot \rangle = -\langle \cdot, \cdot \rangle$). We call (E, J) half integrable if one of the eigenbundles of J is closed under the Courant bracket, and integrable if both are closed under the Courant bracket. In the case where both eigenbundles are integrable, we refer to (E, J) as a para-Hermitian algebroid.*

Courant algebroids are anchored to the tangent bundle of the underlying manifold, and so we are lead to consider para-holomorphic algebroids, which occur when a para-Hermitian algebroid, (E, J_E) , is anchored to a para-Hermitian manifold, (M, J_{TM}) , and the anchor map, $\rho : E \rightarrow TM$, is para-holomorphic. Another program in this chapter will be to generalize the construction in Example 2.1 by considering connections $A : TM \rightarrow E$. As we will see, connections are themselves para-complex structures, and just as in the case of para-holomorphic algebroids, it makes sense to consider the case where they commute with another para-complex structure on E . We complete this generalization in Example 5.1 in section 5, which one can think of as containing the companion examples to this section.

4.1 Integrability and the Para-Kähler Condition

In order to measure the integrability of the eigenbundles, one can construct the Nijenhuis tensor as in equation (2). Immediately, we see from Proposition 2.3 that if the eigenbundles are integrable, then they are maximally isotropic, and so by Theorem 3.2,

$E = E_+ \oplus E_-$ and (E_+, E_-) is a Lie bialgebroid. By the discussion in the previous section, both E_+ and E_- form Lie algebroids under the restricted brackets. This allows us to define the derivative operators

$$d_{\pm} : C^{\infty}(M) \rightarrow \Gamma(E_{\pm}^*) \quad (13)$$

with the property that $d_E = d_+ + d_-$. These operators then give us a type decomposition of forms. If we identify $E_{\pm}^* \cong E_{\mp}$ under $\langle \cdot, \cdot \rangle$ and consider them as subbundles of E , then

$$\begin{aligned} \bigwedge^{(1,0)} E^* &:= \{\omega \in E^* \mid \omega(\sigma^*) = 0 \ \forall \sigma^* \in \Gamma(E_-)\}, \\ \bigwedge^{(0,1)} E^* &:= \{\omega \in E^* \mid \omega(\sigma) = 0 \ \forall \sigma \in \Gamma(E_+)\}. \end{aligned} \quad (14)$$

The two derivations d_+ and d_- are only traditionally defined on forms of type $\bigwedge^{(p,0)} E^*$ and $\bigwedge^{(0,q)} E^*$, respectively, but we see that under this splitting,

$$d_E : \bigwedge^{(p,q)} E^* \rightarrow \bigwedge^{(p+1,q)} E^* \oplus \bigwedge^{(p,q+1)} E^*, \quad (15)$$

and so we can in general define d_+ to be the projection onto the $(p+1, q)$ subbundle and d_- to be the projection onto the $(p, q+1)$ subbundle, as in the classical para-Hermitian case. Note also that this decomposition is only possible when both eigenbundles are integrable, and so we have an analogous result to Theorem 2.2. It is also important to note that the derivative operators act by derivations on the induced Schouten brackets associated to the dual Lie algebroid, so for any $e_1^{\pm}, e_2^{\pm} \in \bigwedge E_{\pm}^*$,

$$d_{\pm}[e_1^{\mp}, e_2^{\mp}]_{\mp} = [d_{\pm}e_1^{\mp}, e_2^{\mp}] + [e_1^{\mp}, d_{\pm}e_2^{\mp}]. \quad (16)$$

Each of these operators can be expressed in local form. Consider the local basis $\{e_1^+, \dots, e_n^+, e_1^-, \dots, e_n^-\}$, and suppose that

$$a_+(e_i^+) = A_i^j(x) \frac{\partial}{\partial x^j}, \quad a_-(e_i^-) = A_i^j(x) \frac{\partial}{\partial x^j}, \quad (17)$$

where $a_{\pm} = \rho \circ \pi_{E_{\pm}}$, and

$$[e_i^+, e_j^+] = C_{ij}^k(x) e_k^+ \quad [e_i^-, e_j^+] = C_{ij}^k(x) e_k^+ + C_{ij}^{\bar{k}}(x) e_k^- \quad [e_i^-, e_j^-] = C_{ij}^{\bar{k}}(x) e_k^- \quad (18)$$

From these two facts, one can readily compute that for $\sigma^\pm \in \Gamma(E_\pm)$ locally expressed as $\sum_i \sigma_i^\pm e_i^\pm$,

$$\begin{aligned}
d_+(\sigma^+) &= - \sum_{i,j} \left(A_j^k(x) \frac{\partial \sigma_i^+}{\partial x^k} + C_{ij}^k(x) \sigma_k^+ \right) e_i^+ \wedge e_j^-, \\
d_-(\sigma^+) &= - \sum_{i,j} \left(A_j^k(x) \frac{\partial \sigma_i^+}{\partial x^k} - A_i^k(x) \frac{\partial \sigma_j^+}{\partial x^k} + C_{ij}^{\bar{k}}(x) \sigma_k^+ \right) e_i^+ \wedge e_j^+, \\
d_+(\sigma^-) &= - \sum_{i,j} \left(A_j^k(x) \frac{\partial \sigma_i^-}{\partial x^k} - A_i^k(x) \frac{\partial \sigma_j^-}{\partial x^k} + C_{ij}^k(x) \sigma_k^- \right) e_i^- \wedge e_j^-, \\
d_-(\sigma^-) &= - \sum_{i,j} \left(A_j^k(x) \frac{\partial \sigma_i^-}{\partial x^k} + C_{ij}^{\bar{k}}(x) \sigma_k^- \right) e_i^- \wedge e_j^+.
\end{aligned} \tag{19}$$

Just as in the case of a para-holomorphic vector bundle, we have $d_\pm(\sigma^i e_i^\pm) = \partial_\pm(\sigma^i) \otimes e_i^\pm + \sigma^i d_\pm(e_i^\pm)$. It would be useful at this point to define what we mean by para-holomorphic forms. We consider a section $\sigma \oplus \tau \in \wedge^{(p,q)} E^* \oplus \wedge^{(q,p)} E^*$ to be *para-holomorphic* if $(d_- \oplus d_+)(\sigma \oplus \tau) = 0$. In particular, there are two ways for a section $\sigma \in \Gamma(E)$ to be para-holomorphic. By identifying $E \cong E^*$ under the non-degenerate pairing on E , one can decompose $\sigma = \sigma^+ \oplus \sigma^- \in \wedge^{(0,1)} E^* \oplus \wedge^{(1,0)} E^*$, or $\sigma = \sigma^- \oplus \sigma^+ \in \wedge^{(1,0)} E^* \oplus \wedge^{(0,1)} E^*$. In this way, σ can be para-holomorphic as a $(0,1) \oplus (1,0)$ -form or as a $(1,0) \oplus (0,1)$ -form.

Returning to Theorem 3.3, the notion of pseudo-para-holomorphic structures on almost para-Hermitian algebroids gives an interesting interpretation of the integrability condition for the eigenbundles. Consider the bundle E_+ . The Nijenhuis tensor when restricted to E_+ is given by

$$N^J(e_i^+, e_j^+) = \frac{1}{2} ([e_i^+, e_j^+]_E - J[e_i^+, e_j^+]), \tag{20}$$

which when compared to equation (12) tells us that $\phi(e_i^+, e_j^+, e_k^+) = \langle e_k^+, N^J(e_i^+, e_j^+) \rangle + c.p..$ This can be done for E_- as well, to obtain $\psi(e_i^-, e_j^-, e_k^-) = \langle e_k^-, N^J(e_i^-, e_j^-) \rangle + c.p..$

Stretching this analogy further, we have $d_+\phi = 0$ and $d_-\psi = 0$, and so the obstruction to the integrability of the brackets is a pseudo-para-holomorphic $(3, 0) \oplus (0, 3)$ -form corresponding to the restriction of the Nijenhuis bracket onto each eigenbundle. This was first realized by Svoboda in [36].

In light of this, and Theorem 3.1, we can investigate how the bracket on E is constructed from the given data of the Lie bialgebroid (A, A^*) . In particular, we take notice of $\langle \cdot, \cdot \rangle_-$. In terms of the metric on E , which we denote $\langle \cdot, \cdot \rangle$, and the para-complex structure J with $+1$ -eigenbundle A and -1 -eigenbundle A^* , we define the fundamental 2-form on E by

$$\omega = \langle \cdot, J\cdot \rangle. \quad (21)$$

Note that when the eigenbundles are integrable, this form is of type $(1, 1)$. Recall that in the case where the eigenbundles are integrable, E_+ and E_- are isotropic, meaning $\omega(e_i^\pm, e_j^\pm) = 0$. The ‘‘para-Kähler’’ condition, usually given by $d_E\omega = 0$, can be equivalently stated as $d_\pm\omega = 0$. We can use this expression to obtain a PDE constraining the coefficients of ω . To this end, we have the following.

Lemma 4.1. *The coefficients of the fundamental 2-form ω associated to the para-Hermitian Courant algebroid $E = E_+ \oplus E_-$ satisfy*

$$A_i^l(x) \frac{\partial \omega_{\bar{j}k}}{\partial x^l} - A_j^l(x) \frac{\partial \omega_{i\bar{k}}}{\partial x^l} - C_{ij}^l(x) \omega_{l\bar{k}} + C_{i\bar{k}}^l(x) \omega_{lj} - C_{j\bar{k}}^l(x) \omega_{li} = 0, \quad (22)$$

$$A_i^l(x) \frac{\partial \omega_{j\bar{k}}}{\partial x^l} - A_j^l(x) \frac{\partial \omega_{i\bar{k}}}{\partial x^l} - C_{ij}^l(x) \omega_{l\bar{k}} + C_{i\bar{k}}^l(x) \omega_{lj} - C_{j\bar{k}}^l(x) \omega_{li} = 0. \quad (23)$$

Proof. This follows easily from equation (9), along with the substitutions of vectors from each desired eigenspace. The first equation corresponds to $d_+\omega = 0$ and the second corresponds to $d_-\omega = 0$. \square

4.2 Exact Almost Para-Hermitian Algebroids

Recall that for an almost para-Hermitian algebroid $E = E_+ \oplus E_-$, the projections onto each eigenbundle induce a splitting in the anchor $\rho : E \rightarrow TM$ as the sum of the two operators $a_{\pm} : E_{\pm} \rightarrow TM$. This splitting of the anchor map induces the following sequences:

$$\begin{aligned} 0 &\longrightarrow T^*M \xrightarrow{a_-^*} E_+ \xrightarrow{a_+} TM \longrightarrow 0, \\ 0 &\longrightarrow T^*M \xrightarrow{a_+^*} E_- \xrightarrow{a_-} TM \longrightarrow 0. \end{aligned}$$

Lemma 4.2. *Let $E = E_+ \oplus E_-$ be a Lie bialgebroid, with anchor ρ . If $\rho \circ \rho^* = 0$, then $a_{\pm} \circ a_{\mp}^* + a_{\mp} \circ a_{\pm}^* = 0$*

Proof. To begin, suppose that $E = E_+ \oplus E_-$ is a Lie bialgebroid. We find

$$\begin{aligned} \rho \circ \rho^*(X) &= (a_+ + a_-)(a_+^*(X) + a_-^*(X)) \\ &= a_+(a_+^*(X)) + a_+(a_-^*(X)) + a_-(a_+^*(X)) + a_-(a_-^*(X)) \\ &= a_+(a_-^*(X)) + a_-(a_+^*(X)) \end{aligned}$$

where we have used the fact that $a_{\pm} \circ a_{\pm}^* = 0$, owing to the fact that $E_+ \oplus E_-$ is a Lie bialgebroid. The statement then follows by the fact that $\rho \circ \rho^* = 0$. \square

This gives us the following useful proposition.

Proposition 4.1. *Let (E, J) be an almost para-Hermitian algebroid. If $E = E_+ \oplus E_-$ is exact, then $\ker(a_{\pm}) = \text{Im}(a_{\mp}^*)$.*

Proof. Suppose that E is exact, so that $\ker(\rho) = \text{Im}(\rho^*)$. Explicitly, $\rho(e) = 0$ if and only if $e = \rho^*(\xi)$ for some $\xi \in \Gamma(T^*M)$. Suppose that $a_{\pm}(e^{\pm}) = 0$ for some $e^{\pm} \in \Gamma(E_{\pm})$.

Since e^\pm is a section of the eigenbundle, $a_\pm(e^\pm) = \rho(e^\pm)$, and so $e^\pm = \rho^*(\xi)$ for some $\xi \in T^*M$. However, since $\rho = a_+ + a_-$, we can decompose $e^\pm = a_-^*(\xi) + a_+^*(\xi)$. This tells us that $a_\pm^*(\xi) = 0$, as the image is in the opposite eigenbundle to e^\pm , and so we must conclude that $e^\pm = a_\mp^*(\xi)$. In conclusion, $a_\pm(e^\pm) = 0$ if and only if $e^\pm = a_\mp^*(\xi)$ for some $\xi \in \Gamma(T^*M)$, and so $\ker(a_\pm) = \text{Im}(a_\mp^*)$. \square

4.3 Para-Holomorphic Algebroids

Recall that all para-holomorphic, para-Hermitian vector bundles in the traditional sense have constant transition functions.

Definition 4.2. *Let $E \rightarrow M$ be a para-Hermitian algebroid over a para-Hermitian manifold. We say that E is a para-holomorphic algebroid if $\rho \circ J_E = J_{TM} \circ \rho$.*

One immediate consequence of the condition that the para-complex structures commute with ρ is that the splitting of ρ into $a_+ + a_-$ has the property that the image of the maps land in their respective eigenbundles of the para-Hermitian structure on M . By this we mean that $a_\pm : E_\pm \rightarrow T^\pm M$. If we consider that $d_\pm f(e) = df(a_\pm(e))$, we are led to conclude that the image of e under a_\pm is in $T^\pm M$. In fact, the condition $d_\pm f(e) = \partial_\pm f(a_\pm(e))$ is equivalent to saying that a_\pm is anchored in $T^\pm M$. Since TM is para-Hermitian, we see that $T^{(1,0)}M \cong (T^+M)^*$ and $T^{(0,1)}M \cong (T^-M)^*$. This sets up two sequences:

$$\begin{aligned} 0 &\longrightarrow T^{(0,1)}M \xrightarrow{a_-^*} E_+ \xrightarrow{a_+} T^+M \longrightarrow 0, \\ 0 &\longrightarrow T^{(1,0)}M \xrightarrow{a_+^*} E_- \xrightarrow{a_-} T^-M \longrightarrow 0. \end{aligned}$$

We have the following.

Proposition 4.2. *Suppose that $E = E_+ \oplus E_-$ is a para-holomorphic algebroid. Then E is exact if and only if the previous two sequences are exact.*

Proof. If both of these sequences are exact, then we can simply note that taking the termwise direct sum gives us that E is exact since each of the individual terms are vector bundles whose direct sums are the desired bundles in the sequence for E and $\rho = a_+ + a_-$, so we focus on the other direction. Suppose that E is exact. From Proposition 4.1, we know that $\ker(a_\pm) = \text{Im}(a_\mp^*)$, and so we simply need to check that $\ker(a_\mp^*) = 0$ and that the maps a_\pm are surjective. Surjectivity follows from the fact that the images of a_\pm are disjoint, and the map $\rho = a_+ + a_-$ is surjective by exactness. Additionally, we know that $\ker(\rho^*) = 0$ by exactness, and so if $a_\pm^*(\xi) = 0$, then clearly $\rho^*(\xi) = 0$ as well, and this completes the proof. \square

We immediately have the following corollary.

Corollary 4.1. *If E is an exact para-holomorphic algebroid, then $\dim(E) = 4k$ for some $k \in \mathbb{N}$.*

Continuing, let E be an exact almost para-Hermitian algebroid. We can relate the anchors a_\pm of the eigenbundles E_\pm to a natural bivector field on M . More concretely, there is a natural bivector on M that is Poisson when E is para-Hermitian, and whose vanishing is a requirement for $E \rightarrow M$ to be para-holomorphic. Consider the bivector $\pi \in \mathfrak{X}^2(M)$ given by

$$\iota_\mu \pi = \rho(\text{pr}_{E_-}(\rho^*(\mu))) = -\rho(\text{pr}_{E_+}(\rho^*(\mu))), \quad (24)$$

where the second equality comes from the fact that E is exact. It was shown in [25] that the rank of the bundle map $\pi^\# : T^*M \rightarrow TM$ is given by $\text{rank}(\pi^\#) = \dim(\rho(E_+) \cap \rho(E_-))$ and so this bivector is identically 0 when the images under the anchor ρ of E_+ and E_- are disjoint, as is the case when $E \rightarrow M$ is para-holomorphic. Further, they deduce that the obstruction to π being a Poisson bivector is

$$\frac{1}{2}[\pi, \pi]_s = \rho(\phi) + \rho(\psi), \quad (25)$$

where ϕ and ψ are defined as in equation (12), we have identified $E_{\pm}^* \cong E_{\mp}$ and $[\cdot, \cdot]_s$ is the Schouten bracket derived from the Lie bracket on TM . This allows us to state the following.

Theorem 4.1. *Let E be an exact almost para-Hermitian algebroid over a para-Hermitian manifold M . If E is para-Hermitian, the bivector field π , as defined in equation (24), is Poisson. If $E \rightarrow M$ is para-holomorphic, then $\pi = 0$.*

We can now consider morphisms of para-Hermitian algebroids. In this case, the model is the morphism of para-Hermitian manifolds. A morphism of para-Hermitian manifolds is a smooth map $\Phi : (M, J_M) \rightarrow (N, J_N)$ such that Φ_* is an isometry, and $\Phi_* \circ J_M = J_N \circ \Psi_*$. Note that by the properties of pushforwards, $[\Phi_*(\cdot), \Phi_*(\cdot)]_N = \Phi_*[\cdot, \cdot]_M$. With this in mind, we can define a morphism of para-Hermitian algebroids.

Definition 4.3. *Let (E, J_E) and (F, J_F) be para-Hermitian algebroids. A vector bundle map $\Psi : E \rightarrow F$ is a morphism of para-Hermitian algebroids if Ψ is an isometry, para-holomorphic (meaning $\Psi \circ J_E = J_F \circ \Psi$) and preserves the Courant bracket.*

It is important to note here that the definition of a para-Hermitian algebroid morphism is less strict than a morphism of Courant algebroids because we do not require that these maps preserve the anchor. We will see in Example 5.2 that these maps can allow us to construct interesting examples in cases where Courant algebroid morphisms are too strict of a condition. We can introduce a notion of a para-holomorphic algebroid morphism as well.

Definition 4.4. *Let (E, J_E) and (F, J_F) be para-holomorphic algebroids over (M, g_1, J_1) and (N, g_2, J_2) respectively. Suppose that $\Psi : E \rightarrow F$ is a vector bundle morphism covering the diffeomorphism $\psi : M \rightarrow N$. We say that Ψ is a morphism of para-holomorphic algebroids if Ψ is a para-Hermitian algebroid morphism, and the diffeomorphism $\psi : M \rightarrow N$ is a para-holomorphic isometry.*

4.4 Connections on Para-Hermitian Algebroids

Bressler and Chervov give a notion of a connection on a Courant algebroid in [5] that will be of use to us.

Definition 4.5. *Let E be a Courant algebroid. A connection on E is a vector bundle morphism $A : TM \rightarrow E$ that satisfies:*

1. $\rho \circ A = Id_{TM}$.
2. $\langle A(v_1), A(v_2) \rangle = 0$ for all $v_1, v_2 \in TM$.

In the context of exact Courant algebroids, connections are important in that they provide a decomposition $E = \rho^*(T^*M) \oplus A(TM)$. One can see that the dimensions work out, as ρ^* and A are injective (by exactness and $\rho \circ A = Id_{TM}$, respectively). Finally, we see that $\ker(\rho) = Im(\rho^*)$, but $\rho \circ A = Id_{TM}$, and so we can conclude that these subbundles are disjoint. We can define the para-complex structure K with eigenbundles $A(TM)$ and $\rho^*(T^*M)$. This para-complex structure is always half integrable precisely because E is an exact Courant algebroid. We know that $\ker(\rho) = Im(\rho^*)$, and additionally

$$\rho([\rho^*(\xi), \rho^*(\eta)]) = [\rho(\rho^*(\xi)), \rho(\rho^*(\eta))] = 0,$$

so $[\rho^*(\xi), \rho^*(\eta)] = \rho^*(\zeta)$. This is exactly the condition needed for $\rho^*(T^*M)$ to be closed under the bracket on E . We can now consider the related concept of curvature for these connections.

Definition 4.6. *Let $A : TM \rightarrow E$ be a connection. The curvature of A is a map $R : TM \times TM \rightarrow E$ defined by*

$$R(v_1, v_2) = [A(v_1), A(v_2)]_E - A([v_1, v_2]). \quad (26)$$

In short, R is the measure of the failure of the connection to preserve the bracket on TM and E .

One then has the following theorem that relates the integrability of the para-complex structure K on an exact Courant algebroid E to this concept of curvature.

Theorem 4.2. *Let E be an exact courant algebroid, and let $A : TM \rightarrow E$ be a connection on E . Then $E = A(TM) \oplus \rho^*(T^*M)$ has the structure of a half integrable para-complex algebroid, with $K(A(v)) = A(v)$ and $K(\rho^*(\xi)) = -\rho^*(\xi)$. We call K the standard para-complex structure with respect to A on E . The para-complex structure on E is fully integrable if and only if the connection on E is flat.*

Proof. It is clear that since K is half integrable, one would only need to check that $A(TM)$ is closed under the bracket if and only if A is flat. For this, we suppose that $v_1, v_2 \in \Gamma(TM)$. Then $A(TM)$ is closed if and only if there exists $v_3 \in \Gamma(TM)$ such that

$$[A(v_1), A(v_2)] = A(v_3).$$

We note that ρ preserves the bracket and that $\rho \circ A = Id_{TM}$, and so by applying ρ to both sides, we find that $v_3 = [v_1, v_2]$. This is precisely the condition that A is flat. As for the fact that K is compatible with the metric, we find

$$\begin{aligned} \langle K(A(v) + \rho^*(\xi)), K(A(w) + \rho^*(\eta)) \rangle &= -\langle A(v) + \rho^*(\xi), A(w) + \rho^*(\eta) \rangle, \\ \langle A(v) - \rho^*(\xi), A(w) - \rho^*(\eta) \rangle &= -\langle A(v) + \rho^*(\xi), A(w) \rangle, \\ \langle A(v), A(w) \rangle + \langle \rho^*(\xi), \rho^*(\eta) \rangle &= -\langle A(v), A(w) \rangle - \langle \rho^*(\xi), \rho^*(\eta) \rangle, \\ \langle \rho^*(\xi), \rho^*(\eta) \rangle &= -\langle \rho^*(\xi), \rho^*(\eta) \rangle. \end{aligned}$$

Finally we see that $\langle \rho^*(\xi), \rho^*(\eta) \rangle = 0$ for all $\xi, \eta \in \Gamma(T^*M)$, as $\mathfrak{S}(\rho^*)$ isotropic by exactness. □

Example 4.1. *Consider the standard Courant algebroid $E = TM \oplus T^*M$ for some smooth manifold M , with $\rho : E \rightarrow TM$ given by projection on the first factor. Then ρ^* is closed under this bracket in the trivial sense, as $[\xi, \eta] = 0$. In this case, $Im(\rho^*)$ is isotropic as well. One can check rather easily that all connections on E are in the form*

$A(X) = X \oplus \tilde{\omega}(X)$ for some $\omega \in \Omega^2(M)$. This is the graph of a 2-form in E , and so by the famous result that the Dirac subbundles of $T \oplus T^*$ are closed 2-forms, (E, K) is a para-Hermitian algebroid (i.e., the image of the connection is integrable) if and only if ω is a closed 2-form. So, for $E = TM \oplus T^*M$, para-Hermitian algebroid structures with one eigenbundle equal to $Im(\rho^*)$ are in one-to-one correspondence with closed 2-forms.

Para-complex and para-Hermitian algebroids are more general than connections, but para-Hermitian structures with one eigenbundle equal to $\rho^*(T^*M)$ are in one-to-one correspondence with connections on E , as we justify in the following proposition.

Proposition 4.3. *Given an exact Courant algebroid, we can make a choice of subbundle H such that $E = \rho^*(T^*M) \oplus H$. Then $\rho|_H : H \rightarrow TM$ is an isomorphism. If H is also isotropic (meaning this is a half integrable para-Hermitian algebroid), then H defines a connection on E by $\rho^{-1} : TM \rightarrow H \hookrightarrow E$.*

4.5 Para-Complex Connections

We can now move on to the more general case of a para-complex structure that does not necessarily come from a connection or $Im(\rho^*)$. Let $E \rightarrow M$ be an almost para-Hermitian algebroid over an almost para-Hermitian manifold M . We say that a connection A on E is para-complex if $J_E \circ A = A \circ J_{TM}$. If this is the case, we immediately arrive at the following.

Lemma 4.3. *If A is a para-complex connection, then A maps the ± 1 -eigenspace of J_{TM} into the ± 1 -eigenspace of J_E .*

Proof. Suppose that A is a para-complex connection on $E \rightarrow M$. Then for $v \in T^\pm M$,

$$J_E(A(v)) = A(J_{TM}(v)) = A(\pm v) = \pm A(v).$$

□

One immediately finds that if A preserves the bracket, there is a sense in which the structure of the underlying almost para-Hermitian manifold is determined by the structure of the almost para-Hermitian algebroid that covers it. We can formalize this in the following theorem.

Theorem 4.3. *Let E be a para-Hermitian algebroid over an almost para-Hermitian manifold M . Then M is para-Hermitian if E admits a flat para-complex connection.*

Proof. To begin, suppose that E is a para-Hermitian algebroid, so that both of its eigenbundles are closed under the Courant bracket. Recall that the image of $T^\pm M$ under A is in E_\pm . From this, we can see that for $v_1, v_2 \in \Gamma(T^\pm M)$,

$$\begin{aligned} A(J_{TM}[v_1, v_2]) &= J_E(A([v_1, v_2])) \\ &= J_E([A(v_1), A(v_2)]_E) \\ &= \pm[A(v_1), A(v_2)] \\ &= A(\pm[v_1, v_2]). \end{aligned}$$

We can now use the fact that $\rho \circ A = Id_{TM}$, so that $J_{TM}[v_1, v_2] = \pm[v_1, v_2]$, meaning $[v_1, v_2] \in \Gamma(T^\pm M)$. Therefore, the eigenbundles of J_{TM} are closed under the Lie bracket, and M is a para-Hermitian manifold. \square

If A is para-complex we can decompose A into two bundle maps $A = A_+ + A_-$, given by $A_\pm : T^\pm M \rightarrow E_\pm$. One immediately has the identity $a_\pm \circ A_\pm = Id_{T^\pm M}$. Further, we know that the image of A , and both E_\pm are isotropic, and so given two $v, w \in \Gamma(TM)$, decomposed by $v = v_+ + v_-$, $w = w_+ + w_-$ in the usual way, we have

$$\langle A_+(v_+), A_-(w_-) \rangle = -\langle A_-(v_-), A_+(w_+) \rangle.$$

In any case, the left and right hand sides of this equation are completely independent, and so we can conclude that the images of A_+ and A_- are orthogonal under this metric.

of the para-complex structure on E admit the following decomposition.

$$\begin{aligned} E_+ &= a_-^*(T^{(0,1)}M) \oplus A_+(T^+M), \\ E_- &= a_+^*(T^{(1,0)}M) \oplus A_-(T^-M). \end{aligned}$$

Now, this decomposition could equally have been written

$$E \cong (A_+(T^+M) \oplus A_-(T^-M)) \oplus (a_-^*(T^{(0,1)}M) \oplus a_+^*(T^{(1,0)}M)).$$

Note that in this case, we clearly have $JK = KJ$ where K is the standard para-complex structure with respect to A on E as defined in Theorem 4.2, and so we can construct a third para-complex structure $L = JK$. In the case where $Im(\rho^*)$ is isotropic, then J and K are both para-Hermitian, and so the resulting para-complex structure L is *split-para-complex*, as $\langle L\cdot, L\cdot \rangle = \langle \cdot, \cdot \rangle$. This structure of two commuting para-complex structures (and the induced third para-complex structure) is referred to as a *split-para-complex structure*, and since two of the structures are para-Hermitian, we refer to this as a *split-para-Hermitian structure*. As far as I am aware, this is the first naturally occurring instance of a split-para-Hermitian structure, though similar structures have been considered before. We can conclude with this theorem

Theorem 4.6. *If (E, J) is an exact para-holomorphic algebroid, then E admits a flat para-complex connection if and only if (E, J, K) is split-para-complex.*

Proof. We discussed already the forward direction here, so we focus on proving that a split-para-complex structure (with one para-complex structure being the standard one with one eigenbundle equal to $\rho^*(T^*M)$) is equivalent to defining a para-complex connection. To begin, we know that the choice of para-Hermitian structure K is equivalent to choosing a decomposition $E = \rho^*(T^*M) \oplus H$, where H is half dimensional and isotropic. We also remarked that $\rho|_H$ must be an isomorphism, and so $A := (\rho|_H)^{-1} : TM \rightarrow H \hookrightarrow$

E defines a connection on E . Our goal now is to show that A is a para-complex connection in the case where (E, J, K) is split-para-complex and (E, J) is para-holomorphic. We focus on the subbundle H . Since $JK = KJ$ and H is the -1 -eigenbundle of K , we have $-J|_H = K \circ J|_H$, and so $J|_H : H \rightarrow H$. Further, since $\rho|_H$ is invertible and (E, J) is para-holomorphic, we have $(\rho|_H)^{-1} \circ J|_H = J|_H \circ (\rho|_H)^{-1}$, and so $(\rho|_H)^{-1}$ is a para-complex connection. In particular, the eigenbundle H is integrable if and only if A is a flat connection. \square

Returning briefly to the concept of a para-Hermitian algebroid morphism, given two para-holomorphic algebroids (E, J_E) and (F, J_F) over the para-Hermitian manifold M , then a para-Hermitian algebroid morphism Ψ does not necessarily map connections to connections. Given $A : TM \rightarrow E$, the map $\Psi \circ A : TM \rightarrow F$ does not necessarily have the property that $\rho_F \circ \Psi \circ A = Id_{TM}$, and so we cannot conclude that A is a connection. In fact, the condition for this to be true is precisely that Ψ must preserve the anchor which is equivalent to Ψ being a Courant algebroid morphism. The only obstruction to $\Psi \circ A$ being a connection is the one previously mentioned, and so $\Psi \circ A$ is still a bivector on F . Further, we have the following.

Proposition 4.4. *Let (E, J_E) and (F, J_F) be exact para-Hermitian algebroids over a para-Hermitian manifold M , and let $\Psi : E \rightarrow F$ be an isomorphism of para-Hermitian algebroids. When a connection $A : TM \rightarrow E$ is flat, $\Psi \circ A$ is a Poisson bivector on F .*

Proof. Since Ψ is an isometry, $\langle \Psi \circ A, \Psi \circ A \rangle = \langle A, A \rangle = 0$, so the image of A is isotropic, meaning A is the graph of a bivector on F . Further Since A is flat, $[\Psi \circ A, \Psi \circ A]_F = \Psi([A, A]_E) = \Psi \circ A[\cdot, \cdot]_M$, so $\Psi \circ A$ preserves the Courant bracket and is therefore a Poisson bivector. \square

5 Examples

5.1 The natural structures on $\mathbb{T}M$

Example 5.1. *The natural para-holomorphic structure with compatible para-complex connection on a para-Hermitian manifold.*

Let (M, g, J) be an almost para-Hermitian manifold. The splitting of the tangent bundle into $TM = T^+M \oplus T^-M$ induces a type decomposition on the space of differential forms, and so we have $T^*M = T^{(1,0)}M \oplus T^{(0,1)}M$. We endow $\mathbb{T}M = TM \oplus T^*M$ with the standard Courant algebroid structure as described in the introduction and equation (1). Now, the splitting of the tangent and the cotangent bundles allows us to introduce an almost para-complex structure \tilde{J} on $\mathbb{T}M$, with $+1$ -eigenbundle $T^+M \oplus T^{(0,1)}M$ and -1 -eigenbundle $T^-M \oplus T^{(1,0)}M$. We can note that for any $X^+ \in \Gamma(T^+M)$, $X^- \in \Gamma(T^-M)$, $\xi^{(1,0)} \in \Gamma(T^{(1,0)}M)$ and $\xi^{(0,1)} \in \Gamma(T^{(0,1)}M)$, we have $\iota_{X^+}\xi^{(0,1)} = \iota_{X^-}\xi^{(1,0)} = 0$, so this structure is para-Hermitian as the eigenbundles are isotropic. Further, we can easily see that the para-complex structures J and \tilde{J} commute with respect to the projection onto TM , so we have an almost para-holomorphic structure. Turning now to the obstruction to integrability, we can compute that

$$\begin{aligned} [X^+ \oplus \xi^{(0,1)}, Y^+ \oplus \eta^{(0,1)}] &= [X^+, Y^+] \oplus (\iota_{X^+}d\eta^{(0,1)} + \iota_{Y^+}d\xi^{(0,1)}), \\ [X^- \oplus \xi^{(1,0)}, Y^- \oplus \eta^{(1,0)}] &= [X^-, Y^-] \oplus (\iota_{X^-}d\eta^{(1,0)} + \iota_{Y^-}d\xi^{(1,0)}). \end{aligned} \quad (27)$$

Now, we did not assume that the eigenbundles $T^\pm M$ were integrable, and so not only is $[X^\pm, Y^\pm]$ not guaranteed to be a section of $T^\pm M$, but we fail to satisfy the conditions in Theorem 2.2, and so

$$d\Omega^{(1,0)}(M), d\Omega^{(0,1)}(M) \subset \Omega^{(2,0)}(M) \oplus \Omega^{(1,1)}(M) \oplus \Omega^{(0,2)}(M).$$

One can check that the bivector field on M defined by equation (24) is identically zero, as $\rho(\pi_+(\rho^*(\xi))) = \rho(0 \oplus \xi^{(0,1)}) = 0$, which is to be expected because $\mathbb{T}M$ is a paraholomorphic algebroid. By Theorem 2.2, we can see that if T^+M is integrable, then $d\Omega^{(0,1)}(M) \subset \Omega^{(1,1)}(M) \oplus \Omega^{(0,2)}(M)$. Since in equation (27) we are contracting $d\eta^{(0,1)}$ and $d\xi^{(0,1)}$ with elements of $\Gamma(T^+M)$, the part of this that remains is the $\Omega^{(1,1)}(M)$ -part, and so upon contracting with such a section, we are left with an element of $\Omega^{(0,1)}(M)$, as desired. There was nothing special about choosing T^+M here, so the integrability of the eigenbundle $T^\pm M$ implies the integrability of the ± 1 -eigenbundle of the para-Hermitian structure on $\mathbb{T}M$ and vice versa. All are equivalent to the Nijenhuis tensor vanishing, as discussed in the paragraph after equation (20).

Finally, consider the connection $A_\omega : TM \rightarrow \mathbb{T}M$, given by

$$A_\omega : X \mapsto X \oplus \iota_X \omega, \quad (28)$$

where $\omega = g(\cdot, J_{TM}\cdot)$ is the fundamental 2-form. One can check that the image of A_ω is isotropic (as it is the graph of a 2-form), and that it gives the identity map when composed with the projection onto TM . Then A_ω is a para-complex connection for the following reason. We note that $g(X^\pm, X^\pm) = 0$ as the eigenbundles $T^\pm M$ are isotropic, and so $\iota_{X^+}\omega \in \Omega^{(0,1)}(M)$ and $\iota_{X^-}\omega \in \Omega^{(1,0)}(M)$, which ensures that A_ω is a para-complex connection, as it maps the ± 1 -eigenbundles of TM into the ± 1 -eigenbundles of $\mathbb{T}M$.

The closure of the image of A_ω under the Courant bracket corresponds to $Gr_\omega \subset \mathbb{T}M$ being a Dirac subbundle, and hence corresponds to ω being closed. Therefore the Kähler condition is equivalent to the para-complex connection A_ω being flat. Additionally, one can easily check that if A_σ is a connection generated by the 2-form σ , then A is para-complex if and only if $\sigma(J\cdot, J\cdot) = -\sigma$ (i.e. σ is compatible with J). We can see that the interesting facts about para-Hermitian manifolds can be translated

nicely to the setting of para-holomorphic algebroids and para-complex connections. This construction is due to Svoboda [36], though we have chosen to take a different perspective. We will now look at some non-trivial examples arising from Lie groups.

5.2 Quadratic Lie Groups

Example 5.2. *A para-Hermitian structure on $\mathbb{T}G$ for any Lie group G with a quadratic Lie algebra \mathfrak{g} .*

A class of examples can be derived from the discussion of Dirac structures on Lie groups in [1]. Let G be a Lie group with Lie algebra \mathfrak{g} , and suppose that this Lie algebra carries with it an Ad -invariant inner product B . Let $\theta^L, \theta^R \in \Omega^1(G) \times \mathfrak{g}$ be the left/right-Maurer-Cartan forms, respectively. One can define a bi-invariant (with respect to the adjoint action of G) pseudo-Riemannian metric on G by $B(\theta^L, \theta^L)$. One can also define the bi-invariant 3-form $\eta \in \Omega^3(G)$ by

$$\eta := \frac{1}{12} B(\theta^L, [\theta^L, \theta^L]_{\mathfrak{g}}).$$

The bi-invariance of η implies that it is also closed (as its Jacobiator vanishes on constant sections), and so one can define the η -twisted Courant bracket on $\mathbb{T}G$, $[[\cdot, \cdot]]_{\eta}$. Now, let $D = G \times G$ with Lie algebra $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$. We have a natural smooth action

$$\mathcal{A} : D \rightarrow \text{Diff}(G), \quad \mathcal{A}(g, g') = l_g \circ r_{g^{-1}},$$

for $g, g' \in G$, whose corresponding infinitesimal action

$$\mathcal{A}_* : \mathfrak{d} \rightarrow \mathfrak{X}(G), \quad \mathcal{A}_*(X, X') = X^L - (X')^R,$$

where $X_g^L = l_g(X)$ and $(X')_g^R = r_g(X')$ for any $X, X' \in \mathfrak{g}$. The authors then use this action to define a D -equivariant map

$$\begin{aligned} s : \mathfrak{d} &\rightarrow \Gamma(\mathbb{T}G), & s(X, X') &= s^L(X) + s^R(X'), \\ s^L(X) &= X^L \oplus \frac{1}{2} B(\theta^L, X), & s^R(X') &= -(X')^R \oplus \frac{1}{2} B(\theta^R, X'). \end{aligned}$$

Equipping \mathfrak{d} with the bilinear form $B_{\mathfrak{d}}$, given by $+B$ on the first \mathfrak{g} summand of $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$, and $-B$ on the second summand gives a split signature bilinear form (exactly what it needed for \mathfrak{d} to admit a para-complex structure). The map $s : \mathfrak{d} \rightarrow \Gamma(\mathbb{T}G)$ has the following important properties:

1. $\langle s(e_1), s(e_2) \rangle = B_{\mathfrak{d}}(e_1, e_2)$,
2. $\llbracket s(e_1), s(e_2) \rrbracket_{\eta} = s([e_1, e_2])$,
3. $\Upsilon(s(e_1), s(e_2), s(e_3)) = B_{\mathfrak{d}}(e_1, [e_2, e_3])$,

where $e_i \in \mathfrak{d}$, and $\Upsilon(x_1, x_2, x_3) = -\langle \llbracket x_1, x_2 \rrbracket, x_3 \rangle$. These identities tell us that s is a D -equivariant isometric isomorphism of vector bundles $G \times \mathfrak{d} = \mathbb{T}G$, identifying the twisted Courant bracket $\llbracket \cdot, \cdot \rrbracket_{\eta}$ on $\mathbb{T}G$ with the unique Courant bracket on $G \times \mathfrak{d}$ which agrees with the Lie bracket on \mathfrak{d} for constant sections. Note that s is not necessarily a Courant algebroid isomorphism as it does not preserve the anchors. For this reason, any pair of Lagrangian subalgebras on \mathfrak{d} (called a Manin-triple) is equivalent to defining an integrable para-Hermitian algebroid structure on $\mathbb{T}G$ with the η -twisted bracket, and the map s can be understood as an isomorphism of para-Hermitian algebroids. A classification of Lagrangian subalgebras of Lie algebras in the form $\mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}$ where \mathfrak{g} is a complex semi-simple Lie algebra was completed in [20], and so we have a wealth of examples of para-Hermitian algebroids corresponding to Lie groups with quadratic Lie algebras on the Courant algebroid $\mathbb{T}G$.

5.3 Structures on $\mathbb{T}G$ induced by the Iwasawa Decomposition

Example 5.3. *A example of an exact para-holomorphic algebroid with a compatible flat para-complex connection over $\mathbb{T}G$ where G is a Lie group with a quadratic, semi-simple Lie algebra \mathfrak{g} corresponding to the Cartan-Dirac structure.*

Example 5.2 can be pushed further to yield an exact para-holomorphic algebroid with a para-complex connection. This construction is based on the Iwasawa decomposition, which one can see described in [19]. We will first give the highlights of the Iwasawa decomposition. Let \mathfrak{g} be a complex semi-simple Lie algebra. Then \mathfrak{g} is guaranteed to have a compact real form \mathfrak{k} , with respect to which \mathfrak{g} (when viewed as a real vector space) admits the decomposition $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$. Since \mathfrak{k} is a compact Lie algebra, the Killing form κ on \mathfrak{k} , defined by $\kappa(X, Y) = 2 \dim(\mathfrak{k}) \text{Tr}(ad_X ad_Y)$, is negative definite. We can extend the Killing form on \mathfrak{k} to all of \mathfrak{g} by \mathbb{C} -linearity. One then obtains a non-degenerate symmetric bilinear form on \mathfrak{g} by taking $-\text{Im}(\kappa)$, which makes \mathfrak{g} into a quadratic Lie algebra, for which \mathfrak{k} is a Lagrangian subalgebra.

There exists on \mathfrak{g} another Lagrangian subalgebra that is naturally dual to \mathfrak{k} . To begin, pick a maximal Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k}$, and note that $\mathfrak{a} = i\mathfrak{t}$ is also a commuting subalgebra of \mathfrak{g} . If we consider the root space decomposition with respect to this choice of \mathfrak{a} and define a notion of positivity using the inner product on \mathfrak{g} , then we can define $\mathfrak{n} = \bigoplus_{\Sigma^+} g_\lambda$ to be the direct sum over the positive roots spaces. Then \mathfrak{g} , when viewed as a real Lie algebra, admits the decomposition into the sum of Lagrangian subalgebras $\mathfrak{g} = \mathfrak{k} \oplus (\mathfrak{a} \oplus \mathfrak{n})$, which we refer to as the Iwasawa decomposition of \mathfrak{g} .

Returning to Example 5.2, let G be a Lie group with complex semi-simple Lie algebra \mathfrak{g} . The anchor map on $G \times \mathfrak{d} = G \times (\mathfrak{g} \oplus \bar{\mathfrak{g}})$ is simply given by projection onto the first summand, where $TG \cong G \times \mathfrak{g}$ under left trivialization. One then defines an integrable para-Hermitian structure $J_{\mathfrak{g}}$ on TG by taking the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus (\mathfrak{a} \oplus \mathfrak{n})$, and defining \mathfrak{k} to be the $+1$ eigenbundle, and $(\mathfrak{a} \oplus \mathfrak{n})$ to be the -1 -eigenbundle. This lifts to a natural para-Hermitian structure on $G \times \mathfrak{d}$ by $J_{\mathfrak{d}}(X \oplus Y) = J_{\mathfrak{g}}(X) \oplus J_{\mathfrak{g}}(Y)$. The most natural connection on $\mathbb{T}G$ is the diagonal map $A_{\Delta} : TG \rightarrow G \times \mathfrak{d}$, given by

$A_\Delta(g, X) = s(g, X, X)$ for $(g, X) \in T_g G$. The connection A_Δ is flat with respect to the induced Courant bracket on $G \times \mathfrak{g}$. The map $s \circ A_\Delta$ also preserves η -twisted bracket, and is often referred to as the Cartan-Dirac structure. The Dirac foliation of $s \circ A_\Delta$ corresponds to the conjugacy classes of G .

It's quite easy to see that the connection A_Δ is para-complex and that the anchor commutes with the para-complex structure on $G \times \mathfrak{d}$, so $G \times \mathfrak{d}$ is an example of an exact para-holomorphic algebroid with a flat para-complex connection. Further, the Cartan-Dirac structure on $\mathbb{T}G$ can be realized as the image of this para-complex connection under the para-Hermitian algebroid isomorphism s , which has the form

$$s \circ A_\Delta(g, X) = \left(X^L - X^R, \frac{1}{2}B(\theta^L + \theta^R, X) \right). \quad (29)$$

It is instructive that $\mathbb{T}G$ and $G \times \mathfrak{d}$ are isomorphic as para-holomorphic algebroids under s (as s covers the identity map on G which is a bi-para-holomorphic isometric diffeomorphism), but not as Courant algebroids. This illustrated the utility of considering para-holomorphic algebroids in the first place. As a concrete example, consider the Lie group $SL(n; \mathbb{C})$. The Lie group $SL(n; \mathbb{C})$ admits an Iwasawa decomposition with $\mathfrak{su}(n)$ as the choice of compact real form, and so $K = SU(n)$,

$$A = \left\{ \left[\begin{array}{cccc} t_1 & 0 & \cdots & 0 \\ 0 & t_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_n \end{array} \right], \quad t_i \in \mathbb{R}, \quad \prod_i t_i = 1 \right\},$$

$$N = \left\{ \left[\begin{array}{cccc} 1 & \theta_{12} & \cdots & \theta_{1n} \\ 0 & 1 & \cdots & \theta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{array} \right], \quad \theta_i \in \mathbb{R}, \quad \text{not all } \theta_i = 0 \right\}.$$

Therefore $\mathbb{T}SL(n; \mathbb{C})$ with the η -twisted Courant bracket admits an exact para-holomorphic algebroid structure with an associated flat para-complex connection (induced by the diagonal inclusion of $\mathfrak{sl}(n; \mathbb{C})$ into its double) for every n .

5.4 Poisson-Lie Groups

The connection to Poisson-Lie groups here is exciting. In the above example, all that was required was that the complex semi-simple Lie algebra \mathfrak{g} has a compact real form. The non-degenerate symmetric bilinear form could be derived from the Killing form on the subalgebra corresponding to that compact real form. Poisson-Lie groups allow us to begin with a Lie group K together with a multiplicative Poisson structure Π_K and then find an exact para-holomorphic algebroid structure with a flat para-complex connection on the Drinfeld double D . This was first done by Lu in [29]. Consider the definition:

Definition 5.1. *Let K be a Lie group and Π_K a Poisson structure on K . We say that Π_K is multiplicative if the multiplication map $m : K \times K \rightarrow K$ is a Poisson map, where $K \times K$ is endowed with the product Poisson structure. For $k_1, k_2 \in K$, the multiplicativity condition is equivalent to*

$$(\Pi_K)_{k_1 k_2} = L_{k_1}(\Pi_K)_{k_2} + R_{k_2}(\Pi_K)_{k_1}.$$

An immediate consequence of this definition is that all multiplicative Poisson structures vanish at the identity. The Poisson structure therefore admits a linearization at the identity to give a map $d_e \Pi_K : \mathfrak{k} \rightarrow \mathfrak{k} \wedge \mathfrak{k}$. The condition that the dual map $(d_e \Pi_K)^* : \mathfrak{k}^* \wedge \mathfrak{k}^* \rightarrow \mathfrak{k}^*$ must satisfy in order to be a Lie bracket is the Jacobi identity, which is equivalent to the fact that the bivector Π_K is Poisson. This means that \mathfrak{k}^* is a Lie algebra in its own right. Lu was able to show that there is a unique connected simply-connected Poisson Lie group (K^*, Π_{K^*}) such that $(d_e \Pi_{K^*})^*$ is the Lie bracket on \mathfrak{k} . This gives $(\mathfrak{k} \oplus \mathfrak{k}^*)$ the structure of a Lie bialgebra. Finally, one considers the double

Lie algebra to the pair $(\mathfrak{k}, \mathfrak{k}^*)$ to be $\mathfrak{d} = \mathfrak{k} \oplus \mathfrak{k}^*$ together with the Lie bracket

$$[X \oplus \xi, Y \oplus \eta]_{\mathfrak{d}} = ([X, Y]_{\mathfrak{k}} + ad_{\xi}^*(Y) - ad_{\eta}^*(X)) \oplus ([\xi, \eta]_{\mathfrak{g}^*} + ad_X^*(\eta) - ad_Y^*(\xi)),$$

for $X, Y \in \mathfrak{k}, \xi, \eta \in \mathfrak{k}^*$.

Now, one can consider the unique connected simply-connected Lie group D corresponding to the Lie algebra \mathfrak{d} . It is a fact about Lie bialgebras that the natural pairing $\langle X \oplus \xi, Y \oplus \eta \rangle = \langle X, Y \rangle + \langle \eta, X \rangle$ is *ad*-invariant with respect to the above bracket, and so \mathfrak{d} is a quadratic Lie algebra with \mathfrak{k} and \mathfrak{k}^* as dual Lagrangian subalgebras [24]. There exists two natural inclusions $\mathfrak{k}, \mathfrak{k}^* \hookrightarrow \mathfrak{d}$. These inclusions integrate to two local diffeomorphisms

$$\begin{aligned} \Psi : K \times K^* &\rightarrow D, & \Phi : K^* \times K &\rightarrow D, \\ (k, u) &\mapsto ku & (u, k) &\mapsto uk \end{aligned} \tag{30}$$

The first of these local diffeomorphisms defines a para-complex structure on D , which is clearly compatible with the quadratic Lie algebra structure. Further, given a local basis $\{e_i\}$ of \mathfrak{k} and corresponding dual basis $\{\varepsilon^i\}$ of \mathfrak{k}^* , we can define the classical R -matrix

$$\Lambda = \frac{1}{2} \sum_i e_i \wedge \varepsilon^i. \tag{31}$$

This classical R -matrix defines a natural Poisson structure on D given by

$$\Pi_D^+ = \Lambda^L + \Lambda^R.$$

The Poisson structure Π_D^+ falls short of defining a connection on $\mathbb{T}D$ because it is possible that it is degenerate, and so will not compose with the anchor to give the identity map. There is a condition under which Π_D^+ is non-degenerate however, and it is related to the two diffeomorphisms in equation (30). Following Lu, we define the left

infinitesimal Dressing vector field on K (resp. K^*) induced by the element $\xi \in \mathfrak{k}^*$ (resp $X \in \mathfrak{k}$) by

$$\begin{aligned} \lambda : \mathfrak{k}^* &\rightarrow \mathfrak{X}(K), & \rho : \mathfrak{k} &\rightarrow \mathfrak{X}(K^*) \\ \xi &\mapsto \tilde{\Pi}_K(\xi^L) & X &\mapsto \tilde{\Pi}_{K^*}(X^L) \end{aligned}$$

where $\tilde{\Pi}_K : T^*K \rightarrow TK$ is the bundle map induced by Π_K , and ξ^L is the left-invariant 1-form induced by ξ on K . One can do something similar for the dual Lie group K^* to define the Dressing action of \mathfrak{k} on K^* . We say that (K, Π_K) is complete if the left infinitesimal Dressing action integrates to an action of K^* on K (i.e. when the flows of the vector fields on K are complete). Finally, from Proposition 2.45 in [29], we have the following result.

Theorem 5.1. [29] *Assume that (K, Π_K) is a complete and simply connected Poisson-Lie group. Then the Poisson structure (D, Π_D^+) is non-degenerate (and therefore symplectic) everywhere.*

Relating back to the maps in equation (30), Lu also tells us the following:

Theorem 5.2. [29] *A simply-connected Poisson-Lie group (K, Π_K) is complete if and only if the maps $\Psi : K \times K^* \rightarrow D$ and $\Phi : K^* \times K \rightarrow D$ are diffeomorphisms.*

For this reason, we restrict our attention to complete simply-connected Poisson Lie groups (K, Π_K) . If (K, Π_K) is a complete simply-connected Poisson-Lie group, then the Drinfeld double admits a global diffeomorphism $D \cong K \times K^*$, and an invertible Poisson structure Π_D . If we let the related symplectic structure by ω_D , we know that ω_D is closed because Π_D^+ is Poisson, and so ω_D is symplectic on D . We can now begin the construction of a para-holomorphic algebroid with a flat para-complex connection in a similar way to Example 5.1.

Consider the standard Courant algebroid on $\mathbb{T}D$ induced by the Lie bracket on $\mathfrak{d} = \mathfrak{k} \oplus \mathfrak{k}^*$. We have a connection A_{ω_D} , given by the graph of ω_D . There is a natural para-complex structure J_D on TD coming from the span of the left invariant vector fields on K and K^* , making $\{(e_i)^L, (\varepsilon^i)^L\}$ a global frame for TD . By the non-degeneracy of ω_D , we have a global frame for $\mathbb{T}D$, given by $\{(e_i)^L, (\varepsilon^i)^L, \tilde{\omega}_D((e_i)^L), \tilde{\omega}_D((\varepsilon^i)^L)\}$. If we chose to extend J_D to an almost para-complex structure \tilde{J}_D on $\mathbb{T}D$ by identifying the $+1$ -eigenbundle as the $C^\infty(D)$ -span of $\{(e_i)^L, \tilde{\omega}_D((e_i)^L)\}$, and the -1 -eigenbundle as the $C^\infty(D)$ -span of $\{(\varepsilon^i)^L, \tilde{\omega}_D((\varepsilon^i)^L)\}$, then it is clear that with respect to this almost para-complex structure, A_{ω_D} defines a para-complex connection.

To summarize, we have an almost para-complex structure \tilde{J}_D with a para-complex connection A_{ω_D} on $\mathbb{T}D$. The induced para-Hermitian structure then comes from extending the natural left invariant pairing $\langle \theta^L, \theta^L \rangle$ to $\mathbb{T}D$ by taking $\langle \theta^L \circ \tilde{\Pi}_D, \theta^L \circ \tilde{\Pi}_D \rangle$ on the second factor, which we denote simply by $\langle \cdot, \cdot \rangle_{L, \Pi_D}$. This metric is clearly para-Hermitian with respect to \tilde{J}_D , and the anchor ρ_{TD} given by projection onto the first factor is clearly para-holomorphic.

This construction uses the standard Courant algebroid structure and so by Example 5.1 both the eigenbundles are integrable. Therefore, $(\mathbb{T}D, \tilde{J}_D, \langle \cdot, \cdot \rangle_{L, \Pi_D}, \rho_{TD})$ is an exact para-holomorphic algebroid with a para-complex connection A_{ω_D} , and this is an example that can be done over any complete semi-simple Poisson-Lie group that is different from the one induced by the Cartan-Dirac structure previously, and uses the standard Courant algebroid structure on $\mathbb{T}D$.

References

- [1] Alekseev, A., Bursztyn, H., Meinrenken, E., *Pure Spinors on Lie Groups*, ArXiv Mathematics e-prints, 18 July 2008
- [2] Alekseevsky, D., & Medori, C., & Tomassini, A., *Homogeneous Para-Kähler Einstein Manifolds*, Russian Mathematical Surveys, Volume 64, Number 1, 1 November 2009, Pages 1-44.
- [3] Bejan, C., *Structuri Hiperbolice pe Diverse Spatii Fibrat* (Ph.D. Thesis, Universitatea Al. I. Cuza), 1990.
- [4] Bejan, C. *The Existence Problem Of Hyperbolic Structures on Vector Bundles*. Publications De L'institut Mathématique Vol. 53, 1993, Pages 133-138.
- [5] Bressler, P., Chervov, A. *Courant Algebroids*. Journal of Mathematical Sciences, 2005.
- [6] Bruce, A., *Homological Sections of Lie Algebroids*, ArXiv Mathematics e-prints. 3 November 2020.
- [7] Cai, L., Sheng, Y., *Hom-Big Brackets: Theory and Applications*. Symmetry, Integrability, and Geometry. Methods and Applications, vol. 12, 5 February 2016.
- [8] Courant, T., *Dirac manifolds*, Transactions of the American Mathematical Society. Volume 319, Number 2, June 1990, Pages 631–661.
- [9] Cruceanu, V., & Fortuny, P., & Gadea, P.M. *A Survey on Paracomplex Geometry* Rocky Mountain Journal of Mathematics Vol. 26, Number 1, 1996, Pages 83-115
- [10] da Silva, A. C., *Lectures on Symplectic Geometry*, Springer-Verlag Berlin Heidelberg, 2008.

- [11] Erdem, S., *Paraholomorphic Structures and the Connections of Vector Bundles over Paracomplex Manifolds*. New Zealand J. Math., Vol. 30 (2001), pages 41-50
- [12] Etayo, F., & Santamaría, R., & Trías U. R. *The Geometry of a Bi-Lagrangian Manifold*. ArXiv Mathematics e-prints, March 2004.
- [13] Freidel, L., Rudolph, F. J., Svoboda, D., *A Unique Connection for Born Geometry*. Communications in Mathematical Physics, Volume 372, Pages 119–150, 8 March 2019.
- [14] Gauduchon, P., *Hermitian Connections and Dirac Operators*. Bollettino della Unione Matematica Italiana B Volume 11 B, Issue SUPPL. 2, 1997, Pages 257-288
- [15] Gover, A., R. & Nurowski, P. *Calculus and Invariants on Almost Complex Manifolds, Including Projective and Conformal Geometry*. ArXiv Mathematics e-prints, 3 August 2012.
- [16] Grüzmann, M., *H-Twisted Courant Algebroids*. ArXiv Mathematics e-prints. 15 June 2012
- [17] Hitchin, N., *Generalized Calabi-Yau Manifolds*, Quarterly Journal of Mathematics, Volume 54, Issue 3, September 2003, Pages 281-308.
- [18] Ivanov, S. & Zamkovoy, S., *ParaHermitian and Paraquaternionic Manifolds*, Differential Geometry and its Applications, Volume 23, Issue 2, 2005, Pages 205-234, ISSN 0926-2245.
- [19] Jana, S., *Cartan and Iwasawa Decompositions in Lie Theory* (Masters Thesis, University of British Columbia, Vancouver, Canada). Retrieved from <https://personal.math.ubc.ca/~reichst/Cartan+Iwasawa.pdf>.
- [20] Karolinsky, E., *A Classification of Poisson Homogeneous Spaces of Complex Reductive Poisson-Lie Groups*. ArXiv Mathematics e-prints, 17 January 2000.

- [21] Kosmann-Schwarzbach, Y., *Jacobian Quasi-Bialgebras and Quasi-Poisson Lie Groups*. Mathematical Aspects of Classical Field theory, Volume 132, 1992, Pages 459-489.
- [22] Krahe, M., *Para-Pluriharmonic Maps and Twistor Spaces*. The Quarterly Journal of Mathematics, Volume 60, Issue 3, 1 September 2009, Pages 367–385.
- [23] Kravchenko, O., *Strongly Homotopy Lie Bialgebras And Lie Quasi-Bialgebras*. Letters in Mathematical Physics, Volume 81, 2007, Pages 19-40.
- [24] Laurent-Gengoux, C., Pichereau, A., Vanhaecke, P., *Poisson Structures*. Springer-Verlag Berlin Heidelberg, 2013. Doi: 10.1007/978-3-642-31090-4.
- [25] Li-Bland, D., Meinrenken, E., *Courant Algebroids and Poisson Geometry*. International Mathematics Research Notices, Volume 2009, Issue 11, 2009, Pages 2106–2145.
- [26] Libermann, P., *Sur le Problème d'Équivalence de Certaines Structures Infinitésimales*. Mathématiques [math]. Université de Strasbourg, 1953. Français. fftel-00439921f
- [27] Liu, Z., Weinstein, A., Xu, P., *Manin Triples for Lie Bialgebroids*. Journal of Differential Geometry, Volume 45, Issue 3, 1997, Pages 547-574.
- [28] Loustau, B., & Sanders, A., *Bi-Lagrangian Structures and Teichmüller Theory*. ArXiv Mathematics e-prints, August 2017.
- [29] Lu, J-H., *Multiplicative and Affine Poisson Structures on Lie Groups* (Ph.D. thesis, University of California, Berkeley), 1990.
- [30] Lu, J-H., *Classical Dynamical r -Matrices and Homogeneous Poisson Structures on G/H and K/T* . University of Arizona Tucson. 7 August 2018
- [31] Marotta, V., Szabo, R., *Para-Hermitian Geometry, Dualities and Generalized Flux Backgrounds*. ArXiv Mathematics e-prints. 9 November 2018.

- [32] Meinrenken, E., *Introduction to Poisson Geometry*. Lecture Notes, 2017. Retrieved from math.toronto.edu/mein/teaching/MAT1341_PoissonGeometry/Poisson8.pdf.
- [33] Moroianu, A., *Lectures on Kähler Geometry*. Cambridge University Press, 2007.
- [34] Roytenberg, D., *Courant Algebroids, Derived Brackets and Even Symplectic Supermanifolds*. ArXiv Mathematical e-prints. 15 October 1999.
- [35] Smolentsev, N. K., *Almost Complex and Almost Para-Complex Cayley Structures on Six-Dimensional Pseudo-Riemannian Spheres*. ArXiv Mathematics e-prints. 23 July 2018
- [36] Svoboda, David, *Algebroid Structures on Para-Hermitian Manifolds*. Journal of Mathematical Physics, Volume 59, Issue 12, 21 December 2018.
- [37] Vaintrob, A., *Lie Algebroids and Homological Vector Fields*. Communications of the Moscow Mathematical Society, 18 January 1997.
- [38] Vaisman, I. *Towards a Double Field Theory on Para-Hermitian Manifolds*. Journal of Mathematical Physics, Volume 54, Issue 12, 23 December 2013.
- [39] Vilcu, G. E., *Para-hyperhermitian structures on tangent bundles*. Proceedings of the Estonian Academy of Sciences, Volume 60, Issue 3, 2011, pages 165-173
- [40] Yokota I., *Non-compact Simple Lie Group G'_2 of Type G_2* . Jour. Fac. Sci. Shinshu University, Volume 12, Number 1, 1977, Pages 45–52.
- [41] Yano, K., *Differential geometry on complex and almost complex spaces*. International Series of Monographs in Pure and Applied Mathematics, Vol. 49 A Pergamon Press Book. The Macmillan Co., NewYork 1965.