

Simple Drawings of  $K_n$  from Rotation Systems

by

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### **Author's Declaration**

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## Abstract

A complete rotation system on  $n$  vertices is a collection of  $n$  cyclic permutations of the elements  $[n] \setminus \{i\}$ , for  $i \in [n]$ . If  $D$  is a drawing of a labelled graph, then a rotation at vertex  $v$  is the cyclic ordering of the edges at  $v$ . In particular, the collection of all vertex rotations of a simple drawing of  $K_n$  is a complete rotation system. Can we characterize when a complete rotation system can be represented as a simple drawing of  $K_n$  (a.k.a. realizable)?

This thesis is motivated by two specific results on complete rotation systems. The first motivating theorem was published by Kynčl in 2011, who, using homotopy, proved as a corollary that if all complete 6-vertex rotation systems of a complete  $n$ -vertex rotation system  $H$  are realizable, then  $H$  is realizable. Combined with communications with Aichholzer, Kynčl determined that complete realizable  $n$ -vertex rotation systems are characterized by their complete 5-vertex rotation systems. The second motivating theorem was published by Gioan in 2005, he proved that if two simple drawings of the complete graph  $D$  and  $D'$  have the same rotation system, then there is a sequence of Reidemeister III moves that transforms  $D$  into  $D'$ .

Motivated by these results, we prove both facts combinatorially by sequentially drawing the edge crossings of an edge to form a simple drawing. Such a method can be used to prove both theorems, generate every simple drawing of a complete rotation system, or find a non-realizable complete 5-vertex rotation system in any complete rotation system (when one exists).

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## List of Symbols

$H_n$	5	complete $n$ -vertex rotation system
$\pi(v)$	5	rotation at a vertex $i$
$V(H_n)$	5	vertex set of $H_n$
$H - S$	5	Rotation system $H$ without the vertex set $S$
$\pi_S(v)$	5	$\pi(v)$ restricted to $S$
$\overrightarrow{\gamma_{e_1, \dots, e_k}}$	10	unique simple closed curve on $e_1, \dots, e_k$
$\overrightarrow{(i, j, k)}$	10	an oriented cycle
$\overrightarrow{(i, j, k)}_L$	10	left side of an oriented cycle
$\overrightarrow{(i, j, k)}_R$	10	right side of an oriented cycle
$\Delta_{\{e, f, g\}}$	10	the side of $\gamma_{e, f, g}$ that contains no vertices
$\rho_{\{e, f, g\}}(D)$	11	a Reidemeister III move
$\overrightarrow{(u, v)}$	13	a directed edge
$\prec_e^D$	13	order $e$ crosses edges in $D$
$G[V]$	13	the induced graph on the vertex set $V$
$G[E]$	13	the induced graph on the edge set $E$
$D[S]$	13	the induced drawing on $D$ from the set of edges and vertices $S$
$\mathcal{B}(R)$	14	boundary of region $R$
$<_{\wedge}^e$	29	edge ordering of two adjacent edges on $e$
$<_{\parallel}^e$	29	edge ordering of two non-crossing and non-adjacent edges on $e$
$<_{\Delta}^e$	29	edge ordering of two crossing edges on $e$ certified by a vertex
$<_{K_6}^e$	30	edge ordering of two non-adjacent edges in their $K_6$ with $e$
$\mathcal{A}_\ell$	81	the length of a chain or cycle of relations
$\mathcal{A}_{\parallel}$	81	the number of $<_{\parallel}^e$ relations in a chain or cycle of relations
$\mathcal{A}_w$	81	the sum of $\mathcal{A}_\ell$ and $\mathcal{A}_{\parallel}$

# 1 Introduction

A *complete rotation system*  $H$  on  $n$  vertices is a set of  $n$  cyclic permutations  $\pi(i)$  such that for each  $i$ ,  $\pi(i)$  is a cyclic permutation of  $[n] \setminus \{i\}$ . Such a structure arises naturally in the area of graph drawings as a drawing of a labelled complete graph in the plane naturally has a clockwise (or counter clockwise) rotation around each vertex inducing an associated complete rotation system for that graph as seen in Figure 1.

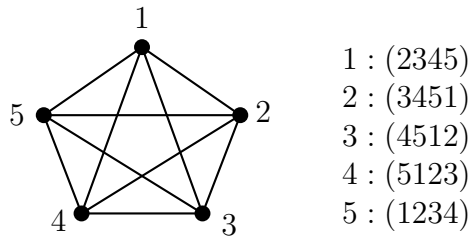


Figure 1: A labelled  $K_5$  and its associated clockwise rotation system.

A drawing  $D$  of a graph is *simple* if all pairs of edges intersect in at most one point in  $D$  and edges are not self crossing in  $D$ .

The Harary-Hill conjecture states that the minimum number of edge crossings in any drawing of the complete graph in the sphere  $cr(K_n)$  is equal to

$$H(n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor.$$

This conjecture has been open for over 50 years and has been verified for  $n \leq 12$ , the most recent results appearing in [10]. There exist simple drawings of  $K_n$  achieving  $H(n)$  crossings (see [12]) and so it is left to prove that  $H(n) \leq cr(K_n)$ . Dan Archdeacon’s combinatorial generalization of the Harary-Hill conjecture to rotation systems in [4] states that in any  $n$ -vertex complete rotation system, the number of induced non-planar  $K_4$ ’s is at least  $H(n)$ . This generalizes the Harary-Hill conjecture to complete rotation systems and turns a geometric problem into a purely combinatorial problem. Archdeacon wrote a hill-climbing program that found for small values of  $n$  the conjecture is true, however the conjecture remains unresolved.

The complete graph is one of the most well studied classes of graphs for simple drawings. In [13], Kynčl finds the existence of simple drawings

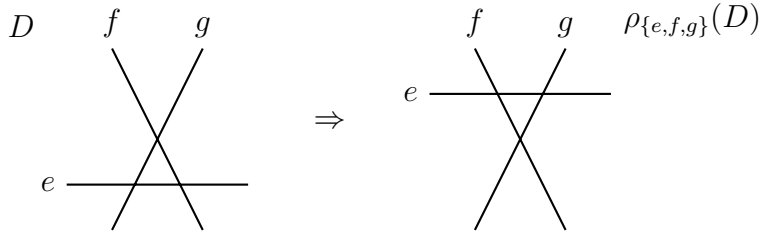


Figure 2: A Reidemeister III move over the edges  $e, f, g$ .

of  $K_n$  with edges having few crossings. In [14], Pach and Tóth estimate the number of drawings of  $K_n$  under various restrictions. In [15], Pach, Solymosi and Tóth show a Ramsey type theorem on simple drawings of  $K_n$ , in particular, for every integer  $r$ , there exists an integer  $n$  such that every simple drawings of  $K_n$  contains one of two simple drawings of  $K_r$ . The same analysis on complete rotation systems gives the same result for the two associated rotation systems of the two simple drawings of  $K_r$ . In [16], Pach, Rubin and Tardos use geometric arguments to show that any simple drawing of  $K_n$  with straight line segments has a set of at least  $n^{1-o(1)}$  edges that are each pairwise crossing.

If  $H$  is a complete  $n$ -vertex rotation system, then there is a drawing  $D_H$  of a graph  $K_n$  in the sphere having  $H$  as its rotation system by simply drawing the vertices in the sphere with their associated rotations and connecting the edges. If we restrict the drawings to be *simple* (i.e. drawings with no loops, edges intersecting in at most one point (such a point being a crossing or a vertex), and no three edges crossings at the same point), then it is not as obvious that complete rotation systems have an associated simple drawing. A rotation system is *realizable* if it has an associated simple drawing. An  $(n, k)$  *complete rotation system*  $H$  is a complete rotation system with  $n$  vertices such that every complete rotation system of size  $k$  inside  $H$  is realizable.

Given a triple of pairwise crossings edges  $(e, f, g)$  in a drawing  $D$  such that there is a face bounded by exactly  $e, f$  and  $g$ , then informally a *Reidemeister III move*  $\rho_{\{e,f,g\}}$  over edges  $(e, f, g)$  is the operation of moving one of the edges over the crossing of the other two without having it pass any other edges of vertices in  $D$  as seen in Figure 2.

The main motivation of this thesis is the following result.

**Theorem 1.1.** *If  $n \geq 6$  and  $H_n$  is a complete  $(n, 5)$ -rotation system, then*

$H_n$  is realizable.

In [1], Kynčl proves Theorem 1.1 holds for  $(n, 6)$ -rotation systems by proving this on complete abstract topological graphs. A *complete abstract topological graph* is a tuple  $(G, X)$  such that  $G$  is a graph and  $X$  is a set of pairs of edges from  $G$ . A complete abstract topological graph  $(G, X)$  is said to be *realizable* if there is a simple drawings of  $G$  in which exactly the pairs in  $X$  cross. Since every complete rotation system is a complete abstract topological graph, complete abstract topological graphs are more general than complete rotation systems.

Comparatively, Kynčl's methods use homotopy and orderings of edge crossings on a fixed star, whereas ours will use combinatorial arguments and orderings of edge crossings on a fixed edge. Through private communications with Aichholzer (see [2],[8]), it is known computationally that complete  $(6, 5)$ -rotation systems are realizable, however such a result has not been published. Combining these two results implies Theorem 1.1.

We fill a hole in literature by giving a formal proof that  $(6, 5)$ -rotation systems are realizable. Seeing as Kynčl chose to prove Theorem 1.1, we give a different proof using a combinatorial approach. Such a proof gives rise to Theorem 1.2.

**Theorem 1.2.** *Let  $n \geq 4$  and  $D$  be a simple drawing of  $K_n$ . If  $c$  is an edge of  $K_n$  and  $P$  is a point of  $D$  in some face, then either:*

1. *There is a sequence (possibly empty) of Reidemeister III moves on  $D$  to a simple drawing  $D'$  such that a non-trivial segment of  $c$  is on the boundary of the face of  $D'$  containing  $P$ ; or*
2. *There is some drawing  $\mathcal{D}$  in  $D$  on a  $K_4$  containing  $c$  such that no face of  $\mathcal{D}$  contains  $P$  and has a non-trivial segment of  $c$  on its boundary.*

Theorem 1.2 is independent of rotation systems and is a easily stated fact of how edges and faces in simple drawings of  $K_n$  are related. Such a fact provides a new structural theorem to a well studied area of graph drawings.

**Theorem 1.3.** *Let  $n$  be a positive integer,  $H_n$  be a realizable complete  $n$ -vertex rotation system. If  $D$  and  $D'$  are two simple drawings realizing  $H_n$ , then one can be obtained from the other through a series of Reidemeister III moves.*

Gioan presents a sketch of the proof for Theorem 1.3 in [3], but a published version of his work has yet to appear. However, a full version of the proof of Theorem 1.3 appears in [4]. We present a proof of the same result from the perspective of drawing the crossings sequentially (similar to our proof of Theorem 1.1). All proofs of Theorem 1.3 follow the inductive arguments found in [3] and use some of the ideas that are in [3].

Our arguments brings an essential simplification to the proof of Theorem 1.3 by noticing that if  $D$  is a simple drawing of  $K_n$  and  $L$  is a subdrawing of  $D$  containing a partial edge  $e_i$ , then the set of edges that  $e_i$  can cross in  $L$  that extend that drawing to a simple drawing of  $K_n$  with the same associated rotation system as  $D$  appear consecutively on the boundary of the face containing  $e_i$  in  $L$ .

The main new contributions of this thesis are:

- A combinatorial proof of Theorem 1.1, improving Kynčl’s results from a result on  $(n, 6)$ -rotation systems to a result on  $(n, 5)$ -rotation systems;
- Theorem 1.2, a new structural theorem relating edges and faces in simple drawings of  $K_n$ ; and
- A simplified perspective on the proof of Theorem 1.3.

We end this section with a description of the thesis. In Section 2, we describe the preliminary work on complete rotation systems to introduce the reader to the literature and basic concepts. Section 3 characterizes how edges and faces interact in simple drawings of  $K_n$ . Section 4 is used to prove Theorem 1.1 for  $n = 6$ , an interesting result in itself as it extends Kynčl’s results. Section 5 is used to find orderings of edge crossings on a fixed edge. This section is broken into three smaller sections for the cases  $n = 7, n = 8$  and  $n \geq 9$ .

Sections 6 completes the proof of Theorem 1.1 by using the ordered edge crossings in Section 5 along with the edges and faces theorem in Section 3 to algorithmically draw simple drawings of a realizable complete rotation system. Section 7 offers a closed proof of Gioan’s Theorem (Theorem 1.3) originally stated in [3].

Appendix A offers an algorithm that takes a complete  $n$ -vertex rotation system  $H$  for  $n \geq 5$  as input, and outputs either a non-realizable complete 5-vertex rotation system in  $H$  or a simple drawing  $D$  realizing  $H$ . Furthermore, given specific choices, the algorithm will output any of the simple drawings realizing  $H$ .

## 2 Groundwork

This section will be dedicated to fundamental definitions and observations made on complete rotation systems and drawings of graphs. Many of the observations found in this section can be found in existing literature like [1], [2], [3], [5] and [7]. Let us start by defining edges and edge segment topologically in drawings of graphs. For the purposes of this thesis, an *edge* is a homeomorph of the closed compact interval  $[0, 1]$  and a *non-trivial segment* of an edge is a closed connected component of an edge that is not a point.

**Definition 2.1.** A *complete  $n$ -vertex rotation system*  $H_n$  is a collection of  $n$  cyclic permutations such that for each  $i \in [n]$ , there exists a unique permutation  $\pi(i)$  in  $H_n$  on the elements of  $[n] \setminus \{i\}$ . Define the vertices of  $H_n$  to be  $V(H_n) = [n]$  and  $H_n^{-1}$  to be the rotation system  $H_n$  with every cyclic permutation reversed.

For the purposes of this thesis, every rotation system will be considered to be a complete rotation system on some set of vertices, we remove complete for simplicity.

**Notation 2.2.** Let  $H$  be a rotation system on vertices  $V(H)$ . For  $S \subseteq V(H)$ , let  $H - S$  be the rotation system contained in  $H$  induced by the vertices  $V(H) \setminus S$ .

**Notation 2.3.** Let  $H_n$  be a complete  $n$ -vertex rotation system. For all  $v \in V(H_n)$  and for any subset  $S$  of  $V(H_n)$  not containing  $v$ , let  $\pi_S(v)$  be the restriction of  $\pi(v)$  to  $S$ .

**Definition 2.4.** A complete  $n$ -vertex rotation system  $H_n$  is *realizable* if there exists a simple drawing  $D$  on a labelled  $K_n$  such that the associated rotation system on  $D$  is  $H_n$ .

**Definition 2.5.** For positive integers  $n$  and  $k$  such that  $n \geq k$ , an  $(n, k)$ -*rotation system*  $H$  is an  $n$ -vertex rotation system such that every rotation system that is a restriction of  $H$  to a set of  $k$  vertices is realizable.

**Observation 2.6** ([7], Sec.2). *If  $H$  is a 4-vertex realizable rotation system, then the rotation of 3 vertices determines the rotation at the fourth.*

Ábrego et al. use this observation in [7] to computationally generate  $(n, 4)$ -rotation systems. We will use this observation in the end of the proof of

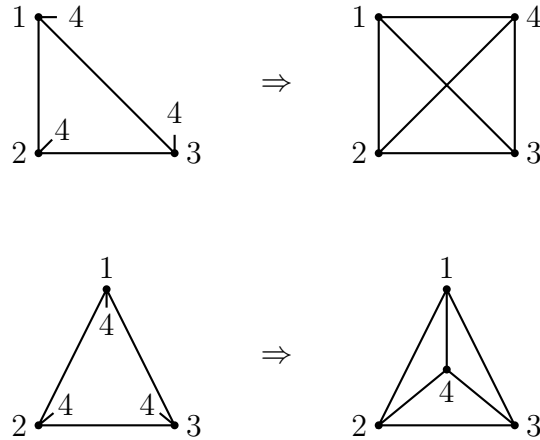


Figure 3: 3-cycles with small edge segments representing realizable rotation systems.

Theorem 1.1 to verify in a generated drawing of the complete graph that the rotation at a specific vertex coincides with the rotation system that produced the drawing.

**Observation 2.7.** *If  $H$  is a realizable 4-vertex rotation system, then there is a unique labelled drawing  $D$  that is a realization of  $H$ . In particular,  $H$  determines the oriented crossings of  $D$ .*

There are  $2^4 = 16$  4-vertex rotation systems. By Observation 2.6, half of these rotation systems are not realizable. Let  $C$  be a 3-cycle having three of the four vertices of  $H$  on it. Draw small segments at each vertex on  $C$  to represent the fourth vertex in the rotation of each of the vertices in  $H$ . If the small segments to the 4th vertex  $v$  all start on one side of  $C$ , then connect them all at a point on that side of  $C$  and call that point the 4th vertex as seen in Figure 3. This results in two rotation systems (depending on the side of  $C$  the edge segments are in) each of which have unique labelled planar representations.

Alternatively, one small segment starts on the opposite side of the other two small segments. There are 6 ways to choose the side of  $C$  that contains the single small segment and the starting vertex for the small segment. For each choice, have the single small segment cross  $C$  on the edge that is not incident to its starting vertex, then connect all the segments at a point on

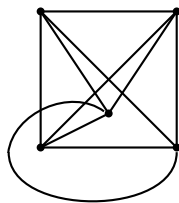


Figure 4: The Harborth drawing of  $K_5$ .

that side of  $C$ . Each choice produces a unique oriented crossing and a unique rotation system. Portions of this observation have appeared in various forms; for example see [5] Lemma 10.

Let us now prove a short lemma of a similar nature.

**Lemma 2.8.** *Let  $n \leq 5$  be a positive integer. If  $H_n$  is a realizable rotation system, then there is a unique labelled simple drawing  $D$  that realizes  $H_n$ .*

*Proof.* For  $n \leq 3$  this is trivial as there is only one rotation system  $H_n$  that corresponds to the unique simple drawing of  $K_n$ . If  $n = 4$ , then Observation 2.7 is our desired result. Therefore, assume without loss of generality  $n = 5$ . Let  $D$  be a simple drawing that is a realization of  $H$ ,  $V(H) = [5]$ , and  $D_i$  be the simple drawing  $D - i$ . By Observation 2.7,  $D_5$  is uniquely determined. Every 3-cycle in  $D_j$  not containing 5 has 5 on a specific side of the cycle, for  $j \in [4]$ . Observe for each of two simple drawings of  $K_4$ , each face is uniquely determined by the intersection of sides of triangles of that  $K_4$ .

It follows that the intersecting sides of the 3-cycles containing 5 is a unique face in  $D_5$ . Therefore, 5 is contained in a unique face in  $D_5$ .

Again observe over all possible simple drawings of  $D_5$ , the edge  $(u, 5)$  in  $D$  is uniquely determined by the location of 5 and the starting of  $(u, 5)$  at the rotation of  $u$ . This implies that there is a unique labelled simple drawing  $D$  that realizes  $H_n$ .  $\square$

For any induction on  $(n, 5)$ -rotation systems, this lemma will be very useful for the base case as it allows us to ignore the rotation system itself, and talk about its associated simple drawing.

A common simple drawing of  $K_5$  that appears frequently is the *Harborth* drawing of  $K_5$  (also known as the twisted graph, see [5]). Such a drawing can be seen in Figure 4.



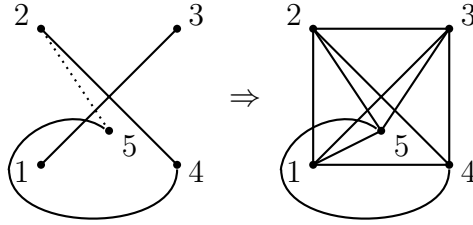


Figure 5: Harborth subdrawings determining vertex rotations.

**Observation 2.9.** *The Harborth drawing of  $K_5$  is the unique simple drawing of  $K_5$  that has a 3-cycle crossed three times by a single edge. Furthermore, the labelled subdrawing of a 3-cycle crossed three times by a single edge uniquely determines the labelled Harborth drawing.*

To prove this observation, draw a labelled 3-cycle crossed three times by a single labelled edge. Each edge not in the drawing can be added in a unique way to form a simple drawing of  $K_5$ . This can be explicitly seen along with the unique rotations in Figure 5 by including the dotted line  $(2, 5)$  in the leftmost drawing.

**Observation 2.10.** *The Harborth drawing of  $K_5$  is the unique simple drawing of  $K_5$  that has an edge  $e$  crossed by two edges  $f$  and  $g$  sharing an endpoint such that  $f$  and  $g$  cross  $e$  from opposite sides when starting at their common endpoint. Furthermore, the labelled subdrawing of  $e, f$  and  $g$  uniquely determines the labelled Harborth drawing.*

Again, this observations follows from drawing the labelled edges  $e, f$  and  $g$ , then extending it to a simple drawing of  $K_5$ . This can be explicitly seen along with the unique rotations in Figure 5 by excluding the dotted line in the leftmost drawing.

**Definition 2.11.** Two  $(n, 4)$ -rotation systems  $H_1$  and  $H_2$  are *weakly isomorphic* if they have the same set of pairwise edge crossings.

Originally, this definition is defined on realizable rotation systems in [2, 5] and is used in [7], however, it can be generalized. In particular, most literature apply it in the context of the following proposition:

**Proposition 2.12.** *Two  $(n, 4)$ -rotation systems  $H_1$  and  $H_2$  are weakly isomorphic if and only if  $H_1 = H_2$  or  $H_1 = H_2^{-1}$ .*

*Proof.* An important fact to be applied in this proof is that adjacent transpositions on a totally ordered set generate the symmetric group on that set of elements. In particular, the bubble sort method (a method which takes an element  $A$  and compares it too another element  $B$  by taking two adjacent entries of  $A$  and swaps them if they are in the incorrect order compared to  $B$ ) shows that every transposition applied is applied at most once.

As a corollary, for two rotation systems  $A$  and  $B$  on the same point set  $P$ , for any  $x \in P$  with the rotation of  $x$  in  $A$  being  $\pi^A(x)$  and the rotation of  $x$  in  $B$  being  $\pi^B(x)$ , there exists a set of adjacent transpositions from  $\pi^A(x)$  to  $\pi^B(x)$  such that no transposition is applied twice. We will prove Proposition 2.12 for 5-vertex rotation systems first.

Let  $A$  and  $B$  be 5-vertex two rotation systems on the same vertex set  $P$  (without loss of generality  $P = [5]$ ), with the same crossings. Without loss of generality, assume  $B \neq A$ . We will show  $B = A^{-1}$ . Let  $A_i$  and  $B_i$  be the rotation systems  $A - i$  and  $B - i$ , respectively, for  $i \in P$ . Since  $B \neq A$ , without loss of generality  $B_1 \neq A_1$ . Let  $\{e_j\}_{j=1}^k$  be a sequence of adjacent transpositions that sends  $A$  to  $B$ .

Since  $B_1 \neq A_1$  and they have the same crossings, it follows that  $B_1 = A_1^{-1}$ . Since  $B_1 = A_1^{-1}$ , four transpositions were applied to  $A_1$  to obtain  $B_1$ . Each of these transpositions uniquely apply to another  $A_i$ . It follows that every  $A_i$  has had at least one transposition applied to it. Since every  $B_i$  has the same crossing as  $A_i$ , it follows that  $B_i = A_i$  or  $B_i = A_i^{-1}$ . Since every  $A_i$  has had at least one transposition applied to it to obtain  $B_i$ , it follows that for all  $i \in P$ ,  $B_i = A_i^{-1}$ , and  $B = A^{-1}$ .

Comparing the 5-vertex rotation systems  $H_5$  and  $H'_5$  of two weakly isomorphic  $(n, 4)$ -rotation system  $H$  and  $H'$  on the same vertex set, we have shown that  $H_5 = H'_5$  or  $H_5 = H'^{-1}_5$ . From here, we follow the arguments of Kynčl for Proposition 6 in [9]

Let  $H$  be an  $(n, 4)$  rotation systems and  $H'$  be any rotation system weakly isomorphic to  $H$ . From our arguments, it is clear that 5-vertex rotation systems of  $H$  and  $H'$  on common vertex sets are the same or inverses. Following Lemma in Proposition 6 of [9], if  $B'$  and  $C'$  are two 5-vertex rotation systems in  $H'$  with exactly 4 common vertices, then  $B'$  uniquely determines  $C'$ . The proof of this fact is the proof of Lemma in Proposition 6 of [9].

As the proof of Proposition 6 in [9] states repeated use of this fact results in every 5-vertex rotation system of  $H'$  being the same as  $H$  or inverse. Since the rotation at a vertex is uniquely determined by its 3-element subsets, it follows that  $H' = H$  or  $H' = H^{-1}$   $\square$

Kynčl had already considered Proposition 2.12 on realizable rotation system in [9] and it was used in various literature (see [2, 5, 9]). We modify Kynčl's arguments and proposition to extend to  $(n, 4)$ -rotation systems. As mentioned before, such rotation systems are interesting as counting crossings is still viable, and Archdeacon has suggested in [4] that the crossing number of  $(n, 4)$ -rotation systems is the same as the crossing number of  $K_n$ . Theorem 1.1 reduces this question to comparing the crossing number of  $(n, 4)$ -rotation systems to the crossing number of  $(n, 5)$ -rotation systems.

**Notation 2.13.** Let  $D$  be a drawing of a graph  $G$ , and  $E = \{e_1, \dots, e_k\}$  be a set of edges. If there is a unique simple closed curve defined on the edges of  $E$  in  $D$ , then we define  $\gamma_{e_1, \dots, e_k}$  to be that curve.

**Notation 2.14.** Let  $n \geq 3$  and  $D$  be a simple drawing of  $K_n$ . For distinct vertices  $i, j, k \in V(K_n)$ , let  $\overrightarrow{(i, j, k)}$  be the directed 3-cycle whose labels appear in clockwise order  $i$ , then  $j$ , then  $k$ . The *left side* (*right side*)  $\overrightarrow{(i, j, k)}_L$  ( $\overrightarrow{(i, j, k)}_R$ ) is the region that is on the left side (right side) of  $\overrightarrow{(i, j, k)}$ .

**Notation 2.15.** Let  $D$  be a simple drawing of a graph  $G$  and  $e, f$  and  $g$  be three edges in  $G$  that pairwise cross in  $D$ . If  $\gamma_{e, f, g}$  has a side that contains no vertices of  $V(\{e, f, g\})$ , then this side is  $\Delta_{\{e, f, g\}}$ .

For three edges  $e, f, g$ , if  $\Delta_{\{e, f, g\}}$  exists and is a face, then there are two drawings  $D$  and  $\bar{D}$  of  $\Delta_{\{e, f, g\}}$  with the same oriented crossings. In both drawings, performing a boundary walk just outside of  $\Delta_{\{e, f, g\}}$ , we find a closed disc  $S$  ( $\bar{S}$ ) that contains  $\gamma_{e, f, g}$  on its interior, contains only non-trivial segments of  $e, f, g$ , and does not contain any vertices. Both  $S$  and  $\bar{S}$  are homotopically equivalent.

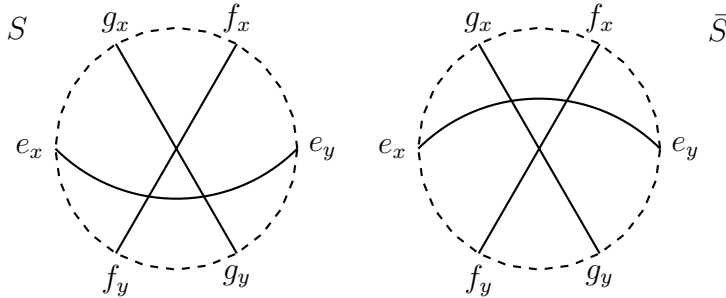


Figure 6: A Reidemeister III move.

**Definition 2.16.** Let  $n \geq 6$ ,  $D$  be a simple drawing of  $K_n$  and  $e, f, g$  be three pairwise crossings edges. Suppose  $\Delta_{\{e,f,g\}}$  exists and is a face. Let  $S$  be a closed disc containing  $\Delta_{\{e,f,g\}}$  in its interior, contains no vertices, and only contains non-trivial segments of  $e, f, g$ . Each of  $e, f, g$  intersect the boundary of  $S$  at two points  $e_x, e_y, f_x, f_y, g_x, g_y$  respectively, such that from  $e_x$  to  $e_y$ ,  $e$  crosses  $f$  than  $g$ .

Let  $\bar{S}$  be a closed disc whose boundary is the boundary of  $S$  that contains three edge segments  $\bar{e}, \bar{f}, \bar{g}$  that pairwise cross once such that the ends of  $\bar{e}, \bar{f}, \bar{g}$  are  $e_x, e_y, f_x, f_y, g_x, g_y$ , respectively, and from  $e_x$  to  $e_y$ ,  $\bar{e}$  crosses  $g$  than  $f$ .

A *Reidemeister III move*  $\rho_{\{e,f,g\}}(D)$  is the drawing  $(D[K_n] \setminus S) \cup \bar{S}$ .

Given a simple drawing  $D$  of  $K_n$ , and the existence of  $\Delta_{\{e,f,g\}}$ , we note the following simple facts about Reidemeister III moves:

- When performing the Reidemeister III move  $\rho_{\{e,f,g\}}$  on  $D$ , we can choose  $\bar{S}$  in such a way that exactly one edge changes when compared to  $S$  and we can choose which edge changes.
- If  $\Delta_{\{e,f,g\}}$  is a face in  $D$ , then,  $\rho_{\{e,f,g\}}(\rho_{\{e,f,g\}}(D)) = D$ ;
- For three edges  $a, b, c$  in  $K_n$ , if  $\Delta_{\{a,b,c\}}$  exists in  $D$ , then  $\Delta_{\{a,b,c\}}$  exists in  $\rho_{\{e,f,g\}}(D)$ ; and
- For three edges  $a, b, c$  in  $K_n$ , if  $\Delta_{\{a,b,c\}}$  exists in  $D$ , is not a face in  $D$  and is a face in  $\rho_{\{e,f,g\}}(D)$ , then  $\rho_{\{a,b,c\}}(\rho_{\{e,f,g\}}(D))$  is a valid simple drawing.

The label Reidemeister III move originates from Knot Theory. In literature this is also known as a triangle mutation (see [3]) or triangle flip (see [4]). Since such a move applied to a simple drawing of a graph does not change the rotation at any vertex, it does not change the associated rotation system of a simple drawing. This motivates Theorem 1.3 and plays a crucial role in Theorem 3.8 and implicitly Theorem 1.1.

We show that if  $\Delta_{\{e,f,g\}}$  contains no vertices in a simple drawing  $D$  of  $K_n$ , then  $\Delta_{\{e,f,g\}}$  can be emptied of edge segments by applying Reidemeister III moves.

**Lemma 2.17.** *Let  $D$  be a simple drawing of the complete graph,  $x, y, z$  be three edges such that  $\Delta_{x,y,z}$  exists and contains no vertices, and  $\bar{x}, \bar{y}$  and  $\bar{z}$*

are the segments of  $x, y, z$ , respectively, on the boundary of  $\Delta_{x,y,z}$ . If there are no edges crossing both  $\bar{y}$  and  $\bar{z}$ , then there exists  $\Delta_{x,y_1,z_1}$  contained in  $\Delta_{x,y,z}$  that is a face.

*Proof.* Define  $\Delta_{x,y_1,z_1}$  to be a triangle contained inside  $\Delta_{x,y,z}$  such that one side of  $\Delta_{x,y_1,z_1}$  is  $\bar{x}_1$  contained in  $x$ , every edge crossing  $\Delta_{x,y_1,z_1}$  crosses  $\bar{x}_1$ , and the number of crossings in  $\Delta_{x,y_1,z_1}$  is minimal (including crossings on the boundary).  $\Delta_{x,y_1,z_1}$  exists as  $\Delta_{x,y,z}$  satisfies the definition containing some finite number of crossings.  $\Delta_{x,y_1,z_1}$  is the desired triangle unless it is non-empty.

Assume by way of contradiction that  $\Delta_{x,y_1,z_1}$  is not empty and let  $\bar{y}_1$  and  $\bar{z}_1$  be the segments of  $y_1$  and  $z_1$ , respectively, bounding  $\Delta_{x,y_1,z_1}$ . Without loss of generality, let  $\bar{y}_1$  be a side of  $\Delta_{x,y_1,z_1}$  that has a crossing apart from its endpoints.

Let  $z_2$  be the edge that crosses both  $\bar{x}_1$  and  $\bar{y}_1$  (other than  $z_1$ ) that is furthest away from the  $(x, y_1)$  crossing on  $\bar{x}_1$ ,  $\bar{x}_2$  be the segment from the  $(x, y_1)$  crossing to  $z_2$  on  $\bar{x}_1$ , and  $\tilde{y}_1$  be the segment from the  $(x, y_1)$  crossing to  $z_2$  on  $\bar{y}_1$ .

By definition of  $z_2$ , every edge that crosses  $\bar{y}_1$  other than  $z_1$  does so on  $\bar{x}_2$ . Since  $\Delta_{x,y_1,z_2}$  is contained in  $\Delta_{x,y,z}$  and  $\Delta_{x,y,z}$  contains no vertices, it follows that  $\Delta_{x,y_1,z_2}$  contains no vertices. In particular, every edge that crosses  $\Delta_{x,y_1,z_2}$  at  $z_2$  also crosses  $\bar{x}_2$ .

Therefore, every edge that crosses  $\Delta_{x,y_1,z_2}$ , crosses  $\bar{x}_2$ . In particular,  $\Delta_{x,y_1,z_2}$  contains less crossings than  $\Delta_{x,y_1,z_1}$ , a contradiction.  $\square$

**Corollary 2.18.** *Let  $D$  be a simple drawing of the complete graph. Suppose  $\Delta_{x,y,z}$  exists with the boundary consisting of segments  $x_1, y_1$  and  $z_1$  of the edges  $x, y$  and  $z$ , respectively. If there are no edges crossing both  $y_1$  and  $z_1$ , then there exists a series of Reidemeister III moves  $\{\rho_{X_i}\}_{i=1}^k$  on  $D$  with  $\rho_{X_i}(D_{i-1}) = D_i$  and  $D = D_0$  such that*

- For all  $i \in [k]$ ,  $x \in X_i$ ;
- $x, y, z \in X_k$ ; and
- $\Delta_{X_i} \subset \Delta_{X_k}$  in  $D_{i-1}$ .

This corollary follows by repeatedly applying Reidemeister III moves to the triangles found in Lemma 2.17 until the final Reidemeister III move made is over the triangle on edges  $x, y$  and  $z$ . We end this section with three final definitions for notational purposes.

**Notation 2.19.** Let  $G$  be a graph. For  $V \subseteq V(G)$  and  $E \subseteq E(G)$ , let  $G[V]$  denote the subgraph induced by the vertex set  $V$  in  $G$  and  $G[E]$  the subgraph containing  $E$  and its endpoints in  $G$ .

**Notation 2.20.** Let  $D$  be a drawing of a graph  $G$ ,  $\bar{V} \subseteq V(G)$ , and  $\bar{E}$  be a set of edges of  $G$ . Let  $D[\bar{V} + \bar{E}]$  be the subdrawing of  $G[\bar{V}] \cup G[\bar{E}]$  in  $D$ .

**Notation 2.21.** Let  $G$  be a graph and  $(u, v)$  an edge in  $E(G)$ . Then  $\overrightarrow{(u, v)}$  is the directed edge from  $u$  to  $v$ .

**Notation 2.22.** Let  $D$  be a simple drawing of a graph  $G$  and  $e, f$  and  $g$  be three edges such that  $e$  is a directed edge and both  $f$  and  $g$  cross  $e$ . Define  $f \prec_D^e g$  if  $e$  crosses  $f$  then  $g$  in  $D$ .

### 3 Characterizing Edges and Faces

The purpose of this section is to prove Theorem 3.8 which is a more technical result of Theorem 1.2 that relates faces to edges in simple drawings of  $K_n$ . In particular, Theorem 3.8 shows for any simple drawing  $D$  of  $K_n$ , point  $P$  not in the  $K_n$  and any edge  $c$ , either there exists a  $K_4$  drawn in  $D$  that separates  $c$  from  $P$  or there is a series of Reidemeister III moves on  $D$  such that the resulting drawing has  $c$  on the boundary of the face containing  $P$ .

This is crucial to the algorithmic proof of Theorem 1.1 as sequentially drawing an edge  $e$  by its crossing segments requires  $c$  the next edge crossed to be on the boundary of the appropriate face. If  $c$  is on the appropriate face, then we continue algorithmically drawing. If  $c$  is not on the appropriate face, then Theorem 3.8 finds a  $K_4$  that separates the edge from the current region which can be used to relate to an associated small non-realizable rotation system or finds a set of Reidemeister III moves that brings the edge we want to the boundary of our desired face.

We start this section by relating faces and sides of 3-cycles. Following this, we will describe how edges intersect boundaries of faces in simple drawings of  $K_n$ . Finally we state and prove Theorem 3.8.

**Notation 3.1.** Let  $D$  be a drawing of a graph  $G$  and let  $R$  be a face in  $D$ . Define  $\mathcal{B}(R)$  to be the boundary of  $R$ .

The following Lemma is a portion of Lemma 4.7 from [11] and is known as Carathéodory's Theorem for simple complete topological graphs (for simple drawings of  $K_n$ ).

**Lemma 3.2.** *Let  $D$  be a simple drawing of  $K_n$  and let  $x$  be a point in the interior of a bounded face of  $D$ . Then there is a 3-cycle  $(u, v, w)$  in  $D$  containing  $x$  in its bounded side.*

An immediate consequence of Lemma 3.2 is Corollary 3.3 by considering two faces  $F_1$  and  $F_2$  one of which is the unbounded face by choice, and the other being bounded. There exists a 3-cycle separating  $F_1$  and  $F_2$ . For each pairing  $(F_i, F_j)$  we do this comparison and consider the intersections of sides of these 3-cycles.

**Corollary 3.3.** *Let  $D$  be a simple drawing of  $K_n$ . If  $R$  is a face in  $D$ , then  $R$  is the unique open intersection of specific sides of all 3-cycles in  $K_n$ .*

*Proof.* Let  $D$  be a simple drawing of  $K_n$  and let  $\mathcal{F} = \{F_1, \dots, F_k\}$  be the set of faces in  $D$ . Let  $F_i$  and  $F_j$  be two different faces in  $D$ . Without loss of generality, assume  $F_i$  is the unbounded face in  $D$ . Let  $x$  be a point in  $F_j$ . By Lemma 3.2 there is a 3-cycle  $C_{i,j}$  in  $K_n$  such that  $F_i$  and  $x$  are on opposite sides. In particular,  $F_i$  and  $F_j$  are on opposite sides of  $C_{i,j}$ .

Let  $\mathcal{C}$  be the set of such 3-cycles, ones for each pair  $(i, j)$ . It follows that each face  $\mathcal{F}$  is uniquely determined by the intersections of sides of 3-cycles in  $\mathcal{C}$ . Since  $\mathcal{C}$  is a subset of the set of all 3-cycles in  $K_n$ , it follows that each face in  $\mathcal{F}$  is uniquely determined by the intersection of sides of 3-cycles in  $K_n$ .  $\square$

As an important note to this corollary, the intersections of sides of 3-cycles in a simple drawing of  $K_n$  does not always determine a face. This corollary will be very useful in helping characterize the relation between edges and faces in any simple drawing of  $K_n$ .

**Lemma 3.4.** *Let  $n \geq 4$ ,  $D$  be a simple drawing of  $K_n + e_i$  where  $e_i$  is a partial edge starting at  $u \in V(G)$  and has  $i$  crossings. If  $R_{e_i}$  is a face in  $D - e_i$ , then  $e_i$  has exactly one non-trivial segment in  $R_{e_i}$ .*

*Proof.* Let  $n \geq 4$ ,  $D$  be a simple drawing of  $K_n + e_i$  where  $e_i$  is a partial edge starting at  $u \in V(G)$  and has  $i$  crossings, and  $R_{e_i}$  be a face in  $D - e_i$ . By way of contradiction, assume  $e_i$  has at least two non-trivial segments in  $R_{e_i}$ . It follows that there is some non-trivial segment  $e_c$  of  $e_i$  starting and ending at  $\mathcal{B}(R_{e_i})$  and is contained in  $S$  the side of  $\mathcal{B}(R_{e_i})$  that does not contain  $R_{e_i}$ .

Let  $f$  be one of the edges on  $\mathcal{B}(R_{e_i})$  that  $e_c$  crosses. Pick  $\mathcal{C}$  to be any closed curved on  $\mathcal{B}(R_{e_i})$  and  $e_c$  that contains  $e_c$ . Since  $R_{e_i}$  is a face, it follows that the ends of  $f$  are on separate sides of  $\mathcal{C}$ .

Let  $u_1$  be the end of  $f$  that is on the opposite side of  $\mathcal{C}$  from  $u$ . In  $D$ ,  $(u, u_1)$  does not cross  $e_c$  and it does not cross  $R_{e_i}$  in  $D_{e_i}$ . It follows that  $(u, u_1)$  does not cross  $\mathcal{C}$ , a contradiction with  $u$  and  $u_1$  on opposite sides of  $\mathcal{C}$ .  $\square$

This Lemma is a tool to describe how an edge is drawn sequentially in simple drawings. It will be used in the inductive proof of Theorem 1.1 and is not relevant to the results in this section. The following Lemma shows a similar result for edges in a simple drawing of  $K_n$ .

**Lemma 3.5.** *Let  $n \geq 4$ ,  $D$  be a simple drawing of  $K_n$ ,  $R$  be a face in  $D$  and  $e$  be an edge of  $K_n$ . If  $e$  has a non-trivial segment on  $\mathcal{B}(R)$ , then  $e \cap \mathcal{B}(R)$  is exactly one non-trivial segment of  $e$*



*Proof.* Let  $n \geq 4$ ,  $D$  be a simple drawing of  $K_n$ ,  $R$  be a face in  $D$  and  $e = (u, v)$  be an edge of  $K_n$ . Suppose  $e$  has a non-trivial segment on  $\mathcal{B}(R)$ . By way of contradiction, assume  $e$  intersects  $\mathcal{B}(R)$  at two separate connected components. It follows that there is some non-trivial segment  $e_1$  of  $e$  starting and ending at  $\mathcal{B}(R)$ , is contained in  $S$  the side of  $\mathcal{B}(R)$  that does not contain  $R$ , and crosses an edge on at least one end.

Let  $f$  be one of the edges on  $\mathcal{B}(R)$  that  $e_1$  crosses. Pick  $\mathcal{C}$  to be any closed curved on  $\mathcal{B}(R)$  and  $e_1$  that uses  $e_1$ . Since  $R$  is a face, it follows that the ends of  $f$  are on separate sides of  $\mathcal{C}$ .

Let  $u_1$  be the end of  $f$  that is on the opposite side of  $\mathcal{C}$  from  $u$ . In  $D$ ,  $(u, u_1)$  does not cross  $e_1$  and it does not cross  $R$ . It follows that  $(u, u_1)$  does not cross  $\mathcal{C}$ , a contradiction with  $u$  and  $u_1$  on opposite sides of  $\mathcal{C}$ .  $\square$

**Lemma 3.6.** *Let  $n \geq 4$  and  $D$  be a simple drawing of  $K_n$ . If  $c$  is an edge of  $K_n$ ,  $P$  is a point of  $D$  in some face and there is a 3-cycle having  $c$  and  $P$  on opposite sides, then there is some drawing  $\mathcal{D}$  in  $D$  on a  $K_4$  containing  $c$  such that no face of  $\mathcal{D}$  contains  $P$  and has a non-trivial segment of  $c$  on its boundary.*

*Proof.* By way of contradiction, assume every drawing in  $D$  on a  $K_4$  containing  $c$  has a face containing  $P$  and has  $c$  on the boundary. Let  $T = (q_1, q_2, q_3)$  be a 3-cycle in  $D$  having  $c$  and  $P$  on opposite sides. Let  $S_c$  be the side of  $T$  that contains  $c$  and  $S_P$  be the side of  $T$  that contains  $P$ .

If some 3-cycle involving  $u$  and two of  $q_1, q_2, q_3$  has  $c$  and  $P$  on opposite sides, then this 3-cycle along with  $v$  will induce a drawing of a  $K_4$  in  $D$  that has  $c$  not on the boundary of the face containing  $P$ , a contradiction.

If the edges of  $u$  to the vertices in  $T$  are drawn inside  $S_c$ , then such a 3-cycle exists. Therefore, one of the edges  $(u, q_i)$  must cross  $T$ , for  $i \in [3]$ . Without loss of generality, let  $(u, q_1)$  cross  $(q_2, q_3)$ .

Similarly,  $c$  and  $P$  are not on opposite sides of  $(u, q_2, q_3)$ . Therefore,  $c$  is in the face bounded by  $(u, q_1, q_2, q_3)$ . Note the face bounded by  $\gamma_{(q_1, q_3), (q_2, q_3), (u, q_1)}$  is symmetric to the face bounded by  $\gamma_{(q_1, q_2), (q_2, q_3), (u, q_1)}$  up to relabelling of  $q_2$  and  $q_3$ .

In particular, since  $P$  is in  $S_P$ , it follows that  $P$  is in one of these faces. Without loss of generality, assume  $P$  is in the face bounded by  $\gamma_{(q_1, q_3), (q_2, q_3), (u, q_1)}$  as in Figure 7. It follows that  $(u, q_1, q_3)$  separates  $P$  from  $c$ , and thus this 3-cycle along with  $v$  will induce a drawing of a  $K_4$  in  $D$  that has  $c$  not on the boundary of the face containing  $P$ , a contradiction.  $\square$

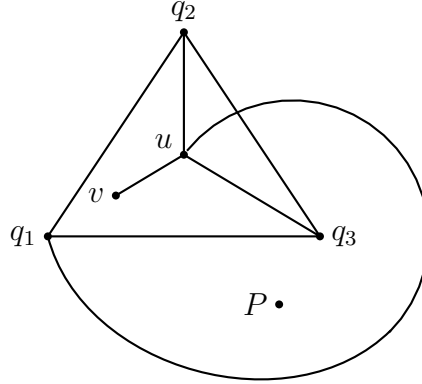


Figure 7: End of proof of Lemma 3.6.

Before stating and proving the main result of this section, we must state one last observation on simple drawings of  $K_4$  to simplify the arguments in the forthcoming proof.

**Observation 3.7.** *Let  $D$  be a simple drawing of a  $K_4$  and let  $u$  be a vertex of  $K_4$ , then there are three distinct faces in  $D$  that each have a distinct pair of edges on the boundary incident with  $u$ .*

The proof of Observation 3.7 follows by checking this fact on the two simple drawings  $K_4$  at any vertex.

**Theorem 3.8.** *Let  $n \geq 4$  and  $D_1$  be a simple drawing of  $K_n$ . If  $c$  is an edge of  $K_n$  and  $P$  is a point of  $D_1$  in some face, then either:*

1. *There is a sequence (possibly empty) of Reidemeister III moves  $\{\rho_{X_i}\}_{i=1}^k$  with sets of edges  $X_i$  such that  $D_{i+1} = \rho_{X_i}(D_i)$  with:*
  - i. *A non-trivial segment of  $c$  is on the boundary of the face of  $D_{k+1}$  containing  $P$ ;*
  - ii.  *$P \notin \Delta_{X_i}, \forall i \in [k]$ ; and*
  - iii. *For  $i \in [k]$ , if  $c \notin X_i$ , then there exists  $j > i$  in  $[k]$  such that  $c \in X_j$  and  $\Delta_{X_i} \subset \Delta_{X_j}$  in  $D_i$ ; or*
2. *There is some drawing  $\mathcal{D}$  in  $D_1$  on a  $K_4$  containing  $c$  such that no face of  $\mathcal{D}$  contains  $P$  and has a non-trivial segment of  $c$  on its boundary.*

For the proof of Theorem 1.1, we add an edge by successively drawing its segments across a face. To do so, we choose a special edge  $c$  that must be crossed in the current drawing on the boundary of some face  $F$ . However, it is not guaranteed that  $c$  is on the boundary of  $F$ . Therefore, Theorem 3.8 offers some structure on the relation of such an edge/face pair in a simple drawing of the complete graph. Note that Theorem 1.2 is a simplification of Theorem 3.8. We end this section with a proof of Theorem 3.8.

*Proof.* Let  $n \geq 4$ ,  $c = (u, v)$  be an edge of  $K_n$ ,  $D_1$  be a simple drawing of  $K_n$ ,  $D_j$  be an arbitrary simple drawing of  $K_n$  derived from applying a sequence of Reidemeister III moves to  $D_1$ , and  $P$  be a point in some face  $R_j$  in  $D_j$  such that  $c$  does not have a non-trivial segment on  $R_j$ .

Our goal is to show that if for all drawings  $\mathcal{D}$  in  $D_1$  on a  $K_4$  containing  $c$ , some face of  $\mathcal{D}$  contains  $R_1$  and has a non-trivial segment of  $c$  on its boundary, then there are sets of three edges  $X_i$  and a sequence of Reidemeister III moves  $\{\rho_{X_i}\}_{i=1}^k$  on  $D_1$  to a drawing  $D_{k+1}$  such that:

- i.  $D_{k+1}$  has a non-trivial segment of  $c$  on the face containing  $P$ ;
- ii. the intersection of each  $\Delta_{X_i}$  with  $P$  is empty; and
- iii. for each  $X_i$  not containing  $c$ , there exists a  $j > i$  such that  $c \in X_j$  and  $\Delta_{X_i}$  is contained in  $\Delta_{X_j}$  in  $D_i$ .

Note that any drawing of a  $K_4$  in  $D_j$  is topologically equivalent to the drawing of that  $K_4$  in  $D_1$  and  $P$  is on the same side of every cycle 3-cycle in  $D_1$  and  $D_j$  (as long as we choose  $S$  and  $\bar{S}$  from Definition 2.16 carefully as to not contain  $P$ ). These two facts combined imply that any  $K_4$  in  $D_j$  having  $P$  in a face and  $c$  not on the boundary of the face containing  $P$  also has this property in  $D_1$ . Without loss of generality, we can assume:

- (1)  $c \cap \mathcal{B}(R_j) \subseteq \{u, v\}$  for;
- (2) for all drawings  $\mathcal{D}$  in  $D_j$  on a  $K_4$  containing  $c$ , some face of  $\mathcal{D}$  contains  $R_j$  and has a non-trivial segment of  $c$  on its boundary.

**Claim 1.**  $c \cap \mathcal{B}(R_j) = \emptyset$ .

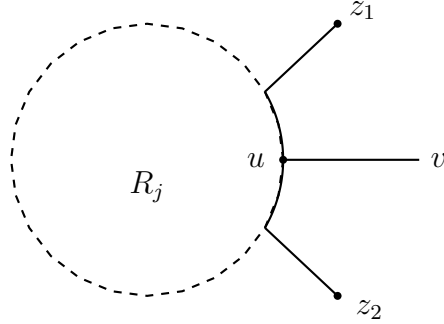


Figure 8: Set-up for Claim 1.

*Proof.* By way of contradiction, assume  $c \cap \mathcal{B}(R_j) \neq \emptyset$ . Up to relabelling and by (1), without loss of generality  $u \in \mathcal{B}(R_j)$ . There are two edges  $(z_1, u)$  and  $(z_2, u)$  that have non-trivial segments more than just their endpoints on the boundary of  $R_j$  as seen in Figure 8. In particular, Observation 3.7 implies that  $c$  and  $R_j$  are in separate regions in  $D_j[\{z_1, z_2, u, v\}]$ , a contradiction with (2). ■

Define  $V_{R_j}$  to be the vertex set of  $K_n$  that induces the face  $R_j$  (the set of vertices that are endpoints of edges that have non-trivial segments on  $R_j$ ).

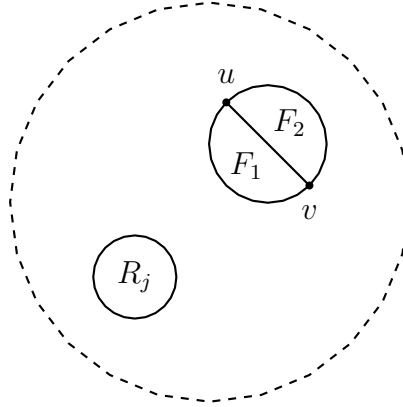


Figure 9: Set-up for Claim 2.

**Claim 2.**  $D_j[V_{R_j} \cup \{u, v\}]$  has an edge that crosses  $c$ .

*Proof.* By way of contradiction, assume  $D_j[V_{R_j} \cup \{u, v\}]$  has no edge that

crosses  $c$ . It follows that  $c$  bounds two faces  $F_1$  and  $F_2$  in  $D_j[V_{R_j} \cup \{u, v\}]$  as seen in Figure 9. Applying Corollary 3.3 to  $F_i$  and  $R_j$  implies that there is a 3-cycle  $T_i$  that separates  $F_i$  from  $R_j$ . If both  $T_1$  and  $T_2$  contain  $c$ , then the union of  $T_1$  and  $T_2$  covers  $c$  as  $F_1$  and  $F_2$  are bounded by opposite sides of  $c$ . Thus, the simple drawings on the  $K_4$  induced by  $T_1$  and  $T_2$  has  $R_j$  separated from  $c$ , a contradiction with (2).

Therefore, one of  $T_1$  or  $T_2$  does not contain  $c$ . Without loss of generality, assume  $T_1$  does not contain  $c$ . Since  $T_1$  does not contain  $c$ ,  $c$  does not cross  $T_1$ , and  $F_1$  and  $R_j$  are separated by  $T_1$ , it follows that  $c$  and  $R_j$  are separated by  $T_1$ , a contradiction with Lemma 3.6 ■

Define  $E(\mathcal{B}(R_j))$  to be the set of edges that have non-trivial segments on  $\mathcal{B}(R_j)$  and  $E(D_j[V_{R_j} \cup \{u, v\}])$  to be the set of edges in the drawing of  $D_j[V_{R_j} \cup \{u, v\}]$ . Notice that  $R_j$  is a face of  $D_j[V_{R_j}]$ .

**Claim 3.** *An edge in  $E(\mathcal{B}(R_j))$  crosses  $c$ .*

*Proof.* By way of contradiction, assume no edge in  $E(\mathcal{B}(R_j))$  crosses  $c$ . It follows by Claim 2 that some edge  $d$  in  $E(D_j[V_{R_j} \cup \{u, v\}]) \setminus E(\mathcal{B}(R_j))$  crosses  $c$ .

**Subclaim 3.1.**  *$d$  has an endpoint on  $\mathcal{B}(R_j)$ .*

*Proof.* By way of contradiction, assume  $d$  does not have an endpoint on  $\mathcal{B}(R_j)$ . Let  $d = (x_1, y_1)$ . Since  $d \in E(D_j[V_{R_j} \cup \{u, v\}])$ , it follows that an edge incident to  $x_1$  and an edge incident to  $y_1$  both have non-trivial segments on  $\mathcal{B}(R_j)$ . Without loss of generality, let  $e = (x_1, x_2)$  and  $f = (y_1, y_2)$  be such edges.

Define  $e_i$  to be the segment of  $e$  that starts at  $i$  and intersects  $\mathcal{B}(R_j)$  only at its end and let  $f_j$  be the segment  $f$  that starts at  $j$  and intersects  $\mathcal{B}(R_j)$  only at its end, for  $i \in \{x_1, x_2\}$  and  $j \in \{y_1, y_2\}$ . Note that Lemma 3.5 implies that every non-trivial segment of  $e$  or  $f$  that is not contained in  $\mathcal{B}(R_j)$  does not have both ends as crossing points. There are two cases to consider, whether  $e_{x_1}$  and  $f_{y_1}$  cross or not.

**Case 1.**  *$e_{x_1}$  and  $f_{y_1}$  do not cross.*

Note that Figure 10 outlines the edges that will be of importance in this case.

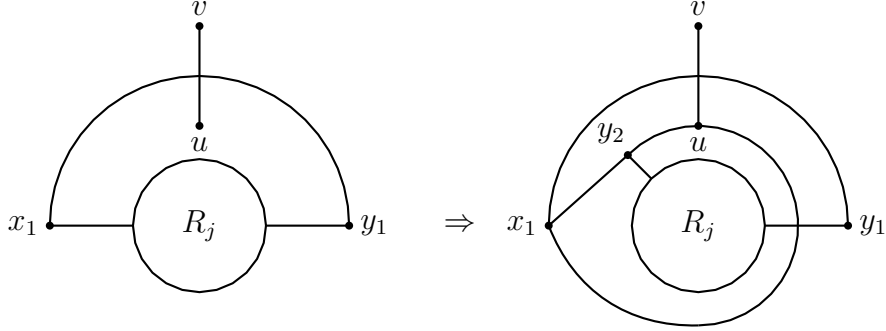


Figure 10: Case 1 of Subclaim 3.1

In  $D_j[\{x_1, y_1, u, v\}]$ ,  $d$  crosses  $c$  and  $R_j$  is in some face  $F$ . It follows that the only faces in  $D_j[\{x_1, y_1, u, v\}]$  that have a non-trivial segments of  $c$  on their boundaries are the faces that have the crossing of  $c$  and  $d$  on their boundary. By (2),  $F$  must be one of these faces.

For this to happen, one of  $(u, x_1), (u, y_1), (v, x_1), (v, y_1)$  must cross one of  $e_{x_1}$  or  $f_{y_1}$ . Since these potential crossings are the same up to relabelling of  $u, v, e_{x_1}$  and  $f_{y_1}$ , assume without loss of generality that  $(u, x_1)$  crosses  $f_{y_1}$  as in Figure 10.

Let  $S$  be the side of  $\gamma_{(u,x_1),(x_1,y_1),f_{y_1}}$  that contains  $R_j$ . Note by the drawing of the edge  $(u, x_1)$ ,  $u$  is in  $S$ . Since  $f_{y_2}$  can not cross  $S$  and starts on  $\mathcal{B}(R_j)$ ,  $f_{y_2}$  is also contained in  $S$ .

Since  $u$  and  $y_2$  are in  $S$  and  $(u, y_2)$  can cross  $\gamma_{(u,x_1),(x_1,y_1),f_{y_1}}$  at most once (at  $(x_1, y_1)$ ), it follows that  $(u, y_2)$  is drawn inside  $S$ . Since  $y_2$  is inside  $S$  and  $(x_1, y_2)$  can not cross  $S$ ,  $(x_1, y_2)$  is also contained in  $S$ . All of these edges have been outlined in Figure 10. The only simple drawing of these edges in  $D_j$  has the 3-cycle  $(u, x_1, y_2)$  separating  $c$  from  $R_j$ , a contradiction with (2).

**Case 2.**  $e_{x_1}$  and  $f_{y_1}$  cross.

Note that it does not matter which direction  $e_{x_1}$  and  $f_{y_1}$  cross as those cases are symmetric to each other in the sphere. Since  $c$  crosses  $(x_1, y_1)$  and not  $f_{y_1}$  or  $e_{x_1}$ , the ends of  $c$  are on opposite side of the simple closed curve  $\gamma_{f_{y_1},(x_1,y_1),e_{x_1}}$ .

Without loss of generality, assume  $u$  is on the side of  $\gamma_{f_{y_1},(x_1,y_1),e_{x_1}}$  that does not contain  $R_j$ . If all the edges in  $D_j[\{x_1, y_1, u, v\}]$  not in  $\gamma_{f_{y_1},(x_1,y_1),e_{x_1}}$ , do not cross  $\gamma_{f_{y_1},(x_1,y_1),e_{x_1}}$ , then  $D_j[\{x_1, y_1, u, v\}]$  has a  $K_4$  separating  $c$  from

$R$ , a contradiction with (2).

Therefore one of these edges crosses  $\gamma_{f_{y_1},(x_1,y_1),e_{x_1}}$ . Up to symmetry, the two cases are  $(u, y_1)$  crosses  $e_{x_1}$  or  $(v, y_1)$  crosses  $e_{x_1}$ .

**Case 2.1.**  $(u, y_1)$  crosses  $e_{x_1}$ .

Since  $f_{y_1}$  crosses  $e_{x_1}$  and  $(u, y_1)$  crosses  $e_{x_1}$ , the region containing  $f_{y_2}$  is determined. Furthermore,  $(u, y_2)$  must cross  $e_{x_1}$  and the 3-cycle  $(u, y_1, y_2)$  separates  $c$  from  $R_j$ , a contradiction with (2).

**Case 2.2.**  $(v, y_1)$  crosses  $e_{x_1}$ .

$e_{x_1}$  and  $f_{y_1}$  partition  $\mathcal{B}(R_j)$  as they each intersect  $\mathcal{B}(R_j)$  at one point. Let  $C_{R_j}$  be a simple closed curve that starts on the crossing of  $e_{x_1}$  and  $f_{y_1}$ , takes the edge segment  $f_{y_1}$  to  $\mathcal{B}(R_j)$ , walks along  $\mathcal{B}(R_j)$  to  $e_{x_1}$ , then takes the edge segment  $e_{x_1}$  back to the crossing of  $e_{x_1}$  and  $f_{y_1}$ .

Both  $e_{x_2}$  and  $f_{y_2}$  must be on the same side of this curve as they can not cross it. If  $f_{y_2}$  and  $(v, y_1)$  are on the opposite sides of  $C_{R_j}$ , then the 3-cycle  $(v, y_1, y_2)$  separates  $c$  from  $R_j$ , a contradiction with (2).

Therefore,  $f_{y_2}$  and  $(v, y_1)$  are on the same side of  $C_{R_j}$ . It follows that  $e_{x_2}$  and  $(v, y_1)$  are on the same side of  $C_{R_j}$ . The 3-cycle  $(v, y_1, x_2)$  separates  $c$  from  $R_j$ , a contradiction with (2).  $\blacksquare$

**Subclaim 3.2.**  $d$  has two endpoints on  $\mathcal{B}(R_j)$ .

*Proof.* By Subclaim 3.1,  $d$  has at least one endpoint on  $\mathcal{B}(R_j)$ . By way of contradiction, assume  $d$  has at exactly one endpoint on  $\mathcal{B}(R_j)$ . Let  $d = (x_1, x_2)$  such that  $x_1$  is on  $\mathcal{B}(R_j)$ . Since  $d \in E(D_j[V_{R_j} \cup \{u, v\}])$ , it follows that an edge incident to  $x_2$  has a non-trivial segment on  $\mathcal{B}(R_j)$ . Without loss of generality, let  $e = (x_2, x_3)$  be such an edge.

Let  $\gamma_{d,e,R_j}$  be one of two unique simple closed curves on edges  $d, e$  and the simple closed curve  $\mathcal{B}(R_j)$ . Since no edge in  $E(\mathcal{B}(R_j))$  crosses  $c$ ,  $e$  does not cross  $c$  and  $c$  does not cross  $\mathcal{B}(R_j)$ . It follows that  $\gamma_{d,e,R_j}$  has  $u$  and  $v$  on opposite sides. Without loss of generality, let  $u$  be on the opposite side of  $x_3$ . Observe that the edge  $(x_2, u)$  is uniquely determined relative to  $\mathcal{B}(R_j), d, e, c$ .

Let  $d_{x_i}$  be the segment of  $d$  from  $x_i$  to the crossing with  $c$ . If  $(x_3, u)$  crosses  $d_{x_1}$ , then the 3-cycle  $(x_2, x_3, u)$  separates  $c$  from  $R_j$ , a contradiction with (2).

Therefore,  $(x_3, u)$  is drawn crossing  $d_{x_2}$ . Since  $(x_1, x_3)$  can cross  $\gamma_{d,e,(e,x_3)}$  or  $R_j$ , it follows that  $(x_1, x_3)$  is uniquely determined in the drawing involving  $d, e, c, (x_1, u), (x_2, u)$  and  $(x_3, u)$ . In particular, the 3-cycle  $(x_1, x_3, u)$  separates  $c$  from  $R_j$ , a contradiction with (2). ■

By Subclaim 3.2,  $w$  has two endpoints on  $\mathcal{B}(R_j)$ . Let  $w = (x_1, x_2)$ . Noting that the edges in the  $K_4$  involving  $x_1, x_2, u, v$  are determined relative to  $R_j$ , it follows that  $D_j[x_1, x_2, u, v]$  has  $c$  separated from  $R_j$ , a contradiction with (2). ■

By Claim 3, there exists an edge  $e^j$  that crosses  $c$  in  $E(\mathcal{B}(R_j))$ . Without loss of generality, let  $e^j = (x_1, x_2)$ .

Since  $e^j$  has a non-trivial segment on the boundary of  $R_j$ , it follows that some segment of  $e^j$ , up to relabelling, starts at  $x_2$  crosses  $c$  and intersects  $\mathcal{B}(R_j)$  only at its end, call this segment  $e_{x_2}^j$ . Define  $e_{x_1}^j$  to be the segment of  $e^j$  that starts at  $x_1$  and intersects  $\mathcal{B}(R_j)$  only at its end. Note, it is possible that  $e_{x_1}^j$  is only the vertex  $x_1$ . Note that by Lemma 3.5, every non-trivial segment of  $e^j$  that is not contained in  $\mathcal{B}(R_j)$  does not have both ends as crossing points on  $\mathcal{B}(R_j)$ .

Without loss of generality, starting from  $x_2$ ,  $e_{x_2}^j$  crosses  $\overrightarrow{(u, v)}$  from right to left as the opposing case is completely analogous. The end of  $e_{x_2}^j$  that is not  $x_2$  is a crossing with an edge  $f^j = (y_1, y_2)$  at the intersection of  $e^j$  with  $\mathcal{B}(R_j)$ . Since  $f^j$  has a crossing on  $\mathcal{B}(R_j)$ , it follows by Lemma 3.5 that  $f^j$  has segments  $f_i^j$  for  $i \in \{y_1, y_2\}$  that starts at  $i$  and only intersect  $\mathcal{B}(R_j)$  at its end.

Without loss of generality, we can assume walking clockwise around  $\mathcal{B}(R_j)$ , we cross  $e_{x_2}^j, f_{y_2}^j, e_{x_1}^j, f_{y_1}^j$ , if not we can relabel the ends of  $f^j$ .

**Claim 4.**  $\Delta_{c,e^j,f^j}$  exists.

*Proof.* By way of contradiction, assume  $f_{y_2}^j$  does not cross  $c$  between  $u$  and the crossing of  $c$  with  $e^j$  and  $f_{y_1}^j$  does not cross  $c$  between  $v$  and the crossing of  $c$  with  $e^j$ .

**Subclaim 4.1.**  $f^j$  crosses  $c$ .

*Proof.* By way of contradiction, suppose  $f^j$  does not cross  $c$ . For the readers convenience, we offer a diagram depicting this case in Figure 11.

It is either  $(x_1, u)$  crosses  $f_{y_1}^j, f_{y_2}^j$ , or it does not cross  $f^j$ . All cases follow a similar argument with the goal to show that one of the 3-cycles  $(y_1, y_2, u)$ ,



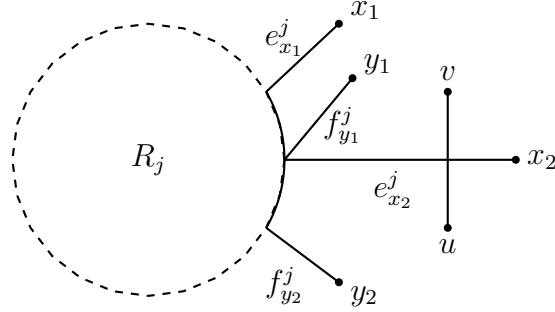


Figure 11: Claim 4 Case 1.

$(x_1, y_1, u)$ , or  $(x_1, y_2, u)$  contradict (2).

**Case 1.**  $(x_1, u)$  does not cross  $f^j$ .

In this case, if the edge  $(y_2, u)$  crosses  $e^j$ , then  $y_2$  and  $R_j$  are contained on the same side of  $\gamma_{(x_1, u), (u, y_2), e^j}$ . In particular, the edge  $(x_1, y_2)$  is drawn on the same side of  $\gamma_{(x_1, u), (u, y_2), e^j}$  as  $R_j$  and the 3-cycle  $(x_1, y_2, u)$  separates  $c$  from  $R_j$ , a contradiction with (2).

It follows that the edge  $(y_2, u)$  does not cross  $e^j$ . If  $\pi_{y_1, y_2, v}(u) = [v, y_1, y_2]$ , then the 3-cycle  $(y_1, y_2, u)$  separates  $c$  from  $R_j$  as  $c$  does not cross  $f$ , a contradiction with (2).

Since  $\pi_{\{y_1, y_2, v\}}(u) = [v, y_2, y_1]$ , the edge  $(y_1, u)$  crosses  $e^j$ . If  $(x_1, y_1)$  does not cross  $c$ , then the 3-cycle  $(x_1, y_1, u)$  separates  $c$  from  $R_j$ , a contradiction with (2).

Therefore,  $(x_1, y_1)$  crosses  $c$ . In particular,  $(y_1, v)$  does not cross  $(x_1, u)$  and  $(x_1, v)$  does not cross  $(y_1, u)$ . It follows that the 4-cycle  $(y_1, v, x_1, u)$  separates  $c$  from  $R_j$ , a contradiction with (2).

**Case 2.**  $(x_1, u)$  crosses  $f_{y_2}^j$ .

This case is the exact same as the previous, except both  $y_2$  and  $(x_1, y_2)$  will be on the opposite side of  $\gamma_{(x_1, u), (u, y_2), e^j}$  when compared to  $R_j$ , and  $c$  and  $R_j$  switch sides in the cycle  $(x_1, y_2, u)$ .

**Case 3.**  $(x_1, u)$  crosses  $f_{y_1}^j$ .

$x_1$  is contained on the left side of the closed curve defined by starting at  $u$ , taking the edge  $c$  to the crossing of  $c$  and  $e^j$ , then taking the edge  $e^j$  to the crossing of  $e^j$  with  $f^j$ , then taking the edge  $f^j$  to the crossing of  $f^j$  with  $(x_1, u)$ , then taking the edge  $(x_1, u)$  to  $u$ . In particular, the edge  $(x_1, y_2)$  is contained on the left side of this curve.

If  $(y_2, u)$  crosses  $e^j$ , then the 3-cycle  $(x_1, y_2, u)$  separates  $c$  from  $R_j$ , a contradiction with (2). Therefore,  $(y_2, u)$  does not cross  $e^j$  and the 3-cycle  $(y_1, y_2, u)$  separates  $c$  from  $R_j$ , a contradiction with (2). ■

By Subclaim 4.1,  $f^j$  crosses  $c$ . If  $e^j$  and  $f^j$  cross  $c$  in opposite directions on segments  $e_i^j$  and  $f_\ell^j$  for  $i \in \{x_1, x_2\}$  and  $\ell \in \{y_1, y_2\}$ , then  $D_j[\{i, \ell, u, v\}]$  has the 4-cycle  $(i, u, \ell, v)$  separating  $c$  from  $R_j$ , a contradiction with (2).

Therefore,  $e^j$  and  $f^j$  cross  $c$  in the same direction, since  $\overrightarrow{(x_1, x_2)}$  crosses  $\overrightarrow{(u, v)}$  from left to right, so does  $\overrightarrow{(y_1, y_2)}$ . Without loss of generality, assume  $f_{y_2}^j$  crosses  $c$  as  $f_{y_1}^j$  crossing  $c$  is symmetric.

Since  $f_{y_2}^j$  does not cross  $c$  between  $u$  and the crossing of  $c$  with  $e^j$ , it follows that  $f_{y_2}^j$  crosses  $c$  between  $v$  and the crossing of  $c$  with  $e^j$ .

Consider the simple drawing  $D_j[E(\mathcal{B}(R_j)) \cup \{e^j, f^j, c\}]$ . By our choice of oriented crossing on  $c$ ,  $(x_1, y_1)$ ,  $(x_1, v)$  and  $(y_1, v)$  are uniquely drawn into  $D_j[E(\mathcal{B}(R_j)) \cup \{e^j, f^j, c\}]$  to keep the drawing  $D_j[E(\mathcal{B}(R_j)) \cup \{e^j, f^j, c, (x_1, y_1), (x_1, v), (y_1, v)\}]$  simple. In particular, the 3-cycle  $(x_1, y_1, v)$  will separate  $c$  from  $R_j$ , a contradiction with (2).

Therefore,  $f_{y_2}^j$  crosses  $c$  between  $u$  and the crossing of  $c$  with  $e^j$ . Note by our choice of oriented crossings, that the ends of  $e^j$ ,  $f^j$  and  $c$  are on the same side of the simple closed curve  $\gamma_{e^j, f^j, c}$ . Therefore,  $\Delta_{e^j, f^j, c}$  exists. ■

Let  $T_j = \Delta_{e^j, f^j, c}$  in  $D_j$  in Claim 4. Note that  $T_j$  and  $R_j$  have empty intersection. It follows that  $T_j$  does not contain  $P$ .

**Claim 5.**  $T_j$  does not contain any vertices.

*Proof.* By way of contradiction, assume there is some vertex  $z$  in  $T_j$  as depicted in Figure 12. If  $(z, v)$  crosses  $e^j$  and  $(z, u)$  crosses  $f^j$ , then one of the drawings  $D_j[\{z, u, v, x_2\}]$  or  $D_j[\{z, u, v, y_2\}]$  has  $c$  separated from  $R_j$ . Note that no simple drawing has  $(z, v)$  crossing  $f^j$  and  $(z, u)$  crossing  $e^j$ . Since  $\mathcal{B}(T_j)$  separates  $z$  from  $u$  and  $v$ , it follows that  $(z, u)$  and  $(z, v)$  cross  $\mathcal{B}(T_j)$  both on  $e^j$  or both on  $f^j$ . By symmetry, without loss of generality assumed  $(z, u)$  and  $(z, v)$  both cross  $e^j$ .

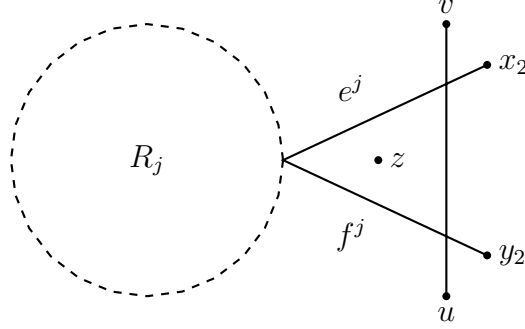


Figure 12: Claim 5.

Since  $(z, u)$  and  $(z, v)$  both cross  $e^j$ , the edges  $(z, x_1)$  and  $(x_1, u)$  are uniquely drawn in  $D_j[E(\mathcal{B}(R_j)) \cup \{e^j, c, (z, u), (z, v)\}]$  to produce a simple drawing  $D_j[E(\mathcal{B}(R_j)) \cup \{e^j, c, (z, u), (z, v), (z, x_1), (x_1, u)\}]$ . In particular, the 3-cycle  $(z, x_1, u)$  separates  $c$  from  $R_j$ , a contradiction with (2).  $\blacksquare$

By Claim 5 there are no vertices in  $T_j$ . Since  $T_j$  contains no vertices, if an edge in  $D_j$  crosses  $\mathcal{B}(T_j)$ , it does so exactly twice.

Define  $N_{D_j}(T_j)$  to be the number of edges that cross  $\mathcal{B}(T_j)$  not at  $c$  in  $D_j$ . Inductively, we will show the following claim.

**Claim 6.** There is a sequence of simple drawings  $D_1, \dots, D_k$  such that:

- 1) For  $i \in [k]$ , each  $D_i$  has an  $e^i, f^i$  and  $T_i$ ;
- 2) For  $i \in [k - 1]$ ,  $N_{D_{i+1}}(T_{i+1}) < N_{D_i}(T_i)$ ;
- 3)  $N_{D_k}(T_k) = 0$ ;
- 4) For  $i \in [k - 1]$ , there are sets of three edges  $X_\ell^i$  and a sequence of Reidemeister III moves  $\{\rho_{X_\ell^i}\}_{\ell=1}^{j_i}$  such that  $D_{i+1}$  is  $\{\rho_{X_\ell^i}\}_{\ell=1}^{n_i}$  on  $D_i$ ;
- 5) For  $i \in [k - 1]$  and  $\ell \in [n_i - 1]$ ,  $\Delta_{X_\ell^i} \subset T_i$  in  $D_i$ ; and
- 6) For  $i \in [k - 1]$ ,  $P \notin T_i$  in  $D_i$ .

*Proof.* If  $N_{D_1}(T_1) = 0$ , then we are done as  $P$  is in  $R_1$  and not  $T_1$ .

Therefore,  $N_{D_1}(T_1) > 0$ . For our induction step, we can assume  $N_{D_i}(T_i) > 0$  for some  $i \geq 1$ . It follows that there exists an edge  $g^i$  such that any edge that crosses  $\Delta_{e^i, f^i, g^i}$  crosses  $g^i$  and  $\Delta_{e^i, f^i, g^i} \subset T_i$ .

Applying Corollary 2.18 implies that there exists a sequence of Reidemeister III moves  $\{\rho_{X_\ell^i}\}_{\ell=1}^{n_i}$  (None of which cross  $P$  by careful selection) from  $D_i$  to a simple drawing  $D_{i+1}$  for some integer  $n_i$  such that:

- for  $\ell \in [n_i]$ ,  $\Delta_{X_\ell^i} \subset T_i$  in  $D_i$ ;
- $X_{n_i}^i = \{e^i, f^i, g^i\}$ ; and
- $g^i \in X_\ell^i$  for all  $\ell \in [n_i]$ .

In  $D_{i+1}$ , a non-trivial segment of  $g^i$  is on the boundary of the face containing  $P$ , also known as  $R_i$ . The result is that  $e^i, g^i, f^i$  have consecutive non-trivial segments on  $R_{i+1}$ . If  $g^i$  does not cross  $c$ , then setting  $g^i = e^{i+1}$  and  $f^i = f^{i+1}$  gives a contradiction to Claim 4.

Therefore,  $g^i$  does cross  $c$ , and does so outside of  $T_i$  by definition of  $g^i$ . By setting  $e^i, f^i$  and  $g^i$  to be the appropriate variables ( $e^{i+1}$  or  $f^{i+1}$ ), it follows by Claims 4 and 5 that  $\Delta_{e^i, g^i, c}^{D_i}$  and  $\Delta_{f^i, g^i, c}^{D_i}$  exist and are empty of vertices. Since they are both empty of vertices, one of them must contain  $T_i$  in  $D_{i+1}$ . Without loss of generality, let  $T_i \subset \Delta_{e^i, g^i, c}^{D_{i+1}}$  in  $D_{i+1}$ .

Set  $e^{i+1} = e^i, f^{i+1} = g^i$  and  $T_{i+1} = \Delta_{e^{i+1}, f^{i+1}, c}^{D_{i+1}}$ . Setting  $j = i + 1$ , we see that  $e^{i+1}, f^{i+1}$  and  $T_{i+1}$  satisfy Claims 1 - 5.

Note that, since  $g^i \in X_\ell^i$  for all  $\ell \in [n_i]$ , we can choose our Reidemeister III moves so that only the edge  $g^i$  is changing. It follows that an edge not  $g^i$  crosses  $\mathcal{B}(T_i)$  not at  $c$  in  $D_i$  if and only if it crosses  $\mathcal{B}(T_i)$  not at  $c$  in  $D_{i+1}$ . Since  $\rho_{X_{n_i}^i}$  was the last Reidemeister III move, every edge that crosses  $\mathcal{B}(T_{i+1})$  not at  $c$  in  $D_{i+1}$  also crosses  $\mathcal{B}(T_i)$  not at  $c$  in  $D_{i+1}$ . Since  $g^i$  does not cross  $\mathcal{B}(T_{i+1})$  in  $D_{i+1}$ , it follows that  $N_{D_{i+1}}(T_{i+1}) < N_{D_i}(T_i)$ .

By applying induction, it is clear that 1) - 3) are satisfied. By our use of Claim 4, 4) and 5) are satisfied. Since  $g^i$  is the only edge moving by the Reidemeister moves  $\{\rho_{X_\ell^i}\}_{\ell=1}^{n_i}$ , we choose the Reidemeister III move carefully so that  $P$  is not in the disc that contains the Reidemeister III move. By this choice,  $P \notin T_{i+1}$  in  $D_{i+1}$ , satisfying 6). ■

To complete our proof, we apply Claim 6 to find a sequence of simple drawing  $D_1, \dots, D_k$ , a sequence of Reidemeister III moves  $\{\rho_{X_\ell^i}\}_{\ell=1}^{n_i}$ , and  $T_i$  satisfying 1) through 6). Finally we apply Corollary 2.18 to find a sequence of Reidemeister III moves  $\{\rho_{X_\ell^k}\}_{\ell=1}^{n_k}$  from  $D_k$  to  $D_{k+1}$  such that  $c \in X_\ell^k$  for all  $\ell \in [n_k]$  and  $\Delta_{X_\ell^k} \subset \Delta_{X_{n_k}^k}$  in  $D_k$ . Since  $T_k = \Delta_{X_{n_k}^k}$  in  $D_k$ , every Reidemeister III

move occurs in some  $T_i$ . Since every  $T_i$  in  $D_i$  does not contain  $P$ , it follows no Reidemeister III move contains  $P$ , satisfying ii.

Again, since every Reidemeister III move between  $D_i$  and  $D_{i+1}$  occurs in  $T_i$ , all we need to show to satisfy iii. is that a Reidemeister move over  $\Delta_{e^i, f^i, c}$  occurs after  $T_i$  is emptied.

In  $D_{k+1}$ ,  $c$  is on the boundary of the face containing  $P$ , and so is also on the boundary of the face containing  $P$  in the simple subdrawing of the  $K_6$  induced by  $\Delta_{e^i, f^i, c}$  in  $D_{k+1}$ .

However,  $c$  is not on the boundary of the face containing  $P$  in the simple subdrawing of the  $K_6$  induced by  $\Delta_{e^i, f^i, c}$  in  $\rho_{X_{n_i}^i}(D_{i+1})$ . This implies a Reidemeister move over  $\Delta_{e^i, f^i, c}$  occurs after  $T_i$  is emptied, as desired.  $\square$

Obviously no simple drawing of  $K_3$  containing  $c$  can separate a point  $P$  from  $c$ , therefore the 4 in Theorem 3.8 is least possible. Moreover, Lemma 3.6 shows that no 3-cycle separates  $c$  from  $P$ , else there is a  $K_4$  in  $D$  containing  $c$  that separates  $c$  from  $P$ .

## 4 (6, 5)-Rotation Systems

The goal of this section is to prove the case  $n = 6$  of Theorem 1.1. Such a case is interesting as it can be combined with results in [1] to prove Theorem 1.1. Although this section contains some flavor of the arguments required to prove Theorem 1.1, it also contains some of the most technical arguments found in this writing. We introduce the notion that orderings of edge crossings on a fixed edge under certain realizability constraints.

**Notation 4.1.** Let  $n \geq 5$ ,  $H_n$  be an  $n$ -vertex rotation system,  $e$  be a directed edge of  $H_n$ , and  $f$  and  $g$  edges of  $H_n$  such that  $e$  crosses  $f$  and  $g$ . Define:

- $f <_{\wedge}^e g$ , if  $H_n$  is an  $(n, 5)$ -rotation system,  $f$  and  $g$  share an endpoint, and  $e$  crosses  $f$ , then  $g$  in the drawing of  $K_5$  induced by  $e, f, g$ ;
- $f <_{\parallel}^e g$ , if  $H_n$  is an  $(n, 6)$ -rotation system,  $f$  and  $g$  do not cross in  $H_n$ , there is no ordering of  $f$  and  $g$  of  $<_{\wedge}^e$  relations in the rotation system induced by  $e, f, g$  from  $H_n$ , and  $e$  crosses  $f$ , then  $g$  in the drawing of the  $K_6$  induced by  $e, f, g$ ; and
- $f <_{\Delta}^e g$ , if  $H_n$  is an  $(n, 7)$ -rotation system,  $\Delta_{\{e, f, g\}}$  exists containing a vertex  $v$ , and  $e$  crosses  $f$ , then  $g$  in the drawings of the  $K_7$  induced by  $e, f, g, v$ .

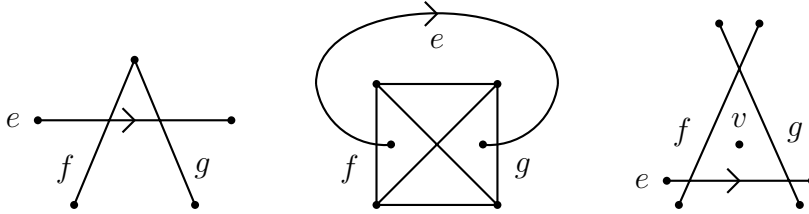


Figure 13: The three relations  $f <_{\wedge}^e g$ ,  $f <_{\parallel}^e g$  and  $f <_{\Delta}^e g$ .

The three relations are outline in Figure 13. In a simple drawing of a  $K_5$  containing a directed edge  $e$  and two edge  $f$  and  $g$ ,  $f <_{\wedge}^e g$  is well defined in that drawing, and by Lemma 2.8, is determined by  $H$ . Looking at  $f <_{\parallel}^e g$  in an associated simple drawing of a  $K_6$  shows that the segments of  $e$  from the crossing with  $f$  to the crossing with  $g$  is uniquely determined by the oriented crossings of  $e$  with  $f$  and  $e$  with  $g$ . By Observation 2.7, this ordering is

uniquely determined by  $H$ . Infrequently, we use  $f <_{K_6}^e g$  to say that two edges  $f$  and  $g$  are ordered on  $e$  with either a  $<_{\parallel}^e$  relation or a chain of  $<_{\wedge}^e$  relations.

Finally, if  $f <_{\Delta}^e g$  occurs in some simple drawing of an associated  $K_7$ , then we will see in Lemma 5.7 that  $e$  must cross  $f$  then  $g$  in every drawing realizing that associated rotation system by some  $<_{\wedge}^e$  and  $<_{\parallel}^e$  relations. In particular,  $f <_{\Delta}^e g$  is in reality a chain of  $<_{\wedge}^e$  and  $<_{\parallel}^e$  relations each determined by  $H$ .

Our first goal is for a fixed edge  $e$ , find a total ordering on the edges it crosses associated with the rotation system we are given. To that end, we want to make sure the relation  $<_{\wedge}^e$  on the edges that  $e$  crosses induces an acyclic directed graph (As described in Lemma 4.6). If such a graph contained a directed cycle, then there exists (up to relabelling) a directed cycle described in Lemmas 4.4 and 4.5. We first make an observation on rotation system information that is provided from the  $<_{\wedge}^e$  relation.

**Definition 4.2.** Let  $n \geq 5$ ,  $D$  be a simple drawing of  $K_n$ , and  $f$  and  $g$  be two edges both incident to the same vertex that both cross a third edge  $e$ . Define  $f$  and  $g$  to *agree* if from their common endpoint, they cross  $e$  in the same direction, otherwise the two edges *disagree*.

**Observation 4.3.** Let  $D$  be a simple drawing of  $K_5$ , and  $f = (v, f_1)$  and  $g = (v, g_1)$  be two edges incident to vertex  $v$  that both cross a third edge  $e$ . Suppose  $f <_{\wedge}^e g$ . If  $f$  and  $g$  agree, then the rotation at  $v$  in  $D$  is determined by the oriented crossings of  $e$  with  $f$  or  $e$  with  $g$ . If  $f$  and  $g$  disagree, then  $D$  is Harborth and the rotation at every vertex is determined by the oriented crossings of  $e$  with  $f$  or  $e$  with  $g$ .

Suppose  $f$  and  $g$  agree and you are given the oriented crossing of  $e$  with  $f$ . Since  $f$  and  $g$  agree, we also have the oriented crossing of  $e$  with  $g$ . Since  $f <_{\wedge}^e g$ , there is a unique way to draw the star at  $v$  with the edge  $e$  in the simple drawing  $D$  and this drawing produces the desired rotation at  $v$ . In particular, if  $e = \overrightarrow{(u, v)}$  is a directed edge that crosses  $f = \overrightarrow{(v, f_1)}$ , then crosses  $g = \overrightarrow{(v, g_1)}$ , both from right to left, then  $\pi_{u, v, f_1, g_1}(v) = [v, g_1, f_1, u]$ .

Similarly, if  $f$  and  $g$  disagree, then by Observation 2.10,  $D$  is Harborth and the drawing is determined by the drawing of  $e, f$  and  $g$ . It is clear that the rotations at each vertex is determined by the oriented crossings of  $e$  with  $f$  or  $e$  with  $g$  given  $f <_{\wedge}^e g$ .

**Lemma 4.4.** *Let  $H$  be a  $(6, 5)$ -rotation system and  $e$  be a directed edge of  $H$ . If  $e_1, e_2, e_3$  are three edges sharing an endpoint and all cross  $e$ , then  $e_1 <_{\wedge}^e e_2 <_{\wedge}^e e_3 <_{\wedge}^e e_1$  does not occur.*

*Proof.* Let  $H$  be a  $(6, 5)$ -rotation system and  $e = \overrightarrow{(u, v)}$  be a directed edge of  $H$ . Define the simple drawings  $D_i$  to be a realization of  $H - \{i\}$  for  $i \in V(H)$ , and  $D_e$  to be a realization of  $H - \{u, v\}$ . Suppose  $e_1, e_2, e_3$  are three edges sharing an endpoint and all cross  $e$ .

By way of contradiction, assume such a cycle  $e_1 <_{\wedge}^e e_2 <_{\wedge}^e e_3 <_{\wedge}^e e_1$  exists. Without loss of generality this cycle is  $(1, 2) <_{\wedge}^e (1, 3) <_{\wedge}^e (1, 4) <_{\wedge}^e (1, 2)$ . Two of three edges must cross  $e$  in the same direction when starting at the vertex 1. Without loss of generality, these two edges are  $(1, 2)$  and  $(1, 3)$ , and up to relabelling of the ends of  $e$ , they cross  $e$  from left to right starting at the vertex 1. Since  $(1, 2) <_{\wedge}^e (1, 3)$ , it follows that in  $D_4$ ,  $\pi_{u,v,2,3}(1) = [v, 3, 2, u]$ .

Assume by way of contradiction that  $(1, 4)$  crosses  $e$  in the same direction as  $(1, 2)$  and  $(1, 3)$  starting at 1. Then from a similar argument  $\pi_{u,v,3,4}(1) = [v, 4, 3, u]$  and  $\pi_{u,v,2,4}(1) = [v, 2, 4, u]$ . Therefore,  $\pi(1)$  contains the three incompatible cyclic subrotations  $[v, 3, 2]$ ,  $[v, 4, 3]$  and  $[v, 2, 4]$ , a contradiction.

Therefore,  $(1, 4)$  crosses  $e$  from the opposite side as  $(1, 2)$  and  $(1, 3)$  starting at 1. By Observation 4.3, both  $D_3$  and  $D_2$  are Harborth drawings of  $K_5$  with  $e$  being crossed three times in each of the drawings. In particular, by Observation 4.3, we learn:

- $\pi_{2,4,v}(1) = [2, 4, v]$  in  $D_3$ ;
- $(1, 4)$  and  $(2, v)$  do not cross in  $D_3$ ;
- $\pi_{v,2,1}(4) = [v, 2, 1]$  in  $D_3$ ;
- $(2, 4)$  does not cross  $(1, v)$  in  $D_3$ ; and
- $\overrightarrow{(1, 4)}$  crosses  $\overrightarrow{(3, v)}$  from right to left in  $D_2$ .

We break this into two cases depending on whether  $(2, v)$  crosses  $(1, 3)$  or not.

**Case 1.**  $(2, v)$  crosses  $(1, 3)$ .

In  $D_4$ , there is a unique direction  $(2, v)$  can cross  $(1, 3)$ , in particular,  $(2, v)$  crosses  $(1, 3)$  from right to left. In  $D_u$ , this determines the drawing of the  $K_4$  on  $1, 2, 3$  and  $v$ . Let  $S_1$  be the side of  $\gamma_{(1,3),(2,v),(3,v)}$  into which  $\overrightarrow{(1, 4)}$  crosses



into as it crosses  $\overrightarrow{(3, v)}$  from right to left. Since  $(1, 4)$  does not cross  $(2, v)$  or  $(1, 3)$ , the vertex 4 is contained in  $S_1$ .

It follows that there is a unique way to draw  $(4, v)$  and it determines  $\pi_{2,3,4}(v)$  to be  $\pi_{2,3,4}(v) = [3, 4, 2]$ . Since  $\pi_{1,2,v}(4) = [v, 2, 1]$ , the start of the edge  $(2, 4)$  at 4 and the vertex 2 are on opposite sides of the 3-cycle  $(1, 4, v)$ . This is not possible as  $(2, 4)$  does not cross  $(1, v)$ .

**Case 2.**  $(2, v)$  does not cross  $(1, 3)$ .

By the symmetry of  $(2, v)$  and  $(3, u)$ ,  $(3, u)$  does not cross  $(1, 2)$ . From  $D_2$  and  $D_3$ , respectively,  $\pi_{1,3,v}(u) = [1, v, 3]$  and  $\pi_{1,2,u}(v) = [2, u, 1]$ . Since  $(2, v)$  does not cross  $(1, 3)$  and  $(3, u)$  does not cross  $(1, 2)$ , it follows that in  $D_4$ ,  $\overrightarrow{(2, v)}$  crosses  $\overrightarrow{(3, u)}$  from right to left.

From these crossings, the edges  $(2, u)$  and  $(3, v)$  are uniquely determined in  $D_4$  to keep the drawing simple. It follows that the 3-cycles  $\overrightarrow{(1, 2, u)}$  and  $\overrightarrow{(1, 3, v)}$  are uniquely determined in the drawing. In particular,  $\overrightarrow{(1, v, 3)}_R \subset \overrightarrow{(1, v, 2)}_R$ .

In  $D_2$ , 4 is in  $\overrightarrow{(1, v, 3)}_R$ . Deleting  $u$  from  $D_4$  and adding 4, we find that 4 is in  $\overrightarrow{(1, v, 3)}_R \subset \overrightarrow{(1, v, 2)}_R$ . In  $D_3$ , 4 is in  $\overrightarrow{(1, v, 2)}_L$ , a contradiction with 4 in  $\overrightarrow{(1, v, 2)}_R$ .  $\square$

**Lemma 4.5.** *If  $H$  is a  $(6, 5)$ -rotation system and  $e = \overrightarrow{(u, v)}$  is a directed edge of  $H$ , then the cycle  $(1, 2) <_{\wedge}^e (2, 3) <_{\wedge}^e (3, 4) <_{\wedge}^e (1, 4) <_{\wedge}^e (1, 2)$  does not occur.*

*Proof.* Define the simple drawings  $D_i$  to be a realization of  $H - \{i\}$  for  $i \in V(H)$ , and  $D_e$  to be a realization of  $H - \{u, v\}$ .

By way of contradiction, assume the cycle  $\mathcal{C}$  of relations defined by  $(1, 2) <_{\wedge}^e (2, 3) <_{\wedge}^e (3, 4) <_{\wedge}^e (1, 4) <_{\wedge}^e (1, 2)$  does occur. Observation 4.3 tell us that two edges agreeing or disagreeing provides rotation system information given we know an oriented crossing with  $e$ .

Note that the number of relations in  $\mathcal{C}$  that agree must be even, else the direction edges cross  $e$  is not well defined. It follows that the number of relations in  $\mathcal{C}$  that disagree is also even since there is an even number of relations in  $\mathcal{C}$ .

We will break this into three cases. Either two consecutive relations disagree, two non-consecutive relations disagree, or all relations agree.

**Case 1.** *Two consecutive relations in  $\mathcal{C}$  disagree.*

Assume without loss of generality that  $(1, 2) <_{\wedge}^e (2, 3)$  and  $(2, 3) <_{\wedge}^e (3, 4)$  disagree, and  $\overrightarrow{(1, 2)}$  crosses  $\overrightarrow{(u, v)}$  from left to right. Since  $(1, 2) <_{\wedge}^e (2, 3)$  and the relation disagrees, by Observation 4.3,  $D_4$  is a unique labelled Harborth drawing. Similarly,  $D_1$  is a unique labelled Harborth drawing. In particular, the drawings have:

1.  $\pi_{3,4,u,v}(2) = [v, 4, 3, u]$  in  $D_1$ ;
2.  $\pi_{2,4,u,v}(3) = [v, 4, u, 2]$  in  $D_1$ ;
3.  $\pi_{2,3,4,v}(u) = [v, 4, 3, 2]$  in  $D_1$ ;
4.  $\pi_{2,3,u,v}(4) = [u, 2, 3, v]$  in  $D_1$ ;
5.  $\pi_{1,3,u,v}(2) = [u, 1, v, 3]$  in  $D_4$ ;
6.  $\pi_{1,2,u,v}(3) = [v, u, 1, 2]$  in  $D_4$ ; and
7.  $\pi_{1,2,3,v}(u) = [1, v, 3, 2]$  in  $D_4$ .

Combining the rotations at 2, 3 and  $u$  we get  $\pi_{1,3,4,u,v}(2) = [u, 1, v, 4, 3]$ ,  $\pi_{1,2,4,u,v}(3) = [v, 4, u, 1, 2]$ , and  $\pi_{1,2,3,4,v}(u) = [v, 4, 3, 2, 1]$ . Observe that in  $D_4$ ,  $\overrightarrow{(3, u)}$  crosses  $\overrightarrow{(1, 2)}$  from left to right. This determines the drawing of the  $K_4$  in  $D_v$ . The rotations of 2 and 3 imply that 4 is in  $\overrightarrow{(1, 3, 2)}_R \cap \overrightarrow{(3, 2, u)}_R$ .

The rotation at  $u$  and the location of 4 in  $D_v$  implies that  $\overrightarrow{(u, 4)}$  crosses  $(1, 2)$ , then  $(1, 3)$ , then  $(2, 3)$ . By Observation 2.9,  $D_v$  is Harborth and the rotation of the vertices are determined. In particular,  $\pi_{1,2,3,u}(4) = [3, u, 1, 2]$ .

Note that in  $D_1$ ,  $\overrightarrow{(u, v)}$  crosses  $\overrightarrow{(2, 4)}$  from right to left. In  $D_4$ ,  $\overrightarrow{(u, v)}$  crosses  $\overrightarrow{(1, 2)}$  from right to left. By definition of  $\mathcal{C}$ ,  $\overrightarrow{(u, v)}$  crosses  $\overrightarrow{(4, 1)}$ , therefore, it does so from left to right (else the 3-cycle  $(1, 2, 4)$  has  $e$  crossing it from the same side three times, a contradiction in a drawing realizing its 5-vertex rotation system).

Furthermore, we have  $(1, 4) <_{\wedge}^e (1, 2)$ , therefore it follows  $(1, 4) <_{\wedge}^e (1, 2) <_{\wedge}^e (2, 4)$  and that  $D_3$  is Harborth by Observation 2.9. In particular,  $\pi_{1,2,u,v}(4) = [1, v, u, 2]$ . Therefore,  $H$  contains the three incompatible rotations at 4  $[u, 2, 3, v]$ ,  $[3, u, 1, 2]$  and  $[1, v, u, 2]$ , a contradiction.

**Case 2.** *Two non-consecutive relation in  $\mathcal{C}$  disagree.*

Assume without loss of generality that  $(1, 2) <_{\wedge}^e (2, 3)$  and  $(3, 4) <_{\wedge}^e (1, 4)$  both disagree. By Observation 4.3, it follows that  $D_2$  and  $D_4$  are both uniquely labelled Harborth drawings up to deciding the oriented crossings of  $(1, 2)$  and  $(3, 4)$  with  $e$ .

It follows in  $D_4$ , independent of the oriented crossing of  $(1, 2)$  with  $e$ ,  $(1, 2) <_{\wedge}^e (1, 3)$ . Similarly in  $D_2$ , independent of the oriented crossing of  $(3, 4)$  with  $e$ ,  $(1, 3) <_{\wedge}^e (1, 4)$ . From  $\mathcal{C}$ ,  $(1, 4) <_{\wedge}^e (1, 2)$ , and so  $(1, 2) <_{\wedge}^e (1, 3) <_{\wedge}^e (1, 4) <_{\wedge}^e (1, 2)$ , a contradiction with Lemma 4.4.

**Case 3.** *All relations in  $\mathcal{C}$  agree.*

Without loss of generality, let  $\overrightarrow{(1, 2)}$  cross  $e$  from left to right. Since all the relations in  $\mathcal{C}$  agree, it follows that every edge in  $\mathcal{C}$  has its oriented crossing with  $e$  determined from the oriented crossing of  $(1, 2)$  with  $e$ . In particular:

- $\pi_{2,4,u,v}(3) = [v, 4, 2, u]$  from  $D_1$ ;
- $\pi_{1,3,u,v}(4) = [v, u, 3, 1]$  from  $D_2$ ;
- $\pi_{2,4,u,v}(1) = [v, 2, 4, u]$  from  $D_3$ ; and
- $\pi_{1,3,u,v}(2) = [v, u, 1, 3]$  from  $D_4$ .

At this point we do analysis on how the edges of  $D_4$  cross each other. These claims will have two crossings that do not occur, however this will require one proof per claim as the edges chosen have a symmetry in  $H$ .

**Claim 1.**  $(1, u)$  crosses  $(2, 3)$  or  $(3, v)$  crosses  $(1, 2)$ .

*Proof.* The two cases are symmetric, so by way of contradiction, we may assume  $(1, u)$  crosses  $(2, 3)$ . In  $D_4$ , there is a unique direction  $(1, u)$  can cross  $(2, 3)$ . Such a crossing determines the simple drawing  $D_4$ . In particular,  $\pi_{1,3,u}(v) = [u, 1, 3]$  and  $\pi_{1,2,u,v}(3) = [v, 2, 1, u]$ . Combining the rotation at 3 in  $D_1$  and  $D_4$ , we get  $\pi_{1,2,4,u,v}(3) = [v, 4, 2, 1, u]$ .

Observe that  $\overrightarrow{(3, 4)}$  crosses  $e$  from left to right. It follows that the drawing of the  $K_4$  induced on  $3, 4$  and  $e$  is determined, in particular  $\overrightarrow{(3, v, u)}_R \cap \overrightarrow{(3, v, 4)}_R$  exists. By the rotations at 3, 4 and  $v$ , it follows that  $v \in \overrightarrow{(3, v, u)}_R \cap$

$\overrightarrow{(3, v, 4)}_R$ . However, as  $\pi_{1,4,u}(3) = [4, 1, u]$ , the edge  $(3, 4)$  can not reach the vertex 4 in  $D_2$ , a contradiction with the existence of  $D_2$ . ■

**Claim 2.**  $(1, 3)$  crosses  $(2, u)$  or  $(1, 3)$  crosses  $(2, v)$ .

*Proof.* As per Claim 1,  $(1, u)$  does not cross  $(2, 3)$  and  $(3, v)$  does not cross  $(1, 2)$ . It follows that  $D_4 - \{(1, 3), (1, v), (3, u)\}$  is uniquely determined.

As one would expect, the two cases are symmetric, so by way of contradiction, we may assume  $(1, 3)$  crosses  $(2, v)$ . In  $D_4$ , there is a unique direction  $(1, 3)$  crosses  $(2, v)$ . Again, such a crossing determines the simple drawing  $D_4$ . It follows from  $D_4$  that:  $\pi_{3,u,v}(1) = [u, 3, v]$ ;  $\pi_{1,3,v}(u) = [1, 3, v]$ ;  $\pi_{1,3,u}(v) = [u, 3, 1]$ ; and  $(1, 2) <_{\wedge}^e (1, 3)$ .

Note that the oriented crossing of  $(1, 4)$  with  $e$  is determined, and so the  $K_4$  induced by 1, 4 and  $e$  is uniquely drawn in  $D_2$ . By the rotations at 1, 4,  $u$  and  $v$  ( $\pi_{1,3,u}(4) = [u, 3, 1]$  determined by  $D_2$ ), determine the drawing of  $D_2$ . In particular,  $(1, 3) <_{\wedge}^e (1, 4)$ .

From  $\mathcal{C}$  we have  $(1, 4) <_{\wedge}^e (1, 2)$ , and thus we have  $(1, 2) <_{\wedge}^e (1, 3) <_{\wedge}^e (1, 4) <_{\wedge}^e (1, 2)$ , a contradiction with Lemma 4.4. ■

Since the crossing in Claims 1 and 2 do not occur, the drawing of  $D_4 - \{(1, v), (3, u)\}$  is uniquely determined. Through the symmetry of this case, these arguments extend to  $D_i$  for  $i \in [4]$ .  $D_1$ ,  $D_2$ , and  $D_4$  each give  $\pi_{2,4,u,v}(3) = [v, 4, 2, u]$ ,  $\pi_{1,4,u}(3) = [1, 4, u]$ , and  $\pi_{1,2,v}(3) = [v, 2, 1]$ , respectively. Combining these rotations results in  $\pi_{1,2,4,u,v}(3) = [v, 4, 2, u, 1]$ . It follows that  $D_4 - \{(1, v)\}$  is uniquely determined, which implies  $D_4$  is uniquely determined by symmetry, and  $D_i$  for  $i \in [4]$  is uniquely determined by symmetry.

$D_1$  and  $D_4$  give  $\pi_{2,3,4,v}(u) = [v, 4, 2, 3]$  and  $\pi_{1,2,3,v}(u) = [v, 2, 1, 3]$ , respectively. Combining these rotation results in  $\pi_{1,2,3,4,v}(u) = [v, 4, 2, 1, 3]$ , a contradiction with  $\pi_{1,3,4,v}(u) = [3, 1, v, 4]$  in  $D_2$ . □

**Lemma 4.6.** *If  $H$  is a  $(6, 5)$ -rotation system, and  $e$  is a directed edge of  $H$ , then there are no cycles comprised of  $<_{\wedge}^e$  relations in  $H$ .*

*Proof.* Let  $H$  be a  $(6, 5)$ -rotation system,  $e$  be a directed edge of  $H$ . By way of contradiction, assume  $\mathcal{C} = (a_0, \dots, a_{k-1}, a_0)$  is a shortest cycle of  $<_{\wedge}^e$  relations in  $H$ . Without loss of generality,  $a_0 = (1, 2)$  and  $a_1 = (2, 3)$ . If there exists an  $i$  such that  $V(\{a_i, a_{i+1}, a_{i+2}\}) \subseteq [4] \setminus j$  for some  $j \in [4]$ , then  $(a_0, \dots, a_i, a_{i+2}, \dots, a_{k-1}, a_0)$  is a shorter cycle of  $<_{\wedge}^e$  relations, a contradiction with  $\mathcal{C}$ .

It follows by Lemma 4.4 and the previous argument that  $a_2 = (3, 4)$ ,  $a_3 = (4, 1)$ , and  $a_5 = (1, 2)$ , a contradiction with Lemma 4.5.  $\square$

Note that the  $<_{\parallel}^e$  relation does not exist in  $(6, 5)$ -rotation systems. Even though the  $<_{\wedge}^e$  relation induces an acyclic graph, it is not known in which order two uncrossed disjoint edges  $f$  and  $g$  cross  $e$ . If the induced rotation system of  $f$  and  $g$  has a planar representation, then for each face, exactly one of  $f$  or  $g$  bounds that face. If the induced rotation system of  $f$  and  $g$  has a realization that is a crossing  $K_4$ , then in Lemma 4.8, we show that the rotation system implies that the oriented crossings of  $e$  with  $f$  and  $e$  with  $g$  are from opposite sides of the uncrossed 4-cycle.

When drawing a realization of a  $(6, 5)$  rotation system, we choose to draw a specific edge  $e$  and depending on the current non-vertex end of our partially drawn edge, it follows that at most one of  $f$  and  $g$  can be crossed at this time. We proceed by proving Lemma 4.7 and using it as a tool to prove Lemma 4.8.

**Lemma 4.7.** *Let  $n \geq 6$ ,  $H_n$  be an  $(n, 5)$ -rotation system, and  $\{x, a, b, c\} \subset V(H_n)$ . If  $\pi_{a,b,c}(x) = [a, b, c]$  and  $e$  is a directed edge of  $H_n$  that crosses  $\overrightarrow{(x, a)}$ ,  $\overrightarrow{(x, b)}$ , and  $\overrightarrow{(x, c)}$  from left to right, then the order of these three crossings on  $e$  is a cyclic permutation of  $[(x, a), (x, b), (x, c)]$ .*

*Proof.* Let  $\pi_{a,b,c}(x) = [a, b, c]$  and suppose  $e = \overrightarrow{(u, v)}$  is a directed edge that crosses all  $(x, i)$  from left to right for  $i \in \{a, b, c\}$ . Define  $D_i$  for  $i \in \{a, b, c\}$  to be the realization of the 5-vertex rotation system defined on  $(\{a, b, c\} \setminus i) \cup \{x, u, v\}$ .

By the symmetry of  $a, b$  and  $c$ , suppose  $e$  crosses  $(x, a)$  first. By way of contradiction, assume  $(x, a) <_{\wedge}^e (x, c) <_{\wedge}^e (x, b)$ . Since the oriented crossing of  $e$  and  $(x, i)$  are determined, and the order  $e$  crosses the edges  $(x, i)$  is determined, Observation 4.3 tells us the rotations at  $x$  in each  $D_i$  are determined. In particular,  $\pi_{a,b,u,v}(x) = [u, a, b, v]$ ,  $\pi_{a,c,u,v}(x) = [u, a, c, v]$ , and  $\pi_{b,c,u,v}(x) = [u, c, b, v]$ . Combining these rotations gives  $\pi_{a,b,c,u,v}(x) = [u, a, c, b, v]$ , a contradiction with  $\pi_{a,b,c}(x) = [a, b, c]$ .  $\square$

**Lemma 4.8.** *Let  $H$  be a  $(6, 5)$ -rotation system,  $e = \overrightarrow{(u, v)}$  be a directed edge of  $H$ , and  $f$  and  $g$  two uncrossed edges from a crossing 4-vertex rotation system  $H_4$  in  $H$  such that  $(u, v)$  crosses both  $f$  and  $g$ . In  $H$ , if there is no chain of  $<_{\wedge}^e$  relations ordering  $f$  and  $g$ , then  $e$  crosses  $f$  and  $g$  from different sides of the uncrossed 4-cycle in  $H_4$ .*

*Proof.* Let  $H$  be a  $(6,5)$ -rotation system,  $e = \overrightarrow{(u,v)}$  be a directed edge of  $H$ ,  $f$  and  $g$  two uncrossed edges from a crossing 4-vertex rotation system  $H_4$  in  $H$  such that  $e$  crosses both  $f$  and  $g$  such that there is not chain of  $<_{\wedge}^e$  relations ordering  $f$  and  $g$ , and set  $V(H_4) = [4]$ . Suppose in  $H$  that there is no chain of  $<_{\wedge}^e$  relations ordering  $f$  and  $g$ .

Without loss of generality, assume  $e = \overrightarrow{(u,v)}$  and  $V(H_4) = [4]$ . For  $i \in [4] \cup \{u,v\}$ , define  $D_i$  to be the realization of  $H_i$  and  $D_e$  to be the realization of  $H - \{u,v\}$ .

Some edge in  $D_e$  crosses another and so without loss of generality, let  $\overrightarrow{(1,4)}$  crosses  $\overrightarrow{(2,3)}$  from left to right in  $D_e$ . Notice that this prescribes the oriented drawing  $D_e$ . Without loss of generality, we set  $f = (1,2)$  and  $g = (3,4)$ . We make use of the following observation.

**Observation 4.9.** *Let  $u$  be in side  $S$  of a 3-cycle. If  $v$  is in  $S$ , then  $e$  crosses out of  $S$  the same number of times it crosses into  $S$ . Similarly, if  $v$  is not in  $S$ , then  $e$  crosses out of  $S$  exactly once more than it crosses into  $S$ .*

For each uncrossed edge  $(i,j)$  in  $D_e$ , define the face that has both the crossing of  $(1,4)$  with  $(2,3)$  and the edge  $(i,j)$  on the boundary to be  $F_{(i,j)}$ . Let the fifth and final face of  $D_e$  be  $F_4$ . To be clear,  $F_4$  is the face that does not have the crossing on the boundary (the side of the uncrossed 4-cycle that does not contain the crossing).

We note the following two facts that will be used extensively in the proofs of the upcoming cases.

**Fact 1:** Let  $y \in \{(1,2), (3,4)\}$ ,  $x \in \{(1,3), (1,4)(2,3)(2,4)\}$  and  $z \in \{(1,2), (3,4)\} \setminus \{y\}$ . If  $x <_{\wedge}^e y$  ( $y <_{\wedge}^e x$ ), then  $x <_{\wedge}^e z$  ( $z <_{\wedge}^e x$ ).

**Fact 2:** Let  $C$  be a 3-cycle. Given the oriented crossings of  $C$  with  $e$ , the locations of  $u$  and  $v$  relative to  $C$ , and a pair of edges of the three cycle the relation  $<_{\wedge}^e$  is known, then the order the edges of the 3-cycle are crossed is determined.

Fact 1 is an immediate consequence of there not being a chain of  $<_{\wedge}^e$  relations ordering  $y$  and  $z$ . Fact 2 follows from the consecutive crossings of a  $C$  come from opposite sides of  $C$ .

To prove the lemma, there are 8 cases up to symmetry of which faces in  $D_e$  contain  $u$  and  $v$  from  $D_v$  and  $D_u$ , respectively. The cases are as follows

with the first five cases having neither  $u$  nor  $v$  in  $F_4$ , while the last three have at least one in  $F_4$ :

1.  $u$  and  $v$  are both in  $F_{(1,2)}$ ;
2.  $u$  is in  $F_{(1,2)}$  and  $v$  is in  $F_{(3,4)}$ ;
3.  $u$  is in  $F_{(1,2)}$  and  $v$  is in  $F_{(1,3)}$ ;
4.  $u$  and  $v$  are both in  $F_{(1,3)}$ ;
5.  $u$  is in  $F_{(1,3)}$  and  $v$  is in  $F_{(2,4)}$ ;
6.  $u$  is in  $F_{(1,3)}$  and  $v$  is in  $F_4$ ;
7.  $u$  is in  $F_{(1,2)}$  and  $v$  is in  $F_4$ ; and
8.  $u$  and  $v$  are in  $F_4$ .

Note that, by way of contradiction, in all cases  $e$  crosses  $(1,2)$  and  $(3,4)$  into  $F_4$  or out of  $F_4$ , respectively. This forms 16 cases in total. Case 1.1 will give a detailed explanation how Observation 4.9 determines the oriented crossings of  $e$  with edges of  $D_e$ . In the cases following Case 1.1, we will apply Observation 4.9 to determine the oriented crossings without explanation.

For each case, we offer Figures 14-29 that on the left describe the faces containing the vertices  $u$  and  $v$  and the direction  $e$  crosses  $(1,2)$  and  $(3,4)$  and on the right the implied crossings of  $e$  with the remaining edges.

**Case 1.1.**  $u$  and  $v$  are both in  $F_{(1,2)}$ , and  $e$  crosses  $(1,2)$  and  $(3,4)$  into  $F_4$ .

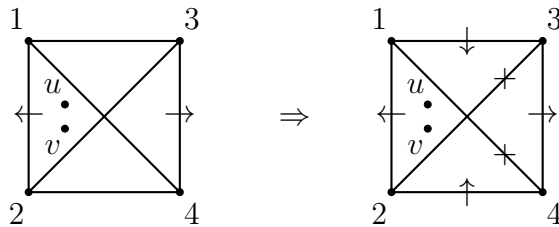


Figure 14: Lemma 4.8 Case 1.1

Because  $u$  and  $v$  are both in  $F_{(1,2)}$ ,  $e$  crosses each 3-cycle in  $D_e$  an even number of times and the orientations of these crossings are known. By Observation 4.9, since  $e$  crosses out of  $\overrightarrow{(1,3,2)}_R$  at  $(1,2)$ , Observation 4.9 shows  $e$  does not cross out of  $\overrightarrow{(1,3,2)}_R$  at  $(2,3)$ .

By Observation 4.9, since  $e$  crosses out of  $\overrightarrow{(2,3,4)}_R$  at  $(3,4)$  and  $u$  and  $v$  are not inside  $\overrightarrow{(2,3,4)}_R$ ,  $e$  does not cross  $(2,3)$  out of  $\overrightarrow{(2,3,4)}_R$ . Combined with the preceding paragraph,  $e$  does not cross  $(2,3)$ . Therefore,  $e$  crosses into  $\overrightarrow{(1,3,2)}_R$  at  $(1,3)$ .

By the same arguments,  $e$  does not cross  $(1,4)$ , and  $e$  crosses  $(1,3)$  and  $(2,4)$  into  $F_{(1,3)}$  and  $F_{(2,4)}$ , respectively. Note that each of the 3-cycles on vertices  $[4] \setminus \{j\}$ , for  $j \in [4]$ , are crossed exactly twice and the oriented crossings are known.

Since each 3-cycle is crossed twice and we know the oriented crossings and the order in which  $e$  crosses the edges of each 3-cycle, Observation 4.3 shows  $\pi_{2,3,u,v}(1) = [u, 2, 3, v]$ ,  $\pi_{1,4,u,v}(2) = [v, 4, 1, u]$ ,  $\pi_{1,4,u,v}(3) = [v, 4, 1, u]$ , and  $\pi_{2,3,u,v}(4) = [u, 2, 3, v]$ . From these rotations and the oriented crossing of  $(1,4)$  with  $(2,3)$ , it follows that  $D_u$  is uniquely determined with  $\pi_{1,2,3,4}(v) = [1, 2, 4, 3]$ .

Since  $e$  crosses  $\overrightarrow{(1,2)}$  from left to right and  $\overrightarrow{(3,4)}$  from right to left, it follows that  $\pi_{1,2,u}(v) = [1, u, 2]$  and  $\pi_{3,4,u}(v) = [4, u, 3]$ . By the existence of the rotation at  $v$ , since we can not combine these rotations at  $v$ , we have a contradiction.

**Case 1.2.**  $u$  and  $v$  are both in  $F_{(1,2)}$ , and  $e$  crosses  $(1,2)$  and  $(3,4)$  into  $F_{(1,2)}$  and  $F_{(3,4)}$ , respectively.

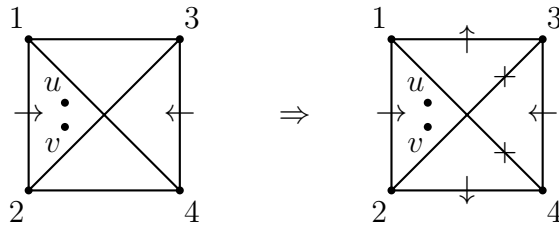


Figure 15: Lemma 4.8 Case 1.2

Redirecting  $e$  to go from  $v$  to  $u$  converts this to Case 1.1, and resolves the case.



**Case 2.1.**  $u$  in in  $F_{(1,2)}$  and  $v$  is in  $F_{(3,4)}$ , and  $e$  crosses  $(1,2)$  and  $(3,4)$  into  $F_4$ .

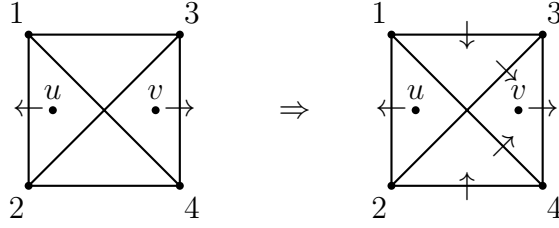


Figure 16: Lemma 4.8 Case 2.1

By Observation 4.9,  $e$  crosses  $(1,3)$ ,  $(2,4)$ ,  $(1,4)$ , and  $(2,3)$  into  $F_{(1,3)}$ ,  $F_{(2,4)}$ ,  $\overrightarrow{(1,3,4)}_R$ , and  $\overrightarrow{(2,3,4)}_R$ , respectively, and none of the other remaining edges.

If  $(2,3) <_{\wedge}^e (3,4)$ , then  $(2,3) <_{\wedge}^e (1,2)$ , by Fact 1. By Fact 2 on  $(1,2,3)$ ,  $(2,3) <_{\wedge}^e (1,3) <_{\wedge}^e (1,2)$ . By Fact 1,  $(1,3) <_{\wedge}^e (3,4)$ . It follows that  $(2,3) <_{\wedge}^e (1,3) <_{\wedge}^e (3,4)$ , a contradiction with  $x = 3$  in Lemma 4.7.

If  $(3,4) <_{\wedge}^e (2,3)$ , then by Fact 2 on  $(2,3,4)$ ,  $(2,4) <_{\wedge}^e (3,4) <_{\wedge}^e (2,3)$ . By Fact 1,  $(2,4) <_{\wedge}^e (1,2)$ . By Fact 2 on  $(1,2,4)$ ,  $(1,4) <_{\wedge}^e (2,4) <_{\wedge}^e (1,2)$ . It follows that  $(1,4) <_{\wedge}^e (2,4) <_{\wedge}^e (3,4)$ , a contradiction with  $x = 4$  in Lemma 4.7.

**Case 2.2.**  $u$  in in  $F_{(1,2)}$  and  $v$  is in  $F_{(3,4)}$ , and  $e$  crosses  $(1,2)$  and  $(3,4)$  into  $F_{(1,2)}$  and  $F_{(3,4)}$ , respectively.

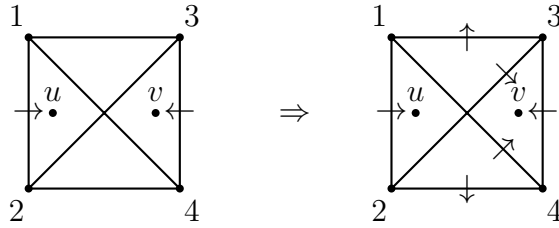


Figure 17: Lemma 4.8 Case 2.2

Redirecting  $e$  to go from  $v$  to  $u$  in Case 2.1 resolves this case.

**Case 3.1.**  $u$  is in  $F_{(1,2)}$  and  $v$  is in  $F_{(1,3)}$ , and  $e$  crosses  $(1,2)$  and  $(3,4)$  into  $F_4$ .

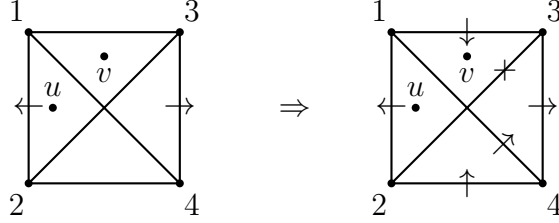


Figure 18: Lemma 4.8 Case 3.1

By Observation 4.9,  $e$  crosses  $(1,3)$ ,  $(2,4)$ , and  $(1,4)$  into  $F_{(1,3)}$ ,  $F_{(2,4)}$ , and  $\overrightarrow{(1,3,4)}_R$ , respectively, and none of the other remaining edges. By Fact 2 on  $(1,2,3)$ ,  $(1,2) <_{\wedge}^e (1,3)$ .

By Fact 1,  $(3,4) <_{\wedge}^e (1,3)$ . By Fact 2,  $(1,4) <_{\wedge}^e (3,4) <_{\wedge}^e (1,3)$ . By Fact 1,  $(1,4) <_{\wedge}^e (1,2)$ . By Fact 2 on  $(1,2,4)$ ,  $(1,4) <_{\wedge}^e (2,4) <_{\wedge}^e (1,2)$ . By Fact 1,  $(2,4) <_{\wedge}^e (3,4)$ . It follows that  $(1,4) <_{\wedge}^e (2,4) <_{\wedge}^e (3,4)$ , a contradiction with  $x = 4$  in Lemma 4.7.

**Case 3.2.**  $u$  is in  $F_{(1,2)}$  and  $v$  is in  $F_{(1,3)}$ , and  $e$  crosses  $(1,2)$  and  $(3,4)$  into  $F_{(1,2)}$  and  $F_{(3,4)}$ , respectively.

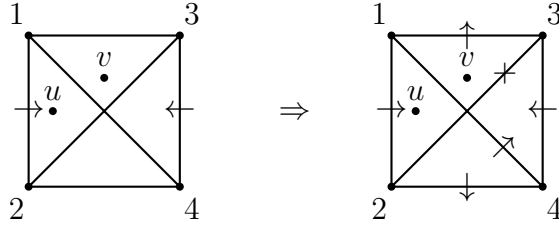


Figure 19: Lemma 4.8 Case 3.2

By Observation 4.9,  $e$  crosses  $(1,3)$ ,  $(2,4)$ , and  $(1,4)$  into  $F_4$ ,  $F_4$ , and  $\overrightarrow{(1,3,4)}_R$ , respectively, and none of the other remaining edges. Fact 2 on  $(2,3,4)$  shows  $(3,4) <_{\wedge}^e (2,4)$ . By Fact 1,  $(1,2) <_{\wedge}^e (2,4)$ . Fact 2 on  $(1,2,4)$  shows  $(1,4) <_{\wedge}^e (1,2) <_{\wedge}^e (2,4)$ .

From the oriented crossings, and  $(1,4) <_{\wedge}^e (1,2) <_{\wedge}^e (2,4)$ , by Observation 2.9,  $D_3$  is determined and is Harborth. In particular,  $\pi_{2,4,v}(1) = [2, 4, v]$ .

It follows that in  $D_u$ , the edge  $(1, v)$  starts in  $F_{(1,2)}$  and ends in  $F_{(1,3)}$ , a contradiction with  $D_u$  being simple.

**Case 4.1.**  $u$  and  $v$  are both in  $F_{(1,3)}$ , and  $e$  crosses  $(1,2)$  and  $(3,4)$  into  $F_4$ .

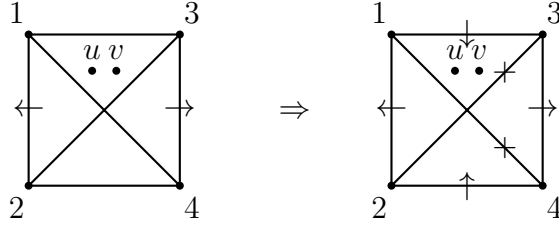


Figure 20: Lemma 4.8 Case 4.1

By Observation 4.9,  $e$  crosses  $(1,3)$  and  $(2,4)$  into  $F_{(1,3)}$  and  $F_{(2,4)}$ , respectively, and none of the other remaining edges. By Fact 2 on  $(1,2,3)$ ,  $(1,2) <_{\wedge}^e (1,3)$ .

Given the ordering  $e$  crosses  $(1,2)$  and  $(1,3)$  along with the oriented crossings of  $e$  with  $(1,2)$  and  $(1,3)$ , the rotation at 1 is determined in  $D_4$ . In particular,  $\pi_{2,3,u,v}(1) = [u, 2, 3, v]$ . The same arguments apply to every 3-cycle with  $e$ . It follows that  $\pi_{1,4,u,v}(2) = [v, 1, 4, u]$ ,  $\pi_{1,4,u,v}(3) = [v, 1, 4, u]$ , and  $\pi_{2,3,u,v}(4) = [u, 2, 3, v]$ .

Given the location of  $v$  in  $D_e$  along with these rotations determines  $D_u$ , in particular  $\pi_{1,2,3,4}(v) = [1, 2, 4, 3]$ . The oriented crossings of  $e$  with  $(1,2)$  and  $(3,4)$  determine that  $\pi_{1,2,u}(v) = [1, u, 2]$  and  $\pi_{3,4,u}(v) = [4, u, 3]$ , respectively. Since the three rotations at  $v$  can not be combined, it follows that  $\pi_{1,2,3,4,u}(v)$  is not well defined, a contradiction.

**Case 4.2.**  $u$  and  $v$  are both in  $F_{(1,3)}$ , and  $e$  crosses  $(1,2)$  and  $(3,4)$  into  $F_{(1,2)}$  and  $F_{(3,4)}$ , respectively, and none of the other remaining edges.

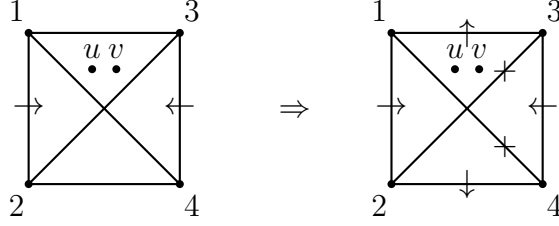


Figure 21: Lemma 4.8 Case 4.2

Redirecting the edge  $e$  from  $v$  to  $u$  and applying the same arguments as Case 4.1 will result in the same conclusion on  $\pi_{1,2,3,4,v}(u)$ .

**Case 5.1.**  $u$  is in  $F_{(1,3)}$  and  $v$  is in  $F_{(2,4)}$ , and  $e$  crosses  $(1,2)$  and  $(3,4)$  into  $F_4$ .

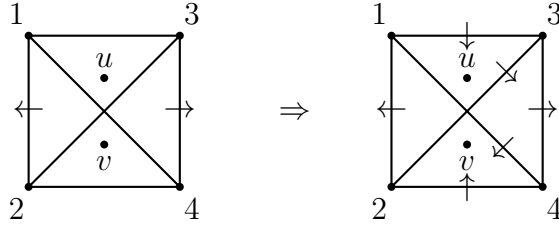


Figure 22: Lemma 4.8 Case 5.1

By Observation 4.9,  $e$  crosses  $(1,3)$ ,  $(2,4)$ ,  $(1,4)$ , and  $(2,3)$  into  $F_{(1,3)}$ ,  $F_{(2,4)}$ ,  $\overrightarrow{(1,4,2)}_R$  and  $\overrightarrow{(2,3,4)}_R$ , respectively.

If  $(2,3) <_{\wedge}^e (3,4)$ , then  $(2,3) <_{\wedge}^e (1,2)$  by Fact 1.Fact 2 on  $(1,2,3)$  shows  $(2,3) <_{\wedge}^e (1,3) <_{\wedge}^e (1,2)$ . Fact 1 then implies  $(1,3) <_{\wedge}^e (3,4)$ . Finally, it follows  $(2,3) <_{\wedge}^e (1,3) <_{\wedge}^e (3,4)$ , a contradiction with  $x = 3$  in Lemma 4.7.

It follows that  $(3,4) <_{\wedge}^e (2,3)$ . This along with the oriented crossings of  $e$  with  $(3,4)$  and  $(2,3)$  imply the rotation at 3 is determined in  $D_1$ , in particular  $\pi_{2,4,u}(3) = [u, 2, 4]$ .  $u$  in  $F_{(1,3)}$  in  $D_v$  and  $(3, u)$  starting in  $F_{(3,4)}$  contradicts the fact that  $D_v$  is a simple drawing.

**Case 5.2.**  $u$  is in  $F_{(1,3)}$  and  $v$  is in  $F_{(2,4)}$ , and  $e$  crosses  $(1,2)$  and  $(3,4)$  into  $F_{(1,2)}$  and  $F_{(3,4)}$ , respectively.

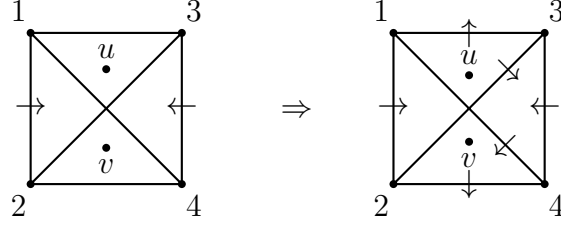


Figure 23: Lemma 4.8 Case 5.2

Redirecting the edge  $e$  from  $v$  to  $u$  and applying the same arguments as Case 5.1 will result in the same conclusion on  $v$  in  $D_u$ .

**Case 6.1.**  $u$  is in  $F_{(1,3)}$  and  $v$  is in  $F_4$ , and  $e$  crosses  $(1,2)$  and  $(3,4)$  into  $F_4$ .

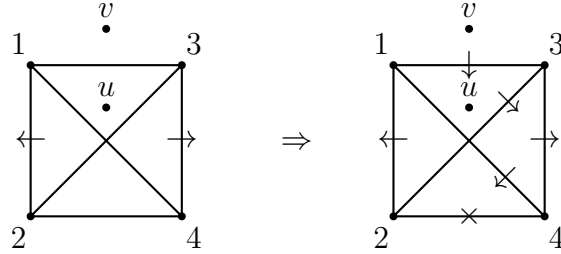


Figure 24: Lemma 4.8 Case 6.1

By Observation 4.9,  $e$  crosses  $(1,3)$ ,  $(1,4)$ , and  $(2,3)$  into  $F_{(1,3)}$ ,  $\overrightarrow{(1,4,2)}_R$  and  $\overrightarrow{(2,3,4)}_R$ , respectively, and none of the other remaining edges.

Applying Fact 2 to  $(1,2,4)$  gives  $(1,4) <_{\wedge}^e (1,2)$ . Fact 1 implies  $(1,4) <_{\wedge}^e (3,4)$ . Applying Fact 2 to  $(1,3,4)$  gives  $(1,4) <_{\wedge}^e (1,3) <_{\wedge}^e (3,4)$ . By Fact 1,  $(1,3) <_{\wedge}^e (1,2)$ . Fact 2 on  $(1,2,3)$  gives  $(2,3) <_{\wedge}^e (1,3) <_{\wedge}^e (1,2)$ . It follows that  $(2,3) <_{\wedge}^e (1,3) <_{\wedge}^e (3,4)$ , a contradiction with  $x = 3$  in Lemma 4.7.

**Case 6.2.**  $u$  is in  $F_{(1,3)}$  and  $v$  is in  $F_4$ , and  $e$  crosses  $(1,2)$  and  $(3,4)$  into  $F_{(1,2)}$  and  $F(3,4)$ , respectively.

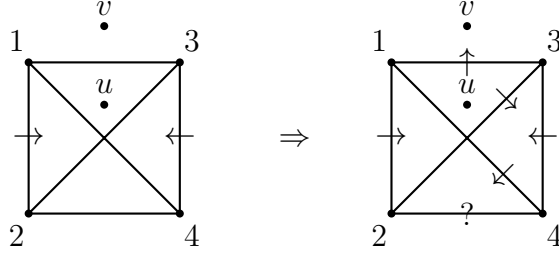


Figure 25: Lemma 4.8 Case 6.2

By Observation 4.9,  $e$  crosses  $(1, 3)$ ,  $(1, 4)$  and  $(2, 3)$  into  $F_4$ ,  $\overrightarrow{(1, 4, 2)}_R$  and  $\overrightarrow{(2, 3, 4)}_R$ , respectively. Observe that  $\overrightarrow{(2, 3, 4)}_R$  has  $e$  crossing into it twice and has both  $u$  and  $v$  on the same side, a contradiction with Observation 4.9.

**Case 7.1.**  $u$  is in  $F_{(1,2)}$  and  $v$  is in  $F_4$ , and  $e$  crosses  $(1, 2)$  and  $(3, 4)$  into  $F_4$ .

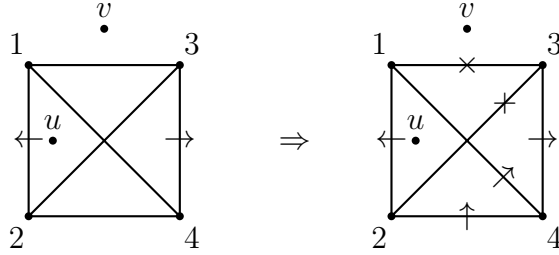


Figure 26: Lemma 4.8 Case 7.1

By Observation 4.9, either  $e$  crosses  $(1, 4)$  and  $(2, 4)$  into  $\overrightarrow{(1, 3, 4)}_R$  and  $F_{(2,4)}$ , respectively, or  $e$  crosses  $(2, 3)$  and  $(1, 3)$  into  $\overrightarrow{(2, 3, 4)}_R$  and  $F_{(1,3)}$ , respectively. A change of labelling of 3 maps to 4 and 1 maps to 2 implies that both cases are the same. Therefore, without loss of generality  $e$  crosses  $(1, 4)$  and  $(2, 4)$  into  $\overrightarrow{(1, 3, 4)}_R$  and  $F_{(2,4)}$ , respectively.

Applying Fact 2 to  $(2, 3, 4)$  gives  $(2, 4) <_{\wedge}^e (3, 4)$ . By Fact 1,  $(2, 4) <_{\wedge}^e (1, 2)$ . Fact 2 on  $(1, 2, 4)$  gives  $(1, 4) <_{\wedge}^e (2, 4) <_{\wedge}^e (1, 2)$ . It follows that  $(1, 4) <_{\wedge}^e (2, 4) <_{\wedge}^e (3, 4)$ , a contradiction with  $x = 4$  in Lemma 4.7.

**Case 7.2.**  $u$  is in  $F_{(1,2)}$  and  $v$  is in  $F_4$ , and  $e$  crosses  $(1, 2)$  and  $(3, 4)$  into  $F_{(1,2)}$  and  $F_{(3,4)}$ , respectively.

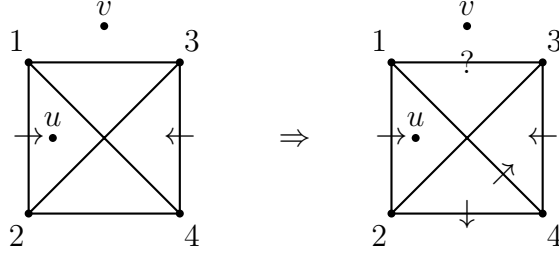


Figure 27: Lemma 4.8 Case 7.2

Applying Observation 4.9 on  $(1, 2, 4)$ , gives  $e$  crosses out of  $\overrightarrow{(1, 4, 2)}_R$  at both  $(1, 4)$  and  $(2, 4)$ . It follows that  $e$  crosses into  $\overrightarrow{(1, 3, 4)}_R$  twice, once at  $(3, 4)$  and once at  $(1, 4)$  implying  $v \in \overrightarrow{(1, 3, 4)}_R$ , a contradiction with  $v$  in  $F_4$ .

**Case 8.1.**  $u$  and  $v$  are in  $F_4$ , and  $e$  crosses  $(1, 2)$  and  $(3, 4)$  into  $F_4$ .

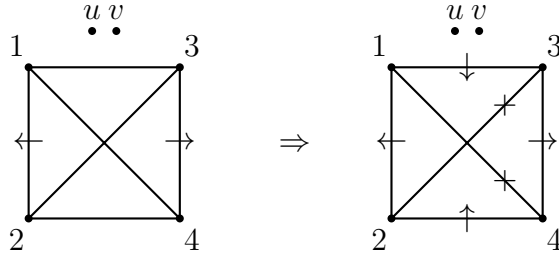


Figure 28: Lemma 4.8 Case 8.1

By Observation 4.9,  $e$  crosses  $(1, 3)$  and  $(2, 4)$  into  $F_{(1,3)}$  and  $F_{(2,4)}$ , respectively. Applying Fact 2 to  $(1, 2, 3)$  gives,  $(1, 3) <_{\wedge}^e (1, 2)$ .

Since the oriented crossings of  $e$  with  $(1, 3)$  and  $(1, 2)$  are known and the order  $e$  crosses them is known, the rotation at 1 is determined in  $D_4$ , in particular  $\pi_{2,3,v}(1) = [3, 2, v]$ . By similar arguments  $\pi_{1,4,v}(2) = [v, 1, 4]$ ,  $\pi_{1,4,v}(3) = [v, 4, 1]$ , and  $\pi_{2,3,v}(4) = [2, 3, v]$ .

Applying these rotations to  $D_e$  determines the simple drawing  $D_u$ , in particular  $\pi_{1,2,3,4}(v) = [1, 2, 4, 3]$ . From the oriented crossings of  $e$  with  $(1, 2)$  and  $(3, 4)$ , it follows that  $\pi_{1,2,u}(v) = [1, u, 2]$  and  $\pi_{3,4,u}(v) = [4, u, 3]$ . The three rotations at  $v$  can not be combined, a contradiction with  $\pi_{1,2,3,4,u}(v)$  being well defined.

**Case 8.2.**  $u$  and  $v$  are in  $F_4$ , and  $e$  crosses  $(1,2)$  and  $(3,4)$  into  $F_{(1,2)}$  and  $F_{(3,4)}$ , respectively.

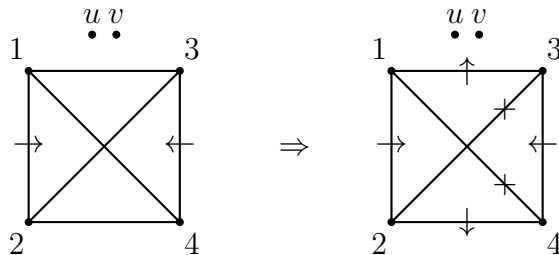


Figure 29: Lemma 4.8 Case 8.2

By Observation 4.9,  $e$  crosses  $(1,3)$  and  $(2,4)$  into  $F_4$  and  $F_4$ , respectively. Since the proof of Case 8.1 did not use the fact that  $(1,2)$  and  $(3,4)$  are not ordered by  $<^e_\wedge$  relations, this case follows by symmetry to the previous case.  $\square$

We use Lemmas 4.6 and 4.8 to create partial realizations of a  $(6,5)$ -rotation system  $H$ . We start with Lemma 4.12 that finds a  $K_4$  with an edge  $e$  whose endpoints are not in the  $K_4$  that is a partial realization of  $H$ . In Lemma 4.13, we extend this result to find a drawing of  $K_5$  with an edge  $e$  with exactly one endpoint in the underlying  $K_5$  that is a partial realization of  $H$ . Before we do so, we must extend the notion of a simple drawing.

**Definition 4.10.** Let  $H_n$  be an  $n$ -vertex rotation system,  $D$  be a simple drawing realizing some  $k$ -vertex rotation system  $H_k$  of  $H_n$  for  $k < n$ , and  $e = (u, v)$  be an edge of  $H_n$  having  $u$  in  $D$  and  $v$  not in  $D$ . If  $e_a$  is a partial arc of  $e$  starting at  $u$  in the sphere, then  $D + e_a$  is *simple* if  $e_a$  has at most one intersection point with any edge in  $D$ .

**Definition 4.11.** If  $D$  is a simple drawing of a graph  $G$  such that  $u, v \in V(G)$ ,  $(u, v) \notin E(G)$ , and  $e_a$  is partial arc of  $e$  in the sphere starting at  $u$ , then  $D + e_a$  is *simple* if  $e_a$  has at most one intersection point with every edge not incident to  $v$  and  $e_a$  has no intersection point with any edge incident to  $v$ .

Luckily, the definition of a simple drawing of a graph extends nicely to a simple drawing of a graph with a partial arc. We proceed by showing how



to find a partial realization of a  $(6, 5)$ -rotation system that is a  $K_4$  with an edge.

**Lemma 4.12.** *Let  $H$  be a  $(6, 5)$ -rotation system on the vertices  $[4] \cup \{u, v\}$ ,  $e = \overrightarrow{(u, v)}$ , and  $E$  be the set of edges that  $e$  crosses as determined by  $H$ . If  $D_e$  is a realization of  $H - \{u, v\}$ , then there exists a simple drawing  $D_e + \{e\}$  that has  $u$  and  $v$  in their respective faces of  $D_e$  as determined by  $H$ , the order  $e$  crosses the edges of  $E$  is consistent with  $<_{\wedge}^e$ , and the oriented crossings on  $e$  are in the prescribed orientations determined by  $H$ .*

*Proof.* Let  $D_j$  be a realization of  $H - \{j\}$  for  $j \in [4] \cup \{u, v\}$ , and  $D_e$  be a realization of  $H - \{u, v\}$ . Let  $e_i$  be a segment of  $e$  starting at  $u$ , crossings exactly  $i$  edges satisfying the partial ordering  $<_{\wedge}^e$  in the correct orientation as determined by  $H$ , and that ends in a face of  $D_e$  for  $i \in \{0\} \cup [cr_H(e)]$ .

Let  $D_{e_i}$  be a drawing of  $D_e + e_i$ , and  $E_i$  be the set of edge  $e_i$  crosses in  $D_{e_i}$ . It is enough to prove that  $D_{e_i}$  exists for all  $i \in \{0\} \cup [cr_H(e)]$  inductively on  $i$ .

Once this is done, we draw  $v$  on the non-vertex end of  $e_{cr_H(e)}$  and call this drawing  $D_e + \{e\}$ . If  $v$  is not in its respective face in  $D_e$  determined by  $H$ , then by Corollary 3.3, there exists some 3-cycle  $T$  that has  $v$  on opposite sides in  $H$  compared to  $D_e + \{e\}$ .

There is a unique way for  $e$  to cross  $T$  as defined by the partial ordering  $<_{\wedge}^e$  and the oriented crossing of  $e$  with each edge of the 3-cycle. Therefore, any realization of the associated 5-vertex rotation system on  $e$  and  $T$  has  $e$  and  $T$  crossing as in  $D_e + e$ , a contradiction with  $v$  being on opposite sides of the 3-cycle in  $H$  compared to  $D_e + \{e\}$ . We continue by proving the inductive statement.

Note that the drawing of  $D_v$  that has all the edges sharing  $u$  as an endpoint deleted (keeping  $u$  in the drawing) satisfies the definition of  $D_{e_0}$ .

Assume  $D_{e_i}$  exists for some  $i \in \{0\} \cup [cr_H(e) - 1]$ . By way of contradiction, assume  $D_{e_{i+1}}$  does not exist. From the partial ordering  $<_{\wedge}^e$  there is a set  $C_i$  of minimal elements in  $E \setminus E_i$ . It follows that no two edges in  $C_i$  share a common endpoint. As  $C_i$  is a set of edges in  $D_e$ ,  $|C_i| \leq 2$ . Let  $v_i$  be the non-vertex end of  $e_i$  and  $R_{v_i}$  be the face that contains  $v_i$  in  $D_{e_{i-1}}$  (if  $i = 0$ , then we let  $D_{e_{i-1}} = D_e$ ).

Extending the drawing  $D_{e_i}$  having  $e_i$  cross one of the elements in  $C_i$  satisfies  $<_{\wedge}^e$  and so we will choose to do so.

For every edge  $c \in C_i$ , and 3-cycle  $T$  containing  $c$  in  $D_e$ , there are two

sides of  $T$ . Define the side  $S_1$  to be the side of  $T$  bounded by the side of  $c$  that  $e$  crosses, and the other side of  $T$  to be  $S_2$ .

If  $v_i$  is in  $S_2$ , then the induced drawing of  $T$  with  $e_i$  in  $D_{e_i}$  can be extended to a realization of its associated  $k$ -vertex rotation system for  $k \leq 5$ . Since  $v_i$  is in  $S_2$ , this drawing has  $e_i$  not crossing  $c$  next, a contradiction with the definition of  $c$  being in  $C_i$ . Therefore, for every edge  $c \in C_i$  and every triangle  $T$  containing  $c$ ,  $e_i$  is in the correct side of  $T$  to cross  $c$ .

If  $D_e$  is planar, then every edge in  $C_i$  is on  $R_{v_i}$  and  $|C_i| = 1$  by the  $<_{\wedge}^e$  relation. In this case, we would cross the one edge in  $C_i$  to find a drawing of  $D_{e_{i+1}}$ .

Therefore,  $D_e$  is a crossing  $K_4$ . If  $v_i$  is on the uncrossed side of the uncrossed 4-cycle in  $D_e$ , then the  $<_{\wedge}^e$  relation along with Lemma 4.8 imply that there is a unique edge in  $C_i$  that  $e_i$  can cross on  $R_{v_i}$ . Again we cross this edge, to find a drawing of  $D_{e_{i+1}}$ .

Therefore,  $v_i$  is on the crossing side of the uncrossed 4-cycle in  $D_e$ . Again if there is a unique edge in  $C_i$  to cross, we do so.

If there are two edges in  $C_i$  to cross on  $R_{v_i}$ , then by the  $<_{\wedge}^e$  relation these edges must be crossing in  $D_e$ . In this case, we choose either edge to cross. If there are no such edges to cross, then for any edge  $c \in C_i$ ,  $e_i$  is on the correct side of any triangle  $T$  containing  $c$ .

It follows that  $|C_i| = 1$ ,  $c_i \in C_i$  is the unique edge in  $D_e$  that has empty intersection with  $R_{v_i}$ , the oriented crossing of  $e$  with  $c_i$  is from the uncrossed side of the uncrossed 4-cycle in  $D_e$  to the crosses side.

Without loss of generality, let  $c_i = (3, 4)$  and let the clockwise labelling of the uncrossed 4-cycle in  $D_e$  from the crossed side be  $\overrightarrow{(1, 3, 4, 2)}$ . It follows that  $\overrightarrow{(1, 4)}$  crosses  $\overrightarrow{(2, 3)}$  from left to right and  $e_i$  is in the unique face  $F_{(1,2)}$  that has  $(1, 2)$  and the crossing of  $(1, 4)$  with  $(2, 3)$  on its boundary in  $D_e$ .

For edges  $(j, k)$  in the uncrossed 4-cycle, define the other 3 symmetric faces in  $D_e$  analogously, and define the face that does not have the crossing of  $(1, 4)$  with  $(2, 3)$  on the boundary as  $F_4$ . Partition the proof into three cases depending on the oriented crossing of  $e$  with  $(1, 2)$  ( $e$  not crossing  $(1, 2)$  being one such case).

**Case 1.**  $e$  crosses  $\overrightarrow{(1, 2)}$  from left to right.

There are two cases to consider, whether  $(1, 2)$  has been crossed by  $e_i$  or not.

**Case 1.1.**  $(1, 2)$  has been crossed by  $e_i$ .

Since  $e_i$  crosses out of  $F_{(1,2)}$  at  $(1, 2)$ , it follows that the last edge  $e_i$  crossed was not  $(1, 2)$ , in particular  $(1, 2) <_{\wedge}^e \cdots <_{\wedge}^e (3, 4)$ . After  $e_i$  crossed  $(1, 2)$  it must get back into  $F_{(1,2)}$ . Without loss of generality, to do so it crosses  $(1, 2)$  into  $F_4$ , then  $(2, 4)$  into  $F_{(2,4)}$ , then crosses  $(2, 3)$  into  $F_{(1,2)}$ . In particular,  $(1, 2) <_{\wedge}^e (2, 4) <_{\wedge}^e (2, 3)$

Note by the oriented crossing on  $(2, 3, 4)$ ,  $v$  is in  $\overrightarrow{(2, 3, 4)}_R$ . Since  $(2, 4) <_{\wedge}^e (2, 3)$  and the fact the edges are already crossed, it follows that  $(2, 4) <_{\wedge}^e (2, 3) <_{\wedge}^e (3, 4)$ . Since the oriented crossings are known, by Observation 2.9, the rotations in  $D_1$  are known, and it is Harborth. In particular,  $\pi_{2,4,u,v}(3) = [4, v, u, 2]$  and  $v$  is in  $\overrightarrow{(2, 3, 4)}_R$ . Combining this with the rotation at 3 in  $D_e$  gives  $\pi_{1,2,4,u,v}(3) = [4, v, u, 2, 1]$ .

Since  $(1, 2) <_{\wedge}^e (2, 3)$  and the oriented crossings are known, it follows in  $D_4$ ,  $\pi_{1,3,u,v}(2) = [v, 3, 1, u]$ . Combining this with the rotation at 2 in  $D_e$  gives  $\pi_{1,3,4,u,v}(2) = [v, 3, 4, 1, u]$ . By the rotation at 2 and the fact  $v \in \overrightarrow{(2, 3, 4)}_R$ , in  $D_u$ ,  $v$  is in  $F_{(3,4)}$ . In  $D_3$ ,  $e$  crosses  $(1, 2)$ , then  $(2, 4)$  into  $\overrightarrow{(1, 4, 2)}_R$  and must end at  $v$  which is outside of  $\overrightarrow{(1, 4, 2)}_R$ , therefore  $e$  crosses  $(1, 4)$  from right to left and  $e_i$  does not cross  $(1, 4)$ .

Since  $e_i$  does not cross  $(1, 4)$ , it follows that  $(3, 4) <_{\wedge}^e (1, 4)$ . By the same analysis on  $(1, 3, 4)$ ,  $e$  crosses  $\overrightarrow{(1, 3)}$  from right to left and  $(3, 4) <_{\wedge}^e (1, 3) <_{\wedge}^e (1, 4)$ . The rotations in  $D_2$  are determined, in particular,  $\pi_{1,4,u,v}(3) = [u, 4, 1, v]$ , a contradiction with  $\pi_{1,2,4,u,v}(3) = [4, v, u, 2, 1]$ .

**Case 1.2.**  $(1, 2)$  has not been crossed by  $e_i$ .

Note that  $(3, 4)$  is the next edge crossed and  $(1, 2)$  has yet to be crossed. It is clear that there is a chain of  $<_{\wedge}^e$  relations from  $(3, 4)$  to  $(1, 2)$ , as if not,  $(1, 2)$  would be an element of  $C_i$ , a contradiction with  $C_i = \{(3, 4)\}$ . By symmetry, without loss of generality, assume  $e$  crosses one of the edges on  $(1, 2, 4)$  between  $(1, 2)$  and  $(3, 4)$ .

Again we can assume without loss of generality that it crosses this edge into  $\overrightarrow{(1, 4, 2)}_R$ . To justify this, as an example, if the edge crossed was edge  $(1, 4)$  out of  $\overrightarrow{(1, 4, 2)}_R$  and  $(3, 4) <_{\wedge}^e (1, 4) <_{\wedge}^e (1, 2)$ , then the oriented crossings on  $(1, 2, 4)$  would imply that  $e$  crosses  $(2, 4)$  into  $\overrightarrow{(1, 4, 2)}_R$  and

$(1, 4) <_{\wedge}^e (2, 4) <_{\wedge}^e (1, 2)$ , in particular  $(3, 4) <_{\wedge}^e (1, 4) <_{\wedge}^e (2, 4) <_{\wedge}^e (1, 2)$ . By Lemma 4.4, it would follow that  $(3, 4) <_{\wedge}^e (2, 4) <_{\wedge}^e (1, 2)$ .

Therefore, either  $e$  crosses  $y \in \{(2, 4), (1, 4)\}$  into  $\overrightarrow{(1, 4, 2)}_R$  such that  $(3, 4) <_{\wedge}^e y <_{\wedge}^e (1, 2)$ . We partition this into the two cases for  $y \in \{(2, 4), (1, 4)\}$ .

**Case 1.2.1.**  $y = (2, 4)$ .

To be clear, in this case  $e$  crosses  $\overrightarrow{(2, 4)}$  from right to left into  $F_{(2,4)}$  and  $(3, 4) <_{\wedge}^e (2, 4) <_{\wedge}^e (1, 2)$ . By the oriented crossings of the edges of  $(2, 3, 4)$  and the fact  $(3, 4) <_{\wedge}^e (2, 4)$ ,  $e$  crosses  $\overrightarrow{(2, 3)}$  from right to left and  $(3, 4) <_{\wedge}^e (2, 3) <_{\wedge}^e (2, 4)$ . By Observation 2.9,  $D_1$  is Harborth and every rotation in  $D_1$  is determined. In particular,  $\pi_{3,4,u,v}(2) = [3, u, v, 4]$ .

Since  $(2, 4) <_{\wedge}^e (1, 2)$  and the oriented crossings of  $e$  with  $(2, 4)$  and  $(1, 2)$  are known, it follows that  $\pi_{1,4,u,v}(2) = [v, 1, 4, u]$ . Finally  $\pi_{1,3,4}(2) = [1, 3, 4]$  from  $D_e$ . All three of the rotations of 2 can not be combined, therefore we have a contradiction with the existence of  $\pi_{1,2,4,u,v}(2)$ .

**Case 1.2.2.**  $y = (1, 4)$ .

In  $D_v$ , either  $u \in \overrightarrow{(1, 4, 2)}_R$  or not.

If  $u \notin \overrightarrow{(1, 4, 2)}_R$ , then  $e_i$  must have crossed  $\overrightarrow{(2, 4)}$ , then  $(2, 3)$  to end in  $F_{(1,2)}$ . In particular,  $e_i$  must cross  $(2, 4)$  into  $\overrightarrow{(1, 4, 2)}_R$ . It follows that  $(2, 4) <_{\wedge}^e (3, 4) <_{\wedge}^e (1, 4) <_{\wedge}^e (1, 2)$ , in particular  $(2, 4) <_{\wedge}^e (1, 4) <_{\wedge}^e (1, 2)$ . In  $D_3$ , by the oriented crossings  $e$  would cross into  $\overrightarrow{(1, 4, 2)}$  at both  $(2, 4)$  and  $(1, 4)$  consecutively, a contradiction.

Therefore,  $u \in \overrightarrow{(1, 4, 2)}_R$ . By the oriented crossings of the edges of  $(1, 2, 4)$  and the fact  $(1, 4) <_{\wedge}^e (1, 2)$ , it would follow that  $e$  crosses  $\overrightarrow{(2, 4)}$  from left to right and  $(2, 4) <_{\wedge}^e (1, 4) <_{\wedge}^e (1, 2)$ . By the oriented crossings of the edges at 4, the fact that  $(2, 4) <_{\wedge}^e (1, 4)$ , and  $(3, 4) <_{\wedge}^e (1, 4)$ , it follows by Lemma 4.7 that  $(2, 4) <_{\wedge}^e (3, 4) <_{\wedge}^e (1, 4)$ .

Since  $(3, 4)$  is the next edge  $e_i$  must cross, it follows that  $e_i$  has already crossed  $(2, 4)$  and  $v_i$  is outside  $\overrightarrow{(1, 4, 2)}_R$ . Since  $F_{(1,2)}$  is contained in  $\overrightarrow{(1, 4, 2)}_R$ ,  $v_i$  is also inside  $\overrightarrow{(1, 4, 2)}_R$ , a contradiction.

**Case 2.**  $e$  crosses  $\overrightarrow{(1, 2)}$  from right to left.

Since  $e$  crosses both  $(1, 2)$  and  $(3, 4)$  towards the crossing in  $D_e$ , it follows by Lemma 4.8 that  $(1, 2)$  and  $(3, 4)$  are ordered by  $<_{\wedge}^e$  relations. If  $(1, 2)$  was the last edge crossed by  $e_i$ , then  $(3, 4)$  would not be the next edge crossed by  $e_i$  as  $(1, 2)$  and  $(3, 4)$  are ordered by  $<_{\wedge}^e$  relations.

It follows that  $(1, 2)$  is not the last edge crossed by  $e_i$ . If  $(1, 2)$  has been crossed by  $e_i$ , then there is a unique drawing up to symmetry of  $D_e + e_i$ . Without loss of generality, this drawings has  $e_i$  crossing  $\overrightarrow{(1, 4)}$ ,  $\overrightarrow{(1, 3)}$ ,  $\overrightarrow{(2, 4)}$ , and  $\overrightarrow{(2, 3)}$  from right to left. The oriented crossings on the edges of  $\overrightarrow{(1, 4, 2)}_R$  and  $\overrightarrow{(1, 3, 4)}_R$  imply that  $v$  is in both regions, however, these two regions have empty intersection, a contradiction with  $D_u$ .

Therefore,  $(1, 2)$  has not been crossed by  $e_i$ . It follows that  $(3, 4) <_{\wedge}^e \dots <_{\wedge}^e (1, 2)$  as  $(3, 4)$  is the next edge that  $e_i$  must cross. By symmetry, without loss of generality, assume  $e$  crosses one of the edges on  $(1, 2, 4)$  between  $(1, 2)$  and  $(3, 4)$ .

Again we can assume without loss of generality that it crosses this edge into  $\overrightarrow{(1, 4, 2)}_R$ . The justification for this is the same as in Case 1.2 with oriented crossings reversed on  $(1, 2, 4)$ .

Therefore, either  $e$  crosses  $y \in \{(1, 4), (2, 4)\}$  into  $\overrightarrow{(1, 4, 2)}_R$  such that  $(3, 4) <_{\wedge}^e y <_{\wedge}^e (1, 2)$ . We partition this into the two cases for  $y \in \{(1, 4), (2, 4)\}$ .

**Case 2.1.**  $y = (1, 4)$ .

By the oriented crossings on  $(1, 3, 4)$  and  $(3, 4) <_{\wedge}^e (1, 4)$  it follows that  $e$  crosses  $\overrightarrow{(1, 3)}$  from right to left and  $(3, 4) <_{\wedge}^e (1, 3) <_{\wedge}^e (1, 4)$ . In particular,  $(1, 3) <_{\wedge}^e (1, 4) <_{\wedge}^e (1, 2)$ , a contradiction with  $x = 1$  in Lemma 4.7.

**Case 2.2.**  $y = (2, 4)$ .

Note that if  $(1, 4)$  is not crossed, then by the oriented crossings of the edges of  $(1, 2, 4)$ ,  $v$  is in  $\overrightarrow{(1, 4, 2)}_R$ . If  $(1, 4)$  is crossed, then by the oriented crossings of the edges of  $(1, 2, 4)$  or  $\overrightarrow{(1, 3, 4)}$ ,  $v$  is in exactly one of  $\overrightarrow{(1, 4, 2)}_R$  or  $\overrightarrow{(1, 3, 4)}_R$ . Therefore,  $v$  is in either  $\overrightarrow{(1, 3, 4)}_R$  or  $\overrightarrow{(1, 4, 2)}_R$ . We consider these two cases separately.

**Case 2.2.1.**  $v \in \overrightarrow{(1, 3, 4)}_R$ .

Since  $v$  is not in  $\overrightarrow{(1, 4, 2)}_R$  and  $u$  is, it follows by the oriented crossings of the edges in  $(1, 2, 4)$  and the fact that  $(2, 4) <^e_{\wedge} (1, 2)$  that  $e$  crosses  $\overrightarrow{(1, 4)}$  from left to right and  $(2, 4) <^e_{\wedge} (1, 2) <^e_{\wedge} (1, 4)$ .

Since  $(3, 4) <^e_{\wedge} (2, 4) <^e_{\wedge} (1, 4)$ , it follows by Lemma 4.4 that  $(3, 4) <^e_{\wedge} (1, 4)$ . These relations along with the oriented crossings of  $(1, 3, 4)$  implies that  $\overrightarrow{(1, 3)}$  is crossed from right to left and that  $(3, 4) <^e_{\wedge} (1, 3) <^e_{\wedge} (1, 4)$ .

By Observation 2.9,  $D_2$  is Harborth and the rotations are determined. In particular,  $\pi_{3,4,u,v}(1) = [3, u, v, 4]$ . If  $u$  was in  $F_{(2,4)}$ , then  $e_i$  crosses  $\overrightarrow{(2, 3)}$  from right to left and  $(2, 3) <^e_{\wedge} (3, 4) <^e_{\wedge} (1, 3)$ , a contradiction with  $x = 3$  in Lemma 4.7.

Therefore,  $u$  is in  $F_{(1,2)}$ . Knowing the rotation at 1 along with the location of  $u$  in  $D_v$ , implies that  $D_v$  is not simple, a contradiction,

**Case 2.2.2.**  $v \in \overrightarrow{(1, 4, 2)}_R$ .

Since both  $u$  and  $v$  are in  $\overrightarrow{(1, 4, 2)}_R$  and not  $\overrightarrow{(1, 3, 4)}_R$ , it follows that  $e$  does not cross  $(1, 4)$  and  $e$  crosses  $\overrightarrow{(1, 3)}$  from right to left. Note that  $(3, 4) <^e_{\wedge} (1, 3)$  as  $\overrightarrow{(1, 3, 4)}_R$  does not contain  $u$  or  $v$  and so  $e$  must cross into the  $\overrightarrow{(1, 3, 4)}_R$  at  $(3, 4)$ , then out at  $(1, 3)$  in  $D_2$ .

By Observation 4.3, it follows that  $\pi_{1,4,u,v}(3) = [u, 4, 1, v]$ . Combining this with the rotation at 3 in  $D_e$  gives  $\pi_{1,2,4,u,v}(3) = [u, 4, 2, 1, v]$ . Applying the same arguments to  $(1, 2, 4)$  and  $(2, 3, 4)$ , and combining the rotations in  $D_e$  gives  $\pi_{1,3,4,u,v}(2) = [u, 4, 3, 1, v]$  and  $\pi_{1,2,3,u,v}(4) = [v, 2, 1, 3, u]$ , respectively.

If  $e$  crosses  $\overrightarrow{(2, 3)}$ , then it does so from left to right as  $u \in F_{(1,2)} \subset \overrightarrow{(1, 3, 2)}_R$ . Since  $(3, 4) <^e_{\wedge} (2, 4)$ , it would follow by the oriented crossings of the edges of  $(2, 3, 4)$  that  $(3, 4) <^e_{\wedge} (2, 4) <^e_{\wedge} (2, 3)$ , in particular  $(2, 4) <^e_{\wedge} (2, 3)$ . Setting  $x = 2$ , Lemma 4.7 implies that  $(2, 4) <^e_{\wedge} (1, 2) <^e_{\wedge} (2, 3)$ . Since  $(1, 2) <^e_{\wedge} (2, 3)$ , it follows by the oriented crossings of the edges of  $(1, 2, 3)$  that  $(1, 3) <^e_{\wedge} (1, 2) <^e_{\wedge} (2, 3)$ . Since order of the edges crossed in  $(1, 2, 3)$  and  $e$  are known, and the oriented crossings of  $e$  with these edges are known, it follows that the rotations of the vertices in  $D_4$  are known. In particular, by Observation 2.9,  $D_4$  is Harborth and  $\pi_{1,3,u,v}(2) = [3, v, u, 1]$ , a contradiction with  $\pi_{1,3,4,u,v}(2) = [u, 4, 3, 1, v]$ .

Therefore,  $e$  does not cross  $(2, 3)$ . It follows by the oriented crossings that both  $u$  and  $v$  are in  $F_{(1,2)}$ . The oriented crossings on  $(1, 2, 4)$  and the positions of  $u$  and  $v$  imply that in  $D_3$ ,  $\pi_{1,4,u,v}(2) = [u, 4, 1, v]$ . Applying the

same arguments to  $(1, 2, 3)$ ,  $(1, 3, 4)$ , and  $(2, 3, 4)$  gives  $\pi_{2,3,u,v}(1) = [v, 2, 3, u]$ ,  $\pi_{1,4,u,v}(3) = [u, 4, 1, v]$ , and  $\pi_{2,3,u,v}(4) = [v, 2, 3, u]$ , respectively. In  $D_u$ ,  $v \in F_{(1,2)}$ , the partial rotation at the vertices  $[4]$ , and the fact that  $D_u$  is simple, implies the rotations at the vertices in  $D_u$  are determined, in particular  $\pi_{1,2,3,4}(v) = [1, 2, 4, 3]$ .

The oriented crossings of  $e$  with  $(2, 4)$  and  $(1, 3)$  give  $\pi_{2,4,u}(v) = [2, u, 4]$  and  $\pi_{1,3,u}(v) = [3, u, 1]$ , respectively. The three rotations at  $v$  can not be combined, a contradiction with the existence of  $\pi_{1,2,3,4,u}(v)$ .

**Case 3.**  $e$  does not cross  $(1, 2)$ .

Up to symmetry, partition this cases into 4 separate cases depending on the number  $i$  of edges  $e_i$  crosses. Note that the Figures from this point on are representative of the information at hand, specifically with the edge  $(3, 4)$  being crosses next by  $\langle_{\wedge}^e$ .

Partition this cases into 5 separate cases up to symmetry on the edges  $e_i$  crosses. If  $i = 0$ , then  $e_i$  is determined in  $D_{e_i}$ . If  $i \geq 1$ , then without loss of generality  $e_i$  crosses  $\overrightarrow{(1, 4)}$  from left to right to enter  $F_{(1,2)}$ . If  $i = 1$ , then  $u$  is in  $F_{(1,3)}$ . If  $i = 2$ , then  $u$  starts in  $F_{(3,4)}$  and  $e_i$  crosses  $(2, 3)$  then  $(1, 4)$  to end in  $F_{(1,2)}$ , or  $u \in F_4$  and  $e_i$  crosses  $(1, 3)$  then  $(1, 4)$  to end in  $F_{(1,2)}$ . If  $i \geq 3$ , then  $e_i$  crosses  $(2, 4)$  then  $(1, 3)$  then  $(1, 4)$  to end in  $F_{(1,2)}$ .

**Case 3.1.**  $i = 0$ .

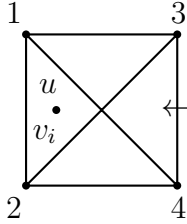


Figure 30:  $i = 0$

This cases gives the least amount of crossing information. At this point we know that  $(3, 4)$  is the next edge  $e$  crosses by  $\langle_{\wedge}^e$  and that  $v_i = v_0 = u$  in this instance is inside  $F_{(1,2)}$  as in Figure 30.

Either  $e$  does not cross  $(1, 4)$  or  $(2, 3)$ , or  $e$  does cross  $(1, 4)$  or  $(2, 3)$ . If  $e$  does cross  $(1, 4)$  or  $(2, 3)$ , then we can choose the labelling so that it

crosses  $(1, 4)$ . In this instance, we split into two cases depending on the oriented crossing and therefore the 3 cases are  $e$  does not cross  $(1, 4)$  or  $(2, 3)$ ,  $e$  crosses  $\overrightarrow{(1, 4)}$  from right to left, or  $e$  crosses  $\overrightarrow{(1, 4)}$  from left to right.

**Case 3.1.1.**  $e$  does not cross  $(1, 4)$  or  $(2, 3)$ .

At this point note that  $e$  must cross one of  $(1, 3)$  or  $(2, 4)$ , as if it did not, then the oriented crossings (or lack there of) would imply that  $v$  is in both  $F_{(3,4)}$  and  $F_{(1,2)}$ , a contradiction. Since  $(1, 3)$  and  $(2, 4)$  are symmetric up to relabelling, we assume without loss of generality  $e$  crosses  $\overrightarrow{(2, 4)}$  from left to right (note the direction  $e$  crosses  $(2, 4)$  is determined). Partition this into two final cases depending on  $e$  crossing  $(1, 3)$  or not.

**Case 3.1.1.1.**  $e$  crosses  $(1, 3)$ .

Since  $u$  is in  $\overrightarrow{(1, 3, 2)}_R$ ,  $e$  must cross  $\overrightarrow{(1, 3)}$  from right to left since it is the only edge crossed on  $(1, 2, 3)$ . Since  $(3, 4)$  is the next edge  $e_i$  must cross, it follows that  $(3, 4) <_{\wedge}^e (1, 3)$  and  $(3, 4) <_{\wedge}^e (2, 4)$ . We can also note that by the oriented crossings that  $v \in F_4$ .

By the oriented crossings of the edges in  $(1, 3, 4)$  and  $(2, 3, 4)$  and the order these edges are crossed by  $e$ , it follows that in  $D_2$  and  $D_1$  that  $\pi_{1,4,u,v}(3) = [u, 4, 1, v]$  and  $\pi_{2,3,u,v}(4) = [v, 2, 3, u]$ , respectively. Combining these with their respective rotations in  $D_e$  gives  $\pi_{1,2,3,u,v}(4) = [v, 2, 1, 3, u]$  and  $\pi_{1,2,4,u,v}(3) = [u, 4, 2, 1, v]$ .

Note that all the oriented crossings of the edges in  $(1, 2, 4)$  with  $e$  are known, along with the rotation at 4. It follows that  $D_3 - \{(1, v), (2, v)\}$  is determined. In particular,  $\pi_{2,4,u}(1) = [4, u, 2]$  and  $\pi_{1,2,4,v}(u) = [1, v, 2, 4]$ . Similarly on  $(1, 2, 3)$  in  $D_4 - \{(1, v), (2, v)\}$ ,  $\pi_{1,3,u}(2) = [1, u, 3]$  and  $\pi_{1,2,3,v}(u) = [2, 3, 1, v]$ .

The oriented crossing of  $e$  with  $(3, 4)$  gives  $\pi_{3,4,v}(u) = [4, v, 3]$ . Combining the rotations at  $u$  gives  $\pi_{1,2,3,4,v}(u) = [1, v, 2, 3, 4]$ . The rotations at 1, 2, 3 and 4 along with  $u \in F_{(1,2)}$  implies that  $D_v$  is determined. In particular,  $\pi_{1,2,3,4}(u) = [1, 2, 4, 3]$ , a contradiction with  $\pi_{1,2,3,4,v}(u) = [1, v, 2, 3, 4]$ .

**Case 3.1.1.2.**  $e$  does not cross  $(1, 3)$ .

Since  $(3, 4)$  is the next edge that  $e_i$  crosses, it is clear that  $(3, 4) <_{\wedge}^e (2, 4)$ .



By the oriented crossings of the edges on  $(2, 3, 4)$  and the order the edges are crossed, it follows that in  $D_1$ ,  $\pi_{2,3,u,v}(4) = [v, 2, 3, u]$ . Combining the rotation of 4 from  $D_e$  results in  $\pi_{1,2,3,u,v}(4) = [v, 2, 1, 3, u]$ . In  $D_3$ , knowing the locations of  $u$  and  $v$  relative to  $(1, 2, 4)$ ,  $e$  only crosses  $(2, 4)$  and the rotation at 4, implies  $D_3 - \{(1, v), (2, v)\}$  is uniquely determined, in particular,  $\pi_{2,4,u}(1) = [2, 4, u]$  and  $\pi_{1,2,4,v}(u) = [1, v, 2, 4]$ .

Apply the same argument to  $(1, 3, 4)$  and  $D_2 - \{(1, u)(3, u)\}$  gives  $\pi(1) = [3, v, 4]$ . From  $D_e$ ,  $\pi_{2,3,4}(1) = [3, 4, 2]$ . Combining the rotations at 1 results in  $\pi_{2,3,4,u,v}(1) = [2, 3, v, 4, u]$ . The rotation at 1 implies that  $D_2 - \{(3, u)\}$  is uniquely determined from  $D_2 - \{(1, u), (3, u)\}$ , in particular  $\pi_{1,4,v}(u) = [1, 4, v]$ , a contradiction with  $\pi_{1,2,4,v}(u) = [1, v, 2, 4]$ .

**Case 3.1.2.**  $e$  crosses  $\overrightarrow{(1, 4)}$  from right to left.

Since  $(3, 4)$  is the next edge  $e_i$  crosses, it follows that  $(3, 4) <_{\wedge}^e (1, 4)$ . Since  $e$  crosses both  $(1, 4)$  and  $(3, 4)$  into  $\overrightarrow{(1, 3, 4)}_R$ , it follows that  $e$  crosses  $\overrightarrow{(1, 3)}$  from right to left and  $(3, 4) <_{\wedge}^e (1, 3) <_{\wedge}^e (1, 4)$ . Since the order that  $e$  crosses the edges on  $(1, 3, 4)$  is known and the oriented crossings are known, it follows that  $D_2$  is uniquely drawn. In particular, by Observation 2.9,  $D_2$  is Harborth and  $\pi_{3,4,u,v}(1) = [3, u, v, 4]$ . Given the rotation at 1 and the fact  $u$  is in  $F_{(1,2)}$  in  $D_e$ , it follows that  $D_v$  is not simple, a contradiction.

**Case 3.1.3.**  $e$  crosses  $\overrightarrow{(1, 4)}$  from left to right.

Since  $u$  is in  $\overrightarrow{(1, 4, 2)}_R$ , it follows that  $e$  crosses  $\overrightarrow{(2, 4)}$  from left to right and  $(2, 4) <_{\wedge}^e (1, 4)$ . Since  $i = 0$  and  $(3, 4)$  is the next edge  $e_i$  crosses, it follows that  $(3, 4) <_{\wedge}^e (2, 4) <_{\wedge}^e (1, 4)$ , a contradiction with  $x = 4$  in Lemma 4.7.

**Case 3.2.**  $i = 1$ .

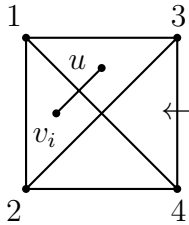


Figure 31:  $i = 1$

In this instance,  $e_i$  has already crossed one of  $(1, 4)$  or  $(2, 3)$  into  $F_{(1,2)}$ . Up to relabelling, these are the same case, therefore we assume without loss of generality that  $e_i$  crosses  $\overrightarrow{(1, 4)}$  from left to right into  $F_{(1,2)}$  as in Figure 31.

By the oriented crossings on the edges of  $(1, 2, 4)$ , either  $(2, 4)$  is not crossed by  $e$ , or  $e$  crosses  $\overrightarrow{(2, 4)}$  from left to right.

**Case 3.2.1.**  $e$  does not cross  $(2, 4)$ .

Since  $(1, 4)$  is crossed by  $e_i$ , it follows that  $(1, 4) <_{\wedge}^e (3, 4)$ . Since the order  $e$  crosses  $(1, 4)$  and  $(3, 4)$  is known along with the oriented crossings of these edges with  $e$ , it follows that in  $D_2$ ,  $\pi_{1,3,u,v}(4) = [v, 3, 1, u]$ . Combining this with the rotation of 4 in  $D_e$  gives  $\pi_{1,2,3,u,v}(4) = [v, 3, 2, 1, u]$ .

The oriented crossings of  $(1, 2, 4)$  imply that  $v \in \overrightarrow{(1, 4, 2)}_R$ . This along with the rotation at 4 imply that  $v \in F_{(1,2)}$  as  $D_u$  is a simple drawing. This implies that  $v$  is not in  $\overrightarrow{(2, 3, 4)}_R$  and by the oriented crossing of the edges in  $(2, 3, 4)$  that  $e$  crosses  $\overrightarrow{(2, 3)}$  from right to left. Applying the same arguments to  $(1, 2, 3)$  shows that  $e$  crosses  $\overrightarrow{(1, 3)}$  from right to left. Since  $u$  and  $v$  are both in  $\overrightarrow{(1, 3, 2)}_R$ , it follows by the oriented crossings of the edges that  $(1, 3) <_{\wedge}^e (2, 3)$ . Since  $e_i$  does not cross  $(1, 3)$  and  $(3, 4)$  is the next edge that  $e_i$  crosses, we have  $(3, 4) <_{\wedge}^e (1, 3)$ . This implies  $(3, 4) <_{\wedge}^e (1, 3) <_{\wedge}^e (2, 3)$ , a contradiction with  $x = 3$  in Lemma 4.7.

**Case 3.2.2.**  $e$  crosses  $\overrightarrow{(2, 4)}$  from left to right.

Since  $e_i$  crosses  $(1, 4)$ ,  $e_i$  does not cross  $(2, 4)$  and the next edge that  $e_i$  crosses is  $(3, 4)$ , it follows that  $(1, 4) <_{\wedge}^e (3, 4) <_{\wedge}^e (2, 4)$ , a contradiction with  $x = 4$  in Lemma 4.7.

**Case 3.3.**  $i = 2$ .

Without loss of generality  $e_i$  crosses  $\overrightarrow{(1, 4)}$  from left to right into  $F_{1,2}$  as in Case 3.2. Before this crossing occurs, there are two options for how  $e_i$  enters  $F_{(1,3)}$ , either  $e_i$  crosses  $\overrightarrow{(2, 3)}$  from right to left, and  $u$  is in  $F_{(3,4)}$  or  $e_i$  crosses  $\overrightarrow{(1, 3)}$  from left to right, and  $u$  is in  $F_4$ .

**Case 3.3.1.**  $e_i$  crosses  $\overrightarrow{(2,3)}$  from right to left, and  $u$  is in  $F_{(3,4)}$ .

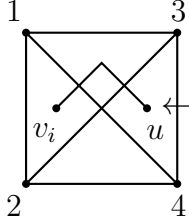


Figure 32:  $i = 2$  and  $e_i$  crosses  $(2,3)$ .

If  $e$  crosses  $\overrightarrow{(2,4)}$ , then it does so from left to right as the oriented crossings of the remaining edges of  $(2,3,4)$  are determined and  $(2,3)$  is the first of the three edges  $e$  crosses. Since  $e_i$  crosses  $(1,4)$ ,  $e_i$  does not cross  $(2,4)$  and the next edge  $e_i$  crosses is  $(3,4)$ , it would follow that  $(1,4) <_{\wedge}^e (3,4) <_{\wedge}^e (2,4)$ , a contradiction with  $x = 4$  with Lemma 4.7.

Therefore,  $e$  does not cross  $(2,4)$ . By the oriented crossings of  $(2,3,4)$ ,  $v \in \overrightarrow{(2,3,4)}_R$  and is not in  $\overrightarrow{(1,3,2)}_R$ . From the oriented crossings of  $(1,2,3)$  and locations of  $u$  and  $v$ , it follows that  $e$  crosses  $\overrightarrow{(1,3)}$  from right to left. Since  $e_i$  crosses  $(2,3)$ ,  $e_i$  does not cross  $(1,3)$  and the next edge  $e_i$  crosses is  $(3,4)$ , it follows that  $(2,3) <_{\wedge}^e (3,4) <_{\wedge}^e (1,3)$ , a contradiction with  $x = 3$  in Lemma 4.7.

**Case 3.3.2.**  $e_i$  crosses  $\overrightarrow{(1,4)}$  from left to right,  $e_i$  crosses  $\overrightarrow{(1,3)}$  from left to right, and  $u$  is in  $F_4$ .

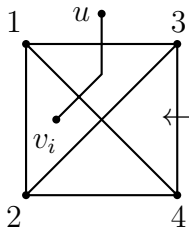


Figure 33:  $i = 2$  and  $e_i$  crosses  $(1,3)$ .

By the oriented crossings of  $\overrightarrow{(1,3,4)}$ ,  $v$  is in  $\overrightarrow{(1,3,4)}_R$ . Since  $\overrightarrow{(1,3,4)}_R$  and  $\overrightarrow{(1,4,2)}_R$  have empty intersection, it follows that  $v$  is not in  $\overrightarrow{(1,4,2)}_R$ . Since

$v$  is not in  $\overrightarrow{(1, 4, 2)}_R$ ,  $e_i$  crosses into  $\overrightarrow{(1, 4, 2)}_R$ , and  $e$  does not cross  $(1, 2)$ , it follows that  $e$  crosses  $\overrightarrow{(2, 4)}$  from left to right.

Since  $e_i$  crosses  $(1, 4)$ ,  $e_i$  does not cross  $(2, 4)$ , and the next edge  $e_i$  crosses is  $(3, 4)$ , it follows that  $(1, 4) <_{\wedge}^e (3, 4) <_{\wedge}^e (2, 4)$ , a contradiction with  $x = 4$  in Lemma 4.7.

**Case 3.4.**  $i \geq 3$ .

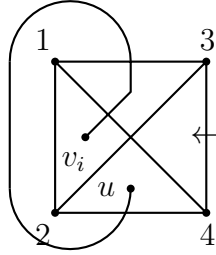


Figure 34:  $i = 3$ .

Since  $i = 3$ , the only possibility for the edges that  $e_i$  crosses is  $e_i$  crossing  $(2, 4)$  then  $(1, 3)$  then  $(1, 4)$  and ends in  $F_{(1,2)}$  as in Figure 34.  $e_i$  crosses  $(2, 4)$ , then  $(1, 4)$  and does not cross  $(3, 4)$ . It follows that  $(2, 4) <_{\wedge}^e (1, 4) <_{\wedge}^e (3, 4)$ , a contradiction with  $x = 4$  in Lemma 4.7.  $\square$

**Lemma 4.13.** *Let  $H$  be a  $(6, 5)$  rotation system on the vertices  $[4] \cup \{u, v\}$ ,  $e = \overrightarrow{(u, v)}$ , and  $E$  be the set of edges that  $e$  crosses determined by  $H$ . If  $D_v$  is a realization of  $H - \{v\}$ , then there exists a simple drawing  $D_v + \{e\}$  that has  $v$  in its respective face in  $D_v$  determined by  $H$ ,  $e$  crosses exactly the edges of  $E$  satisfying  $<_{\wedge}^e$ , the oriented crossings involving  $e$  are in the prescribed orientations determined by  $H$ , and the rotation at  $u$  is the same in both  $H$  and  $D_v + \{e\}$ .*

*Proof.* Let  $H$  be a  $(6, 5)$ -rotation system on the vertices  $[4] \cup \{u, v\}$ ,  $e = \overrightarrow{(u, v)}$ ,  $D_i$  be a realization of  $H - \{i\}$  for  $i \in ([4] \cup \{u, v\})$ ,  $D_e$  be a realization of  $H - \{u, v\}$ ,  $E$  be the set of edges that  $e$  crosses determined by  $H$ , and  $\{c_j\}_{j=1}^{cr_H(e)}$  be the sequence in which  $e$  crosses the edges of  $E$  in Lemma 4.12.

Let  $e_i$  be the segment of  $e$  starting at  $u$  that has exactly  $i$  crossings for  $i \in [cr_H(e)]_0$ , and  $D_{e_i}$  be the drawing of  $D_v + e_i$ . We will prove  $D_{e_i}$  exists for all  $i \in [cr_H(e)]_0$  inductively on  $i$ .

Note that there is no need to talk about the order  $e$  crosses edge related to  $<_{\wedge}^e$  as this is satisfied by the definition of  $\{c_j\}_{j=1}^{cr_H(e)}$ . By adding a small segment at the correct rotation at  $u$  in  $D_v$ , we find a simple drawing of  $D_{e_0}$ .

Assume  $D_{e_i}$  exists for some  $i \in [cr_H(e) - 1] \cup \{0\}$ . By way of contradiction, assume  $D_{e_{i+1}}$  does not exist. Let  $R_{v_i}$  be the face in  $D_v$  containing the non-vertex end of  $e_i$ . There are two cases depending upon the existence of  $c_{i+1}$ .

**Case 1.**  $c_{i+1}$  does not exist.

If  $c_{i+1}$  does not exist, then the edge  $e_i$  has crossed all edges of the sequence  $\{c_j\}_{j=1}^{cr_H(e)}$  in  $D_{e_i}$ . Let  $T$  be the intersection of sides of triangles in  $D_v$  containing  $v$  determined by  $H$ . If  $R_{v_i}$  does not correspond to  $T$ , then by Corollary 3.3, there is some 3-cycle having  $e_i$  and  $v$  on separate sides, a contradiction with the induced 5-vertex rotation system in  $H$  being realizable.

It follows that  $R_{v_i}$  correspond to  $T$ . Drawing the vertex  $v$  at the end of  $e_i$  in  $D_{e_i}$  produces the desired simple drawing  $D_v + \{e\}$ .

**Case 2.**  $c_{i+1}$  exists.

Let  $(z, u_{i+1}, v_{i+1})$  be some 3-cycle containing  $c_{i+1} = (u_{i+1}, v_{i+1})$  in  $D_{e_i}$ ,  $y \in V(H) \setminus \{z, u_{i+1}, v_{i+1}, u, v\}$ , and let  $F_y$  be the induced drawing of  $D_{e_i}$  without  $y$ . Since  $F_y$  is defined on five vertices, it is clear  $F_y$  extends to a realization of  $H_y$ . Since  $c_{i+1}$  is the next edge crossed in  $F_y$ , it follows that  $e_i$  is on the correct side  $S$  of  $(z, u_{i+1}, v_{i+1})$  to cross  $c_{i+1}$  in the correct orientation to extend the drawing. Note that  $R_{v_i}$  is contained in  $S$  since  $c_{i+1}$  is the next edge crossed in  $F_y$  and  $S$  is defined in  $D_{e_i}$ .

By Lemma 3.4, the only segment of  $e_i$  in  $R_{v_i}$  is the segment formed by  $e_{i-1}$  crossing into  $R_{v_i}$ . It follows that if  $c_{i+1}$  is on the boundary of  $R_{v_i}$ , then  $e_i$  can be extended in  $D_{e_i}$  to cross  $c_{i+1}$  in the correct orientation to create  $D_{e_{i+1}}$ , a contradiction.

Therefore,  $c_{i+1}$  is not on the boundary of  $R_{v_i}$ . By Theorem 3.8, since there are no Reidemeister III moves in  $D_v$ , there is some drawing  $\mathcal{D}$  of  $D_v$  on a  $K_4$  containing  $c_{i+1}$  such that no face of  $\mathcal{D}$  contains  $R_{v_i}$  and has  $c_{i+1}$  on its boundary.

Consider the drawing  $F$  of  $D_{e_i}$  containing  $\mathcal{D}$  and  $e_i$ . If both  $u$  and  $v$  are not in  $\mathcal{D}$ , then by definition of  $\{c_j\}_{j=1}^{cr_H(e)}$  and Lemma 4.12,  $F$  can be extended

with  $e_i$  crossing  $c_{i+1}$ , a contradiction with no face of  $\mathcal{D}$  containing  $R_{v_i}$  and having  $c_{i+1}$  on its boundary, and  $\mathcal{D}$  being contained in  $F$ .

Therefore, at least one of  $u$  or  $v$  is in  $\mathcal{D}$ . It follows that  $\mathcal{F}$  is a partial drawing of some realizable 5-vertex rotation system  $Z$  of  $H$  containing  $u, v, u_{i+1}, v_{i+1}$ . Thus  $\mathcal{F}$  extends to a realization of  $Z$ , with  $e_i$  crossing  $c_{i+1}$  next, a contradiction with no face of  $\mathcal{D}$  containing  $R_{v_i}$  and having  $c_{i+1}$  on its boundary.  $\square$

To find a realization of a  $(6, 5)$ -rotation system, we would like to apply Lemma 4.13 to all possible edges incident to one vertex  $v$ , then glue the drawings along their common  $K_5$ . However, such a strategy could create a drawing that is not simple since two edges having  $v$  as an endpoint could cross multiple times. Therefore, we first create a lemma that finds structure in two tangled edges incident to the same vertex.

At this point, I would like to thank Bruce Richter for offering a beautiful proof of the following lemma about structure in tangled edges.

**Lemma 4.14.** *Let  $D$  be a drawing of a path of length 2. If the two edges have a finite positive number of crossings, then there exists a face of  $D$  whose boundary is exactly a non-trivial segment of each edge.*

*Proof.* Let  $D$  be a drawing of a path of length 2 with  $e = (v, u_1)$  and  $f = (v, u_2)$  being the two crossing edges. Traversing  $f$  from  $v$  to  $u_2$ , let the crossings of  $e$  and  $f$  be labelled  $x_1 = v, x_2, \dots, x_k$  for some  $k \in \mathbb{Z}$ . Define  $B_i$  to be the bigon on  $e[x_i, x_{i+1}] \cup f[x_i, x_{i+1}]$  and  $S_i$  to be the closed side of  $B_i$  that does not contain  $u_1$ .

**Claim 1.** *There exists an  $i$  such that  $u_1, u_2, v$  are all in  $S_i$ .*

*Proof.* Either every crossing of  $e$  and  $f$  is in the same direction, or for some  $i$ ,  $x_i$  and  $x_{i+1}$  are crossings of  $e$  and  $f$  in opposing directions.

**Case 1.** *Every crossing of  $e$  and  $f$  is in the same direction.*

Consider  $B_1$ .  $e$  only crosses  $f[x_1, x_2]$  at its ends. For  $f$  to cross  $e[x_1, x_2]$  it would have to do it at least once in the opposite direction of the crossing at  $x_2$ . Since this is not the case,  $f$  does not cross  $e[x_1, x_2]$  and  $B_1$  is our desired bigon.

**Case 2.** *There exists an  $i$  such that  $x_i$  and  $x_{i+1}$  are crossings of  $e$  and*

$f$  in opposing directions.

Topologically, there is one way to draw  $f[x_i, x_{i+1}]$  relative to  $e$ , and that is to have  $v$  not in  $S_i$ . Direct  $e$  from  $v$  to  $u_1$  and let  $e_L$  and  $e_R$  be the sides of  $e$  ( $L$  and  $R$  represent left and right, respectively, or right and left respectively). Without loss of generality  $e_R$  bounds  $B_i$  and there is no path from any point in  $S_i$  to  $e_L$  (if we consider edges to have width  $\epsilon$ , then this is natural). By way of contradiction,  $u_2$  is in  $S_i$ , else  $B_i$  is our desired bigon..

For  $i + 1 < j$ , let  $x_j$  be the next time  $f$  crosses  $e$  in the same direction as the crossing of  $x_1$ . We know such a  $j$  exists since  $f$  must cross  $e_R$  into  $B_i$  to end at  $u_2$ . It follows that  $B_{j-1}$  is a bigon with  $S_{j-1}$  not containing  $v$ . By the orientation of the crossings at  $x_{j-1}$  and  $x_j$  it follows that  $S_{j-1}$  is bounded by  $e_L$  and there is no path from a point in  $S_{j-1}$  to  $e_R$ .

The only way for  $B_i$  to intersect  $B_j$  would be for  $f[x_{j-1}, x_j]$  to intersect  $B_i$ . It is clear this does not happen on  $f$  unless  $x_{i+1} = x_{j-1}$ , in which case the intersection is at  $x_{i+1}$ . Since  $f[x_{j-1}, x_j]$  contains only intersection of  $e$  on  $e_L$ , it follows that  $B_i$  intersects  $B_{j-1}$  on at most  $x_{i+1}$  (Again, we distinguish between  $e_L$  and  $e_R$ ).

If  $u_s$  is not in  $S_{j-1}$ , then  $B_{j-1}$  is our desired bigon. Therefore  $u_2$  is in  $S_{j-1}$ . It follows that one  $u_1$ -empty side of a bigon is contained in the  $u_1$ -empty side of the other bigon. Without loss of generality, assume  $S_{j-1} \subset S_i$ , as the opposing case is completely analogous. It follows that  $u_2$  has a path from itself to both  $e_L$  and  $e_R$  in  $S_i$ , a contradiction. ■

Let  $N_j$  be the number of bigons in  $S_j$  that are described in Claim 1. Let  $B_i$  be a bigon as described in Claim 1 such that  $N_i$  is minimum. If  $e[x_i, x_i + 1]$  has a crossing with  $f$  other than its ends, then it is at some  $x_j$ . Since  $u_2$  is not in  $S_i$ , it follows that  $x_{j+1}$  is determined and  $B_j$  is a bigon as in Claim 1 with  $S_j \subset S_i$  and  $N_j < N_i$ , a contradiction.

Therefore,  $e[x_i, x_i + 1]$  is uncrossed,  $f[x_i, x_i + 1]$  is uncrossed by definition, and  $S_i$  contains none of  $u_1, u_2, v$ . It follows that  $S_i$  is a face with  $B_i$  bounding it, as desired. □

To finish this section and prove Lemma 4.15, we use Lemma 4.13 to form a not necessarily simple drawing  $D$  of  $H$  that has the rotations at each vertex the same as in  $H$ , then use Lemma 4.14 to find the associated simple drawing that realizes  $H$ .

**Lemma 4.15.** *If  $H_6$  is a (6, 5)-rotation system, then  $H_6$  is realizable.*

*Proof.* Let  $H_6$  be a  $(6, 5)$ -rotation system,  $v$  be a vertex of  $V(H_6)$ ,  $H_v = H_6 - v$ ,  $e_i = (v, u_i)$  be the edges in  $H_6$  having  $v$  as an endpoint, and  $D_v$  be a realization of  $H_v$ . Since  $H_v$  is a 5-vertex rotation system,  $D_v$  is the unique simple drawing realizing  $H_v$ .

By Lemma 4.13, there are simple drawings  $D_v + e_i$  such that have  $v$  in its respective face in  $D_v$  determined by  $H$ ,  $e_i$  crosses exactly the edges determined by  $H_6$  in their prescribed orientations in  $D_v + e_i$ , and the rotations at  $u_i$  in each of the respective drawings is the same as in  $H_6$ . Since there is a unique way to draw  $D_v$ , it follows we can glue all the  $D_v + e_i$  along  $D_v$ . Call this new drawing,  $D$ . If  $D$  is simple, then we are done.

Therefore,  $D$  is not simple. Since  $D_v + e_i$  is simple for every  $i$ , it follows that the reason  $D$  is not simple is because two edges  $e_i$  and  $e_j$  cross. By Lemma 4.14, there exists a closed curve  $\delta$  comprised of a non-trivial segment  $e_{i_c}$  of  $e_i$  and a non-trivial segments  $e_{j_c}$  of  $e_j$  such that a side  $S$  of  $\delta$  has empty intersection with  $e_i$  and  $e_j$ . In  $D$ ,  $S$  could contain a vertex.

**Case 1.**  $S$  contains a vertex  $z$ .

Since the intersection of  $S \cap e_i$  and  $S \cap e_j$  are empty, it follows that  $u_i$  and  $u_j$  are not in  $S$ . In  $D$ , it must be the case that  $(z, u_i)$  crosses  $\delta$ . Since  $D$  contains the drawing of  $D_v + e_i$ , it follows that  $(z, u_i)$  crosses  $e_j$  in  $D$ , and thus in  $D_v + e_j$ . Similarly,  $(z, u_j)$  crosses  $e_i$  in  $D$ , and thus in  $D_v + e_i$ . By definition of  $D_v + e_i$  and  $D_v + e_j$ ,  $H_6$  has  $(z, u_i)$  crossing  $e_j$  and  $(z, u_j)$  crossing  $e_i$ . The rotation on  $v, z, u_i$  and  $u_j$  is realizable, a contradiction with simple drawings of  $K_4$  having at most one crossing.

**Case 2.**  $S$  contains no vertices.

Let  $f$  be an edge that crosses  $\delta$ . Since  $S$  contains no vertices, it follows that  $f$  crosses  $\delta$  twice, once at  $e_{i_c}$  and once at  $e_{j_c}$ . It follows that an edge crosses  $e_{i_c}$  if and only if it crosses  $e_{j_c}$ .

Let  $\bar{e}_i$  be the edge  $e_i$  rerouted to take  $e_{j_c}$  instead of  $e_{i_c}$ . Similarly, let  $\bar{e}_j$  to be the edge  $e_j$  rerouted to take  $e_{i_c}$  instead of  $e_{j_c}$ . Since  $e_{i_c}$  and  $e_{j_c}$  cross the same edges, it follows that  $\bar{e}_i$  and  $\bar{e}_j$  cross the same edges as  $e_i$  and  $e_j$ , respectively.

Consider the new drawing of  $\bar{D}$  that has  $\bar{e}_i$  replacing  $e_i$  and  $\bar{e}_j$  replacing  $e_j$  from  $D$  with the crossings on  $\delta$  uncrossed.

If  $\delta$  has  $v$  on the boundary, then the number of crossings between  $(v, u_i)$



and  $(v, u_j)$  has reduced by 1. If  $\delta$  does not have  $v$  on the boundary, then the number of crossings between  $(v, u_i)$  and  $(v, u_j)$  has reduced by 2.

Comparing the rotations of the vertices, only the rotation at  $v$  could have possibly changed. Repeatedly applying this procedure results in a drawing  $\mathcal{D}$  that is simple, and has all the rotations at every vertex the same as  $H$  other than  $v$ . By Observation 2.6, the rotation at  $v$  at a triple of vertices is the same in both  $H$  and  $\mathcal{D}$ . It follows that the rotation at  $v$  is the same in both  $H$  and  $\mathcal{D}$ . Therefore,  $\mathcal{D}$  is a realization of  $H$ , as desired.  $\square$

## 5 Acyclic Orderings from Rotation Systems

The beginning of Section 4 showed that the  $<_{\wedge}^e$  relation on a fixed edge  $e$  induces an acyclic graph for  $(6, 5)$ -rotation systems. The purpose of this section is to show that the  $<_{\wedge}^e$  relation and  $<_{\parallel}^e$  relation on a fixed edge  $e$  induces an acyclic graph on  $(n, n - 1)$ -rotation systems. We partition this into sections depending on the value of  $n$ . In Section 5.1 we show this for  $(7, 6)$ -rotation systems, in Section 5.2 we show this for  $(8, 7)$ -rotation system, and in Section 5.3 we show this on a  $(n, 8)$ -rotation systems for  $n \geq 9$ , which is a stronger conclusion than required.

### 5.1 7-vertex Rotation Systems

The main goal of this section is to prove that  $<_{\wedge}^e$  and  $<_{\parallel}^e$  induce an acyclic directed graph for  $(7, 6)$  rotation systems. To that end, we want to show the following theorem:

**Theorem 5.1.** *If  $H$  is a  $(7, 6)$ -rotation system, and  $e$  is a directed edge of  $H$ , then there are no cycles comprised of  $<_{\wedge}^e$  and  $<_{\parallel}^e$  relations in  $H$ .*

We prove this result by showing if such a cycle existed, then Lemma 5.3 shows no two consecutive relations can be a  $<_{\parallel}^e$  relation. From this, we show that up to relabelling one of the cycles in Lemma 5.4 and 5.5 occurs in the graph for a contradiction. Before we prove these lemmas, we require a small observation to shorten the arguments.

**Observation 5.2.** *Let  $H$  be a  $(7, 6)$ -rotation system,  $e = \overrightarrow{(u, v)}$  be a directed edge of  $H$ ,  $V(H) = [5] \cup \{u, v\}$ , and  $\mathcal{C} = (a_0, \dots, a_{k-1}, a_0)$  be a cycle of  $<_{\wedge}^e$  and  $<_{\parallel}^e$  relations such that  $a_i <^e a_{i+1}$ , for all  $i \in \mathbb{Z}_k$ .*

*For all  $i \in \mathbb{Z}_k$ , if  $a_i$  and  $a_{i+2}$  are adjacent, then either:*

- $V(\{a_i, a_{i+1}, a_{i+2}\}) = [5]$ ; or
- $(a_0, \dots, a_i, a_{i+2}, \dots, a_{k-1}, a_0)$  is a cycle of  $<_{\wedge}^e$  and  $<_{\parallel}^e$  relations.

*Proof.* Suppose there exists some  $i$  such that  $a_i$  and  $a_{i+2}$  are adjacent, and  $V(\{a_i, a_{i+1}, a_{i+2}\}) \subseteq [5] \setminus \{j\}$ , for some  $j \in [5]$ . Let the simple drawing  $D_j$  be a realization of  $H - \{j\}$  for  $j \in V(H)$ . It follows that  $a_i <^e a_{i+1} <^e a_{i+2}$  in  $D_j$ , and that  $a_i <_{\wedge}^e a_{i+2}$ . Therefore, there exists some cycle of relations  $\bar{\mathcal{C}} = (a_0, \dots, a_i, a_{i+2}, \dots, a_{k-1}, a_0)$ , as desired.  $\square$

**Lemma 5.3.** *Let  $H$  be a  $(7, 6)$ -rotation system,  $e = \overrightarrow{(u, v)}$  be a directed edge of  $H$ . If  $\mathcal{C} = (a_0, \dots, a_{k-1}, a_0)$  is a minimal cycle of  $<_{\wedge}^e$  and  $<_{\parallel}^e$  relations such that  $a_i <^e a_{i+1}$ , for all  $i \in \mathbb{Z}_k$ , then for all  $j \in \mathbb{Z}_k$ , one of  $a_j <^e a_{j+1}$  or  $a_{j+1} <^e a_{j+2}$  is a  $<_{\wedge}^e$  relation.*

*Proof.* Define the simple drawings  $D_i$  to be a realization of  $H - \{i\}$  for  $i \in V(H)$ , and  $D_e$  to be a realization of  $H - \{u, v\}$ .

By way of contradiction, suppose both  $a_i <_{\parallel}^e a_{i+1}$  and  $a_{i+1} <_{\parallel}^e a_{i+2}$ . Without loss of generality,  $a_i = (1, 2)$  and  $a_{i+1} = (3, 4)$ . By Observation 5.2, it follows that  $a_{i+2} \in \{(1, 5), (2, 5)\}$ . Since vertices 1 and 5 are symmetric, without loss of generality pick  $a_{i+2} = (1, 5)$ . By minimality of  $\mathcal{C}$ ,  $a_{i+2} <_{\wedge}^e a_i$ . To prove this claim, we must find a contradiction to the relations  $(1, 2) <_{\parallel}^e (3, 4) <_{\parallel}^e (1, 5) <_{\wedge}^e (1, 2)$ . Note that vertex 3 and 4 are symmetric in these relations, therefore there are at most two cases to draw the partial  $K_6$  illustrating the relation  $(1, 2) <_{\parallel}^e (3, 4)$  determined by the direction  $e$  crosses  $\overrightarrow{(1, 2)}$ . In actuality, there is one case as the inverse rotation system flips the oriented crossings of edges and preserves the relation of edge crossings over any fixed edge. By applying the analysis of the first case to the inverse rotation system, the result would be the second case. Without loss of generality assume  $e$  crosses  $\overrightarrow{(1, 2)}$  from right to left.

$D_5$  is determined by the relation  $(1, 2) <_{\parallel}^e (3, 4)$  with  $e$  crossing  $\overrightarrow{(1, 2)}$  from right to left. By the symmetry of vertices 3 and 4 we can assume without loss of generality that  $e$  crosses  $\overrightarrow{(3, 4)}$  from left to right. It follows that  $H - \{u, v, 5\}$  is determined and that  $\overrightarrow{(1, 3)}$  crosses  $\overrightarrow{(2, 4)}$  from right to left.

In  $D_2$ , consider when  $e$  crosses the edges of  $\overrightarrow{(1, 3, 4)}$  and  $(1, 5)$  after crossing the edge  $(3, 4)$ . For  $x \in \{(1, 3), (1, 4)\}$ , if  $(3, 4) \prec_{D_2}^e x \prec_{D_2}^e (1, 5)$ , then  $(1, 2) <_{\parallel}^e (3, 4) <_{\wedge}^e x <_{\wedge}^e (1, 5) <_{\wedge}^e (1, 2)$ . By Observation 5.2, this can be reduced to  $(1, 2) <_{\parallel}^e (3, 4) <_{\wedge}^e x <_{\wedge}^e (1, 2)$ , a contradiction with the existence of  $D_5$ .

It follows that the crossing of  $e$  and  $(1, 5)$  is in  $\overrightarrow{(1, 3, 4)}_R$  as  $e$  crosses  $(3, 4)$  into  $\overrightarrow{(1, 3, 4)}_R$  then crosses  $(1, 5)$ . Since  $(3, 4)$  and  $(1, 5)$  are  $<_{\parallel}^e$  related, they do not cross, and the edge  $(1, 5)$  is contained in  $\overrightarrow{(1, 3, 4)}_R$ , in particular, the vertex 5 is contained in  $\overrightarrow{(1, 3, 4)}_R$ .

In  $D_e$ , the vertex 5 is not in  $\overrightarrow{(1, 2, 3)}_R$  as  $\overrightarrow{(1, 2, 3)}_R \cap \overrightarrow{(1, 3, 4)}_R = \emptyset$ . From  $D_5$ ,  $\pi_{2,3,4}(1) = [2, 3, 4]$  and from  $D_2$ ,  $\pi_{3,4,5}(1) = [3, 5, 4]$ . It follows that

$\pi_{2,3,4,5}(1) = [2, 3, 5, 4]$ . In  $D_4$ , from the rotation at 1 and the directed crossing of  $e$  with  $\overrightarrow{(1, 2)}$ ,  $(2, 3)$  crossing  $(1, 5)$  would imply that  $5 \in \overrightarrow{(1, 2, 3)}_R$ . Since we know that  $5 \notin \overrightarrow{(1, 2, 3)}_R$ ,  $(2, 3)$  does not cross  $(1, 5)$ . This implies that  $(2, 3)$  must cross  $e$ , in particular,  $(2, 3) <^e_{\wedge} (1, 2)$  and  $e$  crosses  $(2, 3)$  from left to right.

In  $D_1$ ,  $e$  crosses into  $\overrightarrow{(2, 3, 4)}_R$  at  $(3, 4)$  and  $(2, 3)$ . It follows that  $e$  crosses out of  $\overrightarrow{(2, 3, 4)}_R$  at  $(2, 4)$  and that  $(3, 4) <^e_{\wedge} (2, 4) <^e_{\wedge} (2, 3)$ . We now have that  $(1, 2) <^e_{\parallel} (3, 4) <^e_{\wedge} (2, 4) <^e_{\wedge} (2, 3) <^e_{\wedge} (1, 2)$ , a contradiction with  $H$  containing realizable 6-vertex rotation systems.  $\square$

**Lemma 5.4.** *If  $H$  is a  $(7, 6)$ -rotation system and  $e = \overrightarrow{(u, v)}$  is a directed edge of  $H$ , then the cycle  $(1, 2) <^e_{\wedge} (2, 3) <^e_{\wedge} (3, 4) <^e_{\wedge} (4, 5) <^e_{\wedge} (1, 5) <^e_{\wedge} (1, 2)$  does not occur.*

*Proof.* Define the simple drawings  $D_i$  to be a realization of  $H - \{i\}$  for  $i \in V(H)$ , and  $D_e$  to be a realization of  $H - \{u, v\}$ .

By way of contradiction, assume such a cycle exists. Three consecutive edges in this cycle form a 6-vertex rotation system with  $e$ . We will show that the second of any three consecutive edges can not be replaced in this cycle with some other edge in the induced 6-vertex rotation system.

By symmetry, it is enough to consider the order  $(1, 2) <^e_{\wedge} (2, 3) <^e_{\wedge} (3, 4)$  and check if  $(2, 3)$  can be replaced. Since  $D_5$  is simple, it is clear that no edge incident to  $e$  crosses  $e$ . Therefore, suppose some edge  $x \in \{(1, 3), (1, 4), (2, 4)\}$  has the property that  $(1, 2) <^e_{\wedge} x <^e_{\wedge} (3, 4)$ . If this were the case, then the cycle  $(1, 2) <^e_{\wedge} x <^e_{\wedge} (3, 4) <^e_{\wedge} (4, 5) <^e_{\wedge} (1, 5) <^e_{\wedge} (1, 2)$  exists.

If  $x \in \{(1, 3), (1, 4)\}$ , then this cycle reduces to  $(1, 5) <^e_{\wedge} x <^e_{\wedge} (3, 4) <^e_{\wedge} (4, 5) <^e_{\wedge} (1, 5)$ , a contradiction with 6-vertex rotation systems being realizable. Similarly, if  $x = (2, 4)$ , then the cycle reduces to  $(1, 2) <^e_{\wedge} x <^e_{\wedge} (4, 5) <^e_{\wedge} (1, 5) <^e_{\wedge} (1, 2)$ , again a contradiction with realizable 6-vertex rotation systems. Therefore, no such edge  $x$  can replace  $(2, 3)$ .

In  $D_1$ , if  $(1, 2)$  and  $(3, 4)$  cross, then  $e$  can not cross  $(1, 2)$ ,  $(2, 3)$ , and  $(3, 4)$  consecutively as after crossing  $(1, 2)$ ,  $e$  would cross  $(2, 3)$  from the crossing side of the uncrossed 4-cycle to the non-crossing side of the uncrossed 4-cycle and would not be able to cross  $(3, 4)$ .

If  $(1, 2)$  and  $(3, 4)$  are not crossing and are in a crossing  $K_4$ , then after  $e$  has crossed  $(1, 2)$ , then  $(2, 3)$ , it would be contained in a face containing crossing edges and an edge that shares an endpoint with  $(1, 2)$ . Thus,  $e$  would not be able to cross  $(3, 4)$  after crossing  $(2, 3)$ . It follows

that a  $H - \{u, v, 5\}$  can only be realized by a planar  $K_4$ . By symmetry,  $H - \{u, v, 1\}$ ,  $H - \{u, v, 2\}$ ,  $H - \{u, v, 3\}$ , and  $H - \{u, v, 4\}$  are also all realized by planar drawings. However, every realization of the 5-vertex rotation system  $H - \{u, v\}$  contains at least one crossing, a contradiction.  $\square$

**Lemma 5.5.** *If  $H$  is a  $(7, 6)$ -rotation system and  $e = \overrightarrow{(u, v)}$  is a directed edge of  $H$ , then the cycle  $(1, 2) \prec_{\parallel}^e (3, 4) \prec_{\wedge}^e (4, 5) \prec_{\wedge}^e (1, 5) \prec_{\wedge}^e (1, 2)$  does not occur.*

*Proof.* Define the simple drawings  $D_i$  to be a realization of  $H - \{i\}$  for  $i \in V(H)$ , and  $D_e$  to be a realization of  $H - \{u, v\}$ .

By way of contradiction, assume such a cycle exists. Notice since  $(1, 2)$  and  $(3, 4)$  are in a crossing  $K_4$  where they do not cross, that one of the crossing diagonals is  $y \in \{(1, 3), (1, 4)\}$ .

Let  $C_i$  be the oriented cycle defined by the vertices  $[4] \setminus \{i\}$  such that in  $D_i$ ,  $e$  crosses into  $C_{i,R}$  at  $(3, 4)$  or  $e$  crosses out of  $C_{i,R}$  at  $(1, 2)$ .

Consider the order  $e$  crosses the edges of  $(1, 3, 4)$  and  $(1, 5)$  after crossing into  $C_{2,R}$  at  $(3, 4)$ . For  $x \in \{(1, 3), (1, 4)\}$ , if  $(3, 4) \prec_{D_2}^e x \prec_{D_2}^e (1, 5)$ , then  $(1, 2) \prec_{\parallel}^e (3, 4) \prec_{\wedge}^e x \prec_{\wedge}^e (1, 5) \prec_{\wedge}^e (1, 2)$ . By Observation 5.2, this can be reduced to  $(1, 2) \prec_{\parallel}^e (3, 4) \prec_{\wedge}^e x \prec_{\wedge}^e (1, 2)$ , a contradiction with the existence of  $D_5$ .

It follows that the crossing of  $e$  and  $(1, 5)$  is inside  $C_{2,R}$  as  $e$  crosses  $(3, 4)$  into  $C_{2,R}$  then crosses  $(1, 5)$ . Since the crossing of  $e$  and  $(4, 5)$  happens between the crossing of  $e$  with  $(3, 4)$  and the crossing of  $e$  with  $(1, 5)$  on  $e$ , the crossing of  $e$  with  $(4, 5)$  must also be in  $C_{2,R}$ .

Since a  $K_4$  contains at most one crossing, one of the edges  $(1, 5)$  or  $(4, 5)$  is contained in  $C_{2,R}$ . It follows that 5 is contained in  $C_{2,R}$ .

By considering the order in which  $e$  crosses the edges of  $C_{3,R}$  and  $(1, 5)$ , the same argument shows that 5 is contained in  $C_{3,R}$ .

If  $y = (1, 4)$ , then  $C_{2,R}$  and  $C_{3,R}$  have empty intersection in  $D_5$ , and thus have empty intersection in  $D_e$ . By the existence of  $D_e$  and  $5 \in C_{2,R} \cap C_{3,R}$ ,  $y \neq (1, 4)$ . Therefore,  $y = (1, 3)$ .

There are two cases to finish this claim depending on whether  $(2, 3)$  is crossed by  $e$  or not.

**Case 1.**  $(2, 3)$  is not crossed by  $e$ .

Consider the order the edges of  $C_{4,R}$  and  $(1, 5)$  are crossed by  $e$ . If  $(1, 5) \prec_{\wedge}^e$

$(1, 3) <_{\wedge}^e (1, 2)$ , then  $(1, 2) <_{\parallel}^e (3, 4) <_{\wedge}^e (4, 5) <_{\wedge}^e (1, 5) <_{\wedge}^e (1, 3) <_{\wedge}^e (1, 2)$ . By Observation 5.2, this cycle can be reduced to  $(3, 4) <_{\wedge}^e (4, 5) <_{\wedge}^e (1, 5) <_{\wedge}^e (1, 3) <_{\wedge}^e (3, 4)$ , a contradiction with the existence of  $D_2$ .

Since  $(2, 3)$  does not cross  $e$ ,  $C_{4,R}$  contains the crossing of  $(1, 5)$  with  $e$ . Note that  $C_{2,R}$  contains the vertex 5 and the intersection of  $C_{2,R}$  and  $C_{4,R}$  is empty in  $D_e$ . It follows that  $C_{4,R}$  does not contain the vertex 5 and that  $(1, 5)$  starts inside  $C_{4,R}$  since the intersection of  $(1, 5)$  and  $e$  is contained in  $C_{4,R}$ . This is a contradiction as  $D_6$  can not be a simple drawing if the edge  $(1, 5)$  starts in  $C_{4,R}$  and  $5 \in C_{2,R} \cap C_{3,R}$ .

**Case 2.**  $e$  crosses  $(2, 3)$ .

Each of  $C_1$  and  $C_4$  have  $e$  crossing from one side of the cycle to the other at  $(3, 4)$  and  $(1, 2)$ , respectively. By our choice of  $C_1$  and  $C_4$ , either  $e$  crosses  $(2, 3)$  from the same side as  $(1, 2)$  in  $C_4$ , or  $e$  crosses  $(2, 3)$  from the same side as  $(3, 4)$  in  $C_1$  (as observed in  $D_1$ ). Without loss of generality, we will assume that  $e$  crosses  $(2, 3)$  from the same side as  $(1, 2)$  in  $C_4$  as the argument for the opposing assumption is completely analogous.

Working in  $D_4$ , we note that  $e$  also crosses  $(1, 3)$  into  $C_{4,R}$  as both  $(1, 2)$  and  $(2, 3)$  are crossed outward by  $e$ . In particular,  $(1, 3)$  is the second edge crossed by  $e$  of the edges in  $C_4$ .

If  $(1, 5) \prec_{D_4}^e (2, 3) \prec_{D_4}^e (1, 2)$ , then  $(1, 5) <_{\wedge}^e (1, 3) <_{\wedge}^e (1, 2)$  as  $(1, 3)$  is the second edge crossed in  $C_4$  by  $e$ . It follows that  $(1, 2) <_{\parallel}^e (3, 4) <_{\wedge}^e (4, 5) <_{\wedge}^e (1, 5) <_{\wedge}^e (1, 3) <_{\wedge}^e (1, 2)$ . By Observation 5.2, these relations can be reduced to  $(1, 3) <_{\wedge}^e (3, 4) <_{\wedge}^e (4, 5) <_{\wedge}^e (1, 5) <_{\wedge}^e (1, 3)$ , a contradiction with the existence of  $D_2$ .

It follows that the crossing of  $(1, 5)$  and  $e$  is contained in  $C_{4,R}$ . Since the vertex 5 is contained in the exterior of  $C_{4,R}$ , it follows that the edge  $(1, 5)$  starts inside  $C_{4,R}$  at 1.

This is a contradiction, as the drawing  $D_e$  is simple, therefore, it can not have  $(1, 5)$  start in  $C_{4,R}$  and end at  $5 \in C_{2,R} \cap C_{3,R}$ .  $\square$

We end this section by applying Observation 5.2 and Lemmas 5.3, 5.4 and 5.5 to prove Theorem 5.1.

*Proof of Theorem 5.1.* Let  $H$  be a  $(7, 6)$ -rotation system and  $e = \overrightarrow{(u, v)}$  be a directed edge of  $H$ . By way of contradiction, let  $\mathcal{C} = (a_0, a_1, \dots, a_{k-1})$  be a minimal cycle of  $<_{\wedge}^e$  and  $<_{\parallel}^e$  relations such that  $a_i <^e a_{i+1} \pmod{k}$  for  $i \in [k]$ .

If  $\mathcal{C}$  does not contain a  $<_{\parallel}^e$  relation, then repeated applications of Observation 5.2 would give us that up to relabelling  $\mathcal{C} = ((1, 2), (2, 3), (3, 4), (4, 5), (5, 1), (1, 2))$ , a contradiction with Lemma 5.4.

Therefore,  $\mathcal{C}$  contains a  $<_{\parallel}^e$  relation  $a_i < a_{i+1}$ , for some  $i \in \mathbb{Z}_k$ . Without loss of generality, let  $a_i = (1, 2)$  and  $a_{i+1} = (3, 4)$ . By Observation 5.2, Lemma 5.3, and by symmetry of  $(3, 4)$ , it must be the case that  $a_{i+1} <_{\wedge}^e a_{i+2}$  and that  $a_{i+2} = (4, 5)$ . Similarly,  $a_{i-1} <_{\wedge}^e a_i$  and  $a_{i-1} = (1, 5)$ .

If  $(1, 5) <_{\wedge}^e (4, 5)$ , then the cycle  $(a_0, \dots, a_i, a_{i-1}, a_{i+2}, \dots, a_{k-1}, a_0)$  exists and  $\mathcal{C}$  is not minimal, a contradiction. Therefore,  $(4, 5) <_{\wedge}^e (1, 5)$  and  $(1, 2) <_{\parallel}^e (3, 4) <_{\wedge}^e (4, 5) <_{\wedge}^e (1, 5) <_{\wedge}^e (1, 2)$  exists, a contradiction with Lemma 5.5, as desired.  $\square$

## 5.2 8-Vertex Rotation Systems

Similar to the previous section, this section proves that  $<_{\wedge}^e$  and  $<_{\parallel}^e$  induce an acyclic directed graph for  $(8, 7)$  rotation systems. To that end, we want to show the following theorem:

**Theorem 5.6.** *If  $H$  is a  $(8, 7)$ -rotation system and  $e$  is a directed edge of  $H$ , then there are no cycles comprised of  $<_{\wedge}^e$  and  $<_{\parallel}^e$  relations in  $H$ .*

Similar to the proof of Theorem 5.1, we restrict the structure of a minimal cycle of relations using Observation 5.9 and show that up to relabelling one of the cycles in Lemmas 5.10 or 5.11 exists for a contradiction. Before we can prove these statements we must first learn more about the  $<_{\Delta}^e$  relation. In particular, the  $<_{\Delta}^e$  relation can always be represented as a chain of  $<_{\wedge}^e$  and  $<_{\parallel}^e$  relations. Many thanks go to Bruce Richter for taking a very crude statement and argument of the following lemma and turning it into what it is today.

**Lemma 5.7.** *Let  $H$  be an  $(n, 7)$ -rotation system,  $a, b$  be edges of  $H$  and  $e$  a directed edge of  $H$  such that  $a <_{\Delta}^e b$ , as certified by the vertex  $v$ . Let  $u_a$  and  $v_a$  be the ends of  $a$  such that traversing  $a$  from  $u_a$  to  $v_a$  the crossings of  $e$  precedes the crossings of  $b$ . Likewise, changing the roles of  $a$  and  $b$ , we get  $u_b$  and  $v_b$ . Then either:*

- *There exists an  $i \in \{a, b\}$  such that  $(u_i, v)$  crosses  $e$ , and does not cross  $a$  or  $b$ ; or*
- $a <_{\parallel}^e (v_b, v) <_{\wedge}^e (u_a, v) <_{\wedge}^e (u_b, v) <_{\wedge}^e (v_a, v) <_{\parallel}^e b$ .

*Proof.* The proof involves drawing the edge  $(u_a, v)$ . It starts at  $v$  on the side  $\Delta$  of  $\gamma_{a,b,e}$  not incident to any vertex in  $V(\{a, b, e\})$  and finishes at  $u_a$  on the other side of  $\gamma_{a,b,e}$ . Evidently, it crosses  $\Delta$  an odd number of times and does not cross  $a$ , therefore it crosses  $\Delta$  exactly once.

As we start from  $v$ , we leave  $\Delta$  by crossing one of  $e$  or  $b$ .

**Case 1.** *The  $\Delta$ -leaving crossing of  $(u_a, v)$  is with  $b$ .*

One side of  $\gamma_{a,b,(u_a,v)}$  contains exactly  $v$ , a non-trivial segment of  $b$  starting at  $u_b$ , and a non-trivial segment of  $e$ . It follows that  $(u_b, v)$  crosses  $\gamma_{a,b,(u_a,v)}$  and even number of times, but can only cross  $a$ . It follows that  $(u_b, v)$  does not cross  $\gamma_{a,b,(u_a,v)}$ . The only thing  $(u_b, v)$  can cross is the segment of  $e$  on the same side of  $\gamma_{a,b,(u_a,v)}$ . This implies that  $(u_b, v)$  leave  $\Delta$  from  $e$  and does not cross  $a$  or  $b$ , as desired.

**Case 2.** *The  $\Delta$ -leaving crossing of  $(u_a, v)$  is with  $e$ .*

Case 1 with the roles of  $a$  and  $b$  interchanged implies that  $(u_b, v)$  also leaves  $\Delta$  by crossing  $e$ . We may assume that  $(u_i, v)$  crosses  $j$  for  $\{i, j\} = \{a, b\}$ , else we are done. In this case,  $\gamma_{e,b,(u_a,v)}$  exists.

Trivially, as  $(u_a, v)$  does not cross  $b$  at  $\gamma_{a,b,e}$ , it crosses  $b$  at one of the two components in  $b \setminus \gamma_{a,b,e}$ .

**Case 2.1.**  *$(u_a, v)$  crosses  $b$  on the component containing  $u_b$  in  $b \setminus \gamma_{a,b,e}$ .*

In this case, one side of  $\gamma_{a,b,(u_a,v)}$  contains exactly  $v, u_b$  and a segment of  $e$ . Evidently,  $(u_b, v)$  crosses  $\gamma_{a,b,(u_a,v)}$  an even number of times, but can only cross  $a$ . It follows that  $(u_b, v)$  does not cross  $\gamma_{a,b,(u_a,v)}$ . Since the only edge segment on the same side of  $\gamma_{a,b,(u_a,v)}$  containing  $v$  and  $u_b$  is  $e$ , it follows that  $(u_b, v)$  crosses  $e$  and not  $a$  or  $b$ , as desired.

**Case 2.2.**  *$(u_a, v)$  crosses  $b$  on the component containing  $v_b$  in  $b \setminus \gamma_{a,b,e}$ .*

Similarly,  $(u_b, v)$  crosses  $a$  on the component containing  $v_a$  in  $b \setminus \gamma_{a,b,e}$ . These crossings uniquely determine the drawing  $D$  of  $a, b, e, (u_a, v)$  and  $(u_b, v)$ . Since every 7-vertex rotation system in  $H$  is realizable, it follows that the drawing  $D$  can be extended to a drawing  $\bar{D}$  containing the induced  $K_4$ s on  $v, v_a, v_b, u_a$  and  $v, v_a, v_b, u_b$ .



There is a unique way to extend  $D$  to  $\bar{D}$  to keep  $\bar{D}$  simple. In  $\bar{D}$ ,  $a <_{\parallel}^e (v_b, v) <_{\wedge}^e (u_a, v) <_{\wedge}^e (u_b, v) <_{\wedge}^e (v_a, v) <_{\parallel}^e b$ , as desired.  $\square$

**Corollary 5.8.** *Let  $H$  be an  $(n, 7)$ -rotation system,  $a, b$  be edges of  $H$  and  $e$  a directed edge of  $H$  such that  $a <_{\Delta}^e b$ , as certified by the vertex  $v$ . Let  $u_a$  and  $v_a$  be the ends of  $a$  such that traversing  $a$  from  $u_a$  to  $v_a$  the crossings of  $e$  proceeds the crossings of  $b$ . Likewise, changing the roles of  $a$  and  $b$ , we get  $u_b$  and  $v_b$ . Then either:*

- $a <_{\wedge}^e (u_a, v) <_{\parallel}^e b$ ;
- $a <_{\parallel}^e (u_b, v) <_{\wedge}^e b$ ; or
- $a <_{\wedge}^e (u_a, v) <_{\wedge}^e (u_b, v) <_{\wedge}^e b$ .

Lemma 5.7 is used primarily in the proof of Lemma 5.10 to find structure in realizations of 7-vertex rotation system that share a common 6-vertex rotation system. As for Corollary 5.8, it is used to simplify the proof of Theorem 5.12. Let us make an observation on the structure of cycles of  $<_{\wedge}^e$  and  $<_{\parallel}^e$  relations in  $(8, 7)$ -rotation systems.

**Observation 5.9.** *Let  $H$  be an  $(8, 7)$ -rotation system,  $e = \overrightarrow{(u, v)}$  be a directed edge of  $H$ ,  $\mathcal{C} = (a_0, \dots, a_{k-1}, a_0)$  be a cycle of  $<_{\wedge}^e$  and  $<_{\parallel}^e$  relations (i.e. for all  $i \in \mathbb{Z}_k$ ,  $a_i <_{\wedge}^e a_{i+1}$  or  $a_i <_{\parallel}^e a_{i+1}$ ).*

*For all  $i, j \in \mathbb{Z}_k$ , if  $a_i$  and  $a_j$  are related by an  $<_{\wedge}^e$  or  $<_{\parallel}^e$  relation and  $j \notin \{i - 1, i + 1\}$ , then either:*

- $V(\{a_i, \dots, a_j\}) = [6]$ ; or
- $(a_0, \dots, a_i, a_j, \dots, a_{k-1}, a_0)$  is a cycle of  $<_{\wedge}^e$  and  $<_{\parallel}^e$  relations.

*Proof.* Suppose there exists some  $i$  and  $j$  such that  $a_i$  and  $a_j$  are related by an  $<_{\wedge}^e$  or  $<_{\parallel}^e$  relation, and  $V(\{a_i, \dots, a_j\}) \subseteq [6] \setminus \{k\}$ , for some  $k \in [6]$ . Let the simple drawing  $D_k$  be a realization of  $H - \{k\}$ . It follows that  $a_i < \dots <^e a_j$  in  $D_k$ , and that  $a_i <^e a_j$  is a  $<_{\wedge}^e$  or  $<_{\parallel}^e$  relation. Therefore, there exists some cycle of relations  $\bar{\mathcal{C}} = (a_0, \dots, a_i, a_j, \dots, a_{k-1}, a_0)$ , as desired.  $\square$

Since the  $<_{\Delta}^e$  relation is always a chain of  $<_{\wedge}^e$  and  $<_{\parallel}^e$  relations, Lemma 5.10 has more substance than one would expect. In particular, it is used in every case of the proof of Theorem 5.6.

**Lemma 5.10.** *If  $H$  is an  $(8, 7)$ -rotation system, and  $e$  is a directed edge of  $H$ , then the cycle order  $(1, 2) <_{\Delta}^e (3, 4) <_{\Delta}^e (1, 2)$  does not occur.*

*Proof.* Define the simple drawings  $D_i$  to be a realization of  $H - \{i\}$  and  $D_e$  to be a realization of  $H - \{u, v\}$ .

Suppose such a cycle relation exists with vertex 5 being the certificate for  $(3, 4) <_{\Delta}^e (1, 2)$  and vertex 6 being the certificate for  $(1, 2) <_{\Delta}^e (3, 4)$ . Up to relabelling, without loss of generality it can be assumed that  $e$  crosses  $\overrightarrow{(1, 2)}$  from right to left,  $e$  crosses  $\overrightarrow{(3, 4)}$  from right to left and  $\overrightarrow{(1, 2)}$  crosses  $\overrightarrow{(3, 4)}$  from right to left. We will break this proof into cases depending on how Lemma 5.7 relates to  $D_6$ .

**Case 1.**  $(3, 4) <_{\parallel}^e (1, 5) <_{\wedge}^e (2, 5) <_{\wedge}^e (4, 5) <_{\wedge}^e (3, 5) <_{\parallel}^e (1, 2)$ .

It follow that  $(3, 4) <_{\wedge}^e (3, 5) <_{\parallel}^e (1, 2)$  and  $(3, 4) <_{\parallel}^e (1, 5) <_{\wedge}^e (1, 2)$ .

If  $(1, 2) <_{K_6}^e (i, 6) <_{K_6}^e (3, 4)$  where  $i \in \{1, 3\}$ , then in  $D_2$ ,  $(i, 6) <_{K_6}^e (3, 4) <_{K_6}^e (i, 5)$ , in particular  $(i, 6) <_{\wedge}^e (i, 5)$ . It would follow that  $(1, 2) <_{K_6}^e (i, 6) <_{\wedge}^e (i, 5) <_{K_6}^e (1, 2)$ , a contradiction with the existence of  $D_3$ .

Therefore, by Lemma 5.7,  $(1, 2) <_{\parallel}^e (4, 6) <_{\wedge}^e (1, 6) <_{\wedge}^e (3, 6) <_{\wedge}^e (2, 6) <_{\parallel}^e (3, 4)$ , in particular,  $(1, 2) <_{\wedge}^e (1, 6) <_{\wedge}^e (3, 6) <_{\wedge}^e (3, 4)$ . Since  $(1, 5) <_{\parallel}^e (1, 2) <_{\wedge}^e (1, 6)$ , it follows that  $(1, 5) <_{\wedge}^e (1, 6)$  and  $(3, 4) <_{\parallel}^e (1, 5) <_{\wedge}^e (1, 6) <_{\wedge}^e (3, 6) <_{\wedge}^e (3, 4)$ , a contradiction with the existence of  $D_2$ .

**Case 2.** *There exists an  $i \in \{2, 4\}$  such that  $(i, 5)$  crosses  $e$ , and does not cross  $(1, 2)$  or  $(3, 4)$ .*

Since the edges  $(2, 5)$  and  $(4, 5)$  are symmetric at this point, without loss of generality the edge  $(2, 5)$  crosses  $e$  and not  $(1, 2)$  and  $(3, 4)$ . Let  $\bar{e}$  be the segment of  $e$  that starts at the crossing of  $e$  and  $(3, 4)$  and ends at the crossing of  $e$  and  $(1, 2)$ . Observe that the edges  $(1, 2)$ ,  $(3, 4)$ ,  $\bar{e}$ ,  $(2, 5)$ ,  $(2, 3)$  and  $(1, 3)$  are uniquely drawn in  $D_6$ . In particular,  $e$  crosses  $(1, 2)$  into  $\overrightarrow{(1, 3, 2)}_R$ , and  $(2, 5)$  is contained in  $\overrightarrow{(1, 2, 3)}_R$ .

The same argument as Case 1 on  $D_5$  shows  $(1, 2) <_{\parallel}^e (4, 6) <_{\wedge}^e (1, 6) <_{\wedge}^e (2, 6) <_{\wedge}^e (3, 6) <_{\wedge}^e (3, 4)$  does not occur. By Lemma 5.7, there exists a  $j \in \{1, 3\}$  such that  $(j, 6)$  crosses  $e$ , and does not cross  $(1, 2)$  or  $(3, 4)$ .

**Case 2.1.**  $(3, 6)$  crosses  $e$  and not  $(1, 2)$  or  $(3, 4)$ .

In  $D_5$ , the edges  $(1, 2), \bar{e}, (3, 4), (3, 6), (1, 3)$  and  $(2, 3)$  are uniquely drawn. In particular,  $\pi_{1,2,6}(3) = [2, 6, 1]$ . In  $D_4$ , since the edge  $(3, 6)$  starts inside  $\overrightarrow{(1, 3, 2)}_R$  and does not cross  $(1, 2)$ , it follows that  $(3, 6)$  is contained in  $\overrightarrow{(1, 3, 2)}_R$ . Since  $(3, 6)$  and  $(2, 5)$  are contained in opposite sides of the 3-cycle  $(1, 2, 3)$ , it follows that  $(2, 5)$  and  $(3, 6)$  do not cross.

In  $D_5$ ,  $(1, 2) <_{K_6}^e (3, 6)$  as they do not cross and  $e$  crosses  $(1, 2)$  first.

Similarly, in  $D_6$ ,  $(3, 4) <_{K_6}^e (2, 5)$ . Since  $(2, 5) <_{\wedge}^e (1, 2) <_{K_6}^e (3, 6)$  and  $(2, 5)$  does not cross  $(3, 6)$ , it follows that in  $D_1$ ,  $(2, 5) <_{K_6}^e (3, 6)$ . This results in the cycle  $(3, 4) <_{K_6}^e (2, 5) <_{K_6}^e (3, 6) <_{\wedge}^e (3, 4)$ , a contradiction with the existence of  $D_1$ .

**Case 2.2.**  $(1, 6)$  crosses  $e$  and not  $(1, 2)$  or  $(3, 4)$ .

We break this into two smaller cases depending on if  $(3, 5)$  crosses  $\bar{e}$  in  $D_6$  or not.

**Case 2.2.1.**  $(3, 5)$  does not cross  $\bar{e}$  in  $D_6$ .

By the symmetry of Case 2.1, we also have that  $(4, 5)$  does not both cross  $e$  and not  $(1, 2)$  and  $(3, 4)$ .

Let  $E_{1,2,3,4}$  be the edge set of the  $K_4$  involving  $(1, 2)$  and  $(3, 4)$ . Observe that the edges of  $E_{1,2,3,4}, (2, 5), (3, 5), (4, 5)$  and  $\bar{e}$  are uniquely drawn in  $D_6$ , in particular that  $\pi_{1,4,5}(3) = [5, 4, 1]$  and  $\pi_{1,3,5}(4) = [1, 3, 5]$ .

Similarly, the edges of  $E_{1,2,3,4}, (1, 6)$  and  $\bar{e}$  are uniquely drawn in  $D_5$ , in particular that  $\pi_{2,3,6}(1) = [3, 6, 2]$ . Let  $C_1$  be the closed curve in  $D_4$  defined by starting at the vertex 1, taking the edge  $(1, 3)$  to the vertex 3, then taking the edge  $(3, 5)$  to the crossing of  $(3, 5)$  with  $(1, 2)$ , and taking the edge  $(1, 2)$  back to the vertex 1. Let  $C_{1,R}$  be the side of  $C_1$  that is bounded by the right side of  $\overrightarrow{(1, 3)}$ . Since  $\pi_{2,3,6}(1) = [3, 6, 2]$ , the edge  $(1, 6)$  starts inside  $C_{1,R}$ .

Let us now go to  $D_4$ . Note that the  $K_4$  on  $(3, 5)$  and  $(1, 2)$  is determined as  $\overrightarrow{(1, 2)}$  crosses  $\overrightarrow{(3, 5)}$  from right to left in  $D_6$ . Since  $\pi_{2,3,6}(1) = [3, 6, 2]$ ,  $(1, 6)$  is either contained on one side of  $\gamma_{(1,3),(3,5),(1,2)}$  or  $\overrightarrow{(1, 6)}$  crosses  $\overrightarrow{(3, 5)}$  from right to left in  $D_4$ .

**Case 2.2.1.1.**  $(1, 6)$  is contained on one side of  $\gamma_{(1,3),(3,5),(1,2)}$  in  $D_4$

It follows that  $(1, 6)$  does not cross  $(3, 5)$ . Observe that the drawing of the induced  $K_4$  on  $1, 2, 3, 5$  along with the edge  $(1, 6)$  is determined. In any realization of these 5-vertices along with  $e$ ,  $e$  crosses  $\overrightarrow{(2, 5)}$  from left to right, then crosses  $(1, 2)$ , then crosses  $(1, 6)$ . In particular,  $e$  crosses into  $\overrightarrow{(3, 2, 5)}_R$  at  $(2, 5)$  and this region does not contain  $(1, 6)$  as it does not contain  $\gamma_{(1,3),(3,5),(1,2)}$ .

It follows that  $e$  crosses  $x \in \{(2, 3), (3, 5)\}$  after crossing  $(2, 5)$  and before crossing  $(1, 6)$ . Note that both these edge share an endpoint with  $(2, 5)$  and neither of these edge crosses  $(1, 6)$  since they are contained on the wrong side of  $\gamma_{(1,3),(3,5),(1,2)}$ . It follows that  $(2, 5) <_{K_6}^e x <_{K_6}^e (1, 6)$ .

Therefore, we have  $(3, 4) <_{\parallel}^e (2, 5) <_{K_6}^e x <_{K_6}^e (1, 6) <_{\parallel}^e (3, 4)$ . Note that  $D_1$ , it follows that  $(3, 4) <_{\wedge}^e x <_{K_6}^e (1, 6) <_{\parallel}^e (3, 4)$ , a contradiction in one of  $D_2$  or  $D_5$ .

**Case 2.2.1.2.**  $\overrightarrow{(1, 6)}$  crosses  $\overrightarrow{(3, 5)}$  from right to left in  $D_4$ .

Now we make some observations on  $D_2$ . Note that the  $K_4$  involving  $(1, 6)$  and  $(3, 5)$  is uniquely determined by their oriented crossings. Since  $\pi_{1,4,5}(3) = [5, 4, 1]$  and  $(3, 4)$  does not cross  $(1, 6)$  in  $D_5$ , it follows that  $(3, 4)$  is one side of  $\gamma_{(1,3),(1,6),(3,5)}$ , in particular on the side  $S$  that is a face in the induced  $K_4$  on  $1, 3, 5, 6$ .

Suppose  $(1, 4)$  is drawn inside  $S$ . It would follow that  $\overrightarrow{(1, 3, 4)}_L$  contains  $(1, 6)$ . After  $e$  crosses  $(1, 6)$  in  $D_2$ , it would have to cross  $(1, 3, 4)$  at  $(1, 3)$  or  $(1, 4)$  to cross  $(3, 4)$  in the correct orientation. It would follow that  $(1, 6) <_{\wedge}^e x <_{\wedge}^e (3, 4)$  for  $x \in \{(1, 3), (1, 4)\}$ . Expanding our cycle, we would have  $(3, 4) <_{\parallel}^e (2, 5) <_{\wedge}^e (1, 2) <_{\wedge}^e (1, 6) <_{\wedge}^e x <_{\wedge}^e (3, 4)$ . Reducing over  $D_5$  would give,  $(3, 4) <_{\parallel}^e (2, 5) <_{\wedge}^e (1, 2) <_{\wedge}^e x <_{\wedge}^e (3, 4)$ , a contradiction with  $D_6$ .

Therefore,  $(1, 4)$  is not drawn inside  $S$ . There is a unique way to draw  $(1, 4)$ , starting at 4 it crosses  $(3, 5)$ , then  $(3, 6)$ , then ends at 1. In particular, there is a closed curve  $\gamma_{(3,4),(1,4),(3,5)}$  that has 5 on one side and the starting of the edge  $(4, 5)$  at 4 on the other (as per  $\pi_{1,3,5}(4) = [1, 3, 5]$ ), a contradiction.

**Case 2.2.2.**  $(3, 5)$  crosses  $\bar{e}$  in  $D_6$ .

In  $D_6$ , the order  $e$  crosses edges is  $(3, 5)$ , then  $(1, 2)$ . Without loss of generality, by deleting the vertex 4 and adding the vertex 6 and its edges, we

obtain  $D_4$  from  $D_5$ . In particular, the order  $e$  crosses edge in  $D_4$  is  $(3, 5)$ , then  $(1, 2)$ , then  $(1, 6)$ . From  $D_4$ , we obtain  $D_2$  in a similar manner with  $e$  crossing  $(3, 5)$  then  $(1, 6)$ . However,  $(3, 4)$  must be crossed after  $(1, 6)$  and before  $(3, 5)$  in  $D_2$ , a contradiction.  $\square$

In the same vein as Lemma 5.10, we show in Lemma 5.11 that there are no short cycles of  $<_{\parallel}^e$  relations.

**Lemma 5.11.** *If  $H$  is an  $(8, 7)$ -rotation system, and  $e$  is a directed edge of  $H$ , then the cycle order  $(1, 2) <_{\parallel}^e (3, 4) <_{\parallel}^e (5, 6) <_{\parallel}^e (1, 2)$  does not occur.*

*Proof.* Let  $e = \overrightarrow{(u, v)}$  be a directed edge of  $H$ . Define the simple drawings  $D_i$  to be a realization of  $H - \{i\}$  for  $i \in V(H)$ , and  $D_e$  to be a realization of  $H - \{u, v\}$ . For all  $i$ , let  $\bar{e}_i$  be the segment of  $e$  in  $D_i$  between the two edges  $e$  crosses in  $\{(1, 2), (3, 4), (4, 5)\}$ . In the drawing of  $D_i$ , define  $\mathcal{D}_i$  to be the unique drawing of  $\bar{e}_i$  with the induced  $K_4$  on the two edges in  $\{(1, 2), (3, 4), (5, 6)\}$  that cross  $e$ .

By way of contradiction, assume the cycle order  $(1, 2) <_{\parallel}^e (3, 4) <_{\parallel}^e (5, 6) <_{\parallel}^e (1, 2)$  does occur. Up to relabelling, we may assume without loss of generality that  $e$  crosses  $\overrightarrow{(1, 2)}$ ,  $\overrightarrow{(3, 4)}$ , and  $\overrightarrow{(5, 6)}$  from right to left.

**Claim 1.** *Let  $W = \{1, 2, 3, 4\}$ . If  $w \in W$ , then  $(5, w)$  does not cross  $\bar{e}_6$  in  $D_6$ .*

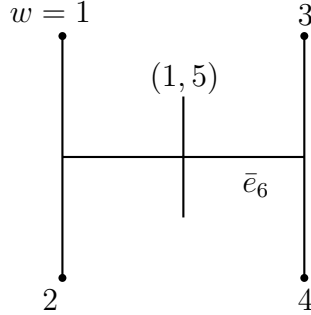


Figure 35: Lemma 5.11 Claim 1.

*Proof.* Without loss of generality, we can assume that  $w = 1$ . By way of contradiction, assume  $(1, 5)$  crosses  $\bar{e}_6$ , in particular,  $e$  crosses  $(1, 5)$  before  $(3, 4)$  and  $(1, 2) <_{\parallel}^e (1, 5)$  as in Figure 35.

By deleting the vertex 2 in  $D_6$  and adding the vertex 6 along with all of its incident edges, we create a realization  $Z$  of  $H - \{2\}$ . In  $Z$ ,  $e$  crosses  $(1, 5)$ , then  $(3, 4)$ , then  $(5, 6)$ . In particular,  $(1, 5) <_{\wedge}^e (5, 6)$ . It follows that  $(1, 2) <_{\wedge}^e (1, 5) <_{\wedge}^e (5, 6) <_{\parallel}^e (1, 2)$ , a contradiction with the existence of  $D_3$ . ■

One remark about Claim 1 is that there is a symmetry. In particular, we can replace  $W$  with any four vertices that are the ends of two of  $(1, 2)$ ,  $(3, 4)$  and  $(5, 6)$ . Furthermore, we can replace 5 with 6, or alternatives with any end vertex  $[6]$  that is not in  $W$  and consider the drawing  $D_i$ , where  $i$  is not used.

**Claim 2.** *Let  $W = \{1, 2, 3, 4\}$ . If  $w \in W$ , then  $(5, w)$  does not cross any edge in  $D_6$ .*

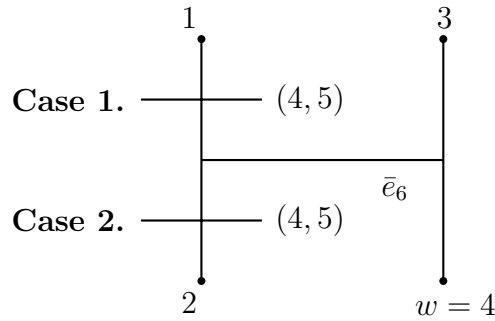


Figure 36: Lemma 5.11 Claim 2.

*Proof.* Without loss of generality, we may assume that  $w = 4$ . Since  $D_6$  is a simple drawing,  $(4, 5)$  and  $(3, 4)$  do not cross. By Claim 1, we need only prove that  $(4, 5)$  does not cross the edge  $(1, 2)$ . By way of contradiction, suppose  $(4, 5)$  does cross  $(1, 2)$ . We will break this into cases depending on the location that  $(4, 5)$  crosses  $(1, 2)$  on  $(1, 2)$  as depicted in Figure 36.

**Case 1.** *Assume  $(4, 5)$  crosses  $(1, 2)$  between the crossing of  $e$  with  $(1, 2)$  and the vertex 1 in  $D_6$ .*

Consider how the edge  $(3, 5)$  and  $(2, 3)$  are drawn with the partial edge  $\bar{e}_6$ ,  $(3, 4)$ ,  $(4, 5)$ , and the segment of  $(1, 2)$  starting at 2 and ending at the crossing of  $\bar{e}_6$ . Since  $D_6$  is a simple drawing, the drawing of these edges is unique, in particular,  $\overrightarrow{(4, 5)}$  crosses  $\overrightarrow{(2, 3)}$  from left to right and  $\pi_{2,4,5}(3) = [2, 4, 5]$ .

In  $\mathcal{D}_1 + (2, 3)$ , by Claim 1,  $(2, 3)$  does not cross  $\bar{e}_1$ . Let  $C_1$  be the unique close curve defined on  $\bar{e}_1$  and segments of  $(3, 4)$ ,  $(3, 5)$  and  $(5, 6)$  and let  $C_{1,R}$  be the side of  $C_1$  that is bounded by the right side of  $\overrightarrow{(3, 5)}$ . In particular, the edge  $(4, 5)$  is outside  $C_{1,R}$ .

Since  $\pi_{2,4,5}(3) = [2, 4, 5]$ ,  $(2, 3)$  starts at 3 in  $C_{1,R}$ . Since  $(2, 3)$  crosses  $(4, 5)$  and does not cross  $\bar{e}_1$ , it must cross  $C_1$  at  $(5, 6)$ , then cross  $(4, 5)$ .

Since there is a unique way to do this, 2 must be in the unique region bounded by the edge  $(3, 5)$  and the crossing of  $(3, 6)$  with  $(4, 5)$ . It follows that  $(2, 6)$  is uniquely determined in  $\mathcal{D}_1 + (2, 3) + (2, 6)$ , in particular,  $\overrightarrow{(4, 5)}$  crosses  $\overrightarrow{(2, 6)}$  from left to right and  $\pi_{2,4,6}(5) = [2, 4, 6]$ .

In  $\mathcal{D}_3 + (4, 5)$ , let the  $C_2 = \overrightarrow{(1, 2, 6, 5)}$ . Since  $\overrightarrow{(4, 5)}$  crosses  $\overrightarrow{(2, 6)}$  from left to right, if  $(4, 5)$  starts inside  $C_{2,R}$ , then of the edges of  $C_2$  it would cross  $(2, 6)$  first. It follows that if  $(4, 5)$  starts inside  $C_{2,R}$ , then it would cross  $\bar{e}_3$ , a contradiction with Claim 1.

Therefore,  $(4, 5)$  starts outside of  $C_{2,R}$ . In particular,  $\pi_{1,4,6}(5) = [6, 4, 1]$ .  $\pi_{2,4,6}(5) = [2, 4, 6]$  implies that  $\pi_{1,2,4,6}(5) = [2, 4, 1, 6]$ . Given this rotation at 5 and  $\overrightarrow{(4, 5)}$  crosses  $\overrightarrow{(2, 6)}$  from left to right, there is a unique drawing of  $(4, 5)$  in  $\mathcal{D}_3 + (4, 5)$ . In particular, 4 is contained in the region bounded by  $(2, 6)$  and segments of  $(2, 5)$  and  $(1, 6)$ . Extending this drawing to a drawing of  $\mathcal{D}_3 + (4, 5) + (1, 4)$  shows that  $(1, 4)$  must cross  $\bar{e}_3$ , a contradiction with Claim 1.

**Case 2.** Assume  $(4, 5)$  crosses  $(1, 2)$  between the crossing of  $e$  with  $(1, 2)$  and the vertex 2 in  $D_6$ .

In  $D_6$ , by deleting 3 and adding the vertex 6 and its incident edges we get a realization  $\bar{D}_3$  of  $H - \{3\}$  with  $(4, 5)$  crossing  $(1, 2)$  between 2 and the crossing of  $e$ . Setting  $\bar{x} = (5, 6)$ ,  $\bar{y} = (1, 2)$ ,  $\bar{j} = 4$  and  $\bar{z} = 5$ , we have a contradiction with Case 1. ■

One remark about Claim 2 is that there is a symmetry. In particular, we can replace  $W$  with any four vertices that are the ends of two of  $(1, 2)$ ,  $(3, 4)$  and  $(5, 6)$ . Furthermore, we can replace 5 with 6, or alternatives with any end vertex [6] that is not in  $W$  and consider the drawing  $D_i$ , where  $i$  is not used.

Now we finish the proof of Lemma 5.11. Since  $D_5$  is a simple drawing, the drawing of  $\mathcal{D}_5 + (1, 3) + (1, 4)$  in  $D_5$  is uniquely determined. In particular,

$2 \in \overrightarrow{(1, 3, 4)}_R$ .

By Claims 1 and 2, the simple drawings of  $\mathcal{D}_4 + (1, 5) + (1, 6)$  in  $D_4$  is uniquely determined. In particular,  $2 \in \overrightarrow{(1, 5, 6)}_L$ .

By Claims 1 and 2, the simple drawings of  $\mathcal{D}_2 + (1, 3) + (1, 4) + (1, 5) + (1, 6)$  in  $D_2$  is uniquely determined. In particular,  $\overrightarrow{(1, 3, 4)}_R \subset \overrightarrow{(1, 5, 6)}_R$ . In  $D_e$ , this must also be the case.

In  $D_e$ , since  $2 \in \overrightarrow{(1, 5, 6)}_L$ ,  $2 \notin \overrightarrow{(1, 5, 6)}_R$ . Since  $\overrightarrow{(1, 3, 4)}_R \subset \overrightarrow{(1, 5, 6)}_R$ ,  $2 \notin \overrightarrow{(1, 3, 4)}_R$ , a contradiction with  $2 \in \overrightarrow{(1, 3, 4)}_R$ . □

We conclude this section with the proof of Theorem 5.6.

*Proof of Theorem 5.6.* Let  $H$  be an  $(8, 7)$ -rotation system and  $e = \overrightarrow{(u, v)}$  be a directed edge of  $H$ . By way of contradiction, let  $\mathcal{C} = (a_0, a_1, \dots, a_{k-1}, a_0)$  be a smallest cycle of  $<_{\wedge}^e$  and  $<_{\parallel}^e$  relations such that  $a_i <^e a_{i+1}$  for  $i \in \mathbb{Z}_k$ .

Define the simple drawings  $D_i$  to be realizations of  $H - \{i\}$  for  $i \in V(H)$ , and  $D_e$  to be a realization of  $H - \{u, v\}$ .

Let  $i \in [k]$ . Without loss of generality, assume  $a_i = (1, 2)$ . Let us partition this proof into three cases depending on if  $\mathcal{C}$  contains two consecutive  $<_{\parallel}^e$  relations, if  $\mathcal{C}$  contains a  $<_{\parallel}^e$  relations and does not contain two consecutive  $<_{\parallel}^e$  relations, or if  $\mathcal{C}$  contains no  $<_{\parallel}^e$  relations.

**Case 1.**  $\mathcal{C}$  contains no  $<_{\parallel}^e$  relation.

Without loss of generality, set  $a_{i+1} = (2, 3)$ . If an end of  $a_{i+2}$  is the vertex 2, then by Observation 5.9,  $\mathcal{C}$  is not a smallest cycle of  $<_{\wedge}^e$  and  $<_{\parallel}^e$  relations.

This shows that any three consecutive edges in  $\mathcal{C}$  do not contain the same vertex. Without loss of generality, it follows that  $a_{i+2} = (3, 4)$ . By repeated use of Observation 5.9, we find that  $\mathcal{C} = (1, 2) <_{\wedge}^e (2, 3) <_{\wedge}^e (3, 4) <_{\wedge}^e (4, 5) <_{\wedge}^e (5, 6) <_{\wedge}^e (6, 1) <_{\wedge}^e (1, 2)$ .  $(1, 2)$  and  $(4, 5)$  can not be ordered in their 6-vertex rotation system with  $e$  as they are crossed in different orders in  $D_6$  and  $D_3$ . Since they are ordered in each of  $D_6$  and  $D_3$ , it follows that  $(1, 2) <_{\Delta}^e (4, 5) <_{\Delta}^e (1, 2)$ , a contradiction with of Lemma 5.10 up to relabelling.

**Case 2.**  $\mathcal{C}$  contains two consecutive  $<_{\parallel}^e$  relations.

Without loss of generality, assume  $a_i <_{\parallel}^e a_{i+1} <_{\parallel}^e a_{i+2}$  and let  $a_{i+1} = (3, 4)$ .



By Observation 5.9, it follows that  $a_{i+2} = (5, 6)$ . If  $a_{i+2} <_{\parallel}^e a_{i+3}$ , then Observation 5.9 implies that  $a_{i+3} = (1, 2)$ , a contradiction with Lemma 5.11. Therefore,  $a_{i+2} <_{\wedge}^e a_{i+3}$  and by Observation 5.9, without loss of generality  $a_{i+3} = (1, 5)$ . If  $(1, 2) <_{\wedge}^e (1, 5)$ , then the cycle  $(a_0, \dots, a_i, a_{i+3}, \dots, a_{k-1})$  is shorter than  $\mathcal{C}$ , a contradiction.

It follows that  $(1, 2) <_{\parallel}^e (3, 4) <_{\parallel}^e (5, 6) <_{\wedge}^e (1, 5) <_{\wedge}^e (1, 2)$ .  $(3, 4)$  and  $(1, 5)$  can not be ordered in their 6-vertex rotation system with  $e$  as they are in different orders in  $D_6$  and  $D_2$ . Since they are ordered in  $D_6$  and  $D_2$ , it follows that  $(3, 4) <_{\Delta}^e (1, 5) <_{\Delta}^e (3, 4)$ , a contradiction with Lemma 5.10 up to relabelling.

**Case 3.**  $\mathcal{C}$  contains a  $<_{\parallel}^e$  relation and does not contain two consecutive  $<_{\parallel}^e$  relations.

Without loss of generality, assume  $a_i <_{\parallel}^e a_{i+1}$  and  $a_{i+1} = (3, 4)$ . By Case 2,  $a_{i-1} <_{\wedge}^e a_i$  and  $a_{i+1} <_{\wedge}^e a_{i+2}$ . By Observation 5.9, without loss of generality  $a_{i-1} = (1, 5)$  and  $a_{i+2} = (3, 6)$ . By Observation 5.9, it is clear that  $a_{i+3} = (x, 5)$  where  $x \in [6] \setminus \{5\}$ . If  $x \in \{1, 3, 4\}$ , then any outcome of Observation 5.9 would produce a smaller cycle than  $\mathcal{C}$ , a contradiction. Therefore,  $x \in \{2, 6\}$ .

**Case 3.1.**  $x = 2$ .

Note that  $(2, 5) <_{\wedge}^e (1, 2) <_{\parallel}^e (3, 4) <_{\wedge}^e (3, 6) <_{\parallel}^e (2, 5)$ .  $(2, 5)$  and  $(3, 4)$  can not be ordered in their 6-vertex rotation system with  $e$  as they are in different orders in  $D_6$  and  $D_1$ . Since they are ordered in  $D_6$  and  $D_1$ , it follows that  $(2, 5) <_{\Delta}^e (3, 4) <_{\Delta}^e (2, 5)$ , a contradiction with Lemma 5.10 up to relabelling.

**Case 3.2.**  $x = 6$ .

Note that  $(2, 5) <_{\wedge}^e (1, 2) <_{\parallel}^e (3, 4) <_{\wedge}^e (3, 6) <_{\wedge}^e (5, 6)$ . If  $(2, 5) <_{\wedge}^e (5, 6)$ , then the cycle  $(a_0, \dots, a_{i-1}, a_{i+3}, \dots, a_{k-i})$  is shorter than  $\mathcal{C}$ , a contradiction. Therefore,  $(5, 6) <_{\wedge}^e (2, 5)$  and  $(2, 5) <_{\wedge}^e (1, 2) <_{\parallel}^e (3, 4) <_{\wedge}^e (3, 6) <_{\wedge}^e (5, 6) <_{\wedge}^e (2, 5)$ .

$(2, 5)$  and  $(3, 4)$  can not be ordered in their 6-vertex rotation system with  $e$  as they are in different orders in  $D_6$  and  $D_1$ . Since they are ordered in  $D_6$  and  $D_1$ , it follows that  $(2, 5) <_{\Delta}^e (3, 4) <_{\Delta}^e (2, 5)$ , a contradiction with

Lemma 5.10 up to relabelling. □

### 5.3 $n$ -Vertex Rotation Systems

We finish Section 5 by proving that  $<_{\wedge}^e$  and  $<_{\parallel}^e$  induce an acyclic directed graph for  $(n, 8)$  rotation systems. To that end, this short section starts with a statement of Theorem 5.12 and is followed by a direct proof.

**Theorem 5.12.** *If  $n \geq 9$ ,  $H$  is an  $(n, 8)$ -rotation system and  $e$  is a directed edge of  $H$ , then there are no cycles comprised of  $<_{\wedge}^e$  and  $<_{\parallel}^e$  relations in  $H$ .*

We obtain a short proof by building a weight function and using Claim 1 to show that a minimum weight cycle  $\mathcal{C}$  does not exist.

*Proof.* Let  $n \geq 9$ ,  $H$  be an  $(n, 8)$ -rotation system and  $e = \overrightarrow{(u, v)}$  be a directed edge of  $H$ . For an arbitrary cycle (string)  $\mathcal{A}$  of relations, define  $\mathcal{A}_{\ell}$  to be length  $\mathcal{A}$ ,  $\mathcal{A}_{\parallel}$  to be the number of  $<_{\parallel}^e$  relations in  $\mathcal{A}$  and  $\mathcal{A}_w = \mathcal{A}_{\ell} + \mathcal{A}_{\parallel}$  to be the weight of  $\mathcal{A}$ .

By way of contradiction, let  $\mathcal{C} = (a_0, a_1, \dots, a_{k-1}, a_0)$  be a cycle of  $<_{\wedge}^e$  and  $<_{\parallel}^e$  relations with minimum cycle weight such that  $a_i <^e a_{i+1}$  for  $i \in \mathbb{Z}_k$ .

**Claim 1.** *There is no string of relations  $\mathcal{A}$  in  $\mathcal{C}$  defined on at most 8 vertices and having  $\mathcal{A}_w \geq 4$ .*

*Proof.* By way of contradiction, assume  $\mathcal{C} = (a_i, \dots, a_j, \dots, a_i)$  and  $\mathcal{A} = (a_i, \dots, a_j)$  such that  $|V(a_i, \dots, a_j, e)| \leq 8$  and  $\mathcal{A}_w \geq 4$ .  $V(a_i, \dots, a_j, e)$  induces a realizable 8-vertex rotation system  $H_1$  in  $H$ . Let  $D$  be a realization of  $H_1$ . In  $D$ ,  $a_i <^e a_j$  for some  $<_{\wedge}^e$ ,  $<_{\parallel}^e$  or  $<_{\Delta}^e$  relation.

If this relation is  $<_{\wedge}^e$  or  $<_{\parallel}^e$ , then the cycle  $(a_i, a_j, \dots, a_{i-1}, a_i)$  exists in  $H$  and has smaller weight than  $\mathcal{C}$ , a contradiction.

It follows that  $a_i <_{\Delta}^e a_j$ . By Corollary 5.8, there is a string of  $<_{\wedge}^e$  and  $<_{\parallel}^e$  relations  $a_i <^e \dots <^e a_j$  in  $H_1$  with at most 7-vertices that contributes 3 to the weight of any cycle containing it. Replacing  $a_i <^e \dots <^e a_j$  with this string of relations reduces the weight of  $\mathcal{C}$ , a contradiction. ■

Let  $i \in [k]$ . Let us partition this proof into three cases depending on if  $\mathcal{C}$  contains two consecutive  $<_{\parallel}^e$  relations, if  $\mathcal{C}$  contains a  $<_{\parallel}^e$  relation and does not contain two consecutive  $<_{\parallel}^e$  relations, or if  $\mathcal{C}$  contains no  $<_{\parallel}^e$  relations.

**Case 1.**  $\mathcal{C}$  contains two consecutive  $<_{\parallel}^e$  relations.

Without loss of generality  $a_i <_{\parallel}^e a_{i+1} <_{\parallel}^e a_{i+2}$ . By Claim 1, this does not occur.

**Case 2.**  $\mathcal{C}$  contains no  $<_{\parallel}^e$  relations.

In this case,  $\mathcal{C}_\ell > 4$  as a cycle of  $<_{\wedge}^e$  relations of length at most 4 is defined on some realizable 8-vertex rotation system in  $H$ , a contradiction. It follows that  $a_i <_{\wedge}^e a_{i+1} <_{\wedge}^e a_{i+2} <_{\wedge}^e a_{i+3} <_{\wedge}^e a_{i+4}$  is a string of relations in  $\mathcal{C}$  with distinct edges defined on at most 8 vertices, a contradiction with Claim 1.

**Case 3.**  $\mathcal{C}$  contains a  $<_{\parallel}^e$  relation, but does not contain two consecutive  $<_{\parallel}^e$  relations.

Without loss of generality, assume  $a_i <_{\parallel}^e a_{i+1}$ . By Case 1, without loss of generality  $a_{i-1} <_{\wedge}^e a_i <_{\parallel}^e a_{i+1} <_{\wedge}^e a_{i+2}$ . It is clear that  $a_{i-1}$  and  $a_{i+2}$  are distinct edges as they share an end with  $a_i$  and  $a_{i+1}$ , respectively. Observe that  $a_{i-1} <_{\wedge}^e a_i <_{\parallel}^e a_{i+1} <_{\wedge}^e a_{i+2}$  is defined on at most 8 vertices, a contradiction with Claim 1, as desired.  $\square$

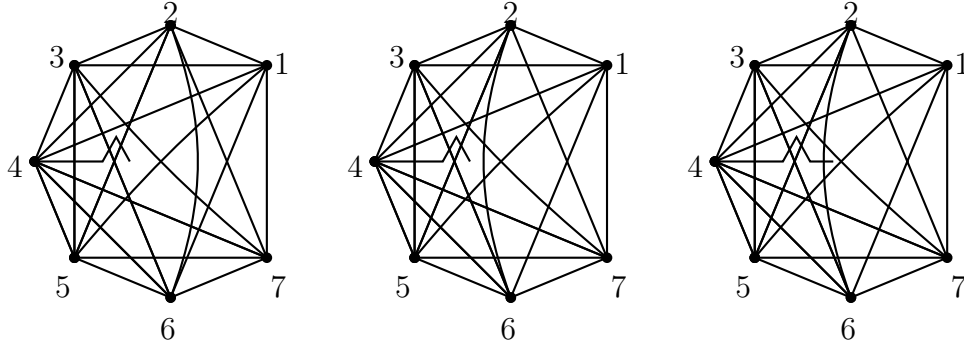


Figure 37: Drawing a partial edge  $(4, 8_4)$  into a drawing of  $K_7$ .

## 6 Simple Drawings of $(n, 6)$ -Rotation Systems

It is important to keep in mind the process in which we will find a realization of an  $(n, n - 1)$  rotation system  $H_n$ . Given such a rotation system, we will draw the smallest known realizable rotation system contained in  $H_n$ , which in this section will be a simple drawing of  $K_6$ . From there, we find a new drawing that contains a single new edge to a new vertex  $v$ . For each edge incident to  $v$  after the first, we will draw that edge from  $v$  sequentially by its crossing segments, possibly choosing to cross other edges incident to  $v$  and possibly choosing to change the underlying drawing without  $v$ . The resulting drawing is not guaranteed to be simple, however, it will have the rotation at every vertex other than  $v$  coincide with its rotation in  $H_n$ . Finally, untangling the edges will result in a simple drawing, and a short argument will show that the simple drawing is a realization of  $H_n$ .

We start this section by showing in Lemma 6.1 that there is a simple drawing of  $K_{n-1} + e$  for some  $(n, n - 1)$ -rotation system. The statement and proof of this is similar to Lemma 4.13, however, Lemma 6.1 will only be used to form a base case for the induction of  $(n, n - 1)$ -rotation systems. After the base case has been established, we have all the tools to make an inductive proof that  $(n, n - 1)$ -rotation systems are realizable for  $n \geq 6$ , an equivalent statement to Theorem 1.1.

The proof of Lemma 6.1 is straight forward. We start with an  $(n, n - 1)$ -rotation system  $H$  and a realization  $D$  of  $K_{n-1}$ . Commence by starting to draw an edge  $e$  from a vertex  $u$  in  $K_{n-1}$  to an  $n$ th vertex  $v$ . Since we know that

$<^e$  is a partial ordering, we can always find an edge  $c$  that must be crossed next. Since no drawing of  $K_4$  has  $c$  separated from the face containing the non-vertex end of our partial edge we are drawing, by Theorem 3.8, there is a set of Reidemeister III moves that results in a drawing of  $c$  on the boundary of the face that our partial edge has crossed into. Crossing  $c$  and applying a simple induction gives us the result.

As an example, we see in Figure 37 that the edge  $(4, 8)$  is being drawn into a simple drawing of  $K_7$ . In the left simple drawing,  $(4, 8)$  needs to cross  $(2, 6)$ , however  $(2, 6)$  does not bound the correct face. The center drawing is obtained from the left by performing a Reidemeister III move on edges  $\Delta_{(2,6),(3,7),(1,5)}$  to move a non-trivial segment of  $(2, 6)$  to the appropriate face. Finally, the right drawing is obtained from the center drawing by having  $(4, 8)$  cross  $(2, 6)$ .

**Lemma 6.1.** *Let  $n \geq 6$ ,  $H$  be an  $(n, n - 1)$ -rotation system, and  $e = \overrightarrow{(u, v)}$  a directed edge of  $H$ . If  $D_v$  is a realization of  $H - \{v\}$ , then there exists a simple drawing  $\bar{D}_v + e$  that has  $v$  in its respective region determined by  $H$ , the rotation at  $u$  is the same as  $H_n$ , and the rotation at every vertex in  $V(H_n - \{u, v\})$  is the same in both  $\bar{D}_v + e$  and  $H_n - v$ .*

*Proof.* By Lemma 4.15, we let  $n \geq 7$ . Let  $H$  be an  $(n, n - 1)$ -rotation system on the vertices  $[n - 2] \cup \{u, v\}$ ,  $e = \overrightarrow{(u, v)}$  be a directed edge of  $H$ ,  $D_i$  be a realization of  $H - \{i\}$  for  $i \in V(H)$ ,  $D_e$  be a realization of  $H - \{u, v\}$ , and  $E$  be the set of edges that  $e$  crosses determined by  $H$ .

Let  $G$  be the directed graph where  $V(G) = E$  and the arc  $\overrightarrow{(f, g)}$  exists if  $f <_\wedge^e g$  or  $f <_\parallel^e g$ . By Theorems 5.1, 5.6, and 5.12,  $G$  contains no directed cycles.

By greedily picking and deleting source vertices of  $G$ , it follows that there exists a sequence of vertices  $\{c_i\}_{i=1}^{cr_H(e)}$  such that for  $i_1 > i_2$ , then there is no directed path from  $c_{i_1}$  to  $c_{i_2}$  in  $G$ . In particular, no chain of  $<_\wedge^e$  or  $<_\parallel^e$  relations has  $c_{i_1} <^e \cdots <^e c_{i_2}$ .

Let  $e_i$  be the segment of  $e$  starting at  $u$  in the correct rotation that has crossed  $c_1$  to  $c_i$  and ends at a point  $v_i$  in some face of  $D_v$ , and  $D_i$  be the drawing of  $D_v + e_i$  with the rotation at  $u$  the same as  $H_n$ . We will prove  $D_i$  exists for all  $i \in \{0\} \cup [cr_H(e)]$  inductively on  $i$ .

For the sake of the reader, we illustrate an example of  $D_v + e_i$  in Figure 38 for  $i = 3$  with  $e_i = (4, 8_3)$  represented by the dashed line in the figure and

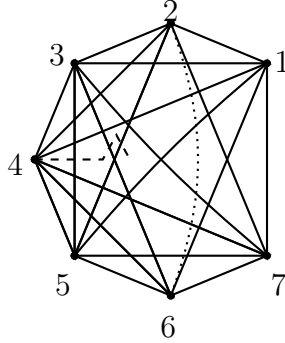


Figure 38:  $D_8 + (4, 8_3)$ .

$c_{i+1} = (2, 6)$  dotted line in the figure representing the next edge in  $\{c_i\}_{i=1}^{cr_H(e)}$  that  $e_i$  crosses.

Note that the drawing of  $D_v$  satisfies the definition of  $D_0$  by starting  $e_0$  in the correct rotation at  $u$ .

Assume  $D_i$  exists for some  $i \in \{0\} \cup [cr_H(e) - 1]$ . By way of contradiction, assume  $D_{i+1}$  does not exist. Let  $R_{v_i}$  be the face in  $D_{i-1}$  containing  $v_i$  (if  $i=0$ , then let  $D_{i-1} = D_v$ ). There are two cases dependent upon  $i = cr_H(e)$  or  $i < cr_H(e)$ .

**Case 1.**  $i = cr_H(e)$ .

In this case,  $e_i$  has crossed all edges of the sequence  $\{c_i\}_{i=1}^{cr_H(e)}$  in  $D_i$ . Let  $T$  be the intersection of sides of triangles in  $D_v$  containing  $v$  determined by  $H$ . If  $R_{v_i}$  does not correspond to  $T$ , then by Corollary 3.3, there is some 3-cycle having  $e_i$  and  $v$  on separate sides.

Let  $D_{T+e}$  be a realization of the 5-vertex rotation system induced on  $T$  and  $e$  in  $H$ . Since  $e$  crosses edges of  $D_v$  in the order of the sequence  $\{c_i\}_{i=1}^{cr_H(e)}$ , and does so in the correct orientation, it follows that the crossings of  $e$  with  $T$  are the same in both  $D_{e_i}$  and  $D_{T+e}$ , a contradiction with  $v$  and  $e_i$  being on opposite sides of  $T$  in  $D_{e_i}$ .

It follows that  $R_{v_i}$  correspond to  $T$ . Drawing the vertex  $v$  at the end of  $e_i$  in  $D_i$  produces the desired simple drawing  $\bar{D}_v + e$ .

**Case 2.**  $i < cr_H(e)$ .

By Lemma 3.4, the only segment of  $e_i$  in  $R_{v_i}$  is the segment formed by  $e_{i-1}$  crossing into  $R_{v_i}$ . It follows that if  $c_{i+1}$  is on the boundary of  $R_{v_i}$ , then  $e_i$  can be extended in  $D_i$  to cross  $c_{i+1}$  in the correct orientation to create  $D_{i+1}$ . If the orientation of this crossing does not follow  $H$ , then for any 3-cycle  $T$  containing  $c_{i+1}$ ,  $e_i$  is on the wrong side of  $T$  to cross  $c_{i+1}$ .

Let  $D_{T+e}$  be a realization of the 5-vertex rotation system induced on  $T$  and  $e$  in  $H$ . Since  $e$  crosses edges of  $D_v$  in the order of the sequence  $\{c_i\}_{i=1}^{cr_H(e)}$ , and does so in the correct orientation, it follows that the crossings of  $e$  with  $T$  are the same in both  $D_{e_i}$  and  $D_{T+e}$ . By definition of  $\{c_i\}_{i=1}^{cr_H(e)}$ ,  $c_{i+1}$  is the next edge  $e_i$  crosses in  $D_{T+e}$ , a contradiction with  $e_i$  being on wrong sides of  $T$  in  $D_{e_i}$  to cross  $c_{i+1}$ .

It follows that  $c_{i+1}$  is not on the boundary of  $R_{v_i}$ , else we would cross it. Theorem 3.8 give two possibilities for  $c_{i+1}$

**Case 2.1.** *There is some drawing  $\mathcal{D}$  of  $D_v$  on a  $K_4$  containing  $c_{i+1}$  such that no face of  $\mathcal{D}$  contains  $R_{v_i}$  and has  $c_{i+1}$  on its boundary.*

By Lemma 4.12, there is some 3-cycle  $T$  in  $\mathcal{D}$  that has  $e_i$  on the wrong side of  $T$  to cross  $c_{i+1}$ . Let  $D_{T+e}$  be a realization of the 5-vertex rotation system induced on  $T$  and  $e$  in  $H$ . Since  $e$  crosses edges of  $D_v$  in the order of the sequence  $\{c_i\}_{i=1}^{cr_H(e)}$ , and does so in the correct orientation, it follows that the crossings of  $e$  with  $T$  are the same in both  $D_{e_i}$  and  $D_{T+e}$ . By definition of  $\{c_i\}_{i=1}^{cr_H(e)}$ ,  $c_{i+1}$  is the next edge  $e_i$  crosses in  $D_{T+e}$ , a contradiction with  $e_i$  being on wrong sides of  $T$  in  $D_{e_i}$  to cross  $c_{i+1}$ .

**Case 2.2.** *There are sets of edges  $X_i$  and a sequence of Reidemeister III moves  $\{\rho_{X_i}\}_{i=1}^k$  in  $D_v$  that places  $c_{i+1}$  on the boundary of the component containing  $v_i$  such that the intersection of each  $\Delta_{X_i}$  with  $v_i$  is empty, and for each Reidemeister III move  $\rho_{X_i}$  not on the edge  $c$ , there exists a  $j$  such that  $\rho_{X_j}$  is on the edge  $c$  and  $\Delta_{X_i}$  is contained in  $\Delta_{X_j}$ .*

If the non-vertex end of  $e_i$  was inside any  $\Delta_{X_i}$ , then  $R_{v_i}$  is in  $\Delta_{X_i}$ , and thus,  $P \in \Delta_{X_i}$ , a contradiction. Therefore, the non-vertex end of  $v_i$  is not in  $\Delta_{X_i}$  for any  $i$ . By Lemma 3.4, it follows that there is at most one non-trivial segment of  $e_i$  in any  $\Delta_{X_i}$  that crosses in and out of the region. Applying a Reidemeister III move over this segment, allows us to perform next apply

$\rho_{X_i}$  and continue.

It follows that  $c_{i+1}$  can be placed on the boundary of  $R_{v_i}$ . Having  $e_i$  cross  $c_{i+1}$  produces the drawing  $D_{i+1}$ , a contradiction with  $D_{i+}$  not existing.  $\square$

We come to the first of two main theorems in this thesis, the proof of which is similar to the proof of Lemma 6.1. To finish this section and prove Theorem 6.2, we use Lemma 6.1 to form a base case for induction. In the inductive step, we draw a new edge  $f = (u_f, v)$  sequentially by its crossing from  $v$  to  $u_f$  and use the partial ordering  $<^e$  found in Section 5 at each iteration to determine which edge  $c$  is crossed next.

To make sure we can produce a drawing in which  $c$  is crossed next, we use Theorem 3.8 from Section 3 to show we can change the underlying drawing without  $v$  (by using Reidemeister III moves) so that our partial edge can cross  $c$  (possibly having to cross other edges incident to  $v$ ). It is important that we check that for each Reidemeister III move on  $\Delta_{x,y,z}$  for arbitrary edges  $x, y, z$ , that  $v \notin \Delta_{x,y,z}$ . To prove this, we use the special conditions ii. and iii. in Theorem 3.8

Once all the crossings have been performed, we argue we can connect this partial edge at  $u_f$  to form  $f$ . The only problem with the resulting drawing is that the edge  $(u_f, v)$  may have crosses other edges incident to  $v$ . We finish by turning the associated drawing into a simple drawing by untangling the edges and argue that the rotation at each vertex is the same as the rotations in  $H_n$ , as desired.

**Theorem 6.2.** *Let  $n \geq 6$ . If  $H_n$  is an  $(n, n - 1)$ -rotation system, then  $H_n$  is realizable.*

*Proof.* Let  $n \geq 6$ ,  $H_n$  be an  $(n, n - 1)$ -rotation system and  $v$  be a vertex in  $V(H_n)$ . By Lemma 4.15, without loss of generality  $n \geq 7$ .

Let  $E_v$  be the set of edges having  $v$  as an endpoint. Let us build a simple drawing  $D$  of  $K_n$  with vertex set  $V(H_n)$  that has the rotation at every vertex other than  $v$  the same as in  $H_n$ . By Observation 2.6 and the fact that  $H_n$  is an  $(n, 4)$ -rotation system, it would follow that the rotation at  $v$  is correct and  $D$  is a simple drawing realizing  $H_n$

To show such a drawing  $D$  exists, we will show there is a simple drawing  $D_i + F_i$  such that  $F_i \subseteq E(v)$ ,  $1 \leq i \leq n - 1$ ,  $|F_i| = i$ ,  $D_i + F_i$  has the same rotations at each vertex as  $H_n - v$ ,  $v$  is in the correct region in  $D_i$  determined by  $H_n$ , and for every edge  $(u_j, v)$  in  $F_i$ , the rotation at  $u_j$  in  $D_i + F_i$  is the same as the rotation at  $u_j$  in  $H_n$ .



Let us prove this by induction on  $i$ . Let  $e = (u_e, v)$  be an edge having  $v$  as an endpoint and  $F_1 = \{e\}$ . By Lemma 6.1, there is a simple drawings  $D_1 + F_1$  that has  $v$  in its respective region determined by  $H_n$ , the rotation at  $u_e$  is the same as in  $H_n$ , and the rotation at every vertex in  $V(H_n - \{u_1, v\})$  is the same in both  $D_1 + F_1$  and  $H_n - v$ .

Assume for some  $i$ ,  $D_i + F_i$  exists. All we need to show it  $D_{i+1} + F_{i+1}$  exists. Let  $f = (u_f, v)$  be an edge in  $E(v) \setminus F_i$  and  $E_f$  be the set of edges  $f$  crosses in  $H_n$ .

Let  $f$  be directed from  $v$  to  $u_f$  and  $G$  be the directed graph where  $V(G) = E_f$  and the arc  $\overrightarrow{(g, h)}$  exists if  $g <_{\wedge}^f h$  or  $g <_{\parallel}^f h$ . By Theorems 5.1, 5.6, and 5.12,  $G$  contains no directed cycles.

By greedily picking and deleting source vertices of  $G$ , it follows that there exists a sequence of vertices  $\{c_k\}_{k=1}^{cr_{H_n}(f)}$  such that for  $k_1 > k_2$  there is no directed path from  $c_{k_1}$  to  $c_{k_2}$  in  $G$ . In particular, no chain of  $<_{\wedge}^f$  or  $<_{\parallel}^f$  relations has  $c_{k_1} <^f \dots <^f c_{k_2}$ .

Similar to the proof of Lemma 6.1, we show by induction there is a simple drawing  $D_i^j + F_i + f_j$  for  $0 \leq j \leq cr_{H_n}(f)$  such that  $f_j$  is an edge starting at  $v$  crossing  $\{c_k\}_{k=1}^j$  in their respective orders and oriented crossings,  $D_i^j + F_i + f_j$  has the same rotations at each vertex as  $H_n - v$ , and for every edge  $(u_j, v)$  in  $F_i \cup f$ , the rotation at  $u_j$  in  $D_i^j + F_i + f_j$  is the same as the rotation at  $u_j$  in  $H_n$ .

For the sake of the reader, we illustrate an example of  $D_i^j + F_i + f_j$  in Figure 39 for  $i = 3$   $j = 2$  with  $F_2 = \{(1, 8), (7, 8)\}$ ,  $f_j = (4, 8_3)$  represented by the dashed line in the figure and  $c_{j+1} = (2, 6)$  dotted line in the figure representing the next edge in  $\{c_k\}_{k=1}^{cr_{H_n}(f)}$  that  $f_j$  crosses.

By drawing a small segment of an edge at  $v$  in  $D_i + F_i$ , it is clear  $D_i^0 + F_i + f_0$  exists.

By induction, assume  $D_i^j + F_i + f_j$  exists. Let  $v_j$  be the non-vertex end of  $f_j$  and let  $R_{v_j}$  be the face in  $D_i^j + F_i + f_j$  that contains  $v_j$ . There are two cases depending on if  $j = cr_H(f)$  or  $j < cr_f(H)$ .

**Case 1.**  $j = cr_H(f)$ .

In this case, the edge  $f_j$  has crossed all edges of  $\{c_k\}_{k=1}^{cr_{H_n}(f)}$  in  $D_i^j + F_i + f_j$ . Let  $T$  be the intersection of sides of triangles in  $D_i^j$  containing  $u_f$  determined by  $H_n$ . If  $R_{v_j}$  does not correspond to  $T$ , then by Corollary 3.3, there is some 3-cycle  $T$  in  $\mathcal{D}$  that has  $f_j$  and  $u_f$  on opposite sides of  $T$ .

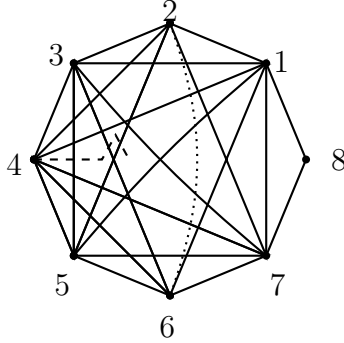


Figure 39:  $D_2^3 + \{(1, 8), (7, 8)\} + (4, 8_3)$ .

Let  $D_{T+f}$  be a realization of the 5-vertex rotation system induced on  $T$  and  $f$  in  $H_n$ . Since  $f$  crosses edges of  $D_i^j + F_i + f_j$  in the order of the sequence  $\{c_k\}_{k=1}^{cr_{H_n}(f)}$ , and does so in the correct orientation, it follows that the crossings of  $f$  with  $T$  are the same in both  $D_i^j + F_i + f_j$  and  $D_{T+f}$ , a contradiction with  $u_f$  and  $f_j$  on opposite sides of  $T$ .

Therefore,  $R_{v_j}$  correspond to  $T$ . It follows that  $u_f$  is in  $R_{v_j}$ . Since  $D_j^i$  is simple,  $f_j$  can be extended to connect  $v_j$  to  $u_f$  (possibly crossing edges of  $F_i$ ) in  $R_{v_j}$  producing a drawing  $D_{i+1} + F_{i+1}$  with  $D_{i+1} + f_j$  being simple. Since  $R_{v_j}$  corresponds to  $T$ , observation 3.7 applied to the drawing of the  $K_4$  involving  $u_f, v$  and the former and ladder consecutive vertices around  $v$  in the rotation at  $u_f$  in  $H_n$  implies that the rotation at  $u_f$  in the drawings corresponds to the rotation at  $u_f$  in  $H_n$ .

This drawing may not be simple, to be exact,  $f$  may cross the edges in  $F_i$  at least once, but finitely many times. Assume this is the case, as if not we are done. We will show there is a simple drawing  $D_{i+1} + \bar{F}_{i+1}$  that preserves the rotation of the vertices in  $D_{i+1} + F_{i+1}$  other than  $v$  such that  $D_{i+1} + \bar{F}_i$  is simple and corresponds to  $H_n$ .

Let  $g \in F_i$  be an edge that crosses  $f$  in  $D_{i+1} + F_{i+1}$  such that  $g = (u_g, v)$ . By Lemma 4.14, there exists a closed curve  $\delta$  comprised of a non-trivial segment  $f^c$  of  $f$  and a non-trivial segments  $g^c$  of  $g$  such that a side  $S$  of  $\delta$  does not contain  $v, u_g$  and  $u_f$ . In  $D_{i+1} + F_{i+1}$ ,  $S$  could contain a vertex.

**Case 1.1.**  $S$  contains a vertex  $z$ .

Let  $x \in \{u_g, u_f\}$ . In  $D_{i+1} + F_{i+1}$ , since  $u_x$  is not in  $S$  and  $z$  is in  $S$ , it must be the case that  $(z, u_x)$  crosses  $\delta$ . Since  $D_{i+1} + (u_x, v)$  is simple, it follows that  $(z, u_x)$  crosses  $(u_y, v)$  for  $y \in \{u_f, u_g\} \setminus \{x\}$  in  $D_{i+1} + F_{i+1}$ , and thus in  $D_{i+1} + (u_x, v)$ .

The rotation system on  $v, u_f, u_g$  and  $z$  in  $H_n$  is realizable, and contains at most one crossing. It follows that at least one of the crossings  $(z, u_x)$  with  $(u_y, v)$  does not occur. By Observation 2.6, one of the rotations at  $u_g, u_f$  or  $z$  in  $D_{i+1} + f$  or  $D_{i+1} + g$  does not match its rotation in  $H_n$ , a contradiction.

**Case 1.2.**  $S$  contains no vertices.

Let  $h$  be an edge that crosses  $\delta$ . Since  $S$  contains no vertices, it follows that  $h$  crosses  $\delta$  twice, once at  $f_c$  and once at  $g_c$ . It follows that an edge crosses  $f_c$  if and only if it crosses  $g_c$ .

Let  $\bar{f}$  be the edge  $f$  rerouted to take  $g_c$  instead of  $f_c$ . Similarly, let  $\bar{g}$  to be the edge  $g$  rerouted to take  $f_c$  instead of  $g_c$ . Since  $f_c$  and  $g_c$  cross the same edges, it follows that  $\bar{f}$  and  $\bar{g}$  cross the same edges as  $f$  and  $g$ , respectively.

Consider the new drawing of  $D_{i+1} + \bar{F}_{i+1}$  that has  $\bar{f}$  replacing  $f$  and  $\bar{g}$  replacing  $g$  in  $D_{i+1} + F_{i+1}$  with the crossings on  $\delta$  uncrossed.

If  $\delta$  has  $v$  on the boundary, then the number of crossings between  $f$  and  $g$  has reduced by 1. If  $\delta$  does not have  $v$  on the boundary, then the number of crossings between  $f$  and  $g$  has reduced by 2.

Comparing the rotations of the vertices, only the rotation at  $v$  could have possible changed. Repeatedly applying this procedure results in a drawing that is simple, and has all the rotations at every vertex the same as in  $H_n$  other than  $v$ , as desired.

**Case 2.**  $j < cr_f(H)$ .

If  $c_{j+1}$  is on the boundary of  $R_{v_j}$ , then we cross edges of  $F_i$  and cross  $c_{j+1}$  to form a drawing  $D_i^{j+1} + F_i + f_{j+1}$ . Such a drawing may not be simple, however, a similar argument as in Case 1.2. using Lemma 4.14, produces a desired simple drawing. Therefore, without loss of generality,  $c_{j+1}$  is not on the boundary of  $R_{v_j}$ .

Suppose there is a drawing  $D_c$  on some  $K_4$  containing  $c_{j+1}$  in  $D_i^j$  that separates  $c_{j+1}$  from  $R_{v_j}$ . The vertices of such a  $K_4$  along with the ends of the edge  $f$  are defined on a 6-vertex rotation system in  $H_n$ .

If both  $u_f$  and  $v$  are not in  $D_c$ , then by definition of  $\{c_j\}_{j=1}^{cr_H(e)}$  and Lemma 4.12,  $D_c$  can be extended with  $f_j$  crossing  $c_{j+1}$ , a contradiction with no face of  $D_c$  containing  $R_{v_i}$  and having  $c_{j+1}$  on its boundary.

Therefore, at least one of  $u_f$  or  $v$  is in  $D_c$ . It follows that  $D_c + f_j$  is a partial drawing of some realizable 5-vertex rotation system of  $H$ . A realization of such a 5-vertex rotation system must contain  $D_c + f_j$  as  $D_c$  is unique to the rotation system and  $f_j$  follows  $\{c_j\}_{j=1}^{cr_H(e)}$ . Therefore,  $f_j$  crosses  $c_{j+1}$  next in the realization and can not do so in  $D_c + f_j$ , a contradiction.

1. There is a sequence (possibly empty) of Reidemeister III moves  $\{\rho_{X_i}\}_{i=1}^k$  with sets of edges  $X_i$  such that  $D_{i+1} = \rho_{X_i}(D_i)$  with:
  - i. A non-trivial segment of  $c$  is on the boundary of the face of  $D_{k+1}$  containing  $P$ ;
  - ii.  $P \notin \Delta_{X_i}, \forall i \in [k]$ ; and
  - iii. For  $i \in [k]$ , if  $c \notin X_i$ , then there exists  $j > i$  in  $[k]$  such that  $c \in X_j$  and  $\Delta_{X_i} \subset \Delta_{X_j}$  in  $D_i$ ; or

It follows by Theorem 3.8, that there are sets of edges  $X_\ell$  and a sequence of Reidemeister III moves  $\{\rho_{X_\ell}\}$  in  $D_i^j$  that places  $c$  on the boundary of the component containing  $v_j$  such that the intersection of each  $\Delta_{X_\ell}$  with  $v_j$  is empty, and for each Reidemeister III move  $\rho_{X_{\ell_1}}$  not on the edge  $c$ , there exists an  $\ell_2 > \ell_1$  such that  $\rho_{X_{\ell_2}}$  is on the edge  $c$  and  $\Delta_{X_{\ell_1}}$  is contained in  $\Delta_{X_{\ell_2}}$ .

There are two cases, either for every  $\rho_{X_\ell}$ ,  $\Delta_{X_\ell}$  does not contain  $v$ , or there exists a  $\rho_{X_\ell}$  such that  $\Delta_{X_\ell}$  contains  $v$ .

**Case 2.1.** *For every  $\rho_{X_\ell}$ ,  $\Delta_{X_\ell}$  does not contain  $v$ .*

Apply the Reidemeister III moves  $\rho_{X_\ell}$  to  $D_i^j + F_i + f_j$  until we find an  $\ell_1$  such that  $\Delta_{X_{\ell_1}}$  is not empty. If  $\ell_1$  does not exist, then we can extend  $f_j$  (crossing edges in  $F_i$ ) to cross  $c_{i+1}$  after applying the Reidemeister III moves to form a drawing  $D_i^{j+1} + F_i + f_{j+1}$ . Such a drawing again can be reduced to be simple, similar to Case 1.2. using Lemma 4.14.

Therefore, such an  $\ell_1$  exists. Let  $\tilde{D}_i^j + F_i + f_j$  be the simple drawings after applying the Reidemeister III moves  $\{\rho_{X_\ell}\}_{\ell=1}^{\ell_1-1}$ . Note  $P, v$ , and nothing in  $\tilde{D}_i^j$  is in  $\Delta_{X_{\ell_1}}$ . It follows that only segments of edges in  $F_i$  and segments of  $f_j$  can be in  $\Delta_{X_{\ell_1}}$ .

Since  $\tilde{D}_i^j + F_i + f_j$  is simple, it follows that these segments are not crossing. Since these segments are not crossings, they each cross the same pair of edges  $g_1$  and  $g_2$  on the boundary of  $\Delta_{X_{\ell_1}}$ .

It follows that there exists an edge segment  $h$  that crosses both  $g_1$  and  $g_2$  such that one side of  $\gamma_{g_1, g_2, h}$  is empty and contained in  $\Delta_{X_{\ell_1}}$ . Applying a Reidemeister III move to this triple of edges over  $\gamma_{g_1, g_2, h}$  reduces the number of edge segments crossing  $\Delta_{X_{\ell_1}}$ .

Repeatedly applying this process results in a drawing with  $\Delta_{X_{\ell_1}}$  empty, and thus  $\rho_{X_{\ell_1}}$  can be applied.

Repeatedly applying the rest of the Reidemeister III moves in  $\{\rho_{X_\ell}\}$  in a similar fashion will produce a simple drawing  $D_v^\rho + F_i + f_j$  with  $c_{i+1}$  on the boundary of the region containing  $v_j$  in  $D_v^\rho$ . Extend  $f_j$  at  $v_j$  in this drawing to cross  $c_{j+1}$  (possibly crossing edges of  $F_i$ ) to form a drawing  $D_i^{j+1} + F_i + f_{j+1}$ . Such a drawing may not be simple, however, a similar argument as in Case 1.2. using Lemma 4.14, produces a desired simple drawing. Therefore, without loss of generality,  $c_{i+1}$  is not on the boundary of  $R_{v_j}$ .

**Case 2.2.** *There exists a  $\rho_{X_\ell}$  such that  $\Delta_{X_\ell}$  contains  $v$ .*

By Theorem 3.8, there exists  $X_c$  such that  $v \in \Delta_{X_c}$  for  $\ell < c$ . We will show the existence of such a drawing contradicts the fact that  $c_{i+1}$  is the next edge to cross as per the sequence  $\{c_j\}_{j=1}^{cr_H(e)}$ . Let  $(v_1, w_1), (v_2, w_2)$  and  $c_{i+1} = (v_c, w_c)$  be the three edges that bound  $\Delta_{X_c}$ .

Consider the simple subdrawing  $D_7 + f_j$  in  $D_i^j + F_i + f_j$  induced on  $f_j$  and the vertices  $v_x, w_x$  and  $u_f$ , for  $x \in \{1, 2, c\}$  and let  $D_6$  be the induced drawing on the vertices  $v_x, w_x$ .

By our assumption,  $v \in \Delta_{X_c}$ . Without loss of generality, label the ends of  $(v_x, w_x)$  in such a way that when traversing  $(v_x, w_x)$ , starting at  $v_x$  the crossing of  $c_{i+1}$  precedes the crossings of  $(v_y, w_y)$ , for  $y \in \{1, 2\} \setminus \{x\}$ .

By Theorem 3.8,  $R_{v_j}$  is not in  $\Delta_{X_c}$ . Since there is exactly one Reidemeister III move in  $D_6$  on  $c$ , it follows that  $R_{v_j}$  is contained on the side  $S_w$  of  $\gamma_{(v_1, w_1), (v_2, w_2), (w_1, w_2)}$  not containing  $v$ , and the face containing  $R_{v_j}$  in  $D_7 + f_j$  has the crossing of  $(v_1, w_1)$  and  $(v_2, w_2)$  on the boundary.

**Claim 1.**  $|V(D_7 + f_j)| < 8$ .

*Proof.* Since  $|V(D_7 + f_j)| \geq 8$ ,  $n \geq 8$ . Consider the simple subdrawing  $D_{u_f}$  on edges  $(v_1, w_1), (v_2, w_2), c, (w_1, w_1)$  and vertex  $v$  in  $D_7 + f_j$ . Such a drawing

is contained in every realization of its associated 7-vertex rotation system. Let  $\bar{D}_f$  be such a realization.

By the position of  $v$ , there is no sequence of Reidemeister III moves that brings  $c_{i+1}$  to the boundary of  $R_{v_j}$  in  $\bar{D}_f$ . By Theorem 3.8, that there is a  $K_4$  in  $\bar{D}_f$  that separates  $c_{i+1}$  from  $R_{v_j}$ . Since there exists a sequence of Reidemeister III moves in  $D_i^j$  that bring  $c_{i+1}$  to  $R_{v_j}$ , it follows that this  $K_4$  contains  $v$ .

Since the drawing of this  $K_4$  is uniquely determined by  $H_n$  and  $f_j$  follows  $\{c_k\}_{k=1}^{cr_{H_n}(f)}$ , it follows that the drawing of this  $K_4$  along with  $f_j$  is contained in any realization of its induced 5-vertex rotation system, a contradiction with  $c_{i+1}$  being the next edge crosses in this  $K_4$  and the  $K_4$  separating  $R_{v_j}$  from  $c_{i+1}$ .  $\blacksquare$

Let  $D_{v_x} + f_j$  be the drawing of  $D_7 + f_j$  without  $v_x$ , for  $x \in \{1, 2\}$ . Since  $v_j$  is the non-vertex end of  $f_j$  and  $f_j$  does not cross  $c_{i+1}$ , it follows up to symmetry on  $(v_1, w_1)$  and  $(v_2, w_2)$  that starting at  $v$ ,  $f_j$  crosses  $(v_1, w_1)$  then crosses  $(w_1, w_2)$  into  $S_w$ , or  $f_j$  crosses  $(v_1, w_1)$  then crosses  $(v_2, w_2)$  into  $S_w$ .

**Case 2.2.1.**  $f_j$  crosses  $(v_1, w_1)$  then crosses  $(w_1, w_2)$ .

It is clear that the edges  $(v_x, w_x)$  all have distinct endpoints for  $x \in \{1, 2, c\}$ , as each pair of edges crosses in a simple drawing.

By Claim 1,  $|V(D_7 + f_j)| < 8$ . By the crossings of  $f_j$ ,  $u_f = v_2$ . Let  $D_f$  be a realization of the induced 5-vertex rotation system on  $v, u_f, w_1, w_2, v_1$  in  $H_n$ . The drawing of the  $K_4$  on  $w_1, w_2, v_1, v_2$  is the same in both  $D_f$  and  $D_7 + f_j$ . Since  $f_j$  crosses edges in the order  $\{c_k\}_{k=1}^{cr_{H_n}(f)}$ , it follows that the drawing of  $w_1, w_2, v_1, v_2$  and  $f_j$  is the same in both  $D_f$  and  $D_7 + f_j$ .

Note that  $f_j$  can not cross out  $\gamma_{(v_1, w_1), (v_2, w_2), (w_1, w_2)}$  as it has crossed two of the three edges already, and is adjacent to the third at  $u_f$  in  $D_f$ . However,  $f$  ends at  $u_f$  and so it does cross this curve, a contradiction.

**Case 2.2.2.**  $f_j$  crosses  $(v_1, w_1)$  then crosses  $(v_2, w_2)$ .

Each pair of edge of  $c, f, (v_1, w_1), (v_2, w_2)$  cross in  $H_n$ , therefore  $|V(D_7 + f_j)| = 8$ , a contradiction with Claim 1.  $\square$

A simple induction using Theorem 6.2 gives way to a proof of Theorem 1.1.

## 7 Gioan's Theorem

This section is dedicated to Theorem 1.3 which states that two simple drawings  $D$  and  $D'$  of the same rotation system are the same up to a set of Reidemeister III moves applied to  $D$ . Together with Theorem 1.1, we have a complete characterization for when two simple drawings of  $K_n$  are different. In particular, two simple drawings of  $K_n$  either differ on their associated rotation systems or there is a sequence of Reidemeister III moves from one to the other. We restate Theorem 1.3 for the readers convenience.

**Theorem 1.3.** *Let  $n$  be a positive integer,  $H_n$  be a realizable complete  $n$ -vertex rotation system. If  $D$  and  $D'$  are two simple drawings realizing  $H_n$ , then one can be obtained from the other through a series of Reidemeister III moves.*

To prove Theorem 1.3, we start by comparing two simple drawings  $D$  and  $D'$  with the same rotation system. Suppose  $D$  and  $D'$  agree on  $K[\{v_1, \dots, v_r\}]$  plus some edges at  $v_{r+1}$  and a partial edge  $e_i$  starting at  $v_{r+1}$ , but they do not agree on  $e_{i+1}$  the extension of  $e_i$  in both drawings. In particular,  $c$  the last edge  $e_{i+1}$  crosses in  $D$  is not the same edge as  $c'$  the last edge  $e_{i+1}$  crosses in  $D'$ . The idea behind the proof of Theorem 7 is that  $e_i$  can cross an edge  $h$  that is one step closer to  $c'$  than  $c$  resulting in a realization  $\bar{D}$  that has the same rotation system as  $D$  and  $D'$  as seen in Figure 40. Furthermore, Lemma 7.2 shows there is a sequence of Reidemeister III moves from  $D$  to  $\bar{D}$  if  $\Delta_{c,e,h}$  exists (where  $e$  is the edge we are drawing). Lemma 7.4 shows that consecutive edges like  $c$  and  $h$  always have the property that  $\Delta_{c,e,h}$  exists. Applying a simple induction proves Theorem 1.3.

**Observation 7.1.** *Let  $D$  be a simple drawing of three directed edges  $e, f$  and  $g$  such that  $\Delta_{\{e,f,g\}}$  exists. If the order that  $e$  crosses  $f$  and  $g$  is known and pairwise two oriented crossings of  $e, f$  and  $g$  are known, then topologically there are exactly two drawings that realize the given information each having different oriented crossings.*

Suppose  $D$  is a simple drawing of  $K_n$ ,  $L$  is a subdrawing of  $D$  with a partial edge  $e_i$  and  $c$  is the next edge  $e_i$  crosses in  $D$ . We proceed by characterizing when a neighbour of  $c$  can be crossed in  $L$  to extend to a simple drawing of  $K_n$  with the same associated rotation system as  $D$ . Furthermore, we connect this new drawing and  $D$  by a series of Reidemeister III moves.

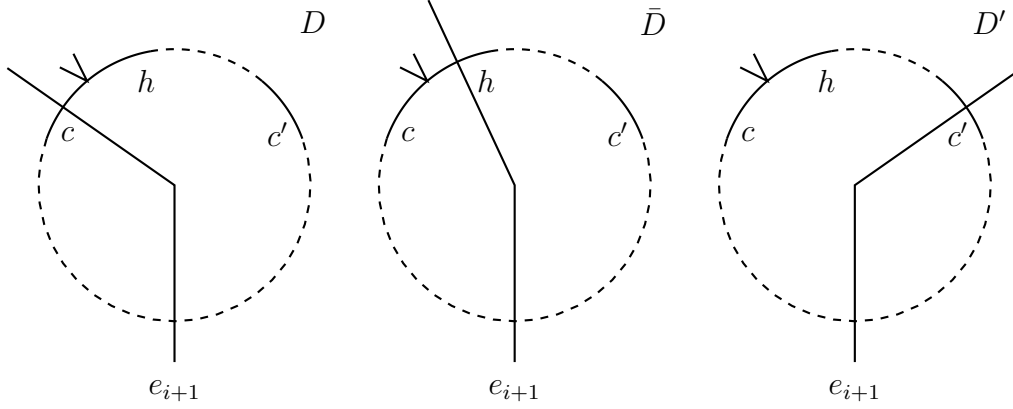


Figure 40: From  $D$  to  $\bar{D}$  and eventually to  $D'$ .

**Lemma 7.2.** *Let  $n \geq 6$ ,  $H_n$  be a realizable  $n$ -vertex rotation system,  $D$  be a simple drawing realizing  $H_n$ ,  $e = (u, v)$  be an edge of  $H_n$  and  $F_v$  be a subset of edges having  $v$  as an endpoint in  $D$ . Let  $r \geq 5$ ,  $L$  be a subdrawing of  $D$  on some complete graph  $K_r$  along with the edges  $F_v$  containing a segment of  $e$  labelled  $e_i$  starting at  $v$  and has  $i$  crossings in  $L$ , and  $R_L$  be the region containing the non-vertex end of  $e_i$  in  $L$ .*

*If  $e_i$  crosses  $f$  next in  $D$ ,  $f$  and  $g$  are consecutive edges on the boundary of  $R_L$ ,  $\Delta_{\{e,f,g\}}$  exists in  $D$  and  $e_i$  has not crossed  $g$  in  $L$ , then there is a simple drawing  $\bar{D}$  containing  $L$  with  $e_i$  crossing  $g$  next that realizes  $H_n$  and there is a sequence of Reidemeister III moves  $\{\rho_j\}_{j=1}^k$  from  $D$  to  $\bar{D}$ .*

*Proof.* The goal of this proof is to find a series of Reidemeister III moves from  $D$  to a drawing  $\bar{D}$  such that each move is on a triangle that contains an edge segment not in  $L$  and the last Reidemeister III move is on  $\Delta_{\{e,f,g\}}$ . Such a drawing  $\bar{D}$  keeps the rotations at the vertices the same, contains the subdrawing  $L$  and has  $e_i$  crossing  $g$  next.

Let  $H_n$  be a realizable  $n$ -vertex rotation system and  $D$  be a simple drawing realizing  $H_n$ , and  $e = (u, v)$  be an edge of  $H_n$ .

Let  $n \geq 6$ ,  $H_n$  be a realizable  $n$ -vertex rotation system,  $D$  be a simple drawing realizing  $H_n$ ,  $e = (u, v)$  be an edge of  $H_n$ ,  $E_v$  be the set of edges having  $v$  as an endpoint in  $D$ , and  $F_v \subset E_v$ . Let  $r \geq 5$ ,  $L$  be a subdrawing of  $D$  on some complete graph  $K_r$  along with the edges  $F_v$  containing a segment of  $e$  labelled  $e_i$  starting at  $v$  and has  $i$  crossings in  $L$ , and  $R_L$  be the region containing the non-vertex end of  $e_i$  in  $L$ . Assume  $e_i$  crosses  $f$  next in  $D$ ,  $f$



and  $g$  are consecutive edges on the boundary of  $R_L$ ,  $\Delta_{\{e,f,g\}}$  exists in  $D$  and  $e_i$  has not crossed  $g$  in  $L$ .

Define  $e'$ ,  $f'$  and  $g'$  to be the segments of  $e$ ,  $f$  and  $g$  respectively, that form the boundary of  $\Delta aef$  in  $D$  and let  $\bar{f}$  and  $\bar{g}$  be the consecutive segments of  $f$  and  $g$  respectively, that are on the boundary of  $R_L$ .

Observe that  $f'$  is the segment of  $\bar{f}$  from the crossings of  $f$  and  $g$  to the crossing of  $e$  and  $f$  in  $D$ . It follows that, other than  $g$ , no edge in  $L$  crosses  $f'$ . In  $D$ , since  $\Delta aef$  contains no vertices, there are three types of segments that could cross  $\Delta aef$ :

- $X$  the set of segments that cross  $f'$  and  $g'$ ;
- $Y$  the set of segments that cross  $f'$  and  $e'$ ; and
- $Z$  the set of segments that cross  $e'$  and  $g'$ .

At this point, note that no edge segment in  $L$  is in  $X$ , and the only edge segment in  $L$  that could be in  $Y$  is a segment of  $g$ . Let  $a_1$  be the first edge segment from the  $f, g$ -crossing on  $f'$  that is in  $X$ , if no such edge segment exists, then we have successfully emptied  $X$ . We will empty  $\Delta_{a_1,e,f}$  of crossings by using Reidemeister III moves on edges that are not in  $L$ , showing we can empty  $\Delta_{\{e,f,g\}}$  of  $a_1$  and thus empty  $\Delta_{\{e,f,g\}}$  of segments in  $X$ .

Note that the segment of  $f$  on  $\Delta_{a_1,f,g}$  has no crossings other than at its ends. By Lemma 2.17, there is a triangle  $\Delta_{f,g_1,h_1}$  contained in  $\Delta_{a_1,f,g}$  that is empty of edges and contains a segment of  $f$ . Since  $\Delta_{f,g_1,h_1}$  is contained in  $\Delta_{a_1,f,g}$ , it follows for both  $g_1$  and  $h_1$  that some segment of  $g_1$  and  $h_1$  is contained  $X \cup Y$ . Since one of these edges is not  $g$ , it follows that one of the segments on the boundary of  $\Delta_{f,g_1,h_1}$  is not in  $L$ .

Applying a Reidemeister III move to  $\Delta_{f,g_1,h_1}$  reduces the number of crossings in  $\Delta_{a_1,f,g}$ , as desired. Using Reidemeister III moves to repeat this process shows we can empty  $\Delta_{a_1,f,g}$ . It follows that there is a series of Reidemeister III moves from  $D$  to a drawing  $D_1$  each having an edge segment not in  $L$  that empties  $\Delta_{\{e,f,g\}}$  of segments of type  $X$ . This implies  $D_1$  contains the subdrawing  $L$ .

Applying the same argument to segments in  $Y$  results in a series of Reidemeister III moves each containing an edge segment not in  $L$  from  $D_1$  to a drawing  $D_2$  that contains the subdrawing  $L$  which has  $\Delta_{\{e,f,g\}}$  empty of segments from  $X \cup Y$ .

Let  $a_2$  be the first edge segment from the  $e, g$ -crossing on  $e'$  that is in  $Z$  in  $D_2$ , if no such edge segment exists, then we have successfully emptied  $Z$ .

Note that  $\Delta_{a_2,e,g}$  is contained in  $\Delta_{\{e,f,g\}}$ . We will empty  $\Delta_{a_2,e,g}$  of crossings by using Reidemeister III moves on edges that are not in  $L$ , showing we can empty  $\Delta_{\{e,f,g\}}$  of  $a_2$  and thus empty  $\Delta_{\{e,f,g\}}$  of segments in  $Z$ .

By definition of  $a_2$ , Lemma 2.17 applies and implies that there is a triangle  $\Delta_{e,g_2,h_2}$  that is contained in  $\Delta_{a_2,e,g}$  that is empty of edges and has a segment of  $e$  on the boundary. Note that the segment of  $e$  on the boundary of  $\Delta_{e,g_2,h_2}$  is contained on the segment of  $e$  on the boundary of  $\Delta_{a_2,e,g}$  which is contained on the segment of  $e$  on the boundary of  $\Delta_{\{e,f,g\}}$ . It follows that the segment of  $e$  on the boundary of  $\Delta_{e,g_2,h_2}$  is not in  $L$ .

Applying a Reidemeister III move to  $\Delta_{e,g_2,h_2}$  reduces the number of crossings in  $\Delta_{a_2,e,g}$  and preserves the drawing of  $L$ . Using Reidemeister III moves to repeat this process shows we can empty  $\Delta_{a_2,e,g}$  and preserve the drawing of  $L$ . Noting again that the segment of  $e$  in  $\Delta_{a_2,e,g}$  is not in  $L$ , we can apply a Reidemeister III move over  $\Delta_{a_2,e,g}$  to reduce the number of segments in  $Z$  and preserve the drawing  $L$ .

It follows that there is a series of Reidemeister III moves from  $D_2$  to a drawing  $D_3$  that empties  $\Delta_{\{e,f,g\}}$  of segments of type  $Z$  and  $D_3$  contains  $L$ . Noting that the segment of  $e$  bounding  $\Delta_{\{e,f,g\}}$  is not in  $L$ , we finally apply a Reidemeister III move to  $\Delta_{\{e,f,g\}}$  resulting in a drawing  $\bar{D}$  obtained from a series of Reidemeister III moves from  $D$  such that  $\bar{D}$  contains  $L$  and has  $e$  crossing  $g$  as the next step, as desired.  $\square$

Now that we can describe when neighbouring edges have extensions to simple drawing of  $K_n$  with the same rotation system. We look at the set of all such edge on the boundary of  $L$  and show that they appear consecutively. To that end, let us define the set of choice edges.

**Notation 7.3.** Let  $n \geq 6$ ,  $H_n$  be a realizable  $n$ -vertex rotation system, and  $D$  a simple drawing that realizes  $H_n$ . Let  $e = (u, v)$  be an edge in  $H_n$ ,  $L$  be a subdrawing of  $D$  containing a segment of  $e$  labelled  $e_i$ , starting at  $v$  and has  $i$  crossings in  $L$ , and  $R_L$  be the region containing the non-vertex end of  $e_i$  in  $L$ . Define  $C_L(e_i)$  to be the set of edges that  $e_i$  can cross on the boundary of  $R_L$  such that the drawing after the crossing of  $e_i$  with the boundary can be extended to a simple drawing that realizes  $H_n$ .

We proceed by showing that the edges in  $C_L(e_i)$  appear consecutively along the boundary of the face in  $L$  containing the non-vertex end of  $e_i$ .

**Lemma 7.4.** *Let  $n \geq 6$ ,  $H_n$  be a realizable  $n$ -vertex rotation system,  $D$  be a simple drawing realizing  $H_n$ ,  $e = (u, v)$  be an edge of  $H_n$ , and  $F_v$  be a subset*

of edge incident to  $v$  in  $D$  not containing  $e$ . For  $r \geq 5$ , let  $L$  be a subdrawing of  $D$  on some  $K_r$  not containing  $v$ , along with the edges  $F_v$ , and a segment of  $e$  starting at  $v$  that has  $i$  crossings in  $L$  labelled  $e_i$ . If  $R_L$  is the region containing the non-vertex end of  $e_i$  in  $L$ , then the elements of  $C_L(e_i)$  appear consecutively on the boundary of  $R_L$ .

*Proof.* Let  $n \geq 6$ ,  $H_n$  be a realizable  $n$ -vertex rotation system,  $D$  be a simple drawing realizing  $H_n$ ,  $e = (u, v)$  be an edge of  $H_n$ ,  $F_v$  be a subset of edge incident to  $v$  in  $D$  that does not contain  $e$ ,  $L$  be a subdrawing of  $D$  containing a segment of  $e$  labelled  $e_i$ , and starting at  $u$  and has  $i$  crossings in  $L$ . Suppose  $R_L$  is the region containing the non-vertex end of  $e_i$  in  $L$  and let  $z$  be the point on  $\mathcal{B}(R_L)$  that  $e_i$  crosses ( $z$  could be the vertex  $v$  if  $i = 0$ ). By way of contradiction, assume  $C_L(e_i)$  contains two edges  $f$  and  $g$  such that  $\bar{f}$  and  $\bar{g}$  are their segments on  $\mathcal{B}(R_L)$  respectively, and there is no path of edges in  $C_L(e_i)$  on  $R_L$  that connects the segments  $\bar{f}$  and  $\bar{g}$ .

**Claim 1.** *Let  $d$  be an edge in  $L$ . If  $d$  has at least two non-trivial segments on  $\mathcal{B}(R_L)$ , then there exists a simple closed curve  $\phi$  on  $d \cup \mathcal{B}(R_L)$  containing  $v$  such that there is a side  $S_\phi$  of  $\phi$  does not contain any vertex and  $\phi \cap R_L = \{\bar{d}_1, \bar{d}_2\}$ , where  $\bar{d}_1$  and  $\bar{d}_2$  are segments of edges  $d_1$  and  $d_2$  that start at vertex  $v$  and end at their respective crossings with  $d$ .*

*Proof.* Let  $d = (u_d, v_d)$ . Since  $d$  has at least two non-trivial segments on  $\mathcal{B}(R_L)$ , it follows that there is a non-trivial segment of  $d$ , call it  $s_d$  that intersects  $\mathcal{B}(R_L)$  only at its ends and these intersections are at crossing edges with  $d$ .

Let  $\phi$  be any simple closed curve defined on  $\mathcal{B}(R_L)$  and  $s_d$  that uses  $s_d$ . If both sides of  $\phi$  contain an end of  $d$ , then without loss of generality there exists a vertex  $w \neq v$  in  $L$  such that  $w$  and  $u_d$  are on opposite sides of  $\phi$ .  $(w, u_d)$  is in  $L$  by definition of  $L$ , and so  $(w, u_d)$  must cross  $\phi$ . Since  $L$  is simple this is not possible, and so some side of  $\phi$ , labelled  $S_\phi$ , contains no end of  $d$ .

Similarly,  $S_\phi$  contains no vertex  $w \neq v$ . The crossings of  $s_d$  with  $\mathcal{B}(R_L)$  on  $\phi$  are at edges  $d_1$  and  $d_2$ . For each of these edges, there is an end that is inside  $S_\phi$  and an end that is outside  $S_\phi$  since  $L$  is simple. It follows that each of  $d_1$  and  $d_2$  have  $v$  as an endpoint.

If  $S_\phi$  contained a segment of an edge not having  $v$  as an end, then one of the ends of that edge would be contained in  $S_\phi$ , a contradiction. This shows

$\bar{d}_1$  and  $\bar{d}_2$  the segments of  $d_1$  and  $d_2$  respectively, in  $S_\phi$  cross no edges. Walking along the boundary of  $R_L$  from the  $(d_1, d)$  crossing to the  $(d_2, d)$  crossing shows that we only walk along  $\bar{d}_1$  and  $\bar{d}_2$ . Therefore,  $\gamma_{d_1, d_2, d}$  is a closed curve defined on  $\mathcal{R}_L$  and  $s_d$  that uses  $s_d$ . Setting  $\phi = \gamma_{d_1, d_2, d}$  completes the proof. ■

By repeated use of Lemma 7.2, there is a maximal set of consecutive edge segments  $B$  that are choices on  $\mathcal{B}(R_L)$  containing  $\bar{f}$  such that crossing any of these segments can complete into a drawing. Without loss of generality, let the clockwise boundary walk of  $\mathcal{B}(R_L)$  be  $(\bar{b}_1, \dots, \bar{b}_k, \dots, \bar{g}, \dots, z, \dots)$  with  $B = \{\bar{b}_1, \dots, \bar{b}_k\}$  and let  $b_i$  be the respective edge of  $\bar{b}_i$ . Since  $g$  has a crossing with each  $b_i$  or they are segments of the same edge, we can orient each  $\bar{b}_i$  towards the crossing of  $b_i$  and  $g$ , or towards the segment  $\bar{g}$ .

**Claim 2.** *For all  $i$ ,  $\bar{b}_i$  is oriented clockwise along  $\mathcal{B}(R_L)$ .*

*Proof.* By way of contradiction, assume  $\bar{b}_i$  is oriented counter clockwise for some  $i$ . Without loss of generality, let  $\bar{b}_i$  be the edge segment in  $B$  that is closest in  $B$  to  $\bar{b}_1$  that is oriented counterclockwise.

**Case 1.**  $b_i = g$ .

It follows by taking a edge walk on  $g$  that from  $\bar{b}_i$  to  $\bar{g}$ , we see  $\bar{b}_i$ , then the crossing of  $g$  with some edge  $g_2$  on the boundary of  $R_L$ , then  $\bar{g}$ . Claim 1 implies that  $g_2$  has  $v$  as an endpoint.

Let  $D_1$  be a realization of  $H_n$  with  $e_i$  crossing  $\bar{b}_i$  in  $L$  and  $D_2$  be a realization of  $H_n$  with  $e_1$  crossings  $\bar{g}$  in  $L$ . In  $D_1$ ,  $e <_g^g g_2$  and in  $D_2$   $g_2 <_g^g e$ , a contradiction with both drawings being a realization of  $H_n$ .

**Case 2.**  $b_i \neq g$ .

Since  $\bar{b}_i$  and  $\bar{g}$  are separated by  $z$ , it must be the case that the head of  $\bar{b}_i$  crosses some segment  $\bar{h}_1$  on  $\mathcal{B}(R_L)$ . Let  $h_1$  be the respective edge for  $\bar{h}_1$ .

**Case 2.1.**  $h_1 = g$ .

Orient  $\bar{g}$  towards the  $(b_i, h_1)$  crossing. By Claim 1, a simple closed curve  $\phi$  on  $g \cup \mathcal{B}(R_L)$  containing  $v$  such that there is a side  $S_\phi$  of  $\phi$  does not contain any vertex and  $\phi \cap R_L = \{\bar{d}_1, \bar{d}_2\}$ , where  $\bar{d}_1$  and  $\bar{d}_2$  are segments of edges  $d_1$  and  $d_2$  that start at vertex  $v$  and end at their respective crossings with  $d$ .

Since  $b_i$  is not incident to  $v$ , it follows that both  $\bar{g}$  and  $\bar{h}_1$  are oriented clockwise on  $\mathcal{B}(R_L)$ . As per Claim 1, up to relabelling, starting at  $\bar{g}$  and following its orientation,  $g$  crosses  $d_1$ , then  $d_2$ , then  $b_i$ .

Let  $D_1$  be a realization of  $H_n$  with  $e_i$  crossing  $\bar{b}_i$  in  $L$  and  $D_2$  be a realization of  $H_n$  with  $e_1$  crossings  $\bar{g}$  in  $L$ . From  $D_2$  we get  $e \prec_{D_2}^g d_1 \prec_{D_2}^g b_i$ . It follows from  $D_2$ , that  $e \prec_{\wedge}^g d_1$  in  $H_n$ .

Since  $e$  crosses  $b_i$  then  $g$  in  $D_1$  and all three edges are in a R3-triangle, it follows that the order  $g$  crosses  $e$  and  $b_i$  is reversed in  $D_1$ . In particular,  $g$  crosses  $d_1 \prec_{D_1}^g b_i \prec_{D_1}^g e$ . It follows from  $D_1$ , that  $d_1 \prec_{\wedge}^g e$ , a contradiction with both drawings being a realization of  $H_n$ .

**Case 2.2.**  $h_1 \neq g$ .

If  $e$  and  $h_1$  are ordered on  $b_i$  or  $e$  has already crossed  $h_1$  in  $L$ , then in any drawing realizing  $H_n$  and containing  $L$ ,  $e <_{b_i} h_1 <_{b_i} g$ . It would follow by Observation 7.1 that the drawing of  $\gamma_{e,b_i,g}$  is determined and one of  $\bar{g}$  or  $\bar{b}_i$  is not a choice. Therefore,  $\Delta_{e,b_i,h_1}$  exists and  $h_1$  has not been crossed by  $e$  in  $L$ .

By Lemma 7.2,  $\bar{h}_1 \in B$  and  $\bar{h}_1$  is oriented clockwise on  $\mathcal{B}(R_L)$  by minimality of  $\bar{b}_i$ . It follows that  $b_i$  crosses  $h_1$  from right to left and  $e$  crosses  $h_1$  from right to left from  $H_n$ .

We will prove that at this moment  $g$  is not an option for a contradiction by looking at the drawing of the  $K_6 + e_i$  in  $L$  induced on the ends of  $h_1, b_i$  and  $g$ . Let  $D_4 + e_i$  be the drawing of the  $K_4$  induced on the ends of  $h_1$  and  $b_i$ . Since  $\mathcal{B}(R_L)$  has the crossing of  $h_1$  and  $b_i$ , the region  $R_4$  containing  $R_L$  in  $D_4$  is determined. Let  $u_h$  and  $u_b$  be the ends of  $h_1$  and  $b_i$  respectively, that are on the boundary of  $R_4$ .

Now extend  $D_4 + e_i$  to include the drawing of  $g$  from  $L$ . By the orientation of  $b_i$  and  $h_1$  in  $L$ , and the definition of  $\Delta_{b_i,h_1,g}$ , it follows that  $g$  does not cross  $R_4$  on  $b_i$  or  $h_1$ . Since  $g$  is on the boundary of  $R_L$ , some segment of  $g$  is in  $R_4$ . It follows that  $g$  crosses  $R_4$  once at  $(u_b, u_h)$ . Let  $u_g$  be the end of  $g$  that is in  $R_4$  (Such an end exists as  $g$  crosses the boundary of  $R_4$  once).

Again extend this drawing to be  $D_4 + e_i + \{(u_g, u_h), (u_g, u_b)\}$ , the original drawing extended to include the drawing of the edges  $(u_g, u_h)$  and  $(u_g, u_b)$  from  $L$ . Since this drawing is simple,  $\gamma_{b_i,g,(u_b,u_h)}$  implies that  $(u_g, u_b)$  is contained in  $R_4$ . Similarly,  $\gamma_{h_1,g,(u_b,u_h)}$  implies that  $(u_g, u_h)$  is contained in  $R_4$ . It follows that there is no facial region in  $R_4$  that both contain a segment of  $g$  the crossing of  $b_i$  and  $h_1$ , a contradiction with the existence of  $R_L$ . ■

By Claim 2, every edge segment in  $B$  is directed clockwise (in particular  $\bar{b}_k$ ). Let  $g_k$  be the edge that the head of  $b_k$  crosses on  $\mathcal{B}(R_L)$ .

If  $g = g_k$ , we orient  $\bar{g}$  towards the  $(b_k, g_k)$  crossing. Since  $b_k$  is not incident to  $v$ , it follows that both  $\bar{g}$  and  $\bar{g}_k$  are oriented counterclockwise on  $\mathcal{B}(R_L)$ . As per Claim 1, up to relabelling, starting at  $\bar{g}$  and following its orientation,  $g$  crosses  $d_3$ , then  $d_4$ , then  $b_i$ , where  $d_3$  and  $d_4$  are edges incident to  $v$ .

Let  $D_1$  be a realization of  $H_n$  with  $e_i$  crossing  $\bar{b}_k$  in  $L$  and  $D_2$  be a realization of  $H_n$  with  $e_1$  crossings  $\bar{g}$  in  $L$ . From  $D_2$  we get  $e \prec_{D_2}^g d_3 \prec_{D_2}^g b_k$ . It follows from  $D_2$ , that  $e \prec_{\wedge}^g d_3$  in  $H_n$ .

Since  $e$  crosses  $b_k$  then  $g$  in  $D_1$  and all three edges are in a R3-triangle, it follows that the order  $g$  crosses  $e$  and  $b_k$  is reversed in  $D_1$ . In particular,  $g$  crosses  $d_3 \prec_{D_1}^g b_k \prec_{D_1}^g e$ . It follows from  $D_1$ , that  $d_3 \prec_{\wedge}^g e$ , a contradiction with both drawings being a realization of  $H_n$ .

Therefore,  $g \neq g_k$ . If  $e$  and  $g_k$  are ordered on  $b_k$  or  $e$  has already crossed  $g_k$  in  $L$ , then in any drawing realizing  $H_n$  and containing  $L$ ,  $e \prec_{b_k} g_k \prec_{b_k} g$ . It would follow by Observation 7.1 that the drawing of  $\gamma_{e,b_k,g}$  is determined and one of  $\bar{g}$  or  $\bar{b}_k$  is not a choice. Therefore,  $\Delta_{e,b_k,g_k}$  exists and  $g_k$  has not been crossed by  $e$  in  $L$ . By Lemma 7.2,  $g_k$  is a choice, a contradiction with the definition of  $b_k$ .  $\square$

Finally, we end this section with a proof of Theorem 1.3 using Lemma 7.2 and Lemma 7.4.

*Proof of Gioan's Theorem* Let  $n$  be a positive integer,  $H_n$  be a realizable  $n$ -vertex rotation system, and  $D$  and  $D'$  be two simple drawings realizing  $H_n$ . If  $n \leq 5$ , then  $H_n$  uniquely determines its associated realizable drawings and  $D = D'$ . Therefore,  $n \geq 6$ .

Let  $r \geq 5$  be the largest integer such that there exists a common drawing  $L_r$  of  $K_r$  in  $D$  and  $D'$ . Since every common  $K_5$  in  $D$  and  $D'$  is uniquely drawn the same, it follows that such an  $L_r$  exists. Since  $D \neq D'$ , there is a vertex  $v$  in  $D$  not in  $L_r$ .

Let  $E_r^v$  be the edges having an endpoint in  $L_r$  and the other endpoint being  $v$ , and  $F_r^v \subset E_r^v$  such that  $L_r + F_r^v$  is a common drawing in both of  $D$  and  $D'$ .  $L_r + F_r^v$  exists because  $L_r$  exists and  $F_r^v$  can be empty.

Since  $L_{r+1}$  does not exist, it follows that  $E_r^v \neq F_r^v$ . Let  $e = (u, v)$  be an edge in  $E_r^v \setminus F_r^v$ , and  $i$  be the largest integer such that there exists a segment of  $e$ , labelled  $e_i$ , starting at  $v$  having  $i$  crossings such that  $L_r + F_r^v + e_i$  is a common drawing in both  $D$  and  $D'$ . Such an  $L_r + F_r^v + e_i$  exists since it is

possible that  $i = 0$  and  $e_0$  is a small segment of  $e$  starting at the correct spot in the rotation at  $v$ .

If we can show that there is a sequence of Reidemeister III moves from  $D$  to some drawing  $D_1$  such that the common drawing of  $D_1$  and  $D'$  is  $L_r + F_r^v + e_{i+1}$ , then induction will imply that there is a sequence of Reidemeister III moves from  $D$  to  $D'$ .

Let  $R_L$  be the region in  $L_r + F_r^v + e_i$  that contains the non-vertex end of  $e_i$ . Let  $f$  be the next edge that  $e_i$  crosses in  $D$  and  $f'$  be the next edge that  $e_i$  crosses in  $D'$ . By Lemma 7.4, the elements of  $C_{L_r + F_r^v + e_i}(e_i)$  appear consecutively on the boundary of  $R_L$ .

Let  $(b_1 = f, \dots, b_k = f')$  be such a path of segments. Starting from  $D = D_1$ , define  $D_j$  for  $2 \leq j \leq k$  to be the simple drawing  $\bar{D}$  produced by Lemma 7.2 with  $e_i$  crossing  $b_j$ . Lemma 7.2 implies there is a sequence of Reidemeister III moves  $\{\rho_{\ell_j}^j\}$  that takes  $D_j$  to  $D_{j+1}$ .

Note that  $D_k$  and  $D'$  have  $L_r + F_r^v + e_{i+1}$  in common (if  $e_{i+1}$  has crossed all its edges, then this is  $L_r + (F_r^v \cup \{e\})$ , and if  $F_r^v \cup \{e\} = E_r^v$ , then this is  $L_{r+1}$ ).

Taking the sequence of Reidemeister III moves  $\{\rho_{\ell_j}^j\}_{j=1}^{k-1}$  from  $D$  to  $D_k$  gives us the desired inductive result.  $\square$

## 8 Concluding Remarks

The proof of Theorem 1.1, provide insight into how to draw a realizable rotation system using a combinatorial algorithm. Restricting this proof to the case  $n = 6$ , offers the first non-computational proof that  $(6, 5)$ -rotation systems are realizable. Taking a combinatorial approach to the proof of Theorem 1.1 resulted in Theorem 3.8, which characterizes edges and faces in simple drawings of  $K_n$  independent of rotation systems. Stand alone such a result is interesting as the Harary-Hill conjecture remains open and the characterization applies directly to every simple drawing of  $K_n$ . Finally, we provide a simplified proof of Theorem 1.3 and provide intuition as to how graphs with the same rotation system are drawn.

We end this thesis with three separate open questions. The first two questions are motivated by Theorems 1.1 and 1.3 respectively. The last questions is the open question posed by Dan Archdeacon cited in [4].

Let us extend the definitions of realizable rotation systems and  $(n, k)$ -rotation systems. Define a complete  $n$ -vertex rotation system  $H$  to be  $g$ -realizable if there is a simple drawing  $D$  of  $K_n$  in a surface of Euler genus  $g$  having the rotations at the vertices the same as  $H$ . Let a rotation system  $H$  be an  $(n, k, g)$ -rotation system if  $H$  is an  $n$ -vertex rotation system and every rotation system  $H_k$  that is obtained from  $H$  restricted to a set of  $k$  vertices is  $g$ -realizable.

**Open Question 1.** *Does there exist an integer function  $k(g)$  such that for every  $n \geq k(g)$ , every  $(n, k(g), g)$ -rotation system is  $g$ -realizable?*

**Open Question 2.** *If  $D$  and  $D'$  are two simple drawings of  $K_n$  in a surface of Euler genus  $g$  have the same rotation system, then does there exist a sequence of Reidemeister III moves applied to  $D$  that transforms  $D$  into  $D'$  (or possibly a more generalized set of operations)?*

**Open Question 3** (Archdeacon, [4]). *Is the number of non-planar  $K_4$ 's in any  $n$ -vertex complete rotation system at least  $H(n)$ ?*



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## Appendix A

We present a rudimentary drawing algorithm for complete rotation systems that takes as input a complete rotation system  $H$ , and outputs either a simple drawing  $D$  that realizes  $H$  or a 5-vertex non-realizable rotation system  $H_5$ .

### Edge Drawing Subroutine $\mathcal{E}(H, D, (u, v))$

- Input: A complete rotation system  $H$ , a desired edge  $(u, v)$  to be drawn, and a partial drawing  $D$  on a complete graph containing  $u$  along with a set of edges incident to  $v$  from  $H$ .
  - Output: A drawing  $\bar{D}$  that is a partial drawing of  $H$  and contains the edges  $D$  and  $(u, v)$ , or a non-realizable at most 8-vertex rotation system in  $H$
- 1) Set  $i = 0$  and start the edge  $(u, v)$  at  $u$  in the appropriate position according to  $H$  in  $D$  and call this new drawing  $D_0$ .
  - 2) Define  $(u, v)_i$  to be the partial edge of  $(u, v)$  in  $D_i$ ,  $v_i$  to be the non-vertex end of  $(u, v)_i$  and the new drawing  $D_i$ . Define  $E_i$  to be a set of minimal edges that  $(u, v)_i$  has yet to cross defined by the  $<_{\wedge}^e$  and  $<_{\parallel}^e$  relations. If such a set does not exist, define  $E_i = \infty$ .
    - a) If  $E_i = \infty$ , then Section 5 implies we can find a non-realizable 8-vertex rotation system  $H_8$  in  $H$ . Output  $H_8$  and stop the subroutine
    - b) If  $E_i = \emptyset$  and  $v$  is currently on the boundary of the region containing  $v_i$ , then extend  $(u, v)_i$  by connecting it to  $v$ , set the new drawing to be  $\bar{D}$ . Output  $\bar{D}$  and stop the subroutine.
    - c) If  $E_i = \emptyset$  and  $v$  is not currently on the boundary of the region containing  $v_i$ , then  $v_i$  and  $v$  are in different faces of  $D_i - v$ , in particular, there is a 3-cycle  $T$  separating  $v_i$  from  $v$ . The induced at most 5-vertex rotation system  $H_5$  on  $T$  and  $(u, v)$  is non-realizable. Output  $H_5$  and stop the subroutine.
    - d) If  $E_i \neq \emptyset$  and some edge of  $c_i \in E_i$  is on the boundary of the region containing  $v_i$  do:

- i) if crossing  $c_i$  forms the correct oriented crossing with  $(u, v)$ , then extend  $(u, v)_i$  in  $D_i$  to cross  $e$ , set the new drawing to be  $D_{i+1}$ ,  $i := i + 1$ , and start at step 2).
  - ii) if crossing  $c_i$  does not form the correct oriented crossing with  $(u, v)$ , then define  $H_4$  to a 4-vertex rotation system in  $H$  induced on the ends of  $c_i$  and  $(u, v)$ .  $H_4$  is not realizable, therefore output  $H_4$  and stop the subroutine.
- e) If  $E_i \neq \emptyset$  and no edge of  $E_i$  is on the boundary of the region containing  $v_i$ , then let  $c_i$  be an edge in  $E_i$  and do:
- i) If Theorem 3.8 finds a  $K_4$  separating  $v_i$  from  $c_i$  in  $D_i - v$ , then the at most 6-vertex rotation system  $H_6$  induced on this  $K_4$  and  $(u, v)$  is non-realizable. Output  $H_6$  and stop the subroutine.
  - ii) If Theorem 3.8 finds a sequence of Reidemeister III moves  $\{\rho_{X_j}\}$  that places  $c_i$  on the boundary of the region containing  $v_i$  in  $D - v$ , then do:
    - 1) For some  $j$ , if a vertex  $v$  is in  $\Delta_{X_j}$ , then there is some  $K_4$  involving  $v$  and  $c_i$  separating  $c_i$  from the region containing  $v_i$ . Such a  $K_4$  along with  $(u, v)$  form a non-realizable  $H_5$ . Output  $H_5$  and stop the subroutine.
    - 2) For all  $j$ , if every  $\Delta_{X_j}$  contains no vertices, then perform the sequence of Reidemeister III moves  $\{\rho_{X_j}\}$  on  $D_i$  and go to step 2.d.i).

**Basic Algorithm  $\mathcal{A}(H)$**

- Input: A complete rotation system  $H$  on the vertices  $[n]$ .
  - Output: A drawing  $D$  that is a realization of  $H$ , or a non-realizable 5-vertex rotation system  $H_5$  in  $H$ .
- 1) Start by drawing the 3-cycle  $(1, 2, 3)$ , set  $i = 3$  and call the drawing  $D$ .
  - 2) For  $i$  from 4 to  $n$  do: For  $j$  from 1 to  $i - 1$  do:
    - i) Process  $\mathcal{E}(H, D, (i, j))$ .

- ii) If the output of the subroutine is some non-realizable  $H_k$  for  $k \leq 8$ , then check every 5-vertex rotation system in  $H_8$  till a non-realizable 5-vertex rotation system  $H_5$  is found. Output  $H_5$  and stop the algorithm.
- iii) Otherwise, the output of the subroutine is  $\bar{D}$ . Set  $D := \bar{D}$ . If  $j = n - 1$ , then output  $D$  and stop the algorithm.