Optimal Inventory Decisions for a Risk-Averse Retailer when Offering Layaway

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Abstract: Layaway allows economically disadvantaged budget-constrained consumers to purchase expensive items through amortized payments and nominal program fees, as opposed to using high-interest financing options such as credit cards and payday loans. We consider a risk-averse retailer's ordering decisions when offering a layaway program. We use the net loss and total loss functions, found in the literature, to determine a risk-averse retailer's optimal order quantity under conditional value-at-risk (CVaR). We next analyze the effects of the model parameters, retailer's risk aversion, the market default rate, enrollment fee, cancellation fee and so on, on the optimal order quantity decisions. We show that the optimal order quantity depends on different loss functions and different demand distribution functions. Further, we show that as market default rate increases or the retailer becomes more risk averse, then a rational retailer will not offer a layaway program.

Keywords: Inventory; Risk-averse retailer; Layaway; Conditional value-at-risk (CVaR); Order quantity

1 Introduction

Layaway programs allow economically disadvantaged budget-constrained consumers to purchase expensive goods by dividing the total price of an item into several installments, instead of paying the entire price at once. However, an item purchased via layaway is received by a consumer only after the item is entirely paid. Layaway programs have been applied by various retailers such as Walmart, Toys R Us, Sears, GameStop, Burlington Coat Factory, Marshalls, Baby depot and Best Buy (Trae Bodge, 2017). According to a survey by AYTM (Ask Your Target Market) in 2012 (Pilon, 2012), 47% of those surveyed said they used layaway programs at stores; another 37% of respondents said they were likely to use layaway when they intended to buy expensive goods; 47% of respondents said that an optional layaway program has definitely influenced their shopping decision at a certain store; Females and those respondents over 34 were more interested in using layaway services than others. According to Deloitte Touche Tohmatsu Limited (Deloitte, 2015),

about 17% of surveyed US shoppers planned to make use of a layaway program during the holiday season (November and December) in 2016; 17% of all shoppers represents a sizable opportunity for retailers trying to grow sales. Retail stores report that layaway programs play a significant role in holiday sales, which accounts for 40% or more of their annual sales; especially in recent years, the economic dilemmas in the United States make it difficult for consumers to bear large purchase costs (Casteele, 2017). Even though layaway is a large opportunity that retailers leverage, there are very few studies that consider the impact of offering layaway on a retailer's inventory decisions, we are only aware of Dimitrov and Ceryan (2018).

There are also different layaway program formats, in terms of fees. For example, some retailers charge a fixed fee no matter how much the items placed on layaway cost, while other retailers charge a fixed percentage of the total items' cost. For the remainder of this paper, we assume that only a single item is placed on layaway by a consumer and that the consumer is charged a fixed program fee (as we observe in Sears (Sears, 2018); K-Mart (Kmart, 2018); Walmart (Walmart, 2018). Next, the consumer selects the payment period length (under normal circumstances, the payment period is between eight and twelve weeks). If the consumer does not want to buy the item during the payment period, then the retailer cancels the consumer's layaway program and returns the consumer's payments minus a fixed cancellation fee. If the consumer makes all of the agreed payments on time, then the consumer receives the layaway item.

A natural question one may ask is:"why do consumers choose layaway programs?" There are two main reasons for consumers' choices. First, a financially constrained consumer can utilize a layaway program to buy an expensive item (jewelry, electronics, and durable goods, just to name some examples) even when that consumer cannot afford to purchase the item outright and does not have access to credit (Kenton, 2019). Second, layaway allows consumers to avoid fees associated with other forms of financing, e.g., credit cards or payday loans (Lynnette Khalfani-Cox, 2015). While most retailers charge a fee for using their layaway program, this fee is less than the interest or upfront fees charged by other lenders.

Offering a layaway program has a critical effect on the retailer decision maker. First, offering a layaway program can expand a retailer's market. The layaway program expands the retailer's market to economically disadvantaged and budget-constrained consumers. 10% of 18-24 year olds say that layaway options are important factor for their purchasing decisions (Corralsolution, 2016). In this way, retailers can seize the market of young people, cultivate their spending habits, and win the loyalty of young consumers. Second, consumers need to pay off all the money before they can pick up the purchased items. Thus, the retailer's capital risk is lower in a layaway program, relative to a credit program.

Even with the benefits to both consumers and retailers, the layaway program has multiple drawbacks. These include, but are not limited to stochastic demand, the market default rate, and retailer's risk tolerance. We now discuss each of these drawbacks in turn.

Demand is stochastic due to consumer specific characteristics, retailer decisions, and item characteristics. In particular: (1) The consumers' purchase behavior is affected by their personal situation, such as income, budget, and item valuation (Nelson, 1970); (2) The market demand is also sensitive to the selling season (high and low-demand periods, e.g., swimwear), the item's price (Francis and Krishnan, 1999), layaway program's enrollment fee and cancellation fee, and other market factors. Collectively we say demand is stochastic due to the aforementioned market characteristics.

A consumer may default on a layaway program, thus not finishing all layaway payments. The market propensity to default greatly impacts the feasibility and profitability of a layaway program. For example, if a consumer defaults on an item, then the retailer loses the opportunity to sell this item at full price to another consumer between the time the item was initially put on layaway and

the time the consumer defaults.

Dimitrov and Ceryan (2018) consider only risk-neutral retailers and use the expected profit function to determine the optimal order quantity for the retailer. However, empirical studies on newsvendor order quantities find that managers order quantities different than those determined via risk-neutral newsvendor models (Fisher and Raman, 1996, Kahn, 1992, Schweitzer and Cachon, 2000). The deviation between the observed order quantity and order quantity maximizing expected profit, is called "decision bias" in the newsvendor problem (Chen et al., 2007, Chen, 2015, Vipin and Amit, 2019, Wang and Webster, 2009). Experimental results find that a combination of risk factors collectively explains ordering decisions made in practice (Schweitzer and Cachon, 2000). However, in our study, similar to what is done in the literature (Du et al., 2018, Xinsheng et al., 2015, Xu et al., 2016), we focus on risk aversion only for model tractability. We acknowledge our study is a needed first step to a fuller analysis of all risk factors that attribute to a retailer's ordering decisions when offering layaway.

As retailers with decision bias exist, in this paper we consider a risk-averse retailer. Retailers often lack full information on market behaviors and usually have risk-averse behavior (Agrawal and Seshadri, 2000). Along those same lines, if a retailer is highly risk averse, then the retailer may choose to order a sufficient number of items to decrease the likelihood of running out of stock due to very high demand. However, a retailer will have to pay additional costs for these additional items that may end up being salvaged due to not being sold or being defaulted on, if sold under the layaway program. From this perspective, minimizing downside risk may be more attractive to risk-averse retailers. On the one hand a risk-averse retailer may want to order more than a risk-neutral retailer; on the other hand, a risk-averse retailer may want to order less. In this paper, we analyze these two counteracting forces.

Hence, the decision of the optimal inventory for the retailer depends on at least two market factors, stochastic demand and market default rate, along with the risk attitude of the retailer. We determine the optimal order quantity of a risk-averse retailer offering layaway faced with stochastic demand. In particular, we answer the following research questions:

(1) How do different Conditional Value at Risk (CVaR) loss functions, defined in Section 4.2, influence the retailer's order quantity? When should a risk-averse retailer offer layaway?

(2) How does the degree of the retailer's risk aversion, overage cost, underage cost, items cost, and cancellation fee influence the optimal order quantity?

(3) What are the effects of the market's layaway default rate on the retailer's behavior in offering a layaway program?

The rest of the paper is structured as follows. In Section 2, we review the related literature and summarize the differences between our study and previous studies. Next, in Section 3, we describe a layaway program and formalize the basic models of a risk neutral retailer offering layaway. In Section 4, we derive the optimal ordering decisions under CVaR with different loss functions, net loss and the total loss; In the same section, we analyze the optimal order quantity sensitivity to model parameters. In Section 5, several numerical examples are given to illustrate the differences among the models. We then present theoretical and practical insights before offering our conclusions.

2 Related Work

Although we are studying a new model characterized by retailers offering a layaway program, the setup of our model is similar to minimization of CVaR in the newsvendor problem under stochastic demand and a risk-averse retailer. Using the only paper in the operations management literature to explicitly consider retailer inventory decisions when offering layaway (Dimitrov and Ceryan, 2018),

our modeling and solution approaches are inspired by studies in two research streams. The first stream is single period stochastic inventory decisions. The second stream is inventory decisions under consumer returns. We next discuss each stream in turn.

The first stream, single period stochastic inventory decisions, referred to as the newsvendor problem, has a long history with the first paper published in 1955 (Whitin, 1955). As a result, there are multiple variants of the newsvendor problem (e.g, see Cachon and Kök (2007), Khang and Fujiwara (2000), Khouja (1999), Li and Petruzzi (2017), Petruzzi and Dada (1999), Prasad et al. (2011), Qin and Kar (2013) for additional details). In the version of the newsvendor we consider, we include budget constraints on consumers. To our knowledge, is not common to have budget-constrained consumers (Dimitrov and Ceryan (2018) considered consumer budget constraints) and instead budget constraints are usually associated with the newsvendor (Moon and Silver, 2000, Niederhoff, 2007, Shi and Zhang, 2010, Vairaktarakis, 2000, Zhang, 2010).

In the papers cited in the previous paragraph, the newsvendor (retailer) is assumed to be risk neutral. However, in practice, retailers tends to be risk-averse (Davis and Hyndman, 2018, Xiao and Yang, 2008, Yang et al., 2018, Zhou et al., 2018). In addition to accounting for a risk-averse retailer, we also account for the same retailer offering a layaway program. As there is only one study that considers a layaway program (Dimitrov and Ceryan, 2018), we now review recent work on the risk-averse newsvendor. Eeckhoudt et al. (1995) find the optimal order quantity of a risk-averse newsvendor. Following this line of study, additional studies applied CVaR, a way to model risk aversion, to inventory models. Gotoh and Takano (2007) consider two types of CVaR measures in the single-period newsvendor problem, and introduce the loss functions of net loss and total loss. Xu and Chen (2007) consider the trade off between expected profit and CVaR using a weighted mean-risk objective. Yang et al. (2009) use CVaR as the objective function to study the coordination of supply chains combined with a risk-neutral supplier and a risk-averse retailer. Chen et al. (2009) use CVaR to study both price and order quantity decisions for a risk-averse newsvendor. Xu and Lu (2013) extend Chen et al.'s model to allow for emergency purchases when demand is greater than the initial order quantity. Wu et al. (2014) also extend the work of Chen et al. (2009) by considering price and inventory quantity competition simultaneously. Dai and Meng (2015) study a risk-averse newsvendor making simultaneous ordering, pricing, and marketing decisions. For other cases of using CVaR in inventory models, please refer to Abdel-Aal and Selim (2017), Choi and Ruszczyński (2011), Tomlin and Wang (2005), Xu et al. (2017, 2015) for additional details. Most papers in this stream of research focus on multi-item risk-averse newsvendor or different CVaR loss functions, but do not consider layaway programs. The main features that distinguish our work from previous work on using CVaR to model risk-aversion in newsvendor problems are: (1) we consider that consumers can buy the items in two ways: immediately or via layaway; (2) we account for consumer defaults during the layaway program, which is an important feature of a layaway program.

The second literature stream we consider is the consumer returns literature. In a layaway program a consumer may default on payments, and will be charged a nominal cancellation fee, all other money is returned. The fact that all money is returned except for a cancellation fee, makes layaways seem related to returns, as indicated by Dimitrov and Ceryan (2018). Many recent studies address inventory models associated with returns (see e.g., Choi and Guo (2017), Hu et al. (2014), Letizia et al. (2018), Yang et al. (2017)). As we consider a risk-averse retailer, we focus on return models with risk-aversion. Su (2009) studies consumer return policies and demand uncertainty on a newsvendor model. Hsieh and Lu (2010) study a manufacturer's return policy and two risk-averse retailers' decisions under a single-period setting with price-sensitive random demand. Yoo (2014) studies the relationship between return policy and item quality in a supply chain with a risk-averse supplier. Ohmura and Matsuo (2016) consider a mean-variance risk-aversion model of a supply chain accepting returns; this work does not use a CVaR model, nor does it consider

consumer budget constraints, something we capture in our paper. Mawandiya et al. (2018) present a centralized production-inventory model when both retailer demand and remanufacturer returns are random. However, the main features that distinguish our work from previous papers in retail return polices is that in our paper, only consumers who purchase items through a layaway program have the opportunity to default on ("return") the item. Further, unlike usual returns, where the returned item is "used" or of lower quality than a new item, a defaulted layaway item once returned to regular inventory is new and no different than all other items.

To our best knowledge, no studies address a risk-averse retailer offering layaway. In this paper, we consider a layaway program in which a risk-averse retailer sells an item with random demand and makes inventory decisions in order to minimize Conditional Value-at-Risk (CVaR).

3 Model

In this section, we first introduce the layaway problem, and then we briefly summarize some basic results of the risk-neutral retailer not offering and offering layaway. In particular, the symbols of the parameters are redefined for later comparison with our risk-averse results.

3.1 Notation

We first introduce the notation we use in Table 1 to use as a reference later in the paper. We will reintroduce the notation used inline, as we build the model.

Table 1: Notation, notation and description is similar to that of Dimitrov and Ceryan (2018)

Term	Meaning
r	is the selling price for the item
f_s	is the layaway program enrollment fee
f_c	is the layaway program cancellation fee, this fee is paid only if the consumer terminates the program before payment completion
v	is the consumer's valuation for the item
b	is the consumer's budget
θ	is the consumer waiting disutility from receiving a layaway good
c	is the purchase cost for the item
p_I	is the probability that a consumer purchases the item immediately
p_L	is the probability that a consumer purchases the item through the layaway program
q	is the firm's inventory
L	is the length of the layaway period
α	is the given confidence level $\alpha \in [0, 1)$
ζ	is the maximum tolerated loss by the retailer
$\mathfrak{s}$	is the holdover value of a single item (may be negative or positive: positive means holding cost, while negative means salvage value)
$\mathfrak{u}$	is the shortage cost for a single unit of demand
O	is the unit loss for order quantity great than the realized demand
U	is the unit loss for order quantity less than the realized demand
V	is the unit profit for a single item
$\widehat{\pi}_{zy}(x\|q)$	is the profit for the retailer, $z \in \{R, \bar{R}\}$, where R represents risk-averse condition and $\bar{R}$ represents risk-neutral condition, $y \in \{L, \bar{L}\}$, where L represents offering layaway and $\bar{L}$ represents not offering layaway
π_L	is the expected profit for the risk-averse retailer who offers the layaway program
q^*_{zyx}	is the optimal order quantity for the retailer, $z \in \{R, \bar{R}\}$, $y \in \{L, \bar{L}\}$, $x \in \{T, N\}$, where T represents choose the total loss as the loss function and T represents choose the net loss as the loss function
$L_{\bar{R}yT}(x\|q)$	is the total loss for the risk-neutral retailer, $y \in \{L, \bar{L}\}$

3.2 Problem description

We now formally describe the layaway program. First, similar to Dimitrov and Ceryan (2018), "we assume that consumer valuations and budgets are independently and uniformly distributed between [0, $2\bar{v}$] and [0, $2\bar{b}$], respectively." If a consumer's valuation, v, for the item is greater than the item price, r, i.e., $v > r$, and the consumer's budget, b, is sufficiently high, $b \geq r$, then the consumer buys the item immediately. If a consumer's budget is less than the item price, $b < r$, but the budget satisfies the relations $b \geq r/L$ (L is the layaway program's payoff length), and the consumer's discounted valuation is greater than the item price and the initial enrollment fee, f_s, i.e., $\theta \cdot v > r + f_s$ (θ is consumers' temporal discount factor, as the layaway item is not received immediately), the consumer then purchases the item through the layaway program. In practice, consumers pay a down payment when entering a layaway program, however, for tractability, we assume the down payment is zero and is instead subsumed into the enrollment fee, f_s. If the consumer purchases the item by the layaway program, but wants to suspend the program before payment completion, they only need to pay a cancellation fee, f_c, to the retailer and receive all

previous payments in return. We denote the default rate for the market as δ, where $\delta < 1$.

The Table 2 shows the operating parameters of layaway programs of some of today's retailers, along with what we consider in this paper.

Table 2: Practices of retailer's layaway program

Vendor	Down payment	Enrollment fee	Cancellation	layaway period
Walmart	\$10 or 10% of the purchase	Non	\$10	8/31/18 -12/10/18
Kmart	10% of the purchase	\$5	\$10	8 weeks
Sears	\$10 or 10% of the purchase	\$5/8 week or \$10/12 week	\$15/8 week or \$25/12 weeks	8 weeks or 12 weeks
Our paper	None	Yes	Yes	Yes

As depicted in Figure 1, the black area represents the section of the market that purchases the item immediately, formally the probability of buying immediately is:

$$p_I = \left(1 - \frac{r}{2\bar{v}}\right)\left(1 - \frac{r}{2\bar{b}}\right). \tag{1}$$

The light gray area represents the section of the market that purchases of the item through the layaway program, formally the probability of a layaway purchase is:

$$p_L = \left(1 - \frac{r + f_s}{2\bar{v}\theta}\right)\left(\frac{r}{2\bar{b}}\right)\left(\frac{L-1}{L}\right). \tag{2}$$

The white area represents the probability that the consumer does not purchase the item, $1 - p_I - p_L$.

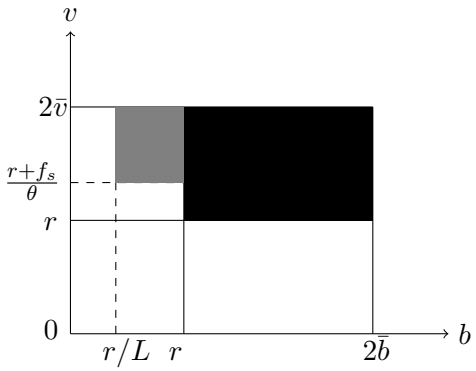


Figure 1: The probability of consumer's purchases behavior.

3.3 The risk-neutral retailer's decision

In this subsection, we determine the optimal order quantity of the risk-neutral retailer, when offering and not offering a layaway program. We include this section to highlight the new modeling perspective we take in the remainder of our paper.

3.3.1 Not offering layaway

For completeness, we write the profit function of a risk-neutral retailer not offering layaway, $\widehat{\pi}_{\bar{R}\bar{L}}(x|q)$, for any given order quantity, q, and market demand, x as:

$$\widehat{\pi}_{\bar{R}\bar{L}}(x|q) = min\{p_I{\cdot}x, q\}{\cdot}r - (q - p_I{\cdot}x)^+{\cdot}\mathfrak{s} - (p_I{\cdot}x - q)^+{\cdot}\mathfrak{u} - c{\cdot}q, \tag{3}$$

where $[Z]^+ = max\{Z, 0\}$, $\mathfrak{s}$ is the holdover value of excess stock (may be negative or positive: positive means holding cost, while negative means salvage value), and $\mathfrak{u}$ is the shortage cost of not being able to service a purchase request. Using classical results (Johnson and Montgomery, 1974, Silver et al., 1998) we find that the optimal order quantity is:

$$q^*_{\bar{R}\bar{L}} = p_I F^{-1}\left(\frac{r + \mathfrak{u} - c}{r + \mathfrak{u} + \mathfrak{s}}\right). \tag{4}$$

Where $F(\cdot)$ is the market demand cumulative distribution function, with a probability density function of $f(\cdot)$.

3.3.2 Offering layaway

Dimitrov and Ceryan (2018) define the profit function for the risk neutral retailer when offering layaway. For any given order quantity, q, the profit of a risk-neutral retailer offering layaway, denoted as $\widehat{\pi}_{\bar{R}L}$, is:

$$\widehat{\pi}_{\bar{R}L}(x|q) = \begin{cases} p_I{\cdot}x{\cdot}r + p_L{\cdot}x{\cdot}(f_s + \delta{\cdot}f_c + (1-\delta){\cdot}r) & \text{if } (p_I + p_L){\cdot}x \leq q \\ -(q - (p_I + p_L){\cdot}x + p_L{\cdot}\delta{\cdot}x){\cdot}\mathfrak{s} - c{\cdot}q, & \\ & \\ \frac{p_I{\cdot}q{\cdot}r}{p_I+p_L} + \frac{p_L{\cdot}q}{p_I+p_L}{\cdot}(f_s + \delta{\cdot}f_c + (1-\delta){\cdot}r) & \text{if } (p_I + p_L){\cdot}x > q \\ -((p_I + p_L){\cdot}x - q){\cdot}\mathfrak{u} - \frac{p_L}{p_I+p_L}{\cdot}q{\cdot}\delta{\cdot}\mathfrak{s} - c{\cdot}q. & \end{cases} \tag{5}$$

Let $p'_I = \frac{p_I}{p_I+p_L}$, $p'_L = \frac{p_L}{p_I+p_L}$, and $r' = p'_I{\cdot}r + p'_L{\cdot}(f_s + \delta{\cdot}f_c + (1-\delta){\cdot}r) - p'_L{\cdot}\delta{\cdot}\mathfrak{s}$, with $\min\{q, x(p_I + p_L)\} = q - (q - x(p_I + p_L))^+$ and $(x(p_I + p_L) - q)^+ = x(p_I + p_L) - q + (q - x(p_I + p_L))^+$, the profit function (5) can be written as:

$$\widehat{\pi}_{\bar{R}L}(x|q) = (r' + \mathfrak{u} - c){\cdot}q - (r' + \mathfrak{u} + \mathfrak{s}){\cdot}(q - x(p_I + p_L))^+ - x(p_I + p_L){\cdot}\mathfrak{u}. \tag{6}$$

Let $O := c + \mathfrak{s}$, $U := r' + \mathfrak{u} - c$, and $V := r' - c = U - \mathfrak{u}$, we rewrite (6) as:

$$\widehat{\pi}_{\bar{R}L}(x|q) = V{\cdot}x(p_I + p_L) - O[q - x(p_I + p_L)]^+ - U[x(p_I + p_L) - q]^+. \tag{7}$$

In regard to the demand distribution function, the risk-neural retailer's optimal order quantity can be obtained by solving $\frac{\partial\pi}{\partial q} = 0$, as:

$$q^*_{\bar{R}L} = (p_I + p_L)F^{-1}\left(\frac{U}{O + U}\right), \tag{8}$$

where the optimal order quantity for the risk neutral condition is similar to Dimitrov and Ceryan (2018).

4 Minimization of CVaR for the risk-averse retailer's optimal order quantity

As discussed in Section 1, retailers may be risk averse. In general, risk averse retailers are modeled as rational individuals minimizing risk (Zhang et al., 2009). One common risk measure is conditional value at risk (CVaR), which is a modification of the value at risk (VaR) (Rockafellar and Uryasev, 2000). In this section, we investigate retailer's optimal order quantity under CVaR. First, we formally introduce VaR and CVaR. Second, we define different loss functions to capture different decision makers' policies. For an optimistic decision maker, we choose the net loss function, which accounts for the upside of decisions by including revenue; for the pessimistic decision maker, we choose total loss, which only considers losses from operating, the pessimistic decision maker receives no reward for revenues, as revenues are not losses. Then we analyze the optimal order quantity for each case. We conclude this section by exploring the sensitivity of the optimal order quantity with respect to model parameters.

4.1 The conditional value at risk (CVaR)

The Value at Risk (VaR) introduced in 1993 (G30, 1993) is used to measure the risk of a financial portfolio. Let $\Psi_{\ell(x|q)}(x)$ be the cumulative distribution function of the retailer's loss, discussed in greater detail below. Given a fixed conditional level α, the α-VaR for the retailer is $\Psi^{-1}_{\ell(x|q)}(\alpha)$ (Rockafellar and Uryasev, 2002). However, VaR does not consider the tail risk (the distribution of $\Psi_{\ell(x|q)}(x|x > \Psi^{-1}_{\ell(x|q)}(\alpha))$), and it lacks some important properties such as sub-additivity and convexity; a risk measure having these properties ensures a lower risk tolerances value, α, corresponds to lower risk. However, as VaR lacks sub-additivity and convexity, cases where even if $\alpha' < \alpha''$, may result in α'-VaR $> \alpha''$-VaR. An augmentation of VaR is conditional value at risk (CVaR), the expected value of the tail of the loss distribution is computed. CVaR is a risk measure that is sub-additive, positively homogeneous, monotonic, and translation invariant (Rockafellar and Uryasev, 2002). Next, we apply results from finance, CVaR in particular, to inventory decisions similar to Abdel-Aal and Selim (2017), Choi and Ruszczyński (2011), Ohmura (2014), Tomlin and Wang (2005), Xu and Lu (2013), Xu et al. (2017, 2015), and we will analogously define CVaR for the layaway problem.

Let $\ell(x|q)$ denote the loss function associated with having realized demand of x, considered for the remainder of this paper, given the retailer's order quantity q. For example, in our setting if the retailer makes an order of 10 items, but only sells 6, due to random demand, then the loss is the 4 extra units ordered, if the demand is 15, then the loss is the opportunity cost of the 5 extra units of unmet demand. We now formally define the cumulative distribution of loss, $\ell(x|q)$, as $\Psi_{\ell(x|q)}(\varsigma|q) := P\{\ell(x|q) \leq \varsigma\}$.

Given some confidence level $\alpha \in [0, 1)$, the lower bound of ζ is $\zeta_\alpha(q) \equiv \min\{\zeta | P\{\ell(\zeta|q) \leq \varsigma\} \geq \alpha\} \equiv \min\{\zeta | \Psi_{\ell(x|q)}(\zeta|q) \geq \alpha\} \equiv \Psi^{-1}_{\ell(x|q)}(\alpha) = \alpha$-VaR (where appropriate, for space consideration, we omit $\zeta_\alpha(q)$ and only write ζ); at confidence level α, $\zeta_\alpha(q)$ is the smallest number, such that the probability that loss, $\ell(x|q)$, exceeds $\zeta_\alpha(q)$ is not larger than $1 - \alpha$.

With the definition of α-VaR, we formally write $\Psi_{\ell(x|q)}(\zeta|q)$ as:

$$\Psi_{\ell(x|q)}(\zeta|q) = \int_{\ell(x|q)\leq\zeta} f(x)dx, \tag{9}$$

where $f(x)$ is the probability density function of x. By the definition of a cumulative distribution

function we know:

$$\zeta_\alpha(q) = \min\{\zeta | \Psi_{\ell(x|q)}(\zeta|q) \geq \alpha\} = \Psi^{-1}_{\ell(x|q)}(\alpha). \tag{10}$$

CVaR is the mean loss value, conditional on loss exceeding the VaR value. For any specified probability level α in [0,1), and fixed q, the α-CVaR is defined as:

$$\begin{aligned} \Phi_\alpha(q) =& E[\ell(x|q)|\ell(x|q) \geq \zeta_\alpha(q)] \\ &= (1-\alpha)^{-1} \int_{\ell(x|q)\geq\zeta_\alpha(q)} \ell(x|q) f(x) dx. \end{aligned} \tag{11}$$

In order to minimize $\Phi_\alpha(q)$, Rockafellar and Uryasev (2002) defined the following convex function, an affine transformation of $\Phi_\alpha(q)$:

$$P_\alpha(q,\zeta) = \zeta + (1-\alpha)^{-1} \int_{x\in\mathbb{R}} [\ell(x|q) - \zeta]^+ f(x) dx, \tag{12}$$

where $\alpha \in [0,1)$ reflects the degree of risk aversion for the retailer (the larger the α, the more risk-averse the retailer). As $P_\alpha(q,\zeta)$ is an affine transformation of $\Phi_\alpha(q)$, minimizing $P_\alpha(q,\zeta)$ is equivalent to minimizing $\Phi_\alpha(q)$ (Boyd and Vandenberghe, 2004).

4.2 Minimization of CVaR with different loss functions

In this subsection, we consider the risk-averse retailer offering and not offering layaway. We derive the optimal ordering decisions under CVaR with two different loss functions, net loss and total loss.

4.2.1 Net loss CVaR minimization

First, we consider the net loss $-\widehat{\pi}_{\bar{R}L}(x|q)$ as the loss function $\ell(x|q)$ (see Gotoh and Takano (2007)):

$$\ell(x|q) = -\widehat{\pi}_{\bar{R}L}(x|q). \tag{13}$$

The retailer considers all profit over the threshold ζ. Then, we rewrite the minimization of equation (12) using definition (13) as:

$$\min_q P_\alpha(q,\zeta) = \min_q \{\zeta + (1-\alpha)^{-1} \int_{x\in\mathbb{R}} [-\widehat{\pi}_{\bar{R}L}(x|q) - \zeta]^+ f(x) dx\}. \tag{14}$$

Substituting (7) into (14), we have:

$$\begin{aligned} P_\alpha(q,\zeta) =& \zeta + (1-\alpha)^{-1} \left[\int_{x\in\mathbb{R}} [-(Vx(p_I+p_L) - O(q - x(p_I+p_L))^+ \right. \\ & \left. -U\left(x(p_I+p_L) - q\right))^+ - \zeta] f(x) dx \right]. \end{aligned} \tag{15}$$

We first compute the derivative of the net loss $-\pi_{\bar{R}L}(x|q)$:

$$\begin{aligned} \frac{d(-\pi_{\bar{R}L}(x|q))}{dq} &= \frac{d}{dq}[\int_0^{\frac{q}{p_I+p_L}} [-(Vx(p_I+p_L) - O(q - x(p_I+p_L)))] f(x) dx \\ &\qquad + \int_{\frac{q}{p_I+p_L}}^{\infty} [-(Vx(p_I+p_L) - U(x(p_I+p_L) - q))] f(x) dx]], \\ &= O \int_0^{\frac{q}{p_I+p_L}} f(x) dx - U \int_{\frac{q}{p_I+p_L}}^{\infty} f(x) dx. \end{aligned}$$

We now find the second order derivative of $-\pi_{\bar{R}L}(x|q)$ with respect to q:

$$\frac{d^2(-\pi_{\bar{R}L}(x|q))}{dq^2} = (O+U)f\left(\frac{q}{p_I+p_L}\right)\frac{1}{p_I+p_L} > 0.$$

The second order derivative is non-negative, which means that the net loss function is convex in q. Hence, there is a unique optimal order quantity, q^*, that minimizes (15).

Proposition 1. *The risk-averse retailer's optimal order quantity and maximum loss threshold* $(q^*_{RLN}, \zeta^*_{RLN})$ *when offering layaway under the* net loss *loss function are:*

$$q^*_{RLN} = (p_I+p_L)\frac{O+V}{O+U}F^{-1}\left[\frac{(1-\alpha)U}{O+U}\right] + (p_I+p_L)\frac{U-V}{O+U}F^{-1}\left[\frac{\alpha O+U}{O+U}\right], \tag{16a}$$

$$\zeta^*_{RLN} = (p_I+p_L)\frac{O(U-V)}{O+U}F^{-1}\left[\frac{\alpha O+U}{O+U}\right] - (p_I+p_L)\frac{U(O+V)}{O+U}F^{-1}\left[\frac{(1-\alpha)U}{O+U}\right]. \tag{16b}$$

Proof: The derivation of all results is presented in Appendix A.

If we consider a risk-averse retailer not offering layaway, then the optimal order quantity, $q^*_{R\bar{L}N}$, is found as the argmin of the following variant of (12), corresponding to loss $\ell(x|q) = -\widehat{\pi}_{\bar{R}\bar{L}}(x|q)$:

$$\min_q P_\alpha(q,\zeta) = \min_q\{\zeta + (1-\alpha)^{-1}\int_{x\in\mathbb{R}}[-\widehat{\pi}_{\bar{R}\bar{L}}(x|q)-\zeta]^+ f(x)dx\}. \tag{17}$$

Corollary 1. *If the risk-averse retailer does not offer layaway, and the retailer considers the* net loss *loss function, then* (17) *has an optimal solution* $(q^*_{R\bar{L}N}, \zeta^*_{R\bar{L}N})$ *of:*

$$q^*_{R\bar{L}N} = \frac{p_I\mathfrak{u}}{\mathfrak{s}+r+\mathfrak{u}}F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right) + \frac{p_I(r+\mathfrak{s})}{\mathfrak{s}+r+\mathfrak{u}}F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right], \tag{18a}$$

$$\begin{aligned}\zeta^*_{R\bar{L}N} = {} & \frac{p_I\mathfrak{u}(\mathfrak{s}-c)}{\mathfrak{s}+r+\mathfrak{u}}F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right) + \\ & \frac{p_I(r+\mathfrak{s})(r+\mathfrak{u}-c)}{\mathfrak{s}+r+\mathfrak{u}}F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right].\end{aligned} \tag{18b}$$

Proof: The derivation of all results is presented in Appendix D.

Please note that the results in equation (18) are the same as those in Gotoh and Takano (2007), but accounting for consumer budgets and valuation constraints. Discussion of our results is found in Section 4.3.

4.2.2 The total loss CVaR minimization

In the previous section we present the optimal order quantities of a risk-averse retailer when offering either layaway or no layaway, but only when considering net loss in determining optimal order quantities. In this section we repeat the results from section 4.2.1, but with the total loss loss function. The total loss loss function, based on profit function (7), is defined as:

$$L_{\bar{R}L}(x|q) = O[q - x(p_I+p_L)]^+ + U[x(p_I+p_L)-q]^+, \tag{19}$$

where $O[q - x(p_I + p_L)]^+$ is the overage cost and $U[x(p_I + p_L) - q]^+$ corresponds to the underage cost. Note that $L_{\bar{R}L}(x|q)$ does not consider lost revenue from sales, unlike (13).

We now use the total loss $L_{\bar{R}L}(x|q)$ as the loss function $\ell(x|q)$ (Gotoh and Takano, 2007):

$$\ell(x|q) = L_{\bar{R}L}(x|q). \tag{20}$$

The retailer wants to reduce total loss, which now is only the overage and underage costs. We rewrite minimizing equation (12) with $\ell(x|q) = L_{\bar{R}L}(x|q)$, defined in (20), as:

$$\min_q P_\alpha(q, \zeta) = \zeta + (1-\alpha)^{-1} \int_{x \in \mathbb{R}} [L_{\bar{R}L}(x|q) - \zeta]^+ f(x) dx. \tag{21}$$

Proposition 2. *If the retailer is risk-averse under α-CVaR with $\alpha \in [0, 1)$, then (21) has an optimal solution $(q^*_{RLT}, \zeta^*_{RLT})$ where:*

$$q^*_{RLT} = (p_I + p_L)\frac{O}{O+U} F^{-1}\left[\frac{(1-\alpha)U}{O+U}\right] + (p_I + p_L)\frac{U}{O+U} F^{-1}\left[\frac{\alpha O + U}{O+U}\right], \tag{22a}$$

$$\zeta^*_{RLT} = (p_I + p_L)\frac{OU}{O+U} F^{-1}\left[\frac{\alpha O + U}{O+U}\right] - (p_I + p_L)\frac{UO}{O+U} F^{-1}\left[\frac{(1-\alpha)U}{O+U}\right]. \tag{22b}$$

Proof: The derivation of all results is presented in Appendix B.

We now consider the total loss of a risk-averse retailer not offering layaway. The loss function for such a retailer is:

$$L_{\bar{R}\bar{L}}(x|q) := (c + \mathfrak{s})[q - x{\cdot}p_I]^+ + (r + \mathfrak{u} - c)[x{\cdot}p_I - q]^+, \tag{23}$$

where $(c + \mathfrak{s})[q - x{\cdot}p_I]^+$ is the overage cost and $(r + \mathfrak{u} - c)[x{\cdot}p_I - q]^+$ is the underage cost. The risk-averse retailer determines the optimal order quantity, $q^*_{R\bar{L}T}$, as the argmin of equation (12) with $\ell(x|q) = L_{\bar{R}\bar{L}}(x|q)$ is represented as:

$$\min_q P_\alpha(q, \zeta) = \min_q \{\zeta + (1-\alpha)^{-1} \int_{x \in \mathbb{R}} [L_{\bar{R}\bar{L}}(x|q) - \zeta]^+ f(x) dx\}. \tag{24}$$

Corollary 2. *If a risk-averse retailer does not offer layaway, and the retailer considers the* total loss *as the loss function, then (24) has an optimal solution $(q^*_{R\bar{L}T}, \zeta^*_{R\bar{L}T})$ of:*

$$q^*_{R\bar{L}T} = \frac{p_I{\cdot}(r + \mathfrak{u} - c)}{r + \mathfrak{u} + \mathfrak{s}} F^{-1}\left(\frac{r + \mathfrak{u} - c + \alpha(c + \mathfrak{s})}{r + \mathfrak{u} + \mathfrak{s}}\right) + \frac{p_I(c + \mathfrak{s})}{r + \mathfrak{u} + \mathfrak{s}} F^{-1}\left[\frac{(r + \mathfrak{u} - c)(1-\alpha)}{\mathfrak{s} + r + \mathfrak{u}}\right], \tag{25a}$$

$$\zeta^*_{R\bar{L}T} = \frac{p_I(r + \mathfrak{u} - c)(c + \mathfrak{s})}{\mathfrak{s} + r + \mathfrak{u}} \left(F^{-1}\left(\frac{r + \mathfrak{u} - c + \alpha(c + \mathfrak{s})}{r + \mathfrak{u} + \mathfrak{s}}\right) + F^{-1}\left[\frac{(r + \mathfrak{u} - c)(1-\alpha)}{\mathfrak{s} + r + \mathfrak{u}}\right]\right). \tag{25b}$$

Proof: The derivation of all results is presented in Appendix E.

Similar to equation (18), equation (25) may be reproduced using the work of Gotoh and Takano (2007) while accounting for consumer budgets and valuations.

4.3 Discussion of risk-averse retailer decisions

In equations (16a) and (22a) the first term corresponds to the case where the retailer's order quantity exceeds total demand from sales and layaway, i.e., $q > (p_I + p_L)x$. The second term is the

case where the retailer's order quantity is less than the market demand. $(p_I + p_L)F^{-1}[\cdot]$ represents the realized demand for sales and layaway. The terms $\frac{O+V}{O+U}$ and $\frac{O}{O+U}$ represent the fraction of the total cost due to overage, for the net loss function and the total loss function, respectively. Similarly, $\frac{U-V}{O+U}$ and $\frac{U}{O+U}$ represent the fraction of the total cost due to underage, for the net loss function and the total loss function, respectively. Compare the first term of equations (16a) and (22a), we find that the optimistic decision maker orders more quantity than the pessimistic decision maker, when $q > (p_I + p_L)x$. Compare the second term of equations (16a) and (22a), we find that the optimistic decision maker orders less quantity than the pessimistic decision maker, when $q < (p_I + p_L)x$.

As a benchmark, we also consider inventory decisions under CVaR when not offering layaway. Using the notation outlined in our paper, we replicate the result of Gotoh and Takano (2007), and it is presented in equations (18) and (25).

The equations (16a) and (22a) show that a risk-averse retailer's optimal order quantity is contingent on the value of the inverse of the demand cumulative distribution function at two points: (1) $F^{-1}\left[\frac{(1-\alpha)U}{O+U}\right]$ and (2) $F^{-1}\left[\frac{\alpha O+U}{O+U}\right]$. We note that, when $\alpha = 0$ (the retailer is risk-neutral), the two points become the same point: $F^{-1}\left[\frac{U}{O+U}\right]$, and the expression for the optimal order quantity changes to the risk-neutral condition given in (8). The risk neutral result is consistent with the results of Dimitrov and Ceryan (2018). Similar to the offering layaway case, when we set $\alpha = 0$, the expression in (18a) and (25a), the expression for the optimal order quantity changes to the risk-neutral condition given in (4).

4.4 Sensitivity analysis for order quantity and profit

In this section, we first analyze how the optimal order quantity for both a risk-neutral and a risk-averse retailer who offer layaway, q_L^*, changes with various parameters. We calculate the partial derivative of q_L^* with respect to each parameters. We conclude this section by investigating how a risk averse retailer's expected profit changes with the default rate.

Proposition 3. *1) The optimal order quantity of a risk-neutral retailer, $q_{\bar{R}L}^*$ increases with $\mathfrak{u}$ and f_c, and decreases with c, $\mathfrak{s}$ and δ.*

2) The optimal order quantity of a risk-averse retailer who considers the net loss as the loss function, q_{RLN}^* increases with $\mathfrak{u}$, and decreases with c, and $\mathfrak{s}$.

3) The optimal order quantity of a risk-averse retailer who consider the total loss as the loss function, q_{RLT}^* increases with $\mathfrak{u}$ and f_c, and decrease with c, $\mathfrak{s}$ and δ.

The derivations of Proposition 3 are presented in Appendix C. We summarize the sign of the partial derivative of each optimal solution q^* with respect to parameters r, α, $\mathfrak{u}$, f_c, f_s, c, $\mathfrak{s}$ and δ in Table 3. For completeness, we present the sensitivity results of the model parameters for a risk-averse retailer not offering layaway, and we also compare our results with Gotoh and Takano (2007).

Table 3: Sign of partial derivative of each condition; where c-b-c denotes case-by-case. Gotoh and Takano (2007) does not allow $\mathfrak{s}$ to be negative.

	$\frac{\partial q^*}{\partial r}$	$\frac{\partial q^*}{\partial \alpha}$	$\frac{\partial q^*}{\partial \mathfrak{u}}$	$\frac{\partial q^*}{\partial c}$	$\frac{\partial q^*}{\partial \mathfrak{s}}$	$\frac{\partial q^*}{\partial f_s}$	$\frac{\partial q^*}{\partial f_c}$	$\frac{\partial q^*}{\partial \delta}$
Total loss CVaR layaway	c-b-c	c-b-c	+	-	-	c-b-c	+	-
Net loss CVaR layaway	c-b-c	c-b-c	+	-	-	c-b-c	c-b-c	c-b-c
Risk neutral layaway	c-b-c	0	+	-	-	c-b-c	+	-
Total loss CVaR no layaway	c-b-c	c-b-c	+	-	-			
Total loss CVaR Gotoh	+	c-b-c	+	-	+			
Net loss CVaR no layaway	c-b-c	c-b-c	+	-	-			
Net loss CVaR Gotoh	c-b-c	c-b-c	+	-	+			

From Table 3, we see that the partials of the optimal order quantity with respect to c, $\mathfrak{s}$, and $\mathfrak{u}$ are the same as those of the risk-neutral condition. The optimal order quantity is increasing with the shortage cost, $\mathfrak{u}$. As $\mathfrak{u}$ increases, the greater the retailer's revenue loss, leading to larger orders by the retailer. For completeness, we compare our results to those of Gotoh and Takano (2007) who consider a retailer not offering layaway. One key difference between the work of Gotoh and Takano (2007) and our paper is that the authors consider demand that is independent of price, while in our paper, by definition of p_I, even if layaway is not offered demand will decrease as price increases. It is this key modeling difference that explains the difference in the partial of optimal order quantity, q^*, with respect to selling price, r, our Total loss CVaR no layaway and Total loss CVaR Gotoh. One addition, though minor difference, between our model and that of Gotoh and Takano (2007) is that the authors only consider a salvage value, $\mathfrak{s}$, that is non-negative, we allow for $\mathfrak{s}$ being any value, and when there is positive salvage, we have the value as negative, leading to opposite signs in $\frac{\partial q^*}{\partial \mathfrak{s}}$. In addition, the optimal order quantity is decreasing with the unit cost, c. As c increases, the retailer's profit decreases and loss increases, in both loss functions. However, those sensitive to r, α, f_s, f_c, and δ are loss-function dependent. The one exception is the cancellation cost, f_c, and default rate, δ, for the total loss function. For these two parameters, the sensitivity of the optimal order quantity with respect to f_c and δ is the same as the risk-neutral case. This follows from the definition of the two loss functions. In the total loss function, f_c and δ appear only in the definition of U, thus allowing us to quickly determine the monotone relationships between the optimal order quantity and those two parameters. However, as f_c and δ appear in V, within the integral of equation (15), the relationship is not as clear. For the remaining parameters we find that for a general demand distribution, the relationship is no longer clear. For example, for $\alpha \in [0, 1)$, the optimal order quantity q^* may be decreasing or increasing with the parameter α, which depends on the market demand probability density function $f(x)$. For example, starting with the optimal order quantity equation (22a) for the retailer who uses the total loss as the loss function, we have:

$$\begin{aligned}\frac{\partial q^*_{RLT}}{\partial \alpha} &= (p_I + p_L)\frac{\partial}{\partial \alpha}\left[\frac{O}{O+U}F^{-1}\left(\frac{(1-\alpha)U}{O+U}\right) + \frac{U}{O+U}F^{-1}\left(\frac{\alpha O+U}{O+U}\right)\right] \\ &= (p_I + p_L)\frac{OU}{(O+U)^2}\left[\frac{1}{f\left(F^{-1}\left(\frac{\alpha O+U}{O+U}\right)\right)} - \frac{1}{f\left(F^{-1}\left(\frac{(1-\alpha)U}{O+U}\right)\right)}\right],\end{aligned} \tag{26}$$

from equation (26), we can see that the sign of the $\frac{\partial q^*_{RLT}}{\partial \alpha}$ is the same as the sign of $\left[\frac{1}{f\left(F^{-1}\left(\frac{\alpha O+U}{O+U}\right)\right)} -\right.$

$\frac{1}{f\left(F^{-1}\left(\frac{(1-\alpha)U}{O+U}\right)\right)}\Big]$. If $f(x)$ is increasing, then if follows from $\frac{\alpha O+U}{O+U} > \frac{(1-\alpha)U}{O+U}$ that $F^{-1}\left(\frac{\alpha O+U}{O+U}\right) > F^{-1}\left(\frac{(1-\alpha)U}{O+U}\right)$, implying $\frac{\partial q^*_{RLT}}{\partial \alpha} < 0$; otherwise, if $f(x)$ is decreasing, we have $\frac{\partial q^*_{RLT}}{\partial \alpha} > 0$. Therefore, the optimal order quantity under layaway depends not only on different loss functions but also on different market demand probability density functions. Next, we will show how the expected profit of retailers varies with consumers' default rate.

A rational retailer maximizes expected profit, regardless if layaway is offered. However, when offering layaway, consumers' default rates directly affect the retailer's expected profit. The retailer may end up losing revenue due to consumer defaults. Therefore, when a retailer chooses whether to offer a layaway program, the consumer's default rate is an important decision parameter. Thus, we investigate how a risk averse retailer's expected profit changes with the market default rate.

Proposition 4. *For any given and fixed confidence level α and the order quantity, q, the expected profit for the retailer who offers layaway, $\pi_L(\delta|\alpha, q)$, is a monotone decreasing function of the market default rate, δ.*

The proof of Proposition 4 is found in Appendix F. From Proposition 4, we know that the higher the market default rate, the lower the expected profit of the retailer. In particular, if $\delta = 0$, i.e., layaway consumers always finish the layaway program, then the retailer has the highest expected profit relative to any other default rates, assuming all other problem parameters are held constant. When $\delta = 1$, all layaway consumers will default during the layaway period, the retailer will achieve the least possible expected profit, again assuming all other parameters do not change.

Corollary 3. *For fixed $\alpha \in [0, 1)$, there exists a threshold value, $\delta^*(\alpha) \in [0, 1]$, such that a risk-averse retailer is better off offering layaway if the market default rate, δ, is less than $\delta^*(\alpha)$, and not offering layaway otherwise.*

The proof of Corollary 3 is found in Appendix G. In the remainder of this paragraph we assume that α is fixed and present a proof-sketch of Corollary 3. From Proposition 4, we know that the expected profit when offering layaway, π_L, is strictly decreasing with the default rate δ. In addition, $\pi_{\bar{L}}$ is constant in δ. Therefore, the expected profit when offering layaway, π_L, will equal the expected profit when not offering layaway, $\pi_{\bar{L}}$, at most once at $\delta^*(\alpha)$. This means for $\delta < \delta^*(\alpha)$ the retailer is better off offering layaway, and is worse off when $\delta > \delta^*(\alpha)$.

5 Numerical illustrations and insights

In this section, we perform numerical studies to illustrate how the optimal order quantity and the expected profit change with model parameters, only for those listed as case-by-case in Table 3. For comparison purposes, we compare six cases: (1) a risk-neutral retailer offering a layaway program, similar to Dimitrov and Ceryan (2018); (2) a risk-neutral retailer not offering layaway, similar to a risk-neutral newsvendor; (3) and (4) a risk-averse newsvendor not offering layaway, using the (3) net-loss and (4) total-loss loss functions; (5) and (6) a risk-averse newsvendor not offering layaway, using the (5) net-loss and (6) total-loss loss functions. The parameter values we consider cannot be unbounded due to parameter relationships derived from our analytical results. In particular, we must ensure the probability of purchase is non-negative. At the end of the section we present theoretical and practical insights of our paper.

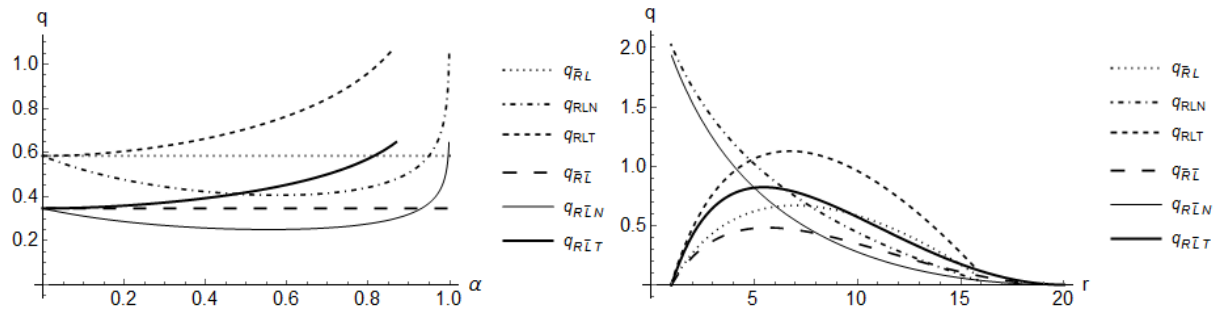
5.1 The optimal order quantity changes with parameters

For a general demand distribution, it is not clear how the optimal order quantity changes with each of the model parameters. To clarify the relationship between q^* and the variables in the different loss functions, several numerical results are presented below. In the literature, consumer demands are modeled using an exponential distribution (see Abdel-Malek and Areeratchakul (2007), Afshar-Nadjafi (2016), Rossi et al. (2014)). We assume that demand obeys the exponential distribution with a parameter of $\lambda = 1$. Further, we assume the retailer has a confidence level $\alpha = 0.8$ of the CVaR. For all other parameters we use the same values as the numerical study of Dimitrov and Ceryan (2018), the item per unit price is $r = 10$, the per unit item cost is $c = 5$, the enrollment fee is $f_s = 0.2$, the cancellation fee is $f_c = 0.5$, the mean of consumer's valuation for the item is $\bar{v} = 10$, the mean of consumer's budget is $\bar{b} = 10$, the discount factor is $\theta = 0.8$, the default rate is $\delta = 0.2$, the length of the layaway period is $L = 8$, the salvage value of a single item is $\mathfrak{s} = -2$, and the shortage cost is $\mathfrak{u} = 4$.

We first discuss why in some cases the trend of the net loss and the total loss is reversed. When comparing the two loss functions, see Section 4.2, the signs of r, f_c, f_s and δ are different (follows from (7), (19)). The opposite signs, for the two loss functions, lead to opposite behavior in the order quantity with an increase in r, f_c, f_s, and δ. Consider r in the next example. If $x(p_I + p_L) - q > 0$, the coefficient of r in the net loss function is $-\left(p'_I + p'_L(1-\delta)\right) \cdot q < 0$, but in the total loss function the coefficient of r is $\left(p'_I + p'_L(1-\delta)\right) \cdot \left(x(p_I + p_L) - q\right) > 0$. Furthermore, when we analyze the sensitivity of the cancellation fee, f_c, the enrollment fee, f_s, the default rate, δ, and the length of layaway period, L, we find that only when layaway is offered will the retailer's profit change, otherwise the profit is constant. In the figures in this section, we use R to denote a risk-averse retailer, and $\bar{R}$ a risk-neutral retailer. We also use L and $\bar{L}$ to denote the presence of a layaway program and no layaway program, respectively. Finally we use T and N to denote the total and net loss loss functions, respectively. All plots are denoted with a subscript indicating which one of six scenarios we consider, the first is the retailer's risk attitude, then if layaway is offered and finally the type of loss function used, only if the retailer is risk averse.

As depicted in Figure 2a, as α tends to 1, the risk averse retailer's optimal order quantity, regardless of whether layaway is offered, and regardless of which loss function is used, q^*_{Ryx} tends to infinity, where R represents risk averse condition, $y \in \{L, \bar{L}\}$ and $x \in \{T, N\}$. The behavior of q^*_{Ryx} follows from the optimal order quantity equations. From (16a) we find that $\frac{\alpha O+U}{O+U}$ tends to 1 as α tends to 1; similarly from (18a) we find that $\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}$ tends to 1 as α tends to 1. In turn, $\lim_{d\to 1} F^{-1}(d) \to \infty$ for the exponential distribution, thus the observed limiting behavior of Figure 2a is expected. As confidence level, α, trends to 1, i.e., the retailer is more sensitive to risk, and as such the optimal order quantity quickly increases.

Figure 2b shows the influence of the item's price on the retailer's optimal order quantity. We can see that as the price increases, the optimal order quantity of both risk-neutral retailer and the risk-averse retailer (modeled by the total loss function) first increases and then decreases. The reason why the optimal order quantity first increases is that a low selling price can satisfy more consumers, including low budget consumers. However, as the price increases, the lower budget consumers cannot afford the item, thus the optimal order quantity will decrease.

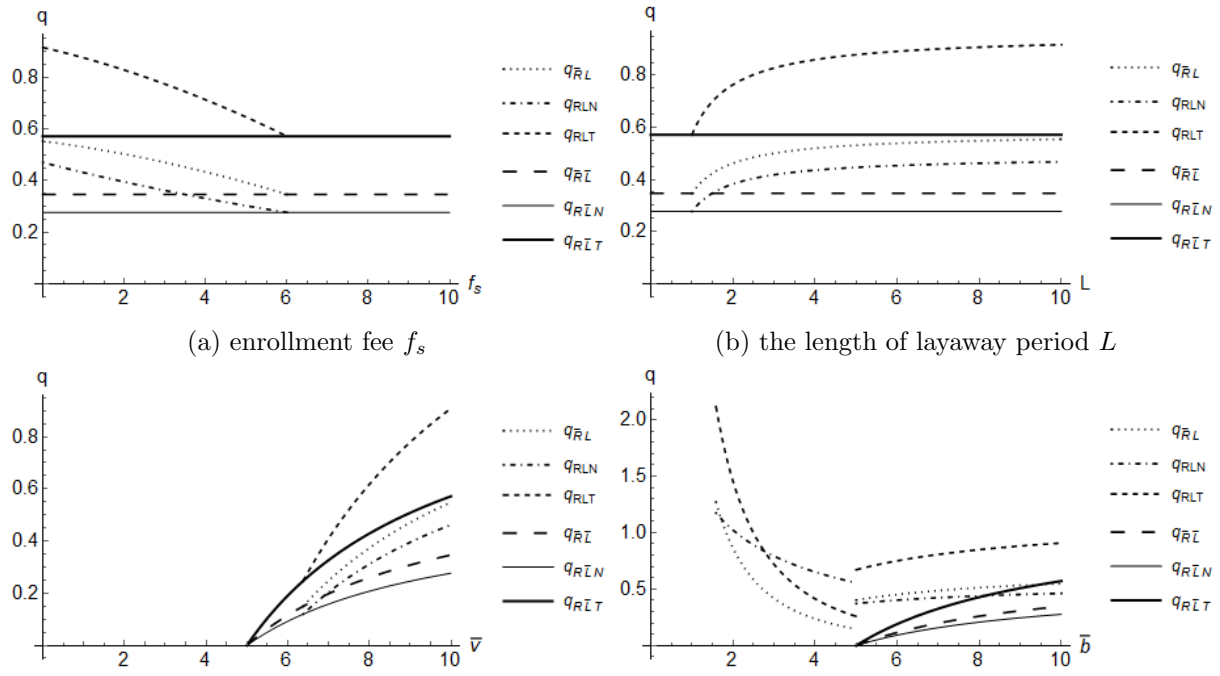


(a) confidence level α (b) selling price r

Figure 2: Sensitivity of the optimal order quantity, q, for different layaway and risk-preference scenarios, with respect to (a) α and (b) r

As shown in Figure 3a, the optimal order quantity decreases as the enrollment fee increases. We note that this figure does not contradict the result in Table 3 as for a lower purchase price and lower default rate the order quantity initially increases then decreases, though slightly (Please see Figure 22, in Appendix I). As shown in Figures 3b and 3c, we see that as consumers' valuations, $\bar{v}$, and layaway length, L, increase, the optimal order quantity of both risk-neutral retailer and the risk-averse retailer also increase. Furthermore, we can see from Figure 3b that as the length of layaway period increases the optimal order quantity increases with diminishing returns. This observation follows from the fact that as length of layaway period increases the budget a consumer needs to purchase the item using layaway decreases, see equation (2). However, eventually the market will reach saturation, thus diminishing returns. We also can see from Figure 3c that more consumers with high valuations may purchase the product when layaway is available; this follows naturally from the fact that under a layaway program budget constrained consumers with high valuations may purchase the product.

Figure 3d shows that, as the average consumer budget, $\bar{b}$, increases, the optimal order quantity first decreases and then, after a discontinuity, increases. This follows from the following observations: prior to the discontinuity only layaway purchases will be observed, and the probability of a layaway purchase decreases with the average consumer budget. Once immediate purchases are observed, after the discontinuity, the probability of an immediate purchase increases with the average consumer budget.



(a) enrollment fee f_s

(b) the length of layaway period L

(c) the mean of consumer's valuation $\bar{v}$

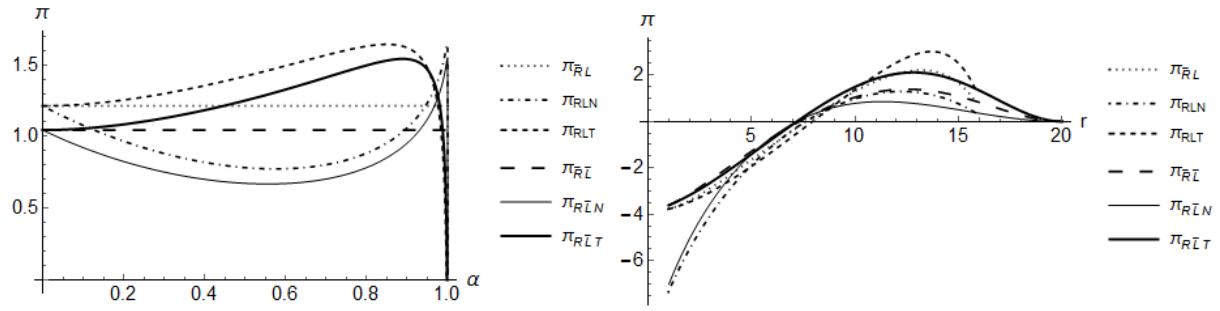
(d) the mean of consumer's budget $\bar{b}$

Figure 3: Sensitivity of the optimal order quantity, q, for different layaway and risk-preference scenarios, with respect to (a) f_s, (b) L, (c) $\bar{v}$, and (d) $\bar{b}$

5.2 Expected profit changes with parameters

We next analyze how expected profit changes with each of the model parameters. Please note that we only set $f_s = 5$, and $f_c = 4$ in Figure 4a. These parameters ensure that the expected profit of offering layaway is greater than the expected profit of not offering layaway. In Figure 4a we can find that as the retailer's risk level, α, tends to 1, there is a threshold risk level, $\tilde{\alpha}$, that makes the expected profit from offering layaway, regardless of the loss function, equal to not offering layaway. If the risk level is less than the threshold, $\alpha < \tilde{\alpha}$, the retailer is better off offering layaway, otherwise, $\alpha > \tilde{\alpha}$ the retailer is better off not offering layaway.

As shown in Figure 4b, we can see that as the selling price increases, the expected profit for all cases initially increases then decreases. A low selling price will attract most potential consumers to purchase the product either directly or via the layaway program. However, as the selling price increases fewer consumer will be able to purchase the product, even through a layaway program. An additional observation is that the selling price that maximizes profit is to the right, i.e., it is higher, when offering layaway, than when not offering layaway.



(a) confidence level α (b) selling price r

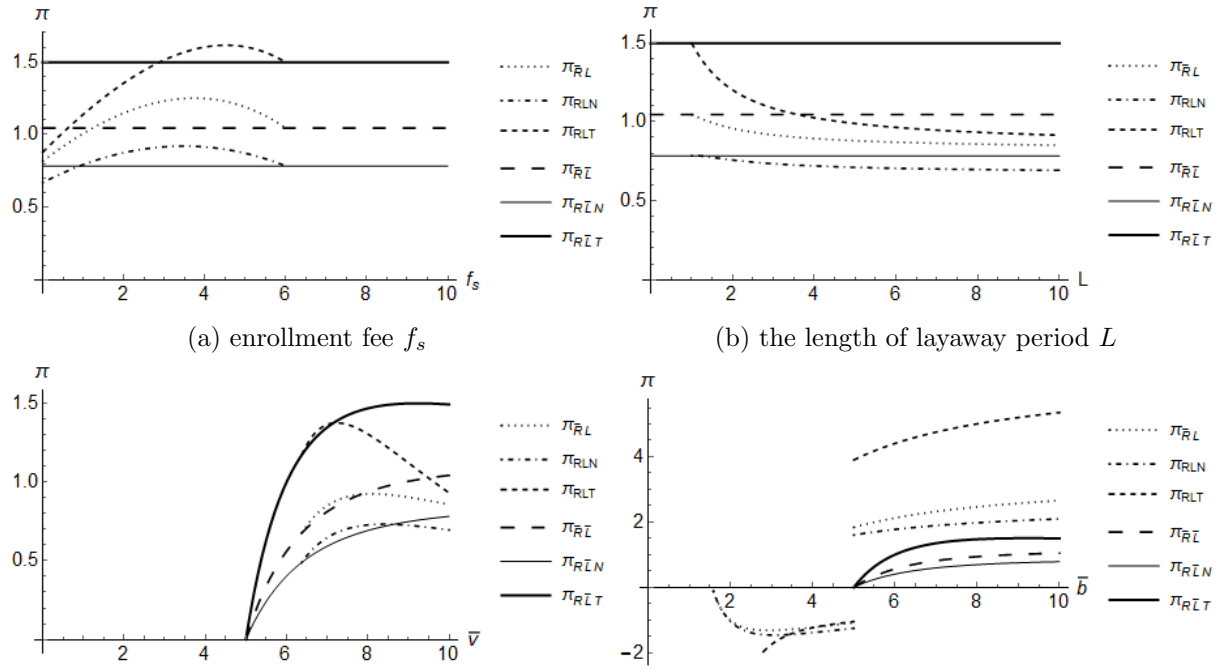
Figure 4: Sensitivity of the expected profit, π, for different layaway and risk-preference scenarios, with respect to (a) α (b) r

As depicted in Figure 5a, as the layaway enrollment fee increases, the expected profit initially increases then decreases. The observed behavior follows from the fact that with an enrollment fee sufficiently low, all consumers with low budgets and sufficiently high valuations of the product will purchase the item. However, as the enrollment fee continues to increase only consumers with higher budgets and higher valuations of the item will be only the ones participating in the layaway program. In fact, as derived from Equation (2), if $f_s > 2\bar{v}\theta - r$ then there will be no layaway purchases.

As shown in Figure 5b, we can see that as the payoff period, L, increases the expected profit monotonically decrease with diminishing returns. The observed behavior follows from the fact that as L increases more purchases will be made using layaway with diminishing returns (see Figure 3b). As more of the retailer's sales are from layaway consumers, the retailer will face more defaults, leading to lower profits.

Figure 5c displays the relationship between expected profit and consumers' expected valuation. We note that the relationship is only defined for valuations greater than or equal to the price, thus we start at $\bar{v} \geq 5$ (Price in our numerical study is set to 10). Considering only cases where layaway is not offered, as consumer valuation grows eventually all consumers will purchase the product, and there will be no consumers remaining to purchase, thus explaining the observed diminishing returns. Moving to the case where layaway is offered, the expected profit first increases with consumers' expected valuation, $\bar{v}$, and then decreases. The expected profit will first increase because as consumer valuation increases more low budget consumers will purchase through layaway. Eventually as more consumers purchase via layaway more consumers will also default leading to a decrease in profit from the defaulting consumers.

As shown in Figure 5d, we note that there is a discontinuity in the relationship at the point where consumers purchasing immediately appear, when consumer's budgets are less than the item price, only layaway purchases will occur and the retailer's profit is negative. A negative expected profit simply means that the retailer will not be in business. However, if item cost, salvage value, and default rate are reduced, then expected profits become positive when only layaway is offered (Please see Figure 23, in Appendix I). Once immediate purchases are made, the expected profit increases with diminishing returns, follows from the fact p_I and p_L increase with diminishing returns as average consumer budget increases.



(a) enrollment fee f_s

(b) the length of layaway period L

(c) the mean of consumer's valuation $\bar{v}$

(d) the mean of consumer's budget $\bar{b}$

Figure 5: Sensitivity of the expected profit, π, for different layaway and risk-preference scenarios, with respect to (a) f_s, (b) L, (c) $\bar{v}$, and (d) $\bar{b}$

5.3 Theoretical contributions and practical contributions

As discussed in the introduction, our work is driven by a layaway program offered by retailers to help economically disadvantaged budget constrained consumers to purchase expensive goods. The theoretical contributions of our paper are as follows. We first reformulate the profit function proposed by Dimitrov and Ceryan (2018) resulting in a simpler form (c.f. (8)). Second, we incorporate two models of a risk-averse retailer into the layaway newsvendor problem, net loss and total loss, we derive the functional forms for each loss function. Next, we determine optimal order quantities under two models of a risk-averse newsvendor and show optimal order quantities respond differently to parameter changes in the two models. Moreover, our paper shows, similar to Dimitrov and Ceryan (2018), that there is a critical market default rate prior to which a retailer is better off offering layaway, and after which the retailer is worse off offering layaway, assuming model parameters remain unchanged.

As previously discussed, retailers tend to be risk averse when making decisions. So far in this paper we derived and discussed the optimal ordering decisions of a risk averse retailer. We now summarize our practical contributions. First, we find that the qualitative properties of Dimitrov and Ceryan (2018) hold for a risk-averse retailer, something not obvious from the current literature. Second, we find that for a given retailer, the risk-preferences are fixed, there exists a critical market default rate below which the retailer is better off offering layaway, and above which the retailer is better off not offering layaway. An immediate implication is a retailer must conduct a market study to understand the market default rate and evaluate if the estimated default rate is worth the risks associated with a layaway program. The market default rate may be learned via a pilot program or randomized multi-store trials. Third, manufacturers may want to work with either risk-neutral or highly risk-averse retailers as those types of retailers will order more than retailers

with moderate risk-aversion, see Figure 2a. Note that extremely risk-averse retailers may be hard to find as expected profits plummet as α tends to one. Fourth, as the layaway period increases the number of consumers participating in the layaway program will increase, due to low payments each payment period. However, the optimal order quantity increases and expected profit decreases as the layaway period increases. This suggests, that there is a maximal layaway length beyond which the retailer should not consider extending the layaway period length. Practically, a retailer may determine this layaway length via market studies and pilot programs. Fifth, if the retailer is extremely risk averse (e.g, α close to 1), the retailer's expected profit will decrease quickly, and is likely better off not offering layaway, potentially explaining why "small" retailers do not offer layaway, and only major retailers offer such programs. Usually larger retailers have more capital and are likely to behave more closely to risk-neutral due to a wealth effect (Cox and Sadiraj, 2006). Finally, through our additional numerical studies, found in Appendix H, the insights we provided qualitatively hold for cases where demand is non-uniform and when there is correlation between consumer valuations and consumer budgets.

6 Conclusions and future work

The results developed in this paper provide insights for retailers offering a layaway program. In particular, we take into account retailer's risk tolerance which constitutes a departure from previous studies. From our derivation, we find that the risk neutral case (Dimitrov and Ceryan, 2018) is only a special case of risk aversion. Our research provide reference for decision makers of a wider class of risk tolerance. Moreover, we consider consumer behavior in our model. We explore consumer valuations and consumer budgets that are independent and relevant, respectively. We also focused on the impact of the market default rate on the risk-averse retailer's decision-making of offering layaway program. We derived the optimal order quantities for a risk-averse retail, modeled using conditional value at risk (CVaR), offering layaway with randomly distributed consumer valuations and budgets. Initially, we construct mathematical models for the retailer's expected profit when facing stochastic market demand. Next, we used CVaR, and two common loss functions found in the literature, to study the optimal order quantity decisions made by a risk-averse retailer. We choose the net loss function to capture the optimistic decision makers' policies, and choose the total loss function to capture the pessimistic decision makers' polices. We conducted sensitivity analysis to associated parameters such as item cost, overage cost, shortage cost, cancellation fee, market default rate, and the retailer's risk aversion coefficient on the decision variables. We moved to numerical studies to illustrate the impact of the parameters on the optimal order quantity of the risk-neutral and risk-averse retailer, in the situations when layaway is and is not offered. Furthermore, we examined the effects of the degree of the retailer's risk aversion and the consumer's default rate on the retailer's expected profit.

This study has limitations and presents opportunities for future research. First, in this paper, we condensed multiple layaway periods into a single period. However, in practice, retailers usually make dynamic decisions. Though we provide insights into the layaway case, it is still unclear if/how a multi-period setting impacts the expected profit and optimal order quantities of risk-neutral and risk-averse retailers. In particular, a multi-period model must account for past decisions when making current and future decisions. Accounting for past, present, and future decisions will make the analytical derivations of this paper not trivial in the multi-period setting. Second, we consider consumer valuations and budgets are independently and uniformly distributed. In fact, this is just a simplifying assumption that facilitated our analysis, and is not necessarily realistic. Although we use numerical analysis to study two forms of correlation between uniformly distributed valuations and

budgets, general distributions and relationships are still analytically unaddressed. For tractabiltiy we made an array of simplifying assumptions; for example, we assumed only risk aversion is present in our model, however empirical research shows that real-world ordering decisions are explained using a menu of risk factors. If retailer data is made available, then we may numerically determine the retailer's optimal order quantities using data analytics. As we have no industry partner in our study, we are not able to take a data-driven approach. Third, we only consider a single item, we assume the down payment is zero and instead subsumed into the enrollment fee. However, if we consider multiple items, the results may change and the zero down payment assumption is not longer made without loss of generality. When we consider the down payment is the percentage of purchase price, as we consider only a single item, the percentage of purchase price is equivalent to a fixed down payment. Percentage down payment will however impact our analysis if we consider multiple items, which we do not in this study and is a limitation of the study that needs consideration in future studies.

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A Proof of proposition 1

Proof. We consider three cases so as to evaluate the net loss CVaR function in (15).

Case 1: $\zeta \leq -Vx$

If the net loss is greater than ζ for any demand x, then equation (15) becomes:

$$\begin{aligned} P_\alpha(q,\zeta) =& \zeta + (1-\alpha)^{-1}\left[\int_0^{\frac{q}{p_I+p_L}} \left[-\left(Vx(p_I+p_L) - O\left(q - x(p_I+p_L)\right)\right) - \zeta\right] f(x)dx \right. \\ &\left. + \int_{\frac{q}{p_I+p_L}}^{\infty} \left[-\left(Vx(p_I+p_L) - U\left(x(p_I+p_L) - q\right)\right) - \zeta\right] f(x)dx\right]. \end{aligned} \tag{27}$$

Given (27), we now compute the first-order derivative of $P_\alpha(q,\zeta)$, i.e., $\frac{\partial}{\partial\zeta}P_\alpha(q,\zeta) = 0$ and $\frac{\partial}{\partial q}P_\alpha(q,\zeta) = 0$.

$$\begin{aligned} \frac{\partial}{\partial\zeta}P_\alpha(q,\zeta) =& \frac{\partial}{\partial\zeta}\left[\zeta + (1-\alpha)^{-1}\left[\int_0^{\frac{q}{p_I+p_L}} \left[-\left(Vx(p_I+p_L) - O\left(q - x(p_I+p_L)\right)\right) - \zeta\right] f(x)dx \right.\right. \\ &\left.\left. + \int_{\frac{q}{p_I+p_L}}^{\infty} \left[-\left(Vx(p_I+p_L) - U\left(x(p_I+p_L) - q\right)\right) - \zeta\right] f(x)dx\right]\right] = 0, \\ \frac{\partial}{\partial q}P_\alpha(q,\zeta) =& \frac{\partial}{\partial q}\left[\zeta + (1-\alpha)^{-1}\left[\int_0^{\frac{q}{p_I+p_L}} \left[-\left(Vx(p_I+p_L) - O\left(q - x(p_I+p_L)\right)\right) - \zeta\right] f(x)dx \right.\right. \\ &\left.\left. + \int_{\frac{q}{p_I+p_L}}^{\infty} \left[-\left(Vx(p_I+p_L) - U\left(x(p_I+p_L) - q\right)\right) - \zeta\right] f(x)dx\right]\right] = 0. \end{aligned}$$

We get:

$$\begin{aligned} q^* &= (p_I+p_L)F^{-1}\left(\frac{U}{O+U}\right), \\ \zeta^* &= -Vq^*. \end{aligned} \tag{28}$$

Case 2: $-Vq < \zeta < Oq$

If ζ is greater than $-Vq$, ζ is less than Oq, and $-\widehat{\pi}(x|q) = \zeta$, then equation (14) becomes:

$$\begin{aligned} P_\alpha(q,\zeta) =& \zeta + (1-\alpha)^{-1}\left[\int_0^{\frac{O\cdot q-\zeta}{(O+V)(p_I+p_L)}} \left[-\left(Vx(p_I+p_L)\right.\right.\right. \\ &\left.\left.\left. -O\left(q - x(p_I+p_L)\right)\right) - \zeta\right] f(x)dx \right. \\ &\left. + \int_{\frac{\zeta+q\cdot U}{(U-V)(p_I+p_L)}}^{\infty} \left[-\left(Vx(p_I+p_L) - U\left(x(p_I+p_L) - q\right)\right) - \zeta\right] f(x)dx\right]. \end{aligned} \tag{29}$$

Given (29), we now compute the first-order derivative of $P_\alpha(q,\zeta)$ (i.e., $\frac{\partial}{\partial\zeta}P_\alpha(q,\zeta) = 0$ and

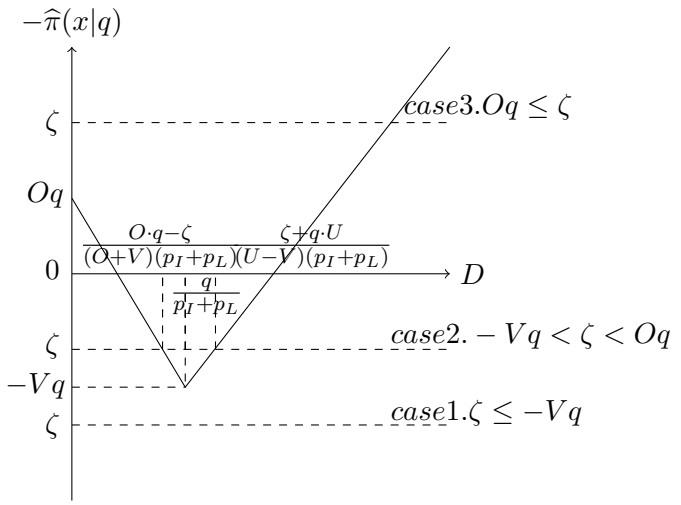


Figure 6: Three cases in minimization of net loss CVaR.

$\frac{\partial}{\partial q}P_\alpha(q,\zeta)=0$),

$$\begin{aligned}\frac{\partial}{\partial\zeta}P_\alpha(q,\zeta)=&\frac{\partial}{\partial\zeta}\left[\zeta+(1-\alpha)^{-1}\left[\int_0^{\frac{O\cdot q-\zeta}{(O+V)(p_I+p_L)}}\left[-\left(Vx(p_I+p_L)\right.\right.\right.\right.\\&\left.\left.\left.\left.-O\left(q-x(p_I+p_L)\right)\right)-\zeta\right]f(x)dx\right.\\&\left.+\int_{\frac{\zeta+q\cdot U}{(U-V)(p_I+p_L)}}^{\infty}\left[-\left(Vx(p_I+p_L)-U\left(x(p_I+p_L)-q\right)\right)-\zeta\right]f(x)dx\right]\\=&1+(1-\alpha)^{-1}\left[-F\left(\frac{O\cdot q-\zeta}{(O+V)(p_I+p_L)}\right)+F\left(\frac{\zeta+q\cdot U}{(U-V)(p_I+p_L)}\right)-1\right].\end{aligned}$$

Setting $\frac{\partial}{\partial\zeta}P_\alpha(q,\zeta)=0$, we find:

$$1+(1-\alpha)^{-1}\left(F\left(\frac{\zeta+q\cdot U}{(U-V)(p_I+p_L)}\right)-1\right)=(1-\alpha)^{-1}F\left(\frac{O\cdot q-\zeta}{(O+V)(p_I+p_L)}\right).\qquad(30)$$

$$\begin{aligned}\frac{\partial}{\partial q}P_\alpha(q,\zeta)=&\frac{\partial}{\partial q}\left[\zeta+(1-\alpha)^{-1}\left(\int_0^{\frac{O\cdot q-\zeta}{(O+V)(p_I+p_L)}}\left[-\left(Vx(p_I+p_L)\right.\right.\right.\right.\\&\left.\left.\left.\left.-O\left(q-x(p_I+p_L)\right)\right)-\zeta\right]f(x)dx\right.\\&\left.\left.+\int_{\frac{\zeta+q\cdot U}{(U-V)(p_I+p_L)}}^{\infty}\left[-\left(Vx(p_I+p_L)-U\left(x(p_I+p_L)-q\right)\right)-\zeta\right]f(x)dx\right)\right],\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial q}P_\alpha(q,\zeta) =&(1-\alpha)^{-1}\frac{\partial}{\partial q}\int_0^{\frac{O\cdot q-\zeta}{(O+V)(p_I+p_L)}}\left[-\left(Vx(p_I+p_L)\right.\right.\\ &\left.\left.-O\left(q-x(p_I+p_L)\right)\right)-\zeta\right]f(x)dx+(1-\alpha)^{-1}\frac{\partial}{\partial q}\int_{\frac{\zeta+q\cdot U}{(U-V)(p_I+p_L)}}^{\infty}\Big[\\ &-\left(Vx(p_I+p_L)-U\left(x(p_I+p_L)-q\right)\right)-\zeta\Big]f(x)dx.\end{aligned}\tag{31}$$

We first calculate the integral of the first term in (31),

$$\begin{aligned}&(1-\alpha)^{-1}\frac{\partial}{\partial q}\int_0^{\frac{O\cdot q-\zeta}{(O+V)(p_I+p_L)}}\left[-Vx(p_I+p_L)+O\left(q-x(p_I+p_L)\right)-\zeta\right]f(x)dx\\ &\quad=(1-\alpha)^{-1}\frac{\partial}{\partial q}\int_0^{\frac{O\cdot q-\zeta}{(O+V)(p_I+p_L)}}\left[-Vx(p_I+p_L)+O\left(q-x(p_I+p_L)\right)-\zeta\right]xF(x)\\ &\quad=(1-\alpha)^{-1}\left[\frac{\partial}{\partial q}\left[-\left(Vx(p_I+p_L)-O\left(q-x(p_I+p_L)\right)\right)-\zeta\right]F(x)\Big|_0^{\frac{O\cdot q-\zeta}{(O+V)(p_I+p_L)}}\right.\\ &\qquad\left.+(O-V)(p_I+p_L)\int_0^{\frac{O\cdot q-\zeta}{(O+V)(p_I+p_L)}}f(x)dx\right]\\ &\quad=(1-\alpha)^{-1}OF\left(\frac{O\cdot q-\zeta}{(O+V)(p_I+p_L)}\right),\end{aligned}\tag{32}$$

and then we calculate the integral of the second term in (31),

$$
\begin{aligned}
&(1-\alpha)^{-1}\frac{\partial}{\partial q}\int_{\frac{\zeta+q\cdot U}{(U-V)(p_I+p_L)}}^{\infty}\left[-\left(Vx(p_I+p_L)-U\left(x(p_I+p_L)-q\right)\right)-\zeta\right]f(x)dx\\
=&(1-\alpha)^{-1}\frac{\partial}{\partial q}\int_{\frac{\zeta+q\cdot U}{(U-V)(p_I+p_L)}}^{\infty}\left[-\left(Vx(p_I+p_L)-U\left(x(p_I+p_L)-q\right)\right)-\zeta\right]dF(x)\\
=&(1-\alpha)^{-1}\frac{\partial}{\partial q}\left[\int_{\frac{\zeta+q\cdot U}{(U-V)(p_I+p_L)}}^{\infty}(U-V)x(p_I+p_L)dF(x)\right.\\
&\left.+\int_{\frac{\zeta+q\cdot U}{(U-V)(p_I+p_L)}}^{\infty}(-Uq-\zeta)dF(x)\right]\\
=&(1-\alpha)^{-1}\frac{\partial}{\partial q}\left[E(x)-(U-V)(p_I+p_L)\int_0^{\frac{\zeta+q\cdot U}{(U-V)(p_I+p_L)}}xdF(x)\right.\\
&\left.+(-Uq-\zeta)\left(1-F\left(\frac{\zeta+q\cdot U}{(U-V)(p_I+p_L)}\right)\right)\right]\\
=&(1-\alpha)^{-1}\frac{\partial}{\partial q}\left[E(x)-(\zeta+q\cdot U)F\left(\frac{\zeta+q\cdot U}{(U-V)(p_I+p_L)}\right)\right.\\
&+(U-V)(p_I+p_L)\int_0^{\frac{\zeta+q\cdot U}{(U-V)(p_I+p_L)}}f(x)dx\\
&\left.+(-Uq-\zeta)\left(1-F\left(\frac{\zeta+q\cdot U}{(U-V)(p_I+p_L)}\right)\right)\right]\\
=&(1-\alpha)^{-1}\left[-U\left(1-F\left(\frac{\zeta+q\cdot U}{(U-V)(p_I+p_L)}\right)\right)\right],
\end{aligned}
\tag{33}
$$

combining the above two results, we obtain:

$$
\begin{aligned}
\frac{\partial}{\partial q}P_\alpha(q,\zeta)=&\frac{\partial}{\partial q}\left[\zeta+(1-\alpha)^{-1}\left[\int_0^{\frac{O\cdot q-\zeta}{(O+V)(p_I+p_L)}}\left[-\left(Vx(p_I+p_L)\right.\right.\right.\right.\\
&\left.\left.\left.\left.-O\left(q-x(p_I+p_L)\right)\right)-\zeta\right]f(x)dx\right.\right.\\
&\left.\left.+\int_{\frac{\zeta+q\cdot U}{(U-V)(p_I+p_L)}}^{\infty}\left[-\left(Vx(p_I+p_L)-U\left(x(p_I+p_L)-q\right)\right)-\zeta\right]f(x)dx\right]\right]\\
=&(1-\alpha)^{-1}\left[OF\left(\frac{O\cdot q-\zeta}{(O+V)(p_I+p_L)}\right)-U\left(1-F\left(\frac{\zeta+q\cdot U}{(U-V)(p_I+p_L)}\right)\right)\right]\\
=&(1-\alpha)^{-1}\left[OF\left(\frac{O\cdot q-\zeta}{(O+V)(p_I+p_L)}\right)+UF\left(\frac{O\cdot q-\zeta}{(O+V)(p_I+p_L)}\right)\right.\\
&\left.-U\left[F\left(\frac{O\cdot q-\zeta}{O+V}\right)+1-F\left(\frac{\zeta+q\cdot U}{(U-V)(p_I+p_L)}\right)\right]\right]\\
=&(1-\alpha)^{-1}\left[(O+U)F\left(\frac{O\cdot q-\zeta}{(O+V)(p_I+p_L)}\right)-U(1-\alpha)\right].
\end{aligned}
\tag{34}
$$

Setting $\frac{\partial}{\partial q}P_\alpha(q,\zeta)=0$, we get:

$$q^*=\frac{\zeta}{O}+\frac{(O+V)(p_I+p_L)}{O}F^{-1}\left[(1-\alpha)\frac{U}{O+U}\right], \tag{35}$$

we also find:

$$(1-\alpha)^{-1}F\left(\frac{O\cdot q-\zeta}{(O+V)(p_I+p_L)}\right)=\frac{U}{O+U}. \tag{36}$$

According to the (34) and (36), we derive:

$$1+(1-\alpha)^{-1}F\left(\frac{\zeta+q\cdot U}{(U-V)(p_I+p_L)}\right)-(1-\alpha)^{-1}=\frac{U}{O+U},$$

$$(1-\alpha)+F\left(\frac{\zeta+q\cdot U}{(U-V)(p_I+p_L)}\right)-1=(1-\alpha)\frac{U}{O+U},$$

$$q^*=\frac{(U-V)(p_I+p_L)}{U}F^{-1}\left[\frac{\alpha O+U}{O+U}\right]-\frac{\zeta}{U}. \tag{37}$$

Therefore, according to the (35) and (37), we determine the optimal order quantity and maximum loss threshold as:

$$\begin{aligned}
q^*_{RLN}&=(p_I+p_L)\frac{O+V}{O+U}F^{-1}\left[\frac{(1-\alpha)U}{O+U}\right]+(p_I+p_L)\frac{U-V}{O+U}F^{-1}\left[\frac{\alpha O+U}{O+U}\right],\\
\zeta^*_{RLN}&=(p_I+p_L)\frac{O(U-V)}{O+U}F^{-1}\left[\frac{\alpha O+U}{O+U}\right]-(p_I+p_L)\frac{U(O+V)}{O+U}F^{-1}\left[\frac{(1-\alpha)U}{O+U}\right].
\end{aligned} \tag{38}$$

Case 3: $Oq\le\zeta$

If ζ is greater than Oq, then equation (14) becomes:

$$\begin{aligned}
P_\alpha(q,\zeta)=&(1-\alpha)^{-1}\frac{\partial}{\partial q}\int_{\frac{\zeta+q\cdot U}{(U-V)(p_I+p_L)}}^{\infty}[-(Vx(p_I+p_L)\\
&-U\left(x(p_I+p_L)-q\right))-\zeta]\,f(x)dx.
\end{aligned} \tag{39}$$

Carrying out analysis similar to Case 2, we show that:

$$\begin{aligned}
\frac{\partial}{\partial q}P_\alpha(q,\zeta)=&(1-\alpha)^{-1}\frac{\partial}{\partial q}\left[\int_{\frac{\zeta+q\cdot U}{(U-V)(p_I+p_L)}}^{\infty}\left[U\left(x(p_I+p_L)-q\right)-\zeta\right]f(x)dx\right]\\
=&(1-\alpha)^{-1}\left[-U\left(1-F\left(\frac{\zeta+q\cdot U}{(U-V)(p_I+p_L)}\right)\right)\right]<0.
\end{aligned} \tag{40}$$

Therefore, there is no optimal order quantity for this case. □

B Proof of proposition 2

Proof. We consider three cases to determine the optimal order quantity for a risk averse retailer using the total loss CVaR function (21).

Case 1: $\zeta \leq 0$

If the total loss is greater than ζ, then equation (21) becomes:

$$\begin{aligned} P_\alpha(q,\zeta) =& \zeta + (1-\alpha)^{-1} \left[\int_0^{\frac{q}{p_I+p_L}} \left[O\left(q - x(p_I+p_L)\right) - \zeta \right] f(x)dx \right. \\ & \left. + \int_{\frac{q}{p_I+p_L}}^{\infty} \left[U\left(x(p_I+p_L) - q\right) - \zeta \right] f(x)dx \right]. \end{aligned} \tag{41}$$

Given (41), we now compute the first-order derivative of $P_\alpha(q,\zeta)$, i.e., $\frac{\partial}{\partial \zeta} P_\alpha(q,\zeta) = 0$ and $\frac{\partial}{\partial q} P_\alpha(q,\zeta) = 0$.

We get:

$$\begin{aligned} q^* &= (p_I + p_L) F^{-1}\left(\frac{U}{O+U}\right), \\ \zeta^* &= 0. \end{aligned} \tag{42}$$

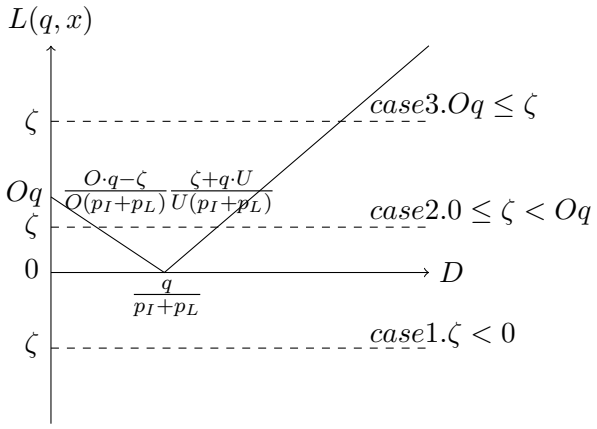


Figure 7: Three cases in minimization of total lost CVaR.

Case 2: $0 < \zeta < O \cdot q$

If the ζ is greater than 0, ζ is less than Oq, and $L(x|q) = \zeta$, then equation (21), $\int_{x\in\mathbb{R}} [L(x|q) - \zeta]^+ f(x)dx$, becomes:

$$\begin{aligned} P_\alpha(q,\zeta) =& \zeta + (1-\alpha)^{-1} \left[\int_0^{\frac{O\cdot q-\zeta}{O(p_I+p_L)}} \left[O\left(q - x(p_I+p_L)\right) - \zeta \right] f(x)dx \right. \\ & \left. + \int_{\frac{\zeta+q\cdot U}{U(p_I+p_L)}}^{\infty} \left[U\left(x(p_I+p_L) - q\right) - \zeta \right] f(x)dx \right]. \end{aligned} \tag{43}$$

Given (43), we now compute the first-order derivative of $P_\alpha(q,\zeta)$:

$$\begin{aligned}\frac{\partial}{\partial\zeta}P_\alpha(q,\zeta)=&\frac{\partial}{\partial\zeta}\left[\zeta+(1-\alpha)^{-1}\left[\int_0^{\frac{O\cdot q-\zeta}{O(p_I+p_L)}}\left[O\left(q-x(p_I+p_L)\right)-\zeta\right]f(x)dx\right.\right.\\&\left.\left.+\int_{\frac{\zeta+q\cdot U}{U(p_I+p_L)}}^{\infty}\left[U\left(x(p_I+p_L)-q\right)-\zeta\right]f(x)dx\right]\right]\\=&1+(1-\alpha)^{-1}\left[-F\left(\frac{O\cdot q-\zeta}{O(p_I+p_L)}\right)+F\left(\frac{\zeta+q\cdot U}{U(p_I+p_L)}\right)-1\right].\end{aligned}$$

Setting $\frac{\partial}{\partial\zeta}P_\alpha(q,\zeta)=0$, we find:

$$1+(1-\alpha)^{-1}F\left(\frac{\zeta+q\cdot U}{U(p_I+p_L)}\right)-(1-\alpha)^{-1}=(1-\alpha)^{-1}F\left(\frac{O\cdot q-\zeta}{O(p_I+p_L)}\right).\tag{44}$$

$$\begin{aligned}\frac{\partial}{\partial q}P_\alpha(q,\zeta)=&\frac{\partial}{\partial q}\left[\zeta+(1-\alpha)^{-1}\left[\int_0^{\frac{O\cdot q-\zeta}{O(p_I+p_L)}}\left[O\left(q-x(p_I+p_L)\right)-\zeta\right]f(x)dx\right.\right.\\&\left.\left.+\int_{\frac{\zeta+q\cdot U}{U(p_I+p_L)}}^{\infty}\left[U\left(x(p_I+p_L)-q\right)-\zeta\right]f(x)dx\right]\right],\\\frac{\partial}{\partial q}P_\alpha(q,\zeta)=&(1-\alpha)^{-1}\frac{\partial}{\partial q}\int_0^{\frac{O\cdot q-\zeta}{O(p_I+p_L)}}\left[O\left(q-x(p_I+p_L)\right)-\zeta\right]f(x)dx\\&+(1-\alpha)^{-1}\frac{\partial}{\partial q}\int_{\frac{\zeta+q\cdot U}{U(p_I+p_L)}}^{\infty}\left[U\left(x(p_I+p_L)-q\right)-\zeta\right]f(x)dx.\end{aligned}\tag{45}$$

We first calculate the integral of the first term in (45),

$$\begin{aligned}&(1-\alpha)^{-1}\frac{\partial}{\partial q}\int_0^{\frac{O\cdot q-\zeta}{O(p_I+p_L)}}\left[O\left(q-x(p_I+p_L)\right)-\zeta\right]f(x)dx\\&\quad=(1-\alpha)^{-1}\frac{\partial}{\partial q}\left[\int_0^{\frac{O\cdot q-\zeta}{O(p_I+p_L)}}\left[O\left(q-x(p_I+p_L)\right)-\zeta\right]dF(x)\right]\\&\quad=(1-\alpha)^{-1}\frac{\partial}{\partial q}\left[\left(O\left(q-x(p_I+p_L)\right)-\zeta\right)F(x)\Big|_0^{\frac{O\cdot q-\zeta}{O(p_I+p_L)}}+O(p_I+p_L)\int_0^{\frac{O\cdot q-\zeta}{O(p_I+p_L)}}f(x)dx\right]\\&\quad=(1-\alpha)^{-1}OF\left(\frac{O\cdot q-\zeta}{O(p_I+p_L)}\right),\end{aligned}\tag{46}$$

and then we calculate the integral of the second term in (45),

$$
\begin{aligned}
&(1-\alpha)^{-1}\frac{\partial}{\partial q}\int_{\frac{\zeta+q\cdot U}{U(p_I+p_L)}}^{\infty}\left[U\left(x(p_I+p_L)-q\right)-\zeta\right]f(x)dx \\
&\quad=(1-\alpha)^{-1}\frac{\partial}{\partial q}\left[\int_{\frac{\zeta+q\cdot U}{U(p_I+p_L)}}^{\infty}Ux(p_I+p_L)dF(x)+\int_{\frac{\zeta+q\cdot U}{U(p_I+p_L)}}^{\infty}\left[-Uq-\zeta\right]dF(x)\right] \\
&\quad=(1-\alpha)^{-1}\frac{\partial}{\partial q}\left[E(x)-U\int_{0}^{\frac{\zeta+q\cdot U}{U(p_I+p_L)}}x(p_I+p_L)dF(x)+(-Uq-\zeta)\left(1-F\left(\frac{\zeta+q\cdot U}{U(p_I+p_L)}\right)\right)\right] \\
&\quad=(1-\alpha)^{-1}\left[-U\left[1-F\left(\frac{\zeta+q\cdot U}{U(p_I+p_L)}\right)\right]\right],
\end{aligned}
\tag{47}
$$

combining the above two results, we obtain:

$$
\begin{aligned}
\frac{\partial}{\partial q}P_\alpha(q,\zeta)=&(1-\alpha)^{-1}\frac{\partial}{\partial q}\int_{0}^{\frac{O\cdot q-\zeta}{O(p_I+p_L)}}\left[O\left(q-x(p_I+p_L)\right)-\zeta\right]f(x)dx \\
&+(1-\alpha)^{-1}\frac{\partial}{\partial q}\int_{\frac{\zeta+q\cdot U}{U(p_I+p_L)}}^{\infty}\left[U\left(x(p_I+p_L)-q\right)-\zeta\right]f(x)dx \\
=&(1-\alpha)^{-1}OF\left(\frac{O\cdot q-\zeta}{O(p_I+p_L)}\right)+(1-\alpha)^{-1}\left[-U\left[1-F\left(\frac{\zeta+q\cdot U}{U(p_I+p_L)}\right)\right]\right] \\
=&(1-\alpha)^{-1}\left[OF\left(\frac{O\cdot q-\zeta}{O(p_I+p_L)}\right)+UF\left(\frac{O\cdot q-\zeta}{O(p_I+p_L)}\right)-U\left[F\left(\frac{O\cdot q-\zeta}{O(p_I+p_L)}\right)\right.\right. \\
&\left.\left.+1-F\left(\frac{\zeta+q\cdot U}{U(p_I+p_L)}\right)\right]\right] \\
=&(1-\alpha)^{-1}\left[(O+U)F\left(\frac{O\cdot q-\zeta}{O(p_I+p_L)}\right)-U(1-\alpha)\right].
\end{aligned}
\tag{48}
$$

Setting $\frac{\partial}{\partial q}P_\alpha(q,\zeta)=0$, we get:

$$
q^*=\frac{\zeta}{O}+(p_I+p_L)F^{-1}\left[(1-\alpha)\frac{U}{O+U}\right]. \tag{49}
$$

We also find that

$$
(1-\alpha)^{-1}F\left(\frac{O\cdot q-\zeta}{O(p_I+p_L)}\right)=\frac{U}{O+U}. \tag{50}
$$

According to the (44) and (50), we derive:

$$
\begin{aligned}
1+(1-\alpha)^{-1}F\left(\frac{\zeta+q\cdot U}{U(p_I+p_L)}\right)-(1-\alpha)^{-1}&=\frac{U}{O+U}, \\
(1-\alpha)+F\left(\frac{\zeta+q\cdot U}{U(p_I+p_L)}\right)-1&=(1-\alpha)\frac{U}{O+U}, \\
q^*=(p_I+p_L)F^{-1}\left[\frac{\alpha O+U}{O+U}\right]&-\frac{\zeta}{U}.
\end{aligned}
\tag{51}
$$

Therefore, according to the (49) and (51), we determine the optimal order quantity and maximum loss threshold as:

$$\begin{aligned} q^*_{RLT} &= (p_I + p_L)\frac{O}{O+U}F^{-1}\left[\frac{(1-\alpha)U}{O+U}\right] + (p_I + p_L)\frac{U}{O+U}F^{-1}\left[\frac{\alpha O+U}{O+U}\right], \\ \zeta^*_{RLT} &= (p_I + p_L)\frac{OU}{O+U}F^{-1}\left[\frac{\alpha O+U}{O+U}\right] - (p_I + p_L)\frac{UO}{O+U}F^{-1}\left[\frac{(1-\alpha)U}{O+U}\right]. \end{aligned} \tag{52}$$

Case 3: $Oq \leq \zeta$
If the ζ is greater than Oq, then equation (21) becomes:

$$P_\alpha(q,\zeta) = (1-\alpha)^{-1}\frac{\partial}{\partial q}\left[\int_{\frac{\zeta+q\cdot U}{U(p_I+p_L)}}^{\infty}\left[U\left(x(p_I+p_L)-q\right)-\zeta\right]f(x)dx\right]. \tag{53}$$

Carrying out analysis similar to Case 2, we show that:

$$\begin{aligned} \frac{\partial}{\partial q}P_\alpha(q,\zeta) &= (1-\alpha)^{-1}\frac{\partial}{\partial q}\left[\int_{\frac{\zeta+q\cdot U}{U(p_I+p_L)}}^{\infty}\left[U(x(p_I+p_L)-q)-\zeta\right]f(x)dx\right] \\ &= U\left[F\left(\frac{\zeta+q\cdot U}{U(p_I+p_L)}\right)-1\right] < 0. \end{aligned} \tag{54}$$

Therefore, there is no optimal order quantity for this case. □

C Proof of proposition 3

C.1 Proof of case 1

Proof. We first compute the sensitivity analysis of the optimal order quantity for the risk neutral retailer who offers layaway.

$$q^*_{\bar{R}L} = (p_I + p_L)F^{-1}\left(\frac{U}{O+U}\right).$$

$$\begin{aligned}
\frac{\partial q^*_{\bar{R}L}}{\partial r} =& (p_I+p_L)\frac{\partial}{\partial r}\left[F^{-1}\left(\frac{U}{O+U}\right)\right]+\left[F^{-1}\left(\frac{U}{O+U}\right)\right]\frac{\partial}{\partial r}(p_I+p_L)\\
=&(p_I+p_L)\frac{1}{f\left(F^{-1}\left(\frac{U}{O+U}\right)\right)}\frac{O}{(O+U)^2}\cdot\left[p'_I+r\cdot\left(\frac{\left(-\frac{1}{2\bar{v}}(1-\frac{r}{2\bar{b}})-(1-\frac{r}{2\bar{v}})\frac{1}{2\bar{b}}\right)(p_I+p_L)}{(p_I+p_L)^2}\right.\right.\\
&\left.-\frac{\left(-\frac{1}{2\bar{v}}(1-\frac{r}{2\bar{b}})-(1-\frac{r}{2\bar{v}})\frac{1}{2\bar{b}}-\left(\frac{1}{2\bar{v}\theta}(\frac{r}{2\bar{b}})-(1-\frac{r+f_s}{2\bar{v}\theta})(\frac{1}{2\bar{b}})\right)(\frac{L-1}{L})\right)\cdot p_I}{(p_I+p_L)^2}\right)\\
&+(f_s+\delta\cdot f_c+(1-\delta)\cdot r-\delta\cdot\mathfrak{s})\cdot\left(\frac{-\left(\frac{1}{2\bar{v}\theta}(\frac{r}{2\bar{b}})-(1-\frac{r+f_s}{2\bar{v}\theta})(\frac{1}{2\bar{b}})\right)(\frac{L-1}{L})\cdot(p_I+p_L)}{(p_I+p_L)^2}\right.\\
&\left.\left.-\frac{\left(-\frac{1}{2\bar{v}}(1-\frac{r}{2\bar{b}})-(1-\frac{r}{2\bar{v}})\frac{1}{2\bar{b}}-\left(\frac{1}{2\bar{v}\theta}(\frac{r}{2\bar{b}})-(1-\frac{r+f_s}{2\bar{v}\theta})(\frac{1}{2\bar{b}})\right)(\frac{L-1}{L})\right)\cdot p_L}{(p_I+p_L)^2}\right)+(1-\delta)p'_L\right]\\
&+\left[F^{-1}\left(\frac{U}{O+U}\right)\right]\cdot\left[-\frac{1}{2\bar{v}}(1-\frac{r}{2\bar{b}})-(1-\frac{r}{2\bar{v}})\frac{1}{2\bar{b}}-\left(\frac{1}{2\bar{v}\theta}(\frac{r}{2\bar{b}})-(1-\frac{r+f_s}{2\bar{v}\theta})(\frac{1}{2\bar{b}})\right)\frac{L-1}{L}\right],
\end{aligned}$$

$$\frac{\partial q^*_{\bar{R}L}}{\partial \alpha} = (p_I+p_L)\frac{\partial}{\partial \alpha}\left[F^{-1}\left(\frac{U}{O+U}\right)\right]=0,$$

$$\frac{\partial q^*_{\bar{R}L}}{\partial \mathfrak{u}} = (p_I+p_L)\frac{\partial}{\partial \mathfrak{u}}\left[F^{-1}\left(\frac{U}{O+U}\right)\right]=(p_I+p_L)\frac{O}{(O+U)^2}\frac{O+V}{f\left(F^{-1}\left(\frac{U}{O+U}\right)\right)}>0,$$

$$\frac{\partial q^*_{\bar{R}L}}{\partial c} = (p_I+p_L)\frac{\partial}{\partial c}\left[F^{-1}\left(\frac{U}{O+U}\right)\right]=-(p_I+p_L)\frac{1}{(O+U)}\frac{1}{f\left(F^{-1}\left(\frac{U}{O+U}\right)\right)}<0,$$

$$\frac{\partial q^*_{\bar{R}L}}{\partial \mathfrak{s}} = (p_I+p_L)\frac{\partial}{\partial \mathfrak{s}}\left[F^{-1}\left(\frac{U}{O+U}\right)\right]=-(p_I+p_L)\frac{Op'_L\delta}{(O+U)^2}\frac{1}{f\left(F^{-1}\left(\frac{U}{O+U}\right)\right)}<0,$$

$$\begin{aligned}
\frac{\partial q^*_{\bar{R}L}}{\partial f_s} =&(p_I+p_L)\frac{\partial}{\partial f_s}\left[F^{-1}\left(\frac{U}{O+U}\right)\right]+\left[F^{-1}\left(\frac{U}{O+U}\right)\right]\frac{\partial}{\partial f_s}(p_I+p_L)\\
=&(p_I+p_L)\frac{1}{f\left(F^{-1}\left(\frac{U}{O+U}\right)\right)}\frac{O}{(O+U)^2}\cdot\left[p'_L-\frac{1}{2\bar{v}\theta}(\frac{r}{2\bar{b}})(\frac{L-1}{L})\frac{(f_s+\delta\cdot(f_c-r-\mathfrak{s}))\,P_I}{(p_I+p_L)^2}\right]\\
&-\left[F^{-1}\left(\frac{U}{O+U}\right)\right]\frac{1}{2\bar{v}\theta}(\frac{r}{2\bar{b}})(\frac{L-1}{L}),
\end{aligned}$$

$$\frac{\partial q^*_{\bar{R}L}}{\partial f_c} = (p_I+p_L)\frac{\partial}{\partial f_c}\left[F^{-1}\left(\frac{U}{O+U}\right)\right]=(p_I+p_L)\frac{1}{f\left(F^{-1}(\frac{U}{O+U})\right)}\frac{Op'_L\cdot\delta}{(O+U)^2}>0,$$

$$\frac{\partial q^*_{\bar{R}L}}{\partial \delta} = (p_I + p_L)\frac{\partial}{\partial \delta}\left[F^{-1}\left(\frac{U}{O+U}\right)\right] = -(p_I + p_L)\frac{1}{f\left(F^{-1}\left(\frac{U}{O+U}\right)\right)}\frac{O\left(p'_L\cdot(r-f_c)+p'_L\cdot\mathfrak{s}\right)}{(O+U)^2} < 0.$$

According to the above analysis, we find that the optimal order quantity of a risk-neutral retailer, $q^*_{\bar{R}L}$ increases with $\mathfrak{u}$ and f_c, and decreases with c, $\mathfrak{s}$ and δ. □

C.2 Proof of case 2

Proof. In this subsection, we compute the sensitivity analysis of the optimal order quantity for the risk averse retailer who offers layaway and chooses the net loss as the loss function.

$$q^*_{RLN} = (p_I + p_L)\frac{O+V}{O+U}F^{-1}\left[\frac{(1-\alpha)U}{O+U}\right] + (p_I + p_L)\frac{U-V}{O+U}F^{-1}\left[\frac{\alpha O+U}{O+U}\right].$$

Setting $a_1 = \frac{(1-\alpha)U}{O+U}$, $a_2 = \frac{\alpha O+U}{O+U}$, we get:

$$\begin{aligned}\frac{\partial q^*_{RLN}}{\partial \alpha} &= (p_I + p_L)\frac{\partial}{\partial \alpha}\left[\frac{O+V}{O+U}F^{-1}(a_1) + \frac{U-V}{O+U}F^{-1}(a_2)\right]\\ &= (p_I + p_L)\left[-\frac{(O+V)U}{(O+U)^2}\frac{1}{f\left(F^{-1}(a_1)\right)} + \frac{(U-V)O}{(O+U)^2}\frac{1}{f(F^{-1}\left(a_2\right))}\right],\end{aligned}$$

$$\begin{aligned}\frac{\partial q^*_{RLN}}{\partial r} =& \left[\frac{O+V}{O+U}F^{-1}(a_1) + \frac{U-V}{O+U}F^{-1}(a_2)\right]\frac{\partial}{\partial r}(p_I + p_L)\\ &+ (p_I + p_L)\frac{\partial}{\partial r}\left[\frac{O+V}{O+U}F^{-1}(a_1) + \frac{U-V}{O+U}F^{-1}(a_2)\right]\\ =& \left[\frac{O+V}{O+U}F^{-1}(a_1) + \frac{U-V}{O+U}F^{-1}(a_2)\right]\\ &\cdot\left[-\frac{1}{2\bar{v}}(1-\frac{r}{2\bar{b}}) - (1-\frac{r}{2\bar{v}})\frac{1}{2\bar{b}} - \frac{1}{2\bar{v}\theta}(\frac{r}{2\bar{b}})(\frac{L-1}{L}) + (1-\frac{r+f_s}{2\bar{v}\theta})(\frac{1}{2\bar{b}})(\frac{L-1}{L})\right]\\ &+ (p_I + p_L)\cdot\left(\frac{U-V}{(O+U)^2}\left(F^{-1}(a_1) - F^{-1}(a_2)\right) + \frac{1}{f\left(F^{-1}(a_1)\right)}\frac{(1-\alpha)O(O+V)}{(O+U)^3}\right.\\ &\left.+\frac{1}{f(F^{-1}(a_2))}\frac{(1-\alpha)O(U-V)}{(O+U)^3}\right)\cdot\left[p'_I + r\cdot(\frac{(-\frac{1}{2\bar{v}}(1-\frac{r}{2\bar{b}}) - (1-\frac{r}{2\bar{v}})\frac{1}{2\bar{b}})(p_I+p_L)}{(p_I+p_L)^2}\right.\\ &\left.-\frac{(-\frac{1}{2\bar{v}}(1-\frac{r}{2\bar{b}}) - (1-\frac{r}{2\bar{v}})\frac{1}{2\bar{b}} - (\frac{1}{2\bar{v}\theta}(\frac{r}{2\bar{b}}) - (1-\frac{r+f_s}{2\bar{v}\theta})(\frac{1}{2\bar{b}}))(\frac{L-1}{L}))\cdot p_I}{(p_I+p_L)^2}\right)\\ &+ (f_s + \delta\cdot f_c + (1-\delta)\cdot r - \delta\cdot\mathfrak{s})\cdot\left(\frac{-(\frac{1}{2\bar{v}\theta}(\frac{r}{2\bar{b}}) - (1-\frac{r+f_s}{2\bar{v}\theta})(\frac{1}{2\bar{b}}))(\frac{L-1}{L}))\cdot(p_I+p_L)}{(p_I+p_L)^2}\right.\\ &\left.-\frac{-\frac{1}{2\bar{v}}(1-\frac{r}{2\bar{b}}) - (1-\frac{r}{2\bar{v}})\frac{1}{2\bar{b}} - (\frac{1}{2\bar{v}\theta}(\frac{r}{2\bar{b}}) - (1-\frac{r+f_s}{2\bar{v}\theta})(\frac{1}{2\bar{b}}))(\frac{L-1}{L}))\cdot p_L}{(p_I+p_L)^2}\right) + (1-\delta)p'_L\Bigg],\end{aligned}$$

$$\frac{\partial q^*_{RLN}}{\partial \mathfrak{u}} = (p_I + p_L)\frac{\partial}{\partial \mathfrak{u}}\left[\frac{O+V}{O+U}F^{-1}(a_1) + \frac{U-V}{O+U}F^{-1}(a_2)\right]$$
$$= (p_I + p_L)\left[\frac{O+V}{(O+U)^2}\left(F^{-1}(a_2) - F^{-1}(a_1)\right) + \frac{(1-\alpha)O}{(O+U)^3}\left(\frac{O+V}{f\left(F^{-1}(a_1)\right)} + \frac{U-V}{f\left(F^{-1}(a_2)\right)}\right)\right] > 0,$$

$$\frac{\partial q^*_{RLN}}{\partial c} = (p_I + p_L)\frac{\partial}{\partial c}\left[\frac{O+V}{O+U}F^{-1}(a_1) + \frac{U-V}{O+U}F^{-1}(a_2)\right]$$
$$= (p_I + p_L)\left[-\frac{(O+V)(1-\alpha)}{(O+U)^2}\frac{1}{f\left(F^{-1}(a_1)\right)} - \frac{(U-V)(1-\alpha)}{(O+U)^2}\frac{1}{f\left(F^{-1}(a_2)\right)}\right] < 0,$$

$$\frac{\partial q^*_{RLN}}{\partial \mathfrak{s}} = (p_I + p_L)\frac{\partial}{\partial \mathfrak{s}}\left[\frac{O+V}{O+U}F^{-1}(a_1) + \frac{U-V}{O+U}F^{-1}(a_2)\right]$$
$$= (p_I + p_L)[\frac{(U-V)(1-p'_L\delta)}{(O+U)^2}\left(F^{-1}(a_1) - F^{-1}(a_2)\right)$$
$$- \frac{O+V}{(O+U)}\frac{1}{f(F^{-1}(a_1))}\left[\frac{(1-\alpha)p'_L\delta}{(O+U)} + \frac{U(1-\alpha)p'_L\delta}{(O+U)^2}\right]$$
$$- \frac{U-V}{(O+U)}\frac{1}{f(F^{-1}(a_2))}\left[\frac{(1-\alpha)(U+Op'_L\delta)}{(O+U)^2}\right] < 0,$$

$$\frac{\partial q^*_{RLN}}{\partial f_s} = (p_I + p_L)\frac{\partial}{\partial f_s}\left[\frac{O+V}{O+U}F^{-1}(a_1) + \frac{U-V}{O+U}F^{-1}(a_2)\right]$$
$$+ \left[\frac{O+V}{O+U}F^{-1}(a_1) + \frac{U-V}{O+U}F^{-1}(a_2)\right]\frac{\partial}{\partial f_s}(p_I + p_L)$$
$$= (p_I + p_L)\left[\frac{(U-V)}{(O+U)^2}\left(F^{-1}(a_1) - F^{-1}(a_2)\right) + \frac{O(1-\alpha)}{(O+U)^3}\left(\frac{O+V}{f(F^{-1}(a_1))} + \frac{U-V}{f(F^{-1}(a_2))}\right)\right]\cdot$$
$$\left[p'_L - \frac{1}{2\bar{v}\theta}(\frac{r}{2\bar{b}})(\frac{L-1}{L})\frac{(f_s + \delta\cdot(f_c - r - \mathfrak{s}))\,P_I}{(p_I + p_L)^2}\right]$$
$$- \left[\frac{O+V}{O+U}F^{-1}(a_1) + \frac{U-V}{O+U}F^{-1}(a_2)\right]\frac{1}{2\bar{v}\theta}(\frac{r}{2\bar{b}})(\frac{L-1}{L}),$$

$$\frac{\partial q^*_{RLN}}{\partial f_c} = (p_I + p_L)\frac{\partial}{\partial f_c}\left[\frac{O+V}{O+U}F^{-1}(a_1) + \frac{U-V}{O+U}F^{-1}(a_2)\right]$$
$$= (p_I + p_L)\left[\frac{p'_L\delta(U-V)}{(O+U)^2}\left(F^{-1}(a_1) - F^{-1}(a_2)\right)\right.$$
$$\left. + \frac{O(1-\alpha)p'_L\delta}{(O+U)^3}(\frac{O+V}{f\left(F^{-1}(a_1)\right)} + \frac{U-V}{f\left(F^{-1}(a_2)\right)})\right],$$

$$
\begin{aligned}
\frac{\partial q^*_{RLN}}{\partial \delta} =&(p_I+p_L)\frac{\partial}{\partial \delta}\left[\frac{O+V}{O+U}F^{-1}(a_1)+\frac{U-V}{O+U}F^{-1}(a_2)\right]\\
=&(p_I+p_L)\left[\frac{(U-V)(p'_L\cdot(r-f_c)+p'_L\cdot\mathfrak{s})}{(O+U)^2}\left(F^{-1}(a_2)-F^{-1}(a_1)\right)\right.\\
&-\frac{O+V}{(O+U)}\frac{1}{f(F^{-1}(a_1))}\frac{O(1-\alpha)\left(p'_L\cdot(r-f_c)+p'_L\cdot\mathfrak{s}\right)}{(O+U)^2}\\
&\left.-\frac{U-V}{(O+U)}\frac{1}{f(F^{-1}(a_2))}\frac{O(1-\alpha)(p'_L\cdot(r-f_c)+p'_L\cdot\mathfrak{s})}{(O+U)^2}\right].
\end{aligned}
$$

According to the above analysis, we find that the optimal order quantity of a risk-averse retailer who considers the net loss as the loss function, q^*_{RLN} increases with $\mathfrak{u}$, and decreases with c and $\mathfrak{s}$. □

C.3 Proof of case 3

Proof. In this subsection, we compute the sensitivity analysis of the optimal order quantity for the risk averse retailer who offers layaway and chooses the total loss as the loss function.

$$
q^*_{RLT}=(p_I+p_L)\frac{O}{O+U}F^{-1}\left[\frac{(1-\alpha)U}{O+U}\right]+(p_I+p_L)\frac{U}{O+U}F^{-1}\left[\frac{\alpha O+U}{O+U}\right],
$$

$$
\begin{aligned}
\frac{\partial q^*_{RLT}}{\partial r} =&\left[\frac{O}{O+U}F^{-1}(a_1)+\frac{U}{O+U}F^{-1}(a_2)\right]\frac{\partial}{\partial r}(p_I+p_L)\\
&+(p_I+p_L)\frac{\partial}{\partial r}\left[\frac{O}{O+U}F^{-1}(a_1)+\frac{U}{O+U}F^{-1}(a_2)\right]\\
=&\left[\frac{O}{O+U}F^{-1}(a_1)+\frac{U}{O+U}F^{-1}(a_2)\right]\left[-\frac{1}{2\bar{v}}(1-\frac{r}{2\bar{b}})-(1-\frac{r}{2\bar{v}})\frac{1}{2\bar{b}}-\frac{1}{2\bar{v}\theta}(\frac{r}{2\bar{b}})(\frac{L-1}{L})\right.\\
&\left.+(1-\frac{r+f_s}{2\bar{v}\theta})(\frac{1}{2\bar{b}})(\frac{L-1}{L})\right]+(p_I+p_L)\cdot\\
&\left[\left(\frac{O}{(O+U)^2}(F^{-1}(a_2)-F^{-1}(a_1))+\frac{1}{f(F^{-1}(a_1))}\frac{(1-\alpha)O^2}{(O+U)^3}+\frac{1}{f(F^{-1}(a_2))}\frac{(1-\alpha)OU}{(O+U)^3}\right)\right.\\
&\cdot\left[p'_I+r\cdot\left(\frac{\left(-\frac{1}{2\bar{v}}(1-\frac{r}{2\bar{b}})-(1-\frac{r}{2\bar{v}})\frac{1}{2\bar{b}}\right)(p_I+p_L)}{(p_I+p_L)^2}\right.\right.\\
&\left.\left.-\frac{\left[-\frac{1}{2\bar{v}}(1-\frac{r}{2\bar{b}})-(1-\frac{r}{2\bar{v}})\frac{1}{2\bar{b}}-\left(\frac{1}{2\bar{v}\theta}(\frac{r}{2\bar{b}})-(1-\frac{r+f_s}{2\bar{v}\theta})(\frac{1}{2\bar{b}})\right)(\frac{L-1}{L})\right]\cdot p_I}{(p_I+p_L)^2}\right)\right.\\
&+(f_s+\delta\cdot f_c+(1-\delta)\cdot r-\delta\cdot\mathfrak{s})\cdot\left(\frac{-(\frac{1}{2\bar{v}\theta}(\frac{r}{2\bar{b}})-(1-\frac{r+f_s}{2\bar{v}\theta})(\frac{1}{2\bar{b}}))(\frac{L-1}{L}))\cdot(p_I+p_L)}{(p_I+p_L)^2}\right.\\
&\left.\left.-\frac{-\frac{1}{2\bar{v}}(1-\frac{r}{2\bar{b}})-(1-\frac{r}{2\bar{v}})\frac{1}{2\bar{b}}-(\frac{1}{2\bar{v}\theta}(\frac{r}{2\bar{b}})-(1-\frac{r+f_s}{2\bar{v}\theta})(\frac{1}{2\bar{b}}))(\frac{L-1}{L}))\cdot p_L}{(p_I+p_L)^2}\right)+(1-\delta)p'_L\right],
\end{aligned}
$$

$$\begin{aligned}\frac{\partial q^*_{RLT}}{\partial \alpha} &= (p_I+p_L)\frac{\partial}{\partial \alpha}[\frac{O}{O+U}F^{-1}(\frac{(1-\alpha)U}{O+U})+\frac{U}{O+U}F^{-1}(\frac{\alpha O+U}{O+U})]\\ &= (p_I+p_L)\frac{OU}{(O+U)^2}\left[\frac{1}{f\left(F^{-1}(\frac{\alpha O+U}{O+U})\right)}-\frac{1}{f\left(F^{-1}(\frac{(1-\alpha)U}{O+U})\right)}\right],\end{aligned}$$

$$\begin{aligned}\frac{\partial q^*_{RLT}}{\partial \mathfrak{u}} &= (p_I+p_L)\frac{\partial}{\partial \mathfrak{u}}\left[\frac{O}{O+U}F^{-1}(a_1)+\frac{U}{O+U}F^{-1}(a_2)\right]\\ &= (p_I+p_L)\frac{O}{(O+U)^2}\left[F^{-1}(a_2)-F^{-1}(a_1)\right]+\frac{(1-\alpha)O}{(O+U)^3}\left[\frac{O}{f\left(F^{-1}(a_1)\right)}+\frac{U}{f\left(F^{-1}(a_2)\right)}\right]>0,\end{aligned}$$

$$\begin{aligned}\frac{\partial q^*_{RLT}}{\partial c} =&(p_I+p_L)\frac{\partial}{\partial c}\left[\frac{O}{O+U}F^{-1}(a_1)+\frac{U}{O+U}F^{-1}(a_2)\right]\\ =&(p_I+p_L)\left[-\frac{1}{(O+U)}\left(F^{-1}(a_2)-F^{-1}(a_1)\right)-\frac{O(1-\alpha)}{(O+U)^2}\frac{1}{f\left(F^{-1}(a_1)\right)}\right.\\ &\left.-\frac{U(1-\alpha)}{(O+U)^2}\frac{1}{f(F^{-1}(a_2))}\right]<0,\\ \frac{\partial q^*_{RLT}}{\partial \mathfrak{s}} =&(p_I+p_L)\frac{\partial}{\partial \mathfrak{s}}\left[\frac{O}{O+U}F^{-1}(a_1)+\frac{U}{O+U}F^{-1}(a_2)\right]\\ =&(p_I+p_L)\left[-\frac{U+Op'_L\delta}{(O+U)^2}\left(F^{-1}(a_2)-F^{-1}(a_1)\right)\right.\\ &-\frac{O}{(O+U)}\frac{1}{f\left(F^{-1}(a_1)\right)}\left[\frac{(1-\alpha)p'_L\delta}{(O+U)}+\frac{U(1-\alpha)p'_L\delta}{(O+U)^2}\right]\\ &\left.-\frac{U}{(O+U)}\frac{1}{f\left(F^{-1}(a_2)\right)}\left[\frac{(1-\alpha)(U+Op'_L\delta)}{(O+U)^2}\right]\right]<0,\end{aligned}$$

$$\begin{aligned}\frac{\partial q^*_{RLT}}{\partial f_s} =&(p_I+p_L)\frac{\partial}{\partial f_s}\left[\frac{O}{O+U}F^{-1}(a_1)+\frac{U}{O+U}F^{-1}(a_2)\right]\\ &+\left[\frac{O}{O+U}F^{-1}(a_1)+\frac{U}{O+U}F^{-1}(a_2)\right]\frac{\partial}{\partial f_s}(p_I+p_L)\\ =&(p_I+p_L)\left[\frac{O}{(O+U)^2}\left(F^{-1}(a_2)-F^{-1}(a_1)\right)+\frac{O(1-\alpha)}{(O+U)^3}\left(\frac{O}{f\left(F^{-1}(a_1)\right)}+\frac{U}{f\left(F^{-1}(a_2)\right)}\right)\right]\cdot\\ &\left[p'_L-\frac{1}{2\bar{v}\theta}(\frac{r}{2\bar{b}})(\frac{L-1}{L})\frac{(f_s+\delta\cdot(f_c-r-\mathfrak{s}))\,P_I}{(p_I+p_L)^2}\right]\\ &-[\frac{O}{O+U}F^{-1}(a_1)+\frac{U}{O+U}F^{-1}(a_2)]\frac{1}{2\bar{v}\theta}(\frac{r}{2\bar{b}})(\frac{L-1}{L}),\end{aligned}$$

$$
\begin{aligned}
\frac{\partial q^*_{RLT}}{\partial f_c} =&(p_I+p_L)\frac{\partial}{\partial f_c}\left[\frac{O}{O+U}F^{-1}(a_1)+\frac{U}{O+U}F^{-1}(a_2)\right]\\
=&(p_I+p_L)\left[\frac{Op'_L\cdot\delta}{(O+U)^2}(F^{-1}(a_2)-F^{-1}(a_1))\right.\\
&\left.+\frac{O}{(O+U)}\frac{1}{f\left(F^{-1}(a_1)\right)}\frac{O(1-\alpha)p'_L\cdot\delta}{(O+U)^2}+\frac{U}{(O+U)}\frac{1}{f\left(F^{-1}(a_2)\right)}\frac{O(1-\alpha)p'_L\cdot\delta}{(O+U)^2}\right]>0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial q^*_{RLT}}{\partial \delta} =&(p_I+p_L)\frac{\partial}{\partial \delta}\left[\frac{O}{O+U}F^{-1}(a_1)+\frac{U}{O+U}F^{-1}(a_2)\right]\\
=&(p_I+p_L)\left[-\frac{O\left(p'_L\cdot(r-f_c)+p'_L\cdot\mathfrak{s}\right)}{(O+U)^2}\left(F^{-1}(a_2)-F^{-1}(a_1)\right)\right.\\
&-\frac{O}{(O+U)}\frac{1}{f\left(F^{-1}(a_1)\right)}\frac{O(1-\alpha)\left(p'_L\cdot(r-f_c)+p'_L\cdot\mathfrak{s}\right)}{(O+U)^2}\\
&\left.-\frac{U}{(O+U)}\frac{1}{f\left(F^{-1}(a_2)\right)}\frac{O(1-\alpha)\left(p'_L\cdot(r-f_c)+p'_L\cdot\mathfrak{s}\right)}{(O+U)^2}\right]<0.
\end{aligned}
$$

According to the above analysis, we find that the optimal order quantity of a risk-averse retailer who considers the total loss as the loss function, q^*_{RLT} increases with $\mathfrak{u}$ and f_c, and decreases with c, $\mathfrak{s}$ and δ. □

C.4 Proof of case 4

Proof. In this subsection, we compute the sensitivity analysis of the optimal order quantity for the risk averse retailer who does not offers layaway and chooses the total loss as the loss function.

$$
q^*_{R\bar{L}T}=\frac{p_I\cdot(r+\mathfrak{u}-c)}{r+\mathfrak{u}+\mathfrak{s}}F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(c+\mathfrak{s})}{r+\mathfrak{u}+\mathfrak{s}}\right)+\frac{p_I(c+\mathfrak{s})}{r+\mathfrak{u}+\mathfrak{s}}F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right],
$$

$$
\begin{aligned}
\frac{\partial q^*_{R\bar{L}T}}{\partial r} =&F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right)\frac{1}{4\bar{v}\bar{b}(r+\mathfrak{s}+u)^2}\left[c\left(-r^2-2r(\mathfrak{s}+u)+2\bar{b}(2\bar{v}+\mathfrak{s}+u)+\right.\right.\\
&\left.2\bar{v}(\mathfrak{s}+u)\right)+2r^3+r^2(-2(\bar{v}+\bar{b})+3\mathfrak{s}+4u)+2r(\mathfrak{s}+u)(u-2(\bar{v}+\bar{b}))+4\bar{v}\mathfrak{s}\bar{b}-\\
&\left.2\bar{v}u(\mathfrak{s}+u)-2\bar{b}u(\mathfrak{s}+u)\right]-\frac{p_I\cdot(r+\mathfrak{u}-c)}{r+\mathfrak{u}+\mathfrak{s}}\frac{1}{f\left(F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right)\right)}\frac{(\alpha-1)(c+\mathfrak{s})}{(r+\mathfrak{s}+u)^2}+\\
&F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right]\frac{(c+\mathfrak{s})\left(r^2+2r(\mathfrak{s}+u)-2\bar{b}(2\bar{v}+\mathfrak{s}+u)-2\bar{v}(\mathfrak{s}+u)\right)}{4\bar{v}\bar{b}(r+\mathfrak{s}+u)^2}-\\
&\frac{p_I(c+\mathfrak{s})}{\mathfrak{s}+r+\mathfrak{u}}\frac{1}{f\left(F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right]\right)}\frac{(\alpha-1)(c+\mathfrak{s})}{(r+\mathfrak{s}+u)^2}
\end{aligned}
$$

$$\frac{\partial q^*_{R\bar{L}T}}{\partial \alpha} = \frac{p_I \cdot (r+\mathfrak{u}-c)}{r+\mathfrak{u}+\mathfrak{s}} \frac{1}{f\left(F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right)\right)} \frac{c+\mathfrak{s}}{r+\mathfrak{s}+u} - \frac{p_I(c+\mathfrak{s})}{r+\mathfrak{u}+\mathfrak{s}} \frac{1}{f\left(F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right]\right)} \frac{-c+r+u}{r+\mathfrak{s}+u}$$

$$\frac{\partial q^*_{R\bar{L}T}}{\partial \mathfrak{u}} = \frac{(c+\mathfrak{s})(r-2\bar{v})(r-2\bar{b})}{4\bar{v}\bar{b}(r+\mathfrak{s}+u)^2} F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right) - \frac{p_I \cdot (r+\mathfrak{u}-c)}{r+\mathfrak{u}+\mathfrak{s}} \frac{1}{f\left(F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right)\right)} \frac{(\alpha-1)(c+\mathfrak{s})}{(r+\mathfrak{s}+u)^2} + - \frac{(c+\mathfrak{s})\left(1-\frac{r}{2\bar{v}}\right)\left(1-\frac{r}{2\bar{b}}\right)}{(r+\mathfrak{s}+u)^2} F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right] - \frac{p_I(c+\mathfrak{s})}{r+\mathfrak{u}+\mathfrak{s}} \frac{1}{f\left(F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right]\right)} \frac{(\alpha-1)(c+\mathfrak{s})}{(r+\mathfrak{s}+u)^2}$$

$$\frac{\partial q^*_{R\bar{L}T}}{\partial c} = - \frac{\left(1-\frac{r}{2\bar{v}}\right)\left(1-\frac{r}{2\bar{b}}\right)}{r+\mathfrak{s}+u} F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right) + \frac{p_I \cdot (r+\mathfrak{u}-c)}{r+\mathfrak{u}+\mathfrak{s}} \frac{1}{f\left(F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right)\right)} \frac{\alpha-1}{r+\mathfrak{s}+u} + \frac{\left(1-\frac{r}{2\bar{v}}\right)\left(1-\frac{r}{2\bar{b}}\right)}{r+\mathfrak{s}+u} F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right] - \frac{p_I(c+\mathfrak{s})}{r+\mathfrak{u}+\mathfrak{s}} \frac{1}{f\left(F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right]\right)} \frac{1-\alpha}{r+\mathfrak{s}+u}$$

$$\frac{\partial q^*_{R\bar{L}T}}{\partial \mathfrak{s}} = - \frac{\left(1-\frac{r}{2\bar{v}}\right)\left(1-\frac{r}{2\bar{b}}\right)(-c+r+u)}{(r+\mathfrak{s}+u)^2} F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right) - \frac{p_I \cdot (r+\mathfrak{u}-c)}{r+\mathfrak{u}+\mathfrak{s}} \frac{1}{f\left(F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right)\right)} \frac{(\alpha-1)(c-r-u)}{(r+\mathfrak{s}+u)^2} - \frac{(r-2\bar{v})(r-2\bar{b})(c-r-u)}{4\bar{v}\bar{b}(r+\mathfrak{s}+u)^2} F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right] - \frac{p_I(c+\mathfrak{s})}{r+\mathfrak{u}+\mathfrak{s}} \frac{1}{f\left(F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right]\right)} \frac{(1-\alpha)(-c+r+u)}{(r+\mathfrak{s}+u)^2}$$

□

According to the above analysis the optimal order quantity, $q^*_{R\bar{L}T}$, change with respect to each model parameter is determined case by case.

C.5 Proof of case 5

Proof. In this subsection, we compute the sensitivity analysis of the optimal order quantity for the risk averse retailer who does not offers layaway and chooses the net loss as the loss function.

$$q^*_{R\bar{L}N} = \frac{p_I \mathfrak{u}}{\mathfrak{s}+r+\mathfrak{u}} F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right) + \frac{p_I(r+\mathfrak{s})}{\mathfrak{s}+r+\mathfrak{u}} F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right]$$

$$\begin{aligned}\frac{\partial q^*_{R\bar{L}N}}{\partial r} =& F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right)\frac{\mathfrak{u}\left(2\left(\mathfrak{u}+\mathfrak{s}\right)\left(-\bar{v}+r-\bar{b}\right)+4\bar{v}\bar{b}+3r^2-4r(\bar{v}+\bar{b})\right)}{4\bar{v}\bar{b}\left(\mathfrak{u}+\mathfrak{s}+r\right)^2} \\ &- \frac{p_I\mathfrak{u}}{\mathfrak{s}+r+\mathfrak{u}}\frac{1}{f\left(F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right)\right)}\frac{(\alpha-1)(c+\mathfrak{s})}{(r+\mathfrak{s}+u)^2} + F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right]\cdot \\ &\frac{2\left(\mathfrak{s}+r\right)^2\left(-\bar{v}+r-\bar{b}\right)+\mathfrak{u}\left(2\mathfrak{s}(-\bar{v}+r-\bar{b})+4\bar{v}\bar{b}+3r^2-4r(\bar{v}+\bar{b})\right)}{4\bar{v}\bar{b}\left(\mathfrak{u}+\mathfrak{s}+r\right)^2} - \\ &\frac{p_I(r+\mathfrak{s})}{\mathfrak{s}+r+\mathfrak{u}}\frac{1}{f\left(F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right]\right)}\frac{(\alpha-1)(c+\mathfrak{s})}{(r+\mathfrak{s}+u)^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial q^*_{R\bar{L}N}}{\partial \alpha} =& \frac{p_I\cdot\mathfrak{u}}{r+\mathfrak{u}+\mathfrak{s}}\frac{1}{f\left(F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right)\right)}\frac{c+\mathfrak{s}}{r+\mathfrak{s}+u} - \\ &\frac{p_I(r+\mathfrak{s})}{r+\mathfrak{u}+\mathfrak{s}}\frac{1}{f\left(F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right]\right)}\frac{-c+r+u}{r+\mathfrak{s}+u}\end{aligned}$$

$$\begin{aligned}\frac{\partial q^*_{R\bar{L}N}}{\partial \mathfrak{u}} =& \frac{(r-2\bar{v})(r+\mathfrak{s})(r-2\bar{b})}{4\bar{v}\bar{b}(r+\mathfrak{s}+u)^2}F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right) - \\ &\frac{p_I\cdot\mathfrak{u}}{r+\mathfrak{u}+\mathfrak{s}}\frac{1}{f\left(F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right)\right)}\frac{(\alpha-1)(c+\mathfrak{s})}{(r+\mathfrak{s}+u)^2} + \\ &-\frac{\left(1-\frac{r}{2\bar{v}}\right)(r+\mathfrak{s})\left(1-\frac{r}{2\bar{b}}\right)}{(r+\mathfrak{s}+u)^2}F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right] - \\ &\frac{p_I(r+\mathfrak{s})}{r+\mathfrak{u}+\mathfrak{s}}\frac{1}{f\left(F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right]\right)}\frac{(\alpha-1)(c+\mathfrak{s})}{(r+\mathfrak{s}+u)^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial q^*_{R\bar{L}N}}{\partial c} =& \frac{p_I\cdot\mathfrak{u}}{r+\mathfrak{u}+\mathfrak{s}}\frac{1}{f\left(F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right)\right)}\frac{\alpha-1}{r+\mathfrak{s}+u} - \\ &\frac{p_I(r+\mathfrak{s})}{r+\mathfrak{u}+\mathfrak{s}}\frac{1}{f\left(F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right]\right)}\frac{1-\alpha}{r+\mathfrak{s}+u}\end{aligned}$$

$$\frac{\partial q^*_{R\bar{L}N}}{\partial \mathfrak{s}} = -\frac{u\left(1-\frac{r}{2\bar{v}}\right)\left(1-\frac{r}{2\bar{b}}\right)}{(r+\mathfrak{s}+u)^2}F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right) - \frac{p_I\cdot\mathfrak{u}}{r+\mathfrak{u}+\mathfrak{s}}\frac{1}{f\left(F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right)\right)}\frac{(\alpha-1)(c-r-u)}{(r+\mathfrak{s}+u)^2} - \frac{u(r-2\bar{v})(r-2\bar{b})}{4\bar{v}\bar{b}(r+\mathfrak{s}+u)^2}F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right] - \frac{p_I(r+\mathfrak{s})}{r+\mathfrak{u}+\mathfrak{s}}\frac{1}{f\left(F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right]\right)}\frac{(1-\alpha)(-c+r+u)}{(r+\mathfrak{s}+u)^2}$$

□

According to the above analysis the optimal order quantity, $q^*_{R\bar{L}N}$, change with respect to each model parameter is determined case by case.

D Proof of Corollary 1

Proof.

$$\min_q P_\alpha(q,\zeta) = \zeta + (1-\alpha)^{-1}\int_{x\in\mathbb{R}}\left[-\widehat{\pi}_{NL}(x|q)-\zeta\right]^+ f(x)dx. \tag{55}$$

We consider three cases so as to evaluate the problem of (55).

Case 1: $\zeta \le -(r-c)q$

If ζ is less than or equal to $-(r-c)q$, then equation (55) becomes:

$$\begin{aligned} P_\alpha(q,\zeta) =& \zeta + (1-\alpha)^{-1}\left[\int_0^{\frac{q}{p_I}}\left[-(p_I\cdot x\cdot r-(q-p_I\cdot x)\cdot\mathfrak{s}-c\cdot q)-\zeta\right]f(x)dx\right. \\ &\left.+\int_{\frac{q}{p_I}}^{\infty}\left[-\left(q\cdot r-(p_I\cdot x-q)\cdot\mathfrak{u}-c\cdot q\right)-\zeta\right]f(x)dx\right]. \end{aligned} \tag{56}$$

Given (56), we now compute the first-order derivative of $P_\alpha(q,\zeta)$, i.e., $\frac{\partial}{\partial\zeta}P_\alpha(q,\zeta)=0$ and $\frac{\partial}{\partial q}P_\alpha(q,\zeta)=0$. We get:

$$\begin{aligned} q^* &= p_I F^{-1}\left(\frac{r+\mathfrak{u}-c}{\mathfrak{s}+r+\mathfrak{u}}\right), \\ \zeta^* &= -(r-c)p_I F^{-1}\left(\frac{r+\mathfrak{u}-c}{\mathfrak{s}+r+\mathfrak{u}}\right). \end{aligned} \tag{57}$$

Case 2: $-(r-c)q < \zeta < (\mathfrak{s}+c)q$

If ζ is greater than $-(r-c)q$, and less than $(\mathfrak{s}+c)q$, and $-\widehat{\pi}(x|q)=\zeta$, then equation (55) becomes:

$$\begin{aligned} P_\alpha(q,\zeta) =& \zeta + (1-\alpha)^{-1}\left[\int_0^{\frac{qc+q\mathfrak{s}-\zeta}{p_I(\mathfrak{s}+r)}}\left[-(p_I\cdot x\cdot r-(q-p_I\cdot x)\cdot\mathfrak{s}-c\cdot q)-\zeta\right]f(x)dx\right. \\ &\left.+\int_{\frac{qr+q\mathfrak{u}+\zeta-cq}{p_I\mathfrak{u}}}^{\infty}\left[-\left(q\cdot r-(p_I\cdot x-q)\cdot\mathfrak{u}-c\cdot q\right)-\zeta\right]f(x)dx\right]. \end{aligned} \tag{58}$$

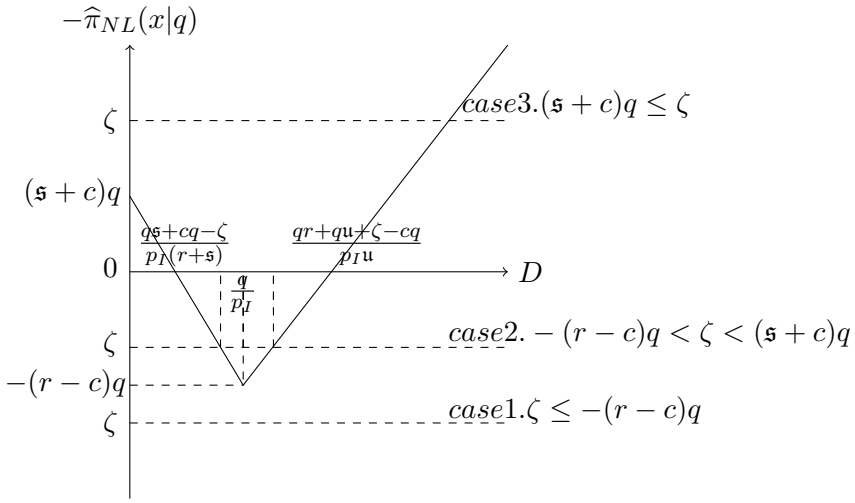


Figure 8: Three cases in minimization of net loss CVaR without layaway.

Given (56), we now compute the first-order derivative of $P_\alpha(q,\zeta)$:

$$\begin{aligned}\frac{\partial}{\partial\zeta}P_\alpha(q,\zeta) =&\frac{\partial}{\partial\zeta}\left(\zeta+(1-\alpha)^{-1}\left[\int_0^{\frac{qc+q\mathfrak{s}-\zeta}{p_I(\mathfrak{s}+r)}}\left[-\left(p_I{\cdot}x{\cdot}r-(q-p_I{\cdot}x){\cdot}\mathfrak{s}-c{\cdot}q\right)\right.\right.\right.\\ &\left.\left.\left.-\zeta\right]f(x)dx+\int_{\frac{qr+q\mathfrak{u}+\zeta-cq}{p_I\mathfrak{u}}}^{\infty}\left[-\left(q{\cdot}r-(p_I{\cdot}x-q){\cdot}\mathfrak{u}-c{\cdot}q\right)-\zeta\right]f(x)dx\right]\right)\\ =&1+(1-\alpha)^{-1}\left[-F\left(\frac{q\mathfrak{s}-\zeta+cq}{p_I(r+\mathfrak{s})}\right)+F\left(\frac{qr+q\mathfrak{u}+\zeta-cq}{p_I\mathfrak{u}}\right)-1\right],\end{aligned}$$

Setting $\frac{\partial}{\partial\zeta}P_\alpha(q,\zeta)=0$, we find that

$$1+(1-\alpha)^{-1}\left[F\left(\frac{qr+q\mathfrak{u}+\zeta-cq}{p_I\mathfrak{u}}\right)-1\right]=(1-\alpha)^{-1}F\left(\frac{q\mathfrak{s}-\zeta+cq}{p_I(r+\mathfrak{s})}\right). \tag{59}$$

$$\begin{aligned}\frac{\partial}{\partial q}P_\alpha(q,\zeta) =&(1-\alpha)^{-1}\left[\frac{\partial}{\partial q}\int_0^{\frac{q\mathfrak{s}-\zeta+cq}{p_I(r+\mathfrak{s})}}\left[-\left(p_I{\cdot}x{\cdot}r-(q-p_I{\cdot}x){\cdot}\mathfrak{s}-cq\right)\right.\right.\\ &\left.\left.-\zeta\right]f(x)dx+\frac{\partial}{\partial q}\int_{\frac{qr+q\mathfrak{u}+\zeta-cq}{p_I\mathfrak{u}}}^{\infty}\left[-\left(q{\cdot}r-(p_I{\cdot}x-q){\cdot}\mathfrak{u}-cq\right)-\zeta\right]f(x)dx\right].\end{aligned} \tag{60}$$

We first calculate the integral of the first term in (60),

$$\begin{aligned}
&(1-\alpha)^{-1}\frac{\partial}{\partial q}\int_0^{\frac{q\mathfrak{s}-\zeta+cq}{p_I(r+\mathfrak{s})}}\left[-(p_I{\cdot}x{\cdot}r-(q-p_I{\cdot}x){\cdot}\mathfrak{s}-cq)-\zeta\right]f(x)dx\\
=&(1-\alpha)^{-1}\frac{\partial}{\partial q}\int_0^{\frac{q\mathfrak{s}-\zeta+cq}{p_I(r+\mathfrak{s})}}\left[-\left(p_I{\cdot}x{\cdot}r-(q-p_I{\cdot}x){\cdot}\mathfrak{s}-cq\right)-\zeta\right]dF(d)\\
=&(1-\alpha)^{-1}(\mathfrak{s}+c)F\left(\frac{q\mathfrak{s}-\zeta+cq}{p_I(r+\mathfrak{s})}\right),
\end{aligned}\tag{61}$$

and then we calculate the integral of the second term in (60),

$$\begin{aligned}
&(1-\alpha)^{-1}\frac{\partial}{\partial q}\int_{\frac{qr+q\mathfrak{u}+\zeta-cq}{p_I\mathfrak{u}}}^{\infty}\left[-\left(q{\cdot}r-(p_I{\cdot}x-q)\cdot\mathfrak{u}-cq\right)-\zeta\right]f(x)dx\\
=&(1-\alpha)^{-1}\frac{\partial}{\partial q}\int_{\frac{qr+q\mathfrak{u}+\zeta-cq}{p_I\mathfrak{u}}}^{\infty}\left[-\left(q{\cdot}r-(p_I{\cdot}x-q){\cdot}\mathfrak{u}-c{\cdot}q\right)-\zeta\right]dF(x)\\
=&(1-\alpha)^{-1}\left[-(r+\mathfrak{u}-c)\left(1-F\left(\frac{qr+q\mathfrak{u}+\zeta-cq}{p_I\mathfrak{u}}\right)\right)\right],
\end{aligned}\tag{62}$$

combining the above three results, we obtain:

$$\begin{aligned}
\frac{\partial}{\partial q}P_\alpha(q,\zeta)=&(1-\alpha)^{-1}\frac{\partial}{\partial q}\int_0^{\frac{q\mathfrak{s}-\zeta+cq}{p_I(r+\mathfrak{s})}}\left[-\left(p_I{\cdot}x{\cdot}r-(q-p_I{\cdot}x){\cdot}\mathfrak{s}-cq\right)\right.\\
&\left.-\zeta\right]f(x)dx+\frac{\partial}{\partial q}\int_{\frac{qr+q\mathfrak{u}+\zeta-cq}{p_I\mathfrak{u}}}^{\infty}\left[-(q{\cdot}r-(p_I{\cdot}x-q){\cdot}\mathfrak{u}-cq)-\zeta\right]f(x)dx]\\
=&(1-\alpha)^{-1}\left[(\mathfrak{s}+r+\mathfrak{u})F\left(\frac{q\mathfrak{s}-\zeta+cq}{p_I(r+\mathfrak{s})}\right)-(r+\mathfrak{u}-c)(1-\alpha)\right],
\end{aligned}\tag{63}$$

Setting $\frac{\partial}{\partial q}P_\alpha(q,\zeta)=0$, we get:

$$q^*=\frac{\zeta}{\mathfrak{s}+c}+\frac{p_I(r+\mathfrak{s})}{\mathfrak{s}+c}F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right].\tag{64}$$

We also find:

$$0=(1-\alpha)^{-1}\left[(\mathfrak{s}+r+\mathfrak{u})F\left(\frac{q\mathfrak{s}-\zeta+cq}{p_I(r+\mathfrak{s})}\right)-(r+\mathfrak{u}-c)(1-\alpha)\right],$$

$$(\mathfrak{s}+r+\mathfrak{u})F\left(\frac{q\mathfrak{s}-\zeta+cq}{p_I(r+\mathfrak{s})}\right)=(r+\mathfrak{u}-c)(1-\alpha),\tag{65}$$

According to the (59) and (65), we derive:

$$(1-\alpha)+\left(F\left(\frac{qr+q\mathfrak{u}+\zeta-cq}{p_I\mathfrak{u}}\right)-1\right)=\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}},$$

$$F\left(\frac{qr+q\mathfrak{u}+\zeta-cq}{p_I\mathfrak{u}}\right)-\alpha=\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}},$$

$$F\left(\frac{qr+q\mathfrak{u}+\zeta-cq}{p_I\mathfrak{u}}\right)=\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}},$$

$$q^*=\frac{p_I\mathfrak{u}}{r+\mathfrak{u}-c}F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right)-\frac{\zeta}{r+\mathfrak{u}-c}, \tag{66}$$

$$\zeta=(p_I\mathfrak{u})F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right)-(r+\mathfrak{u}-c)q^*, \tag{67}$$

$$q^*=\frac{\zeta}{\mathfrak{s}+c}+\frac{p_I(r+\mathfrak{s})}{\mathfrak{s}+c}F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right], \tag{68}$$

$$\begin{aligned}q^*=&\frac{p_I\mathfrak{u}}{\mathfrak{s}+c}F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right)-\frac{r+\mathfrak{u}-c}{\mathfrak{s}+c}q^*\\&+\frac{p_I(r+\mathfrak{s})}{\mathfrak{s}+c}F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right],\end{aligned} \tag{69}$$

$$q^*=\frac{p_I\mathfrak{u}}{\mathfrak{s}+r+\mathfrak{u}}F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right)+\frac{p_I(r+\mathfrak{s})}{\mathfrak{s}+r+\mathfrak{u}}F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right], \tag{70}$$

$$\begin{aligned}\zeta=&\frac{p_I\mathfrak{u}(\mathfrak{s}-c)}{\mathfrak{s}+r+\mathfrak{u}}F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right)\\&+\frac{p_I(r+\mathfrak{s})(r+\mathfrak{u}-c)}{\mathfrak{s}+r+\mathfrak{u}}F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right].\end{aligned} \tag{71}$$

Therefore, according to the (64) and (66), we get the optimal order quantity and maximum loss threshold as:

$$\begin{aligned}q^*_{NL}&=\frac{p_I\mathfrak{u}}{\mathfrak{s}+r+\mathfrak{u}}F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right)+\frac{p_I(r+\mathfrak{s})}{\mathfrak{s}+r+\mathfrak{u}}F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right],\\\zeta_{NL}&=\frac{p_I\mathfrak{u}(\mathfrak{s}-c)}{\mathfrak{s}+r+\mathfrak{u}}F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(\mathfrak{s}+c)}{\mathfrak{s}+r+\mathfrak{u}}\right)+\frac{p_I(r+\mathfrak{s})(r+\mathfrak{u}-c)}{\mathfrak{s}+r+\mathfrak{u}}F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right].\end{aligned} \tag{72}$$

Case 3: $(\mathfrak{s}+c)q\le\zeta$

If the ζ is greater than $(\mathfrak{s}+c)q$, then (55) becomes:

$$P_\alpha(q,\zeta)=(1-\alpha)^{-1}\left[\int_{\frac{qr+q\mathfrak{u}+\zeta-cq}{p_I\mathfrak{u}}}^{\infty}[-(q\cdot r-(p_I\cdot x-q)\cdot\mathfrak{u}-cq)-\zeta]\,f(x)dx.\right. \tag{73}$$

Carrying out analysis similar to Case 2, we show that:

$$\begin{aligned}\frac{\partial}{\partial q}P_\alpha(q,\zeta)=&(1-\alpha)^{-1}\frac{\partial}{\partial q}\left[\int_{\frac{qr+q\mathfrak{u}+\zeta-cq}{p_I\mathfrak{u}}}^{\infty}[-(q\cdot r-(p_I\cdot x-q)\cdot\mathfrak{u}-c\cdot q)-\zeta]\,f(x)dx\right]\\=&(1-\alpha)^{-1}\left[-(r+\mathfrak{u}-c)\left(1-F\left(\frac{qr+q\mathfrak{u}+\zeta-cq}{p_I\mathfrak{u}}\right)\right)\right]<0.\end{aligned} \tag{74}$$

Therefore, there is no optimal order quantity for this case. □

E Proof of Corollary 2

Proof. We consider three cases so as to evaluate the problem of (24).

Case 1: $\zeta < 0$

If ζ is less than 0, then we may substitute (23) into (24) to find:

$$\begin{aligned} P_\alpha(q,\zeta) =& \zeta + (1-\alpha)^{-1} \left[\int_0^{\frac{q}{p_I}} \left[(c+\mathfrak{s})(q - x{\cdot}p_I) - \zeta\right] f(x)dx \right. \\ & \left. + \int_{\frac{q}{p_I}}^{\infty} \left[(r+\mathfrak{u}-c)(x{\cdot}p_I - q) - \zeta\right] f(x)dx \right]. \end{aligned} \tag{75}$$

Given (75), we now compute the first-order derivative of $P_\alpha(q,\zeta)$, i.e., $\frac{\partial}{\partial\zeta}P_\alpha(q,\zeta) = 0$ and $\frac{\partial}{\partial q}P_\alpha(q,\zeta) = 0$. We get:

$$\begin{aligned} q^* &= p_I F^{-1}\left(\frac{r+\mathfrak{u}-c}{\mathfrak{s}+r+\mathfrak{u}}\right), \\ \zeta^* &= -(r-c)p_I F^{-1}\left(\frac{r+\mathfrak{u}-c}{\mathfrak{s}+r+\mathfrak{u}}\right). \end{aligned} \tag{76}$$

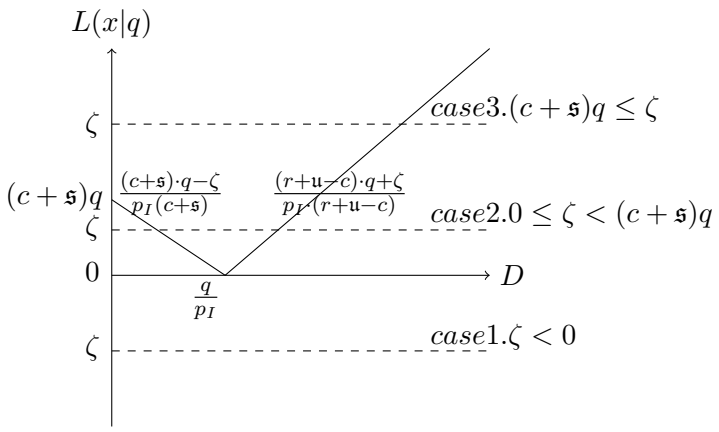


Figure 9: Three cases in minimization of total lost CVaR.

Case 2: $0 \le \zeta < (c+\mathfrak{s})q$

If the ζ is greater than or equal to 0, ζ is less than or equal to $(c+\mathfrak{s})q$, and $L(x|q) = \zeta$, then equation (24), becomes:

$$\begin{aligned} P_\alpha(q,\zeta) =& \zeta + (1-\alpha)^{-1} \left[\int_0^{\frac{(c+\mathfrak{s})\cdot q-\zeta}{p_I(c+\mathfrak{s})}} \left[(c+\mathfrak{s})(q - x{\cdot}p_I) - \zeta\right] f(x)dx \right. \\ & \left. + \int_{\frac{(r+\mathfrak{u}-c)\cdot q+\zeta}{p_I\cdot(r+\mathfrak{u}-c)}}^{\infty} \left[(r+\mathfrak{u}-c)(x{\cdot}p_I - q) - \zeta\right] f(x)dx \right]. \end{aligned} \tag{77}$$

Given (77), we now compute the first-order derivative of $P_\alpha(q,\zeta)$, i.e., $\frac{\partial}{\partial\zeta}P_\alpha(q,\zeta) = 0$ and

$\frac{\partial}{\partial q}P_\alpha(q,\zeta)=0$.

$$\begin{aligned}\frac{\partial}{\partial\zeta}P_\alpha(q,\zeta)=&\frac{\partial}{\partial\zeta}\left[\zeta+(1-\alpha)^{-1}\left[\int_0^{\frac{(c+\mathfrak{s})\cdot q-\zeta}{p_I(c+\mathfrak{s})}}\left[(c+\mathfrak{s})(q-x\cdot p_I)-\zeta\right]f(x)dx\right.\right.\\&\left.\left.+\int_{\frac{(r+\mathfrak{u}-c)\cdot q+\zeta}{p_I\cdot(r+\mathfrak{u}-c)}}^{\infty}\left[(r+\mathfrak{u}-c)(x\cdot p_I-q)-\zeta\right]f(x)dx\right]\right]\\=&1+(1-\alpha)^{-1}\left[-F\left(\frac{(c+\mathfrak{s})\cdot q-\zeta}{p_I(c+\mathfrak{s})}\right)+F\left(\frac{(r+\mathfrak{u}-c)\cdot q+\zeta}{p_I\cdot(r+\mathfrak{u}-c)}\right)-1\right].\end{aligned}$$

Setting $\frac{\partial}{\partial\zeta}P_\alpha(q,\zeta)=0$, we find:

$$1+(1-\alpha)^{-1}\left(F\left(\frac{(r+\mathfrak{u}-c)\cdot q+\zeta}{p_I\cdot(r+\mathfrak{u}-c)}\right)-1\right)=(1-\alpha)^{-1}F\left(\frac{(c+\mathfrak{s})\cdot q-\zeta}{p_I(c+\mathfrak{s})}\right). \tag{78}$$

$$\begin{aligned}\frac{\partial}{\partial q}P_\alpha(q,\zeta)=&\frac{\partial}{\partial q}\left[\zeta+(1-\alpha)^{-1}\left(\int_0^{\frac{(c+\mathfrak{s})\cdot q-\zeta}{p_I(c+\mathfrak{s})}}\left[(c+\mathfrak{s})(q-x\cdot p_I)-\zeta\right]f(x)dx\right.\right.\\&\left.\left.+\int_{\frac{(r+\mathfrak{u}-c)\cdot q+\zeta}{p_I\cdot(r+\mathfrak{u}-c)}}^{\infty}\left[(r+\mathfrak{u}-c)(x\cdot p_I-q)-\zeta\right]f(x)dx\right)\right].\end{aligned} \tag{79}$$

We first calculate the integral of the first term in (79),

$$\begin{aligned}&(1-\alpha)^{-1}\frac{\partial}{\partial q}\int_0^{\frac{(c+\mathfrak{s})\cdot q-\zeta}{p_I(c+\mathfrak{s})}}\left[(c+\mathfrak{s})(q-x\cdot p_I)-\zeta\right]f(x)dx\\=&(1-\alpha)^{-1}\frac{\partial}{\partial q}\int_0^{\frac{(c+\mathfrak{s})\cdot q-\zeta}{p_I(c+\mathfrak{s})}}\left[(c+\mathfrak{s})(q-x\cdot p_I)-\zeta\right]dF(x)\\=&(1-\alpha)^{-1}(c+\mathfrak{s})F\left(\frac{(c+\mathfrak{s})\cdot q-\zeta}{p_I(c+\mathfrak{s})}\right),\end{aligned} \tag{80}$$

and then we calculate the integral of the second term in (79),

$$\begin{aligned}&(1-\alpha)^{-1}\frac{\partial}{\partial q}\int_{\frac{(r+\mathfrak{u}-c)\cdot q+\zeta}{p_I\cdot(r+\mathfrak{u}-c)}}^{\infty}\left[(r+\mathfrak{u}-c)(x\cdot p_I-q)-\zeta\right]f(x)dx\\=&(1-\alpha)^{-1}\left[-(r+\mathfrak{u}-c)(1-F\left(\frac{(r+\mathfrak{u}-c)\cdot q+\zeta}{p_I\cdot(r+\mathfrak{u}-c)}\right)\right],\end{aligned} \tag{81}$$

combining the above two results, we obtain:

$$\begin{aligned}\frac{\partial}{\partial q}P_\alpha(q,\zeta)=&(1-\alpha)^{-1}(c+\mathfrak{s})F\left(\frac{(c+\mathfrak{s})\cdot q-\zeta}{p_I(c+\mathfrak{s})}\right)\\&+(1-\alpha)^{-1}\left[-(r+\mathfrak{u}-c)(1-F\left(\frac{(r+\mathfrak{u}-c)\cdot q+\zeta}{p_I\cdot(r+\mathfrak{u}-c)}\right)\right]\\=&(1-\alpha)^{-1}\left[(r+\mathfrak{u}-\mathfrak{s})F\left(\frac{(c+\mathfrak{s})\cdot q-\zeta}{p_I(c+\mathfrak{s})}\right)-(r+\mathfrak{u}-c)(1-\alpha)\right].\end{aligned} \tag{82}$$

Setting $\frac{\partial}{\partial q}P_\alpha(q,\zeta)=0$, we get:

$$q^*=\frac{\zeta}{c+\mathfrak{s}}+p_I{\cdot}F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{r+\mathfrak{u}+\mathfrak{s}}\right]. \tag{83}$$

We also find:

$$0=(1-\alpha)^{-1}\left[(r+\mathfrak{u}+\mathfrak{s})F\left(\frac{(c+\mathfrak{s}){\cdot}q-\zeta}{p_I(c+\mathfrak{s})}\right)-(r+\mathfrak{u}-c)(1-\alpha)\right],$$

$$(r+\mathfrak{u}+\mathfrak{s})F\left(\frac{(c+\mathfrak{s}){\cdot}q-\zeta}{p_I(c+\mathfrak{s})}\right)=(r+\mathfrak{u}-c)(1-\alpha). \tag{84}$$

According to the (78) and (84), we get:

$$(1-\alpha)+\left(F\left(\frac{(r+\mathfrak{u}-c){\cdot}q+\zeta}{p_I{\cdot}(r+\mathfrak{u}-c)}\right)-1\right)=\frac{(r+\mathfrak{u}-c)(1-\alpha)}{(r+\mathfrak{u}+\mathfrak{s})},$$

$$F\left(\frac{(r+\mathfrak{u}-c){\cdot}q+\zeta}{p_I{\cdot}(r+\mathfrak{u}-c)}\right)=\frac{r+\mathfrak{u}-c+\alpha(c+\mathfrak{s})}{r+\mathfrak{u}+\mathfrak{s}},$$

$$q^*=\frac{p_I{\cdot}(r+\mathfrak{u}-c)}{r+\mathfrak{u}-c}F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(c+\mathfrak{s})}{r+\mathfrak{u}+\mathfrak{s}}\right)-\frac{\zeta}{r+\mathfrak{u}-c}, \tag{85}$$

$$\zeta=p_I{\cdot}(r+\mathfrak{u}-c)F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(c+\mathfrak{s})}{r+\mathfrak{u}+\mathfrak{s}}\right)-(r+\mathfrak{u}-c){\cdot}q^*, \tag{86}$$

$$\begin{aligned}q^*=&\frac{p_I{\cdot}(r+\mathfrak{u}-c)}{r+\mathfrak{u}+\mathfrak{s}}F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(c+\mathfrak{s})}{r+\mathfrak{u}+\mathfrak{s}}\right)\\&+\frac{p_I(c+\mathfrak{s})}{r+\mathfrak{u}+\mathfrak{s}}F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right],\end{aligned} \tag{87}$$

$$\begin{aligned}\zeta=&\frac{p_I(r+\mathfrak{u}-c)(c+\mathfrak{s})}{\mathfrak{s}+r+\mathfrak{u}}F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(c+\mathfrak{s})}{r+\mathfrak{u}+\mathfrak{s}}\right)\\&+\frac{p_I(c+\mathfrak{s})(r+\mathfrak{u}-c)}{\mathfrak{s}+r+\mathfrak{u}}F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right].\end{aligned} \tag{88}$$

Therefore, according to the (83) and (85), we determine the optimal order quantity and maximum loss threshold as:

$$\begin{aligned}q^*_{R\bar{L}T}&=\frac{p_I{\cdot}(r+\mathfrak{u}-c)}{r+\mathfrak{u}+\mathfrak{s}}F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(c+\mathfrak{s})}{r+\mathfrak{u}+\mathfrak{s}}\right)+\frac{p_I(c+\mathfrak{s})}{r+\mathfrak{u}+\mathfrak{s}}F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right],\\ \zeta_{R\bar{L}T}&=\frac{p_I(r+\mathfrak{u}-c)(c+\mathfrak{s})}{\mathfrak{s}+r+\mathfrak{u}}\left(F^{-1}\left(\frac{r+\mathfrak{u}-c+\alpha(c+\mathfrak{s})}{r+\mathfrak{u}+\mathfrak{s}}\right)+F^{-1}\left[\frac{(r+\mathfrak{u}-c)(1-\alpha)}{\mathfrak{s}+r+\mathfrak{u}}\right]\right).\end{aligned} \tag{89}$$

Case 3: $(c+\mathfrak{s})q\le\zeta$

If the ζ is greater than or equal to $(c+\mathfrak{s})q$, then equation (24) becomes:

$$P_\alpha(q,\zeta) = (1-\alpha)^{-1}\left[\int_{\frac{(r+\mathfrak{u}-c)\cdot q+\zeta}{p_I\cdot(r+\mathfrak{u}-c)}}^{\infty}\left[(r+\mathfrak{u}-c)(x\cdot p_I-q)-\zeta\right]f(x)dx\right]. \tag{90}$$

Carrying out analysis similar to Case 2, we show that:

$$\begin{aligned}\frac{\partial}{\partial q}P_\alpha(q,\zeta) =&(1-\alpha)^{-1}\frac{\partial}{\partial q}\left[\int_{\frac{(r+\mathfrak{u}-c)\cdot q+\zeta}{p_I\cdot(r+\mathfrak{u}-c)}}^{\infty}\left[(r+\mathfrak{u}-c)(x\cdot p_I-q)-\zeta\right]f(x)dx\right]\\ =&(1-\alpha)^{-1}\left[-(r+\mathfrak{u}-c)(1-F\left(\frac{(r+\mathfrak{u}-c)\cdot q+\zeta}{p_I\cdot(r+\mathfrak{u}-c)}\right)\right]<0.\end{aligned} \tag{91}$$

Therefore, there is no optimal order quantity for this case. □

F Proof of proposition 4

Proof. For any given α, taking the first order partial derivative of retailer's expected profit $\pi_L(q)$ with respect to δ, we have:

$$\begin{aligned}\frac{\partial}{\partial\delta}\pi_L(q) =&\frac{\partial}{\partial\delta}\left[(r'+\mathfrak{u}-c)\cdot q-E\left[X(p_I+p_L)\right]\cdot\mathfrak{u}\right.\\ &\left.-(r'+\mathfrak{u}+\mathfrak{s})\cdot\int_0^q\left(q-x(p_I+p_L)\right)f(x)dx\right]\\ =&q\cdot p_L'\cdot(f_c-r)-p_L'(f_c-r)\cdot\int_0^q\left(q-x(p_I+p_L)\right)f(x)dx\\ =&p_L'\cdot(f_c-r)\cdot\left(q-\int_0^q(q-x(p_I+p_L))f(x)dx\right)<0.\end{aligned} \tag{92}$$

Where the inequality follows from the assumption that $f_c-r<0$ (the layaway cancellation fee is less than the selling price). □

G Proof of Corollary 3

Proof. Consider any fixed confidence level, α. From Proposition 4 we know that $\pi_{RL}(q^*)$ is decreasing in δ, i.e., $\frac{\partial\pi_{RL}(q^*)}{\partial\delta}<0$. For any retailer not offering layaway, $\pi_{R\bar{L}}(q^*)$ is constant in δ, i.e., $\frac{\partial\pi_{R\bar{L}}(q^*)}{\partial\delta}=0$. Putting the two previous observations together we may say that $\pi_{RL}(q^*)$ crosses $\pi_{R\bar{L}}(q^*)$ once, on $\delta\in\mathbb{R}$, let $\delta^\times$ be such that $\pi_{RL}(q^*)|_{\delta^\times}=\pi_{R\bar{L}}(q^*)|_{\delta^\times}$. As only $\delta\in[0,1]$ is valid for our setting, for any $\delta^\times<0$, we say $\delta^*=0$; for any $\delta^\times>1$, we say $\delta^*=1$; otherwise, $\delta^*=\delta^\times$.

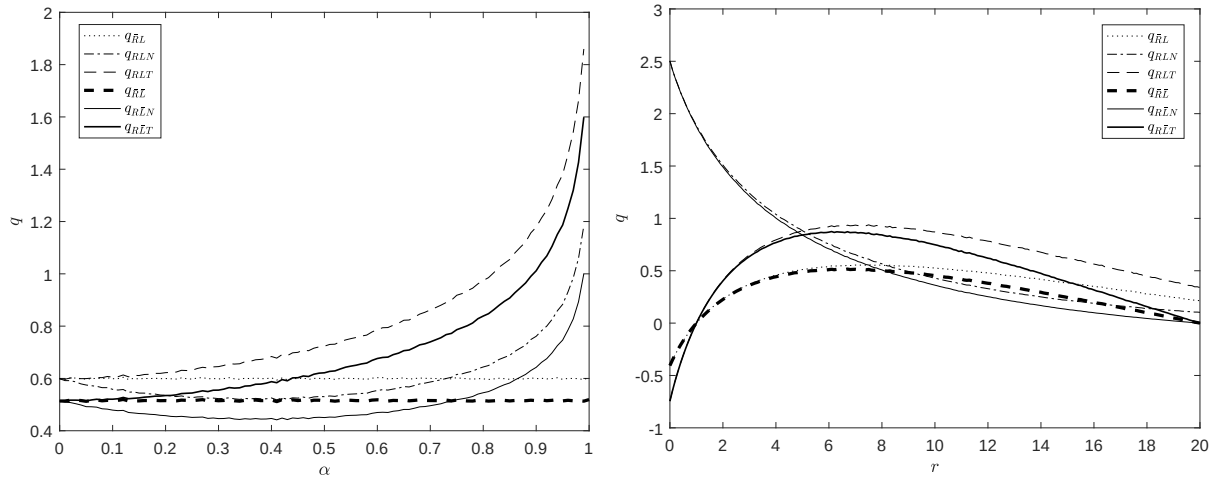
□

H Correlation between valuation and budget and different distributions

In this section, first, we consider consumer valuations and budgets to be dependent. We show that the relationship between optimal order quantity and expected profit are qualitatively the same regardless if valuations and budgets are dependent or independent. Second, we assume that

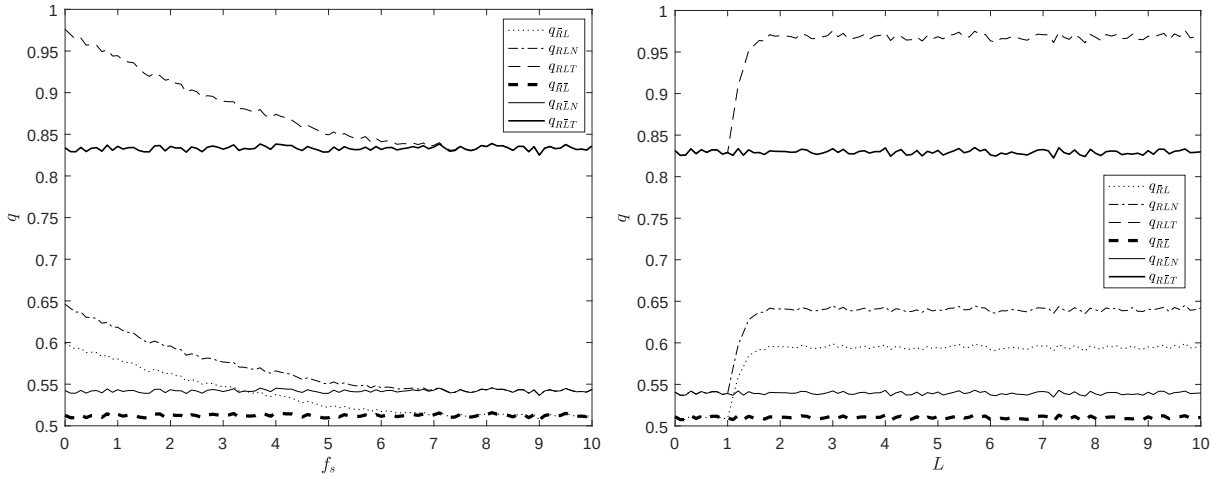
consumer valuations and budgets are independent and obey normal distributions, which differs from our results in Section 5 where we assume an exponential demand distribution.

Case 1: In this case we assume that consumer budgets, b, and valuations v are dependently and uniformly distributed between $[0, 20]$ and $[0, 2b]$, respectively. Our numerical results are depicted in Figures 10–13.



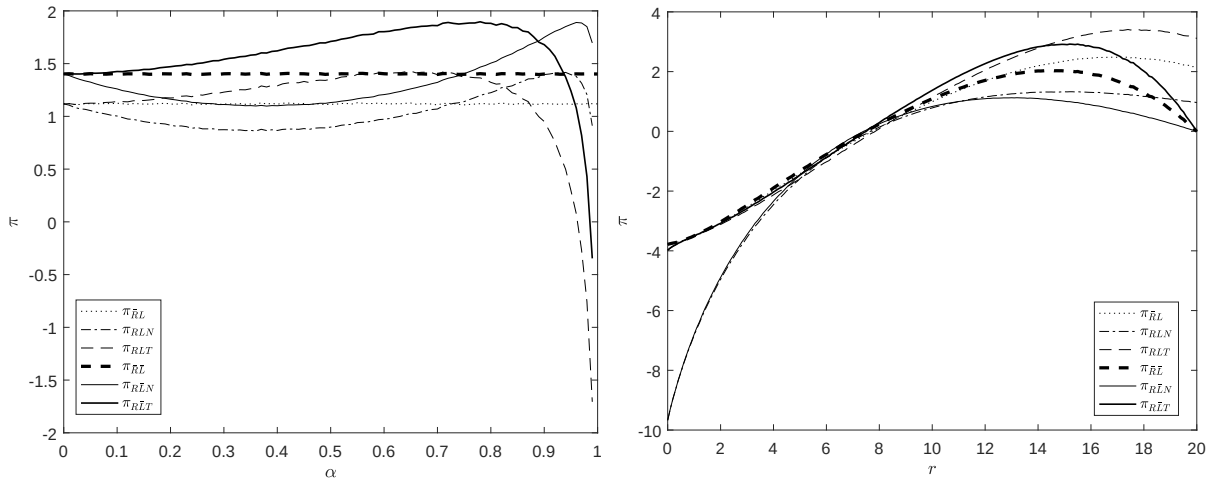
(a) confidence level α (b) selling price r

Figure 10: Sensitivity of the optimal order quantity, q, for different layaway and risk-preference scenarios, with respect to (a) α and (b) r



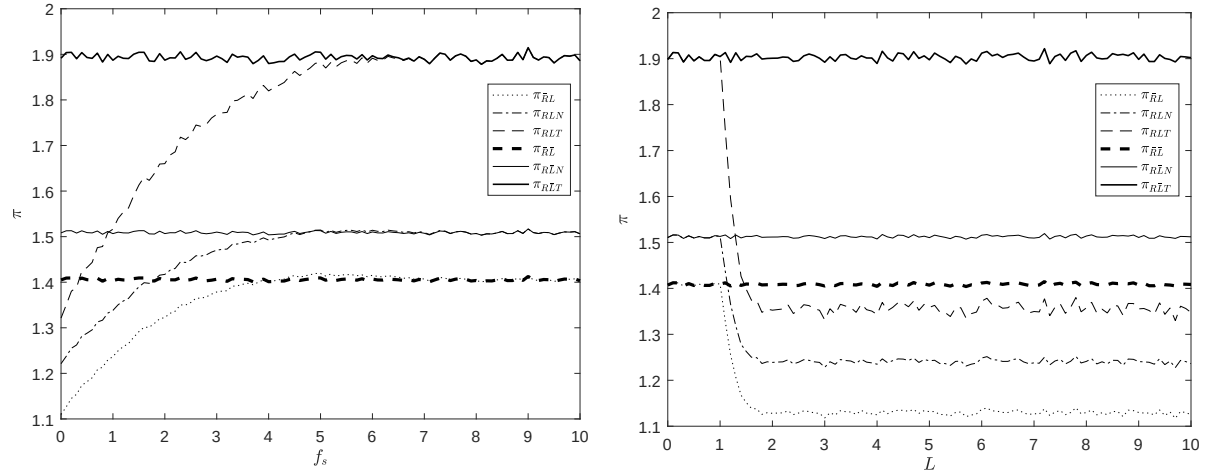
(a) enrollment fee f_s (b) the length of payoff period L

Figure 11: Sensitivity of the optimal order quantity, q, for different layaway and risk-preference scenarios, with respect to (a) f_s and (b) L



(a) confidence level α (b) selling price r

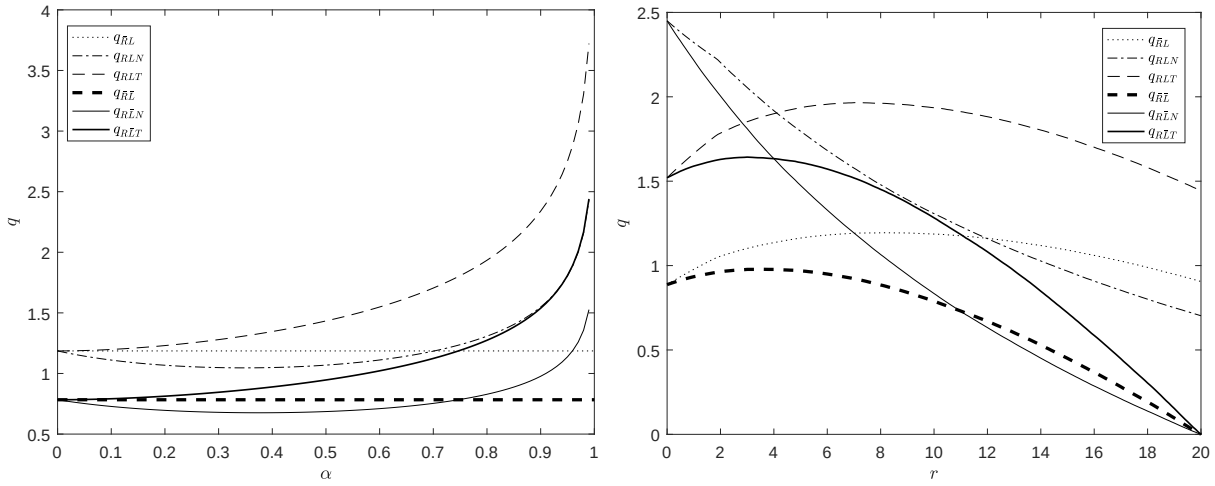
Figure 12: Sensitivity of the expected profit, π, for different layaway and risk-preference scenarios, with respect to (a) α and (b) r



(a) enrollment fee f_s (b) the length of payoff period L

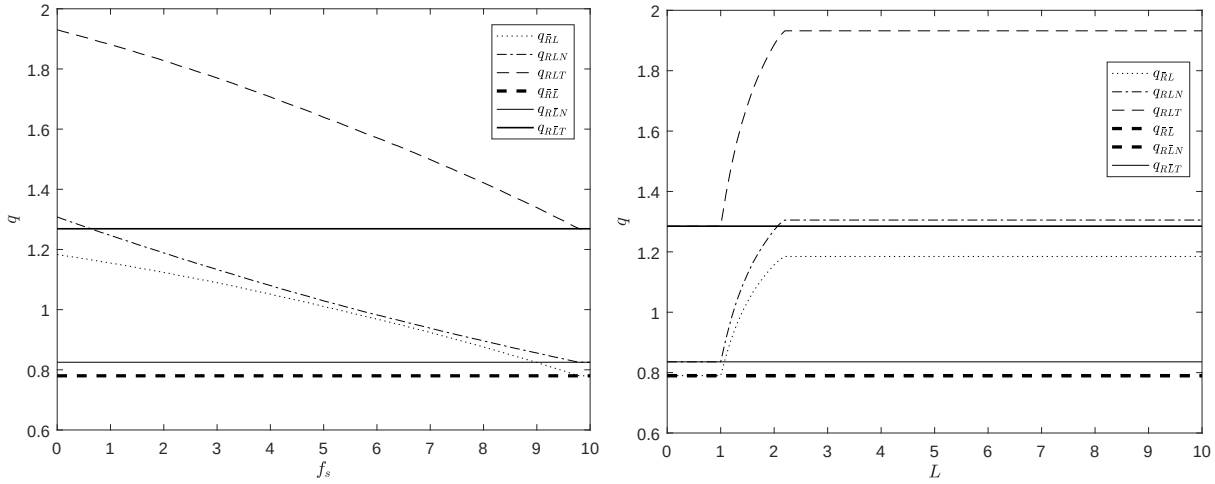
Figure 13: Sensitivity of the expected profit, π, for different layaway and risk-preference scenarios, with respect to (a) f_s and (b) L

Case 2: In this case, we consider a linear relationship between the consumer's budgets, b, and valuations, v. We assume that b is uniformly distributed between $[0, 20]$ and $v = a + m \cdot b$, where a and m are constants. In this simulation, we assume that $a = 2$ and $m = 2$.



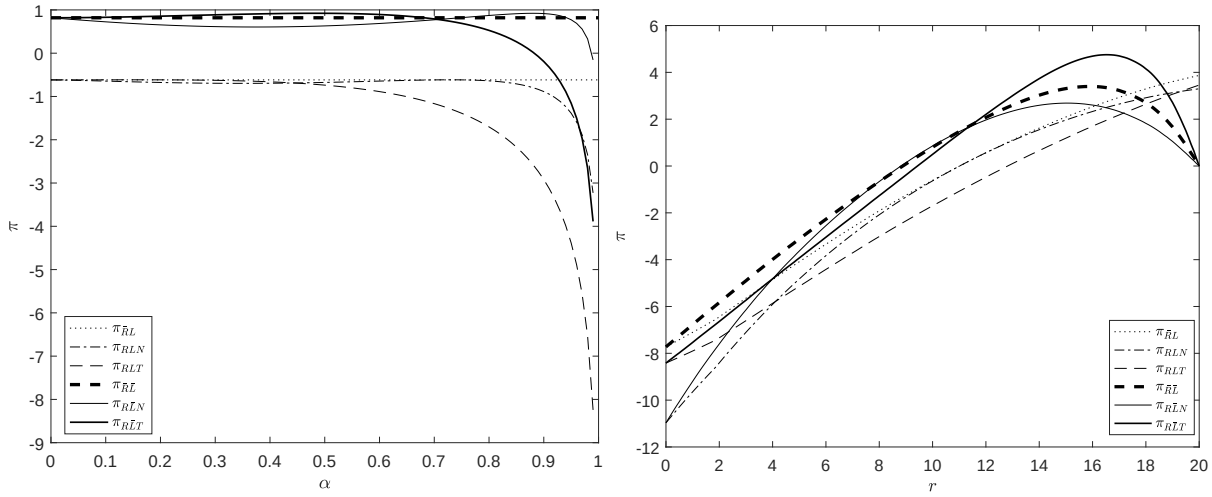
(a) confidence level α (b) selling price r

Figure 14: Sensitivity of the optimal order quantity, q, for different layaway and risk-preference scenarios, with respect to (a) α and (b) r



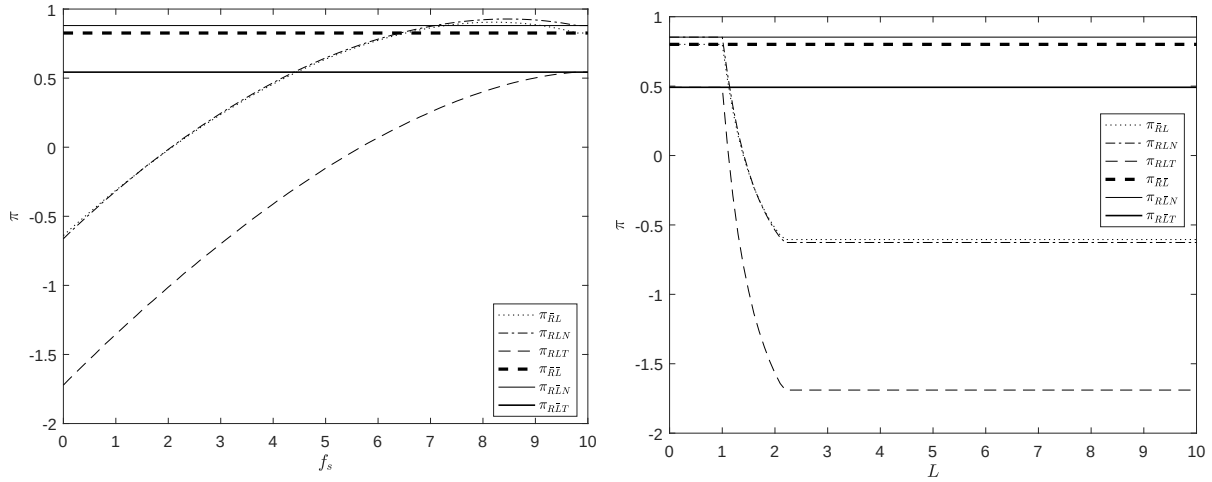
(a) enrollment fee f_s (b) the length of payoff period L

Figure 15: Sensitivity of the optimal order quantity, q, for different layaway and risk-preference scenarios, with respect to (a) f_s and (b) L



(a) confidence level α (b) selling price r

Figure 16: Sensitivity of the expected profit, π, for different layaway and risk-preference scenarios, with respect to (a) α and (b) r

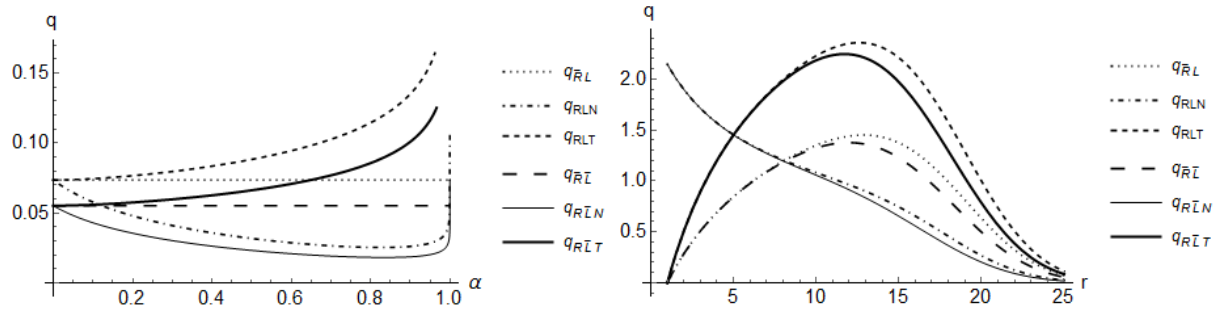


(a) enrollment fee f_s (b) the length of payoff period L

Figure 17: Sensitivity of the expected profit, π, for different layaway and risk-preference scenarios, with respect to (a) f_s and (b) L

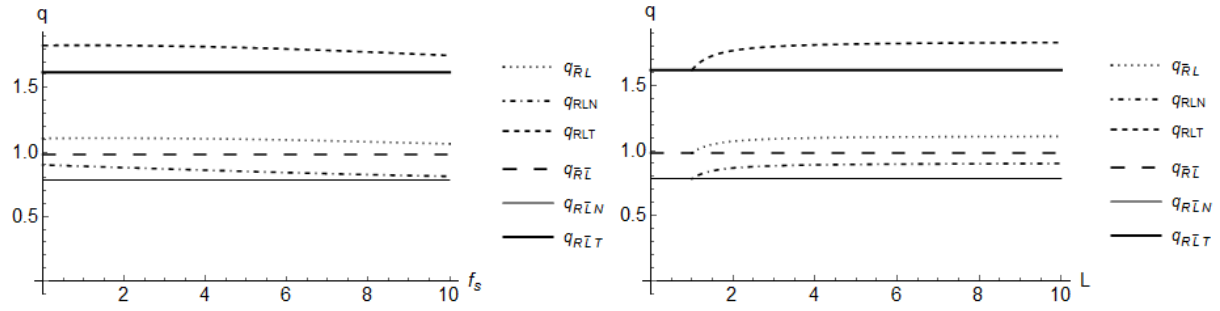
The relationship between optimal order quantity and expected profit with respect to model parameters (α, r, f_s and L) found by our numerical exploration, is depicted in Figure 14−17. We find that for different correlations between consumer valuation and budget, for different parameters, the optimal order quantity and expected profit trends are the same, respectively.

Case 3: In this case we assume that consumer budgets, b, and valuations, v are independent and normal distributed between $N(20, 5)$ and $N(20, 5)$, respectively.



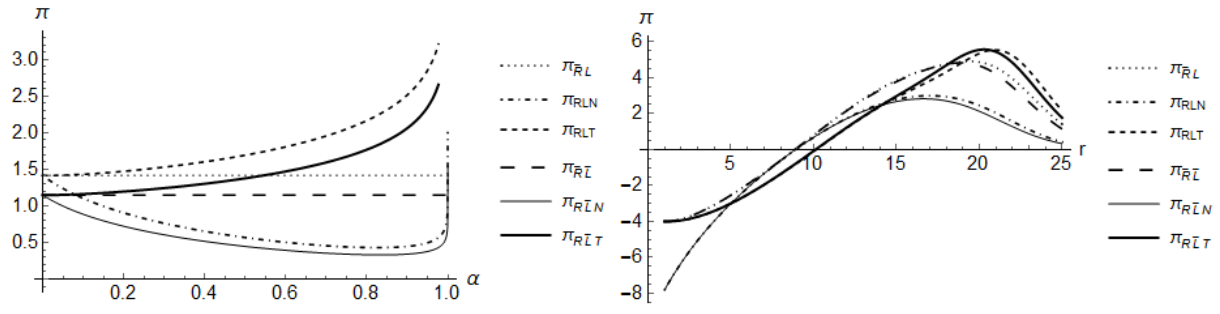
(a) confidence level α (b) selling price r

Figure 18: Sensitivity of the optimal order quantity, q, for different layaway and risk-preference scenarios, with respect to (a) α and (b) r



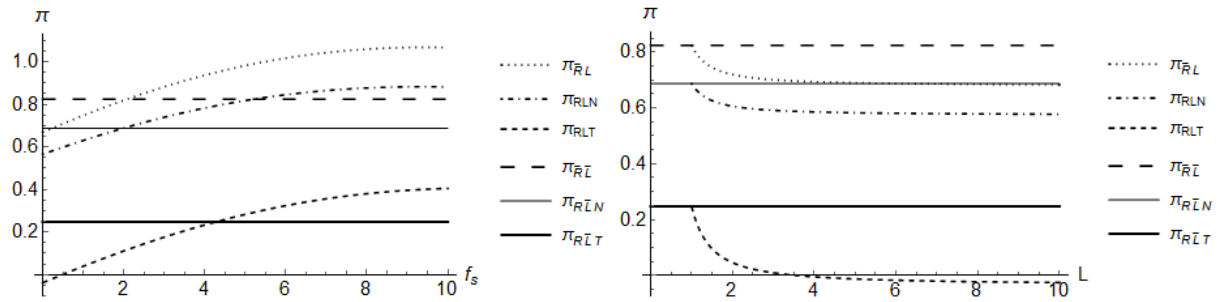
(a) enrollment fee f_s (b) the length of payoff period L

Figure 19: Sensitivity of the optimal order quantity, q, for different layaway and risk-preference scenarios, with respect to (a) f_s and (b) L



(a) confidence level α (b) selling price r

Figure 20: Sensitivity of the expected profit, π, for different layaway and risk-preference scenarios, with respect to (a) α and (b) r



(a) enrollment fee f_s (b) the length of payoff period L

Figure 21: Sensitivity of the expected profit, π, for different layaway and risk-preference scenarios, with respect to (a) f_s and (b) L

Compared with the uniform distribution in our paper, as depicted in Figure 18−21 considering the normal distribution, with the change of each parameter, the retailer's optimal order quantity and expected profit exhibit the same change trends as we found in Section 5.

I Additional numerical illustrations

As shown in Figure 22, we can see that when we assume that $r = 5$, and $\delta = 0.001$, as the enrollment fee increases, the optimal order quantity initially increases and then decreases.

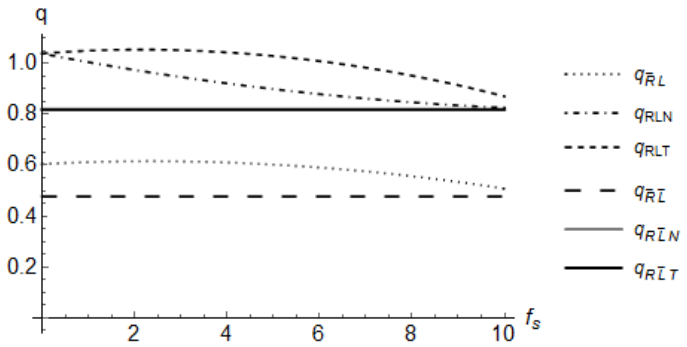


Figure 22: Sensitivity of the optimal order quantity, q, for different layaway and risk-preference scenarios, with respect to f_s

As shown in Figure 5d, we find that when only layaway purchases are made, the retailer's profit is negative. However, negative expected profits mean that a retailer will not operate in such environment. We show that there is a setting in which a retailer will want to only sell using layaway: in order to obtain a positive expected profit as shown in Figure 23, we set $c = 3$, $\mathfrak{s} = -1$, and $\delta = 0.00000000000000000000000001$.

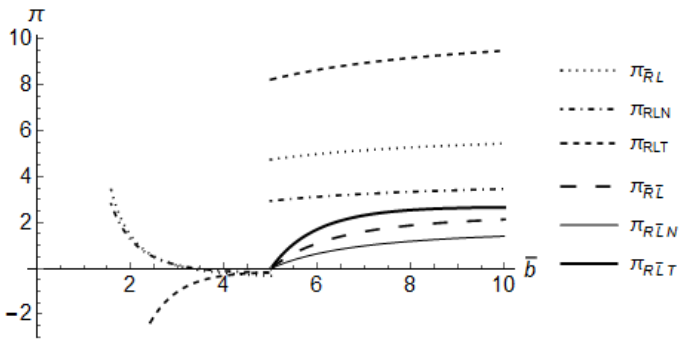


Figure 23: Sensitivity of the expected profit, π, for different layaway and risk-preference scenarios, with respect to $\bar{b}$