To appear in the *International Journal of Production Research*
Vol. 00, No. 00, 00 Month 20XX, 1–26

Optimal Inventory Decisions when Offering Layaway

Stanko Dimitrov[a*] and Oben Ceryan[b]

[a]*Department of Management Sciences, University of Waterloo, Waterloo, ON, Canada;*
[b] *LeBow College of Business, Drexel University, Philadelphia, PA, USA*

(*July 4, 2018*)

This paper presents an inventory management policy for a retailer offering a layaway program. Layaway is a service provided by retailers that allows budget constrained consumers who have sufficiently high valuations to pay for a product in several installments rather than at once and obtain the product that has been reserved for them at the end of the payment period. If a consumer defaults on payments, then the reserved item is released back into store inventory. In this paper, we first determine the retailer's optimal order decisions when layaway is offered. We find that the order quantity under a layaway program decreases with the likelihood of consumers not finishing their layaway plans and that it is not always profitable for a retailer to offer a layaway program. We then identify the market conditions under which the retailer would benefit from a layaway program. Lastly, we consider an extension to capture the influence of the timing of consumer defaults.

Keywords: layaway; inventory management; consumer behavior; order quantity; profit maximization

1. Introduction

A layaway program allows lower budget consumers who have sufficiently high valuations to pay for a product in several installments rather than at once and obtain the product that has been reserved for them at the end of the payment period. In terms of the product categories served, layaways are often associated with the purchase of durable goods, such as home appliances, consumer electronics, and furniture. Studying the range of financial services for low and moderate income families in the Detroit area, Barr (2009) found that, among a sample of 'unbanked' households (i.e., no one in the household has a checking/savings account or access to credit), 21.9% use layaway. According to the Federal Deposit Insurance Corporation (FDIC), in 2009, 7.7% of all households in the US

*Corresponding author. Email: sdimitro@uwaterloo.ca

were unbanked and 7% remained to do so in 2015 (FDIC (2009, 2015). Further, and more recently, a study by Deloitte found that 14% of surveyed US holiday shoppers in 2016 indicated that they planned on taking advantage of layaway. These figures suggest a potentially sizable layaway market in the US.

From a consumer's perspective: when a consumer decides to put an item on layaway, he must first pay a small program fee along with a down payment, which is the greater of a percentage of the purchase price and a flat fee (K-Mart 2017; Sears 2017; Walmart 2017). The consumer then makes equal payments of the remaining price during the payment period (e.g., weekly payments over an eight-week period, though some retailers may also offer a longer payment duration). If the consumer fails to make the agreed payments or informs the retailer that he would like to no longer participate in the layaway program, then the consumer pays the retailer a cancellation fee and receives all of the money he has paid the retailer towards the item so far. From the retailer's perspective: once an item is put on layaway, the retailer removes the item from the sales floor and it is held for the consumer until all payments are made. In the case that the consumer defaults, or chooses to no longer participate in the layaway program, the item is released back to inventory. In a layaway setting, even though a retailer initially stocks out of items, due to layaway defaults excess stock may still remain at the end of the sale period.

Layaway enables more consumers to purchase an item, however, a retailer exposes themselves to consumer defaults due to layaway purchases. According to Rob Oglesby (2012), on average up to 25% of layaway orders are cancelled. This means that stock that was committed to being sold via layaway, may suddenly become available again. For durable goods that have multiple versions, such as the latest mobile phone, losing the opportunity to sell an item for up to two months may mean that the price for the newly released item may be lower than when it was initially put on layaway, resulting in opportunity costs for the retailer. Given that potentially more consumers can now purchase items via a layaway program, but that there is a chance that some consumers may not complete paying for the item, the following question arises: "*How does offering layaway impact the inventory decisions of a retailer?*"

With this paper, we are taking the first step needed to study ordering decisions in the presence of layaway programs, through which budget constrained individuals that may not be able to afford purchasing through a single payment receive the product after paying through multiple smaller installments. To help retailers such as Walmart, Sears, and K-Mart who are actively providing layaway as a service, we characterize the impact of layaway on a retailer's order quantities and investigate conditions under which it is beneficial for a retailer to offer layaway.

To do so, we first propose a model of consumer behavior for a retailer offering layaway and then construct a condensed single period model in a newsvendor framework that captures essential features of and trade-offs in layaway operations. Through the framework, we characterize a retailer's optimal order quantity decision when offering layaway and provide insights on how the optimal ordering decision is influenced by key market parameters such as consumers' default rates, valuations, and budgets; program characteristics such as service and cancellation fees; and retailer's cost structure regarding the salvage, shortage, and purchase costs. Further, we also identify market conditions under which it is optimal for a retailer to offer layaway in the first place. Specifically, we find that there is a critical market default rate below which the firm benefits from offering layaway and above which the firm is better off not offering layaway. We also investigate how this critical default rate changes with other problem parameters. Lastly, we extend the model to a two-period setting to capture the influence of the timing of consumer defaults.

The remainder of the paper is organized in the following manner: In Section 2, we discuss the related work to that presented here. In Section 3, we formalize our model of a firm offering layaway. We present the optimal ordering decisions by the retailer in Section 4 and further investigate the conditions that are conducive to a viable layaway program in Section 5. We consider an extension to our main model to a two-period setting in Section 6 and conclude our paper in Section 7.

2. Related Work

To our knowledge, there is no prior work in operations management that considers the impact of layaway on a retailer's ordering decisions. However, the main set of literature that we are aware of in operations management related to our work is that in returns. In addition to the work on returns, another two streams of literature on renting and rent-to-own is also related to our work. In the remainder of the section, we first discuss the rental and rent-to-own literature then we discuss the returns stream in the literature.

Operations management issues in the rental and rent-to-own industry are studied in the academic literature. The work of Whisler (1967) determines the volume of inventory to hold for rental equipment using a dynamic programming model. Anderson and Sibdari (2012) study the dynamic ordering decisions of a firm offering a rent-to-own product, where the firm balances lost rental opportunities from low inventories with potential excessive inventory due to returns by consumers. Schultz (1986) also considers the inventory policies of a rent-to-own business through a queueing model. The main difference between the work in rental and rent-to-own inventory decisions is that, as noted by Schultz (1986), most rent-to-own items are expected to be returned, the opposite of layaway items.

Retailing returns is more closely related to the problem considered in this paper. In the returns literature, consumers buy goods and after experiencing the good, may decide to return the good back to the retailer. Once a customer returns a product, a retailer usually uses its reverse logistics policies to best deal with the returned product (Meyer 1999). For example, if the product is still functional, then the product may be repackaged and resold through the same channel. However, in cases where the product needs to be remanufactured/refurbished, the resulting product is usually sold through channels other than where the original product was procured at a lower price, sometimes only recouping 20% the original retail price (Meyer 1999). The one thing that all of the returns handling methods share is that they all are costly to the retailer. This cost may be passed down to customers via a percentage of the retail price, *restocking fees*, but these fees only cover part of the return cost (Meyer 1999). Unlike returns, layaways have no post layaway costs, nor do layaways require new supply chain partners to handle reverse logistics via centralized return centers. The only cost layaways present to a retailer is opportunity cost, in the case a retailer stocks out of non-layaway products before a layaway customer defaults on his/her payment plan.

We now consider some of the recent operations management work in consumer returns. One set of papers on consumer returns considers money-back guarantees or very lenient (very small fees for returning items) return policies in retailing. Chen and Chen (2016); Li et al. (2012); Pinçe et al. (2016); Yang et al. (2017) consider various aspect of consumer returns, from optimal pricing and order quantities to selecting clearing channels for returned goods, some even considering competition from other retailers. The main difference between the papers in retail return policies and that presented here is that in our paper, not all consumers are allowed to return an item. Note that returning an item, in our model, is similar to defaulting on the layaway item. In fact, one may think of layaway, with respect to the returns literature, as a return policy in which a consumer must pay a flat fee for the option to return the product later. The set of papers by Chen and Bell (2009, 2011) also consider the impact of consumer returns on a retailer and a retailing supply chain. The authors show that prices may increase and order quantity may decrease when consumer returns increase with number of units sold.

The introduction of returns results in an increase in the demand for products as consumer valuation uncertainty is alleviated. Valuation uncertainty is used to describe the idea that consumers do not know their true value of a product until they have a chance to experience the product. Various researchers argue that offering returns increases the market size for a product when consumers have valuation uncertainty by giving such consumers the option of partially, or fully, recouping their purchase costs (Akçay et al. 2013; Altug and Aydinliyim 2016; Su 2009; Yang et al. 2017). In our paper we do not consider valuation uncertainty, we assume that each consumer will value the product the amount each initially prescribes to the product. However, market size still expands in our model when layaway is offered because we consider budget constrained individuals making purchasing decisions. Layaway allows individuals that would normally not be able to purchase a product outright, to amortize the cost of the product over time, and thus eventually purchase the product. Though we have not seen these terms in the literature, one may argue that work in valuation uncertainty considers valuation market size expansion, while in our paper we consider budgetary market size expansion. This is further reflected by the fact that as shown in the literature, allowing for returns results in a higher retail price (Che 1996). We show that even if price is exogenously set, there may be a net positive benefit for a retailer to offer layaways. (We also note that some of this work, e.g., Altug and Aydinliyim (2016) uses a newsvendor model to determine inventory decisions, the literature in utilizing newsvendor models for inventory decisions is substantial and some recent papers in retail operations include Abdel-Aal et al. (2017); Ma et al. (2017); Sarkar et al. (2018).)

To summarize, even though there are similarities between layaways and returns in terms of opportunity cost of sales that arises from defaulted layaways or returned items, a key distinguishing feature common to layaway programs is the additional low-budget market segment such programs can serve. Specifically, the product return literature generally addresses valuation uncertainty and does not segment the market in terms of consumers' ability to pay as the returns setting requires that a customer has the initial capability to purchase the product if their net utility is positive. On the other hand, the layaway setting particularly allows the study of the benefits and costs of extending the market to consumers with lower income.

3. The Model

A layaway program is characterized by: (*i*) the length of the layaway period, L, during which a layaway consumer pays for a product with price r in equal installments of r/L, (*ii*) an initial enrollment fee f_s for participation in the program, and (*iii*) a cancellation fee f_c if the consumer terminates the program before payment completion. To help in reading the model and results, we include a list of the notation used in this paper in Appendix A. We assume that a consumer's budget, b, is available each of the L periods, something akin to a salary. We also assume that $f_s \leq r$ and $f_c \leq r$, i.e., the program fees are less than the price of the product. In addition, for tractability purposes, we condense all L layaway periods into a single period. A single period model is equivalent to a multi-period model with a single ordering epoch, during the first period of the planning horizon, and consumer defaults only occur at the end of the last period of the planning horizon, meaning that returns from layaways are not sold later in the market. We provide extensions to our model in Section 6 where some layaway defaults may be used to satisfy later arriving immediate purchase demand.

Consumers are heterogeneous in their valuations for the product as well as their available budget for payment. A consumer with valuation v immediately purchases the product only if their valuation exceeds the product price, i.e., $v > r$, and if their budget is sufficient for purchase, i.e., $b \geq r$. If a consumer's budget is insufficient for an immediate purchase, i.e., $b < r$, they can still obtain the product through the layaway program provided that their budget is adequate for each installment payment, i.e., $b \geq r/L$, and that their discounted valuation is greater than the product price and the layaway participation fee, i.e., $\theta v > r + f_s$, where $\theta < 1$ is a discount factor that reflects the disutility from waiting to obtain the product at the end of the layaway period. Note that any consumer with adequate budget prefers to purchase the product immediately rather than through the layaway program. For tractability, we assume that consumer valuations and budgets are independently and uniformly distributed between $[0, 2\bar{v}]$ and $[0, 2\bar{b}]$, respectively. (As a side note, microeconomic theory predicts consumers may have higher willingness to pay when their income increases, see for example Ferber (1962). On the other hand, there is also evidence that consumer budgeting and categorization of expenses results in deviations from predictions, resulting in over consumption of some categories while reducing consumption in others, see for example, Heaths and Soll (1996). Consequently, our assumption on the independence of budgets and willingness to pay are also supported by works such as Flores and Carson (1997).)

Letting p_I denote the probability that a consumer purchases the product immediately, we have $p_I = \left(1 - \frac{r}{2\bar{v}}\right)\left(1 - \frac{r}{2\bar{b}}\right)$, where the first and second terms correspond respectively, to the probability that a consumer's valuation and their budget exceeds the product's price. Similarly, the probability that a consumer purchases the product through the layaway program, p_L, can be expressed as $p_L = \left(1 - \frac{r+f_s}{2\bar{v}\theta}\right)\left(\frac{r}{2\bar{b}}\right)\left(\frac{L-1}{L}\right)$, where the first term is the probability that a consumer's discounted valuation exceeds the product price plus the service fee, and the remaining terms represent the probability that the consumer's budget is inadequate for an immediate purchase but is sufficient for the layaway program. We assume $\bar{v}$ and $\bar{b}$ are sufficiently large such that both segments of the market exist. We let the total market size, denoted by D, to be random with cumulative distribution $F(\cdot)$ that we assume to be invertible. Using D, we can express the demand for immediate sales and for layaway sales as $p_I D$ and $p_L D$, respectively.

The firm makes a single order quantity decision, Q, that is used to satisfy both types of demand across its sales horizon. Whereas an immediately purchased product leaves the firm's inventory at the time of purchase, a product purchased through the layaway program remains in the firm's possession until all payments have been received. The presence of layaway means that previously reserved inventory may be released back onto the sales floor whenever a consumer defaults on a layaway plan. We denote the market default rate with δ, where $\delta < 1$. When a consumer defaults, the firm charges the consumer a cancellation fee f_c in addition to keeping the layaway service fee, f_s, and returns all other payments the consumer has made up to that point in time. If the firm's initial inventory is sufficient to satisfy both types of demand completely, then any remaining inventory, excess or released back due to layaway defaults is salvaged. Our model allows for units to be salvaged either for a profit or at a cost. If the firm's inventory cannot meet all demand, then the firm satisfies the demand proportionally (e.g., on a first come first served basis) assuming that it cannot prioritize either type of consumer. In such an instance, it receives the sales revenue including any layaway fees and incurs a lost sales penalty.

It is important to note that even if the firm cannot initially meet all demand, it may still need to salvage units at the end of the period due to any layaway defaults. Thus, even though the layaway program captures a new market segment and does not impact the size of the regular demand, it may cannibalize regular product sales and force the firm to miss sales and salvage a product due to consumer defaults.

We formulate the problem by first expressing the firm's profit for any given quantity, Q, and demand realization, D, denoted by $\hat{\pi}_L(Q,D)$, and as stated below:

$$\hat{\pi}_L(Q,D) = \begin{cases} p_I \cdot D \cdot r + p_L \cdot D\left(f_s + \delta \cdot f_c + (1-\delta)\cdot r\right) & \text{if } (p_I + p_L)\cdot D \le Q \\ \quad - \left(Q + p_L \delta D - (p_I + p_L)\cdot D\right)\cdot \mathfrak{s} - cQ & \\ & \\ \frac{p_I}{p_I+p_L}\cdot r \cdot Q + \frac{p_L}{p_I+p_L}\cdot Q\left(f_s + \delta\cdot f_c + (1-\delta)\cdot r\right) & \text{if } (p_I + p_L)\cdot D > Q \\ \quad - \left((p_I+p_L)\cdot D - Q\right)\cdot \mathfrak{u} - \frac{p_L}{p_L+p_I}\delta Q \mathfrak{s} - cQ & \end{cases} \tag{1}$$

In (1), the first case where $(p_I + p_L)\cdot D \le Q$ corresponds to firm's profit if its available quantity can meet all demand. The term $p_I \cdot D \cdot r$ is the revenue obtained through immediate sales and the term $p_L \cdot D\left(f_s + \delta \cdot f_c + (1-\delta)\cdot r\right)$ corresponds to revenue generated through the layaway program. Specifically, in addition to the program participation fee, f_s, the firm receives only an additional cancellation fee f_c if a consumer defaults, and receives the full price of the product if a consumer completes full payment. The term $(Q + p_L\delta D - (p_I + p_L)\cdot D)\cdot \mathfrak{s}$ corresponds to the cost due to or revenue from salvaged units. The $\mathfrak{s}$ term captures the net salvage cost after accounting for any holding costs of the units, we assume that the holding cost is the same for layaway and non-layaway items. If the salvage cost is positive ($\mathfrak{s} > 0$), then one may interpret the salvage cost as the cost of scrapping/recycling one unit of the product. However, if the salvage cost is negative

($\mathfrak{s} < 0$) then one may interpret the salvage cost as a discounted sales price. The last term is the purchase cost, where c denotes the unit purchase cost for the product.

If, on the other hand the firm's quantity is not sufficient to meet all demand, i.e., $(p_I + p_L) \cdot D > Q$, then the firm satisfies the demand proportionally as much as its inventory, Q, allows as the first two terms in the corresponding expression denotes. The term $((p_I + p_L) \cdot D - Q) \cdot \mathfrak{u}$ is the cost incurred due to missed sales, where $\mathfrak{u}$ denotes the per unit missed sale penalty (shortage cost). Finally, $\frac{p_L}{p_L + p_I} \delta Q \mathfrak{s}$ is the salvage revenue/cost for unsold units and the last term indicates the ordering cost.

Overall, the firm's objective is to set a quantity decision, Q, that maximizes its expected profit, with respect to a general demand distribution $f(\cdot)$, $\pi_L(Q)$, i.e., $\max_Q \pi_L(Q)$ where $\pi_L(Q) = E_{D \sim f(\cdot)}[\hat{\pi}_L(Q, D)]$. In the following sections, we study the firm's optimal quantity decision taking into account layaways and investigate when it is beneficial for the firm to offer layaways.

4. Layaway Order Quantities

In the previous section we present the profit of a retailer that offers layaway. Using the derived profit function (1), in this section we determine the optimal order quantity of the retailer. We now present the optimal order quantity of the retailer when layaway is offered to consumers.

To start we write $\pi_L(Q)$:

$$\begin{aligned} \pi_L(Q) = \; & \int_0^{Q/(p_I + p_L)} \Big[\underbrace{(p_I \cdot r + p_L \cdot (\delta(f_s + f_c) + (1 - \delta)(f_s + r)))}_{r'} \cdot x \\ & - \mathfrak{s}(Q - (p_I + p_L)x + p_L \delta x) \Big] f(x) dx \\ & + \int_{Q/(p_I + p_L)}^{\infty} \Bigg[\underbrace{\left(\frac{p_I}{p_I + p_L} \cdot r + \frac{p_L}{p_I + p_L} \cdot (\delta(f_s + f_c) + (1 - \delta)(f_s + r)) \right)}_{r''} Q \\ & - \tfrac{p_L}{p_L + p_I} \delta Q \mathfrak{s} - \mathfrak{u}((p_I + p_L)x - Q) \Big] f(x) dx \\ & - cQ \end{aligned} \tag{2}$$

In (2), the first term in the sum corresponds to the case where the firm's order exceeds total demand from sales and layaways, i.e., $Q > (p_I + p_L) \cdot D$. The second term in the sum is the expected profit if fewer items were ordered than demanded by the market. For convenience, we introduce the terms $r' \equiv p_I \cdot r + p_L \cdot (\delta(f_s + f_c) + (1 - \delta)(f_s + r))$ and $r'' \equiv \frac{p_I}{p_I + p_L} \cdot r + \frac{p_L}{p_I + p_L} \cdot (\delta(f_s + f_c) + (1 - \delta)(f_s + r))$ to capture the effective retail price of the sold goods taking into account the market segmentation (layaway purchasers and non-layaway purchasers) and defaults. The third and last term, in the sum, is the retailer's order cost.

We find Q_L^* by solving for Q in the first order conditions of $\pi_L(Q)$.

Proposition 1. *The optimal order quantity of a retailer facing a general demand distribution, with cumulative distribution function $F(\cdot)$, while offering layaway is:*

$$Q_L^* = (p_I + p_L)F^{-1}\left(\frac{r'' - \frac{p_L}{p_I+p_L}\delta\mathfrak{s} + \mathfrak{u} - c}{r'' - \frac{p_L}{p_I+p_L}\delta\mathfrak{s} + \mathfrak{u} + \mathfrak{s}}\right). \tag{3}$$

where $r'' = \frac{p_I}{p_I+p_L} \cdot r + \frac{p_L}{p_I+p_L} \cdot (\delta(f_s + f_c) + (1-\delta)(f_s + r))$.

Proof: The derivation of all results are presented in Appendix B.

In the critical fractile presented in (3), the cost of underage is given by $r'' - \frac{p_L}{p_I+p_L}\delta\mathfrak{s} + \mathfrak{u} - c$, whereas the cost of overage is $\mathfrak{s} + c$. Specifically, the term r'' in the cost of underage expression corresponds to the lost effective revenue as defined earlier. The term $\frac{p_L}{p_I+p_L}\delta\mathfrak{s}$ is an adjustment to incorporate the salvage revenue/cost where $\frac{p_L}{p_I+p_L}\delta$ is the probability that the unit would have been sold as a layaway and defaulted, and $\mathfrak{s}$ is the unit salvage revenue if $\mathfrak{s} < 0$ or salvage cost if $\mathfrak{s} > 0$. The terms $\mathfrak{u}$ and c refer to the unit shortage cost and the ordering cost for the item, respectively. Similarly, the cost of overage terms $\mathfrak{s}$ and c collectively account for the salvage/cost and ordering cost of an excess unit of product. Lastly, we note that the firm orders a fraction $(p_I + p_L)$ of the fractile of the overall demand it chooses to serve.

We next show how the optimal order quantity decision under layaways, Q_L^*, changes with various exogenous parameters. For most parameters, we simply look at the partial derivative of Q_L^* with respect to that parameter in our sensitivity analysis. We now present the sensitivity of Q_L^* with respect to some parameters, while still assuming a general demand distribution.

Proposition 2. *Q_L^* increases with $\mathfrak{u}$ and f_c, and decreases with δ, $\mathfrak{s}$, and c.*

The proof of Proposition 2 is found in Appendix B. We summarize the results of Proposition 2 in Table 1.

For the remaining parameters that influence the values of p_I and p_L $(\theta, \bar{b}, \bar{v}, r, f_s, L)$ we find that there is no clear relationship when we consider a general demand distribution function. Only after we assume a particular functional form of the demand distribution, uniform, are we able to find that for all parameters, except for L, Q_L^* is stationary for only one point. After exploring these relationships numerically, we see that Q_L^* is increasing initially and decreasing thereafter for r and f_s, this is discussed in greater detail in Appendix C. For example, we see that as r increases, while all other parameters remain constant, $Q_L^*(r)$ initially increases and then decreases. The reason $Q_L^*(r)$ increases initially is because a low retail price does not compensate the retailer for the loss of revenue due to layaway defaults, as such the retailer would rather have lost sales than

Variable	Impact on Q_L^*
Cancellation fee, f_c	$\uparrow$
Default rate, δ	$\downarrow$
Shortage cost, $\mathfrak{u}$	$\uparrow$
Salvage cost, $\mathfrak{s}$	$\downarrow$
Purchase cost, c	$\downarrow$

Table 1.: Comparative Statics on the Optimal Order Quantity in the Presence of Layaways ($\uparrow$ indicates weakly increasing, $\downarrow$ indicates weakly decreasing).

too much inventory (we explore the market size for layaway and immediate purchases in greater detail in Appendix E). However, as price increases, the retailer is compensated by each successful sale for each layaway consumer that defaults. As retail price continues to increase we note that fewer consumers are able to afford the product, and as such the amount of products ordered also decreases.

5. The Profitability of Layaways

Offering layaways allows the firm to extend the market it serves by capturing consumers who have high valuation but low budget. However, as we have discussed previously, the benefits from these additional sales may also be offset by losses due to consumer defaults resulting in additionally ordered units to be salvaged at a lower price or at a cost. Thus, it is not immediately clear whether, or under what conditions, it is profitable for the firm to offer layaways in the first place. Our goal in this section is to provide further insights into this decision.

To do so, we first investigate how the firm's expected profit changes with respect to the default rate. We then contrast the firm's expected profit with and without a layaway program to define a critical default rate in the market characteristics that determines whether offering layaways is profitable. For the remainder of this section we assume that $f_c \leq r$, that is, the cancellation fee the consumer pays does not exceed the product's purchase price, and that the salvage price does not exceed $r - f_c$.

Proposition 3. *The firm's optimal expected profit, π, strictly decreases with the market default rate δ.*

The proof of Proposition 3 is provided in Appendix B. As stated in Proposition 3, the firm's expected profit decreases as the default probability among consumers who participate in the layaway program increases. As a result, and as we formally state in Proposition 4, offering layaways may actually hinder a firm's profit compared to what the firm would have been able to achieve without offering layaways. (Note: In Proposition 4, we assume that $-\mathfrak{s} < \mathfrak{u}$ (applicable when $\mathfrak{s} < 0$). This is a technical assumption required for the second order condition of the profit function when no layaway is offered to be concave. Managerially speaking, the constraint guarantees that the benefits of salvaging unsold goods is not so high that the retailer would prefer to order an infinite number of goods even when layaway is not offered.)

Proposition 4. *There exists $\delta_c \in [0, 1]$, indicating a critical default rate, such that, it is profitable to offer layaways if $\delta < \delta_c$ and not to offer layaways otherwise.*

The proof of Proposition 4 is provided in Appendix B. Proposition 4 indicates that there exists a critical default rate, denoted by δ_c, above which a layaway program is not profitable for the firm and below which, offering layaways improves the firm's profitability. Specifically, although the layaway market allows the firm to serve an additional (low budget) segment of the market without cannibalizing the regular demand, this additional low budget segment may in fact reduce the firm's profitability when the default rate among these consumers are high due to the following: Serving this segment of the market induces the firm to order a higher initial quantity and a higher default rate results in a larger number of these units to be salvaged at the end of the period. Moreover, as the firm serves the market in a first-come, first-served fashion, in instances when overall demand is larger than firm's inventory, the firm may miss regular sales and lose revenue even though they may still end the period with on-hand inventory due to layaway defaults.

Next, we numerically explore how this critical default rate is influenced by exogenous parameters assuming a uniform demand function. As depicted in Table 2, we find that the critical default rate decreases with respect to the salvage cost $\mathfrak{s}$ as well as the shortage cost $\mathfrak{u}$. In other words, the firm's

Variable	Impact on critical default rate, δ_c
Shortage cost, $\mathfrak{u}$	$\downarrow$
Salvage cost, $\mathfrak{s}$	$\downarrow$
Purchase cost, c	$\downarrow$
Retail price, r	$\uparrow$
Service fee, f_s	$\uparrow$
Cancellation fee, f_c	$\uparrow$

Table 2.: Comparative statics on the critical default rate, δ_c .

willingness to offer layaways decreases when it is more costly to discard any unsold units or to face shortages. If instead the firm is able to salvage unsold units not at a cost but at a reduced selling price (represented by a negative $\mathfrak{s}$ value), then the firm's willingness to offer layaways increases as the salvage price increases (i.e., a more negative $\mathfrak{s}$ value). The reason δ_c is decreasing in both $\mathfrak{u}$ and $\mathfrak{s}$ follows from the fact that the layaway program increases the likelihood of both inventory shortages and overages. In particular, an item put on layaway may not be sold to a consumer wanting to immediately purchase the item, thus items on layaway may result in further shortages. In addition, defaulted layaway items will be salvaged at the end of the planning horizon, again leading to costs to the retailer. Therefore, as either $\mathfrak{u}$ or $\mathfrak{s}$ increase, the retailer would like to be sure that items on layaway will likely not be defaulted on by participating consumers.

In addition, as shown in Table 2, we also find that layaways become less attractive when the purchase cost of the product increases and more attractive when the product can be sold for a higher margin. As the per-unit procurement cost c increases, layaway becomes less attractive because the market margins are lower and the relative recuperated cost from salvaged items decreases. Using similar logic, as the retail price, r, increases the relative recuperated cost from a successfully sold item increases and layaway becomes more attractive. Finally, both the service fee and the cancellation fee increases the firm's willingness to offer a layaway program. Note that as f_s increases, the likelihood of someone participating in the layaway program tends to zero. We carried out the above numerical analysis for a different demand distribution, lognormal demand, and qualitatively observe the same relationships between δ_c and $\mathfrak{u}$, $\mathfrak{s}$, c, r, f_s, and f_c.

In addition to the single parameter results summarized in Table 2, we also would like to particularly comment on how the firm's decision on whether to offer layaway is influenced by simultaneous variations in the product's price and consumers' budget as these two parameters are among the main ones that determine the relative sizes of the instantaneous purchase and layaway markets. (As we mentioned earlier, a more detailed description on how the market sizes change with respect to various parameters is provided in Appendix E.) In particular, Figure 1 displays the retailer's optimal market strategy regarding whether to offer a layaway (grey region in the figure) or not (white region in the figure) with respect to the product price and budget parameters. Figure 1 shows that, for a given budget level, the firm may shift from not offering the product at all to offering the product only for immediate purchase, and then to offering the product via both immediate and layaway purchases as the retail price of the product gets larger. For a given retail price, the firm may shift from offering the product through both layaway and immediate purchase to only offering the product for immediate purchase as consumers' budgets increase. For a very low retail price, the firm may be better of by only offering the product through immediate purchase or

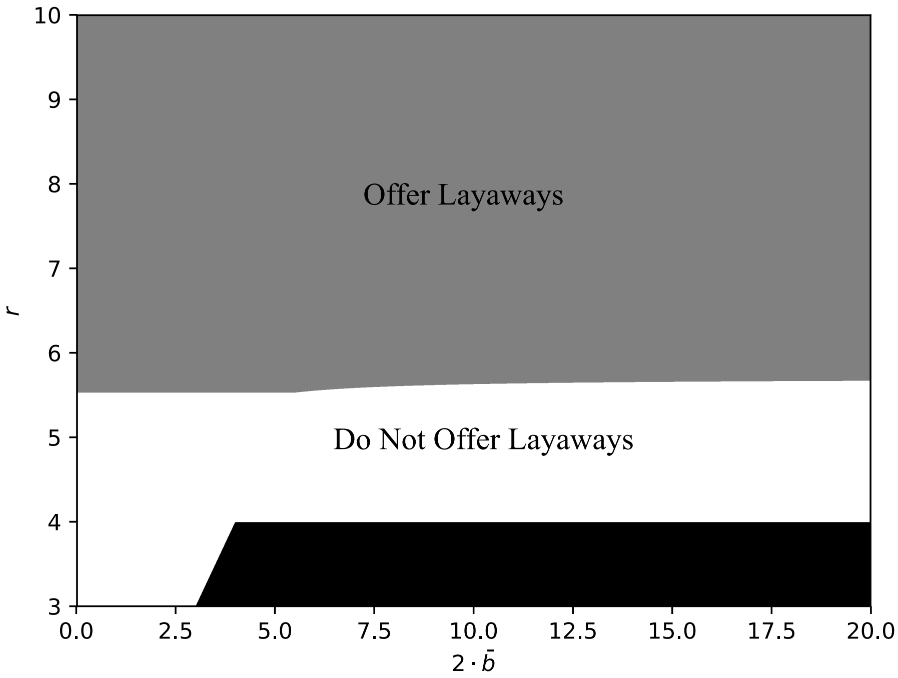

Figure 1.: Profitability of Offering Layaways with respect to retail price and consumers' budget

not offering the product at all. The latter results from most consumers purchasing the product at higher budgets for which the order quantities along with stochastic demand results in a net loss for the retailer due to the shortage and overage costs dominating the retailer's profits from selling items at a low retail price.

6. An Extension to a Two-Period Model

We also would like to provide a brief extension to a two-period model, in which the firm is able to satisfy some late arriving immediate purchasers through units that become available due to early defaults in the layaway program. Specifically, we consider the following sequence of events. At the beginning of the first period, the firm places an order of size Q. While all customers interested in the layaway program arrive in the first period, the immediate purchasers are split between two periods, a fraction $(1-\alpha)$ arriving in the first period, and the remaining fraction α arriving in the second period, where $0 \leq \alpha \leq 1$. Among the layaway defaults, a fraction β (where $0 \leq \beta \leq 1$) occurs in the first period and the remaining fraction $(1-\beta)$ occurs at the end of the second period. Any defaulted units in the first period can be used to satisfy immediate purchase demand during the second period. To aid the exposition and to draw parallels between our initial results, we account for the salvage/holding cost and shortage cost in aggregate for the entire two-period horizon, i.e., holding/salvage is based on the remaining inventory at the end of the second period and shortage corresponds to all missed demand by the end of the second period.

In this extended model, α indicates the relative strength of separation between early arriving layaway customers and late arriving immediate purchasers. For example, $\alpha = 1$ indicates complete separation, and that the two types of demand arrive sequentially. Likewise, β indicates the proportion of defaulted units that can still be used to satisfy late arriving demand. For example, $\beta = 1$ implies all defaults in the layaway program can later be sold to late arriving immediate purchasers. Consequently, the instance with $\alpha = 0$ and $\beta = 0$ reduces the problem to our original formulation.

The model gives rise to two distinct cases based on the relative values of α and β. Apart from having excess units or shortages during both periods that is common to both cases, the first case corresponds to the possibility that the firm has excess inventory in the first period but faces shortages in the second period, and the second case corresponds to the possibility that the firm runs out of stock in the first period but is able to satisfy all demand in the second period (i.e., earlier defaults exceed late arriving demand). The two cases cannot both occur for a given set of problem parameters. While the first case arises when $p_I\alpha \geq p_L\delta\beta$ where δ corresponds to the overall default rate as defined earlier, $p_I\alpha < p_L\delta\beta$ implies the second case. As the analysis of both cases are similar, and also follows similar steps as in our main model, for brevity, we omit the derivation details and only provide the results for the more practical first case where the firm may have excess inventory at the end of the first period but face shortages in the second period. (For brevity, we relegate the description of the expected profit function to Appendix D.) Our main finding is as follows.

Proposition 5. *The optimal order quantity for the two-period model satisfies the following expression:*

$$\begin{aligned}
&(\mathfrak{s} + r + \mathfrak{u})\left[(1-\alpha)p_I + p_L\right] F\left(\tfrac{Q^*}{p_I + (1-\delta\beta)p_L}\right) \\
&+ \left[p_L(f_s + \delta f_c) - (r + \mathfrak{s})p_L\delta(1-\beta) + \mathfrak{u}p_L\delta\beta\right] F\left(\tfrac{Q^*}{(1-\alpha)p_I + p_L}\right) \\
&\qquad = (1-\alpha)p_I r + p_L(f_s + \delta f_c + (1-\delta)r) + p_L\delta\beta r + \mathfrak{u}(1-\alpha)p_I \\
&\qquad\quad + \mathfrak{u}p_L(1+\delta\beta) - \mathfrak{s}p_L\delta(1-\beta) - c(1-\alpha)p_I - cp_L
\end{aligned} \tag{4}$$

The proof of Proposition 5 follows the same procedure as Proposition 1 using the expected profit function defined in Appendix D. The expression in (4) indicates that the optimal order quantity relates the cumulative distribution function of demand at two different points, i.e., $F\left(\frac{Q^*}{p_I+(1-\delta\beta)p_L}\right)$ and $F\left(\frac{Q^*}{(1-\alpha)p_I+p_L}\right)$. We note that, when $\alpha = 0$ and $\beta = 0$, the two cumulative distribution functions are evaluated at the same point, i.e., $F\left(\frac{Q^*}{p_I+p_L}\right)$, and the expression for the optimal order quantity reduces to the result given in Proposition 1. This is expected since the two formulations become equivalent in this case, as discussed previously.

As in our main model, next we would like to comment on the sensitivity of the optimal order quantity with respect to various problem parameters. Our numerical studies demonstrate that the two-period model maintains the monotonicity results we have outlined in Table 1 and Table C1. For example, the optimal order quantity increases in the shortage cost, and the upper bounds for budget and valuations, and decreases in the default rate, salvage cost, and purchase cost. Regarding the sensitivity to the retail price, we again observe that the optimal order quantity first increases and then decreases with the retail price.

One might conjecture that as α or β increases, the optimal order quantity may decrease. The rationale behind this conjecture is that as α increases, more of the immediate demand arrives late and therefore can benefit from any defaulted layaway units earlier in the horizon. In a similar fashion, an increase in β indicates that more of the defaults happen earlier in the horizon, thus increasing the number of units available for the second period. Although we are not able to analytically prove the sensitivity of the optimal order quantity with respect to α and β, extensive numerical tests suggest that the order quantity decreases with α and β, i.e., the firm orders less if more of the immediate purchase demand arrives later in the horizon, or more of the defaults occur earlier in the horizon.

Our formulation in this section did not allow for a subsequent ordering decision at the beginning of the second period. We regard this extension beyond the scope of our current study. Nevertheless, based on our findings, we would expect that an increase in α would cause the subsequent order quantity to increase, while an increase in β would lower the subsequent order. That is, we would expect that the second period order quantity would be higher as more of the instantaneous demand arrives late and would be lower as more of the defaulted layaways are available to satisfy the second period demand. However, we believe that a future study that considers layaways in a more general multiple period setting through a dynamic programming formulation will provide further important contributions to our understanding of the design and operation of layaway programs.

7. Conclusions

This paper discusses layaway programs offered by retailers to serve consumers with high valuations but low budget. We first construct a consumer utility model that captures consumers' heterogeneity in terms of both valuations and budgets. Next, we formulate the firm's quantity selection problem and identify the optimal ordering policy in the presence of layaways. We study the sensitivity of the optimal order quantity with respect to several problem parameters, in terms of (i) market characteristics such as consumers' default rates, valuations, and budgets, (ii) program characteristics such as service and cancellation fees, and (iii) retailer's cost structure regarding the salvage, shortage, and purchase costs. It is important to note that even though offering a layaway program enables the firm to expand the proportion of the market it can serve, and thus generate additional revenue, it also comes with risks arising from consumer defaults in terms of costly salvages and/or potential missed regular sales. Our setting allows us to investigate when a firm would benefit from a layaway program. Specifically, we find that there is a critical market default rate below which the firm benefits from offering layaway and above which the firm is better off not offering layaway. We investigate how this critical default rate changes with other problem parameters. Finally, through a stylized two-period model, we also investigate the influence of the timing of layaway defaults.

The presented work is a much needed first step in studying the operational implications of layaway programs. Given the prevalence of layaway in retail, retailers can benefit from further studies. One example is the fact that customers may put a down payment that is greater than $\frac{r}{L}$, as currently assumed in our model. Another is to relax the assumption that a consumer's valuation and budget are independently distributed. This assumption may not necessarily hold true in practice, it is not inconceivable to think that budget and valuation may be correlated (either positively or negatively). If budgets are negatively correlated with valuations, then the need for layaway and thus the benefit may increase as more customers may participate in a layaway program. Another possible extension would be the consideration of a more general multi-period model. In the presented work we largely focus on a compressed single period model. Though we also briefly study a two period model that considers different times of default and time of purchases, extending the model to more general multiple period settings will be of interest. These are just some of the potential extensions to the work presented here, we look forward to seeing layaway further explored in the operations management literature.

Appendix A. Notation

Q	order quantity
$F(\cdot)$	the distribution of demand, assumed to be invertible
D	the demand for the product
v	valuation of the good $v \sim U[0, 2\bar{v}]$
b	budget of the consumer $b \sim U[0, 2\bar{b}]$
p_I	Probability of an instant purchase: $p_I \equiv \left(1 - \frac{r}{2\bar{b}}\right)\left(1 - \frac{r}{2\bar{v}}\right)$

p_L Probability of a layaway purchase:

$$\begin{aligned} p_L \equiv & \left(1 - \tfrac{r+f_s}{2\bar{v}\theta}\right)\left(\tfrac{Lr-r}{2L\bar{b}}\right) \\ \equiv & \left(1 - \tfrac{r+f_s}{2\bar{v}\theta}\right)\left(\tfrac{r}{2\bar{b}}\right)\left(\tfrac{L-1}{L}\right) \end{aligned}$$

r The $r$etail price of the good

L is the length of the layaway period, usually 8 or 12 weeks. The unit of measure here is weeks

θ is the disutility of waiting to receive the good that is put on layaway

δ is the default rate

f_s is the $s$ervice fee for entering into a layaway contract

f_c is the $c$ancellation fee for cancelling a layaway contract

$\mathfrak{s}$ is the salvage cost/holdover value of a single item (may be negative or positive: positive means holding cost, while negative means salvage value)

$\mathfrak{u}$ is the shortage cost for a single unit of demand

c is the purchase cost for a single item

Appendix B. Proofs

In this section we present the details of the derivations of our results.

Proof of Proposition 1

$$\begin{aligned} \pi_L(Q) = \ & E_{f(\cdot)}\hat{\pi}_L(Q,D) \\ = \ & \int_0^{Q/(p_I+p_L)} [\underbrace{(p_I \cdot r + p_L \cdot (\delta(f_s+f_c) + (1-\delta)(f_s+r)))}_{r'} \cdot x \\ & - \mathfrak{s}(Q - (p_I+p_L)x + p_L\delta x)] f(x)dx \\ & + \int_{Q/(p_I+p_L)}^{\infty} \Bigg[\underbrace{\left(\frac{p_I}{p_I+p_L} \cdot r + \frac{p_L}{p_I+p_L} \cdot (\delta(f_s+f_c) + (1-\delta)(f_s+r)) \right) Q}_{r''} \\ & - \tfrac{p_L}{p_L+p_I}\delta Q\mathfrak{s} - \mathfrak{u}((p_I+p_L)x - Q) \Bigg] f(x)dx \\ & -cQ \\ = \ & \int_0^{Q/(p_I+p_L)} [r'x - \mathfrak{s}(Q - (p_I+p_L)x + p_L\delta x)] f(x)dx \\ & + \int_{Q/(p_I+p_L)}^{\infty} [r''Q - \frac{p_L}{p_L+p_I}\delta Q\mathfrak{s} - \mathfrak{u}((p_I+p_L)x - Q)] f(x)dx - cQ \end{aligned}$$

Now compute the derivative of $\pi_L(Q)$

$$\begin{aligned}
\frac{d\pi_L(Q)}{dQ} &= \frac{1}{p_I+p_L}\left[r'\frac{Q}{p_I+p_L} - \mathfrak{s}(Q - \frac{p_I+p_L}{p_I+p_L}Q + p_L\delta\frac{Q}{p_I+p_L})\right] f\left(\frac{Q}{p_I+p_L}\right) \\
&\quad + \int_0^{Q/(p_I+p_L)} -\mathfrak{s}f(x)dx \\
&\quad - \frac{1}{p_I+p_L}\left[r''Q - \frac{p_L}{p_L+p_I}\delta Q\mathfrak{s} - \mathfrak{u}\left((p_I+p_L)\frac{Q}{p_I+p_L} - Q\right)\right] f\left(\frac{Q}{p_I+p_L}\right) \\
&\quad + \int_{Q/(p_I+p_L)}^{\infty} [r'' - \frac{p_L}{p_L+p_I}\delta\mathfrak{s} + \mathfrak{u}]f(x)dx - c \\
&= \frac{1}{p_I+p_L}\left[r'\frac{Q}{p_I+p_L} - \mathfrak{s}p_L\delta\frac{Q}{p_I+p_L}\right] f(\frac{Q}{p_I+p_L}) - \mathfrak{s}F(\frac{Q}{p_I+p_L}) \\
&\quad - \frac{1}{p_I+p_L}\left[r''Q - \frac{p_L}{p_I+p_L}\delta Q\mathfrak{s}\right] f(\frac{Q}{p_I+p_L}) + [r'' - \frac{p_L}{p_I+p_L}\delta\mathfrak{s} + \mathfrak{u}](1 - F(\frac{Q}{p_I+p_L})) - c \\
&= [r'' - \frac{p_L}{p_I+p_L}\delta\mathfrak{s} + \mathfrak{u}](1 - F(\frac{Q}{p_I+p_L})) - \mathfrak{s}F(\frac{Q}{p_I+p_L}) - c \quad \text{Note: } r'' = \frac{r'}{(p_I+p_L)}
\end{aligned}$$

We can look at the SOCs for $\pi_L(Q)$ with respect to Q:

$$\begin{aligned}
\frac{d^2\pi_L(Q)}{dQ^2} &= -\frac{1}{p_I+p_L}[r'' - \frac{p_L}{p_I+p_L}\delta\mathfrak{s} + \mathfrak{u}]f(\frac{Q}{p_I+p_L}) - \frac{\mathfrak{s}}{p_I+p_L}f(\frac{Q}{p_I+p_L}) \\
&= -\frac{1}{p_I+p_L}[r'' - \frac{p_L}{p_I+p_L}\delta\mathfrak{s} + \mathfrak{u} + \mathfrak{s}]f(\frac{Q}{p_I+p_L})
\end{aligned}$$

$r'' - \frac{p_L}{p_I+p_L}\delta\mathfrak{s} + \mathfrak{u} + \mathfrak{s} \geq 0$ as $\frac{p_L}{p_I+p_L}\delta\mathfrak{s} \leq \mathfrak{s}$ and $r'' \geq 0$. This means that the second order derivative is negative.

We now solve the FOCs of $\pi_L(Q)$ for Q to determine Q_L^*.

$$\begin{aligned}
0 &= [r'' - \frac{p_L}{p_I+p_L}\delta\mathfrak{s} + \mathfrak{u}](1 - F(\frac{Q_L^*}{p_I+p_L})) - \mathfrak{s}F(\frac{Q_L^*}{p_I+p_L}) - c \\
\frac{r'' - \frac{p_L}{p_I+p_L}\delta\mathfrak{s} + \mathfrak{u} - c}{r'' - \frac{p_L}{p_I+p_L}\delta\mathfrak{s} + \mathfrak{u} + \mathfrak{s}} &= F(\frac{Q_L^*}{p_I+p_L})) \\
Q_L^* &= (p_I+p_L)F^{-1}\left(\frac{r'' - \frac{p_L}{p_I+p_L}\delta\mathfrak{s} + \mathfrak{u} - c}{r'' - \frac{p_L}{p_I+p_L}\delta\mathfrak{s} + \mathfrak{u} + \mathfrak{s}}\right)
\end{aligned}$$

□

Proof of Proposition 2

We start with rewriting $Q_L^* = (p_I+p_L)F^{-1}\left(\frac{r'' - \frac{p_L}{p_I+p_L}\delta\mathfrak{s} + \mathfrak{u} - c}{r'' - \frac{p_L}{p_I+p_L}\delta\mathfrak{s} + \mathfrak{u} + \mathfrak{s}}\right)$. We note that for the variables we consider: $f_c, \delta, \mathfrak{u}, \mathfrak{s}$, and c, (p_I+p_L) does not change with their value, in addition, p_I+p_L is always non-negative. We also note that $F^{-1}(x)$ is a positive monotonically increasing function in x, by definition of a inverse cumulative distribution function. Therefore, to determine how Q_L^* changes with respect to the variables of interest, we just have determine how $\frac{r'' - \frac{p_L}{p_I+p_L}\delta\mathfrak{s} + \mathfrak{u} - c}{r'' - \frac{p_L}{p_I+p_L}\delta\mathfrak{s} + \mathfrak{u} + \mathfrak{s}}$ changes with respect to each variable. If we define $C_F = \frac{r'' - \frac{p_L}{p_I+p_L}\delta\mathfrak{s} + \mathfrak{u} - c}{r'' - \frac{p_L}{p_I+p_L}\delta\mathfrak{s} + \mathfrak{u} + \mathfrak{s}}$, then $\text{sign}\left(\frac{\partial Q_L^*}{\partial i}\right) = \text{sign}(\frac{\partial C_F}{\partial i})$, for $i \in \{f_c, \delta, \mathfrak{u}, \mathfrak{s}, c\}$. We omit the calculation of the partials of C_F in the interest of space.

□

Proof of Proposition 3

We first show that for any given Q, the expected profit decreases in the default rate δ. Differentiating the expected profit expression given in (2) with respect to δ, we get:

$$\frac{d\pi(Q)}{d\delta} = p_L(f_c - r - \mathfrak{s}) \int_0^{Q/(P_I+P_L)} xf(x)dx + \frac{Q}{P_I + P_L} p_L(f_c - r - \mathfrak{s})\Big(1 - F(\frac{Q}{P_I + P_L})\Big) < 0$$

where the inequality follows from the assumption that $f_c - r - \mathfrak{s} < 0$. Let $\pi_\delta(Q)$ and $\pi_{\delta'}(Q)$ be the expected profit function based on a quantity selection of Q, corresponding to a default rate of δ and δ', respectively. Therefore, for any given $\delta' > \delta$, and for any particular Q, we have $\pi_\delta(Q) > \pi_{\delta'}(Q)$. Next, let $Q^*(\delta)$ and $Q^*(\delta')$ be the optimal quantity selection corresponding to a default rate of δ and δ', respectively. Then,

$$\pi_\delta(Q^*(\delta)) \geq \pi_\delta(Q^*(\delta')) > \pi_{\delta'}(Q^*(\delta'))$$

where the first inequality is due to the definition of optimality and the second inequality follows from our previous result that the expected profit decreases with respect to the default rate for any particular Q.

$\square$

Proof of Proposition 4

We first define the profit function of the retailer when no layaway is offered, using the notation introduced in Section 3. Letting $\hat{\pi}_{NL}(Q, D)$ denote the profit for the retailer for a particular quantity selection Q, we write:

$$\hat{\pi}_{NL}(Q, D) = \begin{cases} p_I \cdot r \cdot D - (Q - p_I \cdot D) \cdot \mathfrak{s} - cQ & \text{if } p_I \cdot D \leq Q \\ r \cdot Q - (p_I \cdot D - Q) \cdot \mathfrak{u} - cQ & \text{if } p_I \cdot D > Q \end{cases} \tag{B1}$$

Next, let $\pi_{NL}(Q) := E_D(\hat{\pi}_{NL}(Q, D))$ denote the expected profit of the retailer when not offering layaway and ordering Q units. We have:

$$\begin{aligned} \pi_{NL}(Q) = \quad & r\left(\int_0^{Q/p_I} p_I x f(x)dx + \int_{Q/p_I}^{\infty} Qf(x)dx\right) \\ & -\mathfrak{s}\int_0^{Q/p_I} (Q - p_I x)f(x)dx - \mathfrak{u}\int_{Q/p_I}^{\infty} (p_I x - Q)f(x)dx - cQ \end{aligned}$$

Differentiating $\pi_{NL}(Q)$ with respect to Q, yields $\frac{d\pi_{NL}(Q)}{dQ} = (p_I \cdot r + \mathfrak{u})\left(1 - F\left(\frac{Q}{p_I}\right)\right) - \mathfrak{s}F\left(\frac{Q}{p_I}\right) - c$. It can be simply verified that the second order derivative $\frac{d^2\pi_{NL}(Q)}{dQ^2} = -(p_I \cdot r + \mathfrak{u} + \mathfrak{s})F\left(\frac{Q}{p_I}\right)$ is concave whenever the salvage value for the product does not exceed the shortage penalty, i.e., $-\mathfrak{s} < \mathfrak{u}$. The optimal order quantity for the no-layaway case, denoted by Q^*_{NL} is then given by: $Q^*_{NL} = (p_I)F^{-1}\left(\frac{p_I \cdot r + \mathfrak{u} - c}{p_I \cdot r + \mathfrak{u} + \mathfrak{s}}\right)$. Finally, we let $\pi^*_{NL}(Q^*)$ denote the optimal expected profit under the no-layaway case.

Note that the expected optimal profit function for the no-layaway instance $\pi^*_{NL}(Q^*)$ does not depend on δ as the firm does not serve the layaway market and thus is not exposed to the default rate in this instance. We also recall that the expected profit function for the layaway instance, $\pi^*_L(Q^*(\delta))$ strictly decreases with the default rate δ as stated in Proposition 3. Therefore, $\pi^*_L(Q^*(\delta))$ will equal $\pi^*_{NL}(Q^*)$ at most once.

Consider now the comparison between $\pi^*_L(Q^*(\delta))$ and $\pi^*_{NL}(Q^*)$, i.e., the expected profit under the layaway (for any given δ_c) and the no-layaway instances. Let δ_c denote a critical default rate that would cause the firm to be indifferent between offering and not offering layaways. Let $\delta_c \equiv 1$ if $\pi^*_L(Q^*(\delta)) > \pi^*_{NL}(Q^*)$ for all $\delta \in [0, 1]$. Similarly, let $\delta_c \equiv 0$ if $\pi^*_L(Q^*(\delta)) < \pi^*_{NL}(Q^*)$ for all $\delta \in [0, 1]$. Otherwise, δ_c is such that $\pi^*_L(Q^*(\delta_c)) = \pi^*_{NL}(Q^*)$. Hence, it is optimal for the firm to offer layaways if the market default rate $\delta \in [0, 1)$ satisfies $\delta < \delta_c$ and do not offer layaways if otherwise, i.e., $\delta \geq \delta_c$.

□

Appendix C. Computing Sensitivity Analysis of Q^*_L

In this section we present the method we used to compute the sensitivity analysis of the optimal order quantity when layaway is offered for $\bar{b}$, f_s, $\bar{v}$, θ, L, and r. We note that the following holds from the chain rule:

$$\frac{d}{dy} g_1(y)F^{-1}(g_2(y)) = g_1(y)\frac{1}{F'(F^{-1}(g_2(y)))}g_2'(y) + g_1'(y)F^{-1}(g_2(y)) \tag{C1}$$

As we see in (C1), it is no longer sufficient to consider $g_2'(y)$ as even if $g_2'(y)$ were negative in the relationship between the two terms in the sum that dictate if $g_1(y)F^{-1}(g_2(y))$ is increasing or not. Note that in our case $g_1(y) = (p_I + p_L)$ and $g_2(y) = \frac{r'' - \frac{p_L}{p_I + p_L}\delta\mathfrak{s} + \mathfrak{u} - c}{r'' - \frac{p_L}{p_I + p_L}\delta\mathfrak{s} + \mathfrak{u} + \mathfrak{s}}$. For a general distribution function, it is difficult to determine how Q^*_L changes with each of the variables of interest. Therefore, to explore the relationship between Q^*_L and the variables of interest, we consider demand that is distributed uniformly on $[0, K]$. For a given demand distribution, we use $F^{-1}(\cdot)$ and vary only one parameter, holding all other parameters constant in equation (3). We generate multiple plots, but in the interest of space we only qualitatively describe the plots in this section.

Variable	Q_L^* (under Uniform Demand)	Q_L^* (under Lognormal Demand)
f_s	$\uparrow$ then $\downarrow$	$\downarrow$
r	$\uparrow$ then $\downarrow$	$\uparrow$ then $\downarrow$
$\bar{b}$	$\uparrow$	$\uparrow$
$\bar{v}$	$\uparrow$	$\uparrow$
c	$\downarrow$	$\downarrow$
L	$\uparrow$	$\uparrow$
θ	$\uparrow$	$\uparrow$

Table C1.: Additional Comparative Statics on the Optimal Order Quantity in the Presence of Layaways.

We present the relationship we found in our numerical exploration of the value of Q_L^* with changes in $f_s, r, \bar{b}, \bar{v}, c, \delta,$ and θ as depicted in Table C1. What we see in Table C1 is that for both uniform and lognormal demand distributions, the relationship between the optimal order quantity, Q_L^* and the parameters we consider is qualitatively the same for both of the distributions. The one difference is the relationship between f_s and Q_L^*. When we consider uniform demand, we have an initially increasing value of Q_L^* and then a decreasing value of Q_L^* (the relationship looks like an inverted 'U'); for lognormal demand, we notice that Q_L^* is decreasing, the right-hand side of an inverted 'U.' This we think is an artifact of the distribution and may mean the demand for layaway products is highly sensitive to the service fee for the lognormal demand distribution. In the uniform demand setting, the inverted 'U' is observed because on the one hand the retailer would like to collect more layaway program fees from consumers, thus more items are ordered; on the other hand, as the service fee increases the likelihood of someone participating in the layaway program decreases, $p_L = \left(1 - \frac{r+f_s}{2\bar{v}\theta}\right)\left(\frac{r}{2\bar{b}}\right)\left(\frac{L-1}{L}\right)$, thus the need for additional products decreases. As we see initially the desire for increased revenue from layaway purchasers is the dominating effect, but as f_s increases the latter, likelihood of participation decreasing ends up being the dominating effect. The optimal order quantity is increasing with $\bar{b}$ and $\bar{v}$. As $\bar{b}$ increases, a larger fraction of consumers can purchase the good. Similarly, as $\bar{v}$ increases, a larger fraction of consumers will be willing to purchase the good, potentially via layaway for a fixed price. The last relationship we would like to highlight is the relationship between the optimal order quantity and θ, the waiting discount. In our exploration, we first notice that the optimal order quantity does not change for low values of θ, but then increases as θ increases. The reason follows from the choice model we consider in which a consumer will purchase if $\theta v > r + f_s$. Specifically, θ must be sufficiently large for any customer to purchase.

Appendix D. Formulation for the Two-Period Model

Below, we provide the expected profit function, denoted by $\pi_2(Q)$ for the two-period layaway model as presented in Section 6. The formulation corresponds to instances in which $p_I\alpha \geq p_L\delta\beta$, i.e., the firm will never run out of products in period 1 yet fully meet demand in period 2 due to defaulted units at the end of the first period (for details, please see the description in Section 6).

$$
\begin{aligned}
\pi_2(Q) = \;& \int_0^{Q/(p_I+(1-\delta\beta)p_L)} \Bigg[(p_I \cdot r + p_L \cdot (f_s + \delta f_c + (1-\delta)r)) \cdot x \\
&\qquad - \mathfrak{s}(Q - (p_I + p_L)x + p_L\delta x) \Bigg] f(x)dx \\
& + \int_{Q/(p_I+(1-\delta\beta)p_L)}^{Q/((1-\alpha)p_I+p_L)} \Bigg[((1-\alpha)\cdot p_I \cdot r + p_L \cdot (f_s + \delta f_c + (1-\delta)r)) \cdot x \\
&\qquad r \cdot \Big(Q - ((1-\alpha)p_I + p_L)\cdot x + \delta\beta p_L x \Big) \\
&\qquad - \mathfrak{u}\Big(\alpha p_I x - Q + ((1-\alpha)p_I + p_L)\, x - \delta\beta p_L x \Big) \\
&\qquad - \mathfrak{s} p_L (1-\delta\beta)x \Bigg] f(x)dx \\
& + \int_{Q/((1-\alpha)p_I+p_L)}^{\infty} \Bigg[\left(\frac{(1-\alpha)p_I}{(1-\alpha)p_I + p_L} \cdot r + \frac{p_L}{(1-\alpha)p_I + p_L} \cdot (f_s + \delta f_c + (1-\delta)r) \right) Q \\
&\qquad + \frac{p_L}{(1-\alpha)p_I + p_L} \delta\beta Q r \\
&\qquad - \mathfrak{u} \left((p_I + p_L)x - Q - \frac{p_L\delta\beta}{(1-\alpha)p_I + p_L} Q \right) \\
&\qquad - \mathfrak{s} \left(\frac{p_L\delta(1-\beta)}{(1-\alpha)p_I + p_L} Q \right) \Bigg] f(x)dx \\
& - c\, Q
\end{aligned}
$$

The first integral in the expression of the expected profit function for the two-period model corresponds to demand realizations that allows the firm to satisfy all incoming demand in both periods. Specifically, the term $(p_I \cdot r + p_L \cdot (f_s + \delta f_c + (1-\delta)r)) \cdot x$ corresponds to the revenue generated across both periods through immediate purchases, and through the service fee, cancellation fee, and full sales revenues generated by layaways. The term $\mathfrak{s}(Q - (p_I + p_L)x + p_L\delta x)$ is the cost associated with salvaging left over inventory; salvage is costly if $\mathfrak{s} > 0$ and the firm can capture some revenue from salvages if $\mathfrak{s} < 0$. (As a side note, we do not apply discounting on the second period in order to aid in the exposition of the model and highlight the main drivers of the ordering quantity, however, a discount factor can be implemented.)

The second integral corresponds to demand realizations that allows the firm to satisfy all incoming first period demand, but only partially satisfy the second period demand based on the number of units available at the end of period 1 (i.e., units left over after satisfying first period demand plus units that become available for period 2 due to some layaway defaults in period 1). Specifically, the term $((1-\alpha)\cdot p_I \cdot r + p_L \cdot (f_s + \delta f_c + (1-\delta)r))\cdot x$ refers to revenues due to immediate purchases in the first period and the layaway sales. We remind the reader that we assume all layaway customers arrive in the first period. The second term, $r \cdot \Big(Q - ((1-\alpha)p_I + p_L)\cdot x + \delta\beta p_l x\Big)$, corresponds to revenue from immediate purchases that were only partially satisfied as permitted by the available inventory. The inventory consists of all unsold units at the end of period 1, i.e., $Q - ((1-\alpha)p_I + p_L)\cdot x$, and the units that become available for period 2 due to defaults in period 1, $\delta\beta p_L x$. The term $\mathfrak{u}\Big(\alpha p_I x - Q + ((1-\alpha)p_I + p_L)\, x - \delta\beta p_L x\Big)$ captures the costs associated with missed immediate sales in period 2, and the term $\mathfrak{s} p_L(1-\delta\beta)x$ refers to the salvage cost associated with the items that were defaulted in period 2.

Lastly, the third integral corresponds to demand instances where the firm is unable to meet demand in either of the periods. The term $\left(\frac{(1-\alpha)p_I}{(1-\alpha)p_I+p_L}\cdot r + \frac{p_L}{(1-\alpha)p_I+p_L}\cdot (f_s + \delta f_c + (1-\delta)r)\right) Q$ refers to revenues associated with first period immediate purchases and the layaway customers (we assume the products are distributed in a first come first serve fashion). The term $\frac{p_L}{(1-\alpha)p_I+p_L}\delta\beta Q r$ represents the revenues associated with second period sales for immediate purchase customers. In the remaining terms, $\mathfrak{u}\left((p_I + p_L)x - Q - \frac{p_L\delta\beta}{(1-\alpha)p_I+p_L}Q\right)$ and $\mathfrak{s}\left(\frac{p_L\delta(1-\beta)}{(1-\alpha)p_I+p_L}Q\right)$, correspond to missed sales and salvage costs, respectively and $c\,Q$ is the cost of ordering Q units at the beginning of the first period.

Appendix E. Market segment size with model parameters

The market size for immediate and layaway purchases is captured by p_I and p_L. We now consider how these segments change with different model parameters.

From Table E1 we see that p_I and p_L change with different model parameters. We find that depending on value of r, initially more consumers will purchase via layaway, but then this trend will reverse and more will purchase immediately. We also find that the behavior of p_I and p_L with respect to $\bar{b}$ and $\bar{v}$ is not symmetric. This is not surprising given the definition of p_L in which the signs of the $\bar{b}$ and $\bar{v}$ terms are different. A retailer may use the findings in Table E1 to inform itself of what the makeup of its consumers will be as model parameters change.

Variable	p_I	p_L	p_I/p_L
r	$\downarrow$	$\uparrow$ then $\downarrow$	$\downarrow$ then $\uparrow$
f_s	$\leftrightarrow$	$\downarrow$	$\uparrow$
$\bar{b}$	$\uparrow$	$\downarrow$	$\uparrow$
$\bar{v}$	$\uparrow$	$\uparrow$	$\downarrow$
L	$\leftrightarrow$	$\uparrow$	$\downarrow$
θ	$\leftrightarrow$	$\uparrow$	$\downarrow$

Table E1.: Change in probability of purchase, layaway or immediate, with model parameters. ($\uparrow$ means weakly increasing, $\downarrow$ means weakly decreasing, and $\leftrightarrow$ means constant)

References

Abdel-Aal, M. A., Syed, M. N., and Selim, S. Z. (2017). Multi-product selective newsvendor problem with service level constraints and market selection flexibility. *International Journal of Production Research*, 55(1):96–117.

Akçay, Y., Boyacı, T., and Zhang, D. (2013). Selling with Money-Back Guarantees: The Impact on Prices, Quantities, and Retail Profitability. *Production and Operations Management*, 22(4):777–791.

Altug, M. S. and Aydinliyim, T. (2016). Counteracting Strategic Purchase Deferrals: The Impact of Online Retailers' Return Policy Decisions. *Manufacturing & Service Operations Management*, 18(3):376–392.

Anderson, M. H. and Sibdari, S. (2012). Investment decisions in the rent-to-own industry in the absence of inventory. *Journal of the Operational Research Society*, 63(1):89–106.

Che, Y.-K. (1996). Customer Return Policies for Experience Goods. *The Journal of Industrial Economics*, 44(1):17–24.

Chen, J. and Bell, P. C. (2009). The impact of customer returns on pricing and order decisions. *European Journal of Operational Research*, 195(1):280–295.

Chen, J. and Bell, P. C. (2011). Coordinating a decentralized supply chain with customer returns and price-dependent stochastic demand using a buyback policy. *European Journal of Operational Research*, 212(2):293–300.

Chen, J. and Chen, B. (2016). Competing with customer returns policies. *International Journal of Production Research*, 54(7):2093–2107.

FDIC (2009). FDIC National Survey of Unbanked and Underbanked Households. *Federal Deposit Insurance Corporation*, pages 1–160.

FDIC (2015). FDIC National Survey of Unbanked and Underbanked Households. *Federal Deposit Insurance Corporation*, pages 1–88.

Ferber, R. (1962). Research on Household Behavior . *American Economic Review*, 52:19–63.

Flores, N. E. and Carson, R. T. (1997). The Relationship between the Income Elasticities of Demand and Willingness to Pay. *Journal of Environmental Economics and Management*, 33(3):287–295.

Heaths, C. and Soll, J. (1996). Mental Budgeting and Consumer Decisions . *Journal of Consumer Research*, 23:40–52.

K-Mart (2017). K-Mart Layaway.

Li, X., Li, Y., and Cai, X. (2012). Quantity decisions in a supply chain with early returns remanufacturing. *International Journal of Production Research*, 50(8):2161–2173.

Ma, S., Jemai, Z., Sahin, E., and Dallery, Y. (2017). The news-vendor problem with drop-shipping and resalable returns. *International Journal of Production Research*, 55(22):6547–6571.

Meyer, H. (1999). Many Happy Returns. *Journal of Business Strategy*, 20(4):27–31.

Pinçe, Ç., Ferguson, M., and Toktay, B. (2016). Extracting Maximum Value from Consumer Returns: Allocating Between Remarketing and Refurbishing for Warranty Claims. *Manufacturing & Service Operations Management*, 18(4):475–492.

Rob Oglesby (2012). The Layaway Challenge: Optimizing Layaway Programs to Benefit the Retailer's Bottom Line. Technical report, The Parker Avery Group.

Sarkar, B., Zhang, C., Majumder, A., Sarkar, M., and Seo, Y. W. (2018). A distribution free newsvendor model with consignment policy and retailer's royalty reduction. *International Journal of Production Research*, pages 1–20.

Schultz, C. R. (1986). An inventory investment model for a rent-to-own firm. *Omega*, 14(3):270–272.

Sears (2017). Sears Layaway.

Su, X. (2009). Consumer Returns Policies and Supply Chain Performance. *Manufacturing & Service Operations Management*, 11(4):595–612.

Walmart (2017). Walmart Layaway.

Whisler, W. D. (1967). A Stochastic Inventory Model for Rented Equipment. *Management Science*, 13(9):640–647.

Yang, H., Chen, J., Chen, X., and Chen, B. (2017). The impact of customer returns in a supply chain with a common retailer. *European Journal of Operational Research*, 256(1):139–150.