McCarthy, P. J., Nielsen, C., & Smith, S. L. (2014). Cardinality constrained robust optimization applied to a class of interval observers. 2014 American Control Conference, 5337–5342. https://doi.org/10.1109/ACC.2014.6859149

Cardinality Constrained Robust Optimization Applied to a Class of Interval Observers

Philip James McCarthy   Christopher Nielsen   Stephen L. Smith

***Abstract*— We propose a linear programming-based method of interval observer design for systems with uncertain but bounded model parameters and initial conditions. We assume that each uncertain parameter in the system model is bounded by conservative guaranteed bounds, and tighter conditional bounds. We define a class of systems by the cardinality of conservative bounds required to bound all uncertain parameters. Using robust optimization, we solve only a single linear program per class of systems to obtain gains for the interval observer. A conservative upper bound on the worst case steady-state performance of the interval observers over the specified class of systems is minimized.**

I. Introduction

In scenarios such as high-volume manufacturing of low-cost devices, there are many plants characterized by similar dynamics, whose parameters vary within known bounds. Each plant is characterized by the same dynamical model with uncertain parameters. We consider scenarios wherein the originally conservative bounds on specific parameters can be tightened for certain plants. For example, if higher-quality components are used to construct a batch of devices. We address the problem of state estimator design for such plants. We propose a method of optimal *interval observer (IO)* design using *robust optimization*. We define a class of systems using the cardinality of conservative bounds required to bound all uncertain parameters of the plant dynamics. Only a single optimization is performed per class of systems.

An IO comprises an *upper* and *lower observer*, whose trajectories bound those of the plant states from above and below, respectively. IOs are useful when the plant dynamics are highly uncertain, making classic observers (e.g., Luenberger, high-gain) unreliable. One of the earliest dynamical IOs was described in [1], wherein they were designed for a wastewater treatment management system. IOs have also been applied to population dynamics [1], algae cultures [2], and pharmacokinetics [3]. IOs are attractive in biotechnological applications due to the large parametric and measurement uncertainty inherent to biological systems. IOs can be applied to a large class of dynamical systems, e.g., any system with bounded state trajectories [1], and any stable linear system with additive disturbances, by using a time-varying change of coordinates [4]. Necessary and sufficient conditions for the existence of IOs, yielding systematic optimal design procedures, were identified for positive linear systems in terms of matrix inequalities in [5], and for a more general class of nonlinear systems with bounded uncertainties in terms of linear constraints in [6]. The linear constraints identified in [6] were used to develop the linear programs (LPs), which we extend in this paper. Specifically, we account for uncertainty in the coefficients of these LPs.

Given an instance of a convex optimization problem with uncertain coefficients, if even relatively small deviations from nominal coefficient values, called *perturbations*, are not accounted for, classic optimization methods may generate solutions that are far-from-optimal, or even highly infeasible when implemented [7]. Robust optimization generates deterministically feasible solutions under deterministic set-based models of uncertainty, at the expense of increased cost in the nominal problem. Associated with each model of uncertainty is a budget of uncertainty [7], which characterizes the class of problem instances for which the robust solution will be deterministically feasible, as well as the potential deviation from optimality of the solution to the nominal problem. For LPs, several models of uncertainty have been considered. The ellipsoidal [8] and norm [9] models of uncertainty consider various norms of perturbed vectors and matrices of coefficients. Notwithstanding a special case in [9], the robust problem formulations under these models of uncertainty are second order cone programs. The cardinality constrained model of uncertainty [10], used in this paper, considers the cardinality of coefficients that are perturbed from their nominal values. The robust formulation is implemented as an LP. Robust optimization has seen varied application, including antenna [11] and circuit [12] design, constrained stochastic linear-quadratic control [13], and wireless channel power control [14]. The reader is referred to [7] for a comprehensive review of the robust optimization literature.

We propose a method of IO design using the linear programming-based method of [6] and the robust optimization method of [10]. We define a class of systems in terms of the number of perturbed constraint coefficients in the robust formulation of the LP of [6]. The designer can tune the robustness of the solution to improve the bounds on the steady state error of the proposed IO.

The contributions of this paper are: 1) we provide a novel application of cardinality constrained robust optimization to the design of dynamical observers; 2) we develop a tunably robust interval observer design method; 3) we statistically characterize the performance of the proposed observer; 4) we illustrate our results in simulation.

This research is partially supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

The authors are with the Dept. of Electrical and Computer Engineering, University of Waterloo, Waterloo ON, N2L 3G1 Canada. {philip.mccarthy;cnielsen;stephen.smith}@uwaterloo.ca

A. Notation and Terminology

Given a matrix $M \in \mathbb{R}^{m\times n}$, the notation M_j denotes the jth column of M and $M^{-\top}$ is the transpose of its inverse. Given two vectors $v_1 \in \mathbb{R}^m$ and $v_2 \in \mathbb{R}^n$, define the column vector $\mathbf{col}(v_1, v_2) \coloneqq \begin{bmatrix} v_1^\top & v_2^\top \end{bmatrix}^\top \in \mathbb{R}^{m+n}$; the **col** function extends to an arbitrary number of arguments; applied to a matrix $M \in \mathbb{R}^{m\times n}$, $\mathbf{col}(M) \coloneqq \mathbf{col}(M_1, \dots, M_n) \in \mathbb{R}^{mn}$. The *Kronecker delta* δ_{ij} equals 0 if $i \neq j$, and 1 if $i = j$. A matrix $M \in \mathbb{R}^{n\times n}$ is said to be *Metzler* if all its off-diagonal elements are nonnegative. Given a matrix $M \in \mathbb{R}^{n\times n}$ and a locally Lipschitz function $\xi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, the dynamical system $\dot{x}(t) = Mx(t)+\xi(x, t)$ is said to be *positive* if $x(0) \geq 0$ implies $x(t) \geq 0$ for all $t \geq 0$. This is true if M is Metzler and $\xi(x, t) \geq 0$ for all $t \geq 0$. When applied to vectors or matrices, the relations $>, <, \geq, \leq$ are taken elementwise. Given matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times n}$, the relation $A \preceq B$ is defined as $A \leq B$, with $A_{ij} < B_{ij}$ for at least one pair (i, j). Given a scalar $c \in \mathbb{R}$ or vector $v \in \mathbb{R}^n$, the operator $|\cdot|$ is the elementwise absolute value. Given a set S, its cardinality is denoted by $|S|$.

II. Problem Statement and Proposed Approach

A. Problem Statement

We consider systems of the form

$$\dot{x} = Ax + \xi(x, t), \quad y = Cx, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $A \in \mathbb{R}^{n\times n}$ is uncertain but bounded, the nonlinear function $\xi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is uncertain but bounded, $y \in \mathbb{R}^p$ is the output, and $C \in \mathbb{R}^{p\times n}$. We assume existence and uniqueness of solutions for $x(t)$ for all $t \geq 0$. We impose the following standing assumptions on the model (1).

Assumption 1. Given a system of the form (1) there exists a known constant $\varkappa \in \mathbb{R}^n$ such that $\varkappa \geq |\sup_{t\geq 0} x(t)|$.

Assumption 2. Given a system of the form (1), there exist known constants $\underline{x}_0, \overline{x}_0 \in \mathbb{R}^n$, such that $\underline{x}_0 \leq x(0) \leq \overline{x}_0$.

Assumption 3. Given a system of the form (1) that satisfies Assumption 1, there exist known constants $\underline{\xi}, \overline{\xi} \in \mathbb{R}^n$, such that $\underline{\xi} \leq \xi(x, t) \leq \overline{\xi}$, for all $t \geq 0$ and all $-\varkappa \leq x \leq \varkappa$.

Assumption 4. Given a system of the form (1), there exist known matrices $\underline{A}$, $\underline{A}^\uparrow$, $\overline{A}^\downarrow$, $\overline{A} \in \mathbb{R}^{n\times n}$, such that $\underline{A} \leq \underline{A}^\uparrow \leq \overline{A}^\downarrow \leq \overline{A}$ and $\underline{A} \leq A \leq \overline{A}$.

The matrices $(\underline{A}, \overline{A})$, and $(\underline{A}^\uparrow, \overline{A}^\downarrow)$ are called the *outer bounds* and *inner bounds* of A, respectively. The tighter inner bounds are such that $A_{ij} \leq \overline{A}^\downarrow_{ij}$ and $A_{uv} \geq \underline{A}^\uparrow_{uv}$ holds for a subset of pairs (i, j), (u, v). The pairs (i, j), (u, v) for which these inequalities hold are unknown at design time, but the *cardinality* of pairs is known.

Definition II.1. System (1) belongs to uncertainty class $\mathcal{U}(\mathfrak{u}, \mathfrak{l})$ if there exist matrices (A^u, A^l) such that $A^l \leq A \leq A^u$, $\overline{A}^\downarrow \leq A^u \leq \overline{A}$, and $\underline{A} \leq A^l \leq \underline{A}^\uparrow$ with $\mathfrak{u}$ elements of A^u satisfying $A^u_{ij} > \overline{A}^\downarrow_{ij}$ and $\mathfrak{l}$ elements of A^l satisfying $A^l_{uv} < \underline{A}^\uparrow_{uv}$.

An *interval observer (IO)* comprises an *upper observer* $\hat{x}^u$ that bounds the true states x from above, and a *lower observer* $\hat{x}^l$ that bounds the true states from below, i.e., for all $t \geq 0$, $\hat{x}^l(t) \leq x(t) \leq \hat{x}^u(t)$. This is called the *interval property*, and for it to be satisfied, the *interval error* $e \coloneqq \hat{x}^u - \hat{x}^l$ must satisfy positivity. The observer pair must also be initialized such that $\hat{x}^l(0) \leq x(0) \leq \hat{x}^u(0)$. If the evolution of the interval error is governed by positive dynamics, then Assumption 2 means we can choose $\hat{x}^{u,l}(0) \coloneqq \big(\hat{x}^u(0), \hat{x}^l(0)\big)$ to satisfy the interval property.

Problem II.2. Given constants $\mathfrak{u}$ and $\mathfrak{l}$ for a system of form (1), design an IO whose maximum steady state interval error, over the uncertainty class $\mathcal{U}(\mathfrak{u}, \mathfrak{l})$, is minimized with respect to the ℓ_1-norm, i.e., $\|\bar{e}\|_1 \coloneqq \limsup_{t\to\infty} \|e(t)\|_1$.

B. Proposed Approach

An exact solution to Problem II.2 is challenging; even in the simplest case of $\mathfrak{u}, \mathfrak{l} = n^2$, approximations are solved [6, Theorem 4.2]. Therefore, we too solve an approximation of Problem II.2, drawing upon the theory of [6] and [10]. As explained in Section IV-B, we consider an uncertainty class generated in terms of how the uncertainty in A appears in the proposed IO design LP; this class contains conservative characterizations of uncertainty that are not physically realizable. Consequently, we minimize a conservative upper bound on $\|\bar{e}\|_1$, which is at least as great as the maximum $\|\bar{e}\|_1$ over the class $\mathcal{U}(\mathfrak{u}, \mathfrak{l})$.

We use the linear programming-based IO design method of [6] to address optimality, and apply the robust optimization method of [10] to optimize over the given uncertainty class. To ensure the interval property, at runtime we must be given a pair of matrices (A^u, A^l) such that $A^l \leq A \leq A^u$, $\overline{A}^\downarrow \leq A^u \leq \overline{A}$ and $\underline{A} \leq A^l \leq \underline{A}^\uparrow$. But since we use robust optimization, we need not reoptimize for the specific pair (A^u, A^l).

The bound on $\|\bar{e}\|_1$ of the proposed IO, although conservative, is shown to be strictly less than that (2). If (2) is constructed using the same (A^u, A^l) as the proposed IO, then the proposed IO's bound on $\|\bar{e}\|_1$ is at least as tight as that of (2).

III. Background

A. Linear Programming-Based Interval Observers

In [6], an IO is constructed for system (1) with dynamics

$$\begin{aligned} \dot{\hat{x}}^u &= \overline{A}\hat{x}^u + L(y - C\hat{x}^u) - (\overline{A} - \underline{A})\phi(\hat{x}^u) + \overline{\xi} \\ \dot{\hat{x}}^l &= \overline{A}\hat{x}^l + L(y - C\hat{x}^l) - (\overline{A} - \underline{A})\psi(\hat{x}^l) + \underline{\xi} \\ \hat{x}^{u,l}(0) &= (\overline{x}_0, \underline{x}_0), \end{aligned} \tag{2}$$

where $\psi(x) \coloneqq \frac{1}{2}\big(x + |x|\big)$ $\big[\phi(x) \coloneqq \frac{1}{2}\big(x - |x|\big)\big]$ retains the positive [negative] elements of x and maps the negative [positive] elements to 0.

System (2) is an IO for (1), if and only if the following LP is feasible [6, Theorems 3.1, 4.3].

Problem III.1.

$$\begin{aligned}\text{minimize: } & \left[2(\overline{A}-\underline{A})\varkappa+\overline{\xi}-\underline{\xi}\right]^{\top}\lambda\\ \text{subject to: } & \overline{A}^{\top}\lambda - C^{\top}Z\mathbf{1}_n = -\mathbf{1}_n\\ & \underline{A}^{\top}\mathbf{diag}(\lambda) - C^{\top}Z + \beta I \geq \mathbf{0}_{n\times n}\\ & \lambda > \mathbf{0}_n\end{aligned}$$

The gain matrix L of (2) is defined as

$$L \coloneqq \mathbf{diag}(\lambda)^{-1}Z^{\top}. \tag{3}$$

The first and second constraints ensure that $\big(A - LC\big)$ is Hurwitz for all $A \leq \overline{A}$, thereby ensuring stability of the linear interval error dynamics, and Metzler for all $A \geq \underline{A}$, thereby ensuring positivity of the linear interval error dynamics, respectively. We hereinafter refer to the first and second constraint as the *Hurwitz* and *Metzler constraint*, respectively. The third constraint is a technical requirement related to the first two constraints, derived from the theory of positive linear systems [6]. Note that, in implementation, $\lambda > \mathbf{0}_n$ must be replaced with $\lambda \geq \varepsilon\mathbf{1}_n$, where $\varepsilon > 0$ is arbitrarily small, to effect a closed feasible region.

The cost function of Problem III.1 is an upper bound on the ℓ_1-norm of the steady state supremum of the interval error [6, Theorem 4.2],

$$\|\bar{e}\|_1 \leq -\left[2(\overline{A}-\underline{A})\varkappa+\overline{\xi}-\underline{\xi}\right]^{\top}(\overline{A}-LC)^{-\top}\mathbf{1}_n. \tag{4}$$

The Hurwitz constraint can be rearranged, using (3) to eliminate λ and Z, to obtain

$$\|\bar{e}\|_1 \leq \left[2(\overline{A}-\underline{A})\varkappa+\overline{\xi}-\underline{\xi}\right]^{\top}\lambda. \tag{5}$$

B. Cardinality Constrained Robust Optimization

In [10], a method is proposed for protecting against varying numbers of perturbed coefficients, given an LP of the form

Problem III.2.

$$\begin{aligned}\text{minimize: } & c^{\top}q\\ \text{subject to: } & Eq \leq b\\ & l \leq q \leq u,\end{aligned}$$

where $E \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $q, l, u, c \in \mathbb{R}^n$.

All uncertainty is assumed to be in the constraint coefficient matrix E and cost coefficient vector c. The sets J_0 and J_i, $i \in \{1,\ldots,m\}$, contain the indices of the uncertain coefficients in the cost and ith constraints, respectively. If $j \in J_0$, then the jth cost coefficient lies in the interval $[c_j, c_j+d_j]$; if $j \in J_i$, $i \in \{1,\ldots,m\}$, then the jth coefficient of the ith constraint lies in the interval $[E_{ij}-\widehat{E}_{ij}, E_{ij}+\widehat{E}_{ij}]$. The values of the perturbation terms $d_j, \widehat{E}_{ij} \geq 0$ are known for all i, j. The *protection levels* $\Gamma_0 \in \mathbb{Z}_{\geq 0}$ and $\Gamma_i \in \mathbb{R}_{\geq 0}$, $i \in \{1,\ldots,m\}$, specify the number of perturbed coefficients to protect against in the cost and ith constraints, respectively. The vector $\Gamma \coloneqq \mathbf{col}(\Gamma_0, \Gamma_1, \ldots, \Gamma_m)$ specifies only the cardinalities of the sets of protected coefficients — it does not specify individual coefficients to be protected.

The robust formulation of Problem III.2, as developed in [10], is given in Problem III.3.

Problem III.3.

minimize: $c^{\top}q + \Omega_0(q, \Gamma_0, d)$

subject to:

$$\begin{aligned}&\sum_{j=1}^{n} E_{ij}q_j + \Omega_i\big(q, \Gamma_i, (\widehat{E}^{\top})_i\big) \leq b_i \qquad i \in \{1,\ldots,m\}\\ & l \leq q \leq u,\end{aligned}$$

where

$$\Omega_0(q,\Gamma_0,d) \coloneqq \max_{\{S_0 | S_0 \subseteq J_0, |S_0| \leq \Gamma_0\}} \left\{\sum_{j\in S_0} d_j|q_j|\right\}, \tag{6}$$

and $\Omega_i\big(q, \Gamma_i, (\widehat{E}^{\top})_i\big)$, $i \in \{1,\ldots,m\}$, is defined as

$$\max_{\{S_i\cup\{t_i\} | S_i\subseteq J_i, |S_i|\leq\lfloor\Gamma_i\rfloor, t_i\in J_i\setminus S_i\}} \left\{\sum_{j\in S_i}\widehat{E}_{ij}|q_j| + \big(\Gamma_i - \lfloor\Gamma_i\rfloor\big)\widehat{E}_{it_i}|q_{t_i}|\right\}. \tag{7}$$

The set $S_0 \subseteq J_0$ in (6) contains the indices of perturbed cost coefficients. Maximizing over S_0 identifies the set of coefficients which, when perturbed, result in the greatest cost for a given solution q. By augmenting the cost with Ω_0, minimizing the cost yields an optimal solution q^* that minimizes the maximum cost over the class of perturbed cost functions defined by Γ_0.

If $q_j > 0$ [$q_j < 0$], then E_{ij} is perturbed in the positive [negative] direction. Notice that Ω_i is nonnegative for $i \in \{0,\ldots,m\}$. The set of indices $S_i \cup \{t_i\}$ identifies the $\lfloor\Gamma_i\rfloor$ constraint coefficients E_{ij} to be perturbed by $\widehat{E}_{ij}$, $j \in S_i$, and the constraint coefficient E_{it_i} to be perturbed by $\big(\Gamma_i - \lfloor\Gamma_i\rfloor\big)\widehat{E}_{it_i}$, that maximizes the increase in $\sum_j E_{ij}q_j$.

Note that if $\Gamma_i = 0$, $i \in \{0,\ldots,m\}$, then $\Omega_0(q,\Gamma_0,d) = 0$ and $\Omega_i\big(q,\Gamma_i,(\widehat{E}^{\top})_i\big) = 0$, $i \in \{1,\ldots,m\}$, which renders Problem III.3 equivalent to Problem III.2.

Problem III.3 has an equivalent linear formulation [10, Theorem 1], which we use later in this paper, to illustrate our proposed design method.

IV. Robust Interval Observers

If the inner bounds $(\underline{A}^{\uparrow}, \overline{A}^{\downarrow})$ are used in Problem III.1 instead of the outer bounds $(\underline{A}, \overline{A})$, then the attainable optimum cost is minimized. However, the inner bounds cannot be used unless $\underline{A}^{\uparrow} \leq A \leq \overline{A}^{\downarrow}$. By casting Problem III.1 in the framework of Problem III.3, we develop a tunably *robust interval observer (RIO)*. The designer specifies the level of robustness of the solution by manipulating a tuning parameter, which is defined in terms of the cardinality of entries of A that do not satisfy $A_{ij} \leq \overline{A}^{\downarrow}_{ij}$, and the cardinality of entries of A that do not satisfy $A_{uv} \geq \underline{A}^{\uparrow}_{uv}$. As the number of elements of A that do not satisfy $A_{ij} \leq \overline{A}^{\downarrow}_{ij}$ and $A_{uv} \geq \underline{A}^{\uparrow}_{uv}$ increases, the optimal cost also increases.

A. Robust Formulation of the Interval Observer Problem

In the proposed robust formulation of Problem III.1, we define the inner bounds $(\overline{A}^{\downarrow}, \underline{A}^{\uparrow})$ to be the nominal bounds on A. Therefore, casting Problem III.1 in the robust framework of Problem III.3, the inner bounds $(\overline{A}^{\downarrow}, \underline{A}^{\uparrow})$ define the nominal cost and constraint coefficients. Define $\Delta\overline{A} \coloneqq \overline{A} - \overline{A}^{\downarrow}$, $\Delta\underline{A} \coloneqq \underline{A}^{\uparrow} - \underline{A}$; these matrices define the perturbations to the coefficients of the robust LP. Coefficients corresponding to $\overline{A}^{\downarrow}_{ij}$ are perturbed to $\overline{A}^{\downarrow}_{ij} + \Delta\overline{A}_{ij} = \overline{A}_{ij}$, and coefficients corresponding to $\underline{A}^{\uparrow}_{ij}$ are perturbed to $\underline{A}^{\uparrow}_{ij} - \Delta\underline{A}_{ij} = \underline{A}_{ij}$.

We wish to allow for perturbations to be applied independently to each of the $2n^2$ elements of $(\overline{A}^{\downarrow}, \underline{A}^{\uparrow})$. This requires that the robust formulation of Problem III.1 have exactly one decision variable corresponding to each element of $\overline{A}^{\downarrow}$ and $\underline{A}^{\uparrow}$. We introduce the dummy variables $\overline{\lambda}^{(i)}, \underline{\lambda}^{(i)} \in \mathbb{R}^n, i \in \{1,\dots,n\}$ and define the new vector of decision variables as $\Lambda \coloneqq \mathbf{col}\left(\overline{\lambda}^{(1)},\dots,\overline{\lambda}^{(n)},\underline{\lambda}^{(1)},\dots,\underline{\lambda}^{(n)}\right) \in \mathbb{R}^{2n^2}$. The cost coefficient perturbation vector is given by

$$d \coloneqq \mathbf{col}\big(2\Delta\overline{A}\mathbf{diag}(\varkappa), 2\Delta\underline{A}\mathbf{diag}(\varkappa)\big) \in \mathbb{R}^{2n^2}. \quad (8)$$

Since Λ is $2n^2$-dimensional, the sets of indices of uncertain coefficients is $J_i \subseteq \{1,\dots,2n^2\}$, $i \in \{0,\dots,n^2+n\}$. The maximum cardinality of J_i is equal to n. Since perturbations to the cost and constraint coefficients have the same physical interpretation, we stipulate $\Gamma_0 = \sum_{i\neq 0}\lceil\Gamma_i\rceil$, to ensure that the same number of coefficients are perturbed in the cost as in the constraints.

Remark IV.1. All elements of $(\overline{A}^{\downarrow}, \underline{A}^{\uparrow})$ appear in both the cost and constraints of Problem III.1. In the robust formulation, Γ_0 elements of these matrices are perturbed in the cost function, and $\sum_{i\neq 0}\lceil\Gamma_i\rceil$ are perturbed in the constraints. The specific elements of $(\overline{A}^{\downarrow}, \underline{A}^{\uparrow})$ perturbed in the cost and constraints are not necessarily the same. This results in the robust formulation optimizing over an uncertainty class whose members are not all physically realizable. ♦

Perturbations to the coefficients corresponding to $\overline{\lambda}^{(i)}$ [$\underline{\lambda}^{(i)}$], $i \in \{1,\dots,n\}$, are interpreted as upward [downward] perturbations of $\overline{A}^{\downarrow}$ [$\underline{A}^{\uparrow}$].

Since the Hurwitz constraint of Problem III.1 is an equality, we characterize the effect of the protection process of [10] on strict equality constraints with uncertain coefficients.

Proposition IV.2. *If constraint i in Problem III.3 is an equality constraint with at least one uncertain coefficient ($J_i \neq \varnothing$) and a nonzero protection level ($\Gamma_i > 0$), then Problem III.3 is infeasible.*

Proof. An equality constraint $\sum_j E_{ij}q_j = b_i$ can be formulated as two inequality constraints: $\sum_j E_{ij}q_j \leq b_i$ and $-\sum_j E_{ij}q_j \leq -b_i$. Denoting the index of the second constraint by i', the robust formulation is $\sum_j E_{ij}q_j + \Omega_i\big(q,\Gamma_i,(\widehat{E}^\top)_i\big) \leq b_i$, $-\sum_j E_{ij}q_j + \Omega_{i'}\big(q,\Gamma_{i'},(\widehat{E}^\top)_i\big) \leq -b_i$. Summing these constraints, we have $\Omega_i\big(q,\Gamma_i,(\widehat{E}^\top)_i\big) + \Omega_{i'}\big(q,\Gamma_{i'},(\widehat{E}^\top)_i\big) \leq 0$, which is feasible only if $\Gamma_i = 0$. □

In light of Proposition III.3, in the robust version of Problem III.1 we replace the equality Hurwitz constraint with a non-strict inequality. We show that the non-strict inequality is always satisfied with equality, provided that Problem III.1 is feasible.

With these definitions and notation, we propose the following cardinality constrained robust version of the IO design problem from [6].

Problem IV.3.

$$\begin{aligned}
&\text{minimize: } \left[2(\overline{A}^{\downarrow} - \underline{A}^{\uparrow})\varkappa + \overline{\xi} - \underline{\xi}\right]^\top \lambda + \Omega_0(\Lambda,\Gamma_0,d)\\
&\text{subject to:}\\
&\overline{A}_i^{\downarrow\top}\overline{\lambda}^{(i)} - C_i^\top Z\mathbf{1}_n + \Omega_i(\overline{\lambda}^{(i)},\Gamma_i,\Delta\overline{A}_i) \leq -1\\
&-(\underline{A}^{\uparrow}_{ji}\underline{\lambda}^{(i)}_j - C_i^\top Z_j + \delta_{ij}\beta) + \Omega_i(\underline{\lambda}^{(i)},\Gamma_r,\Delta\underline{A}_i) \leq 0\\
&\lambda > \mathbf{0}_n\\
&\overline{\lambda}^{(i)} = \underline{\lambda}^{(i)} = \lambda \qquad\qquad i,j \in \{1,\dots,n\},\ r = ni + j.
\end{aligned}$$

Remark IV.4. The linear implementation of Problem IV.3 has $N = \big(3n^2 + pn + n + 1 + \sum_{i=0}^m |J_i|\big)$ decision variables [10], where $|J_0| \leq 2n^2$, $|J_i| \leq n$ for the n Hurwitz constraints, $|J_i| \leq 1$ for the n^2 Metzler constraints; we assume $p \leq n$.

There exist algorithms that can solve LPs in $O\big(\frac{N^3}{\ln N}U\big)$ time in the worst case [15], where U is the bit length of the binary encoding of the vectors c, b, and matrix E. Therefore, Problem IV.3 can be solved in $O\left(\frac{n^6}{\ln n}U\right)$ time. ♦

Denote by $\mathbf{\Gamma} \coloneqq \mathbf{col}(\Gamma_1,\dots,\Gamma_{n^2+n})$ the vector of constraint protection levels. Denote by $J \coloneqq \{J_0,\dots,J_{n^2+n}\}$ the set of sets of indices of uncertain coefficients, and define $|J| \coloneqq \mathbf{col}\big(|J_0|,\dots,|J_{n^2+n}|\big)$. Denote by $\mathbf{J} \coloneqq \{J_1,\dots,J_{n^2+n}\}$ the set of indices of uncertain constraint coefficients, and define $|\mathbf{J}| \coloneqq \mathbf{col}\big(|J_1|,\dots,|J_{n^2+n}|\big)$.

Theorem IV.5. *If $\Gamma = |J|$, then Problems III.1 and IV.3 are equivalent.*

To prove Theorem IV.5, we first prove that if $\Gamma = |J|$, then the cost and constraint coefficients in both problems are equal. We then prove that if Problem III.1 is feasible, then the Hurwitz constraint in Problem IV.3 is satisfied with equality.

Lemma IV.6. *If $\Gamma = |J|$, then Problems III.1 and IV.3 have the same cost and constraint coefficients.*

Proof. Setting $\Gamma_0 = |J_0|$ results in Problem IV.3 optimizing over all uncertain cost coefficients, i.e., $S_0 = J_0$, yielding $\left[2(\overline{A}^{\downarrow} - \underline{A}^{\uparrow})\varkappa + \overline{\xi} - \underline{\xi}\right]^\top \lambda + \sum_{j\in J_0}\left\{[2\varkappa_j\Delta\overline{A}_j^\top]\overline{\lambda}^{(j)}\right\} + \sum_{j\in J_0}\left\{[2\varkappa_j\Delta\underline{A}_j^\top]\underline{\lambda}^{(j)}\right\}$. Applying the definitions of $\overline{\lambda}^{(i)}$, $\underline{\lambda}^{(i)}$, $i \in \{1,\dots,n\}$, $\overline{A}^{\downarrow}$, $\underline{A}^{\uparrow}$, $\Delta\overline{A}$, and $\Delta\underline{A}$, this simplifies to $\left[2(\overline{A} - \underline{A})\varkappa + \overline{\xi} - \underline{\xi}\right]^\top \lambda$ which is exactly the cost function of Problem III.1.

Setting $\mathbf{\Gamma} = |\mathbf{J}|$ results in Problem IV.3 satisfying feasibility over all admissible constraint coefficient perturbations, i.e., $S_i = J_i$, $i \in \{1,\dots,n^2+n\}$. We first examine the Hurwitz constraint: $\overline{A}_i^{\downarrow\top}\overline{\lambda}^{(i)} - C_i^\top Z\mathbf{1}_n \leq -1$, $i \in \{1,\dots,n\}$. Again, by applying only definitions, this simplifies to $\overline{A}^\top\lambda -$

$C^\top Z\mathbf{1}_n \leq -\mathbf{1}_n$, which has the same coefficients as the Hurwitz constraint of Problem III.1.

We next examine the Metzler constraint: $-(\underline{A}^{\uparrow}_{ji}\lambda^{(i)}_j - C^\top_i Z_j + \delta_{ij}\beta) + \Delta\underline{A}_{ji}\lambda^{(i)}_j \leq 0$. Again, applying only definitions, this simplifies to $\underline{A}^\top\lambda - C^\top Z + \beta I \geq \mathbf{0}_{n\times n}$, which is equivalent to the Metzler constraint of Problem III.1. ☐

Lemma IV.7. *Using a nonstrict inequality relation in the Hurwitz constraint of Problem IV.3 instead of equality, as in Problem III.1, results in* $\left[2(\overline{A}-\underline{A})\varkappa + \overline{\xi} - \underline{\xi}\right]^\top\lambda \leq \left[2(\overline{A}-\underline{A})\varkappa + \overline{\xi} - \underline{\xi}\right]^\top\lambda'$, *where* (λ, Z, β) *and* $(\lambda', \Lambda', Z', \beta')$ *are optimal solutions to Problems III.1 and IV.3, respectively.*

Proof. Consider the two equations: $\lambda = -(\overline{A} - LC)^{-\top}\mathbf{1}_n$ and $\lambda' = -(\overline{A} - LC)^{-\top}\alpha\mathbf{1}_n$, where $\alpha := \mathbf{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n), \alpha_i \geq 1, i \in \{1, \ldots, n\}$. Since $(\overline{A} - LC)^\top$ is Hurwitz and Metzler by construction, we have that $-(\overline{A} - LC)^{-\top}$ is nonnegative [16], and since $\alpha\mathbf{1}_n \geq \mathbf{1}_n$, we have $\lambda' \geq \lambda$, with equality holding if and only if $\alpha = I$. Since α is arbitrary, it follows that $\lambda = -(\overline{A} - LC)^{-\top}\alpha\mathbf{1}_n$ is equivalent to $\lambda \leq -(A - LC)^{-\top}\mathbf{1}_n$. ☐

Before proving that the robust Hurwitz constraint is always satisfied with equality, we characterize the feasibility of Problem IV.3, and the effect of inequality on optimality.

Proposition IV.8. *Any feasible solution to Problem III.3 for given* Γ_i, $i \in \{1, \ldots, m\}$, *is also a feasible solution for any* $\Gamma'_i \leq \Gamma_i$, $i \in \{1, \ldots, m\}$.

Proof. By construction, a constraint perturbation (7) increases the left-hand side of constraints of the form $\sum_j E_{ij}q_j \leq b_i$. Therefore, as Γ_i increases, thereby increasing the magnitude of $\Omega_i(q, \Gamma_i, (\widehat{E}^\top)_i)$, solutions that satisfy constraint i with insufficient slack become infeasible. Conversely, as Γ_i decreases, $\sum_j E_{ij}q_j$ decreases, thereby expanding the set of feasible solutions. Consequently, for fixed E, $\widehat{E}$, b, l, u, the set of feasible solutions to Problem III.3 using the protection levels $\Gamma'_i \leq \Gamma_i$, $i \in \{1, \ldots, m\}$ is a superset of the feasible solutions to Problem III.3 using the constraint protection levels Γ_i, $i \in \{1, \ldots, m\}$. ☐

Corollary IV.9. *Any feasible solution to Problem III.1 is also a feasible solution to Problem IV.3.*

Proof. By Lemma IV.6, when $\Gamma = |J|$, Problem III.1 can be viewed as an instance of Problem IV.3, where the Hurwitz constraint is a nonstrict inequality, instead of an equality. If $\sum_j E_{ij}q_j = b_i$ is feasible, then $\sum_j E_{ij}q_j \leq b_i$ can be satisfied with equality. Therefore, if $\Gamma = |J|$ and Problem III.1 is feasible, then Problem IV.3 is also feasible, and its Hurwitz constraint can be satisfied with equality. Therefore, by Proposition IV.8, any feasible solution to Problem III.1 is also a feasible solution to Problem IV.3 for any $\mathbf{\Gamma} \preceq |\mathbf{J}|$. ☐

Lemma IV.10. *If Problem III.1 is feasible, then the Hurwitz constraint in Problem IV.3 is always satisfied with equality.*

Proof. By Corollary IV.9, if Problem III.1 is feasible, then satisfying the Hurwitz constraint in Problem IV.3 is feasible. By Lemma IV.7, satisfying the Hurwitz constraint with equality minimizes the cost. ☐

Proof of Theorem IV.5. By Lemma IV.6, if $\Gamma = |J|$, then Problems III.1 and IV.3 have the same cost and constraint coefficients, and by Lemma IV.10, the Hurwitz constraint of Problem IV.3 is always satisfied with equality. Therefore, if $\Gamma = |J|$, then Problems III.1 and IV.3 are equivalent. ☐

B. Class of Systems Considered

We now clarify the class of systems considered by Problem IV.3.

Define the set of all admissible sets of indices of uncertain constraint coefficients $\mathcal{J} := \{\mathcal{P}_{\Gamma_1}(J_1), \ldots, \mathcal{P}_{\Gamma_{n^2+n}}(J_{n^2+n})\}$, where $\mathcal{P}_{\Gamma_k}(J_k)$ is the set $\{J'_k \mid J'_k \subseteq J_k, |J'_k| \leq \Gamma_k\}$, and $\mathcal{P}_0(J_k) := \varnothing$. Given pairs $(\overline{A}^{\downarrow}, \underline{A}^{\uparrow})$, $(\overline{A}, \underline{A}) \in \mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n}$ that satisfy Assumption 4, a set of sets of indices of uncertain constraint coefficients $\mathbf{J}$, and a set of sets of indices of perturbed constraint coefficients $\mathbf{S} := \{\{S_1 \cup \{t_1\} \subseteq J_1\}, \ldots, \{S_{n^2+n} \cup \{t_{n^2+n}\} \subseteq J_{n^2+n}\}\}$, we generate a pair $(A^u, A^l) \in \mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n}$ via the mapping $\Xi : \mathbf{J} \to \mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n}$, $\mathbf{S} \mapsto (A^u, A^l)$, where

$$A^u_{ji} = \begin{cases} \overline{A}_{ji} & \text{if } i \leq n,\ j \in S_i \\ \overline{A}^{\downarrow}_{ji} + (\lfloor\Gamma_i\rfloor - \Gamma_i)\Delta\overline{A}_{ji} & \text{if } i \leq n,\ j \in \{t_i\} \\ \overline{A}^{\downarrow}_{ji} & \text{otherwise,} \end{cases}$$

$$A^l_{ji} = \begin{cases} \underline{A}_{ji} & \text{if } i, j \leq n,\ j \in S_{ni+j} \\ \underline{A}^{\uparrow}_{ji} - \Gamma_i\Delta\underline{A}_{ji} & \text{if } i, j \leq n,\ j \in \{t_{ni+j}\} \\ \underline{A}^{\uparrow}_{ji} & \text{otherwise.} \end{cases}$$

A pair $A^{u,l}$ defines a set in $\mathbb{R}^{n\times n}$ by

$$\{A \in \mathbb{R}^{n\times n} : A^l \leq A \leq A^u\}. \tag{9}$$

Further, given constraint protection levels $\mathbf{\Gamma}$, using Ξ we define the mapping

$$\begin{aligned} &\mathcal{A} : \mathbf{J} \times \mathbb{R}^{n^2+n}_{\geq 0} \to \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \\ &(\mathbf{J}, \mathbf{\Gamma}) \mapsto \{A^{u,l} \mid (\exists\, \mathbf{S} \in \mathcal{J})\ (\Xi(\mathbf{S}) = A^{u,l})\}\,. \end{aligned} \tag{10}$$

The set (10) comprises all pairs of upper and lower state matrices effected by perturbing no more than $\mathbf{\Gamma}$ elements of the boundaries of the interval induced by the pair $(\underline{A}^{\uparrow}, \overline{A}^{\downarrow})$ as defined by (9).

C. Robust Interval Observer Dynamics and Performance

In this section, we define the dynamics of the proposed IO and characterize its performance.

Define $\mathcal{F}^L, \mathcal{F}^L_R(J, \Gamma) \subset \mathbb{R}^{n\times p}$ to be the sets of all feasible observer gain matrices constructed using optimal solutions to Problems III.1 and IV.3, respectively. By Corollary IV.9, any feasible solution to Problem IV.3 is also a feasible solution to Problem III.1, i.e., $\mathcal{F}^L_R(J, \Gamma) \supseteq \mathcal{F}^L$. Define the set $\mathcal{L}_R(J, \Gamma) \subset \mathcal{F}^L_R(J, \Gamma)$ to be the set of all matrices L constructed using optimal solutions to Problem IV.3 for a given J and Γ. Similarly, define the set $\mathcal{L} \subset \mathcal{F}^L$ to be the set of all matrices L constructed using optimal solutions to Problem III.1.

We propose the following *robust interval observer (RIO)*, whose gain matrix (3) is constructed using an optimal solution to Problem IV.3 for a given J and Γ.

$$\begin{aligned}\dot{\hat{x}}^u &= A^u\hat{x}^u + L(y - C\hat{x}^u) - (A^u - A^l)\phi(\hat{x}^u) + \overline{\xi}\\ \dot{\hat{x}}^l &= A^u\hat{x}^l + L(y - C\hat{x}^l) - (A^u - A^l)\psi(\hat{x}^l) + \underline{\xi}\\ \hat{x}^{u,l}(0) &= (\overline{x}_0, \underline{x}_0), \quad A^{u,l} \in \mathcal{A}(\mathbf{J}, \mathbf{\Gamma}).\end{aligned} \tag{11}$$

Define the mapping

$$\begin{aligned}&\bar{e}_{\ell_1} : \mathbb{R}^{n\times p} \times \mathcal{A}\big(\mathbf{J}, |\mathbf{J}|\big) \to \mathbb{R}_{\geq 0}\\ &(L, A^{u,l}) \mapsto -\big[2(A^u - A^l)\varkappa + \overline{\xi} - \underline{\xi}\big]^\top (A^u - LC)^{-\top}\mathbf{1}_n.\end{aligned} \tag{12}$$

By (4), the mapping (12) upper bounds $\|\bar{e}\|_1$ of an IO with dynamics (11), constructed with the state matrix pair $A^{u,l}$.

Proposition IV.11. *The optimal cost of Problem IV.3 is greater than or equal to the tightest bound* (5) *on* $\|\bar{e}\|_1$ *of the RIO* (11).

Proof. Given the optimal state matrix pair (A^u, A^l) constructed using the mapping Ξ, by construction of (5), the optimal cost is equal to the tightest bound (5) if and only if the optimal perturbed cost vector is equal to

$$\big[2(A^u - A^l)\varkappa + \overline{\xi} - \underline{\xi}\big]. \tag{13}$$

Since we stipulate that $\Gamma_0 = \sum_{i\neq 0}\lceil\Gamma_i\rceil$, exactly as many cost coefficients will be perturbed as constraint coefficients, so (13) is always a feasible perturbed cost vector. By construction, Problem IV.3 perturbs $\big[2(\overline{A}^{\downarrow} - \underline{A}^{\uparrow})\varkappa + \overline{\xi} - \underline{\xi}\big]$ such that the maximum cost is minimized. Therefore, the optimal cost cannot be less than the bound (5), as this would violate optimality. However, if (13) does not effect the greatest maximum cost, then (13) will not be the optimal perturbed cost vector. □

Theorem IV.12. *Given a system of the form* (1) *that satisfies Assumptions 1, 2, 3, 4, with* $\underline{A} \preceq \overline{A}$*, indices of uncertain cost coefficients* J_0*, indices of uncertain constraint coefficients* $\mathbf{J}$*, cost protection level* $\Gamma_0 < |J_0|$*, and constraint protection levels* $\mathbf{\Gamma} \preceq |\mathbf{J}|$*, the proposed RIO* (11) *effects a smaller upper bound on* $\|\bar{e}\|_1$ *than the IO* (2)*, and a maximum upper bound on* $\|\bar{e}\|_1$ *over the set of state matrices* $\mathcal{A}(\mathbf{J}, \mathbf{\Gamma})$ *no greater than that if it were constructed instead using* $L \in \mathcal{L}$*, i.e.,*

$$\begin{aligned}&\max_{A^{u,l}\in\mathcal{A}(\mathbf{J},\mathbf{\Gamma})} \bar{e}_{\ell_1}(L_R, A^{u,l})\\ &\quad\leq \max_{A^{u,l}\in\mathcal{A}(\mathbf{J},\mathbf{\Gamma})} \bar{e}_{\ell_1}\big(L, A^{u,l}\big) < \bar{e}_{\ell_1}\big(L, (\overline{A}, \underline{A})\big).\end{aligned} \tag{14}$$

Proof. By (4), for a fixed L, choosing any state matrices $\widetilde{A}^{u,l} := (\widetilde{A}^u, \widetilde{A}^l)$ such that $(\widetilde{A}^u - \widetilde{A}^l) \preceq (A^u - A^l)$, necessarily reduces the upper bound on $\|\bar{e}\|_1$. Since $(\overline{A}, \underline{A}) \notin \mathcal{A}(\mathbf{J}, \mathbf{\Gamma})$ for any $\mathbf{\Gamma} \preceq |\mathbf{J}|$, we have

$$\begin{aligned}&\big(\forall\, \Gamma_0 < |J_0|\big)\big(\forall\, \mathbf{\Gamma} \preceq |\mathbf{J}|\big)\big(\widetilde{A}^{u,l} \in \mathcal{A}(\mathbf{J}, \mathbf{\Gamma})\big)\\ &\qquad \implies \Big(\bar{e}_{\ell_1}(L, \widetilde{A}^{u,l}) < \bar{e}_{\ell_1}(L, \big(\overline{A}, \underline{A}\big)\big)\Big).\end{aligned} \tag{15}$$

Therefore, constructing the observer (2) with state matrices $A^{u,l} \in \mathcal{A}(\mathbf{J}, \mathbf{\Gamma})$, $\mathbf{\Gamma} \preceq |\mathbf{J}|$, instead of $(\overline{A}, \underline{A})$, effects a smaller upper bound on $\|\bar{e}\|_1$.

By Proposition IV.11, the optimal cost of Problem IV.3 is no less than $\max_{A^{u,l}\in\mathcal{A}(\mathbf{J},\mathbf{\Gamma})} \bar{e}_{\ell_1}(L, A^{u,l})$, i.e., the maximum upper bound on the ℓ_1-norm of the steady state supremum of the interval error over all state matrix pairs $A^{u,l} \in \mathcal{A}(\mathbf{J}, \mathbf{\Gamma})$. By Corollary IV.9, the set of feasible L matrices admitted by Problem IV.3 is a superset of the feasible L matrices admitted by Problem III.1, and by the optimality of $L \in \mathcal{L}_R$, we have that there exists no $L \in \mathcal{L}$ that effects a smaller maximum upper bound on $\|\bar{e}\|_1$ over the set of state matrix pairs $\mathcal{A}(\mathbf{J}, \mathbf{\Gamma})$. Combining this with (15), we verify (14). □

Remark IV.13. When only $x(0)$ is uncertain, i.e., $\underline{A} = \overline{A}$ and $\xi(x, t) \equiv \xi(y, t)$, an interval observer with dynamics similar to (11) can be constructed [17], such that $\lim_{t\to\infty} e = 0$. An LP similar to Problem IV.3, with the same constraints but different cost function, is used to design the L matrix. The transient behaviour of the observer is optimized, specifically, $\|e\|_1 = \int_0^\infty \|e(t)\|_1 \mathrm{d}t$. ♦

D. Implementation

In this section, we delineate and illustrate the design process of the proposed RIO (11).

1) Identify $\overline{x}_0$, $\underline{x}_0$, $\overline{A}$, $\underline{A}$, $\overline{A}^{\downarrow}$, $\underline{A}^{\uparrow}$, $\varkappa$; 2) set $\mathbf{\Gamma}$ to the elementwise smallest value such that for each possible state matrix A, there exists a state matrix pair $A^{u,l} \in \mathcal{A}(\mathbf{J}, \mathbf{\Gamma})$, which contains A as defined in (9); 3) using $c := \big[2(\overline{A}^{\downarrow} - \underline{A}^{\uparrow})\varkappa + \overline{\xi} - \underline{\xi}\big]$ and d as defined in (8), solve Problem IV.3 using the constraint $\lambda \geq \varepsilon\mathbf{1}_n$, where $\varepsilon \in \mathbb{R}_{>0}$ is an arbitrarily small constant; 4) using the optimal λ and Z, construct the observer gain matrix $L := \mathbf{diag}(\lambda)^{-1}Z^\top$; 5) construct the RIO (11) with the pair $A^{u,l} \in \mathcal{A}(\mathbf{J}, \mathbf{\Gamma})$ that induces the elementwise smallest interval (9) such that $A^l \leq A \leq A^u$, for all possible A.

The resultant RIO with gain matrix L, is optimal in the sense that the maximum cost, which upper bounds $\|\bar{e}\|_1$, over all state matrix pairs $A^{u,l} \in \mathcal{A}(\mathbf{J}, \mathbf{\Gamma})$ is minimized, and is optimal in this sense for any state matrix pair in $\mathcal{A}(\mathbf{J}, \mathbf{\Gamma})$. By stipulating that $A^{u,l}$ be chosen such that the set (9) contains A, we ensure the interval property; by setting $\mathbf{\Gamma}$ as small as is possible while ensuring the interval property, we minimize the attainable optimum cost.

For a class of systems $\mathcal{A}(\mathbf{\Gamma}, \mathbf{J})$, we perform only a single optimization, i.e., design only a single L. This is advantageous when the dynamical parameters of the plant are guaranteed to lie within some range, but under certain circumstances, this range can be refined. The original IO (2) of [6], is optimal only for a single $A^{u,l}$, and would need to be optimized for each state matrix pair in $\mathcal{A}(\mathbf{J}, \mathbf{\Gamma})$.

Example IV.14. To illustrate the proposed approach, we construct and implement a RIO for a three stage fish population model, based on that in [1],

$$\begin{aligned}A &= \begin{bmatrix} -1.4 & 0 & 0\\ 0.9 & -1.3 & 0\\ 0 & 0.8 & -0.1\end{bmatrix}\\ \xi &= \begin{bmatrix}\frac{\big(0.1\sin(t)+0.3\big)x_3}{0.1+x_3} & 0 & -\big(0.05\cos(t)+0.15\big)x_3\end{bmatrix}^\top\\ C &= \begin{bmatrix}0 & 0 & 1\end{bmatrix},\end{aligned}$$

where the states x_1, x_2, and x_3, are the biomass of larval, juvenile, and adult stocks, respectively. The measured output y is taken to be the adult stock, as these fish are larger, and therefore are easier to detect and count.

We define the constants $\overline{x}_0 = \begin{bmatrix}0.9 & 0.8 & 0.55\end{bmatrix}^\top$, $\underline{x}_0 = \begin{bmatrix}0.5 & 0.4 & 0.45\end{bmatrix}^\top$, $\overline{\xi} = \begin{bmatrix}0.35455 & 0 & -0.033\end{bmatrix}^\top$, $\underline{\xi} = \begin{bmatrix}0.15349 & 0 & -0.156\end{bmatrix}^\top$, $\varkappa = \begin{bmatrix}0.9 & 0.8 & 0.78\end{bmatrix}^\top$. Define $\overline{A} = 1.25A$, $\underline{A} = 0.75A$, $\overline{A}^\downarrow = 1.125A$, $\underline{A}^\uparrow = 0.875A$, which induces the cost vector and cost perturbations $c = \begin{bmatrix}0.83106 & 0.925 & 0.482\end{bmatrix}^\top$, $d_1 = d_{10} = 0.315$, $d_2 = d_{11} = 0.2025$, $d_5 = d_{14} = 0.26$, $d_6 = d_{15} = 0.16$, $d_9 = d_{18} = 0.0195$, $d_j = 0$, $j \in \{3, 4, 7, 8, 12, 13, 16, 17\}$. The indices of uncertain cost coefficients are those corresponding to nonzero elements of d, $J_0 = \{1, 2, 5, 6, 9, 10, 11, 12, 14, 15, 18\}$. The indices of uncertain constraint coefficients are those corresponding to nonzero elements of the constraint coefficient perturbation matrix, which correspond to the elements of $\Delta\overline{A} = 0.125A$ and $\Delta\underline{A} = 0.125A$, which yield $J_1 = \{1, 2\}$, $J_2 = \{5, 6\}$, $J_3 = \{9\}$, $J_i = \varnothing$, $i \in \{6, 7, 10, 11\}$, $J_i = \{6 + i\}$, $i \in \{4, 5, 8, 9, 12\}$. Lastly, we set $\varepsilon = 10^{-3}$.

The IO of [6] is constructed with $A^{u,l} = (\overline{A}, \underline{A})$. We compare the cost of the optimal solution to the proposed robust problem, Problem IV.3, to that of the original IO problem, Problem III.1, as well as $\|\bar{e}\|_1$ of the RIO (11) and the original IO (2). These values, and the percent reductions thereof effected by using the proposed approach, are presented in Table I for various $\boldsymbol{\Gamma}$. For ease of exposition, define the simulation parameters $\Gamma_H, \Gamma_M \in \mathbb{R}_{\geq 0}$. The protection levels for the Hurwitz constraints, i.e., Γ_i, $i \in \{1, 2, 3\}$, are all set to $\min(\Gamma_H, |J_i|)$, e.g., $\Gamma_H = 2$ effects $\Gamma_1 = 2, \Gamma_2 = 2$, and $\Gamma_3 = 1$; the protection levels for the Metzler constraints, i.e., Γ_i, $i \in \{4, \ldots, 12\}$, are all set to $\min(\Gamma_M, |J_i|)$, e.g., $\Gamma_M = 1$ effects $\Gamma_i = 1$, $i \in \{4, 5, 8, 9, 12\}$, and $\Gamma_i = 0$, $i \in \{6, 7, 10, 11\}$. Recall that $\Gamma = |J|$ yields the same solution as Problem III.1. Plots of the trial $\Gamma_H, \Gamma_M = 0.5$ are presented in Figure 1. The reductions in cost and $\|\bar{e}\|_1$ are

TABLE I

COMPARISONS FOR VARIOUS Γ.

Γ_H	Γ_M	Γ_0	Cost	$\|\bar{e}\|_1$	% Reduction	
					Cost	$\|\bar{e}\|_1$
0.5	0.5	8	4.27	0.475	12.9	21.2
0	0	0	2.10	0.397	57.1	34.2
	1	5	3.71	0.423	24.3	29.9
1	0	3	4.05	0.539	17.3	10.6
	1	8	4.74	0.564	3.27	6.47
2	0	5	4.69	0.578	4.29	4.15
	1	10	4.90	0.603		

modest to significant, ranging from 3.27% to 57.1%, and 4.15% to 34.2% respectively; the data suggest a positive correlation.

A reduction in the upper bound on $\|\bar{e}\|_1$ also occurs by constructing the IO of [6] with tighter upper and lower state matrices. For comparison, we identify the greatest upper bound on $\|\bar{e}\|_1$ effected by constructing the IO of [6] with $A^{u,l} \in \mathcal{A}(\mathbf{J}, \boldsymbol{\Gamma})$, $\boldsymbol{\Gamma} \preceq |\mathbf{J}|$, and compute the percent reduction in the maximum upper bound on $\|\bar{e}\|_1$ effected by using the RIO instead of the IO, i.e., $100\left(1 - \frac{\max_{A^{u,l} \in \mathcal{A}(\mathbf{J},\boldsymbol{\Gamma})} \bar{e}_{\ell_1}\left(L_R, A^{u,l}\right)}{\max_{A^{u,l} \in \mathcal{A}(\mathbf{J},\boldsymbol{\Gamma})} \bar{e}_{\ell_1}\left(L, A^{u,l}\right)}\right)$.

TABLE II

WORST-CASE COMPARISONS.

Γ_H	Γ_M	Γ_0	$\bar{e}_{\ell_1}$		$\|\bar{e}\|_1$		% Reduction	
			RIO	IO	RIO	IO	$\bar{e}_{\ell_1}$	$\|\bar{e}\|_1$
0.5	0.5	8	3.19	3.67	0.474	0.474	13.1	0.00
0	1	5	2.99	3.78	0.423	0.423	20.9	0.00
1	0	3	3.44	3.57	0.539	0.539	3.64	0.00
	1	8	4.53	4.69	0.564	0.564	3.41	0.00
2	0	5	3.78	3.78	0.578	0.577	0.00	−0.173

We see in Table II that the reduction in cost over the set of state matrix pairs $\mathcal{A}(\mathbf{J}, \boldsymbol{\Gamma})$ is modest to significant, ranging from 0.00% to 20.9%. The difference in $\|\bar{e}\|_1$ is negligible, but for $\Gamma_H = 2$, $\Gamma_M = 0$, the $\|\bar{e}\|_1$ increases by 0.173%. This is possibly a consequence of the conservativeness of the cost of Problem IV.3, as described in Proposition IV.11. ▲

V. MONTE CARLO ANALYSIS

We conduct a Monte Carlo analysis to characterize the reduction in cost effected by using the proposed RIO (11), instead of the IO (2) of [6].

The elements of the matrices $\overline{A}, \underline{A}, \overline{A}^\downarrow, \underline{A}^\uparrow \in \mathbb{R}^{n\times n}$ are seeded by uniform random variables on the intervals $\widetilde{A}_{ij} \in [0, 1]$, $i, j \in \{1, \ldots, n\}, i \neq j$, $\widetilde{A}_{ii} \in \big[-2n, -n\big]$ $i \in \{1, \ldots, n\}$, which ensures the Hurwitz property. Although we do not simulate a specific dynamical system, we assume that ξ is such that the solutions of $\dot{x} = Ax + \xi(x, t)$ are bounded, which is sufficient to satisfy Assumption 1. The seed matrix $\widetilde{A} \in \mathbb{R}^{n\times n}$ is modified to generate the state matrices such that $\|\overline{A} - \underline{A}\|_1 = 1$, $\overline{A}^\downarrow - \underline{A}^\uparrow \geq \frac{2}{n^3}\mathbf{1}_{n\times n}$, which provides consistency in the sizes of the state matrix intervals (9) across trials. The matrix $C \in \mathbb{R}^{p\times n}$ is generated as a uniform random variable on the interval $C \in [0, 1]^{p\times n}$. We define the constants $\overline{\xi} = 0.1\mathbf{1}_n$, $\underline{\xi} = -0.1\mathbf{1}_n$, $\varkappa = 2\max(|\overline{x}_0|, |\underline{x}_0|)$, where max is taken elementwise.

Define the simulation parameter $\Gamma^\star \in \mathbb{Z}_{\geq 0}$. The protection levels for the Hurwitz constraints are set to $\Gamma_i = \min\big(\Gamma^\star, |J_i|\big)$, $i \in \{1, \ldots, n\}$, and a randomly populated set $\mathcal{I} \subseteq \{n+1, \ldots, n^2+n\}$ of cardinality $n\Gamma^\star$, contains the indices of the Metzler constraints that have their protection levels set to 1. This causes $n\Gamma^\star$ elements of both $\overline{A}^\downarrow$, and $n\Gamma^\star$ elements of $\underline{A}^\uparrow$, to be perturbed in each trial. The cost protection level is set to $\Gamma_0 = \sum_{i\neq 0}\Gamma_i$.

Five thousand trials are conducted for each of several combinations of n, p, and $\Gamma^\star$. In each trial, Problems III.1 and IV.3 are solved for the same parameters, and the relative difference between their costs is recorded. The arithmetic means μ and standard deviations σ of these values for representative values of n, p, and $\Gamma^\star$ are presented in Table III.

The data in Table III suggest that the reduction in cost correlates positively with n and negatively with p, making the proposed RIO (11) increasingly attractive as the number of states increases, and as the number of outputs decreases. The negative correlation with p can be interpreted as the robust formulation compensating for the reduction in the number

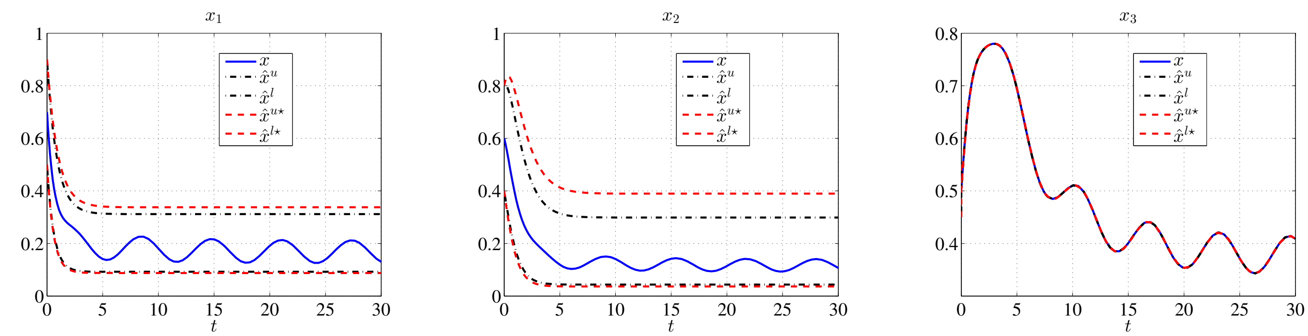

Fig. 1. Comparison of proposed the RIO $\hat{x}$ and the original IO $\hat{x}^{\star}$.

TABLE III

RELATIVE DECREASE IN BOUND ON $\|\bar{e}\|_1$ FOR SELECT n, p, AND Γ_0.

n	p	$\Gamma^{\star}$	μ	σ
2	1	1	0.076	0.062
3			0.15	0.080
4			0.19	0.083
5	1	1	0.22	0.080
	2		0.20	0.051
	3		0.17	0.056
	4		0.074	0.043
5	4	2	0.011	0.013
		3	0.0052	0.0062
		4	0.0025	0.0044

of measurements. A practical implication is that the robust formulation can be used to justify using fewer sensors. For many combinations of n, p, and $\Gamma^{\star}$, the standard deviation is greater than the mean, which suggests that the reduction is highly dependent upon the specific plant being observed.

VI. CONCLUSIONS

We applied the robust optimization method of [10] to the linear programming-based IO design procedure of [6]. We proved that the cost of the proposed RIO is strictly less than that of the original IO. The Monte Carlo analysis suggests that the cost reduction effected by the proposed RIO correlates positively with the number of states, and negatively with the number of outputs. However, the standard deviations of the cost reduction were high, indicating that the reduction is highly dependent upon the specific plant being observed.

Future work should identify analytic bounds on the cost reduction effected by using the proposed RIO over the IO of [6]. Also, an objective function should be identified for simultaneous optimization of transient and steady state performance. The cost of the proposed RIO problem is conservative, as the cost and constraint perturbations may be mismatched; a method should be identified for coupling the perturbing of coefficients that correspond to the same dynamical parameters.

REFERENCES

[1] J.-L. Gouzé, A. Rapaport, and M. Z. Hadj-Sadok, "Interval observers for uncertain biological systems," *Ecological Modelling*, vol. 133, no. 1-2, 2000.

[2] G. Goffaux, A. Vande Wouwer, and O. Bernard, "Improving continuous-discrete interval observers with application to microalgae-based bioprocesses," *Journal of Process Control*, vol. 19, no. 7, pp. 1182–1190, 2009.

[3] M. A. Rami, F. Tadeo, and U. Helmke, "Positive observers for linear positive systems, and their implications," *International Journal of Control*, vol. 84, no. 4, 2011.

[4] F. Mazenc and O. Bernard, "Interval observers for linear time-invariant systems with disturbances," *Automatica*, vol. 47, no. 1, 2011.

[5] Z. Shu, J. Lam, H. Gao, B. Du, and L. Wu, "Positive Observers and Dynamic Output-Feedback Controllers for Interval Positive Linear Systems," *IEEE Trans. Circuits Syst. Regul. Pap.*, vol. 55, no. 10, 2008.

[6] M. A. Rami, C. Cheng, and C. de Prada, "Tight robust interval observers: An LP approach," in *IEEE Conference on Decision and Control*, Cancun, Mexico, 2008, pp. 2967–2972.

[7] D. Bertsimas, D. B. Brown, and C. Caramanis, "Theory and Applications of Robust Optimization," *SIAM Review*, vol. 53, no. 3, pp. 464–501, 2011.

[8] A. Ben-Tal and A. Nemirovski, "Robust solutions of uncertain linear programs," *Operations Research Letters*, vol. 25, no. 1, pp. 1–13, 1999.

[9] D. Bertsimas, D. Pachamanova, and M. Sim, "Robust linear optimization under general norms," *Oper. Res. Letters*, vol. 32, no. 6, 2004.

[10] D. Bertsimas and M. Sim, "Robust discrete optimization and network flows," *Mathematical Programming*, vol. 98, no. 1-3, pp. 49–71, 2003.

[11] A. Ben-Tal and A. Nemirovski, "Robust optimization - methodology and applications," *Math. Program.*, vol. 92, no. 3, pp. 453–480, 2002.

[12] S. P. Boyd, S.-J. Kim, D. D. Patil, and M. A. Horowitz, "Digital Circuit Optimization via Geometric Programming," *Operations Research*, vol. 53, no. 6, pp. 899–932, 2005.

[13] D. Bertsimas and D. B. Brown, "Constrained Stochastic LQC: A Tractable Approach," *IEEE Transactions on Automatic Control*, vol. 52, no. 10, pp. 1826–1841, 2007.

[14] K.-L. Hsiung, S.-J. Kim, and S. P. Boyd, "Power control in lognormal fading wireless channels with uptime probability specifications via robust geometric programming," in *Proceedings of the American Control Conference*, Portland, OR, 2005, pp. 3955–3959.

[15] K. Anstreicher, "Linear Programming in O([n3/ln n]L) Operations," *SIAM Journal on Optimization*, vol. 9, no. 4, pp. 803–812, 1999.

[16] K. J. Arrow, *Studies in resource allocation processes*, L. Hurwicz, Ed. Cambridge: Cambridge University Press, 1977.

[17] P. J. McCarthy, "Cardinality Constrained Robust Optimization Applied to a Class of Interval Observers," M.A.Sc. thesis, University of Waterloo, 2013. [Online]. Available: http://hdl.handle.net/10012/7907