Valuation and Risk Management of Hedge Fund Investments under Alternative Fee Structures

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

A version of Chapter 2 and 3 of this thesis has been prepared as a research paper to be submitted, co-authored with my supervisor, David Saunders. A version of Chapter 4 has been published in *Journal of Alternative Investments*, co-authored with David Saunders and Luis Seco. A version of Chapter 5 has been published in the book *Innovations in Insurance, Risk and Asset Management*, co-authored with Mohammad Shakourifar, Ranjan Bhaduri, Ben Djerroud, David Saunders and Luis Seco.

Abstract

Hedge funds have specialized fee structures, often including performance fees designed to align the incentives of investors and fund managers. However, hedge funds have faced intense scrutiny since the financial crisis, as the fees they charge investors have been outsized compared to the returns. Consequently, innovative fee structures have emerged aiming at better alignment between investors' interests and the hedge fund business objective. In this thesis, we present mathematical and numerical analyses of many aspects of hedge fund investments with three fee structures, first-loss, shared-loss and negative fee structure. The motivation for this is to understand an important new investment type and its implications for investors and managers, as well as the mathematical problems that it poses.

In Chapter 2 and 3, we investigate the optimal withdrawal time of a first-loss or shared-loss hedge fund fee structure from an investor's perspective. Given that a hedge fund dynamic follows a geometric Brownian, calculating the optimal withdrawal time entails solving an optimal stopping problem with a continuous piece-wise linear payoff function. In particular, we explicitly solve the problem in the infinite horizon case. Next, we show that there exist two monotonic and continuous early exercise boundaries and derive an early exercise premium integral representation in the finite horizon case. Finally, we analyze the asymptotic behavior of the early exercise boundaries near maturity.

In Chapter 4, we test the hypothesis of fee diversification. In particular, we study the optimization problem of an investor who may choose any combination of the first loss and classical fee structures, and seeks to maximize either the Sharpe ratio or the Sortino ratio of their final payoff, evaluated using real-world probabilities. We demonstrate that for the vast majority of fund mean returns and volatilities, there is no fee diversification effect: either the first-loss structure or the classical structure is optimal for the investor.

In Chapter 5, we present an analysis of the value and risks for negative fee structure. We begin by employing risk-neutral valuation, using both Black-Scholes and regime-switching models. We then proceed to analyze the risks inherent in investments in hedge funds with negative fee structures. Given the resemblance of these investments to asset-backed securities, we in particular study probability of default and loss given default under the real-world measure for both geometric Brownian motion and regime-switching models.

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Dedication

To the memory of my dearest grandfather, Jianwu Chen.

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Chapter 1

Introduction

1.1 Background

The global hedge fund industry has increased tremendously in size and popularity over the past decades. According to the database of BarclayHedge, a total of 3194 billion dollars of assets under management is reported to the database at the first Quarter 2019 (BarclayHedge, 2019). Investors embraced hedge funds due to their ability to generate absolute return, their outstanding historical performance, and their diversification potential with respect to an existing portfolio (see, e.g. Vogt (2010) and Meucci (2007)). As a consequence, both academics and practitioners have studied several aspects of hedge fund investments.

Hedge funds are private investment vehicles controlled by a small number of partners with limited partnership. They exploit possible investment opportunities by taking long and short positions, using leverage, hedging with financial derivatives and investing in various markets. In most cases, performance-based fees are included to align the managers' and the investors' interests (Lhabitant, 2007; Vogt, 2010). As such, modern hedge funds have deviated significantly from the original purpose of the investment type, which was hedging risks (Lhabitant, 2001). Nevertheless, hedge funds have increased in popularity due to several important factors.

First, as Liang (1999) points out, hedge fund managers are absolute performers, who target benchmarks, such as the T-bill rate (with a possible premium) or LIBOR rate (with a possible premium). These benchmarks are often viewed as risk-free assets with positive returns. Thus, investors anticipate that hedge funds will earn positive returns regardless of the market environment. On the other hand, traditional mutual fund managers are relative performers as they measure their performance with respect to market benchmarks or indices, such as the S&P 500 index (Liang, 1999; Vogt, 2010). Generally, negative returns are acceptable for these managers as long as they outperform the corresponding benchmarks.

Second, hedge funds are more flexible in their investment strategies than many other investment vehicles. In order to protect investors, many potential competitors for investment capital, such as mutual funds, are strictly regulated by government agencies. In contrast, hedge funds are only subject to light regulations in most jurisdictions (Fung and Hsieh, 1999). For example, hedge fund managers can take long and short positions, while traditional mutual fund managers are often only permitted long-only strategies. Moreover, hedge fund managers are able to use substantial leverage to increase investment capital and to purchase various financial derivatives, such as options and swaps (Vogt, 2010).

Third, investors can further diversify a stock/bond portfolio by treating hedge funds as a possible asset class. For example, Kooli (2007) finds that by combining a hedge fund or fund of funds (FOF) with bonds and stocks there is a significant statistical improvement of the global minimum-variance portfolio. Furthermore, even when investors expand their asset universe to include commodities, fixed income, and international assets, the FOF can still provide a statistically significant diversification potential. These results are intuitive since Fung and Hsieh (1997) demonstrate that hedge fund returns are often negatively correlated or weakly correlated to traditional "buy-and-hold" mutual fund returns.

Last but not least, hedge funds have specialized fee structures, often including fees designed to align the incentives of investors and fund managers. Traditionally, the fee structure of a hedge fund consists of two parts: a management fee and a performance fee. The management fee is charged as a percentage of assets under management, and the performance fee is charged as a percentage of profits. For instance, a common fee structure is "two and twenty" (a fee consisting of 2% of assets under management and 20% of profits). The performance fee is supposed to motivate the hedge fund manager to seek high absolute returns rather than following passive trading strategies or purchasing indexed funds. There has been significant research into the question of whether performance fees indeed align the interests of investors and fund managers in practice. Because the performance-based fee induces a call option-like contract on the assets that the manager controls, it seems that the manager will invest in a riskier portfolio. However, Carpenter (2000) demonstrates that the risk of a portfolio can decrease when a manager with option-like compensation chooses to maximize their expected power utility. On the contrary, Kouwenberg and Ziemba (2007) show that the manager can increase the fund's risk when their preference is modeled by prospect theory, depending on the level of the manager's own investment in the fund.

Hedge funds have faced intense scrutiny since the financial crisis, as the fees they charge investors have been outsized compared to the returns they are posting as a group.[1] Consequently, innovative fee structures have emerged aiming at better alignment between investors' interests and the hedge fund business objectives. An example is the class of first-loss fee structures (Banzaca, 2012). In these structures, in return for a higher performance fee, the hedge fund manager provides some downside protection to the investors on their losses. The loss coverage is typically around 10%, while the performance fee can reach 40% or even 50% of profits. He and Kou (2018) present a description, and a mathematical analysis of the incentives that such a fee structure provokes, and study the impact on the utility of both the hedge fund manager and investor. Djerroud et al. (2016) analyze the first-loss fee structure using an option-pricing perspective, leading to the identification of fair levels of the performance fee. Fee innovation has led to discussions and negotiations between investors and managers on the optimal fee to be used in certain situations, amplifying the universe of available fee structures by mixing different structures together. One example is the shared-loss structure, which can be considered as a mixture of the classical structure and the first-loss structure. Under a shared-loss agreement, rather than covering *all* investor losses up to a certain limit, the manager will provide compensation for a proportion of the investor's losses (again subject to a ceiling). To draw an (imperfect) analogy with insurance, in the first loss structure, the manager provides full insurance on losses (up to a preset limit), while in the shared loss structure the manager is providing partial insurance.

In this thesis, we present mathematical and numerical analyses of many aspects of hedge fund investments with shared-loss fee structures. The motivation for this is to understand an important new investment type and its implications for investors and managers, as well as the mathematical problems that it poses. It further serves as a case study of a situation in which compensation is provided through a portfolio of options, and an opportunity to analyze the resulting payoffs and the incentives that they provoke. Similar situations, with related payoffs, arise for example in variable annuities offered by insurers (Lin et al., 2017).

The remainder of this introductory chapter is structured as follows. In the second section, we discuss the payoffs of first-loss and shared-loss fee structures. In the third section, we summarize the problems studied and the novel research contributions of this thesis, and place them in the context of the existing academic literature.

[1] https://www.bloomberg.com/news/articles/2017-06-07/new-york-illinois-pension-funds-say-hedge-funds-fees-too-high

1.2 Hedge Fund Fee Structures: Investor and Manager Payoffs

In this section, we summarize the payoffs and present visualizations (Figure 1.1-1.6) for each of the shared-loss and first-loss fee structures that have been considered in the literature. The manager has two ways to provide downside protection for the investor. First, a separate escrow account can be established, from which the investor's losses are reimbursed. Second, the manager can invest their own money in the fund and insure the investor's losses from their own share. Let α be the constant of proportionality for the performance fees, which are a fixed proportion of the net profits obtained by the fund. The typical value of α ranges from 20% to 50%, depending on the fee structures that the hedge fund manager employs.[2] In addition to the initial investment and the running fee for assets under management, investors may also pay an upfront fee to the fund manager. The fair value for this upfront fee is discussed later in this chapter.

Throughout the thesis, we assume that the value of the investor's initial investment in the hedge fund is 1. This assumption holds even when the current (i.e. time zero) value of the hedge fund assets, typically denoted by x, is not equal to 1, and the initial investment was made in the past. The value 1 will then still appear in the payoff functions of investors and managers, as a threshold to determine whether the investor has made a profit or a loss (and thus whether a performance fee is due, or the insurance component of a novel hedge fund fee structure is activated). This assumption is made without loss of generality. If the initial investment in the fund was instead $y \neq 1$, then rather than receiving the payoff $g(x)$ as analyzed in the main body of the thesis, the investor would receive the payoff $g^*_y(x) = yg(\frac{x}{y})$. This point is discussed in detail in Appendix A.

1.2.1 Compensation from an Escrow Account

In this case, the hedge fund manager sets up an escrow account,[3] which is used to cover the investor's losses. Here, we use c, $0 < c < 1$, a percentage of the initial investment, to

[2] Slight variations on the assumptions presented in this section are employed in Chapters 4 and 5, when it is more convenient for performing a mathematical analysis of the relevant financial arrangements. Details are presented therein.

[3] An escrow account is an account operated by a highly credible third party, such as a major bank, on behalf of the transacting parties. The hedge fund manager deposits the amount c at the initial time in the escrow account. The assets in the account would typically be invested in very safe securities, such as government bonds. Any interest/return on this deposit is payable to the manager, so the amount available as insurance to cover the investor's losses remains constant at c. The key points are that the escrow

denote the escrow amount ($c = 0.1$ would be a typical value in practice).

First-Loss

For the first-loss case, the hedge fund manager will compensate all of the investor's losses until the escrow amount is exhausted. The payoff function to the investor is

$$g(x) = \begin{cases} \alpha + (1-\alpha)x, & x \geq 1, \\ 1, & (1-c) < x < 1, \\ c + x, & x \leq 1 - c. \end{cases} \tag{1.1}$$

The upper piece of the payoff (when $x \geq 1$) gives the investor's payoff as the value to which the investment has grown (x), less the payment of the performance fee to the manager of α times the profit ($\alpha \cdot (x-1)$), i.e. $x - \alpha(x-1) = \alpha + (1-\alpha)x$. The middle component of the payoff reflects the fact that when the value of the fund declines, but by an amount less than the total value of losses insured (c), the investor simply ends up with the value of the initial investment. The lower piece shows that when the fund value declines by more than c, the investor keeps the residual fund value x, and receives the insurance payment c.

account is segregated from the other assets of the fund (the hedge fund manager does not have access to it), and its value does not decline when the hedge fund loses money.

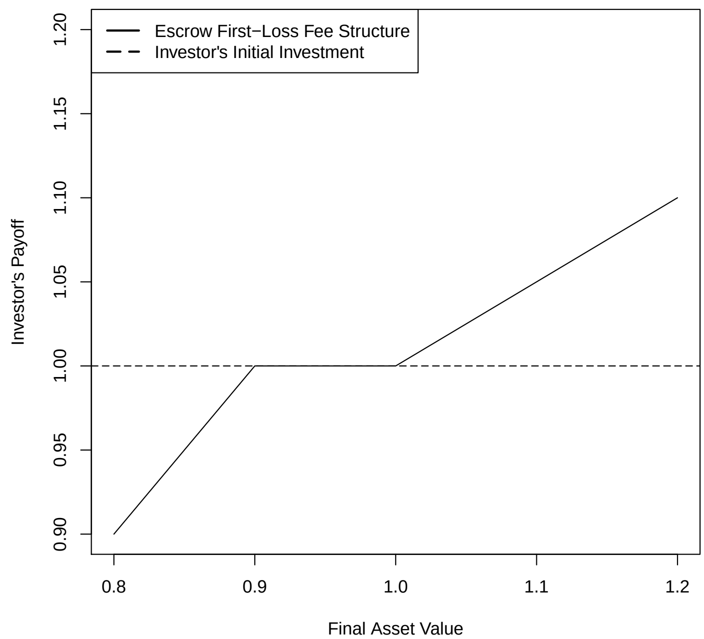

Figure 1.1: Investor's Payoffs: Escrow First-Loss Fee Structure with parameters $c = 0.1$ and $\alpha = 0.5$.

Shared-Loss

For the shared-loss case, the manager covers a proportion θ of the investor's losses from an escrow account. If $c \geq \theta$, which indicates the escrow account cannot be exhausted, then the payoff to the investor is

$$g(x) = \begin{cases} \alpha + (1-\alpha)x, & x \geq 1, \\ \theta + (1-\theta)x, & x < 1. \end{cases} \tag{1.2}$$

The upper piece of the payoff reflects the payment of the performance fee to the fund manager, as described above. In the lower piece, where the fund has lost money, the investor receives the remainder of the fund's assets (x), plus the fraction θ of the amount lost (i.e. $\theta \cdot (1-x)$), for a total payoff of $x + \theta(1-x) = \theta + (1-\theta)x$.

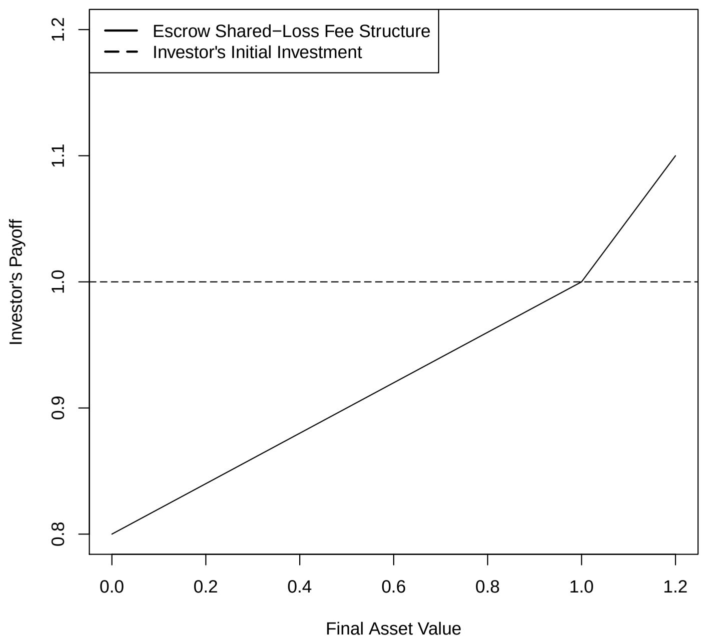

Figure 1.2: Investor's Payoffs: Escrow Shared-Loss Fee Structure ($c \geq \theta$) with parameters $\theta = 0.8$ and $\alpha = 0.5$.

If $c < \theta$, then the payoff to the investor is

$$g(x) = \begin{cases} \alpha + (1-\alpha)x, & x \geq 1, \\ \theta + (1-\theta)x, & (1-\frac{c}{\theta}) < x < 1, \\ c + x, & x \leq (1-\frac{c}{\theta}). \end{cases} \tag{1.3}$$

The difference between this payment and the previous one is that here losses are only compensated until the amount c in the escrow account is exhausted, which occurs when the insurance payment $\theta \cdot (1-x)$ equals c, i.e. when $x = 1 - \frac{c}{\theta}$ (or, equivalently, when the loss equals $\frac{c}{\theta}$). When losses exceed $\frac{c}{\theta}$, the investor receives the residual assets x together with the insurance payment of c.

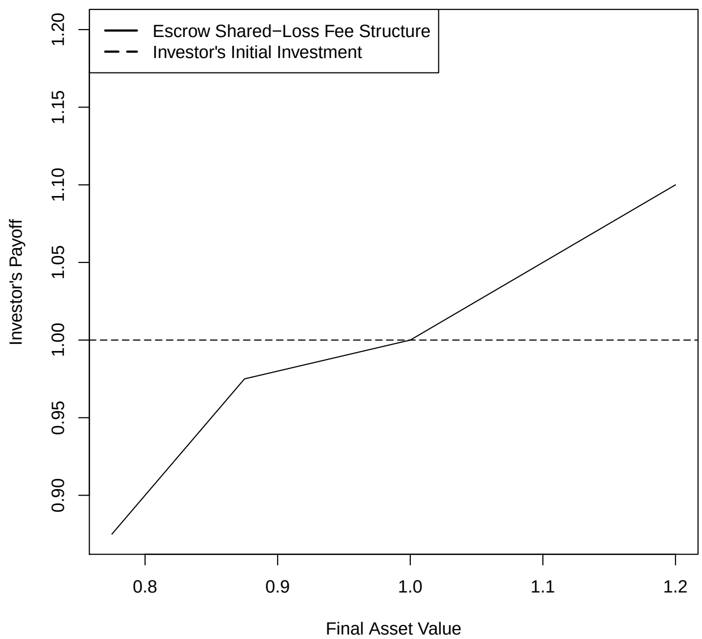

Figure 1.3: Investor's Payoffs: Escrow Shared-Loss Fee Structure ($c < \theta$) with parameters $c = 0.1$, $\theta = 0.8$ and $\alpha = 0.5$.

1.2.2 Compensation from the Manager's Own Investment

In this arrangement, the manager invests their own capital into the fund. Let $\omega \in (0, 1)$ be the proportion of the investor's initial investment contributed by the manager. The total initial investment is thus $(1 + \omega)$. Again, when the portfolio suffers a loss, the manager can use their own share to compensate the investor's loss until the investor is made whole, or the manager's capital is exhausted.

First-Loss

In first-loss structures, the manager's share of the fund is used to completely cover the investor's losses (until the manager's share of the fund is exhausted). The payoff to the investor is

$$g(x) = \begin{cases} \alpha + (1-\alpha)x, & x \geq 1, \\ 1, & \frac{1}{1+\omega} < x < 1, \\ (1+\omega)x, & x \leq \frac{1}{1+\omega}. \end{cases} \tag{1.4}$$

As always, the upper piece reflects the payment of the performance fee to the fund manager. In the event of a loss, if the investor's initial investment of 1 has declined to $x < 1$, then the fund manager's initial investment of ω has declined to ωx, and this amount is available to make the insurance payment to the investor. If $\omega x > (1-x)$ (or, equivalently, $x > \frac{1}{1+\omega}$), then the manager's remaining funds are sufficient to cover the investor's losses, and the investor is made whole with payoff 1. Otherwise, the investor receives the residual amounts from both their own investment (x) and the manager's allocated investment (ωx), i.e. $(1+\omega)x$.

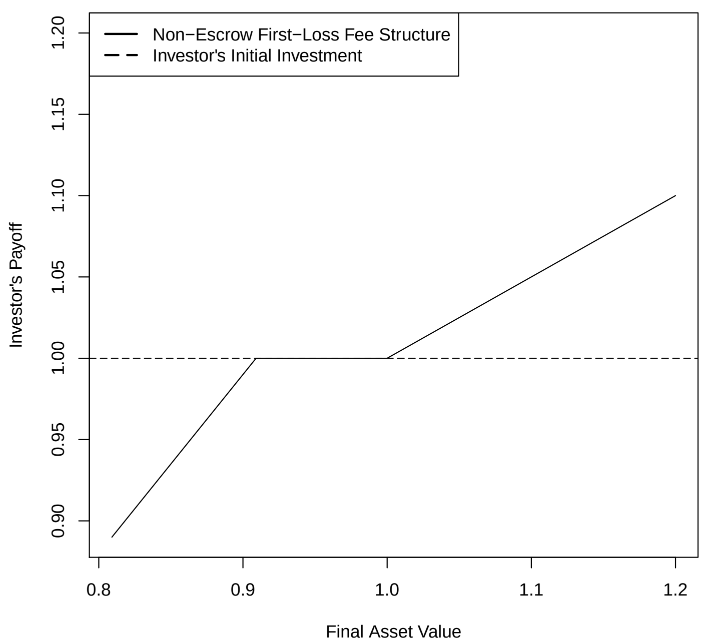

Figure 1.4: Investor's Payoffs: Non-Escrow First-Loss Fee Structure with parameters $\omega = 0.1$ and $\alpha = 0.5$.

Shared-Loss

Finally, in the shared-loss case the manager has invested the amount ω in the fund and covers a proportion θ of the investor's losses using their own share. The investor's payoff is

$$g(x) = \begin{cases} \alpha + (1-\alpha)x, & x \geq 1, \\ \theta + (1-\theta)x, & \frac{\theta}{\omega+\theta} < x < 1, \\ (1+\omega)x, & x \leq \frac{\theta}{\omega+\theta}. \end{cases} \tag{1.5}$$

The upper piece gives the investor's residual share after the performance fee has been paid to the fund manager. As with the first-loss structure, when a loss occurs and the investor's funds have declined to the amount $x < 1$, then the manager's funds have decreased to ωx. This amount is used to cover the fraction $\theta < 1$ of the investor's losses $(1-x)$. The amount to be paid to the investor is thus $\theta(1-x)$. However, this payment is only possible when $\omega x \geq \theta(1-x)$ (or, upon rearrangement, when $x \geq \frac{\theta}{\omega+\theta}$). If this holds, then the investor receives the residual amount of their investment x plus the insurance payment $\theta(1-x)$, for a total payoff of $x + \theta(1-x) = \theta + (1-\theta)x$. Otherwise, the investor receives the residual amounts from both their own investment (x) and the manager's allocated investment (ωx), for a total of $(1+\omega)x$.

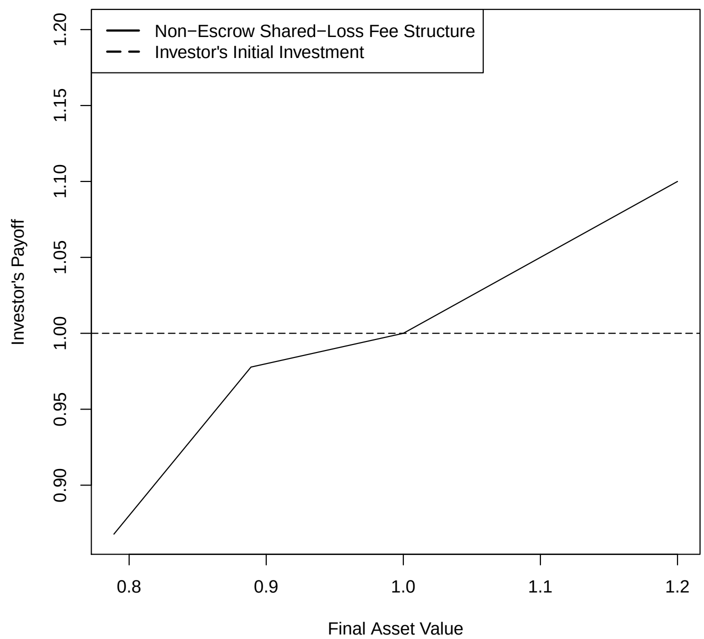

Figure 1.5: Investor's Payoffs: Non-Escrow Shared-Loss Fee Structure with parameters $\omega = 0.1$, $\theta = 0.8$ and $\alpha = 0.5$.

1.2.3 General Formulation

As noted in Chen et al. (2020), under both the first-loss and shared-loss fee structures, the payoff function $g(x)$ can be written in the following form:

$$g(x) = \begin{cases} A + Bx, & 0 \leq x \leq \kappa \\ q + (1-q)x, & \kappa \leq x \leq 1, \\ p + (1-p)x, & 1 \leq x, \end{cases} \tag{1.6}$$

where $B \geq 1 \geq q \geq A \geq 0$, $p \in (0,1)$ and $\kappa = (B - (1-q))^{-1}(q - A)$. The following table provides the detailed parameterization for each fee structure mentioned above.

Payoff	A	B	q	p	κ
Escrow, First-Loss	c	1	1	α	$1-c$
Escrow, Shared-Loss, $c \geq \theta$	θ	1	θ	α	0
Escrow, Shared-Loss, $c < \theta$	c	1	θ	α	$1-\frac{c}{\theta}$
Non-Escrow, First-Loss	0	$1+\omega$	1	α	$(1+\omega)^{-1}$
Non-Escrow, Shared-Loss	0	$1+\omega$	θ	α	$\theta(\omega+\theta)^{-1}$

Table 1.1: Parameterization of the general payoff function (1.6) for all fee structures introduced in Section 1.2.1 and Section 1.2.2.

It is important for the intuitive understanding of the fee arrangements that the above payoff function (1.6) can be interpreted in terms of portfolios of options. In particular, we may regard the position of the investor in the hedge fund as equivalent to the following:

- Owning the assets of the underlying hedge fund (payoff to the investor: x).
- Having given p call options on the hedge fund assets with strike price 1 to the hedge fund manager. These options constitute the performance fee (payoff to the investor: $-p(x-1)_+$).
- Having received from the hedge fund manager q put options on the hedge fund assets with strike price 1. These options constitute the insurance inherent in the shared/first loss fee structures (payoff to the investor $q(1-x)_+$).
- Having given to the hedge fund manager $F = B-(1-q)$ put options with strike price κ. These options represent the limit on the insurance component of the shared/first loss fee structures (payoff to the investor $-F(\kappa - x)_+$).

The total payoff to the hedge fund investor is then:

$$x - p(x-1)_+ + q(1-x)_+ - (B-(1-q))(\kappa - x)_+ = g(x). \tag{1.7}$$

1.3 Summary of the Thesis and its Contributions to Research

1.3.1 Early Withdrawal from Hedge Fund Investments and Optimal Stopping Problems

A critical aspect of the management of investments in hedge funds with first-loss fee structures is the ability of the investor to time the withdrawal of their money. Barr (2011) states that "[t]he downside for managers is that, if they suffer a big monthly loss, they lose their own capital quickly. And first-loss capital providers can pull their money fast to protect their investment". Weiss (2018) notes that billionaire hedge fund manager John Paulson resorted to employing first-loss fee structures in order to attract investors; again, the possibility of early withdrawal of investors' funds was crucial: "[w]hile Prelude and its two peers supply most of the capital in first-loss strategies, they almost never lose any of it. That's because they can shut down an account once most of the hedge fund manager's capital is gone."

In Chapters 2 and 3 of this thesis, we analyze the investor's problem of determining the optimal time to withdraw from an investment in a hedge fund with a first-loss fee structure.[4] In particular, we assume that the investor's share of the hedge fund assets X_t, under the risk-neutral measure $\mathbb{Q}$, satisfies the dynamics:

$$\begin{aligned} dX_t^x &= (r-\delta)X_t^x dt + \sigma X_t^x dW_t, \quad X_0^x = x, \quad t \geq 0, \\ X_t^x &= x \exp\{(r-\delta-\frac{1}{2}\sigma^2)t + \sigma W_t\}. \end{aligned} \tag{1.8}$$

where r is the risk-free rate, $\delta > 0$ is the fee for assets under management (paid continuously), $\sigma > 0$ is the volatility and W_t is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$, the standard augmentation of the filtration generated by W, satisfying the usual conditions. It is worth mentioning that X_0^x denotes the position of investor's share in the hedge fund at current time. The initial investment (which may have taken place before time 0) is assumed to be 1 for Chapters 2 and 3, and this is without loss of generality, as discussed above. From the investor's perspective, we assume the optimal withdrawal time is determined by maximizing the expected discounted payoff

[4] Parts of Chapters 2 and 3 are contained in the paper Meng and Saunders (2019), which has been submitted for publication.

in this risk-neutral world. Then the value function for the infinite horizon case is

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}[e^{-r\tau} g(X_\tau^x)], \tag{1.9}$$

where $\mathcal{T}$ is the set of all stopping times. On the other hand, if the investor has a finite investment horizon T, then the value function at the current time is

$$v(x, T) = \sup_{\tau \in \mathcal{T}_{[0,T]}} \mathbb{E}[e^{-r\tau} g(X_\tau^x)], \tag{1.10}$$

where $\tau \in \mathcal{T}_{[0,T]}$ is the set of all stopping times such that $0 \leq \tau \leq T$, and $g(x)$ is the investor's payoff function (1.6), with different parametrization corresponding to different fee structures. As we will see in Chapters 2 and 3, for interesting values of the fee structure parameters ($q > p$), $V(1) > v(1, T) > 1$. The fair value for an upfront payment (in addition to the investment amount) by the hedge fund investor to the fund manager is $V(1) - 1$ in the infinite horizon case, and $v(1, T) - 1$ when T is the time to maturity at contract initiation, in the finite horizon case.

The value functions with an arbitrary initial investment amount $y \neq 1$ can be obtained using (1.9) and (1.10), in particular as $V(x; y) = y \cdot V(\frac{x}{y})$, and $v(x, T; y) = y \cdot v(\frac{x}{y}, T; y)$. Detailed derivations can be found in Appendix A.

The use of risk-neutral valuation is typically justified in the case of American options based on arbitrage arguments (see, for example Detemple (2006)). In the case of hedge fund investments, some of the assumptions for these arguments, such as the ability to trade in the underlying (and observe its price) continuously do not hold as closely. The motivation for employing risk-neutral valuation is threefold. First of all, there is a clear gain in mathematical convenience and tractability (essentially, working with linear utility functions rather than, e.g. S-shaped utility functions as in He and Kou (2018)). Secondly, equations (1.9) and (1.10) can be interpreted as providing a "fair" price for the hedge fund investment, which gives investors a sense of whether the hedge fund manager is taking advantage of them with the new fee structures or not. Finally, in practice, it is difficult to find a utility function which can describe (or prescribe) different investors' behaviour, and subsequently to determine appropriate parameters for that function. This is significant, as Cremers et al. (2005) show that different specifications of investors' preferences (power utility, bilinear utility and S-shaped utility functions) imply significant divergence in hedge fund investment allocations.

There is a large literature on optimal stopping problems for one-dimensional diffusion processes and their applications in finance, dating back to the seminal work of McKean

(1965). Standard references include the books Detemple (2006), Pham (2009), Peskir and Shiryaev (2006). In the literature, a particularly prominent role is played by the relationship between optimal stopping problems and variational inequalities/free-boundary problems for partial differential operators. This approach to the analysis of optimal stopping problems is also foremost in our work. An optimal stopping problem with a piecewise linear payoff similar to g arises in the analysis of the capped American call option studied in Broadie and Detemple (1995). An optimal stopping game involving piecewise linear payoffs is analyzed in Gapeev (2005). The paper closest to the current work is Chen et al. (2020). In that paper, rather than considering the fund manager's problem of optimally structuring the investments of the hedge fund assets given a specified fee structure, the authors studied the investor's (optimal stopping) problem of when to withdraw the investment from the fund under the assumption that the fund assets follow a geometric Brownian motion. In that paper, no penalty for withdrawal, or fee for assets under management was considered. The infinite horizon optimal stopping problem was solved explicitly, and various properties of the finite horizon problem, including the existence and convexity properties of optimal stopping boundaries were studied. However, the authors mention that they do not investigate other aspects of fee agreements such as the management fee or the penalty for early redemption.

The Case of an Infinite Investment Horizon

The optimal withdrawal time problem for an investor in a hedge fund with a shared-loss fee structure with an infinite horizon is studied in Chapter 2. We follow the established approach of characterizing the value function V of (1.9) as the unique viscosity solution of the variational inequality:

$$\min\left(V - g, rV - LV\right) = 0 \tag{1.11}$$

satisfying appropriate boundary conditions. Here L is the generator of the diffusion (1.8), a differential operator that acts on smooth functions f as:

$$Lf = (r - \delta)\frac{\partial f}{\partial x} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 f}{\partial x^2}. \tag{1.12}$$

Based on financial considerations, and the related work of Chen et al. (2020), we are able to conjecture and then prove properties about the shape of the continuation and stopping regions. In particular, the continuation region is a bounded interval, with the lower stopping boundary being strictly below the initial investment level (normalized to be 1), and the upper boundary strictly above it. The traditional "smooth fit" condition holds at the upper boundary point, but may fail at the lower boundary point. Based on these results, we are subsequently able to derive the value function for the optimal stopping problem explicitly. Computation of the value function requires only the solution of a simple system of nonlinear equations in order to determine the boundaries of the stopping region.[5] Finally, we present examples of the solution of the optimal stopping problem with different values for the model parameters, analyze the sensitivity of the value function to the model parameters, and discuss the financial interpretation of our results.

The Case of a Finite Investment Horizon

In the third chapter, we analyze the optimal stopping problem (1.10), with finite investment horizon. This problem is significantly more difficult mathematically than the infinite horizon problem studied in Chapter 2, and no explicit solution is available. However, we are able to derive several mathematical properties of its solution. As in Chapter 2, we apply the established approach of characterizing the value function as the unique viscosity solution of a variational inequality:

$$\min\left(v - g, rv - Lv + \frac{\partial v}{\partial T}\right) = 0 \tag{1.13}$$

together with appropriate initial and boundary conditions. Again, we are able to show that there are two (time-dependent) stopping boundaries, with the upper boundary being increasing and the lower boundary decreasing, with the limits as $T \to \infty$ being the boundaries from the infinite horizon problem in each case. We then proceed to apply techniques from Detemple (2006) and Peskir (2005) to derive a pair of coupled integral equations for the two stopping boundaries in the case when the smooth fit condition applies. A similar set of integral equations has also been derived by Ciurlia and Roko (2005) in the case of the pricing of American installment options. Finally, following the probabilistic strategy initiated by Lamberton (1995), we derive the asymptotic behaviour of the stopping boundaries in small time.

1.3.2 Hybrid Fee Structures: The Myth of Fee Diversification

In this section, we describe the contributions of Chapter 4 of this thesis.[6] As noted above, in recent years, innovative fee structures have emerged aiming at better alignment between

[5]The solution of the nonlinear system also determines whether the smooth fit condition holds at the lower boundary of the continuation region.

[6]The research in Chapter 4 has been published in Meng et al. (2019).

investors' interests and the hedge fund's business objectives. These range from first-loss agreements, in which the hedge fund manager provides complete insurance for losses (up to some limit), to shared-loss structures, in which losses are borne by both the hedge fund manager and the investor. In all cases, the hedge fund manager is compensated for providing downside protection in terms of a higher performance fee (see Banzaca (2012) for a description). In analogy with the concept of diversification from portfolio theory, some investors and managers began to express the hypothesis[7] that the optimal fee structure from the fund investor's point of view should be a shared-loss structure, i.e. a combination of the extremes.[8]

In Chapter 4, we test the hypothesis of fee diversification, and find that it largely does not hold up to careful scrutiny. In particular, we study the optimization problem of an investor who may choose any combination of the first loss and classical fee structures, and seeks to maximize either the Sharpe ratio or the Sortino ratio of their final payoff, evaluated using real-world probabilities.[9] We demonstrate that for the vast majority of fund mean returns and volatilities, there is no fee diversification effect: either the first-loss structure or the classical structure is optimal for the investor. In the case of the Sortino ratio, this is investigated numerically, while for the Sharpe ratio, we also characterize mathematically the combinations of model parameters for which fee diversification is possible, and present a financial argument for why one should expect this region to be small. Furthermore, in the regions of the parameter space where fee diversification prevails we demonstrate that its impact is not large.

1.3.3 A New Type of Compensation Arrangement: Hedge Funds with Negative Fees

Chapter 5 discusses valuation and risk measurement for investors in a hedge fund with a novel fee structure, referred to as a negative fee arrangement. Negative fee structures provide hedge fund investors with a fixed promised return; as compensation for this promised payment, the hedge fund manager keeps the profits from the investments of the fund's assets. The return profile of an investment in a hedge fund with negative fees then resembles that of an investment in a fixed-income instrument, or an asset-backed security where the underlying pool of assets is the hedge fund's investments. In Chapter 5, we present an

[7] Luis Seco, personal communication, based on experiences at several industry conferences.

[8] The payouts with different fee structures on the same fund are comonotonic, but not perfectly correlated.

[9] We continue to employ the assumption of normally distributed log-returns that was used in earlier chapters.

analysis of the value and risks of investments in such products. We begin by employing risk-neutral valuation, using both Black-Scholes and regime-switching models. The use of option pricing models allows one to calculate a "fair" fee rate, for which the value of the investor's payoff exactly equals their initial investment. We then proceed to analyze the risks inherent in investments in hedge funds with negative fee structures. Given the resemblance of these investments to asset-backed securities, we in particular study probability of default and loss given default under the real-world measure for both geometric Brownian motion and regime-switching models. The impact of the model parameters on the resulting risks of the investment is studied in detail through numerical examples.

The research presented in Chapter 5 has been published in Bhaduri et al. (2018). At the time when the research was undertaken and published, hedge fund investments with negative fee structures were a hypothetical product that we were interested in analyzing as an extension of the shared-loss concept. Since the publication of this research, products with related fee structures have appeared in the European market (Sigma Analysis and Management Ltd., 2020).

1.3.4 Conclusions and Directions for Future Research

The sixth chapter concludes the thesis, and presents several possible extensions of the results in this work, and directions for future research.

Chapter 2

Optimal Withdrawal from Shared-Loss Fee Structures: The Infinite Horizon Case

2.1 Introduction

In this chapter, we analyze the problem of determining the optimal withdrawal time from an investment in a hedge fund with a shared-loss fee structure when there is no investment horizon (i.e. $T = \infty$). The techniques employed are similar to those used for the infinite horizon American put (e.g. Pham (2009)). The value function of the optimal stopping problem is characterized as the unique solution of a variational inequality involving the generator of the diffusion process. A conjecture for the shape of the continuation region is then made based on financial intuition, and it is proved that the continuation region does indeed have this shape. The variational inequality can then be solved using ODE methods and the solution of a pair of nonlinear equations.

We find that the continuation region is an interval, and there are two stopping boundaries, one corresponding to large losses, and the other to large gains. The situation is similar to that which arises with American continuous installment options (see, e.g. Ciurlia and Roko (2005), Kimura (2009)). There is an obvious incentive to withdraw when there are substantial losses, as the insurance against investor losses has basically been exhausted (this is analogous to exercising the option when it is in the money). However, when gains are very large, the cost of the performance fee, which may be interpreted as an option that the fund investor has provided to the fund manager, becomes too large, and the probability of the investor's downside protection taking effect becomes miniscule. At this point, the investor would be better off under another fee structure (this is analogous to the situation when a continuous installment option is exercised/cancelled because the ongoing installment fees are too expensive when the option is very deep out of the money, and unlikely to expire with a positive payoff).

The remainder of the chapter is structured as follows. The second section presents the formulation of the problem of determining the best time to withdraw from the hedge fund investment as an optimal stopping problem, and the characterization of the value function as the unique solution of a variational inequality. The third section discusses basic properties of the stopping and continuation regions. The fourth section gives a detailed derivation of the value function, and the fifth section presents numerical results.

2.2 Problem Formulation and Characterization of the Value Function

Recall from Chapter One that we assume that the investor's share of the hedge fund assets satisfies the dynamics

$$\begin{aligned} dX_t^x &= (r-\delta)X_t^x dt + \sigma X_t^x dW_t, \quad X_0^x = x, \quad t \geq 0, \qquad (2.1) \\ X_t^x &= x\exp\{(r-\delta-\frac{1}{2}\sigma^2)t + \sigma W_t\}, \end{aligned}$$

where $r \geq 0$ is the risk-free rate, $\delta > 0$ is the fee for assets under management (paid continuously), $\sigma > 0$ is the volatility and W_t is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$, the standard augmentation of the filtration generated by W, satisfying the usual conditions.[1] The value function for the investor's problem of determining the optimal time to withdraw from the hedge fund in the infinite

[1]Assuming r and σ to be constant over an infinite horizon, or even a long-dated finite horizon is a simplification that yields a tractable mathematical model, but obviously cannot but justified based on empirical evidence. Extension of the results of this Chapter to more realistic mathematical models is a potential direction for future research. We do note, however, that numerical experiments show that the boundaries for the finite horizon model with constant parameters do converge rather quickly to the infinite horizon boundaries as time to maturity grows (see Chapter 6).

horizon case is

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}[e^{-r\tau} g(X_\tau^x)], \tag{2.2}$$

where $\mathcal{T}$ is the set of all stopping times.

It is well-known that under quite general assumptions, the value function V of the infinite horizon problem (2.2) is a viscosity solution of the following variational inequality:

$$\min\Big(rV - LV, V - g\Big) = 0, \tag{2.3}$$

where L is the infinitesimal generator of the process X. For convenience, we recall here the definition of a viscosity solution in this context (see, e.g. Reikvam (1998), Pham (2009, Definition 4.2.1, Page 63) or Touzi (2013, Definition 6.3, Page 68)). Let L be the infinitesimal generator of the process X, which operates on smooth functions W as

$$LW(x) = (r-\delta)x\frac{\partial W}{\partial x} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 W}{\partial x^2}.$$

Definition 2.2.1. *Let $W \in C([0,\infty),\mathbb{R})$. Then,*

1. *W is a viscosity super-solution of (2.3) if*

$$\min\Big(rW(x_0) - L\varphi(x_0), W(x_0) - g(x_0)\Big) \geq 0 \tag{2.4}$$

for all smooth functions φ and all $x_0 \in (0,\infty)$ such that $W - \varphi$ attains a local minimum at x_0.

2. *W is a viscosity sub-solution of (2.3) if*

$$\min\Big(rW(x_0) - L\psi(x_0), W(x_0) - g(x_0)\Big) \leq 0 \tag{2.5}$$

for all smooth functions ψ and all $x_0 \in (0,\infty)$ such that $W - \psi$ attains a local maximum at x_0.

W is called a viscosity solution of (2.3) if it is both a viscosity super-solution and a viscosity sub-solution.

The results in the following proposition follow from standard techniques, and indeed most of them are special cases of results that can be found in the literature (e.g. monotonicity is immediate, while Lipschitz continuity follows from Pham (2009), Lemma 5.2.1, page 96). They are collected here for completeness and convenience, with no claim to originality intended.

Proposition 2.2.1. *Consider the payoff function $g(x)$ in (1.6) and the value function $V(x)$ in (2.2). The following properties hold:*

a. For $x \in [0, \infty)$, $V(x)$ is non-decreasing, Lipschitz continuous, and $\lim_{x\to\infty} \frac{V(x)}{g(x)} = 1$.

b. If $p \geq q$, then $V(x) = g(x)$.

Proof. a. For $x \geq 0$ and $\tau \in \mathcal{T}$, define

$$J(x, \tau) = \mathbb{E}[e^{-r\tau} g(X_\tau^x)].$$

Then, for $y \geq x$,

$$J(y, \tau) - J(x, \tau) = \mathbb{E}[e^{-r\tau}\big(g(X_\tau^y) - g(X_\tau^x)\big)] \geq 0,$$

because $g(x)$ is an increasing function and $X_\tau^y - X_\tau^x = (y-x)\exp\{(r-\delta-\frac{1}{2}\sigma^2)\tau + \sigma W_\tau\} \geq 0$ for all stopping times $\tau \in \mathcal{T}$. Thus $V(y) = \sup_{\tau\in\mathcal{T}} \mathbb{E}[e^{-r\tau} g(X_\tau^y)] \geq \sup_{\tau\in\mathcal{T}} \mathbb{E}[e^{-r\tau} g(X_\tau^x)] = V(x)$. Hence, $V(x)$ is an increasing function. For Lipschitz continuity, we use that for $r \geq 0, y \geq x$, $e^{-rt}(X_t^y - X_t^x) = (y-x)e^{-\delta t}\exp(-\frac{\sigma^2}{2}t + W_t)$ is a positive, continuous supermartingale so that by the Optional Stopping Theorem (Revuz and Yor (1994) Theorem 3.3, pages 66-67) and the Lipschitz continuity of g:

$$\begin{aligned} |V(y) - V(x)| &= |\sup_{\tau\in\mathcal{T}} \mathbb{E}[e^{-r\tau} g(X_\tau^y)] - \sup_{\theta\in\mathcal{T}} \mathbb{E}[e^{-r\theta} g(X_\theta^x)]| \\ &\leq \sup_{\tau\in\mathcal{T}} \mathbb{E}[e^{-r\tau}|g(X_\tau^y) - g(X_\tau^x)|] \\ &\leq M \sup_{\tau\in\mathcal{T}} \mathbb{E}[e^{-r\tau}|X_\tau^y - X_\tau^x|] \leq M|x-y| \end{aligned} \tag{2.6}$$

where M is the Lipschitz constant of g. Taking $\tau = 0 \in \mathcal{T}$ implies $V(x) \geq g(x)$, so $\liminf_{x\to\infty} V(x)/g(x) \geq 1$. Also, $g(x) \leq A + Bx$ and the fact that $e^{-r\tau}X_\tau^x$ is a positive supermartingale imply, by the Optional Stopping Theorem, $\mathbb{E}[e^{-r\tau} g(X_\tau^x)] \leq \mathbb{E}[A + Be^{-r\tau}X_\tau^x] \leq A + Bx$ for any stopping time $\tau \in \mathcal{T}$, which implies $V(x) \leq A + Bx$. Next, let $\theta_x = \inf\{t \geq 0 : X_t^x = 1\}$. Then, by the Dynamic Programming Principle (Pham, 2009, Page 97), for $x > 1$,

$$\begin{aligned} V(x) &= \sup_{\tau \in \mathcal{T}} \mathbb{E}[e^{-r\tau} g(X_\tau^x) \mathbf{1}_{\tau < \theta_x} + e^{-r\theta_x} V(X_{\theta_x}^x) \mathbf{1}_{\tau \geq \theta_x}] \\ &\leq \sup_{\tau \in \mathcal{T}} \mathbb{E}[e^{-r\tau}(p + (1-p)X_\tau^x) + V(1)e^{-r\theta_x}] \\ &\leq p + (1-p)x + V(1)\mathbb{E}[e^{-r\theta_x}] \\ &\leq g(x) + (A+B). \end{aligned} \tag{2.7}$$

From (2.7), it is easy to verify that $\limsup_{x\to\infty} V(x)/g(x) \leq 1$. Thus, $\lim_{x\to\infty} V(x)/g(x) = 1$.

b. In this case, g is concave, and so Jensen's inequality implies that for $t > s$:

$$\mathbb{E}[g(X_t)e^{-rt}|\mathcal{F}_s] \leq e^{-rt} g(\mathbb{E}[X_t|\mathcal{F}_s]) \leq e^{-rt} g(X_s) \leq e^{-rs} g(X_s) \tag{2.8}$$

where the expectations all exist because of the bound $g(x) \leq A + Bx$. The result then follows from the Optional Stopping Theorem. $\square$

Mathematically, part b of the above result follows from the concavity of g. Financially, the intuition is that in such circumstances the size of the performance fee paid to the hedge fund manager is too large to justify, and in particular is always worth more than the value of the insurance component of the fee structure; it is better for investors to withdraw immediately from the ill-posed contract. Consequently, for the rest of this chapter, we only consider the case when $q > p$.

Theorem 2.2.1. *The value function $V(x)$ in (2.2) is the unique viscosity solution of*

$$\min\Big(rV - LV, V - g\Big) = 0 \tag{2.9}$$

satisfying $V(0) = A$ and $V \sim g$ as $x \to \infty$.

Proof. From part (a) of Proposition 2.2.1, we can verify that $V(x)$ satisfies a linear growth condition. Then, by Pham (2009, Theorem 5.2.1, Page 97-99), $V(x)$ is the unique viscosity solution of (2.9). $\square$

By Theorem 2.2.1, we know that solving (2.2) is equivalent to finding a function satisfying (2.9). Therefore, the next two sections are devoted to finding a more explicit form for $V(x)$. Moreover, given Proposition 2.2.1, we only consider the case when $q > p$.

2.3 Properties of the Stopping and Continuation Regions

Recalling that $V(x) \geq g(x)$, we can divide the interval $[0, \infty)$ into two disjoint sets,

$$\mathcal{S} = \{x | V(x) = g(x)\}, \quad \mathcal{C} = \{x | V(x) > g(x)\},$$

where $\mathcal{S}$ is the stopping region and $\mathcal{C}$ is the continuation region. By Theorem 2.2.1, and $V(x) > g(x)$ in $\mathcal{C}$, we can easily deduce that $V(x)$ satisfies the Cauchy-Euler equation

$$LW - rW = 0, \tag{2.10}$$

in $\mathcal{C}$ (Pham (2009, Lemma 5.2.2, Page 100)). Consider this equation with initial conditions $W(x_0) = z_0$ and $W'(x_0) = z_1$. Let $\beta = \frac{2r}{\sigma^2} > 0$ and $\gamma = \frac{2\delta}{\sigma^2} > 0$. The general solution of (2.10) is of the form

$$W(x) = C_1 x^{z_1} + C_2 x^{z_2}, \tag{2.11}$$

where

$$z_1 = \frac{-(\beta - \gamma - 1) + \sqrt{(\beta - \gamma - 1)^2 + 4\beta}}{2}, \quad z_2 = \frac{-(\beta - \gamma - 1) - \sqrt{(\beta - \gamma - 1)^2 + 4\beta}}{2},$$

and C_1, C_2 are constants depending on the initial conditions. Next, note that since $\beta > 0$ and $\gamma > 0$, the following inequalities must hold:

$$\begin{aligned}
&(\beta - \gamma - 1)^2 + 4\beta > (\beta - \gamma - 1)^2 + 4\beta - 4\gamma \\
&\Rightarrow (\beta - \gamma - 1)^2 + 4\beta > (\beta - \gamma - 1)^2 + 4(\beta - \gamma - 1) + 4 \\
&\Rightarrow (\beta - \gamma - 1)^2 + 4\beta > [(\beta - \gamma - 1) + 2]^2.
\end{aligned} \tag{2.12}$$

If $\beta-\gamma-1+2 \geq 0$, then (2.12) implies

$$\begin{aligned}
&\sqrt{(\beta-\gamma-1)^2+4\beta} > (\beta-\gamma-1)+2 \\
&\Rightarrow -(\beta-\gamma-1)+\sqrt{(\beta-\gamma-1)^2+4\beta} > 2 \\
&\Rightarrow z_1 = \frac{-(\beta-\gamma-1)+\sqrt{(\beta-\gamma-1)^2+4\beta}}{2} > 1.
\end{aligned}$$

On the other hand, if $\beta-\gamma-1+2 < 0$, then

$$\begin{aligned}
&-(\beta-\gamma-1) > 2 \\
&\Rightarrow -(\beta-\gamma-1)+\sqrt{(\beta-\gamma-1)^2+4\beta} > 2 \\
&\Rightarrow z_1 = \frac{-(\beta-\gamma-1)+\sqrt{(\beta-\gamma-1)^2+4\beta}}{2} > 1.
\end{aligned}$$

So, we have proved that $z_1 > 1$. Moreover, $(\beta-\gamma-1)^2+4\beta > (\beta-\gamma-1)^2 > 0$ implies $-(\beta-\gamma-1)-\sqrt{(\beta-\gamma-1)^2+4\beta} < 0$ so that $z_2 < 0$. Finally, the initial conditions $W(x_0)=z_0$ and $W'(x_0)=z_1$ give us the unique solutions for C_1 and C_2 as follows,

$$C_1 = x_0^{-z_1}\frac{x_0 z_1 - z_0 z_2}{z_1-z_2}, \quad C_2 = x_0^{-z_2}\frac{z_0 z_1 - x_0 z_1}{(z_1-z_2)}. \tag{2.13}$$

Remark 2.3.1. *From (2.13), we can easily find,*

$$W''(x) = C_1 z_1(z_1-1)x^{z_1-2} + C_2 z_2(z_2-1)x^{z_2-2}.$$

Since $z_1 > 1$, $z_2 < 0$, we can observe if either $C_1 > 0, C_2 \geq 0$ or $C_1 \geq 0, C_2 > 0$ holds, then $W(x)$ is strictly convex on $(0,\infty)$.

Before deriving the form of $V(x)$ in the next section, we present and prove the following properties of the stopping region and the continuation region.

Proposition 2.3.1. *Suppose that $q > p$, then,*

a. $[0,\kappa] \subseteq \mathcal{S}$.

b. $1 \in \mathcal{C}$.

c. *If $a < 1$ and $a \in \mathcal{S}$, then $[0,a] \subseteq \mathcal{S}$.*

d. *If $b > 1$ and $b \in \mathcal{S}$, then $[b,\infty] \subseteq \mathcal{S}$.*

Proof. a. Taking $\tau = 0 \in \mathcal{T}$, we have $g(x) \leq V(x)$. That $g(x) \leq A + Bx$ and $e^{-rt}X_t^x$ is a supermartingale give us $V(x) \leq A + Bx$. Thus, for $x \in [0, \kappa]$, we conclude $V(x) = A + Bx$.

b. Suppose $1 \in \mathcal{S}$. Consider test functions of the form $\varphi(x) = 1 - M_n + M_n \exp(n(x-1))$, where $M_n = \frac{\xi}{n}$ and $\xi \in (1-q, 1-p)$. Clearly, $\varphi(1) = 1$, $\varphi'(1) = \xi$, and $\varphi''(1) = n\xi$. So, $\varphi(x) < g(x)$ for x close to 1. By the super-solution property, we know $r\varphi(1) - L\varphi(1) \geq 0$, but

$$r\varphi(1) - L\varphi(1) = r - (r-\delta)\xi - \frac{1}{2}\sigma^2 n\xi = r(1-\xi) + \delta\xi - \frac{1}{2}\sigma^2 n\xi < 0$$

for n large enough, which contradicts the super-solution property. So $1 \in \mathcal{C}$.

c. If $a \leq \kappa$, $[0,\kappa] \subseteq \mathcal{S}$ implies $[0,a] \subseteq \mathcal{S}$. So we only consider the case when $\kappa < a < 1$. Suppose there exists an $\tilde{x} \in (\kappa, a]$ such that $V(\tilde{x}) > g(\tilde{x})$. Let $h(x) = V(x) - g(x)$. We have $h(\kappa) = h(a) = 0$ since κ and a are in the stopping region. Noting that $h(\tilde{x}) > 0$, then we must have some point(s) $x^* \in (\kappa, a)$ at which h attains a strictly positive local maximum. In other words, we have the following relations,

$$h(x^*) > 0, x^* \in \mathcal{C},\ h'(x^*) = 0,\ \text{and } h''(x^*) \leq 0.$$

Now, it can be easily obtained that,

$$rh(x^*) - Lh(x^*) = rh(x^*) - \frac{\sigma^2(x^*)^2}{2}h''(x^*) - (r-\delta)xh'(x^*) \geq 0. \tag{2.14}$$

However, $x^* \in \mathcal{C}$ implies that $rV(x^*) - LV(x^*) = 0$ and it is easy to see that:

$$rg(x) - Lg(x) = \begin{cases} rA + \delta xB, & x \in (0,\kappa), \\ rq + \delta x(1-q), & x \in (\kappa, 1), \\ rp + \delta x(1-p), & x \in (1,\infty), \end{cases}$$

so that $rg - Lg > 0$ for $x \in (0,\kappa) \cup (\kappa,1) \cup (1,\infty)$. Therefore, we must have $rh(x^*) - Lh(x^*) = r(V(x^*) - g(x^*)) - L(V(x^*) - g(x^*)) < 0$, a contradiction.

d. Suppose $\tilde{x} = \inf\{x > b | x \in \mathcal{C}\}$ exists. Then $V(\tilde{x}) = p + (1-p)\tilde{x}$ and $V'(\tilde{x}) = (1-p)$. Again, $V(x) = C_1 x^{z_1} + C_2 x^{z_2}$ in $\mathcal{C}$. It is easy to verify that $C_1 > 0$, $C_2 > 0$ and hence $V(x)$ is strictly convex. Again, by the continuity and strict convexity of $V(x)$, it follows that $x \in \mathcal{C}$ for all $x > \tilde{x}$. But,

$$\lim_{x\to\infty} \frac{V(x)}{g(x)} = \lim_{x\to\infty} \frac{C_1 x^{z_1} + C_2 x^{z_2}}{p + (1-p)x} = \lim_{x\to\infty} \frac{C_1 z_1 x^{z_1 - 1} + C_2 z_2 x^{z_2 - 1}}{1-p} = \infty.$$

This contradicts the fact that $\lim_{x\to\infty} V(x)/g(x) = 1$ when x approaches infinity.

□

From Proposition 2.3.1, we can define the stopping boundaries as follows:

$$S_1 := \inf\{x \in [\kappa, 1) | V(x) > g(x)\}, \quad S_2 := \sup\{x > 1 | V(x) > g(x)\}.$$

Then, $\kappa \le S_1 < 1$, $S_2 > 1$, $\mathcal{C} = (S_1, S_2)$ and $\mathcal{S} = [0, S_1] \cup [S_2, \infty]$ with the possible special case $\mathcal{C} = (S_1, \infty)$ and $\mathcal{S} = [0, S_1]$ if $S_2 = \infty$. In the following section, we will show that under all choices of parameters, $S_2 < \infty$, and present the analytic solution of $V(x)$ when $q > p$.

2.4 Derivation of the Value Function

For $V(x)$ with no fee for assets under management (i.e. $\delta = 0$), Chen et al. (2020) proved that the continuation region starts either at $S_1, \kappa \le S_1 < 1$ with the smooth-fit condition or at κ without the smooth-fit condition. These two cases are determined by the parameter values. Following their result, we will seek criteria that distinguish these two cases.

Proposition 2.4.1. *(Smooth-Fit Condition) If $S_1 = \kappa$, then $V'(S_1) \in [1-q, B]$. If $\kappa < S_1 < 1$, then $V'(S_1) = g'(S_1)$. Moreover, if $S_2 < \infty$, $V'(S_2) = g'(S_2)$.*

Proof. The proof mimics the technique from Pham (2009, Proposition 5.2.1). Since $V(x) \ge g(x)$ and $V(S_1) = g(S_1)$, we have $g'(S_1+) \le V'(S_1) \le g'(S_1-)$ and $V'(S_1-) \le g'(S_1) \le V'(S_1+)$. Therefore, if $S_1 = \kappa$, then $V'(S_1) \in [1-q, B]$. If $\kappa < S_1 < 1$, we assume $V'(S_1)$ does not exist. Then, we must have $V'(S_1+) > g'(S_1+) = g'(S_1-) = V'(S_1-)$. Now, taking some $a_1 \in (V'(S_1-), V'(S_1+))$, then we have $V'(S_1+) - a_1 > 0$ and $V'(S_1-) - a_1 < 0$. Next, for $\varepsilon > 0$, we define the test function:

$$\varphi_\varepsilon(x) = V(S_1) + a_1(x - S_1) + \frac{1}{2\varepsilon}(x - S_1)^2. \tag{2.15}$$

Differentiating $V(x) - \varphi_\varepsilon(x)$ for $x \neq S_1$, we obtain

$$V'(x) - \varphi'_\varepsilon(x) = V'(x) - a_1 - \frac{1}{\varepsilon}(x - S_1). \tag{2.16}$$

Then, from (2.16) it is easy to check

$$V'(S_1+) - \varphi'_\varepsilon(S_1+) > 0 \text{ and } V'(S_1-) - \varphi'_\varepsilon(S_1-) < 0,$$

for x close to S_1. Therefore, $V(x) - \varphi_\varepsilon(x)$ attains a local minimum at S_1 and $V(S_1) - \varphi_\varepsilon(S_1) = 0$. By the super-solution property, we must have

$$r\varphi_\varepsilon(x) - L\varphi_\varepsilon(x) = rV(S_1) - (r-\delta)a_1 S_1 - \frac{\sigma^2 S_1^2}{2\varepsilon} \geq 0.$$

But, for ε sufficiently small, $rV(S_1) - (r-\delta)a_1 S_1 - \frac{\sigma^2 S_1^2}{2\varepsilon} < 0$, contradicting the above inequality. Thus, $V'(S_1) = g'(S_1)$.

Finally, to prove $V'(S_2) = g'(S_2)$, we use exactly the same arguments as above to show $V'(S_2-) \leq g'(S_2) \leq V'(S_2+)$ and assume $V'(S_2)$ does not exist. Then, define the test function

$$\varphi_\varepsilon(x) = V(S_2) + a_2(x - S_2) + \frac{1}{2\varepsilon}(x - S_2)^2, \tag{2.17}$$

where $a_2 \in (V'(S_2-), V'(S_2+))$ and $\varepsilon > 0$. Then, $V(x) - \varphi_\varepsilon(x)$ must attain a local minimum at S_2 and $V(S_2) - \varphi_\varepsilon(S_2) = 0$. But, when ε is sufficiently small,

$$r\varphi_\varepsilon(x) - L\varphi_\varepsilon(x) = rV(S_2) - (r-\delta)a_2 S_2 - \frac{\sigma^2 S_2^2}{2\varepsilon} < 0, \tag{2.18}$$

which contradicts the super-solution property. Thus, $V'(S_2) = g'(S_2)$. $\square$

Next, we introduce some notation. Let $W(x; x_0, v_0)$, $C_1(x_0, v_0)$, and $C_2(x_0, v_0)$ denote $W(x) = C_1 x^{z_1} + C_2 x^{z_2}$, the solution to (2.10) with initial values $W(x_0) = q + (1-q)x_0$ and $W'(x_0) = v_0$.

Lemma 2.4.1. *a. For $x_0 \in (0, \infty)$, $C_1(x_0, 1-q)$ is decreasing in x_0, $C_2(x_0, 1-q)$ is increasing in x_0 and $W(x; x_0, 1-q)$ is a strictly convex function of x on $(0, \infty)$. Thus, for $0 < x_1 < x_2$, $W(x; x_1, 1-q) > W(x; x_2, 1-q)$ for all $x \geq x_2$.*

b. For $\frac{(q+(1-q)\kappa)z_2}{\kappa} \leq v_0 \leq \frac{(q+(1-q)\kappa)z_1}{\kappa}$, $W(x; \kappa, v_0)$ is a strictly convex function of x on

$(0,\infty)$. *In particular,* $W(x;\kappa,\frac{qz_1+(1-q)\kappa z_1}{\kappa}) = \kappa^{-(z_1-1)}(1-q+\frac{q}{\kappa})x^{z_1} > g(x)$. *Moreover, for* $x>\kappa$, $W(x;\kappa,v_0)$ *is increasing in* v_0.

The proof of this lemma can be found in Appendix B. Now, define

$$h(x) := W(x;\kappa,1-q) - (p+(1-p)x). \tag{2.19}$$

Then,

$$\begin{aligned}
&h(x) = C_1(\kappa,1-q)x^{z_1} + C_2(\kappa,1-q)x^{z_2} - (p+(1-p)x).\\
&h'(x) = z_1C_1(\kappa,1-q)x^{z_1-1} + z_2C_2(\kappa,1-q)x^{z_2-1} - (1-p),\\
&h''(x) = W''(x;\kappa) > 0.
\end{aligned}$$

Note that $h'(\kappa) = 1-q-1+p = p-q < 0$, $\lim_{x\to\infty} h'(x) = \infty$ and $h''(x) > 0$ for all $x>0$. So, there must exist a unique $x_* > \kappa$ such that $h'(x_*) = 0$. Now, if $h(x_*) > 0$, then the convexity of $h(x)$ implies that $W(x;\kappa,1-q)$ never touches the line $p+(1-p)x$ from above for $x>\kappa$. Since $W(x;\kappa,1-q)$ is always greater than its tangent line $q+(1-q)x$ for $x>\kappa$ as well, $W(x,\kappa,1-q)$ never touches $g(x)$ for $x>\kappa$. Consequently, $W(x;\kappa,v_0)$ never touches $g(x)$ for $x>\kappa$ and $v_0 > 1-q$ by Lemma 2.4.1 Part (b). However, we know that $V(x)$ is continuous and the smooth fit condition should hold at S_2. Hence, we are seeking solutions with initial value $S_1\in[\kappa,1)$ such that

$$\begin{cases} W(S_2;S_1,1-q) = p+(1-p)S_2,\\ W'(S_2;S_1,1-q) = 1-p. \end{cases} \tag{2.20}$$

On the other hand, when $h(x_*) < 0$, it is easy to check that $W(x_*;x_0,1-q)-(p+(1-p)x) < h(x_*) < 0$ for $\kappa < x_0 < 1$. Thus, by the convexity of $W(x;x_0,1-q)$, there does not exist an $S_2 > 1$ such that the smooth-fit condition holds at S_2. Therefore, we shall seek solutions on $v_0\in[1-q,\infty)$ such that

$$\begin{cases} W(S_2;\kappa,v_0) = p+(1-p)S_2,\\ W'(S_2;\kappa,v_0) = 1-p. \end{cases} \tag{2.21}$$

The following proposition shows that there indeed exists a unique solution to (2.20) and (2.21).

Proposition 2.4.2. *Suppose* $p<q$. *Consider the function* $h(x)$ *in equation* (2.19) *and let* $x_*>\kappa$ *be the unique root such that* $h'(x_*)=0$.

a. If $h(x_*)\ge 0$, *then there exists a unique solution* $(S_1,S_2)\in[\kappa,1)\times(1,\infty)$ *to the* *following system of equations*

$$C_1(S_1, 1-q)S_2^{z_1} + C_2(S_1, 1-q)S_2^{z_2} = p + (1-p)S_2, \tag{2.22}$$

$$z_1C_1(S_1, 1-q)S_2^{z_1-1} + z_2C_2(S_1, 1-q)S_2^{z_2-1} = 1-p. \tag{2.23}$$

b. *If $h(x_*) < 0$, then there exists a unique solution $(v_0, S_2) \in \mathbb{R} \times (1, \infty)$ to the following system of equations*

$$C_1(\kappa, v_0)S_2^{z_1} + C_2(\kappa, v_0)S_2^{z_2} = p + (1-p)S_2, \tag{2.24}$$

$$z_1C_1(\kappa, v_0)S_2^{z_1-1} + z_2C_2(\kappa, v_0)S_2^{z_2-1} = 1-p. \tag{2.25}$$

Proof. a. (1) Existence. Let

$$S_1 = \inf\{0 < s < 1 | W(x; s, 1-q) = g(x) \text{ for some } x > 1\}.$$

For $S_1 \leq s < 1$, define $x(s) = \inf\{x > 1 | W(x; s, 1-q) = g(x)\}$. Note that by Lemma 2.4.1, if $z_2 < z_1$, $W(x; z_2, 1-q) > W(x; z_1, 1-q)$ for $x \leq x(z_1)$. Therefore, $x(z_2) > x(z_1)$. Let $s_n \downarrow S_1$. Then, $x(s_n) \uparrow x(S_1) = S_2$, provided that S_2 is finite. Now, let

$$f_1(x; s) = W(x; s, 1-q) - g(x),$$

$$f_1'(x; s) \coloneqq \frac{d}{dx} f_1(x; s).$$

Suppose $f_1'((x(s_n); s_n) > 0$. Since $f_1(x(s_n); s_n) = 0$, $f_1(x(s_n) - \varepsilon; s_n) < 0$ for ε small enough. Note that $f_1'('(x; s) = W''(x; s, 1-q) > 0$ and $f_1(1; s_n) > 0$ implies that there exists $x \in (1, x(s_n))$ such that $f_1(x; s_n) = 0$, which contradicts the definition of $x(s_n)$. Hence $f_1'(x(s_n); s_n) \leq 0$, so $f_1'(S_2; S_1) \leq 0$. Next, suppose $f_1'(S_2; S_1) < 0$, then for ε small enough, we have $f_1(S_2 + \varepsilon; S_1) < 0$. By continuity of $f_1(x; s)$, there exists some $\delta > 0$ such that $f_1(S_2 + \varepsilon; S_1 - \delta) < 0$, which contradicts the definition of S_1. Therefore, $f_1'(S_2; S_1) = 0$ and (S_1, S_2) is a solution of the following equations,

$$\begin{cases} C_1(S_1, 1-q)S_2^{z_1} + C_2(S_1, 1-q)S_2^{z_2} = p + (1-p)S_2, \\ z_1C_1(S_1, 1-q)S_2^{z_1-1} + z_2C_2(S_1, 1-q)S_2^{z_2-1} = 1-p. \end{cases} \tag{2.26}$$

Also, note that $\lim_{x\to\infty} f_1'(x; S_1) = \infty$ and $f_1''(x; S_1) > 0$. Thus, there exists a C large enough such that $f_1'(C; S_1) > 0$, so S_2 is bounded by C.
(2) Uniqueness. Suppose there exists another solution (S_1^*, S_2^*), $S_1^* \neq S_1$ to (2.22) and (2.23). First, we assume $S_1^* < S_1$. Then by Lemma 2.4.1 Part (a), $W(x; S_1^*, 1-q) >$

$W(x; S_1, 1-q) \geq g(x) > p + (1-p)x$ for $x \geq S_1$. The convexity of $W(x; S_1^*, 1-q)$ implies $W(x; S_1^*, 1-q) \geq q + (1-q)x > p + (1-p)x$ for $S_1^* \leq x < S_1$. Therefore, no such $S_2^* \in [S_1^*, \infty)$ could satisfy (2.22). Also, from the convexity of $W(x; S_1^*, 1-q)$, we know $W'(x; S_1^*, 1-q) < 1-q < 1-p$ for $0 < x < S_1^*$. Therefore, no such $S_2^* \in [0, S_1^*)$ could satisfy (2.23). Thus, $S_1^* > S_1$. However, we know $p+(1-p)S_2 = W(S_2; S_1, 1-q) > W(S_2; S_1^*, 1-q)$. The strict convexity of $W(x, S_1^*, 1-q)$ implies that there must exist $0 < a_1 < a_2$ such that $W(a_1; S_1^*; 1-q) = p + (1-p)a_1$ and $W(a_2; S_1^*; 1-q) = p + (1-p)a_2$. The Mean Value Theorem implies there must exist a point $c \in (a_1, a_2)$ satisfying $W'(c; S_1^*, 1-q) = 1-p$. Again, the strict convexity of $W(x; S_1^*; 1-q)$ implies that $W'(a_1; S_1^*; 1-q) < 1-p$ and $W'(a_2; S_1^*; 1-q) > 1-p$. Thus, no such S_2^* exists when $S_1^* > S_1$ as well. So, $S_1^* = S_1$. Thus, we have $f_1(S_2^*; S_1) = 0$ and $f_1'(S_2^*; S_1) = 0$. Since $f_1(x; S_1)$ is a strictly convex function of x on $[1, \infty)$, S_2^* is the unique point on $[1, \infty)$ satisfying $f_1(S_2^*; S_1) = 0$ and $f_1'(S_2^*; S_1) = 0$. But, we also have shown $f_1(S_2; S_1) = 0$ and $f_1'(S_2; S_1) = 0$ with $S_2 > 1$ above. Therefore, $S_2^* = S_2$.
b. (1) Existence. Let

$$\mathcal{X} \coloneqq \left\{0 < v < \frac{qz_1 + (1-q)\kappa z_1}{\kappa} | W(x; \kappa, v) = g(x) \text{ for some } x > 1\right\}, \quad v_0 \coloneqq \sup \mathcal{X}.$$

Note that $h(x_*) < 0$ implies $1-q \in \mathcal{X}$, so $\mathcal{X}$ is non-empty. For $0 \leq v < v_0$, define $x(v) = \inf\{x > 1 | W(x; \kappa, v) = g(x)\}$. Note that for $v_1 < v_2$, $W(x; \kappa, v_2) > W(x; \kappa, v_1)$ for $x \leq x(v_1)$. Therefore, $x(v_2) > x(v_1)$. Let $v_n \uparrow v_0$. Then, $x(v_n) \uparrow x(v_0) \equiv S_2$, provided that S_2 is finite. Define

$$\begin{aligned} f_2(x; v) &= W(x; \kappa, v) - g(x), \\ f_2'(x; v) &\coloneqq \frac{d}{dx} f_2(x; v). \end{aligned}$$

Suppose $f_2'(x(v_n); v_n) > 0$. Since $f_2(x(v_n); v_n) = 0$, $f_2(x(v_n) - \varepsilon; v_n) < 0$ for ε small enough. Note that $f_2''(x; v) = W''(x; \kappa, v) > 0$ and $f_2(1; v_n) > 0$ implies that there exists $x \in (1, x(v_n))$ such that $f_2(x; v_n) = 0$, which contradicts the definition of $x(v_n)$. Hence $f_2'(x(v_n); v_n) \leq 0$, so $f_2'(S_2; v_0) \leq 0$. Next, suppose $f_2'(S_2; v_0) < 0$. Then for ε small enough, we have $f_2(S_2 + \varepsilon; S_1) < 0$. By continuity of $f_2(x; v)$, there exists some $\delta > 0$ such that $f_2(S_2 + \varepsilon; S_1 - \delta) < 0$, which contradicts the definition of S_1. Therefore, $f_2'(S_2; v_0) = 0$ and (v_0, S_2) is the solution of the following equations:

$$\begin{cases} C_1(\kappa; v_0) S_2^{z_1} + C_2(\kappa; v_0) S_2^{z_2} = p + (1-p) S_2, \\ z_1 C_1(\kappa; v_0) S_2^{z_1 - 1} + z_2 C_2(\kappa; v_0) S_2^{z_2 - 1} = 1 - p. \end{cases} \tag{2.27}$$

Also, $\lim_{x\to\infty} f_2'(x; v_0) = \infty$ and $f_2''(x; v_0) > 0$. Thus, there exists some b large enough such that $f_2'(b; v_0) > 0$, so S_2 is bounded by b from above.
(2) Uniqueness. Suppose there exists another solution (v_0^*, S_2^*) to (2.24) and (2.25). First, we assume $v_0^* > v_0$. Then by Proposition 2.4.1 Part (b), $W(x; \kappa, v_0^*) > W(x; \kappa, v_0) \geq g(x) > p + (1-p)x$ for $x \geq \kappa$. Therefore, no such $S_2^* \in (1, \infty)$ could satisfy (2.24). Thus, $v_0^* \leq v_0$. For $\frac{(q+(1-q)\kappa)z_2}{\kappa} \leq v_0^* < v_0$, we know $p + (1-p)S_2 = W(S_2; \kappa, v_0) > W(S_2; \kappa, v_0^*)$. The strict convexity of $W(x, \kappa, v_0^*)$ implies that there must exist $0 < a_1 < a_2$ such that $W(a_1; \kappa, v_0^*) = p + (1-p)a_1$ and $W(a_2; \kappa, v_0^*) = p + (1-p)a_2$. The Mean Value Theorem implies that there must exist a point $c \in (a_1, a_2)$ satisfying $W'(c; \kappa, v_0^*) = 1 - p$. So, $W'(a_1; \kappa, v_0^*) < 1 - p$ and $W'(a_2; \kappa, v_0^*) > 1 - p$. Thus, there does not exist an S_2^* satisfying (2.24) and (2.25) at the same time. Finally, when $v_0^* < 0$, $C_1(\kappa, v_0^*) < 0$. We can calculate

$$W(x; \kappa, v_0^*) = z_1 C_1(\kappa, v_0^*) x^{z_1 - 1} + z_2 C_2(\kappa, v_0^*) x^{z_2 - 1} < 0,$$

for all $x > 0$. Thus, no S_2^* exists that satisfies (2.25). Therefore, we have proved that $v_0 = v_0^*$. Thus, we have $f_2(S_2^*; v_0) = 0$ and $f_2'(S_2^*; v_0) = 0$. Since $f_2(x; v_0)$ is a strictly convex function of x on $[1, \infty)$, S_2^* is the unique point on $[1, \infty)$ satisfying $f_2(S_2^*; v_0) = 0$ and $f_2'(S_2^*; v_0) = 0$. But, we have also shown that $f_2(S_2; v_0) = 0$ and $f_2'(S_2; v_0) = 0$ with $S_2 > 1$. Therefore, $S_2^* = S_2$. ☐

Finally, in the following theorem, we propose the solution for $V(x)$ and show it is indeed the viscosity solution to (2.9).

Theorem 2.4.1. *Suppose $p < q$. Consider the function $h(x)$ in equation (2.19) and let $x_* > \kappa$ be the unique root such that $h'(x_*) = 0$.*

a. If $h(x_*) \geq 0$, then the value function $V(x)$ is

$$V(x) = \begin{cases} g(x), & x \in [0, S_1], \\ W(x; S_1, 1-q), & x \in (S_1, S_2), \\ g(x), & x \in [S_2, \infty), \end{cases} \tag{2.28}$$

where $S_1 \in [\kappa, 1)$ and $S_2 > 1$ are the unique solutions of the system of equations,

$$\begin{cases} C_1(S_1, 1-q) S_2^{z_1} + C_2(S_1, 1-q) S_2^{z_2} = p + (1-p) S_2, \\ z_1 C_1(S_1, 1-q) S_2^{z_1 - 1} + z_2 C_2(S_1, 1-q) S_2^{z_2 - 1} = 1 - p. \end{cases} \tag{2.29}$$

b. If $h_1(x_*) < 0$, then the value function $V(x)$ is

$$V(x) = \begin{cases} g(x), & x \in [0, \kappa], \\ W(x; \kappa, v_0), & x \in (\kappa, S_2), \\ g(x), & x \in [S_2, \infty), \end{cases} \tag{2.30}$$

where $v_0 \in (1-q, \infty)$ and $S_2 > 1$ are the unique solutions of the system of equations,

$$\begin{cases} C_1(\kappa, v_0) S_2^{z_1} + C_2(\kappa, v_0) S_2^{z_2} = p + (1-p) S_2, \\ z_1 C_1(\kappa, v_0) S_2^{z_1 - 1} + z_2 C_2(\kappa, v_0) S_2^{z_2 - 1} = 1 - p. \end{cases} \tag{2.31}$$

Proof. (a) By Proposition 2.4.2 Part (a), we can find a unique pair $(S_1, S_2) \in [\kappa, 1) \times (1, \infty)$ such that the smooth fit condition holds at S_2. Now, all we need to show is that (2.28) is the viscosity solution of (2.9).

Sub-solution. Suppose ψ is a smooth test function satisfying $\psi \geq V$ and $\psi(x) = V(x)$. For $x \in [0, S_1] \cup [S_2, \infty)$, $V(x) = g(x)$ implies the viscosity sub-solution property of $V(x)$. For $x \in (S_1, S_2)$, $r\psi(x) - L\psi(x) \leq rV(x) - LV(x) = 0$. It follows that the viscosity sub-solution property holds for $V(x)$.
Super-solution. Suppose φ is a smooth test function satisfying $\varphi \leq V$ and $\varphi(x) = V(x)$. For $x \in (S_1, S_2)$, $r\varphi(x) - L\varphi(x) \geq rV(x) - LV(x) = 0$ and $V(x) \geq g(x)$ implies that the viscosity super-solution property holds. For $x \in (0, \kappa) \cup (\kappa, S_1) \cup (S_2, \infty)$, $r\varphi(x) - L\varphi(x) = rg(x) - Lg(x) \geq 0$. Thus, the viscosity super-solution inequality holds. Next, at $x = \kappa$, the assumptions on φ give:

$$\begin{aligned} B = g'(\kappa^-) = \lim_{x \to \kappa^-} \frac{g(x) - g(\kappa)}{x - \kappa} \leq \lim_{x \to \kappa^-} \frac{\varphi(x) - \varphi(\kappa)}{x - \kappa} = \varphi'(\kappa^-) = \varphi'(\kappa) \\ 1 - q = g'(\kappa^+) = \lim_{x \to \kappa^+} \frac{g(x) - g(\kappa)}{x - \kappa} \geq \lim_{x \to \kappa^+} \frac{\varphi(x) - \varphi(\kappa)}{x - \kappa} = \varphi'(\kappa^+) = \varphi'(\kappa), \end{aligned}$$

which leads to the inequality $B \leq \varphi(\kappa) \leq 1 - q$. However, $B > 1 - q$. Therefore, no such smooth function φ exists and the super-solution condition holds vacuously. Finally, at $S_i, i = 1, 2, S_1 \neq \kappa$, since $V - \varphi \geq 0$ and $V(S_i) - \varphi(S_i) = 0$, then $V'(S_i) = \varphi'(S_i)$. Moreover, by Taylor's Remainder Theorem,

$$\begin{aligned} V(x) - \varphi(x) &= V(S_i) - \varphi(S_i) + (V'(S_i) - \varphi'(S_i))(x - S_i) + \frac{1}{2}(V''(c) - \varphi''(c))(x - S_i)^2 \\ &= \frac{1}{2}(V''(c) - \varphi''(c))(x - S_i)^2, \end{aligned}$$

where c is an interior point between x and S_i. So, for $x > S_i$ and $x < S_i$, we can obtain,

$$\liminf_{x \to S_i^+} V^{''}(x) \geq \varphi^{''}(S_i), \quad \liminf_{x \to S_i^-} V^{''}(x) \geq \varphi^{''}(S_i),$$

so $r\varphi(S_i) - L\varphi(S_i) \geq \limsup_{x \to S_i} rV(x) - LV(x) \geq 0$. Thus, the viscosity super-solution property is satisfied at $S_i, i = 1, 2$.
(b) By Proposition 2.4.2 Part (b), we can find a unique pair $(v_0, S_2) \in [1-q, \infty) \times (1, \infty)$ such that the smooth fit condition holds at S_2. Now, all we need to show is that (2.30) is the viscosity solution of (2.9).

Sub-solution. Suppose ψ is a smooth test function satisfying $\psi \geq V$ and $\psi(x) = V(x)$. For $x \in [0, \kappa] \cup [S_2, \infty)$, $V(x) = g(x)$ implies the viscosity sub-solution property of $V(x)$. For $x \in (\kappa, S_2)$, $r\psi(x) - L\psi(x) \leq rV(x) - LV(x) = 0$. It follows that the viscosity sub-solution property holds for $V(x)$.
Super-solution. Suppose φ is a smooth test function satisfying $\varphi \leq V$ and $\varphi(x) = V(x)$. For $x \in (\kappa, S_2)$, $r\varphi(x) - L\varphi(x) \geq rV(x) - LV(x) = 0$, and $V(x) \geq g(x)$ implies that the viscosity super-solution property holds. For $x \in (0, \kappa) \cup (S_2, \infty)$, $rV(x) - LV(x) = rg(x) - Lg(x) \geq 0$ and the viscosity super-solution inequality holds. Next, at $x = \kappa$, the assumptions on φ give:

$$\begin{aligned}
B = g'(\kappa^-) = \lim_{x \to \kappa^-} \frac{g(x) - g(\kappa)}{x - \kappa} \leq \lim_{x \to \kappa^-} \frac{\varphi(x) - \varphi(\kappa)}{x - \kappa} = \varphi'(\kappa^-) = \varphi'(\kappa) \\
v_0 = g'(\kappa^+) = \lim_{x \to \kappa^+} \frac{g(x) - g(\kappa)}{x - \kappa} \geq \lim_{x \to \kappa^+} \frac{\varphi(x) - \varphi(\kappa)}{x - \kappa} = \varphi'(\kappa^+) = \varphi'(\kappa),
\end{aligned}$$

which leads to the inequality $B \leq \varphi(\kappa) \leq v_0$. However, the strict convexity of $W(x; \kappa, v)$ implies $v_0 \leq 1 - p$. So, $v_0 \leq 1 - p < 1 \leq B$. Therefore, no such smooth function φ exists and the super-solution condition holds vacuously. Finally, at S_2, since $V - \varphi \geq 0$ and $V(S_2) - \varphi(S_2) = 0$, then $V'(S_2) = \varphi'(S_2)$. Moreover, by Taylor's Remainder Theorem,

$$\begin{aligned}
V(x) - \varphi(x) &= V(S_2) - \varphi(S_2) + (V'(S_2) - \varphi'(S_2))(x - S_2) + \frac{1}{2}(V''(c) - \varphi''(c))(x - S_2)^2 \\
&= \frac{1}{2}(V''(c) - \varphi''(c))(x - S_2)^2,
\end{aligned}$$

where c is an interior point between x and S_2. So, for $x > S_2$ and $x < S_2$, we obtain

$$\liminf_{x \to S_2^+} V^{''}(x) \geq \varphi^{''}(S_2), \quad \liminf_{x \to S_2^-} V^{''}(x) \geq \varphi^{''}(S_2).$$

Then, we can deduce that $\varphi''(S_2) \leq \liminf_{x\to S_2} V''(x)$, so $r\varphi(S_2) - L\varphi(S_2) \geq \limsup_{x\to S_2} rV(x) - LV(x) \geq 0$. Thus, the viscosity super-solution property holds at S_2. $\square$

2.5 Numerical Results

In this section, we apply the above results to see how the parameters affect the stopping boundaries and the value function for different fee structures. By Theorem 2.4.1 we conclude with the following steps to solve for $V(x)$ when $q > p$ for the infinite horizon case.

1. Solve $h'(x_*) = 0$ and calculate $h(x_*)$.
2. If $h(x_*) \geq 0$, then $V(x)$ is of the form (2.28) and (S_1, S_2) are solutions of equation (2.29).
3. If $h(x_*) < 0$, then $V(x)$ is of the form (2.30) and (v_0, S_2) are solutions of equation (2.31).

By applying the above algorithm we are able to provide the value function for different fee structures. In particular, the following figures are obtained by fixing any two of the three parameters r, δ, σ and letting the remaining parameter vary in its reasonable range. From the plots, we can observe that the value function increases and the continuation region becomes wider as r decreases or σ increases in all the fee structures, while δ has a relatively small impact on the value function.

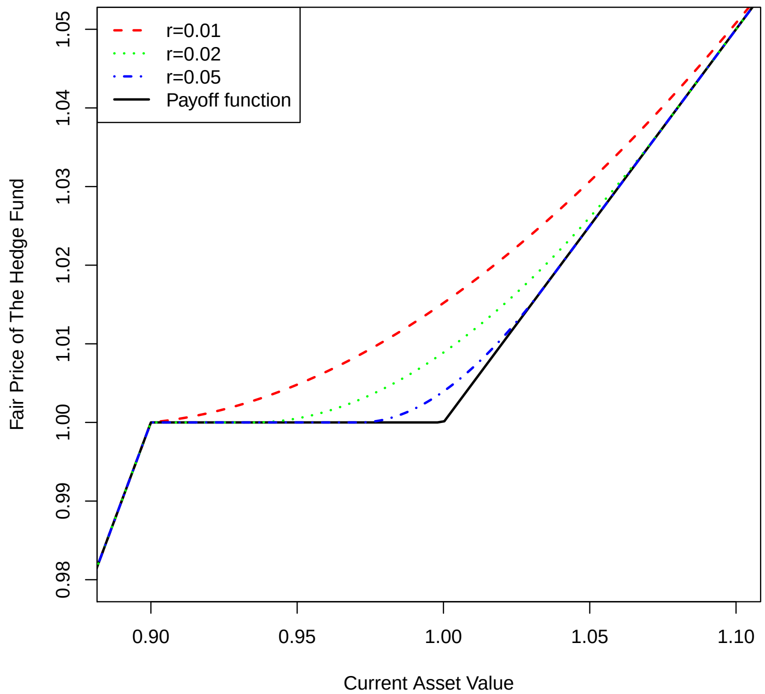

Figure 2.1: Fair Price of the Investor's Payoffs: Escrow First-Loss Fee Structure with parameters $x = 1, \delta = 0.01, \sigma = 0.1, A = 0.1, B = 1, p = 0.5, q = 1$. The solid black curve is the payoff function.

We begin by considering the case in which the loss insurance is implemented by the hedge fund manager placing cash in an escrow account (rather than investing in the fund's assets). In Figure 2.1 we vary the interest rate r, while holding all other parameters at their benchmark values. The value function is a decreasing function of the interest rate. This implies that the lower boundary increases with the interest rate, and the upper boundary decreases with the interest rate. For low values of the interest rate (e.g. $r = 0.01$), we see that $S_1 = \kappa = 0.9$, and the smooth pasting condition fails to hold. This corresponds to the hedge fund investor waiting until the entire escrow account has been consumed before exiting the fund. We observe that while the upper boundary decreases with the interest rate, even for low values of the interest rate it is not very high. Intuitively, once the hedge fund assets have increased significantly, the downside protection provided by the insurance component of the fee contract is worth far less to the investor than the cost that they are paying in terms of the high performance fee.

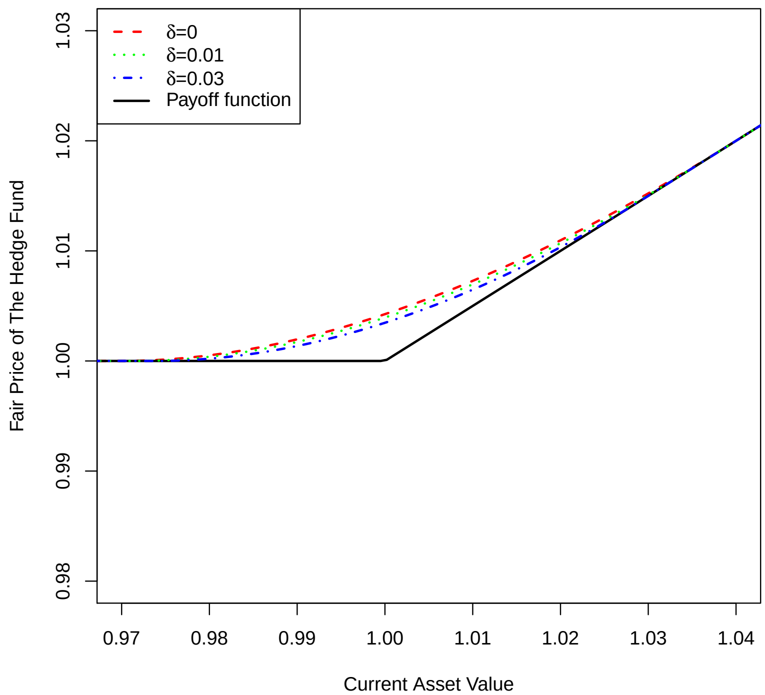

Figure 2.2: Fair Price of the Investor's Payoffs: Escrow First-Loss Fee Structure with parameters $x = 1, r = 0.05, \sigma = 0.1, A = 0.1, B = 1, p = 0.5, q = 1$. The solid black curve is the payoff function.

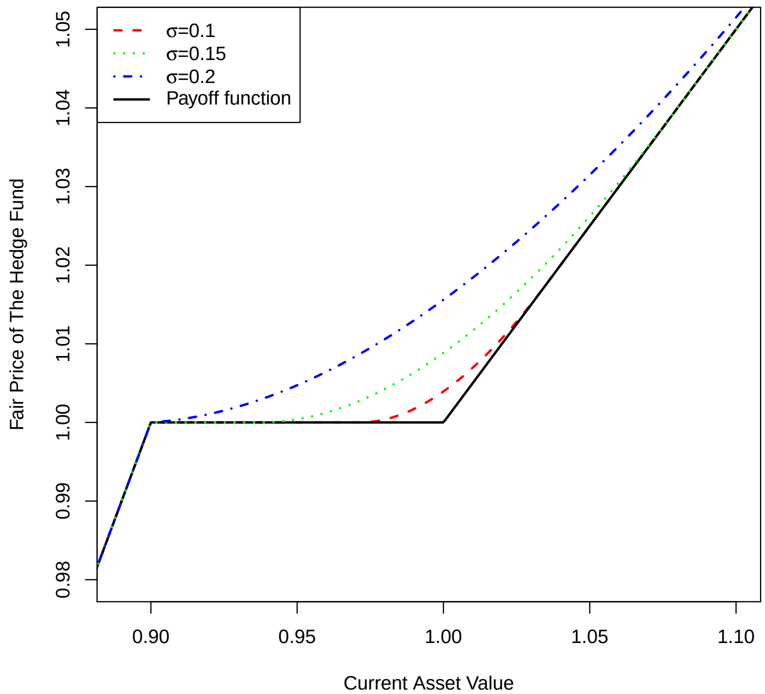

Figure 2.3: Fair Price of the Investor's Payoffs: Escrow First-Loss Fee Structure with parameters $x = 1, r = 0.05, \delta = 0.01, A = 0.1, B = 1, p = 0.5, q = 1$. The solid black curve is the payoff function.

In Figure 2.2, we consider the impact of varying the fee for assets under management δ. Although it is difficult to see from the figure, the value function is (not surprisingly) a decreasing function of the fee for assets under management. The higher the fee the investor has to pay for the hedge fund manager's services, the less the contract is worth for them. That being said, in a reasonable range of values both the value function and the exercise boundaries are relatively insensitive to variations in the parameter δ.

The same cannot be said for the volatility parameter σ. As we see from Figure 2.3, the value of σ can have a significant effect on both the value of the hedge fund investment and the positions of the exercise boundaries. The hedge fund value function is an increasing function of the volatility. This is a non-trivial observation, as the fee structure represents a portfolio of options consisting of both long and short positions. The lower boundary decreases and the upper boundary increases when σ increases. Again, this corresponds to financial intuition. The insurance component of the contract is particularly valuable for high levels of volatility, and the probability of a drop back into the region corresponding to losses from a high value of the hedge fund assets is greater for higher levels of volatility.

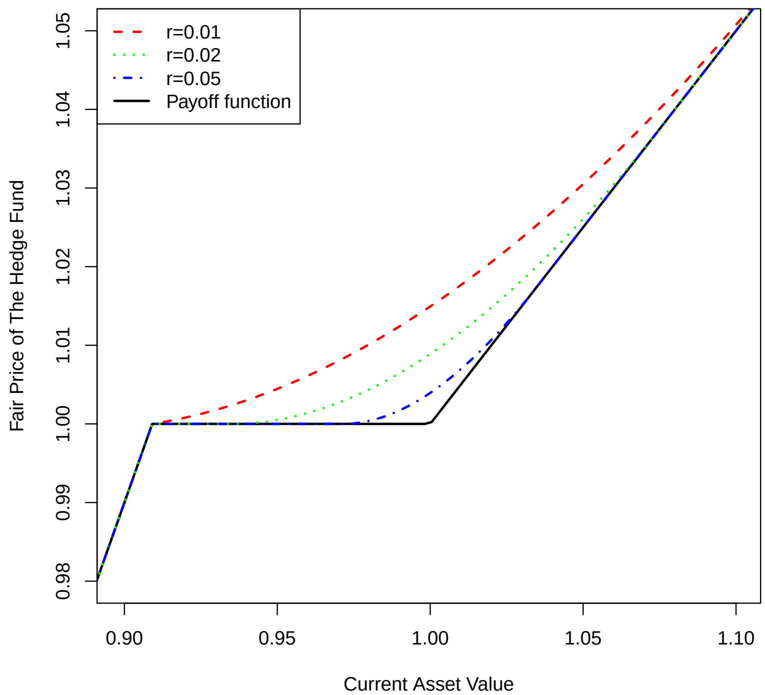

Figure 2.4: Fair Price of the Investor's Payoffs: Non-escrow First-Loss Fee Structure with parameters $x = 1, \delta = 0.01, \sigma = 0.1, A = 0.1, B = 1, p = 0.5, q = 1$. The solid black curve is the payoff function.

Next, we consider the case in which the hedge fund manager invests funds directly in the fund assets (the "non-escrow case"). Investor losses are covered from the fund manager's share of the remaining assets, until that share is exhausted. Figure 2.4 considers different values of the interest rate r, while Figures 2.5 and 2.6 consider varying the fee for assets under management δ and the asset volatility σ respectively. We see that the behaviour and levels of the value functions and the stopping boundaries are very similar to the escrow case.

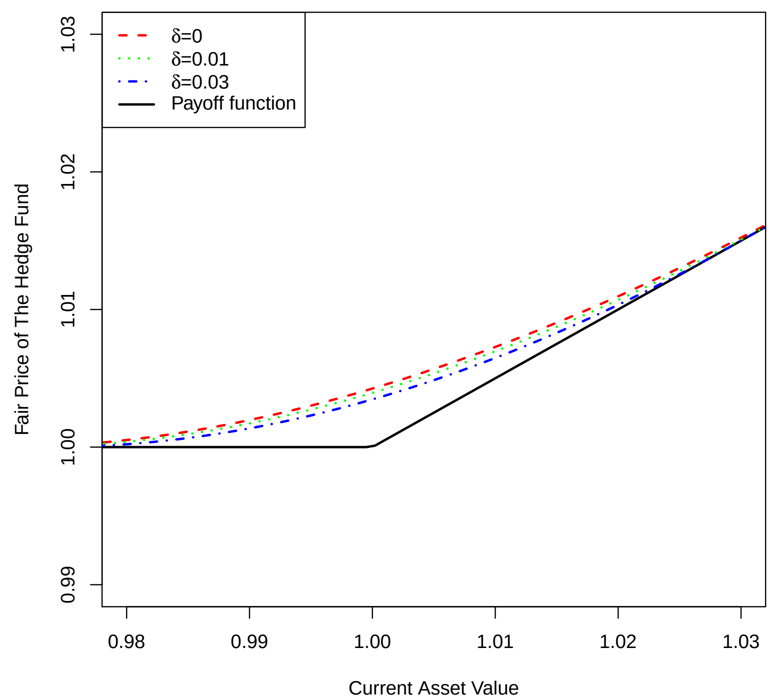

Figure 2.5: Fair Price of the Investor's Payoffs: Non-escrow First-Loss Fee Structure with parameters $x = 1, r = 0.05, \sigma = 0.1, A = 0.1, B = 1, p = 0.5, q = 1$. The solid black curve is the payoff function.

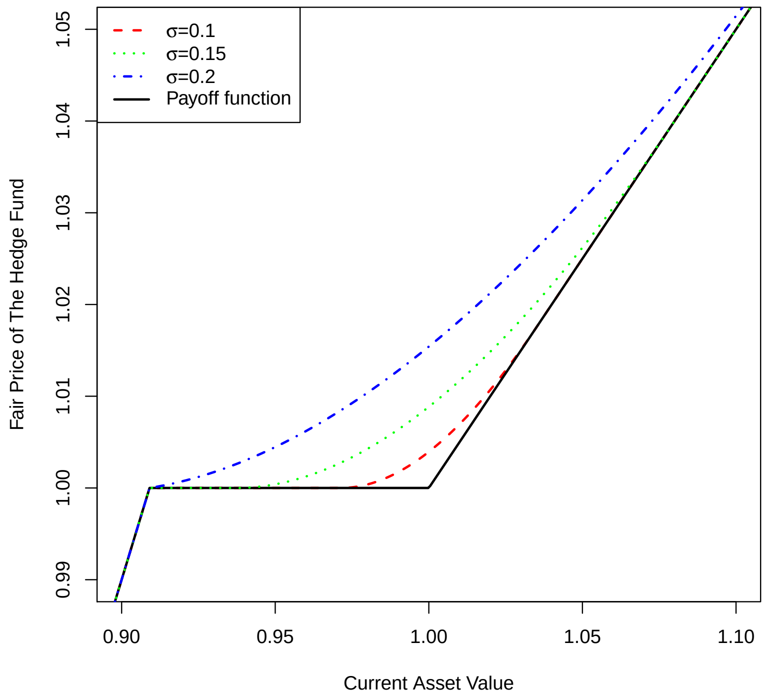

Figure 2.6: Fair Price of the Investor's Payoffs: Non-escrow First-Loss Fee Structure with parameters $x = 1, r = 0.05, \delta = 0.01, A = 0.1, B = 1, p = 0.5, q = 1$. The solid black curve is the payoff function.

Table 2.1 presents all the exercise boundary estimates S_1 and S_2 for Figures 2.1-2.6. Moreover, in section 2.3, a different parameterization $\beta = \frac{2r}{\sigma^2}$ and $\gamma = \frac{2\delta}{\sigma^2}$ is introduced in order to solve the optimal withdrawal problem. These intermediate parameters are also presented in the table as well.

Table 2.1: Summary of intermediate parameters and estimates of Figures 2.1-2.6

Figure 2.1	Red line	$\beta = 2, \gamma = 2, S_1 = 0.9 = \kappa, S_2 = 1.131$
	Green line	$\beta = 4, \gamma = 2, S_1 = 0.935 > \kappa, S_2 = 1.078$
	Blue line	$\beta = 10, \gamma = 2, S_1 = 0.971 > \kappa, S_2 = 1.035$
Figure 2.2	Red line	$\beta = 10, \gamma = 0, S_1 = 0.970 > \kappa, S_2 = 1.040$
	Green line	$\beta = 10, \gamma = 2, S_1 = 0.971 > \kappa, S_2 = 1.035$
	Blue line	$\beta = 10, \gamma = 6, S_1 = 0.973 > \kappa, S_2 = 1.029$
Figure 2.3	Red line	$\beta = 40, \gamma = 8, S_1 = 0.993 > \kappa, S_2 = 1.009$
	Green line	$\beta = 10, \gamma = 2, S_1 = 0.971 > \kappa, S_2 = 1.035$
	Blue line	$\beta = 2.5, \gamma = 0.5, S_1 = 0.9 = \kappa, S_2 = 1.150$
Figure 2.4	Red line	$\beta = 2, \gamma = 2, S_1 = 0.909 = \kappa, S_2 = 1.130$
	Green line	$\beta = 4, \gamma = 2, S_1 = 0.935 > \kappa, S_2 = 1.078$
	Blue line	$\beta = 10, \gamma = 2, S_1 = 0.971 > \kappa, S_2 = 1.035$
Figure 2.5	Red line	$\beta = 10, \gamma = 0, S_1 = 0.970 > \kappa, S_2 = 1.040$
	Green line	$\beta = 10, \gamma = 2, S_1 = 0.971 > \kappa, S_2 = 1.035$
	Blue line	$\beta = 10, \gamma = 6, S_1 = 0.974 > \kappa, S_2 = 1.029$
Figure 2.6	Red line	$\beta = 40, \gamma = 8, S_1 = 0.993 > \kappa, S_2 = 1.009$
	Green line	$\beta = 10, \gamma = 2, S_1 = 0.971 > \kappa, S_2 = 1.035$
	Blue line	$\beta = 2.5, \gamma = 0.5, S_1 = 0.909 = \kappa, S_2 = 1.149$

Chapter 3

Optimal Withdrawal from Shared Loss Fee Structures: The Finite Horizon Case

3.1 Introduction

In this chapter we study the problem of determining the optimal withdrawal time from an investment in a hedge fund with a shared-loss fee structure when there is a finite investment horizon T. In parallel with the standard American put option (see, e.g. Peskir and Shiryaev (2006)), there is no longer a simple semi-analytical solution to the problem. Nonetheless, we can derive various mathematical properties of the value function and optimal stopping time, and employ numerical methods to calculate their solutions. The basic properties of the continuation and stopping regions are inherited from the infinite horizon case, i.e. there is an upper stopping boundary and a lower stopping boundary, with the lower stopping boundary corresponding to the situation in which most or all of the insurance value of the shared loss structure has been exhausted, and the upper stopping boundary to the situation in which the insurance is nearly worthless (as the fund is unlikely to generate significant losses), and the performance fee has become too expensive. However, in the finite horizon case, these boundaries are time-dependent. When considered as a function of time to expiration, both boundaries start at 1, and then tend to the infinite horizon boundaries as time to expiration tends to infinity.[1] The boundaries

[1]This essentially follows from the convergence of the value functions, $\lim_{T\to\infty} v(x,T) = V(x)$, see Proposition 3.2.1, and Lemma 3.2.1. Preliminary numerical experiments (see Chapter 6) indicate that this

are both monotone, with the upper boundary being increasing and the lower boundary being decreasing. The mathematical structure of the solution is similar to that which arises for American installment options (see Ciurlia and Roko (2005) and Kimura (2009)). Recall that for an installment option, the option premium is paid continuously, rather than upfront. For an installment put, for example, this again leads to the situation in which there are two monotone stopping boundaries; the lower boundary corresponds to early exercise, as with the standard American put, while the upper boundary corresponds to the situation when the option premium is too expensive given the probability that the option will end up in the money.

Chen et al. (2020) studied the optimal stopping problem for a hedge fund fee structure with no fee for assets under management. In this chapter, we derive new results, and extend some of their results to the situation in which the contract features a fee for assets under management. In particular, we prove basic properties of the stopping boundaries that extend the results of Chen et al. (2020). We also present a new result deriving an early exercise representation and a pair of coupled integral equations for the stopping boundaries.

Finally, following the probabilistic strategy of Lamberton (1995), we derive the asymptotic behaviour of the optimal stopping boundaries as the time to expiration tends to zero. Aside from their intrinsic mathematical interest, these estimates can be helpful in approximating the stopping boundaries as the starting point for analytical approximations (see, e.g. Chen and Chadam (2007)), or for determining initial behaviour in numerical solutions of the integral equations for the stopping boundaries (which are highly singular for small times).

The remainder of this chapter is structured as follows. The second section presents the formulation of the mathematical problem of determining the best time to withdraw from a hedge fund investment with a shared-loss fee structure with a positive fee for assets under management and a finite investment horizon, and the characterization of the value function as the solution of a variational inequality. The third section discusses the basic shape of the stopping and continuation regions, in analogy with Chapter 2, as well as basic properties (monotonicity and continuity) of the stopping boundaries. The fourth section derives the early exercise premium representation of the investment value, as well as a pair of coupled integral equations for the stopping boundaries. The fifth section presents the asymptotic analysis of the stopping boundaries when time to expiration is small.

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convergence appears to be quite fast. Use of the infinite horizon value function to approximate the finite horizon value function and approximations of the boundary based on interpolating between the known small-time behaviour and infinite horizon solution (similar to Chen and Chadam (2007)) are potential topics for future research.

3.2 Problem Formulation and Characterization of the Value Function

In this section, we recall the characterization of the value function of the optimal stopping problem for optimal withdrawal from a hedge fund investment with a shared-loss fee structure as the solution of a variational inequality. Recall that for all the optimal stopping problems in this thesis, we assume that the underlying assets of the hedge fund satisfy:

$$\begin{aligned} dX_t^x &= (r-\delta)X_t^x dt + \sigma X_t^x dW_t, \quad X_0^x = x, \quad t \geq 0, \\ X_t^x &= x \exp\{(r-\delta-\frac{1}{2}\sigma^2)t + \sigma W_t\}, \end{aligned} \tag{3.1}$$

where $r \geq 0$ is the risk-free rate, $\delta > 0$ is the fee for assets under management (paid continuously), $\sigma > 0$ is the volatility and W_t is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$, the standard augmentation of the filtration generated by W, satisfying the usual conditions. Then, the value function for the investor's problem of determining the optimal time to withdraw from the hedge fund in the finite horizon $T > 0$ is:

$$v(x,T) = \sup_{\tau \in \mathcal{T}_{[0,T]}} \mathbb{E}[e^{-r\tau} g(X_\tau^x)], \tag{3.2}$$

where $\mathcal{T}_{[0,T]}$ is the set of all stopping times such that $0 \leq \tau \leq T$, and $g(x)$ is the payoff function for the first-loss and shared-loss fee structures:

$$g(x) = \begin{cases} A + Bx, & 0 \leq x \leq \kappa \\ q + (1-q)x, & \kappa \leq x \leq 1, \\ p + (1-p)x, & 1 \leq x, \end{cases} \tag{3.3}$$

where $B \geq 1 \geq q \geq A \geq 0$, $p \in (0,1)$ and $\kappa = (B - (1-q))^{-1}(q-A)$.

Similar to the infinite horizon case, $v(x,T)$ solves the variational inequality

$$\min\left(rv - Lv + \frac{\partial v}{\partial T}, v - g\right) = 0 \tag{3.4}$$

in the viscosity sense, for which we use the following standard definition (e.g. Pham (2009)).

Definition 3.2.1. *Let $W \in C([0,\infty) \times [0,\infty), \mathbb{R})$. Then,*

1. *W is a viscosity super-solution of (3.4) if*

$$\min\Big(rW(x_0,t_0) - L\varphi(x_0,t_0) + \frac{\partial \varphi}{\partial T}(x_0,t_0), W(x_0,t_0) - g(x_0)\Big) \geq 0, \tag{3.5}$$

for all smooth functions φ and all $(x_0,t_0) \in (0,\infty)\times(0,\infty)$ such that $W-\varphi$ attains a local minimum at (x_0,t_0).

2. *W is a viscosity sub-solution of (3.4) if*

$$\min\Big(rW(x_0,t_0) - L\psi(x_0,t_0) + \frac{\partial \psi}{\partial T}(x_0,t_0), W(x_0,t_0) - g(x_0)\Big) \leq 0, \tag{3.6}$$

for all smooth functions ψ and all $(x_0,t_0) \in (0,\infty)\times(0,\infty)$ such that $W-\psi$ attains a local maximum at (x_0,t_0).

W is called a viscosity solution of (3.4) if it is both a super-solution and sub-solution.

The following proposition summarizes basic properties of the value function. While we expect that it is a special case of a more general proposition, we are unaware of a precise reference, and therefore include the proof for completeness. Similar results, based on the same arguments, are given by Karatzas and Shreve (1998) for the American put and Chen et al. (2020) for our problem with no fee for assets under management (the proof is identical to that of the corresponding result in Chen et al. (2020), apart from the inclusion of the parameter $\delta \geq 0$ giving the fee for assets under management).

Proposition 3.2.1. *For $x \in [0,\infty)$, the value function $v(x,T)$ is increasing in x, increasing in T and $\lim_{T\to\infty} v(x,T) = V(x)$, where V is the value function (2.2) for the infinite horizon problem.*

Proof. Define

$$J(x,T,\tau) = \mathbb{E}[e^{-r\tau} g(X_\tau^x)],$$

where $x \geq 0$ and $\tau \in \mathcal{T}_{[0,T]}$. Then, for $y \geq x$,

$$J(y,T,\tau) - J(x,T,\tau) = \mathbb{E}[e^{-r\tau}\big(g(X_\tau^y) - g(X_\tau^x)\big)] \geq 0,$$

because $g(x)$ is an increasing function and $X^y_\tau - X^x_\tau \geq 0$ for all stopping times $\tau \in \mathcal{T}_{[0,T]}$. Thus $V(y,T) = \sup_{\tau \in \mathcal{T}_{[0,T]}} \mathbb{E}[e^{-r\tau} g(X^y_\tau)] \geq \sup_{\tau \in \mathcal{T}_{[0,T]}} \mathbb{E}[e^{-r\tau} g(X^x_\tau)] = v(x,T)$. Hence, $v(x,T)$ is increasing in x.

Since $T_1 \leq T_2$ implies that $\mathcal{T}_{[0,T_1]} \subset \mathcal{T}_{[0,T_2]} \subset \mathcal{T}$, the fact that $g(x) = v(x,0) \leq v(x,T_1) \leq v(x,T_2) \leq V(x)$ is immediate and $\tilde{V}(x) = \lim_{T\to\infty} v(x,T) \leq V(x)$ is well-defined. Let $\tau^*_x = \inf\{t \geq 0, X^x_t \notin (S_1, S_2)\}$ be the optimal stopping time for the perpetual problem. Then, by Fatou's Lemma:

$$\begin{aligned}\lim_{T\to\infty} v(x,T) &\geq \varliminf_{T\to\infty} \mathbb{E}[e^{-r(\tau^*_x \wedge T)} g(X^x_{\tau^*_x \wedge T})] \\ &\geq \mathbb{E}[\varliminf_{T\to\infty} e^{-r(\tau^*_x \wedge T)} g(X^x_{\tau^*_x \wedge T})] = \mathbb{E}[e^{-r\tau^*_x} g(X^x_{\tau^*_x})] = V(x). \quad (3.7)\end{aligned}$$

□

Recall that $V(x) = g(x)$ when $p \geq q$. Then, Proposition 3.2.1 implies $g(x) = V(x) \geq v(x,T) = g(x)$. Again, we only need to consider the case when $q > p$. The following Theorem connects (3.4) to our optimal stopping problem (3.2).

Theorem 3.2.1. *The value function is the unique viscosity solution of*

$$\min\left(rv - Lv + \frac{\partial v}{\partial T}, v - g\right) = 0, \quad (3.8)$$

satisfying $v(x,0) = g(x)$, $v(0,T) = A$, $v(x,T) \sim g(x)$ as $x \to \infty$ for all $T > 0$ and $\lim_{T\to\infty} v(x,T) = V(x)$.

Proof. From Proposition 3.2.1, we can easily obtain $v(x,T)$ is locally bounded for all $(x,T) \in [0,\infty) \times [0,\infty)$. By Touzi (2013, Theorem 7.7, Pages 96-99), $v(x,T)$ is the unique viscosity solution of (3.8). □

Next, define the sections of the stopping region and continuation region at each time T as follows:

$$\mathcal{S}_T = \{x | v(x,T) = g(x)\}, \quad \mathcal{C}_T = \{x | v(x,T) > g(x)\}. \quad (3.9)$$

Notice that if $T_1 \leq T_2$ then $x \in \mathcal{S}_{T_2}$ implies that $g(x) = v(x,T_2) \geq v(x,T_1) \geq g(x)$ so that

$$\mathcal{S}_{T_2} \subseteq \mathcal{S}_{T_1}. \quad (3.10)$$

The following properties of the sets in (3.9) generalize the case $\delta = 0$ from Chen et al. (2020).

Proposition 3.2.2. *Suppose that $q > p$, then,*

a. $[0, \kappa] \subseteq \mathcal{S}_T$.

b. $1 \in \mathcal{C}_T$.

c. If $a < 1$ and $a \in \mathcal{S}_T$, then $[0, a] \subseteq \mathcal{S}_T$.

d. If $b > 1$ and $b \in \mathcal{S}_T$, then $[b, \infty] \subseteq \mathcal{S}_T$.

Proof. a. Taking $\tau = 0 \in \mathcal{T}_{[0,T]}$, we have $g(x) \leq v(x, T)$. Since $g(x) \leq A + Bx$ and $e^{-rt} X_t^x$ is a supermartingale we have $v(x, T) \leq A + Bx$. Thus, for $x \in [0, \kappa]$, we conclude $v(x, T) = A + Bx$.

b. Suppose $1 \in \mathcal{S}_T$. Since $v(1, T)$ is increasing in T, for any $T^* < T$, we have $v(1, T^*) = g(1)$. Consider the test functions with the form $\varphi(x, t) = -(t - T^*)^2 + M_n \exp(n(x - 1)) + (1 - M_n)$, where $t \in (T^* - \varepsilon, T^* + \varepsilon)$, $M_n = \frac{\xi}{n}$ and $\xi \in (1 - q, 1 - p)$. Clearly, $\varphi(1, T^*) = 1$, and $\varphi(x, t) < g(x)$ for x close to 1. Also, $\frac{\partial \varphi}{\partial x}(1, t) = \xi$, $\frac{\partial \varphi}{\partial x^2}(1, t) = n\xi$ and $\frac{\partial \varphi}{\partial t}(x, T^*) = 0$. By the super-solution property, we know $r\varphi(1, T^*) - L\varphi(1, T^*) + \frac{\partial \varphi}{\partial t}(1, T^*) \geq 0$, but

$$\begin{aligned} r\varphi(1, T^*) - L\varphi(1, T^*) + \frac{\partial \varphi}{\partial t}(1, T^*) &= r - (r - \delta)\xi - \frac{1}{2}\sigma^2 n\xi + 0 \\ &= r(1 - \xi) + \delta\xi - \frac{1}{2}\sigma^2 n\xi < 0 \end{aligned}$$

for n large enough, which contradicts the super-solution property. So $1 \in \mathcal{C}_T$.

c. If $a \leq \kappa$, $[0, \kappa] \subseteq \mathcal{S}_T$ implies $[0, a] \subseteq \mathcal{S}_T$. So we only consider the case when $\kappa < a < 1$. Suppose $a < 1$ and $a \in \mathcal{S}_T$. If $[0, a] \subseteq \mathcal{S}_T$ is not true, then there exists a $\tilde{x}_T \in (\kappa, a]$ such that $v(\tilde{x}_T, T) > g(\tilde{x}_T)$. Let $h(x, T) = v(x, T) - g(x)$. On $x \in [\kappa, a]$, we have $h(\kappa, T) = h(a, T) = 0$ since κ and a are in the stopping region. Noting that $h(\tilde{x}_T, T) > 0$, for any fixed T, we must have some points $x_T^* \in (\kappa, a)$ such that $h(x_T^*, T) > 0$ attains a local maximum. In other words, we have the following relations,

$$h(x_T^*, T) > 0, x_T^* \in \mathcal{C}_T,\ h_x(x_T^*, T) = 0,\ \text{and } h_{xx}(x_T^*, T) \leq 0.$$

Now, it can be easily obtained that

$$rh(x_T^*,T) - Lh(x_T^*,T) + h_T(x_T^*,T) = rh(x_T^*,T) - \frac{\sigma^2(x^*)^2}{2}h_{xx}(x_T^*,T) - (r-\delta)xh_x(x_T^*,T) + h_T(x_T^*,T) \geq 0. \tag{3.11}$$

However, $x_T^* \in \mathcal{C}_T$ implies that $rV(x_T^*,T) - LV(x_T^*,T) + V_T(x_T^*,T) = 0$ and we have shown $rg - Lg > 0$ for $x \in (0,\kappa) \cup (\kappa,1) \cup (1,\infty)$. Therefore, we must have $rh(x_T^*,T) - Lh(x_T^*,T) + h_T(x_T^*,T) = r(v(x_T^*,T) - g(x_T^*,T)) - L(V(x_T^*,T) - g(x_T^*,T)) + h_T(x_T^*,T) < 0$, which contradicts (3.11).

d. Recall that $[S_2,\infty)$ is the stopping region for the infinite horizon case. So, if $b \geq S_2$, then we have $g(x) = V(x) \geq v(x,T) = g(x)$ for all $x \geq S_2$ by Proposition 3.2.1. Therefore, we only need to consider the case when $1 < b < S_2$. Now, suppose $1 < b < S_2$ and $b \in \mathcal{S}_T$. If $[b,\infty] \subseteq \mathcal{S}_T$ is not true, then there exists a $\tilde{x}_T \in (b,S_2)$ such that $v(\tilde{x}_T,T) > g(\tilde{x}_T)$. Then, similar to Part c), we let $h(x,T) = v(x,T) - g(x)$. On $x \in [b,S_2]$, we have $h(b,T) = h(S_2,T) = 0$ since b and S_2 are in the stopping region. Note that $h(\tilde{x}_T,T) > 0$, then for any fixed T, we must have some points $x_T^* \in (b,S_2)$ such that $h(x_T^*,T) > 0$ attains a local maximum. In other words, we have the following relations,

$$h(x_T^*,T) > 0, x_T^* \in \mathcal{C}_T,\ h_x(x_T^*,T) = 0, \text{ and } h_{xx}(x_T^*,T) \leq 0.$$

Now, it can be easily obtained that

$$rh(x_T^*,T) - Lh(x_T^*,T) + h_T(x_T^*,T) = rh(x_T^*,T) - \frac{\sigma^2(x^*)^2}{2}h_{xx}(x_T^*,T) - (r-\delta)xh_x(x_T^*,T) + h_T(x_T^*,T) \geq 0. \tag{3.12}$$

However, $x_T^* \in \mathcal{C}_T$ implies that $rv(x_T^*,T) - Lv(x_T^*,T) + v_T(x_T^*,T) = 0$ and in Proposition 2.2.1 we have shown $rg - Lg > 0$ for $x \in (0,\kappa) \cup (\kappa,1) \cup (1,\infty)$. Therefore, we must have $rh(x_T^*,T) - Lh(x_T^*,T) + h_T(x_T^*,T) = r(v(x_T^*,T) - g(x_T^*,T)) - L(v(x_T^*,T) - g(x_T^*,T)) + h_T(x_T^*,T) < 0$, which contradicts (3.12).

□

By Proposition 3.2.2, we can define the two stopping boundaries at time to maturity

T as the following,

$$S_{-}(T) := \inf\{x | v(x,T) > g(x)\}, \quad S_{+}(T) := \sup\{x | v(x,T) > g(x)\}.$$

We have from (3.10) that $S_{-}(T)$ is decreasing in T and $S_{+}(T)$ is increasing in T.

Lemma 3.2.1. *1. $S_{-}(T)$ and $S_{+}(T)$ are continuous functions.*

2. $\lim_{T\downarrow 0} S_{-}(T) = \lim_{T\downarrow 0} S_{+}(T) = 1$.

3. The smooth-fit condition holds on the upper boundary, $\lim_{x\to S_{+}(T)} V(x,T) = g'(S_{+}(T)) = 1-p$. Furthermore, if $S_1 > \kappa$, the smooth fit condition holds on the lower boundary as well, i.e. $\lim_{x\to S_{-}(T)} V_x(x,T) = g'(S_{-}(T)) = 1-q$.

4. $\lim_{T\to\infty} S_{-}(T) = S_1$, $\lim_{T\to\infty} S_{+}(T) = S_2$.

Proof. The first two results can be proved using the argument in Theorem 3.1 in De Angelis (2015), while the third result can be proved following the same strategy as for the American put, see Peskir and Shiryaev (2006, pages 381–382). For the fourth result, let $\tilde{S}_2 := \lim_{T\to\infty} S_{+}(T)$. Clearly $\tilde{S}_2 \in [1, S_2]$. Suppose that $\tilde{S}_2 < S_2$. Because $S_{+}(T)$ is increasing, $\tilde{S}_2 \in \mathcal{S}_T$ for all T. Thus $V(\tilde{S}_2) = \lim_{T\to\infty} v(\tilde{S}_2, T) = \lim_{T\to\infty} g(\tilde{S}_2) = g(\tilde{S}_2)$, contradicting the definition of S_2. The proof for the other case is similar. $\square$

3.3 Early Exercise Representation and Integral Equations

We now derive the early exercise representation for v, and a pair of coupled integral equations for $S_{\pm}(T)$. Throughout, we assume that $q > p$, and $S_1 > \kappa$. From the above, we have that v solves:

$$\begin{cases} v(x,0) = g(x), \\ \lim_{x\to S_{-}(T)} v(x,T) = q + (1-q)S_{-}(T), \\ \lim_{x\to S_{-}(T)} v_x(x,T) = 1-q, \\ \lim_{x\to S_{+}(T)} v(x,T) = p + (1-p)S_{+}(T), \\ \lim_{x\to S_{+}(T)} v_x(x,T) = 1-p. \end{cases} \tag{3.13}$$

Along with the information that $S_{-}(T)$ and $S_{+}(T)$ never intersect with each other and are locally bounded and continuous, we can find a representation for $v(x,T)$ by applying the change-of variable formula on curves Peskir (2005).

Theorem 3.3.1. *v defined by (3.2) satisfies:*

$$
\begin{aligned}
v(x,T) =& v_e(x,T) + \delta \int_0^T x e^{-\delta(T-\tau)} \Big(B + (1-q-B)\Phi(d_1(x,\kappa,T-\tau)) \\
&- (1-q)\Phi(d_1(x,S_-(\tau),T-\tau)) + (1-p)\Phi(d_1(x,S_+(\tau),T-\tau)) \Big) d\tau \\
&+ r\int_0^T e^{-r(T-\tau)} \Big(A(1-\Phi(d_2(x,\kappa,T-\tau))) \\
&+ q(\Phi(d_2(x,\kappa,T-\tau)) - \Phi(d_2(x,S_-(\tau),T-\tau)) + p\Phi(d_2(x,S_+(\tau),T-\tau))) \Big) d\tau,
\end{aligned}
\tag{3.14}
$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function,

$$
d_1(x,y,t) = \frac{\log(x/y) + (r-\delta+\sigma^2/2)t}{\sigma\sqrt{t}}, \quad d_2(x,y,t) = d_1(x,y,t) - \sigma\sqrt{t}.
$$

and $v_e(x,T) := \mathbb{E}[e^{-rT} g(X_T^x)]$ is the corresponding European-style value function.

Proof. Note that the variable T is time to maturity, which implies that our time is running backward. Since it is convenient to apply the change-of-variable formula when time is running forward, we introduce the following notation. Define

$$
\begin{aligned}
&\tilde{v}(x,t;T) := v(x,T-t), \quad \tilde{Z}(x,t;T) := e^{-rt} v(x,t;T), \\
&\tilde{S}_-(t) := S_-(T-t), \quad \tilde{S}_+(t) := S_+(T-t),
\end{aligned}
$$

where $\tilde{v}(X_t,t;T)$ is the value process at the current time t, $0 \le t \le T$ and $\tilde{Z}(X_t,t;T)$ is the discounted value process at time t, $0 \le t \le T$. Applying Peskir's change-of-variable formula (Peskir, 2005, Theorem 2.1, Remark 2.3 and Remark 2.5) on $\tilde{Z}(X_t, t; T)$ leads to

$$
\begin{aligned}
\tilde{Z}(X_T, T; T) =& \tilde{Z}(X_0, 0; T) + \int_0^T \frac{\partial \tilde{Z}(X_t, t; T)}{\partial t} \mathbf{1}_{\{X_t \notin \{\tilde{S}_-(t), \tilde{S}_+(t)\}\}} dt \\
&+ \int_0^T \frac{\partial \tilde{Z}(X_t, t; T)}{\partial x} \mathbf{1}_{\{X_t \notin \{\tilde{S}_-(t), \tilde{S}_+(t)\}\}} dX_t \\
&+ \frac{1}{2} \int_0^T \sigma^2 X_t^2 \frac{\partial^2 \tilde{Z}(X_t, t; T)}{\partial x^2} \mathbf{1}_{\{X_t \notin \{\tilde{S}_-(t), \tilde{S}_+(t)\}\}} dt \\
&+ \frac{1}{2} \int_0^T (\frac{\partial Z(X_t+, t; T)}{\partial x} - \frac{\partial Z(X_t-, t; T)}{\partial x}) \mathbf{1}_{\{X_t = \tilde{S}_-(t)\}} d\ell_t^{\tilde{S}_-} \\
&+ \frac{1}{2} \int_0^T (\frac{\partial Z(X_t+, t; T)}{\partial x} - \frac{\partial Z(X_t-, t; T)}{\partial x}) \mathbf{1}_{\{X_t = \tilde{S}_+(t)\}} d\ell_t^{\tilde{S}_+}
\end{aligned} \tag{3.15}
$$

where

$$
\ell_t^{\tilde{S}_\pm} := \mathbb{P}\left[\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{\tilde{S}_\pm(s) - \varepsilon < X_s < \tilde{S}_\pm(s) + \varepsilon\}} \sigma^2 X_s^2 ds\right], \quad i = 1, 2 \tag{3.16}
$$

is the local time of X_s at the curve $\tilde{S}_\pm(s)$ for $s \in [0, t]$. Furthermore, note that

$$
\begin{aligned}
&\frac{\partial \tilde{Z}(X_t, t; T)}{\partial x} = e^{-rt} \frac{\partial \tilde{v}(X_t, t; T)}{\partial x}, \quad \frac{\partial^2 \tilde{Z}(X_t, t; T)}{\partial x^2} = e^{-rt} \frac{\partial^2 \tilde{v}(X_t, t; T)}{\partial x^2}, \\
&\frac{\partial \tilde{Z}(X_t, t; T)}{\partial t} = -re^{-rt} \tilde{v}(X_t, t; T) + e^{-rt} \frac{\partial \tilde{v}(X_t, t; T)}{\partial t}.
\end{aligned}
$$

Substituting the above into (3.15), we can verify that

$$
\begin{aligned}
e^{-rT} \tilde{v}(X_T, T; T) =& \tilde{v}(X_0, 0; T) + \int_0^T e^{-rt} \frac{\partial \tilde{v}(X_t, t; T)}{\partial x} \mathbf{1}_{\{X_t \notin \{\tilde{S}_-(t), \tilde{S}_+(t)\}\}} dX_t \\
&+ \int_0^T e^{-rt} (\frac{\sigma^2 X_t^2}{2} \frac{\partial^2 \tilde{v}(X_t, t; T)}{\partial x^2} - r\tilde{v}(X_t, t; T) + \frac{\partial \tilde{v}(X_t, t; T)}{\partial t}) \mathbf{1}_{\{X_t \notin \{\tilde{S}_-(t), \tilde{S}_+(t)\}\}} dt.
\end{aligned} \tag{3.17}
$$

Next, knowing that $\tilde{v}(X_t, t; T) = g(X_t)$ on the stopping region,

$$
\begin{aligned}
\tilde{v}(X_t, t; T) =& \mathbf{1}_{\{X_t \le \kappa\}} (A + BX_t) + \mathbf{1}_{\{\kappa < X_t \le \tilde{S}_-(t)\}} (q + (1-q) X_t) \\
&+ \mathbf{1}_{\{\tilde{S}_-(t) < X_t < \tilde{S}_+(t)\}} V(X_s, s) + \mathbf{1}_{\{X_t \ge \tilde{S}_+(t)\}} (p + (1-p) X_t).
\end{aligned}
$$

For simplicity, we define

$$f_1(x,t) := \mathbf{1}_{\{x<\kappa\}}(A+Bx) + \mathbf{1}_{\{\kappa<x<\tilde{S}_-(t)\}}(q+(1-q)x) + \mathbf{1}_{\{x>\tilde{S}_+(t)\}}(p+(1-p)x)$$

$$\frac{\partial f_1(x,t)}{\partial x} := \mathbf{1}_{\{x<\kappa\}}B + \mathbf{1}_{\{\kappa<x<\tilde{S}_-(t)\}}(1-q) + \mathbf{1}_{\{x>\tilde{S}_+(t)\}}(1-p).$$

Then it is easy to obtain the following expressions,

$$\tilde{v}(X_t,t;T)\mathbf{1}_{\{X_t\notin\{\tilde{S}_-(t),\tilde{S}_+(t)\}\}} = f_1(X_t,t) + \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}\tilde{v}(X_t,t;T), \tag{3.18}$$

$$\begin{aligned}\frac{\partial\tilde{v}(X_t,t;T)}{\partial x}\mathbf{1}_{\{X_t\notin\{\tilde{S}_-(t),\tilde{S}_+(t)\}\}} &= \frac{\partial f_1(X_t,t)}{\partial x} + \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}\frac{\partial\tilde{v}(X_t,t;T)}{\partial x}\\ &= \mathbf{1}_{\{X_t<\kappa\}}B + \mathbf{1}_{\{\kappa<X_t<\tilde{S}_-(t)\}}(1-q)\\ &\quad + \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}\frac{\partial\tilde{v}(X_t,t;T)}{\partial x} + \mathbf{1}_{\{X_t>\tilde{S}_+(t)\}}(1-p),\end{aligned} \tag{3.19}$$

$$\frac{\partial^2\tilde{v}(X_t,t;T)}{\partial x^2}\mathbf{1}_{\{X_t\notin\{\tilde{S}_-(t),\tilde{S}_+(t)\}\}} = \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}\frac{\partial^2\tilde{v}(X_t,t;T)}{\partial x^2}, \tag{3.20}$$

$$\frac{\partial\tilde{v}(X_t,t;T)}{\partial t}\mathbf{1}_{\{X_t\notin\{\tilde{S}_-(t),\tilde{S}_+(t)\}\}} = \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}\frac{\partial\tilde{v}(X_t,t;T)}{\partial t}. \tag{3.21}$$

Substituting (3.18), (3.19), (3.20) and (3.21) into (3.17), we have

$$
\begin{aligned}
e^{-rT}\tilde{v}(X_T,T;T) =&\tilde{v}(X_0,0;T)+\int_0^T e^{-rt}\left(\frac{\partial f_1(X_t,t)}{\partial x}+\mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}\frac{\partial\tilde{v}(X_t,t;T)}{\partial x}\right)dX_t\\
&+\int_0^T e^{-rt}\Big(\frac{\sigma^2X_t^2}{2}\mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}\frac{\partial^2\tilde{v}(X_t,t;T)}{\partial x^2}\\
&-r\big(f_1(X_t,t)+\mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}\tilde{v}(X_t,t;T)\big)+\mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}\frac{\partial\tilde{v}(X_t,t;T)}{\partial t}\Big)dt\\
=&\tilde{v}(X_0,0;T)+\int_0^T e^{-rt}\alpha dX_t\\
&+\int_0^T \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}e^{-rt}(r-\delta)X_t\frac{\partial\tilde{v}(X_t,t;T)}{\partial x}dt\\
&+\int_0^T \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}e^{-rt}\sigma X_t\frac{\partial\tilde{v}(X_t,t;T)}{\partial x}dW_t\\
&+\int_0^T \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}e^{-rt}(\frac{\sigma^2X_t^2}{2}\frac{\partial^2\tilde{v}(X_t,t;T)}{\partial x^2}-r\tilde{v}(X_t,t;T)+\frac{\partial\tilde{v}(X_t,t;T)}{\partial t})dt\\
&-\int_0^T re^{-rt}f_1(X_t,t)dt\\
=&\tilde{v}(X_0,0;T)+\int_0^T e^{-rt}(r-\delta)X_t\alpha dt+\int_0^T e^{-rt}\sigma X_t\alpha dW_t\\
&+\int_0^T \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}e^{-rt}\Big((r-\delta)X_t\frac{\partial\tilde{v}(X_t,t;T)}{\partial x}\\
&+\frac{\sigma^2X_t^2}{2}\frac{\partial^2\tilde{v}(X_t,t;T)}{\partial x^2}-r\tilde{v}(X_t,t;T)+\frac{\partial\tilde{v}(X_t,t;T)}{\partial t}\Big)dt\\
&+\int_0^T \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}e^{-rt}\sigma X_t\frac{\partial\tilde{v}(X_t,t;T)}{\partial x}dW_t-\int_0^T re^{-rt}f_1(X_t,t)dt.
\end{aligned}
\tag{3.22}
$$

Note that on the continuation region, $v(x,T)$ satisfies the PDE $Lv-rv-v_T=0$, so

$\tilde{v}_t + L\tilde{v} - r\tilde{v} = 0$. Thus, we can further simplify (3.22) as follows,

$$
\begin{aligned}
e^{-rT}\tilde{v}(X_T, T; T) =& \tilde{v}(X_0, 0; T) + \int_0^T e^{-rt}(r-\delta)X_t\alpha dt + \int_0^T e^{-rt}\sigma X_t \alpha dW_t \\
&+ \int_0^T \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}} e^{-rt}\sigma X_t \frac{\partial \tilde{v}(X_t, t; T)}{\partial x} dW_t - \int_0^T re^{-rt} f_1(X_t, t)dt.
\end{aligned} \tag{3.23}
$$

Taking expectations and applying Fubini's Theorem on (3.23), we obtain,

$$
\begin{aligned}
E[e^{-rT}\tilde{v}(X_T, T; T)] =& \tilde{v}(X_0, 0; T) + \int_0^T (r-\delta)e^{-rt} E\left[\alpha X_t\right] dt - \int_0^T re^{-rt} E[f_1(X_t, t)]dt \\
=& \tilde{v}(X_0, 0; T) - \delta \int_0^T e^{-rt} E\left[\alpha X_t\right] dt \\
&+ r\int_0^T e^{-rt}\left(E\left[\alpha X_t\right] - E[f_1(X_t, t)]\right) dt.
\end{aligned}
$$

Note that $\tilde{v}(X_T, T; T) = g(X_T)$, so $E[e^{-rT}\tilde{v}(X_T, T; T)] = E[e^{-rT}g(X_T)]$. After rearranging terms, the value function $v(x, 0; T)$ has the early exercise premium integral representation:

$$
\begin{aligned}
\tilde{v}(x, 0; T) =& v_e(x, T) + \delta \int_0^T e^{-rt} E\left[\alpha X_t\right] dt \\
&+ r\int_0^T e^{-rt}\left(E[f_1(X_t, t)] - E\left[\alpha X_t\right]\right) dt.
\end{aligned} \tag{3.24}
$$

Now, we can calculate the expectations in (3.24) separately. First, we write down each expectation explicitly

$$
\begin{aligned}
v_e(x, T) =& e^{-rT}(E[1_{\{X_T<\kappa\}}(A + BX_T)] + E[\mathbf{1}_{\{\kappa<X_T<1\}}(q + (1-q)X_T)] \\
&+ E[\mathbf{1}_{\{X_T>1\}}(p + (1-p)X_T)]),
\end{aligned} \tag{3.25}
$$

$$
E\left[\alpha X_t\right] = E[\mathbf{1}_{\{X_t<\kappa\}}BX_t + \mathbf{1}_{\{\kappa<X_t<\tilde{S}_-(t)\}}(1-q)X_t + \mathbf{1}_{\{X_t>\tilde{S}_+(t)\}}(1-p)X_t], \tag{3.26}
$$

$$
\begin{aligned}
E[f_1(X_t, t)] =& E[\mathbf{1}_{\{X_t<\kappa\}}(A + BX_t) + \mathbf{1}_{\{\kappa<X_t<\tilde{S}_-(t)\}}(q + (1-q)X_t) + \mathbf{1}_{\{X_t>\tilde{S}_+(t)\}}(p + (1-p)X_t)] \\
=& E[\mathbf{1}_{\{X_t<\kappa\}}A + \mathbf{1}_{\{\kappa<X_t<\tilde{S}_-(t)\}}q + \mathbf{1}_{\{X_t>\tilde{S}_+(t)\}}p] + E\left[\alpha X_t\right].
\end{aligned} \tag{3.27}
$$

Note that since X_t follows a log-normal distribution, we can easily obtain:

$$E\left[\alpha X_t\right] = xe^{(r-\delta)t}\Big(B + (1-q-B)\Phi(d_1(x,\kappa,t) - (1-q)\Phi(d_1(x,\tilde{S}_-(t),t)) + (1-p)\Phi(d_1(x,\tilde{S}_+(t),t))\Big), \quad (3.28)$$

$$\begin{aligned} E[f_1(X_t,t)] - E\left[\alpha X_t\right] = &A(1-\Phi(d_2(x,\kappa,t))) \\ &+ q(\Phi(d_2(x,\kappa,t)) - \Phi(d_2(x,\tilde{S}_-(t),t)) \\ &+ p\Phi(d_2(x,\tilde{S}_+(t),t))). \end{aligned} \quad (3.29)$$

Next, substituting (3.28) and (3.29) into (3.24) yields the following integral representation for $\tilde{v}(X_0,0;T)$:

$$\begin{aligned} \tilde{v}(X_0,0;T) = &v_e(x,T) + \delta\int_0^T xe^{-\delta t}\Big(B + (1-q-B)\Phi(d_1(x,\kappa,t) \\ &- (1-q)\Phi(d_1(x,\tilde{S}_-(t),t)) + (1-p)\Phi(d_1(x,\tilde{S}_+(t),t))\Big)dt \\ &+ r\int_0^T e^{-rt}\Big(A(1-\Phi(d_2(x,\kappa,t))) \\ &+ q(\Phi(d_2(x,\kappa,t)) - \Phi(d_2(x,\tilde{S}_-(t),t)) + p\Phi(d_2(x,\tilde{S}_+(t),t)))\Big)dt. \end{aligned} \quad (3.30)$$

After reverting to our original notation, this is the desired result. □

By Theorem 3.3.1 and the boundary conditions (3.13), the optimal stopping boundaries

$S_-(T)$ and $S_+(T)$ satisfy the following coupled pair of integral equations:

$$\begin{aligned}
q + (1-q)S_-(T) =& v_e(S_-(T), T) + \delta \int_0^T S_-(T) e^{-\delta(T-\tau)} \Big(B + (1-q-B)\Phi(d_1(S_-(T), \kappa, T-\tau) \\
&- (1-q)\Phi(d_1(S_-(T), S_-(\tau), T-\tau)) + (1-p)\Phi(d_1(S_-(T), S_+(\tau), T-\tau)) \Big) d\tau \\
&+ r \int_0^T e^{-r(T-\tau)} \Big(A(1 - \Phi(d_2(S_-(T), \kappa, T-\tau))) \\
&+ q(\Phi(d_2(S_-(T), \kappa, T-\tau)) - \Phi(d_2(S_-(T), S_-(\tau), T-\tau)) \\
&+ p\Phi(d_2(S_-(T), S_+(\tau), T-\tau)) \Big) d\tau
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
p + (1-p)S_+(T) =& v_e(S_+(T), T) + \delta \int_0^T S_+(T) e^{-\delta(T-\tau)} \Big(B + (1-q-B)\Phi(d_1(S_+(T), \kappa, T-\tau) \\
&- (1-q)\Phi(d_1(S_+(T), S_-(\tau), T-\tau)) + (1-p)\Phi(d_1(S_+(T), S_+(\tau), T-\tau)) \Big) d\tau \\
&+ r \int_0^T e^{-r(T-\tau)} \Big(A(1 - \Phi(d_2(S_+(T), \kappa, T-\tau))) \\
&+ q(\Phi(d_2(S_+(T), \kappa, T-\tau)) - \Phi(d_2(S_+(T), S_-(\tau), T-\tau)) \\
&+ p\Phi(d_2(S_+(T), S_+(\tau), T-\tau)) \Big) d\tau,
\end{aligned} \tag{3.32}$$

along with the conditions $\lim_{T\to 0+} S_-(T) = 1$ and $\lim_{T\to 0+} S_+(T) = 1$, by Lemma 3.2.1.

3.4 Exercise Boundaries Near Maturity

In this section, we study the asymptotic behaviour of the stopping boundaries $S_\pm(T)$ for small T. In particular, we show that as $T \searrow 0$:

$$S_\pm(T) \sim 1 \pm \sigma\sqrt{T(-\log T)} \tag{3.33}$$

Throughout, we assume that the parameters are such that $S_1 > \kappa$. We follow the strategy employed by Lamberton (1995) in the case of the American put. Translated into our context, this consists of the following steps:

- Show that for the European option with payoff g, and price $v_e(x,T) = \mathbb{E}_\mathbb{Q}[e^{-rT}g(X_T^x)]$, and for $T > 0$ small enough, there exist two boundaries $S_-^e(T) < 1 < S_+^e(T)$ such that $v_e(S_-^e(T), T) = g(S_-^e(T))$, $v_e(S_+^e(T), T) = g(S_+^e(T))$.

- Derive the small-time behaviour of $S^e_\pm(T)$.
- Show that for small T, the boundaries $S_\pm(T)$ are close to $S^e_\pm(T)$. In particular, for T small enough, there exists a $C > 0$ such that:

$$0 \leq S^e_-(T) - S_-(T) \leq C\sqrt{T}, \quad 0 \leq S_+(T) - S^e_+(T) \leq C\sqrt{T}. \tag{3.34}$$

- Infer the asymptotic behaviour of $S_\pm(T)$ from that of $S^e_\pm(T)$.

Implementing the strategy in this case is significantly more complicated than in the case of the American put covered by Lamberton (1995), for two main reasons. First of all, we need to deal with two boundaries rather than a single one. Secondly, our payoff function is more complicated; in particular it lacks the convexity that aids in the analysis of the American put.

We need the following simple results, whose proofs are contained in Appendix B.

Lemma 3.4.1. *Let $X^x_t = x\exp\{(r - \delta + \frac{\sigma^2}{2})t + \sigma W_t\}$. Then, for $0 \leq a \leq b < \infty$,*

$$\mathbb{Q}[a < X^x_t < b] = \Phi(d_2(x,a,t)) - \Phi(d_2(x,b,t)), \tag{3.35}$$

$$\mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{\{a<X^x_t<b\}} X^x_t] = xe^{(r-\delta)t}(\Phi(d_1(x,a,t)) - \Phi(d_1(x,b,t)), \tag{3.36}$$

where Φ is the standard normal cumulative distribution function,

$$d_1(x,y,t) = \frac{\log(x/y) + (r - \delta + \sigma^2/2)t}{\sigma\sqrt{t}}, \quad \text{and} \quad d_2(x,y,t) = d_1(x,y,t) - \sigma\sqrt{t}.$$

Lemma 3.4.2. *Let $C(x,K,T) = \mathbb{E}[e^{-rT}\max(X^x_T - K, 0)]$ denote the price of a European call option with strike price K, maturity date T, and current stock value x. Then as $T \searrow 0$:*

$$C(x,K,T) - \max(x - K, 0) = \begin{cases} O(\sqrt{T}), & \text{if } x = K, \\ o(\sqrt{T}), & \text{if } x \neq K. \end{cases} \tag{3.37}$$

Lemma 3.4.3. *a. Suppose the function $f_1(x)$ is smooth on the interval $[a_1, b_1]$. If the following conditions are satisfied*

1. $f_1(a_1) < 0$, $f_1(b_1) > 0$, $f_1'(a_1) > 0$, $f_1'(b_1) > 0$.
2. *$f_1(x)$ has a unique inflection point $x_2 \in (a_1, b_1)$ satisfying $f_1''(x_2) = 0$. Moreover, $f_1''(x) < 0$ for $x < x_2$ and $f_1''(x) > 0$ for $x > x_2$.*

3. *For any point $x_1 \in (a_1, b_1)$ such that $f_1'(x_1) = 0$, $f_1(x_1) < 0$.*

Then $f_1(x) = 0$ has a unique solution on the interval (a_1, b_1).

b. *Suppose the function $f_2(x)$ is smooth and strictly convex on the interval $[a_2, \infty)$ with $f_2(a_2) > 0$. If there exists a constant b_2 such that $f_2(x) \leq 0$ for all $x \geq b_2$, then the equation $f_2(x) = 0$ has a unique solution on the interval (a_2, ∞).*

Proposition 3.4.1. *There exists $T_e > 0$ such that for all $T \in (0, T_e]$, $v_e(1, T) > 1$, and there exists a unique $S_-^e(T) \in (\kappa, 1)$ and a unique $S_+^e(T) \in (1, \infty)$ satisfying $v_e(S_-^e(T), T) = g(S_-^e(T))$ and $v_e(S_+^e(T), T) = g(S_+^e(T))$ respectively.*

Proof. Noting that $g(x) = (A + Bx) - (B - 1 + q)(x - \kappa)_+ + (q - p)(x - 1)_+$, we have:

$$v_e(x, T) = Ae^{-rT} + xe^{-\delta T}B - (B - 1 + q)C(x, \kappa, T) + (q - p)C(x, 1, T). \tag{3.38}$$

Let $u_T(x) = v_e(x, T) - q - (1 - q)x$. Recalling that $\kappa = (q - A)(B - 1 + q)^{-1}$ and using (3.37), we have:

$$\begin{aligned} u_T(\kappa) &= Ae^{-rT} + \kappa Be^{-\delta T} - q - (1 - q)\kappa - (B - 1 + q)C(\kappa, \kappa, T) + (q - p)C(\kappa, 1, T) \\ &\leq Ae^{-rT} + \kappa B - q - (1 - q)\kappa - (B - 1 + q)C(\kappa, \kappa, T) + (q - p)C(\kappa, 1, T) \\ &= Ae^{-rT} - q + \kappa(B - 1 + q) - (B - 1 + q)C(\kappa, \kappa, T) + (q - p)C(\kappa, 1, T) \\ &= A(e^{-rT} - 1) - (B - 1 + q)C(\kappa, \kappa, T) + (q - p)C(\kappa, 1, T). \end{aligned} \tag{3.39}$$

By (3.37) and the fact that $e^{-rT} - 1 = O(T)$, it can be verified that $C(\kappa, \kappa, T) = O(\sqrt{T})$ converges slower than the other terms for T small. Since $C(\kappa, \kappa, T)$ is always positive, we must have $u_T(\kappa) < 0$ for some T small enough. Similarly, we also have

$$u_T(1) = Ae^{-rT} + (Be^{-\delta T} - 1) - (B - 1 + q)C(1, \kappa, T) + (q - p)C(1, 1, T) > 0 \tag{3.40}$$

for T sufficiently small (implying $V_e(1, T) > 1$). Therefore, we can conclude that there must exist a T_1 small enough such that for all $T \leq T_1$, $u_T(\kappa) < 0$ and $u_T(1) > 0$. Next, differentiating $u_T(x)$ with respect to x, we obtain

$$\begin{aligned} u_T'(x) = B(e^{-\delta T} - 1) &+ (B - 1 + q)(1 - e^{-\delta T}\Phi(d_1(x, \kappa, T)) \\ &+ (q - p)e^{-\delta T}\Phi(d_1(x, 1, T)), \end{aligned} \tag{3.41}$$

from which it immediately follows that $u_T'(\kappa) > 0$ and $u_T'(1) > 0$ for T small enough. Now, differentiating with respect to x again we have

$$u_T''(x) = -(B-1+q)\frac{e^{-\delta T}}{\sigma x\sqrt{T}}\varphi(d_1(x,\kappa,T)) + (q-p)\frac{e^{-\delta T}}{\sigma x\sqrt{T}}\varphi(d_1(x,1,T)). \tag{3.42}$$

Noting that $d_1(x,1,T) = d_1(x,\kappa,T) + C_1(T)$, where $C_1(T) = \frac{\log\kappa}{\sigma\sqrt{T}} < 0$, we have:

$$u_T''(x) = \frac{e^{-\delta T}\varphi(d_1(x,\kappa,T))}{\sigma x\sqrt{T}}\Big(-(B-1+q) + (q-p)e^{-C_1(T)d_1(x,\kappa,T)-\frac{1}{2}C_1(T)^2}\Big). \tag{3.43}$$

Let $h_T(x) = -(B-1+q) + (q-p)e^{-C_1 d_1(x,\kappa,T)-\frac{1}{2}C_1^2}$, so that the roots of h_T are the same as those of u_T''. It can easily be verified that $\lim_{x\to 0+} h_T(x) = -(B-1+q) < 0$, $\lim_{x\to\infty} h_T(x) = \infty$ and $h_T(x)$ is strictly increasing in x. So for each fixed T we must have a unique root $x_2(T) \in (0,\infty)$ such that $h_T(x_2(T)) = 0$. Letting $C_2 = \frac{q-p}{B-1+q}$, a simple calculation yields

$$x_2(T) = \kappa^{\frac{1}{2}} \exp\left(-\left(r - \delta + \left(\frac{1}{2} - \frac{\log C_2}{\log\kappa} \right)\sigma^2 \right) T \right). \tag{3.44}$$

Moreover, since $\kappa < \kappa^{\frac{1}{2}} < 1$ and $x_2(T)$ converges to $\kappa^{\frac{1}{2}}$ as $T \to 0$, there must exist a T_2 small enough such that $x_2(T) \in (\kappa, 1)$ for all $T \le T_2$. Taking T_e small enough, we have that $u_T(\kappa) < 0$, $u_T(1) > 0$, $u_T'(\kappa) > 0$, $u_T'(1) > 0$, and u_T'' is strictly negative on $(\kappa, x_2(T))$, and strictly positive on $(x_2(T), 1)$ for $T \le T_e$. Finally, substituting $u_T'(x_1(T)) = 0$ into the definition of $u_T(x)$ and simplifying yields:

$$\begin{aligned} u_T(x_1(T)) = q(e^{-rT}-1) - (q-A)(1-\Phi(d_2(x_1(T),\kappa,T))) \\ - (q-p)e^{-rT}\Phi(d_2(x_1(T),1,T)) < 0 \end{aligned} \tag{3.45}$$

Lemma 3.4.3 (a) implies that $u_T(x) = 0$ must attain a unique root on $(\kappa, 1)$ for every fixed $T \le T_e$, i.e. there is a unique $S_-^e(T)$ such that $v_e(S_-^e(T), T) = g(S_-^e(T))$.

To prove $v_e(x,T) = g(x)$ attains a unique root on $(1,\infty)$, we let $v_T(x) = v_e(x,T) - p - (1-p)x$. Note that $u_T(1) = v_T(1)$. So, $v_T(1) > 0$ for T small enough. Since $v_T''(x) = u_T''(x)$, $v_T''(x) > 0$ on $(1,\infty)$ and $v_T(x)$ is strictly convex on $(1,\infty)$ for T small enough. Moreover, since $v_e(x,T) \le v(x,T) = g(x)$ for $x \ge S_+(T)$ and $1 < S_+(T) \le S_2 < \infty$, it can be easily verified that $v_T(x) = v_e(x,T) - p - (1-p)x \le 0$ for all $x \ge S_+(T)$. By Lemma 3.4.3 (b), we obtain that there exists a unique root $S_+^e(T) \in (1,\infty)$ satisfying $V_e(S_+^e(T), T) = g(S_+^e(T))$ for small enough T. □

The following depends only on the fact that $v(x,T) \geq v_e(x,T)$.

Lemma 3.4.4. *For $T \leq T_e$:*

$$S_-(T) \leq S^e_-(T) \leq S^e_+(T) \leq S_+(T) \tag{3.46}$$

Proof. Since $v(x,T) \geq v_e(x,T)$, we have $g(S_-(T)) = v(S_-(T),T) \geq v_e(S_-(T),T)$ and $g(S_+(T)) = v(S_+(T),T) \geq v_e(S_+(T),T)$. Since $v_e > g$ on $(S^e_-(T), S^e_+(T))$, we must have $S^e_-(T) \geq S_-(T)$ and $S^e_+(T) \leq S_+(T)$. □

We next give a rough result on the rate of convergence of the boundaries $S^e_-(T)$ and $S^e_+(T)$ to one in small-time (Lamberton (1995, Lemma 2.2) proves an analogous property for the American put).

Lemma 3.4.5.

$$\lim_{T\to 0+} \frac{S^e_-(T) - 1}{\sqrt{T}} = -\infty, \qquad \lim_{T\to 0+} \frac{S^e_+(T) - 1}{\sqrt{T}} = \infty.$$

Proof. By the previous Lemma, we have

$$1 = \lim_{T\to 0+} S_-(T) \leq \lim_{T\to 0+} S^e_-(T) \leq \lim_{T\to 0+} S^e_+(T) \leq \lim_{T\to 0+} S_+(T) = 1. \tag{3.47}$$

Next, note that for T small enough, we have a unique $S^e_-(T)$ such that $q + (1-q)S^e_-(T) = V_e(S^e_-(T),T)$. Then, by (3.38), and a simple rearrangement, we obtain

$$\begin{aligned} A(e^{-rT} - 1) + S^e_-(T)B(e^{-\delta T} - 1) + (q-p)C(S^e_-(T), 1, T) \\ - (B - 1 + q)\Big(C(S^e_-(T), \kappa, T) - (S^e_-(T) - \kappa)\Big) = 0. \end{aligned} \tag{3.48}$$

Using put-call parity (with dividends), we obtain from (3.48):

$$\begin{aligned} \frac{1 - S^e_-(T)}{\sqrt{T}} e^{-\delta T} = \frac{A(e^{-rT} - 1)}{(q-p)\sqrt{T}} + \frac{BS^e_-(T)(e^{-\delta T} - 1)}{\sqrt{T}} + \frac{\mathbb{E}\left[e^{-rT} \max\left(1 - X_T^{S^e_-(T)}, 0\right)\right]}{\sqrt{T}} \\ + \frac{e^{-\delta T} - e^{-rT}}{\sqrt{T}} - \frac{(B - 1 + q)\left(C(S^e_-, \kappa, T) - (S^e_-(T) - \kappa)\right)}{(q-p)\sqrt{T}}. \end{aligned} \tag{3.49}$$

By Lemma 3.4.2 and elementary calculus, all the terms on the right hand side of the above equation tend to zero, except for the "put-option" term. Thus:

$$
\begin{aligned}
\eta_1 := \liminf_{T\to 0+} \frac{1-S^e_-(T)}{\sqrt{T}} &= \liminf_{T\to 0+} \frac{e^{-rT}}{\sqrt{T}} \mathbb{E}\left[\max\left(1 - X_T^{S^e_-(T)}, 0\right)\right] \\
&= \liminf_{T\to 0} \frac{1}{\sqrt{T}} \mathbb{E}\left[\max\left(1 - S^e_-(T)\exp\left((r-\delta-\tfrac{\sigma^2}{2})T + \sigma\sqrt{T}\cdot Z\right), 0\right)\right] \\
&\geq \mathbb{E}\left[\max\left(\liminf_{T\to 0+} \frac{1-S^e_-(T)}{\sqrt{T}} + \lim_{T\to 0+} \frac{S^e_-(T)\left(1-\exp\left((r-\delta-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\cdot Z\right)\right)}{\sqrt{T}}, 0\right)\right]
\end{aligned}
$$

where $Z \sim N(0,1)$, and we have used Fatou's Lemma. If $\eta_1 \in [0,\infty)$ then we get $\eta_1 \geq \mathbb{E}[\max(\eta_1 - \sigma Z, 0)]$, which leads to a contradiction as $\mathbb{Q}(\{\eta_1 - \sigma Z < 0\}) > 0$ implies $\mathbb{E}[\max(\eta_1 - \sigma Z, 0)] > \eta_1 - \sigma\mathbb{E}[Z] = \eta_1$. Thus $\eta_1 = \infty$. The proof for $S^e_+(T)$ is similar. □

Theorem 3.4.1. *For $T \geq 0$, let*

$$\psi_1(T) = \frac{-\log S^e_-(T)}{\sigma\sqrt{T}}, \quad \psi_2(T) = \frac{\log S^e_+(T)}{\sigma\sqrt{T}}.$$

As $T \searrow 0$:

$$\psi_1(T)^2 e^{\frac{\psi_1(T)^2}{2}} \sim \frac{(q-p)\sigma}{(rq+(1-q)\delta)\sqrt{2\pi T}}, \quad \psi_2(T)^2 e^{\frac{\psi_2(T)^2}{2}} \sim \frac{(q-p)\sigma}{(rq+(1-q)\delta)\sqrt{2\pi T}}, \tag{3.50}$$

and furthermore,

$$1 - S^e_-(T) \sim \sigma\sqrt{T(-\log T)}, \quad \text{and} \quad S^e_+(T) - 1 \sim \sigma\sqrt{T(-\log T)}. \tag{3.51}$$

Proof. Let $y(T) = S^e_-(T)$ for convenience. A simple rearrangement of $v_e(y(T),T) = q + (1-q)y(T)$ yields:

$$
\begin{aligned}
\mathbb{E}[e^{-rT}g(X_T^{y(T)})] = qe^{-rT} + (1-q)y(T)e^{-\delta T} + \mathbb{E}[e^{-rT}\big(\mathbf{1}_{\{X_T^{y(T)}\leq\kappa\}}(A-q+(B-1+q)X_T^{y(T)})\big)] \\
+ \mathbb{E}[e^{-rT}\big(\mathbf{1}_{\{X_T^{y(T)}\geq 1\}}(p-q+(q-p)X_T^{y(T)})\big)].
\end{aligned} \tag{3.52}
$$

So:

$$q(1-e^{-rT})+(1-q)y(T)(1-e^{-\delta T})=\mathbb{E}[e^{-rT}\big(\mathbf{1}_{\{X_T^{y(T)}\leq\kappa\}}(A-q+(B-1+q)X_T^{y(T)})\big)]$$
$$+\mathbb{E}[e^{-rT}\big(\mathbf{1}_{\{X_T^{y(T)}\geq 1\}}(p-q+(q-p)X_T^{y(T)})\big)]. \quad (3.53)$$

Noting that $q(1-e^{-rT})+(1-q)y(T)(1-e^{-\delta T})\sim(rq+(1-q)\delta)T$ and $\lim_{T\to 0^+}\mathbb{E}[e^{-rT}\big(\mathbf{1}_{\{X_T\leq\kappa\}}(A-q+(B-1+q)X_T^{y(T)})\big)]/T=0$, (3.53) becomes

$$\frac{(rq+(1-q)\delta)T}{q-p}\sim\mathbb{E}[e^{-rT}(X_T^{y(T)}-1)^+]. \quad (3.54)$$

Now, let

$$\alpha_1(T)=\frac{-\log S_-^e(T)-(r-\delta-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}.$$

By Lemma 3.4.5, we have $\lim_{T\to 0^+}\alpha_1(T)=\infty$ and $\lim_{T\to 0^+}\sqrt{T}\alpha_1(T)=0$, from which we obtain:

$$\frac{(rq+(1-q)\delta)T}{q-p}\sim\mathbb{E}[(e^{\sigma\sqrt{T}Z}-e^{\sigma\sqrt{T}\alpha_1(T)})^+], \quad (3.55)$$

with $Z\sim N(0,1)$. Next, let

$$f(T):=\mathbb{E}[(e^{\sigma\sqrt{T}Z}-e^{\sigma\sqrt{T}\alpha_1(T)})^+]=\mathbb{E}[\big(e^{\sigma\sqrt{T}Z}-e^{\sigma\sqrt{T}\alpha_1(T)}\big)\mathbf{1}_{\{Z>\alpha_1(T)\}}]. \quad (3.56)$$

Using the inequality $|e^x-1-x|\leq\frac{x^2}{2}e^{|x|}$, we get:

$$|f(T)-\mathbb{E}[\sigma\sqrt{T}(Z-\alpha_1(T))\mathbf{1}_{\{Z>\alpha_1(T)\}}]|$$
$$\leq\frac{\sigma^2T}{2}\mathbb{E}[Z^2e^{\sigma\sqrt{T}|Z|}\mathbf{1}_{\{Z>\alpha_1(T)\}}]+\frac{\sigma^2T\alpha_1(T)^2}{2}e^{\sigma\sqrt{T}|\alpha_1(T)|}\mathbb{Q}[Z>\alpha_1(T)]. \quad (3.57)$$

Since $\lim_{T\to 0^+}\alpha_1(T)=\infty$ and $\lim_{T\to 0^+}\sqrt{T}\alpha_1(T)=0$, both terms in (3.57) are $o(T)$ and we have $|f(T) - \mathbb{E}[\sigma\sqrt{T}(Z-\alpha_1(T))\mathbf{1}_{\{Z>\alpha_1(T)\}}]| = o(T)$. Then, from (3.55), we have:

$$\begin{aligned}\frac{(rq+(1-q)\delta)T}{(q-p)} &\sim \mathbb{E}[\sigma\sqrt{T}(Z-\alpha_1(T))\mathbf{1}_{\{Z>\alpha_1(T)\}}] \\ &= \frac{\sigma\sqrt{T}}{\sqrt{2\pi}\alpha_1(T)^2 e^{\alpha_1(T)^2/2}}\int_0^\infty x e^{-x-\frac{x^2}{2\alpha_1(T)^2}}dx. \end{aligned} \tag{3.58}$$

Since $\lim_{T\to 0^+}\int_0^\infty xe^{-x-\frac{x^2}{2\alpha_1(T)^2}}dx = \int_0^\infty xe^{-x}dx = 1$,

$$\frac{(rq+(1-q)\delta)T}{(q-p)} \sim \frac{\sigma\sqrt{T}}{\sqrt{2\pi}\alpha_1(T)^2 e^{\alpha_1(T)^2/2}}. \tag{3.59}$$

So:

$$\psi_1(T)^2 e^{\frac{\psi_1(T)^2}{2}} \sim \alpha_1(T)^2 e^{\frac{\alpha_1(T)^2}{2}} \sim \frac{(q-p)\sigma}{(rq+(1-q)\delta)\sqrt{2\pi T}}, \tag{3.60}$$

where we have used that $\psi_1(T) - \alpha_1(T) = O(\sqrt{T})$, and $\lim_{T\to 0^+}\alpha_1(T) \to \infty$. Since $\lim_{T\to 0^+}\frac{\psi_1(T)}{\log\psi_1(T)} = \infty$, it is then elementary that $1-y(T) \sim -\log(y(T)) \sim \sigma\sqrt{T(-\log T)}$. The proof for $S_+^e(T)$ is similar. $\square$

To conclude, we must show that $S_-(T), S_+(T)$ are sufficiently close (within $\sqrt{T}$) to the corresponding "European" boundaries $S_-^e(T), S_+^e(T)$. To accomplish this, we need the following bounds on the derivative of v, whose proof is given in the appendix.

Lemma 3.4.6. *For small $T > 0$:*

$$\sup_{(S_-(T),S_-^e(T))}\frac{\partial}{\partial x}v(\cdot,T) \le (1-q) + o(1) \tag{3.61}$$

$$\sup_{(S_+^e(T),S_+(T))}\frac{\partial}{\partial x}v(\cdot,T) \le (1-p) + o(1) \tag{3.62}$$

Proposition 3.4.2. *There exist constants $C > 0$ and $T' > 0$ such that for all $0 < T \le T'$,*

$$0 \le S_-^e(T) - S_-(T) \le C\sqrt{T}, \tag{3.63}$$

and

$$0 \le S_+(T) - S_+^e(T) \le C\sqrt{T}. \tag{3.64}$$

Proof. Note that $v(x,T)$ is a classical solution of $v_T = Lv - rv$ on $(S_-(T), S_+(T))$ and $\frac{\partial v}{\partial x}(S_-(T),T) = 1-q$. Taylor's formula yields:

$$\begin{aligned} v(S_-^e(T),T) &= q + (1-q)S_-^e(T) + \frac{(S_-^e(T) - S_-(T))^2}{2}\frac{\partial^2 v}{\partial x^2}(\xi_1(T),T), \\ &= v_e(S_-^e(T),T) + \frac{(S_-^e(T) - S_-(T))^2}{2}\frac{\partial^2 v}{\partial x^2}(\xi_1(T),T), \end{aligned}$$

where $\xi_1(T) \in (S_-(T), S_-^e(T))$. The early exercise premium representation (3.14), together with the facts that $0 \leqslant \Phi \leqslant 1$ and $\kappa < S_-^e(T) \leq 1$ yields:

$$\frac{(S_-^e(T) - S_-(T))^2}{2}\frac{\partial^2 v}{\partial x^2}(\xi_1(T),T) = v(S_-^e(T),T) - v_e(S_-^e(T),T) \leq KT \tag{3.65}$$

for some $K \geq 0$. Using $S_-(T) < \xi_1(T) < S_-^e(T)$ and $\frac{\partial v}{\partial T}(x,T) \geq 0$ gives:

$$\begin{aligned} \frac{\partial^2 v}{\partial x^2}(\xi_1(T),T) &= \frac{2}{\sigma^2 \xi_1(T)^2}\left(\frac{\partial v}{\partial T}(\xi_1(T),T) - (r-\delta)\xi_1(T)\frac{\partial v}{\partial x}(\xi_1(T),T) + rv(\xi_1(T),T)\right) \\ &\geq \frac{2r}{\sigma^2 \xi_1(T)^2}((v(\xi_1(T),T) - \xi_1(T)\frac{\partial v}{\partial x}(\xi_1(T),T))) \\ &\geq \frac{2r}{\sigma^2 \xi_1(T)^2}(q + (1-q)\xi_1(T) - \xi_1(T)\frac{\partial v}{\partial x}(\xi_1(T),T)) \geq c > 0, \end{aligned}$$

for T small enough, by Lemma 3.4.6. Thus

$$(S_-^e(T) - S_-(T))^2 \leq C_1 T \Rightarrow S_-^e(T) - S_-(T) \leq C_1\sqrt{T}, \tag{3.66}$$

for some $C_1 > 0$ and $0 < T \leq T_1$. The proof of (3.64) is similar. □

Finally, by Theorem 3.4.1, we can easily obtain

$$\sigma\sqrt{T(-\log T)} \sim 1 - S_-(T), \quad \sigma\sqrt{T(-\log T)} \sim S_+(T) - 1. \tag{3.67}$$

3.5 Conclusion

Chapters 2 and 3 analyze optimal stopping problems for a investor with a piecewise linear payoff function, where the underlying follows a geometric Brownian motion, corresponding to a hedge fund with a continuous fee for assets under management deducted (or, equivalently, the price process for a stock paying a continuous dividend yield). We present a complete solution of the problem in the infinite horizon case. In the finite horizon case, we describe the shape of the continuation region, characterize the stopping boundaries using a coupled pair of integral equations, and present an asymptotic analysis of the boundaries in small time.

Chapter 4

Performance Measures of Hedge Fund Investments and the Myth of Fee Diversification

4.1 Introduction

When considering the fee structure of any investment management relationship, both sides must be concerned with the incentives the contract provides to their counterparty. Managers often want to entice further investment in order to increase assets under management, and thus must ensure that the fee structure is appealing to investors (while still being lucrative for themselves). Investors face a classical principal-agent problem, and must ensure that managers are properly motivated to run the portfolio in the best interests of the investors for whom they work.

Typically, the fee structure of a hedge fund consists of two parts: a management fee and a performance fee (perhaps restricted by a high-water mark provision). The management fee is charged as a percentage of assets under management and the performance fee is charged as a percentage of the investor's profits. For instance, a traditional fee structure consists of a management fee of 2% of assets under management and a performance fee of 20% of net profits.

Hedge funds have faced intense scrutiny since the financial crisis, as the fees they charge investors have been outsized compared to the returns they are posting as a group.[1]

[1]https://www.bloomberg.com/news/articles/2017-06-07/new-york-illinois-pension-funds-say-hedge-

Furthermore, it is unclear whether the option-like structure of the manager's payoff under traditional schemes might lead managers to undertake riskier strategies that may be at odds with the risk aversion of investors.[2] In recent years, innovative fee structures have emerged aiming at better alignment between investors' interests and the hedge fund's business objectives. An example is the class of first-loss fee structures (see Banzaca (2012) for a description). In these structures, in return for a higher performance fee, the hedge fund manager provides some downside protection to the investors on their losses. Fee innovation has led to discussions and negotiations between investors and managers on the optimal fee to be used in certain situations, amplifying the universe of available fee structures by mixing different structures together. One example is the shared-loss structure, which can be considered as a mixture of the classical structure and the first-loss structure. Under a shared-loss agreement, rather than covering *all* investor losses up to a certain limit, the manager will provide compensation for a proportion of the investor's losses (again subject to a ceiling).

In analogy with the concept of diversification from portfolio theory, it may be posited that the optimal fee structure from the fund investor's point of view should be a shared-loss structure, i.e. a combination of the extremes. In particular, the hypothesis[3] is that by making investments subject to different fee structures (even on identical or highly correlated underlying assets), diversification would reduce risk and therefore lead to an improved risk-return tradeoff.[4] Were such a diversification effect to exist, it would show up not in the (risk-neutral) valuation of the fee structures, but rather when examining the risk-return tradeoff among fee structures.

In Djerroud et al. (2016), the fairness of the prevailing levels of hedge fund fees was investigated in terms of risk-neutral valuation. In this chapter we put the hypothesis of fee diversification to the test by examining the problem of an investor who may choose any combination of the first loss and classical fee structures, and seeks to maximize either the Sharpe ratio or the Sortino ratio of their final payoff, evaluated using real-world probabilities. We demonstrate that for the vast majority of fund mean returns and volatilities, there is no fee diversification effect: either the first-loss structure or the classical structure is optimal for the investor. We also identify the regions of the parameter space in which

funds-fees-too-high.

[2]For further discussion on this point, see He and Kou (2018), Kouwenberg and Ziemba (2007), and Hodder and Jackwerth (2007).

[3]Luis Seco, personal communication, based on ideas expressed by market practitioners at several industry conferences.

[4]The payouts with different fee structures on the same fund are comonotonic, but not perfectly correlated.

fee diversification prevails, and demonstrate that even in these regions the effect of fee diversification is not large.

The remainder of the chapter is organized as follows. Section 2 describes the payouts to hedge fund investors under mixed fee structures. Section 3 summarizes analytical results regarding the Sharpe ratio (and its maximization) and the Sortino ratio for mixed fee structures. Section 4 presents numerical examples. Section 5 concludes. Technical details and derivations of results are presented in Appendix C.

4.2 Portfolios of Fee Structures

Let X_t^x be the value of the hedge fund's assets at time t, assuming a value of x at the initial date. Further, let m_1, and α_1 denote the management fee and performance fee for the traditional fee structure, and m_2, and α_2 denote the management fee and performance fee for the first-loss fee structure. We assume the management fees are proportional to the initial investment x and performance fees are proportional to the final hedge fund value X_T^x.[5] Thus the payoff function for the investor in the traditional fee structure at maturity is:

$$g_1(X_T^x) = \begin{cases} X_T^x - m_1 x - \alpha_1(X_T^x - m_1 x - x), & X_T^x \geq x + m_1 x, \\ X_T^x - m_1 x, & X_T^x < x + m_1 x, \end{cases} \tag{4.1}$$

while the payoff function under the first-loss fee structure at maturity is:

$$g_2(X_T^x) = \begin{cases} X_T^x - m_2 x - \alpha_2(X_T^x - m_2 x - x), & X_T^x \geq x + m_2 x, \\ x, & x + m_2 x - cx \leq X_T^x < x + m_2 x, \\ X_T^x - m_2 x + cx, & X_T^x < x + m_2 x - cx, \end{cases} \tag{4.2}$$

where c is the deposit amount, giving the fraction of the initial investment that is insured (see Djerroud et al. (2016)).

[5] Note that in this chapter, we assume that the management fee is paid upfront, rather than on a continuous basis, as in Chapters 2 and 3. Although management fees are paid according to a determined schedule (usually monthly or quarterly Djerroud et al. (2016)), for simplicity, we will assume a single payment at the end of a fixed term T.

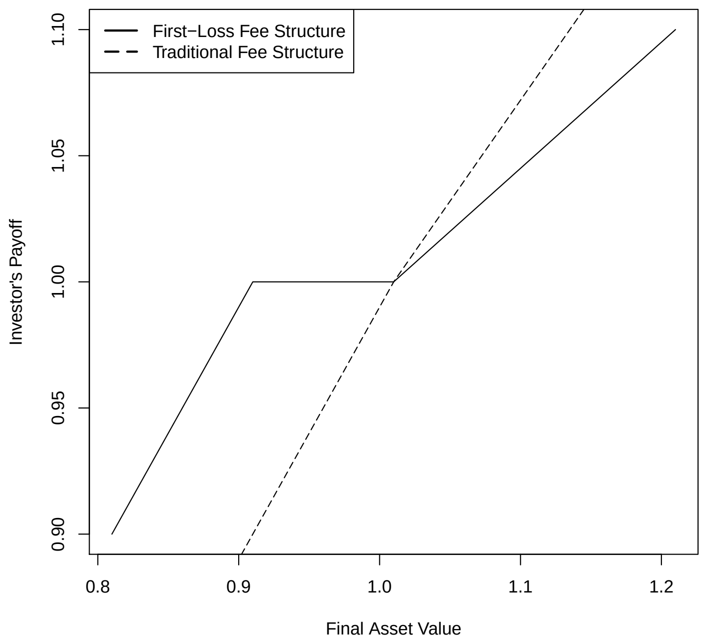


Figure 4.1: Investor's Payoffs: Escrow First-Loss Fee Structure versus Traditional Fee Structure with parameters $x = 1, m_1 = m_2 = 0.01, c = 0.1, \alpha_1 = 0.4$ and $\alpha_2 = 0.5$.

Now, let $\omega_1 \geq 0$ and $\omega_2 \geq 0$ denote the proportion of the fund invested in the traditional fee structure and first-loss fee structure respectively, where $\omega_1 + \omega_2 = 1$. Then, the investor's final payoff becomes

$$g(X_T^x) = g_1(\omega_1 X_T^x) + g_2(\omega_2 X_T^x) = \omega_1 g_1(X_T^x) + \omega_2 g_2(X_T^x), \tag{4.3}$$

where the second equation follows since $g_i(\omega_i X_T^x) = g_i(X_T^{\omega_i x}) = \omega_i g_i(X_T^x)$, for $i = 1, 2$, $\omega_i \geq 0$.

4.3 Performance Ratio Maximization with Portfolios of Fee Structures

Assuming no early withdrawals, and an investment horizon of T, the investor's Sharpe ratio becomes:

$$SR(\boldsymbol{\omega}) := \frac{\mathbb{E}[r(T)] - r}{\sqrt{\mathrm{Var}[r(T)]}} = \frac{\mathbb{E}[g(X_T^x)] - (1+r)x}{\sqrt{\mathrm{Var}[g(X_T^x)]}}, \tag{4.4}$$

where $r(T) = g(X_T^x)/x - 1$ is the fund investor's return, r is the risk-free interest rate and $\boldsymbol{\omega} = (\omega_1, \omega_2)$. Note that maximizing the investor's SR in terms of $g_2(\omega_2 X_T^x)$ alone will lead to $c = 1$, which is unrealistic. Therefore, c should be held in a reasonable range and a combination of the traditional fees structure and first-loss fee structure is a reasonable approach.

We assume that the investor seeks to maximize the Sharpe Ratio $SR(\boldsymbol{\omega})$. Let Y_1 be the excess profit (above an investment at the risk-free rate) of an investment in the classical fund scheme ($\omega_1 = 1$), and Y_2 be the excess profit of an investment in the first-loss structure ($\omega_2 = 1$). Let $\tilde{\mu}_i = \mathbb{E}[Y_i]$, $i = 1, 2$, $\tilde{\sigma}_i^2 = \mathrm{Var}[Y_i], i = 1, 2$ and $\tilde{\sigma}_{12} = \mathrm{Cov}[Y_1, Y_2]$.

Theorem 4.3.1. *Suppose $\tilde{\mu}_1 \geq 0$ and $\tilde{\mu}_2 \geq 0$. Let $\boldsymbol{\omega}^* = (\omega_1^*, \omega_2^*)$ be the optimal solution for (4.4). If*

$$\tilde{\sigma}_{12} \leq \min\left\{\frac{\tilde{\mu}_1}{\tilde{\mu}_2}\tilde{\sigma}_2^2,\ \frac{\tilde{\mu}_2}{\tilde{\mu}_1}\tilde{\sigma}_1^2\right\}, \tag{4.5}$$

is satisfied, then

$$\omega_1^* = \frac{\tilde{\mu}_1\tilde{\sigma}_2^2 - \tilde{\mu}_2\tilde{\sigma}_{12}}{C^*} > 0, \quad \omega_2^* = \frac{\tilde{\mu}_2\tilde{\sigma}_1^2 - \tilde{\mu}_1\tilde{\sigma}_{12}}{C^*} > 0, \tag{4.6}$$

where $C^* = (\tilde{\mu}_1\tilde{\sigma}_2 - \tilde{\mu}_2\tilde{\sigma}_1)^2 + 2\tilde{\mu}_1\tilde{\mu}_2(\tilde{\sigma}_1\tilde{\sigma}_2 - \tilde{\sigma}_{12})$. Otherwise,

$$\boldsymbol{\omega}^* = \begin{cases} (1/\tilde{\mu}_1, 0), & \textit{if } \frac{\tilde{\mu}_1^2}{\tilde{\sigma}_1^2} \geq \frac{\tilde{\mu}_2^2}{\tilde{\sigma}_2^2}, \\ (0, 1/\tilde{\mu}_2), & \textit{o.w.} \end{cases} \tag{4.7}$$

The proof can be found in Appendix C. By the above results, one can distinguish between the cases for which there exists a non-trivial asset allocation for the traditional fee structure and the first-loss fee structure ($\omega_i^* > 0$ for $i = 1, 2$), and when full investment in one fee structure is optimal in terms of the Sharpe ratio. The results can be understood in terms of the Sharpe ratios $SR_i = \frac{\tilde{\mu}_i}{\tilde{\sigma}_i}$, $i = 1, 2$ of the two payoffs, and their correlation $\rho = \text{corr}(Y_1, Y_2) = \tilde{\sigma}_{12}/\tilde{\sigma}_1\tilde{\sigma}_2$.[6] Assuming that $\tilde{\mu}_i > 0$, if the correlation is less than the threshold $H = \min\{\frac{SR_1}{SR_2}, \frac{SR_2}{SR_1}\}$, then there is enough potential for diversification that the optimal portfolio will have a positive investment in each fee structure. Otherwise, all of the wealth will be invested in the fee scheme with the higher Sharpe ratio. The greater the difference between the Sharpe ratios of the two stand-alone fee structures Y_1 and Y_2, the lower the threshold H, and consequently the lower the correlation required to prompt investors to put some money in the fee structure with the lower Sharpe ratio. Since Y_1 and Y_2 are both increasing piecewise linear functions of the same random variable X_T^x, they will tend to be highly correlated, and for most parameter sets we would expect $\rho > H$, and full investment in the fee structure with the highest Sharpe ratio.

The above discussion indicates a Catch-22 of seeking diversification in fee structures, due to the typically high correlation between Y_1 and Y_2. Either the threshold H is low, in which case it is likely that investing all wealth in one or other fee structure is optimal, or H is high, in which case the Sharpe ratios of the two fee structures are very close, and the improvement due to diversification (if it exists) is small.

Real-world hedge fund return distributions are often asymmetric, with fat tails. The Sharpe-ratio, which is based on the use of variance as a measure of risk, may be misleading when used for such distributions. Consequently, we also consider the maximization of the investor's Sortino ratio, which instead uses downside variance as the measure of risk:

$$SOR(\boldsymbol{\omega}) := \frac{\mathbb{E}[r(T)] - l}{\sigma_d}, \tag{4.8}$$

where l is the minimal acceptable return and $\sigma_d^2 := \mathbb{E}[\min\{r(T) - l, 0\}^2]$[7] is called the lower partial moment of $r(T)$.

4.4 Numerical Results

In this section, we present numerical results analyzing the optimal fee structure for hedge fund investors in terms of maximizing both the Sharpe Ratio and the Sortino Ratio, discussing the limited potential extent and impact of "fee diversification."

[6]Explicit mathematical expressions for $\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\sigma}_1, \tilde{\sigma}_2$ and $\tilde{\sigma}_{12}$ can be found in Appendix C, equations (C.18) to (C.22).

[7]A detailed mathematical derivation for σ_d^2 is provided in the Appendix C.

Throughout this section, we assume the hedge fund's dynamics under the real-world measure follow:

$$dX_t^x = \mu X_t^x dt + \sigma X_t^x dW_t, \quad X_0^x = x, \tag{4.9}$$

where $\mu > 0$ is the annual growth rate, $\sigma > 0$ is the annual volatility, W_t is a standard Brownian Motion, and the management fees for traditional fee structure and first-loss fee structure are equal. i.e. $m = m_1 = m_2$.

It should be noted that in all the examples in this chapter we assume that the parameters of the processes are known with certainty. This is obviously not the case in the real world (indeed, in practice we do not even know the process driving the returns). In practical applications, the added uncertainty due to the need to estimate the parameters (and the structure of the process driving returns) must be taken into consideration. In particular, the drift parameter μ, which has a significant impact on the preferred fee structure, is notoriously difficult to estimate in practice (see, e.g. Merton (1980)).

In Figures 4.2 and 4.3, we fix the six parameters μ, σ, T, m, c, r and let the remaining parameters α_1 and α_2 (the performance fee rates for the traditional and first-loss fee structures respectively) vary in their reasonable ranges. For each valid point (α_1, α_2), we check condition (4.5) and color the point black if the condition is satisfied. The black regions indicate that there exists a portfolio with non-zero allocation to both fee structures $\boldsymbol{\omega}$ which achieves the optimal Sharpe Ratio (an 'interior maximum'). In the white regions, there is no interior maximum, and the optimal allocation is 100% in one fee structure or the other (in the area above the black region the traditional fee structure is preferred, while below the black region the first-loss fee structure is preferred). We see that the black regions for Figures 4.2 and 4.3 are very narrow, implying that in most cases there is no benefit due to 'fee diversification'; either the traditional fee structure or the first-loss fee structure makes the investor achieve the optimal Sharpe Ratio when α_1 and α_2 are changing. The first-loss fee structure appears to be preferred for the bulk of the performance fee combinations that we consider (i.e. the area below the black region is larger than the area above the black region). The black region follows a nearly linear path for all combinations of (μ, σ) that we considered. Looking at the different graphs, we see that the traditional fee structure is preferable (requires a lower performance fee to be the structure selected) when the underlying hedge fund assets have higher expected returns, and lower volatilities, with the impact of low volatilities being particularly pronounced. This is intuitive, as with higher means and lower volatilities, the put-option-like downside protection provided to the investor in the first-loss structure is less likely to be exercised (and will tend to pay off less when it is exercised).

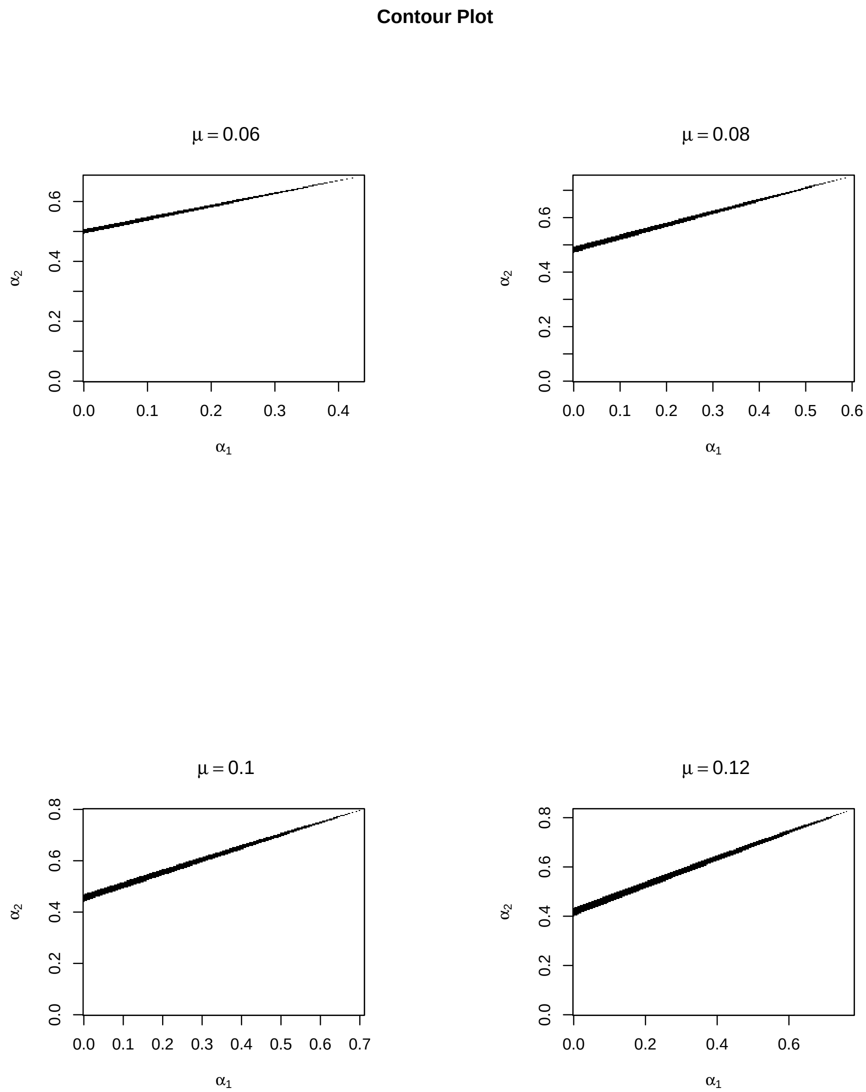


Figure 4.2: Sharpe Ratio Contour Plots: The two axes are α_1 and α_2, the performance fees of the traditional fee structure and the first-loss fee structure. The black region indicates that the condition (4.5) for an interior maximum is satisfied. The remaining parameter values are fixed at $x = 1$, $\sigma = 0.1$, $T = 1$, $m = 0.01$, $c = 0.1$, $r = 0.02$, and $\mu = 0.06, 0.08, 0.1, 0.12$.

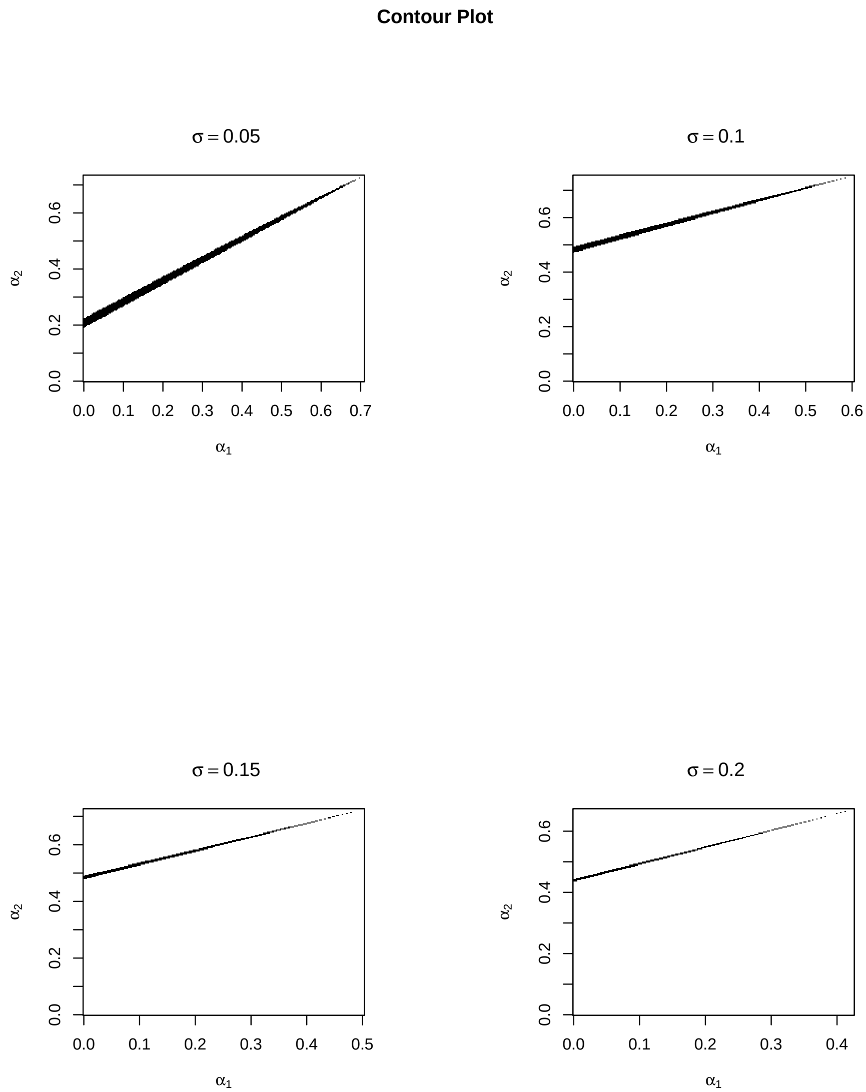


Figure 4.3: Sharpe Ratio Contour Plots: The two axes are α_1 and α_2, the performance fees of the traditional fee structure and the first-loss fee structure. The black region indicates that the condition (4.5) for an interior maximum is satisfied. The remaining parameter values are fixed at $x = 1$, $\mu = 0.08$, $T = 1$, $m = 0.01$, $c = 0.1$, $r = 0.02$, and $\sigma = 0.05, 0.1, 0.15, 0.2$.

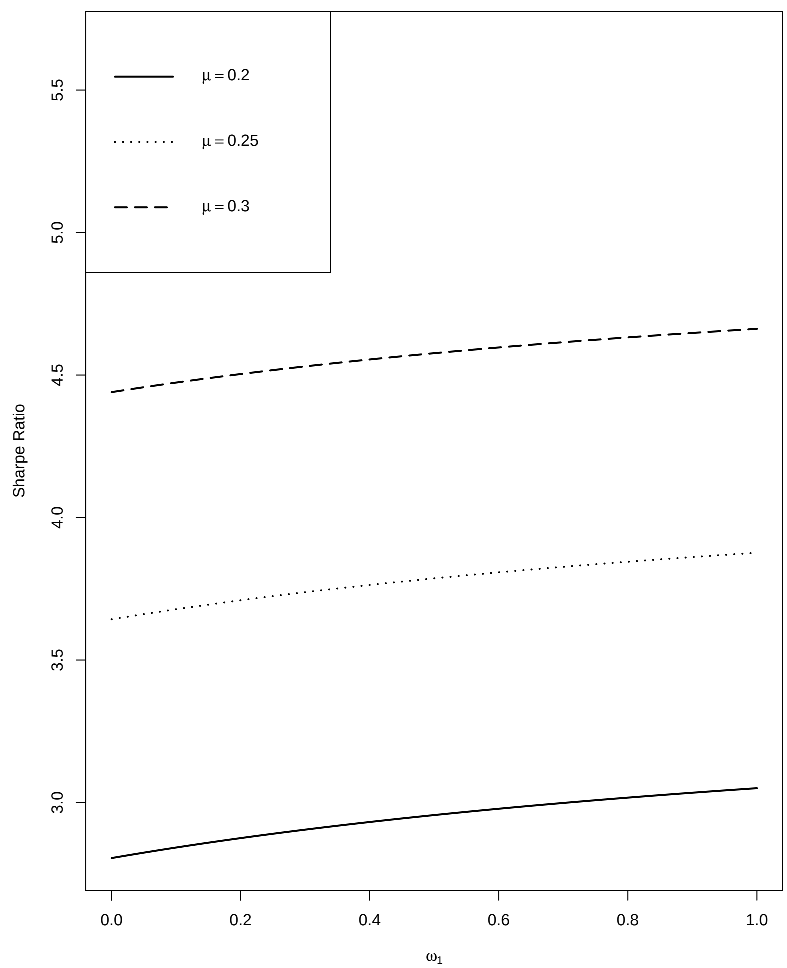


Figure 4.4: Sharpe Ratio vs the proportion the investor puts in the traditional fee structure fund, ω_1. The solid, dotted, and dashed lines are the cases $\mu = 0.2, 0.25$, and 0.3 respectively. The remaining parameter values are chosen as: $x = 1, \sigma = 0.05, T = 1, m = 0.01, c = 0.1$, and $r = 0.02$.

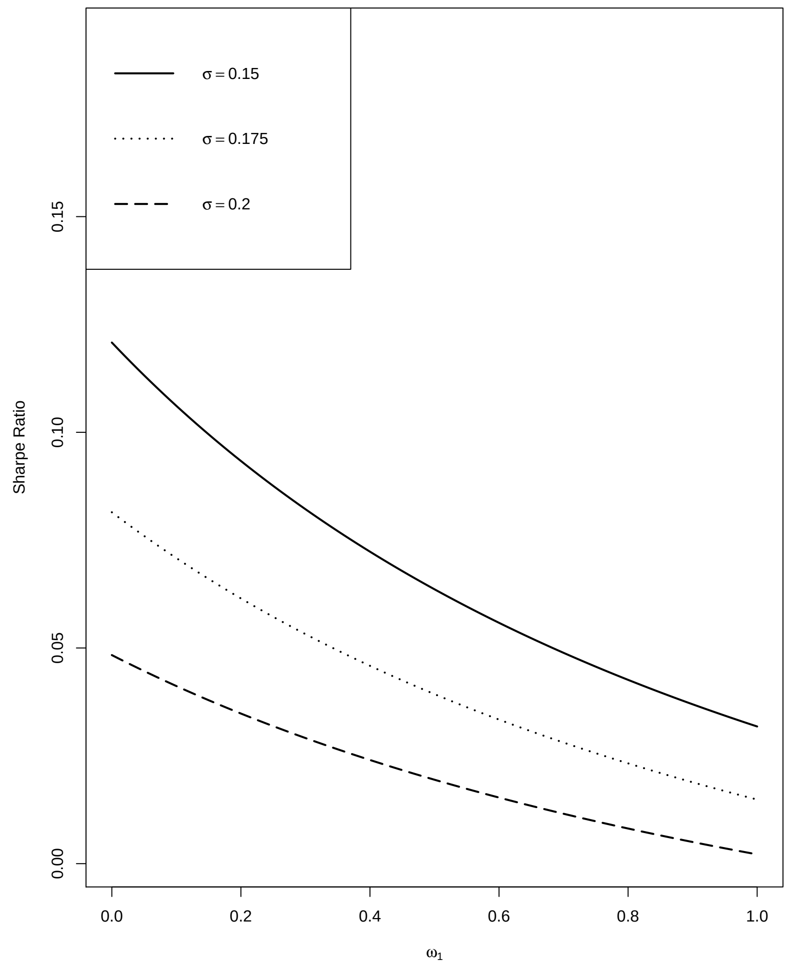


Figure 4.5: Sharpe Ratio vs the proportion the investor puts in the traditional fee structure fund, ω_1. The solid, dotted, and dashed lines are the cases $\sigma = 0.15, 0.175$, and 0.2 respectively. The remaining parameter values are chosen as: $x = 1, \mu = 0.05, T = 1, m = 0.01, c = 0.1$, and $r = 0.02$. Note that when $\sigma \approx 0.205$, $\alpha_2^* \approx 0.5$.

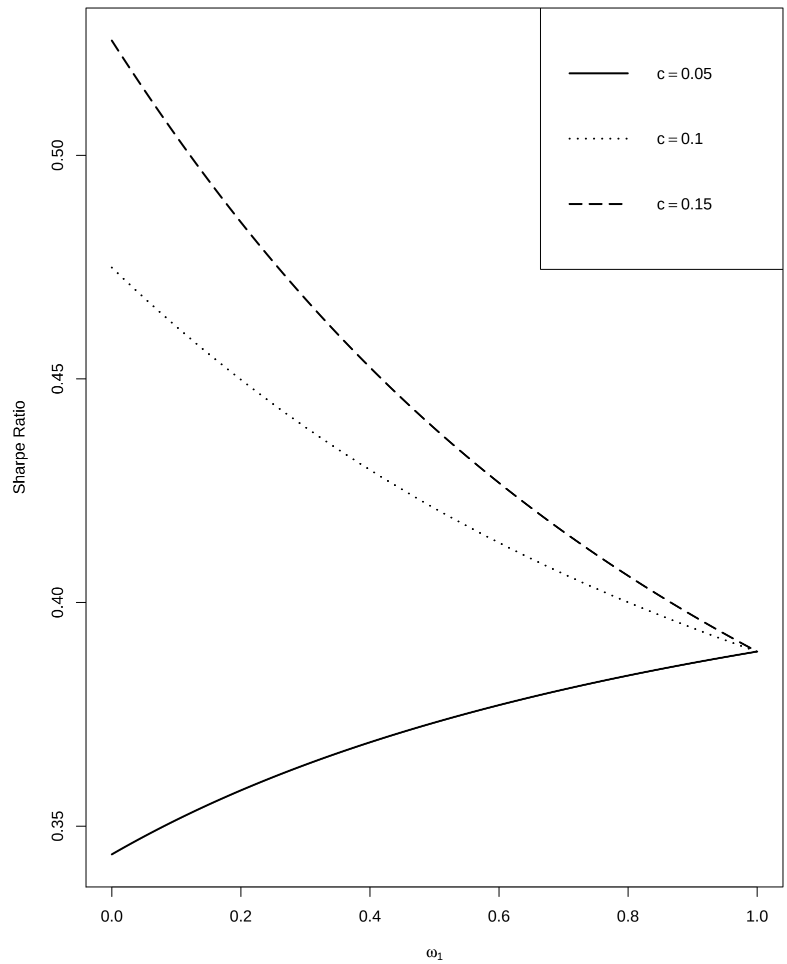


Figure 4.6: Sharpe Ratio vs the proportion the investor puts in the traditional fee structure fund, ω_1. The solid, dotted, and dashed lines are the cases $c = 0.05, 0.1$, and 0.15 respectively. The remaining parameter values are chosen as: $x = 1, \mu = 0.08, \sigma = 0.1, T = 1, m = 0.01, c = 0.1$, and $r = 0.02$.

Next, we fix $\alpha_1 = 0.2$ and $\alpha_2 = 0.5$ (typical market values) and consider all possible portfolios (ω_1, ω_2) for different values of the parameters μ, σ, and c. In particular, we consider funds with high expected return and low volatility, low expected return and high volatility, and different deposit amounts, for which results are given in Figures 4.4, 4.5, and 4.6 respectively. The results in Figures 4.4, 4.5, and 4.6 are rather intuitive, and reinforce the observations made above. As high expected return and low volatility indicate the hedge fund is more likely to make a profit, and the downside protection afforded by the first-loss structure will not be relevant, the investor will favour the fee structure with the lower performance fee (the classical fee structure). However, we notice that the Sharpe Ratio curves in Exhibit 4.4, corresponding to a low volatility regime, are rather flat; while the classical fee structure is preferred, there is not much significant difference between the two fee structures in terms of Sharpe Ratio. On the other hand, low expected return and high volatility make the fund have a higher chance of suffering a loss, leading the investor to prefer the the first-loss fee structure. Although Exhibit 4.5 presents steeper curves, the Sharpe Ratios are very close to zero. Finally, Exhibit 4.6 indicates that the first-loss fee structure is more heavily favoured by investors when the deposit amount increases. This is entirely intuitive; the greater the protection against losses offered, the more appealing the first-loss fee structure appears.

Figures 4.7 and 4.8 present detailed results on the nature of the investor's optimal fee structure for different values of μ and σ. In Exhibit 4.7, $\alpha_2 = 0.4$, while in Exhibit 4.8, $\alpha_2 = 0.5$. The results demonstrate that the investor will always choose the traditional fee structure given the hedge fund expected return is high enough. The significance of the downside protection effect of the first-loss fee structure diminishes as the expected return increases. Essentially, fee structures with lower performance fees result in higher Sharpe Ratios. Furthermore, we see that, in contrast to what we observed earlier, the classical fee structure is preferred for extremely high levels of volatility. As noted before, when volatility increases, both the probability that the guarantee will be triggered increases, and the expected payoff given a loss is larger. However, when volatility is very high, the possibility of an extremely high payoff in the classical structure (under which the investor keeps a greater share of the profits compared to the first-loss structure) outweighs the higher expected payoff of the insurance. Very low expected returns can also produce negative Sharpe ratios, for which our Sharpe ratio maximization framework is not appropriate. Comparing the two figures, we observe the unsurprising result that the classical fee structure is preferred more often when α_2 (the manager's participation rate in the first-loss structure) is higher.

Finally, we let S_{max} denote the optimal Sharpe Ratio and S_{min} denote the lowest possible Sharpe Ratio for the investor. In Figures 4.9 and 4.10, we plot the ratios of S_{max} to S_{min} for some parameter values for which there exists an interior optimal point. Note that the lowest Sharpe Ratio is always attained on one of the boundary points, i.e.

either the traditional fee structure or the first-loss fee structure admits the lowest Sharpe Ratio. For all the cases ($\sigma = 0.1, \sigma = 0.15$ and $\sigma = 0.2$), the ratio plots exhibit valley-like behaviours. The reason is that the ratios of S_{max} to SR_1 decrease as the ratios of S_{max} to SR_2 increase when μ increases. In other words, the decreasing parts of the graphs are cases when the traditional fee structure admits lowest Sharpe ratio. And when μ is high enough, the first-loss fee structure admits the lowest Sharpe Ratio so that the graphs increase again. Generally, we find that the optimal Sharpe Ratio is not significantly larger than the worst Sharpe Ratio of these two fee structures. In fact, the largest ratio of S_{max} to S_{min} is only around 1.006 when $\mu \approx 0.15$ and $\sigma = 0.1$, indicating very little benefit from "fee diversification." This makes sense in light of our theoretical results. Since $\rho = \mathrm{corr}(Y_1, Y_2)$ is high, in order to meet the condition for an interior maximum $\rho \leq H = \min\{\frac{SR_1}{SR_2}, \frac{SR_2}{SR_1}\}$, we must have $SR_1 \approx SR_2$, i.e. the Sharpe ratios of the two fee structures must be nearly identical, and diversification has little impact.

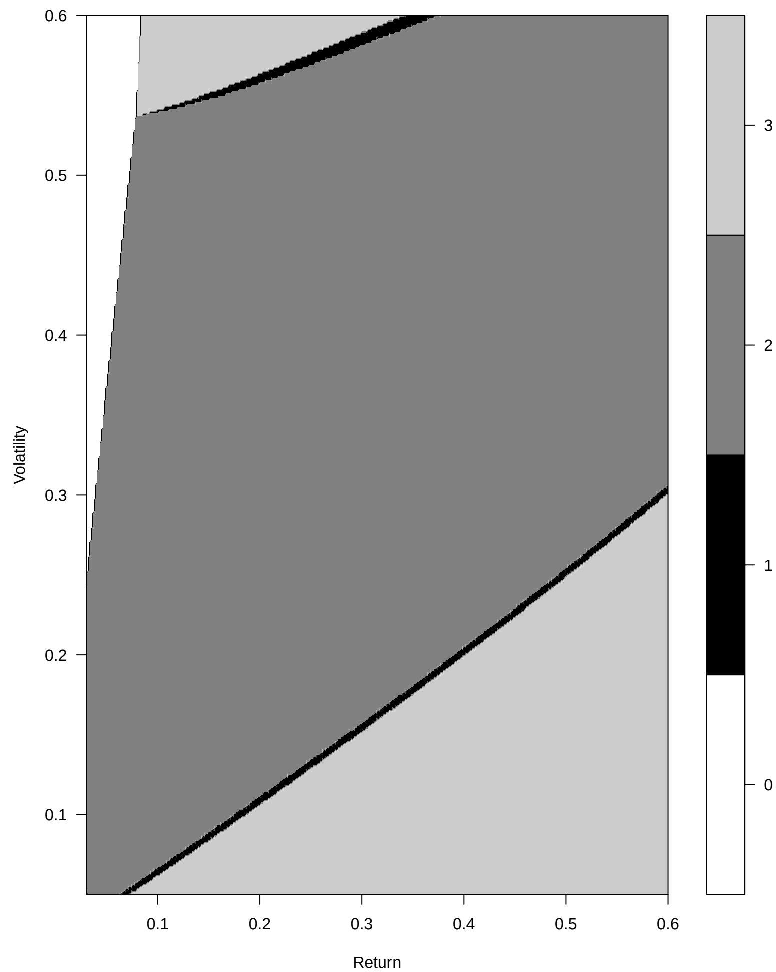


Figure 4.7: Sharpe Ratio Contour Plots: The two axes are μ and σ, the hedge fund expected return and volatility. We set the performance fees $\alpha_1 = 20\%$ and $\alpha_2 = 40\%$. In the white regions, both the first-loss and traditional fee structures have negative Sharpe ratios. The dark grey region indicates that the investor favours the first-loss fee structure. The light grey regions indicate that the investor favours the traditional fee structure. The black region indicates that there exists a non-trivial asset allocation. The remaining parameter values are chosen as: $x = 1, T = 1, m = 0.01, c = 0.1$, and $r = 0.02$.

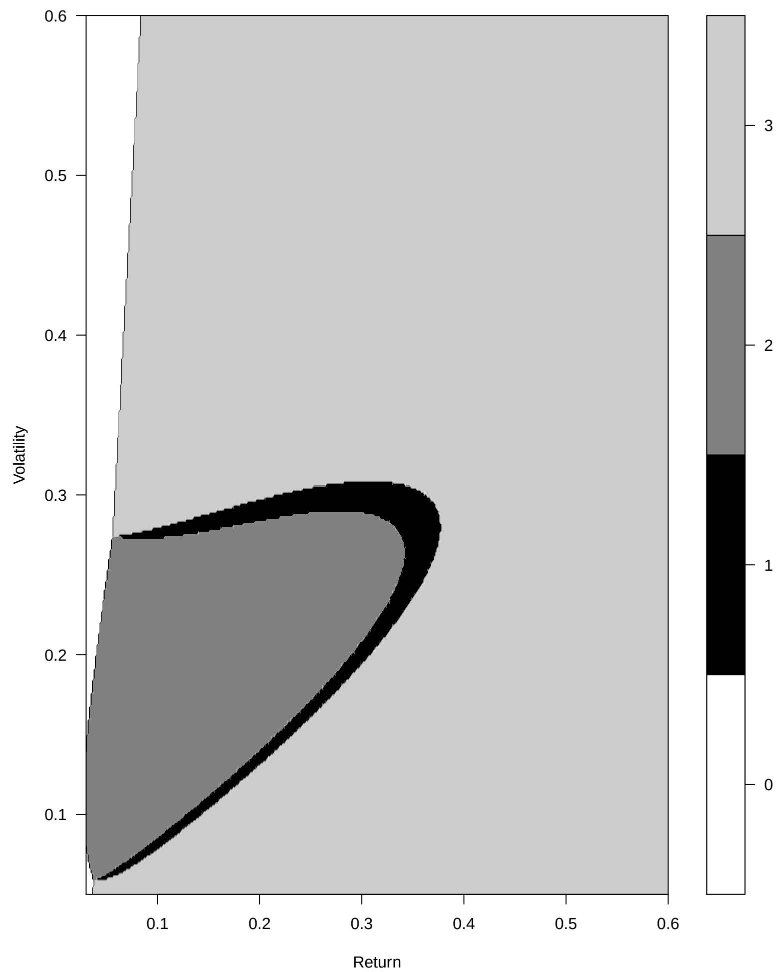


Figure 4.8: Sharpe Ratio Contour Plots: The two axes are μ and σ, the hedge fund expected return and volatility. We set the performance fees $\alpha_1 = 20\%$ and $\alpha_2 = 50\%$. In the white regions, both the first-loss and traditional fee structures have negative Sharpe ratios. The dark grey region indicates that the investor favours the first-loss fee structure. The light grey regions indicate that the investor favours the traditional fee structure. The black region indicates that there exists a non-trivial asset allocation. The remaining parameter values are chosen as: $x = 1, T = 1, m = 0.01, c = 0.1$, and $r = 0.02$.

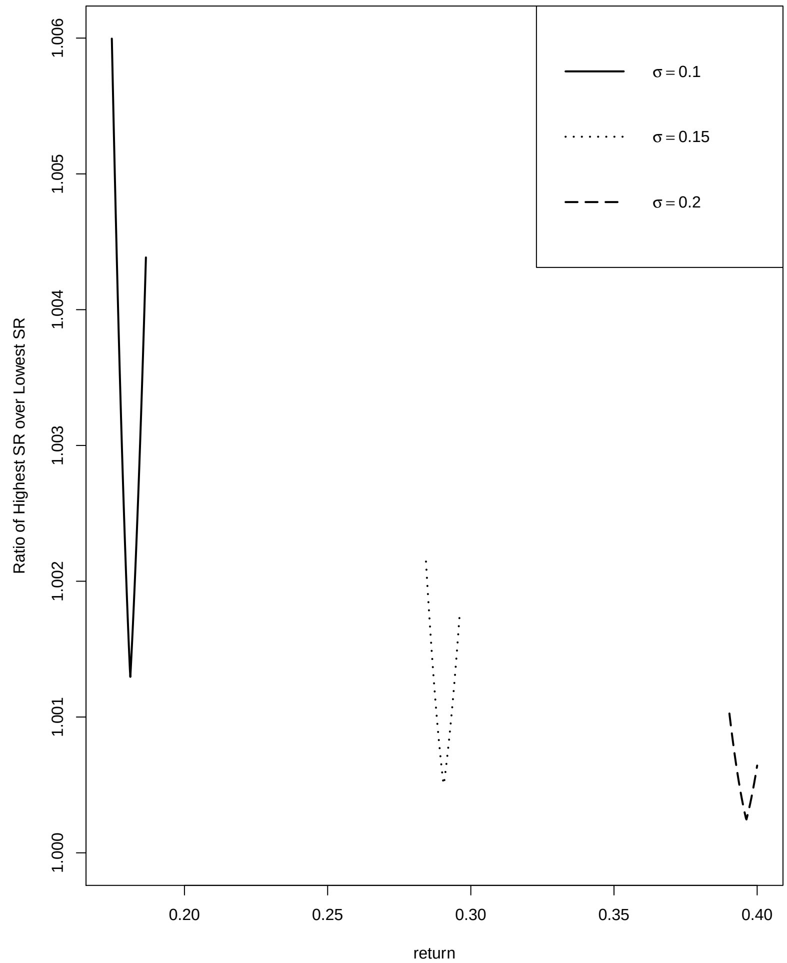


Figure 4.9: Ratio S_{max}/S_{min} vs Return (μ): We set the performance fees $\alpha_1 = 20\%$ and $\alpha_2 = 40\%$. The solid, dotted, and dashed lines represent the ratios when $\sigma = 0.1, 0.15$, and 0.2 repectively. The remaining parameter values are chosen as: $x = 1, T = 1, m = 0.01, c = 0.1$, and $r = 0.02$.

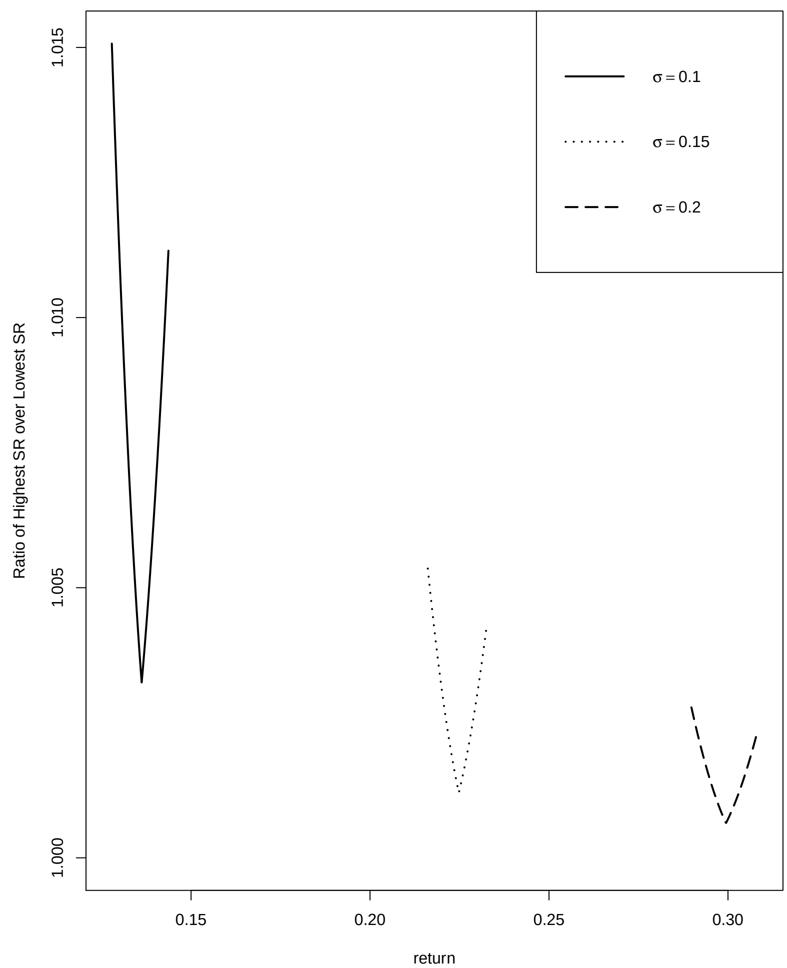


Figure 4.10: Ratio S_{max}/S_{min} vs Return (μ): We set the performance fees $\alpha_1 = 20\%$ and $\alpha_2 = 50\%$. The solid, dotted, and dashed lines represent the ratios when $\sigma = 0.1, 0.15$, and 0.2 repectively. The remaining parameter values are chosen as: $x = 1, T = 1, m = 0.01, c = 0.1$, and $r = 0.02$.

Next, we consider the results of maximizing the Sortino ratio. Figures 4.11 and 4.12 present detailed results on the nature of the investor's optimal fee structure for different values of μ and σ, with $\alpha_2 = 0.4$ and $\alpha_2 = 0.5$ respectively. The results here again admit an intuitive interpretation. Fixing a level of return $\mu > 0$, consider beginning at $\sigma = 0$, and then increasing the volatility. For zero or very low volatility, the positive mean return will dominate, and the insurance component of the first-loss structure will almost never be triggered. As a result, the classical fee structure will be preferred as the one in which investors keep a larger percentage of the gains, since $\alpha_1 < \alpha_2$. As volatility increases, so do both the probability that the insurance portion of the first-loss structure will come into effect, and the expected size of the protected losses given that losses occur. Hence, the first-loss fee structure becomes more appealing. There is a brief transition period of limited "fee diversification" before investing all the funds in the first-loss fee structure becomes optimal. As the volatility increases still further, the expected benefit of the insurance component is limited (the downside protection is capped), and the potential for very large upside gains (due to extremely high volatility) becomes increasingly important. Since investors in the classical fee structure keep a higher percentage of these extreme gains, for very high levels of the volatility, the classical fee structure is preferred. Again, there is a small black transition region, in which limited "fee diversification" exists.

To assess the potential for diversification benefits when performance is measured using the Sortino ratio rather than the Sharpe ratio, we plot the ratios of the maximal Sortino ratio (SOR_{max}) to the minimal Sortino ratio (SOR_{min}) in Figures 4.13 and 4.14 for cases in which there exists an interior optimal point. Similar to Figures 4.9 and 4.10, the decreasing and increasing parts of the graphs are where the traditional fee structure and first-loss fee structure attain SOR_{min} respectively. The benefits from diversification for the Sortino ratio, while still confined to a subset of the possible parameter values, are more significant than in the case of the Sharpe ratio, and this effect is larger at higher levels of volatility. Again, considering a fixed μ, and increasing σ, we see the case when volatilities are high and diversification is possible corresponds to the black region transitioning between exclusive investment in the first-loss fee structure (the dark grey region in Figures 4.11 and 4.12) and the high volatility region when exclusive investment in the classical fee structure is optimal (the upper light grey region). The scope for diversification here is more pronounced because as we move into the classical structure, we are only being penalized in the Sortino ratio for its downside performance (rather than its variance, which is affected by the potential for extremely good returns, as well as extremely bad ones).

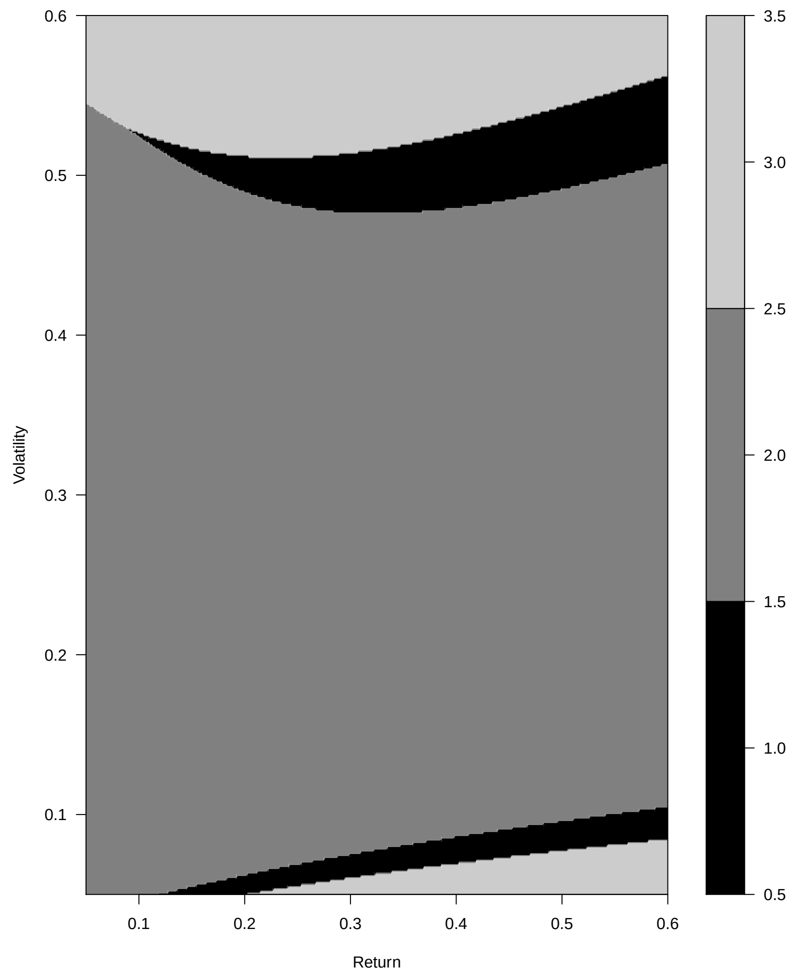


Figure 4.11: Sortino Ratio Contour Plots: The two axes are μ and σ, the hedge fund expected return and volatility. We set the performance fees $\alpha_1 = 20\%$ and $\alpha_2 = 40\%$. The dark grey region indicates that the investor favours the first-loss fee structure. The light grey regions indicate that the investor favours the traditional fee structure. The black regions indicate that there exists a non-trivial asset allocation. The remaining parameter values are chosen as: $x = 1, T = 1, m = 0.01, c = 0.1$, $r = 0.02$, and $l = r$.

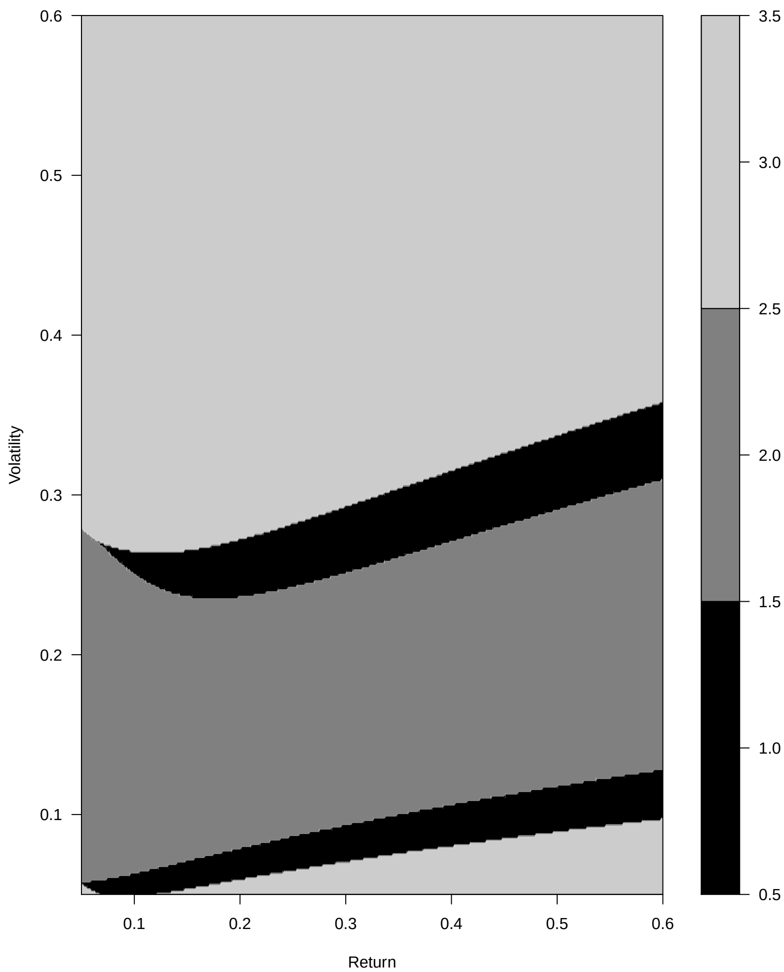


Figure 4.12: Sortino Ratio Contour Plots: The two axes are μ and σ, the hedge fund expected return and volatility. We set the performance fees $\alpha_1 = 20\%$ and $\alpha_2 = 50\%$. The dark grey region indicates that the investor favours the first-loss fee structure. The light grey regions indicate that the investor favours the traditional fee structure. The black regions indicate that there exists a non-trivial asset allocation. The remaining parameter values are chosen as: $x = 1, T = 1, m = 0.01, c = 0.1$, $r = 0.02$, and $l = r$.

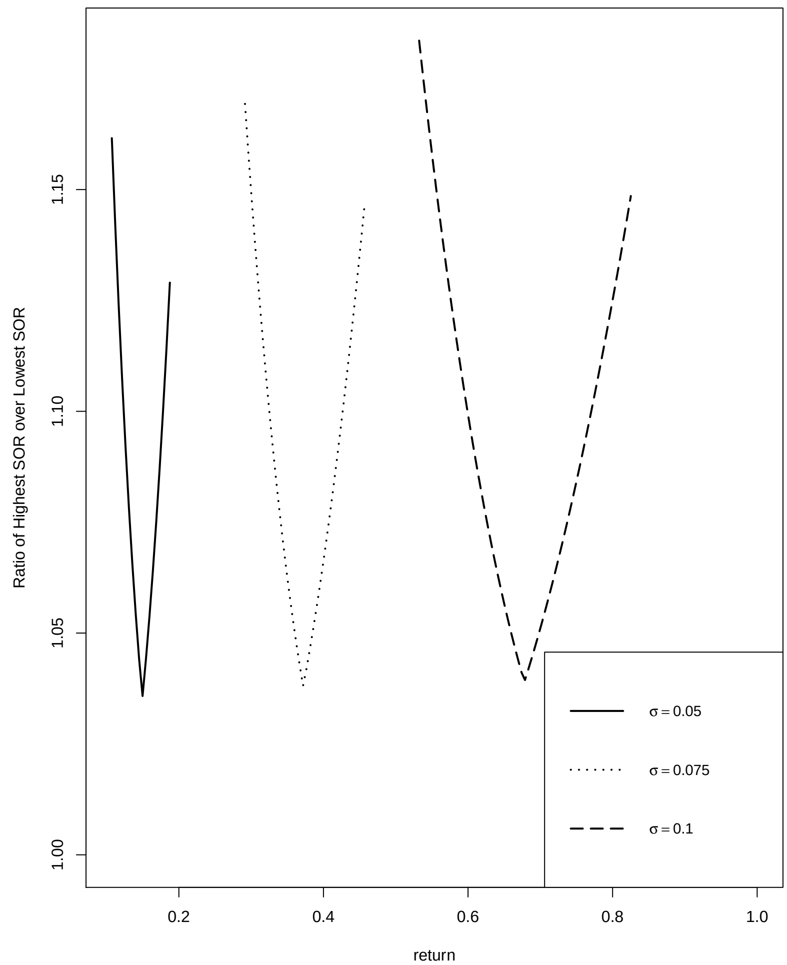


Figure 4.13: Sortino Ratio vs Return(μ). The solid, dotted, and dashed lines are the cases $\sigma = 0.05, 0.075$, and 0.1 respectively. The remaining parameter values are chosen as: $x = 1, T = 1, m = 0.01, c = 0.1$, $r = 0.02$, and $l = r$.

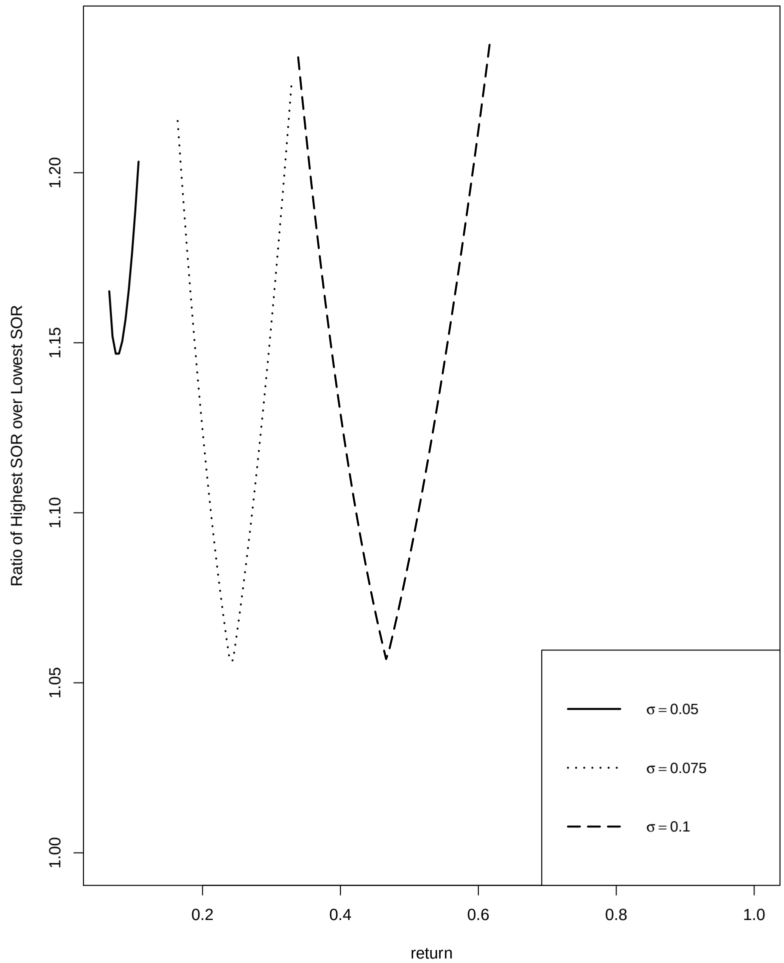


Figure 4.14: Sortino Ratio vs Return(μ). The solid, dotted, and dashed lines are the cases $\sigma = 0.05, 0.075$, and 0.1 respectively. The remaining parameter values are chosen as: $x = 1, T = 1, m = 0.01, c = 0.1$,$r = 0.02$, and $l = r$.

Figures 4.15 and 4.16 present detailed results on the nature of the investor's optimal fee structure for different values of μ and σ, with $\alpha_1 = 0.2$, and α_2 set so that a risk-neutral investor is indifferent between the first-loss and classical fee structures.[8] The traditional

[8]Details are provided in Appendix C.

fee structure is preferred for various choices of (μ, σ). The reason is that the performance fee α_2 needs to be extremely high ($\approx 75\%$) in order for the first-loss fee structure to have the same risk-neutral value as the traditional fee structure (a similar observation can be found in Djerroud et al. (2016)). Hence, the downside protection effect of the first-loss fee structure is offset by the high performance fee. Furthermore, the interest rate does not seem to have a significant impact on the results, as Figures 4.15 and 4.16 illustrate with risk-free rates $r = 0.02$ and $r = 0.05$ respectively.

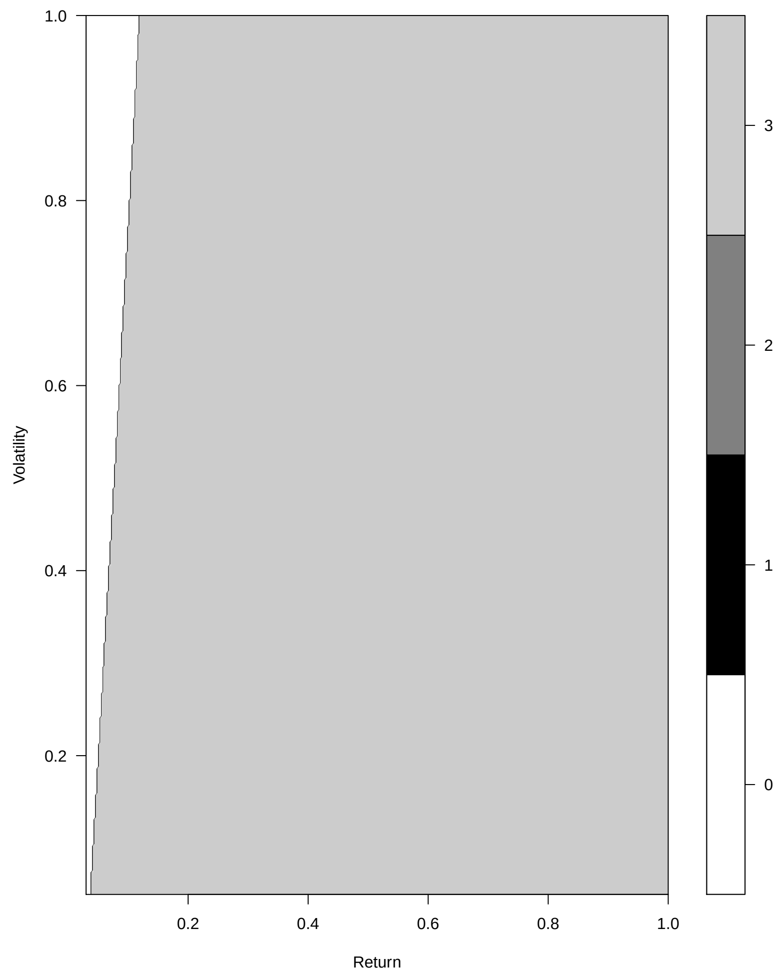


Figure 4.15: Sharpe Ratio Contour Plots: The two axes are μ and σ, the hedge fund expected return and volatility. We set the performance fee $\alpha_1 = 20\%$, α_2 so that the risk-neutral values of the two fee structures are equal. In the white regions, both the first-loss and traditional fee structures have negative Sharpe ratios. The light grey regions indicate that the investor favours the traditional fee structure. The remaining parameter values are chosen as: $x = 1, T = 1, m = 0.01, c = 0.1$, and $r = 0.02$.

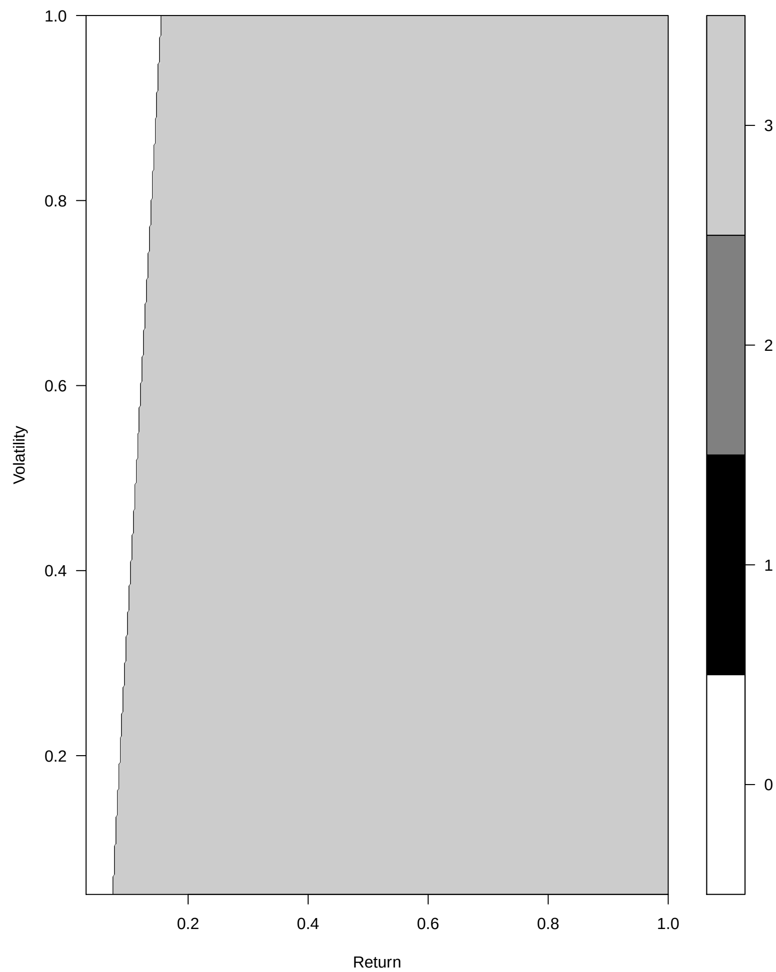


Figure 4.16: Sharpe Ratio Contour Plots: The two axes are μ and σ, the hedge fund expected return and volatility. We set the performance fee $\alpha_1 = 20\%$, and α_2 so that the risk-neutral values of the two fee structures are equal. In the white regions, both the first-loss and traditional fee structures have negative Sharpe ratios. The light grey regions indicate that the investor favours the traditional fee structure. The remaining parameter values are chosen as: $x = 1, T = 1, m = 0.01, c = 0.1$, and $r = 0.05$.

4.5 Conclusion

In this chapter, we study and compare the Sharpe Ratios and Sortino Ratios of hedge fund investors facing combinations of the first-loss fee structure and traditional fee structure. In particular, we maximize the Sharpe Ratio or the Sortino Ratio of a portfolio that combines payoffs of the two fee structures at maturity T. A criterion (4.5) is presented to distinguish whether or not there exists an interior optimal weight allocation between the two fee structures for the Sharpe Ratio. Numerical examples are presented for both the Sharpe and Sortino Ratios, assuming a geometric Brownian motion process for the hedge fund assets. In most cases we find that the optimal weights are on the boundary points, indicating that one extreme fee structure is preferred compared to the other, and all possible mixtures. Typically, the classical fee structure is preferred at very low volatilities, where the insurance in the first-loss contract is unlikely to be triggered, and at extremely high volatilities, for which the share of a very large potential upside is important. At intermediate volatilities (often covering most of the range typically seen in practice), the first-loss structure is preferred, owing to the importance of its insurance component. We find that there is negligible benefit to the investor due to "fee diversification," even in the case when a combination of fee structures is optimal.

Chapter 5

Valuation and Risk Analysis for Returns in Hedge Funds with Negative Fee Structures

5.1 Introduction

According to FitchRatings, the total of sovereign debt with negative yields increased to \$11.7 trillion as of June 27, 2016, up \$1.3 trillion from the total at the end of May.[1] Major institutional investors have approximately 30% to 50% of their assets allocated to fixed income, which makes them increasingly vulnerable to the interest rate environment (OECD (2016)). The low-rate environment has also impacted the manner in which hedge fund managers are compensated since the investment opportunity set for hedge funds has shrunk in general. Moreover, investors accept paying the traditional fees to hedge fund managers only if the underlying trading strategy generates superior returns (or alpha). However, the lukewarm performance of hedge funds in recent years has pressured the fees as investors need to maintain an acceptable share of gross returns to meet their investment thresholds. Hedge funds often pursue trading strategies that involve significant short positions in securities. These short positions are typically implemented through reverse repo transactions, in which the hedge fund purchases a security from a counterparty today (and resells it in the market), and commits to sell the security back to the counterparty (after buying it back in the market, hopefully at a lower price) at a specified future date and fixed (higher) future price. The difference in the price paid to the counterparty today for

[1] https://www.fitchratings.com/site/pr/1008156

the security, and the price received for it in the future is referred to as the "short rebate," and represents the interest component of the transaction (from the counterparty's point of view, the transaction is a repo agreement, which is essentially a loan collateralized by the securities). In addition to depressing returns on fixed-income assets, the low-rate environment has significantly trimmed the short rebates that managers used to receive on their short book resulting in lower performance of trading strategies in general. This has further undermined the acceptability of traditional 2&20 fee structures (see Lorin (2017)) and has encouraged investors to seek innovative fee methodologies.

Investors' demand for yield, combined with the difficult market environment and the challenges faced by many hedge fund managers in raising assets, has led institutional investors and fund managers to embrace new fee structures featuring an element of downside protection. In these fee structures, commonly referred to as 'first-loss' or 'shared-loss' structures, the fund manager insures a portion of the investor's losses.

In this chapter, we extend the concept of first-loss fee structures by considering a guarantee not just against losses but providing a minimum return guarantee from the manager to the investor. In this regard, the investment starts to look to the investor like a bond with a coupon payment that contains two parts: a fixed component, coming from the return guarantee offered by the hedge fund, and a variable one, arising from the performance of the hedge fund investment net of performance fees. Figure 5.1 illustrates a spectrum of fee structures from the traditional to the first-loss family of structures and beyond. In the traditional '2&20' structure, the investor's return varies with the performance of the hedge fund strategy; the investor can experience periods of losses as seen in the leftmost bar in the figure. A simple first-loss structure involves a higher share of the strategy performance allocated to the manager in return for downside protection for the investor. The investor will be less likely to experience periods of losses under this structure; however, the investor return could be zero. A first variant of the first-loss fee structure is the one in which the investor requires a minimum return coupled with a smaller share of the strategy performance in exchange for a higher performance fee paid to the manager. The rightmost bar illustrates a first-loss structure in which the investor 'swaps' the performance of the strategy on its capital for a promised fixed 'coupon'. These last two fee structures both provide bond-like payment, which are achieved by a management fee waiver (Rosenbaum (2019)). When the waived amount exceeds the management fee charged, a negative fee, in other words, a cash rebate is provided to the investor. In fact, the SEC (U.S. Securities and Exchange Commission) approved such a fee structure for an ETF (exchange-traded fund) in May 2019 (Walker (2019)) for the first time in history. Hence, we refer the two last structures as 'negative fee structures.' From left to right in Figure 5.1, the upside to the investor is gradually reduced in exchange for downside protection, provided by the fund manager. In addition, the investor is more certain to receive a higher minimum return or a larger 'coupon.' It should be noted that the performance fees on the horizontal axis are for illustration purposes only, and the size of the fixed coupon is dependent on the underlying strategy.

Figure 5.1: Schematic representation of hedge fund fee structures from traditional to first-loss fee structures.

The remainder of the chapter is structured as follows. The second section discusses hedge fund fee structures. The third section analyzes negative fee structures from an option pricing perspective under a regime-switching model using risk-neutral valuation. The fourth section analyzes the risks of the investor's returns under a negative fee structure, now using the real-world measure. The fifth section concludes.

5.2 Hedge Fund Fee Structures: From Traditional Fee Structures to Negative Fees

5.2.1 Traditional Fee Structures

Traditionally, a hedge fund manager charges two types of fees to the fund's investors:

- A fixed management fee, usually ranging from 1% to 2% of the fund's net asset value.
- A performance fee, most commonly equal to 20% of net profits obtained by the fund.

In this chapter we assume a single investor and a single share issued by the fund. The extension to the case of multiple investors and multiple shares is straightforward. Although fees are paid according to a determined schedule (usually monthly or quarterly for management fees and annually for performance fees) we will assume a single payment at the end of a fixed term T.

The initial fund supplied by the investor is denoted by X_0. The hedge fund manager then invests the fund assets to create the future gross values X_t, for $t > 0$. The gross fund value X_t is split between the investor's share I_t (the net asset value) and the manager's fee M_t:

$$X_t = I_t + M_t.$$

At time 0, $I_0 = X_0$ and $M_0 = 0$.

There are countless variations on the basic framework, including hurdles, clawbacks, etc. (for more details on first-loss arrangements, see the previous chapters, or Banzaca (2012)). We will ignore these and assume the commonly used version of a management fee equal to $m \cdot X_0$ (m represents a fixed percentage of the initial investment), and a performance fee of $\alpha \cdot (X_T - (1+m)X_0))_+$, so that the performance fee is payable only when the investor's return is positive, and is zero when it is negative. Hence, the manager's payoff due to fees is:

$$M_T = m \cdot X_0 + \alpha \cdot (X_T - (1+m)X_0)_+ .$$

In other words, while the management fee is a fixed future liability to the investor, the performance fee is a contingent claim on the part of the manager. As a consequence, we will be pricing the management fee simply as a fixed guaranteed fee with a predetermined future cash value, and we will be valuing the performance fee as the value of a certain call option. In our setting, we will assume a regime-switching process for the invested assets X_t, which allows us to value the performance fee using known results. It is worth mentioning that hedge fund managers can speculate on volatility, credit risks, etc. and in contrast to traditional money managers, they can go long and short. The diversity in investment styles and the different levels of gross and net exposure that they can employ often result in leptokurtic returns, for example through the potential for large negative returns in the left tail of the return distribution. A regime-switching process including a 'stress regime' with high volatility can reproduce these properties of hedge fund returns. Generalization of the current framework to other models of hedge fund returns, for example using stochastic volatility by employing generalized autoregressive conditional heteroskedasticity (GARCH) models, could be a subject for future research.

From a business perspective, it is important to note that the investor has a say in the fees paid to the fund manager: sometimes, as in the case of managed account investments, through a direct negotiation of the fees, at other times, such as in a normal fund structure, through the right not to invest in the fund in the first place. However, when it comes to the choice of the portfolio, the manager has full discretion, within the limits existing in the offering memorandum, without seeking investors' permission or input. This consideration will play a role if one tries to extrapolate the results of this chapter to real investment situations.

5.2.2 From First-Loss to Negative Fee Structures

While the first-loss fee structure protects investors from downside moves in the market, if the manager does not generate returns the investor does not make any profits. A negative fee structure results from modifying the first-loss structure to provide a fixed level of promised return to investors, while maintaining some level of downside protection. The cost of this bond-like return for investors is the increase of the performance fee paid to the manager; we refer to this framework as the 'high-yield bond like' framework. In the limit, the investor has a guaranteed return and pays 100% of the performance beyond the guarantee to the manager. As such, the return profile provided to the investor resembles that of an investor in an asset-backed security, with the underlying portfolio being the assets of the hedge fund; we refer to this framework as the 'swap' framework. In the swap framework, at the end of each period, all returns generated by the strategy are allocated to the investor up to the 'return hurdle' which is negotiated with the hedge fund manager. The remaining returns above the return hurdle are fully allocated to the manager as a performance fee. If the fund return is less than the return hurdle, the manager's deposit is used to make up the difference. In subsequent periods, profits are first used to replenish the manager's deposit, before either the investor's return or the performance fee is paid.

A close look at the negative fee structure reveals that the positions of the investor and the hedge fund manager can be formulated as portfolios of options. The first-loss fee structure was analyzed from an option-pricing perspective using the Black–Scholes model in Djerroud et al. (2016). In the next section, we extend that analysis to the negative fee structure under a regime-switching model. Given the bond-like payoff of the negative fee structure, this setting is very similar to the classical Merton model for credit risk (see Merton (1974)), with the difference coming from the additional downside protection provided to investors by the hedge fund manager.

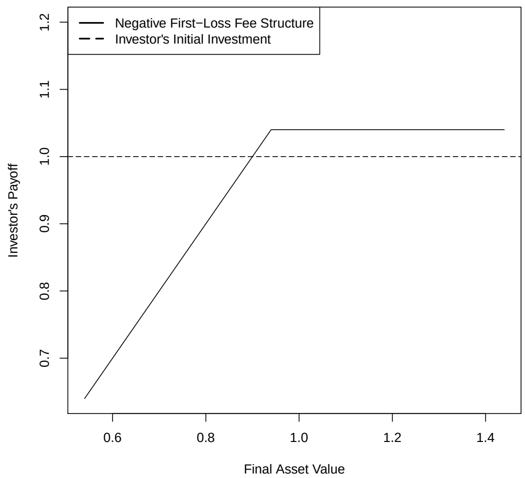
Denoting the return threshold by H, the payoff functions of the investor and the manager at the terminal time T are respectively:

$$I_T = \begin{cases} X_0(1+H) & \text{when } (X_T - HX_0) \geq (1-c)X_0, \\ X_T + cX_0 & \text{when } (X_T - HX_0) < (1-c)X_0, \end{cases} \tag{5.1}$$

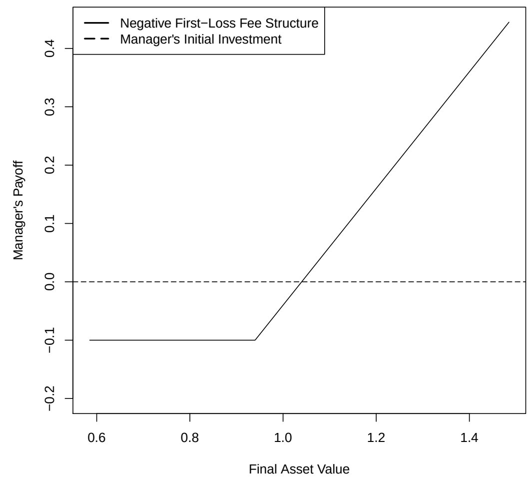
$$M_T = \begin{cases} X_T - X_0(1+H) & \text{when } (X_T - H\,X_0) \geq (1-c)X_0, \\ -cX_0 & \text{when } (X_T - H\,X_0) < (1-c)X_0. \end{cases}$$

Writing these payoff functions more compactly, we obtain:

$$\begin{aligned} I_T &= X_0(1+H) - ((1-c)X_0 - X_T + H\,X_0)_+, \\ M_T &= X_T - X_0(1+H) + ((1-c)X_0 - X_T + H\,X_0)_+. \end{aligned} \tag{5.2}$$



(a)



(b)

Figure 5.2: Investor's and manager's payoff: Figure (a) and Figure (b) are the investor's and manager's payoff respectively with parameters: $X_0 = 1, c = 0.1$ and $H = 0.04$.

From the above formulas, we see that the investor (manager) has a short (long) position in a put option on the fund assets, with strike price $(1-c)X_0+H\,X_0$. Risk-neutral valuation can be applied to derive the price of the positions.[2]

In particular, the value of the investor's position is:

$$V_I(0) = \exp(-rT)X_0(1+H) - P(X_0, T, (1-c)X_0 + H\,X_0, r),$$

where $P(X, T, K, r)$ is the price of a put option on a non-dividend paying asset with current value of the underlying X, time to expiration T, strike price K, and where the risk-free interest rate is r. The above framework can be easily extended to the case in which the investor receives a portion of the excess return above the return threshold H.

[2]It should be noted that, similarly to Merton (1974), some of the assumptions used to justify arbitrage-free pricing methods do not hold in practice in the context in which we are applying the model here. In particular, it is typically not possible for the investor to trade in (or even directly observe) the hedge fund assets X_t.

5.3 Pricing the Payoffs

We assume a regime-switching model, in which the coefficients of a diffusion process for the value of the hedge fund assets themselves follow continuous-time Markov chains. Regime-switching models have found many applications in finance since the seminal work of Hamilton (1989). They are able to reproduce many features of real-world return distributions, including skewness, volatility clustering, and fat tails. For applications of regime-switching models to insurance products with investment guarantees, similar in sprit to the hedge-fund guarantees considered in this paper, see Hardy (2003). For many other financial applications, see the papers in the volumes Mamon and Elliott (2007), and Zeng and Wu (2013).

We assume the regime is governed by a finite state continuous-time Markov chain $\epsilon(t)$ with state space $\mathbb{S} = \{1, 2\}$, where state 1 represents the 'normal' regime and state 2 represents the 'stress' regime. The generator of $\epsilon(t)$ is the matrix:

$$Q = \begin{bmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{bmatrix},$$

where λ_1 and λ_2 are the transition rates of leaving states 1 and 2 respectively. The value of the hedge fund assets X_t follows a geometric Brownian motion, except that the coefficients of X_t change with the regime:

$$dX_t = \mu_{\epsilon(t)} X_t dt + \sigma_{\epsilon(t)} X_t dZ_t$$

where Z_t is a standard Brownian motion, independent of $\epsilon(t)$, and $\mu_{\epsilon(t)}$ and $\sigma_{\epsilon(t)}$ are constants in each state. For simplicity, when $\epsilon(t) = 1, 2$, we use μ_1, μ_2 and σ_1, σ_2 to denote the growth rates and volatilities in each regime. Finally, the risk-free asset B satisfies $B_t = e^{rt}$. The value of the investor's position can be determined using an expectation under a risk-neutral measure (see Elliott et al. (2005)) to be:

$$V_I^i = \mathbb{E}_{\mathbb{Q}}[I(T)|\epsilon(0) = i]. \tag{5.3}$$

Then, we have

$$V_I^i = \exp(-rT)X_0(1 + H) - P_i(X_0, T, (1 - c)X_0 + H\, X_0, r), \tag{5.4}$$

where $P_i(X, T, K, r) = \mathbb{E}_{\mathbb{Q}}[e^{-rT}(K - S_T)_+|\epsilon(0) = i], i = 1, 2$ is the European put option price under the Markov-modulated geometric Brownian motion model. Moreover, from

Guo (2001) and Fuh et al. (2012), we obtain:

$$\begin{aligned}P_i(X,T,K,r) &= \mathbb{E}_{\mathbb{Q}}[e^{-rT}(K-S_T)_+|\epsilon(0)=i] \\ &= e^{-rT}\int_0^{K-1}\int_0^T \frac{y}{K-y}\phi(\ln(K-y),m(t),v(t)f_i(t,T)dtdy \end{aligned} \tag{5.5}$$

where:

$$\begin{aligned}
&m(t) = \ln(X) + (rT - \frac{1}{2}v(t)), \\
&v(t) = (\sigma_1^2 - \sigma_2^2)t + \sigma_1^2 T, \\
&f_1(t,T) = e^{-\lambda_1 T}\delta_0(T-t) + e^{-\lambda_2(T-t)-\lambda_1 t}[\lambda_1 I_0(2(\lambda_1\lambda_2 t(T-t))^{1/2}) \\
&\qquad + (\frac{\lambda_1\lambda_2 t}{T-t})^{1/2} I_1(2(\lambda_1\lambda_2 t(T-t))^{1/2})], \\
&f_2(t,T) = e^{-\lambda_2 T}\delta_0(t) + e^{-\lambda_2(T-t)-\lambda_1 t}[\lambda_2 I_0(2(\lambda_1\lambda_2 t(T-t))^{1/2}) \\
&\qquad + (\frac{\lambda_1\lambda_2 (T-t)}{t})^{1/2} I_1(2(\lambda_1\lambda_2 t(T-t))^{1/2})],
\end{aligned}$$

where $\phi(x,m(t),v(t))$ is the normal density function with mean $m(t)$ and variance $v(t)$, I_0 and I_1 are the modified Bessel functions,

$$I_a(z) = \left(\frac{z}{2}\right)^a \sum_{k=0}^{\infty} \frac{(a/2)^{2k}}{k!\Gamma(k+a+1)}, \tag{5.6}$$

and δ_0 is a delta function with a mass at 0.

Figures 5.3, 5.4, and 5.5 illustrate the sensitivity of the value of the investor's payoff to the model parameters. Figure 5.3 is generated assuming that the market is initially in the normal state ($\varepsilon(0)=1$). Figure 5.4 repeats the analysis assuming that the market is initially in the stressed state ($\varepsilon(0)=2$). Finally, Figure 5.5 assumes that $\varepsilon(0)$ is random, generated according to the stationary distribution of the Markov chain $\varepsilon(t)$, i.e., $\varepsilon(0)=1$ with probability $\pi_1=\lambda_2/(\lambda_1+\lambda_2)$, and $\varepsilon(0)=2$ with probability $\pi_2=1-\pi_1$. The parameters are set to $T=\frac{1}{12}$ (the investment horizon is one month), $c=10\%$[3] (the manager deposit), $r=1\%$ (annual risk-free interest rate), $X_0=\$1$ (the initial investment), and $H=4\%$ (annual return threshold). The volatility and transition rate in a normal market are $\sigma_1=10\%$ and $\lambda_1=1$, while the corresponding parameters in a stressed market are $\sigma_2=20\%$ and $\lambda_2=12$.

[3] We adapt the value $c=10\%$ from first-loss fee structure in Chapter 4.

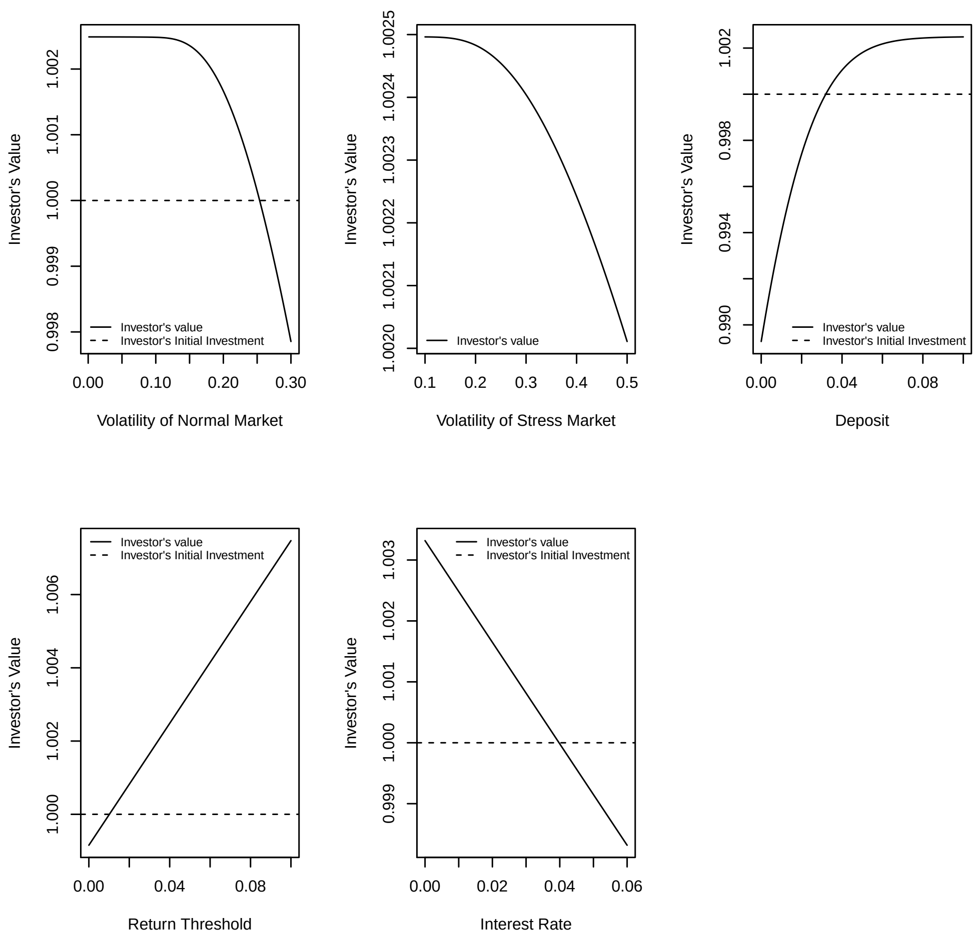


Figure 5.3: Sensitivity of the value of the investor's payoff to various parameters, given that the market is initially in the normal state ($\varepsilon(0) = 1$). Benchmark parameter values are $T = \frac{1}{12}$ (the investment horizon is one month), $c = 10\%$ (the manager deposit), $r = 1\%$ (annual risk-free interest rate), $\sigma_1 = 10\%$ (the annual volatility in a normal market), $\sigma_2 = 20\%$ (the annual volatility in a stressed market), $\lambda_1 = 1$ (the transition rate in a normal market), $\lambda_2 = 12$ (the transition rate in a stressed market), $X_0 = 1$ (the initial investment), and $H = 4\%$ (the annual return threshold).

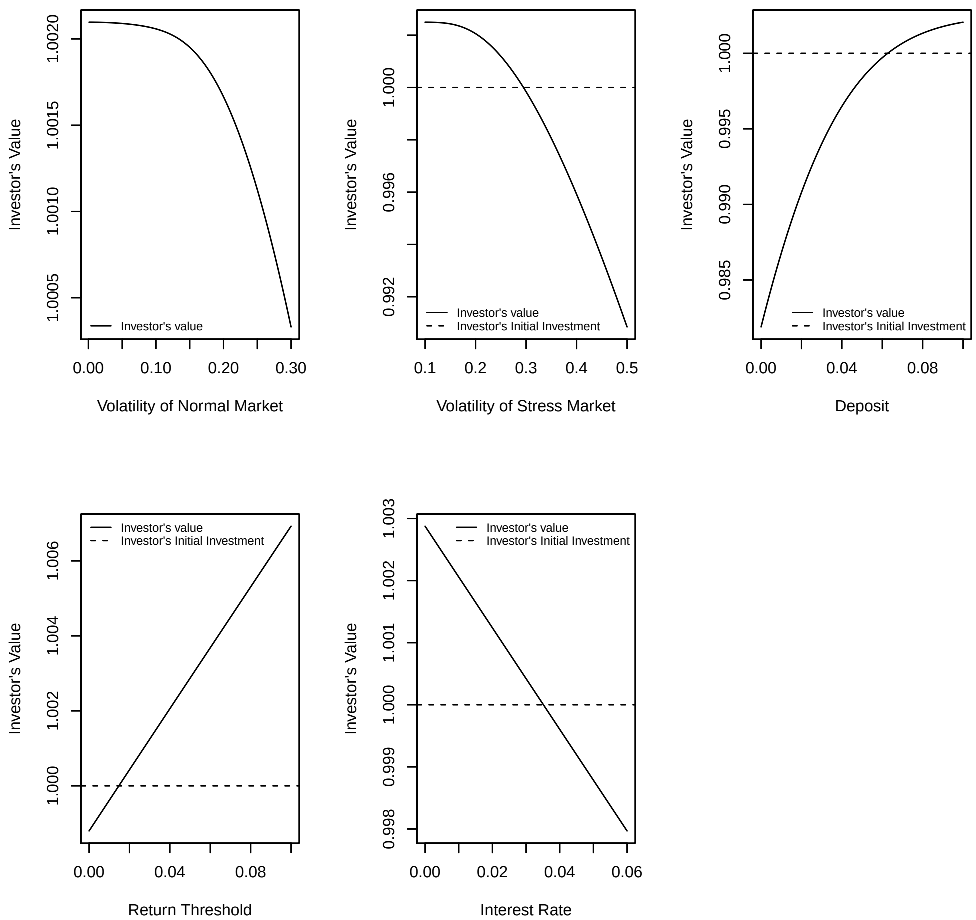


Figure 5.4: Sensitivity of the value of the investor's payoff to various parameters, given that the market is initially in the stressed state ($\varepsilon(0) = 2$). Benchmark parameter values are $T = \frac{1}{12}$ (the investment horizon is one month), $c = 10\%$ (the manager deposit), $r = 1\%$ (annual risk-free interest rate), $\sigma_1 = 10\%$ (the annual volatility in a normal market), $\sigma_2 = 20\%$ (the annual volatility in a stressed market), $\lambda_1 = 1$ (the transition rate in a normal market), $\lambda_2 = 12$ (the transition rate in a stressed market), $X_0 = 1$ (the initial investment), and $H = 4\%$ (the annual return threshold).

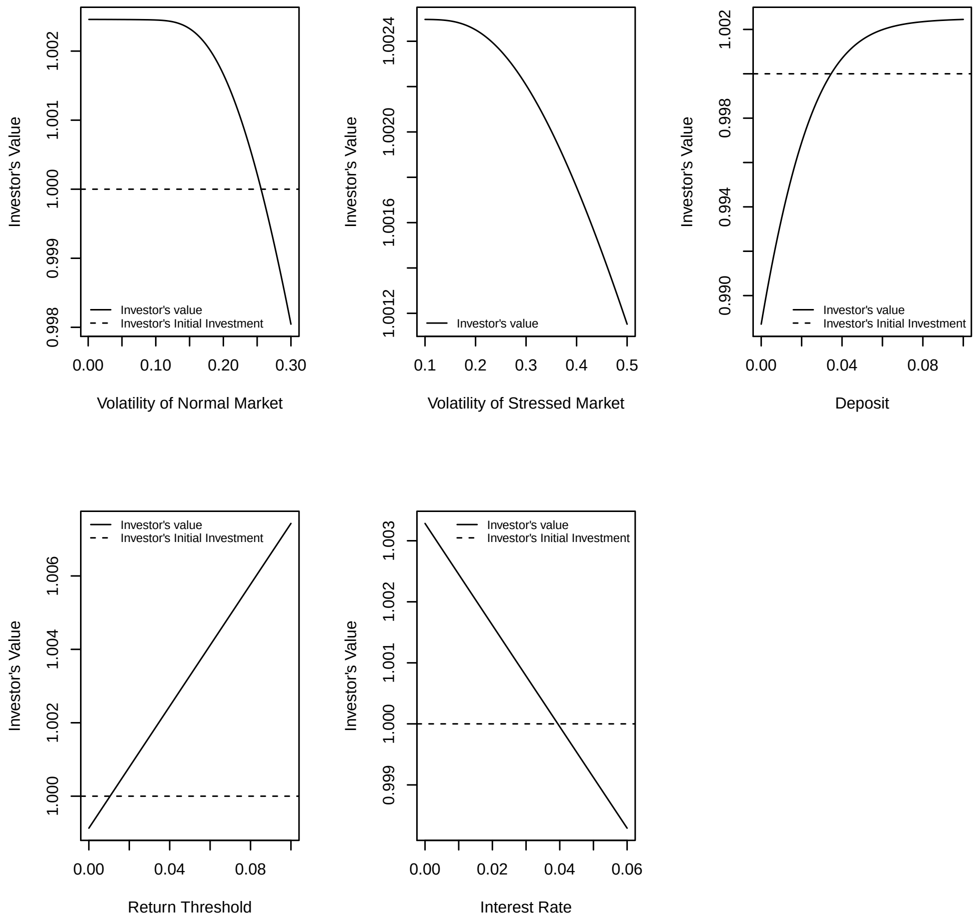


Figure 5.5: Sensitivity of the value of the investor's payoff to various parameters, given that $\varepsilon(0)$ is chosen randomly from its stationary distribution. Benchmark parameter values are $T = \frac{1}{12}$ (the investment horizon is one month), $c = 10\%$ (the manager deposit), $r = 1\%$ (annual risk-free interest rate), $\sigma_1 = 10\%$ (the annual volatility in a normal market), $\sigma_2 = 20\%$ (the annual volatility in a stressed market), $\lambda_1 = 1$ (the transition rate in a normal market), $\lambda_2 = 12$ (the transition rate in a stressed market), $X_0 = 1$ (the initial investment), and $H = 4\%$ (the annual return threshold).

The same basic patterns emerge when looking at the three sets of figures. The volatility sub-figures show that the value of the investor's position is generally a decreasing function of the volatility parameters of the underlying fund. As the volatility becomes very large, the value of the investor's position starts to decline steeply as the hedge fund's put option (in which the investor has a short position) becomes more valuable. However, the scales on the y-axes are extremely narrow, which indicates very limited sensitivity. The deposit sub-figures (varying c) illustrate that, as expected, the value of the investor's position is an increasing function of the deposit level. The return threshold sub-figures show the intuitive monotonic relationship between the value of the investor's position and the return threshold. The value of the investor's position is also a decreasing function of the risk-free rate r, in accord with the bond-like nature of the investor's payoff.

The value of the investor's payoff is lower in a stressed market than in a normal market. The stressed market starts with a higher volatility ($\sigma_2 > \sigma_1$), thus increasing the value of the put option in which the investor has a short position. It is further important to note that given the short time horizon ($T = 1/12$), there is a significant probability that the market will remain in the high volatility, stressed regime over the entire life of the contract. For longer-lived contracts, the discrepancy between the investor's values given that the market is in either the stressed or normal state will be less pronounced. Finally, we note that the figures for when the initial market state is random with distribution equal to the stationary distribution of $\varepsilon(t)$ are close to those for when the market is started in the normal state. This is due to the fact that with our benchmark parameters the stationary distribution assigns a high probability to the market being in a normal state ($\pi_1 = 12/13 \approx 92.3\%$), as is common in financial applications of regime-switching models (see the references cited above).

For each figure, one can look for the point where the curve crosses the value 1.0 (if it exists). This allows us to identify the parameter values for which the contract (in terms of risk-neutral valuation in the regime-switching model) favours either the investor or the manager. Parameter values for which the curve is above 1.0 show that the contract favours the investor, while for parameter values where the curve is below 1.0 the contract favours the manager. The point at which the curve crosses 1.0 is the break-even, or indifference, point.

5.4 Risk Analysis of the Investor's Position as a Bond

As noted above, the position of the investor is analogous to a bond, with a promised return of H (received if the hedge fund assets perform sufficiently well). In the event of default, there is a random amount of recovery (again determined by the level of the hedge fund assets). In this section, we examine the properties of the investor's payoff from the perspective of this analogy with a fixed income investment. In particular, we compute default probabilities and expected recovery rates.

In this section, we consider the manager's and investor's expected payoffs under the real world measure. In order to obtain numerical results, one can discretize the Markov-modulated geometric Brownian motion process as follows:

$$X_{t+\Delta t} = X_t + \mu_{\tilde{\varepsilon}(t)} X_t \Delta t + \sigma_{\tilde{\varepsilon}_t} X_t \sqrt{\Delta t} \cdot \eta_t$$
$$R_t := \frac{X_{t+\Delta t}}{X_t} - 1 = \mu_{\tilde{\varepsilon}(t)} \Delta t + \sigma_{\tilde{\varepsilon}(t)} \sqrt{\Delta t} \cdot \eta_t$$

where the η_t are i.i.d. standard normal random variables, and $\tilde{\varepsilon}(t)$ is a discretized version of the continuous-time Markov chain $\varepsilon(t)$, with transition matrix:

$$P = \begin{bmatrix} 1 - p_{1,2} & p_{1,2} \\ p_{2,1} & 1 - p_{2,1} \end{bmatrix},$$

where $p_{1,2}$ is the probability of transitioning from state 1 to state 2, and $p_{2,1}$ is the probability of transitioning from state 2 to state 1. The stationary distribution for this Markov chain is $\pi_0 = p_{2,1}(p_{1,2} + p_{2,1})^{-1}$, $\pi_1 = p_{1,2}(p_{1,2} + p_{2,1})^{-1}$.

We simulated 5000 scenarios of the returns of the hedge fund using the above model with an annual time horizon and daily time steps. Recall that the payoffs to the hedge fund manager ($\tilde{M}$) and investor ($\tilde{I}$) for the traditional fee structure are:

$$\tilde{M}(T) = \begin{cases} \alpha(X_T - X_0), & X_T \geq X_0, \\ 0, & X_T < X_0, \end{cases} \qquad \tilde{I}(T) = \begin{cases} X_0 + (1-\alpha)(X_T - X_0), & X_T \geq X_0, \\ X_T, & X_T < X_0, \end{cases}$$

and the payoffs for the negative loss structure are:

$$M(T) = \begin{cases} X_T - X_0(1+H), & X_T \geq (1-c+H)X_0, \\ -cX_0, & X_T < (1-c+H)X_0, \end{cases}$$
$$I(T) = \begin{cases} X_0(1+H), & X_T \geq (1-c+H)X_0, \\ X_T + cX_0, & X_T < (1-c+H)X_0. \end{cases}$$

Table 5.1 presents simulated expected returns and standard deviations (with standard errors of the estimates in parentheses) for both the traditional fee structure and the negative fee structure. The volatility in normal markets is set to $\sigma_1 = 0.1$, while in stressed markets it is $\sigma_2 = 0.2$. We consider two different possible growth rates for each state, $\mu_1 = 0.1, 0.15$ and $\mu_2 = -0.05, 0.0$. The probability $p_{1,2}$ is set to 0.01, while $p_{2,1}$ is set to

0.05, indicating a high level of persistence in both states, and a stationary distribution of $(\frac{5}{6}, \frac{1}{6})^T$. In this section, we assume that the initial state of the Markov chain is 1 (normal market). We see that for investors, expected payoffs are slightly higher for the traditional fee structure, but standard deviations are also significantly higher in this case. This is consistent with the analogy that the negative fee structure more closely resembles a fixed income investment, while the traditional fee structure gives a more 'equity-like' payoff. In contrast, the manager's expected payoff and standard deviation are higher under the negative fee structure, and lower under the traditional fee structure. As is to be anticipated, expected payoffs are larger when the growth parameters are larger; the standard deviations of payoffs do not change significantly when the μ_i's are varied.

Table 5.1: Expected payoffs (and standard errors in parentheses) for the traditional fee structure and negative fee structure. $X_0 = 1, T = 1, \alpha = 20\%, p_{1,2} = 0.01, p_{2,1} = 0.05, \sigma_1 = 0.1, \sigma_2 = 0.2, N_{sim} = 5000, H = 0.04$, and $c = 0.1$.

(μ_0, μ_1)	$\mathbb{E}_{\mathbb{P}}[\tilde{M}(T)]$	$\mathbb{E}_{\mathbb{P}}[\tilde{I}(T)]$	$\mathbb{E}_{\mathbb{P}}[M(T)]$	$\mathbb{E}_{\mathbb{P}}[I(T)]$
$\mu_0 = 0.1,\ \mu_1 = -0.05$	0.0209 (0.0003)	1.0641 (0.0016)	0.0526(0.0017)	1.0323(0.0003)
$\mu_0 = 0.15,\ \mu_1 = -0.05$	0.0296 (0.0004)	1.1077 (0.0017)	0.097 (0.0019)	1.036 (0.0003)
$\mu_0 = 0.1,\ \mu_1 = 0$	0.0218 (0.0003)	1.069 (0.0016)	0.063 (0.0017)	1.034 (0.0003)
$\mu_0 = 0.15,\ \mu_1 = 0$	0.0308 (0.0003)	1.1142 (0.0016)	0.1081 (0.0019)	1.0369(0.0002)
	S.D. of $\tilde{M}(T)$	S.D. of $\tilde{I}(T)$	S.D. of $M(T)$	S.D. of $I(T)$
$\mu_0 = 0.1,\ \mu_1 = -0.05$	0.0215 (0.0003)	0.1121 (0.0013)	0.1202 (0.0013)	0.0269 (0.0009)
$\mu_0 = 0.15,\ \mu_1 = -0.05$	0.0251 (0.0003)	0.1167 (0.0012)	0.1345 (0.0014)	0.0178 (0.0008)
$\mu_0 = 0.1,\ \mu_1 = 0$	0.0218 (0.0002)	0.1118 (0.0011)	0.1216 (0.0012)	0.0244(0.0009)
$\mu_0 = 0.15,\ \mu_1 = 0$	0.0247 (0.0003)	0.1131 (0.0012)	0.1312 (0.0013)	0.0169(0.0009)

Tables 5.2 and 5.3 repeat the analysis with $p_{1,2} = 0.1, p_{2,1} = 0.05$, and $p_{1,2} = 0.01, p_{2,1} = 0.1$ respectively. Comparing to the figures in Table 5.1, Table 5.2 was generated assuming a significantly higher (by a factor of 10) probability of transitioning from the normal state to the stressed state, and Table 5.3 was generated assuming a significantly higher (by a factor of 2) probability of transitioning from the stressed state to the normal state. The stationary distribution for the simulation in Table 5.2 is $(\frac{1}{3}, \frac{2}{3})^T$ (so that the 'stressed' state is more prevalent), while the stationary distribution for the simulation in Table 5.3 is

$(\frac{10}{11}, \frac{1}{11})^T$. The expected returns and standard deviations of the different payoff structures appear to be relatively insensitive to the choices of the parameters $p_{1,2}, p_{2,1}$.

Table 5.2: Expected payoffs (and standard errors in parentheses) for the traditional fee structure and negative fee structure. $X_0 = 1, T = 1, \alpha = 20\%, p_{1,2} = 0.1, p_{2,1} = 0.05, \sigma_1 = 0.1, \sigma_2 = 0.2, N_{sim} = 5000, H = 0.04$, and $c = 0.1$.

(μ_0, μ_1)	$\mathbb{E}_{\mathbb{P}}[\tilde{M}(T)]$	$\mathbb{E}_{\mathbb{P}}[\tilde{I}(T)]$	$\mathbb{E}_{\mathbb{P}}[M(T)]$	$\mathbb{E}_{\mathbb{P}}[I(T)]$
$\mu_0 = 0.1,\ \mu_1 = -0.05$	0.0163(0.0004)	1.0057(0.0022)	0.0164(0.0020)	1.0056(0.0009)
$\mu_0 = 0.15,\ \mu_1 = -0.05$	0.0181(0.0004)	1.0201(0.0022)	0.0277(0.0021)	1.0104 (0.0009)
$\mu_0 = 0.1,\ \mu_1 = 0$	0.0197(0.0004)	1.0294(0.0023)	0.0374(0.0021)	1.0117(0.0008)
$\mu_0 = 0.15,\ \mu_1 = 0$	0.0223(0.0004)	1.0452(0.0023)	0.0520(0.0023)	1.0155(0.0008)
	S.D. of $\tilde{M}(T)$	S.D. of $\tilde{I}(T)$	S.D. of $M(T)$	S.D. of $I(T)$
$\mu_0 = 0.1,\ \mu_1 = -0.05$	0.0248 (0.0004)	0.1576 (0.0017)	0.1426 (0.0021)	0.0633 (0.0011)
$\mu_0 = 0.15,\ \mu_1 = -0.05$	0.0258 (0.0004)	0.1575 (0.0017)	0.1467 (0.0020)	0.0602 (0.0012)
$\mu_0 = 0.1,\ \mu_1 = 0$	0.0266 (0.0004)	0.1597 (0.0016)	0.1507 (0.0020)	0.0581(0.0011)
$\mu_0 = 0.15,\ \mu_1 = 0$	0.0289 (0.0004)	0.1643 (0.0017)	0.1615 (0.0021)	0.0549 (0.0012)

Table 5.3: Expected payoffs (and standard errors in parentheses) for the traditional fee structure and negative fee structure. $X_0 = 1, T = 1, \alpha = 20\%, p_{1,2} = 0.01, p_{2,1} = 0.1, \sigma_1 = 0.1, \sigma_2 = 0.2, N_{sim} = 5000, H = 0.04$, and $c = 0.1$.

(μ_0, μ_1)	$\mathbb{E}_{\mathbb{P}}[\tilde{M}(T)]$	$\mathbb{E}_{\mathbb{P}}[\tilde{I}(T)]$	$\mathbb{E}_{\mathbb{P}}[M(T)]$	$\mathbb{E}_{\mathbb{P}}[I(T)]$
$\mu_0 = 0.1$, $\mu_1 = -0.05$	0.0227(0.0003)	1.0771(0.0015)	0.0645 (0.0017)	1.0354(0.0003)
$\mu_0 = 0.15$, $\mu_1 = -0.05$	0.0308(0.0003)	1.1169(0.0015)	0.1095 (0.0018)	1.0382(0.0002)
$\mu_0 = 0.1$, $\mu_1 = 0$	0.0225(0.0003)	1.0768(0.0015)	0.0637 (0.0017)	1.0356(0.0003)
$\mu_0 = 0.15$, $\mu_1 = 0$	0.0324(0.0003)	1.1235(0.0015)	0.1179 (0.0018)	1.0381(0.0002)
	S.D. of $\tilde{M}(T)$	S.D. of $\tilde{I}(T)$	S.D. of $M(T)$	S.D. of $I(T)$
$\mu_0 = 0.1$, $\mu_1 = -0.05$	0.0214 (0.0002)	0.1049 (0.0011)	0.1178 (0.0012)	0.0187 (0.0008)
$\mu_0 = 0.1$, $\mu_1 = -0.05$	0.0235 (0.0002)	0.1048 (0.0011)	0.1245 (0.0013)	0.0107 (0.0007)
$\mu_0 = 0.1$, $\mu_1 = 0$	0.0212 (0.0002)	0.1042 (0.0011)	0.1172 (0.0012)	0.0192 (0.0009)
$\mu_0 = 0.15$, $\mu_1 = 0$	0.0244 (0.0003)	0.1084 (0.0012)	0.1288 (0.0014)	0.0117 (0.0008)

Given the similarity of the investor's payoff to the payoff of a fixed income investment, it is interesting to examine the probability of default (i.e. the probability that the investor's return will be lower than the promised hurdle rate H), and the recovery rate (i.e. the fraction of the promised amount $X_0(1 + H)$ that is expected to be recovered conditional upon default having occurred). Simulation results under the regime-switching model for these quantities are provided in Tables 5.4 and 5.5 (with standard errors of the estimates in parentheses). Probabilities of default are quite high, ranging from 18% under the best parameter combination to nearly 30% under the worst parameter set. However, these high probabilities of default are mitigated by very high expected recovery rates, in the range of 95-96%.

Table 5.4: Probabilities of default under different parameter assumptions for the regime-switching model. $X_0 = 1, T = 1, \alpha = 20\%, \sigma_1 = 0.1, \sigma_2 = 0.2, N_{sim} = 5000, H = 0.04$, and $c = 0.1$. Standard errors are in parentheses.

	Probability of Default		
(μ_0, μ_1)	$p = 0.01, q = 0.05$	$p = 0.1, q = 0.05$	$p = 0.01, q = 0.1$
$\mu_0 = 0.1,\ \mu_1 = -0.05$	0.1338 (0.0048)	0.3544 (0.0067)	0.0962 (0.0042)
$\mu_0 = 0.15,\ \mu_1 = -0.05$	0.0736 (0.0037)	0.3116 (0.0065)	0.0414 (0.0028)
$\mu_0 = 0.1,\ \mu_1 = 0$	0.1248 (0.0047)	0.2968 (0.0065)	0.0914 (0.0041)
$\mu_0 = 0.15,\ \mu_1 = 0$	0.0586 (0.0033)	0.2604 (0.0062)	0.0430 (0.0029)

Table 5.5: Expected recovery rates under different parameter assumptions for the regime-switching model. $X_0 = 1, T = 1, \alpha = 20\%, \sigma_1 = 0.1, \sigma_2 = 0.2, N_{sim} = 5000, H = 0.04$, and $c = 0.1$. Standard errors are in parentheses.

	Recovery Rate		
(μ_0, μ_1)	$p = 0.01, q = 0.05$	$p = 0.1, q = 0.05$	$p = 0.01, q = 0.1$
$\mu_0 = 0.1,\ \mu_1 = -0.05$	0.9825 (0.0507)	0.9429 (0.0722)	0.9922 (0.0399)
$\mu_0 = 0.15,\ \mu_1 = -0.05$	0.9888 (0.0432)	0.9452 (0.0737)	0.9975 (0.0328)
$\mu_0 = 0.1,\ \mu_1 = 0$	0.9844 (0.0455)	0.9447 (0.0707)	0.9919 (0.0439)
$\mu_0 = 0.15,\ \mu_1 = 0$	0.9866 (0.0469)	0.9460 (0.0713)	0.9964 (0.0374)

Moreover, the probability of default and the recovery rate can also be solved using a PDE approach. Define the default probability and expected recovery rate respectively as follows:

$$p(t, x, i) = \mathbb{E}_{\mathbb{P}}[\mathbb{I}(X_t \leq (1 + H)x)|\epsilon(0) = i], \tag{5.7}$$

$$R(t, x, i) = \mathbb{E}_{\mathbb{P}}\Big[\frac{\mathbb{E}_{\mathbb{P}}[X_T|X_T \leq (1 + H)X_0]}{(1 + H)X_0}|\epsilon(0) = i\Big]. \tag{5.8}$$

By Zhu et al. (2015), Elliott and Siu (2011), and Kennedy (2007), one can obtain a system of PDEs for (5.7),

$$p_t(t,x,1) + \mu_1 x p_x(t,x,1)\frac{1}{2}\sigma_1 x^2 p_{xx}(t,x,1) + \lambda_2\big(p(t,x,2) - p(t,x,1)\big) = 0, \tag{5.9}$$

$$p_t(t,x,2) + \mu_2 x p_x(t,x,2)\frac{1}{2}\sigma_2 x^2 p_{xx}(t,x,2) + \lambda_1\big(p(t,x,1) - p(t,x,2)\big) = 0, \tag{5.10}$$

with boundary condition

$$p(T,x,1) = p(T,x,2) = \mathbb{I}(x \leq (1+H)X_0). \tag{5.11}$$

For (5.8), a simple mathematical derivation yields,

$$\begin{aligned}
R(t,x,i) &= \mathbb{E}_{\mathbb{P}}\Big[\frac{\mathbb{E}_{\mathbb{P}}[X_T|X_T \leq (1+H)X_0]}{(1+H)X_0}|\epsilon(0) = i\Big] \\
&= \mathbb{E}_{\mathbb{P}}\Big[\frac{\mathbb{E}_{\mathbb{P}}[X_T\mathbb{I}(X_T \leq (1+H)X_0)]}{(1+H)X_0\mathbb{E}_{\mathbb{P}}[\mathbb{I}(X_T \leq (1+H)X_0)]}|\epsilon(0) = i\Big] \\
&= \frac{\mathbb{E}_{\mathbb{P}}[X_T\mathbb{I}(X_T \leq (1+H)X_0)|\epsilon(0) = i]}{(1+H)X_0\mathbb{E}_{\mathbb{P}}[\mathbb{I}(X_T \leq (1+H)X_0)|\epsilon(0) = i]} \\
&= \frac{\mathbb{E}_{\mathbb{P}}[X_T\mathbb{I}(X_T \leq (1+H)X_0)|\epsilon(0) = i]}{p(t,x,i)(1+H)X_0}.
\end{aligned} \tag{5.12}$$

Again, if we let

$$q(t,x,i) = \mathbb{E}_{\mathbb{P}}[X_T\mathbb{I}(X_T \leq (1+H)x)|\epsilon(0) = i], \tag{5.13}$$

then, a similar system of PDEs can be obtained as follows:

$$q_t(t,x,1) + \mu_1 x q_x(t,x,1)\frac{1}{2}\sigma_1 x^2 q_{xx}(t,x,1) + \lambda_2\big(q(t,x,2) - q(t,x,1)\big) = 0, \tag{5.14}$$

$$q_t(t,x,2) + \mu_2 x q_x(t,x,2)\frac{1}{2}\sigma_2 x^2 q_{xx}(t,x,2) + \lambda_1\big(q(t,x,1) - q(t,x,2)\big) = 0, \tag{5.15}$$

with boundary condition

$$q(T,x,1) = q(T,x,2) = x\mathbb{I}(x \leq (1+H)X_0). \tag{5.16}$$

5.4.1 Impact of the Manager's Deposit c

A key parameter for first-loss and negative loss fee structures is the manager's deposit c, as it determines the amount of downside protection provided to the investor by the fund manager. In this section, we investigate the impact of this parameter on the payoffs for the fund investor and manager. Figures 5.6 and 5.7 present the expected payoffs of the investor and manager respectively, as the parameter c varies, under the benchmark parameter set used to generate Table 5.1. We see that for large levels of downside protection, the investor's return quickly approaches the promised value H. For lower levels of insurance, the investor's expected return becomes negative. The manager's expected payoff follows the opposite pattern. Expected payoffs are high for low levels of c, but decrease rapidly as c increases. Similarly, as illustrated in Figures 5.8 and 5.9, the volatility of the investor's payoff decreases quickly as the level of downside protection c increases, and the volatility of the manager's payoff increases accordingly. The investor's Sharpe Ratio as a function of c is given in Figure 5.10 (the risk-free interest rate is set at $r = 1\%$). For very high levels of protection c, the Sharpe ratio grows very quickly (as $H > r$ and a very large level of downside protection virtually guarantees that the investor will receive the return H).

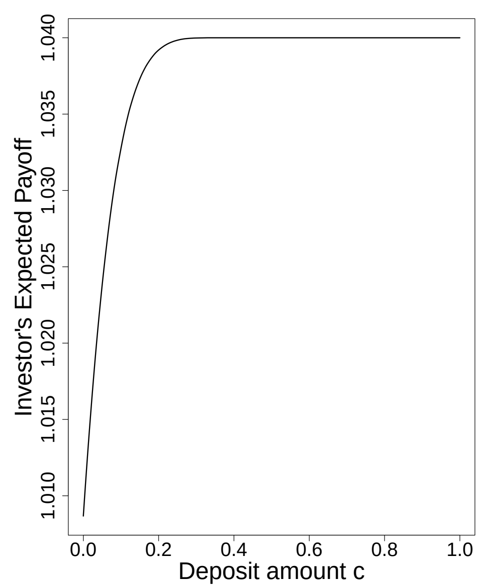


Figure 5.6: Investor's expected payoff as a function of the manager's deposit c. $X_0 = 1, T = 1, \alpha = 20\%, p_{1,2} = 0.01, p_{2,1} = 0.05, \sigma_1 = 0.1, \sigma_2 = 0.2, N_{sim} = 5000, H = 0.04$.

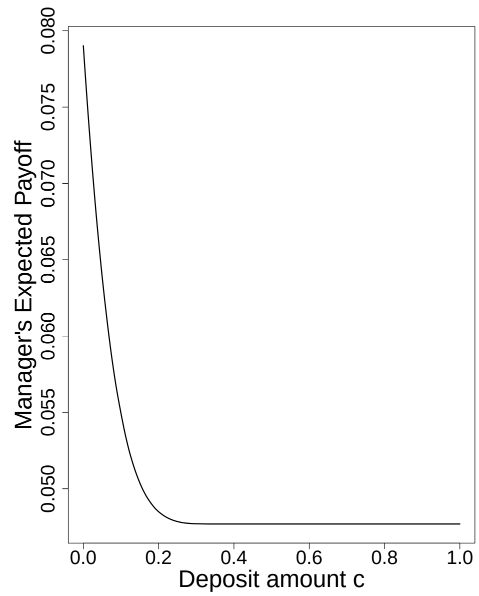


Figure 5.7: Manager's expected payoff as a function of the manager's deposit c. $X_0 = 1, T = 1, \alpha = 20\%, p_{1,2} = 0.01, p_{2,1} = 0.05, \mu_1 = 0.1, \mu_2 = -0.05, \sigma_1 = 0.1, \sigma_2 = 0.2, N_{sim} = 5000, H = 0.04$.

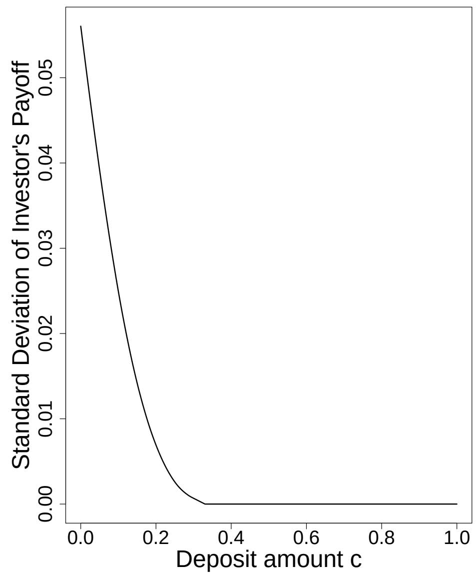


Figure 5.8: Standard deviation of the investor's payoff as a function of the manager's deposit c. $X_0 = 1, T = 1, \alpha = 20\%, p_{1,2} = 0.01, p_{2,1} = 0.05, \mu_1 = 0.1, \mu_2 = -0.05, \sigma_1 = 0.1, \sigma_2 = 0.2, N_{sim} = 5000, H = 0.04$.

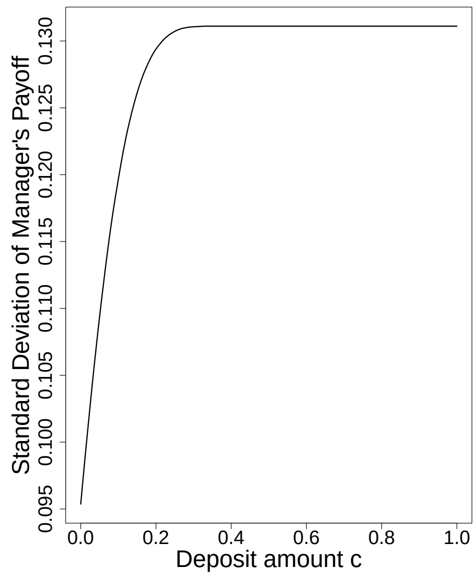


Figure 5.9: Standard deviation of the manager's payoff as a function of the manager's deposit c. $X_0 = 1, T = 1, \alpha = 20\%, p_{1,2} = 0.01, p_{2,1} = 0.05, \mu_1 = 0.1, \mu_2 = -0.05, \sigma_1 = 0.1, \sigma_2 = 0.2, N_{sim} = 5000, H = 0.04$.

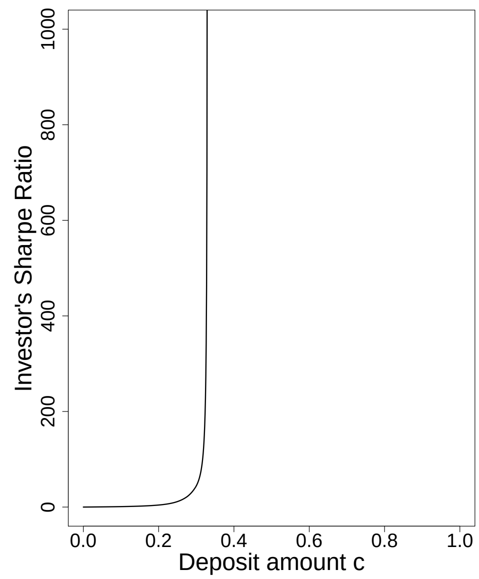


Figure 5.10: Sharpe Ratio of the investor's payoff as a function of the manager's deposit c. $X_0 = 1, T = 1, \alpha = 20\%, p_{1,2} = 0.01, p_{2,1} = 0.05, \mu_1 = 0.1, \mu_2 = -0.05, \sigma_1 = 0.1, \sigma_2 = 0.2, N_{sim} = 5000, H = 0.04$.

As with many collateralized products, the credit risk of the investor's payoff is intimately related to the market risk of the underlying reference portfolio. In the preceding analysis, we have measured risk using the standard deviations of payoffs and returns. While this is appropriate for normal distributions, the payoffs of the hedge fund manager and investor are non-normal, especially in the context of the regime-switching framework. As a consequence, it is important to consider the tail risks faced by the investor. We will do this by considering the investor's expected shortfall (also known as conditional Value-at-Risk, or conditional tail expectation), the expectation of losses given that the losses are above a given confidence level of their distribution.

Let

$$L^I = -(I(T) - X_0),$$

so that we have ‘positive’ loss. Define

$$\mathrm{ES}_\beta(L^I) = \frac{1}{1-\beta}\int_\beta^1 \mathrm{VaR}_u(L^I)du, \tag{5.17}$$

where

$$\mathrm{VaR}_\beta(L^I) = \inf\{x \in \mathbb{R} : \mathbb{P}[L^I \leq x] \geq \beta\}. \tag{5.18}$$

Note that we have a probability mass at the point $-HX_0$. The estimator for expected shortfall can be expressed as:

$$\widehat{\mathrm{ES}}_\beta(L^I) = w\frac{\sum_{i=1}^N \mathbb{I}_{\{L_i^I > \widehat{\mathrm{VaR}}_\beta(L^I)\}} L_i^I}{\sum_{i=1}^N \mathbb{I}_{\{L_i^I > \widehat{\mathrm{VaR}}_\beta(L^I)\}}} + (1-w)\widehat{\mathrm{VaR}}_\beta(L^I),$$

where

$$w = \frac{\sum_{i=1}^N \mathbb{I}_{\{L_i^I > \widehat{\mathrm{VaR}}_\beta(L^I)\}}}{N \cdot (1-\beta)}.$$

We increase the number of scenarios in the simulation to 1,000,000 in order to have more scenarios in the tail and a more accurate estimate of expected shortfall. Figure 5.11 shows the investor’s expected shortfall as a function of the manager’s deposit c for $\beta = 0.95, 0.99$. As expected, lower levels of the manager’s deposit are associated with higher levels of risk. In particular, for manager deposits near our benchmark level of $c = 10\%$, expected shortfall can exceed 20% of the initial investment, indicating significant losses for investors under extreme scenarios. Because of the large number of scenarios used, the confidence intervals for the estimates are quite small (the lengths of the confidence intervals are around 1.5% of the estimated values).

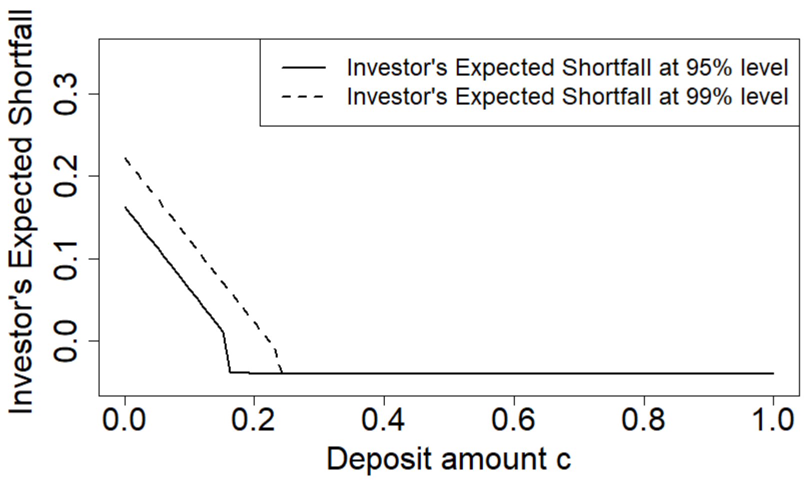


Figure 5.11: Expected shortfall of the investor's losses as a function of the manager's deposit c. $X_0 = 1, T = 1, \alpha = 20\%, p_{1,2} = 0.01, p_{2,1} = 0.05, \mu_1 = 0.1, \mu_2 = -0.05, \sigma_1 = 0.1, \sigma_2 = 0.2, N_{sim} = 1,000,000, H = 0.04$.

5.5 Conclusion

Recently, market pressures have led to the introduction of innovative hedge fund fee structures, in which the fund manager receives higher performance fees in return for providing downside protection to fund investors. These arrangements are referred to by the general name of shared-loss fee structures. An extreme version is the negative fee structure, in which the manager receives all profits above a pre-defined hurdle rate, and for which the investor's position resembles that of an investment in an asset-backed security, with the underlying assets being the hedge fund's portfolio. In this chapter we analyzed the negative fee structure in a regime-switching model, both by pricing it using risk-neutral valuation, and performing a risk analysis under the real-world measure (including examining the probability of default and expected recovery rate).

There are a number of important questions that could be considered for future research. The fee structure could be analyzed under other mathematical models, including those that allow more general stochastic behaviour of volatility. The incentives of the manager, in terms of the structuring of the hedge fund portfolio, and the investor, in terms of the decision to withdraw from the fund, could both be studied, either in isolation (as a stochastic control problem and an optimal stopping problem respectively), or together (in a stochastic game of control and stopping). Finally, the limitations of the assumptions underlying risk-neutral valuation (particularly the ability to observe the value of, and dynamically trade in, the underlying assets of the hedge fund) could be investigated, perhaps through models that more realistically represent the bargaining process between principal (investor) and agent (manager).

Chapter 6

Conclusion and Directions for Future Work

In this final chapter, we summarize the original research contributions of this thesis, and suggest several directions for future research.

6.1 Optimal Stopping Problems for Withdrawal from Hedge Funds with Alternative Fee Structures

We study the optimal stopping problem arising from an investor determining the best time to withdraw from a hedge fund investment with a shared-loss fee structure and a positive fee for assets under management. To be precise, we analyze an optimal stopping problem for an investor with a piecewise linear payoff function, where the underlying follows a geometric Brownian motion, corresponding to a hedge fund with a continuous fee for assets under management deducted (or, equivalently, the price process for a stock paying a continuous dividend yield). The optimal solution is characterized as the first exit time of the fund value from a bounded region with upper and lower stopping boundaries. We present a complete solution of the problem in the infinite horizon case. In the finite horizon case, we describe the shape of the continuation region, characterize the stopping boundaries using a coupled pair of integral equations, and present an asymptotic analysis of the boundaries in small time.

6.2 Fee Structures

We study and compare the Sharpe Ratios and Sortino Ratios of hedge fund investors facing combinations of the first-loss fee structure and traditional fee structure. In particular, we maximize the Sharpe Ratio or the Sortino Ratio of a portfolio that combines payoffs of the two fee structures at maturity T. A criterion (4.5) is presented to distinguish whether or not there exists an interior optimal weight allocation between the two fee structures for the Sharpe Ratio. Numerical examples are presented for both the Sharpe and Sortino Ratios, assuming a geometric Brownian motion process for the hedge fund assets. In most cases we find that the optimal weights are on the boundary points, indicating that one extreme fee structure is preferred compared to the other, and all possible mixtures. Typically, the classical fee structure is preferred at very low volatilities, where the insurance in the first-loss contract is unlikely to be triggered, and at extremely high volatilities, for which the share of a very large potential upside is important. At intermediate volatilities (often covering most of the range typically seen in practice), the first-loss structure is preferred, owing to the importance of its insurance component. We find that there is negligible benefit to the investor due to "fee diversification," even in the case when a combination of fee structures is optimal.

Finally, an extreme version, the negative fee structure is investigated, in which the manager receives all profits above a pre-defined hurdle rate, and for which the investor's position resembles that of an investment in an asset-backed security, with the underlying assets being the hedge fund's portfolio. We analyzed the negative fee structure in a regime-switching model, both by pricing it using risk-neutral valuation, and performing a risk analysis under the real-world measure (including examining the probability of default and expected recovery rate).

6.3 Directions for Future Work

6.3.1 Integral Equations

In Chapter 3, we have proven that the exercise boundaries $S_-(T)$ and $S_+(T)$ should decrease and increase in T respectively. However, the numerical solutions of the equations appear to be unstable. In particular, when applying several different numerical methods, the solutions of (3.31) and (3.32) become ill-behaved, and even non-monotonic near the infinite horizon exercise boundaries.

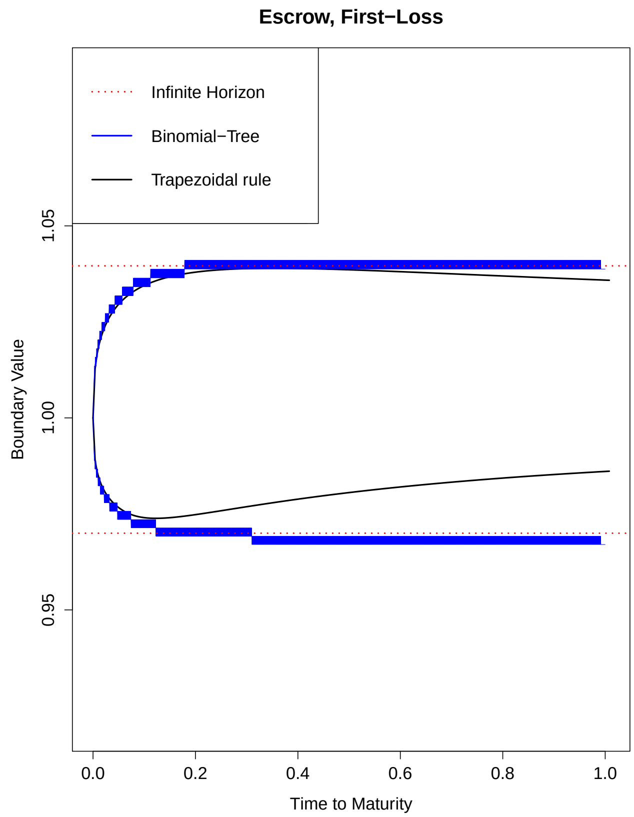


Figure 6.1: Exercise Boundaries of the Investor: Escrow First-Loss Fee Structure with parameters $x = 1, r = 0.05, \delta = 0, \sigma = 0.1, T = 1, A = 0, B = 1, p = 0.5, q = 1, N = 252, N_{bino} = 5000$, where N is the number of time steps for the trapezoidal rule method and N_{bino} is the number of time steps for the binomial-tree method.

Figure 6.1 illustrates this loss of monotonicity, comparing numerical solution of the integral equations using the trapezoid rule against the boundaries calculated from a binomial tree method, which provides rough estimates for our exercise boundaries. Other numerical methods, such as Simpson's rule (Chiarella and Ziogas (2005)) and the gaussian quadrature rule with polynomial interpolation to smooth the exercise boundaries Kim et al. (2013) produce similar results. Thus, in the future, we are going to approach the exercise boundary value problem differently. For instance, we can implement the penalty technique to solve the variational inequality (3.8) numerically. Furthermore, all of our figures indicate that the exercise boundaries $S_{-}(T)$ and $S_{+}(T)$ seem to go to their infinite horizon counterparts S_1 and S_2 very quickly (within 0.25 years). Consequently, an alternative might be to employ semi-analytic methods that solve a numerical equation that interpolates between the asymptotic formula for short time and the infinite horizon solution, as done by Chen and Chadam (2007) in the case of the American put. Once we have a reliable numerical method, we then can study how the exercise boundaries depend on the parameters. In particular, we can investigate the robustness of the parameters.

6.3.2 Empirical Analysis

Another way to extend our work is performing analysis with empirical data. Our motivation is that the investors generally do not know the portfolio positions because of the low transparency of hedge funds. Moreover, instead of reporting daily returns to the investors like mutual funds, the hedge managers typically only report monthly returns to the investors Vogt (2010). Therefore, it is very difficult for investors to keep track of how successfully the hedge funds are performing. More importantly, investors cannot decide when to withdraw from the fund based on perfect knowledge of the current value of the hedge fund assets.

In the future, we would like to come up with a withdrawal rule based on the data provided by the hedge fund managers and other public sources, so that the investors can have an objective tool to determine the withdrawal time. Furthermore, we can take the fee structures into account to see whether various fee structures have an impact on investors' withdrawal decisions. For instance, one can study whether the first-loss fee structure can lead investor to a better withdrawal time when the economy is bearish.

6.3.3 Fee Structures

There are a number of important questions related to innovative fee structures that could be considered for future research. The fee structure could be analyzed under other mathematical models, including those that allow more general stochastic behaviour of volatility. The incentives of both the manager, in terms of the structuring of the hedge fund portfolio, and the investor, in terms of the decision to withdraw from the fund, could both be studied, either in isolation (as a stochastic control problem and an optimal stopping problem respectively), or together (in a stochastic game of control and stopping). For instance, by Bayraktar and Huang (2013), we can define the stochastic game as follows. If the investor acts first:

$$U(x,t) = \inf_{c} \sup_{\tau} \mathbb{E}[e^{-r\tau} g(X_{\tau}^{x,c})]. \tag{6.1}$$

On the other hand, if the manager acts first:

$$V(x,t) = \sup_{\pi} \inf_{c} \mathbb{E}[e^{-r\tau} g(X^{x,c}_{\pi[c]})], \tag{6.2}$$

where $c \in \mathcal{C}$ is the set of all admissible controls, $\tau \in \mathcal{T}_{[0,t]}$ is the set of all stopping times if the investor acts first, $\pi[c] \in \pi_{[0,t]}$ is the set of all stopping times if the manager acts first, and $X^{x,c}_t$ is the hedge fund dynamic with control strategy c. Generally speaking, such stochastic game problems are difficult to analyze and solve. However, by assuming some regularity conditions, we can claim $U(x,t) = V(x,t)$ and the solution of (6.1) or (6.2) satisfies so-called Hamilton-Jacobi-Bellman Variational Inequality (Øksendal and Sulem, 2007, Chapter 4, Page 65) in the following:

$$\min\{-V_t - \inf_{c}\{LV - rV\}, g - V\} = 0. \tag{6.3}$$

It is worth mentioning that, it requires rigorous and thorough mathematical analysis to prove that $U(x,t)$ or $V(x,t)$ is indeed a unique viscosity solution to the equation (6.3). Moreover, if such claim can be proved true, then a numerical solution can be obtained. For example, Forsyth and Labahn (2007) and Huang et al. (2012) proposed piece-wise constant policy and fixed point-policy methods to solve stochastic control and game problems. However, the authors mention that the convergence for stochastic games is not guaranteed, which makes it an interesting research topic. Finally, the limitations of the assumptions underlying risk-neutral valuation (particularly the ability to observe the value of, and dynamically trade in, the underlying assets of the hedge fund) could be investigated, perhaps through models that more realistically represent the bargaining process between principal (investor) and agent (manager) than those we have discussed this thesis.

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APPENDICES

Appendix A

Appendix for Chapter 1

A.1 Derivation of Payoff Functions and Value Functions with Arbitrary Initial Investment

For our analysis in Chapters 1, 2, and 3, we assume the initial investment is 1 and claim that it is without of loss generality. Appendix A.1 is devoted to further clarify our claim. Let $g_y^*(x)$ denote the payoff function and initial investment amount $y > 0$.

A.1.1 Compensation from an Escrow Account

In this case, the hedge fund manager sets up an escrow account, which is used to cover the investor's losses. Here, we use c, $0 < c < 1$, a percentage of the initial investment.

First-Loss

For the first-loss case, the hedge fund manager will compensate all of the investor's losses until the escrow amount is exhausted. The payoff function to the investor is

$$g_y^*(x) = \begin{cases} y + (1-\alpha)(x-y), & x \geq y, \\ y, & (1-c)y < x < y, \\ cy + x, & x \leq (1-c)y, \end{cases} \tag{A.1}$$

$$= \begin{cases} \alpha y + (1-\alpha)x, & x \geq y, \\ y, & (1-c)y < x < y, \\ cy + x, & x \leq (1-c)y. \end{cases} \tag{A.2}$$

Shared-Loss

For the shared-loss case, the manager covers a proportion θ of the investor's losses from an escrow account. If $c \geq \theta$, which indicates the escrow account cannot be exhausted, then the payoff to the investor is

$$g_y^*(x) = \begin{cases} y + (1-\alpha)(x-y), & x \geq y, \\ y + (1-\theta)(x-y), & x < y, \end{cases} \tag{A.3}$$

$$= \begin{cases} \alpha y + (1-\alpha)x, & x \geq y, \\ \theta y + (1-\theta)x, & x < y. \end{cases} \tag{A.4}$$

If $c < \theta$, then the payoff to the investor is

$$g_y^*(x) = \begin{cases} \alpha y + (1-\alpha)x, & x \geq y, \\ \theta y + (1-\theta)x, & (1-\frac{c}{\theta})y < x < y, \\ cy + x, & x \leq (1-\frac{c}{\theta})y. \end{cases} \tag{A.5}$$

A.1.2 Compensation from the Manager's Own Investment

In this arrangement, the manager invests their own capital into the fund. Let $\omega \in (0,1)$ be the proportion of the investor's initial investment contributed by the manager. The total initial investment is thus $(1+\omega)y$.

First-Loss

In first-loss structures, the manager's share of the fund is used to completely cover the investor's losses (until the manager's share of the fund is exhausted). The payoff to the investor is

$$g_y^*(x) = \begin{cases} \alpha y + (1-\alpha)x, & x \geq y, \\ y, & \frac{1}{1+\omega} y < x < y, \\ (1+\omega)x, & x \leq \frac{1}{1+\omega} y. \end{cases} \tag{A.6}$$

Shared-Loss

Finally, in the shared-loss case the manager has invested the amount ωy in the fund and covers a proportion θ of the investor's losses using their own share. The investor's payoff is

$$g_y^*(x) = \begin{cases} \alpha y + (1-\alpha)x, & x \geq y, \\ \theta y + (1-\theta)x, & \frac{\theta}{\omega+\theta} y < x < y, \\ (1+\omega)x, & x \leq \frac{\theta}{\omega+\theta} y. \end{cases} \tag{A.7}$$

A.1.3 General Formulation

Again, under both the first-loss and shared-loss fee structures, the payoff function $g_y^*(x)$ can be written in the following form:

$$g_y^*(x) = \begin{cases} Ay + Bx, & 0 \leq x \leq \kappa y \\ qy + (1-q)x, & \kappa y \leq x \leq y, \\ py + (1-p)x, & y \leq x, \end{cases} \tag{A.8}$$

where $B \geq 1 \geq q \geq A \geq 0$, $p \in (0,1)$ and $\kappa = (B-(1-q))^{-1}(q-A)$. The detailed parameterization for each fee structure can be found in Table 1.1. Observe that $g_y^*(x) = yg(\frac{x}{y})$.

Next, we present the definitions of value functions similar to (1.9) and (1.10), but with arbitrary initial investment y. First, assume that the investor's share of the hedge fund assets, under the risk-neutral measure $\mathbb{Q}$, satisfies the dynamics:

$$d\tilde{X}_t^x = (r-\delta)\tilde{X}_t^x dt + \sigma \tilde{X}_t^x dW_t, \quad \tilde{X}_0^x = x, \quad t \geq 0, \tag{A.9}$$
$$\tilde{X}_t^x = x \exp\{(r-\delta-\frac{1}{2}\sigma^2)t + \sigma W_t\}.$$

Furthermore, let the value function for the infinite horizon case be

$$V^*(x;y) = \sup_{\tau \in \mathcal{T}} \mathbb{E}[e^{-r\tau} g_y^*(\tilde{X}_\tau^x)], \tag{A.10}$$

where $\mathcal{T}$ is the set of all stopping times. On the other hand, if the investor has a finite investment horizon T, then the value function at the current time is

$$v^*(x,T;y) = \sup_{\tau \in \mathcal{T}_{[0,T]}} \mathbb{E}[e^{-r\tau} g_y^*(\tilde{X}_\tau^x)], \tag{A.11}$$

where $\tau \in \mathcal{T}_{[0,T]}$ is the set of all stopping times such that $0 \leq \tau \leq T$, and $g_y^*(x)$ is the investor's payoff function. The key observation is that $g_y^*(x) = yg(\frac{x}{y})$ where $g(\cdot)$ is the payoff function with initial investment 1 of the form (1.6). Recalling that the process with initial investment 1 is X_t^x, then for any stopping time $\tau \in \mathcal{T}$,

$$\begin{aligned}\mathbb{E}[e^{-r\tau} g_y^*(\tilde{X}_\tau^x)] &= y\mathbb{E}\left[e^{-r\tau} g\left(\frac{\tilde{X}_\tau^x}{y}\right)\right] \\ &= y\mathbb{E}\left[e^{-r\tau} g\left(X_\tau^{\frac{x}{y}}\right)\right].\end{aligned}$$

Finally, by (1.9) and (A.10),

$$V^*(x;y) = yV\left(\frac{x}{y}\right). \tag{A.12}$$

Similarly, we can easily obtain

$$v^*(x,T;y) = yv\left(\frac{x}{y}, T\right). \tag{A.13}$$

Appendix B

Appendix for Chapters 2 and 3

B.1 Proof of Lemma 2.4.1

a. From (2.13),

$$\begin{aligned}
C_1(x_0, 1-q) &= \frac{x_0^{-m_1}}{m_1 - m_2}\big((1-q)x_0 - (q + (1-q)x_0)m_2\big) \\
&= \frac{x_0^{-m_1}}{m_1 - m_2}\big((1-q)(1-m_2)x_0 - qm_2\big), \qquad \text{(B.1)} \\
C_2(x_0, 1-q) &= \frac{x_0^{-m_2}}{m_1 - m_2}\big((q + (1-q)x_0)m_1 - (1-q)x_0\big) \\
&= \frac{x_0^{-m_2}}{m_1 - m_2}\big((1-q)(m_1 - 1)x_0 + qm_1\big). \qquad \text{(B.2)}
\end{aligned}$$

Then, we can get

$$C_1'(x_0, 1-q) = \frac{1}{m_1 - m_2}((1-q)(1-m_1)(1-m_2)x_0^{-m_1} + qm_1m_2x_0^{-m_1-1}) < 0, \qquad \text{(B.3)}$$

$$C_2'(x_0, 1-q) = \frac{1}{m_1 - m_2}((1-q)(m_1-1)(1-m_2)x_0^{-m_2} - qm_1m_2x_0^{-m_2-1}) > 0. \qquad \text{(B.4)}$$

Therefore, $C_1(x_0, 1-q)$ is decreasing in x_0 and $C_2(x_0, 1-q)$ is increasing in x_0. Moreover, observe that $\lim_{x_0 \to 0} C_1(x_0, 1-q) = \infty$, $\lim_{x_0 \to \infty} C_1(x_0, 1-q) = 0$, $\lim_{x_0 \to 0} C_2(x_0, 1-q) = 0$ and $\lim_{x_0 \to \infty} C_2(x_0, 1-q) = \infty$. Then, we can verify that $C_1(x_0, 1-q) > 0$ and $C_2(x_0, 1-q) > 0$ for all $x_0 \in (0, \infty)$. By Remark 2.3.1, we can verify that $W(x; x_0, 1-q)$ is a strictly convex function on $(0,\infty)$. Next, consider

$$\begin{aligned} D(x;x_1,x_2) &= W(x;x_1,1-q) - W(x;x_2,1-q) \\ &= C_1(x_1,1-q)x^{m_1} + C_2(x_1,1-q)x^{m_2} - C_1(x_2,1-q)x^{m_1} - C_2(x_2,1-q)x^{m_2} \\ &= (C_1(x_1,1-q) - C_1(x_2,1-q))x^{m_1} + (C_2(x_1,1-q) - C_2(x_2,1-q))x^{m_2}. \end{aligned} \tag{B.5}$$

By the strict convexity of $W(x;x_1,1-q)$, $W(x_2;x_1,1-q) > q+(1-q)x_2 = W(x_2;x_2,1-q)$. So, $D(x_2;x_1,x_2) > 0$. Furthermore, from (B.5),

$$\begin{aligned} D'(x;x_1,x_2) =& m_1(C_1(x_1,1-q) - C_1(x_2,1-q))x^{m_1-1} \\ &+ m_2(C_2(x_1,1-q) - C_2(x_2,1-q))x^{m_2-1} > 0. \end{aligned} \tag{B.6}$$

This implies that $D(x;x_1,x_2) > 0$ for all $x \geq x_2$. That is $W(x;x_1) > W(x;x_2)$.
b. Again, from (2.13),

$$C_1(\kappa,v_0) = \frac{\kappa^{-m_1}}{m_1-m_2}\big(\kappa v_0 - (q+(1-q)\kappa)m_2\big), \tag{B.7}$$

$$C_2(\kappa,v_0) = \frac{\kappa^{-m_2}}{m_1-m_2}\big((q+(1-q)\kappa)m_1 - \kappa v_0\big). \tag{B.8}$$

Note that $C_1(\kappa,v_0) \geq 0$ if and only if $v_0 - (q+(1-q)\kappa)m_2 \geq 0$ and $C_2(\kappa,v_0) \geq 0$ if and only if $(q+(1-q)\kappa)m_1 - \kappa v_0 \geq 0$. Then we can easily derive that when

$$\frac{(q+(1-q)\kappa)m_2}{\kappa} \leq v_0 \leq \frac{(q+(1-q)\kappa)m_1}{\kappa},$$

$W(x;\kappa,v_0)$ is a strictly convex function on $(1,\infty)$. Next, when $v_0 = \frac{qm_1+(1-q)\kappa m_1}{\kappa}$,

$$\begin{aligned} W\Big(x;\kappa,\frac{qm_1+(1-q)\kappa m_1}{\kappa}\Big) &= \kappa^{-m_1}\frac{(1-q)m_1\kappa + qm_1 - (q+(1-q)\kappa)m_2}{m_1-m_2}x^{m_1} \\ &= \kappa^{-m_1}\frac{(1-q)(m_1-m_2)\kappa + q(m_1-m_2)}{m_1-m_2}x^{m_1} \\ &= \kappa^{-(m_1-1)}\Big(1-q+\frac{q}{\kappa}\Big)x^{m_1} > g(x), \end{aligned} \tag{B.9}$$

since $\kappa^{-(m_1-1)} > 1$ and $(1 - q + \frac{q}{\kappa}) > 1$. Last, for $v_0 < v_1$, consider

$$\begin{aligned} D(x; v_1, v_0) &= W(x; \kappa, v_1) - W(x; \kappa, v_0) \\ &= (C_1(\kappa, v_1) - C_1(\kappa, v_0))x^{m_1} + (C_2(\kappa, v_1) - C_2(\kappa, v_0))x^{m_2} \\ &= \frac{\kappa^{-m_1}}{m_1 - m_2}\kappa(v_1 - v_0)x^{m_1} + \frac{\kappa^{-m_2}}{m_1 - m_2}\kappa(v_0 - v_1)x^{m_2}. \end{aligned} \tag{B.10}$$

Then we can easily find

$$D'(x; v_1, v_0) = \frac{m_1\kappa^{-m_1}}{m_1 - m_2}\kappa(v_1 - v_0)x^{m_1-1} + \frac{m_2\kappa^{-m_2}}{m_1 - m_2}\kappa(v_0 - v_1)x^{m_2-1} > 0.$$

Note that $D(\kappa; v_1, v_0) = 0$, so $W(x; \kappa, v_1) > W(x; \kappa, v_0)$ for $x > \kappa$. $\square$

B.2 Proof of Lemma 3.4.1

Considering the inequality $a < X_t < b$ and $W_t = \sqrt{t}Z$ where Z is a standard normal random variable, we can derive

$$\begin{aligned} &\log(a/x) < (r - \delta - \sigma^2/2) - \sigma\sqrt{t}Z < \log(b/x) \\ &\Rightarrow \frac{\log(x/b) + (r - \delta - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} < Z < \frac{\log(x/a) + (r - \delta - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \\ &\Rightarrow d_2(x, b, t) < Z < d_2(x, a, t). \end{aligned} \tag{B.11}$$

Clearly, from (B.11), we have

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{\{a<X_t<b\}}] &= \mathbb{Q}[d_2(x, b, t) < Z < d_2(x, a, t)] \\ &= \Phi(d_2(x, a, t)) - \Phi(d_2(x, b, t)). \end{aligned}$$

Hence, (3.35) is proved. Moreover,

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{\{a<X_t<b\}}X_t] &= \int_{d_2(x,b,t)}^{d_2(x,a,t)} X_t f_Z(z)dz \\
&= \int_{d_2(x,b,t)}^{d_2(x,a,t)} X_t \exp\{(r-\delta-\frac{1}{2}\sigma^2)t - \sigma\sqrt{t}Z\}\exp\{-\frac{1}{2}z^2\}dz \\
&= xe^{(r-\delta)t}\int_{d_2(x,b,t)}^{d_2(x,a,t)} \exp\{-\frac{1}{2}(z^2+2\sigma\sqrt{t}z+\sigma^2 t)\}dz \\
&= xe^{(r-\delta)t}\int_{d_2(x,b,t)}^{d_2(x,a,t)} \exp\{-\frac{1}{2}(z+\sigma\sqrt{t})^2\}dz \\
&= xe^{(r-\delta)t}\int_{d_2(x,b,t)+\sigma\sqrt{t}}^{d_2(x,a,t)+\sigma\sqrt{t}} \exp\{-\frac{1}{2}y^2\}dy \\
&= xe^{(r-\delta)t}(\Phi(d_1(x,a,t)) - \Phi(d_1(x,b,t))).
\end{aligned}
$$

□

B.3 Proof of Lemma 3.4.2

When $x < K$, we have

$$C(x,K,T) - \max(x-K,0) = xe^{-\delta T}\Phi(d_1(x,K,T)) - Ke^{-rT}\Phi(d_2(x,K,T)) \tag{B.12}$$

Noting that $\lim_{T\to 0+} d_1(x,K,T) = \lim_{T\to 0+} d_2(x,K,T) = -\infty$, we can easily obtain,

$$\lim_{T\to 0+}\frac{\Phi(d_1(x,K,T))}{\sqrt{T}} = \lim_{T\to 0+}\frac{\Phi(d_2(x,K,T))}{\sqrt{T}} = 0. \tag{B.13}$$

So, $C(x,K,T) - \max(x-K,0) = o(\sqrt{T})$. Next, when $x = K$, we can apply the Mean Value Theorem and write,

$$
\begin{aligned}
C(x,K,T) - \max(x-K,0) &= xe^{-\delta T}\Phi(d_1(x,K,T)) - Ke^{-rT}\Phi(d_2(x,K,T)) \\
&= xe^{-\delta T}\phi(\xi(T))\sigma\sqrt{T} + \Phi(d_2(x,K,T))(xe^{-\delta T} - Ke^{-rT}).
\end{aligned}
$$

for some $\xi(T) \in (d_2(x,K,T), d_1(x,K,T))$. Since $x = K$, $0 = \lim_{T\to 0+} d_2(x,K,T) \leq \lim_{T\to 0+} \xi(T) \leq \lim_{T\to 0+} d_1(x,K,T) = 0$. In other words,

$$\lim_{T\to 0+} \frac{C(K,K,T)}{\sqrt{T}} = K\phi(0)\sigma + \lim_{T\to 0+} \Phi(d_2(K,K,T)) \frac{Ke^{-\delta T} - Ke^{-rT}}{\sqrt{T}}$$
$$= \frac{K\sigma}{\sqrt{2\pi}} + 0 = O(\sqrt{T}).$$

Finally, $x > K$ implies

$$C(x,K,T) - \max(x-K,0) = xe^{-\delta T}\Phi(d_1(x,K,T)) - Ke^{-rT}\Phi(d_2(x,K,T)) - x + K$$
$$= x(e^{-\delta T}\Phi(d_1(x,K,T)) - 1) + K(1 - e^{-rT}\Phi(d_2(x,K,T)).$$

Again, it can be shown that

$$\lim_{T\to 0+} = \frac{e^{-\delta T}\Phi(d_1(x,K,T)) - 1}{\sqrt{T}} = \lim_{T\to 0+} = \frac{1 - e^{-rT}\Phi(d_1(x,K,T))}{\sqrt{T}} = o(\sqrt{T}). \tag{B.14}$$

□

B.4 Proof of Lemma 3.4.3

Part (a). Since $f''(x) < 0$ for $x < x_2$ and $f''(x) > 0$ for $x > x_2$, $f_1'(x)$ is strictly decreasing on the interval $[a_1, x_2]$ and strictly increasing on the interval $[x_2, b_1]$. In other words, $f_1'(x_2)$ attains a local minimum on the interval $[a_1, b_1]$. Now, we can prove the uniqueness.

First, suppose no critical point exists on the interval (a_1, b_1). Then $f_1'(a_1) > 0$, $f_1'(b_1) > 0$ and the fact that $f_1'(x_2)$ is a local minimum point implies that $0 < f_1'(x_2) \leq f_1'(x)$ for all $x \in [a_1, b_1]$. So, $f_1(x)$ must be strictly increasing. In other words, $f_1(x)$ must have a unique root on the interval (a_1, b_1) since $f_1(a_1) < 0$ and $f_1(b_1) > 0$.

Next, suppose there exist points x_1 satisfying $f_1'(x_1) = 0$. Since $f_1'(x_2)$ is a local minimum, we must have $f_1'(x_2) \leq 0$. If $f_1'(x_2) = 0$, then condition 2 implies $f_1'(x) > 0$ for $x \in [a_1, x_2) \cup (x_2, b_1]$, which means $f_1(x)$ is strictly increasing for $x \in [a_1, b_1]$. So, $f_1(x)$ must have a unique root in the interval (a_1, b_1).

Last, if $f_1'(x_2) < 0$, we must have two critical points. Let x_1^* and x_1^{**} denote the two critical points, and without loss of generality, we assume $x_1^* < x_1^{**}$. Then the Mean-Value

Theorem implies there must exist some point $c_1 \in (x_1^*, x_1^{**})$ such that $f_1''(c_1) = 0$. Since the inflection point is unique, we must have $c_1 = x_2$. Therefore, condition 2 implies that $f_1''(x_1^*) < 0$. So, $f_1(x_1)$ attains a local maximum on the interval $[a_1, x_1^{**}]$. Moreover, by condition 3, it can be easily verified that for all $a_1 \le x \le x_1^{**}$, $f(x) \le f(x_1^*) < 0$. Therefore, we can conclude that there is no root for $f(x) = 0$ between a_1 and x_1^{**}. Meanwhile, the fact that $x_2 < x_1^{**}$ implies that $f_1''(x) > 0$ for $x \ge x_1^{**}$. In other words, we must have $f'(x) > 0$ for $x > x_1^{**}$. Also, recalling that $f(x_1^{**}) < 0$ and $f(b_1) > 0$, $f(x) = 0$ must attain a unique root on (x_1^{**}, b_1).

Part (b). Suppose there is no root for $x \in (a_2, \infty)$. Then, we must have $f(x) > 0$ for all $x \ge a_2$, which contradicts that $f(x) \le 0$ for $x \ge b_2$. Thus, we can claim that $f(x) = 0$ must have at least one root. Next, suppose x_2^* and x_2^{**} are two distinct roots on the interval (a_2, ∞) such that $f(x_2^*) = 0$, $f(x_2^{**}) = 0$ and $x_2^* < x_2^{**}$. Then, the Mean Value Theorem implies that there exists $f'(c_2) = 0$ for some $c_2 \in (x_2^*, x_2^{**})$. Noting that $f(x)$ is strictly convex, it can be easily verified that $f'(x_2^{**}) > 0$. Thus, for some constant $b_2^* > x_2^{**}$, we have $f(x) > 0$ for all $x \ge b_2^*$. This contradicts the fact that $f(x) \le 0$ for all $x \ge b_2$. Thus, we must have a unique root x^* on the interval (a_2, ∞) satisfying $f(x^*) = 0$. ☐

B.5 Proof of Lemma 3.4.6

Proof. Note that V is increasing and Lipschitz continuous in x (uniformly on any $[0, \bar{T}]$) by Touzi (2013, Proposition 4.7, pages 46-47). Let $x, y \in (S_-(T), S_-^e(T))$ with $x \ge y$. By the Dynamic Programming Principle (Touzi (2013, page 41)), for any stopping time $\theta \in \mathcal{T}_{[0,T]}$

$$v(x, T) = \sup_{\tau \in \mathcal{T}_{[0,T]}} \mathbb{E}[\mathbf{1}_{\tau < \theta} e^{-r\tau} g(X_\tau^x) + \mathbf{1}_{\tau \ge \theta} e^{-r\theta} V(T - \theta, X_\theta^x)]$$

so that

$$\begin{aligned} V(x, T) - V(y, T) \le \sup_{\tau \in \mathcal{T}_{[0,T]}} & \mathbb{E}[\mathbf{1}_{\tau < \theta} e^{-r\tau} (g(X_\tau^x) - g(X_\tau^y)) \\ & + \mathbf{1}_{\tau \ge \theta} e^{-r\theta} (V(T - \theta, X_\theta^x) - V(T - \theta, X_\theta^y))]. \end{aligned} \tag{B.15}$$

Define:

$$\theta = \inf\{t > 0,\ X_t^x = 1 \text{ or } X_t^y = \kappa\}, \tag{B.16}$$

to obtain:

$$V(x,T)-V(y,T) \leq (x-y)(1-q)+C(x-y)\mathbb{Q}(\theta \leq \tau) \leq (x-y)(1-q)+C(x-y)\mathbb{Q}(\theta \leq T).$$

where C is the Lipschitz constant of V, and we have suppressed the dependence of θ on x and y. Take $\eta > \kappa$. Then since $S_-(T) \to 1$, for T small enough

$$\theta \geq \bar{\theta} = \inf\{t > 0,\ X_t^{S_-^e(T)} = 1 \text{ or } X_t^{\eta} = \kappa\} \tag{B.17}$$

and $\bar{\theta}$ does not depend on the choice of x, y. Thus:

$$\frac{V(x,T) - V(y,T)}{x - y} \leq (1-q) + C\mathbb{Q}(\bar{\theta} \leq T). \tag{B.18}$$

The final term is bounded by the constant C multiplied by the sum of the probabilities that the process X started at η hits κ before T, and that X started at $S_-^e(T)$ hits 1 before T. Both of these probabilities can be shown to be $o(1)$ using the explicit form of the hitting time distribution of a geometric Brownian motion (the first trivially, and the second using the estimate (3.51)). The proof for $x, y \in (S_+^e(T), S_+(T))$ is similar. $\square$

Appendix C

Appendix for Chapter 4

C.1 Derivation of the Condition for an Interior Optimum in the Sharpe Ratio Maximization Problem

Introduce the notation

$$\begin{aligned}
&\mu_1 = \mathbb{E}[g_1(X_T^x)], \quad \mu_2 = \mathbb{E}[g_2(X_T^x)], \\
&\sigma_1 = \mathrm{Var}[g_1(X_T^x)], \quad \sigma_2 = \mathrm{Var}[g_2(X_T^x)], \\
&\sigma_{12} = \mathrm{Cov}[g_1(X_T^x),\ g_2(X_T^x)].
\end{aligned}$$

From (4.3), we can write,

$$\mathbb{E}[g(X_T^x)] = \boldsymbol{\omega}'\boldsymbol{\mu}, \quad \mathrm{Var}[g(X_T^x)] = \boldsymbol{\omega}'\Sigma\boldsymbol{\omega}, \tag{C.1}$$

where

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_1 & \sigma_{12} \\ \sigma_{12} & \sigma_2 \end{bmatrix}. \tag{C.2}$$

Here, the investor seeks to maximize the Sharpe Ratio $SR(\boldsymbol{\omega})$ in (4.4). As a result, the investor is solving the following optimization problem:

$$\begin{aligned} &\max_{\boldsymbol{\omega}} \frac{\boldsymbol{\omega}'\boldsymbol{\mu} - (1+r)x}{\sqrt{\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}}} \\ &\text{subject to } \boldsymbol{\omega}'\mathbf{1} = 1, \\ &\qquad \boldsymbol{\omega} \geq 0. \end{aligned} \tag{C.3}$$

First, we can simplify (C.3) by considering the payoff functions:

$$\tilde{g}_1(X_T^x) = g_1(X_T^x) - (1+r)x, \tag{C.4}$$

$$\tilde{g}_2(X_T^x) = g_2(X_T^x) - (1+r)x. \tag{C.5}$$

Then, we can easily verify that

$$\begin{aligned} &\tilde{\mu}_1 = \mathbb{E}[\tilde{g}_1(X_T^x)] = \mu_1 - (1+r)x, \quad \tilde{\mu}_2 = \mathbb{E}[\tilde{g}_2(X_T^x)] = \mu_2 - (1+r)x, \\ &\tilde{\sigma}_1^2 = \mathrm{Var}[\tilde{g}_1(X_T^x)] = \sigma_1^2, \quad \tilde{\sigma}_2^2 = \mathrm{Var}[\tilde{g}_2(X_T^x)] = \sigma_2^2, \quad \tilde{\sigma}_{12} = \mathrm{Cov}[\tilde{g}_1(X_T^x), \tilde{g}_2(X_T^x)] = \sigma_{12}. \end{aligned}$$

So, the problem (C.3) can be expressed as follows,

$$\begin{aligned} &\max_{\boldsymbol{\omega}} \frac{\boldsymbol{\omega}'\tilde{\boldsymbol{\mu}}}{\sqrt{\boldsymbol{\omega}'\tilde{\Sigma}\boldsymbol{\omega}}} \\ &\text{subject to } \boldsymbol{\omega}'\mathbf{1} = 1, \\ &\qquad \boldsymbol{\omega} \geq 0, \end{aligned} \tag{C.6}$$

where

$$\tilde{\boldsymbol{\mu}} := \begin{bmatrix} \tilde{\mu}_1 \\ \tilde{\mu}_2 \end{bmatrix}, \quad \tilde{\Sigma} = \begin{bmatrix} \tilde{\sigma}_1 & \tilde{\sigma}_{12} \\ \tilde{\sigma}_{12} & \tilde{\sigma}_2 \end{bmatrix}. \tag{C.7}$$

Now, we can rewrite (C.6) as a standard quadratic programming problem if we assume $\tilde{\mu}_1 > 0$ and $\tilde{\mu}_2 > 0$. This is a natural assumption here because the investor anticipates a higher expected return than the risk-free rate when investing in the hedge fund. If we let $f(\boldsymbol{\omega})$ denote the objective function in (C.6), then it can be verified that for any real number $\lambda > 0$, $f(\boldsymbol{\omega}) = f(\lambda\boldsymbol{\omega})$. Thus, the problem (C.6) is equivalent to the optimization problem:

$$\begin{aligned}
&\max_{\boldsymbol{\omega}} \frac{1}{\sqrt{\boldsymbol{\omega}'\tilde{\Sigma}\boldsymbol{\omega}}}\\
&\text{subject to } \boldsymbol{\omega}'\tilde{\boldsymbol{\mu}} = 1,\\
&\boldsymbol{\omega} \geq 0.
\end{aligned} \tag{C.8}$$

Clearly, (C.8) can be rewritten as an equivalent minimization problem:

$$\begin{aligned}
&\min_{\boldsymbol{\omega}} \boldsymbol{\omega}'\tilde{\Sigma}\boldsymbol{\omega}\\
&\text{subject to } \boldsymbol{\omega}'\tilde{\boldsymbol{\mu}} = 1,\\
&\boldsymbol{\omega} \geq 0.
\end{aligned} \tag{C.9}$$

which is a standard quadratic programming problem.

Proof of Theorem 4.3.1:

Proof. By Best (2010, Chapter 9, Page 192), we can obtain the following optimality conditions for (C.9),

$$\begin{cases}
\boldsymbol{\omega}'\tilde{\boldsymbol{\mu}} = 1,\\
\mathbf{I}\lambda - \mathbf{2}\boldsymbol{\Sigma}\boldsymbol{\omega} = \nu\boldsymbol{\mu},\\
\lambda'\mathbf{I}\boldsymbol{\omega} = \mathbf{0},\\
\boldsymbol{\omega} \geq 0,\\
\lambda \geq 0.
\end{cases} \tag{C.10}$$

where ν is the multiplier for the constraint $\boldsymbol{\omega}'\tilde{\boldsymbol{\mu}} = 1$ and λ is the vector of multipliers for the constraints $\boldsymbol{\omega} \geq 0$. More explicitly, this leads to the linear system:

$$\omega_1\tilde{\mu}_1 + \omega_2\tilde{\mu}_2 = 1, \tag{C.11}$$

$$\lambda_1 - 2\omega_1\tilde{\sigma}_1^2 - 2\omega_2\tilde{\sigma}_{12} = \nu\tilde{\mu}_1, \tag{C.12}$$

$$\lambda_2 - 2\omega_1\tilde{\sigma}_{12} - 2\omega_2\tilde{\sigma}_2^2 = \nu\tilde{\mu}_2, \tag{C.13}$$

$$\lambda_1\omega_1 + \lambda_2\omega_2 = 0, \tag{C.14}$$

with constraints $\boldsymbol{\omega} \geq 0$ and $\lambda \geq 0$. The constraints imply the solution to (C.14) must satisfy one of the following three cases: (i) $\lambda_1 = 0$ and $\lambda_2 = 0$, (ii) $\lambda_1 > 0$ and $\lambda_2 = 0$, (iii) $\lambda_1 = 0$ and $\lambda_2 > 0$.

It is easy to see check that case (i) can be viewed as an optimization problem:

$$\min_{\boldsymbol{\omega}} \boldsymbol{\omega}'\tilde{\Sigma}\boldsymbol{\omega}$$
$$\text{Subject to } \boldsymbol{\omega}'\tilde{\boldsymbol{\mu}} = 1. \tag{C.15}$$

Let $\boldsymbol{\omega}^{**} = (w_1^{**}, w_2^{**})$ be the solution to (C.15). A simple calculation gives:

$$w_1^{**} = \frac{\tilde{\mu}_1\tilde{\sigma}_2^2 - \tilde{\mu}_2\tilde{\sigma}_{12}}{C^*}, \quad w_2^{**} = \frac{\tilde{\mu}_2\tilde{\sigma}_1^2 - \tilde{\mu}_1\tilde{\sigma}_{12}}{C^*}, \tag{C.16}$$

where $C^* = (\tilde{\mu}_1\tilde{\sigma}_2 - \tilde{\mu}_2\tilde{\sigma}_1)^2 + 2\tilde{\mu}_1\tilde{\mu}_2(\tilde{\sigma}_1\tilde{\sigma}_2 - \tilde{\sigma}_{12})$. This is the solution to (C.9) without the constraint $\boldsymbol{\omega} \geq 0$. Therefore, $\boldsymbol{\omega}^* = \boldsymbol{\omega}^{**}$ when $\boldsymbol{\omega}^{**}$ is a feasible solution to (C.9). Next, note that $C^* > 0$, so $\boldsymbol{\omega}^{**}$ is feasible for (C.9) if and only if $\tilde{\mu}_1\tilde{\sigma}_2^2 - \tilde{\mu}_2\tilde{\sigma}_{12} \geq 0$ and $\tilde{\mu}_2\tilde{\sigma}_1^2 - \tilde{\mu}_1\tilde{\sigma}_{12} \geq 0$. That is:

$$\tilde{\sigma}_{12} \leq \min\left\{\frac{\tilde{\mu}_1}{\tilde{\mu}_2}\tilde{\sigma}_2^2, \ \frac{\tilde{\mu}_2}{\tilde{\mu}_1}\tilde{\sigma}_1^2\right\}. \tag{C.17}$$

Next, if (C.17) is not satisfied, then the optimal solution is the solution in either case (ii) or case (iii). Clearly, case (ii) leads to $\omega_2 = 0$, and from (C.11), it is easy to calculate $\omega_1 = 1/\tilde{\mu}_1$. On the other hand, $\omega_1 = 0$ and $\omega_2 = 1/\tilde{\mu}_2$ is the solution for case (iii). Finally, we substitute the solutions in case (ii) and (iii) back into the objective function in (C.9) and compare their values to obtain the optimal solution. This yields $\boldsymbol{\omega}^* = (1/\tilde{\mu}_1, 0)$ if $\tilde{\sigma}_1^2/\tilde{\mu}_1^2 \leq \tilde{\sigma}_2^2/\tilde{\mu}_2^2$, otherwise, $\boldsymbol{\omega}^* = (0, 1/\tilde{\mu}_2)$. □

To investigate the nature of the investor's optimal strategy in terms of the original model parameters, we need to write $\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_{12}$ explicitly. Following Djerroud et al. (2016) we assume $m_1 = m_2 = m$. Introducing the notation:

$$\begin{aligned}
C_1 &:= \mathbb{E}[(X_T^x - mx - x)_+], & C_2 &:= \mathbb{E}[(X_T^x - mx - x)_+^2], \\
P_1 &:= \mathbb{E}[(x + mx - X_T^x)_+], & P_2 &:= \mathbb{E}[(x + mx - X_T^x)_+^2], \\
P_{1,c} &:= \mathbb{E}[((1-c)x + mx - X_T^x)_+], & P_{2,c} &:= \mathbb{E}[((1-c)x + mx - X_T^x)_+^2],
\end{aligned}$$

we obtain:

$$
\begin{aligned}
\tilde{\mu}_1 &= (1-\alpha_1)C_1 - P_1 - rx, && \text{(C.18)}\\
\tilde{\mu}_2 &= (1-\alpha_2)C_1 - P_{1,c} - rx, && \text{(C.19)}\\
\tilde{\sigma}_1^2 &= \mathrm{Var}[(1-\alpha_1)(X_T^x - mx - x)_+ - (x + mx - X_T^x)_+ - rx]\\
&= (1-\alpha_1)^2(C_2 - C_1^2) + P_2 - P_1^2 + 2(1-\alpha_1)C_1P_1, && \text{(C.20)}\\
\tilde{\sigma}_2^2 &= \mathrm{Var}[(1-\alpha_2)(X_T^x - mx - x)_+ - ((1-c)x + mx - X_T^x)_+ - rx]\\
&= (1-\alpha_2)^2(C_2 - C_1^2) + P_{2,c} - P_{1,c}^2 + 2(1-\alpha_2)C_1P_{1,c}. && \text{(C.21)}\\
\tilde{\sigma}_{12} &= \mathbb{E}[\tilde{g}_1(X_T^x)\tilde{g}_2(X_T^x)] - \tilde{\mu}_1\tilde{\mu}_2\\
&= (1-\alpha_1)(1-\alpha_2)C_2 - (1-\alpha_1)rxC_1 - (1-\alpha_2)rxC_1 + P_{2,c} + cxP_{1,c}\\
&\quad + rxP_1 + rxP_{1,c} + r^2x^2 - \tilde{\mu}_1\tilde{\mu}_2. && \text{(C.22)}
\end{aligned}
$$

Hence, SR_1, SR_2 and ρ can be obtained. Next, Recalling that $\tilde{\mu}_1 > 0$ and $\tilde{\mu}_2 > 0$, we can obtain the valid ranges for α_1 and α_2 from (C.18) and (C.19):

$$
\alpha_1 < \frac{C_1 - P_1 - rx}{C_1} := \alpha_1^*, \quad \alpha_2 < \frac{C_1 - P_{1,c} - rx}{C_1} := \alpha_2^*. \tag{C.23}
$$

By (C.23) and noting that $P_1 \geq P_{1,c}$, we can easily deduce that $\alpha_1^* \leq \alpha_2^*$. This is reasonable, because the first-loss fee structure provides downside protection for the investor. In return, the investor can tolerate a higher performance fee.

C.2 Sortino Ratio Maximization

From (4.8) and noting that $r(T) = g(X_T^x)/x - 1$, we can easily obtain

$$
\begin{aligned}
\sigma_d^2 &= \mathbb{E}[\min\{r(T) - l, 0\}^2]\\
&= \mathbb{E}[\min\{\frac{g(X_T^x)}{x} - 1 - l, 0\}^2]\\
&= x^{-2}\mathbb{E}[\min\{g(X_T^x) - (1+l)x, 0\}^2]. && \text{(C.24)}
\end{aligned}
$$

As a result, we have

$$
SOR(\boldsymbol{\omega}) = \frac{\mathbb{E}[\frac{g(X_T^x)}{x}] - l}{\sigma_d} = \frac{\mathbb{E}[g(X_T^x)] - (1+l)x}{\sqrt{\mathbb{E}[\min\{g(X_T^x) - (1+l)x, 0\}^2]}}. \tag{C.25}
$$

Recall that $g(X_T^x) = \omega_1 g_1(X_T^x) + \omega_2 g_2(X_T^x)$ and let

$$\tilde{g}_{1,l}(X_T^x) = g_1(X_T^x) - (1+l)x \quad \text{and} \quad \tilde{g}_{2,l}(X_T^x) = g_2(X_T^x) - (1+l)x.$$

Thus, we can further simplify equation (C.25) as follows,

$$SOR(\boldsymbol{\omega}) = \frac{\mathbb{E}[\tilde{g}_l(X_T^x)]}{\sqrt{\mathbb{E}[\min\{\tilde{g}_l(X_T^x), 0\}^2]}}, \tag{C.26}$$

where $\tilde{g}_l(X_T^x) = \omega_1 \tilde{g}_{1,l}(X_T^x) + \omega_2 \tilde{g}_{2,l}(X_T^x)$. Now, introduce the notation:

$$\begin{aligned}
&\mu_{1,l} := \mathbb{E}[\tilde{g}_{1,l}(X_T^x)], \quad \mu_{1,l} := \mathbb{E}[\tilde{g}_{2,l}(X_T^x)], \\
&\sigma_{1,l}(\boldsymbol{\omega}) := \mathbb{E}[\tilde{g}_{1,l}(X_T^x)^2 \mathbf{1}_{\{\tilde{g}_l(X_T^x) \le 0\}}], \\
&\sigma_{2,l}(\boldsymbol{\omega}) := [\tilde{g}_{2,l}(X_T^x)^2 \mathbf{1}_{\{\tilde{g}_l(X_T^x) \le 0\}}], \\
&\sigma_{12,l}(\boldsymbol{\omega}) := \mathbb{E}[\tilde{g}_{1,l}(X_T^x) \tilde{g}_{2,l}(X_T^x) \mathbf{1}_{\{\tilde{g}_l(X_T^x) \le 0\}}],
\end{aligned} \tag{C.27}$$

and note that

$$\begin{aligned}
\mathbb{E}[\min\{\tilde{g}_l(X_T^x), 0\}^2] =& \mathbb{E}[\tilde{g}_l(X_T^x)^2 \mathbf{1}_{\{\tilde{g}_l(X_T^x) \le 0\}}] \\
=& \mathbb{E}[(\omega_1 \tilde{g}_{1,l}(X_T^x) + \omega_2 \tilde{g}_{2,l}(X_T^x))^2 \mathbf{1}_{\{\tilde{g}_l(X_T^x) \le 0\}}] \\
=& \omega_1^2 \mathbb{E}[\tilde{g}_{1,l}(X_T^x)^2 \mathbf{1}_{\{\tilde{g}_l(X_T^x) \le 0\}}] \\
& + 2\omega_1 \omega_2 \mathbb{E}[\tilde{g}_{1,l}(X_T^x) \tilde{g}_{2,l}(X_T^x) \mathbf{1}_{\{\tilde{g}_l(X_T^x) \le 0\}}] \\
& + \omega_2^2 \mathbb{E}[\tilde{g}_{2,l}(X_T^x)^2 \mathbf{1}_{\{\tilde{g}_l(X_T^x) \le 0\}}].
\end{aligned} \tag{C.28}$$

We can rewrite (C.26) as

$$SOR(\boldsymbol{\omega}) = \frac{\omega_1 \mu_{1,l} + \omega_2 \mu_{1,l}}{\omega_1^2 \sigma_{1,l}(\boldsymbol{\omega}) + 2\omega_1 \omega_2 \sigma_{12,l}(\boldsymbol{\omega}) + \omega_2^2 \sigma_{2,l}(\boldsymbol{\omega})}. \tag{C.29}$$

Similar to the Sharpe Ratio maximization framework in the previous section, the investor's goal is to maximize $SOR(\boldsymbol{\omega})$ at maturity T. In general, the expression (C.29) is difficult to optimize analytically.

C.3 Derivation of Explicit Formulas

Assuming $m_1 = m_2 = m$, we rewrite the investor payoffs $\tilde{g}_1(X_T^x)$ and $\tilde{g}_2(X_T^x)$ in the following more compact forms:

$$\begin{aligned}\tilde{g}_1(X_T^x) =& X_T^x - mx - \alpha_1(X_T^x - mx - x)_+ - (1+r)x.\\ \tilde{g}_2(X_T^x) =& X_T^x - mx - \alpha_2(X_T^x - mx - x)_+ + (x + mx - X_T^x)_+\\ &- ((1-c)x + mx - X_T^x)_+ - (1+r)x.\end{aligned}$$

Then, using the equality $X_T^x - mx - x = (X_T^x - mx - x)_+ - (x + mx - X_T^x)_+$, we obtain that:

$$\tilde{g}_1(X_T^x) =(1-\alpha_1)(X_T^x - mx - x)_+ - (x + mx - X_T^x)_+ - rx, \tag{C.30}$$

$$\tilde{g}_2(X_T^x) =(1-\alpha_2)(X_T^x - mx - x)_+ - ((1-c)x + mx - X_T^x)_+ - rx. \tag{C.31}$$

The expressions for $\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\sigma}_1^2$, and $\tilde{\sigma}_2^2$ then follow immediately. Moreover, noting that $(x+mx-X_T^x)_+((1-c)x+mx-X_T^x)_+ = ((1-c)x+mx-X_T^x)_+^2 + cx((1-c)x+mx-X_T^x)_+$ yields:

$$\begin{aligned}\tilde{\sigma}_{12} &= \mathbb{E}[\tilde{g}_1(X_T^x)\tilde{g}_2(X_T^x)] - \tilde{\mu}_1\tilde{\mu}_2\\ &=(1-\alpha_1)(1-\alpha_2)C_2 - (1-\alpha_1)rxC_1 - (1-\alpha_2)rxC_1 + P_{2,c} + cxP_{1,c}\\ &\quad + rxP_1 + rxP_{1,c} + r^2x^2 - \tilde{\mu}_1\tilde{\mu}_2,\end{aligned} \tag{C.32}$$

$$\begin{aligned}\tilde{g}_{1,l}(X_T^x) =& X_T^x - mx - \alpha_1(X_T^x - mx - x)_+ - (1+l)x,\\ \tilde{g}_{2,l}(X_T^x) =& X_T^x - mx - \alpha_2(X_T^x - mx - x)_+ + (x + mx - X_T^x)_+\\ &- ((1-c)x + mx - X_T^x)_+ - (1+l)x.\end{aligned}$$

Similarly, we can write $\tilde{g}_{1,l}(X_T^x)$ and $\tilde{g}_{2,l}(X_T^x)$ as follows,

$$\tilde{g}_{1,l}(X_T^x) =(1-\alpha_1)(X_T^x - mx - x)_+ - (x + mx - X_T^x)_+ - lx, \tag{C.33}$$

$$\tilde{g}_{2,l}(X_T^x) =(1-\alpha_2)(X_T^x - mx - x)_+ - ((1-c)x + mx - X_T^x)_+ - lx. \tag{C.34}$$

It is easy to check $\tilde{g}_l(X_T^x) \le 0 \Rightarrow \omega_1 \tilde{g}_{1,l}(X_T^x) + \omega_2 \tilde{g}_{2,l}(X_T^x) \le 0$, which implies that

$$\begin{aligned}&(\omega_1(1-\alpha_1) + \omega_2(1-\alpha_2))\,(X_T^x - mx - x)_+ \\ &\le \omega_1(x + mx - X_T^x)_+ + \omega_2((1-c)x + mx - X_T^x)_+ + lx.\end{aligned} \tag{C.35}$$

When $X_T^x \le (1+m)x$, the inequality (C.35) always holds. On the other hand, when $X_T^x > (1+m)x$, we can easily obtain

$$\begin{aligned}&(\omega_1(1-\alpha_1) + \omega_2(1-\alpha_2))\,(X_T^x - mx - x) \le lx \\ &\Rightarrow X_T^x \le \frac{l}{\omega_1(1-\alpha_1) + \omega_2(1-\alpha_2)} x + (1+m)x = (1+a+m)x,\end{aligned} \tag{C.36}$$

where $a = l/\omega_1(1-\alpha_1) + \omega_2(1-\alpha_2)$. Therefore, we have that $\tilde{g}_l(X_T^x) \le 0 \iff X_T^x \le (1+a+m)x$. It follows that

$$\begin{aligned}\mathbb{E}[\min\{\tilde{g}_l(X_T^x), 0\}^2] =& \mathbb{E}[\tilde{g}_l(X_T^x)^2 \mathbf{1}_{\{\tilde{g}_l(X_T^x) \le 0\}}] \\ =& \mathbb{E}[(\omega_1 \tilde{g}_{1,l}(X_T^x) + \omega_2 \tilde{g}_{2,l}(X_T^x))^2 \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}] \\ =& \omega_1^2 \mathbb{E}[\tilde{g}_{1,l}(X_T^x)^2 \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}] \\ &+ 2\omega_1\omega_2 \mathbb{E}[\tilde{g}_{1,l}(X_T^x)\tilde{g}_{2,l}(X_T^x) \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}] \\ &+ \omega_2^2 \mathbb{E}[\tilde{g}_{2,l}(X_T^x)^2 \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}].\end{aligned} \tag{C.37}$$

Introducing the notation:

$$\begin{aligned}C_0^l &:= \mathbb{E}[\mathbf{1}_{\{X_T^x \le (1+a+m)x\}}], \\ C_1^l &:= \mathbb{E}[(X_T^x - mx - x)_+ \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}], \\ C_2^l &:= \mathbb{E}[(X_T^x - mx - x)_+^2 \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}], \\ P_{1,c}^l &:= \mathbb{E}[((1-c)x + mx - X_T^x)_+ \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}], \\ P_{2,c}^l &:= \mathbb{E}[((1-c)x + mx - X_T^x)_+^2 \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}],\end{aligned}$$

by (C.33) and (C.34), we can obtain

$$
\begin{aligned}
\mu_{1,l} =&(1-\alpha_1)C_1 - P_1 - lx, \quad \mu_{1,l} = (1-\alpha_2)C_1 - P_{1,c} - lx,\\
\sigma_{1,l}(\boldsymbol{\omega}) =&\mathbb{E}[\tilde{g}_{1,l}(X_T^x)^2 \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}]\\
=&\mathbb{E}[((1-\alpha_1)(X_T^x - mx - x)_+ - (x+mx-X_T^x)_+ - lx)^2 \, \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}].\\
=&(1-\alpha_1)^2 C_2^l + P_{2,0}^l + l^2x^2C_0^l - 2(1-\alpha_1)lxC_1^l + 2lxP_{1,0}^l.\\
\sigma_{2,l}(\boldsymbol{\omega}) =&\mathbb{E}[\tilde{g}_{2,l}(X_T^x)^2 \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}]\\
=&\mathbb{E}[((1-\alpha_2)(X_T^x - mx - x)_+ - ((1-c)x+mx-X_T^x)_+ - lx)^2 \, \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}]\\
&= (1-\alpha_2)^2 C_2^l + P_{2,c}^l + l^2x^2C_0^l - 2(1-\alpha_2)lxC_1^l + 2lxP_{1,c}^l\\
\sigma_{12,l}(\boldsymbol{\omega}) =&\mathbb{E}[\tilde{g}_{1,l}(X_T^x)\tilde{g}_{2,l}(X_T^x)\mathbf{1}_{\{X_T^x \le (1+a+m)x\}}]\\
&= (1-\alpha_1)(1-\alpha_2)C_2^l - (1-\alpha_1)lxC_1^l - (1-\alpha_2)lxC_1^l + P_{2,c}^l + cxP_{1,c}^l\\
&\quad + lxP_{1,0}^l + lxP_{1,c}^l + l^2x^2C_0^l.
\end{aligned}
$$

In order to derive explicit formulas for $C_i, P_i, P_{i,c}, P_{i,c}^l$, $i = 1,2$ and C_j^l, $j = 0,1,2$, we need the following result.

Proposition C.3.1. *Let $X_t^x = x\exp\{(\mu - \frac{1}{2}\sigma^2)t - \sigma W_t\}$. Then, for $0 \le a \le b$,*

$$
\mathbb{E}[(X_t^x)^2 \mathbf{1}_{\{a < X_t^x < b\}}] = x^2 e^{2\mu t + \sigma^2 t}(\Phi(\tilde{d}_1(x,a,t)) - \Phi(\tilde{d}_1(x,b,t))). \tag{C.38}
$$

Proof. Let $Z = t^{-1/2}W_t \sim N(0,1)$,

$$
d_1(x,y,t) = \frac{\log(x/y) + (\mu + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}, \quad d_2(x,y,t) = d_1(x,y,t) - \sigma\sqrt{t}.
$$

Then

$$
\begin{aligned}
&\log(a/x) < (\mu - \sigma^2/2) - \sigma\sqrt{t}Z < \log(a/x)\\
\iff& \frac{\log(X_t/b) + (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} < Z < \frac{\log(x/a) + (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\\
\iff& d_2(x,b,t) < Z < d_2(x,a,t).
\end{aligned} \tag{C.39}
$$

$$
\begin{aligned}
\mathbb{E}[(X_t^x)^2 \mathbf{1}_{\{a<X_t^x<b\}}] &= \int_{d_2(x,b,t)}^{d_2(x,a,t)} x^2 \exp\{2(\mu - \frac{1}{2}\sigma^2)t - 2\sigma\sqrt{t}Z\} \exp\{-\frac{1}{2}z^2\} dz \\
&= x^2 e^{2\mu t} \int_{d_2(x,b,t)}^{d_2(x,a,t)} \exp\{-\frac{1}{2}(z^2 + 4\sigma\sqrt{t}z + 2\sigma^2 t)\} dz \\
&= x^2 e^{2\mu t} \int_{d_2(x,b,t)}^{d_2(x,a,t)} \exp\{-\frac{1}{2}(z + 2\sigma\sqrt{t})^2\} \exp\{\sigma^2 t\} dz \\
&= x^2 e^{2\mu t + \sigma^2 t} \int_{d_2(x,b,t)+2\sigma\sqrt{t}}^{d_2(x,a,t)+2\sigma\sqrt{t}} \exp\{-\frac{1}{2}y^2\} dy \\
&= x^2 e^{2\mu t + \sigma^2 t} (\Phi(\tilde{d}_1(x,a,t)) - \Phi(\tilde{d}_1(x,b,t))),
\end{aligned}
$$

where

$$
\tilde{d}_1(x,y,t) = \frac{\log(x/y) + (\mu + \frac{3}{2}\sigma^2)t}{\sigma\sqrt{t}} \tag{C.40}
$$

□

To simplify notation, we define

$$
\begin{aligned}
\tilde{d}_1(x,bx,T) &:= \tilde{d}_1(b), & d_1(x,bx,T) &:= d_1(b), \\
d_2(x,bx,T) &:= d_2(b).
\end{aligned}
$$

Then, we can explicitly write C_0^l, C_1, C_1^l, C_2, C_2^l, P_1, $P_{1,0}^l$, P_2, $P_{2,0}^l$, $P_{1,c}$, $P_{1,c}^l$, $P_{2,c}$ and $P_{2,c}^l$

as follows,

$$
\begin{aligned}
C_0^l &= \mathbb{E}[\mathbf{1}_{\{X_T^x \le (1+a+m)x\}}] = \Phi(-d_2((1+m+a))) \\
C_1 &= \mathbb{E}[(X_T^x - x - mx)_+] = xe^{\mu T}\Phi(d_1(1)) - (1+m)x\Phi(d_2(1)), \\
C_1^l &= \mathbb{E}[(X_T^x - mx - x)_+ \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}] \\
&= xe^{\mu T}(\Phi(d_1((1+m)) - \Phi(d_1((1+m+a))) + (1+m)x(\Phi(d_2((1+m)) - \Phi(d_2((1+m+a))) \\
C_2 &= \mathbb{E}[(X_T^x - x - mx)_+^2] \\
&= x^2 e^{2\mu T + \sigma^2 T}\Phi(\tilde{d}_1(1)) - 2(1+m)x^2 e^{\mu T}\Phi(d_1(1)) + (1+m)^2 x^2 \Phi(d_2(1)), \\
C_2^l &= \mathbb{E}[(X_T^x - mx - x)_+^2 \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}] \\
&= x^2 e^{2\mu T + \sigma^2 T}(\Phi(\tilde{d}_1((1+m)) - \Phi(\tilde{d}_1((1+m+a))) \\
&\quad - 2(1+m)x^2 e^{\mu T}(\Phi(d_1((1+m)) - \Phi(d_1((1+m+a)))) \\
&\quad + (1+m)^2 x^2(\Phi(d_2((1+m)) - \Phi(d_2((1+m+a))) \\
P_1 &= P_{1,0} = (1+m)x\Phi(-d_2(1)) - xe^{\mu T}\Phi(-d_1(1)), \\
P_{1,0}^l &= P_{1,0} = (1+m)x\Phi(-d_2(1)) - xe^{\mu T}\Phi(-d_1(1)) \\
P_2 &= P_{2,0} = (1+m)^2 x^2 \Phi(-d_2(1)) - 2(1+m)x^2 e^{\mu T}\Phi(-d_1(1)) + x^2 e^{2\mu T + \sigma^2 T}\Phi(-\tilde{d}_1(1)), \\
P_{2,0}^l &= P_{2,0} = (1+m)^2 x^2 \Phi(-d_2(1)) - 2(1+m)x^2 e^{\mu T}\Phi(-d_1(1)) + x^2 e^{2\mu T + \sigma^2 T}\Phi(-\tilde{d}_1(1)), \\
P_{1,c} &= \mathbb{E}[((1-c)x + mx - X_T^x)_+] \\
&= (1-c+m)x\Phi(-d_2(1-c)) - xe^{\mu T}\Phi(-d_1(1-c)), \\
P_{1,c}^l &= P_{1,c} \\
&= (1-c+m)x\Phi(-d_2(1-c)) - xe^{\mu T}\Phi(-d_1(1-c)), \\
P_{2,c} &= \mathbb{E}[((1-c)x + mx - X_T^x)_+^2] \\
&= (1-c+m)^2 x^2 \Phi(-d_2(1-c)) - 2(1-c+m)x^2 e^{\mu T}\Phi(-d_1(1-c)) + x^2 e^{2\mu T + \sigma^2 T}\Phi(-\tilde{d}_1(1-c)). \\
P_{2,c}^l &= P_{2,c} \\
&= (1-c+m)^2 x^2 \Phi(-d_2(1-c)) - 2(1-c+m)x^2 e^{\mu T}\Phi(-d_1(1-c)) + x^2 e^{2\mu T + \sigma^2 T}\Phi(-\tilde{d}_1(1-c)).
\end{aligned}
$$

C.4 Derivation of Risk-Neutral Indifferent Performance Fee

Let

$$V_1 = \mathbb{E}_{\mathbb{Q}}[e^{-rT} g_1(X_T^x)] \tag{C.41}$$

be the risk-neutral value of the investor for the traditional fee structure, and let

$$V_2 = \mathbb{E}_{\mathbb{Q}}[e^{-rT} g_2(X_T^x)] \tag{C.42}$$

be the risk-neutral value of the investor for the first-loss fee structure. A simple derivation yields

$$g_1(X_T^x) = (1-\alpha_1)(X_T^x - mx - x)_+ - (x + mx - X_T^x)_+ + x, \tag{C.43}$$

$$g_2(X_T^x) = (1-\alpha_2)(X_T^x - mx - x)_+ - ((1-c)x + mx - X_T^x)_+ + x. \tag{C.44}$$

Then, the Black-Scholes formula can be used to derive V_1 and V_2 as follows:

$$V_1 = (1-\alpha_1)C(x,(1+m)x) - P(x,(1+m)x) + e^{-rT}x, \tag{C.45}$$

$$V_2 = (1-\alpha_2)C(x,(1+m)x) - P(x,(1-cx+m)x) + e^{-rT}x, \tag{C.46}$$

where $C(x, K)$ is the Black-Scholes price of a call option on a non-dividend paying asset with current value of the underlying x and strike price K, and $P(x, K)$ is the Black-Scholes put option price with the same parameters as arguments.

Finally, we can set $V_1 = V_2$. Then, a risk-neutral indifferent performance fee α_2 can be obtained:

$$\begin{aligned}
&(1-\alpha_1)C(x,(1+m)x) - P(x,(1+m)x) + e^{-rT}x \\
&= (1-\alpha_2)C(x,(1+m)x) - P(x,(1-cx+m)x) + e^{-rT}x \\
&\Rightarrow (\alpha_2-\alpha_1)C(x,(1+m)x) = P(x,(1+m)x) - P(x,(1-cx+m)x) \\
&\alpha_2 = \alpha_1 + \frac{P(x,(1+m)x) - P(x,(1-cx+m)x)}{C(x,(1+m)x)}.
\end{aligned} \tag{C.47}$$