Born Reciprocal Representations of the Quantum Algebra

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Chapters 2, 3, 4 and 5 are based on the publication [59], of which I am the sole author. The publication [59] was in turn based on the unpublished early work in my Perimeter Scholars Internationals essay written under the supervision of Laurent Freidel at the Perimeter Institute.

Chapters 6 and 7 are based on the publication [60], co-authored with Marc Geiller.

Abstract

This thesis addresses the representations of the Weyl algebra of quantum mechanics with an explicitly Born reciprocal structure and the dynamics of physical systems in these representations. In this context, we discuss both the regular modular representations based on Aharonov's modular variables, as well as the irregular modular polymer representations inspired by the spin network representation in loop quantum gravity.

We introduce the modular space as the quantum configuration space corresponding to the modular representation and treat it as an alternative to the classical configuration space. The modular representations have a built-in Abelian gauge symmetry. We discuss the singular limits in which the modular representation converges to the Schrödinger and momentum representations, and show that these limits require two distinct fixings of the modular gauge symmetry.

In order to explore the propagation of quantum states in the modular space, we construct a Feynman path integral for the harmonic oscillator explicitly as the transition amplitude between two modular states. The result is a functional integral over the compact modular space, which shows novel features of the dynamics such as winding modes, an Aharonov-Bohm phase, and a new action on the modular space. We compare the stationary trajectories that extremize this modular action to the phase space solution of Hamilton's equations in the Schrödinger representation and find a new translation symmetry. We also identify the other symmetries of the harmonic oscillator in the modular action and show their correspondence to the classical case.

We generalize the relationship between the classical Hamiltonian function and the modular action by postulating a new modular Legendre transform. For demonstration we apply this transformation on the Kepler problem and reformulate it in terms of modular variables.

We then switch our focus to the representations of the Weyl algebra that violate the assumptions of the Stone–von Neumann theorem. In a similar fashion to the polymer representations studied as a toy model for quantum gravity, we polymerize the modular representation to obtain an inequivalent representation called the modular polymer (MP) representation.

The new MP representation lacks both position and momentum operators, therefore the Hamiltonian of the harmonic oscillator is regularized with the Weyl operators and the non-separable MP Hilbert space is split into superselection sectors. The solutions to the Schrödinger equation fall into two categories depending on whether the ratio between the modular scale and the polymeric regularization scale is a rational or irrational number. We find a discrete and finite energy spectrum in each superselection sector in the former case, and a continuous, bounded one in the latter.

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Dedication

To my parents, who are my source of inspiration in overcoming all difficulties of life.

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Chapter 1

Introduction

The aim of this thesis is to elucidate the role of Born reciprocity in the representations of the Weyl algebra, the physical meaning of dual representations, and the dynamical properties of physical systems therein.

1.1 Background

The fundamental nature of space and time has been a puzzling topic for physicists since at least the birth of modern quantum mechanics in the 1920s. The contrast between the local, background-independent dynamics on a curved, Lorentzian spacetime manifold suggested by general relativity and the propagation of quantum states or fields embellished with entanglement and probability distributions raises many questions that have complicated the search for a unified theory of quantum gravity for over a century. One such question is the status of Born reciprocity [11], named after Max Born, which suggests that the duality-symmetry found between position and momentum variables in quantum mechanics hints at the dualistic nature of the background space in quantum gravity.

The problem with Born reciprocity is that space and momentum play very different roles in general relativity, which displays no such duality. Born proposed in 1938 [11] that the momentum space should have an independent metric that can be curved as well. This idea lead to the development of the principle of relative locality [5, 4] a decade ago, which suggests that the non-linearity in the momentum space results in different spacetime projections constructed by observers at different locations who may disagree on the locality of events.

Unlike general relativity, string theory does offer a duality-symmetry in the form of T-duality. It has been proposed [23, 24] that Born reciprocity is the basis for T-duality. This idea culminated in the development of metastring theory by Freidel, Leigh and Minic [25, 26, 27, 21], where this duality is realized explicitly on the target space. Another approach to a dual space from string theory is double field theory [39, 35, 36], which is proposed as a non-stringy low-energy regime for string theory that realizes T-duality explicitly as an $O(d, d)$ symmetry on its background space [3]. The requirement that the generalized diffeomorphisms in double field theory form a closed group imposes a constraint on the theory, which is often solved by breaking the duality [37, 30]. The dual spacetime arising from metastring theory and double field theory is also studied as a geometric structure in its own right, which is known as Born geometry [28, 29, 51].

Born reciprocity appears in its simplest form in the Weyl algebra of quantum mechanics. The generic representations of this algebra were first studied by Mackey [45]. Among them, the modular representations, named after Aharonov's modular variables [1], were of particular interest because they do not discard the position or momentum variables in favor of the other. The modular representations are used for describing the wave function in the interferometer experiments [2]. They are related to the Schrödinger representation by a Zak transform [61, 62], which is used in condensed matter theory for electrons in a lattice.

Feynman's path integral formulation of quantum mechanics [19] gives a spacetime interpretation to the propagation of quantum states in the Schrödinger representation, therefore it is valuable to us in studying the relationship between the dual space and the representations. A path integral over entangled states was constructed in [33]. Different representations of the states in the Hilbert space can be interpreted as corresponding to the perspective of different quantum observers. This idea leads to quantum reference frames [32, 58], in which locality is a relative concept between quantum observers.

The representations of the Weyl algebra have also attracted interest in loop quantum gravity as a toy model for Planck-scale physics [14, 15]. The spin-network representation of the holonomy-flux algebra can be translated verbatim to quantum mechanics where it corresponds to the polymer representation [7], which violates the assumptions of the Stone–von Neumann theorem [53, 57] and is thus inequivalent to the Schrödinger representation [34, 54]. Similar uniqueness theorems are also known for the holonomy-flux algebra [41, 20]. The representations that violate the assumptions of these uniqueness theorems and exhibit Born reciprocity [13, 60] would be of physical interest for quantum gravity.

1.2 Outline and main results

We begin Chapter 2 by reviewing the Weyl algebra in quantum mechanics. We introduce a double space notation that will be convenient for writing down all of our expressions on a dual space. We present the modular representations in this notation and we study their gauge symmetry in detail. We find that the Schrödinger and momentum representations correspond to two opposite limits of modular representations, but they require a different gauge fixing.

The modular space is the corresponding configuration space for the modular representation, and it is a dual space with explicit Born reciprocity. In Chapter 3, we construct a path integral for the harmonic oscillator in the modular space, which sheds light on the dynamics on this space. The result contains a sum over winding numbers of the path accompanied by an Aharonov-Bohm-type phase factor between the trajectories. This path integral also gives rise to a new modular action in the modular space, different from the standard action obtained by a Legendre transform, which is studied in the next two chapters.

In Chapter 4, we analyze the modular action for its stationary solutions and symmetries on this double-dimensional space. The results are compared and contrasted with the Hamiltonian formalism for the Schrödinger action. We extend this correspondence in Chapter 5, where we propose a modular Legendre transform that converts a generic Hamiltonian function to a modular Lagrangian. This proposal is then demonstrated on the Kepler problem, giving an alternative formulation of Newtonian gravity.

We return to the Weyl algebra in Chapter 6 to discuss its irregular representations and introduce a new set of inequivalent representations that we called modular polymer (MP) representations, which are inspired by the spin-network representation in loop quantum gravity. We analyze the dynamics of the harmonic oscillator in an MP representation in Chapter 7. The ratio between the modular scale and the polymerization scale is of particular interest to us, because we find that the spectrum of solutions depends on this ratio.

The proofs for most of the features of modular and MP representations are omitted in the main text to enhance its readability, but assembled together as endnotes in Appendix A. The reader can find hyperlinks of the form $^{\{n\}}$ in the text, which link to the relevant endnotes.

Finally, Appendix B contains a brief review of the Jacobi theta function, which appears in the construction of the modular path integral.

Chapter 2

Representations of the Weyl algebra

We base our discussion on the simple mechanical system of a non-relativistic particle living on the d-dimensional Euclidean space $\mathbb{R}^d$. In the case of classical mechanics, the state of this particle is described by a position vector $q \in \mathbb{R}^d$ and a momentum vector $p \in \mathbb{R}^d$, which together form a vector $(q, p) \in \mathcal{P} = \mathbb{R}^{2d}$ in the phase space. These variables satisfy the Poisson bracket $\{q^a, p_b\} = \delta^a_b$. In the case of quantum mechanics by canonical quantization, we promote the position and momentum variables to quantum operators $\hat{q}^a$ and $\hat{p}_a$, which satisfy Heisenberg's commutation relation $[\hat{q}^a, \hat{p}_b] = i\hbar\, \delta^a_b\, \hat{\mathbb{1}}$.

In differential geometry, the phase space is constructed as the cotangent bundle over the space manifold. However, we assume here a flat geometry, and consider the position and momentum variables on equal footing. That is, the position and momentum spaces are considered as two abstract vector spaces that are dual to each other. The vectors in one space are identified as covectors in the other, and vice versa.

This simplistic picture is the basis for *Born reciprocity* [11]. Born reciprocity is the observation that the position and momentum operators in quantum mechanics are on an equal footing, in the sense that a complete description of the system can be given in terms of either set of variables. Heisenberg's commutation relation is invariant under any symplectomorphism, therefore there is no reason to treat the position variables as more fundamental than the momentum variables as far as the kinematical part of a quantum theory is concerned.

This simple observation about quantum mechanics is obscured when we treat the Schrödinger representation as the bridge for its correspondence to a classical picture. Although the classical configuration space is directly linked to the Schrödinger representation as the common eigenspace of the chosen commutative subalgebra, which appears intrinsically as arbitrary, our classical notions of locality and propagation are meaningful only on this particular choice of space, which seems to be in conflict with the Born reciprocity. This dilemma persists even when the Hamiltonian is symmetric in position and momentum variables, such as the case of a harmonic oscillator, therefore it cannot be attributed simply to the system's dynamics.

Instead, we will explore here another class of representations that do not discard the position or momentum variables in favor of the other, but keep the Born reciprocity as manifest. Although this might be counter-intuitive, keeping both position and momentum variables is possible despite the uncertainty principle, and the price is a compact configuration space. We will compare and contrast the alternative classical picture that arises from this approach.

2.1 Double space notation

As we will develop representations that are covariant in the phase space $\mathcal{P} = \mathbb{R}^{2d}$ and treat the position and the momentum on an equal footing, it will be a lot simpler to use phase space vectors rather than continuing to write the position and the momentum separately. For this reason, we are now going to introduce a "double-space notation", which is adapted from [27] and also parallels the notation in the literature of double field theory [3], and we will use it for the most of this work.

In our double-space notation, the double-stroke, capital letters such as $\mathbb{X}$ denote a pair of position and momentum variables, $\mathbb{X} = (x, \tilde{x}) \in \mathbb{R}^{2d}$, which are represented by the corresponding lowercase letters without and with a tilde $\sim$, respectively. The individual components of these parameters are written as $\mathbb{X}^A$ with a capital letter index $A = 1, ..., 2d$, or as x^a and $\tilde{x}_a$ with a lowercase letter index $a = 1, ..., d$, respectively.

Since the momentum space is dual to the position space, any momentum vector also serves as a covector to the position vectors. Therefore, a phase space vector consists of a position vector and a position covector (i.e. a momentum vector), which we write as

$$\{X^A\}_{A=1,...,2d} = (\{x^a\}_{a=1,...,d}, \{\tilde{x}_a\}_{a=1,...,d})$$

with superscript and subscript indices, respectively. We will often omit the curly brackets and write this simply as $\mathbb{X}^A = (x^a, \tilde{x}_a) \in \mathcal{P}$.

We will later also introduce some phase space covectors, where the index placements will be reversed. For example, $\mathbb{P}_A = (p_a, \tilde{p}^a) \in \mathcal{P}^*$ is a phase space covector, which consists of a position covector (i.e. momentum vector) and a momentum covector.

We use the Einstein summation convention for both the capital and the lowercase indices. We also use the dot $\cdot$ as a short-hand for the product of a vector and a dual vector. For example, $\mathbb{X} \cdot \mathbb{P} \equiv \mathbb{X}^A\, \mathbb{P}_A$ and $x \cdot \tilde{x} \equiv x^a\, \tilde{x}_a$.

The Heisenberg operators can be combined into $\hat{\mathbb{Q}} = (\hat{q}, \hat{\tilde{q}})$, where $\hat{\tilde{q}} \equiv \hat{p}$. As an exception to our general rule, we will continue using $\hat{p}$ for the momentum operator instead of $\hat{\tilde{q}}$, since the latter is cluttered and uncommon in the literature.

2.2 Geometric structures on the phase space

The phase space $\mathcal{P}$ is more than a linear vector space. It is endowed with a symplectic structure that comes from the Poisson bracket, and we can also introduce metric structures on it to accommodate various geometric relations.

Symplectic form

The symplectic 2-form ω on $\mathcal{P}$ can be written as

$$\begin{aligned} \omega &= \frac{1}{2}\, \omega_{AB}\, \mathrm{d}\mathbb{X}^A \wedge \mathrm{d}\mathbb{X}^B \\ &= \mathrm{d}\tilde{x}_a \wedge \mathrm{d}x^a \ , \end{aligned} \tag{2.1}$$

where

$$\omega_{AB} = \begin{pmatrix} 0 & -\delta_a{}^b \\ \delta^a{}_b & 0 \end{pmatrix} \tag{2.2}$$

in the Darboux coordinates. We can also use the the symplectic form as a bilinear map $\omega : \mathcal{P} \otimes \mathcal{P} \to \mathbb{R}$ with

$$\omega(\mathbb{X}, \mathbb{Y}) = \mathbb{X}^A\, \omega_{AB}\, \mathbb{Y}^B = \tilde{x} \cdot y - x \cdot \tilde{y} \ , \tag{2.3}$$

or as a linear isomorphism $\omega : \mathcal{P} \to \mathcal{P}^*$ with

$$(\omega \mathbb{X})_A = \omega_{AB} \mathbb{X}^B \ . \tag{2.4}$$

We will use the same symbol ω for each of these functionalities, and how we use the symplectic structure will generally be clear from the context. From the latter perspective, the symplectic map has an inverse $\omega^{-1} : \mathcal{P}^* \to \mathcal{P}$ with $(\omega^{-1}\mathbb{P})^A = (\omega^{-1})^{AB}\, \mathbb{P}_B$, where

$$\left(\omega^{-1}\right)^{AB} = \begin{pmatrix} 0 & \delta^a{}_b \\ -\delta_a{}^b & 0 \end{pmatrix} . \tag{2.5}$$

This inverse symplectic map is particularly useful for writing the canonical commutation relations in the double-space notation,

$$\left[\hat{\mathbb{Q}}^A, \hat{\mathbb{Q}}^B\right] = i\hbar \left(\omega^{-1}\right)^{AB} \hat{\mathbb{1}} \,, \tag{2.6}$$

and for writing Hamilton's equations of motion,

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \mathbb{X}(t) = \left(\omega^{-1}\right)^{AB} \frac{\partial \mathcal{H}_{\mathrm{cl}}(\mathbb{X}(t))}{\partial \mathbb{X}^B(t)} \,, \tag{2.7}$$

for some Hamiltonian function $\mathcal{H}_{\mathrm{cl}}$ on the phase space $\mathcal{P}$.

Polarization metric

Another useful structure on $\mathcal{P}$ is an $O(d,d)$ metric η with split signature that we call the *polarization metric*. In a chosen set of (Darboux) coordinates, we introduce this metric with the components

$$\eta_{AB} = \begin{pmatrix} 0 & \delta_a{}^b \\ \delta^a{}_b & 0 \end{pmatrix} . \tag{2.8}$$

The main functionality of this metric for our purposes is that, together with ω, it creates a *bilagrangian* or *para-Hermitian structure* on $\mathcal{P}$ [28]. In other words, the pair (ω, η) dictates a unique splitting of the phase space $\mathcal{P}$ into position and momentum subspaces.

First of all, we can combine ω and η to define the linear map $K \equiv \omega^{-1} \circ \eta : \mathcal{P} \to \mathcal{P}$, which is an involution, $K^2 = \mathrm{id}_{\mathcal{P}}$. It has the components

$$K^A{}_B = \begin{pmatrix} \delta^a{}_b & 0 \\ 0 & -\delta_a{}^b \end{pmatrix} . \tag{2.9}$$

We will always assume that ω and η are compatible such that K can be brought to this form. Using K we can define the projectors $P \equiv \frac{1}{2}(\mathbb{1} + K)$ and $\tilde{P} \equiv \frac{1}{2}(\mathbb{1} - K)$ on $\mathcal{P}$, which satisfy $P^2 = P$, $\tilde{P}^2 = \tilde{P}$, and $P\tilde{P} = 0$. The images of these projectors, $L \equiv P(\mathcal{P})$ and $\tilde{L} \equiv \tilde{P}(\mathcal{P})$, can be identified as the position and momentum subspaces of $\mathcal{P}$, respectively. Hence, we can write the phase space as the direct sum $\mathcal{P} = L \oplus \tilde{L}$.

Introducing η for the splitting of the phase space may seem counter-intuitive in regard to our motivation for treating position and momentum variables on the same level. However, the metric η will turn out to be essential for the modular representations in Section 2.6 as it is required the consistency of the translation operations. Therefore, formalizing any possible splitting of the phase space by a corresponding $O(d,d)$ metric turns out to be a more feasible strategy when we cannot avoid the splitting altogether.

It is worth mentioning that the same metric η also appears in closed string theory with toroidal compactifications, where the T-duality can be generalized to an $O(d,d,\mathbb{Z})$ symmetry [3]. Though this motivation for η is unrelated to the scope of our work, there are some attempts in the literature to establish a connection between these different approaches [23, 28].

The symmetry transformations on $\mathcal{P}$ that preserve both the symplectic form ω and the polarization metric η make up the general linear group $Sp(2d) \cap O(d,d) = GL(d)$.

Positive-definite metric

A final structure on $\mathcal{P}$ is a positive-definite metric G with the symmetry group $O(2d)$. In a set of coordinates where ω and η take the forms (2.2) and (2.8) respectively, we can choose G with the components

$$G_{AB} = \begin{pmatrix} \gamma g_{ab} & 0 \\ 0 & \gamma^{-1} g^{ab} \end{pmatrix}, \tag{2.10}$$

where g_{ab} are the components of the Euclidean metric on the position space, g^{ab} are the components of its inverse, and γ is a constant. If x and $\tilde{x}$ are dimensionful with units of length and momentum, and the metric g is dimensionless, then γ should have the unit of mass/time for the metric G to be consistent. Hence, the square-distance $G(\mathbb{X}-\mathbb{Y}, \mathbb{X}-\mathbb{Y})$ between two points in the phase space $\mathcal{P}$ has the units of $\hbar$.

This positive-definite metric has the following properties:

- Unit determinant: $\det G = 1$
- Compatibility with ω : $\left(G^{-1} \circ \omega\right)^2 = -\mathbb{1}$
- Compatibility with η : $\left(G^{-1} \circ \eta\right)^2 = \mathbb{1}$

The first property is a convenient normalization. We will use the second property explicitly in our calculations in Section 3.3, though our results can also be generalized to the case when this property does not hold. We will not use the third property in this work, though it is worth mentioning that this is an essential property of the generalized metric in the context of double field theory [36]. The phase space $\mathcal{P}$ equipped with all three structures (ω, η, G) satisfying these properties is an example of Born geometry [51].

This metric preserves the Born reciprocity between the position and momentum variables, such that $x^a \mapsto \gamma^{-1} g^{ab} \tilde{x}_b$, $\tilde{x}_a \mapsto -\gamma g_{ab} x^b$, and $\eta \mapsto -\eta$. Therefore, it is useful for writing a quadratic Hamiltonian function that holds position and momentum variables on equal footing. For example, the Hamiltonian function for a classical harmonic oscillator with mass m and frequency Ω can be written in terms of G with $\gamma = m\Omega$ as

$$\begin{aligned}\mathcal{H}_{\text{h.o.}}(\mathbb{X}) &= \frac{1}{2}\,\Omega\, G(\mathbb{X},\mathbb{X}) \\ &= \frac{1}{2m}\, g^{-1}(\tilde{x},\tilde{x}) + \frac{1}{2}\, m\Omega^2\, g(x,x)\ .\end{aligned} \tag{2.11}$$

We will use this quadratic Hamiltonian for the dynamical system that we consider in the next chapters.

The symmetry transformations on $\mathcal{P}$ that preserve both the symplectic form ω and the positive-definite metric G make up the unitary group $Sp(2d) \cap O(2d) = U(d)$. Similarly, the transformations on $\mathcal{P}$ that preserve both the polarization metric and the positive-definite metric are in the symmetry group $O(d,d) \cap O(2d) = O(d) \times O(d)$. Finally, the symmetry transformations that preserve all three structures are the rotations $O(d) = Sp(2d) \cap O(d,d) \cap O(2d)$ [27].

2.3 Weyl algebra

Now that we introduced the geometry of the phase space, it is time to discuss the quantum algebra and its representations. As mentioned, the position and momentum operators, to which we will refer collectively as the *Heisenberg operators*, satisfy the canonical commutation relations (2.6).

Since the Heisenberg operators are unbounded, it is advantageous to consider their exponentiated versions. Given any $\mathbb{X} = (x, \tilde{x}) \in \mathcal{P}$, we define the *Weyl operator* $\hat{W}_{\mathbb{X}}$ by

$$\hat{W}_{\mathbb{X}} \equiv e^{i\omega(\mathbb{X},\hat{\mathbb{Q}})/\hbar} = e^{i(\tilde{x}\cdot\hat{q} - x\cdot\hat{p})/\hbar}\ . \tag{2.12}$$

The Weyl operators generate the *Weyl algebra* $\mathcal{W}$ (sometimes also referred to as the Heisenberg–Weyl algebra) where the involution and the product are given by the relations{1}

$$\hat{W}_{\mathbb{X}} = \hat{W}^{\dagger}_{-\mathbb{X}}\ , \tag{2.13}$$

$$\hat{W}_{\mathbb{X}}\,\hat{W}_{\mathbb{Y}} = e^{\frac{i}{2}\omega(\mathbb{X},\mathbb{Y})/\hbar}\,\hat{W}_{\mathbb{X}+\mathbb{Y}}\ . \tag{2.14}$$

As such, the Weyl algebra is a non-commutative C*-algebra.

We may consider these relations as *defining* the Weyl algebra of some abstract operators $\hat{W}_{\mathbb{X}}$, and thus forget about their origin in the Heisenberg operators. This will be important in Chapter 6 when we discuss certain representations in which the Heisenberg operators do not exist.

We will now discuss the representations of the Weyl algebra $\mathcal{W}$.

2.4 Stone-von Neumann theorem

The Stone–von Neumann uniqueness theorem [53, 57] (see [49, 55] for more recent and pedagogical treatments) states that any irreducible representation of $\mathcal{W}$ which is weakly continuous[1] in the argument $\mathbb{X}$ is unitarily equivalent to the Schrödinger representation, which we will review in the next section. We call such representations *regular*, while those that are inequivalent to the Schrödinger representation are called *irregular*.

The choice of a representation for the Weyl algebra $\mathcal{W}$ is intrinsically tied to a choice of an underlying configuration space and its topology. In order to construct a representation of the Weyl algebra $\mathcal{W}$, one generally selects a commutative subalgebra of $\mathcal{W}$ that becomes diagonalized in this representation. Once a commutative C*-subalgebra is chosen, the Gelfand-Naimark theorem [31] provides an associated topological space, such that the subalgebra is isometrically *-isomorphic to an algebra of complex functions on this space. We view this space provided by the Gelfand-Naimark theorem as the *(quantum) configuration space* for the chosen representation of the Weyl algebra.

Hence, there is potential value in investigating not only the inequivalent representations, but also the regular alternatives of the Schrödinger representation for the geometric properties of their corresponding configuration space. One special set of regular representations, called *modular representations*, displays a rich geometry as it inherits all the structures that we introduced in Section 2.2 from the phase space. These modular representations will be the main topic of this chapter. In Chapter 6, we will revisit the representations of the Weyl algebra with an irregular alternative.

2.5 Schrödinger representation

Before we introduce the modular representations, it is helpful to review the well-known Schrödinger and momentum representation, since comparing the modular representations

[1] Irreducibility and weak continuity imply that the Hilbert space of the representation must be separable.

to them will be one of our main objectives. We are going to use Dirac's bracket notation for the regular representations.

The Schrödinger representation of the Weyl algebra $\mathcal{W}$ is based on a commutative subalgebra of $\mathcal{W}$ that is spanned by the elements $\{\hat{W}_{(0,\tilde{y})} \,|\, \tilde{y} \in \mathbb{R}^d\}$. In other words, the position operators $\hat{q}^a$ and their exponentials are diagonalized in this representation. Their common eigenvectors are denoted by $|x\rangle_{\text{Sch}}$, $x \in \mathbb{R}^d$, and they satisfy $\hat{q}^a \,|x\rangle_{\text{Sch}} = x^a \,|x\rangle_{\text{Sch}}$ and $\hat{W}_{(0,\tilde{y})} \,|x\rangle_{\text{Sch}} = e^{i\tilde{y}\cdot x/\hbar} \,|x\rangle_{\text{Sch}}$.

Using the position eigenvectors as the basis, a general quantum state can be written in the Schrödinger representation as

$$|\psi\rangle = \int_{\mathbb{R}^d} \mathrm{d}^d x \; \psi(x) \,|x\rangle_{\text{Sch}} \;, \tag{2.15}$$

where $\psi \in \mathcal{H}_{\text{S}} \equiv L^2(\mathbb{R}^d, \mathrm{d}^d x)$ is a Schrödinger wave function, and $\mathcal{H}_{\text{S}}$ is the Schrödinger Hilbert space. The momentum operators act on a Schrödinger wave function as

$$\hat{p}_a \,|\psi\rangle = \int_{\mathbb{R}^d} \mathrm{d}^d x \left(-i\hbar \,\frac{\partial}{\partial x^a}\, \psi(x) \right) |x\rangle_{\text{Sch}} \;, \tag{2.16}$$

which is often written shortly as[2] $\hat{p}_a \psi(x) \simeq -i\hbar \frac{\partial}{\partial x^a} \psi(x)$. The Weyl operators act unitarily in the Schrödinger Hilbert space $\mathcal{H}_{\text{S}}$, such that

$$\hat{W}_{\mathbb{Y}} \,|\psi\rangle = \int_{\mathbb{R}^d} \mathrm{d}^d x \left(e^{-\frac{i}{2} y\cdot\tilde{y}/\hbar} \, e^{i\tilde{y}\cdot x/\hbar} \, \psi(x-y) \right) |x\rangle_{\text{Sch}} \;. \tag{2.17}$$

In fact, we could reverse the logic here and define the Schrödinger representation by (2.17) referring only to the Weyl algebra. We would then argue that the map $\mathbb{Y} \mapsto \hat{W}_{\mathbb{Y}}$ is weakly continuous and define the Heisenberg operators from the Weyl operators. This prescription works for the regular representations in this chapter, but not for the irregular representations that we will discuss in Chapter 6.

The momentum representation is similarly constructed from the commutative subalgebra of $\mathcal{W}$ spanned by $\{\hat{W}_{(y,0)} \,|\, y \in \mathbb{R}^d\}$. The momentum eigenvectors $|\tilde{x}\rangle_{\text{mom}}$, $\tilde{x} \in \mathbb{R}^d$, satisfy $\hat{p}_a \,|\tilde{x}\rangle_{\text{mom}} = \tilde{x}_a \,|\tilde{x}\rangle_{\text{mom}}$, and they are related to the position eigenvectors by a Fourier transform,

$${}_{\text{Sch}}\langle x|\tilde{x}\rangle_{\text{mom}} = (2\pi\hbar)^{-d/2} \, e^{ix\cdot\tilde{x}/\hbar} \;. \tag{2.18}$$

[2] Note that this expression does not use the Dirac notation, but the functions alone stand for the quantum states. We use the symbol $\simeq$ when we switch to this alternative notation.

2.6 Modular representations

After having considered two standard examples, we may now look for more generic commutative subalgebras of the Weyl algebra $\mathcal{W}$. The commutator of two Weyl operators can be written as

$$\left[\hat{W}_{\mathbb{X}}, \hat{W}_{\mathbb{Y}}\right] = \left(e^{i\omega(\mathbb{X},\mathbb{Y})/\hbar} - 1\right) e^{-\frac{i}{2}\omega(\mathbb{X},\mathbb{Y})/\hbar}\, \hat{W}_{\mathbb{X}+\mathbb{Y}} \; . \tag{2.19}$$

This implies that

$$\left[\hat{W}_{\mathbb{X}}, \hat{W}_{\mathbb{Y}}\right] = 0 \qquad \Leftrightarrow \qquad \frac{1}{2\pi\hbar}\, \omega(\mathbb{X},\mathbb{Y}) \in \mathbb{Z} \; . \tag{2.20}$$

Since this relation is bilinear, the arguments $\mathbb{X} \in \mathcal{P}$ of the Weyl operators in a generic commutative subalgebra of $\mathcal{W}$ are supported on a lattice in the phase space.

A lattice $\Lambda \subset \mathcal{P}$ is called a *symplectic lattice* if it is a maximal subset that satisfies

$$\omega(\Lambda, \Lambda) = 2\pi\hbar\, \mathbb{Z} \; . \tag{2.21}$$

Our discussion above shows that the Weyl operators $\hat{W}_{\mathbb{K}}$ and $\hat{W}_{\mathbb{K}'}$ commute if $\mathbb{K}, \mathbb{K}' \in \Lambda$ are elements of the same symplectic lattice.

For reasons that will soon become clear, we also need the symplectic lattice Λ to be compatible with the polarization metric η. We call $\Lambda \subset \mathcal{P}$ a *modular lattice* (with respect to ω and η) if it is a symplectic lattice that satisfies

$$(\eta + \omega)(\Lambda, \Lambda) = 4\pi\hbar\, \mathbb{Z} \; . \tag{2.22}$$

This condition is equivalent to $\Lambda = P(\Lambda) \oplus \tilde{P}(\Lambda)$ for the projectors $P, \tilde{P}$ defined in Section 2.2. Hence, if we choose a set of coordinates on $\mathcal{P}$ such that ω and η take their canonical forms (2.2) and (2.8), we can write any modular lattice as $\Lambda = \bar{\Lambda}\mathbb{Z}^{2d} \subset \mathcal{P}$ with

$$\bar{\Lambda}^A{}_B = \begin{pmatrix} \lambda^a{}_b & 0 \\ 0 & \tilde{\lambda}_a{}^b \end{pmatrix} , \tag{2.23}$$

where λ and $\tilde{\lambda}$ are two $d \times d$-matrices that satisfy $\lambda^c{}_a \tilde{\lambda}_c{}^b = 2\pi\hbar\, \delta_a^b$. Furthermore, we can make a $GL(d)$ coordinate transformation on $\mathcal{P}$ to bring these matrices to their simplest form $\lambda = \ell \mathbb{1}_{(d)}$ and $\tilde{\lambda} = \tilde{\ell} \mathbb{1}_{(d)}$, where ℓ and $\tilde{\ell}$ are some position and momentum scales with $\ell\tilde{\ell} = 2\pi\hbar$.

Recall that the Schrödinger representation is generated by the commutative subalgebra $\{\hat{W}_{\mathbb{Y}} \,|\, \mathbb{Y} \in \tilde{L}\}$ of the Weyl operators on the momentum space $\tilde{L} = \mathbb{R}^d$. The momentum space satisfies $\omega(\tilde{L}, \tilde{L}) = \{0\} \subset 2\pi\hbar\,\mathbb{Z}$, therefore it is consistent with the above discussion. Although $\tilde{L}$ is not a lattice, it corresponds to a certain limit of modular lattices, which we will discuss in Section 2.8.

Given a modular lattice Λ, we define the commutative *-subalgebra $\mathcal{W}_\Lambda = \{\hat{W}_{\mathbb{K}} \,|\, \mathbb{K} \in \Lambda\} \subset \mathcal{W}$. The common eigenvectors of $\mathcal{W}_\Lambda$ are called the *modular vectors*. A modular vector $|\mathbb{X}\rangle_\Lambda$, $\mathbb{X} \in \mathcal{P}$, can be expressed in terms of Schrödinger's position eigenvectors through a Zak transform [61], such that[{2}]

$$|\mathbb{X}\rangle_\Lambda \equiv \left(\det \tilde{\lambda}\right)^{-1/2} e^{\frac{i}{2} x\cdot\tilde{x}/\hbar} \sum_{n\in\mathbb{Z}^d} e^{i\tilde{x}\cdot\lambda n/\hbar} \, |x + \lambda n\rangle_{\mathrm{Sch}} \ . \tag{2.24}$$

Equivalently, they can be expressed through a Fourier transform in terms of momentum eigenvectors as[{3}]

$$|\mathbb{X}\rangle_\Lambda = (\det \lambda)^{-1/2} e^{-\frac{i}{2} x\cdot\tilde{x}/\hbar} \sum_{\tilde{n}\in\mathbb{Z}^d} e^{-ix\cdot\tilde{\lambda}\tilde{n}/\hbar} \, |\tilde{x} + \tilde{\lambda}\tilde{n}\rangle_{\mathrm{mom}} \ . \tag{2.25}$$

These modular vectors satisfy the eigenvalue equation

$$\hat{W}_{\mathbb{K}} \, |\mathbb{X}\rangle_\Lambda = e^{\frac{i}{2} k\cdot\tilde{k}/\hbar} \, e^{i\omega(\mathbb{K},\mathbb{X})/\hbar} \, |\mathbb{X}\rangle_\Lambda \ , \qquad \mathbb{K} \in \Lambda \ , \ \mathbb{X} \in \mathcal{P} \ . \tag{2.26}$$

The action of a generic Weyl operator on a modular vector is given by[{4}]

$$\hat{W}_{\mathbb{Y}} \, |\mathbb{X}\rangle_\Lambda = e^{\frac{i}{2}\omega(\mathbb{Y},\mathbb{X})/\hbar} \, |\mathbb{X} + \mathbb{Y}\rangle_\Lambda \ , \qquad \mathbb{X}, \mathbb{Y} \in \mathcal{P} \ . \tag{2.27}$$

Moreover, the modular vectors are quasi-periodic under discrete translations along the modular lattice, such that[{5}]

$$|\mathbb{X} + \mathbb{K}\rangle_\Lambda = e^{\frac{i}{2} k\cdot\tilde{k}/\hbar} \, e^{\frac{i}{2}\omega(\mathbb{K},\mathbb{X})/\hbar} \, |\mathbb{X}\rangle_\Lambda \ , \qquad \mathbb{K} \in \Lambda \ , \ \mathbb{X} \in \mathcal{P} \ . \tag{2.28}$$

Note that we use here and always the canonical coordinates, in which ω and η take the forms (2.2) and (2.8) respectively. In general, $x = P\mathbb{X}$ and $\tilde{x} = \tilde{P}\mathbb{X}$ are the projections of $\mathbb{X}$ with respect to the bilagrangian structure defined by (ω, η). Therefore, the quasi-periodicity relation (2.28) reads in arbitrary coordinates as

$$|\mathbb{X} + \mathbb{K}\rangle_\Lambda = \exp\left(\frac{i}{4\hbar}\,\eta(\mathbb{K},\mathbb{K}) + \frac{i}{2\hbar}\,\omega(\mathbb{K},\mathbb{X})\right) \, |\mathbb{X}\rangle_\Lambda \ . \tag{2.29}$$

The fact that the phase is not linear in $\mathbb{K}$ puts constraints on the modular lattice Λ. For $\mathbb{K}, \mathbb{K}' \in \Lambda$, we can use this relation on $|\mathbb{X}+\mathbb{K}+\mathbb{K}'\rangle_\Lambda$ in multiple different ways, which requires for consistency

$$e^{\frac{i}{4\hbar}\eta(\mathbb{K}',\mathbb{K}')+\frac{i}{2\hbar}\omega(\mathbb{K}',\mathbb{X}+\mathbb{K})}\, e^{\frac{i}{4\hbar}\eta(\mathbb{K},\mathbb{K})+\frac{i}{2\hbar}\omega(\mathbb{K},\mathbb{X})} = e^{\frac{i}{4\hbar}\eta(\mathbb{K}+\mathbb{K}',\mathbb{K}+\mathbb{K}')+\frac{i}{2\hbar}\omega(\mathbb{K}+\mathbb{K}',\mathbb{X})} \,, \tag{2.30}$$

or equivalently

$$\eta(\mathbb{K},\mathbb{K}') + \omega(\mathbb{K},\mathbb{K}') \in 4\pi\hbar\,\mathbb{Z} \,. \tag{2.31}$$

This is the reason why we had to impose the condition (2.22) on the modular lattice Λ.

In the following, we will usually drop the subscript Λ on the modular vectors for better readability.

The quasi-periodicity implies that not all modular vectors are linearly independent. In order to construct a basis from the modular vectors, we consider the quotient $T_\Lambda \equiv \mathcal{P}/\Lambda$, which is called a *modular space*. Each element[3] $\mathbb{X} \in T_\Lambda$ of the modular space is an equivalence class of the points $(\mathbb{X}+\Lambda) \subset \mathcal{P}$, which can be represented by any of those points. The modular space is topologically a torus in $2d$ dimensions and it has the volume $(2\pi\hbar)^d$. It is the associated Gelfand-Naimark space for a modular representation, which we will regard as a quantum configuration space.

We define a *modular cell* $M_\Lambda \subset \mathcal{P}$ as any set of representatives of the modular space T_Λ. Then, the vectors $\{|\mathbb{X}\rangle_\Lambda \,|\, \mathbb{X} \in M_\Lambda\}$ form a complete and orthonormal basis of the Hilbert space. The orthogonality relation reads

$$\langle\mathbb{X}|\mathbb{Y}\rangle = \delta^{2d}(\mathbb{X}-\mathbb{Y}) \,, \qquad \mathbb{X},\mathbb{Y} \in M_\Lambda \,, \tag{2.32}$$

where δ^{2d} denotes the $2d$-dimensional Dirac delta distribution. While the relation (2.32) gives the inner product of two modular vectors from the same modular cell, the inner product of two generic modular vectors is given by

$$\langle\mathbb{X}|\mathbb{Y}\rangle = \sum_{\mathbb{K}\in\Lambda} e^{\frac{i}{2}k\cdot\tilde{k}/\hbar}\, e^{\frac{i}{2}\omega(\mathbb{K},\mathbb{X})/\hbar}\, \delta^{2d}(\mathbb{X}-\mathbb{Y}+\mathbb{K}) \,, \qquad \mathbb{X},\mathbb{Y} \in \mathcal{P} \,. \tag{2.33}$$

The completeness relation for the modular vectors reads{6}

$$\mathbb{1} = \int_{T_\Lambda} \mathrm{d}^{2d}\mathbb{X}\; |\mathbb{X}\rangle\langle\mathbb{X}| \;. \tag{2.34}$$

[3]We abuse the notation by using the same symbol $\mathbb{X}$ both for the elements of $\mathcal{P}$ as well as for the corresponding equivalence classes on T_Λ.

Note that writing the integration in (2.34) over the modular space T_Λ employs the fact that $|\mathbb{X}\rangle\langle\mathbb{X}|$ is periodic on T_Λ and therefore independent of the choice of the modular cell.

We can write a general quantum state in the modular basis as

$$|\phi\rangle = \int_{T_\Lambda} \mathrm{d}^{2d}\mathbb{X}\ \phi(\mathbb{X})\, |\mathbb{X}\rangle_\Lambda\ , \tag{2.35}$$

where $\phi(\mathbb{X})$ is called a *modular wave function*[4]. This integral is well-defined only when the integrand $\phi(\mathbb{X})\, |\mathbb{X}\rangle_\Lambda$ is periodic on T_Λ. Therefore, we require the modular wave functions to be also quasi-periodic, such that

$$\phi(\mathbb{X}+\mathbb{K}) = e^{-\frac{i}{2}k\cdot\tilde{k}/\hbar}\, e^{-\frac{i}{2}\omega(\mathbb{K},\mathbb{X})/\hbar}\, \phi(\mathbb{X})\ , \qquad \mathbb{K}\in\Lambda\ ,\ \mathbb{X}\in\mathcal{P}\ . \tag{2.36}$$

In order to reformulate this statement in a more abstract way as in [27], one may define a $U(1)$-bundle $E_\Lambda \to T_\Lambda$ over the modular space together with the identification

$$E_\Lambda:\quad (\theta,\mathbb{X}) \sim \Big(\theta\, e^{\frac{i}{2}k\cdot\tilde{k}/\hbar}\, e^{\frac{i}{2}\omega(\mathbb{K},\mathbb{X})/\hbar}, \mathbb{X}+\mathbb{K}\Big)\ ,\quad \mathbb{K}\in\Lambda\ ,\ \mathbb{X}\in\mathcal{P}\ ,\ \theta\in U(1)\ . \tag{2.37}$$

Then, the modular wave functions $\phi\in\mathcal{H}_\Lambda = L^2(E_\Lambda)$ correspond to the square-integrable sections of E_Λ, and they are elements of the modular Hilbert space $\mathcal{H}_\Lambda$.

Finally, we examine the action of Heisenberg operators $\hat{q}^a$ and $\hat{p}_a$ on a quantum state in the modular representation. After some calculation, we find[{7}]

$$\hat{q}^a\,|\phi\rangle = \int_{T_\Lambda} \mathrm{d}^{2d}\mathbb{X}\left(i\hbar\,\frac{\partial}{\partial\tilde{x}_a}\,\phi(\mathbb{X}) + \frac{1}{2}\,x^a\,\phi(\mathbb{X})\right)|\mathbb{X}\rangle_\Lambda\ , \tag{2.38a}$$

$$\hat{p}_a\,|\phi\rangle = \int_{T_\Lambda} \mathrm{d}^{2d}\mathbb{X}\left(-i\hbar\,\frac{\partial}{\partial x^a}\,\phi(\mathbb{X}) + \frac{1}{2}\,\tilde{x}_a\,\phi(\mathbb{X})\right)|\mathbb{X}\rangle_\Lambda\ . \tag{2.38b}$$

These equations can be expressed more compactly in terms of an Abelian connection $\mathbb{A} = \mathbb{A}_A(\mathbb{X})\,\mathrm{d}\mathbb{X}^A$ on E_Λ, given by

$$\mathbb{A}_A(\mathbb{X}) \equiv \left(\frac{1}{2}\,\tilde{x}_a, -\frac{1}{2}\,x^a\right)\ . \tag{2.39}$$

The key property of this *modular connection* $\mathbb{A}$ is that its curvature form coincides with the symplectic 2-form, i.e. $\mathrm{d}\mathbb{A} = \omega$. Using the modular connection, we can define a covariant derivative ∇, which acts on the modular wave functions as

$$\nabla_A\phi(\mathbb{X}) \equiv \partial_A\phi(\mathbb{X}) + \frac{i}{\hbar}\,\mathbb{A}_A(\mathbb{X})\,\phi(\mathbb{X})\ , \tag{2.40}$$

[4]This function can be thought of as mapping $\phi:\mathcal{P}\to\mathbb{C}$ under the restriction (2.36), while a more rigorous definition is given below in terms of E_Λ.

where $\partial_A \equiv \left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial \tilde{x}_a}\right)$. Using $\hat{\mathbb{Q}}^A = (\hat{q}^a, \hat{p}_a)$, we can write the action of the Heisenberg operators in the modular representation as $\hat{\mathbb{Q}}^A \phi(\mathbb{X}) \simeq i\hbar\,(\omega^{-1})^{AB}\,\nabla_B \phi(\mathbb{X})$.

One can check that the actions of the Weyl operators $\hat{W}_{\mathbb{Y}}$ and the Heisenberg operators $\hat{\mathbb{Q}}^A$ on a modular wave function preserve the condition (2.36), therefore these are well-defined operators on the modular Hilbert space $\mathcal{H}_\Lambda = L^2(E_\Lambda)$. [{8}]

2.7 Modular gauge transformation

There is a $U(1)$-gauge freedom in defining the modular vectors, which we will discuss here. For any real, smooth function $\alpha \in C^\infty(\mathcal{P})$ on the phase space, we may redefine the modular vectors as

$$|\mathbb{X}\rangle_\Lambda \to |\mathbb{X}\rangle_\Lambda^\alpha \equiv e^{i\alpha(\mathbb{X})}\,|\mathbb{X}\rangle_\Lambda \ . \tag{2.41}$$

While the eigenvalue equation (2.26) is unaffected by this gauge transformation, the action (2.27) of a generic Weyl operator on a modular vector becomes

$$\hat{W}_{\mathbb{Y}}\,|\mathbb{X}\rangle_\Lambda^\alpha = e^{i\alpha(\mathbb{X})-i\alpha(\mathbb{X}+\mathbb{Y})}\,e^{\frac{i}{2}\omega(\mathbb{Y},\mathbb{X})/\hbar}\,|\mathbb{X}+\mathbb{Y}\rangle_\Lambda^\alpha \ , \qquad \mathbb{X},\mathbb{Y} \in \mathcal{P} \ . \tag{2.42}$$

Similarly, the gauge transformation changes the quasi-periodicity relation (2.28) to

$$|\mathbb{X}+\mathbb{K}\rangle_\Lambda^\alpha = e^{i\beta_\alpha(\mathbb{X},\mathbb{K})}\,|\mathbb{X}\rangle_\Lambda^\alpha \ , \qquad \mathbb{K} \in \Lambda \ , \ \mathbb{X} \in \mathcal{P} \ , \tag{2.43}$$

where

$$\beta_\alpha(\mathbb{X},\mathbb{K}) \equiv \alpha(\mathbb{X}+\mathbb{K}) - \alpha(\mathbb{X}) + \frac{1}{2\hbar}\,k\cdot\tilde{k} + \frac{1}{2\hbar}\,\omega(\mathbb{K},\mathbb{X}) \ . \tag{2.44}$$

Hence, it changes the condition (2.36) accordingly. The $U(1)$-bundle is modified to $E_\Lambda^\alpha \to T_\Lambda$ defined by the identification

$$E_\Lambda^\alpha : \quad (\theta,\mathbb{X}) \sim \left(\theta\,e^{i\beta_\alpha(\mathbb{X},\mathbb{K})},\mathbb{X}+\mathbb{K}\right) \ , \quad \mathbb{K}\in\Lambda \ , \ \mathbb{X}\in\mathcal{P} \ , \ \theta \in U(1) \ . \tag{2.45}$$

The inner product of two modular vectors in an arbitrary gauge is then given by

$${}^{\alpha}_{\Lambda}\langle\mathbb{X}|\mathbb{Y}\rangle_\Lambda^\alpha = \sum_{\mathbb{K}\in\Lambda} e^{i\beta_\alpha(\mathbb{X},\mathbb{K})}\,\delta^{2d}(\mathbb{X}-\mathbb{Y}+\mathbb{K}) \ , \qquad \mathbb{X},\mathbb{Y}\in\mathcal{P} \ . \tag{2.46}$$

The modular connection $\mathbb{A}$ also transforms under this gauge transformation such that

$$\mathbb{A}_A(\mathbb{X}) \to \mathbb{A}_A(\mathbb{X}) + \hbar\, \partial_A \alpha(\mathbb{X}) \ . \tag{2.47}$$

Note that the curvature $\omega = \mathrm{d}\mathbb{A}$ of the modular connection is invariant under the gauge transformations.

For modular vectors $|\mathbb{X}\rangle_\Lambda^\alpha$ in a generic gauge α, we can write the components of the modular connection as

$$\mathbb{A}_A(\mathbb{X}) = \frac{1}{2}\, \mathbb{X}^B\, \omega_{BA} + \hbar\, \partial_A \alpha(\mathbb{X}) \ . \tag{2.48}$$

While the modular vectors defined in the last section had their gauge fixed as $\alpha = 0$, we will consider an arbitrary choice of gauge hereafter, even though we often omit the label α for better readability. We will also find out in the next section that a specific gauge fixing is required to obtain the Schrödinger and momentum representations as singular limits of the modular ones.

2.8 Singular limits of modular representations

Roughly speaking, the Schrödinger and momentum representations correspond to the limits of the set of modular representations when the spacing of the modular lattice goes to infinity or to zero, respectively. In this section, we will discuss the details of this limiting process.

Consider the 1-parameter family of modular lattices $\Lambda = \ell\mathbb{Z}^d \oplus \tilde{\ell}\mathbb{Z}^d$, where ℓ and $\tilde{\ell}$ are length and momentum scales such that $\ell\tilde{\ell} = 2\pi\hbar$. The modular space T_Λ has the size $\ell^d \times \tilde{\ell}^d$. Recall also that the Heisenberg operators are represented in the modular representations by

$$(\hat{q}^a, \hat{p}_a) \sim \left(\frac{1}{2}\, x^a - \hbar\, \frac{\partial \alpha(\mathbb{X})}{\partial \tilde{x}_a} + i\hbar\, \frac{\partial}{\partial \tilde{x}_a} \,,\, \frac{1}{2}\, \tilde{x}_a + \hbar\, \frac{\partial \alpha(\mathbb{X})}{\partial x^a} - i\hbar\, \frac{\partial}{\partial x^a} \right) \ . \tag{2.49}$$

Now, let's consider the limit $\ell \to \infty$. As the position part of the modular space T_Λ grows to infinite size and becomes decompactified, its momentum part shrinks to a point. This has two consequences for the representation of the Heisenberg operators: Firstly, the term $\partial/\partial\tilde{x}_a$ drops, since the wave functions cannot depend non-trivially on momentum. Secondly, the terms $\hbar\, \partial\alpha(\mathbb{X})/\partial\tilde{x}_a$ and $\tilde{x}^a/2 + \hbar\, \partial\alpha(\mathbb{X})/\partial x^a$ must be independent of momentum, otherwise they would become ill-defined in the limit. This implies that α must be of the form $\alpha(\mathbb{X}) = -\frac{1}{2\hbar}\, x \cdot \tilde{x} + f(x)$ for the limit $\ell \to \infty$ to be well-defined. Comparing the representation of the momentum operator to the one in the Schrödinger representation, we find that the gauge choice

$$\alpha_{\rm Sch}(\mathbb{X}) = -\frac{1}{2\hbar}\, x \cdot \tilde{x} + \text{const.} \,, \tag{2.50}$$

is needed to obtain the Schrödinger representation, in which $(\hat{q}^a, \hat{p}_a) \sim (x^a, -i\hbar\, \frac{\partial}{\partial x^a})$. We name (2.50) the *Schrödinger gauge*. One can check with this gauge fixing that (2.42) also resembles the action of Weyl operators on Schrödinger eigenvectors given by $\hat{W}_{\mathbb{Y}}\, |x\rangle_{\rm Sch} = e^{\frac{i}{2} y\cdot\tilde{y}/\hbar}\, e^{ix\cdot\tilde{y}/\hbar}\, |x+y\rangle_{\rm Sch}$.

Our argument for (2.50) is also supported by the quasi-periodicity phase function $\beta_{\alpha_{\rm Sch}}(\mathbb{X}, \mathbb{K}) = -k \cdot \tilde{x}/\hbar$. Note that this is independent of the momentum winding number $\tilde{k}$ as it should be, since the momentum part of the modular space shrinks to a point and any dependency on the momentum winding number would result in an ill-defined phase. A winding number k in the position directions, on the other hand, becomes irrelevant as the configuration space is decompactified.

In the limit $\ell \to \infty$, the modular lattice transitions to the momentum space. This transition can be understood in a coarse-graining approximation to the momentum space, although it is in fact a singular transition from a discrete set in $2d$ dimensions to a continuous set in d dimensions. The continuous momentum space is qualified as a modular lattice by definition, since it is a maximal subset $\Lambda \subseteq \mathcal{P}$ satisfying $\omega(\Lambda, \Lambda) \subset 2\pi\hbar\,\mathbb{Z}$, although in fact $\omega(\Lambda, \Lambda) = \{0\}$. The modular space T_Λ also changes its topology as it becomes the Schrödinger configuration space.

In order to see how the modular vectors behave in the Schrödinger limit, one can expand them in terms of momentum eigenvectors as in (2.25). We find

$$\begin{aligned}
\lim_{\ell\to\infty} \left(\det \tilde{\lambda}\right)^{1/2}\, |\mathbb{X}\rangle^{\alpha_{\rm Sch}}_{\Lambda} &= \lim_{\tilde{\ell}\to 0}\, (2\pi\hbar)^{-d/2} \left(\det \tilde{\lambda}\right) \sum_{\tilde{n}\in\mathbb{Z}^d} e^{-ix\cdot(\tilde{x}+\tilde{\lambda}\tilde{n})/\hbar}\, |\tilde{x}+\tilde{\lambda}\tilde{n}\rangle_{\rm mom} \\
&= (2\pi\hbar)^{-d/2} \int_{\mathbb{R}^d} \mathrm{d}^d\tilde{x}\; e^{-ix\cdot(\tilde{x}+\tilde{\lambda}\tilde{n})/\hbar}\, |\tilde{x}+\tilde{\lambda}\tilde{n}\rangle_{\rm mom} \\
&= |x\rangle_{\rm Sch} \ .
\end{aligned} \tag{2.51}$$

Hence, up to a normalization factor, the modular vectors converge to the position eigenvectors. This concludes our analysis: Although the limit $\ell \to \infty$ is a singular one in which the topology of the (modular) configuration space changes, we have enough evidence to identify the Schrödinger representation with this limit of modular representations.

A similar discussion applies to the momentum representation in the limit $\ell \to 0$. However, this limit requires a different choice of gauge fixing, namely

$$\alpha_{\rm mom}(\mathbb{X}) = +\frac{1}{2\hbar}\, x \cdot \tilde{x} + \text{const.}\,. \tag{2.52}$$

Note that we can write these two gauge choices in terms of the polarization metric η as $\alpha_{\rm mom/Sch}(\mathbb{X}) = \pm\frac{1}{4\hbar}\,\eta(\mathbb{X},\mathbb{X}) + \text{const.}$ This supports the motivation for η as the geometric construct that singles out the position and momentum spaces.

The observation that the Schrödinger and momentum representations require different gauge fixings might be useful for developing quantum theories that break the Born reciprocity. If the modular $U(1)$ gauge symmetry is somehow broken, this may be used as a mechanism to fix the quantum configuration space.

Alternatively, we can speculate that the nature has a fundamental modular scale $(\ell, \tilde{\ell})$ in the far infrared, such as at the cosmological scale, so that it can be treated as $\ell \to \infty$ as far as most laboratory experiments are concerned. This would break the Born reciprocity in favor of the position variables spanning the configuration space, and there could be correction terms as the measurement scale approaches the cosmological scale. Furthermore, identifying the modular scale with the cosmological scale would have important consequences for the Early Universe and can be incorporated into a Bounce model. This speculated scenario is not possible in the simple system we discuss here because of the Stone-von Neumann theorem, however it is conceivable in a similar but different quantum system, perhaps one with a dynamical symplectic structure. We can at least conclude that the modular gauge is an interesting subject for exploration in the quest for quantum gravity.

Chapter 3

Path Integrals

In the previous chapter, we introduced the mathematical details underlying the modular representations of the Weyl algebra and their relationship with the Schrödinger representation. We can finally use these modular representations to construct a path integral and compare this path integral to Feynman's original path integral in the Schrödinger representation. This will be the goal of this section. We focus here on the special example of a quantum harmonic oscillator for its simplicity, since it is possible and simple to evaluate Gaussian integrals analytically.

3.1 Harmonic oscillator

The Hamiltonian function for a classical harmonic oscillator in d dimensions is given by

$$\mathcal{H}_{\rm cl}(x,\tilde{x}) = \frac{1}{2m}\,\tilde{x}^2 + \frac{1}{2}\,m\,\Omega^2\,x^2\ , \tag{3.1}$$

where m is the mass and Ω is the angular frequency of the oscillator. We can write this quadratic function in a more compact way by introducing the unit-determinant metric

$$G_{AB} \equiv \begin{pmatrix} m\Omega\, g_{ab} & 0 \\ 0 & (m\Omega)^{-1}\, g^{ab} \end{pmatrix} \tag{3.2}$$

on the phase space $\mathcal{P} = \mathbb{R}^{2d}$, where g_{ab} is the flat Euclidean metric on the position space $\mathbb{R}^d$, and g^{ab} is its inverse, which is also a metric on the dual momentum space $\mathbb{R}^d$. Then, we can write (3.1) as

$$\mathcal{H}_{\text{cl}}(\mathbb{X}) = \frac{1}{2}\,\Omega\,G(\mathbb{X},\mathbb{X})\ . \tag{3.3}$$

The general solution to Hamilton's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{X}(t) = \left(\omega^{-1}\right)^{AB}\frac{\partial\mathcal{H}_{\text{cl}}(\mathbb{X}(t))}{\partial\mathbb{X}^B(t)} \tag{3.4}$$

for the Hamiltonian (3.3) reads

$$\mathbb{X}_{\text{cl}}(t) = \xi\sin\left(\Omega t\right) - \omega^{-1}G\,\xi\cos\left(\Omega t\right) \tag{3.5}$$

with an arbitrary phase space vector $\xi\in\mathcal{P}$. Here, we used the following property of the matrix G in (3.2):

$$\left(\omega^{-1}\circ G\right)^2 = -\mathrm{id}_{T\mathcal{P}}\ , \tag{3.6}$$

where $\omega^{-1}:T^*\mathcal{P}\to T\mathcal{P}$ and $G:T\mathcal{P}\to T^*\mathcal{P}$ are treated as maps between vector bundles.

The Hamilton operator for the quantum harmonic oscillator corresponding to (3.3) can be written similarly as

$$\hat{H} = \frac{1}{2}\,\Omega\,G(\hat{\mathbb{Q}},\hat{\mathbb{Q}})\ . \tag{3.7}$$

The solutions to the time-independent Schrödinger equation $\hat{H}\,|\psi\rangle = E\,|\psi\rangle$ are well-known. The ground state is given in the Schrödinger representation by

$$|\psi_0\rangle = \int_{\mathbb{R}^d}\mathrm{d}^dx\ \psi_0(x)\,|x\rangle_{\text{Sch}}\ ,\qquad \psi_0(x) = \left(\frac{m\Omega}{\pi\hbar}\right)^{d/4}e^{-\frac{m\Omega}{2\hbar}x^2}\ ,\qquad E_0 = \frac{d}{2}\,\hbar\,\Omega\ . \tag{3.8}$$

This ground state can be expressed in the modular representation using the Zak transform (2.24) as

$$|\psi_0\rangle = \frac{2^{d/4}}{(2\pi\hbar)^{d/2}}\int_{T_\Lambda}\mathrm{d}^{2d}\mathbb{X}\ e^{-i\alpha(\mathbb{X})}\,e^{-\frac{1}{4\hbar}G(\mathbb{X},\mathbb{X})}\,\vartheta\!\left(\frac{i}{2\pi\hbar}\,\bar{\Lambda}^T\left(G-i\omega\right)\mathbb{X}\,,\ \frac{i}{2\pi\hbar}\,\bar{\Lambda}^TG\,\bar{\Lambda}\right)^{1/2}|\mathbb{X}\rangle^{\alpha}_{\Lambda}\ . \tag{3.9}$$

The excited states can be created by the ladder operators $\hat{a}^{a\dagger} = \sqrt{\frac{m\Omega}{2\hbar}}\left(\hat{q}^a - \frac{i}{m\Omega}\,g^{ab}\,\hat{p}_b\right)$ in either representation. The energy spectrum is discrete and it is independent of the representation.

3.2 Schrödinger-Feynman path integral

In this section, we will review some key results from Feynman's path integral in the Schrödinger representation for the Hamiltonian in (3.7). These are well-known in the literature, but they will serve later as a reference when we compare them to our new path integral in the modular representation.

The transition amplitude between two position eigenvectors over a finite time interval $[t_0, t_f] \subset \mathbb{R}$ can be expressed via the path integral

$$ {}_{\text{Sch}}\langle x_f|\, e^{-i(t_f-t_0)\hat{H}/\hbar}\, |x_0\rangle_{\text{Sch}} = \int_{x(t_0)=x_0}^{x(t_f)=x_f} \mathcal{D}x\, \exp\!\left(\frac{i}{\hbar}\, S_{\text{Sch}}[x]\right) \,. \tag{3.10} $$

On the right-hand side, the functional integral runs over all paths from x_0 to x_f on the configuration space $\mathbb{R}^d$. The path measure $\mathcal{D}x$ is defined as

$$ \mathcal{D}x \equiv \lim_{N\to\infty} \left(\left(\frac{-i}{2\pi\hbar}\, \frac{mN}{t_f - t_0} \right)^{d/2} \sqrt{\det g} \right)^{N} \prod_{n=1}^{N-1} \mathrm{d}^d x_n \,. \tag{3.11} $$

The action S_{Sch} is given by

$$ S_{\text{Sch}}[x] = \int_{t_0}^{t_f} \mathrm{d}t\, \mathcal{L}_{\text{Sch}}(x(t), \dot{x}(t)) \,, \tag{3.12a} $$

$$ \mathcal{L}_{\text{Sch}}(x, \dot{x}) = \frac{1}{2}\, m\, g(\dot{x}, \dot{x}) - \frac{1}{2}\, m\, \Omega^2\, g(x, x) \,, \tag{3.12b} $$

where the dot $\dot{}$ over a variable denotes its time derivative. We can make a Legendre transformation on the Lagrangian $\mathcal{L}_{\text{Sch}}$ to recover the classical Hamiltonian function $\mathcal{H}_{\text{Sch}}$, given by

$$ \mathcal{H}_{\text{Sch}}(x, \tilde{x}) = \frac{1}{2m}\, g^{-1}(\tilde{x}, \tilde{x}) + \frac{1}{2}\, m\, \Omega^2\, g(x, x) = \mathcal{H}_{\text{cl}}(x, \tilde{x}) \,, \tag{3.13} $$

where $\tilde{x} = \partial\mathcal{L}_{\text{Sch}}/\partial\dot{x}$. Defining $\mathbb{X}^A \equiv (x^a, \tilde{x}_a)$, we can write Hamilton's equation as

$$ \dot{\mathbb{X}}(t) = \Omega\, \omega^{-1} G\, \mathbb{X}(t) \,. \tag{3.14} $$

We will see later that this equation is slightly different in the modular representation.

3.3 Modular path integral construction

In this section, we will construct, step by step, a path integral formulation for the transition amplitude ${}^{\alpha}_{\Lambda}\langle \mathbb{X}_f |\, e^{-i(t_f-t_0)\hat{H}/\hbar}\, |\mathbb{X}_0\rangle^{\alpha}_{\Lambda}$ between two modular vectors over a finite time interval $[t_0, t_f]$. We assume here that the gauge α is arbitrary, and that the modular lattice is of the form $\Lambda = \lambda\mathbb{Z}^d \oplus \tilde{\lambda}\mathbb{Z}^d$, where λ and $\tilde{\lambda}$ are diagonal $d\times d$-matrices that satisfy $\lambda^c{}_a \tilde{\lambda}_c{}^b = 2\pi\hbar\, \delta^b_a$, as mentioned previously in Section 2.6. We will mostly omit the labels Λ and α on the modular vectors. The Hamiltonian operator is that of a quantum harmonic oscillator given in (3.7). Once we have this path integral, we will be able to discuss the dynamics on the modular space as a configuration space through this example.

3.3.1 Decomposition of paths

Following the idea in Feynman's original derivation [19], we pick a large integer $N \in \mathbb{N}$ and split the interval $[t_0, t_f]$ into N equal pieces $[t_n, t_n + \delta t]$, $n = 0, ..., N-1$, where

$$\delta t \equiv \frac{t_f - t_0}{N} = t_{n+1} - t_n \;, \qquad t_n \equiv t_0 + n\,\delta t \;, \qquad t_N \equiv t_f \;. \tag{3.15}$$

We decompose the unitary evolution operator into a product of N operators, such that $e^{-i(t_f-t_0)\hat{H}/\hbar} = e^{-i\delta t\,\hat{H}/\hbar} \cdots e^{-i\delta t\,\hat{H}/\hbar}$. Next, we insert the resolution of the identity (2.34) before each of these N unitary operators,

$$\begin{aligned}\langle \mathbb{X}_f |\, e^{-i(t_f-t_0)\hat{H}/\hbar}\, |\mathbb{X}_0\rangle &= \langle \mathbb{X}_f | \left(\int_{T_\Lambda} \mathrm{d}^{2d}\mathbb{X}_N\, |\mathbb{X}_N\rangle\langle\mathbb{X}_N| \right) e^{-i\delta t\,\hat{H}/\hbar} \cdots \\ &\qquad \cdots e^{-i\delta t\,\hat{H}/\hbar} \left(\int_{T_\Lambda} \mathrm{d}^{2d}\mathbb{X}_1\, |\mathbb{X}_1\rangle\langle\mathbb{X}_1| \right) e^{-i\delta t\,\hat{H}/\hbar}\, |\mathbb{X}_0\rangle \\ &= \int_{T_\Lambda} \mathrm{d}^{2d}\mathbb{X}_N \cdots \mathrm{d}^{2d}\mathbb{X}_1\, \langle \mathbb{X}_f | \mathbb{X}_N\rangle \prod_{n=0}^{N-1} \langle \mathbb{X}_{n+1} |\, e^{-i\delta t\,\hat{H}/\hbar}\, |\mathbb{X}_n\rangle \;. \end{aligned} \tag{3.16}$$

Each of the integrals in (3.16) are over the modular space T_Λ, which means that they are over arbitrary modular cells in the phase space. We are free to specify their integration domains as any modular cell. Since we are going to identify the variables $\mathbb{X}_n$ later as points on a continuous path in $\mathcal{P}$, we make the choice that each integral over $\mathbb{X}_n$ (for $n = 1, ..., N$) is taken over $M_\Lambda(\mathbb{X}_{n-1}) \subset \mathcal{P}$, which is a box-shaped modular cell centered at the previous point $\mathbb{X}_{n-1}$. Hence, we write

$$\int_{T_\Lambda} \mathrm{d}^{2d}\mathbb{X}_N \cdots \mathrm{d}^{2d}\mathbb{X}_1 = \int_{M_\Lambda(\mathbb{X}_0)} \mathrm{d}^{2d}\mathbb{X}_1 \int_{M_\Lambda(\mathbb{X}_1)} \mathrm{d}^{2d}\mathbb{X}_2 \cdots \int_{M_\Lambda(\mathbb{X}_{N-1})} \mathrm{d}^{2d}\mathbb{X}_N \;. \tag{3.17}$$

We can simplify this expression by changing the integration variables. We define $\delta\mathbb{X}_j$ through

$$\mathbb{X}_n \equiv \mathbb{X}_0 + \sum_{j=0}^{n-1} \delta\mathbb{X}_j \tag{3.18}$$

for $n = 1, ..., N$, and we change the integration variables from $\mathbb{X}_n \in M_\Lambda(\mathbb{X}_{n-1})$ to $\delta\mathbb{X}_{n-1} \in M_\Lambda(0)$, where $M_\Lambda(0) = \lambda\left[-\frac{1}{2},\frac{1}{2}\right)^d \oplus \tilde{\lambda}\left[-\frac{1}{2},\frac{1}{2}\right)^d \subset \mathcal{P}$ is a box-shaped modular cell centered at the origin. Then, we get

$$\begin{aligned}\langle\mathbb{X}_f|\, e^{-i(t_f-t_0)\hat{H}/\hbar}\,|\mathbb{X}_0\rangle = &\int_{M_\Lambda(0)} \mathrm{d}^{2d}\delta\mathbb{X}_0 \cdots \int_{M_\Lambda(0)} \mathrm{d}^{2d}\delta\mathbb{X}_{N-1} \\ &\times \langle\mathbb{X}_f|\mathbb{X}_N\rangle \prod_{n=0}^{N-1} \langle\mathbb{X}_{n+1}|\, e^{-i\,\delta t\,\hat{H}/\hbar}\,|\mathbb{X}_n\rangle\ ,\end{aligned} \tag{3.19}$$

together with the definitions (3.18).

3.3.2 Infinitesimal transition amplitude

We focus on calculating the infinitesimal transition amplitudes $\langle\mathbb{X}_{n+1}|\, e^{-i\,\delta t\,\hat{H}/\hbar}\,|\mathbb{X}_n\rangle$ in (3.19) up to linear order in δt. Using the Lie-Trotter product formula, we can split the unitary evolution operator as

$$e^{-i\,\delta t\,\hat{H}/\hbar} = \exp\left[-\frac{i}{\hbar}\,\delta t\,\frac{1}{2m}\,g^{-1}(\hat{p},\hat{p})\right]\exp\left[-\frac{i}{\hbar}\,\delta t\,\frac{1}{2}\,m\,\Omega^2\,g(\hat{q},\hat{q})\right] + \mathcal{O}(\delta t^2)\ . \tag{3.20}$$

We will also expand the modular vectors in terms of Schrödinger and momentum eigenvectors, respectively. Namely, we have

$$|\mathbb{X}_n\rangle = \left(\det\tilde{\lambda}\right)^{-1/2} e^{i\alpha(\mathbb{X}_n)}\, e^{\frac{1}{2}ix_n\cdot\tilde{x}_n/\hbar} \sum_{k\in\lambda\mathbb{Z}^d} e^{ik\cdot\tilde{x}_n/\hbar}\,|x_n+k\rangle_{\mathrm{Sch}}\ , \tag{3.21a}$$

$$\langle\mathbb{X}_{n+1}| = (\det\lambda)^{-1/2}\ e^{-i\alpha(\mathbb{X}_{n+1})}\, e^{\frac{1}{2}ix_{n+1}\cdot\tilde{x}_{n+1}/\hbar} \sum_{\tilde{k}\in\tilde{\lambda}\mathbb{Z}^d} e^{i\tilde{k}\cdot x_{n+1}/\hbar}\,\langle\tilde{x}_{n+1}+\tilde{k}|_{\mathrm{mom}}\ . \tag{3.21b}$$

Using these expressions and omitting the $\mathcal{O}(\delta t^2)$ terms in (3.20), we find

$$\begin{aligned}\langle\mathbb{X}_{n+1}|\, e^{-i\,\delta t\,\hat{H}/\hbar}\,|\mathbb{X}_n\rangle = &(2\pi\hbar)^{-d}\, e^{-i\alpha(\mathbb{X}_{n+1})+i\alpha(\mathbb{X}_n)}\, e^{\frac{1}{2}i\tilde{x}_{n+1}\cdot(x_{n+1}-x_n)/\hbar-\frac{1}{2}ix_n\cdot(\tilde{x}_{n+1}-\tilde{x}_n)/\hbar} \\ &\times \sum_{k\in\lambda\mathbb{Z}^d}\sum_{\tilde{k}\in\tilde{\lambda}\mathbb{Z}^d} e^{-\frac{i}{\hbar}\delta t\left(\frac{1}{2m}g^{-1}(\tilde{x}_{n+1}+\tilde{k},\tilde{x}_{n+1}+\tilde{k})+\frac{1}{2}m\Omega^2 g(x_n+k,x_n+k)\right)} \\ &\times e^{i\tilde{k}\cdot(x_{n+1}-x_n)/\hbar-ik\cdot(\tilde{x}_{n+1}-\tilde{x}_n)/\hbar}\ .\end{aligned} \tag{3.22}$$

By defining $\mathbb{K} \equiv (k, \tilde{k}) \in \Lambda$ and $\mathbb{X}_n^* \equiv (x_n, \tilde{x}_{n+1}) \in \mathcal{P}$, we can formulate the last expression more compactly as

$$\begin{aligned}\langle \mathbb{X}_{n+1}| \, e^{-i\delta t \hat{H}/\hbar} \, |\mathbb{X}_n\rangle = (2\pi\hbar)^{-d} \, e^{-i\alpha(\mathbb{X}_{n+1})+i\alpha(\mathbb{X}_n)} \, e^{\frac{i}{2\hbar}\omega(\mathbb{X}_n^*,\delta\mathbb{X}_n)} \, e^{-\frac{i}{2\hbar}\Omega\,\delta t\, G(\mathbb{X}_n^*,\mathbb{X}_n^*)} \\ \times \sum_{\mathbb{K}\in\Lambda} e^{-\frac{i}{2\hbar}\Omega\,\delta t\, G(\mathbb{K},\mathbb{K})} \, e^{-\frac{i}{\hbar}\Omega\,\delta t\, G(\mathbb{K},\mathbb{X}_n^*)} \, e^{\frac{i}{\hbar}\omega(\mathbb{K},\delta\mathbb{X}_n)} \,. \end{aligned} \tag{3.23}$$

It is easier to handle the infinite sum in this expression if we express it in terms of Jacobi's theta function, whose properties are well-studied. Jacobi's theta function (in $2d$ dimensions), $\vartheta : \mathbb{C}^{2d} \times \mathfrak{H}_{2d} \to \mathbb{C}$, is defined over a complex vector space $\mathbb{C}^{2d}$ and the Siegel upper-half space[1] $\mathfrak{H}_{2d}$ by

$$\vartheta(z, \tau) \equiv \sum_{n\in\mathbb{Z}^{2d}} \exp\left[i\pi\, n^T \tau\, n + 2\pi i\, n^T z\right] \,. \tag{3.24}$$

Some important properties of this function are presented in Appendix B.

In our case, we have a sum over the modular lattice $\Lambda = \bar{\Lambda}\mathbb{Z}^{2d}$, where $\bar{\Lambda}^A{}_B \equiv \lambda^a{}_b \oplus \tilde{\lambda}_a{}^b$. The matrix $\Xi \equiv -\frac{\Omega\,\delta t}{2\pi\hbar}\,\bar{\Lambda}^T G\,\bar{\Lambda}$ is however real, and thus not in the Siegel upper-half space $\mathfrak{H}_{2d}$. In order to avoid this problem, we add a small imaginary part to Ξ and consider $\Xi_\epsilon \equiv \Xi + i\epsilon$ instead, where ϵ is a positive definite matrix[2]. Hence, we can express (3.23) as

$$\begin{aligned}\langle \mathbb{X}_{n+1}| \, e^{-i\delta t \hat{H}/\hbar} \, |\mathbb{X}_n\rangle = (2\pi\hbar)^{-d} \, e^{-i\alpha(\mathbb{X}_{n+1})+i\alpha(\mathbb{X}_n)} \, e^{\frac{i}{2\hbar}\omega(\mathbb{X}_n^*,\delta\mathbb{X}_n)} \, e^{-\frac{i}{2\hbar}\Omega\,\delta t\, G(\mathbb{X}_n^*,\mathbb{X}_n^*)} \\ \times \vartheta\left(\Xi_\epsilon\,\bar{\Lambda}^{-1}\,\mathbb{X}_n^* + \frac{1}{2\pi\hbar}\,\bar{\Lambda}^T\omega\,\delta\mathbb{X}_n,\, \Xi_\epsilon\right) \,. \end{aligned} \tag{3.25}$$

One important feature of Jacobi's theta function is the inversion identity (B.6), which is included in Appendix B. Using this identity, we get

$$\begin{aligned}\vartheta\left(\Xi_\epsilon\,\bar{\Lambda}^{-1}\,\mathbb{X}_n^* + \frac{1}{2\pi\hbar}\,\bar{\Lambda}^T\omega\,\delta\mathbb{X}_n,\, \Xi_\epsilon\right) = (i\Omega\,\delta t)^{-d} \; e^{-\frac{i}{\hbar}\omega(\mathbb{X}_n^*,\delta\mathbb{X}_n)} \, e^{\frac{i}{2\hbar}\Omega\,\delta t\, G_\epsilon(\mathbb{X}_n^*,\mathbb{X}_n^*)} \\ \times \vartheta\left(\bar{\Lambda}^{-1}\,\mathbb{X}_n^* + \frac{1}{2\pi\hbar}\,\Xi_\epsilon^{-1}\,\bar{\Lambda}^T\,\omega\,\delta\mathbb{X}_n,\, -\Xi_\epsilon^{-1}\right) \\ \times \exp\left[\frac{i}{2\hbar}\,(\Omega\,\delta t)^{-1}\,\delta\mathbb{X}_n^T\,\omega^T G_\epsilon^{-1}\omega\,\delta\mathbb{X}_n\right] \,. \end{aligned} \tag{3.26}$$

[1]The *Siegel upper-half space* $\mathfrak{H}_{2d}$ is defined as the set of symmetric, complex $2d \times 2d$-matrices whose imaginary parts are positive definite.

[2]This is a common trick that is also used in the Schrödinger-Feynman path integral.

where G_ϵ is defined through $\Xi_\epsilon = -\frac{\Omega\,\delta t}{2\pi\hbar}\,\bar{\Lambda}^T G_\epsilon\,\bar{\Lambda}$. The $i\epsilon$ scheme is needed only inside the theta function and we consider here the limit $\epsilon \to 0$, therefore we can replace G_ϵ with G. Inserting this equation back into (3.25) and noting that $\omega^T G^{-1}\omega = G$, we find

$$\begin{aligned}\langle \mathbb{X}_{n+1}|\, e^{-i\,\delta t\,\hat{H}/\hbar}\,|\mathbb{X}_n\rangle = (2\pi i\hbar\,\Omega\,\delta t)^{-d}\, e^{-i\alpha(\mathbb{X}_{n+1})+i\alpha(\mathbb{X}_n)}\, e^{-\frac{i}{2\hbar}\,\omega(\mathbb{X}_n^*,\delta\mathbb{X}_n)}\, e^{\frac{i}{2\hbar}\,\frac{1}{\Omega\,\delta t}\,G(\delta\mathbb{X}_n,\delta\mathbb{X}_n)} \\ \times\,\vartheta\!\left(\bar{\Lambda}^{-1}\,\mathbb{X}_n^* + \frac{1}{2\pi\hbar}\,\Xi_\epsilon^{-1}\,\bar{\Lambda}^T\,\omega\,\delta\mathbb{X}_n, -\Xi_\epsilon^{-1}\right)\,. \end{aligned} \tag{3.27}$$

Finally, we can use this expression to write the transition amplitude in (3.19) as

$$\begin{aligned}\langle \mathbb{X}_f|\, e^{-i(t_f-t_0)\hat{H}/\hbar}\,|\mathbb{X}_0\rangle = \int_{M_\Lambda(0)} \mathrm{d}^{2d}\delta\mathbb{X}_0 \cdots \int_{M_\Lambda(0)} \mathrm{d}^{2d}\delta\mathbb{X}_{N-1}\,(2\pi i\hbar\,\Omega\,\delta t)^{-Nd}\,\langle \mathbb{X}_f|\mathbb{X}_N\rangle \\ \times \prod_{n=0}^{N-1}\Bigg(e^{-i\alpha(\mathbb{X}_{n+1})+i\alpha(\mathbb{X}_n)}\, e^{-\frac{i}{2\hbar}\,\omega(\mathbb{X}_n^*,\delta\mathbb{X}_n)}\, e^{\frac{i}{2\hbar}\,\frac{1}{\Omega\,\delta t}\,G(\delta\mathbb{X}_n,\delta\mathbb{X}_n)} \\ \times\,\vartheta\!\left(\bar{\Lambda}^{-1}\,\mathbb{X}_n^* + \frac{1}{2\pi\hbar}\,\Xi_\epsilon^{-1}\,\bar{\Lambda}^T\,\omega\,\delta\mathbb{X}_n, -\Xi_\epsilon^{-1}\right)\Bigg)\,. \end{aligned} \tag{3.28}$$

3.3.3 Limit $N \to \infty$

In order to convert (3.28) into a path integral, we need to take the limit $N \to \infty$, or equivalently $\delta t \to 0$. For this limit to be well-defined, we need to hold the ratio

$$\dot{\mathbb{X}}_n \equiv \frac{\delta\mathbb{X}_n}{\delta t} \tag{3.29}$$

fixed during the limiting process. The variable $\dot{\mathbb{X}}_n$ will be interpreted as the velocity along a trajectory $\mathbb{X} : [t_0, t_f] \to \mathcal{P}$ at the time t_n.

This limit has several consequences for the expression (3.28). Firstly, we can make a Taylor expansion around $\delta t = 0$ to get

$$-i\alpha(\mathbb{X}_{n+1}) + i\alpha(\mathbb{X}_n) - \frac{i}{2\hbar}\,\omega(\mathbb{X}_n^*, \delta\mathbb{X}_n) = -\frac{i}{\hbar}\,\delta t\,\dot{\mathbb{X}}_n^A\,\mathbb{A}_A(\mathbb{X}_n) + \mathcal{O}(\delta t^2)\,. \tag{3.30}$$

As usual, we only keep the terms up to linear order in δt in the exponent. Secondly, since $\Xi_\epsilon^{-1} \propto \delta t^{-1}$, the theta function in (3.28) converges to 1 as $\delta t \to 0$ due to the property (B.7) of Jacobi's theta function, which is included in the Appendix B. Finally, we change the integration variables once again from $\delta\mathbb{X}_n$ to $\dot{\mathbb{X}}_n$. Hence, up to terms of order $\mathcal{O}(\delta t^2)$, we get

$$\langle \mathbb{X}_f | \, e^{-i(t_f-t_0)\hat{H}/\hbar} \, |\mathbb{X}_0\rangle = \int_{\frac{1}{\delta t}M_\Lambda(0)} \mathrm{d}^{2d}\dot{\mathbb{X}}_0 \cdots \int_{\frac{1}{\delta t}M_\Lambda(0)} \mathrm{d}^{2d}\dot{\mathbb{X}}_{N-1} \left(\frac{\delta t}{2\pi i\hbar\,\Omega}\right)^{Nd} \langle \mathbb{X}_f | \mathbb{X}_N\rangle \\ \times \prod_{n=0}^{N-1} \exp\left[\frac{i}{\hbar}\,\delta t \left(-\dot{\mathbb{X}}_n^A \, \mathbb{A}_A(\mathbb{X}_n) + \frac{1}{2\Omega}\, G(\dot{\mathbb{X}}_n, \dot{\mathbb{X}}_n)\right)\right] \, . \tag{3.31}$$

The inner product $\langle \mathbb{X}_f | \mathbb{X}_N\rangle$ in this expression can be evaluated using (2.46) as

$$\langle \mathbb{X}_f | \mathbb{X}_N\rangle = \sum_{\mathbb{W}\in\Lambda} e^{i\beta_\alpha(\mathbb{X}_f,\mathbb{W})}\, \delta^{2d}(\mathbb{X}_f + \mathbb{W} - \mathbb{X}_N) \, , \tag{3.32}$$

where $\beta_\alpha(\mathbb{X}_f, \mathbb{W}) = \alpha(\mathbb{X}_f + \mathbb{W}) - \alpha(\mathbb{X}_f) + \frac{1}{2\hbar}\, w \cdot \tilde{w} + \frac{1}{2\hbar}\, \omega(\mathbb{W}, \mathbb{X}_f)$ was defined in (2.44). The new parameter $\mathbb{W} \equiv (w, \tilde{w}) \in \Lambda$ that enters the modular path integral here will soon play an important role.

We can finally take the limit $N \to \infty$ and write (3.31) as a path integral in $\mathcal{P}$. We introduce the path function $\mathbb{X} : [t_0, t_f] \to \mathcal{P}$ as

$$\mathbb{X}(t_n) \equiv \mathbb{X}_n = \mathbb{X}_0 + \delta t \sum_{j=0}^{n-1} \dot{\mathbb{X}}_j \, . \tag{3.33}$$

The Dirac delta term $\delta^{2d}(\mathbb{X}_f + \mathbb{W} - \mathbb{X}_N)$ restricts the endpoint of these paths to $\mathbb{X}_N = \mathbb{X}_f + \mathbb{W}$. In the space of all paths in $\mathcal{P}$ from $\mathbb{X}_0$ to $\mathbb{X}_f + \mathbb{W}$, we define the modular path measure[3]

$$\mathcal{D}\mathbb{X} \equiv \lim_{N\to\infty} \left(\frac{\delta t}{2\pi i\hbar\,\Omega}\right)^{Nd} \delta^{2d}(\mathbb{X}_f + \mathbb{W} - \mathbb{X}_N) \prod_{n=0}^{N-1} \mathrm{d}^{2d}\dot{\mathbb{X}}_n \, . \tag{3.34}$$

We also introduce the *modular action*

$$S_{\mathrm{mod}}[\mathbb{X}] \equiv \int_{t_0}^{t_f} \mathrm{d}t\, \mathcal{L}_{\mathrm{mod}}(\mathbb{X}(t), \dot{\mathbb{X}}(t)) \, , \tag{3.35a}$$

$$\mathcal{L}_{\mathrm{mod}}(\mathbb{X}, \dot{\mathbb{X}}) \equiv -\dot{\mathbb{X}} \cdot \mathbb{A}(\mathbb{X}) + \frac{1}{2\Omega}\, G(\dot{\mathbb{X}}, \dot{\mathbb{X}}) \, , \tag{3.35b}$$

[3]In this expression, $\mathbb{X}_N$ is defined implicitly in terms of $\dot{\mathbb{X}}$ such that $\mathbb{X}_N \equiv \mathbb{X}_0 + \int_{t_0}^{t_f} \mathrm{d}t\, \dot{\mathbb{X}}(t)$.

where $\dot{\mathbb{X}}(t) \equiv \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{X}(t)$ is the velocity function. Combining all of these definitions, we are finally able to express the transition amplitude between two modular vectors by the path integral

$$\boxed{\langle \mathbb{X}_f | \, e^{-i(t_f - t_0)\hat{H}/\hbar} \, |\mathbb{X}_0\rangle = \sum_{\mathbb{W}\in\Lambda} e^{i\beta_\alpha(\mathbb{X}_f,\mathbb{W})} \int_{\mathbb{X}(t_0)=\mathbb{X}_0}^{\mathbb{X}(t_f)=\mathbb{X}_f+\mathbb{W}} \mathcal{D}\mathbb{X} \, \exp\left(\frac{i}{\hbar} S_{\mathrm{mod}}[\mathbb{X}]\right)} \, . \tag{3.36}$$

This *modular path integral* is clearly different from Feynman's path integral (3.10) as their domains consist of trajectories on two different spaces with a different dimensionality. Moreover, the modular path integral displays at least three new features:

1. The expression (3.36) contains a sum over the modular lattice, which is due to the topology of the modular space. The parameter $\mathbb{W} \in \Lambda$ should be interpreted as a **winding number** for each path around the modular space.

2. The paths of each winding number $\mathbb{W}$ around the modular space obtain an additional phase $\beta_\alpha(\mathbb{X}_f, \mathbb{W})$ depending on their winding number. This phase can be interpreted as analogous to the **Aharonov-Bohm phase**.

3. The **modular action** (3.35) is different from the usual action (3.12), especially through its dependence on the time derivatives of both position x and momentum $\tilde{x}$ variables. As we will see in Section (4.3), this signifies a larger *modular phase space* with twice the number of dimensions.

We will discuss these points and their implications in Chapter 4. In the next section, we will check the transformation of the expression (3.36) under modular lattice translations and gauge transformations to confirm its consistency.

3.3.4 Consistency

Throughout this section, we will frequently use an alternative formulation of the modular action (3.35),

$$S_{\mathrm{mod}}[\mathbb{X}] = -\hbar\,\alpha(\mathbb{X}(t_f)) + \hbar\,\alpha(\mathbb{X}(t_0)) + \int_{t_0}^{t_f} \mathrm{d}t \left(-\frac{1}{2}\,\omega(\mathbb{X},\dot{\mathbb{X}}) + \frac{1}{2\Omega}\,G(\dot{\mathbb{X}},\dot{\mathbb{X}})\right) \, , \tag{3.37}$$

which follows from (2.48).

Discrete translations

Here, we will show that the modular path integral (3.36) is consistent under a discrete translation of its endpoints. $\mathbb{K} \in \Lambda$ denotes an arbitrary lattice point in this section. The proof consists of two parts.

Firstly, we examine a shift in the final point, i.e.

$$\begin{aligned} \langle \mathbb{X}_f + \mathbb{K}| \, e^{-i(t_f - t_0)\hat{H}/\hbar} \, |\mathbb{X}_0\rangle &= \sum_{\mathbb{W}\in\Lambda} e^{i\beta_\alpha(\mathbb{X}_f+\mathbb{K},\mathbb{W})} \int_{\mathbb{X}(t_0)=\mathbb{X}_0}^{\mathbb{X}(t_f)=\mathbb{X}_f+\mathbb{K}+\mathbb{W}} \mathcal{D}\mathbb{X} \, \exp\left[\frac{i}{\hbar} \, S_{\rm mod}[\mathbb{X}]\right] \\ &= \sum_{\mathbb{W}\in\Lambda} e^{i\beta_\alpha(\mathbb{X}_f+\mathbb{K},\mathbb{W}-\mathbb{K})} \int_{\mathbb{X}(t_0)=\mathbb{X}_0}^{\mathbb{X}(t_f)=\mathbb{X}_f+\mathbb{W}} \mathcal{D}\mathbb{X} \, \exp\left[\frac{i}{\hbar} \, S_{\rm mod}[\mathbb{X}]\right] \ , \end{aligned} \tag{3.38}$$

where we redefined the summation variable $\mathbb{W}$ in the second line. We have

$$\beta_\alpha(\mathbb{X}_f + \mathbb{K}, \mathbb{W} - \mathbb{K}) = \beta_\alpha(\mathbb{X}_f, \mathbb{W}) - \beta_\alpha(\mathbb{X}_f, \mathbb{K}) + \frac{1}{\hbar} \, (k - w) \cdot \tilde{k} \ . \tag{3.39}$$

Since $e^{\frac{i}{\hbar}(k-w)\cdot\tilde{k}} = 1$, we find

$$\langle \mathbb{X}_f + \mathbb{K}| \, e^{-i(t_f - t_0)\hat{H}/\hbar} \, |\mathbb{X}_0\rangle = e^{-i\beta_\alpha(\mathbb{X}_f,\mathbb{K})} \, \langle \mathbb{X}_f| \, e^{-i(t_f - t_0)\hat{H}/\hbar} \, |\mathbb{X}_0\rangle \ . \tag{3.40}$$

This is consistent with the quasi-periodicity (2.43) of the modular vector.

The second part of the proof consists of examining a shift in the initial point, i.e.

$$\langle \mathbb{X}_f| \, e^{-i(t_f - t_0)\hat{H}/\hbar} \, |\mathbb{X}_0 + \mathbb{K}\rangle = \sum_{\mathbb{W}\in\Lambda} e^{i\beta_\alpha(\mathbb{X}_f,\mathbb{W})} \int_{\mathbb{X}(t_0)=\mathbb{X}_0+\mathbb{K}}^{\mathbb{X}(t_f)=\mathbb{X}_f+\mathbb{W}} \mathcal{D}\mathbb{X} \, \exp\left[\frac{i}{\hbar} \, S_{\rm mod}[\mathbb{X}]\right] \ . \tag{3.41}$$

For any path $t \in [t_0, t_f] \mapsto \mathbb{X}(t)$, let $\mathbb{X} + \mathbb{K}$ denote the parallel path shifted by the constant $\mathbb{K}$. We can shift the integration variable in the path integral and write

$$\begin{aligned} \langle \mathbb{X}_f| \, e^{-i(t_f - t_0)\hat{H}/\hbar} \, |\mathbb{X}_0 + \mathbb{K}\rangle &= \sum_{\mathbb{W}\in\Lambda} e^{i\beta_\alpha(\mathbb{X}_f,\mathbb{W})} \int_{\mathbb{X}(t_0)=\mathbb{X}_0}^{\mathbb{X}(t_f)=\mathbb{X}_f+\mathbb{W}-\mathbb{K}} \mathcal{D}\mathbb{X} \, \exp\left[\frac{i}{\hbar} \, S_{\rm mod}[\mathbb{X}+\mathbb{K}]\right] \\ &= \sum_{\mathbb{W}\in\Lambda} e^{i\beta_\alpha(\mathbb{X}_f,\mathbb{W}+\mathbb{K})} \int_{\mathbb{X}(t_0)=\mathbb{X}_0}^{\mathbb{X}(t_f)=\mathbb{X}_f+\mathbb{W}} \mathcal{D}\mathbb{X} \, \exp\left[\frac{i}{\hbar} \, S_{\rm mod}[\mathbb{X}+\mathbb{K}]\right] , \end{aligned} \tag{3.42}$$

where we redefined the summation variable $\mathbb{W}$ in the second line. Using the expression (3.37), we find that the action transforms as

$$\begin{aligned} S_{\text{mod}}[\mathbb{X}+\mathbb{K}] &= S_{\text{mod}}[\mathbb{X}] - \frac{1}{2}\,\omega(\mathbb{K},\mathbb{X}(t_f)-\mathbb{X}(t_0)) - \hbar\,\alpha(\mathbb{X}(t_f)+\mathbb{K}) + \hbar\,\alpha(\mathbb{X}(t_f)) \\ &\quad + \hbar\,\alpha(\mathbb{X}(t_0)+\mathbb{K}) - \hbar\,\alpha(\mathbb{X}(t_0)) \\ &= S_{\text{mod}}[\mathbb{X}] - \hbar\beta_\alpha(\mathbb{X}(t_f),\mathbb{K}) + \hbar\beta_\alpha(\mathbb{X}(t_0),\mathbb{K})\ . \end{aligned} \tag{3.43}$$

Finally, we note that

$$\beta_\alpha(\mathbb{X}_f,\mathbb{W}+\mathbb{K}) - \beta_\alpha(\mathbb{X}_f+\mathbb{W},\mathbb{K}) = \beta_\alpha(\mathbb{X}_f,\mathbb{W}) + \frac{1}{\hbar}\,k\cdot\tilde{w}\ . \tag{3.44}$$

Since $e^{\frac{i}{\hbar}k\cdot\tilde{w}} = 1$, we obtain

$$\langle\mathbb{X}_f|\,e^{-i(t_f-t_0)\hat{H}/\hbar}\,|\mathbb{X}_0+\mathbb{K}\rangle = e^{i\beta_\alpha(\mathbb{X}_0,\mathbb{K})}\,\langle\mathbb{X}_f|\,e^{-i(t_f-t_0)\hat{H}/\hbar}\,|\mathbb{X}_0\rangle\ . \tag{3.45}$$

Once again, this is consistent with the quasi-periodicity (2.43) of the modular vector.

Gauge transformations

Here, we will show that the modular path integral (3.36) transforms covariantly under a gauge transformation $\mathbb{A}_A \to \mathbb{A}_A + \hbar\,\partial_A\bar{\alpha}$. The modular action (3.37) and the phase factor transform as

$$S_{\text{mod}}[\mathbb{X}] \to S_{\text{mod}}[\mathbb{X}] - \hbar\,\bar{\alpha}(\mathbb{X}_f+\mathbb{W}) + \hbar\,\bar{\alpha}(\mathbb{X}_0) \tag{3.46a}$$

$$\beta_\alpha(\mathbb{X}_f,\mathbb{W}) \to \beta_\alpha(\mathbb{X}_f,\mathbb{W}) + \bar{\alpha}(\mathbb{X}_f+\mathbb{W}) - \bar{\alpha}(\mathbb{X}_f)\ . \tag{3.46b}$$

Combining these two expressions, we get

$$\langle\mathbb{X}_f|\,e^{-i(t_f-t_0)\hat{H}/\hbar}\,|\mathbb{X}_0\rangle \to e^{-i\bar{\alpha}(\mathbb{X}_f)+i\bar{\alpha}(\mathbb{X}_0)}\,\langle\mathbb{X}_f|\,e^{-i(t_f-t_0)\hat{H}/\hbar}\,|\mathbb{X}_0\rangle\ , \tag{3.47}$$

which is consistent with (2.41).

Chapter 4

Analysis of the modular action

In this chapter, we are going to analyse the new modular action (3.35) and compare it to the better-known Schrödinger action (3.12).

4.1 Stationary paths

The variation of the modular action (3.35) with respect to the path $\mathbb{X}$ is given by

$$\delta S_{\text{mod}} = \int_{t_0}^{t_f} \mathrm{d}t \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(-\mathbb{A} \cdot \delta\mathbb{X} + \frac{1}{\Omega}\, G(\dot{\mathbb{X}}, \delta\mathbb{X}) \right) - \delta\mathbb{X}^A \left(\omega_{AB}\, \dot{\mathbb{X}}^B + \frac{1}{\Omega}\, G_{AB}\, \ddot{\mathbb{X}}^B \right) \right) \ . \quad (4.1)$$

The Euler-Lagrange equation of motion can be read from the (second) bulk term. Using $G^{-1}\omega = -\omega^{-1}G$, we write it as

$$\ddot{\mathbb{X}}(t) = \Omega\, \omega^{-1} G\, \dot{\mathbb{X}}(t) \ . \quad (4.2)$$

This Lagrangian equation of motion is similar to the Hamilton equations (3.14) in the Schrödinger case, but it has an additional time derivative overall. If we integrate (4.2), we get

$$\dot{\mathbb{X}}(t) = \Omega\, \omega^{-1} G\, (\mathbb{X}(t) - \chi) \quad (4.3)$$

with a new integration constant $\chi \in \mathcal{P}$.

In order to solve the equation of motion (4.2), we note that $\omega^{-1}G$ is a complex structure on $\mathcal{P}$, i.e. it is a $2d \times 2d$ matrix that satisfies $(\omega^{-1}G)^2 = -\mathbb{1}$, where $\mathbb{1}$ is the identity matrix.

Combining (4.2) and (4.3) gives $\ddot{\mathbb{X}}(t) = -\Omega^2 \left(\mathbb{X}(t) - \chi\right)$. Hence, the solutions to (4.3) are of the form

$$\mathbb{X}(t) = \chi + \xi \sin(\Omega t) - \omega^{-1} G \, \xi \cos(\Omega t) \ , \tag{4.4}$$

where $\xi \in \mathcal{P}$ is another integration constant. These integration constants, χ and ξ, are fixed by the boundary conditions of a path $\mathbb{X}(t)$, and generally they can take arbitrary values. If we require $\mathbb{X}(t_0) = \mathbb{X}_0$ and $\mathbb{X}(t_f) = \mathbb{X}_f + \mathbb{W}$ as in the path integral (3.36), the stationary paths $\mathbb{X}^{\mathrm{s}}_{\mathbb{W}}$ are explicitly given by[1]

$$\mathbb{X}^{\mathrm{s}}_{\mathbb{W}}(t) = \chi + \xi \sin\left(\Omega \left(t - \frac{t_f + t_0}{2}\right)\right) - \omega^{-1} G \, \xi \cos\left(\Omega \left(t - \frac{t_f + t_0}{2}\right)\right) \ , \tag{4.5a}$$

$$\chi = \frac{1}{2}\left(\mathbb{X}_0 + \mathbb{X}_f + \mathbb{W}\right) + \frac{1}{2}\omega^{-1} G \left(\mathbb{X}_f + \mathbb{W} - \mathbb{X}_0\right) \cot\left(\frac{1}{2}\Omega\left(t_f - t_0\right)\right) \ , \tag{4.5b}$$

$$\xi = \frac{1}{2}\left(\mathbb{X}_f + \mathbb{W} - \mathbb{X}_0\right) \csc\left(\frac{1}{2}\Omega\left(t_f - t_0\right)\right) \ . \tag{4.5c}$$

There are several important differences between these stationary paths and the usual result in the Schrödinger representation:

- Firstly, these two sets of paths are defined on different spaces. Schrödinger paths run over the corresponding configuration space $\mathbb{R}^d$, whereas the modular paths as in (4.5) run over the universal cover of the modular space T_Λ, which is $\mathbb{R}^{2d}$. They are also associated with different phase spaces. The phase space for Schrödinger paths is $\mathcal{P} = \mathbb{R}^{2d}$, whereas the phase space for modular paths is $\mathcal{P}_{\mathrm{mod}} = \mathbb{R}^{4d}$, as we will discuss in Section 4.3.
- The second difference is the number of stationary paths. For any boundary conditions $x(t_0) = x_0$ and $x(t_f) = x_f$, there is a unique classical solution[2] to the harmonic oscillator in the Schrödinger representation. One the other hand, there is one solution (4.5) for each winding number $\mathbb{W} \in \Lambda$ in the modular representation, meaning that there are infinitely many stationary paths in total. This result originates from the compact topology of the modular space.
- Heuristically, we can match the phase space $\mathcal{P}$ of the Schrödinger representation with the universal cover $\mathbb{R}^{2d}$ of the modular space, despite their different physical interpretations. Then, we can compare the solutions in both representations on this common space. The phase space diagram for the Schrödinger solution is an ellipse centered at the origin. On the other hand, the paths (4.5) are infinitely many ellipses which intersect at the point $\mathbb{X}_0$, see Figure 4.1.

[1] We assume here that $\Omega(t_f - t_0) \notin 2\pi\mathbb{Z}$, since otherwise $\mathbb{X}(t_0) = \mathbb{X}(t_f)$.

[2] Again, we assume $\Omega(t_f - t_0) \notin \pi\mathbb{Z}$, since otherwise $x(t_0) = \pm x(t_f)$.

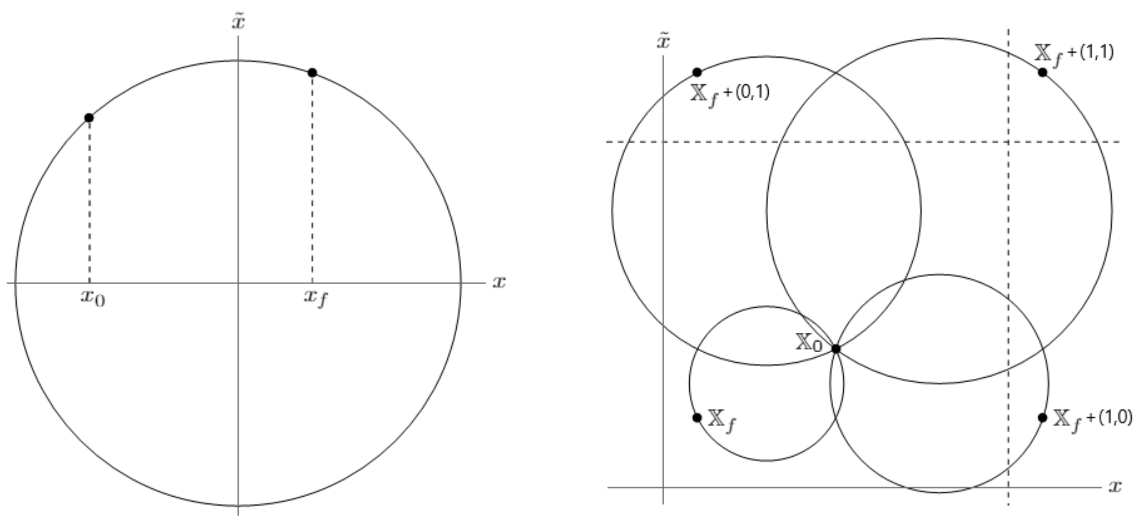

Figure 4.1: On the left, we have the phase space diagram for a stationary solution to the quantum harmonic oscillator in the Schrödinger representation. On the right, four stationary trajectories with different winding numbers in the modular representation are illustrated. These two figures demonstrate the contrast between the trajectories on $\mathcal{P} \sim \mathbb{R}^{2d}$ for the two representations.

- The identification of the "momentum" variable $\tilde{x}$ with the change in position, $m\dot{x}$, is missing in the modular representation. Hence, $\tilde{x}$ in the modular representation should not be understood as the momentum in the physical sense, but rather as a distinct and independent variable.
- Unlike in the Schrödinger representation, the Hamilton operator

$$\hat{H} = \frac{1}{2m}\,\hat{p}^2 + \frac{1}{2}\,m\Omega^2\,\hat{q}^2 \tag{4.6}$$

  cannot be understood in the modular representation as such that the first term is the kinetic and the second term is the potential term. Considering (2.49), both terms contain derivation operators, and (depending on the gauge) both terms may contain potential-like pieces. This explains why the trajectories (4.5) are not necessarily centered around the origin.

4.2 Semi-classical approximation

For each such stationary path in (4.5), the value of the on-shell modular action is

$$\begin{aligned} S_{\text{mod}}[\mathbb{X}^{\text{s}}_{\mathbb{W}}] = &-\hbar\,\alpha(\mathbb{X}_f + \mathbb{W}) + \hbar\,\alpha(\mathbb{X}_0) - \frac{1}{2}\,\omega(\mathbb{X}_0, \mathbb{X}_f + \mathbb{W}) \\ &+ \frac{1}{4}\cot\left(\frac{1}{2}\,\Omega\,(t_f - t_0)\right) G(\mathbb{X}_f + \mathbb{W} - \mathbb{X}_0, \mathbb{X}_f + \mathbb{W} - \mathbb{X}_0)\;. \end{aligned} \tag{4.7}$$

In the semi-classical approximation $\hbar \to 0$, the path integral is dominated by the stationary paths. Moreover, since the second functional derivative of the modular action is independent of the winding number, as in

$$\frac{\delta^2 S_{\text{mod}}}{\delta\mathbb{X}^A(t)\,\delta\mathbb{X}^B(t')} = -\,\omega_{AB}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\delta(t-t') - \frac{1}{\Omega}\,G_{AB}\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\delta(t-t')\;, \tag{4.8}$$

each stationary path contributes to the path integral with equal weight. Hence, we find that in the semi-classical approximation the transition amplitude becomes

$$\begin{aligned} \langle\mathbb{X}_f|\,e^{-i\,(t_f-t_0)\hat{H}/\hbar}\,|\mathbb{X}_0\rangle \underset{\hbar\to 0}{\sim} &\sum_{\mathbb{W}\in\Lambda} e^{i\beta_\alpha(\mathbb{X}_f,\mathbb{W})}\,e^{\frac{i}{\hbar}S_{\text{mod}}[\mathbb{X}^{\text{s}}_{\mathbb{W}}]} \\ &= e^{-i\alpha(\mathbb{X}_f)+i\alpha(\mathbb{X}_0)}\,e^{-\frac{i}{2\hbar}\,\omega(\mathbb{X}_0,\mathbb{X}_f)}\sum_{\mathbb{W}\in\Lambda} e^{\frac{i}{2\hbar}\,w\cdot\tilde{w}}\,e^{\frac{i}{2\hbar}\,\omega(\mathbb{W},\mathbb{X}_0+\mathbb{X}_f)} \\ &\qquad\times e^{\frac{i}{4\hbar}\cot\left(\frac{\Omega}{2}(t_f-t_0)\right)G(\mathbb{W}+\mathbb{X}_f-\mathbb{X}_0,\mathbb{W}+\mathbb{X}_f-\mathbb{X}_0)} \end{aligned} \tag{4.9}$$

up to a constant factor. We can rewrite this expression in terms of Jacobi's theta function that is defined in (3.24) and discussed in Appendix B, such that

$$\begin{aligned}
&\langle \mathbb{X}_f | \, e^{-i(t_f-t_0)\hat{H}/\hbar} \, |\mathbb{X}_0\rangle \\
&\underset{\hbar\to 0}{=} e^{-i\alpha(\mathbb{X}_f)+i\alpha(\mathbb{X}_0)} \, e^{-\frac{i}{2\hbar}\,\omega(\mathbb{X}_0,\mathbb{X}_f)} \, e^{\frac{i}{4\hbar}c\,G(\mathbb{X}_f-\mathbb{X}_0,\mathbb{X}_f-\mathbb{X}_0)} \\
&\qquad \times \vartheta\!\left(\frac{1}{4\pi\hbar}\,\bar{\Lambda}^T\left(\omega\left(\mathbb{X}_0+\mathbb{X}_f\right)+c\,G\left(\mathbb{X}_f-\mathbb{X}_0\right)\right), \, \frac{1}{4\pi\hbar}\,\bar{\Lambda}^T\left(\eta+c\,G\right)\bar{\Lambda}+i\epsilon\right) \, ,
\end{aligned} \tag{4.10}$$

where $c \equiv \cot\big(\frac{\Omega}{2}(t_f-t_0)\big)$, and η is the polarization metric that was introduced in Section 2.2. We used here the $i\epsilon$ prescription to make the sum converge, where ϵ is a positive-definite matrix.

By the Born rule, the probability for a quantum state to transition from $|\mathbb{X}_0\rangle$ to $|\mathbb{X}_f\rangle$ in the time interval $[t_0, t_f] \subset \mathbb{R}$ is given by the quantity

$$\begin{aligned}
&\left|\langle \mathbb{X}_f | \, e^{-i(t_f-t_0)\hat{H}/\hbar} \, |\mathbb{X}_0\rangle\right|^2 \\
&\underset{\hbar\to 0}{=} \left|\vartheta\!\left(\frac{1}{4\pi\hbar}\,\bar{\Lambda}^T\left(\omega\left(\mathbb{X}_0+\mathbb{X}_f\right)+c\,G\left(\mathbb{X}_f-\mathbb{X}_0\right)\right), \, \frac{1}{4\pi\hbar}\,\bar{\Lambda}^T\left(\eta+c\,G\right)\bar{\Lambda}+i\epsilon\right)\right|^2 .
\end{aligned} \tag{4.11}$$

Since $|\mathbb{X}_f\rangle$ and $|\mathbb{X}_f+\mathbb{W}\rangle$ are the same state up to a phase for any $\mathbb{W}\in\Lambda$, this probability has to be invariant under the transformation $\mathbb{X}_f \to \mathbb{X}_f+\mathbb{W}$. In order to check this, we first note that $\bar{\Lambda}^T\eta\bar{\Lambda} = 2\pi\hbar\,\eta$ and $\bar{\Lambda}^T\omega\bar{\Lambda} = 2\pi\hbar\,\omega$. We have

$$\begin{aligned}
&\frac{1}{4\pi\hbar}\,\bar{\Lambda}^T\left(\omega\left(\mathbb{X}_0+\mathbb{X}_f+\mathbb{W}\right)+c\,G\left(\mathbb{X}_f+\mathbb{W}-\mathbb{X}_0\right)\right) \\
&= \frac{1}{4\pi\hbar}\,\bar{\Lambda}^T\left(\omega\left(\mathbb{X}_0+\mathbb{X}_f\right)+c\,G\left(\mathbb{X}_f-\mathbb{X}_0\right)\right) \\
&\quad + \frac{1}{4\pi\hbar}\,\bar{\Lambda}^T\left(\eta+c\,G\right)\bar{\Lambda}\left(\bar{\Lambda}^{-1}\mathbb{W}\right)+\frac{1}{2}\left(\omega-\eta\right)\left(\bar{\Lambda}^{-1}\mathbb{W}\right) \, .
\end{aligned} \tag{4.12}$$

Note that $\bar{\Lambda}^{-1}\mathbb{W}\in\mathbb{Z}^{2d}$ by definition, and $\frac{1}{2}\left(\omega-\eta\right)\left(\bar{\Lambda}^{-1}\mathbb{W}\right)\in\mathbb{Z}^{2d}$ because $\frac{1}{2}\left(\omega-\eta\right)$ is an integer matrix. Hence, $\left|\langle \mathbb{X}_f+\mathbb{W}| \, e^{-i(t_f-t_0)\hat{H}/\hbar} \, |\mathbb{X}_0\rangle\right|^2 = \left|\langle \mathbb{X}_f| \, e^{-i(t_f-t_0)\hat{H}/\hbar} \, |\mathbb{X}_0\rangle\right|^2$ follows from (B.2) and (B.3), confirming the consistency of (4.12) with the quasi-periodicity relation (2.43).

4.3 Canonical analysis

We introduce conjugate momenta $\mathbb{P} \in \mathbb{R}^{2d}$ to the coordinates $\mathbb{X} \in \mathcal{P}$ with respect to the modular action (3.35). These are defined as

$$\mathbb{P}_A \equiv \frac{\delta S_{\text{mod}}}{\delta \dot{\mathbb{X}}^A} = -\mathbb{A}_A(\mathbb{X}) + \frac{1}{\Omega}\, G_{AB}\, \dot{\mathbb{X}}^B \; . \tag{4.13}$$

The *modular phase space* $\mathcal{P}_{\text{mod}} = \mathbb{R}^{4d}$ consists of the pairs of variables $(\mathbb{X}, \mathbb{P})$. Note that this has twice the number of dimensions compared to its Schrödinger counterpart $\mathcal{P}$.

The symplectic potential Θ on the modular phase space $\mathcal{P}_{\text{mod}}$ can be read from the boundary term in the variation of the modular action (4.1) as

$$\Theta = \mathbb{P}_A\, \mathrm{d}\mathbb{X}^A \; . \tag{4.14}$$

The exterior derivative of this symplectic potential gives the *modular symplectic form*

$$\omega_{\text{mod}} = \mathrm{d}\mathbb{P}_A \wedge \mathrm{d}\mathbb{X}^A \; . \tag{4.15}$$

We can also perform a Legendre transform on the modular Lagrangian (3.35b) to get the *modular Hamiltonian*

$$\mathcal{H}_{\text{mod}}(\mathbb{X}, \mathbb{P}) = \frac{1}{2}\, \Omega\, G^{-1}(\mathbb{P} + \mathbb{A}(\mathbb{X}), \mathbb{P} + \mathbb{A}(\mathbb{X})) \; . \tag{4.16}$$

Hamilton's principal function $S^{\text{s}}_{\text{mod}}(\mathbb{X}, t)$ for this system can be read from the on-shell modular action (4.7) as

$$\begin{aligned} S^{\text{s}}_{\text{mod}}(\mathbb{X}, t) = -\hbar\, \alpha(\mathbb{X}) + \hbar\, \alpha(\mathbb{X}_0) - \frac{1}{2}\, \omega(\mathbb{X}_0, \mathbb{X}) \\ + \frac{1}{4} \cot\left(\frac{1}{2}\, \Omega\, (t - t_0)\right) G(\mathbb{X} - \mathbb{X}_0, \mathbb{X} - \mathbb{X}_0) \; . \end{aligned} \tag{4.17}$$

This function satisfies the Hamilton-Jacobi equation for the Hamiltonian (4.16).

4.4 Symmetries

In this section, we will discuss some of the symmetries of the modular action in (3.35). While the symmetries (e.g., rotation and time translation) of the old action (3.12) are still present with formally different currents, we find here a whole new set of symmetries that correspond to translations over the phase space, i.e. spatial translations and momentum translations.

In the following, we will discuss some symmetries of the modular harmonic oscillator together with their corresponding Noether currents.

4.4.1 $U(1)$ gauge symmetry

First of all, the modular action (3.35) is invariant under the $U(1)$ gauge symmetry in (2.47). When the modular connection is transformed as $\mathbb{A}_A \to \mathbb{A}_A + \hbar\, \partial_A \alpha$ for a scalar function α on $\mathcal{P}$, the modular Lagrangian changes by a total derivative,

$$\delta \mathcal{L}_{\text{mod}} = \frac{\mathrm{d}}{\mathrm{d}t} \left(-\hbar\, \alpha(\mathbb{X}) \right) \, . \tag{4.18}$$

4.4.2 Position and momentum translations

Spatial translations, including both the position x and momentum $\tilde{x}$ variables, are not among the symmetries of the Schrödinger harmonic oscillator, but we will show here that they are a new set of symmetries for the modular action (3.35). Consider an infinitesimal translation of the phase space coordinates by a constant vector $\mathcal{E} \in \mathcal{P}$, such that

$$\delta \mathbb{X}^A = \mathcal{E}^A \, . \tag{4.19}$$

The modular Lagrangian changes by a total derivative,

$$\delta \mathcal{L}_{\text{mod}} = \frac{\mathrm{d}}{\mathrm{d}t} \left(-\mathcal{E} \cdot \mathbb{A}(\mathbb{X}) + \omega(\mathbb{X}, \mathcal{E}) \right) \, . \tag{4.20}$$

We find that the Noether current for this transformation is given by

$$\chi = \mathbb{X}(t) + \frac{1}{\Omega}\, \omega^{-1} G\, \dot{\mathbb{X}}(t) \, , \tag{4.21}$$

which is no different than the integration constant we found in (4.3). This quantity is conserved on-shell and it denotes the midpoint of the elliptical trajectories we found in (4.5).

The new conserved current χ vanishes in the Schrödinger limit where the classical Hamilton equations (3.14) are imposed. Therefore, it has no analog in the Schrödinger mechanics.

4.4.3 Time translation

Consider an infinitesimal shift of the time parameter $t \to t + \epsilon$, which results in

$$\delta \mathbb{X}^A = \epsilon \, \dot{\mathbb{X}}^A \ , \qquad \delta \mathcal{L}_{\rm mod} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\epsilon \, \mathcal{L}_{\rm mod} \right) \ . \tag{4.22}$$

The associated Noether current is the total energy, given by

$$E = \frac{1}{2\Omega} \, G(\dot{\mathbb{X}}(t), \dot{\mathbb{X}}(t)) \ . \tag{4.23}$$

Although they are formally different, this expression for conserved energy recovers the usual formula $E = \frac{1}{2} m g(\dot{x}, \dot{x}) + \frac{1}{2} m \Omega^2 g(x, x)$ when the classical Hamilton equations (3.14) are imposed.

4.4.4 Rotation

For any infinitesimal, anti-symmetric 2-tensor L_{ab} on $\mathbb{R}^d$, we consider the rotation as

$$\delta x^a = -g^{ab} L_{bc} \, x^c \ , \qquad \delta \tilde{x}_a = -L_{ab} \, g^{bc} \, \tilde{x}_c \ . \tag{4.24}$$

The modular Lagrangian changes by a total derivative,

$$\delta \mathcal{L}_{\rm mod} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathbb{A}_a(\mathbb{X}) \, L^a{}_b \, x^b + \mathbb{A}^a(\mathbb{X}) \, L_a{}^b \, \tilde{x}_b - \tilde{x}_a \, L^a{}_b \, x^b \right) \ , \tag{4.25}$$

where $\mathbb{A}_A = (\mathbb{A}_a, \mathbb{A}^a)$. Only in this subsection, we are using the metric g on $\mathbb{R}^d$ to raise the indices of L_{ab} and $\tilde{x}_a$. The conserved Noether current is given by

$$J^{ab} = \tilde{x}^{[a} x^{b]} - m \, \dot{x}^{[a} x^{b]} - \frac{1}{m\Omega^2} \, \dot{\tilde{x}}^{[a} \tilde{x}^{b]} \ . \tag{4.26}$$

Once again, although they are formally different, this expression recovers the angular momentum $J^{ab} = x^{[a} \tilde{x}^{b]}$ when the classical Hamilton equations (3.14) are imposed.

4.4.5 Symplectic transformation

Finally, we consider an infinitesimal transformation of the form $\delta \mathbb{X} = \epsilon \, \omega^{-1} G \, \mathbb{X}$, or equivalently,

$$\delta x^a = \frac{\epsilon}{m\Omega} \, g^{ab} \, \tilde{x}_b \ , \qquad \delta \tilde{x}_a = -\epsilon \, m \, \Omega \, g_{ab} \, x^b \ . \tag{4.27}$$

The modular Lagrangian changes again by a total derivative,

$$\delta\mathcal{L}_{\text{mod}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\epsilon\,\mathbb{A}(\mathbb{X})\,\omega^{-1}G\,\mathbb{X} + \frac{\epsilon}{2}\,G(\mathbb{X},\mathbb{X})\right)\ . \tag{4.28}$$

We find the conserved Noether current

$$\kappa = \frac{1}{\Omega}\,\omega(\mathbb{X},\dot{\mathbb{X}}) - \frac{1}{2}\,G(\mathbb{X},\mathbb{X})\ . \tag{4.29}$$

This quantity is not independent of the previous conserved currents and it can be written as

$$\kappa = \frac{1}{\Omega}\,E - \frac{1}{2}\,G(\chi,\chi)\ . \tag{4.30}$$

Note that this symmetry mixes the variables x and $\tilde{x}$, therefore it is a hidden symmetry for the Schrödinger action. As in this example, the modular action can promote hidden symmetries to explicit symmetries.

4.4.6 Discussion

Looking at the above examples, we can draw the following conclusions:

1. The symmetries of the standard action are maintained in the modular action. The corresponding Noether currents can be formally different in the new modular formulation, but they recover their standard expressions under the classical equations of motion.

2. The modular action has a new set of translation symmetries for both position and momentum variables. The corresponding Noether currents vanish under the classical equations of motion.

3. Since the modular action is formulated on the classical phase space, the hidden symmetries that mix the configuration variable x with the conjugate momentum $\tilde{x}$ can be expressed as explicit symmetries of the action for the composite configuration variable $(x, \tilde{x})$.

We conjecture that these three conclusions hold in general for any modular action, i.e. any action that is derived in the same way from the modular representation of an arbitrary physical system. In particular, revealing the hidden symmetries through the modular action can be a useful tool to find all symmetries of physical systems in general.

4.5 Schrödinger limit

We discussed previously in Section 2.8 that the Schrödinger representation of the Weyl algebra can be identified with the limit of the modular representations as the length scale ℓ of the modular lattice goes to infinity. This limit is a singular one, in which the topology of the configuration space changes, but it is well-defined nevertheless.

In this section, we show a similar result for the modular path integral (3.36). Considering the 1-parameter family of modular lattices $\Lambda = \ell\mathbb{Z}^d \oplus \tilde{\ell}\mathbb{Z}^d$, where ℓ is a length scale and $\tilde{\ell} \equiv 2\pi\hbar/\ell$ is a momentum scale, we demonstrate how the path integral (3.36) in modular space can be identified with the Feynman path integral in Schrödinger space (see Section 3.2) in the limit $\ell \to \infty$.

As discussed in Section 2.8, the Schrödinger limit $\ell \to \infty$ can be well-defined only in the Schrödinger gauge (2.50). Therefore, we fix the modular gauge in this section as such, i.e.

$$\mathbb{A}(\mathbb{X}) = (0, -x) \ . \tag{4.31}$$

In this gauge, we have

$$\beta_{\alpha_{\mathrm{Sch}}}(\mathbb{X}_f, \mathbb{W}) = -\frac{1}{\hbar}\, w \cdot \tilde{x}_f \tag{4.32}$$

and

$$S_{\mathrm{mod}}[\mathbb{X}] = \int_{t_0}^{t_f} \mathrm{d}t \left(x(t) \cdot \dot{\tilde{x}}(t) + \frac{m}{2}\, g(\dot{x}(t), \dot{x}(t)) + \frac{1}{2m\Omega^2}\, g^{-1}(\dot{\tilde{x}}(t), \dot{\tilde{x}}(t)) \right) \ . \tag{4.33}$$

We remark that the term $-\dot{\mathbb{X}} \cdot \mathbb{A} = x \cdot \dot{\tilde{x}}$ in the above expression is reminiscent of relative locality [5].

We consider the expression

$$\sum_{w \in \ell\mathbb{Z}^d} \sum_{\tilde{w} \in \tilde{\ell}\mathbb{Z}^d} e^{-\frac{i}{\hbar} w \cdot \tilde{x}_f} \int_{\mathbb{X}(t_0)=\mathbb{X}_0}^{\mathbb{X}(t_f)=\mathbb{X}_f+\mathbb{W}} \mathcal{D}\mathbb{X}\, \exp\left(\frac{i}{\hbar}\, S_{\mathrm{mod}}[\mathbb{X}] \right) \ , \tag{4.34}$$

where $\mathbb{W} = (w, \tilde{w})$, $\tilde{\ell} = 2\pi\hbar/\ell$, and $S_{\mathrm{mod}}[\mathbb{X}]$ is given in (4.33). As we change the parameter ℓ in (4.34), the functional integral is not affected (except for its boundaries), since it is on $\mathcal{P}$, which is independent of ℓ.

In the limit $\ell \to \infty$ and $\tilde{\ell} \to 0$, the modular lattice Λ converges to the momentum space in a coarse-graining approximation. Note that this is a singular transition from a countable set in $2d$ dimensions to an uncountable set in d dimensions. The sum over $\tilde{w} \in \tilde{\ell}\mathbb{Z}^d$ approaches an integral over $\tilde{w} \in \mathbb{R}^d$. Recall that the Dirac delta term $\delta^{2d}(\mathbb{X}_f + \mathbb{W} - \mathbb{X}_N)$ inside the modular path measure (3.34) restricts both the position and momentum endpoints of the paths. An integral over $\tilde{w} \in \mathbb{R}^d$ cancels with $\delta^d(\tilde{x}_f + \tilde{w} - \tilde{x}_N)$ and sets the momentum endpoints of the paths free. Then, the expression in (4.34) approaches

$$\sum_{w \in \ell\mathbb{Z}^d} e^{-\frac{i}{\hbar} w \cdot \tilde{x}_f} \int_{x(t_0)=x_0}^{x(t_f)=x_f+w} \mathcal{D}x \int \mathcal{D}\tilde{x} \, \exp\left(\frac{i}{\hbar} S_{\text{mod}}[\mathbb{X}]\right) , \tag{4.35}$$

up to a constant factor. Here, $\mathcal{D}x$ and $\mathcal{D}\tilde{x}$ are the standard path measures on the Schrödinger and momentum spaces, respectively.

We note that the action (4.33) can be written as

$$S_{\text{mod}}[\mathbb{X}] = S_{\text{Sch}}[x] + \frac{1}{2m\Omega^2} \int_{t_0}^{t_f} \mathrm{d}t \, g^{-1}\big(\dot{\tilde{x}}(t) + m\Omega^2 g x(t), \dot{\tilde{x}}(t) + m\Omega^2 g x(t)\big) , \tag{4.36}$$

where $S_{\text{Sch}}[x]$ is given in (3.12). The integral

$$\int \mathcal{D}\tilde{x} \, \exp\left(\frac{i}{\hbar} \frac{1}{2m\Omega^2} \int_{t_0}^{t_f} \mathrm{d}t \, g^{-1}(\dot{\tilde{x}} + m\Omega^2 g x, \dot{\tilde{x}} + m\Omega^2 g x)\right) \tag{4.37}$$

is equal to an irrelevant constant. Hence, (4.35) becomes

$$\sum_{w \in \ell\mathbb{Z}^d} e^{-\frac{i}{\hbar} w \cdot \tilde{x}_f} \int_{x(t_0)=x_0}^{x(t_f)=x_f+w} \mathcal{D}x \, \exp\left[\frac{i}{\hbar} S_{\text{Sch}}[x]\right] , \tag{4.38}$$

up to constant factors. Finally, as $\ell \to \infty$, the winding modes w become unattainable as they require infinite action. Therefore we set $w = 0$ and remove the sum, getting

$$\int_{x(t_0)=x_0}^{x(t_f)=x_f} \mathcal{D}x \, \exp\left[\frac{i}{\hbar} S_{\text{Sch}}[x]\right] \tag{4.39}$$

from the modular path integral as $\ell \to \infty$. Regarding the left-hand side in (3.36), we already found in (2.51) that modular vectors converge to the corresponding position eigenvectors in the limit $\ell \to \infty$. Hence, we conclude that the modular path integral (3.36) recovers the Schrödinger-Feynman path integral (3.10) in this limit.

Chapter 5

Beyond the harmonic oscillator

5.1 Modular Legendre transform

In the previous chapters, we discussed in detail the modular path integral formulation for the quantum harmonic oscillator. One particular feature of the path integral is that it transforms the Hamiltonian operator of a system to a Lagrangian function. The transformation from the Hamiltonian to the Schrödinger Lagrangian, which is found in the Feynman path integral (3.10), is well-known and formulated in general as the Legendre transform. However, the modular Lagrangian (3.35b) found in the modular path integral is different from the Schrödinger Lagrangian (3.12b), even though it starts from the same Hamiltonian (3.7). This result raises the question of whether we can formulate the transformation from the Hamiltonian to the modular Lagrangian in general as a recipe, analogously to the standard Legendre transform. We call this new transformation the *modular Legendre transform*.

We will decipher the modular Legendre transform by following the construction for the harmonic oscillator in Section 3.3. In this example, the classical Hamiltonian function was given by $\mathcal{H} = \frac{1}{2}\,\Omega\, G(\mathbb{Q}, \mathbb{Q})$, where $\mathbb{Q} = (q, p) \in \mathbb{R}^{2d}$.

The first task is to identify the conjugate variables. In the standard Legendre transform, these are the configuration variable $q \in \mathbb{R}^d$ and the conjugate momentum $p \in \mathbb{R}^d$. In the modular framework, Aharonov's modular variables $\mathbb{X} \in T_\Lambda = \mathbb{R}^{2d}/\Lambda$ replace the configuration variables. We consider the representation[1] of the variables $\mathbb{X}$ on an arbitrary

[1] We will use here the same symbol for the equivalence classes $\mathbb{X} \in T_\Lambda$ and their representatives $\mathbb{X} \in M_\Lambda$, abusing the notation.

modular cell $M_\Lambda \subset \mathbb{R}^{2d}$. Once a modular lattice Λ is chosen, we can split the variable $\mathbb{Q} \in \mathbb{R}^{2d}$ into two parts,

$$\mathbb{Q} = \mathbb{X} + \mathbb{K} , \tag{5.1}$$

where $\mathbb{X} \in M_\Lambda$ is a periodic variable and $\mathbb{K} \in \Lambda$ is a discrete variable. Hence, we identify $\mathbb{X}$ as the configuration variable and $\mathbb{K}$ as the conjugate variable in the modular framework. Then, the classical Hamiltonian function for the harmonic oscillator can be expressed as

$$\mathcal{H}(\mathbb{X}, \mathbb{K}) = \frac{\Omega}{2} G(\mathbb{X} + \mathbb{K}, \mathbb{X} + \mathbb{K}) . \tag{5.2}$$

With an inspiration from the standard Legendre transform, we make the ansatz that the modular Legendre transform can be written in the form

$$\mathcal{L}(\mathbb{X}, \dot{\mathbb{X}}) = \mathcal{B}(\mathbb{X}, \dot{\mathbb{X}}, \mathbb{K}(\mathbb{X}, \dot{\mathbb{X}})) - \mathcal{H}(\mathbb{X}, \mathbb{K}(\mathbb{X}, \dot{\mathbb{X}})) , \tag{5.3}$$

where $\mathcal{B}$ is the symplectic 1-form, also known as the Berry phase [40], which is to be determined, and the function $\mathbb{K}(\mathbb{X}, \dot{\mathbb{X}})$ is also to be determined.

The second task is to find the Berry phase $\mathcal{B}$. For this purpose, we analyze the step (3.23) of the construction in Section 3.3.[2] We identify the summation parameter $\mathbb{K}$ in the said equation with our conjugate variable $\mathbb{K}$ here, since they both represent the remainder part in $\mathbb{Q}$. Moreover, we identify $\frac{i}{\hbar} \delta t \, \mathcal{L}$ with the exponent in the right-hand side of (3.23) in the limit $\delta t \to 0$. We get

$$\mathcal{L} = -\dot{\mathbb{X}} \cdot \mathbb{A}(\mathbb{X}) + \omega(\mathbb{X} + \mathbb{K}, \dot{\mathbb{X}}) - \frac{\Omega}{2} G(\mathbb{X} + \mathbb{K}, \mathbb{X} + \mathbb{K}) . \tag{5.4}$$

Hence, we find that the Berry phase is given by

$$\mathcal{B} = -\dot{\mathbb{X}} \cdot \mathbb{A}(\mathbb{X}) + \omega(\mathbb{X} + \mathbb{K}, \dot{\mathbb{X}}) . \tag{5.5}$$

Note that this Berry phase recovers the standard expression $\mathcal{B} = \tilde{x} \cdot \dot{x}$ if we use the Schrödinger gauge fixing as in (4.31) and set $\mathbb{K} = 0$.

The final task is to determine the function $\dot{\mathbb{X}}(\mathbb{X}, \mathbb{K})$, which shall give $\mathbb{K}(\mathbb{X}, \dot{\mathbb{X}})$ upon inversion. Recall that this step is given in the standard Legendre transform by $\dot{q} = \partial \mathcal{H}(q, p)/\partial p$. We would like to imitate this formula by taking the derivative of the Hamiltonian function

[2] Integrating out the momentum in $\int \mathcal{D}q \, \mathcal{D}p \, e^{\int p\dot{q} - \mathcal{H}(q,p)}$ is analogous to the inversion trick we used in (3.26). Therefore, the Berry phase has to be identified before the inversion.

$\mathcal{H}(\mathbb{X},\mathbb{K})$ with respect to the conjugate variable $\mathbb{K}$. However, $\mathbb{K}$ is a discrete variable and the said derivative is not well-defined.

Recall that the Hamiltonian $\mathcal{H}$ is originally a function of $\mathbb{Q} = \mathbb{X} + \mathbb{K}$. Therefore, the missing derivative with respect to $\mathbb{K}$ can equivalently be expressed as a partial derivative with respect to $\mathbb{X}$ (or $\mathbb{Q}$). Hence, we postulate

$$\dot{\mathbb{X}}^A = (\omega^{-1})^{AB} \frac{\partial \mathcal{H}(\mathbb{X},\mathbb{K})}{\partial \mathbb{X}^B} \,. \tag{5.6}$$

For the harmonic oscillator, we find $\dot{\mathbb{X}} = \Omega\, \omega^{-1} G\, (\mathbb{X} + \mathbb{K})$ and subsequently $\mathbb{K} = -\mathbb{X} - \Omega^{-1}\omega^{-1} G\, \dot{\mathbb{X}}$. Inserting this expression for $\mathbb{K}(\mathbb{X}, \dot{\mathbb{X}})$ into the Lagrangian (5.4) gives the modular Lagrangian function (3.35b) that we found in Section 3.3. In conclusion, this reconstruction of the modular Legendre transform produces the correct result that we found through the path integral construction of the harmonic oscillator.

We conjecture that the modular Legendre transform that we found here by inspecting the example of the harmonic oscillator holds in general for all systems. To summarize, we found the following prescription for the modular Legendre transform:

1. Start from a Hamiltonian function $\mathcal{H}(\mathbb{Q})$ on the phase space.
2. Calculate
$$\dot{\mathbb{X}}^A \equiv (\omega^{-1})^{AB} \frac{\partial \mathcal{H}(\mathbb{Q})}{\partial \mathbb{Q}^B} \,. \tag{5.7}$$
3. Invert the relation $\dot{\mathbb{X}}(\mathbb{Q})$ found in (5.7) to obtain $\mathbb{Q}(\dot{\mathbb{X}})$.
4. Evaluate the modular Lagrangian by
$$\boxed{\mathcal{L}_{\text{mod}}(\mathbb{X}, \dot{\mathbb{X}}) = -\dot{\mathbb{X}} \cdot \mathbb{A}(\mathbb{X}) + \omega(\mathbb{Q}(\dot{\mathbb{X}}), \dot{\mathbb{X}}) - \mathcal{H}(\mathbb{Q}(\dot{\mathbb{X}}))} \,. \tag{5.8}$$

This prescription can be applied to most physical systems in their Hamiltonian formalism to produce a modular Lagrangian function as in (5.8). As an example, we will demonstrate the modular Legendre transform on the Kepler problem in the next section.

5.2 Kepler problem

The Kepler problem is the mechanical system of a single, non-relativistic particle which moves in a Newtonian central potential. The Hamiltonian function of this system is given by

$$\mathcal{H}(q,p) = \frac{1}{2m}\, p^2 - \frac{\nu}{|q|} \;, \tag{5.9}$$

where m is the mass of the particle, ν is a constant, and $|q| = \sqrt{q^2}$ is the Euclidean norm. The Schrödinger Lagrangian function can be found as usual by the Legendre transform,

$$p = m\dot{q} \;, \qquad \mathcal{L}_{Sch}(q,\dot{q}) = \frac{1}{2}\, m\dot{q}^2 + \frac{\nu}{|q|} \;. \tag{5.10}$$

Now, instead of this standard Legendre transform, we will perform a modular Legendre transform on the Kepler Hamiltonian (5.9) following the prescription we found in the last section. We define

$$\dot{x}^j \equiv \frac{\partial \mathcal{H}(q,p)}{\partial p_j} = \frac{1}{m}\, p_j \tag{5.11a}$$

and

$$\dot{\tilde{x}}_j \equiv -\frac{\partial \mathcal{H}(q,p)}{\partial q^j} = -\frac{\nu}{|q|^3}\, q^j \;. \tag{5.11b}$$

We invert these equations and get

$$p_j = m\dot{x}^j \;, \qquad q^j = -\sqrt{\nu}\, \left|\dot{\tilde{x}}\right|^{-3/2} \dot{\tilde{x}}_j \;. \tag{5.12}$$

Let's consider the modular gauge $\alpha(\mathbb{X}) = \frac{2\gamma-1}{4\hbar}\, \eta(\mathbb{X},\mathbb{X})$ for some $\gamma \in \mathbb{R}$. This gives the Schrödinger gauge for $\gamma = 0$ and the momentum gauge for $\gamma = 1$. Then, we have $\mathbb{A}(\mathbb{X}) = (\gamma\tilde{x}, (\gamma-1)\, x)$. From (5.8), we finally get the modular Lagrangian

$$\mathcal{L}_{\text{mod}}(\mathbb{X},\dot{\mathbb{X}}) = -\gamma\, \eta(\mathbb{X},\dot{\mathbb{X}}) + x \cdot \dot{\tilde{x}} + \frac{1}{2}\, m\,\dot{x}^2 + 2\sqrt{\nu\, |\dot{\tilde{x}}|} \;. \tag{5.13}$$

The first term is a total derivative as expected from the gauge invariance. The Euler-Lagrange equations for this Lagrangian are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\, (\tilde{x} - m\dot{x}) = 0 \;, \tag{5.14a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x + \sqrt{\nu}\, \left|\dot{\tilde{x}}\right|^{-3/2} \dot{\tilde{x}}\right) = 0 \;. \tag{5.14b}$$

We see here the same trend as in Section (4.1) that the Lagrangian equations of motion for the modular Lagrangian have an additional time derivative compared to the Hamiltonian equations for the classical Hamiltonian. This results in an additional translation symmetry on $\mathcal{P}$. The equations (5.14) can be integrated with some arbitrary integration constants x_0, $\tilde{x}_0$ as

$$(\tilde{x} - \tilde{x}_0) - m\dot{x} = 0 \, , \tag{5.15a}$$

$$(x - x_0) + \sqrt{\nu} \, \left|\dot{\tilde{x}}\right|^{-3/2} \dot{\tilde{x}} = 0 \, . \tag{5.15b}$$

The constants x_0 and $\tilde{x}_0$ can be understood as corresponding to the lattice coordinates $-k$ and $-\tilde{k}$ in the decomposition $\mathbb{Q} = \mathbb{X} + \mathbb{K}$, though this interpretation is only heuristic because $(x_0, \tilde{x}_0)$ is not restricted to be an element of Λ. The equations (5.15) are equivalent to (5.11) under this identification.

This example demonstrates how the modular Legendre transform can be applied to more general systems beyond the harmonic oscillator. The modular Kepler Lagrangian (5.13) results in a set of equations of motion that are equivalent to the standard ones from the Schrödinger polarization. Nonetheless, the modular Lagrangian (5.13) can be a worthy starting point for alternative theories. Since the last term corresponding to the Newtonian potential is a function of the acceleration, the Modified Newtonian Dynamics [46] or similar proposals, where this is a desired feature, can be more naturally incorporated into a modular Lagrangian. This is subject for future research and beyond the scope of this thesis.

Chapter 6

Representations of the Weyl algebra - revisited

Now that we investigated some constructions based on the modular representation of the Weyl algebra $\mathcal{W}$, it is time to go back and introduce another set of representations. Unlike the previous ones, the representations that we will deal with in the following two chapters are irregular, i.e. inequivalent to the Schrödinger representation.

We will abandon Dirac's braket notation for the irregular representations and simply represent the quantum states with the corresponding functions such as φ.

6.1 Polymer representations

Inequivalent representations to the Schrödinger (or more generally the modular) representation can be obtained by relaxing one or several assumptions of the Stone–von Neumann uniqueness theorem. The polymer representations are obtained by relaxing the condition of regularity, i.e. of weak continuity of the Weyl operators $\hat{W}_{\mathbb{Y}}$ in either y or $\tilde{y}$ [54]. More precisely, relaxing the weak continuity in y produces the p-polymer representation, while relaxing the weak continuity in $\tilde{y}$ results in the q-polymer representation [15]. These two polymer representations are inequivalent to each other and to the Schrödinger representation. In addition, for each inequivalent polymer representation one has the freedom of working with the equivalent position or momentum polarizations.

For the sake of definiteness, let us focus on the p-polymer representation in the position polarization. This is the polymer representation inspired by LQG [7]. The lack of weak continuity implies that there exists no self-adjoint operator $\hat{p}$ such that $\hat{W}_{(y,0)} = e^{-iy\cdot\hat{p}/\hbar}$. The non-existence of the operator $\hat{p}_a = -i\hbar\,\partial/\partial q^a$ is what one could naturally expect from theories of quantum gravity such as LQG, where the notion of continuum space breaks down (in the sense at least that it is not fundamental but rather emergent).

In this polymer representation, the Weyl operators act in the same way as in the Schrödinger representation (2.17), i.e.

$$\hat{W}_{\mathbb{Y}}\, f(q) = e^{-\frac{i}{2}y\cdot\tilde{y}/\hbar}\, e^{i\tilde{y}\cdot q/\hbar}\, f(q-y). \tag{6.1}$$

The Hilbert space $\mathcal{H}_{\mathrm{P}}$, however, consists of functions $f(q)$ on $\mathbb{R}^d$ which vanish away from a countable subset $S_f \subset \mathbb{R}^d$, and are *square-summable* in the sense

$$\sum_{q\in S_f} |f(q)|^2 < \infty\ , \tag{6.2}$$

with an inner product given by

$$\langle f, g\rangle = \sum_{q\in S_f\cap S_g} f(q)^* g(q)\ . \tag{6.3}$$

This non-separable Hilbert space is also sometimes denoted by $\mathcal{H}_{\mathrm{P}} = L^2(\mathbb{R}^d_{\mathrm{d}}, \mathrm{d}\mu_{\mathrm{d}})$, where $\mathbb{R}^d_{\mathrm{d}}$ is the d-dimensional Euclidean space equipped with the discrete topology, and $\mathrm{d}\mu_{\mathrm{d}}$ is the associated discrete measure. In this polymer representation, another important feature is that the position operator actually possesses a complete set of *normalizable* eigenvectors $\left\{\varphi_q \middle|\ q\in\mathbb{R}^d\right\}$ such that

$$\hat{W}_{(0,\tilde{y})}\,\varphi_q = e^{i\tilde{y}\cdot q/\hbar}\,\varphi_q\ , \qquad \hat{W}_{(y,0)}\,\varphi_q = \varphi_{q+y}\ , \qquad \langle\varphi_q, \varphi_{q'}\rangle = \delta_{q,q'}\ , \tag{6.4}$$

where the right-hand side is the Kronecker delta. From this, it is indeed straightforward to see that $\hat{W}_{(0,\tilde{y})}$ is weakly continuous, so that there exists a self-adjoint operator $\hat{q}$ such that $\hat{q}\varphi_q = q\varphi_q$. However, we have

$$\lim_{y\to 0}\left\langle \varphi_q, \hat{W}_{(y,0)}\varphi_q\right\rangle = 0\ , \tag{6.5}$$

whereas $\hat{W}_{(0,0)} = 1$ and $\langle\varphi_q,\varphi_q\rangle = 1$. This means precisely that $\hat{W}_{(y,0)}$ fails to be weakly continuous in y, and therefore that $\hat{p}$ itself does not exist.

This polymer representation can also be obtained from a Gelfand–Naimark–Segal construction using a positive linear functional on $\mathcal{W}$ [7]. Furthermore, notice that we have presented here a polymer representation which is irregular in y and written in the position polarization. However, we can also change polarization, and also for both polarizations consider the polymer representation which is irregular in $\tilde{y}$ instead. These possibilities are presented and studied at length in [15], together with the corresponding GNS constructions. Finally, let us point out that in [13] the authors have given a Stone–von Neumann uniqueness theorem for the irregular polymer representations.

Now that we have reviewed the known Schrödinger, modular, and polymer representations of the Weyl algebra, we turn to the new result of this work and introduce the modular polymer representations.

6.2 Modular polymer representations

In this section, we "polymerize" the modular representation and obtain a new family of irregular representations of the Weyl algebra $\mathcal{W}$, which we call the *"modular polymer (MP) representations"*. From now on, we consider the Weyl algebra as an abstract object without reference to the position and momentum operators.

Just like the modular representations in a general modular gauge in Section (2.7), an MP representation is defined with respect to a modular lattice Λ and a smooth, real function $\alpha \in C^\infty(\mathcal{P})$ on the phase space. For simplicity, we will introduce these representations here in the Schrödinger gauge (2.50), $\alpha = \alpha_{\text{Sch}}$. We will comment on other gauge choices in Section 6.3.

Let Λ be a modular lattice as defined in (2.21). We denote by $\mathcal{H}_{\text{MP}}^{\Lambda} = l^2(E_\Lambda)$ the non-separable Hilbert space of square-summable sections of the $U(1)$-bundle $E_\Lambda \to T_\Lambda$ on the modular space, which is defined in (2.45). This means that each element $f \in \mathcal{H}_{\text{MP}}^{\Lambda}$ is supported on a countable subset S_f of T_Λ and satisfies

$$\|f\| = \sum_{\mathbb{X} \in S_f} |f(\mathbb{X})|^2 < \infty \, . \tag{6.6}$$

Moreover, the domain of each element $f \in \mathcal{H}_{\text{MP}}^{\Lambda}$ can be extended from T_Λ to $\mathcal{P}$ by the section condition (2.45), i.e. such that

$$f(\mathbb{X} + \mathbb{K}) = e^{ik\cdot\tilde{x}/\hbar} f(\mathbb{X}) \, , \qquad \mathbb{X} \in \mathcal{P} \, , \quad \mathbb{K} \in \Lambda \, . \tag{6.7}$$

On the Hilbert space $\mathcal{H}_{\text{MP}}^{\Lambda}$ we consider the inner product

$$\langle f, g \rangle = \sum_{\mathbb{X} \in S_f \cap S_g} f(\mathbb{X})^* g(\mathbb{X}) \, . \tag{6.8}$$

For each $\mathbb{X} \in \mathcal{P}$, we then define the function $\varphi_{\mathbb{X}}^{(\Lambda)} : \mathcal{P} \to \mathbb{C}$ as

$$\varphi_{\mathbb{X}}^{(\Lambda)}(\mathbb{Y}) = \begin{cases} e^{i(y-x)\cdot\tilde{x}/\hbar} \ , & \text{if } \mathbb{Y} - \mathbb{X} \in \Lambda \ , \\ 0 \ , & \text{otherwise} \ , \end{cases} \tag{6.9}$$

for any $\mathbb{Y} \in \mathcal{P}$. We will drop the superscript (Λ) from $\varphi_{\mathbb{X}}^{(\Lambda)}$ unless there is an ambiguity about which modular lattice we refer to. Note that the functions $\varphi_{\mathbb{X}}$ are supported on a single point $\mathbb{X} \in T_\Lambda$ in the modular space. Moreover, they satisfy $\varphi_{\mathbb{X}}(\mathbb{Y}+\mathbb{K}) = e^{ik\cdot\tilde{y}/\hbar}\,\varphi_{\mathbb{X}}(\mathbb{Y})$ for any $\mathbb{K} \in \Lambda$,{9} which is consistent with (6.7). Hence, we have $\varphi_{\mathbb{X}} \in \mathcal{H}_{\text{MP}}^{\Lambda}$ for every $\mathbb{X} \in \mathcal{P}$. Another useful property of the functions $\varphi_{\mathbb{X}}$ is that{10}

$$\varphi_{\mathbb{X}}(\mathbb{Y} + \mathbb{Y}') = e^{i(y+y'-x)\cdot\tilde{y}'/\hbar}\,\varphi_{\mathbb{X}-\mathbb{Y}'}(\mathbb{Y}) \tag{6.10}$$

for any $\mathbb{X}, \mathbb{Y}, \mathbb{Y}' \in \mathcal{P}$. Finally, for any $\mathbb{K} \in \Lambda$, we also have the identity{11}

$$\varphi_{\mathbb{X}+\mathbb{K}} = e^{-ik\tilde{x}/\hbar}\,\varphi_{\mathbb{X}} \ . \tag{6.11}$$

We therefore conclude that the set $\{\varphi_{\mathbb{X}} : \mathbb{X} \in T_\Lambda\}$ is an orthonormal basis for $\mathcal{H}_{\text{MP}}^{\Lambda}$.

It is instructive to compare the functions $\varphi_{\mathbb{X}}^{(\Lambda)} \in \mathcal{H}_{\text{MP}}^{\Lambda}$ to the modular vectors $|\mathbb{X}\rangle_\Lambda$. For example, one can see that identity (6.11) is analogous to the relation (2.43) for $\alpha = \alpha_{\text{Sch}}$. In what follows, our strategy will be to push this analogy forward. For each $\mathbb{Y} \in \mathcal{P}$, we define the action of $\hat{W}_{\mathbb{Y}}$ on the set $\{\varphi_{\mathbb{X}} : \mathbb{X} \in \mathcal{P}\}$ by

$$\hat{W}_{\mathbb{Y}}\,\varphi_{\mathbb{X}} = e^{\frac{i}{2}y\cdot\tilde{y}/\hbar}\,e^{i\tilde{y}\cdot x/\hbar}\,\varphi_{\mathbb{X}+\mathbb{Y}} \ . \tag{6.12}$$

The motivation for this definition comes from the equation (2.42). A straightforward calculation shows that{12}

$$\hat{W}_{\mathbb{Y}}\hat{W}_{\mathbb{Y}'}\,\varphi_{\mathbb{X}} = e^{\frac{i}{2}\omega(\mathbb{Y},\mathbb{Y}')/\hbar}\,\hat{W}_{\mathbb{Y}+\mathbb{Y}'}\,\varphi_{\mathbb{X}} \ , \tag{6.13}$$

and{13}

$$\left\langle \hat{W}_{\mathbb{Y}}\,\varphi_{\mathbb{X}}, \varphi_{\mathbb{X}'} \right\rangle = \left\langle \varphi_{\mathbb{X}}, \hat{W}_{-\mathbb{Y}}\,\varphi_{\mathbb{X}'} \right\rangle \ . \tag{6.14}$$

Hence, we confirmed that the action of the Weyl operators on $\mathcal{H}_{\text{MP}}^{\Lambda}$ as defined in (6.12) constitutes a representation of the Weyl algebra. We note that this representation diagonalizes the subalgebra $\mathcal{W}_\Lambda$, since for any $\mathbb{K} \in \Lambda$,

$$\hat{W}_{\mathbb{K}}\,\varphi_{\mathbb{X}} = e^{\frac{i}{2}k\cdot\tilde{k}/\hbar}\,e^{i\omega(\mathbb{K},\mathbb{X})/\hbar}\,\varphi_{\mathbb{X}} \ . \tag{6.15}$$

Finally, from

$$\left\langle \varphi_{\mathbb{X}}, \hat{W}_{\mathbb{Y}}\, \varphi_{\mathbb{X}} \right\rangle = \begin{cases} 0\,, & \text{when } \mathbb{Y} \notin \Lambda\,, \\ e^{\frac{i}{2} y\cdot\tilde{y}/\hbar}\, e^{i\omega(\mathbb{Y},\mathbb{X})/\hbar}\,, & \text{when } \mathbb{Y} \in \Lambda\,, \end{cases} \tag{6.16}$$

it is clear that $\mathbb{Y} \mapsto \hat{W}_{\mathbb{Y}}$ is not weakly continuous. Therefore, there are no self-adjoint operators $\hat{q}$ and $\hat{p}$ such that $\hat{W}_{\mathbb{Y}} = e^{i(\tilde{y}\cdot\hat{q} - y\cdot\hat{p})/\hbar}$ for all $\mathbb{Y} \in \mathcal{P}$. This result shows that the MP representations cannot be unitarily equivalent to any of the Schrödinger, modular, or polymer representations.

6.3 MP representations in a general gauge

We have so far introduced the MP representations in the Schrödinger gauge $\alpha = \alpha_{\text{Sch}}$. Much like the modular representations, the MP representations can be constructed with different gauge functions.

Given a modular lattice Λ and a real function $\alpha \in C^\infty(\mathcal{P})$, the corresponding MP Hilbert space $\mathcal{H}^{\Lambda,\alpha}_{\text{MP}} = l^2(E^\alpha_\Lambda)$ is the space of square-summable sections of the $U(1)$-bundle $E^\alpha_\Lambda \to T_\Lambda$ defined in (2.45). Each element $f \in \mathcal{H}^{\Lambda,\alpha}_{\text{MP}}$ can be extended to $\mathcal{P}$ by $f(\mathbb{X} + \mathbb{K}) = e^{-i\beta_\alpha(\mathbb{X},\mathbb{K})} f(\mathbb{X})$ for $\mathbb{K} \in \Lambda$ and $\mathbb{X} \in \mathcal{P}$, where $\beta_\alpha(\mathbb{X}, \mathbb{K})$ is defined in (2.44). The definition (6.9) of the MP basis vectors is generalized to

$$\varphi^{(\Lambda,\alpha)}_{\mathbb{X}}(\mathbb{Y}) = \begin{cases} e^{-i\beta_\alpha(\mathbb{X},\mathbb{Y}-\mathbb{X})}\,, & \text{if } \mathbb{Y} - \mathbb{X} \in \Lambda\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \tag{6.17}$$

for any $\mathbb{Y} \in \mathcal{P}$. These functions satisfy

$$\hat{W}_{\mathbb{Y}}\, \varphi^{(\Lambda,\alpha)}_{\mathbb{X}} = e^{i\alpha(\mathbb{X}) - i\alpha(\mathbb{X}+\mathbb{Y})}\, e^{\frac{i}{2}\omega(\mathbb{Y},\mathbb{X})/\hbar}\, \varphi^{(\Lambda,\alpha)}_{\mathbb{X}+\mathbb{Y}} \tag{6.18}$$

and

$$\varphi^{(\Lambda,\alpha)}_{\mathbb{X}+\mathbb{K}} = e^{i\beta_\alpha(\mathbb{X},\mathbb{K})}\, \varphi^{(\Lambda,\alpha)}_{\mathbb{X}} \tag{6.19}$$

for any $\mathbb{X}, \mathbb{Y} \in \mathcal{P}$ and $\mathbb{K} \in \Lambda$.

Now we can show that the MP representations with the same lattice but different gauge functions are unitarily equivalent to each other. The same holds for two MP representations whose lattices have the same bilagrangian structure (ω, η).

Let Λ and Λ' be two modular lattices on $\mathcal{P}$ that are related to each other by $\Lambda' = M\Lambda$ with $M^T\omega M = \omega$ and $M^T\eta M = \eta$. Let $\alpha, \alpha' \in C^\infty(\mathcal{P})$. Then, there is a unitary map $\mathcal{U} : \mathcal{H}_{\rm MP}^{\Lambda,\alpha} \to \mathcal{H}_{\rm MP}^{\Lambda',\alpha'}$ given by

$$\mathcal{U}\varphi_{\mathbb{X}}^{(\Lambda,\alpha)} = e^{-i\alpha'(M\mathbb{X})+i\alpha(\mathbb{X})}\,\varphi_{M\mathbb{X}}^{(\Lambda',\alpha')}\ , \qquad \mathcal{U}\hat{W}_{\mathbb{X}}\mathcal{U}^\dagger = \hat{W}_{M\mathbb{X}}\ . \tag{6.20}$$

This shows that the MP representations corresponding to two modular lattices with respect to the same bilagrangian structure (ω, η) are unitarily equivalent to each other. In particular, this implies that spatial rotations in $O(d)$ preserve the equivalence class of the MP representation.

Chapter 7

Dynamics of the MP harmonic oscillator

A natural next step after developing the new modular polymer representations is to investigate the physical consequences of our construction. For this, we turn our attention once again to the harmonic oscillator. We would like to consider the dynamics of a quantum harmonic oscillator in one dimension with the Hamiltonian operator

$$\hat{H} = \frac{1}{2m}\,\hat{p}^2 + \frac{1}{2}\,m\,\omega^2\,\hat{q}^2 \tag{7.1}$$

in an MP representation with the modular lattice $\Lambda = \{(\ell n, \tilde{\ell}\tilde{n}) \in \mathbb{R}^2 : n, \tilde{n} \in \mathbb{Z}\}$ and the Schrödinger gauge (2.50). However, the operators $\hat{p}$ and $\hat{q}$ (let alone their squares) do not exist in an MP representation. Therefore, we have to construct approximants using the Weyl operators.

The usual procedure in the literature on polymer quantization begins with choosing a "coarse-graining scale". For the MP representation we need two scales: a length scale λ to approximate $\hat{p}$ and a momentum scale $\tilde{\lambda}$ to approximate $\hat{q}$.[1] In fact, we already have a pair of scales ℓ and $\tilde{\ell} = 2\pi\hbar/\ell$ naturally available in an MP representation, but it will be more natural and general to consider independent coarse-graining scales. There are multiple reasons for this. First, the modified operators $\hat{q}$ and $\hat{p}$ at the scale ℓ and $\tilde{\ell}$ will belong to the subalgebra $\mathcal{W}_\Lambda$ and therefore commute, which is undesirable. Second, we want to be able to take the limits $\lambda \to 0$ and $\tilde{\lambda} \to 0$ independently, but the scales ℓ and

[1] The coarse-graining scales $\lambda, \tilde{\lambda}$ are unrelated to the matrices with the same symbols that we introduced in Section (2.6). We will not use the latter in this chapter.

$\tilde{\ell}$ are constrained to satisfy $\ell\tilde{\ell} = 2\pi\hbar$. Finally, for the sake of generality, we would like to consider an arbitrary scale pair $(\lambda, \tilde{\lambda})$, which can be taken at the end of the calculations to be equal to $(\ell, \tilde{\ell})$ if it turns out to be desirable.

Let us therefore introduce an arbitrary length scale λ and an arbitrary momentum scale $\tilde{\lambda}$, in addition to the "modular scales" $(\ell, \tilde{\ell})$. Next, we introduce a lattice $\mathcal{L}_{\mathbb{X}_0}^{\Lambda,(\lambda,\tilde{\lambda})} \subset T_\Lambda$ in the modular space for any $\mathbb{X}_0 \in \mathbb{R}^2$ by

$$\mathcal{L}_{\mathbb{X}_0}^{\Lambda,(\lambda,\tilde{\lambda})} \equiv \left\{ \mathbb{X} \in \mathbb{R}^2 : \exists\, n, \tilde{n} \in \mathbb{Z}, \exists\, \mathbb{K} \in \Lambda : \mathbb{X} = \mathbb{X}_0 + (n\lambda, \tilde{n}\tilde{\lambda}) + \mathbb{K} \right\} / \Lambda \,. \tag{7.2}$$

There are then three cases to consider: [14]

1. Both λ/ℓ and $\tilde{\lambda}/\tilde{\ell}$ are **irrational** numbers. In this case, $\mathcal{L}_{\mathbb{X}_0}^{\Lambda,(\lambda,\tilde{\lambda})}$ is an infinite set, which is dense in T_Λ with respect to the continuum topology.

2. Both λ/ℓ and $\tilde{\lambda}/\tilde{\ell}$ are **rational** numbers. In this case, $\mathcal{L}_{\mathbb{X}_0}^{\Lambda,(\lambda,\tilde{\lambda})}$ is a finite set.

3. A combination of the previous two cases.

These three cases have different physical consequences and we will discuss them separately.

In each case, we consider the elements $\psi \in \mathcal{H}_{\mathrm{MP}}^{\Lambda}$ of the MP Hilbert space which are supported on the lattice $\mathcal{L}_{\mathbb{X}_0}^{\Lambda,(\lambda,\tilde{\lambda})} \subset T_\Lambda$. These elements belong to a *separable* Hilbert space $\mathcal{H}_{\mathrm{MP}}^{\Lambda,(\lambda,\tilde{\lambda}),\mathbb{X}_0}$, which is a *superselection sector* of the full MP Hilbert space. The MP Hilbert space can be written as a direct sum

$$\mathcal{H}_{\mathrm{MP}}^{\Lambda} = \bigoplus_{\mathbb{X}_0 \in T_\Lambda / \mathcal{L}_0^{\Lambda,(\lambda,\tilde{\lambda})}} \mathcal{H}_{\mathrm{MP}}^{\Lambda,(\lambda,\tilde{\lambda}),\mathbb{X}_0} \tag{7.3}$$

over the superselection sectors labeled by $\mathbb{X}_0 = (x_0, \tilde{x}_0)$. Finally, we define the operators $\widehat{q^2}$ and $\widehat{p^2}$ on each superselection sector by [15]

$$\widehat{q^2_{\tilde{\lambda}}} \equiv \frac{\hbar^2}{\tilde{\lambda}^2} \left(2 - \hat{W}_{(0,\tilde{\lambda})} - \hat{W}_{(0,-\tilde{\lambda})} \right) \,, \tag{7.4a}$$

$$\widehat{p^2_{\lambda}} \equiv \frac{\hbar^2}{\lambda^2} \left(2 - \hat{W}_{(\lambda,0)} - \hat{W}_{(-\lambda,0)} \right) \,. \tag{7.4b}$$

These definitions are based on approximations that are valid in the regimes $q \ll \hbar/\tilde{\lambda}$ and $p \ll \hbar/\lambda$, but we will consider them as fundamental definitions at all scales. Note that the operators $\widehat{q^2_{\tilde{\lambda}}}$ and $\widehat{p^2_{\lambda}}$ map each superselection sector onto itself. Finally, let us also point out that here we are considering the regularized operators corresponding to the square of position and momentum, and not the operators squared such as $\hat{q}^2$ and $\hat{p}^2$. While this latter choice also leads to a well-defined regularization of the Hamiltonian, we have made the choice which is usually followed in the literature on polymer quantization [7, 10].

Using the redefined position and momentum operators, we obtain a regularized Hamiltonian of the form

$$\hat{H}_{(\lambda,\tilde{\lambda})} \equiv \frac{\hbar^2}{2m\lambda^2}\left(2 - \hat{W}_{(\lambda,0)} - \hat{W}_{(-\lambda,0)}\right) + \frac{m\omega^2\hbar^2}{2\tilde{\lambda}^2}\left(2 - \hat{W}_{(0,\tilde{\lambda})} - \hat{W}_{(0,-\tilde{\lambda})}\right) . \tag{7.5}$$

In the following, we will make the additional assumption

$$\lambda\tilde{\lambda} \in 2\pi\hbar\,\mathbb{Z} , \tag{7.6}$$

and analyze the spectrum of the Hamiltonian (7.5) in the first two cases of scale ratios. We will motivate this assumption in the following subsection by investigating the solutions under a power-law ansatz, but the generic case $\lambda\tilde{\lambda} \notin 2\pi\hbar\,\mathbb{Z}$ remains open.

Notice also that the third case mentioned above (i.e. the combination of rational and irrational ratios) does not arise under this assumption. Finally, because λ and $\tilde{\lambda}$ are related by (7.6), in particular $\lambda\tilde{\lambda} \geq 2\pi\hbar$, it is not possible to take the limits $\lambda \to 0$ and $\tilde{\lambda} \to 0$ simultaneously under this assumption. This issue disappears in the classical limit, where $\hbar \to 0$.

7.1 Irrational scale ratios

Let's start with the case when both λ/ℓ and $\tilde{\lambda}/\tilde{\ell}$ are irrational numbers. An element $\psi \in \mathcal{H}_{\text{MP}}^{\Lambda,(\lambda,\tilde{\lambda}),\mathbb{X}_0}$ of the superselection sector can be written uniquely$^{\{16\}}$ as

$$\psi = \sum_{n,\tilde{n}\in\mathbb{Z}} A_{n,\tilde{n}}\,\varphi_{\mathbb{X}_0+(n\lambda,\tilde{n}\tilde{\lambda})} , \tag{7.7}$$

where $A_{n,\tilde{n}}$ are complex numbers. From the action of the Hamilton operator (7.5) on this state, we get

$$\begin{aligned}\hat{H}_{(\lambda,\tilde{\lambda})}\,\psi = \sum_{n,\tilde{n}\in\mathbb{Z}} \Bigg(&\frac{\hbar^2}{2m\lambda^2}\left(2A_{n,\tilde{n}} - A_{n-1,\tilde{n}} - A_{n+1,\tilde{n}}\right) \\ &+ \frac{m\omega^2\hbar^2}{2\tilde{\lambda}^2}\left(2A_{n,\tilde{n}} - e^{i\tilde{\lambda}(x_0+n\lambda)/\hbar}\,A_{n,\tilde{n}-1} - e^{-i\tilde{\lambda}(x_0+n\lambda)/\hbar}\,A_{n,\tilde{n}+1}\right)\Bigg)\,\varphi_{\mathbb{X}_0+(n\lambda,\tilde{n}\tilde{\lambda})} ,\end{aligned} \tag{7.8}$$

where we used the equation (6.12), and shifted the summation variables n and $\tilde{n}$. Hence, if ψ is an eigenvector of the Hamilton operator $\hat{H}_{(\lambda,\tilde{\lambda})}$ with eigenvalue $E_{(\lambda,\tilde{\lambda})}$, the coefficients $A_{n,\tilde{n}}$ are required to satisfy the linear recurrence relation

$$\begin{aligned} E_{(\lambda,\tilde{\lambda})}\, A_{n,\tilde{n}} &= \frac{\hbar^2}{2m\lambda^2}\left(2A_{n,\tilde{n}} - A_{n-1,\tilde{n}} - A_{n+1,\tilde{n}}\right) \\ &\quad + \frac{m\omega^2\hbar^2}{2\tilde{\lambda}^2}\left(2A_{n,\tilde{n}} - e^{i\tilde{\lambda}(x_0+n\lambda)/\hbar}\, A_{n,\tilde{n}-1} - e^{-i\tilde{\lambda}(x_0+n\lambda)/\hbar}\, A_{n,\tilde{n}+1}\right) . \end{aligned} \tag{7.9}$$

Before we analyze the spectrum of the Hamiltonian under the assumption (7.6), we are going to motivate this assumption by examining a power-law ansatz. Consider

$$A_{n,\tilde{n}} = a^n\, b^{\tilde{n}}\, c^{n\tilde{n}} , \tag{7.10}$$

where $a, b, c \in \mathbb{C}$ are three complex numbers that are independent of n and $\tilde{n}$. If we substitute this ansatz in the recurrence relation (7.9), we get

$$\begin{aligned} E_{(\lambda,\tilde{\lambda})} &= \frac{\hbar^2}{2m\lambda^2}\left(2 - a\, c^{\tilde{n}} - \frac{1}{a\, c^{\tilde{n}}}\right) \\ &\quad + \frac{m\omega^2\hbar^2}{2\tilde{\lambda}^2}\left(2 - \left(b\, e^{-ix_0\tilde{\lambda}/\hbar}\right)\left(c\, e^{-i\lambda\tilde{\lambda}/\hbar}\right)^n - \left(b\, e^{-ix_0\tilde{\lambda}/\hbar}\right)^{-1}\left(c\, e^{-i\lambda\tilde{\lambda}/\hbar}\right)^{-n}\right) . \end{aligned} \tag{7.11}$$

The left-hand side of (7.11) is independent of n and $\tilde{n}$, thus the right-hand side must also be independent of these variables. This requires

$$c = 1 \qquad \text{and} \qquad \frac{\lambda\tilde{\lambda}}{2\pi\hbar} \in \mathbb{Z} . \tag{7.12}$$

Hence, the solutions of the recurrence relation (7.9) that follow the power-law ansatz (7.10) exist only under the conditions (7.12). This motivates us to consider the condition $\lambda\tilde{\lambda} \in 2\pi\hbar\mathbb{Z}$ in general without the power-law ansatz to simplify (7.9).

Let's multiply both sides of the equation (7.9) with a factor of $e^{-i(n\tilde{r}-\tilde{n}r)}$ for arbitrary $r, \tilde{r} \in \mathbb{R}$, and sum the resulting expression over all $n, \tilde{n} \in \mathbb{Z}$. Assuming that the sum

$$\phi(r,\tilde{r}) \equiv \sum_{n,\tilde{n}\in\mathbb{Z}} e^{-i(n\tilde{r}-\tilde{n}r)}\, A_{n,\tilde{n}} \tag{7.13}$$

has a finite value, we obtain

$$E_{(\lambda,\tilde{\lambda})}\, \phi(r,\tilde{r}) = \frac{\hbar^2}{m\lambda^2}\left(1 - \cos\tilde{r}\right)\phi(r,\tilde{r}) + \frac{m\omega^2\hbar^2}{\tilde{\lambda}^2}\left(1 - \cos\!\left(r + \tilde{\lambda}x_0/\hbar\right)\right)\phi(r,\tilde{r}) . \tag{7.14}$$

One can notice that this is not a differential equation, unlike in the case of standard polymer quantization [7, 10]. The factor $\phi(r,\tilde{r})$ therefore simply drops from this equation and we find

$$E_{(\lambda,\tilde{\lambda})} = \frac{\hbar^2}{m\lambda^2}\,(1-\cos\tilde{r}) + \frac{m\omega^2\hbar^2}{\tilde{\lambda}^2}\left(1-\cos\!\left(r+\tilde{\lambda}x_0/\hbar\right)\right)\ . \tag{7.15}$$

Hence, the spectrum of the Hamiltonian (7.5) is continuous, and it is bounded from both below and above, such that

$$0 \le E_{(\lambda,\tilde{\lambda})} \le \frac{2\hbar^2}{m\lambda^2} + \frac{2m\omega^2\hbar^2}{\tilde{\lambda}^2}\ . \tag{7.16}$$

The upper bound diverges as $\lambda \to 0$ or $\tilde{\lambda} \to 0$.

As the state $\phi(r,\tilde{r})$ cancelled out from the equation (7.14), we found the energy spectrum of the system without finding the states that realize these energy eigenvalues. In fact, the solution

$$\psi = \sum_{n,\tilde{n}\in\mathbb{Z}} e^{i(n\tilde{r}-\tilde{n}r)}\,\varphi_{\mathbb{X}_0+(n\lambda,\tilde{n}\tilde{\lambda})} \tag{7.17}$$

for arbitrary $r,\tilde{r} \in \mathbb{R}$ would be an eigenstate of the Hamiltonian (7.5) with the energy eigenvalue given in (7.15). However, the state in (7.17) is not normalizable, and therefore it is not an element of the MP Hilbert space.

This situation is familiar from Schrödinger quantum mechanics, where the plane waves

$$|\Psi_k\rangle = \int_{\mathbb{R}} \mathrm{d}x\ e^{-ikx}\,|x\rangle\ ,\quad k\in\mathbb{R}\ , \tag{7.18}$$

are not in $L^2(\mathbb{R})$, nevertheless they serve as a useful tool to build normalizable states. One can treat the solutions in (7.17) analogously to the plane waves. In this analogy, the local MP basis vectors $\varphi_{\mathbb{X}_0+(n\lambda,\tilde{n}\tilde{\lambda})}$ match the Schrödinger position eigenvectors $|x\rangle$, the discrete labels $n,\tilde{n}\in\mathbb{Z}$ match the continuous position variable $x\in\mathbb{R}$, and the parameters $r,\tilde{r}\in\mathbb{R}$ match the wave vector $k\in\mathbb{R}$.

In order to investigate the plane wave limit of the energy in (7.15), consider setting $\omega=0$, so that the second term in the Hamiltonian (7.5) disappears, and taking the limit $\lambda\to 0$ while the ratio $\tilde{r}/\lambda$ is held constant. In this limit the energy becomes

$$E_{(\lambda,\tilde{\lambda})} \to \frac{(\tilde{r}\hbar/\lambda)^2}{2m}\ , \tag{7.19}$$

where $\tilde{r}\hbar/\lambda\in\mathbb{R}$ can be interpreted as the momentum and we obtain the well-known formula for kinetic energy.

7.2 Rational scale ratios

Next, we consider the case when both λ/ℓ and $\tilde{\lambda}/\tilde{\ell}$ are rational numbers, and once again under the assumption $\lambda\tilde{\lambda} \in 2\pi\hbar\mathbb{Z}$. For the sake of definiteness, let us write

$$N\lambda = M\ell \qquad \text{and} \qquad \tilde{N}\tilde{\lambda} = \tilde{M}\tilde{\ell}\,, \tag{7.20}$$

for $N, \tilde{N}, M, \tilde{M} \in \mathbb{Z}_+$ where the pairs (N, M) and $(\tilde{N}, \tilde{M})$ are coprime.

The key difference with the first case treated above is that the lattice $\mathcal{L}_{\mathbb{X}_0}^{\Lambda,(\lambda,\tilde{\lambda})} \subset T_\Lambda$ is now finite. Hence, the expansion of an element $\psi \in \mathcal{H}_{\mathrm{MP}}^{\Lambda,(\lambda,\tilde{\lambda}),\mathbb{X}_0}$ as in (7.7) contains only finitely many terms, and can be written as

$$\psi = \sum_{n=0}^{N-1}\sum_{\tilde{n}=0}^{\tilde{N}-1} A_{n,\tilde{n}}\,\varphi_{\mathbb{X}_0+(n\lambda,\tilde{n}\tilde{\lambda})}\ . \tag{7.21}$$

In this expression, the coefficients $A_{n,\tilde{n}}$ are defined a priori only for $0 \leq n \leq N-1$ and $0 \leq \tilde{n} \leq \tilde{N}-1$. This prevents us from freely rearranging the summation variables in order to factor out φ when acting with the Hamiltonian operator.

A lengthy rewriting of the action of the Hamiltonian on the state (7.21) shows however that we can consistently define the coefficients $A_{n,\tilde{n}}$ for all $n, \tilde{n} \in \mathbb{Z}$ by imposing[{17}]

$$A_{n+N,\tilde{n}} \equiv e^{iN\lambda\,\tilde{x}_0/\hbar}\,A_{n,\tilde{n}} \qquad \text{and} \qquad A_{n,\tilde{n}+\tilde{N}} \equiv A_{n,\tilde{n}}\ . \tag{7.22}$$

This implies in particular that

$$A_{n+N,\tilde{n}}\,\varphi_{\mathbb{X}_0+((n+N)\lambda,\tilde{n}\tilde{\lambda})} = A_{n,\tilde{n}}\,\varphi_{\mathbb{X}_0+(n\lambda,\tilde{n}\tilde{\lambda})} \quad \text{and} \quad A_{n,\tilde{n}+\tilde{N}}\,\varphi_{\mathbb{X}_0+(n\lambda,(\tilde{n}+\tilde{N})\tilde{\lambda})} = A_{n,\tilde{n}}\,\varphi_{\mathbb{X}_0+(n\lambda,\tilde{n}\tilde{\lambda})}\ . \tag{7.23}$$

With this definition, we find

$$\begin{aligned}\hat{H}_{(\lambda,\tilde{\lambda})}\,\psi = \sum_{n=0}^{N-1}\sum_{\tilde{n}=0}^{\tilde{N}-1}\bigg(&\frac{\hbar^2}{2m\lambda^2}\,(2A_{n,\tilde{n}} - A_{n-1,\tilde{n}} - A_{n+1,\tilde{n}}) \\ &+ \frac{m\omega^2\hbar^2}{2\tilde{\lambda}^2}\Big(2A_{n,\tilde{n}} - e^{i\tilde{\lambda}(x_0+n\lambda)/\hbar}\,A_{n,\tilde{n}-1} - e^{-i\tilde{\lambda}(x_0+n\lambda)/\hbar}\,A_{n,\tilde{n}+1}\Big)\bigg)\,\varphi_{\mathbb{X}_0+(n\lambda,\tilde{n}\tilde{\lambda})}\ .\end{aligned} \tag{7.24}$$

For the eigenstates ψ of the Hamiltonian $\hat{H}_{(\lambda,\tilde{\lambda})}$, we obtain the relation

$$
\begin{aligned}
E_{(\lambda,\tilde{\lambda})}\, A_{n,\tilde{n}} = & \frac{\hbar^2}{2m\lambda^2} \left(2A_{n,\tilde{n}} - A_{n-1,\tilde{n}} - A_{n+1,\tilde{n}}\right) \\
& + \frac{m\omega^2\hbar^2}{2\tilde{\lambda}^2} \left(2A_{n,\tilde{n}} - e^{i\tilde{\lambda}x_0/\hbar}\, A_{n,\tilde{n}-1} - e^{-i\tilde{\lambda}x_0/\hbar}\, A_{n,\tilde{n}+1}\right) . \qquad (7.25)
\end{aligned}
$$

This equation is identical to the recurrence relation (7.9) in the previous case (using the assumption (7.6)), but its solutions are also constrained by (7.22). If we consider the solutions of the form

$$
A_{n,\tilde{n}} = e^{i(n\tilde{r}-\tilde{n}r)} \qquad (7.26)
$$

for $r, \tilde{r} \in \mathbb{R}$ as before, we find the restrictions

$$
\frac{\tilde{N}r}{2\pi} \in \mathbb{Z} \qquad \text{and} \qquad \frac{N\tilde{r}}{2\pi} - \frac{N\lambda\tilde{x}_0}{2\pi\hbar} \in \mathbb{Z}\, . \qquad (7.27)
$$

Since these solutions are invariant under shifting the parameters $r, \tilde{r}$ by a multiple of 2π, we find exactly $N \times \tilde{N}$ different solutions of the form (7.26). Hence, we can write

$$
r = \frac{2\pi k}{\tilde{N}} \qquad \text{and} \qquad \tilde{r} = \frac{2\pi\tilde{k}}{N} + \frac{\lambda\tilde{x}_0}{\hbar} \qquad (7.28)
$$

for $k = 1, ..., \tilde{N}$ and $\tilde{k} = 1, ..., N$. We obtain the energy eigenstates

$$
\psi = \sum_{n=0}^{N-1} \sum_{\tilde{n}=0}^{\tilde{N}-1} e^{2\pi i(\tilde{k}n/N - k\tilde{n}/\tilde{N})}\, e^{in\lambda\tilde{x}_0/\hbar}\, \varphi_{\mathbb{X}_0+(n\lambda,\tilde{n}\tilde{\lambda})} \qquad (7.29)
$$

with the corresponding energy eigenvalues

$$
E_{(\lambda,\tilde{\lambda})} = \frac{\hbar^2}{m\lambda^2} \left(1 - \cos\left[2\pi\left(\frac{\tilde{k}}{N} + \frac{\tilde{x}_0}{\tilde{\lambda}}\right)\right]\right) + \frac{m\omega^2\hbar^2}{\tilde{\lambda}^2} \left(1 - \cos\left[2\pi\left(\frac{k}{\tilde{N}} + \frac{x_0}{\lambda}\right)\right]\right) . \quad (7.30)
$$

The spectrum of the Hamiltonian operator in the superselection sector $\mathcal{H}_{\mathrm{MP}}^{\Lambda,(\lambda,\tilde{\lambda}),\mathbb{X}_0}$ consists of these $N \times \tilde{N}$ discrete values. {18}

If one chooses the coarse-graining scale for the Hamiltonian to be equal to the MP scale, i.e. $\lambda = \ell$ and $\tilde{\lambda} = \tilde{\ell}$, then there is a single energy eigenstate in each superselection sector that is given by $\psi = e^{in\lambda\tilde{x}_0/\hbar}\, \varphi_{\mathbb{X}_0}$ with the energy

$$
E_{(\ell,\tilde{\ell})} = \frac{2\hbar^2}{m\ell^2} \sin^2\left(\frac{\pi\tilde{x}_0}{\tilde{\ell}}\right) + \frac{2m\omega^2\hbar^2}{\tilde{\ell}^2} \sin^2\left(\frac{\pi x_0}{\ell}\right) . \qquad (7.31)
$$

There are two important differences between our results here (when λ/ℓ and $\tilde{\lambda}/\tilde{\ell}$ are rational numbers) and those in the previous subsection (when the ratios are irrational numbers). Firstly, we find here only a finite number of distinct elements in the energy spectrum. Secondly, the plane-wave-like solutions (7.29) are normalizable, unlike (7.17) in the previous case. The existence of normalizable plane waves is a new feature of the MP representation, which has no analog in the Schrödinger or polymer representations.

Chapter 8

Conclusions

In this thesis we discussed the representations of the Weyl algebra of quantum mechanics with a focus on the duality between position and momentum operators, their relationship with the Schrödinger picture, and the dynamics that arises in this framework. Let us now highlight the main results of our study and the interesting open questions with an outlook for future research.

In Chapter 2 we drew a distinction between the regular and irregular representations in light of the Stone–von Neumann theorem. The regular modular representations, which we introduced in this chapter, reveal the Born reciprocity in the Weyl algebra and carry it to the corresponding configuration space. We analyzed how this duality disappears in the singular limit giving rise to the Schrödinger picture and the role of the modular gauge in this process. In particular, we found that the Schrödinger and the dual momentum representations require a different fixing of the modular gauge – a result that may explain the absence of Born reciprocity between position and momentum variables in the classical world if the modular gauge symmetry is somehow broken.

Chapters 3 and 4 focused on the dynamics of the quantum harmonic oscillator in the modular representation. We constructed the path integral for this system in Chapter 3, which revealed some interesting features of the modular space and the dynamics on it. First of all, the result was a sum over the winding number around the modular space. Thus the stationary trajectories are infinitely many, rather than unique, and they have an Aharonov-Bohm-type phase between them. In other words, the propagation between two quantum states that are localized in the modular space cannot be viewed as a unique sequence of "modularly" localized states in the classical-like limit $\hbar \to 0$, but at best as a sum of such sequences. Hence, we found that the dynamics on the modular space differs fundamentally from the classical dynamics since it has no analog of a localized propagation.

The second interesting feature of the path integral in the modular space was the new action that arises from this construction, which we analyzed in detail in Chapter 4. The Euler-Lagrange equations to the modular action are comparable to the Hamilton equations to the classical Hamiltonian function, but they contain an extra time derivative overall. This difference results in a new translation symmetry in both position and momentum directions, which can be interpreted as corresponding to the freedom for choosing the modular cell.

Although we constructed the modular action explicitly from the path integral only for the harmonic oscillator, it was possible trace back its emergence and postulate a general recipe analogous to the Legendre transform, which converts any classical Hamiltonian function into a modular Lagrangian. We did this in Chapter 5 and applied our recipe to the Kepler problem for demonstration. One interesting outcome of this application was that the Newtonian potential is replaced by a term parametrized by the time derivative of the momentum, which is a feature sought after in the MOND-type modified gravity theories. Our modular Kepler action was consistent with Newtonian gravity by construction, but we argued that it can motivate such alternative theories for modified gravity.

In Chapters 6 and 7, we turned our attention to the irregular representations of the Weyl algebra. We polymerized the modular representations in Chapter 6 to create the new modular polymer (MP) representations, which are inequivalent to all known representations of the Weyl algebra.

To investigate how the dynamics plays out in the MP representation, we focused once again on the harmonic oscillator in Chapter 7. The result depends on the ratio between the coarse-graining scale used for the approximation of the polymerized operators and the modularization scale. In the case when this ratio is a rational number, we found normalizable plane-wave-like solutions, which is a new feature specific to the MP representations.

There is plenty of possible directions for future research in this subject. The most pressing question is how to understand the role of modular variables within a realistic theory of the Universe. We will discuss a "purist attitude" and a "radical attitude" to answer this question from two different perspectives.

From the purist perspective, the modularization is simply a mathematical tool to construct new representations of a complex algebra. Though the regular modular representation reveals interesting features of a dynamical quantum system, it is essentially not more or less fundamental than the Schrödinger representation, since they are unitarily equivalent by the Stone–von Neumann theorem. The inequivalent MP representations, on the other hand, may or may not become a part of a realistic theory.

One can construct similar modular (or MP) representations for complex algebras other than the simple Weyl algebra by using the same tricks that we presented here. For example, the field operators and the conjugate momentum operators in any given QFT satisfy a Weyl-like algebra, for which modular representations can be built. This process of modularization would be similar to the spin network representation in LQG based on the holonomy rather than the connection. There is no known uniqueness theorem for the representations of all QFTs in general (while it exists for some specific cases [41, 20]). Therefore, the modular representations of QFT algebras may possibly contain new physical effects, such as the case for the polymer QFT [38]. Particularly for gauge theories, it would be interesting to analyze how the gauge symmetry plays out in the modular space.

Alternatively, one may adopt a radical attitude and seek to incorporate the modular space into the spacetime picture of our Universe. From the perspective of QFT, this approach corresponds to taking the underlying space(time) as a modular one and building a field theory on top of it, whereas the purist approach is concerned with modularizing the field operators defined on a conventional spacetime.

There are several conceptual questions to be answered in any such radical approach. The first is the question of time. On one hand, space and time variables are tied together conceptually by the empirically known existence of Lorentz symmetry. On the other hand, time plays a different role than space in quantum mechanics: It is not an observable, and making it periodic would violate causality. In our opinion, it is conceivable that the Lorentz symmetry emerges in the Schrödinger limit $\ell \to \infty$ of a modular QFT, whereas it is approximate or modified for finite ℓ. Therefore, we suggest a modular space paired with real time in this approach, rather than a modular spacetime.

The other question is the consistency with our current physical theories. We argued in this thesis that a fundamental modular scale ℓ has to be in the far IR, such as near the cosmological scale, since the modular space converges to the non-compact position space in this limit. If this is the case, treating the background as the classical space would be a justified approximation for laboratory-scale experiments. New physical effects would be expected to appear in the IR, such as in the dark energy or dark matter spectrum [22].

Regardless of whether one adopts the purist or radical attitude, the modular and MP representations are well-motivated mathematical constructs in quantum gravity. We hope that this study will contribute to the understanding of Born reciprocity in dynamical systems and to the reconciliation of the quantum and classical pictures of our world.

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APPENDICES

Appendix A

Notes and proofs

Notes for section 2.3

{1} Even though we will usually consider the Weyl algebra as standing on its own, rather than being defined in reference to the Heisenberg operators, it is important that the product relation (2.14) is compatible with the canonical commutation relations (2.6). From the BCH formula, we have

$$\begin{aligned}\hat{W}_{\mathbb{X}}\,\hat{W}_{\mathbb{Y}} &= \exp\Big(\frac{i}{\hbar}\,\omega(\mathbb{X},\hat{\mathbb{Q}}) + \frac{i}{\hbar}\,\omega(\mathbb{Y},\hat{\mathbb{Q}}) - \frac{1}{2\hbar^2}\,\big[\omega(\mathbb{X},\hat{\mathbb{Q}}),\omega(\mathbb{Y},\hat{\mathbb{Q}})\big]\Big) \\ &= \exp\Big(\frac{i}{\hbar}\,\omega(\mathbb{X}+\mathbb{Y},\hat{\mathbb{Q}}) + \frac{i}{2\hbar}\,\omega(\mathbb{X},\mathbb{Y})\Big) \\ &= e^{\frac{i}{2}\omega(\mathbb{X},\mathbb{Y})/\hbar}\,\hat{W}_{\mathbb{X}+\mathbb{Y}}\ . \end{aligned} \tag{A.1}$$

The exponentiated position and momentum operators often used in the literature are

$$\hat{U}_{(a)} \equiv \hat{W}_{(0,a)} = e^{ia\hat{q}/\hbar}\ , \qquad \hat{V}_{(a)} \equiv \hat{W}_{(a,0)} = e^{-ia\hat{p}/\hbar}\ , \tag{A.2}$$

and they obey evidently the relations $\hat{U}^\dagger_{(a)} = \hat{U}_{(-a)}$ and $\hat{V}^\dagger_{(a)} = \hat{V}_{(-a)}$ as well as the product rules

$$\hat{U}_{(a)}\hat{U}_{(b)} = \hat{U}_{(a+b)}\ , \qquad \hat{V}_{(a)}\hat{V}_{(b)} = \hat{V}_{(a+b)}\ , \qquad \hat{U}_{(a)}\hat{V}_{(b)} = e^{iab/\hbar}\,\hat{V}_{(b)}\hat{U}_{(a)}\ . \tag{A.3}$$

Notes for section 2.6

{2} The Zak transform is a unitary isomorphism $Z_\Lambda : \mathcal{H}_{\rm S} \to \mathcal{H}_\Lambda$ between the Schrödinger Hilbert space $\mathcal{H}_{\rm S} = L^2(\mathbb{R}^d)$ and the modular Hilbert space $\mathcal{H}_\Lambda = L^2(E_\Lambda)$, which is defined below equation (2.36). For $\psi \in \mathcal{H}_{\rm S}$, it gives

$$(Z_\Lambda\psi)(\mathbb{X}) = \big(\det\tilde{\lambda}\big)^{-1/2}\,e^{-\frac{i}{2}x\cdot\tilde{x}/\hbar}\sum_{n\in\mathbb{Z}^d} e^{-i\tilde{x}\cdot\lambda n/\hbar}\,\psi(x+\lambda n)\ .$$

Its inverse can be written for any $\phi \in \mathcal{H}_\Lambda$ as

$$(Z_\Lambda^{-1}\phi)(x) = \big(\det\tilde\lambda\big)^{-1/2} \int_{\mathbb{R}^d/\tilde P(\Lambda)} \mathrm{d}^d\tilde x\ e^{\frac{i}{2}x\cdot\tilde x/\hbar}\,\phi(x,\tilde x)\ .$$

The unitarity of this map can be shown as follows:

$$\begin{aligned}
\langle Z_\Lambda\psi, Z_\Lambda\psi'\rangle_{\mathcal{H}_\Lambda} &= \int_{T_\Lambda} \mathrm{d}^{2d}\mathbb{X}\ (Z_\Lambda\psi)(\mathbb{X})^*\,(Z_\Lambda\psi')(\mathbb{X}) \\
&= \big(\det\tilde\lambda\big)^{-1} \sum_{m,n\in\mathbb{Z}^d} \int_{T_\Lambda} \mathrm{d}^{2d}\mathbb{X}\ e^{i\tilde x\cdot\lambda(m-n)/\hbar}\,\psi(x+\lambda m)^*\,\psi'(x+\lambda n) \\
&= \sum_{m,n\in\mathbb{Z}^d} \int_{\mathbb{R}^d/P(\Lambda)} \mathrm{d}^dx\ \delta_{m-n,0}\,\psi(x+\lambda m)^*\,\psi'(x+\lambda n) \\
&= \int_{\mathbb{R}^d} \mathrm{d}^dx\ \psi(x)^*\,\psi'(x) = \langle\psi,\psi'\rangle_{\mathcal{H}_{\rm S}}\ .
\end{aligned}$$

{3} We show that (2.25) is equivalent to (2.24) as follows:

$$\begin{aligned}
|\mathbb{X}\rangle_\Lambda &= \big(\det\tilde\lambda\big)^{-1/2}\, e^{\frac{i}{2}x\cdot\tilde x/\hbar} \sum_{n\in\mathbb{Z}^d} e^{i\tilde x\cdot\lambda n/\hbar}\,|x+\lambda n\rangle_{\rm Sch} \\
&= \big(\det\tilde\lambda\big)^{-1/2}\,(2\pi\hbar)^{-d/2} \int \mathrm{d}^dp\ e^{\frac{i}{2}x\cdot\tilde x/\hbar} \sum_{n\in\mathbb{Z}^d} e^{i\tilde x\cdot\lambda n/\hbar}\,e^{-ip\cdot(x+\lambda n)/\hbar}\,|p\rangle_{\rm mom} \\
&= \big(\det\tilde\lambda\big)^{-1/2}\,(2\pi\hbar)^{-d/2} \int \mathrm{d}^dp\ e^{\frac{i}{2}x\cdot\tilde x/\hbar}\,e^{-ip\cdot x/\hbar}\,|p\rangle_{\rm mom} \sum_{\tilde n\in\mathbb{Z}^d} \delta^d\Big(\frac{1}{2\pi\hbar}\,\lambda^T(\tilde x-p)+\tilde n\Big) \\
&= \big(\det\lambda\big)^{-1/2} \int \mathrm{d}^dp\ e^{\frac{i}{2}x\cdot\tilde x/\hbar}\,e^{-ip\cdot x/\hbar}\,|p\rangle_{\rm mom} \sum_{\tilde n\in\mathbb{Z}^d} \delta^d\big(\tilde x+\tilde\lambda\tilde n-p\big) \\
&= \big(\det\lambda\big)^{-1/2}\, e^{-\frac{i}{2}x\cdot\tilde x/\hbar} \sum_{\tilde n\in\mathbb{Z}^d} e^{-ix\cdot\tilde\lambda\tilde n/\hbar}\,|\tilde x+\tilde\lambda\tilde n\rangle_{\rm mom}\ .
\end{aligned}$$

{4} We show $\hat{W}_{\mathbb{Y}}\,|\mathbb{X}\rangle_\Lambda = e^{\frac{i}{2}\omega(\mathbb{Y},\mathbb{X})/\hbar}\,|\mathbb{X}+\mathbb{Y}\rangle_\Lambda$ as follows:

$$
\begin{aligned}
\hat{W}_{\mathbb{Y}}\,|\mathbb{X}\rangle_\Lambda &= e^{-\frac{i}{2}y\cdot\tilde{y}/\hbar}\,e^{i\tilde{y}\cdot\hat{q}/\hbar}\,e^{-iy\cdot\hat{p}/\hbar}\,(\det\lambda)^{-1/2}\,e^{-\frac{i}{2}x\cdot\tilde{x}/\hbar}\sum_{\tilde{n}\in\mathbb{Z}^d} e^{-ix\cdot\tilde{\lambda}\tilde{n}/\hbar}\,|\tilde{x}+\tilde{\lambda}\tilde{n}\rangle_{\rm mom} \\
&= e^{-\frac{i}{2}y\cdot\tilde{y}/\hbar}\,e^{i\tilde{y}\cdot\hat{q}/\hbar}\,e^{-\frac{i}{2}y\cdot\tilde{x}/\hbar}\,(\det\lambda)^{-1/2}\,e^{-\frac{i}{2}(x+y)\cdot\tilde{x}/\hbar}\sum_{\tilde{n}\in\mathbb{Z}^d} e^{-i(x+y)\cdot\tilde{\lambda}\tilde{n}/\hbar}\,|\tilde{x}+\tilde{\lambda}\tilde{n}\rangle_{\rm mom} \\
&= e^{-\frac{i}{2}y\cdot\tilde{y}/\hbar}\,e^{i\tilde{y}\cdot\hat{q}/\hbar}\,e^{-\frac{i}{2}y\cdot\tilde{x}/\hbar}\,\big(\det\tilde{\lambda}\big)^{-1/2}\,e^{\frac{i}{2}(x+y)\cdot\tilde{x}/\hbar}\sum_{n\in\mathbb{Z}^d} e^{i\tilde{x}\cdot\lambda n/\hbar}\,|x+y+\lambda n\rangle_{\rm Sch} \\
&= e^{\frac{i}{2}\tilde{y}\cdot x/\hbar}\,e^{-\frac{i}{2}y\cdot\tilde{x}/\hbar}\,\big(\det\tilde{\lambda}\big)^{-1/2}\,e^{\frac{i}{2}(x+y)\cdot(\tilde{x}+\tilde{y})/\hbar}\sum_{n\in\mathbb{Z}^d} e^{i(\tilde{x}+\tilde{y})\cdot\lambda n/\hbar}\,|x+y+\lambda n\rangle_{\rm Sch} \\
&= e^{\frac{i}{2}\omega(\mathbb{Y},\mathbb{X})/\hbar}\,|\mathbb{X}+\mathbb{Y}\rangle_\Lambda \ .
\end{aligned}
$$

{5} We show $|\mathbb{X}+\mathbb{K}\rangle_\Lambda = e^{\frac{i}{2}k\cdot\tilde{k}/\hbar}\,e^{\frac{i}{2}\omega(\mathbb{K},\mathbb{X})/\hbar}\,|\mathbb{X}\rangle_\Lambda$ as follows:

$$
\begin{aligned}
|\mathbb{X}+\mathbb{K}\rangle_\Lambda &= \big(\det\tilde{\lambda}\big)^{-1/2}\,e^{\frac{i}{2}(x+k)\cdot(\tilde{x}+\tilde{k})/\hbar}\sum_{n\in\mathbb{Z}^d} e^{i(\tilde{x}+\tilde{k})\cdot\lambda n/\hbar}\,|x+k+\lambda n\rangle_{\rm Sch} \\
&= \big(\det\tilde{\lambda}\big)^{-1/2}\,e^{\frac{i}{2}(x+k)\cdot(\tilde{x}+\tilde{k})/\hbar}\sum_{n\in\mathbb{Z}^d} e^{i(\tilde{x}+\tilde{k})\cdot(\lambda n-k)/\hbar}\,|x+\lambda n\rangle_{\rm Sch} \\
&= e^{\frac{i}{2}k\cdot\tilde{k}/\hbar}\,e^{\frac{i}{2}\omega(\mathbb{K},\mathbb{X})/\hbar}\,|\mathbb{X}\rangle_\Lambda \ .
\end{aligned}
$$

{6} We show $\mathbb{1} = \int_{T_\Lambda}{\rm d}^{2d}\mathbb{X}\ |\mathbb{X}\rangle\langle\mathbb{X}|$ as follows:

$$
\begin{aligned}
\int_{T_\Lambda}{\rm d}^{2d}\mathbb{X}\ |\mathbb{X}\rangle\langle\mathbb{X}| &= \big(\det\tilde{\lambda}\big)^{-1}\int_{T_\Lambda}{\rm d}^{2d}\mathbb{X}\sum_{m,n\in\mathbb{Z}^d} e^{i\tilde{x}\cdot\lambda(m-n)/\hbar}\,|x+\lambda m\rangle\langle x+\lambda n| \\
&= \int_{\mathbb{R}^d/(\lambda\mathbb{Z}^d)}{\rm d}^d x\sum_{m,n\in\mathbb{Z}^d}\delta_{m,n}\,|x+\lambda m\rangle\langle x+\lambda n| \\
&= \int_{\mathbb{R}^d}{\rm d}^d x\,|x\rangle\langle x| = \mathbb{1} \ .
\end{aligned}
$$

{7} Since the action of the Weyl operators on a modular state,

$$
\hat{W}_{\mathbb{Y}}\,|\psi\rangle = \int_{T_\Lambda}{\rm d}^{2d}\mathbb{X}\ e^{\frac{i}{2}\omega(\mathbb{Y},\mathbb{X})/\hbar}\,\phi(\mathbb{X}-\mathbb{Y})\,|\mathbb{X}\rangle_\Lambda \ ,
$$

is weakly continuous in $\mathbb{Y}$, we can write

$$\hat{q}^a\,|\psi\rangle = \lim_{\tilde{y}_a\to 0}\frac{-i\hbar}{\tilde{y}_a}\Big(\hat{W}_{\tilde{y}_a}\,|\psi\rangle - |\psi\rangle\Big)\ ,$$
$$\hat{p}_a\,|\psi\rangle = \lim_{y^a\to 0}\frac{+i\hbar}{y^a}\Big(\hat{W}_{y^a}\,|\psi\rangle - |\psi\rangle\Big)\ ,$$

from which (2.38) can be obtained.

{8} Let's show that the Weyl and Heisenberg operators preserve the condition (2.36). For the Weyl operators,

$$\begin{aligned}(\hat{W}_{\mathbb{Y}}\,\phi)(\mathbb{X}+\mathbb{K}) &= e^{\frac{i}{2}\omega(\mathbb{Y},\mathbb{X}+\mathbb{K})/\hbar}\,\phi(\mathbb{X}+\mathbb{K}-\mathbb{Y}) \\ &= e^{\frac{i}{2}\omega(\mathbb{Y},\mathbb{X}+\mathbb{K})/\hbar}\,e^{-\frac{i}{2}k\cdot\tilde{k}/\hbar}\,e^{-\frac{i}{2}\omega(\mathbb{K},\mathbb{X}-\mathbb{Y})/\hbar}\,\phi(\mathbb{X}-\mathbb{Y}) \\ &= e^{-\frac{i}{2}k\cdot\tilde{k}/\hbar}\,e^{-\frac{i}{2}\omega(\mathbb{K},\mathbb{X})/\hbar}\,(\hat{W}_{\mathbb{Y}}\,\phi)(\mathbb{X})\ .\end{aligned}$$

For the Heisenberg operators,

$$\begin{aligned}(\hat{\mathbb{Q}}^A\,\phi)(\mathbb{X}+\mathbb{K}) &= i\hbar\,(\omega^{-1})^{AB}\bigg(\partial_B\phi(\mathbb{X}+\mathbb{K}) + \frac{i}{\hbar}\,\mathbb{A}_B(\mathbb{X}+\mathbb{K})\,\phi(\mathbb{X}+\mathbb{K})\bigg) \\ &= i\hbar\,(\omega^{-1})^{AB}\bigg(e^{-\frac{i}{2}k\cdot\tilde{k}/\hbar}\,e^{-\frac{i}{2}\omega(\mathbb{K},\mathbb{X})/\hbar}\bigg(\partial_B\phi(\mathbb{X}) - \frac{i}{2\hbar}\,\mathbb{K}^C\omega_{CB}\,\phi(\mathbb{X})\bigg) \\ &\qquad\qquad + \frac{i}{\hbar}\bigg(\mathbb{A}_B(\mathbb{X}) + \frac{1}{2}\,\mathbb{K}^C\omega_{CB}\bigg)\Big(e^{-\frac{i}{2}k\cdot\tilde{k}/\hbar}\,e^{-\frac{i}{2}\omega(\mathbb{K},\mathbb{X})/\hbar}\,\phi(\mathbb{X})\Big)\bigg) \\ &= e^{-\frac{i}{2}k\cdot\tilde{k}/\hbar}\,e^{-\frac{i}{2}\omega(\mathbb{K},\mathbb{X})/\hbar}\,(\hat{\mathbb{Q}}^A\,\phi)(\mathbb{X})\ .\end{aligned}$$

Hence, both Weyl and Heisenberg operators are well-defined on the modular Hilbert space $L^2(E_\Lambda)$.

Notes for section 6.2

{9} We show $\varphi_{\mathbb{X}}(\mathbb{Y}+\mathbb{K}) = e^{ik\cdot\tilde{y}/\hbar}\,\varphi_{\mathbb{X}}(\mathbb{Y})$ as follows:

$$\begin{aligned}\varphi_{\mathbb{X}}(\mathbb{Y}+\mathbb{K}) &= \begin{cases} e^{i(y+k-x)\cdot\tilde{x}/\hbar}\ , & \text{if } \mathbb{Y}+\mathbb{K}-\mathbb{X}\in\Lambda\ , \\ 0\ , & \text{otherwise}\ ,\end{cases} \\ &= \begin{cases} e^{ik\cdot\tilde{x}/\hbar}\,e^{i(y-x)\cdot\tilde{x}/\hbar}\ , & \text{if } \mathbb{Y}-\mathbb{X}\in\Lambda\ , \\ 0\ , & \text{otherwise}\ ,\end{cases}\end{aligned}$$

$$= \begin{cases} e^{ik\cdot\tilde{y}/\hbar}\, e^{i(y-x)\cdot\tilde{x}/\hbar} \ , & \text{if } \mathbb{Y}-\mathbb{X}\in\Lambda \ , \\ 0 \ , & \text{otherwise} \ , \end{cases}$$
$$= e^{ik\cdot\tilde{y}/\hbar}\, \varphi_{\mathbb{X}}(\mathbb{Y}) \ .$$

{10} We show $\varphi_{\mathbb{X}}(\mathbb{Y}+\mathbb{Y}') = e^{i(y+y'-x)\cdot\tilde{y}'/\hbar}\, \varphi_{\mathbb{X}-\mathbb{Y}'}(\mathbb{Y})$ as follows:

$$\varphi_{\mathbb{X}}(\mathbb{Y}+\mathbb{Y}') = \begin{cases} e^{i(y+y'-x)\cdot\tilde{x}/\hbar} \ , & \text{if } \mathbb{Y}+\mathbb{Y}'-\mathbb{X}\in\Lambda \ , \\ 0 \ , & \text{otherwise} \ , \end{cases}$$
$$= \begin{cases} e^{i(y+y'-x)\cdot\tilde{y}'/\hbar}\, e^{i(y-(x-y'))\cdot(\tilde{x}-\tilde{y}')/\hbar} \ , & \text{if } \mathbb{Y}-(\mathbb{X}-\mathbb{Y}')\in\Lambda \ , \\ 0 \ , & \text{otherwise} \ , \end{cases}$$
$$= e^{i(y+y'-x)\cdot\tilde{y}'/\hbar}\, \varphi_{\mathbb{X}-\mathbb{Y}'}(\mathbb{Y}) \ .$$

{11} We show $\varphi_{\mathbb{X}+\mathbb{K}} = e^{-ik\cdot\tilde{x}/\hbar}\, \varphi_{\mathbb{X}}$ as follows:

$$\varphi_{\mathbb{X}+\mathbb{K}}(\mathbb{Y}) = \begin{cases} e^{i(y-x-k)\cdot(\tilde{x}+\tilde{k})/\hbar} \ , & \text{if } \mathbb{Y}-(\mathbb{X}+\mathbb{K})\in\Lambda \ , \\ 0 \ , & \text{otherwise} \ , \end{cases}$$
$$= \begin{cases} e^{i(y-x-k)\cdot\tilde{k}/\hbar}\, e^{-ik\cdot\tilde{x}/\hbar}\, e^{i(y-x)\cdot\tilde{x}/\hbar} \ , & \text{if } \mathbb{Y}-\mathbb{X}\in\Lambda \ , \\ 0 \ , & \text{otherwise} \ , \end{cases}$$
$$= e^{-ik\cdot\tilde{x}/\hbar}\, \varphi_{\mathbb{X}}(\mathbb{Y}) \ .$$

{12} We show $\hat{W}_{\mathbb{Y}}\hat{W}_{\mathbb{Y}'}\, \varphi_{\mathbb{X}} = e^{\frac{i}{2}\omega(\mathbb{Y},\mathbb{Y}')/\hbar}\, \hat{W}_{\mathbb{Y}+\mathbb{Y}'}\, \varphi_{\mathbb{X}}$ as follows:

$$\begin{aligned} \hat{W}_{\mathbb{Y}}\hat{W}_{\mathbb{Y}'}\, \varphi_{\mathbb{X}} &= e^{\frac{i}{2}y'\cdot\tilde{y}'/\hbar}\, e^{i\tilde{y}'\cdot x/\hbar}\, \hat{W}_{\mathbb{Y}}\, \varphi_{\mathbb{X}+\mathbb{Y}'} \\ &= e^{\frac{i}{2}y'\cdot\tilde{y}'/\hbar}\, e^{i\tilde{y}'\cdot x/\hbar}\, e^{\frac{i}{2}y\cdot\tilde{y}/\hbar}\, e^{i\tilde{y}\cdot(x+y')/\hbar}\, \varphi_{\mathbb{X}+\mathbb{Y}+\mathbb{Y}'} \\ &= e^{\frac{i}{2}\omega(\mathbb{Y},\mathbb{Y}')/\hbar}\, e^{\frac{i}{2}(y+y')\cdot(\tilde{y}+\tilde{y}')/\hbar}\, e^{i(\tilde{y}+\tilde{y}')\cdot x/\hbar}\, \varphi_{\mathbb{X}+\mathbb{Y}+\mathbb{Y}'} \\ &= e^{\frac{i}{2}\omega(\mathbb{Y},\mathbb{Y}')/\hbar}\, \hat{W}_{\mathbb{Y}+\mathbb{Y}'}\, \varphi_{\mathbb{X}} \ . \end{aligned}$$

{13} We show $\left\langle \hat{W}_{\mathbb{Y}}\, \varphi_{\mathbb{X}}, \varphi_{\mathbb{X}'} \right\rangle = \left\langle \varphi_{\mathbb{X}}, \hat{W}_{-\mathbb{Y}}\, \varphi_{\mathbb{X}'} \right\rangle$ as follows:

$$\left\langle \hat{W}_{\mathbb{Y}}\, \varphi_{\mathbb{X}}, \varphi_{\mathbb{X}'} \right\rangle = \sum_{\mathbb{Y}'} \left((\hat{W}_{\mathbb{Y}}\, \varphi_{\mathbb{X}})(\mathbb{Y}') \right)^* \varphi_{\mathbb{X}'}(\mathbb{Y}')$$

$$
\begin{aligned}
&= \sum_{\mathbb{Y}'} \left(e^{\frac{i}{2} y\cdot\tilde{y}/\hbar}\, e^{i\tilde{y}\cdot x/\hbar}\, \varphi_{\mathbb{X}+\mathbb{Y}}(\mathbb{Y}') \right)^* \varphi_{\mathbb{X}'}(\mathbb{Y}') \\
&\overset{(6.10)}{=} \sum_{\mathbb{Y}'} \left(e^{-\frac{i}{2} y\cdot\tilde{y}/\hbar}\, e^{i\tilde{y}\cdot y'/\hbar}\, \varphi_{\mathbb{X}}(\mathbb{Y}'-\mathbb{Y}) \right)^* \varphi_{\mathbb{X}'}(\mathbb{Y}') \\
&= \sum_{\mathbb{Y}'} \left(e^{\frac{i}{2} y\cdot\tilde{y}/\hbar}\, e^{i\tilde{y}\cdot y'/\hbar}\, \varphi_{\mathbb{X}}(\mathbb{Y}') \right)^* \varphi_{\mathbb{X}'}(\mathbb{Y}'+\mathbb{Y}) \\
&= \sum_{\mathbb{Y}'} \varphi_{\mathbb{X}}(\mathbb{Y}')^*\, e^{-\frac{i}{2} y\cdot\tilde{y}/\hbar}\, e^{-i\tilde{y}\cdot y'/\hbar}\, \varphi_{\mathbb{X}'}(\mathbb{Y}'+\mathbb{Y}) \\
&\overset{(6.10)}{=} \sum_{\mathbb{Y}'} \varphi_{\mathbb{X}}(\mathbb{Y}')^*\, e^{\frac{i}{2} y\cdot\tilde{y}/\hbar}\, e^{-ix'\cdot\tilde{y}/\hbar}\, \varphi_{\mathbb{X}'-\mathbb{Y}}(\mathbb{Y}') \\
&= \sum_{\mathbb{Y}'} \varphi_{\mathbb{X}}(\mathbb{Y}')^*\, \hat{W}_{-\mathbb{Y}}\, \varphi_{\mathbb{X}'}(\mathbb{Y}') \\
&= \left\langle \varphi_{\mathbb{X}}, \hat{W}_{-\mathbb{Y}}\, \varphi_{\mathbb{X}'} \right\rangle \ .
\end{aligned}
$$

Notes for section 7.0

{14} Let's visualize the definition (7.2) with an example. We consider a single direction for simplicity.

Case 1 - irrational scale ratio: Let $\ell = 1$ and $\lambda = 1/\sqrt{2}$. Then, the lattice points have coordinates of the form $x_n = x_0 + n/\sqrt{2} \pmod 1$ for $n \in \mathbb{Z}$. This is an infinite lattice on a compact space (w.r.t. continuum topology).

Case 2 - rational scale ratio: Let $\ell = 1$ and $\lambda = 2/5$. Then, we get a finite lattice at the points $x_n = x_0 + n/5 \pmod 1$ for $n = 0, ..., 4$. This is a finite lattice.

{15} The definitions (7.4) are motivated by the Taylor expansion of trigonometric functions. For example,

$$
\begin{aligned}
\frac{\hbar^2}{\lambda^2}\left(2 - \hat{W}_{(\lambda,0)} - \hat{W}_{(-\lambda,0)}\right) \text{ “}=\text{” } &\frac{\hbar^2}{\lambda^2}\left(2 - e^{-i\lambda\hat{p}/\hbar} - e^{i\lambda\hat{p}/\hbar}\right) \\
&= \frac{2\hbar^2}{\lambda^2}\left(1 - \cos\left(\lambda\hat{p}/\hbar\right)\right) \\
&= \widehat{p^2} + \mathcal{O}((\lambda p/\hbar)^2) \ .
\end{aligned}
$$

We used the symbol “ = ” to indicate that the corresponding equation is only a formal writing, since the operator $\hat{p}$ does not exist. We also infer from this calculation that the approximation is valid only in the regime $p \ll \hbar/\lambda$.

Notes for section 7.1

{16} We shall prove the uniqueness of the expression (7.7). Let's define a lattice $\bar{\mathcal{L}}^{(\lambda,\tilde{\lambda})}_{\mathbb{X}_0} \subset \mathbb{R}^2$ for any $\mathbb{X}_0 \in \mathbb{R}^2$ as

$$\bar{\mathcal{L}}^{(\lambda,\tilde{\lambda})}_{\mathbb{X}_0} \equiv \left\{ \mathbb{Y} \in \mathbb{R}^2 : \exists\, n, \tilde{n} \in \mathbb{Z} : \mathbb{Y} = \mathbb{X}_0 + (n\lambda, \tilde{n}\tilde{\lambda}) \right\} \ .$$

Assuming that λ/ℓ and $\tilde{\lambda}/\tilde{\ell}$ are irrational numbers, there is a bijective mapping between the lattices $\bar{\mathcal{L}}^{(\lambda,\tilde{\lambda})}_{\mathbb{X}_0} \subset \mathbb{R}^2$ and $\mathcal{L}^{\Lambda,(\lambda,\tilde{\lambda})}_{\mathbb{X}_0} \subset T_\Lambda$ that is given by

$$\begin{aligned} \Pi : \bar{\mathcal{L}}^{(\lambda,\tilde{\lambda})}_{\mathbb{X}_0} &\to \mathcal{L}^{\Lambda,(\lambda,\tilde{\lambda})}_{\mathbb{X}_0} \\ \mathbb{Y} &\mapsto \mathbb{Y} \ (\mathrm{mod}\,\Lambda) \ . \end{aligned}$$

Moreover, the basis elements satisfy the relation (6.11), i.e. the states $\varphi_{\mathbb{Y}}$ and $\varphi_{\Pi(\mathbb{Y})}$ are linearly dependent for any $\mathbb{Y} \in \mathbb{R}^2$. Since $\left\{ \varphi_{\mathbb{Y}} : \mathbb{Y} \in \mathcal{L}^{\Lambda,(\lambda,\tilde{\lambda})}_{\mathbb{X}_0} \right\}$ is a basis of the superselection sector $\mathcal{H}^{\Lambda,(\lambda,\tilde{\lambda}),\mathbb{X}_0}_{\mathrm{MP}}$, the set $\left\{ \varphi_{\mathbb{Y}} : \mathbb{Y} \in \bar{\mathcal{L}}^{(\lambda,\tilde{\lambda})}_{\mathbb{X}_0} \right\}$ is also a basis. Hence, the expression (7.7) is simply a decomposition of a state in this basis, which is unique.

Notes for section 7.2

{17} For this lengthy rewriting, we act with the regularized Hamiltonian operator (7.5) on the state (7.21), then rearrange and relabel the terms of the sum, and finally use the quasi-periodicity of φ. This gives

$$\begin{aligned} \hat{H}_{(\lambda,\tilde{\lambda})}\,\psi = \sum_{n=0}^{N-1} \sum_{\tilde{n}=0}^{\tilde{N}-1} A_{n,\tilde{n}} \Bigg(& \frac{\hbar^2}{2m\lambda^2} \left(2\varphi_{\mathbb{X}_0+(n\lambda,\tilde{n}\tilde{\lambda})} - \varphi_{\mathbb{X}_0+((n+1)\lambda,\tilde{n}\tilde{\lambda})} - \varphi_{\mathbb{X}_0+((n-1)\lambda,\tilde{n}\tilde{\lambda})} \right) \\ & + \frac{m\omega^2\hbar^2}{2\tilde{\lambda}^2} \Big(2\varphi_{\mathbb{X}_0+(n\lambda,\tilde{n}\tilde{\lambda})} - e^{i\tilde{\lambda}(x_0+n\lambda)/\hbar}\,\varphi_{\mathbb{X}_0+(n\lambda,(\tilde{n}+1)\tilde{\lambda})} \\ & \qquad - e^{-i\tilde{\lambda}(x_0+n\lambda)/\hbar}\,\varphi_{\mathbb{X}_0+(n\lambda,(\tilde{n}-1)\tilde{\lambda})} \Big) \Bigg) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{N-1} \sum_{\tilde{n}=0}^{\tilde{N}-1} A_{n,\tilde{n}} \left(\frac{\hbar^2}{m\lambda^2} + \frac{m\omega^2\hbar^2}{\tilde{\lambda}^2} \right) \varphi_{\mathbb{X}_0+(n\lambda,\tilde{n}\tilde{\lambda})} \\
&\quad - \frac{\hbar^2}{2m\lambda^2} \left(\sum_{n=0}^{N-2} \sum_{\tilde{n}=0}^{\tilde{N}-1} A_{n,\tilde{n}}\, \varphi_{\mathbb{X}_0+((n+1)\lambda,\tilde{n}\tilde{\lambda})} + A_{N-1,\tilde{n}}\, \varphi_{\mathbb{X}_0+(N\lambda,\tilde{n}\tilde{\lambda})} \right) \\
&\quad - \frac{\hbar^2}{2m\lambda^2} \left(\sum_{n=1}^{N-1} \sum_{\tilde{n}=0}^{\tilde{N}-1} A_{n,\tilde{n}}\, \varphi_{\mathbb{X}_0+((n-1)\lambda,\tilde{n}\tilde{\lambda})} + A_{0,\tilde{n}}\, \varphi_{\mathbb{X}_0+(-\lambda,\tilde{n}\tilde{\lambda})} \right) \\
&\quad - \frac{m\omega^2\hbar^2}{2\tilde{\lambda}^2} \Bigg(\sum_{n=0}^{N-1} \sum_{\tilde{n}=0}^{\tilde{N}-2} A_{n,\tilde{n}}\, e^{i\tilde{\lambda}(x_0+n\lambda)/\hbar}\, \varphi_{\mathbb{X}_0+(n\lambda,(\tilde{n}+1)\tilde{\lambda})} \\
&\qquad\qquad + A_{n,\tilde{N}-1}\, e^{i\tilde{\lambda}(x_0+n\lambda)/\hbar}\, \varphi_{\mathbb{X}_0+(n\lambda,\tilde{N}\tilde{\lambda})} \Bigg) \\
&\quad - \frac{m\omega^2\hbar^2}{2\tilde{\lambda}^2} \Bigg(\sum_{n=0}^{N-1} \sum_{\tilde{n}=1}^{\tilde{N}-1} A_{n,\tilde{n}}\, e^{-i\tilde{\lambda}(x_0+n\lambda)/\hbar}\, \varphi_{\mathbb{X}_0+(n\lambda,(\tilde{n}-1)\tilde{\lambda})} \\
&\qquad\qquad + A_{n,0}\, e^{-i\tilde{\lambda}(x_0+n\lambda)/\hbar}\, \varphi_{\mathbb{X}_0+(n\lambda,-\tilde{\lambda})} \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=0}^{N-1} \sum_{\tilde{n}=0}^{\tilde{N}-1} A_{n,\tilde{n}} \left(\frac{\hbar^2}{m\lambda^2} + \frac{m\omega^2\hbar^2}{\tilde{\lambda}^2} \right) \varphi_{\mathbb{X}_0+(n\lambda,\tilde{n}\tilde{\lambda})} \\
&\quad - \frac{\hbar^2}{2m\lambda^2} \left(\sum_{n=1}^{N-1} \sum_{\tilde{n}=0}^{\tilde{N}-1} A_{n-1,\tilde{n}}\, \varphi_{\mathbb{X}_0+(n\lambda,\tilde{n}\tilde{\lambda})} + A_{N-1,\tilde{n}}\, \varphi_{\mathbb{X}_0+(0,\tilde{n}\tilde{\lambda})+(M\ell,0)} \right) \\
&\quad - \frac{\hbar^2}{2m\lambda^2} \left(\sum_{n=0}^{N-2} \sum_{\tilde{n}=0}^{\tilde{N}-1} A_{n+1,\tilde{n}}\, \varphi_{\mathbb{X}_0+(n\lambda,\tilde{n}\tilde{\lambda})} + A_{0,\tilde{n}}\, \varphi_{\mathbb{X}_0+((N-1)\lambda,\tilde{n}\tilde{\lambda})+(-M\ell,0)} \right) \\
&\quad - \frac{m\omega^2\hbar^2}{2\tilde{\lambda}^2} \left(\sum_{n=0}^{N-1} \sum_{\tilde{n}=1}^{\tilde{N}-1} A_{n,\tilde{n}-1}\, e^{i\tilde{\lambda}(x_0+n\lambda)/\hbar}\, \varphi_{\mathbb{X}_0+(n\lambda,\tilde{n}\tilde{\lambda})} \right. \\
&\qquad\qquad\qquad \left. + A_{n,\tilde{N}-1}\, e^{i\tilde{\lambda}(x_0+n\lambda)/\hbar}\, \varphi_{\mathbb{X}_0+(n\lambda,0)+(0,\tilde{M}\tilde{\ell})} \right) \\
&\quad - \frac{m\omega^2\hbar^2}{2\tilde{\lambda}^2} \left(\sum_{n=0}^{N-1} \sum_{\tilde{n}=0}^{\tilde{N}-2} A_{n,\tilde{n}+1}\, e^{-i\tilde{\lambda}(x_0+n\lambda)/\hbar}\, \varphi_{\mathbb{X}_0+(n\lambda,\tilde{n}\tilde{\lambda})} \right. \\
&\qquad\qquad\qquad \left. + A_{n,0}\, e^{-i\tilde{\lambda}(x_0+n\lambda)/\hbar}\, \varphi_{\mathbb{X}_0+(n\lambda,(\tilde{N}-1)\tilde{\lambda})+(0,-\tilde{M}\tilde{\ell})} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=0}^{N-1}\sum_{\tilde{n}=0}^{\tilde{N}-1} A_{n,\tilde{n}} \left(\frac{\hbar^2}{m\lambda^2} + \frac{m\omega^2\hbar^2}{\tilde{\lambda}^2}\right) \varphi_{\mathbb{X}_0+(n\lambda,\tilde{n}\tilde{\lambda})} \\
&\quad - \frac{\hbar^2}{2m\lambda^2}\left(\sum_{n=1}^{N-1}\sum_{\tilde{n}=0}^{\tilde{N}-1} A_{n-1,\tilde{n}}\,\varphi_{\mathbb{X}_0+(n\lambda,\tilde{n}\tilde{\lambda})} + A_{N-1,\tilde{n}}\, e^{-iN\lambda(\tilde{x}_0+\tilde{n}\tilde{\lambda})/\hbar}\,\varphi_{\mathbb{X}_0+(0,\tilde{n}\tilde{\lambda})}\right) \\
&\quad - \frac{\hbar^2}{2m\lambda^2}\left(\sum_{n=0}^{N-2}\sum_{\tilde{n}=0}^{\tilde{N}-1} A_{n+1,\tilde{n}}\,\varphi_{\mathbb{X}_0+(n\lambda,\tilde{n}\tilde{\lambda})} + A_{0,\tilde{n}}\, e^{iN\lambda(\tilde{x}_0+\tilde{n}\tilde{\lambda})/\hbar}\,\varphi_{\mathbb{X}_0+((N-1)\lambda,\tilde{n}\tilde{\lambda})}\right) \\
&\quad - \frac{m\omega^2\hbar^2}{2\tilde{\lambda}^2}\left(\sum_{n=0}^{N-1}\sum_{\tilde{n}=1}^{\tilde{N}-1} A_{n,\tilde{n}-1}\, e^{i\tilde{\lambda}(x_0+n\lambda)/\hbar}\,\varphi_{\mathbb{X}_0+(n\lambda,\tilde{n}\tilde{\lambda})}\right. \\
&\qquad\qquad\qquad\qquad \left. + A_{n,\tilde{N}-1}\, e^{i\tilde{\lambda}(x_0+n\lambda)/\hbar}\,\varphi_{\mathbb{X}_0+(n\lambda,0)}\right) \\
&\quad - \frac{m\omega^2\hbar^2}{2\tilde{\lambda}^2}\left(\sum_{n=0}^{N-1}\sum_{\tilde{n}=0}^{\tilde{N}-2} A_{n,\tilde{n}+1}\, e^{-i\tilde{\lambda}(x_0+n\lambda)/\hbar}\,\varphi_{\mathbb{X}_0+(n\lambda,\tilde{n}\tilde{\lambda})}\right. \\
&\qquad\qquad\qquad\qquad \left. + A_{n,0}\, e^{-i\tilde{\lambda}(x_0+n\lambda)/\hbar}\,\varphi_{\mathbb{X}_0+(n\lambda,(\tilde{N}-1)\tilde{\lambda})}\right) . \qquad \text{(A.4)}
\end{aligned}
$$

With this rewriting of the action of the Hamiltonian, we can see that the coefficients $A_{n,\tilde{n}}$ can be defined to satisfy

$$A_{-1,\tilde{n}} \equiv A_{N-1,\tilde{n}}\, e^{-iN\lambda(\tilde{x}_0+\tilde{n}\tilde{\lambda})/\hbar}, \qquad \text{(A.5a)}$$

$$A_{N,\tilde{n}} \equiv A_{0,\tilde{n}}\, e^{iN\lambda(\tilde{x}_0+\tilde{n}\tilde{\lambda})/\hbar}, \qquad \text{(A.5b)}$$

$$A_{n,-1} \equiv A_{n,\tilde{N}-1}, \qquad \text{(A.5c)}$$

$$A_{n,\tilde{N}} \equiv A_{n,0}, \qquad \text{(A.5d)}$$

which can be written more compactly as in the general definition (7.22). Then, (A.4) leads to (7.24).

{18} Since the shift of the parameter $\mathbb{X}_0$ by a lattice vector maps the solutions we found to the same superselection sector of the Hilbert space, one can question whether we correctly identified all possible solutions under the given assumptions, or whether these solutions should have $M \times \tilde{M}$ different copies. Let's name our solutions as

$$\psi_{\mathbb{X}_0,k,\tilde{k}} \equiv \sum_{n=0}^{N-1}\sum_{\tilde{n}=0}^{\tilde{N}-1} e^{2\pi i(\tilde{k}n/N - k\tilde{n}/\tilde{N})}\, e^{in\lambda\tilde{x}_0/\hbar}\,\varphi_{\mathbb{X}_0+(n\lambda,\tilde{n}\tilde{\lambda})} \, . \qquad \text{(A.6)}$$

Precisely, the question we investigate here is whether or not the sets

$$\left\{\psi_{\mathbb{X}_0,k,\tilde{k}}\,\middle|\,\tilde{k}=1,...,N\ ;\ k=1,...,\tilde{N}\right\}\quad\text{and}\quad\left\{\psi_{\mathbb{X}_0+\mathbb{M},k,\tilde{k}}\,\middle|\,\tilde{k}=1,...,N\ ;\ k=1,...,\tilde{N}\right\}$$

are linearly dependent for arbitrary values of $\mathbb{M}\in\Lambda$. The answer is positive as one finds

$$\psi_{\mathbb{X}_0+(m\ell,\tilde{m}\tilde{\ell}),k,\tilde{k}}=e^{-im\ell\tilde{x}_0/\hbar}\,\psi_{\mathbb{X}_0,k+m\tilde{M},\tilde{k}+\tilde{m}M}\ , \tag{A.7}$$

for any $m,\tilde{m}\in\mathbb{Z}$. Hence, shifting the anchoring point on the modular space by a lattice vector can be interpreted as a permutation of the solutions up to a phase. This confirms that the $N\times\tilde{N}$ solutions we found are complete.

Appendix B

Jacobi's theta function

We give here a brief introduction for Jacobi's theta function. We refer the reader to [47] for proofs and more details.

For $D \in \mathbb{N}$, let $\mathfrak{H}_D$ denote the set of symmetric $D \times D$ complex matrices whose imaginary part is positive definite. $\mathfrak{H}_D$ is an open subset in $\mathbb{C}^{D(D+1)/2}$ called the *Siegel upper-half space*. Jacobi's theta function $\vartheta : \mathbb{C}^D \times \mathfrak{H}_D \to \mathbb{C}$ is defined as

$$\vartheta(z, \tau) \equiv \sum_{n \in \mathbb{Z}^D} \exp\bigl(i\pi\, n^T \tau\, n + 2\pi i\, n^T z\bigr) \tag{B.1}$$

for any $z \in \mathbb{C}^D$ and $\tau \in \mathfrak{H}_D$. Some important properties of this function are listed in the following.

Lemma 1 (Periodicity)**.** *For all $m \in \mathbb{Z}^D$, $z \in \mathbb{C}^D$ and $\tau \in \mathfrak{H}_D$,*

$$\vartheta(z + m, \tau) = \vartheta(z, \tau)\ . \tag{B.2}$$

Lemma 2 (Quasi-periodicity)**.** *For all $m \in \mathbb{Z}^D$, $z \in \mathbb{C}^D$ and $\tau \in \mathfrak{H}_D$,*

$$\vartheta(z + \tau m, \tau) = \exp\bigl(-i\pi\, m^T \tau\, m - 2\pi i\, m^T z\bigr)\, \vartheta(z, \tau)\ . \tag{B.3}$$

Lemma 3. *For all $A \in GL(D, \mathbb{Z})$,*[1] *and for all $z \in \mathbb{C}^D$ and $\tau \in \mathfrak{H}_D$,*

$$\vartheta(A^T z, A^T \tau A) = \vartheta(z, \tau)\ . \tag{B.4}$$

[1] $GL(D, \mathbb{Z})$ is defined as the group of invertible $D \times D$ matrices with integer entries, whose inverses are also integer matrices.

Lemma 4. *For all integer, even-diagonal*[2] *and symmetric $D \times D$ matrices B, and for all $z \in \mathbb{C}^D$ and $\tau \in \mathfrak{H}_D$,*

$$\vartheta(z, \tau + B) = \vartheta(z, \tau) \; . \tag{B.5}$$

Lemma 5 (Inversion identity)**.** *For all $z \in \mathbb{C}^D$ and $\tau \in \mathfrak{H}_D$,*

$$\vartheta(\tau^{-1} z, -\tau^{-1}) = \det[-i\tau]^{1/2} \exp\big[i\pi z^T \tau^{-1} z\big]\, \vartheta(z, \tau) \; . \tag{B.6}$$

Lemma 6. *The following limit holds for all $z \in \mathbb{C}^D$ and $\tau \in \mathfrak{H}_D$,*

$$\lim_{a \to +\infty} \vartheta(z, a\tau) = 1 \; , \tag{B.7}$$

where $a \in \mathbb{R}_+$. The convergence is stronger than quadratic, i.e. $\vartheta(z, a\tau) = 1 + \mathcal{O}(a^{-2})$.

[2] An even-diagonal matrix B is one for which $n^T B\, n$ is an even integer for all $n \in \mathbb{Z}^D$.