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On the Road to Making Science of "Art": Risk Bias in Market Scoring Rules

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Abstract: We study market scoring rules (MSRs) prediction markets in the presence of risk averse or risk seeking agents that have unknown, yet bounded risk preferences. It is well known that if agents can be pre-screened, then MSRs can be corrected to elicit agent's beliefs. However, agents cannot always be screened, and instead, an online MSR mechanism is needed. We show that agents' submitted reports always deviate from their beliefs, unless their beliefs are identical to the current market estimate. This means, in most cases it is impossible for a MSR prediction market to elicit an individual agent's exact belief. To analyze this issue, we introduce a measure to calculate the deviation between an agent's reported belief, and personal belief. We further derive the necessary and sufficient conditions for a MSR to yield a lower deviation relative to another MSR. We find that the deviation of a MSR prediction market is related to the liquidity provided in the MSR's corresponding cost-function prediction market. We use the relation between deviation and liquidity to present a systematic approach to help determine the amount of liquidity required for cost-function prediction markets, an activity that up to this point has been described as "art" in the literature.

Key words: probability forecast, decision analysis, prediction markets, market scoring rules, risk aversion, cost function market makers, market depth, market liquidity

1. Introduction

Prediction Markets are a management tool used in firms to elicit employees' beliefs on the outcome of a future event by incentivizing employees to provide probability estimates. Schlack (2015), Filios (2016) argue that prediction markets are a form of predictive analytics for scenarios with no historical data available. Prediction markets have successfully been used in project management (Ortner 1998), new product sales and development (Plott and Chen 2002) and disease spread forecasting (Polgreen et al. 2007), just to name a few examples. Prediction markets are also used privately within corporations. Prediction markets appear in corporations like General Electric, Google, Hewlett-Packard, IBM, Intel, Microsoft, Siemens, and Yahoo! (Arrow et al. 2008). However, in recent times a number of popular media outlets are stating that prediction markets are failing (Ledbetter 2008, D.R. 2016). Besides the natural argument that prediction markets provide probability estimates on the event, and not a deterministic estimate, popular media provides arguments such as cognitive biases, thin markets, low stakes, and slow to react to events as reasons behind prediction market failure. These may all be viable reasons, but one reason not mentioned is risk bias that induces rational individuals to change their behavior (i.e., reported probability estimates) in prediction markets due to their risk preferences.

In subsidized prediction markets, those markets used in corporate settings, risk bias may arise due to the small number of participants[1]. The impact of risk bias is studied by Dimitrov et al. (2015), in which the authors show that the reward given in a subsidized prediction market must decrease exponentially. However, the results of Dimitrov et al. (2015) are contingent on agents' having unbounded risk preferences, such as an agent who prefers receiving \$0.01 for certain to a lottery that pays \$1,000,000 with probability 0.99 and \$0 otherwise. Such agents are not observed in practice (Cox and Harrison 2008), and there is often a bound on the maximum risk-averse and risk-seeking preferences of a population. In this paper, we study the impact of risk bias on agents' behavior in subsidized prediction markets encountered in practice known as strictly proper market scoring rule prediction markets, when a population's risk preferences are unknown yet bounded. Sethi and Vaughan (2016) show that the price of a subsidized prediction market with risk-averse traders who are budget-limited is a good approximation of the agents' aggregated belief. The result of Sethi and Vaughan (2016) suggests that characteristics of a subsidized prediction market, namely liquidity, can be adjusted such that the market organizer can make better inferences about agents' beliefs. Abernethy et al. (2014) analytically draw the connection between a risk-averse agent, with an exponential utility function, and exponential belief distribution, and a subsidized prediction market's prices. In their work, Abernethy et al. (2014) show how a market's liquidity can be adjusted to alter the market's belief elicitation. Abernethy et al. (2014) use inverse liquidity to measure a market's liquidity. A recent paper by Dudík et al. (2017) considers agents with exponential utilities and beliefs drawn from an exponential distribution and characterizes four components of error (difference between market price and ground truth). By definition, inverse liquidity can only compare the provided liquidity of different market makers from the same family as defined in Section 2.1. The work of Sethi and Vaughan (2016), Abernethy et al.

[1] Corporate prediction markets are subsidized to circumvent no-trade situations that exist when there are few traders in such markets.

(2014), and Dudík et al. (2017) attempt to bridge the gap in the interpretation of prediction markets' prices in terms of agents' beliefs, and the characteristics of subsidized prediction markets. In our work, we extend this stream of literature in five ways.

Our five extensions of the current literature are as follow. First, we do not make any assumption on the agents' belief distribution, for example the work of Abernethy et al. (2014) assumed agents' beliefs are exponentially distributed. Second, we consider both risk-averse and risk-seeking individuals in our study, not only risk-averse as is currently assumed. Third, we do not make any assumption about the type of risk-aversion or risk-seeking utility of agents other than the facts that: an agent cannot be risk-averse and risk-seeking at the same time, and the Arrow-Pratt (Pratt 1964) measure of risk-aversion/risk-seeking preference of agents is bounded above and below. Our setting is an extension of those considered in the literature that in the most general case, only considers risk-averse agents (Sethi and Vaughan 2016). Fourth, we use two measures of liquidity, inverse liquidity and market depth, to show that our results are consistent regardless of how the liquidity is measured, this allows our analysis to be carried across different prediction markets from varying market maker families, while current literature only considers inverse liquidiy (Abernethy et al. 2014). Last, we do not consider bounded-budget agents, and therefore the behavior of an agent considered in our study is the same behavior exhibited by the same agent in the presence of learning, as considered by Abernethy et al. (2014), Sethi and Vaughan (2016).

Lambert (2011), Schlag and van der Weele (2013) show that no deterministic scoring rule can truthfully elicit the beliefs of agents with unknown risk preferences. For completeness, we show that the same result holds for deterministic prediction market mechanism, that is, no deterministic prediction market mechanism can truthfully elicit the beliefs of agents with unknown, yet bounded, risk preferences. Fortunately, a group of papers shows that if probabilistic payment mechanisms are used instead of deterministic mechanisms (all market scoring rule and cost function subsidized prediction markets are deterministic mechanisms), then incentive compatibility may be restored in scoring rules (Allen 1987, Karni 2009). Unfortunately, in practice, individuals inherently dislike probabilistic payments (Wakker et al. 1997), and we are not aware of any probabilistic prediction market mechanisms in use today. It is results such as Wakker et al. (1997) that motivate us to only consider deterministic subsidized prediction market mechanisms in our paper.

In this article, we are particularly interested in the impact of risk bias in corporate prediction markets in which the number of traders may be low (Cowgill and Zitzewitz 2015), also known as thin markets. We cannot eliminate risk bias in corporate prediction markets, and thin markets in general, using deterministic mechanisms; therefore, our only alternative is to minimize risk bias in subsidized corporate prediction markets. As we consider corporate prediction markets for the remainder of the paper, we will use the term prediction markets to reference corporate or thin markets. Risk bias, as measured by *deviation*, the absolute difference in agent belief and report, may lead a market to converge to a market price not indicative of agents' beliefs. One way to combat risk bias is to reduce market liquidity, making the potential loss of any one agent smaller than when liquidity is higher, thereby making it more likely for even risk-averse agents to report their true beliefs. Low liquidity, however, may result in large price oscillations. As such, finding the maximum liquidity parameter possible without changing the worst-case deviation is of interest to any corporate market organizer. In fact, we find the analytical relationship that exists between risk bias and market liquidity for the subsidized prediction markets we consider.

With the analytical relationship between market liquidity and maximum agent deviation, we address a practical problem that has plagued subsidized prediction markets for years: "how much liquidity should be provided in a subsidized market?" This question has until now been answered with what is described by some prediction market researchers as the "art" of prediction markets (Pennock 2010). Our results enable practitioners to move out of the realm of "art" and into the realm of science, by carefully trading off market liquidity with the maximum agent deviation. With a bound on a population's risk preferences (Babcock et al. 1993) and our analytical results, we solve a series of mathematical programs for the Logarithmic MSR (LMSR), Quadratic MSR (QMSR), and Spherical MSR (SMSR) subsidized prediction markets, the type of markets used most frequently in practice, to determine the trade-off made in setting the market liquidity and the maximum agent deviation. It is this trade-off that a market organizer, the individual interested in running a market, must make when designing the market. We note that our results generalize to more than just the LMSR, QMSR, and SMSR prediction markets, and may be applied to any strictly proper MSR prediction market.

The main contribution of our study, considering binary events, are as follows: 1) We introduce the notion of an agent's deviation in a prediction market. 2) We present the necessary and sufficient conditions of the structural properties of two MSRs such that one MSR yields lower deviation relative to another MSR for all agents in a population. 3) We present the relationship between deviation and liquidity for cost-function market makers, defined in Section 2.3, using two different measures of liquidity, namely inverse liquidity and market depth (Chen and Pennock 2007, Abernethy et al. 2014). 4) We show that for each MSR in a family of MSRs, with LMSRs included, higher (lower) deviation implies higher (lower) liquidity. 5) Using our derived relationship between deviation and liquidity, we present an optimization problem one may use to determine the desired liquidity when running a subsidized prediction market.

Managers, practitioners, and prediction market designers will find this work valuable in that we propose a systematic method to set up prediction market mechanisms that elicit probability estimates closer to agents' beliefs than those used in practice today, and gain a new understanding of the relationship between agents' deviation and prediction markets' liquidity. More practically, our results on the LMSR may be used for practitioners to set their market depth to their liking, knowing the maximum deviation that they may expect from a population of agents.

2. Related Work and Background

There is a broad range of literature on prediction markets' accuracy, information elicitation and information aggregation[2]. Prediction markets' performance and information aggregation capability is widely studied in application areas such as politics (Mellers et al. 2015, Atanasov et al. 2016, Chen et al. 2008, 2004), economics (Ostrovsky 2012), finance (Bossaerts et al. 2002, Palfrey and Wang 2012), health (Polgreen et al. 2007), and corporations (Cowgill and Zitzewitz 2015, Csaszar and Eggers 2013, Arrow et al. 2008, Healy et al. 2010, Berg et al. 2009). In this section, we present the related work and background on MSR prediction markets, and discuss the literature on risk in MSR prediction markets. We start with scoring rules in Section 2.1 and explain how MSRs are derived from scoring rules. A summary of the literature on risk in MSRs is presented in Section 2.2. In Section 2.3, we present the related work on cost-function market makers, another type of prediction market market makers that are closely related to MSRs, along with most recent findings involving MSR prediction markets and liquidity. We then formulate our base model in Section 2.5.

2.1. Scoring Rules and Market Scoring Rules

Brier (1950) introduced the *quadratic proper scoring rule* as a reward mechanism designed to induce truthful reporting from weather forecasters. Proper scoring rules were later generalized to include a larger class of functions applied to subjective probability elicitation (Good 1952, Winkler 1969, Savage 1971, McCarthy 1956). Formally, a *scoring rule* is a function $S(\cdot):\Delta_{|\Omega|-1}\times\Omega\mapsto\mathbb{R}$, where Ω is the discrete outcome space of a future event represented by a random variable ω, and $\Delta_{|\Omega|-1}$ is the $|\Omega|-1$-simplex. Consider a prediction market designed to elicit a set of myopic, expected-utility-maximizing agents' beliefs on $\omega\in\Omega$. An incentive compatible scoring rule is called a *proper* scoring rule. In other words, a scoring rule $S(\cdot)$ is proper for a risk neutral agent, when the following is satisfied:

$$\forall \mathbf{q}\in\Delta_{|\Omega|-1}: E_{\mathbf{p}}[S(\mathbf{p},\omega)]\geq E_{\mathbf{p}}[S(\mathbf{q},\omega)], \tag{1}$$

[2] Tziralis and Tatsiopoulos (2012) provide a literature review on prediction markets' history and success, as well as challenges that prediction markets face.

where $\mathbf{p}$ is the individual forecaster's belief on ω, and

$$E_{\mathbf{p}}[S(\mathbf{q},\omega)] \triangleq \sum_{\omega=1}^{\omega=|\Omega|} q_\omega S(\mathbf{q},\omega), \tag{2}$$

is the expected score of reporting a feasible report of $\mathbf{q}$. A scoring rule is called *strictly proper* when it satisfies (1) strictly, whenever $\mathbf{q} \neq \mathbf{p}$.

For example, the following scoring rule, called the Quadratic scoring rule (Brier 1950), is strictly proper:

$$S(\mathbf{q},\omega) = 2q_\omega - \sum_{\omega\in\Omega} q_\omega^2.$$

Another example of a strictly proper scoring rule is Spherical scoring rule Gneiting and Raftery (2007), Savage (1971), McCarthy (1956)

$$S(\mathbf{q},\omega) = \frac{q_\omega}{\sqrt{\sum_{i\in\Omega} q_i^2}}$$

Another popular class of strictly proper scoring rule is logarithmic scoring rule (Savage 1971, McCarthy 1956), defined as follows:

$$S(\mathbf{q},\omega) = \ln\left(q_\omega\right).$$

We can verify that if the score function $S(\cdot)$ is strictly proper, then the function $b\,S(\cdot)$ is also strictly proper for any given positive scalar b[3]. We say two scoring functions are from the same *family*, if one is a positive scalar of the other.

MSRs are derived from proper scoring rules and are used to elicit the belief of each individual in a group and *aggregate* the group's beliefs into a single estimate (Hanson 2003). A MSR takes two sequential reports, $\mathbf{q}^{(t)}$ and $\mathbf{q}^{(t-1)}$, and the observed outcome to determine the score of each agent's report. Similar to Hanson (2003), we define the MSR functions as follows: an agent who reports $\mathbf{q}^{(t)}$, at step $t \geq 1$, will receive:

$$\mathrm{MSR}(\mathbf{q}^t, \mathbf{q}^{t-1}, \omega) = S(\mathbf{q}^t, \omega) - S(\mathbf{q}^{t-1}, \omega), \tag{3}$$

where $S(\cdot)$ is a proper scoring rule and $\mathbf{q}^{(t-1)}$ is the previously submitted report. The initial report, $\mathbf{q}^{(0)}$, is made by the market maker and is referred to as market's *initial estimate*. Similar to proper scoring rules, in a MSR, a risk-neutral, myopic, and expected utility-maximizing individual reports truthfully. That is:

$$\mathbf{p} \in \arg\max_{\mathbf{q}^{(t)}\in\Delta_{|\Omega|-1}} E_{\mathbf{p}}\left[\mathrm{MSR}\left(\mathbf{q}^{(t)}, \mathbf{q}^{(t-1)}, \omega\right)\right] \tag{4}$$

where $\mathbf{p}$ is the agent's belief on ω. A MSR is *strictly proper* if it is proper and $\mathbf{p}$ in (4) is unique. We say two MSRs are from the same *family*, if one is a positive scalar of the other.

[3] In general if $S(\cdot)$ is strictly proper, then any positive affine transformation of $S(\cdot)$ is also strictly proper. However, the shifting factor of an affine transformation has no effect on the properties that we are interested in this paper.

2.2. Risk Aversion in Scoring Rules and Market Scoring Rules

Kadane and Winkler (1988) and Murphy and Winkler (1970) show that when a myopic agent that is not risk neutral, is asked about her belief on the outcome of an event and is rewarded using a proper scoring rule, she may not report truthfully. Such an agent may hedge her expected losses by under or over reporting her belief on ω. As shown by Kadane and Winkler (1988) and also by Lambert (2011), when agent's preferences are known, the scoring rule can be corrected to retain its incentive compatibility. Following a similar line of thought, Offerman et al. (2009) propose a new mechanism that includes a two-stage process. In the first stage, individual agents are prescreened, and their risk preferences are elicited. In the second stage, each agent is scored with a tailored made proper scoring rule. In contrast, we do not prescreen agents with unknown risk preferences, in particular, we propose an online mechanism, similar to the work of Dimitrov et al. (2015). As previously discussed, there are no deterministic markets to elicit the beliefs of agents with unknown risk preferences, and in practice randomized mechanisms are disliked. We remind the reader that we only consider deterministic MSR prediction markets in this paper.

2.3. Market Scoring Rules and Cost Function Market Makers

In addition to having agents report probability estimates, agents may also buy and sell shares on securities with values contingent on a future event. For example if agents are trading on a binary event, E, they trade one or two securities, one "Yes" security and one "No" security. With the "Yes" security having a value of $1 if E occurs, and $0 otherwise, the "No" security is similarly defined. The process of buying and selling is considered to be more intuitive to prediction market participants relative to reporting probability estimates (Chen and Pennock 2007). Chen and Vaughan (2010), later extended by Abernethy et al. (2013), show that for every given MSR market maker, there exist a cost-function prediction market, such that the two markets are equivalent, same prices, reward, etc. , and vice versa. In other words, every MSR prediction market can be transformed into a buying/selling share market via a cost-function and for every cost-function prediction market, there exists a MSR such that an agent's transactions can be interpreted in terms of reporting probability estimates.

The new market maker, called a cost function market maker, operates via a cost function $C : \mathbb{R}^{|\Omega|} \mapsto \mathbb{R}$ that determines the cost of transactions. In particular let $\mathbf{s}_0 = \left(s_{01}, s_{02}, \cdots, s_{0|\Omega|}\right)$ be the current bundle of outstanding shares of all securities for all mutually exclusive outcomes on the traded event. When an agent enters the market and changes the outstanding shares to $\mathbf{s} = \left(s_1, s_2, \cdots, s_{|\Omega|}\right)$, the agent is required to pay $C(\mathbf{s}) - C(\mathbf{s}_0)$. The (instantaneous) price for each security ω, given an outstanding bundle of shares, is also defined by the following partial derivative:

$$\Pr_\omega(\mathbf{s}) = \frac{\partial}{\partial s_\omega} C(\mathbf{s}), \omega \in \Omega.$$

As defined by Abernethy et al. (2013), a cost function is valid when it satisfies five properties. Though all five properties[4] must hold for there to be a cost-function for a given MSR, in this paper we only require the following three properties, plus an additional property, to hold for our results:

$$\begin{array}{cc} \text{(path independent)} & \forall \mathbf{s}, \mathbf{s}', \mathbf{s}'' \in \mathbb{R}^{|\Omega|} : \mathbf{s} = \mathbf{s}' + \mathbf{s}'' \implies C(\mathbf{s}) = C(\mathbf{s}') + C(\mathbf{s}''), \\ \text{(price existence)} & \text{the function } C(\cdot) \text{ is continuous and differentiable everywhere on } \mathbb{R}^{|\Omega|}, \\ \text{(no arbitrage)} & \forall \omega \in \Omega, \mathbf{s} \in \mathbb{R}^{|\Omega|} : \Pr_\omega(\mathbf{s}) \geq 0, \sum_{\omega \in \Omega} \Pr_\omega(\mathbf{s}) = 1. \end{array}$$

The additional property that we require to hold is for the cost-function, $C(\mathbf{s})$, to be twice differentiable. Abernethy et al. (2013) show an equivalence relationship between valid cost functions, and Hanson's MSRs. In other words, there exist a cost function market maker for a large class of MSRs and vice versa. In particular for a MSR $\mathcal{X}$, we can find $C^{\mathcal{X}}(\cdot)$, the corresponding cost function, by solving the following problem:

$$C^{\mathcal{X}}(\mathbf{s}) = \max_{\mathbf{q} \in \Delta_{|\Omega|-1}} \mathbf{s}^T \mathbf{q} - \sum_{\omega \in \Omega} q_i X(\mathbf{q}, \omega). \tag{5}$$

Accordingly the price function is defined as:

$$\Pr_\omega^{\mathcal{X}}(\mathbf{s}) = \frac{\partial}{\partial s_\omega} C^{\mathcal{X}}(\mathbf{s}) = \arg\max_{\mathbf{q} \in \Delta_{|\Omega|-1}} \mathbf{s}^T \mathbf{q} - \sum_{\omega \in \Omega} q_i X(\mathbf{q}, \omega). \tag{6}$$

For instance, the LMSR for a binary outcome event:

$$\text{LMSR}(q, r^{(0)}, \omega) = \begin{cases} b\log(q) - b\log(r^{(0)}) & , \omega = 1 \\ b\log(1-q) - b\log(1-r^{(0)}) & , \omega = 0 \end{cases}$$

in which b is a positive scalar, has the following *logarithmic cost function* (Chen and Vaughan 2010):

$$C_{\text{LMSR}}(\underbrace{\mathbf{s}}_{(s_1, s_0)}) = b \log\left(\exp\left(\frac{s_1}{b}\right) + \exp\left(\frac{s_0}{b}\right)\right). \tag{7}$$

Despite the equivalence relation between MSRs and cost function market makers, some issues are unique to the latter and may not be directly applied to the former. For instance, the ability to handle limit orders, an order to buy or sell a bundle of securities at a given price, is one issue that is directly applied to cost function market makers (Heidari et al. 2015) but has no well-defined equivalence in MSRs. Another issue that is unique to cost-function market makers is determining the appropriate amount of liquidity to provide by the market maker. As discussed in the introduction

[4] A valid cost function is defined as a function that satisfies the following conditions: path independence, existence of instantaneous prices, information incorporation, no arbitrage, and expressiveness.

section, Abernethy et al. (2014) and Sethi and Vaughan (2016) study such liquidity issue using the belief elicitation capabilities of the cost function prediction markets.

Apart from the work of Abernethy et al. (2014) and Sethi and Vaughan (2016), there is another stream of studies related to our work that establish a connection between market parameters to belief aggregation capabilities of cost function markets. Assuming budget-constrained agents, and a certain underlying belief distribution, Frongillo et al. (2012) find a relationship between the budget-weighted average of beliefs of risk-averse traders and the equilibrium market prices[5]. In particular, Frongillo et al. (2012) show that the prices generated in cost function markets are stochastic approximations of equilibrium market prices. Furthermore, the authors show that the prices generated in cost function markets converge toward the equilibrium market prices in time. Premachandra and Reid (2013), Frongillo and Reid (2015) study this convergence in greater details and draw a close relationship between cost-function market liquidity, and the rate at which market prices approach the equilibrium market prices.

In contrast, we do not consider any specific belief distribution, and we assume that agents do not have bounded budgets. Moreover, since we are interested in settings where the number of agents may be low, increasing liquidity may have a different effect on information elicitation capabilities of prediction markets. In particular, increasing liquidity while agents have bounded budgets may lead to the market not capturing all of the agents' beliefs when the number of agents is low, as agents may all exhaust their budgets before moving the market to the consensus price. In this paper, we are interested in the behavior of risk-averse and risk-seeking agents in MSR prediction markets and their equivalent trading behavior in cost-function prediction markets. We show that, unlike a risk-neutral agent, an agent Alice who is risk-averse or risk-seeking, does not reveal her true belief in a MSR prediction market. Thus when other agents observe the market's estimate they do not observe other agent's exact beliefs. We do not consider bounded budget agents, and therefore if Alice participates in a cost function prediction market, her transaction is not affected by any budget constraint, and instead, is a factor of her belief and risk preferences. Hence her transaction does not move the current market price to the price corresponding to her exact belief. It is this difference between Alice's belief and her report that we are interested in. We refer to this difference as the deviation, precisely described in Definition 2. Our main motivation in this paper is to study the magnitude of this deviation and how it relates to the choice of the underlying MSR function.

[5] This line of research is a continuation of the work by Manski (2006) and Wolfers and Zitzewitz (2006) in which the authors draw an analytical relationship between prediction market prices and the equilibrium market prices in absence of any market maker.

2.4. Liquidity Measures in Cost-function Prediction Markets

Regardless of the type of prediction market used, MSR or cost-function, a market maker must determine how to facilitate trade within the market, especially if the market is thin. As such, the market market must provide some liquidity into a market. We now discuss liquidity, and various measures proposed in the literature. Liquidity of a market is defined as the market's price responsiveness to trade. The idea of liquidity is that: in a more liquid market, larger trade volume is required to change the market's price compared to the trade volume that is required in a less liquid market. Prediction market literature uses the following three approaches to measure cost-functions' market liquidity:

- *Inverse liquidity* (Frongillo et al. 2012, Brahma et al. 2012, Abernethy et al. 2011, Wah et al. 2016, Abernethy et al. 2014, Slamka et al. 2013, Othman et al. 2013) Given a base valid cost-function market maker $C(\cdot)$, as defined in Section 2.1, one can verify that the cost-function:

$$C_b(\mathbf{s}) \triangleq b\, C\left(\frac{1}{b}\mathbf{s}\right), \tag{8}$$

is a valid cost-function market maker. The parameter b determines the liquidity of the market. The higher the parameter b, the lower prices change given a fixed volume of trade. This definition is adopted to compare different cost-function market makers from the same family. For instance, when two Logarithmic cost-functions, see (7), with different b parameters, $C_{\mathrm{LMSR}_{b_1}}$ and $C_{\mathrm{LMSR}_{b_2}}$ in which $b_2 > b_1$, are compared; $C_{\mathrm{LMSR}_{b_2}}$ is considered more liquid than $C_{\mathrm{LMSR}_{b_1}}$. Given a base cost-function market maker, a family of cost-function market makers can be generated using different values of b, and liquidity of each market maker can be easily measured by simply comparing the market makers' b parameters. By definition, inverse liquidity is unable to compare two cost-function market makers from different families.

- *Market depth* (Chen and Pennock 2007, Li and Vaughan 2013) Given the cost-function $C^{\mathcal{X}}(\mathbf{s})$, a security's market depth, also known as instantaneous liquidity, is defined as follows:

$$\rho^{\mathcal{X}}_{\omega}(\mathbf{s}) = \frac{1}{\frac{\partial}{\partial s_\omega} Pr^{\mathcal{X}}_{\omega}(\mathbf{s})} \tag{9}$$

in which $Pr^{\mathcal{X}}_{\omega}(\mathbf{s})$ is the security ω's price function, defined in (6). Comparing two cost-function market makers' liquidity using market depth is not as straightforward as using inverse liquidity. To compare two cost-function market makers using market depth, one should examine the rate of change in each market's prices when both markets have equal prices. Definition 1 describes this comparison more precisely.

Definition 1 *Let $C^{\mathcal{X}}$ and $C^{\mathcal{Y}}$ define the cost-function market makers of the two MSRs $\mathcal{X}$ and $\mathcal{Y}$, respectively. We say $C^{\mathcal{Y}}$ has more market depth relative $C^{\mathcal{X}}$ when:*

$$\forall \omega \in \Omega, p \in [0,1] : \rho^{C^{\mathcal{X}}}_{\omega}(\mathbf{s}) \leq \rho^{C^{\mathcal{Y}}}_{\omega}(\mathbf{s}') \tag{10}$$

in which $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^{|\Omega|}$ *such that* $Pr^{\mathcal{Y}}_{\omega}(\mathbf{s}) = Pr^{\mathcal{X}}_{\omega}(\mathbf{s}') = p$.

Definition 1 says: the change in price for all possible prices in one cost-function market maker, $C^{\mathcal{Y}}$, must be less than the change in price for the same prices in another market, $C^{\mathcal{X}}$, for a given security. We note that we compare the depth of each market to one another for a fixed price, instead of number of outstanding shares. We use price because for a fixed number of outstanding shares, there may be different price estimates in two different markets, meaning that the projected likelihood of the security is not the same in each market. Figure 1 illustrates a comparison of the market depth between three well-known cost function market makers.

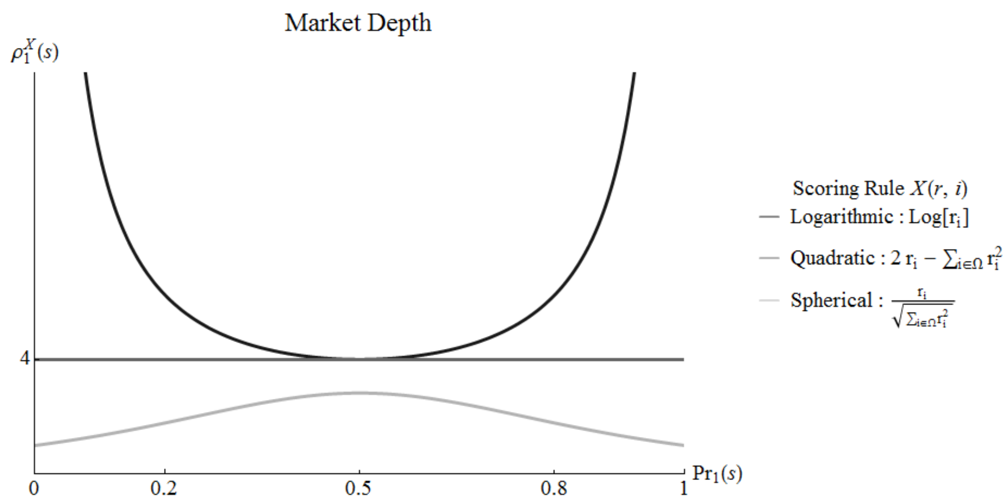

Figure 1 Market depth comparison across three cost functions corresponding to the three well-known scoring rules with $b = 1$.

2.5. Model Set-up

Consider a binary event E, with two possible outcomes in $\{0, 1\}$. Let ω represent the corresponding random variable where $\omega \in \{0, 1\}$. In this paper, we only analyze the behavior of myopic agents one at a time and therefore we can focus on two consecutive reports at a time. Thus we use the similar, but simpler notation of $\mathcal{X}(r, r^{(0)}, \omega)$ instead of $\mathcal{X}(r^{(t)}, r^{(t-1)}, \omega)$, where r is the agents report and $r^{(0)}$ is the market's current estimate, at the time of making report[6]. An agent with personal belief of p on ω, is asked to submit her probability estimates denoted by r. When the outcome of the event is observed, the prediction market rewards according to a market scoring rule function:

$$\mathcal{X}\left(r, r^{(0)}, \omega\right) = X(r, \omega) - X\left(r^{(0)}, \omega\right), \tag{11}$$

[6] Note that since the outcome space is binary, and we use coherent probabilities, we may use the variable r, and $r^{(0)}$ to represent the tuples like $\mathbf{r} = (r, 1-r)$, and $\mathbf{r^{(0)}} = (r^{(0)}, 1 - r^{(0)})$.

where the strictly proper score function $X(\cdot)$ has the following properties:

$$\begin{aligned} X : \Delta_1 \times \{0,1\} \mapsto \mathbb{R} \text{ is smooth on } \Delta_1 \text{ for any given } \omega, \\ \text{and } X\left(\tfrac{1}{2},1\right) = X\left(\tfrac{1}{2},0\right). \end{aligned} \tag{12}$$

In addition to being myopic and expected utility maximizers, we further assume agents may be risk-averse or risk-seeking. As for risk preference, we assume agents have a concave or convex, utility function. We also refer to maximum risk aversion or risk seeking as to the utility of the most risk-averse or risk-seeking agent. We also assume such utility function always exist. We assume that the market organizer has a bound on the Arrow-Pratt measure of the agents risk preferences, but does not know any individuals' risk preferences. An agent with a utility function of the form $u \in \mathcal{U}$ is rational, and thus maximizes her expected utility given her belief of p on ω. The set $\mathcal{U} \subset \mathbb{R}^{\mathbb{R}}$ is the set of all utility functions that are monotonically increasing, twice differentiable, and are either convex, representing risk-seeking agents, or concave, representing risk-averse agents. Such an agent's expected utility from reporting any feasible report of q is as follows:

$$E_p\left[u\left(\mathcal{X}(q, r^{(0)}, \omega)\right)\right] = p\ u\left(X(q,1) - X\left(r^{(0)},1\right)\right) + (1-p)u\left(X(q,0) - X\left(r^{(0)},0\right)\right) \tag{13}$$

In any strictly proper MSR, a risk neutral agent will report truthfully (i.e., the utility maximizing report of a risk-neutral agent is the same as her belief). Given a strictly proper scoring rule $X(\cdot)$, and its corresponding MSR $\mathcal{X}$, we formally define the expected utility maximizing report of $r^u_{\mathcal{X}}\left(p, r^{(0)}\right)$ as:

$$r^u_{\mathcal{X}}\left(p, r^{(0)}\right) \triangleq \underset{q\in[0,1]}{\arg\max}\, E_p\left[u\left(\mathcal{X}\left(q, r^{(0)}, \omega\right)\right)\right]. \tag{14}$$

We note that as we assume all agents are myopic and rational, all reports made by agents will be those that satisfy (14) and we thus refer to all reports adhering to (14) simply as reports. As we show in Section 3, for an arbitrary agent with belief p and a non-linear utility function, the value of $r^u_{\mathcal{X}}(\cdot)$ almost always differs from p (i.e., an agent might under or over report her belief). As a consequence, in MSRs, beliefs are under and over reported. In this paper, we compare these under/over reporting in different MSRs and we find necessary and sufficient conditions for a MSR to yield lower under/over reporting relative to another MSR. Accordingly, we refine our definition of *deviation* to be the difference between an agent's report and belief.

3. Deviation

In this section we show that in presence of agents with unknown and non-linear utilities, risk averse or risk seeking, no Hanson's MSRs with deterministic rewards can be incentive compatible. This result is similar to the work of Lambert (2011) and Schlag and van der Weele (2013), in which they prove that no proper scoring rule with deterministic rewards can be incentive compatible. We also note that for the impossibility result to hold, the assumption of bounded risk-preferences is not necessary. Our impossibility result is similar to the result of Lambert (2011). Lambert (2011) shows that when agents' risk preferences are not known, no scoring rule can elicit agents' beliefs. In Lemma 2, we show that when an agent has unknown utility, concave or convex, no Hanson's MSR can elicit her belief, unless her belief is identical to the market's current estimate. An implication of this finding is that in a MSR, the market's current estimate acts as a reference point. That is, if a risk-averse/risk-seeking agent with belief of p on ω, participates in a MSR with a current estimate of $r^{(0)}$; the only information that can be extracted from her report is either $p = r^{(0)}$, $p < r^{(0)}$, or $p > r^{(0)}$. Being able to extract this information is a new finding and was not explicitly discussed by Lambert (2011). We note that with an affine transformation of the scoring rules used by Lambert (2011), one may be able to recreate this feature. As the main focus of this section we analyze an agent's optimal report in MSRs, and we find that two or more MSRs may be compared to each other, to conclude which one provides a better estimate on agents subjective probability estimates. In other words, which market induces a lower deviation for the same agent for any given market estimate and any given belief. In Theorem 1, we provide the necessary and sufficient conditions for a MSR to yield a lower deviation compared to another MSR.

We define deviation as the value of an agent's under or over-reporting. The deviation thus may be captured by simply getting the difference of the agent's report and her belief.

Definition 2 (Deviation) *For a given market estimate, $r^{(0)}$, in a MSR $\mathcal{X}$, the following is defined as an agent's deviation function of $\mathcal{X}$ for any belief and current market estimate:*

$$\mathcal{F}^u_{\mathcal{X}}\left(p, r^{(0)}\right) = \left| r^u_{\mathcal{X}}\left(p, r^{(0)}\right) - p \right| \tag{15}$$

Theorem 1 below, shows that to compare the deviation in two MSRs, it is necessary and sufficient to compare the projection of the two MSRs' functions under the first derivative of an agent's utility. The optimal report of an agent, required for deviation function (15), is calculated by solving the first order equation of the agent's expected utility maximizing problem (14). By Lemma 2, see Appendices, we may directly relate an agent's belief to her optimal report. Intuitively, Lemma 2 states that an agent's report is a non-linear scale of her belief, where the non-linear scale is a function of the agent's utility and the market's current estimate. Theorem 1 states that for a given agent, given the definition of deviation, in order to compare two MSRs using deviation, one only needs to compare the non-linear scale factor, derived in Lemma 2, see Appendices. The comparison between an agent's reports, in different MSRs, to her belief, provides the necessary and sufficient conditions for a MSR to yield a lower deviation relative to another MSR.

Theorem 1 *Let $\mathcal{X}(\cdot)$ and $\mathcal{Y}(\cdot)$ be two market scoring rules. Also let $u \in \mathcal{U}$ be a concave (convex) function. The following holds for all $r^{(0)} \in [0,1]$:*

$$\forall p \in [0,1] : \mathcal{F}^u_{\mathcal{X}}\left(p, r^{(0)}\right) \leq \mathcal{F}^u_{\mathcal{Y}}\left(p, r^{(0)}\right) \iff \begin{cases} \frac{u'\left(\mathcal{Y}\left(q,r^{(0)},1\right)\right)}{u'\left(\mathcal{Y}\left(q,r^{(0)},0\right)\right)} \leq (\geq) \frac{u'\left(\mathcal{X}\left(q,r^{(0)},1\right)\right)}{u'\left(\mathcal{X}\left(q,r^{(0)},0\right)\right)}, \ \forall q \in \left[0, r^{(0)}\right] \\ \frac{u'\left(\mathcal{Y}\left(q,r^{(0)},1\right)\right)}{u'\left(\mathcal{Y}\left(q,r^{(0)},0\right)\right)} \geq (\leq) \frac{u'\left(\mathcal{X}\left(q,r^{(0)},1\right)\right)}{u'\left(\mathcal{X}\left(q,r^{(0)},0\right)\right)}, \ \forall q \in \left[r^{(0)}, 1\right] \end{cases} \tag{16}$$

Corollary 1 is a special case of Theorem 1, as not all MSR pairs satisfy (17). However, if (17) is satisfied, then we no longer need condition (16) that is a function of an agent's utility. With (17) we are able to only compare the underlying MSR function, as MSRs and the utility functions we consider are monotone. Corollary 1 states that for a family of MSRs, including LMSRs, the deviation can be compared between the two MSRs without considering agents' utility.

Corollary 1 *Let $\mathcal{X}(\cdot)$ and $\mathcal{Y}(\cdot)$ be two market scoring rules that satisfy:*

$$\forall r^{(0)} \in [0,1] : \begin{cases} \mathcal{Y}\left(q, r^{(0)}, 1\right) \leq \mathcal{X}\left(q, r^{(0)}, 1\right) \leq \mathcal{X}\left(q, r^{(0)}, 0\right) \leq \mathcal{Y}\left(q, r^{(0)}, 0\right), \forall q \in \left[0, r^{(0)}\right] \\ \mathcal{Y}\left(q, r^{(0)}, 0\right) \leq \mathcal{X}\left(q, r^{(0)}, 0\right) \leq \mathcal{X}\left(q, r^{(0)}, 1\right) \leq \mathcal{Y}\left(q, r^{(0)}, 1\right), \forall q \in \left[r^{(0)}, 1\right] \end{cases} . \tag{17}$$

Therefore the following holds:

$$\forall u \in \mathcal{U}, \forall p \in [0,1] : \mathcal{F}^u_{\mathcal{X}}\left(p, r^{(0)}\right) \leq \mathcal{F}^u_{\mathcal{Y}}\left(p, r^{(0)}\right).$$

In addition to LMSRs, two other families of MSRs satisfy (17), QMSRs, and SMSRs (Hanson 2003, Gneiting and Raftery 2007), with different b parameters. Thus, when comparing the deviation between two LMSRs it is sufficient to compare their b parameters, as the LMSR with the larger b will have a larger deviation relative to the other LMSR. Similarly, when we compare the deviation between two MSRs from different families, a LMSRs and a Quadratic MSRs for instance, it is sufficient to compare their MSR functions as both MSRs satisfy (17).

4. Deviation and Liquidity

In this section, we present the analytical results that show the amount of liquidity provided by a valid cost-function prediction market, as defined in Section 2.3, is closely related to the deviation of reports in its corresponding MSR prediction market. We use two measures of liquidity: inverse liquidity, and market depth, to illustrate the relation between deviation and liquidity. In Section 4.1, we show that when inverse liquidity is used to measure the amount of provided liquidity, a higher deviation implies a higher liquidity. In Section 4.2, we show that when market depth is being used to measure liquidity, the same results holds and a higher deviation implies a higher liquidity. In Section 4.3, we discuss the other, perhaps more practical, implication of our results that can help a market organizer determine the amount of market liquidity required for a desired level of beleif elicitation. Given a bound on a prediction market populations' risk preferences, a market organizer can optimize the market's liquidity constrained by the maximum allowed deviation. The established optimization problem introduces an analytical criterion to determine the amount of market liquidity for a MSR cost-function market maker while maintaining bounded subsidy.

4.1. Inverse Liquidity and Deviation

Recall that for a valid cost-function, say $C^{\mathcal{X}}$, another valid cost-function $C_b^{\mathcal{X}}(\mathbf{s}) \triangleq b\, C^{\mathcal{X}}\left(\frac{1}{b}\mathbf{s}\right)$ can be generated for any positive scalar b. It is also straightforward to show that the corresponding MSR for the cost-function $C_b^{\mathcal{X}}$ is $b\,\mathcal{X}(\cdot)$, in which $\mathcal{X}(\cdot)$ is the underlying MSR of $C^{\mathcal{X}}$. Thus, given a cost-function, a family of cost-functions can be generated using different b parameters.

Theorem 2 shows that when two cost-functions from the same family are compared using inverse liquidity; higher liquidity is equivalent to higher deviation. To see this, recall that, by definition, higher b parameter equals to higher liquidity. Moreover, a higher b parameter also increases the value of the underlying MSR function which by Corollary 1, implies more deviation.

Theorem 2 *Let $C^{\mathcal{X}}$ and $C^{\mathcal{Y}}$ be the cost function market makers of the two MSRs $\mathcal{X}$ and $\mathcal{Y}$ respectively, in which $C^{\mathcal{Y}}(\mathbf{s}) = b\, C^{\mathcal{X}}(\mathbf{s}/b)$ for some $b > 1$. The market maker $C^{\mathcal{Y}}$ has more inverse liquidity if and only if $\mathcal{Y}$ has more deviation compared to $\mathcal{X}$.*

Given the definition of inverse liquidity, the implication of Theorem 2 is simple: a higher b parameter presents a tension between two desirable properties of less deviation and higher liquidity. Moreover, when inverse liquidity is used to measure the market's liquidity, deviation and liquidity are equivalent. However, by definition of inverse liquidity, (8), we are unable to compare two MSRs from different families, leading us to consider market depth when comparing MSRs from different families.

4.2. Market Depth and Deviation

When we use market depth to measure liquidity, we can compare cost-functions from different families. For instance, we can conclude that the Logarithmic cost-function market maker with b parameter of

1, has more market depth relative to a Quadratic cost-function market maker (Chen and Pennock 2007) with b parameter of 1. Theorem 3 shows that a higher deviation implies a higher market depth. To prove Theorem 3, we need to find the relation between a share quantity and the price function. Fortunately (5) presents the unconstrained optimization problem we need to solve to find the relation. In order to solve the unconstrained optimization problem we consider the first order conditions of the objective function with respect to q and derive the relation between the price and the share quantity, referred to as the *price-share* relation. The price-share relationship allows us to implicitly calculate the derivative of the price w.r.t. the quantity of outstanding shares for a given MSR. It follows from the definition of market depth, Definition 1, that computing the price-share relation allows us to compare the market depths of the cost-functions of two MSRs. In particular, we show that for a cost-function $C^{\mathcal{X}}$, the depth of the market is equal to $\frac{\partial}{\partial q} X(q,1) - X(q,0)\Big|_{q=Pr_1^{\mathcal{X}}(\mathbf{s})}$. Using the previous equation, we relate the depth of a cost-function market maker to its underlying proper scoring rule. Using Corollary 1, and the simple definition of the first derivative, we can conclude that more deviation in a MSR implies more market depth in its corresponding cost-function market maker.

Theorem 3 *Let $C^{\mathcal{X}}$ and $C^{\mathcal{Y}}$ be the cost-function market makers of the two MSRs $\mathcal{X}$, and $\mathcal{Y}$ respectively. If $\mathcal{X}$ has more deviation compared to $\mathcal{Y}$, then $C^{\mathcal{X}}$ has more market depth relative to $C^{\mathcal{Y}}$.*

Similar to Theorem 2, Theorem 3 also shows that higher deviation implies higher market depth. As both market depth and inverse liquidity are the two measures of liquidity used in the literature, the two theorems, Theorem 2 and Theorem 3, collectively state the same result: higher deviation implies higher liquidity. In the next section, we discuss how this result can be used to determine an optimal amount of liquidity given a desired level of belief elicitation.

4.3. Optimizing Market Depth and Report Deviation

As discussed in Section 2.1, setting the market depth of a market maker is often described as "art." In this section, we apply the results of Theorem 2 to show how a market organizer can optimize inverse liquidity to bound a maximum amount of deviation. Our result is similar to the work of Abernethy et al. (2014), where authors show that lower inverse liquidity increases the difference between agents' belief and the market price, what we call deviation. The results of Abernethy et al. (2014), Sethi and Vaughan (2016) suggest that a market organizer can change the market's inverse liquidity parameter to achieve a desirable value of belief elicitation. Unfortunately, neither of these two papers formalize the relationship that exists between inverse liquidity and deviation. As mentioned earlier, we fill in this gap by introducing an optimization problem that can determine a minimum amount of inverse liquidity for a desired level of deviation.

We can verify that for two binary LMSRs with parameter b_1 and b_2 in which $b_2 > b_1$, the following holds:

$$\forall r^{(0)} \in [0,1]: \begin{cases} \mathrm{LMSR}_{b_2}\left(q, r^{(0)}, 0\right) \leq \mathrm{LMSR}_{b_1}\left(q, r^{(0)}, 0\right) \leq \mathrm{LMSR}_{b_1}\left(q, r^{(0)}, 1\right) \leq \mathrm{LMSR}_{b_2}\left(q, r^{(0)}, 1\right), \, q \geq r^{(0)} \\ \mathrm{LMSR}_{b_2}\left(q, r^{(0)}, 1\right) < \mathrm{LMSR}_{b_1}\left(q, r^{(0)}, 1\right) < \mathrm{LMSR}_{b_1}\left(q, r^{(0)}, 0\right) < \mathrm{LMSR}_{b_2}\left(q, r^{(0)}, 0\right), \, q < r^{(0)} \end{cases}. \tag{18}$$

Thus by Corollary 1, the $LMSR_{b_2}$ has more deviation compared to $LMSR_{b_1}$, By Theorem 2, $C^{\mathrm{LMSR}_{b_2}}$ has more inverse liquidity compared to $C^{\mathrm{LMSR}_{b_1}}$. Now assume that $\bar{u(\cdot)}$ is the utility function with the largest absolute Arrow-Pratt measure, as discussed in Section 2.2. We define the function $DevMax(b) \triangleq \max_{r^{(0)}, p \in [0,1]} \mathcal{F}^{\bar{u}}_{\mathrm{LMSR}_b}(p, r^{(0)})$ to be the maximum possible deviation. The following optimization problem can determine the maximum amount of market depth while guaranteeing the maximum market maker losses to $\bar{B}$, and a maximum deviation of $\bar{D}$.

$$\begin{aligned} \max \quad & b \\ \text{subject to:} \quad & \\ & DevMax(b) \leq \bar{D} \quad \text{(deviation constraint)} \\ & b \ln(2) \leq \bar{B} \\ & b \geq 0 \end{aligned} \tag{CP}$$

The budget constraint ensures that the liquidity parameter b does not exceed the maximum subsidy requires to run a market (Chen and Pennock 2007). Using the same analysis, we can set similar optimization problems to (CP), for other MSRs such as QMSRs, and SMSRs. Figure 1 shows the value of parameter b as a function of maximum deviation, $\bar{D}$, accepted by the market maker for three different MSRs. In the example of Figure 2, the value of b is determined by program (CP) for agents with risk preferences determined by Babcock et al. (1993)[7].

However, the results of Theorem 2, similar to the results of Abernethy et al. (2014), Sethi and Vaughan (2016), only allow us to compare MSRs that belong to the same family (for example, two LMSRs or two Quadratic MSRs, etc.). The fact that Figure 2 was derived from the results of Theorem 2, it follows that the lines in Figure 2 cannot be used to compare different MSR families to one another. Fortunately, the results of Theorem 3 allows us to compare MSRs from different families to one another. We note that for a comparison of two MSRs to be valid our definition of one market having more market depth than another, we assume the property, described in Definition 1, holds for all possible market prices. It is not always possible to guarantee that one market will have a larger market depth than another market for all market prices. We would like to point out that the results

[7] Babcock et al. (1993) models individuals' risk attitudes using the constant absolute risk-aversion (CARA) utility function, that is $u(x) = 1 - e^{-\alpha x}$, in which α determines the amount of absolute risk aversion for an individual's risk preference. Babcock et al. (1993) show that for a gamble sizes of less than or equal to \$100, the value of the parameter α ranges from 0.0002 to 0.046204. In the example of Figure 2, we use the value of $\bar{\alpha} = 0.046204$ to to determine the maximum deviation of $\bar{D}$

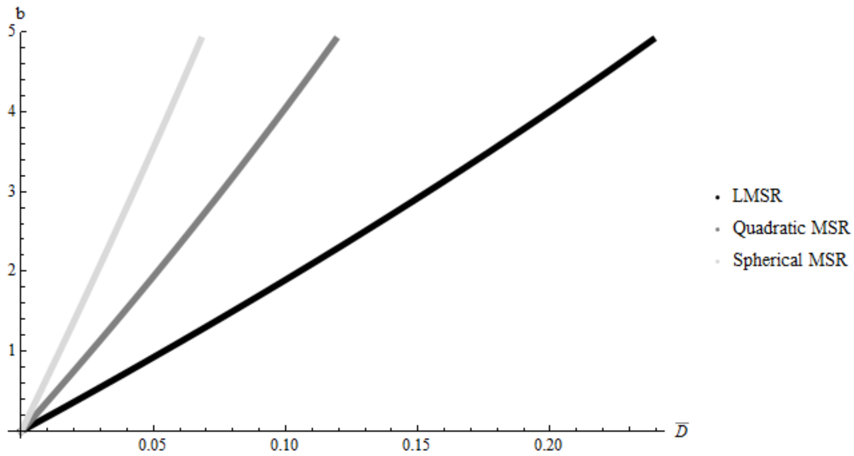

Figure 2 Approximations of parameter b as a function of $\bar{D}$ for different MSRs.

of Theorem 3 may be extended to a locally defined variation of market depth. However, we choose not to use this definition as the comparison of MSRs with a locally defined variation of market depth will require the market maker to have prior knowledge on the distribution of the market prices. If a market maker, has knowledge on the distribution of market prices, then it is no longer clear why the market should exist, as a distribution on market prices is equivalent to a distribution on the outcome of the traded event.

To illustrate an application of Theorem 3, consider a set of MSRs $\mathfrak{X}$, not necessarily from the same families of MSRs, and the set of their corresponding cost-functions $\mathfrak{C}^{\mathfrak{X}}$. Given a maximum amount of deviation, say $\bar{D}$, we may choose the MSR $\mathcal{X}^\star \in \mathfrak{X}$, in which: $\mathcal{X}^\star$ has a maximum deviation of $\bar{D}$, and $C^{\mathcal{X}^\star}(\cdot)$ has more market depth relative to all other cost-functions in $\mathfrak{C}^{\mathfrak{X}}$, if comparable. Figure 3 illustrates the head-to-head comparison of the three popular variants of MSRs.

Using the depth comparison illustrated in Figure 3, we can numerically show that when $\mathfrak{X}$ includes three families of MSRs, Logarithmic, Quadratic, and Spherical MSRs (Chen and Pennock 2007), for any given $\bar{D}$, $\mathcal{X}^\star$ is a LMSR. This is because a logarithmic cost-function either: has more depth relative to any quadratic or spherical cost-function market makers, or is not comparable to a quadratic or spherical cost-function market maker. Moreover, the fact that $\mathcal{X}^\star$ is a LMSR is independent of the maximum risk preference. Using a similar numerical analysis, and assuming any population has the same maximum risk preference found by Babcock et al. (1993), we can show that when $\mathfrak{X}$ includes only two families of Quadratic, and Spherical MSRs (Chen and Pennock 2007), $\mathcal{X}^\star$ is a Quadratic MSR for any given $\bar{D}$. We note that, unlike the case with three families of MSRs, Logarithmic, Quadratic, and Spherical MSRs, when $\mathfrak{X}$ includes only two families of Quadratic, and Spherical MSRs, the utility function of the most risk-averse/risk seeking agent determines if $\mathcal{X}^\star$ is a Quadratic or a Spherical MSR. Moreover, the class of MSRs where $\mathcal{X}^\star$ belongs to, may also depend on $\bar{D}$.

(a) Logarithmic and Quadratic Cost-functions. (b) Logarithmic and Spherical Cost-functions. (c) Spherical and Quadratic Cost-functions.

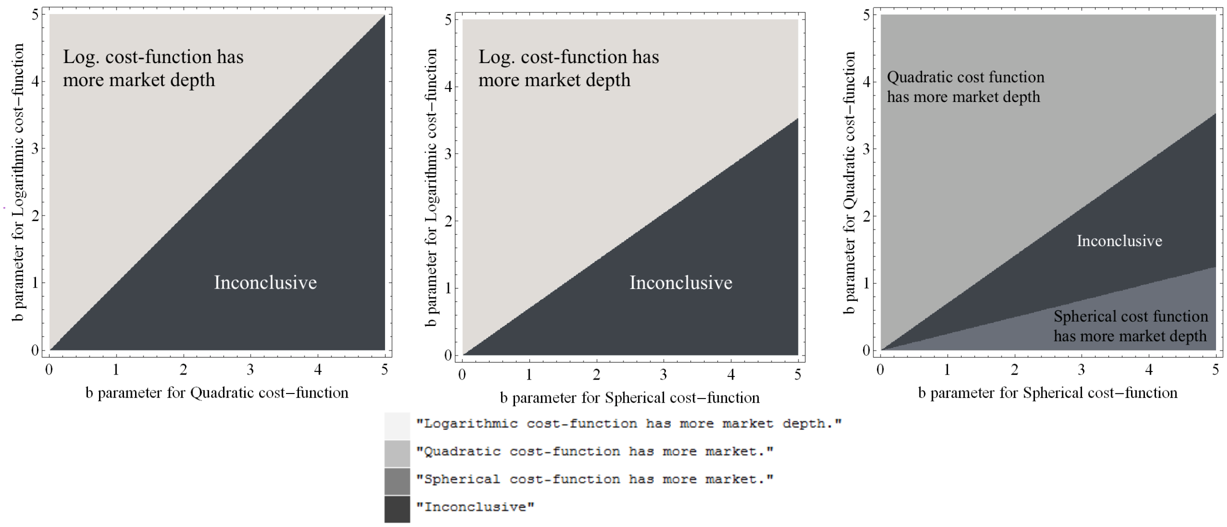

Figure 3 Depth comparison between Logarithmic, Quadratic, and Spherical cost-function market makers. Inconclusive means that the conditions in the definition of Market Depth, Definition 1, are violated.

5. Conclusion

In this paper we study the effect of agent risk bias on belief reporting in MSR prediction markets and characterized the close relation with cost-functions' market liquidity and risk bias in MSRs. Our analytical results suggest that myopic, utility maximizing agents do not report their exact beliefs, unless their beliefs are identical to the current market estimate. We introduced the concepts of deviation to measure the difference between an agent's reported belief and personal belief. Our first finding is that we can compare the amount of deviation across all MSRs by comparing the reward functions provided to agents. The higher the reward functions, the higher the deviation. We also showed that for all MSRs, decreasing the deviation of a MSR implies decreasing the liquidity of the MSR's corresponding cost-function market maker. We used this relation to introduce an analytical approach to determine the amount of liquidity to use to ensure a desirable belief elicitation. In the future, we would like to extend our results with other risky choice theories such as prospect theory. The results assumed that deviation is measured as defined in (15), in the future, we are interested in determining the robustness of this definition by seeing if using different norm measures will impact the results found in (CP). In addition, the presented work only shows how to choose the most liquid market maker amongst three popular MSR families. As a future direction of study, we would like to optimize over the set of all possible MSRs to find the one that has the optimal liquidity across all MSRs. Further, we assumed that agents' budgets are unbounded, an assumption needed for the mapping between probability reports in MSRs and buying and selling shares in a cost-function prediction market. One area of future research is to explore this mapping when agents have budget-bounded reports. Finally, we would like to test the liquidity-deviation relationship in laboratory settings.

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Appendix A: Proofs

Recall that in this paper we only consider binary outcomes.

Lemma 1 *Let $S(\cdot)$ be a strictly proper scoring rule.*

i)

$$\forall r \in (0,1): q\frac{\partial}{\partial q} S(q,1) + (1-q)\frac{\partial}{\partial q} S(q,0)\bigg|_{q=r} = 0. \tag{A1}$$

ii)

$$\forall r \in (0,1): \frac{\partial}{\partial q} S(q,1)\,|_{q=r} > 0 \text{ and } \frac{\partial}{\partial q} S(q,0)\,|_{q=r} < 0. \tag{A2}$$

Proof. i) Given the Schervish representation of strictly proper scoring rules (Gneiting and Raftery 2007, Page 363) we have:

$$\begin{aligned} S(q,1) &= G(q) + (1-q)\,G'(q), \\ S(q,0) &= G(q) - qG'(q), \end{aligned} \tag{A3}$$

in which $G(q)$ is a univariate strictly convex function of q. By first order differentiation and (A3) we get:

$$\begin{aligned} \tfrac{\partial}{\partial q} S(q,1) &= (1-q)\,G''(q), \\ \tfrac{\partial}{\partial q} S(q,0) &= -qG''(q). \end{aligned} \tag{A4}$$

Thus by (A4) we have

$$\left(q\frac{\partial}{\partial q} S(q,1) + (1-q)\frac{\partial}{\partial q} S(q,0)\right)\bigg|_{q=r} = r\,(1-r)\,G''(r) - (1-r)\,rG''(r) = 0,$$

that proves the claim in part (i).

ii) Since the function $G(\cdot)$ is strictly convex in q, (A4) shows that $\frac{\partial}{\partial q} S(q,1)$ is positive, and $\frac{\partial}{\partial q} S(q,0)$ is negative, which proves the claim in part (ii). □

Lemma 2 *Let $r^u_{\mathcal{X}} \triangleq r^u_{\mathcal{X}}\left(p, r^{(0)}\right)$, be the expected utility-maximizing report, defined in* (14)*, of an agent in the Hanson MSR $\mathcal{X}$, in which $\mathbf{p} = (p, 1-p)$ is the agent's belief on ω and $\mathbf{r^{(0)}} = (r^{(0)}, 1-r^{(0)})$ is the market's current estimate.*

i)

$$\frac{1-p}{p} = \frac{1-r^u_{\mathcal{X}}}{r^u_{\mathcal{X}}} \frac{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^{(0)}, 1\right)\right)}{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^{(0)}, 0\right)\right)}. \tag{A5}$$

ii)

$$\begin{aligned} &\textit{(risk averse agent) if } u \textit{ is concave} \implies \begin{cases} p < r^u_{\mathcal{X}} < r^{(0)} & : p < r^{(0)} \\ p = r^u_{\mathcal{X}} = r^{(0)} & : p = r^{(0)} \\ r^{(0)} < r^u_{\mathcal{X}} < p & : p > r^{(0)} \end{cases}. \\ &\textit{(risk seeking agent) if } u \textit{ is convex} \implies \begin{cases} p < r^u_{\mathcal{X}} < r^{(0)} & : p > r^{(0)} \\ p = r^u_{\mathcal{X}} = r^{(0)} & : p = r^{(0)} \\ r^{(0)} < r^u_{\mathcal{X}} < p & : p < r^{(0)} \end{cases}. \end{aligned} \tag{A6}$$

Proof. i) Since X, the underlying score function of $\mathcal{X}$ is smooth, $\mathcal{X}\left(\cdot, r^{(0)}, \omega\right)$ is also smooth, thus $r^u_{\mathcal{X}}$ is an interior point and it satisfies the first order condition. That is:

$$\nabla\left(E_p\left[u\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^{(0)}, \omega\right)\right)\right]\right) = 0.$$

By chain rule we get:

$$p\frac{\partial}{\partial q}\mathcal{X}\left(r^u_{\mathcal{X}}, r^{(0)}, 1\right)u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^{(0)}, 1\right)\right) + (1-p)\frac{\partial}{\partial q}\mathcal{X}\left(r^u_{\mathcal{X}}, r^{(0)}, 0\right)u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^{(0)}, 0\right)\right) = 0.$$

By definition of a MSR (11) we further have:

$$p\frac{\partial}{\partial q}X\left(r^u_{\mathcal{X}}, 1\right)u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^{(0)}, 1\right)\right) + (1-p)\frac{\partial}{\partial q}X\left(r^u_{\mathcal{X}}, 0\right)u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^{(0)}, 0\right)\right) = 0.$$

Rearrangement gives:

$$\frac{1-p}{p} = \frac{\frac{\partial}{\partial q}X\left(r^u_{\mathcal{X}}, 1\right)}{\frac{\partial}{\partial q}X\left(r^u_{\mathcal{X}}, 0\right)}\frac{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^{(0)}, 1\right)\right)}{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^{(0)}, 0\right)\right)}. \tag{A7}$$

By a rearrangement of Lemma 1-i, (A7) can be reduced to the followings:

$$\frac{1-p}{p} = \frac{1-r^u_{\mathcal{X}}}{r^u_{\mathcal{X}}}\frac{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^{(0)}, 1\right)\right)}{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^{(0)}, 0\right)\right)}. \tag{A8}$$

This proves the claim in part (i).

ii) By (A5), to compare $r^u_{\mathcal{X}}$ to p, we need to determine how $\frac{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^{(0)}, 1\right)\right)}{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^{(0)}, 0\right)\right)}$ compares to 1. Without loss of generality, let $r^u_{\mathcal{X}} > r^{(0)}$ and u be a concave function. By Lemma 1-ii we get:

$$\begin{cases} \mathcal{X}\left(r^u_{\mathcal{X}}, r^{(0)}, 1\right) = X\left(r^u_{\mathcal{X}}, 1\right) - X\left(r^{(0)}, 1\right) > 0 \\ \mathcal{X}\left(r^u_{\mathcal{X}}, r^{(0)}, 0\right) = X\left(r^u_{\mathcal{X}}, 0\right) - X\left(r^{(0)}, 0\right) < 0 \end{cases}. \tag{A9}$$

Since u is concave and non-decreasing, the function u' is decreasing and therefore (A9) implies:

$$\frac{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^{(0)}, 1\right)\right)}{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^{(0)}, 0\right)\right)} < 1. \tag{A10}$$

Therefore by (A10) and (A5) we get:

$$r^u_{\mathcal{X}} < p.$$

Similarly we can show the claim holds in the cases in which $r^u_{\mathcal{X}} = r^{(0)}$, $r^u_{\mathcal{X}} < r^{(0)}$, or u is convex. □

Lemma 3 *The image of the univariate function* $r^u_{\mathcal{X}}\left(\cdot, r^{(0)}\right)$ *is* $[0,1]$.

Proof. By definition, we have:

$$Im\left(r^u_{\mathcal{X}}\left(p, r^{(0)}\right)\right) \subseteq [0,1]. \tag{A11}$$

Also

$$r^u_{\mathcal{X}}\left(p, r^{(0)}\right)\Big|_{p=1} = \underset{q\in[0,1]}{\arg\max}\, E_1\left[u\left(\mathcal{X}\left(q, r^{(0)}, \omega\right)\right)\right] = 1$$

and

$$r_{\mathcal{X}}^{u}\left(p, r^{(0)}\right)\Big|_{p=0} = \underset{q\in[0,1]}{\arg\max}\, E_0\left[u\left(\mathcal{X}\left(q, r^{(0)}, \omega\right)\right)\right] = 0$$

$$\{0,1\} \in Im\left(r_{\mathcal{X}}^{u}\left(\cdot, r^{(0)}\right)\right). \tag{A12}$$

The function $r^u_{\mathcal{X}r^{(0)}}(p)$ is continuous and since the continuous image of a compact set is compact, (A11) and (A12) gives:

$$[0,1] \subseteq Im\left(r_{\mathcal{X}}^{u}\left(\cdot, r^{(0)}\right)\right),$$

which completes the proof. □

Theorem 1 *Let $\mathcal{X}(\cdot)$ and $\mathcal{Y}(\cdot)$ be two market scoring rules. Also let $u \in \mathcal{U}$ be a concave (convex) function. The following holds for all $r^{(0)} \in [0,1]$:*

$$\forall p \in [0,1] : \mathcal{F}_{\mathcal{X}}^{u}\left(p, r^{(0)}\right) \le \mathcal{F}_{\mathcal{Y}}^{u}\left(p, r^{(0)}\right) \iff \begin{cases} \frac{u'\left(\mathcal{Y}\left(q,r^{(0)},1\right)\right)}{u'\left(\mathcal{Y}\left(q,r^{(0)},0\right)\right)} \le (\ge) \frac{u'\left(\mathcal{X}\left(q,r^{(0)},1\right)\right)}{u'\left(\mathcal{X}\left(q,r^{(0)},0\right)\right)}, \ \forall q \in \left[0, r^{(0)}\right] \\ \frac{u'\left(\mathcal{Y}\left(q,r^{(0)},1\right)\right)}{u'\left(\mathcal{Y}\left(q,r^{(0)},0\right)\right)} \ge (\le) \frac{u'\left(\mathcal{X}\left(q,r^{(0)},1\right)\right)}{u'\left(\mathcal{X}\left(q,r^{(0)},0\right)\right)}, \ \forall q \in \left[r^{(0)}, 1\right] \end{cases} \tag{A13}$$

Proof. Let $r_{\mathcal{X}}^{u} \triangleq r_{\mathcal{X}}^{u}\left(p, r^{(0)}\right)$ and $r_{\mathcal{Y}}^{u} \triangleq r_{\mathcal{Y}}^{u}\left(p, r^{(0)}\right)$, be the expected utility-maximizing report, defined in (14), of an agent in the Hanson MSR $\mathcal{X}$ and $\mathcal{Y}$, respectively. By the definition of deviation, Definition 2, we have:

$$\mathcal{F}_{\mathcal{X}}^{u}\left(p, r^{(0)}\right) \le \mathcal{F}_{\mathcal{Y}}^{u}\left(p, r^{(0)}\right) \iff \left|r_{\mathcal{X}}^{u} - p\right| \le \left|r_{\mathcal{Y}}^{u} - p\right|.$$

Without loss of generality let $p < r^{(0)}$ and u be a concave utility function. By Lemma 2-ii, we have $p < r_{\mathcal{X}}^{u}, r_{\mathcal{Y}}^{u} < r^{(0)}$ and thus we get:

$$\mathcal{F}_{\mathcal{X}}^{u}\left(p, r^{(0)}\right) \le \mathcal{F}_{\mathcal{Y}}^{u}\left(p, r^{(0)}\right) \iff r_{\mathcal{X}}^{u} - p \le r_{\mathcal{Y}}^{u} - p \iff r_{\mathcal{X}}^{u} \le r_{\mathcal{Y}}^{u}.$$

By Lemma 2-i we have:

$$\frac{1-r_{\mathcal{X}}^{u}}{r_{\mathcal{X}}^{u}} \frac{u'\left(\mathcal{X}\left(r_{\mathcal{X}}^{u},r^{(0)},1\right)\right)}{u'\left(\mathcal{X}\left(r_{\mathcal{X}}^{u},r^{(0)},0\right)\right)} = \frac{1-p}{p} = \frac{1-r_{\mathcal{Y}}^{u}}{r_{\mathcal{Y}}^{u}} \frac{u'\left(\mathcal{Y}\left(r_{\mathcal{Y}}^{u},r^{(0)},1\right)\right)}{u'\left(\mathcal{Y}\left(r_{\mathcal{Y}}^{u},r^{(0)},0\right)\right)}. \tag{A14}$$

By simplifying (A14) we get:

$$\frac{1-r_{\mathcal{X}}^{u}}{r_{\mathcal{X}}^{u}} \frac{u'\left(\mathcal{X}\left(r_{\mathcal{X}}^{u},r^{(0)},1\right)\right)}{u'\left(\mathcal{X}\left(r_{\mathcal{X}}^{u},r^{(0)},0\right)\right)} = \frac{1-r_{\mathcal{Y}}^{u}}{r_{\mathcal{Y}}^{u}} \frac{u'\left(\mathcal{Y}\left(r_{\mathcal{Y}}^{u},r^{(0)},1\right)\right)}{u'\left(\mathcal{Y}\left(r_{\mathcal{Y}}^{u},r^{(0)},0\right)\right)}. \tag{A15}$$

By (A15) and simple comparison we get:

$$r_{\mathcal{X}}^{u} \le r_{\mathcal{Y}}^{u} \iff \frac{1-r_{\mathcal{X}}^{u}}{r_{\mathcal{X}}^{u}} \ge \frac{1-r_{\mathcal{Y}}^{u}}{r_{\mathcal{Y}}^{u}} \iff \frac{u'\left(\mathcal{X}\left(r_{\mathcal{X}}^{u},r^{(0)},1\right)\right)}{u'\left(\mathcal{X}\left(r_{\mathcal{X}}^{u},r^{(0)},0\right)\right)} \le \frac{u'\left(\mathcal{Y}\left(r_{\mathcal{Y}}^{u},r^{(0)},1\right)\right)}{u'\left(\mathcal{Y}\left(r_{\mathcal{Y}}^{u},r^{(0)},0\right)\right)}. \tag{A16}$$

Consider $q \in \left[0, r^{(0)}\right]$ and assume:

$$\frac{u'\left(\mathcal{X}\left(q,r^{(0)},1\right)\right)}{u'\left(\mathcal{X}\left(q,r^{(0)},0\right)\right)} \le \frac{u'\left(\mathcal{Y}\left(q,r^{(0)},1\right)\right)}{u'\left(\mathcal{Y}\left(q,r^{(0)},0\right)\right)}. \tag{A17}$$

To prove (A13), in the case wehre $q \in \left[0, r^{(0)}\right]$, we need to show (A16) $\iff$ (A17).

- (A16) $\Longrightarrow$ (A17): By (A16), $r^u_{\mathcal{X}} \le r^u_{\mathcal{Y}}$. By Lemma 2-ii, and the fact that u' is decreasing, we get:

$$\begin{cases} \mathcal{Y}\left(r^u_{\mathcal{X}}, r^{(0)}, 1\right) \le \mathcal{Y}\left(r^u_{\mathcal{Y}}, r^{(0)}, 1\right) \\ \mathcal{Y}\left(r^u_{\mathcal{Y}}, r^{(0)}, 0\right) \le \mathcal{Y}\left(r^u_{\mathcal{X}}, r^{(0)}, 0\right) \end{cases} \Longrightarrow \frac{u'\left(\mathcal{Y}\left(r^u_{\mathcal{Y}}, r^{(0)}, 1\right)\right)}{u'\left(\mathcal{Y}\left(r^u_{\mathcal{Y}}, r^{(0)}, 0\right)\right)} \le \frac{u'\left(\mathcal{Y}\left(r^u_{\mathcal{X}}, r^{(0)}, 1\right)\right)}{u'\left(\mathcal{Y}\left(r^u_{\mathcal{X}}, r^{(0)}, 0\right)\right)}. \tag{A18}$$

By (A16), the right hand side of (A18) implies:

$$\frac{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^{(0)}, 1\right)\right)}{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^{(0)}, 0\right)\right)} \le \frac{u'\left(\mathcal{Y}\left(r^u_{\mathcal{X}}, r^{(0)}, 1\right)\right)}{u'\left(\mathcal{Y}\left(r^u_{\mathcal{X}}, r^{(0)}, 0\right)\right)}. \tag{A19}$$

By Lemma 3, (A19) can be rewritten as:

$$\frac{u'\left(\mathcal{X}\left(q, r^{(0)}, 1\right)\right)}{u'\left(\mathcal{X}\left(q, r^{(0)}, 0\right)\right)} \le \frac{u'\left(\mathcal{Y}\left(q, r^{(0)}, 1\right)\right)}{u'\left(\mathcal{Y}\left(q, r^{(0)}, 0\right)\right)},$$

which proves (A16) $\Longrightarrow$ (A17).

- (A17) $\Longrightarrow$ (A16): Assume to the contrary that there exists $p^{(0)} \in \left[0, r^{(0)}\right]$ and let $a^u_{\mathcal{X}} \triangleq r^u_{\mathcal{X}}\left(p^{(0)}, r^{(0)}\right)$ and $b^u_{\mathcal{X}} \triangleq r^u_{\mathcal{Y}}\left(p^{(0)}, r^{(0)}\right)$, such that:

$$\frac{u'\left(\mathcal{Y}\left(b^u_{\mathcal{Y}}, r^{(0)}, 1\right)\right)}{u'\left(\mathcal{Y}\left(b^u_{\mathcal{Y}}, r^{(0)}, 0\right)\right)} < \frac{u'\left(\mathcal{X}\left(a^u_{\mathcal{X}}, r^{(0)}, 1\right)\right)}{u'\left(\mathcal{X}\left(a^u_{\mathcal{X}}, r^{(0)}, 0\right)\right)}. \tag{A20}$$

By (A15), we get:

$$b^u_{\mathcal{X}} < a^u_{\mathcal{X}}. \tag{A21}$$

By (A17), (A21), and the fact that u' is decreasing, we get:

$$\begin{cases} \mathcal{X}\left(a^u_{\mathcal{X}}, r^{(0)}, 1\right) \le \mathcal{X}\left(b^u_{\mathcal{Y}}, r^{(0)}, 1\right) \\ \mathcal{X}\left(b^u_{\mathcal{Y}}, r^{(0)}, 0\right) \le \mathcal{X}\left(a^u_{\mathcal{X}}, r^{(0)}, 0\right) \end{cases} \Longrightarrow \frac{u'\left(\mathcal{X}\left(a^u_{\mathcal{X}}, r^{(0)}, 1\right)\right)}{u'\left(\mathcal{X}\left(a^u_{\mathcal{X}}, r^{(0)}, 0\right)\right)} \le \frac{u'\left(\mathcal{X}\left(b^u_{\mathcal{Y}}, r^{(0)}, 1\right)\right)}{u'\left(\mathcal{X}\left(b^u_{\mathcal{Y}}, r^{(0)}, 0\right)\right)}. \tag{A22}$$

By (A20), the right-hand side of (A22) gives:

$$\frac{u'\left(\mathcal{Y}\left(b^u_{\mathcal{Y}}, r^{(0)}, 1\right)\right)}{u'\left(\mathcal{Y}\left(b^u_{\mathcal{Y}}, r^{(0)}, 0\right)\right)} < \frac{u'\left(\mathcal{X}\left(b^u_{\mathcal{Y}}, r^{(0)}, 1\right)\right)}{u'\left(\mathcal{X}\left(b^u_{\mathcal{Y}}, r^{(0)}, 0\right)\right)}. \tag{A23}$$

Since u is concave, $p^{(0)} \in \left[0, r^{(0)}\right]$ and $b^u_{\mathcal{X}} \triangleq r^u_{\mathcal{Y}}\left(p^{(0)}, r^{(0)}\right)$, we conclude that $b^u_{\mathcal{X}} \in \left[0, r^{(0)}\right]$. Thus (A23) is a contradiction with (A17). $\square$

Corollary 1 *Let $\mathcal{X}(\cdot)$ and $\mathcal{Y}(\cdot)$ be two market scoring rules that satisfy:*

$$\forall r^{(0)} \in [0,1]: \begin{cases} \mathcal{Y}\left(q, r^{(0)}, 1\right) \le \mathcal{X}\left(q, r^{(0)}, 1\right) \le \mathcal{X}\left(q, r^{(0)}, 0\right) \le \mathcal{Y}\left(q, r^{(0)}, 0\right), \forall q \in \left[0, r^{(0)}\right] \\ \mathcal{Y}\left(q, r^{(0)}, 0\right) \le \mathcal{X}\left(q, r^{(0)}, 0\right) \le \mathcal{X}\left(q, r^{(0)}, 1\right) \le \mathcal{Y}\left(q, r^{(0)}, 1\right), \forall q \in \left[r^{(0)}, 1\right] \end{cases}. \tag{A24}$$

Hence the following holds:

$$\forall u \in \mathcal{U}, \forall p \in [0,1]: \mathcal{F}^u_{\mathcal{X}}\left(p, r^{(0)}\right) \le \mathcal{F}^u_{\mathcal{Y}}\left(p, r^{(0)}\right).$$

Proof. Let (A24) hold, $u(\cdot)$ be concave and without loss of generality let $q \in \left[r^{(0)}, 1\right]$. By Theorem 1, to prove the claim we need to prove the following:

$$\frac{u'\left(\mathcal{X}\left(q, r^{(0)}, 1\right)\right)}{u'\left(\mathcal{X}\left(q, r^{(0)}, 0\right)\right)} \leq \frac{u'\left(\mathcal{Y}\left(q, r^{(0)}, 1\right)\right)}{u'\left(\mathcal{Y}\left(q, r^{(0)}, 0\right)\right)}. \tag{A25}$$

Since $u(\cdot)$ is concave and monotonically increasing, u', the first derivative of the function $u(\cdot)$, is monotonically decreasing. (A24) and Lemma 1-ii gives:

$$\begin{cases} \mathcal{Y}\left(q, r^{(0)}, 1\right) \leq \mathcal{X}\left(q, r^{(0)}, 1\right) \\ \mathcal{X}\left(q, r^{(0)}, 0\right) \leq \mathcal{Y}\left(q, r^{(0)}, 0\right) \end{cases} \Longrightarrow \frac{u'\left(\mathcal{X}\left(q, r^{(0)}, 1\right)\right)}{u'\left(\mathcal{X}\left(q, r^{(0)}, 0\right)\right)} \leq \frac{u'\left(\mathcal{Y}\left(q, r^{(0)}, 1\right)\right)}{u'\left(\mathcal{Y}\left(q, r^{(0)}, 0\right)\right)}, \tag{A26}$$

which completes the proof in this case.

When $u(\cdot)$ is convex, by Theorem 1 we need to prove the reverse of (A25). However, since $u(\cdot)$ is convex and monotonically increasing, $u'(\cdot)$ is monotonically increasing, and thus (A24) is enough to show the reverse of (A25). □

Theorem 2 *Let $C^{\mathcal{X}}$ and $C^{\mathcal{Y}}$ be the cost function market makers of the two MSRs $\mathcal{X}$ and $\mathcal{Y}$ respectively, in which $C^{\mathcal{Y}}(\mathbf{s}) = b\, C^{\mathcal{X}}(\mathbf{s}/b)$ for some $b > 1$. The market maker $C^{\mathcal{Y}}$ has more inverse liquidity if and only if $\mathcal{Y}$ has more deviation compared to $\mathcal{X}$.*

Proof.

$\Rightarrow$: Since $C^{\mathcal{Y}}(\mathbf{s}) = b\, C^{\mathcal{X}}(\mathbf{s}/b)$, by definition we have $\mathcal{Y} = b\, \mathcal{X}$; in which $\mathcal{Y}$ and $\mathcal{X}$ are the underlying MSRs of the two cost functions $C^{\mathcal{Y}}$ and $C^{\mathcal{X}}$ respectively. Since $b > 1$ we have:

$$\forall r^{(0)} \in [0,1]: \begin{cases} b\,\mathcal{X}\left(q, r^{(0)}, 1\right) \leq \mathcal{X}\left(q, r^{(0)}, 1\right) \leq \mathcal{X}\left(q, r^{(0)}, 0\right) \leq b\,\mathcal{X}\left(q, r^{(0)}, 0\right), \ \forall q \in \left[0, r^{(0)}\right] \\ b\,\mathcal{X}\left(q, r^{(0)}, 0\right) \leq \mathcal{X}\left(q, r^{(0)}, 0\right) \leq \mathcal{X}\left(q, r^{(0)}, 1\right) \leq b\,\mathcal{X}\left(q, r^{(0)}, 1\right), \ \forall q \in \left[r^{(0)}, 1\right] \end{cases}.$$

Thus by Corollary 1, $\mathcal{Y} = b\mathcal{X}$ has more deviation compared to $\mathcal{X}$.

$\Leftarrow$: Since $b > 1$, by definition of inverse liquidity (8), $C^{\mathcal{Y}}$ has more liquidity compared to $C^{\mathcal{X}}$.

□

Theorem 3 *Let $C^{\mathcal{X}}$ and $C^{\mathcal{Y}}$ be the cost function market makers of the two MSRs $\mathcal{X}$ and $\mathcal{Y}$ respectively. $C^{\mathcal{X}}$ has more market depth relative to $C^{\mathcal{Y}}$, only if $\mathcal{X}$ has more deviation compared to $\mathcal{Y}$.*

Proof. By Equivalence relation (5) we have:

$$C^{\mathcal{X}}(\mathbf{s}) = \max_{q \in [0,1]} s_1 q + s_0(1-q) - \left(qX(q,1) + (1-q)X(q,0)\right). \tag{A27}$$

By definition of the price function (6), (A27), and the first order condition of (A27); we have:

$$\frac{\partial}{\partial q}\left(s_1 q + s_0(1-q) - \left(qX(q,1) + (1-q)X(q,0)\right)\right)\Big|_{q = Pr_1^{\mathcal{X}}(\mathbf{s})} = 0. \tag{A28}$$

Expanding (A28) gives:

$$s_1 - \big(X(q,1) - X(q,0)\big) + q\frac{\partial}{\partial q}X(q,1) + (1-q)\frac{\partial}{\partial q}X(q,0)\Big|_{q=Pr_1^{\mathcal{X}}(\mathbf{s})} = 0. \tag{A29}$$

By Lemma 1-i, $q\frac{\partial}{\partial q}X(q,1) + (1-q)\frac{\partial}{\partial q}X(q,0)\Big|_{q=Pr_1^{\mathcal{X}}(\mathbf{s})} = 0$, thus (A29) reduces to:

$$s_1 - \left(X\left(Pr_1^{\mathcal{X}}(\mathbf{s},1)\right) - X\left(Pr_1^{\mathcal{X}}(\mathbf{s},0)\right)\right) = 0. \tag{A30}$$

From (A30), we can implicitly derive the rate of change in $Pr_1^{\mathcal{X}}$ w.r.t. s_1. By implicit differentiation we get:

$$\frac{\partial}{\partial s_1}Pr_1^{\mathcal{X}}(\mathbf{s}) = \frac{1}{\frac{\partial}{\partial q}\left(X(q,1) - X(q,0)\right)\Big|_{q=Pr_1^{\mathcal{X}}(\mathbf{s})}}. \tag{A31}$$

Thus by definition of market depth, (9), we get:

$$\rho_1^{\mathcal{X}}(\mathbf{s}) = \frac{\partial}{\partial q}\big(X(q,1) - X(q,0)\big)\Big|_{q=Pr_1^{\mathcal{X}}(\mathbf{s})} \tag{A32}$$

Carrying our the procedure above, after replacing $C^{\mathcal{Y}}$ for $C^{\mathcal{X}}$ we find:

$$\rho_1^{\mathcal{Y}}(\mathbf{s}) = \frac{\partial}{\partial q}\big(Y(q,1) - Y(q,0)\big)\Big|_{q=Pr_1^{\mathcal{Y}}(\mathbf{s})}. \tag{A33}$$

Thus by Definition 1, $C^{\mathcal{Y}}$ has more market depth relative to $C^{\mathcal{X}}$, only if[8]:

$$\forall a\in[0,1]: \frac{\partial}{\partial q}\big(X(q,1) - X(q,0)\big)\Big|_{q=a} \le \frac{\partial}{\partial q}\big(Y(q,1) - Y(q,0)\big)\Big|_{q=a}. \tag{A34}$$

Therefore, to show that $C^{\mathcal{Y}}$ has more market depth relative to $C^{\mathcal{X}}$ it is enough to show that (A34) is satisfied.

By assumption, $\mathcal{Y}$ has more deviation compared to $\mathcal{X}$. Corollary 1, for a given $a\in[0,1]$ gives:

$$\begin{cases} \mathcal{Y}(q,a,1) \le \mathcal{X}(q,a,1) \le \mathcal{X}(q,a,0) \le \mathcal{Y}(q,a,0),\ q\le a \\ \mathcal{Y}(q,a,0) \le \mathcal{X}(q,a,0) \le \mathcal{X}(q,a,1) \le \mathcal{Y}(q,a,1),\ q> a \end{cases} \tag{A35}$$

Equation (A35) gives:

$$\begin{cases} \mathcal{X}(q,a,0) - \mathcal{X}(q,a,1) \le \mathcal{Y}(q,a,0) - \mathcal{Y}(q,a,1) : q\le a \\ \mathcal{X}(q,a,1) - \mathcal{X}(q,a,0) \le \mathcal{Y}(q,a,1) - \mathcal{Y}(q,a,0) : q> a \end{cases}. \tag{A36}$$

Thus by definition of a MSR (11), (A36) can be expanded to:

$$\begin{cases} 0 \le X(q,0) - X(a,0) - \big(X(q,1) - X(a,1)\big) \le Y(q,0) - Y(a,0) - \big(Y(q,1) - Y(a,1)\big) : q\le a \\ 0 \le X(q,1) - X(a,1) - \big(X(q,0) - X(a,0)\big) \le Y(q,1) - Y(a,1) - \big(Y(q,0) - Y(a,0)\big) : q> a \end{cases}. \tag{A37}$$

Rearranging (A37) gives:

$$\begin{cases} \frac{Y(q,1)-Y(q,0)-\left(Y(a,1)-Y(a,0)\right)}{q-a} \le \frac{X(q,1)-X(q,0)-\left(X(a,1)-X(a,0)\right)}{q-a} \quad q\le a \\[2ex] \frac{X(q,1)-X(q,0)-\left(X(a,1)-X(a,0)\right)}{q-a} \le \frac{Y(q,1)-Y(q,0)-\left(Y(a,1)-Y(a,0)\right)}{q-a} \quad q> a \end{cases}. \tag{A38}$$

By (A38) and the limit definition of the first derivative we get:

$$\frac{\partial}{\partial q}\big(X(q,1) - X(q,0)\big)\Big|_{q=a} \le \frac{\partial}{\partial q}\big(Y(q,1) - Y(q,0)\big)\Big|_{q=a}. \tag{A39}$$

Note that the above equation holds since both score functions $X(\cdot)$ and $Y(\cdot)$ are differentiable and continuous. (A39) shows that (A34) is satisfied and the proof is complete. □

[8] Note that since $|\Omega| = 2$, we only require to show (A34) to conclude that $C^{\mathcal{Y}}$ has more market depth relative to $C^{\mathcal{X}}$.

References

Abernethy, Jacob, Yiling Chen, Jennifer Wortman Vaughan. 2013. Efficient market making via convex optimization, and a connection to online learning. *ACM Transactions on Economics and Computation* **1**(2) 12.

Abernethy, Jacob, Yiling Chen, Jennifer Wortman Vaughan. 2011. An optimization-based framework for automated market-making. *Proceedings of the 12th ACM conference on Electronic Commerce*. ACM, 297–306.

Abernethy, Jacob, Sindhu Kutty, Sébastien Lahaie, Rahul Sami. 2014. Information aggregation in exponential family markets. *Proceedings of the fifteenth ACM conference on Economics and computation*. ACM, 395–412.

Allen, Franklin. 1987. Notes-discovering personal probabilities when utility functions are unknown. *Management Science* **33**(4) 542–544.

Arrow, Kenneth J, Robert Forsythe, Michael Gorham, Robert Hahn, Robin Hanson, John O Ledyard, Saul Levmore, Robert Litan, Paul Milgrom, Forrest D Nelson, et al. 2008. The promise of prediction markets. *Science* **320**(5878) 877–878.

Atanasov, Pavel, Phillip Rescober, Eric Stone, Samuel A Swift, Emile Servan-Schreiber, Philip Tetlock, Lyle Ungar, Barbara Mellers. 2016. Distilling the wisdom of crowds: Prediction markets vs. prediction polls. *Management Science* to appear.

Babcock, Bruce A, E Kwan Choi, Eli Feinerman. 1993. Risk and probability premiums for cara utility functions. *Journal of Agricultural and Resource Economics* **18**(1) 17–24.

Berg, Joyce E, George R Neumann, Thomas A Rietz. 2009. Searching for google's value: Using prediction markets to forecast market capitalization prior to an initial public offering. *Management Science* **55**(3) 348–361.

Bossaerts, Peter, Leslie Fine, John Ledyard. 2002. Inducing liquidity in thin financial markets through combined-value trading mechanisms. *European Economic Review* **46**(9) 1671–1695.

Brahma, Aseem, Mithun Chakraborty, Sanmay Das, Allen Lavoie, Malik Magdon-Ismail. 2012. A bayesian market maker. *Proceedings of the 13th ACM Conference on Electronic Commerce*. ACM, 215–232.

Brier, Glenn W. 1950. Verification of forecasts expressed in terms of probability. *Monthly Weather Review* **78**(1) 1–3.

Chen, Kay-Yut, Leslie R Fine, Bernardo A Huberman. 2004. Eliminating public knowledge biases in information-aggregation mechanisms. *Management Science* **50**(7) 983–994.

Chen, M Keith, Jonathan E Ingersoll Jr, Edward H Kaplan. 2008. Modeling a presidential prediction market. *Management Science* **54**(8) 1381–1394.

Chen, Yiling, David M Pennock. 2007. A utility framework for bounded-loss market makers. *In Proceedings of the 23rd Conference on Uncertainty in Artificial Intelligence*.

Chen, Yiling, Jennifer Wortman Vaughan. 2010. A new understanding of prediction markets via no-regret learning. *Proceedings of the 11th ACM conference on Electronic Commerce*. ACM, 189–198.

Cowgill, Bo, Eric Zitzewitz. 2015. Corporate prediction markets: Evidence from google, ford, and firm x. *The Review of Economic Studies* **82**(4) 1309–1341.

Cox, James C, Glenn W Harrison. 2008. *Risk Aversion in Experiments*, vol. 12. Emerald Group Publishing.

Csaszar, Felipe A, JP Eggers. 2013. Organizational decision making: An information aggregation view. *Management Science* **59**(10) 2257–2277.

Dimitrov, Stanko, Rahul Sami, Marina A Epelman. 2015. Subsidized prediction mechanisms for risk-averse agents. *ACM Transactions on Economics and Computation* **3**(4) 24.

D.R. 2016. Polls versus prediction markets: Who said brexit was a surprise? *The Economist* .

Dudík, Miroslav, Sébastien Lahaie, Ryan Rogers, Jennifer Wortman Vaughan. 2017. A decomposition of forecast error in prediction markets. Working Paper, Microsoft Research.

Filios, Matt. 2016. Identifying black swans. URL `http://www.predictful.com/category/prediction-markets/`.

Frongillo, Rafael, Mark D Reid. 2015. Convergence analysis of prediction markets via randomized subspace descent. *Advances in Neural Information Processing Systems*. 3034–3042.

Frongillo, Rafael M, Nicholás Della Penna, Mark D Reid. 2012. Interpreting prediction markets: a stochastic approach. *Advances in Neural Information Processing Systems*. 3266–3274.

Gneiting, Tilmann, Adrian E Raftery. 2007. Strictly proper scoring rules, prediction, and estimation. *Journal of the American Statistical Association* **102**(477) 359–378.

Good, I. J. 1952. Rational decisions. *Journal of the Royal Statistical Society. Series B (Methodological)* **14**(1) 107–114. URL `http://www.jstor.org/stable/2984087`.

Hanson, Robin. 2003. Combinatorial information market design. *Information Systems Frontiers* **5**(1) 107–119.

Healy, Paul J, Sera Linardi, J Richard Lowery, John O Ledyard. 2010. Prediction markets: alternative mechanisms for complex environments with few traders. *Management Science* **56**(11) 1977–1996.

Heidari, Hoda, Sébastien Lahaie, David M Pennock, Jennifer Wortman Vaughan. 2015. Integrating market makers, limit orders, and continuous trade in prediction markets. *Proceedings of the Sixteenth ACM Conference on Economics and Computation*. ACM, 583–600.

Kadane, Joseph B, Robert L Winkler. 1988. Separating probability elicitation from utilities. *Journal of the American Statistical Association* **83**(402) 357–363.

Karni, Edi. 2009. A mechanism for eliciting probabilities. *Econometrica* **77**(2) 603–606.

Lambert, Nicholas. 2011. Probability elicitation for agents with arbitrary risk preferences. Working Paper, Stanford University, Graduate School of Business.

Ledbetter, James. 2008. The fizz-dom of crowds: If prediction markets are so great, whey have they been so wrong lately? *Slate* .

Li, Xiaolong, Jennifer Wortman Vaughan. 2013. An axiomatic characterization of adaptive-liquidity market makers. *Proceedings of the fourteenth ACM conference on Electronic Commerce*. ACM, 657–674.

Manski, Charles F. 2006. Interpreting the predictions of prediction markets. *Economics Letters* **91**(3) 425–429.

McCarthy, John. 1956. Measures of the value of information. *Proceedings of the National Academy of Sciences of the United States of America* **42**(9) 654–655.

Mellers, Barbara, Eric Stone, Pavel Atanasov, Nick Rohrbaugh, S Emlen Metz, Lyle Ungar, Michael M Bishop, Michael Horowitz, Ed Merkle, Philip Tetlock. 2015. The psychology of intelligence analysis: Drivers of prediction accuracy in world politics. *Journal of Experimental Psychology: Applied* **21**(1) 1.

Murphy, Allan H, Robert L Winkler. 1970. Scoring rules in probability assessment and evaluation. *Acta Psychologica* **34** 273–286.

Offerman, Theo, Joep Sonnemans, Gijs Van de Kuilen, Peter P Wakker. 2009. A truth serum for non-bayesians: Correcting proper scoring rules for risk attitudes. *The Review of Economic Studies* **76**(4) 1461–1489.

Ortner, Gerhard. 1998. Forecasting markets – an industrial application. Universität Wien Working Paper.

Ostrovsky, Michael. 2012. Information aggregation in dynamic markets with strategic traders. *Econometrica* **80**(6) 2595–2647.

Othman, Abraham, David M Pennock, Daniel M Reeves, Tuomas Sandholm. 2013. A practical liquidity-sensitive automated market maker. *ACM Transactions on Economics and Computation* **1**(3) 14.

Palfrey, Thomas R, Stephanie W Wang. 2012. Speculative overpricing in asset markets with information flows. *Econometrica* **80**(5) 1937–1976.

Pennock, D. M. 2010. Why automated market makers? `blog.oddhead.com/2010/07/08/why-automated-market-makers/`. Accessed 03/06/2015.

Plott, Charles R, Kay-Yut Chen. 2002. Information aggregation mechanisms: Concept, design and implementation for a sales forecasting problem. Social Science Working Paper, 1131, California Institute Of Technology, Division Of The Humanities And Social Sciences.

Polgreen, Philip M, Forrest D Nelson, George R Neumann, Robert A Weinstein. 2007. Use of prediction markets to forecast infectious disease activity. *Clinical Infectious Diseases* **44**(2) 272–279.

Pratt, John W. 1964. Risk aversion in the small and in the large. *Econometrica: Journal of the Econometric Society* 122–136.

Premachandra, Mindika, Mark Reid. 2013. Aggregating predictions via sequential mini-trading. *Asian Conference on Machine Learning*. 373–387.

Savage, Leonard J. 1971. Elicitation of personal probabilities and expectations. *Journal of the American Statistical Association* **66**(336) 783–801.

Schlack, Julie Wittes. 2015. Ask your customers for predictions, not preferences. *Harvard Business Review, January* **5**.

Schlag, Karl H, Joël J van der Weele. 2013. Eliciting probabilities, means, medians, variances and covariances without assuming risk neutrality. *Theoretical Economics Letters* **3**(1) 38–42.

Sethi, Rajiv, Jennifer Wortman Vaughan. 2016. Belief aggregation with automated market makers. *Computational Economics* **48**(1) 155–178.

Slamka, Christian, Bernd Skiera, Martin Spann. 2013. Prediction market performance and market liquidity: A comparison of automated market makers. *IEEE Transactions on Engineering Management* **60**(1) 169–185.

Tziralis, George, Ilias Tatsiopoulos. 2012. Prediction markets: An extended literature review. *The Journal of Prediction Markets* **1**(1) 75–91.

Wah, Elaine, Sébastien Lahaie, David M Pennock. 2016. An empirical game-theoretic analysis of price discovery in prediction markets. *IJCAI*. 510–516.

Wakker, Peter, Richard Thaler, Amos Tversky. 1997. Probabilistic insurance. *Journal of Risk and Uncertainty* **15**(1) 7–28.

Winkler, Robert L. 1969. Scoring rules and the evaluation of probability assessors. *Journal of the American Statistical Association* **64**(327) 1073–1078.

Wolfers, Justin, Eric Zitzewitz. 2006. Interpreting prediction market prices as probabilities. Tech. rep., National Bureau of Economic Research.