Edge coloring multigraphs without small dense subsets

P.E. Haxell* H.A. Kierstead†

June 9, 2015

Abstract

One consequence of a long-standing conjecture of Goldberg and Seymour about the chromatic index of multigraphs would be the following statement. Suppose G is a multigraph with maximum degree Δ, such that no vertex subset S of odd size at most Δ induces more than $(\Delta+1)(|S|-1)/2$ edges. Then G has an edge coloring with $\Delta+1$ colors. Here we prove a weakened version of this statement.

1 Introduction

In this note we study edge colorings of (loopless) multigraphs. We use the standard notation $\chi'(G)$ to denote the chromatic index of the multigraph G, that is, the smallest number of matchings needed to partition the edge set of G. It is clear that the maximum degree $\Delta(G)$ is a lower bound for $\chi'(G)$ for every graph G. The classical upper bounds for $\chi'(G)$ are $\chi'(G) \leq 3\Delta(G)/2$ (Shannon's Theorem [15]) and $\chi'(G) \leq \Delta(G) + \mu(G)$ (Vizing's Theorem [18]), where $\mu(G)$ denotes the maximum edge multiplicity of G.

For a multigraph G, a subset $S \subseteq V(G)$, and a subgraph $H \subseteq G$, we denote by $G[S]$ the subgraph induced by S, by $\|H\|$ the number of edges in H, and by $|H|$ the number of vertices in H. We also set $G[H] = G[V(H)]$ and $\|S\| = \|G[S]\|$. Let $\rho(S)$ be the quantity $\frac{\|S\|}{\lfloor |S|/2 \rfloor}$. The parameter $\rho(G)$ is defined by

$$\rho(G) = \max\{\rho(S) : S \subseteq V(G)\}.$$

*Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. pehaxell@uwaterloo.ca; Partially supported by NSERC.

†School of Mathematical and Statistical Sciences, Arizona State University, Tempe, AZ, USA 85287. kierstead@asu.edu. The authors gratefully acknowledge the support of the Mittag-Leffler Institute (Djursholm, Sweden), where this work was done.

Then $\lceil\rho(G)\rceil$ is a lower bound on $\chi'(G)$, since for a set S on which $\rho(G)$ is attained, each matching in $G[S]$ has size at most $\lfloor|S|/2\rfloor$ and therefore at least $\lceil\frac{\|S\|}{\lfloor|S|/2\rfloor}\rceil$ colors are needed to color the edges of $G[S]$. On the other hand, when $\rho(G) \geq \Delta(G)$ the chromatic index can also be bounded above in terms of $\lceil\rho(G)\rceil$. Kahn [7] gave the bound $\chi'(G) \leq (1+o(1))\lceil\rho(G)\rceil$, which was recently improved by Plantholt [10] to

$$\chi'(G) \leq (1 + \frac{\log_{3/2}\lceil\rho(G)\rceil}{\lceil\rho(G)\rceil})\lceil\rho(G)\rceil.$$

The focus of this paper is the long-standing conjecture due to Goldberg [3] (see also [4]) and independently Seymour [14] which states that the chromatic index of G should be essentially determined by either $\rho(G)$ or $\Delta(G)$.

Conjecture 1 *For every multigraph G*

$$\chi'(G) \leq \max\{\Delta(G)+1, \lceil\rho(G)\rceil\}.$$

Goldberg [4] also proposed the following sharp version for multigraphs with $\rho(G) \leq \Delta(G) - 1$.

Conjecture 2 *For every multigraph G, if $\rho(G) \leq \Delta(G) - 1$ then $\chi'(G) = \Delta(G)$.*

Conjecture 1 implies that if $\chi'(G) > \Delta + k$, $k \geq 1$, then G must contain a set S of vertices for which $\rho(S) > \Delta + k$, certifying this inequality. Thus S induces a very dense subgraph in G. As $\|S\| \leq \Delta(G)|S|/2$, if $|S|$ is even then $\rho(S) \leq \Delta(G)$; so $|S|$ is odd and $\rho(S) \leq \Delta(G)|S|/(|S|-1) = \Delta(G) + \Delta(G)/(|S|-1)$. We say S is *small* in the sense that its size depends only on Δ and not on the number of vertices of G. In particular $|S| \leq \Delta(G)$. Conjecture 2 gives a similar statement for $k = 0$, but the corresponding set S need not be small.

We can therefore think of Conjecture 1 as providing structural information about multigraphs for which $\chi'(G) > \Delta + 1$, namely, that they must contain small sets S that are very dense. Our aim in this note is to prove a result of this form. Unfortunately we cannot make such a conclusion about all G with $\chi'(G) > \Delta + 1$, but we show that when k is bounded below by a logarithmic function of Δ then a structural result of this type for multigraphs G satisfying $\chi'(G) > \Delta + k$ is possible.

Conjecture 1 has inspired a significant body of work, with contributions from many researchers, see for example [16] or [6] for an overview. Here we mention just the results that directly relate to this note. The best known approximate version is as follows, due to Scheide [11] (independently proved by Chen, Yu and Zang [1], see also [12] and [2]), who proved that the conjecture is true when $\lceil\rho(G)\rceil \geq \Delta + \sqrt{\frac{\Delta-1}{2}}$.

Theorem 3 *For every multigraph G*

$$\chi'(G) \le \max\{\Delta(G) + \sqrt{\frac{\Delta(G)-1}{2}}, \lceil \rho(G) \rceil\}.$$

Since $\lceil \rho(S) \rceil > \Delta + \sqrt{\frac{\Delta-1}{2}}$ implies $|S| < \sqrt{\frac{2\Delta^2}{\Delta-1}} + 1$, the following corollary about multigraphs without small dense subsets is implied by Theorem 3.

Corollary 4 *Let G be a multigraph with maximum degree Δ. If $\lceil \rho(S) \rceil \le \Delta + \sqrt{\frac{\Delta-1}{2}}$ for every $S \subseteq V(G)$ with $|S| < \sqrt{\frac{2\Delta^2}{\Delta-1}} + 1$ then $\chi'(G) \le \Delta + \sqrt{\frac{\Delta-1}{2}}$.*

The main theorem of this note states that if the density of small vertex subsets S is restricted somewhat further then a substantially better upper bound can be given for $\chi'(G)$, in which the quantity $\sqrt{\frac{\Delta-1}{2}}$ in the conclusion of Corollary 4 is replaced by a logarithmic function of Δ. It can also be viewed as a weakened version of the statement of Conjecture 2.

Theorem 5 *Let G be a multigraph with maximum degree Δ, and let ε be given where $0 < \varepsilon < 1$. Let $k = \lfloor \log_{1+\varepsilon} \Delta \rfloor$. If $\rho(S) \le (1-\varepsilon)(\Delta + k)$ for every $S \subseteq V(G)$ with $|S| < \Delta/k + 1$ then $\chi'(G) \le \Delta + k$.*

For example, this implies that $\chi'(G) < \Delta + 101 \log \Delta$ unless G contains a set S of vertices with $|S| < \frac{\Delta}{100 \log \Delta}$ with density parameter $\rho(S) > 0.99(\Delta + 100 \log \Delta)$.

Our proof uses the technique of Tashkinov trees, developed by Tashkinov in [17]. In the next section we give a brief introduction to this technique together with the main tools we use, including our main technical lemma, Lemma 8. The proof of Theorem 5 appears in Section 3.

2 Tools

The method of Tashkinov trees, due to Tashkinov [17], is a sophisticated generalization of the method of alternating paths. It is based on an earlier approach from [8]. See [16] for a comprehensive account of this technique.

Let G be a multigraph with $\chi'(G) \ge \Delta + 2$, and let ϕ be a partial edge coloring of G that uses at most $\chi' - 1$ colors. We say ϕ is a *t-coloring* if the the codomain of ϕ is $\{1, \dots, t\}$. We normally assume ϕ is *maximal*, that is, the maximum possible number of edges of G are colored by ϕ. For a vertex v of G, color α is said to be *missing* at v if no edge incident to v is colored α by ϕ. Let $T = (p_0, e_0, p_1, \dots, e_n - 1, p_n)$ be a sequence of distinct vertices p_i and edges e_i of G, such that the vertices of each e_i are p_{i+1} and p_r for some $r \in \{0, \ldots, i\}$. Observe that the vertices and edges of T form a tree. We say that T is a *Tashkinov tree* with respect to ϕ if e_0 is uncolored, and for all $i > 0$, the color $\phi(e_i)$ is missing at p_j for some $j < i$. Thus T is a Tashkinov tree if its first edge is uncolored, and each subsequent edge is colored with a color that is missing at some previous vertex. The key property of Tashkinov trees is captured in the following theorem, due to Tashkinov [17].

Theorem 6 *Let ϕ be a maximal partial edge coloring of G with at most $\chi'(G) - 1$ colors, and let T be a Tashkinov tree with respect to ϕ. Then no two vertices of T are missing the same color.*

For a color ω we denote by $\partial_\omega(T)$ the set of edges of color ω that have exactly one vertex in T. Every vertex $v \in V(T)$ is incident to an edge of $G[T]$ of color ω, or is incident to an edge of $\partial_\omega(T)$, or is incident to no edge of color ω. Let $m_\omega(T)$ be the number of vertices missing color ω in T and $q_\omega(T) = |\partial_\omega(T)| + m_\omega(T)$. (Thus $q_\omega(T)$ counts the number of vertices in T that are not incident with an edge of $G[T]$ of color ω.) By Theorem 6, $m_\omega(T)$ is at most 1; so we have the following corollary.

Corollary 7 *Let ϕ be a maximal partial edge coloring of G with at most $\chi'(G) - 1$ colors, and let T be a Tashkinov tree with respect to ϕ. If $|T|$ is odd then for every color ω, the quantity $|\partial_\omega(T)|$ is even if and only if ω is missing at some vertex of T.*

Let T be a Tashkinov tree with respect to some maximal coloring ϕ. If a color α is missing on $v \in V(T)$ and not used by ϕ on an edge of T we say that α is *free* for T. The number of colors missing at v that are free for T is denoted by $f_T(v)$, or simply $f(v)$ if there is no danger of confusion. Set $f^*(T) = \min\{f(v) : v \in T\}$. It was observed by e.g. [2] that if T is a Tashkinov tree with respect to ϕ such that $\rho(G)$ is not attained on $V(T)$, and if $f^*(T) > 0$, then by (possibly) replacing ϕ by another maximal coloring it is possible to construct a Tashkinov tree that is larger than T. This technical fact was used in several results using Tashkinov trees, for example [1, 5, 11]. Our main lemma, Lemma 8, is also based on this parameter. For technical reasons we will work with the slightly modified parameter $f^k(T) = \min\{f^*(T), k\}$.

Lemma 8 *Let G be a multigraph with maximum degree Δ and suppose $\chi'(G) \geq \Delta+2$. Let ϕ be a maximal $(\Delta+k)$-coloring of G, where $\Delta+1 \leq \Delta+k \leq \chi'(G)-1$, and let T be a Tashkinov tree with respect to ϕ such that $f^k(T) > 0$. Let $\omega \in \{1, \ldots, \Delta+k\}$ be a color. Then there exists a maximal $(\Delta+k)$-coloring ψ and a Tashkinov tree T' with respect to ψ such that*

- $T \subset T'$

- $f^k(T') \geq f^k(T) - 1$,
- $|T'| \geq |T| + q_\omega(T) - 1$.

Proof. To simplify notation we let $f(T) = f^k(T)$ throughout this proof.

If ω is missing on a vertex of T then we may simply add the edges in $\partial_\omega(T)$ to T, forming a Tashkinov tree T' with $|T'| = |T| + q_\omega(T) - 1$. Clearly $f_T(v) \geq f_{T'}(v)$ for every vertex v of T. Since T' is a Tashkinov tree, each color used on an edge of T' is missing at some vertex of T. Thus by Theorem 6, each color missing at a vertex of $T' - T$ is not used on an edge of T', and so each such color is free for T'. As at least k colors are missing at each vertex in $V(T') \setminus V(T)$ and they are all free, we have $f(T') \geq \min\{f(T) - 1, k\}$ for all vertices in $V(T') \setminus V(T)$. Hence by definition of $f(T) = f^k(T)$ we find $f(T') \geq f(T) - 1$ and so $\psi = \phi$ satisfies the theorem.

We may therefore assume that ω is not missing on T. Set $q = q_\omega(T) = |\partial_\omega(T)|$. There are at least $k \geq 1$ colors missing on each vertex of T. By Theorem 6, these $\geq k|T|$ colors are distinct. As T has only $|T| - 2$ colored edges, there are at least k free colors missing on some vertex v; let γ be one of these.

We consider the (γ, ω)-alternating path P beginning at v. The other end z of P is not a vertex of T, since ω is not missing in T and by Theorem 6 no $x \in V(T)$ different from v can be missing γ. Let y be the last vertex of P in T and denote by Q the (y, z)-segment of P. Then $E(Q) \cap E(T) = \emptyset$. Since $f(T) > 0$ there exists a color α missing on y that is not used on T. In the case $v = y$ we choose $\alpha = \gamma$, otherwise $\alpha \neq \gamma$ by Theorem 6. See Figure 1 for a general picture of P.

For $i \geq 0$ we now define a sequence of Tashkinov trees T_i with respect to ϕ, together with colors α_i, vertices z_i and segments Q_i of Q satisfying the following properties.

1. $T_0 \subset \cdots \subset T_i$,
2. α_i is missing on z_i and not used on T_i,
3. $f(T_i) \geq f(T) - 1$ for each $i \geq 1$,
4. for $i \geq 1$, every edge of $E(T_i) \setminus E(T_{i-1})$ is of color γ or α_{i-1},
5. Q_i is the (z_i, z)-segment of Q, and the length of Q_i is positive but less than the length of Q_{i-1}.

We begin the construction by setting $T_0 = T$, $\alpha_0 = \alpha$, $z_0 = y$, and $Q_0 = Q$. Then (1)-(5) hold for $i = 0$.

Suppose $i \geq 0$ and that we have completed the construction up to i. We now consider two cases according to whether any (α_i, γ)-component intersects both T_i

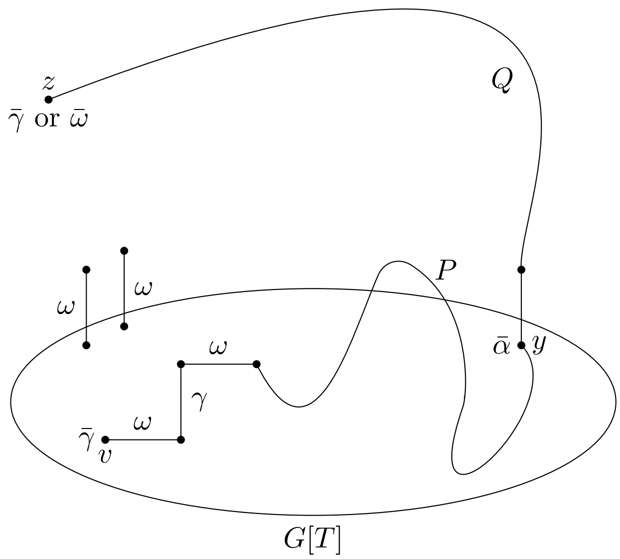

Figure 1: The path P in the proof of Lemma 8. Here $\bar{c}$ means color c is missing at the vertex indicated.

and $E(Q_i)$. If there is such a component then we show that either ϕ itself satisfies the theorem, or that we can extend our sequence. If no such component exists then we will terminate the sequence and find a recoloring ψ that satisfies the theorem.

Case 1: Some (α_i, γ)-component R contains an edge of Q_i and a vertex of T_i.

Note that Case 1 occurs if the edge e of Q incident to y has color γ because e is in Q and has a vertex y in T. Also, Case 1 does not occur if $v = y$ since then an (α_0, γ)-component is an edge colored γ, which cannot be incident to v.

In Case 1 we define T_{i+1} to be the Tashkinov tree obtained by extending T_i to a spanning tree of $T_i \cup R$. This is a valid Tashkinov tree for ϕ because α_i and γ are both missing on T_i. Then (1) and (4) are satisfied for $i+1$. We let z_{i+1} be the vertex of T_{i+1} that is closest to z on Q_i, and note that the (z_{i+1}, z)-segment Q_{i+1} is shorter than Q_i because R contained an edge of Q_i, verifying the second condition in (5) for $i+1$. Let α_{i+1} be any color missing on z_{i+1}; then (2) is satisfied for $i+1$.

To verify Condition (3) for $i+1$, first note that every vertex of $V(T_{i+1}) \setminus V(T)$ has at least k missing colors that are not used on T. Observe that by (4), every edge of $E(T_{i+1}) \setminus E(T)$ has one of the colors γ or α_j for some $0 \le j \le i$. By (2), the colors α_j for $j \ge 1$ are missing on the vertices z_j, and since the z_j are all distinct (by the second part of (5)), no other color missing on z_j is used on T_{i+1}. So $f_{T_{i+1}}(z_j) \geq k-1$. By the choice of γ, which is missing on v, we know $f_{T_{i+1}}(v) \geq k-1 \geq f(T)-1$ since no other colors missing on v were used. Therefore the only new color used that may affect $f(T_{i+1})$ is $\alpha_0 = \alpha$, and hence $f(T_{i+1}) \geq f(T) - 1$.

Finally we turn to the first condition in (5). If this condition holds, in other words $z_{i+1} \neq z$, then we extend our sequence using the above definitions. If $z_{i+1} = z$, then we claim that ϕ satisfies the lemma in this case. Note that if γ is missing at z then we have a contradiction to Theorem 6, because γ is also missing at $v \neq z$. Therefore ω is missing at z. Then we may construct T' by adding all remaining edges of $\partial_\omega(T)$ that join a vertex of T_{i+1} to a vertex outside T_{i+1}. By the existence of R this in fact gives us $|T'| \geq |T|+q$. By (3) for $i+1$ we have $f(T_{i+1}) \geq f(T)-1$, and the only new color used in the construction of T' from T_{i+1} is ω, which is missing on z_{i+1}. But no other color missing on z_{i+1} appears on an edge of T', so $f_{T_{i+1}}(z_{i+1}) \geq k-1 \geq f(T)-1$. Thus $f(T') \geq f(T) - 1$.

Case 2: No (α_i, γ)-component contains an edge of Q_i and a vertex of T_i.

In this case we modify ϕ. Note that (as observed in Case 1) if $i = 0$ then the edge e of Q incident to y has color ω. First we interchange α_i and γ on every (α_i, γ)-component containing an edge of Q_i. Since we are in Case 2, this change does not affect the color of any edge induced by $V(T_i)$. Therefore T_i is a Tashkinov tree with respect to the new coloring. The path Q_i becomes an (α_i, ω)-path from z_i to z, which (as before) is disjoint from all of $\partial_\omega(T)$ except possibly for e, if it has color ω. (Note that if $i = 0$ and $v = y$ then none of these steps caused any change.) We complete the construction of ψ by interchanging ω and α_i on Q_i. Then ω is missing on z_i. We construct T' by adding to T_i all the edges of $\partial_\omega(T) \setminus \{e\}$ that join $V(T_i)$ to its complement. Then $|T'| \geq |T| + q - 1$ (and if $i \geq 1$ then $|T'| \geq |T| + q$). The only new color used that was not used on T_i is ω, which is missing on z_i. If $i = 0$ then trivially $f(T') \geq f(T) - 1$. If $i \geq 1$ then by (3) and the fact that no other color on z_i is used on T' we have $f(T') \geq f(T) - 1$. This completes the proof. □

3 Proof of Theorem 5

The proof of Theorem 5 follows by a sequence of applications of Lemma 8. Since k is fixed, we may again set $f(T) = f^k(T)$ for simplicity of notation.

Proof. The theorem is trivially true for $\Delta = 1$ so we may assume $\Delta \geq 2$, and hence $k \geq 1$. If $\chi'(G) \leq \Delta + k$ then the conclusion of the theorem holds so we may assume on the contrary that $\Delta + k \leq \chi'(G) - 1$. Let ϕ be a maximal $(\Delta + k)$-coloring of G. Since $\chi'(G) > \Delta + k$, there is an uncolored edge e_0 with vertices p_0 and p_1.

For the proof of Theorem 5 we provide a construction consisting of a series of steps. We begin with the partial coloring $\psi_1 = \phi$. At each step $i \geq 2$ an application of Lemma 8 is used to construct a new maximal $(\Delta+k)$-coloring ψ_i with e_0 uncolored and a new Tashkinov tree T_i with $|T_i| \geq 1 + (1+\varepsilon)^i$, where $|T_i|$ is odd.

Step 1. Set $\psi_1 = \phi$ and let p_2 be a vertex joined to p_1 by an edge whose color α is missing at p_0. Then $\{p_0, p_1, p_2\}$ forms a Tashkinov tree T_1 with respect to ψ_1, and $f(T_1) \geq k$ because there were at most $\Delta - 1$ colored edges incident to p_0. Note that $|T_1| = 3 \geq 1 + (1+\varepsilon)$, since $\varepsilon < 1$.

Step i. Suppose that the Tashkinov tree T_{i-1} and coloring ψ_{i-1} have been defined for some $2 \leq i \leq k+1$, such that $f(T_{i-1}) \geq k - i + 2$, $|T_{i-1}| \geq 1 + (1+\varepsilon)^{i-1}$, and $|T_{i-1}|$ is odd. Choose a color ω such that $q = q_\omega(T_{i-1})$ is largest. Since $|T_{i-1}|$ is odd, we know by Corollary 7 that q is odd. Consider two cases:

Case 1: $q = q_\omega(T_{i-1}) \leq \varepsilon|T_{i-1}| + 1 - \varepsilon$.

Then each color occurs on at least $(|T_{i-1}| - q)/2 \geq (1-\varepsilon)(|T_{i-1}| - 1)/2$ edges of T_{i-1}. As $e_0 \in T_{i-1}$ is uncolored,

$$\|T_{i-1}\| \geq (\Delta + k)(1-\varepsilon)(|T_{i-1}| - 1)/2 + 1.$$

Therefore $S = V(T_{i-1})$ is such that

$$\rho(S) > (1-\varepsilon)(\Delta + k).$$

Moreover $|S| < \Delta/k + 1$ by Theorem 6, because at least $k|S| + 2$ colors are missing on the vertices of S. This contradicts the assumptions of Theorem 5.

Case 2: $q = q(T_{i-1}) > \varepsilon|T_{i-1}| + 1 - \varepsilon$.

As $|T_{i-1}| \geq 3$ and q is an odd integer, $q \geq 3$. Let ψ_i be the maximal coloring and T' be the Tashkinov tree given by Lemma 8. Then by that lemma $f(T') \geq k - i + 1$, and

$$|T'| \geq |T_{i-1}| + q - 1 > 1 + (|T_{i-1}| - 1)(1+\varepsilon) \geq 1 + (1+\varepsilon)^i.$$

If $|T'|$ is odd (e.g. if $|T'| = |T_i| + q - 1$) then we set $T_i = T'$. If $|T'|$ is even then choose an arbitrary color β that is used by ψ_i on an edge of T'. Then Theorem 6 implies that some edge e colored β has exactly one vertex in T'. We define T_i to be the Tashkinov tree formed by adding e to T', so that $|T_i|$ is odd and $f(T_i) = f(T') \geq k - i + 1$.

It suffices to show that eventually Case 1 occurs. Otherwise, we construct a maximal coloring ψ_{k+1} and a Tashkinov tree T_{k+1} with $|T_{k+1}| \geq 1 + (1+\varepsilon)^{k+1}$. By Theorem 6 this implies that the number of colors that are missing on the vertices of T_k is at least $k(1 + (1+\varepsilon)^{k+1}) + 2$. Then using the definition of k we derive the contradiction

$$\Delta + k > k(1 + (1+\varepsilon)^{k+1}) > k + k\Delta.$$

□

For each $0 < \epsilon < 1$ and $k = \lfloor \log_{1+\epsilon} \Delta \rfloor$, the proof of Theorem 5 shows the existence of either an edge coloring of G with $\Delta + k$ colors or a small, dense set S with $|S| \leq \Delta/k + 1$ and $\rho(S) > (1-\epsilon)(\Delta + k)$. In fact this yields a procedure for constructing one of these structures in time polynomial in $|E(G)|$. We start by greedily coloring the edges of G with colors $\{1, \ldots, \Delta + k\}$. If we get stuck before finishing then as in the proof of Theorem 5 we attempt to construct a large Tashkinov tree T. If we halt in Case 1 then we have constructed a small, dense set $S = V(T)$. Otherwise at some point in Case 2, some color is missed at distinct vertices of T. In this case, the proof of Theorem 6 (which gives a polynomial time algorithm [17], see e.g. [9] or [13]) allows us to recolor G so that there is an additional colored edge. Then we start over using this new coloring. After fewer than $|E(G)|$ restarts our procedure halts with a small, dense subset or a proper edge coloring with $\Delta + k$ colors.

References

[1] G. Chen, X. Yu, and W. Zang, Approximating the chromatic index of multigraphs, *J. Comb. Optim.* 21 (2011), 219–246.

[2] L. M. Favrholdt, M. Stiebitz and B. Toft, Graph edge coloring: Vizing's Theorem and Goldberg's Conjecture, Preprint 2006 No. 20, IMADA, The University of Southern Denmark, 91 pages.

[3] M. K. Goldberg, On multigraphs with almost-maximal chromatic class, *Diskret. Analiz.* 23 (1973) 3–7. [In Russian]

[4] M. K. Goldberg, Edge-coloring of multigraphs: recoloring technique, *J. Graph Th.* 8 (1984), 123–137.

[5] P. E. Haxell and J. M. McDonald, On characterizing Vizing's edge-coloring bound, *J. Graph Th.* 69 (2012), 160–168.

[6] T. Jensen and B. Toft, Graph Coloring Problems, Wiley-Interscience, New York, 1995.

[7] J. Kahn, Asymptotics of the chromatic index for multigraphs, *J. Combin. Theory Ser. B* 68 (1996) 233–254.

[8] H. A. Kierstead, On the chromatic index of multigraphs without large triangles, *J. Combin. Theory Ser. B* 36 (1984) 156–160.

[9] J. M. McDonald, Multigraphs with high chromatic index, PhD thesis, University of Waterloo (2009).

[10] M. Plantholt, A combined logarithmic bound on the chromatic index of multigraphs, *J. Graph Theory* 73 (2013) 239–259.

[11] D. Scheide, Graph edge coloring: Tashkinov trees and Goldberg's conjecture, *J. Combin. Theory Ser. B* 100:1 (2009) 68–96.

[12] D. Scheide, A polynomial-time $\Delta + \sqrt{\frac{\Delta-1}{2}}$-edge coloring algorithm, Preprints 2009 No. 4, IMADA, The University of Southern Denmark, 15 pages.

[13] D. Scheide, Kantenfärbungen von Multigraphen. Diploma Thesis, TU Ilmenau, Ilmenau (2007)

[14] P. D. Seymour, On multi-colorings of cubic graphs, and conjectures of Fulkerson and Tutte, *Proc. London Math. Soc.* 3 (1979) 423–460

[15] C. E. Shannon, A theorem on coloring the lines of a network, *J. Math. Phys.* 28 (1949), 148–151

[16] M. Stiebitz, D Scheide, B. Toft, L. Favrholdt, Graph Edge Coloring: Vizing's Theorem and Goldberg's Conjecture, Wiley, 2012.

[17] V. A. Tashkinov, On an algorithm to color the edges of a multigraph, *Diskret. Analiz.* *7* (2000) 72–85. [in Russian]

[18] V. G. Vizing, On an estimate of the chromatic class of a p-graph, *Discret. Analiz.* 3 (1964), 25–30