Correspondence Colouring and its Applications to List Colouring and Delay Colouring

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of contributions

The original results in Chapters 3, and 4 were obtained jointly with my supervisor, Penny Haxell.

Abstract

In this thesis, we study correspondence colouring and its applications to list colouring and delay colouring. We give a detailed exposition of the paper of Dvořák, and Postle introducing correspondence colouring.

Moreover, we generalize two important results in delay colouring. The first is a result by Georgakopoulos, stating that cubic graphs are 4-delay colourable. We show that delay colouring can be formulated as an instance of correspondence colouring. Then we show that the modified line graph of a cubic bipartite graph is generally 4-correspondence colourable, using a Brooks' type theorem for correspondence colouring. This allows us to give a more simple proof of a stronger result. The second result is one by Edwards and Kennedy, which states that quartic bipartite graphs are 5-delay colourable. We introduce the notion of p-cyclic correspondence colouring which is a type of correspondence colouring that generalizes delay colouring. We then prove that the modified line graph of a quartic bipartite graph is 5-cyclic correspondence colourable using the Combinatorial Nullstellensatz.

We also show that the maximum DP-chromatic number of any cycle plus triangles (CPT) graph is 4. We construct a CPT graph with DP-chromatic number at least 4. Moreover, the upper bound follows easily from the Brooks' type theorem for correspondence colouring. Finally, we do a preliminary investigation into using parity techniques in correspondence colouring to prove that CPT graphs are 3-choosable.

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Chapter 1

Introduction

1.1 Correspondence Colouring

Correspondence colouring is an important notion of colouring which was introduced in 2015 by Zdeněk Dvořák and Luke Postle [17]. It was originally formulated as a generalization of list colouring, to prove a list colouring result on planar graphs without cycle lengths in a certain range. Correspondence colouring allowed them to use techniques, used to prove an analogous colouring result, which do not generalize to a list colouring setting. We present their proof in Chapter 5. In correspondence colouring every vertex v is assigned a list of colours $L(v)$ and the goal is to choose a colour from each list such that no two adjacent vertices have corresponding colours. For a pair of adjacent vertices u, v each colour in $L(u)$ corresponds to at most one colour in $L(v)$, and the correspondences are arbitrary. The lists and correspondences are called a correspondence assignment. See Definition 2.2 for a formal definition of correspondence colouring. Correspondence colouring generalizes list colouring as in list colouring every vertex u is assigned a list $L(u)$ and for a pair of adjacent vertices u, v the colour c in $L(u)$ corresponds to c' in $L(v)$ if and only if $c = c'$. We note that list colouring is also known as choosability and we use both terms interchangeably.

Correspondence colouring turned out to have significant links to other well-studied topics in colouring. It is often useful to place restrictions on correspondence assignments to prove generalizations of results for other variants of colouring. As previously mentioned, in [17], Dvořák and Postle show that every planar graph without cycles of length 4-8 is 3-correspondence colourable for correspondence assignments which are consistent (see Definition 2.7) on walks of length 3. Such correspondence assignments generalize list colouring. Furthermore, Dvořák and Postle note that placing such a restriction is necessary for their proof. In Chapter 4 we show that quartic bipartite graphs are 5-correspondence colourable for cyclic correspondence assignments (see Definition 4.1). Cyclic correspondence assignments generalize the notion of delay colouring (see Definition 3.1, Section 3.1) and they allow us to use the Combinatorial Nullstellensatz, a result that usually does not translate to correspondence colouring. This is further discussed in Section 1.2. Correspondence colouring is also recognised as a significant variation of colouring that is studied for its own sake. In particular, correspondence colouring with no restrictions on the correspondence is called DP colouring in recognition of Dvořák and Postle who introduced it. Quite a large body of theory of DP colouring has been built up, with major contributions from Bernshteyn, Bonamy, Delcourt, Dvořák, Kaul, Kim, Kostochka, Lang, Molloy, Mudrock, Ozeki, Pelsmajer, Postle, Pron, and many others. Some examples of advances in this theory are [5, 6, 8, 7, 9, 17, 30, 33, 28, 29, 10, 32].

1.2 Contributions of This Thesis

This thesis consists of an expository component and an original component. In Chapter 5, we present an exposition of the original result of Dvořák and Postle, correspondence colouring was introduced to prove. We aim to highlight the advantages of using correspondence colouring to prove list colouring results. Moreover, we show some of the strengths of the notion of correspondence colouring. We also exhibit some of the original techniques used in correspondence colouring.

Chapters 3 and 4 contain original work motivated by the topic of delay colouring. Delay colouring was introduced in 2004 to model a specific problem coming from a real-world application. The Time-domain Wavelength Interleaved Network (TWIN) is a large-scale optical network that operates over long distances including coast-to-coast in North America. Scheduling of time slots for the TWIN optical network has a complication due to speed-of-light delays since signals travelling long distances are delayed by the amount of time it takes for light to travel that distance. Hence time slots in the repeating schedule of a receiver might interfere with each other if some signals are coming from afar while others are emitted by nearby transmitters. This translates into an abstract colouring problem for edges of a bipartite graph, in which certain colours "interfere" with each other and thus such correspondences must be avoided. See Section 3.1 for the precise definition of delay colouring.

We demonstrate that delay colouring is captured by the notion of correspondence colouring in line graphs with a special type of correspondence assignment. Using this perspective, we are able to give substantial generalizations of two main results on delay colouring

(Chapters 3 and 4). In Chapter 3, we show correspondence colouring can be used to show that cubic bipartite multigraphs are 4 delay colourable using a Brooks type theorem for correspondence colouring, due to Bernshteyn, Kostochka, and Pron (see Theorem 2.24, Section 2.3). This proof is simpler and more general than the existing proof using only delay colouring.

In Chapter 4, we introduce cyclic and anticyclic correspondence colouring, which are special types of correspondence assignments that generalize delay colouring. We use the Combinatorial Nullstellensatz to prove the existence of correspondence colourings for these special assignments in Theorems 4.10, 4.12. Theorems 4.10, 4.12 are obtained by modifying a well known application of the Combinatorial Nullstellensatz in list colouring (see Theorem 4.9). This work generalizes another established theorem on delay colouring that proves quartic bipartite graphs are 5 delay colourable. These proofs are of particular interest since list colouring results proved using the Combinatorial Nullstellensatz in general do not seem to translate to the correspondence colouring setting, as discussed by Kostochka and Bernsteyn in [8] (see Section 2.2.2) and in Section 4.3.2.

Chapter 6 also contains original material, in which we study correspondence colouring in CPT graphs, an important problem in list colouring. A cycle plus triangles (CPT) graph is a graph on $3n$ vertices that admits a decomposition into a Hamiltonian cycle and n vertex disjoint triangles (see Definition 6.1). CPT graphs were introduced by Du, Hsu, and Hwang in 1986 to model certain computer networks and multiprocessor systems [16]. In 1990, Erdős conjectured that every CPT graph is 3-colourable. Fleischner and Stiebitz showed, in 1992, that CPT graphs are 3-choosable using the Combinatorial Nullstellensatz [20]. In 1994, Sachs gave an elementary proof that every cycle plus triangles graph is 3-colourable using an argument based on parity [34]. There is no similar proof for the list colouring result. It is therefore natural to ask what correspondence colouring results can be proved for CPT graphs. In Chapter 6, we prove that the maximum DP chromatic number for a CPT graph is 4 thus showing that the CPT Theorem for list colouring does not generalize to DP-colouring. We do so by constructing a CPT graph with DP-chromatic number at least 4. Moreover, we obtain the upper bound using the Brooks type theorem for correspondence colouring (see Theorem 2.24, Section 2.3). We also consider what restrictions on the correspondences might allow the CPT Theorem to generalize.

We end this thesis with a chapter of concluding remarks and pointers to future work.

Chapter 2

Background

In Section 2.1, we introduce basic definitions, and concepts relating to correspondence colouring. We will discuss the concepts of renaming colours and consistency and their implications. These concepts pertain to the relation between correspondence colouring and list colouring and give some insight into the advantage of using correspondence colouring. The material in Section 2.1 is from the paper of Dvořák and Postle [14]. In Section 2.2, we will compare previously established upper bounds on the list chromatic number and the correspondence chromatic number of different classes of graphs. In Section 2.3, we will review a Brooks' type theorem for correspondence colouring, due to Bernshteyn and Kostochka.

2.1 Basics

The main source of the material in this section is [17].

2.1.1 Correspondence Colouring Definition

In correspondence colouring we assign each vertex a list of colours, and to each edge uv we assign a correspondence between some colours at u and some colours at v. Our goal is to pick a colour from the list assigned to each vertex such that no two adjacent vertices have colours that correspond to each other.

Example 2.1. Figure 2.1 gives an example of a correspondence assignment for the edge uv. We assign u the list $L(u) = \{1_u, 2_u, 3_u\}$ and v the list $L(v) = \{1_v, 2_v, 4_v\}$. Moreover 1_u

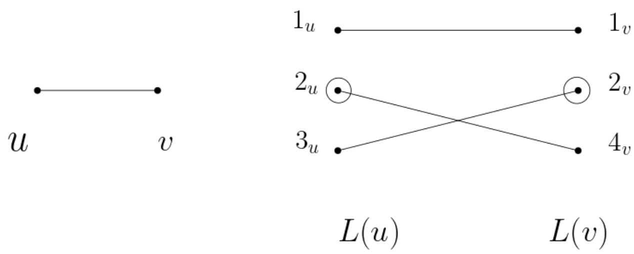

Figure 2.1: Example of a correspondence assignment.

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corresponds to 1_v, 2_u corresponds to 4_v, and 3_u corresponds to 2_v. We see that $f(u) = 2_u$ and $f(v) = 2_v$ is proper correspondence colouring of uv given the described correspondence since 2_u does not correspond to 2_v.

Formally we define a correspondence assignment and correspondence colouring as follows.

Definition 2.2. Let G be a graph.

- A *Correspondence assignment* for G, (L, C), consists of a list assignment L, and a function C which assigns to every edge $e = uv$ in G a partial matching C_e between $u \times L(u)$ and $v \times L(v)$.
- An (L, C)-*colouring* of G is a function ϕ which assigns to each vertex v in $V(G)$ a colour in $L(v)$ such that for each edge $e = uv$ $(u, \phi(u))$ and $(v, \phi(v))$ are not adjacent in C_e.
- Given a correspondence assignment (L, C), we say that G is (L, C)-*colourable* if an (L, C)-colouring of G exists.

Remark: For a correspondence assignment (L, C), C assigns a matching to each edge of G. Therefore the existence of multiple edges is consequential in correspondence colouring, whereas it is usually not in colouring or list colouring.

Correspondence colouring generalizes list colouring as follows. Given a list assignment L for a graph G, we can define a function C such that, for any edge $e = uv$, C_e matches exactly the common colours of $L(u)$ and $L(v)$. We see that an (L, C)-colouring of G gives an L-colouring of G.

2.1.2 Renaming of Colours

In correspondence colouring we are mainly concerned with the correspondences between a colour at a vertex and the colours at adjacent vertices. Hence we can arbitrarily rename the colours while updating the correspondence assignment as follows. Let (L, C) be a correspondence assignment for a graph G, and let v be a vertex in G. If $c_1 \in L(v)$ and $c_2 \notin L(v)$, then we can define a correspondence assignment (L', C') where

$$L'(u) = \begin{cases} L(u) & \text{if } u \neq v \\ \big(L(v) \backslash \{c_1\}\big) \cup \{c_2\} & \text{if } u = v \end{cases}$$

and for each e in $E(G)$, and C'_e is obtained from C_e by replacing the vertex (v, c_1) by (v, c_2), if e is incident with v. We say (L', C') is obtained from (L, C) by *renaming (at a vertex v)*.

Definition 2.3. We say that two correspondence assignments (L, C) and (L', C') are *equivalent* if one can be obtained from the other by a sequence of renaming.

The following are some useful facts about equivalent correspondence assignments that will be used later in Chapter 5.

Fact 2.4. Let (L, C) and (L', C') be equivalent correspondence assignments for a graph G. Then G is (L, C)-colourable if and only if it is (L', C')-colourable.

Fact 2.5. Let (L, C) be a correspondence assignment for a graph G such that for each vertex v, $|L(v)| = k$. Then there exists a correspondence (L', C'), equivalent to (L, C), such that L' assigns the list $[k]$ to each vertex.

Since we can assign every vertex the same list we can perform vertex identification which is usually not possible in a list colouring settings. For example, in Lemma 5.13 (Chapter 5), we perform vertex identification to get a smaller counter example of a planar graph G without cycles of length 4-8 and a consistent 3-correspondence assignment such that G is not C-colourable.

We often study correspondence assignments where every list has size k for some integer k. In this case, we may assume that each vertex is assigned the list $[k]$ (see Fact 2.5). Under Definition 2.6, we give some definitions for such correspondence assignments. Furthermore, we define the notion of correspondence chromatic number.

Definition 2.6.

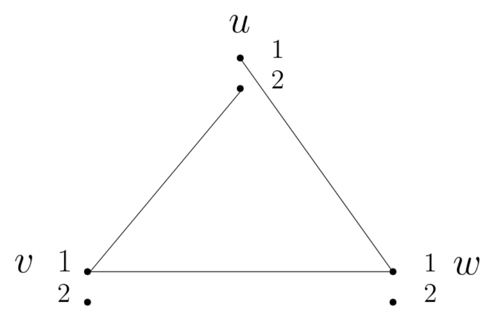

Figure 2.2: For example the walk uwvu is inconsistent

- A *k-correspondence assignment* for G is a function C that to each edge $e = uv \in E(G)$ assigns a partial matching between $\{u\} \times [k]$ and $\{v\} \times [k]$.
- Given a k-correspondence assignment C, let L be the list assignment which assigns the list $[k]$ to each vertex. Then we call an (L, C)-colouring of G a *C-colouring* of G.
- A graph G is *k-correspondence colourable* if it is C-colourable for every k-correspondence assignment C.
- The *correspondence chromatic number* of G, $\chi_{DP}(G)$, is the smallest integer k such that G is k-correspondence colourable.

Correspondence colouring is also known as DP-colouring in honour of Dvořák and Postle who introduced it. It also has an equivalent definition which we describe in Section 2.2.1.

2.1.3 Consistency of Correspondence assignments

The concept of consistency plays an important role in illustrating the relationship between list colouring and correspondence colouring.

Definition 2.7. Let G be a graph.

- Let (L, C) be a correspondence assignment for G, and let $W = v_1 v_2 ... v_m$ be a closed walk on G with $v_1 = v_m$. We say that (L, C) is *inconsistent* on W if there exist colours $c_1, c_2, ..., c_m$ such that $c_i \in L(v_i)$ for each i, $(v_i, c_i)(v_{i+1}, c_{i+1})$ is an edge in $C_{v_i v_{i+1}}$ for $i = 1, ..., m-1$, and $c_1 \neq c_m$. (See Figure 2.2)
- We say (L, C) is *consistent* on W otherwise

- We say (L, C) is *consistent* on G if it is consistent on every closed walk in G.

Note that the notion of inconsistency is invariant under renaming of colours.

Fact 2.8. Let (L, C) and (L', C') be equivalent correspondence assignments for a graph G. For every closed walk W in G, the correspondence assignment (L, C) is consistent on W if and only if (L', C') is.

The following lemma shows that list colouring is equivalent to correspondence colouring with consistent assignments.

Lemma 2.9. *A graph G is k-choosable if and only if G is C-colourable for every consistent k-correspondence assignment C.*

Proof. Suppose that G is C-colourable for any consistent k-correspondence assignment C. Let L be a list assignment of G, such that $L(v)$ has size k for each v in G. Define C' such that for each edge $e = uv$ in G, C'_{uv} consists of the edges $(u, c)(v, c)$ for $c \in L(u) \cap L(v)$ (i.e C'_{uv} matches exactly the common colours of u and v). Let $W = v_1 ... v_m$ be a closed walk in G such that $v_1 = v_m$. Suppose there are colours $c_1, ..., c_m$ such that, for each i, $c_i \in L(v_i)$. Moreover for $i = 1, ..., m-1$, $(v_i, c_i)(v_{i+1}, c_{i+1}) \in E(C'_{v_i v_{i+1}})$. Then by the definition of C' $c_1 = c_2 = ... = c_m$. It follows that C' is consistent on W. Since W was an arbitrary closed walk, C' is consistent on G. By fact 2.5, there exists a k-correspondence assignment C which is equivalent to (L, C'). By fact 2.8, since (L, C') is consistent on G, C is consistent on G. Therefore, there is a C-colouring of G. By Fact 2.4, there is a (L, C')-colouring of G, hence G is L-colourable.

Conversely, suppose that G is k-choosable. Let C be a consistent k-correspondence assignment on G. Let H be the graph with the vertex set $V(G) \times [k]$ and the edge set $\cup_{e \in E(G)} E(C_e)$. We claim that for every v in $V(G)$, each component of H intersects $\{v\} \times [k]$ in at most one vertex. Suppose for a contradiction that there is a vertex v in G, and a component of H which contains (v, i) and (v, j) for $i \neq j$. Then there is a path P in H connecting (v, i) to (v, j). Let $P = (v_1, c_1) ... (v_m, c_m)$ where $v_1 = v_m = v$. Then G has a closed walk $W = v_1 ... v_m$ and colours $c_1, ..., c_m$ such that for each i, $c_i \in L(v_i)$, for $i = 1, ..., m-1$, $(v_i, c_i)(v_{i+1} c_{i+1}) \in E(C_{v_i v_{i+1}})$, and $c_1 \neq c_m$. Therefore G is inconsistent on W, which leads to a contradiction. As a result, every component of H intersects $\{v\} \times [k]$ in at most one vertex.

For each vertex v in G let $L(v)$ be the set of components in H which intersect $\{v\} \times [k]$.

Then $|L(v)| = k$, since every vertex (v, i) is in a component of H. Since G is k-choosable, G has a L-colouring ϕ. We can use ϕ to define a C-colouring of G as follows. For each vertex v in G let $\phi'(v)$ be such that $(v, \phi'(v))$ is in the component $\phi(v)$. Then for any edge $e = uv$, $\phi(u)$ and $\phi(v)$ are different components, since ϕ is a L-colouring of G. Therefore $(u, \phi'(u))$ and $(v, \phi'(v))$ are not adjacent in H. Since e was an arbitrary edge, ϕ' is a C-colouring of G. Furthermore since C was arbitrary, G is C-colourable, for any consistent k-correspondence assignment C. □

2.2 On Differences Between DP-colouring and List colouring

For a graph G, the list chromatic number of G is at least the chromatic number of G. Moreover the gap between the chromatic number and the list chromatic number can be arbitrarily large. For example, $\chi(K_{n,n}) = 2$, while $\chi_l(K_{n,n}) = (1 + o(1)) \log_2(n)$ [19]. Furthermore it was conjectured that for a simple graph G the chromatic index of G is equal to list chromatic number of G. This is known as the List Colouring Conjecture and it is a long standing problem in edge colouring. It was suggested independently by many researchers including Vizing, Albertson, Collins, Tucker and Gupta (See [1] for a history of the LCC). In 1995, Galvin proved that the List Colouring Conjecture holds for bipartite graphs [21]. Moreover, Kahn showed that the List Colouring Conjecture holds asymptotically [27].

In this section, we make some comparisons between list colouring and DP-colouring. The main results of this section are from a paper by Bernshteyn and Kostochka [8]. The authors note that many upper bounds for the list chromatic number extend to the DP-chromatic number but not all of them. In particular, the authors construct a planar bipartite graph with DP-chromatic number 4. Whereas every planar bipartite graph is 3-choosable [4]. Moreover, Bernshteyn and Kostochka show that the chromatic index of a d-regular graph is at least $d + 1$ for $d \geq 2$. Thus the aforementioned result of Galvin does not hold for correspondence edge colouring.

2.2.1 Alternative Definition of Correspondence Colouring

In [8] Bernshteyn and Kostochka give the following alternative definition of correspondence assignments and correspondence colouring. The authors define correspondence assignments using graph covers and define correspondence colouring using independent sets. Given a graph G, every vertex in G corresponds to a clique in a cover, and every edge corresponds to a matching. More formally

Definition 2.10. Let G be a graph. A *cover* of G is a pair $\mathcal{H} = (L, H)$ consisting of a graph H and a function $L : V(G) \longrightarrow 2^{V(H)}$ satisfying the following requirements:

(C1) The sets $\{L(u) : u \in V(G)\}$ partition $V(H)$.

(C2) For every $u \in V(G)$, $L(u)$ forms a clique in H.

(C3) If $E_H(L(u), L(v)) \neq \emptyset$, and $u \neq v$, then $uv \in E(G)$.

(C4) If $uv \in E(G)$, then $E_H(L(u), L(v))$ is a matching.

A cover $\mathcal{H} = (L, H)$ of G is *k-fold* if $|L(u)| = k$ for each $u \in V(G)$.

Let G be a graph. Given a correspondence assignment (L, C) of G, as described in Definition 2.2, we can construct a cover $\mathcal{H} = (L', H)$ of G as follows. Let $V(H) = \cup_{u \in V(G)} \big(\{u\} \times L(u)\big)$. Let $L' : V(G) \longrightarrow V(H)$ be such that $L'(u) = \{u\} \times L(u)$. Then the set $\{L'(u) : u \in V(G)\}$ partitions $V(H)$. For $(u, c_1), (v, c_2) \in V(H)$, (u, c_1) is adjacent to (u, c_2) in H if and only if $u = v$ or u is adjacent to v in G and (u, c_1) corresponds to (v, c_2) in C_{uv}. This fulfills conditions (C2-4) of Definition 2.10.

Conversely given a cover $\mathcal{H} = (L, H)$ of G, we can construct a correspondence assignment of G, (L, C), as follows. For every edge $e = uv$ in G, for $c_1 \in L(u)$ and $c_2 \in L(v)$, (u, c_1) is adjacent to (v, c_2) in C_e if and only if c_1 is adjacent to c_2 in H. Thus the definition of a cover (Definition 2.10) is equivalent to the definition of a correspondence assignment (Definition 2.2).

Definition 2.11. Let G be a graph, and $\mathcal{H} = (L, H)$ be a cover of G. An $\mathcal{H}-$*colouring* of G is an independent set of vertices in H of size $|V(G)|$.

Note that an independent set in H intersects $L(u)$ in at at most one vertex, for each vertex u in G, since $L(u)$ is a clique. Furthermore an independent set of size $|V(G)|$ intersects $L(u)$ exactly once for each $u \in V(G)$. Therefore the $\mathcal{H}$-colourings of G are precisely the (L, C)-colourings of G in the equivalence described above. Hence the following is an alternative definition of the correspondence chromatic number of G.

Definition 2.12. Let G be a graph. The *DP-chromatic number* of G, $\chi_{DP}(G)$, is the smallest k in $\mathbb{N}$ such that G admits a $\mathcal{H}$-colouring for every k-fold cover $\mathcal{H}$ of G.

Some list chromatic results also hold for the DP-chromatic number. It is easy to show $\chi_{DP}(G) \leq d+1$ for any d-degenerate graph G. The simple argument used for the analogous colouring, and list colouring results applies easily to a DP-colouring context.

Definition 2.13. We say a graph G is *d-degenerate* if every subgraph of G has a vertex of degree at most d.

Lemma 2.14. *If G is a d-degenerate graph, then $\chi_{DP}(G) \leq d+1$.*

Proof. Let G be a d-degenerate graph. Then there is an ordering of $V(G)$ such that every vertex v of G has at most d neighbours u such that $u \leq v$. Thus given a $(d+1)$-correspondence assignment C of G we use the greedy algorithm to find a C-colouring of G. □

In the following example we show that C_4 has correspondence chromatic number 3. This argument easily generalizes to all even cycles. Note that all even cycles are 2-choosable and this can be easily shown using an elementary proof.

Example 2.15. $\mathcal{H}$ is a 2-fold cover of C_4 where $L(u) = \{1_u, 2_u\}$ for each vertex u in G (see Figure 2.3). We claim that C_4 is not $\mathcal{H}$-colourable.

Suppose for a contradiction that C_4 is $\mathcal{H}$-colourable, and let I be an independent subset of $V(H)$ of size $|V(G)|$. Then I intersects $L(u)$ in exactly one vertex, for each vertex u in G. Suppose WLOG that $I \cap L(a) = \{1_a\}$. Since 1_a is adjacent to 1_b, I intersects $L(b)$ in 2_b. Similarly I intersects $L(d)$ in 2_d. Since 2_b is adjacent to 2_c, I intersects $L(c)$ at 1_c. Note that 1_c and 2_d are adjacent, hence I is not independent. This leads to a contradiction. By symmetry, if I intersects $L(a)$ at 2_a, then I intersects $L(d)$ at 1_d and $L(c)$ at 2_c. Since 1_d and 2_c are adjacent, we arrive at a contradiction. Thus C_4 does not admit a $\mathcal{H}$-colouring.

Since C_4 is not $\mathcal{H}$-colourable, C_4 is not 2-DP-colourable. Since C_4 is 2-degenerate it has DP-chromatic number at most 3. Therefore C_4 has DP-chromatic number 3.

Note that the reasoning in Example 2.15 can be used to show that a cycle of any even length ≥ 4 has DP-chromatic number 3. Since all odd length cycles have list chromatic number 3, they have DP-chromatic number 3. As a result, all cycles have DP-chromatic number 3.

As noted earlier, correspondence colouring is a generalization of list colouring. Given a list assignment L of G, we can construct a cover of G, $\mathcal{H} = (L, H)$, where the vertices of

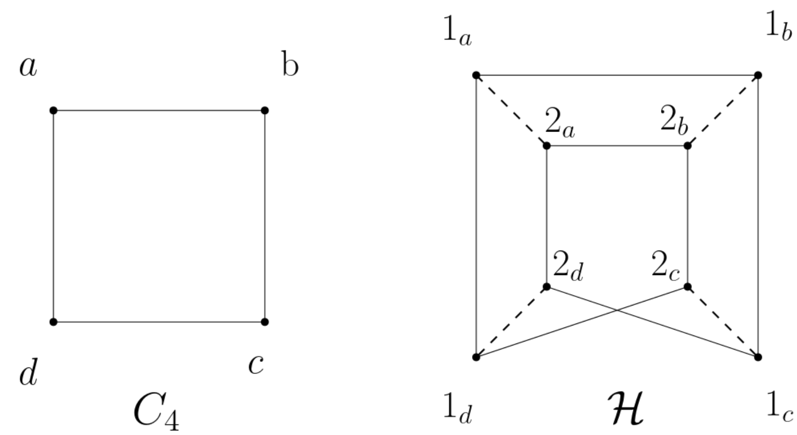

Figure 2.3: $\mathcal{H}$ is a 2-cover of C_4 (the cliques connected by dotted edges correspond to the vertices of C_4). We call $\mathcal{H}$ the twisted cube.

H are $u \times L(u)$ for $u \in V(G)$, and the matching corresponding to each edge uv matches exactly the common colours of $L(u)$ and $L(v)$. Any $\mathcal{H}$-colouring of G corresponds uniquely to a L-colouring of G.

2.2.2 DP-Colouring vs. List Colouring

Dvořák and Postle observed that the proofs of the following list colouring results of Thomassen [37, 38] easily translate to a DP-colouring setting. For any planar graph G, $\chi_{DP}(G) \leq 5$, and $\chi_{DP}(G) \leq 3$ if G has girth at least 5.

Furthermore, there are list colouring results that can only be proved using DP-colouring. For example, Dvořák and Postle introduced DP-colouring to prove that every planar graph without cycles of length 4-8 is 3-list colourable [17]. Their proof is given in Chapter 5. Moreover, Bernshteyn and Kostochka used DP-colouring to give a classification of graphs that satisfy Dirac's bound for list critical graphs with equality, answering a question of Kostochka and Stiebitz [7].

We say a graph G is $(k+1)$-*critical* if $\chi(G) = k+1$ and $\chi(G-u) = k$, for every vertex u in G. Given a k-list assignment L of G, we say that G is L-*critical* if G is not L-colourable but $G-u$ is L-colourable for every vertex u of G. Kostochka and Stiebitz proved the following list colouring generalization of Dirac's lower bound for the number of edges in a k-critical graph. If G is a graph, not containing a clique of size $k+1$, and L is a k-list assignment of G such that G is L-critical, then

$$2m \geq kn + k - 2$$

where $m := |E(G)|$ and $n := |V(G)|$.

There are also some prominent differences between DP-colouring and list colouring. For example, the DP-chromatic number of every graph, with average degree d, is $\Omega(d/\log d)$ [5], while the list chromatic number of such graphs is $\Omega(\log d)$ [2]. However, asymptotic upper bounds on list chromatic numbers tend to have the same order of magnitude as the DP-chromatic number. Note that the list chromatic bound is tight since $\chi_l(K_{d,d}) = O(\log d)$ [19]. For instance, both $\chi_l(G)$ and $\chi_{DP}(G)$ are $O(\Delta/\log \Delta)$ for triangle free graphs G, with max degree Δ [5]. Recently, Molloy refined the previous result to $\chi_l(G) \leq (1 + o(1))(\Delta/\ln \Delta)$ [31] and Bernshteyn proved this refinement also generalizes to DP-colouring [6].

Furthermore the orientation results of Alon and Tarsi, known as the Combinatorial Nullstellensatz (see Theorem 4.7), and closely related Bondy-Boppana-Siegel Lemma are important tools in list colouring which do not generalize to a DP-colouring setting [4]. For example, they can be used to show that even cycles are 2-list colourable, but the DP-chromatic number of any cycle is 3 (see Example 2.15). We will discuss the Alon-Tarsi method in detail in Section 4.3. Here we note one more list colouring result obtained using the Alon-Tarsi method that does not generalize to correspondence colouring.

The following result is about the list chromatic number of bipartite planar graphs. It is an application of the Alon-Tarsi orientation method.

Theorem 2.16. *Every planar bipartite graph is 3-list colourable.*

In the next subsection we describe a construction by Bernshteyn and Kostochka which shows the following.

Theorem 2.17. *There exists a planar bipartite graph G with $\chi_{DP}(G) = 4$.*

2.2.3 Proof of Theorem 2.17

The planar bipartite graph G, with $\chi_{DP}(G) = 4$, and the problematic 3-fold cover of G are constructed using the graph Q and its cover $\mathcal{F} = (L, F)$. Let Q be the skeleton of the 3d cube and let $\mathcal{F}$ be the cover shown in Figure 2.4. Note that Q is bipartite with partition (X, Y) where $X = \{a, b, d_1, d_3\}$ and $Y = \{c_1, c_2, d_2, d_4\}$

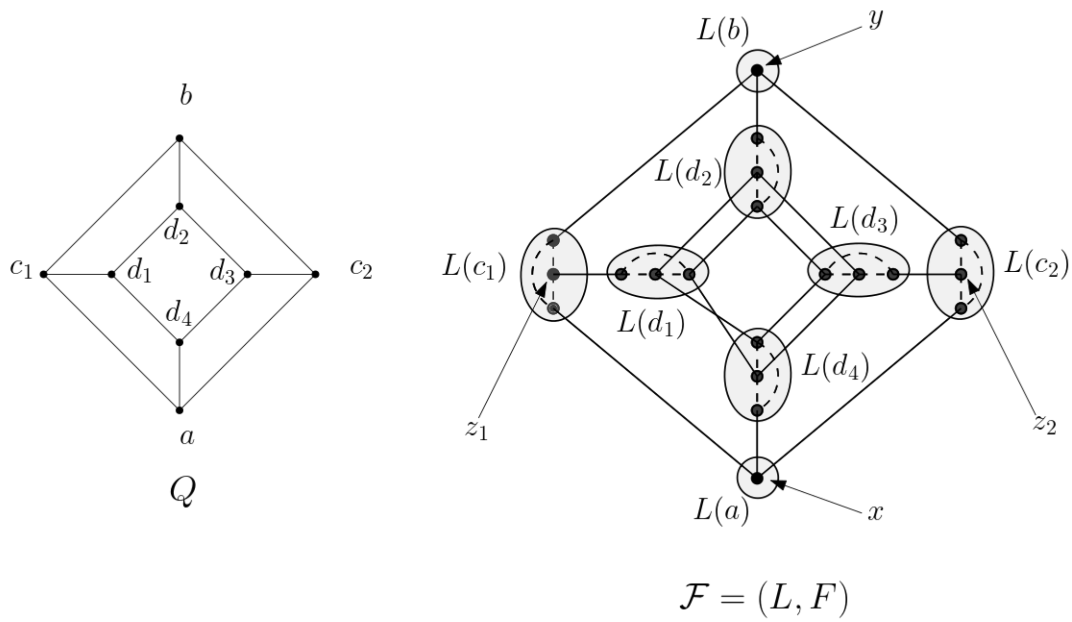

Figure 2.4: Q is a skeleton of a 3d cube and $\mathcal{F}$ is a cover of Q. The dashed lines represent the edges of the cliques corresponding to the vertices of Q.

Lemma 2.18. *Q is not $\mathcal{F}$-colourable.*

Proof. Suppose for a contradiction that I is a $\mathcal{F}$-colouring of Q. Recall that I is an independent set of vertices, of size $|V(Q)|$ in F. Since x, y are the only vertices in $L(a), L(b)$ respectively, $x, y \in I$. Since z_1 is the only vertex in $L(c_1)$ not adjacent to x or y, $z_1 \in I$. Similarly $z_2 \in I$. Thus the vertices corresponding to d_1, d_2, d_3, d_4, that are not adjacent to a vertex in I, span a twisted cube in F (see Figure 2.3). Since the twisted cube does not contain an independent set of size 4, we arrive at a contradiction. □

Now we prove Theorem 2.17.

Proof.
Let Q_{ij}, $1 \le i, j \le 3$, be 9 pairwise disjoint copies of Q. For each vertex u in $V(Q)$, let its copy in Q_{ij} be u_{ij}. For each i, j, let $\mathcal{F}_{ij} = (L_{ij}, F_{ij})$ be a cover of Q_{ij} isomorphic to $\mathcal{F}$. Note that F_{ij} are pairwise disjoint. For u in $V(F)$ let u_{ij} be its copy in F_{ij}.

Let G be obtained from the disjoint union of Q_{ij}, $1 \le i, j \le 3$, by identifying $a_{11}, ..., a_{33}$ to form a vertex a^*, and $b_{11}, ..., b_{33}$ to form a vertex b^*. Note that G is bipartite with partition

(X^*, Y^*) with $Y^* := \cup_{1\leq i,j\leq 3} Y_{ij}$ and $X^* := (\cup_{1\leq i,j\leq 3} X_{ij} \setminus \{a_{ij}, b_{ij}\}) \cup \{a^*, b^*\}$. Moreover G is planar since the graphs Q_{ij} can be embedded in the plane side by side and stretched so that $a_{11}, , , a_{33}$ and $b_{11}, ..., b_{33}$ can be identified without edges crossing. Therefore G is a planar bipartite graph.

Let H be the graph obtained from the unions of the graphs F_{ij} as follows. For each $1 \leq i, j \leq 3$, identify the vertices $x_{i1}, ..., x_{i3}$ to a new vertex x_i, and the vertices $y_{1j}, ..., y_{3j}$ to form a vertex y_j. Place edges between $x_1, ..., x_3$ so that they form a triangle and edges between $y_1, ..., y_3$ so that they form a triangle. Let L^* be defined as follows

$$L^*(u) = \begin{cases} L_{ij}(u) & \text{if } u \in V(Q_{ij}) \\ \{x_1, x_2, x_3\} & \text{if } u = a^* \\ \{y_1, y_2, y_3\} & \text{if } u = b^* \end{cases}$$

Then (L^*, H) is a 3-fold cover of G.

We claim that G is not (L^*, H)-colourable. Suppose for a contradiction that G is (L^*, H)-colourable and let I be a (L^*, H)-colouring of G. Let i, j be such that $x_i, y_j \in I$. Then $\{x_{ij}, y_{ij}\} \cup (I \cap V(F_{ij}))$ induces a $\mathcal{F}_{ij}$-colouring of Q_{ij} which leads to a contradiction. □

2.3 Brooks' Theorem

Since every graph G is $\Delta(G)$-degenerate (see Definition 2.13), $\chi(G) \leq \Delta(G) + 1$. Brooks showed that every graph is G is $\Delta(G)$ colourable unless G is a complete graph or an odd cycle. In this section we state Brooks' Theorem and analogous results for list colouring and correspondence colouring.

Theorem 2.19 (Brooks' Theorem)**.** *If G is connected then $\chi(G) \leq \Delta(G)$, unless G is a complete graph or an odd cycle.*

Also, as noted previously, for any graph G, $\chi_l(G) \leq \Delta(G) + 1$. A Brooks' type result for list colouring was shown by Borodin[11, 13] and independently by Erdős, Rubin, and Taylor[19]. To state the result we need the following definition.

Definition 2.20. Let G be a graph and L a list assignment of G.

- We say L is a *degree-list assignment* if $\deg(v) \leq |L(v)|$ for all v in $V(G)$.

- We say that G is *degree-choosable* if and only if G has an L-colouring for every degree-list assignment L.

Theorem 2.21. *A connected graph G is not degree choosable if and only if each block of G is isomorphic to K_n for some integer n or C_n for some odd integer n.*

Recall Lemma 2.14 states that $\chi_{DP}(G) \leq d + 1$ for any d-degenerate graph G. Therefore $\chi_{DP}(G) \leq \Delta(G) + 1$ for any graph G. Bernshteyn, Kostochka, and Pron proved a Brooks' type theorem for the colouring of multigraphs [9].

Definition 2.22. Let G be a multigraph, then G is said to be *degree-DP-colourable* if G admits a (L, C)-colouring for every degree-list assignment L and each correspondence assignment over L.

Definition 2.23. For a multigraph G and an integer t, G^t is the graph obtained from G by replacing each edge by t parallel edges.

Theorem 2.24. *A connected multigraph G is not degree-DP-colourable if and only if each block of G is K_n^t or C_n^t for some n, t.*

Theorem 2.24 is used in Section 3.4.3 to show that the modified line graph of every cubic bipartite graph with maximum degree 3 is 4-correspondence colourable (see Theorem 3.14). This generalizes a distortion colouring result stated in Theorem 3.10.

In [30] Kim and Ozeki also characterize the type of degree-DP-assignments which do not admit a colouring.

Chapter 3

Delay colouring

In this chapter we review the notion of delay colouring and some important results of Alon, Asodi, Edwards, Georgakopoulos, Kennedy, Haxell, Wilfong, and Winkler. Furthermore we formulate delay colouring as an instance of correspondence colouring, and use correspondence colouring to generalize some of the aforementioned results.

3.1 Definitions and Background

Delay colouring was introduced by Haxell, Wilfong, and Winkler in [24]. As outlined in Chapter 1, delay colouring is motivated by a problem in Time-domain Wavelength Interleaved Networking (TWIN) architecture, which arose in work by Saniee and Widjaja to model the problem of speed-of-light delays in optical networks operating over long distances [35].

Definition 3.1. Let G be a bipartite (multi)graph and let (X, Y) be a partition of $V(G)$. Let $k > 0$ be an integer.

- Let $r : E(G) \longrightarrow \mathbb{N}$ be a function. We call r a *delay function*.
- A *delay graph* (G, r) consists of a bipartite (multi)graph G and a delay function r on G.
- A *k-delay colouring* of (G, r) is a function f from $E(G)$ to the integers mod k such that for each $x \in X$ the elements in the collection $(f_e : e \in E, x \in e)$ are distinct and for each $y \in Y$ the elements in the collection $(f_e + r_e \pmod{k} : e \in E, y \in e)$ are distinct.

- If f is a k-delay colouring of (G, r), then for $e \in E$ the *effective colour* of e is $f_e + r_e (\text{mod } k)$.

Remark: In the definition above two different edges with the same ends can be assigned different delays.

A delay graph (G, r) models TWIN network architecture as follows. We let $G = (X, Y, E)$ be a bipartite graph where the vertices of X correspond to the transmitter nodes of a network, and the vertices of Y correspond to the receiver nodes. For $x \in X$ and $y \in Y$ there is an edge between x and y iff x will send data to y. For an edge e with ends $x \in X$ and $y \in Y$, r_e is the transmission delay between x and y. The formulation allows for parallel edges because there can be several routes from x to y through the network. In practice, it does not often occur that two such routes have substantially different lengths but the theoretical model allows this property.

Definition 3.2. Let G be a graph.

- We say G is *k-delay colourable* if there is a k-delay colouring of (G, r) for each delay function r.
- The *delay chromatic number* of G, $\chi_d(G)$, is the minimum k such that G is k-delay colourable.

A simple greedy colouring argument shows that $\chi_d(G)$ is at most $2\Delta(G) - 1$.

Notation: In this chapter we will use $[n]$ to denote the set $\{0, 1, 2, ..., n\}$, and in every other chapter we will use it in a standard way to denote the set $\{1, 2, ..., n\}$.

3.2 Previous Results

Let m be an integer. An important problem in delay colouring is finding the minimum $k = k(m)$ such that any delay graph (G, r), where G has maximum degree m, has a k-delay colouring. Haxell, Wilfong and Winkler made the following conjecture.

Conjecture 3.3. *Let G be a bipartite (multi)graph with maximum degree m. Then $\chi_d(G) \leq m + 1$.*

Let $G = (X, Y, E)$ be a bipartite (multi)graph with maximum degree m such that $|X| = |Y| = 1$, and let r be a delay function on G. Then G has m edges and Conjecture 3.3 reduces to finding a permutation π of the set $[m]$ such that ($\pi_e + r_e \pmod{m+1}$: $e \in E$) are all distinct (mod $m + 1$). As shown in [24], this is possible by the following theorem of Marshall Hall [23].

Theorem 3.4. *Let $A = \{a_1, ..., a_n\}$ be an abelian group of order n, and let $b_1, ..., b_n \in A$, not necessarily distinct, be such that $\sum_{i=1}^{n} b_i = 0$. Then there exists a permutation π of A such that the list of elements $(\pi a_i + b_i : 1 \leq i \leq n)$ are distinct.*

We can add an edge e' to G to obtain a graph G'. Furthermore we define a delay function r' on G' as follows.

$$r'_e = \begin{cases} r_e & \text{if } e \in E \\ -\sum_{e \in E} r_e \pmod{m+1} & \text{if } e = e' \end{cases}$$

Thus we have a delay graph (G', r') where the delays sum up to 0 (mod $m+1$). By Theorem 3.4 (G', r') has a $(m + 1)$-delay colouring, hence (G, r) has a $(m + 1)$-delay colouring. Therefore Conjecture 3.3 holds for bipartite graphs $G = (X, Y, E)$ where $|X| = |Y| = 1$.

Furthermore Georgakopoulos showed that Conjecture 3.3 holds for $m = 3$ [22], and Edwards and Kennedy showed that it holds for $m = 4$ [18].

Haxell, Wilfong and Winkler also proved the following more general theorem.

Theorem 3.5. *Let integers m and t be given where $2m \geq 3t^2 + 4t - 2$. Let (G, r) be a delay graph, where G has maximum degree m. Then (G, r) is n-delay colourable where $n = 2m - t + 1$, and such a colouring can be found efficiently.*

Alon and Asodi proved the following asymptotic result using the probabilistic method [3].

Theorem 3.6. *For every $\epsilon > 0$ there is $m_0 = m_0(\epsilon)$ such that the following holds. Let (G, r) be a delay graph where G is simple and has maximum degree $m \geq m_0$. Then for $k = \lceil (1 + \epsilon)m \rceil$ there is a k-delay colouring of G.*

In [31] Molloy proved that for every simple graph G with maximum degree Δ, $\chi'_{DP}(G) = \Delta + o(\Delta)$. In Section 3.3, we show that delay colouring can be formulated as an instance of correspondence colouring. Therefore Theorem 3.6 follows from the aforementioned result of Molloy. Moreover the result of Molloy extends a result of Kahn, which shows that the List Colouring Conjecture (see Section 2.2 for discussion of the LCC) holds asymptotically [27], to DP-colouring.

3.3 Correspondence Colouring Formulation

In this section, we will show that delay colouring is a special case of correspondence colouring.

Definition 3.7. Let $G = (X, Y, E)$ be a bipartite (multi)graph. The *modified line graph* of G is a graph $ML(G) = (V', E')$ where $V' = E(G)$ and the E' is defined as follows. For $e, f \in E$, $(ef)_X$ is in E' if and only if e and f share an endpoint in X, and $(ef)_Y$ is in E' if and only if e and f share an endpoint in Y.

Remark: The definition above differs from the usual definition of a line graph only when G has parallel edges. If e, f are parallel edges in G, then they are joined by two parallel edges, $(ef)_X$ and $(ef)_Y$, in the modified line graph instead of one edge in the usual line graph.

Let (G, r) be a delay graph, then $ML(G)$ is the modified line graph of G. Let $k > 0$ be an integer. We will define a k-correspondence assignment, $C(r)$, of $ML(G)$ as follows. For each $e \in V(ML(G))$, let $L(e) = [k-1]$. Let e, f be edges which share an end in G. If $(ef)_X$ is an edge in $ML(G)$, then let $C(r)_{(ef)_X}$ be the perfect matching that matches (e, c) to (f, c), for each $c \in [k-1]$ (we say the matching is *full* and *straight*, as in Definitions 5.4 and 5.5). If $(ef)_Y$ is an edge in $ML(G)$, then let $C(r)_{(ef)_Y}$ be a full matching such that for $c_1, c_2 \in [k-1]$, (e, c_1) corresponds to (f, c_2) if and only if $c_1 = c_2 + r_f - r_e \pmod k$.

Lemma 3.8. *(G, r) is k-delay colourable if and only if $ML(G)$ is $C(r)$-colourable.*

Proof. Let ϕ be a k-delay colouring of (G, r). Let $\phi' : V(ML(G)) \longrightarrow [k-1]$ be defined as follows. For each $e' \in V(ML(G))$, $\phi'(e) = \phi(e)$. Note that for each $e \in V(ML(G))$, $\phi'(e) \in L(e)$. Let $e, f \in V(ML(G))$ be adjacent, then e, f are two edges that share an endpoint in G.

If e, f share an endpoint in X then $\phi(e) \neq \phi(f)$, by the definition of delay colouring, thus $\phi'(e) \neq \phi'(f)$. Moreover (e, c_1) is adjacent to (f, c_2) in $C(r)_{(ef)_X}$ if and only if $c_1 = c_2$ for any $c_1, c_2 \in [k-1]$. Therefore $(e, \phi'(e))$ is not adjacent to $(f, \phi'(f))$ in $C(r)_{(ef)_X}$.

If e, f share an endpoint in Y then $\phi(e) + r_e$ and $\phi(f) + r_f$ are distinct (mod k), by the definition of delay colouring. Hence $\phi'(e) + r_e$ and $\phi'(f) + r_f$ are distinct (mod k). Moreover (e, c_1) is adjacent to (f, c_2) in $C(r)_{(ef)_Y}$ if and only if $c_1 = c_2 + r_f - r_e \pmod k$, for any $c_1, c_2 \in [k-1]$. Therefore, $(e, \phi'(e))$ is not adjacent to $(f, \phi'(f))$ in $C(r)_{(ef)_Y}$. In any case, ϕ' does not violate any correspondence dictated by $C(r)$. As a result ϕ' is a $C(r)$-colouring of $ML(G)$.

Conversely, let ϕ' be a $C(r)$-colouring of $ML(G)$. Let $\phi : E(G) \longrightarrow [k-1]$ be defined as follows. For each $e \in E(G)$, $\phi(e) = \phi'(e)$. Let e, f be two edges in G which share an endpoint. If e, f share an endpoint in X then $(e, \phi'(e))$ is adjacent to $(f, \phi'(e))$ in $C(r)_{(ef)_X}$. As a result $\phi(e) = \phi'(e) \neq \phi'(f) = \phi(f)$. If e, f share an endpoint in Y then $(e, \phi'(e))$ is adjacent to (f, c) in $C(r)_{(ef)_Y}$, where $c = \phi'(e) + r_e - r_f \pmod k$.

As a result $\phi'(f) \neq \phi'(e) + r_e - r_f$ hence $\phi(f) \neq \phi(e) + r_e - r_f \pmod k$. Therefore ϕ is a k-delay colouring of G. □

3.4 Delay Colouring in Cubic Graphs

3.4.1 Distortion Colouring

In his paper on delay colouring in cubic graphs, Georgakopoulos introduced the more general notion of distortion colouring [22]. Given a bipartite (multi)graph $G = (X, Y, E)$, and an integer $k > 0$, every edge $e \in E$ is assigned a permutation π_e of $[k-1]$. We aim to find an edge colouring, $\phi : E \longrightarrow [k-1]$, such that for any pair of edges e, f, which share an end in X, $\phi(e) \neq \phi(f)$, and for any pair of edges e, f, which share an end in Y, $\pi_e\phi(e) \neq \pi_f\phi(f)$. For a definition of distortion colouring we will use the following notation.

Notation: Let $S_{[n]}$ denote the set of permutations of $[n]$.

Definition 3.9.

- Let $G = (X, Y, E)$ be a bipartite graph and let $k > 0$ be an integer, then a *k-distortion function* on G is a function $\pi : E \longrightarrow S_{[k-1]}$.
- Given a bipartite graph $G = (X, Y, E)$, and a k-distortion function π on G, a *distortion colouring* ϕ of (G, π) is a function, $\phi : E \longrightarrow [k-1]$, such that for $x \in X$ the collection of elements $(\phi(e) : e \in E, x \in e)$ are distinct and for $y \in Y$ the elements of the collection $(\pi_e\phi(e) : e \in E, y \in e)$ are distinct.

Distortion colouring is a generalization of delay colouring since for an integer $k > 0$ a constant delay is a permutation of the set $[k-1]$. Georgakopoulos proved the following result for distortion colouring.

Theorem 3.10. *For every bipartite graph G with maximum degree 3 and any 4-distortion function π on G, there is a distortion colouring of (G, π).*

Theorem 3.10 implies that a bipartite graph G with maximum degree 3 is 4-delay colourable.

3.4.2 Correspondence formulation of Distortion Colouring

In this subsection, we will show that distortion colouring can also be formulated as a correspondence colouring problem. Let G be bipartite (multi)graph, $k > 1$ be an integer, and π a k-distortion function on G.

We will define a k-correspondence assignment, $C(\pi)$, of $ML(G)$ as follows. For each $e \in V(ML(G))$ let $L(e) = [k-1]$. Let e, f be edges which share an end in G. If $(ef)_X$ is an edge in $ML(G)$, then let $C(\pi)_{(ef)_X}$ be the perfect matching that matches (e, c) to (f, c), for each $c \in [k-1]$. If $(ef)_Y$ is and edge in $ML(G)$, then let $C(\pi)_{(ef)_Y}$ be a full matching such that for $c_1, c_2 \in [k-1]$, (e, c_1) corresponds to (f, c_2) if and only if $\pi_e c_1 = \pi_f c_2$.

Lemma 3.11. *Given a bipartite (multi)graph G and a k-distortion function π on G, (G, π) has a distortion colouring if and only if $ML(G)$ is $C(\pi)$-colourable.*

The proof is similar to that of Lemma 3.8.

3.4.3 The Application of Correspondence Colouring to Delay Colouring

Recall Bernshteyn and Kostochka proved a version of Brooks' Theorem for correspondence colouring, Theorem 2.24. The theorem states that a (multi)graph is not degree-DP-colourable if and only if each block of G is K_n^t or C_n^t for some n, t. In this section, we will show that Theorem 3.10 follows as an easy consequence of Theorem 2.24. In fact, we will prove a substantial generalization of Theorem 3.10 in Theorem 3.14. We will need the following lemmas.

Lemma 3.12. *Let G be a connected cubic bipartite (multi)graph. Then the modified line graph of G is 2-connected.*

Proof. Suppose for a contradiction that $ML(G)$ is not 2-connected. Then $ML(G)$ has a cut vertex, thus G has a cut edge e. Let x, y be the ends of e and suppose without loss of generality $x \in X$. Let C be the component of $G - e$ containing x. Let $X_C = V(C) \cap X$ and $Y_C = V(C) \cap Y$.

Note that y is not in C in $G - e$, therefore x is the only vertex in $V(C)$ which is adjacent to y in G. As a result every vertex in Y_C and $X_C \backslash \{x\}$ has degree 3 in C. Thus we have

$$\begin{aligned} 3(|X_C| - 1) + 2 &= \sum_{v \in X_C} \deg_c v \\ &= \sum_{v \in Y_C} \deg_c v \\ &= 3|Y_C|. \end{aligned}$$

As a result $2 = 3|Y_C| - 3(|X_C| - 1)$, which leads to a contradiction since 2 is not divisible by 3. □

Consider K_2^3, the graph consisting of two vertices and 3 parallel edges between them. The modified line graph of K_2^3 is K_3^2. Note that K_2^3 is 4-delay colourable by Theorem 3.4.

Lemma 3.13. *Let G be a connected cubic bipartite (multi)graph and let $ML(G)$ be the modified line graph of G. $ML(G)$ is isomorphic to K_n^t or C_n^t for some integers n, t if and only if G is isomorphic to K_2^3.*

Proof. Let (X, Y) be a bipartition of G. Suppose $ML(G) \cong K_n^t$ for some n, t. Suppose for a contradiction that $|X| > 1$. Then $|Y| > 1$ since $3|X| = 3|Y|$. Let x be a vertex in X which has more than one neighbour in Y. Note that x exists since G is connected and cubic. Let y be a neighbour of x such that x and y have only one edge between them. Note that y exists because otherwise x would have degree at least 4. Let y' be a neighbour of x distinct from y, and let x' be a neighbour of y distinct from x. Let e be an edge between x and y' and let f be an edge between x' and y. Then e, f do not share an end in G. Since $ML(G) \cong K_n^t$, e is adjacent to f in $ML(G)$, hence e and f share an end point in G. This leads to a contradiction. It follows that $|X| = |Y| = 1$. Since G is cubic, $G \cong K_2^3$.

Suppose that $ML(G) \cong C_n^t$ for some t, n. By the definition of a modified line graph (see Definition 3.7), $t \leq 2$. Let e be an edge in G, then e has two neighbours in $ML(G)$, since $ML(G) \cong C_n^t$. Let x, y be the ends of e in G and let f_1, f_2 be the neighbours of e in $ML(G)$. If $t = 1$, then f_i is incident with exactly one of x, y, for each i. It follows that x or y has degree strictly less than 3. This leads to a contradiction, therefore $t = 2$. Note that the neighbours of e in $ML(G)$ are $\{f \in E(G) : x \in f \text{ or } y \in f\}$. Thus there are only 3 edges incident with x or y in G, f_1, f_2 and e. Since G is cubic, x, y are both incident with all three edges. Moreover x, y don't have neighbours in $G - \{x, y\}$. Since G is connected $V(G) = \{x, y\}$, thus $G \cong K_2^3$.

Conversely suppose that G is isomorphic to K_2^3. Then $ML(G) \cong K_3^2$. □

Now we can prove the main theorem of this section.

Theorem 3.14. *Let G be a bipartite (multi)graph with maximum degree 3. Let $ML(G)$ be the modified line graph of G. Then $ML(G)$ is 4-correspondence colourable if and only if G does not have a component isomorphic to K_2^3.*

Proof. Suppose no component of G is isomorphic to K_2^3. Every bipartite (multi)graph with maximum degree 3 is a subgraph of a cubic bipartite (multi)graph. Moreover since G does not contain a component isomorphic to K_2^3 we can find a cubic bipartite (multi)graph G', which does not contain a component isomorphic to K_2^3, and contains G as a subgraph. Thus we may assume that G is cubic. Let C' be a component of $ML(G)$, then C' is the modified line graph of a component C of G. By Lemma 3.12 C' is 2-connected. By Lemma 3.13, since C is not isomorphic to K_2^3, C' is not isomorphic to K_n^t or C_n^t. Thus by Theorem 2.24 C is degree-DP-colourable. As a result C' is 4-correspondence colourable. Therefore $ML(G)$ is 4-correspondence colourable.

Conversely suppose that $ML(G)$ is 4-correspondence colourable. Let C be a component of G. Then the modified line graph of C, C' is a component of $ML(G)$. By Lemma 3.12 C' is 2-connected, hence C' is a block of $ML(G)$. By Theorem 2.24 C' is not isomorphic to K_n^t or C_n^t for some t, n. As a result C is not isomorphic to K_2^3. □

Now we derive Theorem 3.10 as a corollary of Theorem 3.14, except in the special case of K_2^3 which is easily verified by hand (see Appendix A).

Corollary 3.15. *Let G be a bipartite (multi)graph, with maximum degree 3, such that no component of G is isomorphic to K_2^3. For any 4-distortion function π on G, there is a distortion colouring of (G, π).*

Proof. Given G and π construct the the 4-correspondence assignment $C(\pi)$ on $ML(G)$. Since no component of G is isomorphic to K_2^3, by Theorem 3.14 $ML(G)$ is 4-correspondence colourable. Therefore $ML(G)$ is $C(\pi)$-colourable. By Lemma 3.11 (G, π) has a 4-distortion colouring. □

Chapter 4

Cyclic and Anticyclic Correspondence Colouring

In this chapter, we introduce cyclic correspondence colouring, an instance of correspondence colouring which generalizes delay colouring. We also introduce the related notion of anticyclic correspondence colouring. Moreover, we use the Combinatorial Nullstellensatz to prove a result, analogous to an orientation theorem of Alon and Tarsi (see Theorem 4.9), for cyclic and anticyclic correspondence colouring (see Theorems 4.10 and 4.12). We use Theorem 4.10 to generalize a delay colouring result of Edwards and Kennedy. Our proof uses the same main ideas in the proof of Edwards and Kennedy.

4.1 Cyclic Correspondence Colouring

Definition 4.1. Let $G = (V, E)$ be a (multi)graph, and let $p > 0$ be an integer. A correspondence assignment C on G is a *p-cyclic correspondence assignment* (p-CCA) if it satisfies the following (see Figure 4.1).

i. For each $v \in V$, $L(v) = \mathbb{Z}_p$.

ii. There exists a function $\rho : E \longrightarrow \mathbb{Z}$ and an ordering of $V = \{v_1, ..., v_n\}$ such that for each edge e with ends v_i, v_j, where $i < j$, (v_i, c_1) is adjacent to (v_j, c_2) in C_e if and only if $c_1 = c_2 - \rho(e) \pmod p$, for any $c_1, c_2 \in \mathbb{Z}_p$.

We call ρ a *difference function* on G. Note that for an edge e $\rho(e)$ determines the perfect matching C_e.

Let $G = (V, E)$ be a (multi)graph and let C be a p-CCA on G. Note that C depends on an ordering of the vertices of G and a difference function on G. However the following lemma shows that we are not bound to a specific ordering of V, as given any ordering of V, there is a difference function on G that gives C.

Lemma 4.2. *Let $G = (V, E)$ be a (multi)graph on n vertices, and let C be a p-CCA of G where $p > 0$ is an integer. Let $\pi : V \longrightarrow \{1, ..., n\}$ be an ordering of V. Then there is a difference function ρ on G such that C satisfies condition (ii) of the definition of a p-CCA with respect to π and ρ.*

Proof. By the definition of a p-CCA, there exists an ordering of V, $\pi' : V \longrightarrow \{1, ..., n\}$ and a difference function ρ' on G such that C satisfies condition (ii) with respect to ρ' and π'. Let $\rho : E \longrightarrow \mathbb{Z}$ be defined as follows. For $e \in E$ with ends u, v

$$\rho(e) = \begin{cases} \rho'(e) \text{ if } \operatorname{sign}(\pi(u) - \pi(v)) = \operatorname{sign}(\pi'(u) - \pi'(v)) \\ -\rho'(e) \text{ otherwise.} \end{cases}$$

Let C' be a p-CCA with respect to π and ρ. Let e be an edge in G with ends u, v and suppose without loss of generality that $\pi'(u) < \pi'(v)$.

Case 1:

Suppose $\pi(u) < \pi(v)$. Then $\rho(e) = \rho'(e)$. It follows that $C_e = C'_e$.

Case 2:

Suppose that $\pi(u) > \pi(v)$, then $\rho(e) = -\rho(e)$. Since $\pi(u) > \pi(v)$, then for any colours $c_u, c_v \in \mathbb{Z}_p$, (u, c_u) is adjacent to (v, c_v) in C'_e if and only if $c_v = c_u - \rho(e) = c_u + \rho'(e)$. Therefore (u, c_u) is adjacent to (v, c_v) in C'_e if and only if $c_u = c_v - \rho'(e)$.

Moreover since $\pi'(u) < \pi'(v)$, for any colours $c_u, c_v \in \mathbb{Z}_p$, (u, c_u) is adjacent to (v, c_v) in C_e if and only if $c_u = c_v - \rho'(e)$. As a result $C_e = C'_e$.

In any case, $C_e = C'_e$. Therefore $C = C'$. □

Definition 4.3. We say a graph G is *p-cyclic correspondence colourable* (p-CCC) if it has a C-colouring, for every p-CCA C.

The following lemma shows that p-cyclic correspondence colouring generalizes delay colouring.

Lemma 4.4. *Let (G, r) be a delay graph and let $p > 0$ be an integer. Then the p-correspondence assignment $C(r)$ defined on $ML(G)$ is a p-CCA.*

Proof. Note that for every vertex $e \in ML(G), L(e) = \mathbb{Z}_p$, where L is the list assignment of $C(r)$. Let $e_1, ..., e_n$ be an ordering of the vertices of $ML(G)$ and let $\rho : E(ML(G)) \longrightarrow \mathbb{Z}_p$ be defined as follows. Let e_i, e_j be edges which share an end in G, with $i < j$.

- If e_i, e_j share an end in X then $\rho((e_i e_j)_X) = 0$.
- If e_i, e_j share an end in Y then $\rho((e_i e_j)_Y) = r_{e_i} - r_{e_j}$.

If e_i, e_j are edges in G which share an end in X, then (e_i, c_1) corresponds to (e_j, c_2) in $C(r)_{(e_i e_j)_X}$ if and only if

$$\begin{aligned} c_1 &= c_2 \pmod{p} \\ &= c_2 - \rho((e_i e_j)_X) \pmod{p}. \end{aligned}$$

If e_i, e_j are edges in G which share an end in Y, then (e_i, c_1) corresponds to (e_j, c_2) in $C(r)_{(e_i e_j)_Y}$ if and only if

$$\begin{aligned} c_1 &= c_2 + r_{e_j} - r_{e_i} \pmod{p} \\ &= c_2 - \rho((e_i e_j)_Y) \pmod{p}. \end{aligned}$$

In any case, for an edge f in $ML(G)$ with ends e_i, e_j such that $i < j$, (e_i, c_1) is adjacent to (e_j, c_2) in C_e if and only if $c_1 = c_2 - \rho(f) \pmod{p}$. As a result $C(r)$ is a p-CCA. □

Note that p-cyclic correspondence colouring generalizes delay colouring as it is defined for general (multi)graphs and not just the modified line graphs of bipartite (multi)graphs. Moreover the difference for each edge is arbitrary and does not come from the differences determined by a delay function on a bipartite (multi)graph.

P-cyclic correspondence colouring is a special case of group colouring, a notion introduced by Jaeger, Linial, Payan and Tarsi in 1992 [26]. Given a graph G, an orientation D of G, and an abelian group Γ, let $\rho : E(D) \longrightarrow \Gamma$ be a function which assigns each arc $a = (u, v)$ in D, an element of Γ $\rho(a)$. A Γ-colouring of (D, ρ) is a function $f : V(G) \longrightarrow \Gamma$ such that for an arc $a = (u, v)$ in D, $f(u) - f(v) \neq \rho(a)$. If $\Gamma = \mathbb{Z}_p$ we have p-Cyclic Correspondence Colouring.

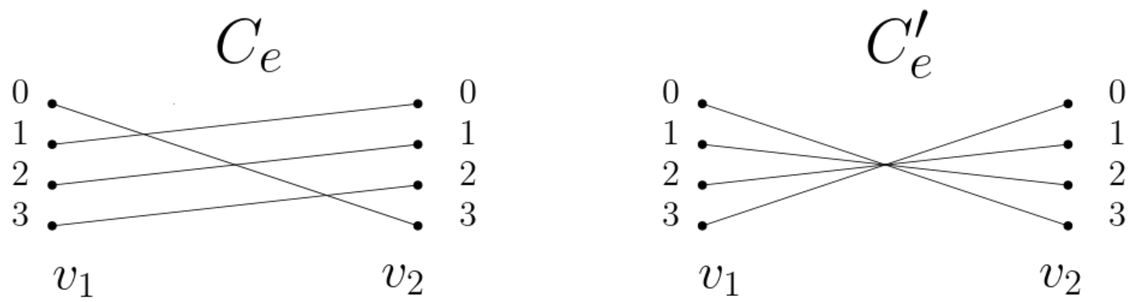

Figure 4.1: C_e is 4-cyclic correspondence on e with $\rho(e) = 3$ and C'_e is a 4-anticyclic correspondence assignment on e with $\sigma(e) = 3$.

4.2 Anticyclic Correspondence Colouring

Now we will introduce another family of correspondence assignments similar to cyclic correspondence assignments.

Definition 4.5. Let $G = (V, E)$ be a (multi)graph, and let $p > 0$ be an integer. A correspondence assignment C on G is a *p-anticyclic correspondence assignment* (p-ACA) if it satisfies the following (see Figure 4.1).

- For each $v \in V$, $L(v) = \mathbb{Z}_p$.
- There exists a function $\sigma : E \longrightarrow \mathbb{Z}$ such that for each edge e with ends v_i, v_j, (v_i, c_1) is adjacent to (v_j, c_2) in C_e if and only if $\sigma(e) = c_1 + c_2 \pmod p$, for any $c_1, c_2 \in \mathbb{Z}_p$.

We call σ the *sum function* of C.

Definition 4.6. We say a graph G is *p-anticyclic correspondence colourable* (p-ACC) if it has a C-colouring, for every p-ACA C.

Some of the results we derive for p-cyclic correspondence colouring can be easily shown for anticyclic correspondence colouring.

4.3 Alon and Tarsi

4.3.1 Preliminaries

Alon and Tarsi proved the following result known as the Combinatorial Nullstellensatz [4].

Theorem 4.7. *Let F be a field and $P = P(x_1, ..., x_n)$ be a polynomial in $F[x_1, ..., x_n]$. Suppose the degree of P is $\sum_{i=1}^{n} t_i$, where t_i is a non-negative integer for each i, and suppose $\prod_{i=1}^{n} x_i^{t_i}$ has a nonzero coefficient in P. If $S_1, ..., S_n$ are subsets of F with $|S_i| > t_i$, there are $s_1 \in S_1, ..., s_n \in S_n$ such that $P(s_1, ..., s_n) \neq 0$.*

Moreover Alon and Tarsi showed the result stated in Theorem 4.9 as a consequence of Theorem 4.7. To state Theorem 4.9 we need the following definitions.

Definition 4.8. Let G be an undirected graph and let $V = \{v_1, ..., v_n\}$ be an ordering of the vertices of G. Let $\tau : V(G) \longrightarrow \mathbb{N}$.

- An orientation of the edges of G *obeys* τ if $d^+(v) = \tau(v)$ for each $v \in V(G)$.
- $\mathrm{D}_G(\tau)$ denotes the number of orientations of G obeying τ.
- Given an orientation of the edges of G a directed edge (v_i, v_j) is *reversed* if $i > j$.
- The *parity* of an orientation is the parity of the number of its reversed edges.
- $\mathrm{DE}_G(\tau)$ and $\mathrm{DO}_G(\tau)$ denote the number of even and odd orientations of G obeying τ respectively.

Theorem 4.9. *Let G be a graph on an ordered set of vertices V. Let F be a field and let $L = (L(v) : v \in V)$ be a collection of subsets of F. Let $\tau : V \longrightarrow \mathbb{N}$ be such that $\tau(v) < |L(v)|$ for each v. If $DE_G(\tau) \neq DO_G(\tau)$ then G has an L-colouring.*

4.3.2 Previous Results

Theorem 4.9 is an important result with many applications in list colouring. Alon and Tarsi used it to show that every planar bipartite graph is 3-choosable [4]. Moreover Fleischner and Steibitz also used it to show that every cycle plus triangles graph is 3-choosable [20]. Cycle plus triangles graphs are formally defined in Section 6.1 (see Definition 6.1).

However the Combinatorial Nullstellensatz does not always work in a correspondence colouring setting. Kostochka and Bernshteyn gave an example of a planar bipartite graph with DP-chromatic number 4 in [8]. Their example is described in Section 2.2.3. Furthermore, in Section 6.2, we presented a cycle plus triangles graph with DP-chromatic number 4. In the following section, we will prove an analogous version of Theorem 4.9 for p-cyclic correspondence colouring.

4.3.3 Analogous Theorems for p-CCC and p-ACC

The following theorem is a result, analogous to Theorem 4.9, for p-cyclic correspondence colouring.

Theorem 4.10. *Let $G = (V, E)$ be a (multi)graph and let $V = \{v_1, ..., v_n\}$ be an ordering of the vertices of G. Let $\tau : V \longrightarrow \mathbb{N}$ be a map and let $p > \max_{v \in V} \tau(v)$ be prime. If $DE_G(\tau) \not\equiv DO_G(\tau)$ (mod p) then G is p-cyclic correspondence colourable.*

Let C be a p-CCA of G. Then, by Lemma 4.2, there is a difference function ρ such that C satisfies the definition of a p-CCA with respect to ρ and the given ordering. Let

$$P(x_1, ..., x_n) = \prod_{\substack{e \in E \\ v_i, v_j \in e, i<j}} (x_i - x_j + \rho(e)).$$

Let $c_1, ..., c_n \in \mathbb{Z}_p$ be such that $P(c_1, ..., c_n) \not\equiv 0 \pmod{p}$. Then for every edge e, with ends v_i, v_j such that $i < j$, $c_i - c_j + \rho(e) \not\equiv 0 \pmod{p}$, hence $c_i \not\equiv c_j - \rho(e) \pmod{p}$. Therefore $c_1, ..., c_n$ is a C-colouring of G. Thus a non-zero solution to $P \pmod{p}$ gives a C-colouring of G.

To prove Theorem 4.10 we will use Theorem 4.7 to find non-zero solution to $P \pmod{p}$. For $1 \leq i \leq n$ let $\tau(v_i) = t_i$.

We will verify that $\sum_{i=1}^{n} t_i = |E|$. If there are no orientations of G obeying τ then we have $DE_G(\tau) = DO_G(\tau) = 0$, hence $DE_G(\tau) \equiv DO_G(\tau) \pmod{p}$. Since $DE_G(\tau) \not\equiv DO_G(\tau)$ (mod p), one of $DE_G(\tau)$, $DO_G(\tau)$ is non-zero. Thus there exists an orientation of G obeying τ. As a result

$$\begin{aligned} |E| &= \sum_{v \in V} \tau(v) \\ &= \sum_{i=1}^{n} t_i. \end{aligned}$$

Moreover we need the following claim, which counts the monomials of the form $\prod_{i=1}^{n} x_i^{t_i}$ in the expansion of P.

Claim 4.11. *(After expanding P but before adding like terms) There is a 1-1 correspondence between the monomial terms of P of the form $\prod_{i=1}^{n} x_i^{t_i}$ and orientations of G obeying τ. Moreover the sign of each such monomial is positive if and only if the parity of the corresponding orientation, with respect to given ordering of the vertices, is even.*

Proof. Note that P has degree at most $|E|$. Let f be a monomial of P after expansion and before adding like terms. Then from each term $(x_i - x_j + \rho(e))$, where $e \in E$, $v_i, v_j \in e$ and $i < j$, f picks up exactly one of $x_i, -x_j, \rho(e)$. If f has degree $|E|$, then f picks up x_i or $-x_j$. Given such a monomial f we can define an orientation D_f of G as follows. For each edge e, $e \in d^+_{D_f}(v)$ if and only if $v \in e$ and f picked up the variable corresponding to v from the term corresponding to e.

Choosing a starting vertex for every edge uniquely determines an orientation of G. Moreover, for each orientation of G the expansion of P has one monomial of degree $|E|$. Therefore, there is a 1-1 correspondence between orientations of G and monomials of degree $|E|$ in the expansion of P.

Let f be a monomial, of the expansion of P, of degree $|E|$. For each $i \in \{1, ..., n\}$ the exponent of x_i in f is $|d^+_{D_f}(v_i)|$. Thus D_f obeys τ if and only if it is of the form $\prod_{i=1}^n x_i^{t_i}$. Let e be an edge with ends v_i, v_j where $i < j$. Then the arc corresponding to e in D_f is reversed if and only if f picks up $-x_j$. It follows that f is positive if and only if D_f has an even number of reversed arcs. □

We may now complete the proof of Theorem 4.10.

Proof. Note that P has degree at most $|E|$, hence P has degree $\sum_{i=1}^n t_i$.

By Claim 4.11, the coefficient of $\prod_{i=1}^n x_i^{t_i}$ is $DE_G(\tau) - DO_G(\tau)$. Since $DE_G(\tau) \not\equiv DO_G(\tau)$ (mod p), $\prod_{i=1}^n x_i^{t_i}$ has a non-zero coefficient in P.

Since P has degree $\sum_{i=1}^n t_i$, $\prod_{i=1}^n x_i^{t_i}$ has a non-zero coefficient in P, and $\tau(v) < p = |\mathbb{Z}_p|$, for each vertex v, by Theorem 4.7, there is $c_1, ..., c_n$ such that $P(c_1, ..., c_n) \not\equiv 0$ (mod p). As a result, G has a C-colouring. □

We also give an analogous result for p-anticyclic correspondence colouring. The proof of the following theorem is very similar to that of Theorem 4.10.

Theorem 4.12. *Let $G = (V, E)$ be a (multi)graph, and let $V = \{v_1, ..., v_n\}$ be an ordering of the vertices of G. Let $\tau : V \longrightarrow \mathbb{N}$ be a map, and let $p > \max_{v \in V} \tau(v)$ be prime. If $D_G(\tau) \not\equiv 0$ (mod p) then G is p-anticyclic correspondence colourable.*

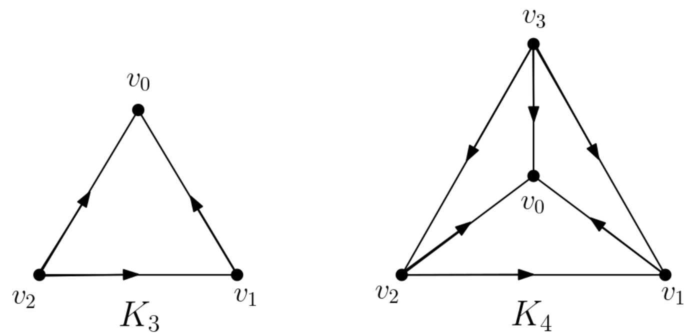

Figure 4.2: Examples of a transitive tours of K_3 and K_4. Note that for each i, vertex v_i has out degree i.

4.4 p-Cyclic Correspondence Colouring in Line Graphs of Quartic Bipartite Graphs

Edwards and Kennedy used the Combinatorial Nullstellensatz to show that every bipartite graph with maximum degree 4 is 5-delay colourable [18]. In this section, we will use a similar approach to prove the following generalization of their result.

Theorem 4.13. *Let $G = (X, Y)$ be a bipartite (multi)graph, on $2n$ vertices, with maximum degree 4. Then $ML(G)$ is 5-cyclic correspondence colourable.*

Since any bipartite (multi)graph with maximum degree 4 is a subgraph of a 4-regular bipartite (multi)graph we may assume G is 4-regular. Let the vertices of $G = \{v_1, ..., v_{2n}\}$. Since $ML(G)$ is a modified line graph it consists of cliques $C_1, ..., C_{2n}$ where each clique C_i corresponds to a vertex v_i of G. Moreover the edges of $C_1, ..., C_{2n}$ partition the edges of $ML(G)$.

To see that, let f be an edge in $ML(G)$ with ends e_1, e_2. Then e_1, e_2 are edges which share an end in G. Note that e_1, e_2 share at most two ends in G. If e_1, e_2 share exactly one end v_i then f is in $E(C_i)$. If e_1, e_2 share two ends then one of them is in X and one of them is in Y. If $f = (e_1e_2)_X$ then f is in the clique corresponding to the end in X. If $f = (e_1e_2)_Y$ then f is in the clique corresponding to the end in Y. In any case, there is a unique clique in G which contains f. We will need the following definitions and notation to count the orientations of $ML(G)$.

Definition 4.14.

- Let $t > 0$ be an integer. An orientation D of K_t is a *transitive tour* if the out degrees of the vertices of K_t in D are exactly $0, 1, ..., t-1$ (see Figure 4.2).
- Let G be a graph and let $G_1, ..., G_m$ be cliques in G. We write $G = \bigoplus_{i=1}^{m} G_i$ if the edges of $G_1, ..., G_m$ partition the edges of G.
- Let $G = \bigoplus_{i=1}^{m} G_i$ be a graph where G_i is a complete graph for each i. A *clique transitive* orientation of G is an orientation where, for each i, the induced orientation of G_i is a transitive tour.
- Given a map $\tau : V(G) \longrightarrow \mathbb{N}$, and a vertex ordering of G, let $DE'_G(\tau)$ and $DO'_G(\tau)$ denote the even and odd clique transitive orientations of G obeying τ, respectively (see Definition 4.8 for odd and even orientations).

Remark: We use the term transitive tour as defined by Edwards and Kennedy in [18]. Moreover the definition of a transitive tour is equivalent to the definition of a transitive tournament.

We will use the following lemmas, proved by Häggkvist and Janssen in [25]. The first lemma is about the number of orientations of a complete graph.

Lemma 4.15 (See Lemma 2.2 in [25])**.** *Let K_n be the complete graph on n vertices. Then for any map $\tau : V(K_n) \longrightarrow \mathbb{N}$, $DE_{K_n}(\tau) \neq DO_{K_n}(\tau)$ only when $\tau(V(G)) = \{0, 1, ..., n-1\}$. In this case the only orientation obeying τ is a transitive tour hence $|DE_{K_n}(\tau) - DO_{K_n}(\tau)| = 1$.*

The second lemma allows us to only count clique transitive orientations.

Lemma 4.16 (See Prop 2.3 in [25])**.** *Let $G = \bigoplus_{i=1}^{n} G_i$ where each G_i is a complete graph, and fix an ordering of the vertices of G. Then for each map $\tau : V(G) \longrightarrow \mathbb{N}$ the number of even orientations obeying τ, which are not clique transitive, is equal to the number of odd such orientations.*

To prove their theorem Edwards and Kennedy showed that there is a map $\tau : E(G) \longrightarrow \mathbb{N}$ such that the number of clique transitive orientations of $ML(G)$ obeying τ is a power of 2. Moreover all clique transitive orientations of $ML(G)$ obeying τ are of the same parity. We will now give their construction of τ.

By Kőnig's Theorem, the edges of G can be partitioned into 4 matchings, as a result $ML(G)$ is 4-colourable. Fix a colouring $\phi : E(G) \longrightarrow \{\alpha, \beta, \delta, \gamma\}$. Let $\tau : V(ML(G)) \longrightarrow \mathbb{N}$ be defined as follows. For each vertex $e \in ML(G)$

$$\tau(e) = \begin{cases} 4 & \text{if } \phi(e) \in \{\alpha, \beta\} \\ 2 & \text{if } \phi(e) \in \{\delta, \gamma\}. \end{cases}$$

Lemma 4.17. *Let $E(G) = \{e_1, ..., e_{4n}\}$ be an ordering of the edges of G. Then the number of clique transitive orientations of $ML(G)$ obeying τ is a power of 2, moreover, all such orientations have the same parity.*

Proof. Let D be a clique transitive orientation of $ML(G)$ obeying τ. Recall that $ML(G)$ decomposes into $2n$ edge disjoint cliques isomorphic to K_4, $C_1, ..., C_{2n}$, where each clique C_i corresponds to the vertex v_i in G.

Moreover every vertex in $ML(G)$ is contained in exactly two of these cliques. Note that in each clique C_i, the vertices of C_i have out degrees $0, 1, 2, 3$ in the orientation induced by D on C_i. For each i, $1 \leq i \leq 2n$, letting e_k be the vertex of $ML(G)$ with out degree k in C_i, we define the *colour sequence* of C_i to be $\phi(C_i) = (\phi(e_3), \phi(e_2), \phi(e_1), \phi(e_0))$. Moreover, we define $\tau(C_i) = (\tau(e_3), \tau(e_2), \tau(e_1), \tau(e_0))$.

The following claim shows that there are 4 possible colour sequences for each i, $1 \leq i \leq 2n$.

Claim 4.18. *Fix i between 1 and $2n$. Let e be an edge in G incident with v_i. The out degree of e in C_i, with respect to D, is 1 or 3 if and only if $\phi(e) \in \{\alpha, \beta\}$, and e has out degree 0 or 2 in C_i, with respect to D, if and only if $\phi(e) \in \{\delta, \gamma\}$.*

Proof. Let (A, B) be a bipartition of the vertices of G such that A contains v_i. Let $\mathcal{A} = \{C_j : v_j \in A\}$, and $\mathcal{B} = \{C_j : v_j \in B\}$. Note that each clique C_j in $\mathcal{B}$ contains exactly one vertex e_{j3} that has out degree 3 in C_j. Moreover e_{j3} has out degree 1 in the clique containing it in $\mathcal{A}$. Since G is bipartite, two different cliques in $\mathcal{B}$ can not share a vertex. Thus for two distinct cliques C_j and C_l in $\mathcal{B}$, $e_{j3} \neq e_{l3}$. Furthermore e_{j3} and e_{l3} are not contained in the same clique in $\mathcal{A}$ since each clique contains exactly one vertex that has out degree 1 in that clique. Thus, since $|\mathcal{B}| = |\mathcal{A}|$, every vertex that has out degree 1 in some clique in $\mathcal{A}$ has out degree 3 in the clique containing it in $\mathcal{B}$. Similarly we see that every vertex that has out degree 2 in a clique in $\mathcal{A}$ has out degree 0 in the clique containing it in $\mathcal{B}$.

Now let e_0, e_1, e_2, e_3 be the vertices in C_i such that for each $k \in \{0, 1, 2, 3, \}$, e_k has out degree k in C_i. Since D obeys τ, $d^+_D(e_3) = \tau(e_3)$. Moreover $d^+_D(e_3) \geq 3$ since e_3 has out degree 3 in C_i. As a result $\tau(e_3) = 4$. Similarly since e_1 has out degree 3 in the clique containing it in $\mathcal{B}$, $\tau(e_1) = 4$

Note that e_0 is contained in exactly one clique other than C_i. Since e_0 has out degree 0 in C_i and out degree at most 3 in the other clique, $d_D^+(e_0) \leq 3$. As a result $\tau(e_0) = d_D^+(e_0) = 2$. Similarly since e_2 has out degree 0 in the clique containing it in $\mathcal{B}$, $\tau(e_2) = 2$.

As a result $\tau(C_i) = (4, 2, 4, 2)$. It follows that $\phi(e_3), \phi(e_1) \in \{\alpha, \beta\}$, and $\phi(e_2), \phi(e_0) \in \{\delta, \gamma\}$. Moreover since ϕ is a proper colouring of $ML(G)$, the converse holds. □

Claim 4.18 shows that for each i, C_i has four possible colour sequences $(\alpha, \delta, \beta, \gamma)$, $(\alpha, \gamma, \beta, \delta)$, $(\beta, \delta, \alpha, \gamma)$ and $(\beta, \gamma, \alpha, \delta)$. Moreover the colour sequence of C_i is uniquely determined by the first two entries.

Let $G'_{\alpha\beta}$ denote the subgraph of $ML(G)$ induced by the vertices of $ML(G)$ coloured with α or β, and define $G'_{\delta\gamma}$ similarly. Let $H^1_{\alpha\beta}, ..., H^r_{\alpha\beta}$ be the components of $G'_{\alpha\beta}$, and let $H^1_{\delta\gamma}, ..., H^s_{\delta\gamma}$ be the components of $G'_{\delta\gamma}$. By Lemma 4.15 assigning the out degrees, corresponding to a transitive tour, to each clique C_i in $ML(G)$ gives a unique clique transitive orientation of $ML(G)$. The following claim counts the ways to assign such out degrees to create a clique transitive orientation obeying τ.

Claim 4.19. *The number of clique transitive orientations of $ML(G)$ obeying τ is 2^{r+s}.*

Proof. Let $\overrightarrow{ML(G)}$ be an arbitrary orientation of $ML(G)$ obeying τ. Note that $G'_{\alpha\beta}$ contains exactly two vertices from each clique. It follows that every vertex in $G'_{\alpha\beta}$ has degree 2 in $G'_{\alpha\beta}$. Hence every component of $H_{\alpha\beta}$ is a cycle where each two adjacent vertices are in a common clique in $ML(G)$. Similarly every component of $H_{\delta\gamma}$ is a cycle where each two adjacent vertices are in a common clique in $ML(G)$.

For all i, $1 \leq i \leq r$, let $\mathcal{C}(H^i_{\alpha\beta}) = \{C_j | C_j \cap H^i_{\alpha\beta}\}$. For i, $1 \leq i \leq s$, define $\mathcal{C}(H^i_{\delta\gamma})$ similarly. Let i be such that $1 \leq i \leq r$. Let C_j be in $\mathcal{C}(H^i_{\alpha\beta})$ and let e be a vertex in $C_j \cap H^i_{\alpha\beta}$. Let C_l be the other clique containing e. By Claim 4.18, e has out degree 1 or 3 in C_j, and C_l. Moreover the sum of the out degree of e in C_j and C_l is $\tau(e) = 4$. Thus the out degree of e in C_j forces the out degree of e in C_l. Let e_j be the neighbour of e in $C_j \cap H^i_{\alpha\beta}$. In a clique transitive orientation C_j contains exactly one vertex with out degree 3 and exactly one vertex with out degree 1. Therefore the out degree of e in C_j also determines the out degree of e_j in C_j. It is easy to see that the out degree of e in C_j and C_l determines the out degree of every vertex in $H^i_{\alpha\beta}$ in each clique containing it. Since there are two options for the out degrees of e, 1 or 3, there are two options for the out degrees of every vertex in $H^i_{\alpha,\beta}$. Therefore there are 2^r possibilities for the out degrees of the vertices in $H_{\alpha\beta}$ in their cliques. By a similar argument there are 2^s possibilities for the out degrees of the vertices in $H_{\delta\gamma}$ in their cliques. Since each vertex in $ML(G)$ is in exactly one of $G'_{\alpha\beta}$ and $G'_{\delta\gamma}$ there at most 2^{r+s} orientations of $ML(G)$.

By Lemma 4.15, assigning the out degrees, corresponding to a transitive tour, in each clique gives a unique clique transitive orientation of $ML(G)$. Thus there are 2^{r+s} clique transitive orientations of $ML(G)$ that obey τ. □

Now we will show that any two such orientations of $ML(G)$ have the same parity.

Claim 4.20. *Any two clique transitive orientations of $ML(G)$ obeying τ have the same parity.*

Proof. Any two clique transitive orientations of $ML(G)$, obeying τ, have to follow the structure described in Claim 4.19. Thus for any two τ obeying clique orientations, one can be obtained from the other by a series of the following operations:

i. Switching the out degrees of each vertex in $H^i_{\alpha,\beta}$ for some i, in both of the cliques containing it (from 1 to 3 or from 3 to 1).

ii. Switching the out degrees of each vertex in $H^i_{\delta,\gamma}$ for some i, in both of the cliques containing it (from 2 to 0 or from 0 to 2).

Note that changing the out degree of a vertex in a clique from 3 to 1 or 2 to 0 changes the orientation of two edges. Therefore operation [i] preserves the parity of an orientation. Similarly operation [ii] preserves the parity of an orientation. As a result all clique transitive orientations of $ML(G)$, obeying τ have the same parity. □

Thus the number of clique transitive orientations, obeying τ, is a power of 2, and they all have the same parity. □

Now we can prove Theorem 4.13.

Proof. By Lemma 4.17, there exists a map $\tau : V(ML(G)) \longrightarrow \mathbb{N}$ such that $\mathrm{DE}'_{ML(G)}(\tau) \not\equiv \mathrm{DO}'_{ML(G)}G(\tau) \pmod 5$. Moreover, by Lemma 4.16, $\mathrm{DE}_{ML(G)}(\tau) \not\equiv \mathrm{DO}_{ML(G)}(\tau) \pmod 5$. Since $5 > 4 = \max_{e \in V(ML(G))} \tau(e)$, by Theorem 4.10, $ML(G)$ is 5-cyclic correspondence colourable. □

Chapter 5

List Colouring Planar Graphs Without Cycles of Length 4 to 8

Correspondence colouring was introduced by Dvořák and Postle, in 2015, to prove that every planar graph without cycles of length 4 to 8 is 3-choosable, answering a problem that had been open for 15 years. A preprint appeared on arXiv on in 2015 but their paper was published in 2018. This problem was motivated by Steinberg's conjecture which states that every planar graph without cycles of length 4 and 5 is 3-colourable [36]. Steinberg's conjecture was recently disproved by Cohen-Addad, Hebdige, Král, Li, Salgado [15]. Whereas in 2005 Borodin, Glebov, Raspaud, and Salavatipour showed that planar graphs without cycles of length 4-7 are 3-colourable [14]. Moreover an earlier proof of Borodin which shows that planar graphs without cycles of length 4-9 are 3-colourable also implies 3-choosability of such graphs [12]. In this chapter, we discuss the proof of Dvořák and Postle in depth and learn correspondence colouring techniques. For convenience we will provide an outline of the proof of each lemma or claim in Section 5.2 before giving a more detailed proof.

In [14] Borodin et al. proved the following result.

Theorem 5.1. *Every planar graph without cycles of length 4-7 is 3-colourable.*

The proof of Theorem 5.1 does not translate to list colouring, as it depends on vertex identification which is not always possible in a list colouring setting. To bypass this obstacle Dvořák and Postle showed an analogous result for a special type of correspondence assignment [17].

Theorem 5.2. *Every planar graph without cycles of length 4-8 is C-colourable for every correspondence assignment C which is consistent on closed walks of length 3.*

Correspondence colouring allows for vertex identification since we can rename colours at vertices so that each vertex has the same list of colours, as discussed in Section 2.1.2. Furthermore recall, by Lemma 2.9, list colouring is equivalent to correspondence colouring using consistent correspondence assignments. Thus Theorem 5.2 gives a stronger result implying every planar graph without cycles of length 4-8 is 3-choosable. We highlight this important point in the following remark, for future reference.

Remark 5.3. List colouring is equivalent to correspondence colouring for consistent correspondence assignments. Here we only require the correspondence assignment to be consistent on closed walks of length 3.

The proof of Theorem 5.2 is quite long and technical, so here we give a brief overview. The proof of Theorem 5.2 has a similar structure to the proof of Theorem 5.1. It is a discharging proof using faces and vertices. We assume Theorem 5.2 is false and consider a minimal counterexample, where the notion of minimal is carefully chosen and involves assumptions on the correspondence assignment as well as the graph itself (see Definition 5.8). We assign the vertices and faces initial charges, depending on their lengths and degrees respectively, which sum up to -12. Then we redistribute the charges, according to a set of rules (R1)-(R4). Finally, using the structure of the counterexample, we show that the final charges sum up to at least -11, thereby arriving at a contradiction.

We analyze the structure of the minimal counterexample in Lemmas 5.9, 5.10, 5.11, 5.13, and Corollary 5.15. The proof of Lemma 5.9 is straightforward and does not require any special properties of correspondence colouring. We use Lemma 5.9 and discharging rules to prove that all but a bounded number of vertices have a non-negative final charge in Claim 5.16. The final charge of a face f depends on the number of light vertices (see Definition 5.14) incident with f. Corollary 5.15 limits the number of light vertices incident with a face f. This is done by showing that too many consecutive vertices on the face of a graph give rise to a tetrad (see Definition 5.12), which is a forbidden subgraph. We show that G does not contain a tetrad in Lemma 5.13 by performing vertex identification to get a smaller counterexample. Thus we can conclude that almost every face of G has a non-negative final charge in Claim 5.17.

The minimal counterexample consists of a graph G and a correspondence assignment C which contains a maximal number of correspondences. Intuitively the more correspondences there are in C the harder it is to find a C-colouring of G. However Lemma 5.6

states that if C is full and consistent (see Definitions 5.4 and 2.7) on a subgraph of H of G then we may assume that C is straight (see Definition 5.5) on H. Correspondence colouring using a straight and full assignment is equivalent to colouring. Thus we use Lemma 5.6 to force C to be locally equivalent to colouring. This enables us to use the techniques in the proof of Theorem 5.1 by Borodin et al. in Lemmas 5.11 and 5.13.

In Lemmas 5.10 and 5.11 we add as many correspondences as possible. This is done by adding correspondences whenever possible and checking that they do not cause any inconsistencies on closed walks of length 3. Note that by Lemma 2.9 list colouring is equivalent to correspondence colouring with consistent correspondence assignments. However, if we require C to be consistent on every closed walk in G then the proof of Lemmas 5.10, and 5.11 would be considerably harder. Thus we see the benefit of using correspondence colouring.

5.1 Preliminaries

In this section, we will prove some results needed for Theorem 5.2 (proved in a stronger form in Theorem 5.7) and cite some results mentioned in earlier sections. We also point out specific places in the proof where the advantage of using correspondence colouring can be seen (see Remark 5.3). Recall Lemma 2.9 states the relationship between list colouring and correspondence colouring.

Lemma 2.9. *A graph G is k-choosable if and only if G is C-colourable for every consistent k-correspondence assignment C.*

Lemma 2.9 is essential to understanding the use of correspondence colouring, as opposed to list colouring for Theorem 5.7. Theorem 5.7 applies to correspondence assignments that are consistent on walks of length 3, while list colouring is equivalent to correspondence colouring with consistent (see Definition 2.7) correspondence assignments. It is much easier to check that a correspondence assignment satisfies the former condition. For example, in Lemma 5.10 we add edges to a correspondence assignment and check if it is still consistent on walks of length 3. Similarly in Lemma 5.11 we modify the correspondence assignment and check that the modified version is consistent on triangles. In Lemma 5.13 we identify two vertices to obtain a new graph G', and correspondence assignment C'. We also show that the identification does not create any cycles of length at most 8. Since no new triangles are created we know that C' is still consistent on all walks of length 3.

For the next result we will need the following definitions. Let G be a graph and let (L, C) be a correspondence assignment of G as defined in Definition 2.2.

Definition 5.4.

- Let e be an edge in G. We say that e is *full* in C if C_e is a perfect matching.
- Let H be a subgraph of G. We say that H is *full* in C if C_e is a perfect matching for every edge e in H.

Definition 5.5.

- Let e be an edge in G. We say that G is *straight* on $e = uv$ if, for any $c_1 \in L(u)$ and $c_2 \in L(v)$, if $c_1 \neq c_2$ then (u, c_1) is not adjacent to (v, c_2) in C_e.
- Let H be a subgraph of G. We say that H is *straight* in C if e is straight in C for every edge e in H.

The following lemma allows us to force a correspondence assignment to be locally equivalent to ordinary colouring.

Lemma 5.6. *Let G be a graph with a k-correspondence assignment C. Let H be a subgraph of G such that for every cycle K in H, the assignment C is consistent on K and all edges of K are full. Then there exists a k-correspondence assignment C' for G equivalent to C such that all edges of H are straight in C', and C' can be obtained from C by renaming on vertices of H.*

Lemma 5.6 is used in the proof of Lemma 5.11 and Lemma 5.13. In both cases, we have a minimal counterexample consisting of a graph G and a correspondence assignment C. We modify a subgraph H of G or we modify C on H and show that the modified graph and correspondence assignment is not a counterexample. Then we show that, as a result, G and C are not a counterexample. Lemma 5.6 is used to straighten the edges of H in C. This makes colouring H much simpler. Note that Lemma 5.6 applies whenever H is a forest. Indeed we use this for the proof of Lemma 5.11.

Proof. Without loss of generality, we can assume that H is connected because otherwise, we can rename the components of H separately. Let T be a spanning tree of H rooted at a vertex v. We will perform renaming on the vertices of T in depth first search order, without renaming at v. Let u be a vertex in T and suppose we have already renamed the colours at w, the parent of u in T. Rename the colours at u so that the edge uw is straight.

Let C' be the correspondence assignment resulting from the renaming process. By construction, all the edges of T are straight in C'. Let $e \in E(H) \backslash E(T)$, and let K be the unique cycle in $T + e$. Note that the edges of K, except for e are straight in C' since they are all in T. Furthermore the edges of K are full by assumption. By Lemma 2.8, C' is consistent on K. As a result e is straight in C'. Therefore all the edges of H are straight in C'. □

5.2 Main Result

To facilitate the induction argument we will prove a stronger statement that generalizes Theorem 5.2. The modified statement allows for some precoloured vertices.

Theorem 5.7. *Let G be a plane graph without cycles of length 4 to 8. Let S be a set of vertices of G such that either $|S| \leq 1$ or S is the set of vertices incident with a face of G. Let C be a 3-correspondence assignment for G such that C is consistent on every closed walk of length 3 in G. If $|S| \leq 12$, then any C-colouring ϕ_0 of $G[S]$ extends to a C-colouring of G.*

Suppose for a contradiction that Theorem 5.7 is false. We give a definition of a minimal counterexample below.

Definition 5.8. Let $B = (G, S, C, \phi_0)$ where

- G is a plane graph without cycles of lengths 4 to 8,
- $S \subseteq G$ consists of at most one vertex incident with the outer face of G or all the vertices incident with the outer face of G,
- C is a 3-correspondence assignment which is consistent on every closed walk of length 3,
- ϕ_0 is a C-colouring of $G[S]$.

Furthermore let $e(B) = |E(G)| - |E(G[S])|$ (i.e, the number of edges with at least one end not in S), and let $c(B) = \sum_{e \in G} |E(C_e)|$ (i.e, the total number of edges in the correspondence matchings).

- B is a *target* if $|S| \leq 12$.

- A target B is a *counterexample* if ϕ_0 does not extend to a C-colouring of G.
- B is a *minimal counterexample* if $|V(G)|$ is minimized, with respect to that $e(B)$ is minimized and with respect to that $-c(B)$ is minimized.

The following lemma establishes some basic properties of a minimal counterexample assuming one exists.

Lemma 5.9. *If $B = (G, S, C, \phi_0)$ is a minimal counterexample, then*

(a) $V(G) \neq S$*,*

(b) G *is 2-connected,*

(c) for any cycle K in G that does not bound the outer face, if K has length at most 12, then the open disk bounded by K does not contain any vertex,

(d) if e_1 and e_2 are chords of a cycle K in G of length at most 12, then there does not exist a triangle containing both e_1 and e_2,

(e) all vertices of G of degree at most 2 are contained in S,

(f) the outer face F of G is bounded by an induced cycle and $S = V(F)$, and

(g) if P is a path in G of length 2 or 3 with both ends in S and no internal vertices in S, then no edge of P is contained in a triangle the intersects S in at most one vertex.

The proof of Lemma 5.9 does not rely on any special properties of correspondence colouring.

Proof. (a) Since ϕ_0 does not extend to a C-colouring of G, $V(G) \neq S$.

(b) Suppose for a contradiction that G is not 2-connected. If G is not connected, then each component of G has a C-colouring extending the restriction of ϕ_0 to the component, by the minimality of B. As a result, G has a C-colouring which extends ϕ_0, which leads to a contradiction. Thus we may assume that G is connected.

Since G is not two connected $G = G_1 \cup G_2$ where G_1 and G_2 are proper induced subgraphs that intersect in exactly one vertex v. Suppose $v \in S$. Then each G_i has a C-colouring ϕ_i which extends the restriction of ϕ_o to $G_i[S]$. Moreover, ϕ_1 and ϕ_2 agree on v. Let ϕ be the colouring which is equal to ϕ_i on the vertices of G_i. Then ϕ is a C-colouring of G extending ϕ_0, which is a contradiction.

Suppose $v \notin S$. Since G is connected, the vertices of S are either entirely in G_1 or entirely in G_2. We can assume without loss of generality that $S \subseteq V(G_1)$. By the minimality of B, there exists a C-colouring ϕ_1 which extends ϕ_0 to G_1. Let C' be the restriction of C to G_2, let $S' = \{v\}$, and let ϕ'_0 be the C'-colouring of S' such that $\phi'_0(v) = \phi_1(v)$. Since G_2 has strictly less vertices than G, (G_2, S', C', ϕ_0) is a smaller target than B. Therefore G_2 has a C'-colouring ϕ_2 which extends ϕ'_0 to G_2. Since ϕ_1 and ϕ_2 agree on v, ϕ_1 and ϕ_2 give a C-colouring of G which extends ϕ_0. This leads to a contradiction. It follows that G is 2-connected.

(c) Let K be a cycle of length at most 12 which does not bound the outer face of G, and let Λ be the open disk bounded by K. Suppose for a contradiction that Λ contains a vertex. Let G_1 be the subgraph of G induced by the vertices drawn in the complement of Λ, and let G_2 consist of the edges and vertices drawn in the closure of Λ. Since Λ contains at least one vertex, G_1 has strictly less vertices than G. Thus, by the minimality of B, G_1 has a C-colouring ϕ_1 which extends ϕ_0. Let $S' = V(K)$, ϕ'_0 be ϕ_1 restricted to the vertices of K. Since $|K| \leq 12$, (G_2, S', C, ϕ'_0) is a target.

Since K does not bound the outer face, G_2 has strictly less vertices than G. As a result G_2 has a C-colouring, ϕ_2, which extends ϕ'_0. Since ϕ_1 and ϕ_2 agree on the vertices of K, ϕ_1 and ϕ_2 give a C-colouring of G which extends ϕ_0. This leads to a contradiction.

(d) Let $K = v_1v_2...v_t$ be a cycle in G of length at most 12, and suppose K has a chord e. By symmetry we may assume $e = v_1v_j$ for $3 \leq j \leq \frac{t}{2} + 1$. Since G does not have cycles of length 4 to 8, and $\frac{t}{2}+1 \leq 7$ it follows that $j = 3$. Therefore e is in the triangle $v_1v_2v_3$. Since G does not contain cycles of length 4, e is not incident with any other triangle. Therefore no triangle contains e and another chord of K.

(e) Suppose for a contradiction that G has a vertex $v \in V(G)\backslash S$ which has degree at most 2. Let $G' = G - v$. Since G' has strictly less vertices than G, there is a C-colouring ϕ of G' which extends ϕ_0. Since v has two neighbours, we can assign v a colour c such that for each neighbour u of v, $(v, c)(u, \phi(u))$ is not an edge in C_{uv}. Thus ϕ_0 can be extended to a C-colouring of G, leading to a contradiction.

(f) Suppose $|S| \leq 1$. If $S = \emptyset$, then we can include an arbitrary vertex which is incident with the outer face. Thus we may assume that $S = \{v\}$ for some vertex v which is incident with the outer face of G.

Suppose v is contained in a cycle of length at most 12. Let E be the shortest cycle containing v. Then E is induced and has length at most 12. If E does not bound the outer face, then by part c, the open disk bounded by E does not contain any vertices. Therefore we can redraw G so that E bounds the outer face. Thus we may assume that E bounds the outer face of G. Let $S' = V(E)$, and let ϕ_0' be a C-colouring of E which extends ϕ. We can extend ϕ to a C-colouring of E since cycles are 3-correspondence colourable. Let $B' = (G, S', C, \phi_0')$. Since $e(B') < e(B)$, B' is a smaller target than B. As a result, G has a C-colouring which extends ϕ_0'. Since ϕ_0' is an extension of ϕ_0, we arrive at a contradiction. It follows that all cycles containing v have length at least 13.

Let x, y be the neighbours of v in the outer face, and let $G' = G + xy$. Since v is not in any cycles of length at most 12, G' does not any cycles of length 4 to 8. Furthermore we can redraw G' such that xyv is the outer face. let $S' = \{v, x, y\}$, and let C' be defined as follows.

$$C'_e = \begin{cases} C_e & \text{if } e \neq xy \\ \text{The empty matching} & \text{if } e = xy \end{cases}$$

Let ϕ_0' be a C'-colouring of the triangle xyz extending ϕ_0, and let $B' = (G', S', C', \phi_0')$. Since $e(B') < e(B)$, B' is a smaller target than B. It follows that G' has a C'-colouring which extends ϕ_0'. Since ϕ_0' is an extension of ϕ_0 is C' contains the same correspondences as C, G has a C-colouring which extends ϕ_0. This leads to a contradiction. As a result S is the set of all vertices incident with the outer face of G.

Now suppose for a contradiction that the cycle F bounding the outer face of G is not induced. Then F has a chord e. Let F_1 and F_2 be the two cycles e forms with F. Since $V(F) = S$ and S has size at most 12 by assumption, F has length at most 12. It follows that F_1 and F_2 have length at most 12. Furthermore F_1 and F_2 do not bound the outer face of G. Thus by part (c) the open disks bounded by F_1 and F_2 do not contain any vertices. As a result $V(G) = V(F_1) \cup V(F_2) = V(F) = S$. This contradicts part (a). Therefore the outer face of G is bounded by an induced cycle.

(g) Let F be the cycle bounding the outer face of G. Suppose for a contradiction that there is a path P of length $p \in \{2, 3\}$ with both ends in S and no internal vertices in S, and there is an edge in P which is contained in a triangle T which intersects S in at most one vertex. Let K_1 and K_2 be the cycles of $F \cup P$ distinct from F. Now we will show that K_1 and K_2 both have length at least 9. If K_1 is a triangle, then this implies that P has length 2 and K_1 intersects S in two vertices. Since T intersects S in at most one vertex it is distinct from K_1. Furthermore, since T and K_1 share an edge they form a 4 cycle which leads to a contradiction. Therefore K_1 is not a triangle, and similarly K_2 is not a triangle.

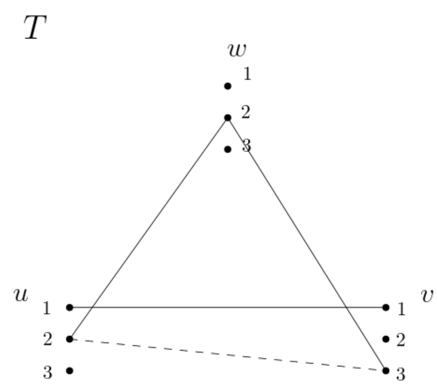

Figure 5.1: For example if $a = 2$, $c_w = 2$, and $b = 3$, then $C^{2,3}$ is consistent.

Since G has no cycles of length 4 to 8, K_1 and K_2 have length at least 9.

Note that $|K_1| + |K_2| = |F| + 2p$, and since F has length at most 12 and p is at most 3, $|K_1| + |K_2| \leq 18$. It follows that $|K_1| = |K_2| = 9$. Thus by part (c), the open discs bounded by K_1 and K_2 are empty. As a result $V(G) = S \cup V(P)$, hence T contains a chord e of K_1 or K_2. The edge e together with a subpath of K_1 or K_2 forms a cycle of length 6,7 or 8 which leads to a contradiction.

□

The following lemma simplifies the correspondence assignment by adding new correspondences where possible.

Lemma 5.10. *Let $B = (G, S, C, \phi_0)$ be a minimal counterexample. If $e \in E(G)$ does not join two vertices in S then $|E(C_e)| \geq 2$. Furthermore if e is not contained in a triangle then e is full in the assignment C (i.e the matching corresponding to e is perfect).*

Note that we can add an edge to C_e if e doesn't join two vertices in S because doing so does not affect ϕ_0. The main idea of the proof is we add an edge to C_e and check that the resulting correspondence assignment is still consistent on any triangle containing e. If the correspondence assignment is still consistent then we have a "smaller" counterexample.

Note that by Remark 5.3 we only have to check that the resulting assignment is consistent on triangles containing e as opposed to all cycles containing e. Furthermore, since G doesn't have cycles of length 4, at most one triangle contains e.

Proof Outline

If e is not included in a triangle and e is not full in the assignment, then we can add edges to C_e since doing so does not create any inconsistent walks of length 3. Therefore we can assume that $e = uv$ is included in a triangle $T = uvw$ and $|E(C_e)| \leq 1$. If $E(C_e)$ is empty then we can remove e since doing so will not affect the correspondence assignment. Since we have reduced the number of edges, where one end is not in S, we have found a smaller counterexample. If $E(C_e)$ has one edge, we try to find a path in the correspondence assignments of uw wv. We can then add an edge to turn that path into a cycle. This gives a consistent correspondence assignment with strictly more edges than C. Thus we have a smaller counterexample.

Proof. Suppose $e = uv$ is not contained in a triangle and e is not full, then there exist colours c_1 and c_2 such that (u, c_1) and (v, c_2) are isolated vertices of C_{uv}. Let C' be obtained from C by adding the edge $(u, c_1)(v, c_2)$ to C_{uv}. Then

$$\sum_{e \in E(G)} |E(C'_e)| > \sum_{e \in E(G)} |E(C_e)|$$

while other parameters, with respect to which B is minimized, remain the same. Furthermore adding $(u, c_1)(v, c_2)$ to C_e does not create any inconsistent walks of length 3 since e is not contained in any triangles. Hence C' is consistent. Furthermore any C'-colouring of G would give a C-colouring of G since C' contains all the correspondence relationships of C. As a result, (G, S, C', ϕ_0) is a counterexample contradicting the minimality of B.

Thus we may assume that e is an edge of a triangle, $T = uvw$, and $|E(C_{uv})| \leq 1$. Note that T is unique since G does not contain cycles of length 4. If $|E(C_{uv})| = 0$ we will show that removing e gives a smaller counterexample. $G - e$ has the same number of vertices as G, and $e(G - e) < e(G)$. Moreover C has not been affected since C_e is empty. As a result C is consistent on $G - e$ any C-colouring of $G - e$ is a C-colouring of G. Hence $(G - e, S, C, \phi_0)$ is a counterexample contradicting the minimality of B.

Now suppose $|E(C_{uv})| = 1$. By Lemma 5.6 we can assume that $(u, 1)(v, 1)$ is the only edge in $E(C_{uv})$. For $a, b \in \{2, 3\}$ let $C^{a,b}$ be the 3-correspondence assignment obtained from C by adding the edge $(u, a)(v, b)$. We will show that $C^{a,b}$ is inconsistent for any a and b. Let $B^{a,b} = (G, S, C^{a,b}, \phi_0)$. Note that we have not changed the number of vertices, or $e(B)$ to obtain $B^{a,b}$. Furthermore the total number of edges in $C^{a,b}$ is strictly greater than that of $C^{a,b}$. Thus $B^{a,b}$ is smaller than B. Moreover any $C^{a,b}$-colouring of G is a C-colouring of G. Since B is a minimal counterexample, $B^{a,b}$ can not be a counterexample. Therefore $C^{a,b}$ must be inconsistent on a closed walk of length 3. Since C and $C^{a,b}$ only differ on T, $C^{a,b}$ must be inconsistent on T.

Now we aim to find colours a, b such that $C^{a,b}$ is consistent, thus arriving at a contradiction. We will do so by showing that $(u, 2)$,$(u, 3)$ are not isolated in C_{uw} and $(v, 2), (V, 3)$ are not isolated in C_{vw}. In that case, by the pigeonhole principle, there exists colours $a, b \in \{2, 3\}$ and c_w such that $(u, a)(w, c_w)(v, b)$ form a path. Thus $C^{a,b}$ is consistent, see Figure 5.1.

Suppose for a contradiction that $(u, 2)$ is isolated in C_{uw}. By a previous observation $C^{2,b}$ is inconsistent on T. There are three closed walks on T, $vuwv$, $wvuw$, and $uvwu$. Suppose $C^{2,b}$ is inconsistent on $vuwv$. Since $(u, 2)$ is isolated in $C_{u,w}$, and $C^{2,b}$ has the same correspondence matching on uw, the inconsistent walk in $C^{2,b}$ does not use the edge $(u, 2)(v, b)$. However C and $C^{2,b}$ only differ on $(u, 2)(v, b)$, thus C is also inconsistent. This leads to a contradiction. Similarly $C^{2,b}$ is not inconsistent on $wvuw$. Therefore $C^{2,b}$ must be inconsistent on $uvwu$. As a result there is a path $(u, 2)(v, b)(w, d_b)(u, c_b)$ in $C^{2,b}$, where $c_b \neq 2$.

Let $P_b = (v, b)(w, d_b)(u, c_b)$, and note that P_2 and P_3 are both in C. Since C consists of matchings, and P_2 and P_3 have different starting points, they must have different end points. Hence $c_2 \neq c_3$. Since $c_2 \neq 2$ and $c_3 \neq 2$, by the pigeonhole principle we must have $c_2 = 1$ or $c_3 = 1$. We can assume, without loss of generality, that $c_2 = 1$. But then $(v, 1)(u, 1)(w, d_2)(v, 2)$ is a path in C, thus C is inconsistent on $vuwv$. This leads to a contradiction since C is consistent on all closed walks of length 3. Therefore $(u, 2)$ is not isolated in C_{uw}. Similarly $(u, 3)$ is not isolated in C_{uw} and $(v, 2)$,$(v, 3)$ are not isolated in C_{vw}. □

The following lemma strengthens Lemma 5.10 in case of triangles containing two vertices of degree 3.

Lemma 5.11. *let $B = (G, S, C, \phi_0)$ be a minimal counterexample, and let T be a triangle in G containing at least two vertices of degree three not belonging to S. Then all the edges of T are full in the assignment C.*

Notes on the Proof

The proof uses Lemma 5.10, and Lemma 5.6. Let $T = v_1v_2v_3$ be a triangle where v_1 and v_2 are not in S and have degree 3. Let x_1 and x_2 be the neighbours of v_1 and v_2 not in T respectively. Let H be the subgraph of G induced by $\{v_1, v_2, v_3, x_1, x_2\}$. Note that H does not have any cycles other than T because otherwise G would have a cycle of length

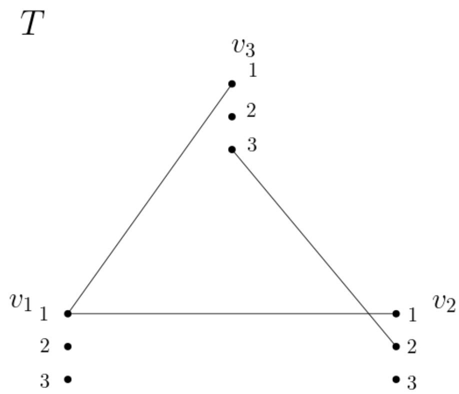

Figure 5.2: The edges of the correspondence assignment C that we have shown to exist.

4. Furthermore C is consistent on T. By Lemma 5.6 if the edges of T are full in C we can obtain a colouring equivalent to C where all the edges of T are straight.

We will assume the edges of T are not full in C for a contradiction. Note that $H - v_2v_3$ contains no cycles, hence the requirements of Lemma 5.6 are vacuously satisfied. Therefore we may assume that the edges of $H - v_2v_3$ are straight in C. Let D be a 3-correspondence assignment which is the same as C on $E(G)\backslash E(T)$ and the edges of T are straight and full in D.

Note that D is consistent on T. By Remark 5.3 we only have to check that D is consistent on closed walks of length 3. Furthermore none of the edges in T are contained in other triangles, because otherwise G would have a cycle of length 4. Thus it is easy to check that D satisfies the hypotheses of Theorem 5.7.

We will examine the differences between D and C. Note that T has two vertices not in S, and thus has no edges which join two vertices in S. Therefore ϕ_0 is still a D-colouring of $G[S]$ Since D has more edges overall, (G, S, D, ϕ_0) is a smaller target than B. As a result, there is a D-colouring ϕ' of G which extends ϕ_0. We will attempt to modify ϕ' on v_1 and v_2 to get a C-colouring of G which leads to a contradiction. Note that this is possible since v_1 and v_2 have small degrees. By Lemma 5.9 e, the minimum degree for a vertex not in S is 3.

Proof. Suppose for a contradiction that not all edges of T are full. Let $T = v_1v_2v_3$ where v_1 and v_2 are not in S and have degree three. Let x_1 and x_2 be the neighbours of v_1 and v_2 outside T respectively. By Lemma 5.6 we can assume that the edges x_1v_1, x_2v_2, v_1v_2 and v_1v_3 are straight in C. By Lemma 5.10 $C_{v_1v_2}$ and $C_{v_1v_3}$ each have at least two edges.

Therefore by the pigeonhole principle, there exists a colour c such that (v_1, c) is not isolated in $C_{v_1v_3}$ or $C_{v_1v_2}$. Without loss of generality, assume $c = 1$. Since the edges are straight, $(v_1, 1)(v_2, 1) \in E(C_{v_1v_2})$ and $(v_1, 1)(v_3, 1) \in E(C_{v_1v_3})$.

Let D be a 3-correspondence assignment on G which is the same as C on $E(G)\backslash E(T)$, and has the property that all edges of T are full and straight. (G, S, D, ϕ_0) is smaller than B, and thus can not be a counterexample. Hence there exists a D-colouring ϕ' which extends ϕ_0 to G. Since B is a counterexample ϕ' is not a C-colouring of G. Note that D and C only differ T. Moreover Since v_1v_2 and v_1v_3 are straight in both C and D and they are full in D, $C_{v_1v_2} \subseteq D_{v_1v_2}$ and $C_{v_1v_3} \subseteq D_{v_1v_3}$. Thus $C_{v_2v_3}$ must have a non-straight edge which is not in $D_{v_2v_3}$. It follows that $(v_2, \phi'(v_2))(v_3, \phi'(v_3))$ is an edge in $C_{v_2v_3}$ and not $D_{v_2v_3}$, and $\phi'(v_2) \neq \phi'(v_3)$.

We will show that $\phi'(v_2)$, $\phi'(v_3) \in \{2, 3\}$. Suppose that $\phi'(v_2) = 1$, then $\phi'(v_3) = c$ for some $c \neq 1$. Hence $(v_2, 1)(v_3, c)$ is an edge in C. As a result C is inconsistent on $v_3v_2v_1v_3$. Similarly if $\phi'(v_3) = 1$ then C is inconsistent on $v_2v_3v_1v_2$. Therefore $\phi'(v_2), \phi'(v_3) \in \{2, 3\}$. By symmetry we may assume $\phi'(v_2) = 2$ and $\phi'(v_3) = 3$. (See Figure 5.2)

Now we will show that $(v_3, 3)$ is not isolated in $C_{v_1v_3}$. Suppose for a contradiction that $(v_3, 3)$ is isolated in $C_{v_1v_3}$. We will recolour v_1 and v_2 to get a C-colouring of G which extends ϕ_0. Recall that the edges v_1x_1 and v_2x_2 are straight in C. Hence each colour in $L(v_1)$ and $L(v_2)$ correspond to the same colours in $L(x_1)$ and $L(x_2)$ respectively, if a correspondence exists. Since $\phi'(v_3) = 3$ and $(v_2, 2)(v_3, 3)$ is an edge in $C_{v_2v_3}$, we can recolour v_2 by a colour c in $\{1, 3\}$ different from $\phi'(x_2)$. Then since $(v_3, 3)$ is isolated in $C_{v_1v_3}$ we can recolour v_1 by a colour different from c and $\phi'(x_1)$.

As a result $(v_3, 3)$ is not isolated in $C_{v_1v_3}$. Since v_1v_3 is straight in C, we have the edge $(v_1, 3)(v_3, 3)$ in $C_{v_1v_3}$. By Lemma 5.10, we have either the edge $(v_1, 3)(v_2, 3)$ or $(v_1, 2)(v_2, 2)$ in $E(C_{v_1v_2})$. If $(v_1, 2)(v_2, 2)$ is an edge in $E(C_{v_1v_2})$, then C is inconsistent on $v_1v_2v_3v_1$. If $(v_1, 3)(v_2, 3)$ is an edge in $E(C_{v_1v_2})$, then C is inconsistent on $v_2v_1v_3v_2$. In either case C is not consistent on all closed walks of length 3 in G which leads to a contradiction. □

The next lemma deals with the main reducible configuration of this proof.

Definition 5.12. A *tetrad* in a plane graph is a path $v_1v_2v_3v_4$ of vertices of degree 3 contained in the boundary of a face, such that both v_1v_2 and v_3v_4 are edges of triangles. (See Figure 5.3)

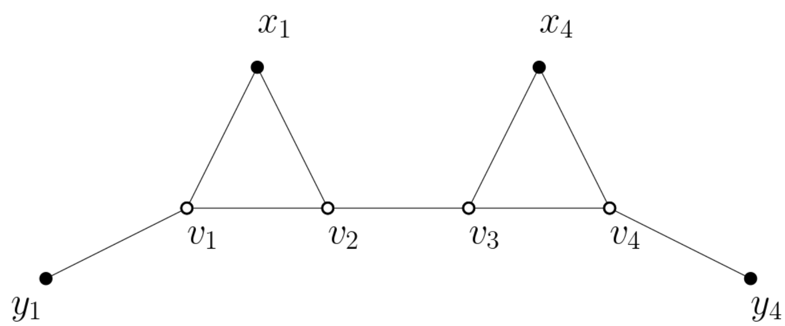

Figure 5.3: A tetrad

Lemma 5.13. *If $B = (G, S, C, \phi_0)$ is a minimal counterexample, then every tetrad in G contains a vertex in S.*

The proof of Lemma 5.13 uses Lemmas 5.9,5.10, and 5.11. Moreover the proof involves a vertex identification used to get a smaller counterexample. Vertex identification is not possible in a list colouring setting but it is possible in a correspondence colouring setting.

Proof Outline
We proceed by contradiction. Suppose that G has a tetrad $v_1v_2v_3v_4$ which does not have a vertex in S. Let v_1v_2 be in the triangle $v_1v_2x_1$, and let v_3v_4 be in the triangle $v_3v_4x_4$. For $i = 1, 4$ let y_i be the neighbour of v_i, distinct from x_i which is not in the tetrad. We will identify y_1 and x_4 in the graph $G - \{v_1, ..., v_4\}$ to obtain G'. Furthermore we will define C' to be the restriction of C to G'. We will use Lemma 5.9 to show that this identification does not create any edges between the vertices of S. Lemma 5.9 also shows that there is no path of length at most 8 between x_1 and y_4 in $G - \{v_1, ..., v_4\}$. Thus the identification will not create any cycles of length 4-8 in G'. Hence (G', S, C', ϕ_0) is a target, which is smaller than B. As a result there is a C'-colouring of G' which extends ϕ_0. This colouring gives a C-colouring ϕ of $G - \{v_1, ..., v_4\}$ where y_1 and x_4 have the same colour. We will then find colours for $v_1, ..., v_4$ to extend ϕ to a C-colouring of G, to arrive at a contradiction. Using Lemma 5.10 and 5.11 we can show that the edges incident with $\{v_1, ..., v_4\}$ are full. Consequently we can use Lemma 5.6 to straighten out the edges between $\{v_1, ..., v_4\}$. This will make colouring easier.

Proof. Suppose for a contradiction that $v_1v_2v_3v_4$ is a tetrad in G disjoint from S. Let v_1v_2 be in the triangle $v_1v_2x_1$, and let v_3v_4 be in the triangle $v_3v_4x_4$. For $i = 1, 4$ let y_i be the neighbour of v_i, distinct from x_i which is not in the tetrad. Note that all these vertices are pairwise distinct since G does not have cycles of length 4-8.

We want to identify x_4 and y_1, or x_1 and y_4, in $G - \{v_1, ..., v_4\}$. Moreover, we need to show that the identification would not create any new edges between the vertices of S. The identification of x_4 and y_1 would create an edge between the vertices of S if and only if $x_4 \in S$ and y_1 has a neighbour in S, or if $y_1 \in S$ and x_4 has a neighbour in S. Similarly the identification of x_1 and y_4 would create an edge between the vertices of S if and only if $x_1 \in S$ and y_4 has a neighbour in S, or if $y_4 \in S$ and x_1 has a neighbour in S. We will show that $x_4 \notin S$, and either $y_1 \notin S$ or x_4 has no neighbours in S, or we will show the analogous statement for y_4 and x_1.

Recall Lemma 5.9 g states that if P is a path in G of length 2 or 3 with both ends in S and no internal vertices in S, then no edge of P is contained in a triangle that intersects S in at most one vertex. Consider the path $x_1v_2v_3v_4x_4$, none of the internal vertices are in S. If x_1 and x_4 are both in S then by Lemma 5.9 g, the triangle $x_1v_1v_2$ must contain at least two vertices in S. Therefore one of v_1 and v_2 is in S. This leads to a contradiction. It follows that at most one of x_1 and x_4 are in S. Assume $y_1 \in S$, then by applying Lemma 5.9 g to the path $y_1v_1x_1$, we see that $x_1 \notin S$. Furthermore let z be an arbitrary neighbour of x_1 other than v_1. If z in in S then we can apply Lemma 5.9 g to $y_1v_1x_1z$ to get a contradiction. It follows that if y_1 is in S, then no neighbour of x_1 is in S. Similarly if $y_4 \in S$, then $x_4 \notin S$, and no neighbour of x_4 is in S. Since x_1 and x_4 can not be both in S we will assume, without loss of generality that $x_4 \notin S$. Now if $y_1 \notin S$ we are done. Suppose y_1 is in S. If $y_4 \in S$, then no neighbour of x_4 is in S and we are done. If $y_4 \notin S$, recall that y_1 is in S and thus x_1 is not in S. As a result $x_1 \notin S$ and $y_4 \notin S$ so we can use x_1 and y_4 instead of x_4 and y_1. By symmetry we can assume that $x_4 \notin S$ and either $y_1 \notin S$ or x_4 has no neighbours in S.

Now we want to show that the identification of x_4 and y_1 in $G - \{v_1, ..., v_4\}$ does not create any cycles of length at most 8. This happens if and only if $G - \{v_1, ..., v_4\}$ contains a path of length at most 8 between x_1 and y_4. Suppose for a contradiction that such a path P exists. Then P along with $y_1v_1v_2v_3x_4$ form a cycle E of length at most 12. The open disk bounded by E either contains v_4 or x_1, or x_1 is a in P. If the open disk bounded by E contains v_4 or x_1, then this contradicts Lemma 5.9 c. Thus we may assume that x_1 is in P. Note that v_1 and v_2 are not neighbours of x_1 in E since P is in $G - \{v_1, ..., v_4\}$. Therefore v_1x_1 and v_2x_1 are chords of E. It follows that $x_1v_1v_2$ contains two distinct chords of E. This contradicts Lemma 5.9 d. As a result $G - \{v_1, ..., v_4\}$ does not contain a path of length at most 8 between y_1 and x_4.

Note that v_2v_3 is not contained in a triangle, since v_2 and v_3 both have degree 3, and

G does not contain cycles of length 4. Moreover y_1v_1, and y_4v_4 are not contained in triangles since v_1 and v_4 both have degree 3 and G does not contain cycles of length 4. Thus by Lemma 5.10, the edges v_1y_1, v_2v_3 and y_4v_4 are full in C. Furthermore $x_1v_1v_2$ and $x_4v_3v_4$ are triangles which have two vertices not in S each. Hence, by Lemma 5.11, the edges of $x_1v_1v_2$ and $x_4v_3v_4$ are both full in the assignment C. Therefore all the edges incident with $\{v_1, ..., v_4\}$ are full in the assignment C. Moreover by Lemma 5.6, we can assume that the edges incident with $\{v_1, ..., v_4\}$ are all straight.

Let G' be the graph obtained from G by identifying y_1 and x_4 in $G - \{v_1, ..., v_4\}$, and let z be the vertex resulting from the identification of y_1 and x_4. Define C', a correspondence assignment of G', as follows. For $x, y \in V(G')$ such that $x \neq z$ and $y \neq z$ let

$$C'_{xy} = C_{xy}$$

and for $y \neq z \in V(G')$

$$C'_{zy} = \begin{cases} C_{x_4y} & \text{if y is adjacent to } x_4 \text{ in G} \\ C_{y_1y} & \text{if y is adjacent to } y_1 \text{ in G} \end{cases}$$

This is well defined since x_4 and y_1 have no common neighbours in G. Recall that the identification of y_1 and x_4 did not create any new edges between the vertices of S. Therefore ϕ_0 is a C' colouring of $G'[S]$. Furthermore, the identification does not create any cycle of length at most 8, therefore G' does not contain any cycles of length 4 to 8, and C' is consistent on all walks of length three in G'. Thus it remains to show that G' is planar and as a result (G', S, C', ϕ_0) is a smaller target than B.

Now will will show that G' is planar. By the definition of a tetrad, v_1, v_2, v_3, v_4 are on the boundary of a face f. By Lemma 5.9 c, the open disk bounded by $x_1v_1v_2$ does not contain any vertices. Therefore y_1 is also on the boundary of f. Also, the open disk bounded by $v_3v_4x_4$ does not contain any vertices. Place an edge, going through f and the open disk bounded by $v_3v_4x_4$, between y_1 and x_4. Then the edge only crosses v_3v_4. y_1x_4 does not cross any edges in $G - \{v_1, ..., v_4\} + x_4y_1$, therefore $G - \{v_1, ..., v_4\} + x_4y_1$ is planar. Since G' can be obtained from $G - \{v_1, ..., v_4\} + x_4y_1$ by contracting x_4y_1, G' is planar.

Therefore by the minimality of B, G' has a C'-colouring which extends ϕ_0. It follows that there is a C-colouring ϕ of $G - \{v_1, ..., v_4\}$ which extends ϕ_0, and $\phi(x_4) = \phi(x_1)$. We can colour $v_1, ..., v_4$ to get a C-colouring of G as follows. Choose a colour c_{v_4}, for v_4, distinct from $\phi(x_4)$ and $\phi(y_4)$. Then choose a colour c_{v_3}, for v_3, distinct from $\phi(x_4)$ and

c_{v_4}. Now we want to choose a colour c_{v_1} for v_1 such that $\phi(x_1)$, c_{v_3} and c_{v_1} are 2 colours in total. If $\phi(x_1)$ and C_{v_3} are the same then we can choose any colour for v_1 different from $\phi(x_1)$ and $\phi(y_1)$. Otherwise note that $\phi(x_4) = \phi(y_1)$ and we chose c_{v_3} to be different from $\phi(x_4)$. Moreover $\phi(x_1) \neq c_{v_3}$. Thus both neighbours of v_1 have colours different from c_{v_3}. In that case we let $c_{v_1} = c_{v_3}$. Therefore two of $\phi(x_1)$, c_{v_1} and c_{v_3} are the same colours. As a result we can choose a colour c_{v_2} which is different from $\phi(x_1)$, c_{v_1} and c_{v_3}.

Thus we have a C-colouring of G which extends ϕ_0. This leads to a contradiction. □

Definition 5.14. we say a vertex v is *light* if $deg(v) = 3$, v is incident with a triangle and $v \notin S$.

The following is a corollary of Lemma 5.13 and it limits the number of light vertices incident with a face.

Corollary 5.15. *If $B = (G, S, C, \phi_0)$ is a minimal counterexample, then no face of G is incident with five consecutive light vertices. Furthermore, if a face of G is incident with consecutive vertices $v_0, v_1, ..., v_5$ and the vertices $v_1, ..., v_4$ are light, then the edges v_0v_1, v_2v_3, and v_4v_5 are incident with triangles.*

Proof. Suppose for a contradiction that a face f is incident with five consecutive light vertices, $v_1, ..., v_5$. Since v_3 is a light vertex, it is incident with a triangle and has degree 3. Therefore the triangle incident with v_3 must contain the edge v_2v_3 or v_3v_4. By symmetry we can assume v_3v_4 is incident with a triangle. Since G does not contain any cycles of length 4, and v_3 has degree 3, then v_2v_3 is not incident with a triangle. Since v_2 is a light vertex, the triangle incident with v_2 must contain v_1v_2 or v_2v_3. Therefore v_1v_2 is contained in a triangle. Since $v_1, ..., v_4$ are light, they all have degree 3. As a result $v_1v_2v_3v_4$ form a tetrad. Since $v_1, ..., v_4$ are light vertices, none of them are in S. This contradicts Lemma 5.13. Note that if the triangle incident with v_3 contains the edge v_2v_3 instead of v_3v_4, then $v_2v_3v_4v_5$ is a tetrad which does not contain any vertices in S. In any case, we arrive at a contradiction.

Similarly if a face f is incident with consecutive vertices $v_0, v_1, ..., v_5$ and vertices $v_1, ..., v_4$ are light, then either the edges $\{v_1v_2, v_3v_4\}$ or $\{v_0v_1, v_2v_3, v_4v_5\}$ are contained in triangles. In the former case v_1, v_2, v_3, v_4 are light, thus they all have degree 3, and are not in S. It follows that $v_1v_2v_3v_4$ is a tetrad which does not contain a vertex in S. This contradicts Lemma 5.13. Therefore the edges $\{v_0v_1, v_2v_3, v_4v_5\}$ are incident with triangles.

□

With this we are ready to use a discharging argument to prove Theorem 5.7.

Proof of Theorem 1

Suppose for a contradiction that Theorem 5.7 is false and let $B = (G, S, C, \phi_0)$ be a minimal counterexample. We assign the initial charge of $ch_0(v) = 2deg(v) - 6$ for every vertex v and $ch_0(f) = |f| - 6$ for every face f. By Euler's formula the sum of the charges is -12. Then we redistribute the charges according to the following rules.

(R1) Each vertex incident with a non-outer face f of length three sends 1 to f.

(R2) Each face of length at least 9 sends 1/2 to each incident light vertex.

(R3) Let v be a vertex of degree at least 4 and let f be a face of length at least 9 incident with v. If deg(v) $\geq$ 5, then v sends 1/2 to f. If deg(v)=4 and v is incident with exactly one triangle vxy and either edge vx or vy is incident with f, then v also sends 1/2 to f.

(R4) If $v \in S$ has degree two, then the non-outer face incident with v sends 1/2 to v.

Since redistributing charges does not affect the total charge, the sum of charges is still -12. let the final charge be denoted by ch.

Claim 5.16. *For $v \in V(G)$*

i. If $v \notin S$ if v or has degree at least 4, then $ch(v) \geq 0$.

ii. If $v \in S$ and v has degree three then $ch(v) \geq -1$.

iii. If $v \in S$ and v has degree two then $ch(v) \geq -3/2$.

Remarks.

- By Lemma 5.9 b, G is 2-connected and hence has a minimum degree of two. Thus Claim 5.16 is exhaustive.
- part iii. is true if G is not a triangle. G is not a triangle because a triangle would not be a counterexample.

The proof of Claim 5.16 is straight forward does not require anything outside (R1)-(R4) and Lemma 5.9. We can get an upper bound on the charge sent out by a vertex. Based on that upper bound any vertex of degree 5 will have a non-negative final charge. Then we have to do some cases analysis for vertices of degree 3 and 4. For vertices of degree 2 the result follows straight from (R4).

Proof. By Lemma 5.9 (e) if a vertex is not in S then it has degree at least 3. Thus it suffices to prove i for vertices of degree at least 4, and vertices not in S of degree 3. Let v be a vertex in G, and let t be the number triangles containing v. Since G does not have cycles of length 4 to 8, every face incident with v has either length 3 or length at least 9. Therefore v is incident with deg(v)-t faces of length at least 9. By (R1) and (R3) v gives out at most $t+\frac{\deg(v)-t}{2}$ charges. Since G does not contain cycles of length 4 to 8, $t \leq \frac{\deg(v)}{2}$ As a result v gives out at most $\frac{3}{4}$deg(v) charges. Hence if deg(v)= 5, then

$$\mathrm{ch}(v) = \mathrm{ch}_0(v) - \frac{3}{4}\deg(v) = \frac{5}{4}\deg(v) - 6 > 0.$$

Suppose v has degree 4, then $t \leq 2$. If $t = 0$ then $\mathrm{ch}(v) = \mathrm{ch}_0(v) = 2$. If $t = 1$, then v is adjacent to one triangle. Hence by (R1), v gives out one charge. Moreover v is incident with 3 faces of length at least 9, and v is incident with exactly one triangle which shares an edge with 2 of those faces. Thus by (R3) v gives a $\frac{1}{2}$ charge to 2 faces. It follows that

$$\mathrm{ch}(v) = \mathrm{ch}_0(v) - 1 - 2\frac{1}{2} = 0.$$

If $t = 2$, then by (R1) v gives out a charge to each triangle it is incident with. As a result

$$\mathrm{ch}(v) = \mathrm{ch}_0(v) - 2 = 0.$$

In any case a vertex of degree 4, has a final non-negative charge.

Suppose v is not in S and v and has degree 3, then $t \leq 1$. If $t = 0$, then $\mathrm{ch}(v) = \mathrm{ch}_0(v) = 0$. If $t = 1$ then, by (R1), v gives a charge to the triangle it is incident with. However since v is incident with a triangle, has degree 3, $v \notin S$, v is a light vertex. As a result, by (R2), v receives half a charge from each face, of length at least 9, it it is incident with. Therefore

$$\mathrm{ch}(v) = \mathrm{ch}_0(v) - 1 + 2\frac{1}{2} = 0.$$

This concludes the proof of i.

Suppose v is in S and has degree 3, then $t \leq 1$. If $t = 0$, then $\text{ch}(v) = \text{ch}_0(v)$. If $t = 1$, then by (R1)

$$\text{ch}(v) = \text{ch}_0(v) - 1 = -1.$$

Thus if v is in S and has degree 3, then $\text{ch}(v) \geq -1$.

Now suppose v has degree 2, then Lemma 5.9 e, v is in S. Hence, by (R4), v receives half a charge from the non outer face incident with it. Furthermore note that v is not incident with a non-outer face of size 3 because otherwise either G is a triangle or the cycle bounding the outer face has a chord. We know that G is not a triangle as a triangle is not a counterexample. Furthermore by Lemma 5.9 f, the outer face is bounded by an induced cycle. It follows that

$$\text{ch}_0(v) = \text{ch}(v) - \frac{1}{2} = \frac{-3}{2}.$$

□

Claim 5.17. *Every face f of G distinct from the outer face F has non-negative final charge.*

By (R2) the number of charges a face sends out depends on the the number of light vertices it is incident with. Hence the proof of Claim 5.17 uses Corollary 5.15. The proof also uses Lemma 5.9.

Proof Outline

If $|f| = 3$, then the result follows from (R1). Otherwise $|f| \geq 9$. We define R to be the set of light vertices incident with f and s the number of vertices of degree 2 incident with f. Using (R2) and (R4) we find an upper bound for the number of charges sent out by f with respect to $|R|$ and s. Furthermore, using some structural properties of G, we can show that f is guaranteed to have non-negative final charge unless $|f| = 9$ and $|R| + s = 7$. Assuming $|f| = 9$ and $|R| + s = 7$, using our rough upper bound we can show, f has final charge at least -0.5. Thus we aim to find a vertex which satisfies (R3), and sends f half a charge. We break this down into two cases, $s = 0$ and $s > 0$.

Proof. Let f be a non-outer face in G. Since G has no cycles of length 4 to 8 or, either $|f| = 3$ or $|f| \geq 9$. If $|f| = 3$, then by (R1) every vertex incident with f gives f a charge. Therefore

$$\text{ch}(f) = \text{ch}_0(f) + 3 = 0.$$

Thus we may assume that $|f| \geq 9$. Let R be the set of light vertices incident with f, and let s be the number of vertices of degree 2 incident with f. We will find a rough upper bound on the charge given out by f. Note that $|R| + s \leq |f|$. By (R2) f sends half a charge to each incident light vertex, and by (R4) f sends half a charge to each incident vertex of degree 2. As a result f sends out at most $\frac{1}{2}(|R| + s) \leq \frac{|f|}{2}$. Therefore

$$\mathrm{ch}(f) \geq \mathrm{ch}_0(f) - \frac{|R| + s}{2} \geq \mathrm{ch}_0(f) - \frac{|f|}{2} = \frac{|f|}{2} - 6.$$

If $|f| \geq 12$, then $\mathrm{ch}(f) \geq 0$. Furthermore after doing some arithmetic $\mathrm{ch}(f)$ is only negative in one of the following cases

- $|f| = 11 = |R| + s$
- $|f| = 10$ and $|R| + s \geq 9$
- $|f| = 9$ and $|R| + s \geq 7$

We will show that we only have to worry about the last case. Suppose $s > 0$. By Lemma 5.9 (e) all vertices of degree 2, incident in f are in S. Since S is the set of all vertices incident with the outer face, a vertex of degree 2 in S will have both neighbours in S. Moreover since G is 2-connected the boundary of f must have two vertices, of degree at least 3, which are adjacent to vertices of degree 2. Since these two vertices are in S they can not be light vertices. Thus we have two vertices which are not in R and do not contribute to s. As a result $|R| + s \leq |f| - 2$. Furthermore if $s = 0$ then we must have $|R| + s \leq |f| - 2$ because otherwise the boundary of f would have five consecutive light vertices contradicting Corollary 5.15. Thus we may assume $|f| = 9$ and $|R| + s = 7$. Hence

$$ch(f) \geq \mathrm{ch}_0(f) - \frac{7}{2} \geq -0.5.$$

We aim to find a vertex which satisfies (R3), and consequently such a vertex sends f half a charge. As a result we will prove that f has a non final charge. We will consider two subcases, $s > 0$ and $s = 0$.

Case 1. Suppose $s = 0$, then $|R| = 7$.

Label the vertices of f $v_1, ..., v_9$ in order, such that v_1 and v_6 are not in R. Note that we can label the vertices as such because f can not have 5 consecutive light vertices by Corollary 5.15. Furthermore, by Corollary 5.15, v_1v_2, v_3v_4, and v_5v_6 are incident with triangles. Since v_7, v_8, v_9 are light vertices, they have degree 3, and are incident with triangles. Thus for each v_i in $\{v_7, v_8, v_9\}$, v_i in a triangle which contains one of its neighbours in f. Therefore either v_7v_8, and v_9v_1 are in triangles, or v_6v_7 and v_8v_9 are in triangles. By symmetry we can assume that v_7v_8 is incident with a triangle. Since v_7 has degree 3, v_6v_7 can not be in a triangle. Since v_5 and v_7 are light, they are not in S. If v_6 is in S then it must have at least 2 neighbours in S. As a result if v_6 is in S, it must have degree at least 4. Suppose v_6 is not in S. Since v_6 is not in S, and is incident with a triangle, v_6 can not have degree 3, because otherwise it would be a light vertex. Therefore if v_6 is not in S, v_6 has degree at least 4. In any case v_6 has degree at least 4.

We will now show that v_6 satisfies the requirements of (R3). Suppose v_6 has degree 4, then v_6 is incident with only one triangle since G does not contain cycles of length 4 and v_6v_7 is not in a triangle. As a result v_6 has degree 4, and is incident with exactly one triangle which contains an edge incident with f. Therefore by (R3) v_6 sends half a charge to f. If v_6 has degree at least 5, then by (R3), v_6 also sends half a charge to f. In any case the final charge of f is non-negative.

Case 2. Suppose that $s > 0$, then we have $|R| + s = 7$.

We will show that the vertices of R are consecutive on the boundary of f. Any vertex of degree 2 in S has both neighbours in S. It follows that f is incident with at least 2 vertices of degree at least 3 in S. However, since $|R| + s = 7 = |f| - 2$, f is incident with at most 2 vertices of degree at least 3 in S. Therefore f is incident with exactly 2 vertices of degree at least 3 in S. Moreover all vertices of degree 2 are consecutive in the boundary of f, and all the vertices in R are also consecutive in the boundary of f.

Now we will show that $|R| > 0$. Let the two vertices of degree at least 3 in S incident with f be called x, and y. Suppose for a contradiction that $s = 7$, then x and y must be adjacent. By Lemma 5.9 e, all vertices of degree 2 are contained in S. Furthermore x and y are also in S. Therefore $V(f) \subseteq V(F)$. If $V(f) \neq V(F)$, then xy is a chord of F, which contradicts Lemma 5.9 f. if $V(f) = V(F)$ then $V(G) = S$, which contradicts Lemma 5.9 a. Therefore we must have $|R| > 0$.

Let P be the path obtained from the boundary cycle of f by removing the vertices of degree two. Then x and y are the ends of P and all the internal vertices of P belong to R. Let K be the cycle in $F \cup P$ distinct from the boundaries of F and f. Note that $|K| + |f| = |F| + 2|R| + 2$, and thus $|K| = |F| + 2|R| - 7$.

Suppose for a contradiction that $|R| \leq 3$, then $|K| < |F| \leq 12$. It follows by Lemma 5.9 c, that the open disk bounded by K does not contain any vertices. Moreover the open disk bounded by f does not contain any vertices. Therefore $V(G) = V(F) \cup R$. Since all vertices of R have degree 3 they are each incident with a chord of K. Note that they can not be adjacent to any vertices outside K, since $V(G)\backslash V(K)$ all have degree 2. As a result K is not a triangle. If $|R| \leq 2$, then $|K| \leq |F| - 3 \leq 9$. Since K has a chord, it follows that G has a cycle of length 4 to 8.

Suppose $|R| = 3$, and let x be the middle vertex of P. x is light, hence it has degree 3 and is incident with a triangle, T. We will show that T contains two distinct chords of K, thereby contradicting Lemma 5.9 d. T must contain a chord of K since all vertices in $V(G)\backslash V(K)$ have degree 2. Let y be the other end of that chord. Since x has degree 3, the third vertex in T must be adjacent to x in K. Let z be the third vertex in T. Since z is light z, has degree 3. As a result y can not be adjacent to z in K because otherwise z would have degree 2. Therefore T contains two distinct chords of K.

Thus we may assume that $|R| \geq 4$, and by Corollary 5.15 $|R| = 4$. As in case 1 we will find a vertex that fulfills the requirements for (R3). Label the vertices of f by $v_1, ..., v_9$ in order so that $R = \{v_2, v_3, v_4, v_5\}$. By Corollary 5.15, v_1v_2 v_3v_4 and v_5v_6 are incident with triangles $v_1v_2x_1$ v_3v_4y and $v_5v_6x_6$. We will show that either v_1 or v_6 has degree at least 4. Suppose v_1 and v_6 have degree 3, then x_1 and x_6 are both in $V(F)$. Let K_1 be the cycle in $F \cup x_1v_2v_3v_4v_5x_6$ such the open disk bounded by K_1 does not contain f. Note that K_1 is obtained from the cycle bounding F by removing $v_1v_9v_8v_7v_6$ and adding $v_2v_3v_4v_5$. It follows that $|K_1| = |F| - 1 \leq 11$. Thus by Lemma 5.9 c, the open disk bounded by K_1 does not contain any vertices. Consequently, the edges v_3y and v_4y are chords of K_1, which contradicts Lemma 5.9 d.

Thus we may assume v_1 or v_6 has degree at least 4. Without loss of generality assume v_1 has degree at least 4. We will show that v_1 satisfies the conditions for (R3) with respect to f. If v_1 has degree 4 then it has a neighbour, z, distinct from v_2,x_1, and v_9. Note that $z \neq v_8$ since v_8 has degree 2 and it has two neighbours other than v_1. Since v_9 has degree 2 and v_9's other neighbour is not adjacent with v_1, v_9 is not contained in a triangle with v_1. Thus if there is a triangle containing v_1 other than $v_1x_1v_2$ it must include z and one of x_1 and v_2. This would lead to a cycle of length 4. As a result v_1 is incident with exactly 1 triangle and the triangle contains an edge incident with f. Therefore by (R3) v_1 sends half a charge to f. If v_1 has degree at least 5, then by (R3) v_1 also sends half a charge to f. It follows that f has a non-negative final charge.

□

Now the outer face F has charge at least $|F| - 6$. Hence by Claim 5.17 the final sum of charges on the faces of G is at least $|F| - 6$. Since $V(G) \neq S$, and G is connected F has at least one vertex of degree 3. Since G is 2 connected F has at least 2 vertices of degree 3. It follows that F has at most $|F| - 2$ vertices of degree 2. Therefore by Claim 5.16 final sum of charges on the vertices is at least $-3/2 \cdot (|F| - 2) - 1 \cdot 2$. Since F has length at most 12, the final sum of charges is at least

$$|F| - 6 - 3/2(|F| - 2) - 2 = -5 - |F|/2 \geq -11.$$

Thus we arrive at a contradiction.

Chapter 6

Cycle Plus Triangles Problem

In this chapter, we discuss Cycle Plus Triangles Graphs. The main contribution of this chapter is constructing a CPT that has DP-chromatic number 4 (see Section 6.2).

6.1 Introduction

A cycle plus triangles (CPT) graph is a graph, on $3n$ vertices, that admits a decomposition into a Hamilton cycle and n vertex disjoint triangles. We restate this formally, in Definition 6.1, for future reference. CPT graphs were introduced by Du, Hsu, and Hwang in 1986 to model certain computer networks and multiprocessor systems [16]. They also conjectured that a CPT graph, on $3n$, vertices has independence number n. In 1990, Erdős conjectured that every CPT graph is 3-colourable, strengthening the conjecture of Du, Hsu and Hwang. In 1992, Fleischner and Stiebitz showed that CPT graphs are 3-choosable using the Combinatorial Nullstellensatz [20]. In 1994, Sachs gave an elementary proof that every cycle plus triangles graph is 3-colourable [34]. However, there is no similar proof for the list colouring result. As discussed in Sections 2.2.2 and 4.3, the Combinatorial Nullstellensatz does not always generalize to a DP-colouring setting. Indeed, in Section 6.2, we show that there exists a CPT graph with DP-chromatic number 4. Moreover, in Section 6.3, we ask if there is a set of restrictions such that any CPT graph is C-colourable for any 3-correspondence assignment C that meets these restrictions.

Definition 6.1.

1. We say a graph $G = (V, E)$, on $3n$ vertices ($n > 0$), is a cycle plus triangles graph

if there exists a Hamilton cycle H such that $G - E_H$ is a set of pairwise disjoint triangles $T_H(G)$.

2. Let $\mathbb{G}$ denote the set of cycle plus triangle graphs.

3. Let $\mathbb{G}_n := \{G \in \mathbb{G} : |V(G)| = 3n\}$.

Remark: The decomposition of a CPT graph G into H and $T_H(G)$ is not unique.

6.2 Correspondence Colouring Cycle Plus Triangles Graphs

In this section, we show that the maximum DP-chormatic number of any CPT graph is 4. First we show that any CPT graph has DP-chromatic number at most 4 using a Brooks' type theorem for DP-colouring, due to Bernshteyn, Kostochka, and Pron (see Theorem 2.24).

Theorem 6.2. *Let G be a CPT graph on $3n$ vertices with $n > 1$, then $\chi_{DP}(G) \leq 4$.*

Proof. By definition G has a Hamilton cycle, therefore G is 2-connected. Note that G is not isomorphic to C_n^t or K_n^t for any n, t. Thus by Theorem 2.24 G is degree-DP-colourable. Therefore G is 4-DP-colourable. □

Theorem 6.3. *There exists a CPT graph G such that $\chi_{DP}(G) = 4$.*

To prove Theorem 6.3, we consider the line graph of K_4, $L(K_4)$. We will show $\chi_{DP}(L(K_4)) = 4$ using a theorem, of Bernshteyn and Kostochka, proving that any d-regular graph G has DP-chromatic index at least $d+1$ [8]. Furthermore we will give a concrete 3-correspondence assignment C, of $L(K_4)$, such that $L(K_4)$ is not C-colourable. Our motivation for doing this is to show that even if we restrict to correspondence assignments that are consistent on all triangles in $T_H(G)$, there still exist such assignments that do not admit a correspondence colouring (see Corollary 6.5).

Let $\{u, v, a, b, c, d, \}$ be the vertex set of $L(K_4)$ (see Figure 6.1). Note that $L(K_4)$ consists of consists of a Hamilton cycle $H = ubcvda$ and a set of vertex disjoint triangles $T_H(L(K_4)) = \{ucd, vab\}$ (see Figure 6.1). Thus $L(K_4)$ is a cycle plus triangles graph with $n = 2$. Since K_4 is 3-regular, $\chi_{DP}(G) \geq 4$ by the aforementioned result of Bernshteyn and Kostochka.

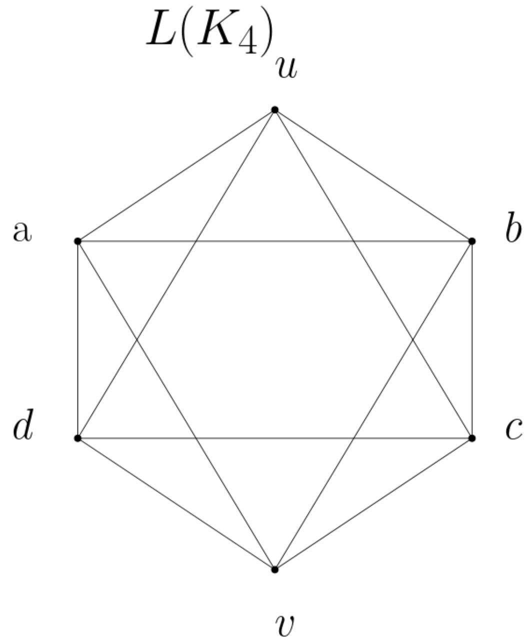

Figure 6.1: An illustration of $L(K_4)$

Let C be a 3-correspondence assignment of $L(K_4)$ defined as follows. The edges uc, ud, ab, av, bv, bc, cd, cv and dv are full and straight (see Figure 6.2, and Definitions 5.4, and 5.5). For the edges ad, ua and cb (see Figure 6.3)

$$
\begin{aligned}
C_{ad} &= \{\{(a,1)(d,2)\},\{(a,2)(d,1)\},\{(a,3),(d,3)\}\} \\
C_{ua} &= \{\{(u,2),(a,1)\},\{(u,1),(a,3)\}\} \\
C_{ub} &= \{\{(u,2),(b,1)\},\{(u,1),(b,3)\}\}
\end{aligned}
$$

We will show that there is no C-colouring of $L(K_4)$.

Suppose for a contradiction that there is a C-colouring ϕ of $L(K_4)$. Let K be a spanning subgraph of $L(K_4)$ whose edges are exactly the set of edges which are straight and full in C (see Figure 6.4). Then ϕ is a C-colouring of K. Since all the edges of K are straight and full in C, ϕ is a 3-colouring of K. We will use this to simplify our reasoning about ϕ.

The following claim is about the structure of ϕ.

Claim 6.4. *Let* $\phi(u) = x \in \{1, 2, 3\}$. *Then*

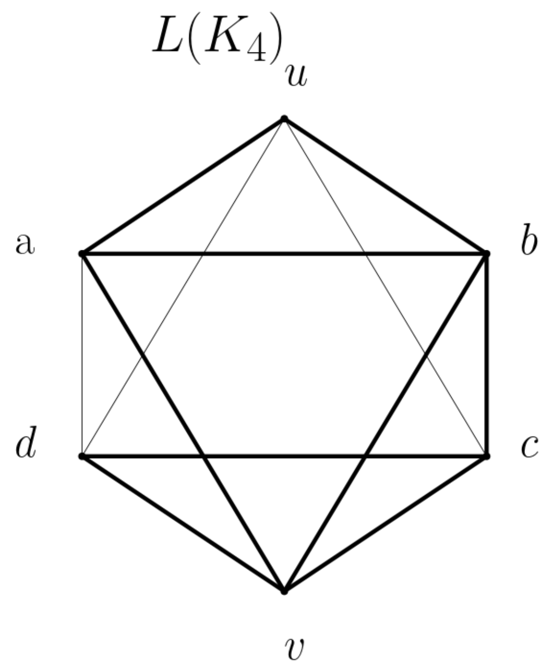

Figure 6.2: The bold edges are full and straight in C

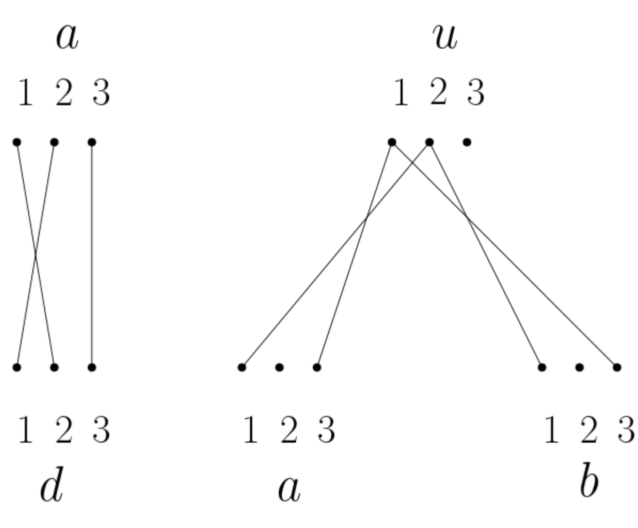

Figure 6.3: The correspondence assignment for the edges ad, ud, cd

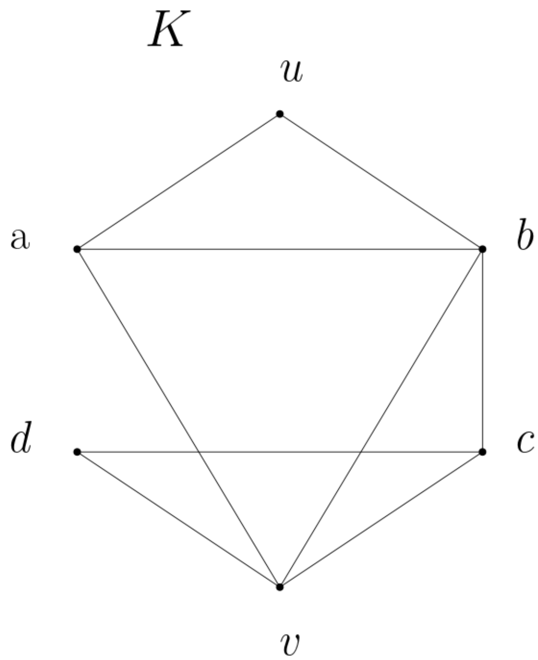

Figure 6.4: An illustration of K

i. $\phi(v) = x$

ii. the colours $\phi(a)$ $\phi(b)$ $\phi(c)$ $\phi(d)$ *alternate between the two colours in* $[3]\backslash\{x\}$.

Proof. Since ucd is a triangle in K, $\phi(c)$ and $\phi(d)$ are two distinct colours in $[3]\backslash\{x\}$. It follows that $\phi(v) = x$, since vcd is a triangle in K.

Since vab is a triangle in K, $\phi(a)$ and $\phi(b)$ are two distinct colours in $[3]\backslash\{x\}$. Moreover, since b is adjacent to c in K, $\phi(b) \neq \phi(c)$. As a result the sequence $\phi(a)$ $\phi(b)$ $\phi(c)$ $\phi(d)$ alternates between the two colours in $[3]\backslash\{x\}$. □

Now we can prove Theorem 6.3.

Proof. We will now analyze the different cases based on $\phi(u)$.

Case 1: Suppose $\phi(u) = 3$.

Then, by part ii of Claim 6.4, a, b, c, d are all coloured with the colours 1,2. However it is not possible to have a C-colouring of the cycle $abcd$ in $L(K_4)$ with the colours 1,2 (see

Example 2.15). This leads to a contradiction.

Case 2: Suppose $\phi(u) = 1$.

Case 2a: Suppose $\phi(a) = 2$.
Then, by Claim 6.4 part ii, $\phi(b) = 3$. Note that $(u, 1)$ is adjacent to $(b, 3)$ in C_{ub}. This leads to a contradiction.
Case 2b: Suppose $\phi(a) = 3$.
Note that $(u, 1)$ is adjacent to $(a, 3)$ in C_{ua}. This leads to a contradiction.

Case 3: Suppose $\phi(u) = 2$.

Case 3a: Suppose $\phi(a) = 1$.
Note that $(u, 2)$ is adjacent to $(a, 1)$ in C_{ua}. This leads to a contradiction.
Case 3b: Suppose $\phi(a) = 3$.
Then, by Claim 6.4 part ii, $\phi(b) = 1$. Note that $(u, 2)$ is adjacent to $(b, 1)$ in C_{ub}. This leads to a contradiction.

In any case, we get a contradiction. It follows that there is no C-colouring of $L(K_4)$.

By Theorem 6.2 $L(K_4)$ is 4-DP-colourable, therefore $\chi_{DP}(L(K_4)) = 4$. □

Note that in counterexample $(L(K_4), C)$, $L(K_4)$ consists of a Hamilton cycle H and a set of n disjoint triangles $T_H(L(K_4))$ and C is consistent on all the triangles in $T_H(L(K_4))$. Thus we get the following corollary of Theorem 6.3.

Corollary 6.5. *There exists a CPT graph G, with a Hamilton cycle H and a set of n vertex disjoint triangles $T_H(G)$, and a 3-correspondence assignment C such that C is consistent on every triangle in $T_H(G)$ and G is not C-colourable.*

6.3 Restrictions on Correspondence Assignments

Due to Fleischner and Steibitz, we know that CPT graphs are 3-choosable. By Lemma 2.9, we know that CPT graphs are 3-correspondence colourable for correspondence assignments that are consistent on all closed walks. It is therefore natural to ask if there is a weaker restriction we can place on 3-correspondence assignments which still allows 3-correspondence colouring of CPT graphs. Furthermore we ask if we can give an elementary proof of the 3-choosability of CPT graphs using correspondence colouring.

In 1994, Sachs gave an elementary proof that every cycle plus triangles graph is 3-colourable [34]. However, there is no similar proof for the list colouring result. In his proof, Sachs uses interesting techniques which rely on parity. Sachs proves that the number of distinct 3-colourings of a CPT graph G, denoted $\pi(G)$, is odd (see Theorem C.2), which implies that every CPT graph is 3-colourable (see Corollary C.3). The proof of Theorem C.2 proceeds by induction on two parameters. The first parameter is the length of the shortest path on the Hamilton cycle between two distinct vertices of one of the added triangles, denoted $S(G)$ (see Definition C.4). We call $S(G)$ the minimum span of G. The second parameter is the number of triangles added to the Hamilton cycle of a graph in $\mathbb{G}$ (see Definition 6.1). Motivated by the questions above, in Appendix C we make a preliminary study of the elementary proof of the 3-colourability of CPT graphs. Furthermore we attempt to translate some of its parity techniques to a correspondence colouring setting using ideas from the proof of Theorem 5.7 by Dvořák and Postle.

Chapter 7

Conclusion

In this thesis, we emphasised some of the important consequences of studying colouring problems within the framework of correspondence colouring. In particular, we saw the benefit of the correspondence colouring perspective in certain problems on list colouring and delay colouring. Dvořák and Postle proved that planar graphs without cycles of length 4 to 8 are 3-correspondence colourable, for correspondence assignments that are consistent on closed walks of length 3. Their result gives the only known proof that planar graphs without cycles of length 4 to 8 are 3-choosable thereby answering a long standing question of Borodin. One of the benefits of correspondence colouring is that it allowed them to do vertex identification, a technique that is generally not generally possible in list colouring.

There are several intriguing questions that remain open or are newly suggested by our work here. In Chapter 6, we saw that the theorem stating 3-choosability of CPT graphs does not generalise to correspondence colouring. However, we know that if sufficiently strong conditions are put on correspondence assignments, that they are consistent, then by Theorem 2.9, every CPT graph is 3-correspondence colourable. This raises a natural question, motivated by the result of Dvořák and Postle. Is there a weaker set of restrictions that would guarantee every CPT graph is 3-correspondence colourable? In particular, this would give new proof that CPT graphs are 3-choosable. Furthermore, is there a fruitful way to translate the parity arguments of Sachs to correspondence colouring?

We have done some preliminary exploration of these questions. By applying some techniques of Dvořák and Postle, we show that some components of Sachs' proof can generalize to correspondence colouring. These investigations are detailed in Appendix C.

Furthermore, the 17 year old delay colouring conjecture of Haxell, Wilfong, and Winkler remains open (see Conjecture 3.3). In this thesis we demonstrate that delay colouring is a natural special case of correspondence colouring. We were able to generalize a result of Georgakopoulos, stating that cubic bipartite graphs are 4-delay colourable, using a Brooks' type theorem for correspondence colouring, due to Bernshteyn and Kostochka. Moreover, we were able to generalize a result of Edwards and Kennedy, stating that quartic bipartite graphs are 5-delay colourable, using the Combinatorial Nullstellenstaz. These developments suggest that further progress on delay colouring could be possible when viewing the problem from the correspondence colouring perspective.

In Sections 2.2.2 and 6.2, we see that list colouring results obtained using the Combinatorial Nullstellensatz often do not generalize to correspondence colouring. However, the generalization of the result of Edwards and Kennedy shows that the Combinatorial Nullstellensatz can be applied in non trivial instances of correspondence colouring. This raises a natural question, when can the Combinatorial Nullstellensatz be applied in correspondence colouring? Some partial results on this question were obtained very recently by Kaul and Mudrock in [28].

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APPENDICES

Appendix A

K_2^3 is 4-Distortion Colourable

In this appendix we show that K_2^3, as defined in Section 3.4.3, is 4-distortion colourable. Recall we define K_2^3 to be the graph consisting of two vertices with 3 parallel edges between them.

Claim A.1. *K_2^3 is 4-distortion colourable.*

Proof. Let x, y be vertices of K_2^3. Let π be a distortion function on K_2^3 (see Definition 3.9). Let e_1, e_2, e_3 be the edges of K_2^3 and for $i \in \{1, 2, 3\}$ let π_i be the permutation assigned to e_i.

Suppose $\pi_i(j) = \pi_k(j)$ for some $i \neq k,\ i, j, k \in \{0, 1, 2, 3\}$. We may assume, without loss of generality, $\pi_1(0) = \pi_2(0) = 0$. In this case let $\phi(e_1) = 0$. If $\pi_3(0) = 0$ as well then let $\phi(e_2) = 1$ and let $\phi(e_3) = a$ with $a \in \{2, 3\}$ such that $\pi_3(a) \neq \pi_2(1)$. Note that $(\phi(e_i) : x \in e_i) = (0, 1, a)$ where $1 \neq a$, and $a \neq 0$. Furthermore $(\pi_i(\phi(e_i)) : y \in e_i) = (0, \pi_2(1), \pi_3(a))$. We know that $\pi_2(1) \neq \pi_3(a)$. Moreover $\pi_2(1) \neq 0$ and $\pi_3(a) \neq 0$ since $\pi_2(0) = \pi_3(0) = 0$.

Suppose $\pi_3(0) \neq 0$, then we may assume, without loss of generality that $\pi_3(0) = 1$. In this case let $\phi(e_2) = a \in \{1, 2, 3\}$, such that $\pi_2(a) = 1$, and let $\phi(e_3) = b \in \{1, 2, 3\}\backslash\{a\}$ such that $\pi_3(b) \neq 0$. Note that $(\phi(e_i) : x \in e_i) = (0, a, b)$ where $a \neq b$ and $a, b \in \{1, 2, 3\}$. Furthermore $(\pi_i(\phi(e_i)) : y \in e_i) = (0, 1, \pi_3(b))$. We know that $\pi_3(b) \neq 0$. Moreover $\pi_3(b) \neq 1$ since $\pi_3(0) = 1$ and $b \neq 0$.

Hence we may assume that $\pi_i(j) \neq \pi_k(j)$ for all $i \neq k$ such that $i, j, k \in \{0, 1, 2, 3\}$. Assume, without loss of generality, that $\pi_1(0) = 0$, $\pi_2(1) = 0$, and $\pi_3(1) = 1$. Let $\phi(e_1) = 0$, $\phi(e_3) = 1$, and $\phi(e_2) = a \in \{2, 3\}$ such that $\pi_2(a) \neq 1$. Note that $(\phi(e_i) : x \in e_i) = (0, a, 1)$

where $a \neq 0$ and $a \neq 1$. Moreover $(\pi_i(\phi(e_i)) : y \in e_i) = (0, \pi_2(a), 1)$. We know that $\pi_2(a) \neq 1$. Furthermore $\pi_2(a) \neq 0$ since $\pi_2(1) = 0$ and $a \neq 1$.

□

Appendix B

Häggvist and Janssen for Modified Line Graphs

Let G be a graph on vertices $v_1, ..., v_n$, and let G' be the line graph of G. Then G' has cliques $G_1, ..., G_n$, where G_i corresponds to v_i for each i, and the edges of $G_1, ..., G_n$ partition the edges of G'. In [25], Häggvist and Janssen denote this decomposition into cliques as $G' = G_1 \oplus ... \oplus G_n$. Then they prove the following lemma about the number of non clique transitive orientations about a graph that admits such a decomposition.

Lemma B.1 (See Prop 2.3 in [25])**.** *Let $G = G_1 \oplus ... \oplus G_n$ be a graph where each G_i is a complete graph and fix an ordering of the vertices of G. Then for each mach $\tau : V(G) \longrightarrow \mathbb{N}$ the number of even orientations obeying τ which are not clique transitive is equal to the number of such odd orientations.*

In this section we will show that Lemma B.1 only requires that $G_1, ..., G_n$ are cliques in G such that the edges of $G_1, ..., G_n$ partition the edges of G. Thus Lemma B.1 holds for modified line graphs.

To prove Lemma B.1 Häggvist and Janssen prove the following lemma, about the orientations of a simple complete graph.

Lemma B.2 (See Lemma 2.2 in [25])**.** *Let K_n be the complete graph on n vertices. Then for any map $\tau : V(K_n) \longrightarrow \mathbb{N}$ $DE_{K_n}(\tau) \neq DO_{K_n}(\tau)$ only when $\tau(V(G)) = \{0, 1, ..., n-1\}$. In this case the only orientation obeying τ is a transitive tournament hence $|DE_{K_n}(\tau) - DO_{K_n}(\tau)| = 1$.*

The proof of Lemma B.1 is as follows

Proof. The graph polynomial of G can be written as follows

$$\begin{aligned} f_G &= \prod_{v_i \sim v_j} (x_i - x_j) \\ &= \prod_{\substack{v_i \sim v_j \\ \text{in } G_1}} (x_i - x_j) \ldots \prod_{\substack{v_i \sim v_j \\ \text{in } G_n}} (x_i - x_j) \\ &= f_{G_1} \cdot \ldots \cdot f_{G_n}. \end{aligned}$$

The second equality is true since the edges of $G_1, ..., G_n$ partition the edges of G. Since the coefficient of the term $x_1^{\tau(v_1)} ... x_n^{\tau(v_n)}$ is $DE_G(\tau) - DO_G(\tau)$ we have

$$DE_G(\tau) - DO(\tau) = \sum_{\tau^1, ..., \tau^n} (DE_{G_1}(\tau^1) - DO_{G_1}(\tau^1))...(DE_{G_n}(\tau^n) - DO_{G_n}(\tau^n))$$

Where the sum is taken over all maps $\tau^1, ..., \tau^n$, where $\tau^i : V(G_i) \longrightarrow \mathbb{N}$ for each i, such that $\sum_{i=1}^n \tau^i(v) = \tau(v)$ for each vertex v. By Lemma B.2 if τ^i does not correspond to a transitive tour then $DE_{G_i}(\tau^i) - DO_{G_i}(\tau^i) = 0$, otherwise $|DE_{G_i}(\tau^i) - DO_{G_i}(\tau^i)| = 1$. Thus if τ^i does not correspond to a transitive tour, for some i, then

$$(DE_{G_1}(\tau^1) - DO_{G_1}(\tau^1))...(DE_{G_n}(\tau^n) - DO_{G_n}(\tau^n)) = 0.$$

If $\tau^1, ..., \tau^n$ all correspond to transitive tours then, by Lemma B.2, $\tau^1, ..., \tau^n$ give a unique clique transitive orientation of G. Moreover

$$(DE_{G_1}(\tau^1) - DO_{G_1}(\tau^1))...(DE_{G_n}(\tau^n) - DO_{G_n}(\tau^n)) = \pm 1$$

where the term is equal to -1 if there is an odd number of is such that the orientation of G_i obeying τ^i is odd. Thus the term is equal to -1 iff $\tau^1, ..., \tau^n$ give an odd orientation of G. Therefore

$$DE_G(\tau) - DO_G(\tau) = DE'_G(\tau) - DO'_G(\tau)$$

. It follows that the number of number of even orientations obeying τ which are not clique transitive is equal to the number of such odd orientations. □

Thus we see that the only condition we need from the ambiguous notation $G = G_1 \oplus ... \oplus G_n$ is that the edges of $G_1, ..., G_n$ partition the edges of G. Thus Lemma B.1 holds for modified line graphs.

Appendix C

Applying Parity Arguments in Correspondence Colouring of CPT graphs

C.1 Preliminaries

Let G be a graph, then every 3-colouring of G induces a partition of $V(G)$ into 3 independent sets.

Definition C.1. Let $\pi(G)$ denote the number of distinct partitions induced by 3-colourings of G.

The following theorem is the main result of Sachs.

Theorem C.2. *$\pi(G)$ is odd for every $G \in \mathbb{G}$.*

As a corollary we get that every cycle plus triangles graph is 3-colourable.

Corollary C.3. *Every graph in $\mathbb{G}$ is 3-colourable.*

The proof of Theorem C.2 proceeds by induction on two parameters. The first parameter is the number of triangles added to the Hamilton cycle of a graph in $\mathbb{G}$. The second parameter is the length of the shortest path on the Hamilton cycle between two distinct vertices of one of the added triangles. We will introduce some new notation and definition for convenience and formality.

Definition C.4. Let $G = (V, E) \in \mathbb{G}$, and let T be a triangle in $T(G)$. Let x, y, z be the vertices of T and let e be the edge of T joining x and y.

- Let $p(e)$ denote the x, y-path of H, not containing z.
- We say the span of e, $\sigma(e)$, is the length of $p(e)$.
- Let $\sigma_1(T), \sigma_2(T), \sigma_3(T)$ denote the spans of the edges of T such that $\sigma_1(T) \leq \sigma_2(T) \leq \sigma_3(T)$.
- We say T is singular if $\sigma_1(T) \leq 2$.
- We define the minimum span of G to be

$$S(G) := \min\{\sigma_1(T) : T \in T(G)\}$$

.

C.2 Simple Reductions

In this section we will define 3 types of reduction possible on a cycle plus triangles graph of minimum span at most 2. The reductions introduced will be used in the inductive step of the proof of Theorem C.2.

Let G be in $\mathbb{G}_n$ (where $n > 1$) and suppose $T(G)$ contains a singular triangle T_0. We can remove T_0, using one of the following transformations, to obtain G' in $\mathbb{G}_{n-1}$. Moreover G' contains the remaining triangles in $T(G) \backslash T_0$. Let H be the Hamilton cycle of G and let x, y, z be the vertices of T_0.

Definition C.5 (A-reduction)**.** Suppose x, y are adjacent to z in H. Let u, v be the other neighbours of x, y in H respectively. Then we remove x, y, z and add a new edge e with ends u and v. (See Figure C.1)

Definition C.6 (B-reduction)**.** Suppose x is adjacent to y in H and neither x nor y are adjacent to z in H. Let u be a neighbour of x in H distinct from y and v a neighbour of y in H distinct from x. Let s, t be the neighbours of z in H. Let t be the neighbour of z in

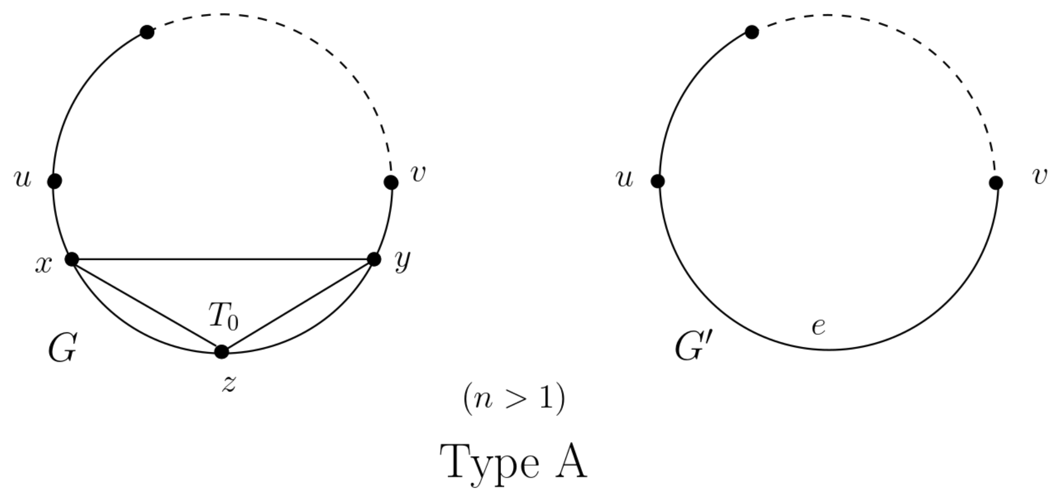

Figure C.1: Type A Reduction. Note that the solid lines signify an edge, the dashed lines signify a path of length at least 1 and the dashed lines signify a path that may not exist.

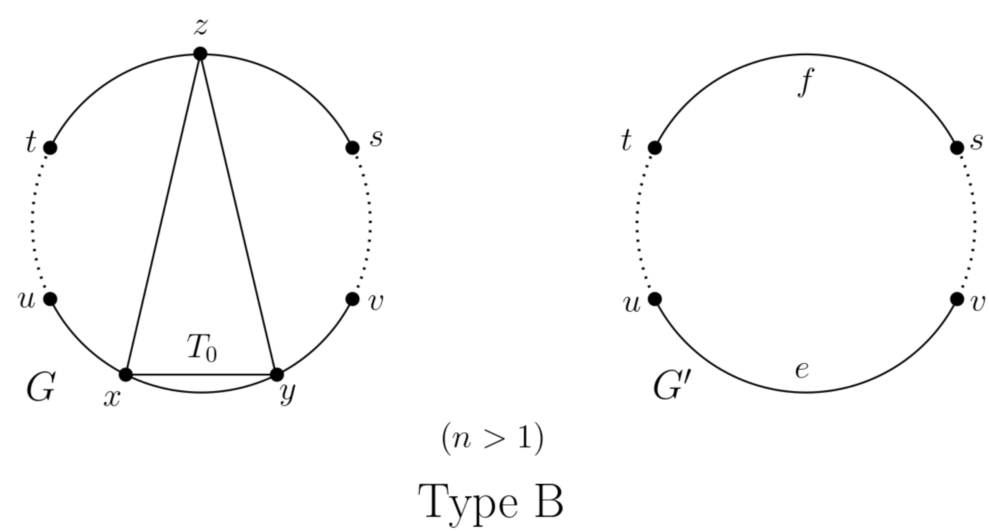

Figure C.2: Type B Reduction

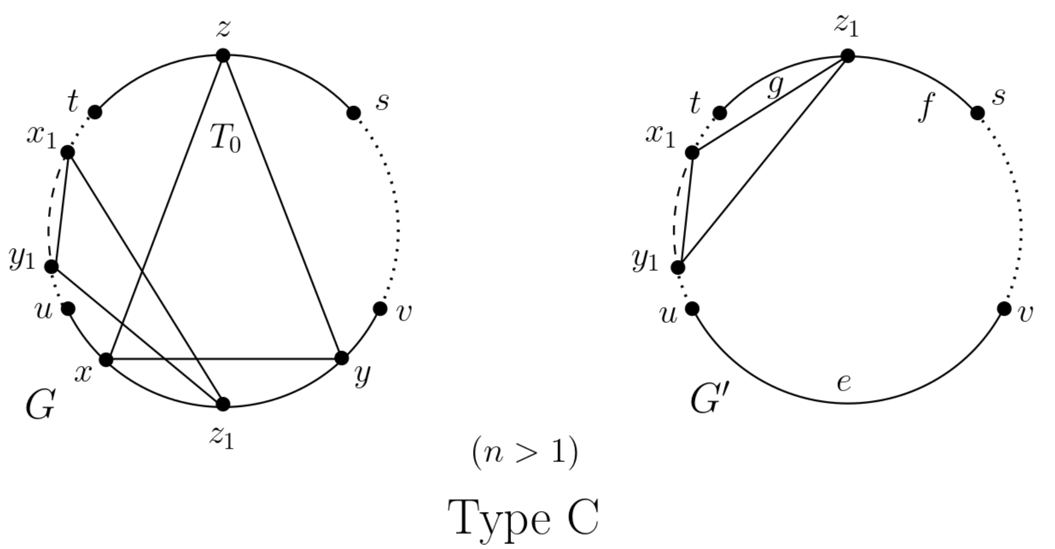

Figure C.3: Type C Reduction

$p(zx)$, and s the neighbour of z in $p(zy)$ (t, u are not necessarily distinct and s, v are not necessarily distinct). Then we remove z, x, y and add a new edge e with ends u and v, and a new edge f with ends t and s. (See Figure C.2)

Definition C.7 (C-reduction)**.** Suppose no two vertices in $\{x, y, z\}$ are adjacent on H. Since T_0 is singular we know that $\sigma(e) \leq 2$ for some edge e of T_0. Suppose, without loss of generality, that $\sigma(xy) \leq 2$. Then $\sigma(xy) = 2$. Let z_1 be the vertex lying between x and y on H. Let $T_1 = x_1y_1z_1$ be the triangle in $T(G)$ containing z_1. Let u, v be the neighbours of x, y distinct from z_1 in H respectively. Let t be the neighbour of z in $p(zx)$ and s the neighbour of z in $p(zy)$. Suppose without loss of generality that y_1 is closer to u on $p(xz)$ than x_1. Note that the pairs s, v, x_1, t and y_2, u are not necessarily distinct. We remove x, y, z and add an edge e with ends u, v, an edge f with ends s, z_2 and an edge g with ends t, z_2. (See Figure C.3)

Sachs proved the following lemma which allowed him to use induction in the proof of Theorem C.2.

Lemma C.8. *Suppose that G' is obtained from G using reduction A, B, or C. Then $\pi(G) \equiv \pi(G')$.*

For the purposes of this section we will prove the following version of Lemma C.8 focusing exclusively on the A-type reduction.

Lemma C.9. *Suppose that G' is obtained from G using an A-reduction. Then $\pi(G) \equiv \pi(G')$.*

Proof. Let e_0 be some edge of G connecting vertices v_1 and V_2 which remains unchanged under the reduction. Note that e_0 exists since either reduction removes only one triangle and $n > 1$. Let C and C' denote the set of colourings c of G and c' of G' such that $c(v_1) = 1, c(v_2) = 2$ and $c'(v_1) = 1, c'(v_2) = 2$ respectively. Thus $|C|$ and $|C'|$ are the number of distinct ways to partition $V(G)$ and $V(G')$, respectively, into independent sets. It follows that $\pi(G) = |C|$ and $\pi(G') = |C'|$.

Note that C is split into two disjoint sets

$$C_= := \{c | \ c \in C \text{ and } c(u) = c(v)\}$$

and

$$C_{\neq} := \{c | \ c \in C \text{ and } c(u) \neq c(v)\}.$$

Therefore $\pi(G) = |C| = |C_=| + |C_{\neq}|$. Note that there are no 3-colouring c' of G' where $c'(u) = c'(v)$ since u and v are adjacent in G'. We will show that

(i) $|C_=| \equiv 0$

(ii) $|C_{\neq}| \equiv |C'|$

As a result we will have $|C| \equiv |C'|$.

We will define some more notation. Given $V^* \subseteq V$ and $\hat{C} \subseteq C$ let $\hat{C}(V^*) := \{c_{|_{V^*}} | \ c \in \hat{C}\}$. Moreover let (i, j, k) be an arbitrary permutation of the colour triple $(1, 2, 3)$. Let $V' = V(G') = V(G) - \{x, y, z\}$.

Note that:
(i) Every partial colouring from $C_=(V')$ can be extended to a colouring of G from $C_=$ in exactly two ways. As a result $|C_=| = 2|C_=(V')| \equiv 0$. (See Figure C.4, table A(i))

(ii) Note that $C_{\neq}(V') \subseteq C'$. Moreover every colouring c' in C' can be extended to a colouring of $C_{\neq}$ in exactly 3 ways. (See Figure C.4, Table A(ii)). It follows that $C' = C_{\neq}(V')$ and $|C_{\neq}| = 3|C'| \equiv |C'|$. □

In the following section we attempt to prove an analogous version of Lemma C.9 thereby extending some of the parity arguments used by Sachs to the context of correspondence colouring.

Table A(i)

u	v	x	y	z
i	i	j	k	i
		k	j	i

Table A(ii)

u	v	x	y	z
i	j	j	i	k
		j	k	i
		k	i	j

Figure C.4: Proof of A reduction case

C.3 Correspondence Colouring

C.3.1 Approach

In the proof of Lemma C.10 we count the number of ways each 3-colouring of G' extends to a 3-colouring of G. There are a few challenges in doing this with DP-colouring. When performing an $A-$reduction we add edges, thus we have to think about the correspondence we assign to the added edge.

In the proof of Theorem 5.7 Dvořák and Postle choose a counter example which has a maximal number of edges in the correspondence assignment. Furthermore they use the Lemma 5.6 to make their correspondence assignment as close to ordinary colouring as possible.

Lemma 5.6. *Let G be a graph with a k-correspondence assignment C. Let H be a subgraph of G such that for every cycle K in H, the assignment C is consistent on K and all edges of K are full. Then there exists a k-correspondence assignment C' for G equivalent to C such that all edges of H are straight in C', and C' can be obtained from C by renaming on vertices of H.*

Note that when the edges of a correspondence assignment are full and straight that is just regular colouring.

We attempt to follow the same approach for CPT graphs. We require that a minimal counter example have a correspondence assignment with as many edges as possible. Moreover we will only consider correspondence assignments that are consistent on cycles of length at most 3. Note that the counter example described in Section 6.2 the correspondence assignment is not consistent on all cycles of length 3. Moreover requiring that the correspondence assignment be consistent on cycles of length 2 will eliminate some of the difficulties caused by parallel edges as follows. Suppose $u, v \in V(G)$ and there are parallel edges e, f with ends u, v. If C is inconsistent on the cycle consisting of e, f then a colour in $L(u)$ can correspond to two different colours in $L(v)$. If C is consistent on all parallel edges that is no longer a problem.

C.3.2 Preliminaries

Let $G \in \mathbb{G}$ and let C be a 3-correspondence assignment of G which is consistent on cycles of length at most 3, such that G is not C-colourable. Choose (G, C) such that $|V(G)|$ is minimal and subject to that $-\sum_{e \in E(G)} |C_e|$ is minimal.

We will add as many edges to the correspondence assignment as possible in Lemmas C.11,C.12, and C.13. The following lemma makes it easier to deal with parallel edges.

Lemma C.10. *Let $e, f \in E(G)$ be parallel edges with ends u, v. Then $(u, c_1)(v, c_2) \in C_e$ for colours $c_1, c_2 \in [3]$ if and only if $(u, c_1)(v, c_2) \in C_f$.*

Proof. Suppose for a contradiction that C_e has an edge $(u, c_1)(v, c_2)$ which is not in C_f. Let C' be obtained from C by adding $(u, c_1)(v, c_2)$ to C_f. Since C' contains strictly more edges than C, (G, C') is smaller than (G, C). By the minimality of (G, C), (G, C') is not a counter example.

Note that G is not C'-colourable since any C'-colouring of G is a C-colouring of G. As a result C' is inconsistent on a cycle W of length at most 3. Moreover, since C and C' only differ on the edge $(u, c_1)(v, c_2)$ in C'_f, W contains f and C' is inconsistent on W due to $(u, c_1)(v, c_2)$ in C'_f. Let W' be $W - f + e$. Since C and C' agree on $W - f$, C is inconsistent on W' due to $(u, c_1)(v, c_2)$ in C_e. □

In Lemma C.11, C.12, and C.13, we try to establish some properties of C. Specifically we want to prove that some edges and structures are full in C. In the proof of Lemma C.11, C.12 we will add edges to C to obtain a smaller correspondence assignment. It is easy to show, using Lemma C.10, that adding new edges does not create inconsistencies on cycles of length 2. Lemma C.10 allows us to treat parallel edges as one edge as opposed to a cycle of length 2.

Let $e \in E(G)$ and let u, v be the ends of e. Suppose there exist colours c_u, c_v in $L(u), L(v)$

respectively that are isolated in C_e. Let C' be obtained from C by adding $(u, c_u)(v, c_v)$ to C_e. Suppose there is $f \in E(G)$ parallel to e. Then by Lemma C.10, $C_e = C_f$, as a result $(u, c_u), (v, c_v)$ are isolated in C_f. Therefore C' is consistent on the cycle consisting of e, f. It follows that C' is consistent on all cycles of length 2 in G.

Lemma C.11. *Let $e \in E(G)$ be an edge not contained in any triangles in G, then e is full in C.*

Proof. Suppose for a contradiction that e is not full in C. Let u, v be the ends of e, then there exist colours $c_1 \in L(u)$ and $c_2 \in L(v)$ such that (u, c_1) and (v, c_2) are isolated in C_e. Let C' be obtained from C by adding $(u, c_1)(v, c_2)$ to C_e.

Since C' has strictly more edges than C, (G, C') is smaller than (G, C). Note that G is not C'-colourable since any C'-colouring of G is a C-colouring of G. Moreover C' is consistent on all triangles since C and C' only differ on e and e is not contained in any triangles. By Lemma C.10, C' is also consistent on all cycles of length 2 in G. As a result (G, C') is a strictly smaller counter example than (G, C). This leads to a contradiction. □

Lemma C.12. *Let $e \in E(G)$. If e is contained in at most one triangle in G, then $|C_e| \geq 2$.*

Proof. Let u, v be the ends of e. If e is not contained in any triangles then by Lemma C.11 e is full in C_e. Thus we may assume that e is contained in exactly one triangle $T = uvw$.

Suppose C_e is empty. If $(u, 1)$ is isolated in C_{uw} and $(v, 1)$ is isolated in C_{vw} then let C' be obtained by adding the edge $(u, 1)(v, 1)$ to C_e. Since $(u, 1)$ and $(v, 1)$ are isolated in C_{uw} and C_{vw} respectively, C' is still consistent on T. Thus C' is consistent on all triangles in G. Moreover by Lemma C.10, C' is consistent on all cycles of length 2. Note that G is not C'-colourable since any C'-colouring of G is a C-colouring of G. It follows that (G, C') is a smaller counter example than (G, C'). This leads to a contradiction. As a result $(u, 1)$ is not isolated in C_{uw} or $(v, 1)$ is not isolated in C_{vw}. Suppose without loss of generality that $(u, 1)$ is not isolated in C_{uw}. Then there exists $a \in [3]$ such that $(u, 1)(w, a)$ is in C_{uw}.

If (w, a) is isolated in C_{vw} then there exists a colour b such that (v, b) is isolated in C_{vw}. Let C' be obtained from C by adding $(u, 1)(v, b)$ to C_e. As in the previous case (G, C') is a smaller counter example than (G, C) which leads to a contradiction.

Therefore (w, a) is not isolated in C_{uw}. Therefore there exists a colour b such that $(w, a)(v, b)$ is in C_{vw}. Let C' be obtained from C by adding $(u, 1)(v, b)$ to C_e. Note that $(u,1)(v,b)(w,a)$ is a cycle in $C'_e \cup C'_{uw} \cup C'_{vw}$. As a result C' is still consistent on T (see Figure 5.1). It follows that C' is consistent on all triangles in G. By Lemma C.10, C' is also consistent on all cycles of length 2 in G. Moreover any C'-colouring of G is a C-colouring of G therefore G is not C'-colourable. As a result (G, C') is a strictly smaller counterexample than (G, C). This leads to a contradiction.

Thus we may assume $|E(C_e)| = 1$. By renaming the colours at u and v, we can assume that $(u,1)(v,1)$ is the only edge in $E(C_e)$. For $a, b \in \{2,3\}$ let $C^{a,b}$ be the 3-correspondence assignment obtained from C by adding the edge $(u,a)(v,b)$ to C_e. We will show that $C^{a,b}$ is inconsistent on T for any a and b. Since the total number of edges in $C^{a,b}$ is strictly greater than that of C, $(G, C^{a,b})$ is smaller than (G, C). Note that any $C^{a,b}$-colouring of G is a C-colouring of G, thus G is not $C^{a,b}$-colourable. Therefore $C^{a,b}$ must be inconsistent on a cycle of length at most 3. By Lemma C.10, C' is consistent on all cycles of length 2 in G. Therefore $C^{a,b}$ is inconsistent on a triangle. Since C and $C^{a,b}$ only differ on e and T is the only triangle containing e, $C^{a,b}$ must be inconsistent on T.

Now we aim to find colours a, b such that $C^{a,b}$ is consistent, thus arriving at a contradiction. We will do so by showing that $(u,2)$,$(u,3)$ are not isolated in C_{uw} and $(v,2),(v,3)$ are not isolated in C_{vw}. In that case, by the pigeonhole principle, there exists colours $a, b \in \{2,3\}$ and $c_w \in [3]$ such that $(u,a)(w,c_w)(v,b)$ form a path. Thus $C^{a,b}$ is consistent (see Figure 5.1).

Suppose for a contradiction that $(u,2)$ is isolated in C_{uw}. By a previous observation $C^{2,b}$ is inconsistent on T. There are three closed walks on T, $vuwv$, $wvuw$, and $uvwu$. Suppose $C^{2,b}$ is inconsistent on $vuwv$. Since $(u,2)$ is isolated in C_{uw}, and $C^{2,b}$ has the same correspondence matching on uw, the inconsistent walk in $C^{2,b}$ does not use the edge $(u,2)(v,b)$. However C and $C^{2,b}$ only differ on $(u,2)(v,b)$, thus C is also inconsistent. This leads to a contradiction. Similarly $C^{2,b}$ is not inconsistent on $wvuw$. Therefore $C^{2,b}$ must be inconsistent on $uvwu$. As a result there is a path $(u,2)(v,b)(w,d_b)(u,c_b)$ in $C^{2,b}$, where $c_b \neq 2$.

Let $P_b = (v,b)(w,d_b)(u,c_b)$, and note that P_2 and P_3 are both in C. Since C consists of matchings, and P_2 and P_3 have different starting points, they must have different end points. Hence $c_2 \neq c_3$. Since $c_2 \neq 2$ and $c_3 \neq 2$, by the pigeonhole principle we must have $c_2 = 1$ or $c_3 = 1$. We can assume, without loss of generality, that $c_2 = 1$. But then $(v,1)(u,1)(w,d_2)(v,2)$ is a path in C, thus C is inconsistent on $vuwv$. This leads to a contradiction since C is consistent on all closed walks of length 3. Therefore $(u,2)$ is not isolated in C_{uw}. Similarly $(u,3)$ is not isolated in C_{uw} and $(v,2)$,$(v,3)$ are not isolated in C_{vw}.

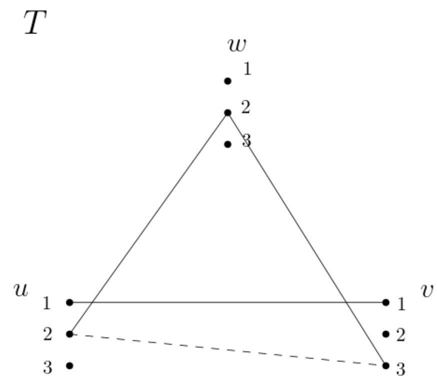

Figure C.5: For example if $a = 2$, $c_w = 2$, and $b = 3$, then $C^{2,3}$ is consistent.

As a result there exists an edge we can add to C to obtain a smaller counter example. Thus we arrive at a contradiction. Therefore $|E(C_e)| \geq 2$. □

Lemma C.13. *Let T be a triangle in $T(G)$ with no edges of span 2. If at least one edge in T has a parallel edge then then edges of T are full in C.*

Note: T contains vertices which are the endpoints of two parallel edges. When we refer to an edge between two vertices in T we are referring to the edge in T.

Proof. Suppose for a contradiction that not all edges of T are full. Let $T = v_1v_2v_3$ where v_1 and v_2 are the ends of parallel edges. Then v_1, v_2 both have at most one neighbour outside of T. Let x_1 and x_2 be the neighbours of v_1 and v_2 outside T respectively. If all the edges of T had parallel edges then G would be in $\mathbb{G}_1$, and hence would not be a counter example. Therefore T has an edge with no parallel edges. Assume without loss of generality that v_2v_3 has no parallel edges.

By Lemma 5.6 we can assume that the edges v_1x_1, v_2x_2, v_1v_2, v_1v_3, and edges parallel to them are straight in C. By Lemma C.12, $C_{v_1v_2}$ and $C_{v_1v_3}$ each have at least two edges. Therefore by the pigeonhole principle, there exists a colour c such that (v_1, c) is not isolated in $C_{v_1v_3}$ or $C_{v_1v_2}$. Without loss of generality, assume $c = 1$. Since the edges v_1v_2 and v_1v_3 are straight, $(v_1, 1)(v_2, 1) \in E(C_{v_1v_2})$ and $(v_1, 1)(v_3, 1) \in E(C_{v_1v_3})$.

Let D be a 3-correspondence assignment on G which is the same as C on $E(G)\backslash E(T)$, and has the property that all edges of T are full and straight. Note that D is consistent on T. Moreover, since none of the edges of T have span 2, none of the edges of T are contained in another triangle. As a result D is consistent on every triangle in G. Since v_2v_3 has no parallel edges and by Lemma C.10, D is consistent on all cycles of length 2. AS a result (G, D) is smaller than (G, C), and thus can not be a counterexample. Hence there exists a D-colouring ϕ' of G.

Since (G, C) is a counterexample ϕ' is not a C-colouring of G. Note that D and C only differ on T. Moreover Since v_1v_2 and v_1v_3 are straight in both C and D and they are full in D, $C_{v_1v_2} \subseteq D_{v_1v_2}$ and $C_{v_1v_3} \subseteq D_{v_1v_3}$. Thus $C_{v_2v_3}$ must have a non-straight edge which is not in $D_{v_2v_3}$. It follows that $(v_2, \phi'(v_2))(v_3, \phi'(v_3))$ is an edge in $C_{v_2v_3}$ and not $D_{v_2v_3}$, and $\phi'(v_2) \neq \phi'(v_3)$.

We will show that $\phi'(v_2)$, $\phi'(v_3) \in \{2, 3\}$. Suppose that $\phi'(v_2) = 1$, then $\phi'(v_3) = c$ for some $c \neq 1$. Hence $(v_2, 1)(v_3, c)$ is an edge in C. As a result C is inconsistent on $v_3v_2v_1v_3$. Similarly if $\phi'(v_3) = 1$ then C is inconsistent on $v_2v_3v_1v_2$. Therefore $\phi'(v_2), \phi'(v_3) \in \{2, 3\}$. By symmetry we may assume $\phi'(v_2) = 2$ and $\phi'(v_3) = 3$. (See Figure C.6)

Now we will show that $(v_3, 3)$ is not isolated in $C_{v_1v_3}$. Suppose for a contradiction that $(v_3, 3)$ is isolated in $C_{v_1v_3}$. We will recolour v_1 and v_2 to get a C-colouring of G. Recall that the edges v_1x_1 and v_2x_2 are straight in C. Hence each colour in $L(v_1)$ and $L(v_2)$ correspond to the same colours in $L(x_1)$ and $L(x_2)$ respectively, if a correspondence exists. Since $\phi'(v_3) = 3$ and $(v_2, 2)(v_3, 3)$ is an edge in $C_{v_2v_3}$, we can recolour v_2 by a colour c in $\{1, 3\}$ different from $\phi'(x_2)$. Then since $(v_3, 3)$ is isolated in $C_{v_1v_3}$ we can recolour v_1 by a colour different from c and $\phi'(x_1)$.

As a result $(v_3, 3)$ is not isolated in $C_{v_1v_3}$. Since v_1v_3 is straight in C, we have the edge $(v_1, 3)(v_3, 3)$ in $C_{v_1v_3}$. By Lemma C.12, we have either the edge $(v_1, 3)(v_2, 3)$ or $(v_1, 2)(v_2, 2)$ in $E(C_{v_1v_2})$. If $(v_1, 2)(v_2, 2)$ is an edge in $E(C_{v_1v_2})$, then C is inconsistent on $v_1v_2v_3v_1$. If $(v_1, 3)(v_2, 3)$ is an edge in $E(C_{v_1v_2})$, then C is inconsistent on $v_2v_1v_3v_2$. In either case C is not consistent on all closed walks of length 3 in G which leads to a contradiction. □

C.3.3 The Parity argument with DP-colouring

In this section we attempt to extend the parity argument to the A-reduction case using DP-colouring.

The A-reduction

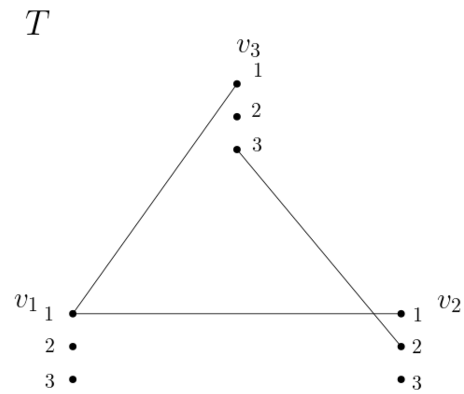

Figure C.6: The edges of the correspondence assignment C that we have shown to exist.

Let H be the Hamilton cycle of G and suppose G contains a triangle T_0 as described in the A-reduction (see Definition C.5), and let x, y, z be the vertices of T_0. T_0 contains a parallel edge. Moreover xz and yz both have span 1, thus xy does not have span 2 because otherwise H would have length 4 which is not possible. Therefore T_0 does not have edges with span two. By Lemma $C.13$ the edges of T_0 are full in C. Moreover the edges parallel to the edges of T_0 are also full in C by Lemma C.10.

We claim that the edges ux and vy are not contained by any triangles in G and hence are full in G. Since the triangles of $T(G)$ are disjoint and x is in T_0, ux is not contained by a triangle in $T(G)$. Thus ux can only be in a triangle if a triangle in $T(G)$ contains uy or uz. Since T_0 includes y, z and does not include u, and the triangles $T(G)$ are vertex disjoint, this can not happen. As a result ux is not contained in a triangle in G. Similarly yv is not contained in any triangle in G. Therefore, by Lemma C.13 yv and ux are both full in C. Let K be the subgraph of G consisting of the edges f T_0, the edges parallel to them, ux and vy. Then the edges of K are full in C. Moreover C is consistent on T_0 and the parallel edges between x, z and z, y. Then by Lemma 5.6 the edges of K are straight in C.

Let G' be obtained from G by performing the A-reduction, and let e denote the added edge between u, v. If there is no edge between u, v then we may choose any assignment for C'_e, specifically we can choose C'_e to be full and straight. Since the edges of K and e are full and straight we can use the exact same counting argument from the proof of Lemma C.9 to show that the parity of the number C-colourings of G is the same as the parity of C'-colourings of G'. However if there is an edge e' between u, v in G then we would have

$C_e = C_{e'}$

There is a way to handle the case of the A-reduction without a parity argument.

Claim C.14. *Let (G, C) be a minimal counterexample. Then G does not contain the triangle described in the A-reduction.*

Proof. Suppose for a contradiction that G contains the triangle $T = xyz$ such that x, y are adjacent to z in H. Let u, v be the other neighbours of x, y in H respectively. Let G' be formed from G by removing x, y, z and adding an edge e between u, v. Let C' a 3-correspondence assignment on G' such that $C' = C$ on $E(G')\backslash\{e\}$, and C_e is empty. Since $|V(G')| < |V(G)|$, (G', C') is strictly smaller than (G, C). Since $C' = C$ on $E(G')\backslash\{e\}$ C' is consistent on all triangles and 2 cycles in G' not containing e. Since C'_e is empty C' is also consistent on all triangles and 2 cycles in G' containing e. Therefore (G', C') is strictly smaller counter example than (G, C). As a result G' has a C'-colouring ϕ'.

We will extend ϕ' to a C-colouring of G as follows. Choose a colour c_x, for x, distinct from $\phi'(u)$. Choose a colour c_y, for y, different from $\phi'(v)$ and c_x. Then choose a colour c_z for, for z, different from c_x, c_y. Thus G is C colourable, which leads to a contradiction. □

Thus we see that we have had some some success in using the techniques of Dvořák and Postle to generalize some aspects of the Sachs argument. Furthermore if we are not trying to prove a stronger result using parity we can proceed using Claim C.14. It is not evident at the moment how to proceed with a parity argument but we do think it is an interesting question.